Probabilistic Representations and Approximations in Stochastic Control and Financial Mathematics

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Mémoire d'habilitation à diriger des recherches

par

Idris KHARROUBI

Représentations et approximations probabilistes en contrôle stochastique et finance mathématique

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Introduction

This document presents a synthesis of most of my research works on probability theory and mathematical finance. I choose to regroup the results among four chapters that can be gathered into two main parts.

The first part concerns the probabilistic representations and approximations of stochastic optimal control problems and is composed by the two first chapters.

Chapter 1 deals with the representation of optimal control problems by Backward Stochastic Differential Equations (BSDEs for short). We first consider some specific control problems where the allowed strategies are sequences of interventions. Namely, the considered problems are impulse control and optimal switching. In the first section, we provide a representation of Feynman-Kac type for the associated PDEs by means of constrained BSDEs with jumps. We then focus in the second section on an existing representation for optimal switching given by BSDEs with oblique reflections. We weaken the assumptions under which the existence and uniqueness of solutions for such equations hold. Unfortunately, the representation by BSDEs with oblique reflections does not cover the general case of a completely controlled diffusion. By considering a family of such BSDEs, we extend this representation for general switching problems. As we get two kinds of equations representing the same control problem, a natural question is to find a link between them. We answer this question by relating the solutions to BSDE with oblique reflections and constrained ones. In the third section we turn to the continuous time control case. We show that the representation by constrained BSDEs still holds for such problems.

In Chapter 2, we focus on the numerical approximation of optimal control problems. We first consider the approximation of optimal switching problems. In the first section, we present a discrete-time approximation result for BSDEs with oblique reflections. We then provide in the second section a discretization algorithm based on quantization methods. In the third section, we focus on the continuous time control case. Using the representation we get in the first chapter, we propose a discrete time approximation. Using the shaking coefficient method of Krylov and the switching approximation of Barles and Jacobsen, we get convergence rates improving those existing in the literature.

The second part of this manuscript focuses on the mathematical modeling of financial markets. It is composed by the two last chapters.

In Chapter 3, we present models on imperfection in financial markets. In the first section we consider an optimal liquidation problem in an illiquid market. This model allows to take

into account the costs due to the lack of liquidity. We characterize the optimal wealth and an associated optimal liquidation strategy. We then consider in the second section the problem of super-replication under investment constraints. More precisely, we look for necessary and sufficient conditions on the model to ensure that the super-replication price of a given contingent claim equalize the replication price of a modification of this contingent claim. This question is actually related to a first order viability for BSDEs. By adapting the existing result on zero order viability to our case, we derive such necessary and sufficient conditions. Finally, we study in the third section an optimal switching problem under state constraints arising from energy management.

Chapter 4 is dedicated to models with progressive enlargement of filtrations. We consider in the first section an optimal investment problem under multiple default risk. Using a decomposition approach we characterize the optimal value and an optimal investment strategy. We then use this decomposition approach in a dynamic way. This allows to solve quadratic BSDEs with jumps in the second section. Finally, in the third section, we consider a mean-variance hedging problem up to a default time arising from the valuation of insurance contracts. Using BMO-properties of BSDEs we characterize an optimal investment strategy and the optimal value.

List of articles

Published or accepted articles:

[1] I. Kharroubi, J. Ma, H. Pham and J. Zhang (2010) "Backward SDEs with constrained jumps and quasi-variational inequalities". *The Annals of Probability*, 38 (2), 794-840.

[2] R. Elie and I. Kharroubi (2010) "Probabilistic representation and approximation for coupled systems of variational inequalities". *Statistics and Probability Letters*, 80 (17-18) 1388-1396.

[3] I. Kharroubi and H. Pham (2010) "Optimal portfolio liquidation with execution cost and risk". *SIAM Journal on Financial Mathematics*, 1, 897-931.

[4] J.-F. Chassagneux, R. Elie and I. Kharroubi (2011) "A note on existence and uniqueness for solutions of multidimensional reflected BSDEs". *Electronic Communication in Probability*, 16, 120-128.

[5] I. Kharroubi (2011) "Comparison theorem for Brownian multidimensional BSDEs via jump processes". *Comptes Rendus Mathématique*, 349 (7-8), 463-468.

[6] J.-F. Chassagneux, R. Elie and I. Kharroubi (2012) "Discrete-time approximation of BSDEs with oblique reflections". *The Annals of Applied Probability*, 22 (3), 971-1007.

[7] P. Gassiat, I. Kharroubi and H. Pham (2012) "Time discretization and quantization methods for optimal multiple switching problem". *Stochastic Processes and their Applications*, 122 (5), 2019-2052.

[8] I. Kharroubi and T. Lim (2012) "Progressive enlargement of filtrations and Backward SDEs with jumps". *Journal of Theoretical Probability*, 27 (3), 683-724.

[9] Y. Jiao, I. Kharroubi and H. Pham (2013) "Optimal investment under multiple defaults risk: a BSDE-decomposition approach". *The Annals of Applied Probability*, 23 (2), 455-491.

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[13] I. Kharroubi and H. Pham (2014) "Feynman-Kac representation for Hamilton-Jacobi-Belllman IPDEs". To appear in *The Annals of Probability*.

[14] J.F. Chassagneux, R. Elie and I. Kharroubi (2014) "When terminal facelift enforces Delta constraints". To appear in *Finance and Stochastics*.

[15] I. Kharroubi N. Langrené and H. Pham (2014) "Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps". To appear in *The Annals of Applied Probability*.

[16] I. Kharroubi, N. Langrené and H. Pham (2014) "A numerical algorithm for fully nonlinear HJB equations: an approach by control randomization". To appear in *Monte Carlo Methods and Applications*.

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[17] I. Kharroubi and T. Lim (2012) "A decomposition approach for the discrete-time approximation of FBSDEs with a jump II: the quadtratic case".

`http://arxiv.org/pdf/1211.6231v1.pdf`.

[18] I. Kharroubi (2014) "Optimal Switching in Finite Horizon under State Constraints".

`http://arxiv.org/pdf/1405.3538v1.pdf`

Notations

For a vector $x \in \mathbb{R}^d$, x^i denotes its i-th component and $|x|$ its euclidean norm. We denote by $\mathbb{R}^{d\times d}$ the set of $d \times d$ real matrices and $\mathcal{S}_d$ the subset of sysmetric elements of $\mathbb{R}^{d\times d}$. For $M \in \mathbb{R}^{d\times d}$, $\mathrm{Tr}[M]$ denotes the trace of M, $M^{\cdot j}$ its j-th column and $M^{i,j}$ the i-th coefficient of $M^{\cdot j}$. The transpose of a vector or a matrix y is denoted by $y^\top$ and the product scalar $x^\top y$ between x and y is denoted by $x \cdot y$ whenever it exists. Finally, for $x \in \mathbb{R}^d$ and $k \in \{1, \ldots, d\}$, $x^{(k)}$ denote the vector of R^k composed by the k first components of x.

For $E \subset \mathbb{R}^d$, we denote by $C^{1,2}([0,T] \times E)$ the set of functions $\varphi : (t,x) \in [0,T] \times E \mapsto \varphi(t,x) \in \mathbb{R}$ that admits a continuous first order derivative $\partial_t \varphi$ w.r.t. the variable t and a continuous second order derivative $D^2\varphi$ w.r.t. the variable x.

In the sequel, we fix a probability space $(\Omega, \mathcal{T}, \mathbb{P})$. We suppose that this probability space is endowed with a d-dimensional Brownian motion W and we denote by $\mathbb{F}$ the complete and right-continuous filtration generated by W.

We also suppose that the probability space is endowed with a Poisson random measure μ defined on $\mathbb{R}_+ \times E$, with E a Borelian subset of $\mathbb{R}^q$, and independent of W. We suppose that μ admits the compensator $\int_0^\cdot \lambda(de)ds$ where λ is a finite measure defined on the Borel algebra $\mathcal{B}(E)$ of E. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ the right-continuous and complete filtration generated by W and μ.

Finally, we suppose that $(\Omega, \mathcal{T}, \mathbb{P})$ is also endowed with a finite ordered sequence of random times $(\tau^k)_{1\leq k\leq n}$:

$$\mathbb{P}\Big(\tau^1 \leq \tau^2 \leq \cdots \leq \tau^n\Big) \;=\; 1,$$

and a sequence of random marks $(\zeta^k)_{1\leq k\leq n}$ valued in E. We denote by $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ the smallest right continuous and complete filtration generated by W and for which $\tau^1, \ldots, \tau^n$ are stopping times and each random mark ζ^k is $\mathcal{H}_{\tau^k}$-measurable for all $k \in \{1, \ldots, n\}$. We denote by η the random measure associated to the sequence $(\tau^k, \zeta^k)_{1\leq k\leq n}$:

$$\eta \;=\; \sum_{k=1}^n \delta_{\tau^k, \zeta^k}$$

with δ the Dirac measure.

For a given filtration $\mathbb{J}$ defined on $(\Omega, \mathcal{T}, \mathbb{P})$, we denote by

- $S^2_{\mathbb{J}}$ (resp. $S^2_{\mathbb{J},c}$) the set of $\mathbb{J}$-adapted *càdlàg* (resp. continuous) processes Y valued in

$\mathbb{R}$ such that

$$\|Y\|_{S^2} \;:=\; \mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t|^2\Big]^{\frac{1}{2}} \;<\; \infty,$$

- $S^\infty_{\mathbb{J}}$ (resp. $S^\infty_{\mathbb{J},c}$) the set of $\mathbb{J}$-adapted *càdlàg* (reps. continuous) processes Y valued in $\mathbb{R}$ such that

$$\|Y\|_{S^\infty} \;:=\; \inf\Big\{c\in\mathbb{R} \;:\; \mathbb{P}\Big(\sup_{t\in[0,T]}|Y_t|\le c\Big) \;=\; 1\Big\} \;<\; \infty,$$

- $L^2_{\mathbb{J}}(W)$ the set of $\mathbb{J}$-predictable processes Z valued in $\mathbb{R}^d$ such that

$$\|Z\|_{L^2(W)} \;:=\; \mathbb{E}\Big[\int_0^T |Z_s|^2 ds\Big]^{\frac{1}{2}} \;<\; \infty,$$

- $K^2_{\mathbb{J}}$ (resp. $K^2_{\mathbb{J},c}$) the set of non-decreasing processes $K\in S^2_{\mathbb{J}}$ (resp. $K\in S^2_{\mathbb{J},c}$) such that $K_0=0$.

We also denote by

- $L^2(\mu)$ the set of $\mathbb{G}$-predictable processes U valued in $\mathbb{R}^d$ such that

$$\|U\|_{L^2(\mu)} \;:=\; \mathbb{E}\Big[\int_0^T\int_E |U_s(e)|^2\lambda(de)ds\Big]^{\frac{1}{2}} \;<\; \infty,$$

- $L^2(\eta)$ the set of $\mathbb{H}$-predictable processes U valued in $\mathbb{R}$ such that

$$\|U\|_{L^2(\eta)} \;:=\; \mathbb{E}\Big[\int_0^T\int_E |U_s(e)|^2\eta(de,ds)\Big]^{\frac{1}{2}} \;<\; \infty,$$

Part I

Probabilistic representations and approximations

Chapter 1

Representation of stochastic control problems

Optimal control of stochastic systems is the study of optimization problems where the underlying controlled system evolves with some uncertainty. It has numerous applications in decision making problems, especially in economics and finance. Therefore, this theory has attracted a lot of interest in the last decades. In particular, numerous papers investigate the numerical approximations of such problems. A possible approach is to use Monte Carlo methods by setting convenient probabilistic representations.

We present in this chapter our results on such representations of optimal control problems. More precisely, we consider representation by BSDEs. In Section 1.1, we focus on a specific class of control problems where the allowed strategies are sequences. For such problems we present a representation by constrained BSDEs. We then consider in Section 1.2, a class of BSDEs related to optimal switching problems for which we present the results we get. Finally, we deal in Section 1.3 with the BSDE representation of classical continuous time control problems.

The content of this chapter follows from the articles [CEK11], [EK14b], [KMPZ10] and [KP14]. For the sake of clarity, we present specific versions of results obtained in [KMPZ10] and [KP14] and we refer to these articles for the general presentation.

1.1 Sequential control and constrained BSDEs

In many decision making problems, continuous time strategies do not fit with the reality of the situation. This is the case for instance in financial markets where investors are subject to frictions which prevent them from trading too many times. An alternative to continuous time control models consists in considering that the allowed strategies are sequences. This is the case of impulse control and optimal switching problems.

In [KMPZ10], we focus on the impulse control problem that is described as follows. We consider a set of strategies $\alpha = (\nu_k, \beta_k)_{k\geq 1}$ with

- $(\nu_k)_{k\geq 0}$ a nondecreasing sequence of stopping times representing the trading times of the investor,

- β_k an $\mathcal{F}_{\nu_k}$-measurable random variable representing the level of intervention.

For such a strategy α, we consider the controlled diffusion X^α defined by

$$X_s^\alpha \;=\; X_0 + \int_0^s b(X_u^\alpha)du + \int_0^s \sigma(X_u^\alpha)dW_u + \sum_{\nu_i \le s} \gamma(X^\alpha_{\nu_i^-}, \beta_i), \tag{1.1.1}$$

We then are given a reward and terminal payoff function f and g and a cost function c. The impulse control problem consists in computing the following quantity

$$\sup_{\alpha=(\nu_i,\beta_i)_i} \mathbf{E}\Big[g(X_T^\alpha) + \int_0^T f(X_s^\alpha)ds + \sum_{\nu_i \le T} c(X^\alpha_{\nu_i^-}, \beta_i)\Big].$$

The value function associated to this impulse control problem is proved to be a solution to the quasi-variational inequality (QVI for short)

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f\ ,\ v - \mathcal{M}v\Big] \;=\; 0, \tag{1.1.2}$$

where $\mathcal{L}$ is the local second order operator defined by

$$\mathcal{L}v(t,x) \;=\; = b(x).Dv(t,x) + \frac{1}{2}\mathrm{tr}[\sigma(x)\sigma(x)^\intercal D^2 v(t,x))]\ ,$$

and $\mathcal{M}$ is the nonlocal semilinear operator

$$\mathcal{M}v(t,x) \;=\; \sup_e \big[v(t, x+\gamma(x,e)) + c(x,e)\big].$$

To construct a BSDE representation for this QVI, we follow the idea of [Bou09] which consisting in relating this PDE to a stochastic target problem with a jump part that cannot be controlled. More precisely, we replace the strategy α by a Poisson random measure μ and we denote by X the new underlying process.

Using the PDE (1.1.2), we get by a formal application of Itô's formula that the processes $Y_t = v(t, X_t)$ satisfies

$$\begin{aligned} Y_t \;&=\; g(X_T) + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s dW_s \\ &\quad - \int_t^T \int_E \big(U_s(e) - c(X_{s^-}, e)\big)\mu(ds, de), \end{aligned} \tag{1.1.3}$$

where $Z_t = \sigma^\top(X_{t^-})Dv(t, X_{t^-})$, $U_t(e) = v(t, X_{t^-} + \gamma(X_{t^-}, e)) - v(t, X_{t^-}) + c(X_{t^-}, e)$, and $K_t = \int_0^t (-\frac{\partial v}{\partial t} - \mathcal{L}v - f)(s, X_s)ds$. Since v satisfies (1.1.2), we see that K is a continuous (hence predictable), nondecreasing process, and U satisfies the constraint:

$$-U_t(e) \;\ge\; 0. \tag{1.1.4}$$

The idea is then to view (1.1.3) and (1.1.4) as a BSDE with jump constraints, and we expect to retrieve $v(t, X_t)$ by solving the "minimal" solution $(Y, Z, U, K) \in S^2_{\mathbb{G}} \times L^2_{\mathbb{G}}(W) \times L^2(\mu) \times K^2_{\mathbb{G}}$ to this constrained BSDE in the following sense: for any other solution $(Y', Z', U', K') \in S^2_{\mathbb{G}} \times L^2_{\mathbb{G}}(W) \times L^2(\mu) \times K^2_{\mathbb{G}}$, we have $Y \le Y'$.

To the best of our knowledge, the first study of constrained BSDEs appear in [BH98b, BH98a] where constrained BSDEs are solved by weak convergence methods. Then, [CKS98] uses a convex duality approach to study constrained BSDEs. The general case is treated by [Pen99].

In the mixed Brownian-Poison case, the same approach has been used to derive existence and uniqueness of a solution in [Roy06] for the study of decomposition theorems for nonlinear expectations. Here, we also follows Peng's approach to study our constrained BSDE and under classical assumptions on the coefficients, we get the existence of a minimal solution as soon as a solution exists. The proof consists introducing a sequence of BSDEs where the jump part is penalized ensure that it satisfy the constraint asymptotically. Then using weak convergence methods, we show that this sequence converges to the minimal solution of the constrained BSDE.

We then relate this minimal solution (Y, Z, U, K) to an extension of PDE (1.1.2) by allowing the function f and the nonlocal term $\mathcal{M}$ to depend on the first order derivative. More precisely, we get from the Markov property of our model a deterministic function v such that

$$Y_t \quad = \quad v(t, X_t) \ , \quad t \in [0, T] \ .$$

We then show that the function v is a viscosity solution to a general version of the QVI (1.1.2). The proof relies on a penalization argument which allows to work in the classical framework. Then using relations between BSDEs with jumps and Integral PDE (see e.g. [BBP97]), and the stability results for PDEs (see [Bar94]), we get the viscosity property.

In [EK09], we extend this representation to another sequential control problem called optimal switching. In this case, the set of admissible strategies corresponds to the sequences $\alpha = (\nu_k, \beta_k)_{k\geq 1}$ where $(\nu_k)_k$ is a nondecreasing sequence of $\mathbb{F}$-stopping times and β_k is an $\mathcal{F}_{\tau_k}$-measurable random variable valued in the finite set $\{1, \ldots, m\}$. For such a strategy, the controlled process X^α is defined by

$$X_t^\alpha \quad = \quad X_0 + \int_0^t b(X_s^\alpha) ds + \int_0^t \sigma(X_s^\alpha) dW_s \ , \quad t \geq 0. \tag{1.1.5}$$

The optimal switching problem is then given by

$$\sup_{\alpha=(\nu_i,\beta_i)_i} \mathbf{E}\Big[g(X_T^\alpha, \alpha_T) + \int_0^T f(X_s^\alpha, \alpha_s) ds + \sum_{\nu_i \leq T} c(X_{\nu_i}^\alpha, , \beta_{i-1}, \beta_i) \Big]. \tag{1.1.6}$$

where f and g are given and represent a reward and terminal payoff function, c is a cost function and $\alpha_s = \sum_{k\geq 0} \beta_k \mathbb{1}_{[\nu_k, \nu_{k+1})}(s)$ for $s \in [0, T]$. This problem can be seen as a mix between the classical control problems and impulse control since the strategy α influences the drift and the volatility of the controlled process and also induces jumps on the gain via the cost function c. Due to the presence of cost for the change of regime, the associated value function depends on the initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$ of the diffusion but also on the initial regime $i \in \{1, \ldots, m\}$. This leads to a multidimensional dynamic programming

equation of the form

$$\min\Big[-\frac{\partial v}{\partial t}-\mathcal{L}v-f, v-\mathcal{M}v\Big]=0 \quad \text{on} \quad [0,T)\times\bar{\mathbb{R}}^d\times\{1,\dots,m\}, \tag{1.1.7}$$

$$\min\Big[v-g, v-\mathcal{M}v\Big]=0 \quad \text{on} \quad \{T\}\times\bar{\mathbb{R}}^d\times\{1,\dots,m\}. \tag{1.1.8}$$

where $\mathcal{L}$ is the second order local operator defined by

$$\mathcal{L}v(t,x,i) \;=\; \Big(b.Dv+\frac{1}{2}\mathrm{tr}[\sigma\sigma^\intercal D^2v)]\Big)(t,x,i)$$

and $\mathcal{M}$ is the nonlocal operator defined by

$$\mathcal{M}v(t,x,i) \;=\; \max_{\substack{1\le j\le m\\ j\neq i}}\big[v(t,x,j)-c(x,i,j)\big]$$

for all $t\in[0,T]$, $x\in\mathbb{R}^d$ and $i\in\{1,\dots,m\}$. To set a representation by constrained BSDE, we first suppose that the random measure μ is defined on $\mathbb{R}_+\times\{1,\dots,m\}$. Then, we replace the switching strategy α by the random measure μ. This leads to consider the process I defined by

$$I_t \;=\; i+\int_0^t\int_{\{1,\dots,m\}}(a-I_{s-})\mu(da,ds)\;,\quad t\ge 0\;,$$

instead of the process $(\alpha_s)_s$. The associated constrained BSDE takes the following form: find the minimal solution $(Y,Z,U,K)\in S^2_{\mathbb{G}}\times L^2_{\mathbb{G}}(W)\times L^2(\mu)\times K^2_{\mathbb{G}}$ to

$$\begin{aligned} Y_t \;=\;& g(X_T,I_T)+\int_t^T f(X_s,I_s)ds+K_T-K_t-\int_t^T Z_s dW_s \\ &-\int_t^T\int_E\big(U_s(e)-c(X_s,I_{s^-},e)\big)\mu(ds,de), \end{aligned} \tag{1.1.9}$$

From the Makov feature of our model, we get the existence of a function v that links Y to X: $Y_t=v(t,X_t,I_t)$. Using a penalization argument we obtain that the function v is a viscosity solution to (1.1.7)-(1.1.8).

1.2 BSDEs with oblique reflexions

We focus in this section on another class of BSDEs that provides a representation for the values of optimal switching problems. These BSDEs are multidimensional and have oblique reflections. More precisely, they take the following form : find $(Y,Z,K)\in[S^2_{\mathbb{F},c}\times L^2_{\mathbb{F}}(W)\times K^2_{\mathbb{F},c}]^d$ such that

$$\begin{cases} Y^i_t=\xi^i+\int_t^T f^i(s,Y_s,Z^i_s)ds-\int_t^T Z^i_s dW_s+K^i_T-K^i_t\;,\quad 0\le t\le T\;,\\ Y^i_t\ge\max_{j\in\mathcal{I}}\{Y^j_t-C^{ij}_t\}\;,\quad 0\le t\le T\;,\\ \int_0^T[Y^i_t-\max_{j\in\mathcal{I}}\{Y^j_t-C^{ij}_t\}]dK^i_t=0\;,\;0\le i\le d\;, \end{cases} \tag{1.2.10}$$

where

- $\xi = (\xi^1, \dots, \xi^d)$ is an $\mathcal{F}_T$-mesurable random variable such that $\mathbb{E}[|\xi|^2] < +\infty$,
- $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ is a progressive function called driver and satisfying the integrability condition

$$\mathbb{E}\Big[\int_0^T |f(t,0,0)|^2 dt\Big] < +\infty,$$

and the Lipschitz condition: there exists a constant $L > 0$ such that

$$|f(t,y,z) - f(t,y',z')| \quad \leq \quad L\big(|y-y'| + |z-z'|\big)$$

for all $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times d}$ et $t \in [0,T]$,

- $C^{i,j}$, $i,j \in \{1,\dots,d\}$, are progressive processes.

As far as we know, these BSDEs appear for the first time in [HJ07] in the particular case of two regimes, *i.e.* $d = 2$, and a driver not depending on (Y,Z). The general case is then studied by Hu and Tang in [HT10] where the existence and uniqueness of solutions is proved in the case of constant cost processes and a diagonal driver, *i.e.* each line of the driver depends only on the corresponding line of the unknown (Y,Z):

$$f^i(y^1,\dots,y^d,z^i) \quad = \quad f(y^i,z^i) \quad \text{pour tout } y, z^i \in \mathbb{R}^d \text{ et } \ i \in \{1,\dots,d\}\ .$$

Then the result is generalized in [HZ10] where the authors allow the generator to depend on all the component of Y but with the following monotonicity condition:

$$\begin{array}{c}\text{the function } y^j \mapsto f(y^1,\dots,y^{j-1},y^j,y^{j+1},\dots,y^d,z^i) \text{ is increasing} \\ \text{for all } (y^1,\dots,y^{j-1},y^{j+1},\dots,y^d) \in \mathbb{R}^{d-1},\ z^i \in \mathbb{R}^d \text{ et } i \in \{1,\dots,d\}.\end{array}$$

We notice that for such BSDEs with general driver, we cannot use a fixed point procedure to prove existence and uniqueness of solutions. The main reason is that the oblique projection operator has a Lipschitz constant greater than 1. Therefore usual arguments to prove that the classical decoupling scheme is contracting do not work anymore.

In [CEK11], we use a new approach that apply a fixed point procedure for Lipschitz drivers. This is done via the introduction of a convenient one dimensional dominating BSDE and the use of a standard comparison theorem. In particular our result get rid of the previous monotonicity assumption.

As we presented above, BSDEs with oblique reflections provide representation for value functions associated to optimal switching problems under the form (1.1.6). More precisely, [HT10] relate such a value function to a weak solution of a BSDE with oblique reflections. To get such a result the authors specify the controlled dynamics (1.1.5) by assuming firstly that the volatility σ is not controlled and secondly that b is of the form $b = \sigma\mu$ where μ is a bounded function from $\mathbb{R}^d$ to $\mathbb{R}^d$.

They consider the following BSDE with oblique reflections: find $(Y^i, Z^i, K^i)_{1\leq i\leq m} \in (S^2_{\mathbb{F},c} \times L^2_{\mathbb{F}}(W) \times K^2_{\mathbb{F},c})^m$ such that

$$\left\{\begin{array}{l} Y^i_t = g(X_T,i) + \int_t^T \big(f(X_s,i) - Z^i_s\mu(X_s,i)\big)ds - \int_t^T Z^i_s dW_s + K^i_T - K^i_t, \\ Y^i_t \geq \max_{j\in\mathcal{I}}\{Y^j_t - c_{i,j}(t)\}, \qquad 0 \leq t \leq T\ , \\ \int_0^T [Y^i_t - \max_{j\in\mathcal{I}}\{Y^j_t - c_{i,j}(t)\}]dK^i_t \ = \ 0, \quad 1 \leq i \leq m\ , \end{array}\right. \tag{1.2.11}$$

where X is the diffusion defined by

$$X_t \;=\; X_0 + \int_0^t \sigma(X_s) dW_s\,, \qquad \forall t \geq 0. \tag{1.2.12}$$

Using a Girsanov transform argument, The authors prove under Lipschitz assumptions on the coefficients that there exists a weak solution $({}^*\mathbb{P}, {}^*W, {}^*X, {}^*Y, {}^*Z)$ to the decoupled FBSDE (1.2.12)-(1.2.11) such that

$${}^*Y_0^{i_0} \;=\; \sup_{\alpha=(\nu_k,\beta_k)_k \,:\, \beta_0=i_0} \mathbf{E}\Big[g(X_T^\alpha,\alpha_T) + \int_0^T f(X_s^\alpha,\alpha_s)ds + \sum_{\nu_k\leq T} c(X^\alpha_{\nu_k},,\beta_{k-1},\beta_k)\Big]\,,$$

for $i_0 \in \{1,\dots,m\}$.

In [EK14b], we consider the general case: we allow the volatility function σ to be controlled by the switching strategy, and we consider a general drift function. In the spirit of [DHH10], we introduce a family of reflected BSDE depending on couple (ν,ξ) with ν an $\mathbb{F}$-stopping time valued in $[0,T]$ and ξ an $\mathcal{F}_\nu$-measurable random variable taking values in $\mathbb{R}^d$. Such a couple represents the initial condition of the underlying diffusion. For each initial condition, we consider a BSDE with oblique reflection driven by a diffusion with initial condition (ν,ξ). Such a family is large enough to allow for the consideration of change of regimes in the diffusion by passing from a reflected BSDE to another one of this family. To do this we prove a flot property of this family of BSDE w.r.t. the initial condition (ν,ξ). Using the reflection times of a well chosen sequence of BSDE, we derive an optimal strategy and the optimal value, which extends the existing representation results.

As detailed in the previous section, optimal switching problems can also be represented by another kind of equations which are one dimensional constrained BSDEs with jumps. Therefore one can wonder if there is a link between constrained and obliquely reflected BSDEs. We address this issue in [EK14a] where we show that BSDEs with oblique reflections can be seen as a particular case of constrained BSDEs with jumps. To prove such a result, we face two main difficulties. Firstly these two kind of BSDEs do not have the same structure since the first ones are multidimensional and only driven by W and the second ones are one dimensional and driven by W and also μ. The second difficulty comes from the minimality that is imposed to the constrained solutions. Indeed, we do not have any analytical characterization, as e.g. in the reflected case with the skhorokhod condition, and we do not precisely know the behavior of the nondecreasing component K.

We address the first difficulty by constructing a jump process from the solution of the BSDE with oblique reflections (1.2.10) as follows. We suppose here that the random measure μ is defined on $\mathbb{R}_+ \times \{1,\dots,d\}$ and has a finite intensity λ. Denote by I the random process taking the value charged by the random measure μ *i.e.* if $\mu = \sum_{k\geq 1} \delta_{\nu_k,\gamma_k}$ then $I_t = \gamma_k$ for $t \in [\nu_k, \nu_{k+1})$. We then consider the solution (Y,Z,K) to the BSDE with oblique reflections (1.2.10) and define the processes $\tilde{Y}$, $\tilde{Z}$ and $\tilde{U}$ by

$$\tilde{Y}_t = Y_t^{I_t}\,, \quad \tilde{Z}_t = Z_t^{I_{t^-}} \;\text{ and }\; \quad \tilde{U}_t(i) = Y_t^i - Y_{t^-}^{I_{t-}}\,, \tag{1.2.13}$$

for $t \in [0,T]$ and $i \in \{1,\dots,m\}$. Such a triple seems to be a good candidate to be minimal solution to a constrained BSDE. We then define the coefficients

$$\begin{aligned} \zeta &= \xi^{I_T}, && (1.2.14)\\ g(t,y,z,u) &= f^{I_{t^-}}(t,(y+u^i\mathbf{1}_{I_{t^-}\neq i})_{i\in\mathcal{I}},z), \quad (t,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m, && (1.2.15)\\ h_i(t,y,z,v) &= v - C^{i,I_{t^-}} \qquad (i,t,y,z,v)\in\mathcal{I}\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}, && (1.2.16) \end{aligned}$$

and consider the constrained BSDE with jumps: find the minimal solution $(Y,Z,U,K)\in S^2_{\mathbb{G}}\times L^2_{\mathbb{G}}(W)\times L^2(\tilde{\mu})\times K^2_{\mathbb{G}}$ to

$$\begin{aligned} Y_t &= \zeta + \int_t^T g(s,Y_s,Z_s,U_s)ds + K_T - K_t \\ &\quad - \int_t^T Z_s.dW_s - \int_t^T\int_{\{1,\dots,m\}} U_s(i)\mu(ds,di), \end{aligned} \tag{1.2.17}$$

for all $t\in[0,T]$ and the constraint

$$h_i(t,Y_{t^-},Z_t,U_t(i)) \geq 0, \quad \text{almost everywhere on } \Omega\times[0,T]\times\{1,\dots,m\}. \tag{1.2.18}$$

The link between constrained and obliquely reflected BSDEs is given by the following result.

Theorem 1.2.1. *Suppose that assumptions ensuring existence and uniqueness of a solutions to* (1.2.10) *are satisfied and let* (Y,Z,K) *be solution to* (1.2.10)*. Then there exists a process* $\tilde{K}\in K^2_{\mathbb{G}}$ *such that* $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K})$ *is the minimal solution to the constrained BSDE with jumps* (1.2.17)-(1.2.18) *with coefficients given by* (1.2.14)-(1.2.15)-(1.2.16)*.*

To deal with the second difficulty, *i.e.* the minimality condition, we use the characterization of the minimal solution as the limit of penalized BSDEs. We first show that the link (1.2.13) remains true for the penalized BSDE associated to constrained and obliquely reflected BSDEs. Then the convergencence gives the minimality of the solution $\tilde{Y}$.

1.3 HJB equations

We turn in this section to the case of continuous time control. More precisely, we consider the following HJB equation:

$$\begin{aligned} \frac{\partial v}{\partial t} + \sup_{a\in A}\big[b(.,a).Dv + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(.,a)D^2v) + f(.,a)\big] &= 0, \ \text{ on } [0,T)\times\mathbb{R}^d, \quad (1.3.19)\\ v(T,.) &= g, \ \text{ on } \mathbb{R}^d, \end{aligned}$$

where A is a subset of $\mathbb{R}^q$. It is well-known (see e.g. [Pha10]) that such nonlinear PDE is the dynamic programming equation associated to the stochastic control problem

$$\sup_\alpha \mathbb{E}\Big[\int_0^T f(X^\alpha_s,\alpha_s)ds + g(X^\alpha_T)\Big],$$

where X^α is the solution to the controlled SDE:

$$dX_s^\alpha \;=\; b(X_s^\alpha,\alpha_s)ds+\sigma(X_s^\alpha,\alpha_s)dW_s,$$

given a predictable control process α valued in A. The study of Feynman-Kac representations for HJB PDEs is an important subject since it opens the way to new numerical approximation methods. Therefore, it has attracted a lot of interest and generated an important literature over the recent years. One can actually decompose this problem into two main cases.

In the first one, we suppose that the volatility function is not controlled. Assume moreover, that $\sigma\sigma^\top$ is of full rank, then the HJB equation (1.3.19) reduces into a semi-linear PDE:

$$\frac{\partial v}{\partial t}+\frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x)D^2v)+F(x,\sigma^\intercal Dv) \;=\; 0, \tag{1.3.20}$$

where F is a function depending on f, b and σ. In this case, we know from the seminal works by Pardoux and Peng [PP90, PP92], that the (viscosity) solution v to the semi-linear PDE (1.3.20) is connected to the BSDE:

$$Y_t \;=\; g(X_T^0)+\int_t^T F(X_s^0,Z_s)ds-\int_t^T Z_s dW_s, \quad t\le T, \tag{1.3.21}$$

through the relation $Y_t=v(t,X_t^0)$, with a forward diffusion process

$$dX_s^0 \;=\; \sigma(X_s^0)dW_s.$$

The second case with controlled diffusion coefficient $\sigma(x,a)$ associated to fully nonlinear PDE is challenging and led to recent theoretical advances. In [Pen05], the author introduces the concepts of G-Brownian motion B, and G-expectation $\mathbb{E}_G$ to represent the PDE associated to optimal investment under volatility uncertainty. Moreover, G-expectation is closely related to second order BSDE studied in [STZ12], namely the process $Y_t=v(t,B_t)$ satisfies a 2BSDE, which is formulated under a nondominated family of singular probability measures. This gives a nice theory and representation for nonlinear PDE, but it requires a non degeneracy assumption on the diffusion coefficient, and does not cover general HJB equation (i.e. control both on drift and diffusion arising for instance in portfolio optimization).

We provide in [KP14] an alternative BSDE representation including general HJB equation, formulated under a single probability measure (thus avoiding nondominated singular measures), and under which the forward process can be simulated. The idea, used in [KMPZ10] for quasi variational inequalities arising in impulse control problems, is the following. Suppose that the random measure μ is defined on $[0,+\infty)\times A$. Let us then consider the BSDE with jumps

$$Y_t \;=\; g(X_T)+\int_t^T f(X_s,I_s)ds-\int_t^T Z_s dW_s-\int_t^T\int_A U_s(a)\tilde{\mu}(ds,da), \tag{1.3.22}$$

with $\tilde{\mu}$ is the compensated measure of μ and X is the forward regime switching diffusion process defined by

$$dX_s \;=\; b(X_s, I_s)ds + \sigma(X_s, I_s)dW_s,$$

where I is the pure jump process valued in A and taking the value charged by the random measure μ. The linear BSDE (1.3.22) is the Feynman-Kac formula for the linear integro-partial differential equation (IPDE):

$$\frac{\partial v}{\partial t} + b(x,a).Dv + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D^2v) \tag{1.3.23}$$
$$+ \int_A (v(t,x,a') - v(t,x,a))\lambda_A(da') + f(x,a) \;=\; 0, \qquad (t,x,a) \in [0,T) \times \mathbb{R}^d \times A,$$
$$v(T,x,a) \;=\; g(x), \qquad (x,a) \in \mathbb{R}^d \times A, \tag{1.3.24}$$

through the relation: $Y_t = v(t, X_t, I_t)$. Moreover, under some additional assumptions, we can also identify the jump component U of BSDE (1.3.22) as follows:

$$U_t(a) \;=\; v(t, X_t, a) - v(t, X_t, I_{t^-}) \;, \quad (t,x,a) \in [0,T] \times \mathbb{R}^d \times A.$$

In particular, if we impose the constraint

$$U \;\leq\; 0 \tag{1.3.25}$$

we obtain that the function v does not depend on the variable $a \in A$. In particular, PDE (1.3.26) becomes

$$\frac{\partial v}{\partial t} + b(x,a).Dv + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D^2v) + f(x,a) \;=\; 0$$

for all $a \in A$. Taking the supremum over a, we get the HJB PDE. Our main result can be stated as follows.

Theorem 1.3.2. *Suppose that the functions b, σ, f and g are Lipschitz continuous. Then constrained BSDE* (1.3.22)-(1.3.25) *admits a unique minimal solution. Moreover, there exists a function $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$Y_t \;=\; v(t, X_t) \;, \quad t \in [0,T]$$

and v is the unique viscosity solution to HJB PDE (1.3.19) *in the class of function with polynomial growth.*

The main task is to derive the key property that v does not actually depend on a, as a consequence of (1.3.25). This issue is a novelty with respect to the sequential framework of [KMPZ10] where there is a positive cost at each change of the regime I, while the cost is identically degenerate to zero in this case. The proof relies on sharp arguments from viscosity solutions, inf-convolution and semiconcave approximation, as we don't know a priori any continuity results on v.

1.4 Perspective

We aim at extending the BSDE representation of HJB equations presented above to Isaac-Hamilton-Jacobi-Bellman (IHJB for short) PDEs. These PDEs correspond to the dynamic programming equations of Markov stochastic differential games (see e.g. [BCR04, BL08, BCQ12]). To this end we first need to generalize the notion of constrained BSDE. We then propose to consider doubly reflected BSDEs by imposing two constraints with two nondecreasing processes operating with an opposite sign. The main chalenge is then to well define a notion of solution for such doubly constrained BSDEs. In particular, we need to replace the minimality by a maxi-minimality *i.e.* a maximal solution in a set of minimal solutions. We then hope to use such a definition of solutions to take into account the nonlinearity of $\min - \max$-type appearing in IHJB PDEs.

Chapter 2

Probabilistic approximations

In numerical analysis, possible alternatives to deterministic algorithms are probabilistic methods. Such alternatives provide new approximations that can sometimes be simpler to implement and even improve the convergence rates of classical methods.

We present in this chapter our results on probabilistic numerical approximations for optimal control problems. Section 2.1 presents a discrete time approximation for BSDEs with oblique reflections which are related to optimal switching problems. In Section 2.2, we consider a discrete time scheme for system of quasi-variational inequalities related to optimal switching problems. Finally, Section 2.3 investigates the continuous time control case. Using the jump constrained BSDE representation presented in the previous chapter, we propose a discrete-time scheme for which we present the convergence results we get.

The content of this chapter follows from the articles [CEK12], [GKP12], [KLP14b] and [KLP14a].

2.1 Discrete-time approximation of BSDEs with oblique reflexions

A major interest of BSDE representations for stochastic optimal control problems or partial differential equations comes from the ability to use them to derive efficient numerical approximation schemes. Such approximations were introduced in [BT08] and [Zha04] and consist in a backward discrete-time computation which involves conditional expectations that can be computed by different methods (regressions, Malliavin calculus or quantization). These schemes have been extended to several cases allowing to consider the addition of a jump component (see [BE08]), reflected solutions (see [BC08, Cha09]) or quadratic drivers (see [Ric11]). However, there still remain important cases that are not covered by the literature.

In [CEK12], we consider a case related to optimal switching. It concerns the discrete time approximation of Markov BSDEs with oblique reflections which take the following

form

$$\begin{cases} Y_t^i = g^i(X_T) + \int_t^T f^i(X_s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dW_s + K_T^i - K_t^i, \quad 0 \le t \le T\,, \\ Y_t^i \ge \max_{1\le j\le d}\{Y_t^j - c^{ij}(X_t)\}, \quad 0 \le t \le T\,, \\ \int_0^T [Y_t^i - \max_{1\le j\le d}\{Y_t^j - c^{ij}(X_t)\}]dK_t^i = 0\,, \quad 1 \le i \le d\,, \end{cases} \tag{2.1.1}$$

where X is the solution of a forward Stochastic Differential Equation (SDE). As in [MZ05, BC08, Cha09], we first introduce a discretely reflected version of (2.1.1), where the reflection occurs only on a deterministic grid $\Re = \{r_0 := 0, \dots, r_\kappa := T\}$ of $[0, T]$:

$$Y_T^\Re = \widetilde{Y}_T^\Re := g(X_T) \in \mathcal{Q}(X_T),$$

and

$$\begin{cases} \widetilde{Y}_t^\Re &= Y_{r_{j+1}}^\Re + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u^\Re, Z_u^\Re)du - \int_t^{r_{j+1}} Z_u^\Re dW_u\ , \\ Y_t^\Re &= \widetilde{Y}_t^\Re \mathbb{1}_{t\notin\Re} + \mathcal{P}(X_t, \widetilde{Y}_t^\Re)\mathbb{1}_{t\in\Re}, \end{cases} \tag{2.1.2}$$

for $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$, where $\mathcal{P}(X_t, .)$ is the oblique projection operator on $\mathcal{Q}(X_t)$, for $t \le T$. The set valued function $\mathcal{Q}$ being defined by

$$\mathcal{Q}(x) \;=\; \big\{y \in \mathbb{R}^d \;:\; y^i \;\ge\; \max_{1\le j\le d}\{y^j - c^{ij}(x)\} \text{ for all } i \in \{1, \dots, d\}\big\}\,.$$

Extending the approach of Hu and Tang [HT10], we observe that the solution to (2.1.2) interprets as the value process of a one-dimensional optimal BSDE switching problem with switching times belonging to $\Re$. This allows to prove a key stability result for this equation. We control the distance between $(Y^\Re, Z^\Re)$ and (Y, Z) in terms of the mesh of the reflection grid. Due to the obliqueness of the reflections, the direct argumentation of [BC08, Cha09] does not apply. Using the reinterpretation in terms of switching BSDEs, we first prove that $Y^\Re$ approaches Y on the grid points with a convergence rate of order $\frac{1}{2} - \varepsilon$, $\varepsilon > 0$ uniformly on $\Re$, whenever the cost function is Lipschitz and f is bounded in z. Imposing more regularity on the cost functions, we control the convergence rate of $(Y^\Re, Z^\Re)$ to (Y, Z) uniformly on the interval $[0, T]$.

We then consider an Euler type approximation scheme associated to the BSDE (2.1.2) defined on $\pi = \{t_0, \dots, t_n\}$ by $Y_T^{\Re,\pi} := g(X_T^\pi)$ and, for $i \in \{n-1, \dots, 0\}$,

$$\begin{cases} \bar{Z}_{t_i}^{\Re,\pi} &:= (t_{i+1} - t_i)^{-1}\mathbb{E}\left[Y_{t_{i+1}}^{\Re,\pi}(W_{t_{i+1}} - W_{t_i})' \mid \mathcal{F}_{t_i}\right]\,, \\ \widetilde{Y}_{t_i}^{\Re,\pi} &:= \mathbb{E}\left[Y_{t_{i+1}}^{\Re,\pi} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i)f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^{\Re,\pi}, \bar{Z}_{t_i}^{\Re,\pi})\,, \\ Y_{t_i}^{\Re,\pi} &:= \widetilde{Y}_{t_i}^{\Re,\pi}\mathbb{1}_{t_i\notin\Re} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_{t_i}^{\Re,\pi})\mathbb{1}_{t_i\in\Re}\,, \end{cases} \tag{2.1.3}$$

where X^π is the Euler scheme associated to X. It is now well known, see e.g. [BT08, Zha04], that the convergence rate of the scheme (2.1.3) to the solution of (2.1.2) is controlled by the regularity of (Y, Z) through the quantities

$$\mathbb{E}\left[\sum_{i<n}\int_{t_i}^{t_{i+1}} |Y_t^\Re - Y_{t_i}^\Re|^2 dt\right] \quad \text{and} \quad \mathbb{E}\left[\sum_{i<n}\int_{t_i}^{t_{i+1}} |Z_t^\Re - \bar{Z}_{t_i}^\Re|^2 dt\right]\,,$$

with $\bar{Z}_{t_i}^\Re = \frac{1}{t_{i+1}-t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_t^\Re dt \mid \mathcal{F}_{t_i}\right]$, for $i \le n$.

Using classical Malliavin differentiation tools, we prove a representation for $Z^{\Re}$, extending the results of [BC08, Cha09] to the system of discretely reflected BSDEs (2.1.2). We deduce the expected regularity results on $(Y^{\Re}, Z^{\Re})$ and obtain in a very general setting the convergence of (2.1.3) to (2.1.2). However, due to the obliqueness of the reflections, the projection operator $\mathcal{P}(X,.)$ is Lipschitz continuous with a Lipschitz constant $L_{\mathcal{P}} > 1$, leading to a convergence rate including the term $|L_{\mathcal{P}}|^{\kappa}|\pi|^{1/4}$, where we recall that κ is the number of points in the reflection grid $\Re$. Since $L_{\mathcal{P}} > 1$, this lead to a poor logarithmic convergence rate for the approximation of (2.1.1). In the particular case where f does not depend on z, we are able to get rid of the $|L_{\mathcal{P}}|^{\kappa}$ term and get polynomial convergence rates. We use an innovative approach relying on comparison results to get a control of the involved quantities.

Combining the previous estimates, we deduce the convergence of the discrete time scheme (2.1.3) to the solution of (2.1.1) with a convergence rate of order $\frac{1}{2} - \varepsilon$, $\varepsilon > 0$, on the grid points, whenever $\Re = \pi$ and f does not depend on Z. Whenever the cost functions are constant, all the previous estimates hold true with $\varepsilon = 0$. Let us emphasize that all these results are obtained without any assumption on the non-degeneracy of the volatility σ.

2.2 Discretization and quantization methods for optimal switching problems

In the previous section, we consider the discrete time approximation of BSDEs with oblique reflections. These equations are related to optimal switching problem and thereby provide an approximation procedure for optimal switching problems. Unfortunately, the general case of controlled diffusion is not covered by BSDEs with oblique reflections and a natural question is to set such a procedure for switching problems with controlled diffusion.

We address this issue in [GKP12] where we consider the discrete time approximation of systems of variational inequalities with inter-connected obstacles related to optimal switching and taking the form:

$$\begin{aligned} \min\Big[-\frac{\partial v_i}{\partial t} - b(x,i).Dv_i - \frac{1}{2}\mathrm{tr}(\sigma(x,i)\sigma(x,i)^{\intercal}D^2 v_i) - f(x,i)\,, \qquad (2.2.4)\\ v_i - \max_{j\neq i}(v_j - c(x,i,j))\Big] \;=\; 0 \quad \text{on } [0,T)\times\mathbb{R}^d, \end{aligned}$$

together with the terminal condition $v_i(T,x) = g(x,i)$, for any $i = 1,\dots,q$. These equations can be solved by analytical methods (finite differences, finite elements, ...), but are known to require heavy computations, especially in high dimension.

We propose probabilistic numerical methods based on dynamic programming and optimal quantization methods combined with a suitable time discretization procedure for computing the solution to optimal multiple switching problem. We first study a time discretization of the optimal switching problem by considering an Euler-type scheme with step $h = T/m$ for the regime-dependent state process (X_t) controlled by the switching strategy α:

$$\bar{X}_{t_{k+1}} \;=\; \bar{X}_{t_k} + b(\bar{X}_{t_k},\alpha_{t_k})h + \sigma(\bar{X}_{t_k},\alpha_{t_k})\sqrt{h}\,\vartheta_{k+1},\;\; t_k = kh,\; k = 0,\dots,m, \qquad (2.2.5)$$

where ϑ_k, $k = 1, \ldots, m$, are iid, and $\mathcal{N}(0, I_d)$-distributed. We then introduce the optimal switching problem for the discrete-time process $(\bar{X}_{t_k})$ controlled by switching strategies with stopping times valued in the discrete time grid $\{t_k, k = 0, \ldots, m\}$. The convergence of this discrete-time problem is analyzed, and we prove that the error is in general of order $(h \log(1/h))^{\frac{1}{2}}$, and of order $h^{\frac{1}{2}}$,as for optimal stopping problems, when the switching costs $c(x, i, j) \equiv c(i, j)$ do not depend on the state process. Arguments of the proof rely on a regularity result of the controlled diffusion with respect to the switching strategy, and moment estimates on the number of switches. This improves and extends the convergence rate result in [CEK12] derived in the case where X is regime-independent.

Next, we propose approximation schemes by quantization for computing explicitly the solution to the discrete-time optimal switching problem. Since the controlled Markov chain $(\bar{X}_{t_k})_k$ cannot be directly quantized as in standard optimal stopping problems, we adopt a Markovian quantization approach in the spirit of [PPP04], by considering an optimal quantization of the Gaussian random vector ϑ_{k+1} arising in the Euler scheme (2.2.5). A quantization tree algorithm is then designed for computing the approximating value function, and we provide error estimates in terms of the quantization errors $\|\vartheta_k - \hat{\vartheta}_k\|_p$ and state space grid parameters. Alternatively, in the case of regime-independent state process, we propose a quantization algorithm in the vein of [BP03] based on marginal quantization of the uncontrolled Markov chain $(\bar{X}_{t_k})_k$ and we establish L^p-error estimates in terms of quantization errors $\|\bar{X}_{t_k} - \hat{X}_k\|_p$.

2.3 Approximation of HJB equations by control randomization

In the two previous sections, we have considered sequential optimization problems. Since we have presented in the first chapter a BSDE representation for HJB equations, one can wonder if such a representation can be used to derive an approximation scheme.

We address this question in [KLP14b] and [KLP14a]. We consider fully nonlinear generalized HJB equation of the form:

$$\left\{ \begin{array}{rcl} \dfrac{\partial v}{\partial t} + \sup\limits_{a \in A} \big[b(.,a).Dv + \dfrac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(.,a) D^2 v) + f(.,a,v) \big] & = & 0, \quad \text{on } [0,T) \times \mathbb{R}^d, \\ v(T,.) & = & g, \quad \text{on } \mathbb{R}^d. \end{array} \right. \tag{2.3.6}$$

We recall that in the particular case where $f(x, a)$ does not depend on v and Dv, this partial differential equation (PDE) is the dynamic programming equation for the stochastic control problem:

$$v(t,x) \;=\; \sup_{\alpha} \mathbb{E}\Big[\int_t^T f(X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| X_t^\alpha = x \Big], \tag{2.3.7}$$

with controlled diffusion in $\mathbb{R}^d$:

$$dX_t^\alpha \;=\; b(X_t^\alpha, \alpha_t) dt + \sigma(X_t^\alpha, \alpha_t) dW_t,$$

and where α is an adapted control process valued in a compact space A of $\mathbb{R}^q$. Numerical methods for parabolic partial differential equations (PDEs) are largely developed in the literature, but remain a big challenge for fully nonlinear PDEs, like the HJB equation (2.3.6), especially in high dimensional cases (see [FTW08] for a review of some deterministic and probabilistic approaches).

We propose a new probabilistic numerical scheme for HJB equation, relying on the following Feynman-Kac formula for HJB equations obtained by randomization of the control process α.

We consider the minimal solution (Y, Z, U, K) to the backward stochastic differential equation (BSDE) with nonpositive jumps:

$$\left\{\begin{array}{rcl} Y_t & = & g(X_T) + \int_t^T f(X_s, I_s, Y_s, Z_s)ds + K_T - K_t \\ & & - \int_t^T Z_s dW_s - \int_t^T \int_A U_s(a)\tilde{\mu}(ds, da), \quad 0 \leq t \leq T, \\ U_t(a) & \leq & 0, \end{array}\right. \tag{2.3.8}$$

with a forward Markov regime-switching diffusion process (X, I) valued in $\mathbb{R}^d \times A$ given by:

$$\begin{array}{rcl} X_t & = & X_0 + \int_0^t b(X_s, I_s)ds + \int_0^t \sigma(X_s, I_s)dW_s \\ I_t & = & I_0 + \int_{(0,t]} \int_A (a - I_{s^-})\mu(ds, da), \end{array}$$

where we suppose that the random measure μ is defined on $\mathbb{R}_+ \times A$. We provide a an approximation of the minimal solution Y to this constrained BSDEs in two steps. The first step consists in the approximation of the constraint $U \leq 0$. Unfortunately, we cannot use the same approach as before for reflected BSDEs, since the jump component U is defined in an L^2-type space. Therefore, imposing the constraint only on a finite time grid does not modify the solution. Instead, we use another approach which relies on the link between the component U and Y. In particular, the constraint $U \leq 0$ is equivalent to impose Y to have nonpositive jumps. We then introduce a new kind of BSDEs called discretely constrained BSDEs where the nonpositive jump constraint operates only at the times of a finite grid $\pi = \{t_0 = 0 < t_1 < \ldots < t_n = T\}$ of $[0, T]$. These equations take the following form

$$Y_T^\pi \;=\; \mathcal{Y}_T^\pi \;=\; g(X_T) \tag{2.3.9}$$

and

$$\begin{array}{rcll} \mathcal{Y}_t^\pi & = & Y_{t_{k+1}}^\pi + \int_t^{t_{k+1}} f(X_s, I_s, \mathcal{Y}_s^\pi, \mathcal{Z}_s^\pi)ds & (2.3.10) \\ & & - \int_t^{t_{k+1}} \mathcal{Z}_s^\pi dW_s - \int_t^{t_{k+1}} \int_A \mathcal{U}_s^\pi(a)\tilde{\mu}(ds, da)\,, & \\ Y_t^\pi & = & \mathcal{Y}_t^\pi \mathbb{1}_{(t_k, t_{k+1})}(t) + \operatorname*{ess\,sup}_{a \in A} \mathbb{E}\Big[\mathcal{Y}_t^\pi \big| X_t, I_t \;=\; a\Big] \mathbb{1}_{\{t_k\}}(t)\,, & (2.3.11) \end{array}$$

for all $t \in [t_k, t_{k+1})$ and all $0 \leq k \leq n-1$, with X a diffusion process. We notice that the component Y^π has negative jumps at times t_k thanks to (2.3.11).

Under additional assumptions, we provide an error estimate for the convergence of the discretely jump-constrained BSDE as the mesh of π goes to zero. More precisely, we get a

rate of $\frac{1}{10}$ and improve it to $\frac{1}{6}$ in the case where f does not depend on the component Y. The proof relies on the combination of BSDE methods and PDE arguments adapted from the shaking coefficients approach of Krylov [Kry00] and switching systems approximation of Barles, Jacobsen [BJ07].

We then consider an Euler type approximation scheme associated to the BSDE (2.3.9)-(2.3.10)-(2.3.11) defined on π by

$$\left\{\begin{array}{rcl} \bar{Y}^\pi_T = \bar{\mathcal{Y}}^\pi_T &=& g(\bar{X}^\pi_T) \\ \bar{\mathcal{Z}}^\pi_{t_k} &=& \mathbb{E}\Big[\bar{Y}^\pi_{t_{k+1}} \frac{W_{t_{k+1}}-W_{t_k}}{t_{k+1}-t_k}\Big|\mathcal{F}_{t_k}\Big] \\ \bar{\mathcal{Y}}^\pi_{t_k} &=& \mathbb{E}\Big[\bar{Y}^\pi_{t_{k+1}}\Big|\mathcal{F}_{t_k}\Big] + (t_{k+1}-t_k)\; f(\bar{X}^\pi_{t_k}, I_{t_k}, \bar{\mathcal{Y}}^\pi_{t_k}, \bar{\mathcal{Z}}^\pi_{t_k}) \\ \bar{Y}^\pi_{t_k} &=& \operatorname*{ess\,sup}_{a\in A} \mathbb{E}\Big[\bar{\mathcal{Y}}^\pi_{t_k}\Big|\mathcal{F}_{t_k}, I_{t_k}=a\Big], \quad k=0,\dots,n-1, \end{array}\right. \tag{2.3.12}$$

To deal with the term $\operatorname*{ess\,sup}_{a\in A}$ appearing in this scheme, we introduce an error operator involving the essential supremum w.r.t. the value of the process I at each time t_k of the grid π. Then using classical arguments on BSDE discretization, we prove that $\bar{Y}^\pi - Y^\pi$ converges to zero as the mesh of π goes to zero with a rate $\frac{1}{2}$.

Using a comparison argument, we prove that the components $\mathcal{Y}^\pi$ and Y^π are upper-bounded by Y. This domination and the previous rates induce an asymmetric convergence rate for the global error: we get a convergence rate of $\frac{1}{2}$ for the negative part of the error between Y and $\bar{Y}^\pi$ and a convergence rate of $\frac{1}{10}$ that can be improved to $\frac{1}{6}$ for the positive part of the same error.

We also notice that a byproduct of this scheme is an approximation of the optimal control given by $\operatorname{argmax}_{a\in A} \mathbb{E}\Big[\bar{\mathcal{Y}}^\pi_{t_k}\Big|\mathcal{F}_{t_k}, I_{t_k}=a\Big]$ at each time t_k.

To get a computable scheme, we approximate the conditional expectations that arise from (2.3.12). We use empirical least-squares regressions, as this method provides a parametric estimate in feedback form of the optimal control. We finally test our scheme on different examples. In particular, we consider the case of a linear quadratic stochastic control problem where the controlled diffusion is defined by

$$dX^\alpha_s = (-\mu_0 X^\alpha_s + \mu_1 \alpha_s)\, dt + (\sigma_0 + \sigma_1 \alpha_s)\, dW_s\,,$$

and the value function is given by

$$v(t,x) = \sup_\alpha \mathbb{E}\left[-\lambda_0 \int_t^T (\alpha_s)^2\, ds - \lambda_1 (X^\alpha_T)^2 \,\Big|\, X^\alpha_t = x\right].$$

We choose this simple example since there exists an analytical solution (see [YZ93]) to which our results can be compared in order to assess the accuracy of our method. An optimal Markov control is then given by

$$\alpha^*(t, X_t) = A(t)\, X_t + B(t)$$

where A and B are two functions depending on the coefficients λ_0, λ_1, μ_0, μ_1, σ_0 and σ_1. We set the parameters to the following values:

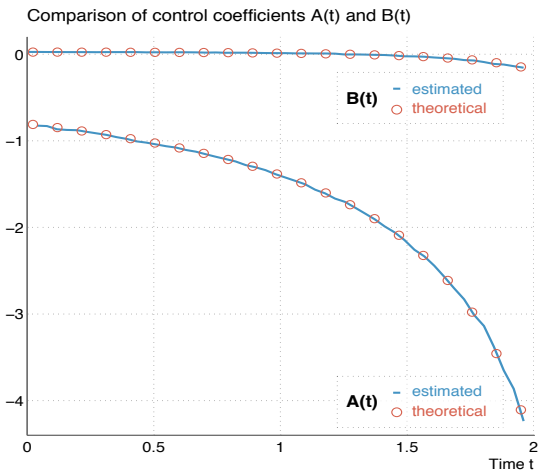

Figure 2.1: Optimal coefficients vs. theoretical values

λ_0	λ_1	μ_0	μ_1	σ_0	σ_1	T
20	200	0.02	0.5	0.2	0.1	2

For the numerical simulation, we use $n = 52$ time-discretization steps, and a sample of $M = 10^6$ Monte Carlo simulations. As can be seen from the comparison on Figure 2.1, our estimates of the control coefficients are very accurate. Regarding the value function, our method provides the estimate $\hat{v}(0,0) = -5.761$. The theoretical value being equal to -5.705, this means a relative error of 1%.

2.4 Perspective

A natural question that appears in mathematical finance is the numerical computation of super-replication prices in the case where the investor is subject to investment constraints. Unfortunately, there is a lack of efficient numerical methods for the computation of such prices. From a probabilistic point of view, this question is related to the numerical approximation of Brownian BSDEs with constraint on the Z-component of the solution. Such a constraint is much more involving than the reflected case where the constraint is only imposed on the Y-component. Indeed, one have to understand how to properly modify the process Y so as for Z to satisfy the constraint. Together with J.-F. Chassagneux and R. Elie, we have obtained partial results on this question and we aim at extending them to the general case.

Part II

Mathematical modeling in finance

Chapter 3

Market imperfection models

In the classical financial theory, the completeness of the market leads, under the no arbitrage assumption, to the definition of a unique fair price for a given continent claim. Unfortunately, this completeness property requires unrealistic assumptions (liquidity of the market, no constraints on the trading strategies, no transaction costs,...). Therefore, one has to take into account the imperfections of the market to get relevant valuations for contingent claims.

In this chapter, we present our results on market imperfection models. We first investigate in Section 3.1 the optimal portfolio liquidation in an illiquid market where the investor is subject to liquidity costs and risks. We then turn to imperfections related to constraints on financial markets. In Section 3.2, we consider an optimal investment problem under portfolio constraints. We provide a necessary and sufficient condition under which the super-replication under portfolio constraints can be reduced to the replication of a modification of the continent claim. Finally, we consider in Section 3.3 an optimal switching model under state constraints that comes from energy management problems.

The content of this chapter follows from the articles [CEK14], [KP10] and [Kha14].

3.1 Optimal portfolio liquidation under liquidity risk

Understanding trade execution strategies is a key issue for financial market practitioners, and has attracted a growing attention from the academic researchers. An important problem faced by stock traders is how to liquidate large block orders of shares. This is a challenging issue due to the following dilemma. By trading quickly, the investor is subject to higher costs due to market impact reflecting the depth of the limit order book. Thus, to minimize price impacts, it is generally beneficial to break up a large order into smaller blocks. However, more gradual trading over time results in higher risks since the asset value can vary more during the investment horizon in an uncertain environment.

We concentrate in [KP10] on this issue and we study the optimal liquidation of a portfolio. In the literature, there are two main formulations for optimal trading problems: the discrete time and the continuous time ones. We choose to adopt an intermediary model by considering an impulse control framework. Such a formulation combines the advantages

of the continuous time models by allowing the use of stochastic calculus techniques, and the realistic modeling of trading strategies by considering discrete-time exchanges.

We consider a financial market composed by two assets. The first one is a non risky asset with interest rate supposed to be equal to zero. The second one is a risky asset P following a Black & Scholes dynamics:

$$dP_t \;=\; P_t\big(bdt + \sigma dW_t\big)\;, \quad t \geq 0\;.$$

We assume that the investor can only trade discretely on $[0,T]$. The trading strategies are then modeled through an impulse control strategy $\alpha = (\nu_n, \beta_n)_{n\geq 0}$ with

- $(\nu_k)_{k\geq 0}$ nondecreasing stopping times representing the trading times of the investor,
- β_k an $\mathcal{F}_{\nu_k}$-measurable random variable valued in $\mathbb{R}$ and giving the number of shares of stock purchased if $\beta_k \geq 0$ or sold if $\beta_k < 0$ at time ν_k.

For such a strategy the number of shares of stock Y_t held by the investor at time t follows the dynamics

$$Y_t \;=\; Y_{\nu_n} \text{ for } \nu_n \leq t < \nu_{n+1} \quad \text{and} \quad Y_{\nu_{n+1}} \;=\; Y_{\nu_n} + \beta_{n+1},$$

for all $n \geq 0$.

To take into account the costs that can appear due to the illiquidity of the market, we suppose that at each intervention time ν_n, the price paid by the investor for β_n shares of stock is not $P_{\nu_n}\zeta_n$ but $P_{\nu_n} f(\beta_n, \nu_{n+1} - \nu_n)$ where f is an impact function. Such a framework allows to take into account the bid-ask spread in the limit order book via the dependence of f on β_n. Moreover, this model also take into account the temporary market price impact penalizing rapid execution trades since f depends on the lag time $\nu_{n+1} - \nu_n$ between two interventions.

In this framework, the amount in the bank account X evolves according to the following dynamics

$$X_t \;=\; X_{\nu_n} \text{ for } \nu_n \leq t \leq \nu_{n+1} \quad \text{and} \quad X_{\nu_{n+1}} \;=\; X_{\nu_n} - \beta_n P_{\nu_n} f(\beta_n, \nu_{n+1} - \nu_n)$$

A key issue in line of the banking regulation and solvency constraints is to define in an economically meaningful way the portfolio value of a position in stock at any time. We address this issue by introducing the liquidation value of a portfolio and imposing

- a no-short selling constraint on the trading strategies, i.e. $Y_t \geq 0$ for $t \in [0,T]$,
- a solvency constraint by asking the strategy α to keep the liquidation value, *i.e.* the wealth the investor gets if he/she immediately sells all the shares of stock he/she holds, positive on the time interval $[0,T]$.

We then consider the optimal liquidation problem

$$\sup_{\alpha\in\mathcal{A}\;:\;Y_T=0} \mathbb{E}\Big[U(X_T)\Big]\;.$$

where $\mathcal{A}$ denotes the set of strategies satisfying the no-short selling and solvency constraints and U is a utility function representing the preferences of the investor. We define a value function v associated to this optimization problem. This function v takes classically the time variable t, the stock price p, the cash amount x and the number of shares of stock y as arguments but also an additional variable θ which represents the initial lag time from the last intervention. We first relate the value function v to a quasi-variational inequality of the form

$$\min\Big[-\frac{\partial v}{\partial t}-\frac{\partial v}{\partial \theta}-\mathcal{L}v, v-\mathcal{M}v\Big] = 0 \quad \text{on } [0,T)\times\bar{\mathcal{S}} \tag{3.1.1}$$

$$\min\Big[v-U_L, v-\mathcal{M}v\Big] = 0 \quad \text{on } \{T\}\times\bar{\mathcal{S}} \tag{3.1.2}$$

where $\mathcal{L}$ is the second order local operator associated to the diffusion P, $\mathcal{M}$ is a nonlocal operator associated to the change of the system (X,Y) at each intervention, and $\bar{\mathcal{S}}$ is the domain associated to the no-short selling and solvency constraints. Using dynamic programming arguments, we first prove the following result.

Theorem 3.1.1. *The value function v is a constrained viscosity solution to* (3.1.1)-(3.1.2).

We then turn to the characterization of the value function as the unique solution to the quasi-variational inequality. Unfortunately, we are not able to prove a comparison result for the PDE (3.1.1)-(3.1.2). One reason is that there is a singularity of the controlled system at $\beta = 0$. More precisely, the impact function f generates a transaction cost that can be very small for very small number of traded shares of stock. In particular, this prevent from using the classical method which consists in the construction of a strict super-solution. Instead, we provide two weaker characterizations of the value function.

We first show that the value function is the minimal constrained viscosity solution to the quasi-variation inequality (3.1.1)-(3.1.2) under some growth and boundary conditions. To prove this minimality result we consider an approximation v^ε of the original problem v consisting in the addition of a positive utility cost ε at each transaction. This leads to a PDE for which we can construct a strict super-solution and then prove a comparison theorem. We show that v^ε is dominated by any constrained solution to (3.1.1)-(3.1.2) and that v^ε converges to v as ε goes to 0.

In the second characterization, we introduce a family $(v_\varepsilon)_{\varepsilon>0}$ of value functions associated to the same model with an additional fixed transaction cost ε. We then show that these value functions are uniquely determined by their associated PDEs and that they converge to the initial value function v as ε goes to 0. Let us mention that this last convergence is used in [GMP13] to set numerical methods for the computation of v.

3.2 Exact replication and investment constraints

After considering liquidity frictions for optimal investment problems, we turn to portfolio constraints. In uncertainty periods for the economy, regulatory institutions are lead to impose more and more restrictions to the investors, in order to reduce systemic risks. The optimal investment problems under constraints have then attracted a lot of interest from

the academic literature. As noticed in [KK96], the uniqueness property of a fair price for a given continent claim fails to be satisfied in the case where the admissible portfolio strategies are subject to additional constraints. Instead, there is a closed interval of arbitrage free prices. A commonly considered prudential pricing methodology consists in selecting the upper bound of this interval. This so-called super-replication price coincides with the minimal initial amount of money required to constitute an admissible portfolio strategy satisfying the constraints and whose terminal value dominates the claim of interest. The super-replication price under convex delta constraints has been thoroughly studied in the literature. In [CK93], the authors obtain a dual representation of the super-replication price in terms of a well chosen set of risk neutral probabilities. In [CKS98], closely related to the previous work, the super-replication price process is shown to be the unique solution of a Backward Stochastic Differential Equation (BSDE) with constraints on the gain process. All these works mainly rely on probabilistic and duality arguments. In a Markovian setting, the super-replication price is characterised using direct dynamic programming arguments and PDE techniques, see [ST03] and [BTM05].

Despite all these studies, the super-replication price remains difficult to compute in practice. A specific case where this computation is easier is the classical one-dimensional Black & Scholes model. In [BCS98], the authors prove that the super-replication price of a derivative $g(S_T)$ under convex delta constraints coincides with the unconstrained replication price of a so-called facelift transform $F_K[g](S_T)$ of this claim. Here the facelift operator is F_K is defined by

$$F_K[g](x) \;=\; \sup_{y\in\mathbb{R}^d} g(x+y) - \delta_K(y)\;, \quad x\in\mathbb{R}^d\;,$$

with δ_K the support function of the convex set K of constraints. In the multidimensional case simple counterexamples can be constructed to prove that this property does not hold anymore. We then focus in [CEK14] on a necessary and sufficient condition under which the noteworthy result of [BCS98] extends to general local volatility models in dimension d of the form

$$dS_t \;=\; \sigma(S_t)dW_t\;, \quad t\geq 0,$$

with $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ the local volatility function. To this end we define the set $\breve{\partial K}$ of points $y\in\partial K$ where there exists a unique normal vector $n(y)$. For such an $y\in\breve{\partial K}$ we associate a family $(n_\ell(y))_{2\leq\ell\leq d}$ such that $\big(n(y), n_2(y),\ldots,n_d(y)\big)$ is an orthonormal basis of $\mathbb{R}^d$. Then we have the following result.

Theorem 3.2.2. *Suppose that the local volatility function σ belongs to $C^1(\mathbb{R}^d,\mathbb{R}^{d\times d})$. Consider the two following conditions.*

(i) For any l.s.c. and lower bounded payoff function h satisfying $\mathbb{E}\big[F_K[h](S_T)\big]<\infty$ for any initial condition in the support of σ, the super-replicating price and strategy of $h(S_T)$ under K-constraint coincides with the exact replicating price and unconstrained strategy of the facelifted claim $F_K[h](S_T)$.

(ii) The following holds true:

$$\partial_x\big[\mathrm{Tr}\big(n_\ell^\top(y)\sigma\sigma^\top(.)n_k(y)\big)\big]n(y) \;=\; 0\,, \qquad 2\le k,\ell\le d\,, \tag{3.2.3}$$

for all $y\in\partial\breve{K}$.

Then (i) implies (ii) and the reverse implication holds whenever σ *is uniformly elliptic.*

To exhibit condition (3.2.3), we use a BSDE representation of the replicating strategy by a multidimensional BSDE with terminal condition $\nabla g(S_T)$. In the case where g is already a facelift transform we get $\nabla g(S_T)\in K$. Thus the problem remains to find necessary and sufficient conditions for the solution of a BSDE to be valued in K as soon as its terminal condition is valued in K. Such a question has been studied in [BQR00], where the authors provide necessary and sufficient conditions on the driver for the viability of the solution of a BSDE in K for any terminal condition valued in K. In our case, the problem is a bit more complicated since the considered terminal conditions are only gradient of functions. However, we get a specific generator corresponding to our model which simplifies the problem. In particular, we take advantage of the linear structure which allows to prove that the problem for general convex sets can be reduced to half spaces.

We also apply this result to financial examples. We show that hypercubes are the only convex sets for which facelifting allows to get rid of the portfolio constraints in a multidimensional Black & Scholes model. This property also extends to multidimensional local volatility models where each asset follows its own dynamics.

3.3 Energy management and optimal switching under state constraint

In the previous section, the imperfection of the market is described by a constraint on the trading strategies. In other situations, the constraint can also be imposed to the state of the considered system. In this situation, the strategies become also constrained since they must keep the system in the constraint domain. This is the case for energy management models, where the limited storage capacities induce a constraint on the quantity of produced energy. Traditionally, the energy markets were only reduced to major energy companies due to the heavy production and storage infrastructures that are required. With the liberalization of the recent years, energy markets become more open since smaller investors are allowed to rent storage facilities and they can trade energy commodities. Therefore, the energy management issue become an important question for financial practitioner.

This question is studied in [CL09] through the optimal switching model. They first model the optimal energy management as an optimal switching problem under state constraints and they provide a verification theorem in the case where the associated PDE admits a smooth solution. Then they provide a numerical algorithm to approximate the value function.

In [Kha14], we concentrate on the optimal switching problem under state constraints to weaken the assumption of existence of a smooth solution. We consider a general switching problem with a controlled underlying diffusion constrained to stay in a given closed set.

The study of optimal control problems under constraints is quite technical in general and leads to impose regularity conditions on the constraint domain as in [Kat94]. Moreover, the characterization of the value function relies on a continuity property (see [Son86a, Son86b, Kat94]) which fails to be true for optimal switching problems under state constraints as simple counterexamples can show.

We use a different approach which allows to characterize the value function associated to the weak formulation of an optimal switching under state constraints. We suppose that for a switching control $\alpha = (\nu_k, \beta_k)_k$, the controlled system evolves according to the following SDE:

$$dX_t^\alpha \;=\; \mu\big(X_t^\alpha, \alpha_t\big)dt + \sigma\big(X_t^\alpha, \alpha_t\big)dW_t\,, \tag{3.3.4}$$

where $\alpha_s = \sum_{k\geq 0} \beta_k \mathbb{1}_{[\nu_k, \nu_{k+1})}(s)$ for $s \in [0,T]$. We then consider the weak formulation of optimal switching under state constraints:

$$\sup_{(\Omega,\mathcal{A},\mathbb{P},W)} \sup_{\alpha\in\mathcal{A}^{\bar{\mathcal{O}}}} \mathbb{E}\Big[g(X_T^\alpha) + \int_0^T f(X_s^\alpha, \alpha_s)ds - \sum_{\nu_k \leq T} c(\beta_{k-1}, \beta_k)\Big]$$

and we define v as the associated value function. Here $\mathcal{A}^{\bar{\mathcal{O}}}$ is the set of strategies α that keep the controlled diffusion X^α in the constraints domain $\bar{\mathcal{O}}$.

In this framework a switching strategy α can be seen as a random variable valued in $\big([0,T]\times\{1,\dots,m\}\big)^{\mathbb{N}}$. From Tychonov theorem, we get the compactness of this space which allows to prove the tightness and hence the convergence in law of a subsequence of a given sequence of strategies. Then from Skorokhod representation theorem, we are able to provide a probability space on which we have an almost sure convergence. We apply this compactness result to an almost optimal sequences associated to unconstrained switching problem penalized out of the constraints domain. This allows firstly to prove that that the limit strategy is optimal and secondly that the value function satisfies a dynamic programming principle. We then relate the value function v to a system of variational inequalities of the form

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f, v - \mathcal{M}v\Big] = 0 \quad \text{on} \quad [0,T)\times\bar{\mathcal{O}}\times\{1,\dots,m\}, \tag{3.3.5}$$

$$\min\Big[v - g, v - \mathcal{M}v\Big] = 0 \quad \text{on} \quad \{T\}\times\bar{\mathcal{O}}\times\{1,\dots,m\}. \tag{3.3.6}$$

where $\mathcal{L}$ is the second order local operator defined by

$$\mathcal{L}v(t,x,i) \;=\; \Big(\mu.Dv + \frac{1}{2}\mathrm{tr}[\sigma\sigma^\intercal D^2 v)]\Big)(t,x,i)$$

and $\mathcal{M}$ is the nonlocal operator defined by

$$\mathcal{M}v(t,x,i) \;=\; \max_{\substack{1\leq j\leq m\\ j\neq i}} \big[v(t,x,j) - c(x,i,j)\big]$$

for all $t\in[0,T]$, $x\in\bar{\mathcal{O}}$ and $i\in\{1,\dots,m\}$. Using this dynamic programming principle we get the following result

Theorem 3.3.3. *Suppose that the value function v is locally bounded. Then it is a constrained viscosity solution to* (3.3.5)-(3.3.6).

We then turn to the uniqueness. Unfortunately, we are not able to prove a comparison result for the system of quasi variational inequalities (3.3.5)-(3.3.6). Instead we provide under some additional assumptions a weaker characterization as follows.

Theorem 3.3.4. *The value function v is the maximal constrained viscosity solutions in the class of function satisfying a polynomial growth condition.*

To prove this maximality result, we use the convergence of unconstrained penalized value functions to v, together with classical uniqueness results for unconstrained systems of variational inequalities.

3.4 Perspective

We present our current research projects related the results presented above.

Portfolio constraints in money amount or wealth proportion. A natural question is wether one can exhibit a necessary and sufficient condition for the result of Section 3.2 in the case where the portfolio constraints are written in terms of amount of money or wealth proportion. Indeed, constraints on the proportion or money amount are more realistic. Unfortunately, our approach fails in such a modelization since the geometric properties used before do not hold anymore. This topic is currently under study with R. Elie and his PhD student R. Dumitrescu.

Numerical approximation for constrained switching problems. Optimal switching problems under state constraints seems to be general enough to represent many financial situations. Unfortunately, there is not any numerical procedure for the approximation of the value function in the general case. We therefore propose to study the discrete time approximation of optimal switching problems under state constraints. A first approach can be based on the approximation unconstrained penalized problems as done in [BPTW12] for BSDEs. We also hope to work directly on the constrained system of variational inequalities to provide a numerical scheme.

Chapter 4

Progressive enlargement of filtrations in finance

An important concern for the financial practitioners is to face the risks resulting from possible jumps of the assets. As these jumps can be dependent of the other parameters of the market, the modelization by continuous processes and independent jumps does not cover all the situations. To take into account this dependence, a growing literature proposes to model financial markets with jumps by the powerful theory of enlargement of filtrations. Since this theory has been extensively studied in the last decades, numerous results can be use d to investigate financial issues.

This chapter presents the results related to financial modeling by progressive enlargement of filtrations. Section 4.1 presents a multiple default risk model in which an optimal investment problem is studied using a decomposition approach. This decomposition approach is dynamically used in Section 4.2 where existence and uniqueness result are provided for general BSDEs in a progressively enlarged filtration. These results are then applied to solve a utility maximization in a market with jumps. Finally Section 4.3 deals with a mean-variance hedging problem up to a default time where an optimal strategy is characterized by an associated BSDE.

The content of this chapter follows form results contained in [JKP13], [KL12] and [KLN13].

4.1 Optimal investment under multiple default risk

In financial instability periods, one of the main concerns of financial practitioners is to manage default phenomena and contagion effects that can follow. In [JKP13], we address, with Y. Jiao and H. Pham, an investment problem in such a framework. We consider multiple default events corresponding for example to the defaults of multi credit names or to counterparty defaults, and contagion effects meaning that defaults on some assets may induce loss or gain on the others.

A usual formulation is to consider the default-free assets price process as an Itô process governed by the d-dimensional Brownian motion W and the jumps are described by

random default times, associated to a marked point process. Then the study of the optimal investment problem leads to Hamilton-Jacobi-Bellman integro-differential equations in a Markov framework, and more generally to Backward Stochastic Differential Equations (BSDEs) with jumps.

We extend the optimal investment problem in this multiple defaults context by using an approach initiated in [JP11] in the single default time case, and further developed in [Pha10] in the multiple defaults with random marks case. This approach consists in considering the progressive enlargement $\mathbb{H}$ of the brownian filtration $\mathbb{F}$ by a sequence $(\tau_k, \zeta_k)_{1\leq k\leq n}$ where $\tau_1, \ldots, \tau_n$ represent the jump times of the assets and ζ_k the size of the jump occurring at time τ_k. To be able to compute expectations in the enlarged filtration $\mathbb{H}$, we suppose the existence of a conditional joint density for the sequence of random times and marks with respect to the filtration $\mathbb{F}$.

(HD) There exists an $\mathbb{F}$-predictable$\otimes\mathcal{B}(\mathbb{R}_+^n)\otimes\mathcal{B}(E^n)$-measurable function α such that

$$\begin{aligned} & \mathbb{P}[(\tau^1,\ldots,\tau^n,\zeta^1,\ldots,\zeta^n)\in d\theta de|\mathcal{F}_t] \\ = \;& \alpha_t(\theta^1,\ldots,\theta^n,e^1,\ldots,e^n)d\theta^1\ldots d\theta^n\nu_1(de^1)\prod_{k=1}^{n-1}\nu_{k+1}(e^{(k)},de^{k+1})\;. \end{aligned}$$

for all $t\geq 0$, with ν_1 a nonnegative Borel measure on E and ν_{k+1} a nonnegative transition kernel on $E^k\times E$.

We recall that for $x\in\mathbb{R}^n$ and $k\in\{1,\ldots,n\}$, $x^{(k)}$ denotes the vector $(x^1,\ldots,x^k)$. We then consider a financial market composed by d assets with values defined by a d-dimensional $\mathbb{H}$-adapted process S. We suppose that the process S has the following decomposed form

$$S_t \;=\; S_t^0\mathbb{1}_{t<\tau^1}+\sum_{k=1}^{n-1}S_t^k(\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k\leq t<\tau^{k+1}}+S_t^n(\tau^{(n)},\zeta^{(n)})\mathbb{1}_{\tau^n\leq t}\;,$$

where S^k is a $\mathbb{F}$-optional$\otimes\mathcal{B}(\mathbb{R}_+^k)\otimes\mathcal{B}(E^k)$ measurable process with dynamics

$$dS_t^k(\theta^{(k)},e^{(k)}) \;=\; \mathrm{Diag}\big(S_t^k(\theta^{(k)},e^{(k)})\big)\big(b_t^k(\theta^{(k)},e^{(k)})dt+\sigma_t^k(\theta^{(k)},e^{(k)})dW_t\big)\qquad t\geq\theta_k\;.$$

Here, b^k and σ^k are $\mathbb{F}$-predictable$\otimes\mathcal{B}(\mathbb{R}_+^k)\otimes\mathcal{B}(E^k)$-measurable processes, valued respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. To take into account the contagion risk in this model, we suppose that each default time may induce a jump in the assets portfolio. This is formalized by considering a family of indexed processes γ^k, $k=0,\ldots,n-1$, which relates the processes S^k, $k=0,\ldots,n-1$, as follows

$$S_{\theta^{k+1}}^{k+1}(\theta^{(k+1)},e^{(k+1)}) \;=\; S_{\theta^{k+1}-}^k(\theta^{(k)},e^{(k)})*\big(\mathbf{1}_d+\gamma_{\theta_{k+1}}^k(\theta^{(k)},e^{(k)},e^{k+1})\big)$$

for each $(\theta,e)\in\mathbb{R}_+^n\times E^n$ where $*$ stands for the product component by component and $\mathbf{1}_d$ denotes the vector in R^d with all components equal to 1.

A trading strategy in this d-assets portfolio model is an $\mathbb{H}$-predictable process π. Thanks to the decomposition result proved originally by T. Jeulin in [Jeu80] and generalized by H.

Pham in [Pha10], such a process π can be written as

$$\pi_t = \pi_t^0 \mathbb{1}_{t\leq\tau^1} + \sum_{k=1}^{n-1} \pi_t^k(\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k<t\leq\tau^{k+1}} + \pi_t^n(\tau^{(n)},\zeta^{(n)})\mathbb{1}_{\tau^n\leq t}\,, \tag{4.1.1}$$

for all $t \geq 0$, with π^0 a $\mathbb{F}$-predictable process and π^k a $\mathbb{F}$-predictable $\otimes\mathcal{B}(\mathbb{R}_+^k) \otimes \mathcal{B}(E^k)$-measurable process for each $k = 1,\ldots,n$. For such a strategy π and an initial endowment x the associated wealth can be written

$$X_t(x) = X_t^0(x)\mathbb{1}_{t<\tau^1} + \sum_{k=1}^{n-1} X^k(x)_t(\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k\leq t<\tau^{k+1}} + X^n(x)_t(\tau^{(n)},\zeta^{(n)})\mathbb{1}_{\tau^n\leq t}\,,$$

for all $t \geq 0$, where X^k is the wealth associated to the strategy π^k is the reference filtration $\mathbb{F}$. In particular the relative jumps γ appears in the dynamics of the processes X^k as follows

$$X_{\theta^{k+1}}^{k+1}(x) = X_{\theta^{k+1}_}^k(x) + \pi_{\theta^{k+1}}\gamma_{\theta^{k+1}}^k.$$

We then consider an investor with preferences described by the utility function $U : x \mapsto -\exp(-px)$, who can trade in the d-assets portfolio following an admissible trading strategy π and has to deliver at maturity T an option of payoff H_T , modeled by a bounded $\mathcal{H}_T$-measurable random variable. The optimal investment problem is then defined by

$$V_0(x) = \sup \mathbb{E}\Big[U\big(X^{x,\pi} - H_T\big)\Big].$$

Our approach relies on the decomposition of this optimization problem into n optimization problems corresponding to the situation on each interval $[\tau_k,\tau_{k+1}]$ as follows. The terminal value function is given by

$$V^n(x,\theta,e) = \operatorname*{ess\,sup}_{\pi^n} \mathbb{E}\Big[U\big(X_T^n(x) - H_T^n)\alpha_T^n(\theta,e)\big|\mathcal{F}_{\theta_n}\Big]$$

and the value function V^k are recursively defined by

$$V^k(\theta^{(k)},e^{(k)}) = \operatorname*{ess\,sup}_{\pi^k} \mathbb{E}\Big[U\big(X_T^k(x) - H_T^k\big)\alpha_T^k(\theta^{(k)},e^{(k)}) +$$

$$\int_{\theta_k}^T \int_E V^{k+1}\big(X_{\theta_{k+1}}^k + \pi_{\theta_{k+1}}^k.\gamma_{\theta^{k+1}}^k(e^{(k+1)}),\theta^{(k+1)},e^{(k+1)}\big)\nu_{k+1}(e^{(k)},de^{k+1})d\theta^{k+1}\big|\mathcal{F}_{\theta_k}\Big].$$

Here,the functions α_t^k are the marginals of α w.r.t. $(\theta^{(k)},e^{(k)})$ on the set $\{\theta^{k+1} > t\}$. They are defined recursively by $\alpha_T^n(\theta,e) = \alpha_T(\theta,e)$ and

$$\alpha_t^k(\theta^{(k)},e^{(k)}) = \int_t^\infty \int_E \alpha_t^k(\theta^{(k)},s,e^{(k)},r)ds\nu_{k+1}(e^{(k)},r)$$

for all (θ,e). We then link this system of value functions to a recursive system of BSDEs in the filtration $\mathbb{F}$. Using the exponential form of the utility function U and the additive property of the wealth process, we prove that the value functions should be written under the form

$$V_t^k(x,\theta^{(k)},e^{(k)}) = U\big(X_t^k(x) - Y_t^k(\theta^{(k)},e^{(k)})\big), \quad \theta_k \leq t \leq T, \tag{4.1.2}$$

for $k = 0, \dots, n$. From the martingale optimality principle, we show that the processes $Y^0, \dots, Y^n$ can be written as solutions to BSDEs. More precisely, (Y^n, Z^n) should satisfy

$$\begin{aligned}(\mathbf{En}) \qquad Y_t^n(\theta, e) &= H_T^n(\theta, e) + \frac{1}{p} \ln \alpha_T(\theta, e) \\ &\quad + \int_t^T f^n(r, Z_r^n, \theta, e) dr - \int_t^T Z_r^n . dW_r, \quad \theta_n \le t \le T,\end{aligned}$$

where the generator f^n is defined by

$$f^n(t, z, \theta, e) = \inf_{\pi} \Big\{ \frac{p}{2} \big| z - \sigma_t^n(\theta, e)' \pi \big|^2 - b^n(\theta, e)' \pi \Big\} \tag{4.1.3}$$

and the processes (Y^k, Z^k) should satisfy the recursive system of BSDEs in $\mathbb{F}$:

$$\begin{aligned}(\mathbf{Ek}) \qquad Y_t^k(\theta^{(k)}, e^{(k)}) &= H_T^k(\theta^{(k)}, e^{(k)}) + \frac{1}{p} \ln \alpha_T^k(\theta^{(k)}, e^{(k)}) \\ &\quad + \int_t^T f^k(r, Y_r^k, Z_r^k, \theta^{(k)}, e^{(k)}) dr - \int_t^T Z_r^k . dW_r, \quad \theta_k \le t \le T,\end{aligned}$$

with a generator f^k defined by

$$\begin{aligned} f^k(t, y, z, \theta^{(k)}, e^{(k)}) &= \\ \inf_{\pi} \Big\{ \frac{p}{2} \big| z - \sigma_t^k(\theta^{(k)}, e^{(k)})' \pi \big|^2 &- b_t^k(\theta^{(k)}, e^{(k)})' \pi \\ + \frac{1}{p} U(y) \int_E U\big(\pi . \gamma_t^k(\theta^{(k)}, e^{(k)}, e^{k+1}) - Y_t^{k+1}(\theta^{(k)}, t, e^{(k)}, e^{k+1})\big) \nu_{k+1}(e^{(k)}, de^{k+1}) \Big\}\end{aligned}$$

The generators of our system of BSDEs **(En)**-**(Ek)** do not satisfy the usual Lipschitz or quadratic growth assumptions. In particular, in addition to the growth condition in z for f^k, there is an exponential term in y via the utility function $U(y)$, which prevents from a direct application of existence results for BSDEs known in the literature.

To deal with this issue, we proceed by truncation. We consider some approximating BSDEs where the generator is penalized in the infimum and exponential terms. More precisely, we introduce the following truncated driver

$$\begin{aligned} f^{k,N}(t, y, z, \theta^{(k)}, e^{(k)}) = \inf_{\pi,\ |(\sigma_t^k)' \pi| \le N} &\Big\{ \frac{p}{2} \big| z - \sigma_t^k(\theta^{(k)}, e^{(k)})' \pi \big|^2 - b_t^k(\theta^{(k)}, e^{(k)})' \pi \\ + \frac{1}{p} U(\max(-N, y)) \int_E U\big(\pi . \gamma_t^k(\theta^{(k)}, e^{(k)}, e^{k+1}) &- Y_t^{k+1}(\theta^{(k)}, t, e^{(k)}, e^{k+1})\big) \nu_{k+1}(e^{(k)}, de^{k+1}) \Big\}\end{aligned}$$

We can use the existing results on quadratic BSDEs (see [Kob00]), and we get the existence and uniqueness of a solution $(Y^{k,N}, Z^{k,N})$ for each $N \ge 1$. Then, the idea consists in proving that for N large enough we can get rid of the truncation. We first prove a uniform bound for the sequence $Y^{k,N}$ which allows to remove the truncation in the utility term U of the driver. To get rid the truncation term on π, we then show that the optimum points $\pi^{k,N}$ are also uniformly bounded. Finally, for N large enough, we get a solution $(Y^{k,N}, Z^{k,N})$ to the BSDE with driver f^k.

We then use these solutions to provide a verification theorem. We show that if (Y^k, Z^k), $k = 0, \dots, n$ is a solution to the recursive system of BSDEs **(En)**-**(Ek)**, then (4.1.2) holds

true for all $k = 0, \dots, n$. Using the BMO-properties of the solutions Z^k we provide an optimal strategy $\hat{\pi}$ defined by the terms appearing in the decomposition (4.1.1) as follows

$$\hat{\pi}_t^k(\theta^{(k)}, e^{(k)}) \in \arg\min_{\pi} \Big\{ \frac{p}{2} \big| Z_t^k(\theta^{(k)}, e^{(k)}) - \sigma_t^k(\theta^{(k)}, e^{(k)})'\pi \big|^2 - b_t^k(\theta^{(k)}, e^{(k)})'\pi$$
$$+ \frac{1}{p} U(Y_t^k(\theta^{(k+1)}, e^{(k)})) \int_E U\big(\pi.\gamma_t^k(\theta^{(k+1)}, e^{(k)}, e^{k+1}) - Y_t^k(\theta^{(k)}, t, e^{(k)}, e^{k+1})\big) \nu_{k+1}(e^{(k)}, de^{k+1}) \Big\},$$

for $k = 0, \dots, n$ and $t \in [0, T]$.

4.2 BSDEs and progressive enlargement of filtrations

In financial mathematics, the study of portfolio management generally leads to solve optimal control problems. A commonly spread approach to deal with such problems is to relate them to a specific BSDE. Then, solving this BSDE provides the optimal value and an optimal strategy. However, for general BSDEs with jumps, the results are far from being numerous as for the continuous case. In particular, quadratic BSDEs with jumps have not been completely studied.

In [KL12], we focus, with T. Lim, on this issue. We consider general BSDEs in the progressive enlargement $\mathbb{H}$ of the filtration $\mathbb{F}$ by the sequence of random times $(\tau^k)_{1 \le k \le n}$ and random marks $(\zeta^k)_{1 \le k \le n}$. We make the following density assumption on the random times and marks.

(HD') There exists an $\mathbb{F}$-predictable$\otimes \mathcal{B}(\mathbb{R}_+^n) \otimes \mathcal{B}(E^n)$-measurable function γ such that

$$\begin{aligned} & \mathbb{P}[(\tau^1, \dots, \tau^n, \zeta^1, \dots, \zeta^n) \in d\theta de | \mathcal{F}_t] \\ = \; & \gamma_t(\theta^1, \dots, \theta^n, e^1, \dots, e^n) d\theta^1 \dots d\theta^n de^1 \dots de^n \; . \end{aligned}$$

for all $t \ge 0$. p

This assumption allows to get basic properties for our progressive enlargement model. It first provides the existence of an absolutely continuous compensator $\lambda_t(e)dedt$ for the random measure $\eta(de, dt)$. Secondly, it allows to extend the stochastic integral w.r.t. W to the space $L^2_{\mathbb{H}}(W)$. With this extension of the stochastic integral, we are able to consider BSDEs of the following form: find $(Y, Z, U) \in S^2_{\mathbb{H}} \times L^2_H(W) \times L^2(\eta)$ such that

$$Y_t \; = \; \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \eta(de, ds), \tag{4.2.4}$$

for all $t \in [0, T]$ where

- ξ is an $\mathcal{H}_T$-measurable random variable of the form:

$$\xi \; = \; \sum_{k=0}^{n} \xi^k(\tau^{(k)}, \zeta^{(k)}) \mathbb{1}_{\tau^k \le T < \tau^{k+1}} \; , \tag{4.2.5}$$

with ξ^0 $\mathcal{F}_T$-measurable and ξ^k $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^k) \otimes \mathcal{B}(E^k)$-measurable for each $k = 1, \dots, n$,

- $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times Bor(E,\mathbb{R}) \to \mathbb{R}$ is an $\mathbb{H}$-predictable$\otimes\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(Bor(E,\mathbb{R}))$-measurable. function. Here $Bor(E,\mathbb{R})$ denotes the set of Borel functions from E to $\mathbb{R}$, and $\mathcal{B}(Bor(E,\mathbb{R}))$ is the borel algebra on $Bor(E,\mathbb{R})$ related to the point wise convergence topology.

We recall that for $x \in \mathbb{R}^n$ and $k \in \{1,\dots,n\}$, $x^{(k)}$ denotes the vector $(x^1,\dots,x^k)$. in particular $\tau^{(k)} = (\tau^1,\dots,\tau^k)$ and $\zeta^{(k)} = (\zeta^1,\dots,\zeta^k)$.

To prove the existence of solutions to such equations, we use the decomposition result which relates $\mathbb{H}$-predictable to $\mathbb{F}$-predictable processes. It says that any $\mathbb{H}$-predictable process X can be written under the form

$$X_t \;=\; X_t^0 \mathbb{1}_{t\le\tau^1} + \sum_{k=1}^{n-1} X_t^k(\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k<t\le\tau^{k+1}} + X_t^n(\tau^{(n)},\zeta^{(n)})\mathbb{1}_{\tau^n<t}\,, \tag{4.2.6}$$

for all $t \ge 0$, with X^0 a $\mathbb{F}$-predictable process and X^k a $\mathbb{F}$-predictable$\otimes\mathcal{B}(\mathbb{R}_+^k) \otimes \mathcal{B}(E^k)$-measurable process for each $k = 1,\dots,n$. We then extend this result to decompose the driver f of BSDE (4.2.4) under the form

$$f(t,y,z,u) \;=\; \sum_{k=0}^{n} f^k(t,y,z,u,\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k<t\le\tau^{k+1}}\,, \tag{4.2.7}$$

with f^0 an $\mathbb{F}$-predictable$\otimes\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(Bor(E,\mathbb{R}))$-measurable function and f^k an $\mathbb{F}$-predictable$\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(Bor(E,\mathbb{R}))\otimes\mathcal{B}(\mathbb{R}_+^k)\otimes\mathcal{B}(E^k)$-measurable function for each $k = 1,\dots,n$. Decompositions (4.2.5) and (4.2.7) lead formally to solve a coupled recursive systme of n BSDEs in the filtration $\mathbb{F}$ defined by

$$\begin{aligned} Y_t^n(\theta,e) \;&=\; \xi^n(\theta,e) + \int_t^T f^n\Big(s, Y_s^n(\theta,e), Z_s^n(\theta,e), 0, \theta, e\Big)ds \\ &\quad - \int_t^T Z_s^n(\theta,e)dW_s\,, \quad \theta^n \wedge T \le t \le T\,, \end{aligned} \tag{4.2.8}$$

and

$$\begin{aligned} Y_t^k(\theta^{(k)},e^{(k)}) \;&=\; \xi^k(\theta^{(k)},e^{(k)}) + \int_t^T f^k\Big(s, Y_s^k(\theta^{(k)},e^{(k)}), Z_s^k(\theta^{(k)},e^{(k)}), \\ &\qquad Y_s^{k+1}(\theta^{(k)},s,e^{(k)},.) - Y_s^k(\theta^{(k)},e^{(k)}), \theta^{(k)}, e^{(k)}\Big)ds \\ &\quad - \int_t^T Z_s^k(\theta^{(k)},e^{(k)})dW_s\,, \quad \theta^k \wedge T \le t \le T\,, \end{aligned} \tag{4.2.9}$$

We then obtain the following general existence result.

Theorem 4.2.5. *Assume that (HD') holds. Suppose that equations* (4.2.8) *and* (4.2.9) *admit solutions* $(Y^n(\theta,e), Z^n(\theta,e)$ *and* $\big(Y^k(\theta^{(k)},e^{(k)}), Z^k(\theta^{(k)},e^{(k)})$ *for all* (θ,e)*. Suppose also that these solutions satisfy*

$$\sup_{(k,\theta,e)} \|Y^k(\theta^{(k)},e^{(k)})\|_{S^\infty} \;<\; \infty\,, \tag{4.2.10}$$

and

$$\mathbb{E}\Big[\int\Big(\int_0^{\theta^1\wedge T}|Z_s^0|^2ds+\sum_{k=1}^n\int_{\theta^k\wedge T}^{\theta^{k+1}\wedge T}|Z_s^k(\theta^{(k)},e^{(k)})|^2ds\Big)\gamma_T(\theta,e)d\theta de\Big] \;<\; \infty\,,$$

then BSDE (4.2.4) *admits a solution* $(Y,Z,U)\in S^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}(W)\times L^2(\eta)$ *given by*

$$\begin{cases} Y_t &= Y_t^0\mathbb{1}_{t<\tau^1}+\sum_{k=1}^n Y_t^k(\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k\le t<\tau^{k+1}}\,,\\ Z_t &= Z_t^0\mathbb{1}_{t\le\tau^1}+\sum_{k=1}^n Z_t^k(\tau^{(k)},\zeta^{(k)})\mathbb{1}_{\tau^k<t\le\tau^{k+1}}\,,\\ U_t(.) &= U_t^0(.)\mathbb{1}_{t\le\tau^1}+\sum_{k=1}^{n-1}U_t^k(\tau^{(k)},\zeta^{(k)},.)\mathbb{1}_{\tau^k<t\le\tau^{k+1}}\,, \end{cases} \tag{4.2.11}$$

with $U_t^k(\tau^{(k)},\zeta^{(k)},.)=Y_t^{k+1}(\tau^{(k)},t,\zeta^{(k)},.)-Y_t^k(\tau^{(k)},\zeta^{(k)})$ *for each* $k=0,\dots,n-1$.

We notice that the jump component U is written as the difference between Y^{k+1} and Y^k at each jump time τ^k. This corresponds to the jump of the process Y thanks to the decomposition appearing in the first one of (4.2.11). The proof of this general existence result consists first in checking the measurability properties for the processes Y, Z and U given by (4.2.11) and showing that this triple satisfies equation (4.2.4).

We apply this general result to the case of a driver with quadratic growth and we get the existence of a solution for BSDEs with bounded terminal condition and uniformly continuous driver with quadratic growth. Under additional assumptions, in particular hypothesis **(H)** *i.e.* any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale, we also get the uniqueness for solutions to quadratic BSDEs with jumps. This uniqueness result is a consequence of a general comparison theorem which allows to compare the solutions of two BSDEs as soon as we can compare their terminal conditions and their drivers.

We then use these existence and uniqueness results to solve an optimal management problem in a market with jumps. We suppose that the market is composed by two assets.

- The first one is a riskless asset with an interest rate supposed to be equal to zero.
- The second one is a risky asset S subjects to some counterparty risks. We suppose that S evolves according to the following dynamics

$$S_t=S_0+\int_0^t S_{u^-}\Big(b_udu+\sigma_udW_u+\int_E\beta_u(e)\eta(de,du)\Big)\,,\quad 0\le t\le T\,.$$

We then consider the following utility maximisation problem

$$V(x) \;:=\; \sup_{\pi\in\mathcal{A}}\mathbb{E}\big[-\exp(-\alpha(X_T^{x,\pi}-B))\big]\,, \tag{4.2.12}$$

with

- B an $\mathcal{H}_T$-measurable random variable representing a contingent claim under the form

$$B \;=\; \sum_{k=0}^{n} B^k(\tau^{(k)}, \zeta^{(k)}) \mathbb{1}_{\tau_k \leq T < \tau_{k+1}} \;,$$

  where B^0 is $\mathcal{F}_T$-measurable and B^k is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^k) \otimes \mathcal{B}(E^k)$-measurable for each $k = 1, \ldots, n$,

- $X^{x,\pi}$ is the wealth of the portfolio with initial endowment x and self financing strategy π. Its dynamics is then given by

$$X_t^{x,\pi} \;=\; x + \int_0^t \pi_s b_s ds + \int_0^t \pi_s \sigma_s dW_s + \int_0^t \int_E \pi_s \beta_s(e) \eta(de, ds)$$

  for all $t \in [0, T]$.

To characterize the maximal expected utility $V(x)$ defined by (4.2.12), we consider the following BSDE

$$Y_t \;=\; B + \int_t^T h(s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds) \;, \quad 0 \leq t \leq T \;,$$

where the driver h is defined by

$$\begin{aligned} h(t, z, u) \;:=\; & \inf_{\pi} \Big\{ \frac{\alpha}{2} \Big| \pi \sigma_t - \Big(z + \frac{b_t}{\alpha \sigma_t} \Big) \Big|^2 + \int_E \frac{\exp(\alpha(u(e) - \pi \beta_t(e))) - 1}{\alpha} \lambda_t(e) de \Big\} \\ & - \vartheta_t z - \frac{|b_t|^2}{2 |\sigma_t|^2 \alpha} \;, \end{aligned}$$

for all $(t, z, u) \in [0, T] \times \mathbb{R} \times Bor(E, \mathbb{R})$. This BSDE falls into the previously studied framework. We therefore get existence and uniqueness of a solution $(Y, Z, U) \in S_{\mathbb{G}}^{\infty} \times L_{\mathbb{G}}^2(W) \times L^2(\eta)$. Using a duality method as in [HIM05], we get the following characterisation of the maximal expected utility

$$V(x) \;=\; -\exp(-\alpha(x - Y_0)) \,, \tag{4.2.13}$$

where Y_0 is the initial value of the unique solution (Y, Z, U) to (4.2.13). The solution to BSDE with data (B, h) also provides a sufficient condition for an investment strategy $\hat{\pi}$ to be optimal for the problem (4.2.12). This sufficient condition writes

$$\hat{\pi}_t \;\in\; arg\min_{\pi} \Big\{ \frac{\alpha}{2} \Big| \pi \sigma_t - \Big(Z_t + \frac{b_t}{\alpha \sigma_t} \Big) \Big|^2 + \int_E \frac{\exp(\alpha(U_t(e) - \pi \beta_t(e))) - 1}{\alpha} \lambda_t(e) de \Big\} \;,$$

for all $t \in [0, T]$.

4.3 Mean-variance hedging up to a default time

In most of financial markets, the completeness assumption fails to be true. In particular, investors cannot always hedge the financial products that they are interested in. One

possible alternative approach to deal with this problem is the mean-variance hedging. For a given contingent claim with payoff H at the maturity T and an initial endowment x, the mean-variance hedging consists in finding an investment strategy π^* such that the terminal value V_T^{x,π^*} of the portfolio with initial capital x and strategy π^* minimizes the mean square error

$$\mathbb{E}\Big[\big|X_T^{x,\pi} - H\big|^2\Big]$$

over all possible investment strategies π. In [KLN13], we focus, with T. Lim and A. Ngoupeyou, on the mean-variance hedging problem over a random horizon. More precisely, we consider a contingent claim H with maturity $T \wedge \tau$ where τ is a random time. We suppose that H can be decomposed as follows:

$$H \quad = \quad H^b \mathbb{1}_{T<\tau} + H_\tau^a \mathbb{1}_{T\geq\tau},$$

where $T < \infty$ is a fixed deterministic terminal time. We then study the mean-variance hedging problem over the horizon $[0, T \wedge \tau]$ defined by

$$\inf_\pi \mathbb{E}\Big[\big|X_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] .$$

Such a framework is particularily adapted to insurance issues. Indeed, if the random time τ represents the loss time of an insured and T the maturity of the contract, the insurer has to deliver an amount H at time $T \wedge \tau$ which takes different values in the cases $T < \tau$ (no loss for the insured) and $\tau \leq T$ (the insured suffered a loss).

The mean-variance hedging problem with deterministic horizon T is one of the classical problems in mathematical finance and has been considered by several authors via two main approaches. The first one is based on martingale theory and projection arguments and the second one considers the problem as a quadratic stochastic control problem and describes the solution using BSDE theory.

A major part of the literature on mean-variance hedging focuses on the continuous case where both approaches are used (see e.g. Delbaen and Schachermayer [DS96], Gouriéroux *et al.* [GLP98], Laurent and Pham [LP99] and Schweizer [Sch96] for the first approach, and Lim and Zhou [LZ02] and Lim [Lim02] for the second one).

In the discontinuous case, the mean-variance hedging problem is considered by several papers. In [Ara05], the author uses a projection approach for general semimartingale price processes model whereas in [Lim06] the problem is considered from the point of view of stochastic control for the case of diffusion price processes driven by a Brownian motion and a Poisson process. The author provides under a so-called "martingale condition" the existence of solutions to the associated BSDEs. In the recent paper [JMSS12], the authors combine tools from both approaches, which allows them to work in a general semimartingale model and to give a description of the optimal solution to the mean-variance hedging via the BSDE theory. Then, they provide an equivalence between the existence of an optimal strategy and the existence of a solution to an associated BSDE. In some specific examples they show the existence of a solution to the BSDE, but the problem remains open in the general case.

In our case, we consider a financial market that is composed by a riskless bond with zero interest rate and a risky asset S.

We denote by $\mathbb{H}$ the enlargement of $\mathbb{F}$ by the (single) random time τ. We suppose that the jump process $N = \mathbb{1}_{[\tau,+\infty)}$ admits a compensator of the form $\int_0^{\cdot\wedge\tau} \lambda_t dt$, and we denote by M the associated compensated martingale. We assume that assumption **(H)** holds true true. In our case this means that W remains a Brownian motion for the filtration $\mathbb{H}$. We then suppose that the price process $(S_t)_{t\geq 0}$ of the risky asset is modeled by the linear stochastic differential equation

$$S_t \;=\; S_0 + \int_0^t S_{s^-}(\mu_s ds + \sigma_s dW_s + \beta_s dM_s)\;, \quad \forall t \in [0,T]\;. \tag{4.3.14}$$

The wealth process $X^{x,\pi}$ corresponding to an initial amount $x \in \mathbb{R}$ and a trading strategy π, is then given by

$$X_t^{x,\pi} \;=\; x + \int_0^t \frac{\pi_r}{S_{r^-}} dS_r\;,$$

for $t \geq 0$. We then consider the problem

$$\inf_{\pi} \mathbb{E}\Big[\big|X_{T\wedge\tau}^{x,\pi} - H\big|^2\Big]\;. \tag{4.3.15}$$

For this problem we propose to look for a solution under the form proposed by [JMSS12]. Then using the martingale optimality principle, we derive a system of three coupled BSDEs in the filtration $\mathbb{H}$. To solve this system, we proceed in two steps. The first one consists in using the decomposition result of Jeulin [Jeu80] to get an equivalent system of BSDEs in the Brownian filtration $\mathbb{F}$. In the second step we make a precise analysis of these BSDEs. Indeed, their generators do not satisfy assumptions under which there are existence and uniqueness results and we have to deal with the specific form of these equations. We proceed by truncation and we establish uniform estimates on the truncated solutions to show that the truncation can be dropped as soon as its coefficient is large enough. Once we get three solutions (Y,Z,U), $(\mathcal{Y},\mathcal{Z},\mathcal{U})$ and (Υ,Ξ,Θ) to our system of three coupled BSDEs in $\mathbb{H}$, we can derive an optimal strategy π^* for the problem (4.3.15) defined by

$$\pi_t^* \;=\; \frac{(\mathcal{Y}_{t^-} - X_{t^-}^*)(\mu_t Y_{t^-} + \sigma_t Z_t + \lambda_t\beta_t U_t) + \sigma_t Y_{t^-}\mathcal{Z}_t + \lambda_t\beta_t\mathcal{U}_t(Y_{t^-} + U_t)}{|\sigma_t|^2 Y_{t^-} + \lambda_t|\beta_t|^2(U_t + Y_{t^-})}\;, \tag{4.3.16}$$

for all $t \in [0,T]$, where X^* is solution to the SDE

$$X_t^* \;:=\; x + \int_0^t \frac{\pi_r^*}{S_{r^-}} dS_r\;, \quad t \geq 0\;.$$

Our verification theorem is stated as follows.

Theorem 4.3.6. *The strategy π^* given by* (4.3.16) *is optimal for the mean-variance problem* (4.3.15) *and we have*

$$\mathbb{E}\Big[\big|X_{T\wedge\tau}^{x,\pi^*} - H\big|^2\Big] \;=\; \min_{\pi}\mathbb{E}\Big[\big|X_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \;=\; Y_0|x - \mathcal{Y}_0|^2 + \Upsilon_0\;.$$

The proof of this verification theorem uses the BMO properties of the solutions to the BSDEs to prove that the process X^* is well-defined. Then the integrability properties of the solution X^* allows to get the optimality of π^*.

4.4 Perspective

With C. Blanchet-Scaillet, E. Chevalier and T. Lim we plan to apply the progressive enlargement of filtration approach to the valuation of insurance products. More precisely, we aim at studying variable annuities contracts with minimum guarantees. These guaranteed contracts are highly demanded on financial markets and their valuation is a challenging issue. An important risk faced by the seller of a variable annuities contract concerns the characteristics of the buyer. In particular, the insurer has to take into account the behavior of the insurer, i.e. how does he/she withdraw money, and his/her exit time, i.e. death time, from the contract. We plan to study an indifference pricing approach that takes into account these risks. We propose to use the progressive enlargement approach to model the death time of the insured. Concerning the risk generated by the investor, we propose to consider the worst case for the seller: we suppose that the buyer chooses the withdrawal strategy that minimizes the expected utility of the seller. We then hope to characterize the indifference price of variable annuities contacts in such a framework.

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