



# Open quantum systems and quantum stochastic processes

Tristan Benoist

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L'ÉCOLE NORMALE SUPÉRIEURE**

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Présentée par

**Tristan BENOIST**

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Sujet de la thèse :

**Open quantum systems and quantum stochastic processes**

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devant le jury composé de :

M. Stéphane Attal	Université Lyon 1	Rapporteur
M. Alberto Barchielli	Université Politecnico di Milano	Rapporteur
M. Denis Bernard	École Normale Supérieure	Directeur de thèse
Mme. Leticia Cugliandolo	Université Pierre et Marie Curie	Présidente du Jury
M. Massimiliano Esposito	Université du Luxembourg	Examineur
M. Francesco Ticozzi	Université di Padova	Examineur



## Résumé

De nombreux phénomènes de physique quantique ne peuvent être compris que par l'analyse des systèmes ouverts. Un appareil de mesure, par exemple, est un système macroscopique en contact avec un système quantique. Ainsi, tout modèle d'expérience doit prendre en compte les dynamiques propres aux systèmes ouverts. Ces dynamiques peuvent être complexes : l'interaction du système avec son environnement peut modifier ses propriétés, l'interaction peut créer des effets de mémoire dans l'évolution du système, ...

Ces dynamiques sont particulièrement importantes dans l'étude des expériences d'optique quantique. Nous sommes aujourd'hui capable de manipuler individuellement des particules. Pour cela la compréhension et le contrôle de l'influence de l'environnement est crucial.

Dans cette thèse nous étudions d'un point de vue théorique quelques procédures communément utilisées en optique quantique. Avant la présentation de nos résultats, nous introduisons et motivons l'utilisation de la description markovienne des systèmes quantiques ouverts. Nous présentons à la fois les équations maîtresses et le calcul stochastique quantique. Nous introduisons ensuite la notion de trajectoire quantique pour la description des mesures indirectes continues. C'est dans ce contexte que l'on présente les résultats obtenus au cours de cette thèse.

Dans un premier temps, nous étudions la convergence des mesures non destructives. Nous montrons qu'elles reproduisent la réduction du paquet d'onde du système mesuré. Nous montrons que cette convergence est exponentielle avec un taux fixé. Nous bornons le temps moyen de convergence. Dans ce cadre, en utilisant les techniques de changement de mesure par martingale, nous obtenons la limite continue des trajectoires quantiques discrètes.

Dans un second temps, nous étudions l'influence de l'enregistrement des résultats de mesure sur la préparation d'état par ingénierie de réservoir. Nous montrons que l'enregistrement des résultats de mesure n'a pas d'influence sur la convergence proprement dite. Cependant, nous trouvons que l'enregistrement des résultats de mesure modifie le comportement du système avant la convergence. Nous retrouvons une convergence exponentielle avec un taux équivalent au taux sans enregistrement. Mais nous trouvons aussi un nouveau taux de convergence correspondant à une stabilité asymptotique. Ce dernier taux est interprété comme une mesure non destructive ajoutée. Ainsi l'état du système ne converge qu'après un temps aléatoire. À partir de ce temps la convergence peut être bien plus rapide. Nous obtenons aussi une borne sur le temps moyen de convergence.

**Mots-clefs :** Systèmes ouverts, Calcul Stochastique Quantique, Trajectoires quantiques, Mesure non destructives, Ingénierie de réservoir.



## Abstract

Many quantum physics phenomena can only be understood in the context of open system analysis. For example a measurement apparatus is a macroscopic system in contact with a quantum system. Therefore any experiment model needs to take into account open system behaviors. These behaviors can be complex: the interaction of the system with its environment might modify its properties, the interaction may induce memory effects in the system evolution, ...

These dynamics are particularly important when studying quantum optics experiments. We are now able to manipulate individual particles. Understanding and controlling the environment influence is therefore crucial.

In this thesis we investigate at a theoretical level some commonly used quantum optics procedures. Before the presentation of our results, we introduce and motivate the Markovian approach to open quantum systems. We present both the usual master equation and quantum stochastic calculus. We then introduce the notion of quantum trajectory for the description of continuous indirect measurements. It is in this context that we present the results obtained during this thesis.

First, we study the convergence of non demolition measurements. We show that they reproduce the system wave function collapse. We show that this convergence is exponential with a fixed rate. We bound the mean convergence time. In this context, we obtain the continuous time limit of discrete quantum trajectories using martingale change of measure techniques.

Second, we investigate the influence of measurement outcome recording on state preparation using reservoir engineering techniques. We show that measurement outcome recording does not influence the convergence itself. Nevertheless, we find that measurement outcome recording modifies the system behavior before the convergence. We recover an exponential convergence with a rate equivalent to the rate without measurement outcome recording. But we also find a new convergence rate corresponding to an asymptotic stability. This last rate is interpreted as an added non demolition measurement. Hence, the system state converges only after a random time. At this time the convergence can be much faster. We also find a bound on the mean convergence time.

**Keywords:** Open systems, Quantum Stochastic Calculus, Quantum trajectories, Non demolition measurement, Reservoir engineering.



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# 1. Introduction

In 1947, in a celebrated experiment, W. E. Lamb and R. C. Rutherford measured a splitting of the hydrogen emission–adsorption rays where the Dirac equation predicted the presence of only one ray [70]. Later the same year this lifting of the hydrogen energy spectrum degeneracy was explained by H. A. Bethe [31]. He showed that it was a consequence of the interaction of the atom with the electromagnetic field. This effect was thereafter named the Lamb shift.

This well known result is an example of the importance of open systems in quantum physics. Actually, since doing an experiment imply performing a measurement, it includes the interaction of a macroscopic apparatus with the system under study. It is intrinsically an open system situation. Any system in an actual experiment is open.

This is particularly true in quantum optics. We have reach the point where individual particles can be experimentally manipulated on the quantum level. The 2012 Nobel price was awarded to S. Haroche and D. J. Wineland for such achievements. One of the goals of these experiments is the preparation of systems in true quantum states. Hence, states including entanglement or superposition of energy states, the so called Schrödinger cat states. The environment has a tendency to destroy such states and unfortunately modeling its influence is not always straightforward.

Consider for example the question of thermalization of an atom in interaction with a heat bath. Its interaction with the electromagnetic field yields a splitting of the atom energy levels, the Lamb shift. Should we expect a thermalization corresponding to the isolated energy spectrum or to the one induced by the Lamb shift?

Also, the system evolution arising from its interaction with the environment includes, in general, memory effects [36]. Thus, from the knowledge of the system state at present time only, we cannot predict the system state at any future time.

In some physically motivated limits this memory effect can be neglected. Under these approximations, the system evolution can be considered Markovian.

Markovian models are also pertinent to describe quantum optics experiments where systems are manipulated through their repeated interaction with auxiliary systems (light pulses, atoms ...) [63].

In this thesis report we introduce the different Markovian descriptions of open quantum systems developed since the seventies. Particularly we present quantum stochastic processes and the theory of continuous indirect measurements, quantum trajectories. We then focus on the results obtained during this thesis on the longtime behavior of two specific indirect measurement setups.

**Markovian approximation for open systems** To the author knowledge, the first complete justification of a Markovian approximation was made by E. B. Davies in the seventies [46]. He proved that when the evolution induced by the system–environment interaction takes much more time than the evolution due to both the system and the environment being taken isolated one from the other, the system evolution is effectively described by a Markovian evolution. This limit is the weak coupling limit. It is valid when the interaction energy is much smaller

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than all energy scales of both the system and the environment being taken isolated one from the other. It is also called the rotating wave Born–Markov approximation.

This result leads to answer our previous question about which energy spectrum should be considered for the thermalization of an atom, the isolated one or the Lamb shifted one. As we will see in section 2.3.2, in the weak coupling limit and under some convergence conditions [45, 55, 91], the atom thermalizes according to its isolated energy spectrum.

In the eighties, a quantum version of classical stochastic equations was developed, especially by R. L. Hudson and K. R. Parthasarathy [64]. This Quantum Stochastic calculus was and still is a tool frequently used to model quantum optics experiments [57]. We will see in chapter 3 that they can be considered as Markovian limits keeping some information on the environment through some quantum noises. These noises are the quantum equivalent of the classical white noise and Poisson processes. From the nineties up to nowadays different limits leading to quantum stochastic evolution for open systems have been found. Let us cite the work of J. Dereziński and W. de Roeck on the weak coupling [48] and the work of S. Attal and Y. Pautrat on the repeated interaction continuous limit [6].

One of the most interesting corollary to the description of open evolutions through quantum stochastic processes is the derivation of stochastic differential equations modeling the continuous indirect measurement of systems. These stochastic differential equations are called quantum trajectories stochastic differential equations and were first derived from quantum stochastic differential equations by V. P. Belavkin in the late eighties and early nineties [26–28]. At the same time, physicists proposed similar stochastic differential equations either as a mean of master equation simulation (J. Dalibard *et al.* [44]) or as a description of continuous indirect monitoring of quantum optics systems (H. Carmichael [40]). Of course previous attempts to model continuous measurements had been made. We can cite for example the instrument formalism approach of E. B. Davies [46] and the work of A. Barchielli *et al.* [14]. Quantum trajectories are an ideal tool to describe typical quantum optics measurements such as interferometry or photon counting [102].

During this thesis we particularly studied two indirect measurement setups: non demolition measurements and state preparation through reservoir engineering.

**Non demolition measurements and wave function collapse** Based on a work by M. Bauer and D. Bernard [21] inspired by S. Haroche’s group experiment [61] we studied the convergence of non demolition measurements [35, 43]. Non demolition measurements are indirect measurement that leave invariant a given set of system states called the pointer states. The pointer states are generally the system energy eigenstates. By indirect we mean the system is not directly connected to a macroscopic apparatus. An intermediate system, a probe, interacts with the system and is then measured by a macroscopic apparatus. The name non demolition may have two meanings in the literature: either it means that the measurement procedure cannot demolish the measured states, or it means that the system is not destroyed through the measurement process since it is not connected to a macroscopic apparatus directly. We are concerned with the first definition. It is sometimes called the standard non demolition condition. One expects that such measurements will push the system to one of the pointer states with a distribution of the pointer states given by von Neumann projection postulate. Hence one expects to reproduce wave function collapse on the system even if the projective measurement was made only on the probe.

We recall the reader that von Neumann projection postulate states that a quantum system

in state  $|\phi\rangle$  is, after a measurement of a non degenerate observable  $A = \sum_{a \in \text{spec}(A)} a|a\rangle\langle a|$ , in a state  $|a\rangle$  with probability  $|\langle a|\phi\rangle|^2$ .  $a$  takes value in the spectrum of  $A$ . The projection of  $|\phi\rangle$  onto  $|a\rangle$  is also called wave function collapse.

In [21] M. Bauer and D. Bernard showed that a repeated non demolition measurements leads to a wave function collapse. They also showed that the convergence is exponential with an explicit rate. In [20] with M. Bauer and D. Bernard we extended this result to any type of non demolition measurement. In particular we allowed the use of different probing methods with any procedure for the choice among the probing methods. Hence we allowed for a possibly highly non Markovian procedure of measurement. We proved collapse in this context and showed that the use of different probing methods can lead to an increased convergence rate toward the collapse. We showed that even if one starts the computation with an estimated state different from the physical one, one obtain the same limit pointer state. Hence we proved estimation stability. We also studied degenerate measurements. All these results are presented in section 4.3. We added a result on the mean time in which the convergence happen. This result is not contained in the article. The main mathematical tools used to obtain these results were bounded martingale convergence and martingale change of measure. To our knowledge it is the first time such a complete study of wave function collapse through repeated indirect measurement was made.

In the same article [20] we introduced a derivation of a diffusive continuous time limit of repeated non demolition measurements. By continuous limit we mean that we search for a continuous time process describing the limit where an increasing number of indirect measurements are performed in a fixed time. The limit stochastic equation we obtained is a diffusive quantum trajectory stochastic differential equation with an added non demolition property. We proved collapse and exponential convergence for these non demolition diffusive quantum trajectories.

Following this work in [24] with M. Bauer and D. Bernard we derived the continuous time limit of general repeated non demolition measurements. We obtained general quantum trajectory stochastic differential equations with an added non demolition property. This work can be seen as a complementary approach to the continuous time approximation of repeated indirect measurement made by C. Pellegrini [84–86]. In our derivation we used martingale change of measure techniques and obtain the continuous limit of the measurement outcomes. The two approaches to the continuous limit are presented in section 5.2.2.

We completed this work on non demolition measurements in [29] with C. Pellegrini. Starting from a continuous time quantum trajectory stochastic differential equation we found the sufficient and necessary conditions for it to correspond to a non demolition measurement. We proved these conditions are equivalent to the pointer probabilities being martingales. Finally using martingale convergence and change of measure techniques we showed collapse, exponential convergence with explicit rate and estimation stability. Part of these results were already known. But to our knowledge the exponential convergence and the stability of the estimated state are new results. This work is presented in section 5.4. We added a result on the mean time of convergence which is not included in the article cited.

**Reservoir engineering and state preparation** In late nineties and early two thousands, the idea of engineering the interaction of a system with its environment to drive it autonomously to a desired pure state emerged [41, 47, 71, 87, 88, 99]. Some experimental realizations of this idea have been made since [19, 68, 72, 77, 90]. In [33, 96, 97] F. Ticozzi *et al.* gave necessary

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and sufficient conditions on the mean Markovian evolution of a system such that it is driven to a specific subspace of the whole state space. If this subspace is of dimension 1, the system converges to a specific pure state.

In a work still in preparation with B. Cloez, C. Pellegrini and F. Ticozzi we investigate such reservoir engineering but when the engineered environment is measured. Some results were obtained on this subject in [94] by F. Ticozzi *et al.*. In our work we show, in any case, that the convergence in mean is equivalent to the almost sure convergence of the state. We find two different convergence rates. One is equivalent to the average convergence rate and one is greater. This increased convergence rate is interpreted as the consequence of an added non demolition measurement. We obtain also a bound on the mean convergence time. This allows us to interpret of mean convergence rate as an actual convergence rate whereas the increase obtained when measurements are recorded corresponds to an asymptotic stability rate. Hence the first characterizes the time after which the state converges, whereas the second characterizes the speed at which it converges when the convergence happen. To our knowledge the convergence for any type of measurement and the characterization of the convergence are both new results.

These early results are presented for the discrete time setup in section 4.4 and for the continuous time setup in 5.5.

**Structure** The thesis report is structured as follow. In a first chapter we present the general description of open quantum systems. We introduce the different models: Spin-boson, repeated interactions, Caldeira-Leggett and path integral approaches. We focus on the evolution of the system once the environment has been traced out. We call this evolution the reduced system evolution. We introduce completely positive trace preserving maps. For the Spin-Boson model we motivate the necessity of an infinite dimensional continuous energy spectrum environment. We highlight the complexity of the evolution induced by the memory of the environment. We present the general expression of Markovian reduced system evolution. Thus we define completely positive trace preserving semi-groups. We discuss both the long time behavior of completely positive trace preserving semi-groups and of repeated action of completely positive trace preserving maps for the repeated interactions model. We finish the chapter by a presentation of the main limits leading to Markovian reduced system evolution.

In the second chapter, we introduce quantum stochastic calculus. We start by a presentation of the quantum noises and the unitary evolutions allowed by quantum stochastic differential equations. Then we present the dilation of completely positive trace preserving semi-groups through quantum stochastic unitary evolution. We also present two limits leading to quantum stochastic evolutions, the weak coupling limit and the continuous limit of repeated interactions. We conclude this chapter with the presentation of the path integral formalism equivalent to the quantum stochastic evolution. We use this path integral formalism to find the classical limit of quantum stochastic evolutions. We recover the usual Langevin equations.

In a third chapter we present results on repeated indirect measurements, also called discrete quantum trajectories. After a presentation of the setup, we introduce the results of B. Kümmerer and H. Maassen on the long time behavior of general discrete quantum trajectories. Namely we present the ergodic theorem for discrete trajectories they obtained [69] and explain how they showed that in the long time limit, in general, a system following a quantum

trajectory tends to purify [75]. We conclude this chapter by the presentation of our results on non demolition measurements and reservoir engineering in discrete time.

In a fourth and last chapter we present continuous time quantum trajectories. After a general definition we present the different derivations we know of these equations. We present a physical phenomenological approach, the continuous limit of repeated interactions, quantum filtering and finally the unraveling of master equations. Then we present the general long time behavior of such trajectories. First we present the continuous time ergodic theorem of B. Kümmerner and H. Maassen [69]. Then we present the purification along the trajectory showed by A. Barchielli and A. M. Paganoni [17]. In the two last sections of this chapter we present our results on non demolition measurements and reservoir engineering in continuous time.



## 2. Open system Markovian description

### 2.1. Open systems

A system is considered open when it is in interaction with an environment. Namely one does not consider only an atom but this atom and its interaction with the environment. Thus the total system we consider is intrinsically bipartite. The first part is the system, the second is the environment. The two parts have, of course, different roles. The environment is

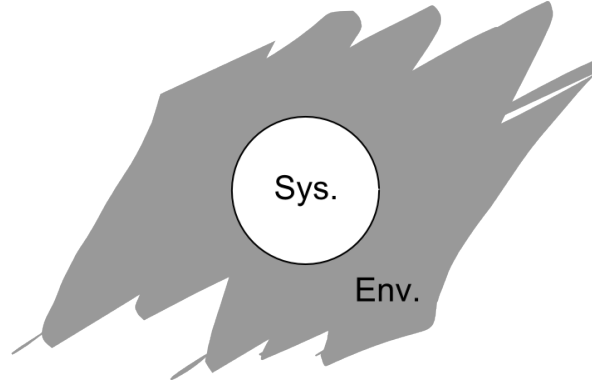


Figure 2.1.: Open system situation.

formally all the world outside the system. Hence it is considered way larger than the system. For example, the system can be a single particle when the environment is composed of a macroscopic number of particles.

Let us recall the basic mathematical formalism of quantum mechanics we will use through this thesis. Once this fixed the relative size criteria between the system and the environment will be much clearer.

**Hilbert spaces and states** A quantum system is described by a Hilbert space  $\mathcal{H}$ . The state of the system is a positive semi definite trace class (hence bounded) operator from the Hilbert space to itself. Its trace is equal to one. So the set of system states is  $\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \text{ s.t. } \rho \geq 0, \text{tr}[\rho] = 1\}$ <sup>1</sup>. We prefer this description to the pure state one where the system state is given by a normalized element of  $\mathcal{H}$ . Indeed the former allow for the description of pure states when the latter is not suited for the description of statistical mixture of pure states. Since we are concerned with open systems, we often encounter mixed states and therefore prefer using density matrices  $\rho$  as states.

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1. Actually this definition is not really suitable for the description of macroscopic systems at strictly positive temperature. One should instead define the state as a positive linear form on an operator algebra. Nevertheless in this thesis we will assume any state can be described by a density matrix  $\rho$ .

## 2. Open system Markovian description

In the traditional formulation of quantum mechanics, physical quantities are represented by selfadjoint operators acting on the Hilbert space. They are not necessarily bounded. These operators are called observables. When performing the measurement of a given observable  $O$  on the system, the outcome  $o(\omega)$  obtained is a random variable. Let us equip the spectrum of  $O$ ,  $\text{spec}(O)$ , with a  $\sigma$ -algebra  $\mathcal{F}$ . The distribution of  $o(\omega)$  is given by  $\mathbb{P}(o(\omega) \in A) = \text{tr}[P_A \rho]$  for any measurable subset  $A$  of  $\text{spec}(O)$ .  $P_A$  is the projector onto the eigensubspace corresponding to the eigenvalues in  $A$ . If  $O$  has discrete non degenerate spectrum,  $\mathbb{P}(o(\omega) = o') = \text{tr}[P_{o'} \rho]$  with  $P_{o'} = |o'\rangle\langle o'|$  where  $|o'\rangle \in \mathcal{H}$  and  $\| |o'\rangle \| = 1$ . After a measurement where we learned that the outcome is in  $A$ , the system state is projected to  $\frac{P_A \rho P_A}{\text{tr}[P_A \rho]}$ . Hence it is a random variable distributed according to  $\mathbb{P}$ . This is the von Neumann projection postulate. The mapping between the observable and the state on one side and the probability space on the other side, is made, in the most general case, thanks to the spectral theorem (see section 5.2.3).

In an open quantum system context we consider two such quantum systems, the actual system and the environment. Let us write their Hilbert spaces  $\mathcal{H}_{\text{sys.}}$  and  $\mathcal{H}_{\text{env.}}$  respectively. The difference between the system and the environment lies in the relative size of  $\mathcal{H}_{\text{env.}}$  with respect to  $\mathcal{H}_{\text{sys.}}$ . It is often assumed that the system Hilbert space is a finite dimensional one, in other words  $\mathcal{H}_{\text{sys.}} = \mathbb{C}^{d_{\text{sys.}}}$ ,  $d_{\text{sys.}} < \infty$ , whereas the environment Hilbert space has infinite dimension. It attempts to translate the macroscopic size of the environment. Actually these conditions are neither sufficient nor the most general.

In many cases the system can be a simple particle, then its Hilbert space is  $L^2(\mathbb{R}^d; \mathbb{C}^k)$ , the space of square integrable functions from  $\mathbb{R}^d$  to  $\mathbb{C}^k$ . It is infinite dimensional. Nevertheless in this thesis we limit ourselves to finite dimensional system Hilbert spaces  $\mathcal{H}_{\text{sys.}} = \mathbb{C}^{d_{\text{sys.}}}$ . They are good representations of the systems we are interested in. For a particle or a field, one can often cut off the higher energy level states. Note that the space of operators on  $\mathcal{H}_{\text{sys.}}$  is then the space of  $d_{\text{sys.}} \times d_{\text{sys.}}$  complex matrices. Thus all the operators on  $\mathcal{H}_{\text{sys.}}$  are bounded. Their adjoint with respect to the canonical scalar product over  $\mathbb{C}^{d_{\text{sys.}}}$  are their Hermitian conjugated  $X^* = X^\dagger$ . We alert the reader whenever the system Hilbert space is not  $\mathbb{C}^{d_{\text{sys.}}}$ .

On the environment side, the condition of an infinite dimensional Hilbert space is not sufficient. A single particle is not really a good model for a whole environment, yet its Hilbert space is infinite dimensional. We need to furthermore assume that the environment is macroscopic, thus we do not bound the possible number of particles. For simplicity we assume the environment has a simple structure. Therefore the environment is assumed to be a field of free bosons. The Hilbert space is then a symmetric Fock space over the one particle space  $\Gamma(L^2(\mathbb{R}^d; \mathbb{C}^k))$ . The one particle states take value in  $\mathbb{C}^k$  the possibility of  $k > 1$  allows for the introduction of spin degrees of freedom. Note that we do not treat cases with fermions in the environment.

Finally the Hilbert space we consider for the description of an open quantum system through out this thesis is

$$\mathcal{H}_{\text{tot.}} = \mathcal{H}_{\text{sys.}} \otimes \mathcal{H}_{\text{env.}} = \mathbb{C}^{d_{\text{sys.}}} \otimes \Gamma(L^2(\mathbb{R}^d; \mathbb{C}^k)). \quad (2.1)$$

It is the Hilbert space of the celebrated spin-boson model. The dimension  $d$  refers to the dimension of space. Thus we should fix it to 3 since we consider we are in a non relativistic situation, but we keep for now this freedom since we will encounter symmetric Fock spaces with  $d = 1$ . We will also encounter environments which are not Fock spaces. We will describe these later.

## 2.1. Open systems

In full generality, the set of states we are concerned with is  $\mathcal{S}(\mathcal{H}_{\text{tot.}})$ . Nevertheless we often are interested only with the system state evolution. Thus from the total state  $\rho_{\text{tot.}} \in \mathcal{S}(\mathcal{H}_{\text{tot.}})$ , we defined a system state, the reduced state, by taking a partial trace over the environment Hilbert space.

$$\rho_{\text{sys.}} = \text{tr}_{\mathcal{H}_{\text{env.}}}[\rho_{\text{tot.}}]. \quad (2.2)$$

The total system  $\rho_{\text{tot.}}$  is thus sometimes referred to as the extended state.

Note that if  $\rho_{\text{tot.}} = \rho \otimes \rho_{\text{env.}}$ , then  $\rho_{\text{sys.}} = \rho$ . We always assume the initial total state has such factorization property. Namely, the system and the environment are initially independent. They are disentangled and statistically uncorrelated.

**Evolution and Poincaré recurrence theorem** The reversible evolution of a state between two times is given by a unitary operator  $U$ . The state after the evolution is  $U\rho U^*$ . In continuous time, the state at any time  $t$  is given by a group of unitaries  $U_t$ . We have  $\rho(t) = U_t \rho U_t^*$ . The group generator is the Hamiltonian  $H$ , the total energy observable<sup>2</sup>:  $U_t = e^{-itH}$ . We usually write this Hamiltonian in three parts

$$H = H_{\text{sys.}} \otimes I_{\text{env.}} + I_{\text{sys.}} \otimes H_{\text{env.}} + H_{\text{int.}}. \quad (2.3)$$

The first two terms correspond to the free evolution of respectively the system and the environment. The term free refers to the evolution when the system and the environment do not interact.  $H_{\text{sys.}}$  is the energy observable of the system alone and  $H_{\text{env.}}$  is the one of the environment. We usually have  $H_{\text{env.}} = \int d\mathbf{p} \omega(\mathbf{p}) a_{\mathbf{p}}^* a_{\mathbf{p}}$  with  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^*$  the annihilation and creation operators acting on the bosonic Fock space. They satisfy the canonical commutation relations (CCR)

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^*] = \delta(\mathbf{p} - \mathbf{p}'). \quad (2.4)$$

Note that we dropped the spin degrees of freedom. For the sake of simplicity for now let us assume  $k = 1$ . We will specify later when  $k > 1$ .  $\omega(\mathbf{p})$  is the dispersion relation giving the energy given the impulsion of a particle.

The interaction Hamiltonian  $H_{\text{int.}}$  is an operator acting on  $\mathcal{H}_{\text{sys.}} \otimes \mathcal{H}_{\text{env.}}$ . It assures that the evolution does not reduce trivially to a free one. Note that we do not allow time dependent Hamiltonian. Formally this would mean that there is an additional classical environment, namely the experimenter adjusting a parameter, which is not described by the present Hamiltonian. This is a generalization beyond the scope of this thesis.

From now on whenever the definition is clear we will not adopt a different notations for  $X$  acting on  $\mathcal{H}_{\text{sys.}}$  and  $X \equiv X \otimes I_{\text{env.}}$  acting on  $\mathcal{H}_{\text{tot.}}$ . Similarly  $Y \equiv I_{\text{sys.}} \otimes Y$  for  $Y$  acting on  $\mathcal{H}_{\text{env.}}$ . Hence for example,  $H_{\text{sys.}} \equiv H_{\text{sys.}} \otimes I_{\text{env.}}$ .

The reduced state evolution is given by a time dependent map  $\Phi_t$  such that

$$\rho_{\text{sys.}}(t) = \Phi_t(\rho_{\text{sys.}}) = \text{tr}_{\mathcal{H}_{\text{env.}}}[U_t \rho_{\text{sys.}} \otimes \rho_{\text{env.}} U_t^*] \quad (2.5)$$

with  $\rho_{\text{sys.}}$  the initial system state.

Often we expect the system state will relax to a stationary state. This adds a new constrain on the spectrum of  $H$ , the energy observable of the total system (system+environment). Its

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2. Through out the text we set  $\hbar = c = 1$ . At one point we introduce back  $\hbar$ . We alert the reader at the time.

## 2. Open system Markovian description

spectrum should at least have a continuous part. If not the total state is quasi periodic and therefore there is good chance that the reduced system state is also periodic. It is a consequence of Poincaré recurrence theorem [32].

Assume  $H$  has a discrete spectrum  $\text{spec}(H) = \{E_n\}_{n \in \mathbb{N}}$  with eigenvectors  $\{|n\rangle\}_{n \in \mathbb{N}} \subset \mathcal{H}_{\text{tot.}}$ . Let  $\rho = \sum_{n,k} c_{nk} |n\rangle\langle k|$  be the initial state of the total system. We note the Frobenius norm  $\|X\|_F^2 = \text{tr}[X^*X]$ . We have explicitly

$$\|\rho_{\text{tot.}}(t) - \rho_{\text{tot.}}\|_F^2 = 2 \sum_{k,n} (1 - \cos(\omega_{nk}t)) |c_{nk}|^2 \quad (2.6)$$

with  $\omega_{nk} = E_k - E_n$ . Since  $\|\rho\|_F \leq 1$  for any state  $\rho$ , the sum converges. Hence for any  $\epsilon > 0$ , it exists a time  $n_0$  such that  $2 \sum_{n > n_0} \sum_k (1 - \cos(\omega_{nk}t)) |c_{nk}|^2 + \sum_{n \leq n_0} \sum_{k > n_0} (1 - \cos(\omega_{nk}t)) |c_{nk}|^2 < \epsilon/2$ . Hence

$$\|(\rho_{\text{tot.}}(t) - \rho_{\text{tot.}})\|_F^2 \leq 2 \sum_{k,n=1}^{n_0} (1 - \cos(\omega_{nk}t)) |c_{nk}|^2 + \epsilon/2. \quad (2.7)$$

We have a finite number of different  $\omega_{nk}$  involved in the remaining sum. Thus using Dirichlet's theorem on diophantine approximations, it exists  $T_0 < \infty$  such that  $|1 - \cos(\omega_{nk}T_0)| \leq \epsilon/2$  for any  $n, k \leq n_0$ . Since  $\text{tr}[\rho_{\text{tot.}}^2] \leq 1$ , we finally have

$$\|\rho_{\text{tot.}}(T_0) - \rho_{\text{tot.}}\|_F^2 \leq \epsilon. \quad (2.8)$$

Thus whatever is the initial total state, we will get back as close as we want to it in a finite time. Thus any total initial state is quasi periodic.

When  $\text{spec}(H)$  has at least a continuous part we can no longer apply Dirichlet's theorem, thus we are not sure we will have quasi periodicity for any initial state. In this case the recurrence time  $T_0$  can be infinite. One can understand this saying the recurrence time is of the order of the inverse of the minimal spacing between two energy levels. Hence if this spacing goes to zero the recurrence time explodes towards infinity.

Let us remark that if both  $H_{\text{sys.}}$  and  $H_{\text{env.}}$  have discrete spectrum, then there is a good chance that  $H$  has a discrete one too. To try to avoid this and for simplicity we assume  $H_{\text{env.}}$  spectrum is the real line:  $\text{spec}(H_{\text{env.}}) = \mathbb{R}$ . Note that since  $\mathcal{H}_{\text{sys.}}$  is finite dimensional, there is no chance  $H_{\text{sys.}}$  has continuous spectrum.

### 2.1.1. System evolution and completely positive maps

We can say a little more on the reduced evolution  $(\Phi_t)_{t \in \mathbb{R}_+}$ . It is actually a family of completely positive (CP) trace preserving (TP) map.

Complete positivity is a stronger property than only positivity. Positivity means that the image of any positive operator through the map  $\Phi_t$  is positive. Complete positivity impose that any extended version of  $\Phi_t$  to a larger finite dimensional Hilbert space is still positive. Namely, for any  $k \geq 0$ ,  $\Phi_t \otimes I_k$  is positive with  $I_k$  the identity on the  $k \times k$  complex matrix space. Counter intuitively, positivity does not imply complete positivity.

Let  $T : \mathcal{B}(\mathbb{C}^2) \rightarrow \mathcal{B}(\mathbb{C}^2)$ ,  $T(X) = I \text{tr}[X] - X$ . This map is positive. Let us now consider its extension to  $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ . Let  $P_-$  be the projector on  $|- \rangle = \frac{1}{\sqrt{2}}(|0 \otimes 1\rangle - |1 \otimes 0\rangle)$  where  $\{|0\rangle, |1\rangle\}$  is an orthonormal basis of  $\mathbb{C}^2$ . Then

$$\langle - | T \otimes I_2 (P_-) - \rangle = \langle - | (\frac{1}{2} I_4 - P_-) - \rangle = -\frac{1}{2} < 0. \quad (2.9)$$

Hence  $T \otimes I_2$  is not positive, therefore  $T$  is not completely positive.

It is clear why we expect quantum evolution to be completely positive and not only positive. A system can be in contact with any other system of arbitrary size. Their joint state can be entangled. But if the evolution only acts on the initial system and not on the second one, the joint state obtained should still be positive. If it is not the case, we could have measurement on the resulting joint state with outcomes of negative probability. To assure that the operator obtained after the evolution is actually a state we must also have  $\text{tr}[\Phi_t(\rho)] = \text{tr}[\rho] = 1$ . Hence the evolution completely positive map must be also trace preserving (TP).

CPTP maps have nice properties [103]. For instance they always accept at least one invariant state  $\rho_{\text{inv.}}$ ,  $\Phi_t(\rho_{\text{inv.}}) = \rho_{\text{inv.}}$ . For a general family of CPTP maps  $\{\Phi_t\}$ ,  $\rho_{\text{inv.}}$  might depend on  $t$ . We also have that  $\Phi : \mathcal{B}(\mathcal{H}_{\text{sys.}}) \rightarrow \mathcal{B}(\mathcal{H}_{\text{sys.}})$  is a CP map if and only if we can find a family of operators  $\{M_j\}_{1 \leq j \leq d_{\text{sys.}}^2} \subset \mathcal{B}(\mathcal{H}_{\text{sys.}})$  such that

$$\Phi(\rho_{\text{sys.}}) = \sum_j M_j \rho_{\text{sys.}} M_j^*. \quad (2.10)$$

It is the Kraus (or Stinespring) decomposition of CP maps. If moreover  $\Phi$  is TP,  $\sum_j M_j^* M_j = I_{\text{sys.}}$ .

Let us now show that the map corresponding to the reduced evolution of the system state up to time  $t$  is always a CPTP map. Actually we prove that the dual of the map is a CP unital map. The dual  $\Phi_t^*$  of  $\Phi_t$  is the map such that for any system state  $\rho_{\text{sys.}}$  and any  $X \in \mathcal{B}(\mathcal{H}_{\text{sys.}})$ ,  $\text{tr}[\Phi_t^*(X) \rho_{\text{sys.}}] = \text{tr}[X \Phi_t(\rho_{\text{sys.}})]$ . It is unital if  $\Phi_t^*(I_{\text{sys.}}) = I_{\text{sys.}}$ . It is clear that this is equivalent to  $\Phi_t$  being TP. Let  $\rho_{\text{env.}} : \mathcal{B}(\mathcal{H}_{\text{tot.}}) \rightarrow \mathcal{B}(\mathcal{H}_{\text{sys.}})$  be the positive map such that for any  $X \in \mathcal{B}(\mathcal{H}_{\text{tot.}})$ ,  $\rho_{\text{env.}}(X) = \text{tr}_{\text{env.}}[X(I_{\text{sys.}} \otimes \rho_{\text{env.}})]$ . Hence it maps the operators on the total Hilbert space to “reduced” operators on the system Hilbert space. Then for any  $X \in \mathcal{B}(\mathcal{H}_{\text{sys.}})$ ,

$$\Phi_t^*(X) = \rho_{\text{env.}}(U_t^* X U_t). \quad (2.11)$$

We decompose  $U_t$  on an orthonormal basis of  $\mathcal{B}(\mathcal{H}_{\text{sys.}})$ ,  $\{B_j\}_{1 \leq j \leq d_{\text{sys.}}^2} : U_t = \sum_j B_j \otimes U_j(t)$  with  $\{U_j(t)\}$  a family of operators on  $\mathcal{H}_{\text{env.}}$ . It follows that

$$\Phi_t^*(X) = \sum_{j,k} \rho_{\text{env.}}(U_j(t)^* U_k(t)) B_j^* X B_k. \quad (2.12)$$

The matrix  $P(t) = (\rho_{\text{env.}}(U_j(t)^* U_k(t)))_{1 \leq j,k \leq d_{\text{sys.}}^2}$  is positive semi definite. Thus it can be diagonalized. Let the unitary  $u(t)$  be such that  $P(t) = u^*(t) D(t) u(t)$  with  $D(t)$  a non negative diagonal matrix. Let  $M_j(t) = \sqrt{D_j(t)} \sum_k u_{jk(t)}(t) B_k$ , then

$$\Phi_t^*(X) = \sum_j M_j(t) X M_j^*(t). \quad (2.13)$$

Moreover since  $U_t^* U_t = I_{\text{tot.}}$ ,  $\Phi_t^*(I_{\text{sys.}}) = I_{\text{sys.}}$ .

Finally the reduced system evolution is given at each time by a time dependent CPTP map

$$\rho_{\text{sys.}}(t) = \Phi_t(\rho_{\text{sys.}}) = \sum_j M_j(t) \rho_{\text{sys.}} M_j^*(t). \quad (2.14)$$

Note that we never used the specific properties of the environment. Actually any reduced evolution starting from a bipartite system leads to a CPTP map for the reduced system

## 2. Open system Markovian description

evolution as soon as the states of the two parts are independent. Namely, as soon as the joint initial state can be factorized into a direct product of states of each part.

Conversely using Stinespring representation of CP unital maps, one can show that for any CPTP map  $\Phi$ , there exists a Hilbert space  $\mathbb{C}^{d'}$ , a unitary operator  $U$  acting on  $\mathbb{C}^{d_{\text{sys.}}} \otimes \mathbb{C}^{d'}$  and a state  $\rho'$  on  $\mathbb{C}^{d'}$  such that

$$\Phi(\rho_{\text{sys.}}) = \text{tr}_{\mathbb{C}^{d'}} [U(\rho_{\text{sys.}} \otimes \rho')U^*]. \quad (2.15)$$

Hence a CPTP map can always be thought as the reduced evolution of a larger system unitary evolution [103]. This is called the CPTP map dilation.

### 2.1.2. Repeated interactions

Besides the Spin–Boson model another open system model has recently gained attention from theoretical physicists and mathematicians [6, 20, 21, 24, 33, 37, 86], the repeated interaction model. It does not really correspond to a natural physical situation, but it has a true experimental relevance [62, 63, 101]. In this model, the environment is not a free boson field but is composed of infinitely many auxiliary systems (or probes) whose Hilbert spaces are finite dimensional.

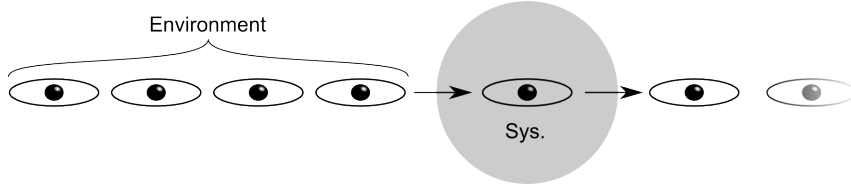


Figure 2.2.: Schematic of a repeated interaction setup.

The setup goes as follows. A set of identical auxiliary systems (probes) described by copies of a finite dimensional Hilbert space  $\mathcal{H}_{\text{probe}} = \mathbb{C}^{d_{\text{probe}}}$  are prepared in the same state  $\rho_{\text{probe}}$ . Thus the total Hilbert space is

$$\mathcal{H}_{\text{sys.}} \otimes \bigotimes_{n=1}^{\infty} \mathcal{H}_{\text{probe}} \quad (2.16)$$

and the initial state is

$$\rho_{\text{tot.}} = \rho_{\text{sys.}} \otimes \bigotimes_n \rho_{\text{env.}} \quad (2.17)$$

The evolution is given by repeated interactions of the system with the probes. First the system interacts with the first probe for a given time. Then the interaction stops and the system interacts with the second probe during the same time. The interaction sequence is then repeated.

Let  $U$  be the unitary interaction operator between the system and one probe. Let  $U_n$  be the corresponding interaction between the system and the  $n^{\text{th}}$  probe. Namely it acts non trivially only on  $\mathcal{H}_{\text{sys.}}$  and the  $n^{\text{th}}$  copy of  $\mathcal{H}_{\text{probe}}$ . We have the recurrence equation

$$\rho_{\text{tot.}}(n+1) = U_{n+1} \rho_{\text{tot.}}(n) U_{n+1}^*. \quad (2.18)$$

Hence at time  $n$  the total state is

$$\rho_{\text{tot.}}(n) = (U_n U_{n-1} \dots U_2 U_1 (\rho_{\text{sys.}} \otimes \bigotimes_{k=1}^n \rho_{\text{env.}}) U_1^* U_2^* \dots U_{n-1}^* U_n^*) \otimes \bigotimes_{k>n} \rho_{\text{env.}}. \quad (2.19)$$

Up until now we did not assume anything on what happens to the probes after they have interacted with the system. We will see that we can either assume they still exist or they can be measured. The former leads, in the continuous time limit, to a quantum stochastic evolution of the system state. In the latter case, the system state evolution is described by a stochastic process called quantum trajectory. For now we will look at what happens when we forget about the probes after their interaction with the system. Namely when we trace out their degrees of freedom.

Since the probes are all initially independent and the interaction happens each time between the system and only one probe, the recurrence equation for the system reduced state evolution can be simplified to

$$\rho_{\text{sys.}}(n+1) = \text{tr}_{\text{probe}}[U \rho_{\text{sys.}}(n) \otimes \rho_{\text{probe}} U^*] = \Phi(\rho_{\text{sys.}}(n)). \quad (2.20)$$

Hence the evolution corresponds to the repeated action of the CPTP map  $\Phi$ .

$$\rho_{\text{sys.}}(n) = \Phi^n(\rho_{\text{sys.}}). \quad (2.21)$$

We can then learn a lot on the long time limit of  $\rho_{\text{sys.}}(n)$  from the properties of CPTP maps. First the spectral radius of a CPTP map is 1, thus its eigenvalues are all on the complex unit disc. Second, let us assume  $\Phi$  is irreducible, namely if there is no Hermitian projector  $P \notin \{0, I_{\text{sys.}}\}$  such that the subset of state  $\mathcal{S}(P\mathcal{H}_{\text{sys.}})$  is stable under  $\Phi$ . We can apply Perron–Frobenius theorem for CP maps [54]. Thus,  $\Phi$  accept a unique positive definite invariant state  $\rho_{\text{inv.}} > 0$ . Still it is not necessarily the only eigenvector with unit norm eigenvalue, we can have eigenvectors  $\rho_{\text{per.}}$  such that  $\Phi(\rho_{\text{per.}}) = e^{i\theta} \rho_{\text{per.}}, \theta \in \mathbb{R}$ . Hence it can exist periodic states.

If  $\Phi$  has no periodic state and a unique invariant state then  $\lim_n \rho_{\text{sys.}}(n) = \rho_{\text{inv.}}$  for any initial system state. The only  $\Phi$  eigenvalue with unit norm is 1. Thus the convergence is exponential. Let  $\lambda_1 = \max\{|\lambda|, \lambda \in \text{spec}(\Phi) \setminus \{1\}\} < 1$ . Then for any  $\epsilon > 0$ ,

$$\rho_{\text{sys.}}(n) = \rho_{\text{inv.}} + O((\lambda_1 + \epsilon)^n). \quad (2.22)$$

Hence as long as  $\lambda_1 + \epsilon < 1$ , we have exponential convergence. This is always allowed since  $\epsilon = (1 - \lambda_1)/2$  is suitable.

Of course  $\Phi$  is not always irreducible. Particularly, for non demolition measurements and reservoir engineering,  $\Phi$  will be reducible.

### 2.1.3. Caldeira–Leggett model

In this subsection let us just cite a celebrated model for the treatment of open quantum systems, the Caldeira–Leggett model [38]. It was introduced by O. Caldeira and A. Leggett as a quantum Brownian particle model. Using a high temperature and weak coupling limit, they obtained stochastic differential equations equivalent to the classical damped Langevin equations. Though this limit has important issues. If they are not corrected they can lead to negative probabilities [53].

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The idea of Caldeira–Leggett model is to let a particle we are interested in interact with infinitely many independent harmonic oscillators. The Hamiltonian is

$$\begin{aligned}
H(x, p, q, \pi) = & \frac{p^2}{2M} + V(x) \\
& + \sum_n \frac{\pi_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2 q_n^2 \\
& - x \sum_n \kappa_n \sqrt{\frac{\hbar}{m_n\omega_n}} q_n \\
& + x^2 \sum_n \frac{\kappa_n}{2m_n\omega_n^2}
\end{aligned} \tag{2.23}$$

where  $x, p$  are the system particle position and impulsion operators respectively.  $\{q_n, \pi_n\}$  are position and impulsion operators of the environment harmonic oscillators. The first line corresponds to the free evolution of the system, the second line to the one of the environment. The third line is the interaction Hamiltonian. We note

$$B = \sum_n \kappa_n \sqrt{\frac{\hbar}{m_n\omega_n}} q_n \tag{2.24}$$

the environment field entering in the interaction. Remark that the interaction is assumed linear in the environment degrees of freedom. The last term is a counter term compensating some renormalization appearing in the high temperature–weak coupling limit. The interaction Hamiltonian can be generalized by replacing  $x$  by any function of  $x$  or of  $p$ .

Note that if we assume that the environment spectrum is actually continuous and that the potential is confining enough such that the isolated system spectrum is discrete, we fall back to the spin–boson model but with an infinite dimensional system Hilbert space.

This model was used, for example, by O. Caldeira and A. Leggett to study the effect of dissipation on tunneling. They did a semi-classical analysis of the path integral corresponding to the thermal equilibrium state of the total system+environment [39].

We refer the interested reader to the book *Theory of Open Quantum Systems* by H.P. Breuer and F. Petruccione [36] for a more general discussion of this model. The book *Quantum Dissipative Systems* by U. Weiss [100] maybe a good choice for a comprehensive review on such approach to open quantum systems.

### 2.1.4. Path integral approach

A widely used computational tool of quantum physics is the celebrated Feynman path integral. In the context of open systems two special applications of path integral may be distinguished.

First, there is the equivalent to the reduced evolution, the Feynman–Vernon influence functional. One assumes that the environment is composed of harmonic oscillators and that the initial environment state is a Gaussian stationary state of the environment free evolution. Then one expresses the evolution of the system in terms of a path integral and traces over the environment degrees of freedom. Hence starting from the Hamiltonian we have given for Caldeira–Leggett model, the path integral giving the probability to measure a particle in  $x_f$

after a time  $t$  knowing it started at position  $x_i$  is

$$P_t(x_f; x_i) = \int_{x_{\pm}(0)=x_i}^{x_{\pm}(t)=x_f} \mathcal{D}[x_{\pm}] e^{i(S_0[x_+] - S_0[x_-]) - \Gamma[x_+, x_-]} \quad (2.25)$$

where

$$S_0[x] = \int_0^t \frac{1}{2} m \dot{x}(s)^2 - V(x(s)) ds. \quad (2.26)$$

If  $B = 0$  for any  $n$  then  $\Gamma = 0$  and we recover the path integral of the isolated system. One could wonder if we should have a non null counter term still present. Actually since  $B = 0$  implies that the counter term is null. We truly recover the free system evolution.

If  $B \neq 0$ , then the influence functional

$$e^{-\Gamma[x_+, x_-]} \quad (2.27)$$

is not null. The expression of  $\Gamma[x_+, x_-]$  only depends on the two time correlation function of  $B(t) = e^{itH_{\text{env}}} B e^{-itH_{\text{env}}}$ :  $G(t-s) = \text{tr}[B(t)B(s)\rho_{\text{env}}]$ . Note that  $\overline{G(t-s)} = \text{tr}[B(s)B(t)\rho_{\text{env}}]$ . We have

$$\begin{aligned} \Gamma[x_+, x_-] = \int_0^t ds \int_0^s du & x_+(s)G(s-u)x_+(u) + x_-(s)\overline{G(s-u)}x_-(u) \\ & - x_+(s)G(s-u)x_-(u) - x_-(s)\overline{G(s-u)}x_+(u). \end{aligned} \quad (2.28)$$

In this case we cannot reduce the number of path involved in the integral. We have a forward path  $x_+$  and a backward one  $x_-$ . The forward one corresponds to the path integral description of the unitary forward evolution  $U_t$  when  $x_-$  corresponds to  $U_t^*$ . Once the environment path are integrated they are correlated with one another.

If one perform a classical limit on this path integral, namely setting  $x_{\pm} = x_c \pm \hbar q$  and taking the limit  $\hbar$  to 0, one recover the classical Martin-Siggia-Rose path integral for diffusive stochastic processes [76].

Second, if one is interested in questions like transport through random media, one would like to keep the information on the evolution of the environment. Then one cannot use the influence functional since the environment degrees of freedom cannot be traced out.

The path integral used to compute the different correlation functions between the different observables of the total system+environment are computed using a full path integral. Let  $\mathcal{O}[x, q] = \hat{x}_1(t_1) \cdots \hat{x}_k(t_k)$  be the observable whose expectation is to be computed. The operators  $\hat{x}_j(t_j)$  are either position operators on the system or the environment. The time parameter  $t_j$  indicate that the measurement of such observable is performed at this time. We assume  $j > k \Leftrightarrow t_j < t_k$ . We could have choose the inverse time ordering. Then the path integral used to compute the average of  $\mathcal{O}[x]$  is

$$\begin{aligned} \langle \mathcal{O}[x] \rangle = \int \mathcal{D}[x_{\pm}, p_{\pm}; q_{\pm}, \pi_{\pm}] & \left( \prod_j x_{+,j}(t_j) \right) \exp \left[ i(S_0[x_+, p_+] - S_0[x_-, p_-]) \right. \\ & + i(S_{\text{env.}}[q_+, \pi_+] - S_{\text{env.}}[q_-, \pi_-]) \\ & + i(S_{\text{int.}}[x_+, q_+] - S_{\text{int.}}[x_-, q_-]) \\ & \left. \times \rho_{\text{sys.}}(x_+(0), x_-(0)) \rho_{\text{env.}}(q_+(0), q_-(0)) \right]. \end{aligned} \quad (2.29)$$

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If the inverse time ordering is chosen, the  $+$  index of  $x_{+,j}(t_j)$  is changed to  $-$ . The different parts of the action are the following.

$$S_0[x, p] = \int_0^\infty p(s)\dot{x}(s) - \frac{p(s)^2}{2M} - V(x(s))ds \quad (2.30)$$

is the free system action.

$$S_{\text{env.}}[q, \pi] = \sum_n \int_0^\infty \pi_n(s)\dot{q}_n(s) - \frac{\pi_n(s)^2}{2m_n} - \frac{1}{2}m_n\omega_n^2 q_n(s)^2 ds \quad (2.31)$$

is the free environment action.

$$S_{\text{int.}}[x, q] = - \int_0^\infty x(s)B(s) - x(s)^2 \frac{\kappa_n}{2m_n\omega_n^2} ds \quad (2.32)$$

is the interaction action. The second term under the integral is the counter term. The two last terms  $\rho_{\text{sys.}}(x_+, x_-)$  and  $\rho_{\text{env.}}(q_+, q_-)$  fix the initial conditions on the paths from the initial states. If they are thermal states they can be expressed through an imaginary time path integral.

The environment part can be changed to an harmonic field depending on the context. Note that the forward and backward paths are not correlated.

Starting from this path integral a common improvement is made using Keldysh formalism. Using an adapted change of variables and the symmetries in the two times correlation functions without interaction, one can simplify diagrammatic expansion based computations. We refer the interested reader to [66] for a presentation of this formalism.

In chapter 3 we will give the path integral formulation for quantum stochastic differential equations and recover through this formalism their classical limit.

## 2.2. Markovian evolution

### 2.2.1. Memory issue

Let us turn back to the system reduced evolution  $\rho_{\text{sys.}}(t) = \Phi_t(\rho_{\text{sys.}})$ . Let us introduce a typical open system Hamiltonian. We assume the interaction is linear in the field operators of the environment. We have

$$H = \lambda^2 H_{\text{sys.}} \otimes I_{\text{env.}} + I_{\text{sys.}} \otimes H_{\text{env.}} + \lambda(D^* \otimes a(g) + D \otimes a^*(g)) \quad (2.33)$$

where  $a(g) = \int_{\mathbb{R}^d} \bar{g}(k)a_k dk$  with  $\bar{g}$  is the complex conjugate of  $g$ . The scaling in  $\lambda$  we introduce here is, for now, purely practical.

We say the interaction Hamiltonian is dipolar like:  $H_{\text{int.}} = \lambda(D^* \otimes a(g) + D \otimes a^*(g))$ . The Hamiltonian describing the interaction of a spin with the electromagnetic field has such expression.

Let  $U_{\text{env.}}(t) = e^{-itH_{\text{env.}}}$  and  $V_t = U_{\text{env.}}^*(t)U_t$ .  $V_t$  is a time dependent unitary operator. From now on the total state evolution is given by  $\rho_{\text{tot.}}(t) = V_t\rho_{\text{tot.}}V_t^*$ . Any system observable  $X$  expectation at time  $t$  is given by  $\text{tr}[X\rho_{\text{tot.}}(t)]$ . Whereas any environment observable  $Y$  expectation is then given by  $\text{tr}[Y(t)\rho_{\text{tot.}}(t)]$  with  $Y(t) = U_{\text{env.}}^*(t)YU_{\text{env.}}(t)$ .

We are only interested in the evolution of the system. So we look at the evolution of the reduced system state.

$$\Phi_t(\rho_{\text{sys.}}) = \text{tr}_{\text{env.}}[V_t(\rho_{\text{sys.}} \otimes \rho_{\text{env.}})V_t^*]. \quad (2.34)$$

Let us also assume that  $\rho_{\text{env.}}$  is a Gaussian stationary state of the evolution given by the group of unitaries  $(e^{-itH_{\text{env.}}})$ . Then  $\text{tr}[a(g)\rho_{\text{env.}}] = 0$ . We note

$$G(t-s) = \text{tr}[e^{i(t-s)H_{\text{env.}}} a^*(g) e^{-i(t-s)H_{\text{env.}}} a(g) \rho_{\text{env.}}] \quad (2.35)$$

and  $G_0(t-s) = \int_{\mathbb{R}^d} d\mathbf{p} e^{i(t-s)\omega(\mathbf{p})} |g(\mathbf{p})|^2$  the two time environment correlation functions. We have  $G(t-s) = \int_{\mathbb{R}^d} d\mathbf{p} e^{i(t-s)\omega(\mathbf{p})} n(\mathbf{p}) |g(\mathbf{p})|^2$  with  $n(\mathbf{p})$  the density of particle with impulsion  $\mathbf{p}$ .

Expanding  $\Phi_t$  in Dyson series up to  $\lambda^2$  terms, we obtain

$$\begin{aligned} \rho_{\text{sys.}}(t) - \rho_{\text{sys.}} &= -i\lambda^2 \int_0^t [H_{\text{sys.}}, \rho_{\text{sys.}}] ds \\ &\quad - \lambda^2 \int_0^t ds \int_0^s du \left\{ G(s-u) \left( DD^* \rho_{\text{sys.}} + \rho_{\text{sys.}} D^* D - D \rho_{\text{sys.}} D^* - D^* \rho_{\text{sys.}} D \right) \right. \\ &\quad \quad \quad + \text{h.c.} \\ &\quad \quad \quad \left. + G_0(s-u) \left( \rho_{\text{sys.}} D^* D - D \rho_{\text{sys.}} D^* \right) + \text{h.c.} \right\} \\ &\quad + \dots \end{aligned} \quad (2.36)$$

The terms in  $\lambda^{2n}$  missing are not necessarily of order lower than  $O(\lambda^2)$  as we will soon see. Already we learn a lot from this first  $\lambda^2$  term. We see that the memory effect due to the bath and given by  $G(t-s)$  and  $G_0(t-s)$  implies a complex time dependency of the dynamic. It involves a double integration of the two times correlation functions. Higher power of  $\lambda$  terms involve even more complex integrals. This complexity is a signature of the memory of previous effect of the system on the environment. One can rely on Nakajima-Zwanzing projection technique [36] to try to simplify the problem, but in general it is not much more simpler than the total system+environment evolution.

In this thesis we are interested in cases where the complexity implied by the memory actually disappear. The memory effect becomes negligible, namely  $G(t-s) \sim \delta(t-s)$ . The reduced system evolution becomes Markovian. The system state at time  $t$  only depends on the state at a previous time  $s$  and not on the whole history from 0 to  $t$ . We have  $\rho_{\text{sys.}}(t) = \Phi_{t-s}(\rho_{\text{sys.}}(s))$  for any  $t \geq s$ .

In next subsection we unravel the differential equation the system state must satisfy if we assume the evolution is Markovian. We study its properties. Afterwards we present some physically meaningful limits leading to Markovian reduced system evolution.

### 2.2.2. The master equation and relaxation to equilibrium

We already saw that every CPTP map can be decomposed in terms of Kraus operators. Namely  $\Phi$  is a CPTP map if and only if  $\Phi(\rho_{\text{sys.}}) = \sum_j M_j \rho_{\text{sys.}} M_j^*$  with  $\{M_j\}$  a family of operators on the system Hilbert space such that  $\sum_j M_j^* M_j = I_{\text{sys.}}$ .

Similarly if the reduced evolution in continuous time is Markovian, then it is given by a continuous semi-group of CPTP maps whose generator  $\mathcal{L}$  has a characteristic expression.

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Namely if for any  $t \geq s$  and any  $\rho_{\text{sys.}}$ ,  $\Phi_t(\rho_{\text{sys.}}) = \Phi_{t-s}(\Phi_s(\rho_{\text{sys.}}))$  then

$$\Phi_t(\rho_{\text{sys.}}) = e^{t\mathcal{L}}\rho_{\text{sys.}} \quad (2.37)$$

with

$$\mathcal{L}(\rho_{\text{sys.}}) = -i[H, \rho_{\text{sys.}}] + \sum_j L_j \rho_{\text{sys.}} L_j^* - \frac{1}{2} \{L_j^* L_j, \rho_{\text{sys.}}\} \quad (2.38)$$

where  $\{L_j\}$  is a family of operators on the system,  $H$  is an Hermitian system operator and  $\{A, B\} = AB + BA$ . This expression can be separated into two parts. The unitary like part  $-i[H, \rho_{\text{sys.}}]$  would correspond to a unitary evolution of the system if the second part is null. The second part,  $\sum_j L_j \rho_{\text{sys.}} L_j^* - \frac{1}{2} \{L_j^* L_j, \rho_{\text{sys.}}\}$ , is some times called the dissipative part. It is responsible for the non reversible evolution. Note that this decomposition is not unique. If we change

$$L_j \rightarrow L_j + c_j I_{\text{sys.}} \quad (2.39)$$

$$H \rightarrow H + \frac{i}{2} \sum_j (\bar{c}_j L_j - c_j L_j^*) \quad (2.40)$$

$\mathcal{L}$  is not modified. Similarly let  $u$  be a unitary matrix then the modification  $L_j \rightarrow \sum_k u_{jk} L_k$  does not modify  $\mathcal{L}$ . Hence we have different  $L_j$ 's and  $H$  giving the same semi-group generator. In particular, we can always choose the  $L_j$  and  $H$  such that all the  $L_j$ 's have null trace.

The operator  $\mathcal{L}$  is called a Lindblad operator or Lindbladian. It is named after one of the researchers who established the general form of CP map semi-group generators [42, 59, 73]. A derivation of the expression of  $\mathcal{L}$  starting from the semi-group property can be found in [36].

Note that actually the characterization of CP map semi-groups through the expression of their generator is not restricted to trace preserving maps. In general a semi-group of maps acting on the space of operators acting on the system Hilbert space,  $\Phi'_t = e^{t\mathcal{L}'^*}$ , is a semi-group of CP maps if and only if it exists a CP map  $\Psi^*$  and a system operator  $K$  such that

$$\mathcal{L}'^*(X) = K^* X + X K + \Psi^*(X) \quad (2.41)$$

If moreover  $K^* + K + \Psi^*(I_{\text{sys.}}) = 0$ , the maps are unital. Namely  $\Phi'_t(I_{\text{sys.}}) = I_{\text{sys.}}$  for any  $t$ . We say the semi-group  $e^{t\mathcal{L}'^*}$  is unital. Therefore the dual maps  $\Phi'_t$  are trace preserving. Note that the expression we gave above for  $\mathcal{L}$  imply directly the trace conservation. We have  $\text{tr}[\mathcal{L}(X_{\text{sys.}})] = 0$  for any system operator  $X_{\text{sys.}}$ . We say the CP semi-group  $e^{t\mathcal{L}}$  is trace preserving if every map  $e^{t\mathcal{L}}$  is trace preserving.

In the physics literature the reduced system evolution differential equation

$$d\rho_{\text{sys.}}(t)/dt = \mathcal{L}(\rho_{\text{sys.}}) \quad (2.42)$$

is often called the master equation in reference to its classical counter part.

From the study of  $\Phi$  we can study  $\mathcal{L}$  spectrum and therefore the large time behavior of  $e^{t\mathcal{L}}\rho_{\text{sys.}}$ . First since  $e^{t\mathcal{L}}$  is trace preserving, the spectrum of  $\mathcal{L}$  is in the non positive real part half plane of the complex plane.  $\mathcal{L}$  has at least one eigenvector  $\rho_{\text{inv.}}$  with eigenvalue 0.

If  $e^{t_0\mathcal{L}}$  is irreducible then  $e^{t\mathcal{L}}$  is irreducible for any  $t$  [103]. So we say the semigroup is irreducible. If  $e^{t\mathcal{L}}$  is irreducible, it accepts a unique positive definite invariant state  $\rho_{\text{inv.}} > 0$

## 2.3. Reduced Markovian approximations

such that  $\mathcal{L}(\rho_{\text{inv.}}) = 0$ . Moreover for CPTP semi-groups, irreducibility implies that the unique invariant state is attractive [103]. Thus for any initial system state  $\lim_{t \rightarrow \infty} e^{t\mathcal{L}} \rho_{\text{sys.}} = \rho_{\text{inv.}}$ . The convergence is exponential with a rate depending on the spectrum of  $\mathcal{L}$  [65]. Let  $-\alpha_0 = \max\{\text{Re}(\alpha)/\alpha \in \text{spec}(\mathcal{L}) \setminus \{0\}\} < 0$ . Then for any  $\epsilon > 0$ ,

$$e^{t\mathcal{L}} \rho_{\text{sys.}} = \rho_{\text{inv.}} + O(e^{-(\alpha_0 - \epsilon)t}). \quad (2.43)$$

### Irreducibility criteria

We recall that  $e^{t\mathcal{L}}$  is irreducible if and only if there is no projector  $P \notin \{0, I_{\text{sys.}}\}$  such that  $\mathcal{S}(P\mathcal{H}_{\text{sys.}})$  is stable under  $e^{t\mathcal{L}}$ . Namely if there is no  $P \notin \{0, I_{\text{sys.}}\}$  such that  $\rho_{\text{sys.}} \in \mathcal{S}(P\mathcal{H}_{\text{sys.}}) \Rightarrow e^{t\mathcal{L}} \rho_{\text{sys.}} \in \mathcal{S}(P\mathcal{H}_{\text{sys.}})$ .

For a general CP semi-group, if  $\Psi$  is irreducible then  $e^{t\mathcal{L}'}$  is irreducible [65]. In particular if it does not exist an Hermitian projector  $P \notin \{0, I_{\text{sys.}}\}$  such that  $(I_{\text{sys.}} - P)L_j P = 0$  for any  $j$  then  $e^{t\mathcal{L}}$  is CPTP semi-group is irreducible.

A sufficient criteria leading to the non existence of such projector is the following (see [91] and references therein, particularly [45, 55]). Let  $\text{linspan}\{L_j\}$  be the space of complex linear combinations (i.e. the linear span) of the operators  $L_j$ . If  $\text{linspan}\{L_j\}$  is Hermitian, namely  $X \in \text{linspan}\{L_j\} \Rightarrow X^* \in \text{linspan}\{L_j\}$ , and if the only system operators commuting with all the  $L_j$ 's are multiples of the identity (i.e.  $\{L_j\}' = \mathbb{C}I_{\text{sys.}}$ ), then it does not exist an Hermitian projector  $P \notin \{0, I_{\text{sys.}}\}$  such that  $(I_{\text{sys.}} - P)L_j P = 0$  for any  $j$ . Hence  $e^{t\mathcal{L}}$  is an irreducible CPTP semigroup and its unique invariant state is attractive. This criteria was originally developed to give a condition under which the invariant state is attractive. But as we saw irreducibility implies attractiveness for CPTP semi-groups.

We see that if we can prove that a system evolution is given by a CPTP semi-group, then the study of relaxation towards a stationary state is simpler. As for repeated interactions it will not be the case for non demolition measurements and reservoir engineering.

## 2.3. Reduced Markovian approximations

Let us now motivate the Markovian description of open quantum systems with the presentation of some physical limits leading to memoryless reduced dynamics. Hence we present some limits where  $G(t) \rightarrow \delta(t)$  and  $G_0(t) \rightarrow \delta(t)$ . For a more comprehensive presentation of Markovian limit of the reduced evolution, we refer the reader to the review of H. Söphn [92]. The only limit not contained in this review is the continuous time limit of repeated interactions.

### 2.3.1. Singular limit

Let us first introduce the singular limit. It is a good first example of Markovian limit. The derivation of this limit is due to P. F. Palmer [81].

For this limit, let us keep the scaling in  $H_{\text{tot.}}$  we introduced in last section.

$$H_{\text{tot.}} = \lambda^2 H_{\text{sys.}} + H_{\text{env.}} + \lambda(D \otimes a^*(g) + D^* \otimes a(g)). \quad (2.44)$$

We are interested in the limit  $\lambda \rightarrow 0$ . In this limit, the system energy becomes much smaller than all environment energy scales. The scaling of the interaction implies that the typical relaxation rate is of the same order as the system energy scale. Hence the typical system evolution time is of the same order as the relaxation time due to the environment. The typical

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environment evolution time is much smaller. Hence we expect that the intrinsic environment evolution will drive it. Thus it should rapidly forgot the effect of the interaction with the system.

The typical system evolution time is modified from  $t$  to  $t\lambda^{-2}$ . We are interested in the system evolution so we scale time such that the new time is coherent with the system typical evolution time. This time scale as the inverse of the energy, thus as  $\lambda^{-2}$ . Hence the new time scale  $t_{\text{new}}\lambda^{-2} = t_{\text{init.}}$  is convenient. Under this new time scale, the Hamiltonian becomes

$$H_{\text{tot.}} = H_{\text{sys.}} + \lambda^{-2}H_{\text{env.}} + \lambda^{-1}(D^* \otimes a(g) + D \otimes a^*(g)). \quad (2.45)$$

This Hamiltonian scaling is more common in the literature on singular coupling than the previous one. It is actually in this scaling that singular coupling is meant. At first glance it could be interpreted as a strong coupling limit. But as we saw it is more a weak coupling limit where the system energy is also considered small. Hence the environment should drive the evolution at short times. This is what is meant by the scaling of the environment Hamiltonian.

Computing the two time correlation functions with this scaling we obtain a scaling  $\lambda^{-2}G(s\lambda^{-2})$  and  $\lambda^{-2}G_0(s\lambda^{-2})$ . If we assume that  $G$  and  $G_0$  are integrable over  $\mathbb{R}_+$ , then

$$\lim_{\lambda \rightarrow 0} \lambda^{-2}G(|s|\lambda^{-2}) = C\delta(s) \quad (2.46)$$

$$\lim_{\lambda \rightarrow 0} \lambda^{-2}G_0(|s|\lambda^{-2}) = c\delta(s) \quad (2.47)$$

with  $C = 2 \int_0^\infty G(s)ds$  and  $c = 2 \int_0^\infty G_0(s)ds$ . In the new time scale and in the limit  $\lambda \rightarrow 0$ , the memory effect disappear.

It remains to show this limit consequence on the reduced evolution. We need an additional slightly more constraining assumption on the two time correlation functions. We assume it exists  $\epsilon > 0$  such that  $s^\epsilon G(s)$  and  $s^\epsilon G_0(s)$  are integrable. Let

$$\mathcal{D}_1(\rho_{\text{sys.}}) = DD^*\rho_{\text{sys.}} + \rho_{\text{sys.}}D^*D - D\rho_{\text{sys.}}D^* - D^*\rho_{\text{sys.}}D \quad (2.48)$$

$$\mathcal{L}_-(\rho_{\text{sys.}}) = D\rho_{\text{sys.}}D^* - \frac{1}{2}\{D^*D, \rho_{\text{sys.}}\} \quad (2.49)$$

$$\mathcal{L}_+(\rho_{\text{sys.}}) = D^*\rho_{\text{sys.}}D - \frac{1}{2}\{DD^*, \rho_{\text{sys.}}\}. \quad (2.50)$$

Remark that  $\mathcal{D}_1(\rho_{\text{sys.}}) + \mathcal{D}_1(\rho_{\text{sys.}})^* = 2(\mathcal{L}_-(\rho_{\text{sys.}})) + \mathcal{L}_+(\rho_{\text{sys.}})$  and  $\mathcal{D}_1(\rho_{\text{sys.}}) - \mathcal{D}_1(\rho_{\text{sys.}})^* = [(DD^* - D^*D), \rho_{\text{sys.}}]$ .

Let  $\Phi_t = \text{tr}_{\text{env.}}[V_t \rho_{\text{sys.}} V_t^*]$  with  $V_t = e^{it\lambda^{-2}H_{\text{env.}}}e^{-itH_{\text{tot.}}}$ . We study the Dyson expansion of  $\Phi_t$ . First let us remark that all odd orders in the field operators  $a(g), a^*(g)$  vanish since  $\text{tr}[\prod_{j=1}^{2n+1} a^\sharp(f_j)\rho_{\text{env.}}] = 0$  where  $\sharp = *$  or nothing. Similarly if we do not have the same number of  $a^*$  and  $a$  operators, the term vanishes. For the non vanishing even orders, let us consider terms where  $H_{\text{sys.}}$  is not present. We use Wick's theorem to decompose the multitime correlation function into a sum of product of two time correlation functions. We encounter two types of terms. First we have terms which we call standard pairing terms. For the  $2n^{\text{th}}$

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order in the expansion the standard pairing term is

$$(-1)^n \int_{0 < t_1 < \dots < t_{2n} < t} \prod_{k=1}^{2n} dt_k \quad (2.51)$$

$$\prod_{j=1}^n \left\{ \lambda^{-2} G((t_{2j} - t_{2j-1})\lambda^{-2}) \mathcal{D}_1 \right. \quad (2.52)$$

$$\left. + \overline{\lambda^{-2} G((t_{2j} - t_{2j-1})\lambda^{-2})} \mathcal{D}_1^* \right. \quad (2.53)$$

$$\left. - (\lambda^{-2} G((t_{2j} - t_{2j-1})\lambda^{-2}) + \overline{\lambda^{-2} G((t_{2j} - t_{2j-1})\lambda^{-2})}) 2\mathcal{L}_- \right. \quad (2.54)$$

$$\left. + (\lambda^{-2} G((t_{2j} - t_{2j-1})\lambda^{-2}) - \overline{\lambda^{-2} G((t_{2j} - t_{2j-1})\lambda^{-2})}) \text{ad} D^* D \right\} (\rho_{\text{sys.}}) \quad (2.55)$$

where  $\text{ad} X(\rho_{\text{sys.}}) = [X, \rho_{\text{sys.}}]$  and  $\mathcal{D}_1^*(\rho_{\text{sys.}}) = \mathcal{D}_1(\rho_{\text{sys.}})^*$ .

They correspond, in the sum over the pairing in the Wick theorem, to pairing of nearest neighbors. One can picture it as cases where only two succeeding interactions are correlated. Let  $C = \gamma + ih$  and  $c = \gamma_0 + ih_0$  with  $\gamma_{(0)}, h_{(0)}$  reals. From the convergence of  $\lambda^{-2} G_{(0)}(|s|\lambda^{-2})$  to a Dirac delta, these standard pairing terms become in the limit

$$\frac{t^n}{n!} \left\{ -i \text{ad} H_{\text{ren.}} + (\gamma + \gamma_0) \mathcal{L}_- + \gamma \mathcal{L}_+ \right\} (\rho_{\text{sys.}}) \quad (2.56)$$

with  $H_{\text{ren.}} = (h + h_0) D^* D - h D D^*$  the change in the system Hamiltonian due to the interaction with the bath.

The other type of terms involves pairing such that at least two two time correlation functions have non neighboring times. Hence we have terms involving integrals such as

$$\int_{0 < t_1 < t_2 < t_3 < t_4 < t} dt_1 dt_2 dt_3 dt_4 \lambda^{-4} G_{(0)}((t_1 - t_3)\lambda^{-2}) G_{(0)}((t_2 - t_4)\lambda^{-2}). \quad (2.57)$$

Since we assumed that it exists  $\epsilon > 0$  such that  $s^\epsilon G(s)$  is integrable, this integral converges to 0 when  $\lambda$  goes to 0. One can interpret this cancellation saying that two interactions between the system and the environment become uncorrelated if they are not following one another. The environment is not perturbed enough for a new interaction with the system to happen before it has relaxed from its last one.

Putting together all the pairing terms, only the standard pairing terms contribute in the limit. If we reintroduce the terms with  $H_{\text{sys.}}$ , then

$$d\rho_{\text{sys.}}(t)/dt = \mathcal{L}(\rho_{\text{sys.}}(t)) \quad (2.58)$$

with the Lindbladian

$$\mathcal{L}(\rho_{\text{sys.}}) = -i[H_{\text{sys.}} + H_{\text{ren.}}, \rho_{\text{sys.}}] + (\gamma + \gamma_0) \mathcal{L}_-(\rho_{\text{sys.}}) + \gamma \mathcal{L}_+(\rho_{\text{sys.}}). \quad (2.59)$$

One can check that  $\mathcal{L}$  has actually the suitable expression for a Lindbladian generating a CPTP semi-group. Therefore the reduced system evolution is given by the CPTP semi-group  $\Phi_t = e^{t\mathcal{L}}$ .

In next section we present the more known weak coupling limit.

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### 2.3.2. Weak coupling limit

The limit we present now has become a standard example of a situation where Markovian limit is relevant. It is also known as the rotating wave Born–Markov approximation.

This limit was first shown by E.B. Davies [46]. Actually the singular coupling limit was obtained as a modification of the weak coupling limit proof. Complete derivations of the rotating wave Born–Markov approximation can be found in [36, 57].

Contrary to the previous limit, here, no scaling is made on the system Hamiltonian. The only assumption is that the interaction with the environment is weak and that the environment behaves well. Namely, the two time correlation functions of the environment must decay rapidly enough as the time interval grows. Hence, the memory of the environment should be short enough. This translates into the same assumption as before for the two time correlation functions. We assume it exists  $\epsilon > 0$  such that

$$\int_0^\infty G_{(0)}(s) < \infty \text{ and } \int_0^\infty G_{(0)}(s)s^\epsilon < \infty. \quad (2.60)$$

#### Born–Markov approximation

The total Hamiltonian is scaled as follows

$$H_{\text{tot.}} = H_{\text{sys.}} + H_{\text{env.}} + \lambda(D^* \otimes a(g) + D \otimes a^*(g)). \quad (2.61)$$

We wish to study the limit  $\lambda \rightarrow 0$ . Hence the limit where both the typical system and environment evolution time are much shorter than the system relaxation time due to the interaction with the environment. Hence, as before, we wish to study the long time behavior of the system. Thus we scale the time such that  $t\lambda^{-2}$  is the initial time and  $t$  is the time scale one which we wish to study the system evolution. The scaling of the Hamiltonian under this new time scale is

$$H_{\text{tot.}} = \lambda^{-2}H_{\text{sys.}} + \lambda^{-2}H_{\text{env.}} + \lambda^{-1}H_{\text{int.}}. \quad (2.62)$$

As before we wish to compute the limit of  $\Phi_t(\rho_{\text{sys.}}) = \text{tr}[V_t \rho_{\text{sys.}} \otimes \rho_{\text{env.}} V_t^*]$  but this time with  $V_t = e^{it\lambda^{-2}(H_{\text{sys.}} + H_{\text{env.}})} e^{-itH_{\text{tot.}}}$ . Hence we subtract the free evolution of both the system and the environment. We have

$$dV_t/dt = -i\lambda^{-1}(D(t\lambda^{-2}) \otimes a^*(e^{it\lambda^{-2}h_{\text{env.}}}g) + D^*(t\lambda^{-2}) \otimes a(e^{it\lambda^{-2}h_{\text{env.}}}g))V_t \quad (2.63)$$

with  $D(t\lambda^{-2}) = e^{it\lambda^{-2}H_{\text{sys.}}} D e^{-it\lambda^{-2}H_{\text{sys.}}}$  and  $h_{\text{env.}}$  is the one particle environment Hamiltonian. Hence  $(h_{\text{env.}}g)(k) = \omega(k)g(k)$ . We want to get rid of the  $D(t)$  time dependency. We thus decompose them on the system Hamiltonian eigenbasis. Let  $(P_E)$  be projectors on the system energy eigenspaces. Namely,  $H_{\text{sys.}} = \sum_E E P_E$ . Let us define

$$D_\omega = \sum_{E, E'; E-E'=\omega} P_{E'} D P_E. \quad (2.64)$$

Thus  $D(t\lambda^{-2}) = \sum_\omega e^{-it\lambda^{-2}\omega} D_\omega$  and  $D^*(t\lambda^{-2}) = \sum_\omega e^{it\lambda^{-2}\omega} D_\omega^*$ . The sum on  $\omega$  runs over the set of all energy differences:  $\text{spec}(H_{\text{sys.}}) - \text{spec}(H_{\text{sys.}})$ . Then

$$dV_t/dt = -i\lambda^{-1} \sum_\omega (D_\omega \otimes a^*(e^{it\lambda^{-2}(h_{\text{env.}} - \omega)}g) + D_\omega^* \otimes a(e^{it\lambda^{-2}(h_{\text{env.}} - \omega)}g))V_t. \quad (2.65)$$

### 2.3. Reduced Markovian approximations

In the Dyson expansion of  $\Phi_t$  as before all odd terms cancel and all even terms with a non standard (non near neighbor time) pairing vanish in the limit. The standard pairing terms involve now integrals over time of the two time correlation functions with different system frequencies  $\omega$  added. Let

$$\mathcal{D}_{\omega_1, \omega_2}(\rho_{\text{sys.}}) = D_{\omega_1} D_{\omega_2}^* \rho_{\text{sys.}} + \rho_{\text{sys.}} D_{\omega_2}^* D_{\omega_1} - D_{\omega_2} \rho_{\text{sys.}} D_{\omega_1}^* - D_{\omega_2}^* \rho_{\text{sys.}} D_{\omega_1} \quad (2.66)$$

$$\mathcal{D}_{-; \omega_1, \omega_2}(\rho_{\text{sys.}}) = D_{\omega_2} \rho_{\text{sys.}} D_{\omega_1}^* - \rho_{\text{sys.}} D_{\omega_2}^* D_{\omega_1}. \quad (2.67)$$

The second non vanishing term of the Dyson expansion of  $\Phi_t$  is,

$$- \sum_{\omega_1, \omega_2} \int_0^t ds \int_0^s du \left\{ e^{-is\lambda^{-2}(\omega_1 - \omega_2)} \lambda^{-2} e^{-i(s-u)\lambda^{-2}\omega_2} G((s-u)\lambda^{-2}) \mathcal{D}_{\omega_1, \omega_2}(\rho_{\text{sys.}}) \right. \quad (2.68)$$

$$+ h.c. \quad (2.69)$$

$$- e^{-is\lambda^{-2}(\omega_1 - \omega_2)} \lambda^{-2} e^{-i(s-u)\lambda^{-2}\omega_2} G_0((s-u)\lambda^{-2}) \mathcal{D}_{-; \omega_1, \omega_2} \quad (2.70)$$

$$+ h.c. \left. \right\} (\rho_{\text{sys.}}) \quad (2.71)$$

Let  $2 \int_0^\infty e^{-is\omega} G_{(0)}(s) ds = \gamma_{(0)}(\omega) + ih_{(0)}(\omega)$  with  $\gamma_{(0)}(\omega)$  and  $h_{(0)}(\omega)$  reals. In addition to the singular coupling limit besides the convergence of  $\lambda^{-2} e^{-i|s|\lambda^{-2}\omega} G_{(0)}(|s|\lambda^{-2})$  to  $(\gamma_{(0)}(\omega) + ih_{(0)}(\omega))\delta(s)$ , the phase  $e^{-is\lambda^{-2}(\omega_1 - \omega_2)}$  set the limit to 0 whenever  $\omega_1 \neq \omega_2$ . This phase comes from the subtraction of the free system evolution. Since its typical time scale is much shorter, the rapid system evolution imposes that the transition happen only between energy eigenstates of the system.

In the limit  $\lambda \rightarrow 0$  this second non vanishing term is equal to

$$t\mathcal{L}(\rho_{\text{sys.}}) \quad (2.72)$$

with

$$\mathcal{L}(\rho_{\text{sys.}}) = -i[H_{\text{ren.}}, \rho_{\text{sys.}}] + \sum_{\omega} (\gamma(\omega) + \gamma_0(\omega)) [D_{\omega} \rho_{\text{sys.}} D_{\omega}^* - \frac{1}{2} \{D_{\omega}^* D_{\omega}, \rho_{\text{sys.}}\}] \quad (2.73)$$

$$+ \gamma(\omega) [D_{\omega}^* \rho_{\text{sys.}} D_{\omega} - \frac{1}{2} \{D_{\omega} D_{\omega}^*, \rho_{\text{sys.}}\}] \quad (2.74)$$

where  $H_{\text{ren.}} = \sum_{\omega} (h(\omega) + h_0(\omega)) D_{\omega}^* D_{\omega} - h(\omega) D_{\omega} D_{\omega}^*$  is the modification of the unitary part of the system evolution due to the interaction with the environment. For an environment corresponding to the electromagnetic field in its vacuum state ( $h(\omega) = 0$ ) this modification corresponds to the Lamb shift. Remark that this Hamiltonian commutes with the system free Hamiltonian.

$$[H_{\text{ren.}}, H_{\text{sys.}}] = 0. \quad (2.75)$$

We can infer that in the limit  $\lambda \rightarrow 0$ ,  $\Phi_t(\rho_{\text{sys.}}) = e^{t\mathcal{L}} \rho_{\text{sys.}}$ . Since  $\mathcal{L}$  has the expression of a Lindbladian generating a CPTP semi-group we further more know that  $\Phi_t$  is a CPTP semi-group. Note that the limit gives a specific expression for the Lindbladian. The part corresponding to the dissipation and the part corresponding to a unitary evolution are fixed by the limit.

## 2. Open system Markovian description

The reduced system state evolution is given by

$$d\rho_{\text{sys.}}(t)/dt = \mathcal{L}(\rho_{\text{sys.}}(t)). \quad (2.76)$$

From the limit we just obtained we can recover Fermi's Golden rule. Let  $\rho_{\text{sys.}} = |E\rangle\langle E|$ . Then transition rate from  $|E\rangle$  to  $|E'\rangle$  with  $E' - E = \omega$  is  $\langle E'|\mathcal{L}(\rho_{\text{sys.}})E'\rangle = (\gamma(\omega) + \gamma_0(\omega))|\langle E'|DE\rangle|^2 + \gamma(\omega)|\langle E|DE'\rangle|^2$ . Assuming  $D$  is Hermitian, set  $\mu(\omega) = 2\gamma(\omega) + \gamma_0(\omega)$ , we have

$$\langle E'|\mathcal{L}(\rho_{\text{sys.}})E'\rangle = \mu(\omega)|\langle E'|DE\rangle|^2. \quad (2.77)$$

The density of states is contained in  $\mu(\omega)$ .

### Thermal relaxation

A nice property of the weak coupling is that it shows that, in this approximation, the system tends to thermalize. If the environment is initially in a thermal equilibrium at inverse temperature  $\beta$ , the system Gibbs state at same inverse temperature,

$$\rho_{\text{therm.}} = \frac{e^{-\beta H_{\text{sys.}}}}{\text{tr}[e^{-\beta H_{\text{sys.}}}]} \quad (2.78)$$

is a stationary state of  $\Phi_t$ .

Let us show that  $\mathcal{L}(\rho_{\text{therm.}}) = 0$ . Assume that the environment is initially at thermal equilibrium. Following KMS condition, we have

$$G(s) = G(s + i\beta) + G_0(s + i\beta). \quad (2.79)$$

Thus  $\gamma(\omega)e^{\beta\omega} = \gamma(\omega) + \gamma_0(\omega)$ . From the definition of  $\rho_{\text{therm.}}$  we have  $\rho_{\text{therm.}}D_\omega = e^{\beta\omega}D_\omega\rho_{\text{therm.}}$ . Moreover  $\rho_{\text{therm.}}$  commutes with  $H_{\text{ren.}}$  and  $D_\omega^*D_\omega$  for each  $\omega$ . Commuting  $\rho_{\text{therm.}}$  with  $D_\omega$  in  $D_\omega\rho_{\text{therm.}}D_\omega^*$  and  $D_\omega^*\rho_{\text{therm.}}D_\omega$ , we obtain

$$\mathcal{L}(\rho_{\text{therm.}}) = 0 + \sum_{\omega} (\gamma(\omega) + \gamma_0(\omega)) [D_\omega^*D_\omega\rho_{\text{therm.}} - \frac{1}{2}\{D_\omega^*D_\omega, \rho_{\text{therm.}}\}] \quad (2.80)$$

$$+ \gamma(\omega) [D_\omega D_\omega^*\rho_{\text{therm.}} - \frac{1}{2}\{D_\omega D_\omega^*, \rho_{\text{therm.}}\}] \quad (2.81)$$

$$= 0. \quad (2.82)$$

Hence  $\rho_{\text{therm.}}$  is an invariant state of  $\Phi_t$ . Moreover since  $D_\omega^*$  and  $D_\omega$  both appear in the sum whenever  $\beta > 0$ ,  $\text{linspan}\{D_\omega, D_\omega^*\}$  is Hermitian. We only need to check that  $\{D_\omega^*, D_\omega\}' = \mathbb{C}I_{\text{sys.}}$  to assure that  $\rho_{\text{therm.}}$  is the unique invariant state and that  $\lim_t \Phi_t(\rho_{\text{sys.}}) = \rho_{\text{therm.}}$  for any initial state  $\rho_{\text{sys.}}$ .

### Multitime correlations

As pointed out by R. Dümcke [51], the convergence of  $\Phi_t$  is not enough to truly have a Markovian evolution. If we turn to Heisenberg picture, we have  $\Phi_t(A) = e^{t\mathcal{L}^*}A$  in the weak coupling limit, with  $\mathcal{L}^*$  the dual of  $\mathcal{L}$  and  $A$  a system operator. The dynamic should be considered Markovian only if  $\text{tr}[V_t^*AV_tV_s^*BV_s\rho_{\text{env.}}]$  converges to  $e^{(t-s)\mathcal{L}^*}Ae^{s\mathcal{L}^*}B$ . For weak coupling, the proof of this convergence is a simple extension of the proof of the convergence of  $\Phi_t$ .

### 2.3.3. Repeated interactions continuous limit

As explained earlier another good open system model is the repeated interaction one. By construction, in discrete time, it is Markovian. Knowing the system state at time  $k$ , the system state at time  $n \geq k$  is simply  $\rho_{\text{sys.}}(n) = \Phi^{n-k}(\rho_{\text{sys.}}(k))$ .

As pointed out by S. Attal and Y. Pautrat [6], this model should also be a good discrete time approximation of continuous time Markovian open system evolutions. They actually showed that, in the continuous time limit, with an adapted scaling, the repeated interaction model (without tracing out the probes degree of freedom) leads to a quantum stochastic evolution. We will discuss this limit in next chapter. For now let us sketch how the continuous time limit of the reduced system dynamic should lead to a CPTP semi-group evolution.

Assume the interaction happen during a time  $1/n$ . Actually this  $1/n$  is simply a scaling of the interaction time. It is an adimensional quantity. The time factor needed to obtain the actual time is included in the definition of the operators involved in the Hamiltonian. Hence the eigenvalues of each part of the Hamiltonian are also adimensional. In this whole thesis we will often consider such adimensional time.

For the interaction to remain relevant at any scale, assume also that the Hamiltonian scales as

$$H_{\text{tot.}} = H_{\text{sys.}} + H_{\text{probe}} + \sqrt{n}H_{\text{int.}}. \quad (2.83)$$

Hence the interaction unitary is  $U = \exp[-i(\frac{1}{n}H_{\text{sys.}} + \frac{1}{n}H_{\text{probe}} + \frac{1}{\sqrt{n}}H_{\text{int.}})]$ . Our time scale is here arbitrary. Assume moreover that the probe state and the interaction Hamiltonian are such that  $\langle \Psi | H_{\text{int.}} | \Psi \rangle = 0$ . Then we can expand  $\Phi$  in series of  $1/\sqrt{n}$ . We have

$$\Phi(\rho_{\text{sys.}}) = \rho_{\text{sys.}} + \frac{1}{n}\mathcal{L}(\rho_{\text{sys.}}) + o(1/n) \quad (2.84)$$

with

$$\mathcal{L}(\rho_{\text{sys.}}) = -i[H_{\text{sys.}}, \rho_{\text{sys.}}] + \sum_j L_j \rho_{\text{sys.}} L_j^* - \frac{1}{2}\{L_j^* L_j, \rho_{\text{sys.}}\} \quad (2.85)$$

where  $L_j = \langle j | H_{\text{int.}} | \Psi \rangle$ .

At time  $t$ , the system has interacted with  $[tn]$  probes where  $[tn]$  the integer part of  $tn$ . Hence we can define

$$\Phi_t = \Phi^{[tn]} = (I_{\text{sys.}} + \frac{1}{n}\mathcal{L} + o(1/n))^{[tn]}. \quad (2.86)$$

In the limit  $n \rightarrow \infty$  we thus have  $\Phi_t = e^{t\mathcal{L}}$ .

We will encounter again this limit two times in the rest of the text. Thus we do not extend the interpretation of this limit here.

### 2.3.4. Low density limit

Before we turn to the presentation of quantum noises and quantum stochastic differential equations, let us cite a last Markovian limit, the low density limit obtained by R. Dümcke [52].

We consider a gas of particles interacting through collisions with the system. Hence the interaction is a scattering one:  $H_{\text{int.}} = D \otimes (a^*(g)a(f) + a^*(f)a(g))$  with  $D^* = D$ . The

## 2. Open system Markovian description

environment is assumed to be in thermal equilibrium at inverse temperature  $\beta$  and chemical potential  $\mu$ . Low density means the fugacity  $\epsilon^2 = e^{\mu\beta}$  should vanish. Hence the limit is obtained as  $\epsilon \rightarrow 0$  with a time scale  $t\epsilon^{-2}$ .

We refer the interested reader to the original article or to the short presentation with the full expression of the generator made in [36].

We just motivated the Markovian description of the reduced dynamic of the system. But, as we will see, one may be interested in keeping some information about the evolution of the environment, say to compute the statistic of some measurement on it. In the next chapter we present quantum stochastic differential equations as a tool for the description of the evolution of Markovian open quantum systems. The quantum noises we introduce hold some information on the environment.

### 3. Markovian environment description – Quantum noises

Stochastic calculus is a common tool of classical statistical physics. One approximates an environment through some noises which obey some constraints enforced to assure it models the right environment.

A famous example is the damped Langevin dynamic

$$\ddot{x}(t) = -\gamma\dot{x}(t) - V'(x(t)) + \sqrt{2\gamma/\beta}\eta(t). \quad (3.1)$$

$V$  is a potential and  $\eta$  a white noise:  $\langle\eta(t)\rangle = 0$ ,  $\langle\eta(s)\eta(t)\rangle = \delta(s-t)$  with  $\langle\cdot\rangle$  the expectation. The environment is modeled by the noise  $\eta$  and  $\gamma$  is the damping due to the environment. The factor  $\sqrt{2\gamma/\beta}$  before  $\eta$  is fixed by the fluctuation dissipation relation.

This equation can be viewed as a shorthand notation for a system of stochastic differential equations(SDE)

$$dx(t) = p(t)dt \quad (3.2)$$

$$dp(t) = -\gamma p(t)dt - V'(x(t))dt + \sqrt{2\gamma/\beta}dB_t \quad (3.3)$$

with  $B_t$  a Wiener process (Brownian motion). Note that we adopt Itô convention for SDE. Hence computation of SDE for  $C^2$  functions of  $p(t)$  and  $x(t)$  can be made using the rule  $dB_tdB_s = dt\delta_{s,t}$ .

One can generalize such equations by introducing Poisson processes  $N_t$  which are at each time Poisson random variables of parameter  $\lambda t$  with  $\lambda$  a parameter called the intensity.

We would like to find an equivalent stochastic formulation of open systems in the quantum world. Thus we need some noises that do not always commute. This is the purpose of Quantum Stochastic Differential Equations(QSDE).

In this chapter we introduce QSDE's and quantum noises. Their complete mathematical formulation is due to R. L. Hudson and K. R. Parthasarathy [64].

The chapter is structured as follow. In first section we present the Fock space on which the quantum noises are defined. We present the quantum noises themselves and their Itô rules. We give the general expression of QSDE for unitary evolutions. In a second subsection we present two physical limits leading to quantum stochastic(QS) dynamics: the weak coupling and the continuous limit of repeated interactions. In a third and last section we present the equivalent formulation of QSDE in terms of path integral and derive their classical limit in this context.

#### 3.1. Quantum noises

We will only deal with bosonic noises. The Fock state on which we define the quantum noises is therefore the symmetric Fock space of particles evolving on a one dimensional positive

### 3. Markovian environment description – Quantum noises

half line.

$$\Gamma(L^2(\mathbb{R}_+)). \quad (3.4)$$

We sometimes call this space or its tensorial combination with the system Hilbert space, the quantum stochastic space. Not surprisingly,  $\mathbb{R}_+$  corresponds to the time set. We do not describe the particle states in energy representation but in time representation. So the one particle states are functions of time and not energy.

In some cases we need to extend the time parameter set to the whole real line. Hence the quantum stochastic space is  $\Gamma(L^2(\mathbb{R}))$ . On this space, in the time representation, the environment one particle Hamiltonian is no longer the multiplication by the energy operator but the generator of a time shift group:  $h_{\text{env.}} = i \frac{d}{dt}$ . This extension of the quantum stochastic space and the explicit representation of the environment one particle Hamiltonian is useful mainly in derivations of stochastic evolutions from Hamiltonian ones. The initial non stochastic evolution is described by a strongly continuous group. As pointed out in [9, 60], this description is not possible if the time set is only the positive real half line. Since we need to describe both the initial and the quantum stochastic evolution on the same space, we need to extend the quantum stochastic space time set to the whole real line. The weak coupling limit we present in section 3.2.2 is an example where such an extension is needed. This extension is, on the contrary, not needed for the stochastic limit of repeated interactions since in this model, the initial evolution is not given by a strongly continuous group.

In any case, the quantum stochastic evolution obtained can be restricted to a quantum stochastic space involving only the positive half real line:  $\Gamma(L^2(\mathbb{R}_+))$ . The limit operators obtained act trivially on  $\Gamma(L^2(\mathbb{R}_- \setminus \{0\}))$ . This factorization of the space will soon be explained.

In this space we define some useful vectors, the exponential vectors. Let  $f \in L^2(\mathbb{R}_+)$ , then the exponential vector of  $f$  is

$$|e(f)\rangle = 1 \oplus \bigoplus_{n=1}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}. \quad (3.5)$$

Each part of the direct sum corresponds to a state of  $n$  similar particles. Note that these vectors are not normalized. Their normalized version  $|c(f)\rangle = e^{-\frac{1}{2}\|f\|^2} |e(f)\rangle$  are the coherent states of the field. The vacuum of the field is  $|0\rangle = |e(0)\rangle = 1 \oplus 0$ .

The set of finite complex linear combinations of exponential vectors is dense in  $\Gamma(L^2(\mathbb{R}_+))$  [3, 9]. All the operators we will be interested in can be defined only through their action on exponential vectors.

The canonical creation annihilation operators on this space are  $a^*(f)$  and  $a(g)$  with  $f, g$  elements of  $L^2(\mathbb{R}_+)$  and  $[a(g), a^*(f)] = \langle g, f \rangle$  where

$$\langle g, f \rangle = \int_0^\infty \overline{g(s)} f(s) ds. \quad (3.6)$$

We have

$$a(g)|e(f)\rangle = \langle g, f \rangle |e(f)\rangle. \quad (3.7)$$

The creation operator is the dual of  $a(g)$ . Hence  $\langle e(f)|a^*(g) = \langle e(f)|\langle f, g \rangle$ .

These operators are more often written  $a(f) = \int_0^\infty f(t) a_t dt$  with  $a_t|e(f)\rangle = f(t)|e(f)\rangle$ .

From these operators we can define the two first quantum noises.

$$A_t = a(1_{[0,t]}) = \int_0^t a_s ds \quad (3.8)$$

$$A_t^* = a^*(1_{[0,t]}) = \int_0^t a_s^* ds. \quad (3.9)$$

They are the quantum equivalent of Brownian motion. We will soon see more clearly why. For now let us just say that since  $a_t^*$  creates a particle at time  $t$ ,  $A_t^*$  creates a particle who lives up until time  $t$ . Let  $|\Psi\rangle \in \Gamma(L^2(\mathbb{R}_+))$ . We can also see the noise  $A_t$  as  $A_t|\Psi\rangle \neq 0$  if there is at least one particle in the environment who survived at least a non vanishing time between time 0 and  $t$ .

Another quantum noise, is somehow the counter part of the Poisson process.

$$\Lambda_t = \int_0^t a_s^* a_s ds. \quad (3.10)$$

Hence  $\langle e(g)|\Lambda_t e(f)\rangle = \langle g, 1_{[0,t]} f \rangle e^{\langle g, f \rangle}$ . This noise is a particle number operator. It counts the number of particles in the field up to time  $t$ .

The noises  $A_t, A_t^*, \Lambda_t$  have a nice time increment independence property. First, their different time increments commute one with the other. We have  $[Z_t - Z_s, Y_s] = 0$  with  $Z$  and  $Y$  any linear combination of noises and  $t, s$  two times. Second,  $\Gamma(L^2(\mathbb{R}_+))$  can be factorized such that we can isolate on which part the increments act non trivially. We have a mapping

$$\Gamma(L^2(\mathbb{R}_+)) \equiv \Gamma(L^2([0, s])) \otimes \Gamma(L^2([s, t])) \otimes \Gamma(L^2([t, \infty])). \quad (3.11)$$

With the following exponential states factorization. Let  $f_{[s,t]}$  be the restriction of  $f$  to  $[s, t]$ . Then

$$|e(f)\rangle \equiv |e(f_{[0,s]})\rangle_{s,0} \otimes |e(f_{[s,t]})\rangle_{t,s} \otimes |e(f_{[t,\infty]})\rangle_{\infty,t} \quad (3.12)$$

with the subscript  $t, s$  indicating that this vector is an element of  $\Gamma(L^2([s, t]))$ .

The corresponding mapping of  $A_t - A_s = a(1_{[s,t]})$  and all the other noise operator same time increments, act non trivially only on  $\Gamma(L^2([s, t]))$ . They act as the identity on the other parts of  $\Gamma(L^2(\mathbb{R}_+))$ .

Hence for any exponential vector  $|e(g)\rangle$  and any Hermitian noise operator  $Z_t$ , we have for any  $s \leq t$

$$\begin{aligned} \langle e(g)|e^{i\lambda_s Z_s + i\lambda_t(Z_t - Z_s)} e(g)\rangle &= {}_{s,0}\langle e(g_{[0,s]})|e^{i\lambda_s Z_s} |e(g_{[0,s]})\rangle_{s,0} \\ &\times {}_{[s,\infty]}\langle e(g_{\infty,s})|e^{i\lambda_t(Z_t - Z_s)} e(g_{[s,\infty]})\rangle_{\infty,s}. \end{aligned} \quad (3.13)$$

Particularly if the environment is in a coherent state  $|c(g)\rangle$  and we note  $\phi_X(\lambda)$  the characteristic function  $\langle c(g)|\exp[i\lambda X]c(g)\rangle$ , we have

$$\begin{aligned} \phi_{Z_s, Z_t - Z_s}(\lambda_s, \lambda_t) &= \langle c(g)|e^{i\lambda_s Z_s + i\lambda_t(Z_t - Z_s)} c(g)\rangle \\ &= \phi_{Z_s}(\lambda_s) \phi_{Z_t - Z_s}(\lambda_t). \end{aligned} \quad (3.14)$$

Hence the increments are independent. This is especially true when the environment is in its vacuum state.

### 3. Markovian environment description – Quantum noises

Let us turn to the definition of quantum stochastic processes using their integral formulation. We note  $dX_t$  the infinitesimal increment  $X_{t+dt} - X_t$ . Let  $(H_t), (G_t), (F_t)$  and  $E_t$  be adapted process, meaning  $H_t, G_t, F_t$  and  $E_t$  act non trivially only on the  $\Gamma(L^2([0, t]))$  part of the quantum stochastic space. They can also act non trivially on any other Hilbert space. Particularly they can act non trivially on  $\mathcal{H}_{\text{sys.}}$ . Let the process  $X_t$  be

$$X_t = \int_0^t H_s ds + F_s dA_s + G_s dA_s^* + H_s d\Lambda_s. \quad (3.15)$$

Then for any  $f, g \in L^2(\mathbb{R}_+)$ ,

$$\begin{aligned} \langle e(f) | X_t e(g) \rangle &= \int_0^t (\langle e(f) | H_s e(g) \rangle + \langle e(f) | F_s e(g) \rangle g(s) \\ &\quad + \langle e(f) | G_s e(g) \rangle \bar{f}(s) + \langle e(f) | H_s e(g) \rangle \bar{f}(s) g(s)) ds. \end{aligned} \quad (3.16)$$

We have defined  $X_t$  through its quantum stochastic integral<sup>1</sup>. This result is due to R. Hudson and K.R. Parthasarathy [64].

As in the classical case, some Itô rules make computations of product of such quantum stochastic processes easier. The following table gives the value of  $dX_t dY_t$  depending on  $X_t$  and  $Y_t$ .

$dX_t \backslash dY_t$	$dA_t$	$dA_t^*$	$d\Lambda_t$	$dt$
$dA_t$	0	$dt$	$dA_t$	0
$dA_t^*$	0	0	0	0
$d\Lambda_t$	0	$dA_t^*$	$d\Lambda_t$	0
$dt$	0	0	0	0

Reading this table, the non commutation of the noises at same times is clear. Using this table, we can compute the differential equation for any product of two quantum stochastic processes. Let  $X_t$  and  $Y_t$  be two quantum stochastic processes, then  $d(X_t Y_t) = dX_t Y_t + X_t dY_t + dX_t dY_t$ .

We have now the definition and the computation rules needed to study quantum stochastic processes.

We focus on two processes which have an interesting classical interpretation. Let us assume the environment is in its vacuum state  $|0\rangle$ .

First let us define  $B_t = A_t + A_t^*$ . The operators  $B_t$  are all Hermitian operators and the different time increments of  $(B_t)$  commute one with another at different times. Thus through spectral theorem there is an isomorphism mapping  $((B_t)_{t \in \mathbb{R}_+}, |0\rangle)$  to a process  $W_t$  on a probability space  $(\Omega, \mathcal{F}, \mu)$  such that  $\mathbb{E}_\mu(f(W_t)) = \langle 0 | f(B_t) | 0 \rangle$  for any bounded function  $f$ .  $\mathbb{E}_\mu$  is the expectation with respect to  $\mu$ . Furthermore, using commutation relations we obtain that

$$\langle 0 | e^{i\lambda(B_t - B_s)} | 0 \rangle = e^{-\frac{1}{2}\lambda^2(t-s)}. \quad (3.17)$$

Hence the characteristic function of  $B_t$  match the characteristic function of a Brownian motion. As we already explained, the increments  $(B_{t_k} - B_{t_{k-1}})_{t_k > t_{k-1}}$  are independent when the environment is in its vacuum state. Hence the probabilistic counter part of  $(B_t)$  through the

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1. We do not discuss here the complete conditions on the adapted process and especially existence and uniqueness of  $X_t$ . In general, whenever it is needed existence and uniqueness are assumed. We refer the interested reader to [64] for the complete definition of quantum stochastic integrals.

isomorphism should be a Wiener process. The only remaining property we would need to check is that  $(B_t)$  is almost surely continuous. It is actually the case. Thus  $(B_t)$  has a clear interpretation as a classical Brownian motion (or Wiener process) using the isomorphism. One can further characterize  $B_t$  as a multiplication operator by a Brownian motion. This interpretation and its proof are due to S. Attal [3].

Second, let us define  $N_t = \Lambda_t + A_t + A_t^* + t$ . We have  $dN_t^2 = dN_t$  and

$$\langle 0 | e^{i\lambda(N_t - N_s)} 0 \rangle = e^{(t-s)(e^{i\lambda} - 1)}. \quad (3.18)$$

Hence, following the same arguments as for  $(B_t)$ ,  $(N_t)$  can be interpreted as a classical Poisson process with unit intensity. As for  $(B_t)$ , one can characterize  $N_t$  as a multiplication operator by a Poisson process. Again this interpretation and its proof are due to S. Attal [3].

One need to be careful. We can interpret independently  $(B_t)$  and  $(N_t)$  as classical processes but we cannot interpret both of them as classical processes together. This limitation is a consequence of the non commutation of  $N_t$  and  $B_t$ . They cannot be mapped together to a probability space.

It is also important to assume the environment is in its vacuum state. If for example it is in a coherent state  $|c(g)\rangle$ , the mean of  $(B_t)$  and the intensity of  $(N_t)$  are modified.

The interpretations as classical process will turn useful when we will study continuous measurements.

### Unitary evolution

Since these quantum noises are meant to describe environment and the evolution of open system keeping the information on the environment, we would like to be able to describe reversible, hence unitary, evolution with them.

The general form of a unitary evolution for a system interacting with an environment described by quantum noises was found by R. L. Hudson and K. R. Parthasarathy [64, 82]. Let  $U_t$  be a family of operators on the quantum stochastic space  $\mathcal{H}_{\text{sys.}} \otimes \Gamma(L^2(\mathbb{R}_+))$ , solution of the QSDE

$$dU_t = -i(H - \frac{i}{2}L^*L)U_t dt + L^*WU_t dA_t - LU_t dA_t^* + (I_{\text{sys.}} - W)U_t d\Lambda_t \quad (3.19)$$

$$U_0 = I_{\text{sys.}} \otimes I_\Gamma \quad (3.20)$$

with  $H, L$  and  $W$  operators on the system Hilbert space where  $H^* = H$ , and  $W^*W = WW^* = I_{\text{sys.}}$ . Then  $(U_t)$  is a family of unitary operators. Since  $\mathcal{H}_{\text{sys.}}$  is finite dimensional, the solution of this QSDE always exists and is unique [82].

We will not take into account unitary evolutions involving  $\Lambda_t$  noises. Hence we set  $W = I_{\text{sys.}}$ . Then one can see  $U_t$  as the family of unitary operators whose time dependent Hamiltonian is

$$H_{\text{tot.}}(t) = H \otimes I_\Gamma + i(L^* \otimes a_t - L \otimes a_t^*). \quad (3.21)$$

We would have  $U_t = \mathcal{T}_> e^{-i \int_0^t H_{\text{tot.}}(s) ds}$  with  $\mathcal{T}_>$  a time ordering such that  $\mathcal{T}_> f(s)g(t) = f(s)g(t)$  if  $s > t$  and  $\mathcal{T}_> f(s)g(t) = g(t)f(s)$  otherwise.

One can remark that  $U_t$  acts non trivially only on  $\mathcal{H}_{\text{sys.}} \otimes \Gamma(L^2([0, t]))$ . Hence  $(U_t)$  is an adapted process. Furthermore,  $U_{t,s} = U_t U_s^*$  acts non trivially only on  $\mathcal{H}_{\text{sys.}} \otimes \Gamma(L^2([s, t]))$  since it is equal to

$$U_{t,s} = I_{\text{sys.}} + \int_s^t -i(H - \frac{i}{2}L^*L)U_{u,s} du + L^*U_{u,s} dA_u - LU_{u,s} dA_u^*. \quad (3.22)$$

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Hence it is the sum of operators acting non trivially only on  $\mathcal{H}_{\text{sys.}} \otimes \Gamma(L^2([s, t]))$ .

The unitary evolution of the system operators under  $U_t$  is  $j_t(X) = U_t^* X U_t$ ,

$$dj_t(X) = j_t(\mathcal{L}^*(X))dt - j_t([L^*, X])dA_t + j_t([L, X])dA_t^* \quad (3.23)$$

with

$$\mathcal{L}^*(X) = i[H, X] + L^*XL - \frac{1}{2}\{L^*L, X\} \quad (3.24)$$

the dual of the Lindbladian we encountered in last chapter.

Assume the environment is in its vacuum state  $|0\rangle$ , then the reduced system dynamic is given by

$$dX(t) = \mathcal{L}^*(X)(t)dt. \quad (3.25)$$

Hence by duality, the system state evolves as

$$d\rho_{\text{sys.}}(t) = \mathcal{L}(\rho(t))dt. \quad (3.26)$$

We recover the Markovian reduced dynamics we encountered in last chapter. Hence such quantum stochastic dynamics should be good candidates for the description of extended Markovian evolutions of open quantum systems. They are the quantum counter part of the Langevin dynamics. Thus unitary QSDE are some times called quantum Langevin equations.

Let us point the interested reader to references [5, 57] for a more thorough definition and presentation of quantum stochastic calculus. The second one may be more suitable for physicists.

In next section we motivate the use of QSDE for the description of Markovian open quantum systems with two example of physical limits leading to quantum stochastic evolutions.

## 3.2. Open systems extended Markovian limits

### 3.2.1. Master equation dilation

In last section, we saw that the reduced dynamic implied by a quantum stochastic (QS) extended one is Markovian. Thus one can expect that any reduced dynamic can be dilated to a unitary QS one. This is the dilation of CP unital semi-groups.

Let us consider the CP unital semi-group  $e^{t\mathcal{L}^*}$  with

$$\mathcal{L}^*(X) = -i[H, X] + \sum_j L_j^* X L_j - \frac{1}{2}\{L_j^* L_j, X\}. \quad (3.27)$$

Let  $j_t(X)$  be the unitary QS dynamic generated by  $U_t$  solution of

$$(dU_t)U_t^* = -i(H - \frac{i}{2} \sum_j L_j^* L_j)dt + \sum_j L_j^* W_j dA_j(t) - L_j dA_j^*(t) + (I - W_j)d\Lambda_j(t) \quad (3.28)$$

$$U_0 = I \quad (3.29)$$

with  $\{A_j(t)\}, \{\Lambda_j(t)\}$  independent quantum noises. By independent we mean they are defined on a tensor product of Fock spaces  $\bigotimes_{j=1}^{d_{\text{sys.}}} \Gamma(L^2(\mathbb{R}_+)) \equiv \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^{d_{\text{sys.}}}))$  with each  $A_j(t)$  or  $\Lambda_j(t)$  acting non trivially only on the  $j^{\text{th}}$  copy. Then  $\langle 0|j_t(X)|0\rangle = e^{t\mathcal{L}^*} X$ .

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Hence for any CP identity preserving semi-group  $e^{t\mathcal{L}^*}$  we can build a QS unitary evolution such that the corresponding reduced evolution match the semi-group. This dilation is not unique. Beside the obvious non uniqueness arising from the non uniqueness of the expression of the Lindbladian we already encountered, we can chose any unitaries  $\{W_j\}$  for the dilation. Hence we can freely add or suppress  $\Lambda_t$  noises from the dilation. The choice of dilation must then be supported by other physical arguments.

In next two subsections we present two limits where the dilation is fixed. No ad-hoc physical assumptions are needed.

#### 3.2.2. Weak coupling

The extended weak coupling limit was obtained, in the form we present here, by Y. Dereziński and W. de Roeck [48]. A first, less complete limit, was obtained before by L. Accardi *et al.* [1]. In [22] M. Bauer and D. Bernard investigated the weak coupling limit for an harmonic oscillator in interaction with a bath of independent harmonic oscillators.

This limit is the extended counter part to the reduced weak coupling limit we derived in section 2.3.2. This time we have a limit for the family of unitaries  $V_t = e^{i(H_{\text{sys.}} + H_{\text{env.}})t\lambda^{-2}} e^{-it\lambda^{-2}H_{\text{tot.}}}$  and not only the reduced dynamic  $\Phi_t$ .

We recall that

$$(dV_t)V_t^* = -i \sum_{\omega} (D_{\omega} \otimes a^*(\lambda^{-1}e^{i(h_{\text{env.}} - \omega)t\lambda^{-2}}g) + D_{\omega}^* \otimes a(\lambda^{-1}e^{i(h_{\text{env.}} - \omega)t\lambda^{-2}}g))dt. \quad (3.30)$$

We assume the environment is in its vacuum state  $|0\rangle$ . This is not a limitation since for any state, one can always find a representation in which the state is a pure one.

From the dilation of the reduced evolution and the fact that the interaction Hamiltonian is dipolar like, we expect the limit QS evolution unitary  $U_t$  to be the solution of

$$(dU_t)U_t^* = -i(H_{\text{ren.}} - \frac{i}{2} \sum_{\omega} \gamma_0(\omega) D_{\omega}^* D_{\omega})dt - iD_{\omega}c_{\omega}dA_{\omega}^*(t) - iD_{\omega}^*\overline{c_{\omega}}dA_{\omega}(t) \quad (3.31)$$

with  $|c_{\omega}|^2 = \gamma_0(\omega)$ . We have as many noises as the number of transition frequencies  $\omega$ . Hence the space on which the noises are defined is

$$\bigotimes_{\omega} \Gamma(L^2(\mathbb{R}_+)) \equiv \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^k)) \quad (3.32)$$

with  $k$  the number of frequencies  $\omega$ . Each  $\Gamma(L^2(\mathbb{R}_+))$  can be called a channel. It has not the same meaning as channels in the context of quantum information.

The operator  $A_{\omega}^*(t)$  creates a particle in the channel corresponding to  $\omega$  up to time  $t$ . Hence the action of  $dA_{\omega}^*(t)$  corresponds to the emission of a  $\omega$  energy quantum in the environment during a time  $dt$ . After this time the energy quantum should be reabsorbed by the system. Hence when integrated over time it will create quanta of energy  $\omega$  in the environment up to the interaction time. The statistic of this emission depends on the interaction with the system and the system state. Each channel keeps track of the emission and adsorption of specific energy quanta by the system into and from the environment.

The solution of (3.31) corresponds to the weak coupling limit of  $V_t$  obtained in [48]. The only additional assumption to the ones made for the reduced weak coupling limit is that  $g$  needs to be integrable. It is not an important constrain and it is usually the case in physics.

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One may just need to add suitable cutoffs in energy to  $g$ . If one consider an environment living in a one dimensional physical space, then  $c_\omega = \sqrt{2\pi}g(\omega)$ . If the environment lives in a higher dimensional physical space, irrelevant degrees of freedom must be kept. Then the notation  $c_\omega dA_\omega^*(t)$  is not clear. The noises should be defined as the infinitesimal time increment of  $a_\omega(1_{[0,t]}\sqrt{2\pi}g_\omega(\mathbf{p}))$  with  $g_\omega(\mathbf{p}) = \int_{\mathbb{R}} d\omega \delta(\omega(\mathbf{p}) - \omega)g(\mathbf{p})$ . If the dispersion relation  $\omega(\mathbf{p})$  depends only on the norm of  $\mathbf{p}$ , then  $g_\omega(\mathbf{p})$  depends only on the orientation of  $\mathbf{p}$ , hence on  $\mathbf{p}/\|\mathbf{p}\|$ .

Let us present the simple case of a two level system in interaction with an environment living in a one dimensional physical space. Hence the total Hilbert space is

$$\mathcal{H}_{\text{tot.}} = \mathbb{C}^2 \otimes \Gamma(L^2(\mathbb{R})). \quad (3.33)$$

The QS Hilbert space will be the same. Particularly, as we indicated earlier, we extend the usual one from  $L^2(\mathbb{R}_+)$  to  $L^2(\mathbb{R})$ . The extension is easily made setting:  $A_{s,t} = a(1_{[s,t]})$  for any  $s < t \in \mathbb{R}$ . The limit QS unitary evolution will act non trivially only on  $\Gamma(L^2(\mathbb{R}_+))$ .

Contrary to [48] we work in time representation. The time dependent Hamiltonian defining  $V_t$  is

$$H_{\text{tot.}}(t) = \sigma_- \otimes a^*(\lambda^{-1}e^{i(h_{\text{env.}} - \omega)t\lambda^{-2}}g) + \sigma_+ \otimes a(\lambda^{-1}e^{i(h_{\text{env.}} - \omega)t\lambda^{-2}}g) \quad (3.34)$$

with  $\sigma_+ = \sigma_-^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  the usual spin  $\frac{1}{2}$  raising operator.

First we define an isometry  $J_\lambda : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$

$$(J_\lambda g)(s) = \lambda e^{-i\omega s} g(s\lambda^2). \quad (3.35)$$

We have

$$(J_\lambda^* g)(s) = \lambda^{-1} e^{i\omega s\lambda^{-2}} g(s\lambda^{-2}) \quad (3.36)$$

and  $J_\lambda^* J_\lambda = J_\lambda J_\lambda^* = I_{L^2}$ .  $J_\lambda$  adapts the time scale of the QS environment to match the initial environment one.  $J_\lambda^*$  does the opposite. In the general case  $J_\lambda^*$  also split the initial environment to match the many Fock spaces of the QS one. The only two properties needed for the mapping  $J_\lambda$  between the QS one particle space and the initial one particle space are

$$J_\lambda J_\lambda^* = I_{L^2(\mathbb{R}^d)} \quad (3.37)$$

$$\lim_{\lambda \rightarrow 0} J_\lambda^* J_\lambda = I_{L^2(\mathbb{R}; \mathbb{C}^k)}. \quad (3.38)$$

Let us study the limit of the function in the creation operator of the interaction under such mapping between the Fock spaces. We have

$$(J_\lambda^* \lambda^{-1} e^{i(h_{\text{env.}} - \omega)t\lambda^{-2}} g)(s) = \lambda^{-2} e^{i\omega(s-t)\lambda^{-2}} g((s-t)\lambda^{-2}). \quad (3.39)$$

since  $g$  is integrable, for any  $L^2(\mathbb{R})$  function  $f$ ,

$$\lim_{\lambda \rightarrow 0} \int_{-\infty}^{\infty} du \int_0^t ds (J_\lambda^* \lambda^{-1} e^{i(h_{\text{env.}} - \omega)s\lambda^{-2}} g)(u) \bar{f}(u) = \sqrt{2\pi} g(\omega) \int_0^t \bar{f}(s) ds. \quad (3.40)$$

We used  $g(\omega) = \int \frac{ds}{\sqrt{2\pi}} e^{i\omega s} g(s)$ . This could seem contradictory with the definition of  $G_0(s)$  but we actually have  $(e^{i h_{\text{env.}}} g)(s) = g(s-t)$ . From the representation of  $h_{\text{env.}}$  as the multiplication

### 3.2. Open systems extended Markovian limits

by  $\omega$ , the change from energy representation to time representation for the interaction function  $g$  is therefore  $g(s) = \int \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega s} g(\omega)$ . Therefore  $g(\omega) = \int \frac{ds}{\sqrt{2\pi}} e^{i\omega s} g(s)$ . Computing explicitly  $\langle g, e^{ith_{\text{env.}}} g \rangle$  we recover  $G_0(t)$ .

We now introduce the exponential (or second quantized) version of  $J_\lambda$ .

$$\Gamma(J_\lambda)|e(f)\rangle = |e(J_\lambda f)\rangle. \quad (3.41)$$

Moreover  $\Gamma(J_\lambda)^* = \Gamma(J_\lambda^*)$ . Thus  $\Gamma(J_\lambda^*)\Gamma(J_\lambda) = \Gamma(J_\lambda)\Gamma(J_\lambda^*) = I_\Gamma$ . We have

$$\Gamma(J_\lambda^*)a(g)\Gamma(J_\lambda) = a(J_\lambda^*g). \quad (3.42)$$

Then from convergence (3.40), we have for any exponential vector  $|e(f)\rangle$ ,

$$\lim_{\lambda \rightarrow 0} \int_0^t ds \Gamma(J_\lambda^*)a(\lambda^{-1}e^{i(h_{\text{env.}} + \omega)s\lambda^{-2}}g)\Gamma(J_\lambda)|e(f)\rangle = \sqrt{2\pi}\bar{g}(\omega)a(1_{[0,t]})|e(f)\rangle. \quad (3.43)$$

Hence  $\Gamma(J_\lambda^*)a(\lambda^{-1}e^{i(h_{\text{env.}} + \omega)s\lambda^{-2}}g)\Gamma(J_\lambda)$  converges strongly to  $\sqrt{2\pi}\bar{g}(\omega)A_t$ . This convergence is a consequence of the convergence of  $J_\lambda^*g$  to a Dirac delta. Hence the initially spread in time interaction, becomes localized in time. Equivalently the interaction localized in energy becomes spread on all environment energy levels.

The strong convergence of  $\Gamma(J_\lambda^*)V_t\Gamma(J_\lambda)$  to  $U_t$  solution of

$$(dU_t)U_t^* = -i(H_{\text{ren.}} - \frac{1}{2}\gamma_0(\omega)\sigma_+\sigma_-)dt - i\sqrt{2\pi}\bar{g}(\omega)\sigma_+dA_t - i\sqrt{2\pi}g(\omega)\sigma_-dA_t^* \quad (3.44)$$

follows from the convergence we just showed and the insertion of the identity  $\Gamma(J_\lambda)\Gamma(J_\lambda^*)$  between each  $a^\sharp$  in the Dyson expansion of  $\Gamma(J_\lambda^*)V_t\Gamma(J_\lambda)$ .

Note that we do not have convergence of  $V_t$  but of the image of  $V_t$  through  $\Gamma(J_\lambda)$ . Actually it corresponds to pushing  $V_t$  to the right space.  $\Gamma(J_\lambda)$  maps states from the QS space to the initial space. Hence  $\Gamma(J_\lambda^*)V_t\Gamma(J_\lambda)$  is a realization of  $V_t$  on the QS space. The condition  $J_\lambda J_\lambda^* = I_{L^2}$  is therefore important. We do not want to lose information on the states by pushing them to the QS space. Hence  $J_\lambda$  should be surjective and therefore  $J_\lambda^*$  should be injective. Several QS states can correspond to the same initial space state but any initial space state should correspond to a unique QS state. Thus, in the general case, the fact that  $J_\lambda^*J_\lambda \neq I_{L^2}$  for all  $\lambda$  and is equal only in the limit is not limiting. The QS space holds many more states than the one we can construct from the initial space through  $\Gamma(J_\lambda^*)$ .

Thanks to this extended limit we keep the information on the exchange of energy between the system and the environment. The noises  $A_\omega(t)$  correspond to limits of environment operators. Using quantum filtering (section 5.2.3) one should be able to describe continuous measurements of the energy exchange. This is possible if we interpret  $\Lambda_\omega(t)$  as the particle counting operator up to time  $t$  in the channel  $\omega$ .

#### 3.2.3. Repeated interactions continuous limit

In [6], S. Attal and Y. Pautrat showed the strong convergence of the repeated interaction unitary evolution we presented in section 2.1.2 to a unitary evolution solution of a QSDE.

In this case the limit is not a limit from non Markovian to Markovian evolution. It is a limit from discrete time evolution to continuous time evolution. Hence the discrete version can be thought as a discrete approximation of the continuous time Markovian evolution. Or,

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the continuous limit can be seen as the approximation of the discrete evolution. We shall encounter a similar limit when we will deal with continuous indirect measurement situations.

Let us present a simpler version of S. Attal and Y. Pautrat general result. We recall that in a repeated interaction situation, the environment Hilbert space is an infinite product of identical finite dimensional Hilbert spaces. Here we consider only two dimensional Hilbert spaces for the environment.

$$\mathcal{H}_{\text{tot.}} = \mathbb{C}^{d_{\text{sys.}}} \otimes \bigotimes_{\mathbb{N}} \mathbb{C}^2. \quad (3.45)$$

The initial state is a tensor product of the initial system state and the identical probe pure states  $|0\rangle$ .

$$\rho_{\text{tot.}} = \rho_{\text{sys.}} \otimes \bigotimes_{\mathbb{N}} |0\rangle\langle 0|. \quad (3.46)$$

The unitary evolution is a product of unitary operators acting non trivially only on the system Hilbert space and one part of the environment. Let  $U_k$  act non trivially only on the system Hilbert space and the  $k^{\text{th}}$  part of the environment. Thus  $U_k = \sum_{j=0,1,2,3} u_j \otimes \bigotimes_{k=1}^{n-1} I_2 \otimes \sigma_j \otimes \bigotimes_{k>n} I_2$  with  $\sigma_j$  the usual Pauli matrices

$$\sigma_0 = I_2, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.47)$$

in the basis  $(|1\rangle, |0\rangle)$ . The evolution up to time  $n$  is given by

$$V(n) = U_n U_{n-1} \dots U_2 U_1. \quad (3.48)$$

We assume, without loss of generality, that each interaction happens during a time  $1/n$ . Hence during a interval of time equal to 1,  $n$  interactions are realized. At time  $t$ ,  $[nt]$  full interactions are realized. We wish to take the limit  $n \rightarrow \infty$ . Hence to have more and more interactions in a fixed time. As for the reduced evolution, the limit is well defined and not trivial only for a well chosen interaction scaling. We assume the unitary interaction  $U_n$  is

$$U_k = \exp\left[-i\frac{1}{n}H_{\text{sys.}} + \frac{1}{\sqrt{n}}(L \otimes \sigma_{+,k} - L^* \otimes \sigma_{-,k})\right] \quad (3.49)$$

with  $L$  an operator on the system Hilbert space and  $\sigma_{\pm,n} = \frac{1}{2}(\sigma_{1,k} \pm i\sigma_{2,k})$ . The index  $k$  indicates that the operator acts non trivially only on the  $k^{\text{th}}$  part of the environment.

A trivial continuous time version of  $V(k)$  is given by  $V(t) = V([nt])$ . But before we take the limit  $n \rightarrow \infty$  we need to represent the discrete evolution in a quantum stochastic space.

Let the quantum stochastic space be  $\mathbb{C}^{d_{\text{sys.}}} \otimes \Gamma(L^2(\mathbb{R}_+))$ . Let us slice  $\Gamma(L^2(\mathbb{R}_+))$  in time intervals of size  $1/n$ . We write  $\Gamma_k = \Gamma(L^2([k/n, (k+1)/n]))$ . We have  $\Gamma(L^2(\mathbb{R}_+)) \equiv \bigotimes_{k \in \mathbb{N}} \Gamma_k$ . We define

$$\begin{aligned} |0\rangle_k &= |e(0)\rangle_k \\ |1\rangle_k &= 0 \oplus |\sqrt{n}1_{[k/n, (k+1)/n]}\rangle_k \bigoplus_{l>1} 0 \\ a_n(k) &= a_k(\sqrt{n}1_{[k/n, (k+1)/n]})P_n^1(k) \\ a_n^*(k) &= P_n^1(k)a_k^*(\sqrt{n}1_{[k/n, (k+1)/n]}). \end{aligned} \quad (3.50)$$

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The index  $k$  everywhere indicates that it is an object on the  $k^{\text{th}}$  part of the environment. For the operator  $a_k(n)$  it means it acts trivially on all the other parts of the environment.  $P_n^{[1]}(k)$  is the projector onto the subspace of 0 or 1 particle of the  $k^{\text{th}}$  part of the Fock space. Hence

$$P_n^{[1]}(k)|e(f)\rangle = |e(f)\rangle_{<k} \otimes (|0\rangle_k + f_n(k)|1\rangle_k) \otimes |e(f)\rangle_{>k} \quad (3.51)$$

where  $f_n(k) = \sqrt{n} \int_{k/n}^{(k+1)/n} f(s)ds$  and  $|e(f)\rangle_{<k}$  is the limitation of  $|e(f)\rangle$  to  $\bigotimes_{l < k} \Gamma_l$ .  $|e(g)\rangle_{>k}$  is defined identically. We furthermore define the projector  $P_n = \prod_k P_n^{[1]}(k)$ . It associates to  $|e(f)\rangle$  a state which can be uniquely identified with a vector of  $\bigotimes_{\mathbb{N}} \mathbb{C}^2$ .

$$P_n|e(f)\rangle = \bigotimes_k (|0\rangle_k + f_n(k)|1\rangle_k). \quad (3.52)$$

We just mapped the ground state of each two level probe in the environment to the ground state of a part of the quantum stochastic space. Similarly we mapped the excited state to a one particle state. The projection allows us to restrict the states of the quantum stochastic space to states we can uniquely identify to vectors of the original environment.

With these definitions we have

$$\begin{aligned} a_n(k)|0\rangle &= 0 \\ a_n(k)|1\rangle_k &= |0\rangle_k \\ a_n^*(k)|0\rangle_k &= |1\rangle_k \\ a_n^*(k)|1\rangle_k &= 0 \end{aligned} \quad (3.53)$$

where  $|0\rangle = \bigotimes_k |0\rangle_k$ . The operators act trivially on all states  $|1\rangle_j, |0\rangle_j$  with  $j \neq k$ . Hence we have a realization of the operators  $\sigma_{\pm,k}$  in the quantum stochastic space.  $a_n(k)$  corresponds to  $\sigma_{-,k}$  and  $a_n^*(k)$  to  $\sigma_{+,k}$ .

We are now equipped to present the convergence of the operators. First let us remark that  $P_n$  and all  $P_n^{[1]}(k)$  converge strongly to the identity. They are self dual,  $P_n^{[1]}(k)^2 = P_n^{[1]}(k)$  and  $\langle e(g)|P_n|e(f)\rangle$  and  $\langle e(g)|P_n^{[1]}(k)|e(f)\rangle$  converge to  $\langle e(g)|e(f)\rangle$  for any  $L^2(\mathbb{R}_+)$   $g$  and  $f$ . The convergence of the operator  $P_n^{[1]}$  is crucial to obtain the convergence of the evolution. At one point we need to map all the states of the quantum stochastic space to states of the initial environment. But this projection should disappear in the limit. If not we would not have convergence of  $V(0) = I$  for example.

Let us define the approximations of  $A_t$  and  $A_t^*$ .

$$a_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]-1} a_n(k) \quad (3.54)$$

$$a_n^*(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]-1} a_n^*(k). \quad (3.55)$$

These approximations converge strongly to respectively  $A_t$  and  $A_t^*$ .

We sketch the proof for  $a_n(t)$ , the proof for  $a_n^*(t)$  is equivalent. For any  $|e(f)\rangle$ , we have

$$\begin{aligned} \|(a_n(t) - A_t)|e(f)\rangle\| &\leq \left\| \int_0^t f(s) (|e(f)\rangle_{\setminus [ns]} - |e(f)\rangle) ds \right\| \\ &\quad + \left\| \int_t^{\frac{[nt]}{n}} f(s) |e(f)\rangle_{\setminus [nt-1]} ds \right\| \end{aligned} \quad (3.56)$$

### 3. Markovian environment description – Quantum noises

with  $|e(f)\rangle_{\setminus k} = |e(f)\rangle_{<k} \otimes |0\rangle_k \otimes |e(f)\rangle_{>k}$ .

We have

$$\left\| \int_t^{\lfloor \frac{nt}{n} \rfloor} f(s) |e(f)\rangle_{\setminus [nt-1]} ds \right\| \leq \int_t^{\lfloor \frac{nt}{n} \rfloor} |f(s)| ds \| |e(f)\rangle \| \quad (3.57)$$

$$\leq \langle 1_{[\frac{[nt]-1}{n}, \frac{[nt]}{n}[}, |f| \rangle ds e^{\frac{1}{2} \|f\|^2} \quad (3.58)$$

$$\leq \frac{1}{n} \|f\| e^{\frac{1}{2} \|f\|^2}. \quad (3.59)$$

And we also have

$$\left\| \int_0^t f(s) (|e(f)\rangle_{\setminus [ns]} - |e(f)\rangle) ds \right\| \leq \int_0^t |f(s)| \| |e(f)\rangle_{\setminus [ns]} - |e(f)\rangle \| ds. \quad (3.60)$$

For any  $s$ ,  $\| |e(f)\rangle_{\setminus [ns]} - |e(f)\rangle \|$  converges to 0 when  $n \rightarrow \infty$  and it is bounded for any  $n$  by  $2\| |e(f)\rangle \|$ .

Thus using dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \| (a_n(t) - A_t) |e(f)\rangle \| = 0. \quad (3.61)$$

Hence  $a_n(t)$  converges strongly to  $A_t$ .

Let us now develop the representation of  $V(n+1) - V(n)$  on the quantum stochastic space in series of  $1/\sqrt{n}$  terms.

$$\begin{aligned} (V(n+1) - V(n))V^*(n) = \\ -i(H_{\text{sys.}} - \frac{i}{2}L^*LP_n^{[0]}(k))\frac{1}{n} + \frac{1}{\sqrt{n}}(L \otimes a_n^*(k) - L^* \otimes a_n(k)) + o(1/n) \end{aligned} \quad (3.62)$$

with  $P_n^{[0]}(k)$  the projector on the vacuum state of  $\Gamma_k$ . We have  $\lim_{n \rightarrow \infty} \| P_n^{[0]}(k) - I_\Gamma |e(f)\rangle \| = 0$ . Thus  $P_n^{[0]}(k)$  converges strongly to the identity.

Set now  $U_t$  the unique solution of

$$\begin{aligned} (dU_t)U_t^* &= -i(H_{\text{sys.}} - \frac{i}{2}L^*L)dt + LdA_t^* - L^*dA_t \\ U_0 &= I. \end{aligned} \quad (3.63)$$

We can infer that  $V([nt])$  converges strongly to  $U_t$  when  $n$  goes to infinity. S. Attal and Y. Pautrat proved this convergence in [6]. Actually to complete the mapping from the initial environment to the quantum stochastic space we should have mapped the operators  $\frac{1}{2}(I_{2,k} + \sigma_{3,k})$  too. Doing it explicitly is not really interesting here. Just note that they are mapped to the process  $\Lambda_t$ . Hence  $\Lambda_t$  counts the number of probes in the excited state up to time  $t$ .

If we summarize, we started from repeated interactions of a system with two levels probes. We mapped the probes states to linear combinations of vacuum and one particle states of a quantum stochastic space  $\Gamma(L^2(\mathbb{R}_+))$ . We identically mapped the operators  $\sigma_{j,k}$ . In the limit we obtain that the raising operators on the probes  $\sigma_{+,k}$  become  $dA_t^*$  similarly,  $\sigma_{-,k}$  becomes  $dA_t$ . Hence the quantum noises keep track of transitions between the energy level in the probes. The tensor product of states of two level probes become multiple bosonic particle states. The presence of one particle corresponds to the excitation of the probes. One can

### 3.3. Corresponding path integral description and classical limit

understand the presence of more than one particle as the excitation of many probes regrouped in a small time interval. If the time interval is small enough the states with more than one particle become irrelevant since the norm of  $|f1_{[t,t+dt]}\rangle \otimes |g1_{[t,t+dt]}\rangle$  is of order  $dt^2$ .

Of course the results of [6] are much more general than what we presented here. They allow for probes with  $\mathbb{C}^{d_{\text{probe}}}$ ,  $2 < d_{\text{probe}} < \infty$  Hilbert spaces. In this case the quantum stochastic space becomes  $\Gamma(L^2(\mathbb{R}_+; \mathbb{C}^k))$  with  $k = d_{\text{probe}} - 1$ . Hence each noise  $A_j(t)$  corresponds to a lowering of the probe state from the excited state  $j$  to the ground state. The operators  $\Lambda_{j,i}(t)$  perform transitions from the state  $i$  to state  $j$ . When  $j = i$  these operators count the number of probes in the excited state  $j$ . The authors also prove the convergence of Hamiltonian including scattering interaction like  $D \otimes (\delta_{i,j+1} + \delta_{j+1,i})_{1 \leq i,j \leq d_{\text{probe}}}$  with  $D = D^*$ . In this case the limit stochastic unitary evolution involves processes  $\Lambda_{j,j+1}(t)$ . When both dipolar like and scattering interactions are present in the Hamiltonian, the unitary stochastic evolution has an unexpected expression. To illustrate it, the corresponding Lindbladian can be

$$\mathcal{L}(\rho_{\text{sys.}}) = -i[K, \rho_{\text{sys.}}] + W\rho_{\text{sys.}}W^* - \frac{1}{2}\{W^*W, \rho_{\text{sys.}}\} \quad (3.64)$$

with

$$K = H_{\text{sys.}} + L^*D^{-2}(\sin(D) - D)L \quad (3.65)$$

$$W = D^{-1}(e^{iD} - I)L. \quad (3.66)$$

Hence the limit evolution is not intuitively the direct limit of the initial one. These type of processes should be more thoroughly investigated.

### 3.3. Corresponding path integral description and classical limit

In this section we formulate QSDE in terms of path integrals. It makes the link with the usual path integral formulation and allows us to compute the classical limit of QSDE.

Since we use path integral formalism we do not seek the most failproof demonstrations. Hence all the result of this section must be though as inferred and not actually proved in the mathematical sense.

#### 3.3.1. Path integral for an environment in its vacuum state

We limit ourselves to the case of one particle in interaction with a bath. We can of course generalize the following derivation to more complex cases, but for now let  $\mathcal{H}_{\text{sys.}} = L^2(\mathbb{R})$ .  $H$  and  $L$  are then functions of  $P$  and  $X$  the canonical impulsions and position operators. We want to compute multi-time correlations of observable functions of  $X$  through a path integral formalism. For the sake of simplicity we start with two time correlations. We want to compute

$$\mathbb{E}(B(s)C(t)) = \text{tr}[U_s^* B U_s U_t^* C U_t \rho_{\text{tot.}}] \quad (3.67)$$

with  $B$  and  $C$  functions of  $X$ .  $\rho_{\text{tot.}}$  is the initial state of the total system+environment. Recall that  $U_t$  is the solution of

$$\begin{aligned} (dU_t)U_t^* &= -i(H - \frac{i}{2}L^*L)dt + L^*dA_t - LdA_t^* \\ U_0 &= I_{\text{sys.}} \end{aligned} \quad (3.68)$$

### 3. Markovian environment description – Quantum noises

In this section we assume the environment is in its vacuum state. We have  $\rho_{tot.} = \rho_{sys.} \otimes |0\rangle\langle 0|$ .

We want to find a formulation like the following for  $\mathbb{E}(\dots)$ :

$$\mathbb{E}(B(s)C(t)) = \quad (3.69)$$

$$\int_{x_+(\infty)=x_-(\infty)} dx_{\pm} dp_{\pm} df B_{\pm}(s) C_{\pm}(t) \rho_{sys.}(x_+(0), x_-(0)) e^{iG[x_{\pm}, p_{\pm}, f]} \quad (3.70)$$

with  $f$  a square integrable function of  $\mathbb{R}_+$ ,  $B_{\pm}(s) = \langle x_{\pm}(s) | B | x_{\pm}(s) \rangle$  and  $\rho_{sys.}(x, x') = \langle x | \rho_{sys.} | x' \rangle$ .  $G$  is some action to be defined. We will refer to the function  $f$  as a “path” on the environment. The indexes  $\pm$  correspond to different time ordering. If both indexes are  $+$  (resp.  $-$ ) then the correlation computed is  $\mathbb{E}(B(s)C(t))$  (resp.  $\mathbb{E}(C(t)B(s))$ ) if  $s > t$  and  $\mathbb{E}(C(t)B(s))$  (resp.  $\mathbb{E}(B(s)C(t))$ ) if  $s < t$ . If the index for  $B$  is  $+$  (resp.  $-$ ) and the one of  $C$  is  $-$  (resp.  $+$ ) then the correlation computed is  $\mathbb{E}(C(t)B(s))$  (resp.  $\mathbb{E}(B(s)C(t))$ ).

The full derivation of the path integral is presented in Appendix B.1. Here we just give the action obtained and a sketch of the main steps of the proof.

The action  $G[x_{\pm}, p_{\pm}, f]$  we obtain is:

$$iG[x_{\pm}, p_{\pm}, f] = -|f|^2 + iS[x_+, p_+, f] - \overline{iS[x_-, p_-, f]} \quad (3.71)$$

$$S[x_{\pm}, p_{\pm}, f] = \int_0^{\infty} ds (p_{\pm}(s) \dot{x}_{\pm}(s) - H_{\pm}(s) + \frac{i}{2} L^* L_{\pm}(s) \quad (3.72)$$

$$+ iL_{\pm}(s) \bar{f}(s)). \quad (3.73)$$

$O_{\pm}(s) = \langle p_{\pm}(s) | O | x_{\pm}(s) \rangle / \langle p_{\pm}(s) | x_{\pm}(s) \rangle$  for any operator  $O$  acting on  $\mathcal{H}$ . If we make a classical analogy,  $-|f|^2$  stands for the “law” of the noises, (3.72) corresponds to the predictable part of the evolution and (3.73) corresponds to the noisy part of the evolution.

This simple formula can, of course, be generalized to any multi time correlation

$$\mathbb{E}(B_n(t_n) \dots B_1(t_1) C_1(s_1) \dots C_k(s_k)) \quad (3.74)$$

with an ordering  $t_n > t_{n-1} > \dots > t_1 > 0$  and  $s_k > s_{k-1} > \dots > s_1 > 0$ .

The proof leading to  $iG$  is based on two important properties that allows us to then use the usual path integral derivation proof. First we use the factorization in time property of  $\Gamma(L^2(\mathbb{R}_+))$  we saw in section 3.1. We can cut it in different time part. We have  $L^2(\mathbb{R}_+) \equiv L^2([0, t]) \oplus L^2([t, \infty[)$  which imply  $\Gamma(L^2(\mathbb{R}_+)) \equiv \Gamma(L^2([0, t])) \otimes \Gamma(L^2([t, \infty[))$ .  $A_t$  acts only on the first half of this decomposition. Since  $U_t$  depends only on noise operators  $A_s, A_s^*$  with  $s < t$ , it also only acts on the first half of the decomposition. Idem,  $U_{t,s} = U_t U_s^*$  only acts on a  $\Gamma(L^2([s, t])$  part of  $\Gamma(L^2(\mathbb{R}_+))$ . Hence when we perform a trace over the environment we can decompose it on each time part.

$$\mathbb{E}(B(s)C(t)) = \text{tr}_{sys.} [C \text{tr}_{[s, \infty[} [U_{t,s} \text{tr}_{[0, s]} [U_s \rho_{tot.} U_s^*] B U_{t,s}^*]] \quad (3.75)$$

Second, the exponential vectors of the environment can be decomposed in the same way. We write  $|e(f)\rangle_{t,s}$  the limitation of  $|e(f)\rangle$  to this  $\Gamma(L^2([s, t])$  part (*i.e.*  $|e(f)\rangle = |e(f)\rangle_{t,0} \otimes |e(f)\rangle_{\infty,t}$ ). The environment vacuum state is  $|0\rangle\langle 0|_{t,0} \otimes |0\rangle\langle 0|_{\infty,t}$ . Since,  $U_t$  acts as the identity on  $|0\rangle\langle 0|_{\infty,t}$  we can put each part of the environment initial state in the corresponding trace.

$$\mathbb{E}(B(s)C(t)) = \text{tr}_{sys.} [C \text{tr}_{[s, \infty[} [U_{t,s} \text{tr}_{[0, s]} [U_s \rho_{sys.} \otimes |0\rangle\langle 0|_{s,0} U_s^*] \otimes |0\rangle\langle 0|_{\infty,s} B U_{t,s}^*]] \quad (3.76)$$

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We can express the trace over the environment degrees of freedom as an integral over square integrable functions using the exponential vectors.

$$\text{tr}_{env}[\dots] = \int_{L^2(\mathbb{R}_+)} df e^{-|f|^2} \langle e(f) | \dots | e(f) \rangle \quad (3.77)$$

with the measure over the function  $f$  such that  $\int df e^{-|f|^2} = 1$ . We use again the decomposition of exponential vectors and it remains to compute the path integral formalism for  $U_{t,s}^f = {}_{t,s}\langle e(f) | U_{t,s} | e(0) \rangle_{t,s}$ . Indeed we have

$$\mathbb{E}(B(s)C(t)) = \int df e^{-|f|^2} \text{tr}_{sys}. [C U_{t,s}^f U_{s,0}^f \rho_{sys}. U_{s,0}^{f*} B U_{t,s}^{f*}] \quad (3.78)$$

The rest of the demonstration is a usual path integral one. As indicated earlier the full derivation can be found in Appendix B.1.

**Characteristic functional** We can resume any multi time correlation function  $\mathbb{E}(\dots)$  into one characteristic functional. Let  $\mathcal{G}[\lambda_{\pm}]$  be

$$\begin{aligned} \mathcal{G}[\lambda_{\pm}] = \int dx_{\pm} dp_{\pm} df \rho_{sys}.(x_+(0), x_-(0)) \delta(x_+(\infty) - x_-(\infty)) \exp \left[ iG[x_{\pm}, p_{\pm}, f] \right. \\ \left. + \int_0^{\infty} ds x_+(s) \lambda_+(s) + x_-(s) \lambda_-(s) \right]. \end{aligned} \quad (3.79)$$

Then  $\mathbb{E}(X(t_n) \dots X(t_1) X(t'_1) \dots X(t'_p)) = \frac{\delta}{\delta \lambda_+(t_n)} \dots \frac{\delta}{\delta \lambda_+(t_1)} \frac{\delta}{\delta \lambda_-(t'_1)} \dots \frac{\delta}{\delta \lambda_-(t'_p)} \mathcal{G}[\lambda_{\pm}] \Big|_{\lambda_{\pm}=0}$

**Example** We illustrate this path integral formalism with a simple example. Let  $H_{sys.} = P^2/2m + V(X)$  and  $L = \alpha X + \beta P$ . If  $L$  is a usual annihilation operator,  $\alpha = \sqrt{\frac{m\omega}{2\hbar}}$  and  $\beta = i\sqrt{\frac{1}{2\hbar m\omega}}$ . We assume  $V(X)$  is polynomial. We write the differential equations for  $X_t = U_t^* X U_t$  and  $P_t = U_t^* P U_t$ . The computation using Itô rules leads to

$$dX_t = P_t/m dt - \gamma X_t dt + dW_t^{\beta} \quad (3.80)$$

$$dP_t = -V'(X_t) dt - \gamma P_t dt - dW_t^{\alpha} \quad (3.81)$$

with  $dW_t^{\mu} = i\mu dA_t^* - i\bar{\mu} dA_t$  and  $\gamma = \text{Im}(\bar{\alpha}\beta)$ . Each one of the  $W_t^{\mu}$  can be individually interpreted as a Brownian motion times  $|\mu|$  but they do not commute unless  $\alpha = c\beta, c \in \mathbb{R}$ . So we cannot replace both with a combination of two independent Brownian motions.

In order to find the corresponding path integral formalism, we start from the unitary description given at the beginning of this section with the operators  $H$  and  $L$  we just defined. We turn to the path integral formulation of this simple one particle case. We find the action directly from the expressions of  $H$  and  $L$ :

$$S[x, p, f] = \int_0^{\infty} ds \left( p(s) \dot{x}(s) - \frac{p(s)^2}{2m} - V(x(s)) + \frac{i}{2} |\alpha x(s) + \beta p(s)|^2 \right. \quad (3.82)$$

$$\left. + i(\alpha x(s) + \beta p(s)) \bar{f}(s) \right). \quad (3.83)$$

We got rid of a  $\alpha\bar{\beta}$  constant which is included in the normalization of the path integral.

### 3. Markovian environment description – Quantum noises

Remark that if one integrates over  $f$ , one recover Feynman–Vernon influence functional. If we compare with usual path integral including the environment the term  $i/2|\alpha x(s) + \beta p(s)|^2$  is unexpected. In the derivation it is directly linked to the presence of noises instead of the usual full interaction. The noise creates additional terms of order  $dt$  in the slicing of the evolution.

#### 3.3.2. Out of vacuum environments

##### Environment state characterization

If the environment is not initially in its vacuum state, from different limits [4, 22], we can infer the Itô rules between  $dA_t^*$  and  $dA_t$  are

$$dA_t dA_s^* = \delta_{s,t} n_t dt, \quad dA_t^* dA_s = \delta_{s,t} (n_t + 1) dt.$$

The other multiplications of infinitesimal increments are still zero.  $n_t$  characterizes the state of the environment. In order to find the corresponding path integral, we want to find a positive semi definite hermitian operator  $\rho_{env.}$  on the environment such that  $\text{tr}[\rho_{env.} dA_t^* dA_s] = \delta_{s,t} n_t dt$ .

Let  $\rho_{env.} \propto e^{\Lambda(-\lambda)}$  with  $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $\Lambda(\lambda) = \int_0^\infty ds \lambda(s) a_s^* a_s$ . Hence it is a deformed version of  $\Lambda_t$ . Suppose  $\text{tr}[\rho_{env.}] = 1$ . We would like to use the commutation relations between  $A_t, A_t^*$  and  $\Lambda(-\lambda)$  to compute the corresponding  $n_t$ . We have  $[dA_t, dA_s^*] = \delta_{s,t} dt$ ,  $[\Lambda(-\lambda), dA_t] = \lambda(t) dA_t$ . From usual relations on exponential of operators, it follows  $e^{\Lambda(-\lambda)} dA_t^* = e^{-\lambda(t)} dA_t^* e^{\Lambda(-\lambda)}$ . Thus we have

$$e^{-\lambda(t)} \text{tr}[(dA_t^* dA_s + \delta_{s,t} dt) \rho_{env.}] = \text{tr}[dA_t^* dA_s \rho_{env.}]. \quad (3.84)$$

Solving this equation for  $n(t, s) dt = \text{tr}[dA_t^* dA_s \rho_{env.}]$ , we obtain  $n(t, s) = (e^{\lambda(t)} - 1)^{-1} \delta_{s,t} dt$ . Hence  $n_t = (e^{\lambda(t)} - 1)^{-1}$ . If we choose  $\lambda(t)$  such that  $n_t$  is the one expected, we find a state fulfilling our expectations. We can even compute its normalization.

$$\text{tr}[e^{\Lambda(-\lambda)}] = \int df e^{-|f|^2} \langle 0 | e^{a(f)} e^{\Lambda(-\lambda)} e^{a^*(f)} | 0 \rangle. \quad (3.85)$$

We use once again usual properties on commutations of operator exponentials and the fact that  $\Lambda(-\lambda)|0\rangle = 0$ .

$$\text{tr}[e^{\Lambda(-\lambda)}] = \int df e^{-(f, (1-e^{-\lambda})f)} = \det(1 - e^{-\lambda})^{-1}. \quad (3.86)$$

Hence  $\rho_{env.} = \det(n+1)^{-1} e^{\Lambda(\ln(\frac{n}{n+1}))}$ . If  $n_t = n_0$  is constant,  $\rho_{env.} = e^{\ln(\frac{n_0}{n_0+1})\Lambda(1)} / (n_0 + 1)$ . Note that actually  $e^{\Lambda(-\lambda)}$  is not necessarily a trace class operator. Hence  $\det(1 - e^{-\lambda})$  is not necessarily well defined. If we wanted to assure that  $\rho_{env.}$  is trace class we should have used the right representation for the environment (GNS construction, Araki–Woods representation).

#### Thermal path integral

Now we want to get the path integral formalism of our QSDE when the initial environment state is a populated state:  $\rho_{tot.} = \rho_{sys.} \otimes \rho_{env.}$ ,  $\rho_{env.} = \det(1 - e^\lambda) e^{\Lambda(\lambda)}$ . The demonstration

### 3.3. Corresponding path integral description and classical limit

follows the same path as earlier, but this time we need two more “paths” on the environment taking into account the fact that the environment state is no longer a coherent one.

$$\mathbb{E}(B(s)C(t)) = \int df dg_{\pm} e^{-|f|^2 - |g_-|^2 - |g_+|^2} \text{tr}_{sys.} \left[ \langle e(f) | BU_{s,t} CU_{t,0} | e(g_+) \rangle \right. \quad (3.87)$$

$$\left. \rho_{sys.} \langle e(g_+) | \rho_{env.} | e(g_-) \rangle \langle e(g_-) | U_{s,0}^* | e(f) \rangle \right]. \quad (3.88)$$

Hence we expect we will find a path integral like

$$\mathbb{E}(B(s)C(t)) = \int_{x_+(\infty)=x_-(\infty)} df dg_{\pm} dx_{\pm} dp_{\pm} B_+(s) C_+(t) \rho_{sys.}(x_+(0), x_-(0)) e^{iG[x_{\pm}, p_{\pm}; f, g_{\pm}]} \quad (3.89)$$

As in the previous section we will not give here a full demonstration but only point out the important changes from the vacuum one. The elements missing here are given in Appendix B.2. The action we obtain for the populated case is

$$iG = -|f|^2 - |g_+|^2 - |g_-|^2 + (g_+, e^{-\lambda} g_-) + (f, g_+) + (g_-, f) \quad (3.90)$$

$$+ iS[x_+, p_+; f, g_+] - i\overline{S[x_-, p_-; f, g_-]} \quad (3.91)$$

with

$$S[x, p; f, g] = \int_0^{\infty} ds \left( \dot{x}(s)p(s) - H(s) + i/2L^*L(s) \right) \quad (3.92)$$

$$+ iL(s)\overline{f(s)} - iL^*(s)g(s). \quad (3.93)$$

Note that contrary to usual Keldysh formalism, the influence of the initial environment state appear directly in the action through the term including  $\lambda$ . We do not have an integral in imaginary time to thermalize the initial state for example. The influence of the state on the “measure” over the environment path is explicit in the action.

The demonstration follows the same path as the one for the vacuum case. The only change is that here we do not have  $|0\rangle\langle 0|$  as an environment state. Instead we have

$$\rho_{env.} = \int dg_{\pm} e^{-|g_+|^2 - |g_-|^2} \rho_{env.}(g_+, g_-) |e(g_+)\rangle \langle e(g_-)|$$

with

$$\rho_{env.}(g_+, g_-) = \det(1 - e^{-\lambda}) \langle 0 | e^{a(g_+)} e^{\Lambda(-\lambda)} e^{a^*(g_-)} | 0 \rangle.$$

Baker–Campbell–Hausdorff formula leads to

$$\rho_{env.}(g_+, g_-) = \det(1 - e^{-\lambda}) e^{(g_+, e^{-\lambda} g_-)}. \quad (3.94)$$

Including this in the full demonstration leads to the result.

As in the vacuum case, any correlation function can be computed with the following characteristic functional. Let  $\mathcal{G}[\lambda_{\pm}]$  be

$$\mathcal{G}[\lambda_{\pm}] = \quad (3.95)$$

$$\int dx_{\pm} dp_{\pm} df dg_{\pm} \rho_{sys.}(x_+(0), x_-(0)) \delta(x_+(\infty) - x_-(\infty)) \quad (3.96)$$

$$\exp \left[ iG[x_{\pm}, p_{\pm}; f, g_{\pm}] + \int_0^{\infty} ds x_+(s) \lambda_+(s) + x_-(s) \lambda_-(s) \right]. \quad (3.97)$$

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Then

$$\mathbb{E}(X(t_n) \dots X(t_1)X(t'_1) \dots X(t'_p)) = \quad (3.98)$$

$$\frac{\delta}{\delta\lambda_+(t_n)} \cdots \frac{\delta}{\delta\lambda_+(t_1)} \frac{\delta}{\delta\lambda_-(t'_1)} \cdots \frac{\delta}{\delta\lambda_-(t'_p)} \mathcal{G}[\lambda_\pm] \Big|_{\lambda_\pm=0} \quad (3.99)$$

#### 3.3.3. Classical limit

An interesting result we can obtain from this path integral formalism is the classical limit of QSDE's. We have little hope to find a limit when  $n_t = 0$  since then the environment is in its vacuum state (*i.e.* is at zero temperature) and the classical world is a strictly positive temperature world. Hence we limit our selves to  $n_t \neq 0$ .

Usually one computes the classical limit  $\hbar \rightarrow 0$  from the influence functional. In other words the limit is taken once the environment degrees of freedom have been traced out. We want to keep the noise induced by the environment in our classical limit. First we need to reintroduce the  $\hbar$  factors in the action  $G$ . We can directly introduce  $1/\hbar$  in the formal solution of  $U_{t,s}$ .

$$U_t = \mathcal{T}_> e^{-i\frac{t}{\hbar}H_{sys.} - \frac{1}{\sqrt{\hbar}}(L \otimes A_t^* - L^* \otimes A_t)}.$$

The  $1/\sqrt{\hbar}$  in front of  $L$  corresponds to the standard expression of  $a_s$ . It is the inverse Fourier transform of a family of standard harmonic oscillator creation annihilation operators  $a_\omega = \frac{1}{\sqrt{2\hbar}}(\sqrt{m\omega}\hat{x} + \frac{1}{\sqrt{m\omega}}\hat{p})$  with  $\hat{x}, \hat{p}$  canonical position and impulsion operators. Hence  $A_t$  includes already a factor  $1/\sqrt{\hbar}$ . Thus it remains only a factor  $1/\sqrt{\hbar}$  in front of  $L$ .

We compute the usual classical limit  $\hbar \rightarrow 0$  (see Appendix B.3 for the proof) with the addition that, following the thermal example  $e^{-\lambda(s)} = 1 - \hbar\tilde{\lambda}(s) + o(\hbar)$  with  $\tilde{\lambda} = \lambda/\hbar$  fixed. We apply the result to our example  $L = \alpha X + \beta P$ . We obtain the action

$$iG_{cl.}[x, p, q, \pi; \eta_1, \eta_2] = \quad (3.100)$$

$$- \int_0^\infty ds \frac{\tilde{\lambda}(s)}{4} (\eta_1(s)^2 + \eta_2(s)^2) \quad (3.101)$$

$$+ i \int_0^\infty ds \pi(s) \left( \dot{x}(s) - p(s)/m + \gamma x(s) - \text{Im}(\beta)\eta_1(s) + \text{Re}(\beta)\eta_2(s) \right) \quad (3.102)$$

$$- i \int_0^\infty ds q(s) \left( \dot{p}(s) + V'(x(s)) + \gamma p(s) + \text{Im}(\alpha)\eta_1(s) - \text{Re}(\alpha)\eta_2(s) \right) \quad (3.103)$$

with  $\gamma = \text{Im}(\bar{\alpha}\beta)$ . If we integrate now over the noises  $\eta_i$  we recover Martin–Siggia–Rose path integral. The corresponding Langevin equation system is

$$\dot{x}(s) = p(s)/m - \gamma x(s) + \text{Im}(\beta)\eta_1(s) - \text{Re}(\beta)\eta_2(s) \quad (3.104)$$

$$\dot{p}(s) = -V(x(s)) - \gamma p(s) - \text{Im}(\alpha)\eta_1(s) + \text{Re}(\alpha)\eta_2(s). \quad (3.105)$$

The noises  $\eta_1$  and  $\eta_2$  are independent white noises with correlation  $\langle \eta_i(s)\eta_j(t) \rangle = \delta(s-t)\delta_{i,j}2\tilde{\lambda}(s)^{-1}$ . If  $\tilde{\lambda}$  correspond to a thermal case were  $\tilde{\lambda}(s) = \beta E(s)$ , the correlation is  $\langle \eta_i(s)\eta_j(t) \rangle = \delta(s-t)\delta_{i,j}\frac{2k_B T}{E(s)}$ . Hence we recovered the usual classical stochastic equations for Markovian diffusive open systems.

If  $L$  is not linear in  $X$  and  $P$  some normal order issues can arise. Although if  $L$  is only a function of  $X$  or of  $P$  but not both it is always possible to derive the classical limit.

### 3.3. Corresponding path integral description and classical limit

Now let us compare with the quantum stochastic differential equations for the observables  $X$  and  $P$ . A simple computation leads to

$$dX_t = P_t/mdt - \gamma X_t dt - i\bar{\beta}dA_t + i\beta dA_t^* \quad (3.106)$$

$$dP_t = -V'(X_t)dt - \gamma P_t dt + i\bar{\alpha}dA_t - i\alpha dA_t^*. \quad (3.107)$$

It is the same expression as in the vacuum case. The only population dependency is in the noises Itô laws. We change the noise setting  $Q_t = A_t + A_t^*$  and  $P_t = -i(A_t - A_t^*)$ . Remark that  $[Q_t, P_s] = 2i \min(t, s)$ ,  $dQ_t^2 = dP_t^2 = (2n_t + 1)dt$  and  $dQ_t dP_t = -dP_t dQ_t = idt$ . With these noises we have

$$dX_t = P_t/mdt - \gamma X_t dt + \text{Im}(\beta)dQ_t - \text{Re}(\beta)dP_t \quad (3.108)$$

$$dP_t = -V'(X_t)dt - \gamma P_t dt - \text{Im}(\alpha)dQ_t - \text{Re}(\alpha)dP_t. \quad (3.109)$$

These equations do look like the classical ones we obtained. Moreover if we take the classical limit for  $2n_t + 1$  we also obtain  $2n_t + 1 = \frac{\bar{\lambda}^{-1}}{\hbar} + 1 + O(\hbar)$ . Hence it must be possible to take the classical limit directly from these stochastic equations. Indeed if we include back the  $\hbar$  factor in the previous equations we have the same equations but with  $dQ_t = \sqrt{\hbar}(dA_t + dA_t^*)$  and  $dP_t = -i\sqrt{\hbar}(dA_t - dA_t^*)$ . Thus in the limit  $\hbar \rightarrow 0$ :  $dQ_t dP_t = dP_t dQ_t = 0$  and  $dQ_t^2 = dP_t^2 = 2\tilde{\lambda}(t)^{-1}dt$  which corresponds to the classical limit we just found.

This result can be thought as a consistency with classical stochastic equation test.

In this chapter we saw that we can describe, in some limits, open quantum systems using quantum stochastic evolutions. From now on we will use either this formalism or the repeated interaction one, without continuous limit, to describe open system settings. The benefit of keeping some information on the environment through the noises will appear clearly when we will deal with the description of the system evolution when continuous measurements on the environment are performed.



## 4. Repeated indirect measurements

As first explained in section 2.1.2, a common technique of system manipulation in quantum optics consists in sending prepared probes towards the system being studied [63]. Once the probes have interacted with the main system they can be measured. Actually this technique is a cornerstone of 2012 Nobel prize awarded to S. Haroche [62] and D. J. Wineland [101]. In this chapter we discuss such repeated indirect measurement procedures.

It is the first time we introduce quantum trajectories. Here this term has no relation with the paths of Feynman path integral. The trajectory is the evolution of the system with respect to the probe measurement results.

The chapter is structured as follows. In a first section we present the general setup of discrete quantum trajectories. The system state evolution is a stochastic process which depends on the indirect measurements outcomes. We identify the mean evolution with the repeated action of a CP map onto the system state. In the second section we present some general asymptotic properties of quantum trajectories obtained by B Kümmerer and H. Maassen [69, 75]. First we present the ergodic behavior of the trajectories. The Cesaro mean of trajectories converges with probability one towards one of the invariant states of the CP map corresponding to the indirect measurement. Then we discuss the purification of the state along the trajectory. If the state is initially mixed, it will, in general, purify along the trajectory. In the long time, the state becomes pure. The third section is devoted to the study of the convergence of discrete trajectories corresponding to non demolition measurements. These are used to perform high precision measurement of a system through the knowledge obtained by the indirect measurements. We explain how these specific indirect measurements reproduce von Neumann projection postulate on the system while we used it only to project the states of the probes. We also give the exponential rate of convergence toward the projection. When different methods of indirect measurement are used during one measurement, we show this rate can be improved. These are results from [20, 21]. The last section deals with attractive subspaces. Based on results concerning the convergence of the system state toward a subspace of the complete system Hilbert space under the repeated action of a CP map [33], we show the convergence property is kept when taking the measurement outcomes into account. The system behavior is nevertheless a bit different. We find two different convergence rates. One corresponds to the mean evolution and another is similar to the action of a non demolition measurement and corresponds to an asymptotic stability rate. We then identify two phases during the evolution. During a random time the state does not show sign of convergence and can even stay far from the limit subspace. After this time, the state converges rapidly toward the limit subspace. The duration of the first phase seem to depends only on the mean convergence rate whereas the speed of convergence after this time seem to depend on asymptotic stability rate.

### 4.1. Discrete time quantum trajectories

In this section we present the general setup of repeated indirect measurements.

## 4. Repeated indirect measurements

### 4.1.1. Indirect measurement

Let us first review one step of the repeated indirect measurement process. We consider a system described by a finite dimensional Hilbert space  $\mathcal{H}_{\text{sys.}}$  and an auxiliary system or probe whose Hilbert space  $\mathcal{H}_{\text{probe}}$  is also finite dimensional. The term probe is here meaningful since we use the auxiliary systems to actually probe the main one. We assume the probe is prepared in a pure state  $|\Psi\rangle$ . We could generalize by taking a mixed state as probe state, but for the sake of simplicity we limit here ourselves to a pure state. It is actually the most relevant situation in quantum optics experiments. We address this potential generalization when needed. In quantum optics, the systems and probes we are talking about can be either atoms, pulses of light, artificial quantum dots (Josephson junctions for example) or photon fields in cavities. For the latter the original infinite dimensional Hilbert space is reduced to a finite dimensional one by assumption since the highest numbers of photon states are often negligible.

An indirect measurement corresponds to a cycle of a unitary interaction between the system and the probe and a measurement on the probe. The interaction entangles the system and the probe and the measurement result determines the resulting evolution of the system. We assume the probe measurement is non degenerate. Let  $U$  be the interaction and  $\{j\}$  the set of probe measurement outcomes. In Serge Haroche's group experiment [61] the interaction would be similar to  $U = e^{i\Delta t \lambda \hat{n} \otimes \sigma_3}$  with  $\hat{n}$  the photon number operator and  $\sigma_3$  the operator differentiating between the two states for the atoms (*i.e.* the third Pauli matrix<sup>1</sup>).  $|\Psi\rangle$  is the initial state the atoms would be prepared in. Hence a superposition of their ground and excited states. The outcomes  $\{j\}$  would be either  $e$  or  $g$  depending on whether the atom is measured in its excited or ground state respectively. The excited and ground states depend of course on the axis along which they are measured.

If the system state before a cycle is  $\rho$ , the system state after one indirect measurement is

$$\rho' = \frac{M_j \rho M_j^*}{\text{tr}[M_j^* M_j \rho]}, \quad M_j = \langle j | U \Psi \rangle \quad (4.1)$$

with outcome  $j$  obtained with probability  $\text{tr}[M_j^* M_j \rho]$ . The  $M_j$ 's are operators acting on the system Hilbert space. They are complex sums of the  $U_{ji}$  blocs of  $U$  in the probe basis  $\{|j\rangle\}$ :  $M_j = \sum_i U_{ji} \langle i | \Psi \rangle$ .

The system update rule is obtained from Born rule. One can remark that any measurement on the system is compatible with the measurement on the probe. It means that we can simultaneously measure the probe and any observable on the system. Therefore, for any system observable we can define a joint probability of its measurement and the measurement on the probe. Let  $X$  be a system observable. From Born's rule the joint distribution of its outcomes and the probe measurement outcomes are

$$p(x, j) = \text{tr}[P_x \otimes P_j U(\rho \otimes |\Psi\rangle\langle\Psi|) U^* P_x \otimes P_j] \quad (4.2)$$

We can use the definition of conditional probability to obtain the distribution of  $X$  outcomes with respect to the probe measurement outcome  $j$ .

$$p(x|j) = \frac{\text{tr}[P_x \otimes P_j U(\rho \otimes |\Psi\rangle\langle\Psi|) U^* P_x \otimes P_j]}{\text{tr}[I_{\text{sys.}} \otimes P_j U(\rho \otimes |\Psi\rangle\langle\Psi|) U^* I_{\text{sys.}} \otimes P_j]} = \text{tr} \left[ P_x \frac{M_j \rho M_j^*}{\text{tr}[M_j^* M_j \rho]} \right]. \quad (4.3)$$

---

1. The Pauli matrices are:  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$  and  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Since this is true for any system observable  $X$ , Born's rule implies that after the probe measurement the system state is given by the update rule we gave. So we did not use the projection postulate. This is possible because we do not keep track of the probe state after the measurement. In most experiment the probe is destroyed or discarded after its measurement [11].

The measurement we just described is usually called positive operator valued measurement (POVM). It means that the outcome distribution is given by a family of CP maps, here  $\rho \mapsto M_j \rho M_j^*$ . If we had not assumed the initial state of the probe was pure, instead of having maps with a single  $M_j$  we would have had a more complex CP maps. Let  $\rho_{\text{probe}} = \sum_a p_a |\Psi_a\rangle\langle\Psi_a|$  be the initial probe state. Let  $\Phi_j(\rho) = \sum_a p_a M_{a,j} \rho M_{a,j}^*$  with  $M_{a,j} = \sum_i U_{ji} |i\rangle\langle\Psi_a|$ . The update rule in this case is  $\rho' = \Phi_j(\rho) / \text{tr}[\Phi_j(\rho)]$  with  $j$  obtained with probability  $\text{tr}[\Phi_j(\rho)]$ . The measurement is still a POVM but this time with the outcome distribution given by the CP maps  $\Phi_j$ . The general description of operator valued measurements is made through the instrument formalism [11]. POVM's include projective measurements. In this case the operators  $M_j$  are just projectors.

Before we start to introduce the repetition of such indirect measurement, let us check a few properties on a unique cycle. First an outcome  $j$  with  $\text{tr}[M_j^* M_j \rho] = 0$  has a null probability to be picked. Hence  $\rho'$  is almost surely always well defined. Then, since  $\sum_j M_j^* M_j = I_{\text{sys.}}$ ,  $\sum_j \text{tr}[M_j^* M_j \rho] = 1$  and  $(\text{tr}[M_j^* M_j \rho])$  is a well defined probability distribution for the outcomes. We also have  $\text{tr}[\rho'] = \text{tr}[\rho] = 1$  hence a state is always mapped onto a state, whatever is the outcome. Therefore, after an indirect measurement, we almost surely stay in the set of system states.

When looking at the average update of  $\rho$ , we find it corresponds to the action of a CP map  $\Phi$  such that  $\Phi(\rho) = \sum_j M_j \rho M_j^*$ . Since  $\sum_j M_j^* M_j = I_{\text{sys.}}$ ,  $\Phi$  is trace preserving. Thus this average evolution matches the criteria for a good deterministic quantum evolution given in section 2.1.1.

We can also see this indirect measurement as an unraveling of a CP map. Given a CP map and a Kraus decomposition of it, we can interpret the CP map as the average action of an indirect measurement whose outcomes would have been the index of the Kraus operators. For this reason, in general, we call the  $M_j$ 's Kraus operators. This idea of unraveling will be more significant when we will deal with continuous quantum trajectories.

#### 4.1.2. Repeated indirect measurements

Discrete quantum trajectories correspond to the repetition of the process just described. An experimenter repeatedly indirectly measures the system. The global process is presented in figure 4.1. If the system state is  $\rho(n)$  at time  $n$ , then at time  $n + 1$ ,

$$\rho(n+1) = \frac{M_{j_{n+1}} \rho(n) M_{j_{n+1}}^*}{\text{tr}[M_{j_{n+1}}^* M_{j_{n+1}} \rho(n)]} \quad (4.4)$$

with the probability of  $j_{n+1} = j$  equal to  $\text{tr}[M_j^* M_j \rho(n)]$ . The state at time  $n$  depends on all  $n$  previous measurements:  $\rho(n) = \rho(n; j_1, \dots, j_n)$ . Hence the system state evolution is a stochastic process depending on the measurements outcomes. A measurement realization or record is an infinite sequence of outcomes  $\omega = j_1, \dots, j_n, \dots$ . The probability space is then  $\Omega = \{(j_1, \dots, j_n, \dots) \text{ s.t. } \forall k, j_k \in \{j\}\}$ . The probability measure  $\mathbb{P}$  with which these outcomes are picked is defined according to quantum physics laws by  $\mathbb{P}(\{\omega \text{ s.t. } \omega_k = j_k \forall k \leq n\}) = \text{tr}[M_{j_1}^* \dots M_{j_n}^* M_{j_n} \dots M_{j_1} \rho(0)]$ . Particularly it depends on the initial system state  $\rho(0)$ .

#### 4. Repeated indirect measurements

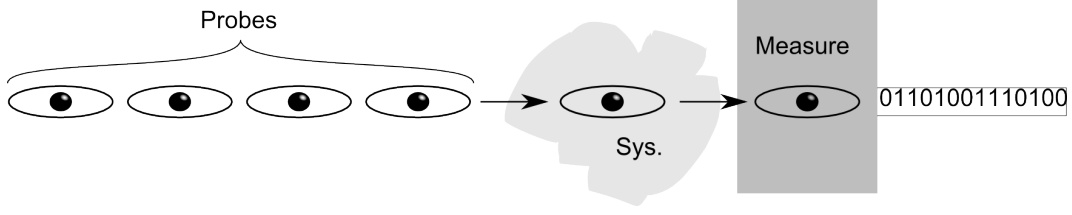


Figure 4.1.: Schematics of the repeated indirect measurements setup. A sequence of probes prepared in a given state is sent to interact with the system. The probes are then measured. The system state is updated with respect to the probe measurement outcomes.

One first important remark is that  $\rho(n)$  is a Markov process since the expectation of any function of  $\rho(n+1)$  with respect to the whole history  $(\rho(k))_{0 \leq k \leq n}$  can be reduced to the expectation with respect to the last known system state:  $\mathbb{E}(f(\rho(n+1)) | (\rho(k))_{0 \leq k \leq n}) = \mathbb{E}(f(\rho(n+1)) | \rho(n))$  where  $\mathbb{E}$  is the expectation under the measure  $\mathbb{P}$ .

Second, the average evolution of the system corresponds to a repeated action of the CP map  $\Phi$ .

$$\mathbb{E}(\rho(n)) = \Phi^n(\rho(0)). \quad (4.5)$$

#### Equivalence between continuous monitoring and complete final measurement

Now let us reintroduce the interpretation of the auxiliary systems as an environment we used in the sections about repeated interaction (2.1.2 and 2.3.3). If we consider the Hilbert space of all the probes at once we have  $\mathcal{H}_{\text{probes}} = \mathcal{H}_{\text{probe}} \otimes \mathcal{H}_{\text{probe}} \otimes \dots \otimes \mathcal{H}_{\text{probe}} \otimes \dots$ . Hence we have an infinite dimensional Hilbert space. We consider it as our environment. Its initial state is simply  $|\psi_0\rangle = |\Psi\rangle \otimes |\Psi\rangle \otimes \dots \otimes |\Psi\rangle \otimes \dots$ .

The projector corresponding to the measurement on the  $n^{\text{th}}$  probe we have to use in Born's rule is

$$P_j^{(n)} = I_{\text{sys.}} \otimes \bigotimes_{k=1}^{n-1} I_{\text{probe}} \otimes |j\rangle\langle j| \otimes \bigotimes_{k=n+1}^{\infty} I_{\text{probe}}. \quad (4.6)$$

The interaction between the system and the  $n^{\text{th}}$  probe is

$$U_n = \sum_{i,j} U_{ji} \otimes \bigotimes_{k=1}^{n-1} I_{\text{probe}} \otimes |j\rangle\langle i| \otimes \bigotimes_{k=n+1}^{\infty} I_{\text{probe}}. \quad (4.7)$$

The projector  $P_j^{(n)}$  commutes with any operator which acts as the identity on the  $n^{\text{th}}$  probe Hilbert space. Hence  $[U_k, P_j^{(n)}] = [P_i^{(k)}, P_j^{(n)}] = 0$  for any  $j, i$  and any different  $k$  and  $n$ . This is this commutation property that allows us to only use Born's rule and not the projection postulate.

Using the commutation we can use Born's rule in two ways that correspond to two different measurement picture. Either, as we have already done, Born's rule is used considering the

probability of each probe outcome is computed just after the system–probe interaction. Then, formally the probability of one indirect measurement realization is

$$\text{tr}[\cdots P_{j_n}^{(n)} U_n P_{j_{n-1}}^{(n-1)} U_{n-1} \cdots P_{j_1}^{(1)} U_1 (\rho \otimes |\psi_0\rangle\langle\psi_0|) U_1^* P_{j_1}^{(1)} \cdots U_{n-1}^* P_{j_{n-1}}^{(n-1)} U_n^* P_{j_n}^{(n)} \cdots]. \quad (4.8)$$

This probability distribution is equal to the one when we consider a measurement on the probes only after all the interactions. This joint measurement is allowed by the commutation between the different projectors on different probes. The equality is allowed because an interaction of the system with a probe acts trivially on any other probe. It is the meaning of the commutation  $[U_k, P_j^{(n)}] = 0$  for  $k \neq n$ . Using commutation relations we have

$$\begin{aligned} & \text{tr}[\cdots P_{j_n}^{(n)} U_n P_{j_{n-1}}^{(n-1)} U_{n-1} \cdots P_{j_1}^{(1)} U_1 (\rho \otimes |\psi_0\rangle\langle\psi_0|) U_1^* P_{j_1}^{(1)} \cdots U_{n-1}^* P_{j_{n-1}}^{(n-1)} U_n^* P_{j_n}^{(n)} \cdots] \\ &= \\ & \text{tr}[P_{\dots, j_n, j_{n-1}, \dots, j_1} \cdots U_n U_{n-1} \cdots U_1 (\rho \otimes |\psi_0\rangle\langle\psi_0|) U_1^* \cdots U_{n-1}^* U_n^* \cdots P_{\dots, j_n, j_{n-1}, \dots, j_1}] \end{aligned} \quad (4.9)$$

with

$$P_{\dots, j_n, j_{n-1}, \dots, j_1} = \prod_{k=1}^{\infty} P_{j_k}^{(k)}. \quad (4.10)$$

Hence it is equivalent to measure the probes along the time or all at once at the end. We will encounter a similar equivalence in chapter 5. We will show that in the context of continuous quantum trajectories, the usual two time measurement is equivalent to the continuous measurement of the environment.

Before we move to the general asymptotic behavior of discrete quantum trajectories let us just highlight that the setup we presented here, even if very general, does not cover all the quantum optics experiments we referred to. Here we lack the possibility to change the probe, the interaction and the measurement on the probes with respect to time and the previous measurement outcomes. It does not include feedback control for example. We will introduce this freedom in section 4.3 for the specific case of non demolition measurements.

## 4.2. Asymptotic behavior

### 4.2.1. Ergodicity

A question often asked, mainly when discrete quantum trajectories are used in numerical simulation, is the equivalence between ensemble average and time average for the system state. Do we have

$$\lim_{n \rightarrow \infty} \mathbb{E}(\rho(n)) \text{ equals } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \rho(k)? \quad (4.11)$$

The second limit is the limit of the Cesàro mean of the quantum trajectory. We will see in next section on non demolition measurements that the answer to this question can also have implications on the characterization of some state convergence. But let us first detail the answer. It was given by B. Kümmerer and H. Maassen in [69] for any quantum trajectory either in discrete time or continuous time. Here we extract the results for discrete time trajectories.

#### 4. Repeated indirect measurements

Let us first remark that the limit of the mean evolution  $\lim_n \mathbb{E}(\rho(n)) = \lim_n \Phi^n(\rho(0))$  is not always well defined. If  $\Phi$  is irreducible it is since it admits a unique invariant state. But if  $\Phi$  is reducible there is no guaranty the limit exists.  $\Phi$  can have several invariant states. A CP map corresponding to a unitary evolution,  $\Phi(\rho) = U\rho U^*$  is a good example where this limit does not always exists.

Every projector onto an eigenstate of  $U$  is an invariant state of  $\Phi$ . But if we start in a state which is not a linear combination of these projectors, the mean state will not converge in general. We are in a finite dimensional Hilbert space so, as we discussed in chapter 2.1, the evolution given by repeated actions of  $\Phi$  is quasi periodic. Let  $\rho$  be the initial system state. Assume  $\rho$  is not an invariant state of  $\Phi$ . For any  $\epsilon > 0$  there is a divergent sequence of times  $(n_k)$  such that  $\|\rho(n_k) - \rho\| < \epsilon$  for any  $k$ . But for the same  $\epsilon$  it exists another divergent sequence of times  $(n'_k)$  such that  $\|\rho(n'_k) - U\rho U^*\| < \epsilon$ . The state will be as close as we want to  $\rho$  an infinite number of times. And it will be as close as we want to  $U\rho U^*$  an infinite number of times. Since by assumption  $\|\rho - U\rho U^*\| > 0$ , this is incompatible with a convergence of  $\Phi^n(\rho)$  as  $n$  tends to infinity.

Fortunately, even if the mean state  $\mathbb{E}(\rho(n))$  does not converge, we will find a convergence of the Cesàro mean of  $(\rho(n))$  towards one of the invariant states of  $\Phi$ .

We define a projector onto the set of these invariant states.

$$P_\Phi(\rho) = \lim_n \frac{1}{n} \sum_{k=1}^n \Phi^k(\rho). \quad (4.12)$$

This projector is well defined since  $\Phi$  always accept at least one invariant state  $\rho_{\text{inv.}}$ . The invariant states are the eigenstates of  $\Phi$  with eigenvalue 1. All other eigenvalues have a norm smaller or equal to 1 and are different from 1. Using the Jordan decomposition on the matrix representation of  $\Phi$  and the expression of geometric sums  $\sum_n^N \lambda^n = (\lambda - \lambda^{N+1})/(\lambda - 1)$  for  $\lambda \neq 1$ , one directly finds the convergence towards the projector  $P_\Phi$  [103].

Of course if  $\Phi$  admits only one invariant state  $\rho_{\text{inv.}}$ , we have  $P_\Phi(\rho) = \rho_{\text{inv.}}$  for any system state  $\rho$ .

Let  $(\rho(n))$  be a quantum trajectory as we defined in previous section. Hence it is a realization of a system state evolution under an indirect measurement process. The evolution depends on the measurement outcomes. In [69] the authors showed that almost surely

$$\lim_n \frac{1}{n} \sum_{k=1}^n \rho(k) = \rho(\infty) \quad (4.13)$$

with  $\rho(\infty)$  one of the invariant states of  $\Phi$ . Moreover the expectation of  $\rho(\infty)$  is equal to the projection of the initial state onto the set of invariant states.

$$\mathbb{E}(\rho(\infty)) = P_\Phi(\rho(0)). \quad (4.14)$$

Hence even if we do not have a unique invariant state for  $\Phi$ , the system state Cesàro mean converges to an invariant state depending on the outcomes.

The two main elements of the demonstration are based on martingale convergence theorems [80]. Let us write the system state evolution according to the Doob decomposition.

$$\rho(n) = \rho(0) + \sum_{k=1}^n \Phi(\rho(k-1)) - \rho(k-1) + \sum_{k=1}^n \frac{M_{j_k} \rho(k-1) M_{j_k}^*}{\text{tr}[M_{j_k}^* M_{j_k} \rho(k-1)]} - \Phi(\rho(k-1)). \quad (4.15)$$

## 4.2. Asymptotic behavior

The first sum over  $k$  is a predictable process and the second one is a martingale with respect to the measurement outcomes. Let  $\mathcal{M}_n = \sum_{k=1}^n \frac{M_{j_k} \rho(k-1) M_{j_k}^*}{\text{tr}[M_{j_k}^* M_{j_k} \rho(k-1)]} - \Phi(\rho(k-1))$ ,

$$\mathbb{E}(\mathcal{M}_n | j_1, \dots, j_{n-1}) = \mathcal{M}_{n-1}. \quad (4.16)$$

Since  $\mathbb{E}((\Delta \mathcal{M}_n)^2 | j_1, j_2, \dots, j_{n-1})$  is almost surely bounded, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}((\Delta \mathcal{M}_n)^2 | j_1, j_2, \dots, j_{n-1}) < \infty \quad (4.17)$$

almost surely. Thus the law of large numbers for martingales apply and  $\lim_n \mathcal{M}_n/n = 0$  almost surely. Then, since  $\rho(n)$  and  $\rho(0)$  are bounded, we obtain

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \Phi(\rho(k)) - \rho(k) = 0 \quad (4.18)$$

almost surely. Hence, the Cesàro mean of the predictable drift process converges to 0.

The other convergent martingale used corresponds to the evolution of the projected state  $P_{\Phi}(\rho(n))$ . From the definition of  $P_{\Phi}$  applied to the Doob decomposition of  $\rho(n)$  we see that no drift part remains in the evolution of  $P_{\Phi}(\rho(n))$ . Hence  $P_{\Phi}(\rho(n))$  is a martingale. If there is a unique invariant state then this martingale is trivially a constant. The system states are bounded, so in any case, this martingale is bounded. Thus it converges almost surely. Since the expectation of a martingale is conserved we have  $\mathbb{E}(P_{\Phi}(\rho(n))) = P_{\Phi}(\rho(0))$ . The last step of the proof consists in showing that  $\lim_n \frac{1}{n} \sum_{k=1}^n \rho(k) - P_{\Phi}(\rho(k)) = 0$ . We refer to the article for this part.

As a corollary of this result we find a law of large numbers for the counting processes of the measurement outcomes.

Let  $N_j(n)$  be the number of times  $j$  was obtained as an outcome up to time  $n$ . The Doob decomposition of  $N_j(n)$  is

$$N_j(n) = \sum_{k=1}^n \text{tr}[M_j^* M_j \rho(k)] + N_j(n) - \sum_{k=1}^n \text{tr}[M_j^* M_j \rho(k)]. \quad (4.19)$$

$N_j(n) - \sum_{k=1}^n \text{tr}[M_j^* M_j \rho(k)]$  is a martingale and as before the law of large numbers for martingales apply. Thus  $\lim_n \frac{1}{n} (N_j(n) - \sum_{k=1}^n \text{tr}[M_j^* M_j \rho(k)]) = 0$ . From the ergodic property of  $\rho(n)$  we know that

$$\lim_n \frac{1}{n} \sum_{k=1}^n \text{tr}[M_j^* M_j \rho(k)] = \text{tr}[M_j^* M_j \rho(\infty)] \quad (4.20)$$

almost surely. It follows directly that

$$\lim_{n \rightarrow \infty} N_j(n)/n = \text{tr}[M_j^* M_j \rho(\infty)] \quad (4.21)$$

almost surely. In the long time limit the frequencies of the outcomes correspond to the frequencies expected when the system is in one of the invariant states of  $\Phi$ . If there is a unique invariant state the frequencies will correspond to it. We will use this property to prove an exponential convergence of the system state in the case of non demolition measurements in section 4.3.

#### 4. Repeated indirect measurements

##### 4.2.2. State purification

One interesting property of quantum trajectories is that in general they preserve pure states. Of course this is not true when the system is subject to an additional dissipation or the probe state is a mixed one. In these cases the state might get mixed. But here we do not take into account these possibilities.

If we look at our state update rule applied to a pure system state, the purity conservation is obvious. Let the system pure state be  $|\phi\rangle$ . After an indirect measurement it becomes

$$|\phi'\rangle = \frac{M_j|\phi\rangle}{\|M_j|\phi\rangle\|} \text{ with probability } \|M_j|\phi\rangle\|^2. \quad (4.22)$$

Hence the state stays pure.

In [75] H. Maassen and B. Kümmerer proved that not only the purity is conserved, but mixed states tend to purify along the trajectory. Of course this is not always true. If all the Kraus operators are proportional to unitary operators ( $M_j = \sqrt{\lambda_j}U_j$ ), then the state after the measurement is unitarily equivalent to the state before. So the eigenvalues of  $\rho'$  are the same as the ones of  $\rho$ . Hence pure states stay pure and mixed states stay mixed.

The result obtained in the cited article, is the following. Either a quantum trajectory purifies with probability one, or it exists a, at least two dimensional, subspace of  $\mathcal{H}_{\text{sys.}}$  on which the action of any  $M_j^*M_j$  corresponds to a multiplication by a positive scalar:  $PM_j^*M_jP = \lambda_jP$  with  $P$  the projection on the subspace. Equivalently, either  $\rho(n)$  purifies when  $n$  goes to infinity with probability one or it exists a mixed state  $\rho$  such that for any  $j$ , it exists a unitary operator  $U_j$  such that  $\rho' = U_j\rho U_j^*$ .

The demonstration of [75] is based on an inequality obtained by M. Nielsen in [79]. For any natural integer  $m$ ,

$$\sum_j \text{tr}[M_j^*M_j\rho] \text{tr} \left[ \left( \frac{M_j\rho M_j}{\text{tr}[M_j^*M_j\rho]} \right)^m \right] \geq \text{tr}[\rho^m]. \quad (4.23)$$

It follows that the processes  $(\text{tr}[\rho(n)^m])$  are submartingales. They are bounded and thus converge almost surely. The value of  $\text{tr}[\rho^m]$  for  $m > 1$  characterizes if  $\rho$  is pure or not. We have  $\text{tr}[\rho^m] = 1$  for  $m > 1$  if and only if  $\rho$  is pure (*i.e.* is a rank one projector). So either  $\text{tr}[\rho(n)^m]$  converges almost surely to one and the state purifies, or there is a possibility it converges to a value strictly smaller than one.

If the second possibility hold, the authors of [75] showed that with a non null probability ( $\rho(n)$ ) will be infinitely many times as close as we want from a mixed state  $\rho_{\text{acc.}}$ . This state is such that for any  $j$ , either  $\text{tr}[M_j^*M_j\rho_{\text{acc.}}] = 0$  and we have  $PM_j^*M_jP = 0$  with  $P$  the projection on the support of  $\rho_{\text{acc.}}$ . Or  $\text{tr}[(\rho'_{\text{acc.}})^m] = \text{tr}[\rho_{\text{acc.}}^m]$  for any  $m \geq 0$  which shows unitary equivalence between  $\rho'_{\text{acc.}}$  and  $\rho_{\text{acc.}}$ . This is actually equivalent to  $PM_j^*M_jP = \lambda_jP$  with  $P$  the projector onto the support of  $\rho_{\text{acc.}}$ . Note that there is no reason for  $\rho'_{\text{acc.}}$  to be supported on  $P\mathcal{H}_{\text{sys.}}$ . Hence the action of the measurement on  $\rho'_{\text{acc.}}$  is not necessarily equivalent to a unitary one.

In next section we illustrate these result for a specific class of discrete quantum trajectories, non demolition measurements. The system state evolution under non demolition measurements is nevertheless not a direct consequence of the results we just presented.

### 4.3. Non demolition indirect measurements and wave function collapse

In this section, we review results on discrete quantum trajectories obtain first in [21] by M. Bauer and D. Bernard and then generalized in [20]. They, in part, correspond to discrete time version of the results on wave function collapse for continuous time quantum trajectories [2,98]. In next chapter we will deal with these continuous time trajectories adding to the cited articles results of [20,29].

This work was inspired by a photon number measurement experiment performed by S. Haroche's group [61]. In this experiment they were able to measure the number of photons inside a super conducting cavity without destroying them. In most cases measuring photons means converting them into an electric signal and therefore destroying them. In this experiment, using a repeated quantum non demolition measurement strategy they were able to obtain an electromagnetic field in a fixed photon number Fock state.

The idea of non demolition (QND) measurements was formulated in 1980 by V. B. Braginsky *et al.* [35]. Non demolition means that a set of states, the pointer states, are invariant under the measurement procedure. If the system is in one of this pointer states, no future measurement will get it out of it. In S. Haroche's group experiment these states would be the fixed photon number Fock states. Non demolition measurements can sometimes be called weak measurements when they also aim at the smallest perturbation of the system by the measurement. The down side being that only partial information is obtained with one measurement.

In Serge Haroche's group experiment, a sequence of two level atoms were sent through the cavity. Their orientation was then measured along a varying axis. Hence it is a typical quantum trajectory implementation. The interaction between the atoms and the photon field was tuned such that the fixed photon number Fock states were invariant under the measurement. After a large number of such indirect measurements, the photon field state was found to be one of the fixed photon number Fock states. The distribution of the limit states corresponded to what is expected by von Neumann projection postulate. Hence the indirect measurement was equivalent to a direct projective measurement of the photon number. But this time the photons were still inside the cavity and not destroyed by the measurement.

In this section we show why repeated QND measurements lead to a wave function collapse following the von Neumann postulate.

The following results may look like a specific case of the previous ergodic and purification results, but they concern actually specific situations where stronger results can be obtained. We are here interested in the convergence of the state itself not of its Cesàro mean. This is allowed by the introduction of the pointer states. Their definition is directly inspired by the previously cited experiments. Certainly, the pointer probabilities martingales  $q_\alpha(n) = \langle \alpha | \rho(n) | \alpha \rangle$  we will introduce correspond to the diagonal elements of  $P_\Phi(\rho(n))$  in the pointer basis. But their introduction allows us to unravel a non degeneracy criteria which leads to the wave function collapse. Finally the exponential convergence of the state we find is not contained in previous results. Even if we use the ergodic theorem in one of the proofs.

The section is structured as follow. In the first subsection, we present a simpler version of our results. We obtain the collapse through a martingale convergence theorem. We show that the convergence is exponential with a rate given by a relative entropy between different distributions of the measurement outcomes. This first part corresponds mainly to [21]. In the

#### 4. Repeated indirect measurements

second subsection we generalize these results to cases where the indirect measurement method can be changed at anytime keeping the non demolition property. We link the wave function collapse with a martingale change of measure. We show how the freedom in the choice of probes allow us to tune the convergence rate. Finally, we analyze degenerate measurements and compare non degeneracy criteria with the purification criteria we gave in previous section.

##### 4.3.1. Introduction to non demolition measurements

The first and only requirement of non demolition measurement is the stability of the pointer basis under the measurement process. It exists an orthonormal basis  $\mathcal{P}$  of  $\mathcal{H}_{\text{sys.}}$  such that for any measurement outcome  $j$  and any vector  $|\alpha\rangle \in \mathcal{P}$ ,

$$\rho = |\alpha\rangle\langle\alpha| \Rightarrow \frac{M_j \rho M_j^*}{\text{tr}[M_j^* M_j \rho]} = |\alpha\rangle\langle\alpha|. \quad (4.24)$$

This is the non demolition condition for  $\mathcal{P}$ . Once we are in one of the pointer states we almost surely stay in this pointer state as would be expected for a measurement process. It does not modify the system state if it is already in one of the state we want to measure. The non demolition condition is fulfilled if and only if all the Kraus operators  $M_j$ 's are diagonal in the basis  $\mathcal{P}$ . Obviously it is fulfilled if the unitary interaction between the system and the probe,  $U$  is bloc diagonal in the basis  $\mathcal{P}$ .

$$\text{non demolition for } \mathcal{P} \left\{ \begin{array}{l} \Leftrightarrow M_j = \sum_{\alpha} m(j|\alpha) |\alpha\rangle\langle\alpha| \\ \Leftrightarrow U = \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U(\alpha) \end{array} \right. . \quad (4.25)$$

With this condition any state diagonal in the pointer basis is stationary under the average evolution. If  $\rho = \sum_{\alpha} q_{\alpha} |\alpha\rangle\langle\alpha|$ ,  $\Phi(\rho) = \sum_j M_j \rho M_j^* = \rho$ .

In the rest of this section we will not make a difference between the set of index of the pointer states and the set of pointer states. We have an obvious bijection between the two sets and the meaning of  $\mathcal{P}$  is never ambiguous.

##### Wave function collapse

We turn to the evolution taking into account the measurement outcomes  $\{j_1, \dots, j_n, \dots\}$ . The study of the diagonal elements of  $\rho(n)$  in  $\mathcal{P}$  will lead us to almost all the results we expect. We define  $q_{\alpha}(n) = \langle\alpha|\rho(n)|\alpha\rangle$ . These processes correspond to the evolution of the probabilities to find the pointers  $\alpha$  as the result of a direct measurement of  $\mathcal{P}$ . If at time  $n$  we stop our indirect measurement and perform a direct von Neumann measurement of  $\mathcal{P}$  on the system, we obtain  $\alpha$  as a result with probability  $q_{\alpha}(n)$ . These  $q_n(\alpha)$  are almost surely positive and bounded by 1. Their sum is equal to 1. Hence  $(q_{\alpha}(n))_{\alpha \in \mathcal{P}}$  is a probability distribution over  $\mathcal{P}$ .

Since the Kraus operators are all diagonal in  $\mathcal{P}$ , the evolution  $(q_{\alpha}(n))$  does not depend on the off diagonal elements.  $q_{\alpha}(n)$  update rule knowing  $\{j_1, \dots, j_n\}$  is

$$q_{\alpha}(n+1) = q_{\alpha}(n) \frac{p(j|\alpha)}{\sum_{\beta \in \mathcal{P}} q_n(\beta) p(j|\beta)} \text{ with probability } \sum_{\beta \in \mathcal{P}} q_n(\beta) p(j|\beta), \quad (4.26)$$

where  $p(j|\alpha) = |m(j|\alpha)|^2$ . This is a direct consequence of the definition of the process  $(\rho(n))$  and the diagonal property of the Kraus operators. The update rule corresponds to a Bayesian

### 4.3. Non demolition indirect measurements and wave function collapse

update.  $q_\alpha(n)$  is the initial distribution,  $p(j|\alpha)$  is the probability of the outcome  $j$  knowing the system is in the state  $\alpha$  and  $\sum_\beta q_\beta(n)p(j|\beta)$  is the probability to obtain the outcome  $j$  when the pointers are distributed by  $(q_\alpha(n))$ . Hence we are reduced to a classical random variable convergence problem.

From the update rule we immediately find that the  $q_\alpha(n)$ 's are martingales. The expectation of  $q_\alpha(n+1)$  knowing the first  $n$  outcomes  $\{j_1, \dots, j_n\}$  is equal to  $q_\alpha(n)$ .

$$\mathbb{E}(q_\alpha(n+1)|\{j_1, \dots, j_n\}) = \sum_j q_\alpha(n)p(j|\alpha) = q_\alpha(n). \quad (4.27)$$

The probabilities of the pointers are conserved in mean. Using bounded martingale convergence theorem [80], we immediately find that all the  $q_\alpha(n)$  converge almost surely when  $n$  tends to infinity. We write  $q_\alpha(\infty)$  this limit. Of course  $q_\alpha(\infty)$  is a random variable. The characterization of this random variable depends on the degeneracy status of our non demolition measurement.

Assume that for two different pointers there is always one outcome which has a different probability either if the system is in one or the other pointer state. Formally

$$\forall \alpha, \beta \in \mathcal{P}, \alpha \neq \beta, \exists j \text{ s.t. } p(j|\alpha) \neq p(j|\beta). \quad (4.28)$$

We call this condition the non degeneracy condition. In [21] M. Bauer and D. Bernard showed that this non degeneracy condition implies that  $q_\alpha(\infty)$  is a random variable which is either 0 or 1 almost surely. Moreover only one  $q_\alpha(\infty)$  can be equal to 1 the others are 0. We call  $\Upsilon$  the pointer corresponding to  $q_\Upsilon(\infty) = 1$ . It is a random variable on the set of pointer states. The probability of  $\Upsilon$  being equal to a specific pointer  $\alpha$  is equal to the initial probability  $q_\alpha(0) = \langle \alpha | \rho(0) | \alpha \rangle$  of this pointer.

The proof is the following. From the expression of the update rule, it is shown in [20], that  $(q_\alpha(\infty))_{\alpha \in \mathcal{P}}$  must be such that

$$q_\alpha(\infty)q_\beta(\infty)(p(j|\alpha) - p(j|\beta)) = 0. \quad (4.29)$$

for any  $j$  and any  $\alpha$  and  $\beta$ . Hence, either  $q_\alpha(\infty)q_\beta(\infty) = 0$  or  $p(j|\alpha) = p(j|\beta)$ . Since the latter is false for at least one outcome, we have that  $q_\alpha(\infty)q_\beta(\infty) = 0$  for any two different pointers. Hence, only one limit pointer probability can be strictly positive. Since  $\sum_\beta q_\beta(\infty) = 1$ , this limit pointer probability must be 1. From the conservation of the expectation of martingales, we have  $\mathbb{E}(q_\alpha(\infty)) = q_\alpha(0)$ . Hence  $q_\alpha(\infty) = 1$  with probability  $q_\alpha(0)$ .

The translation of this pointer distribution convergence to the convergence of the system state is straightforward. We have  $\rho(\cdot) = |\alpha\rangle\langle\alpha|$  if and only if  $q_\alpha(\cdot) = 1$  with the dot equal to any  $n$  or  $\infty$ . We can summarize the results obtained so far: If the non demolition and non degeneracy conditions are fulfilled,

$$\lim_{n \rightarrow \infty} \rho(n) = |\Upsilon\rangle\langle\Upsilon|, \mathbb{P}(\Upsilon = \alpha) = q_\alpha(0). \quad (4.30)$$

Hence in the limit, the system state is projected onto one of the pointer states with a probability corresponding to the von Neumann projection postulate applied to the initial state.

### Exponential convergence

We end this subsection with the derivation of the exponential rate of convergence towards the collapse. We will generalize our setup and give a better understanding of this exponential convergence in next subsection.

#### 4. Repeated indirect measurements

The following elements of proof are different than what was done in [20]. They correspond more to [21]. Let us study the limit of  $\ln(q_\alpha(n))/n$ . From now on we assume  $p(j|\alpha) > 0$  for any outcome and any pointer to avoid infinities.

First off, we have an equivalence between the limit of  $\ln(q_\alpha(n))/n$  and  $\ln(q_\alpha(n)/q_\Upsilon(n))/n$  since  $q_\Upsilon(n)$  converges almost surely to 1. Let us give an explicit expression of  $\ln(q_\alpha(n)/q_\Upsilon(n))$  that we deduce from the update rule of  $q_\alpha(n)$ .

$$\ln(q_\alpha(n)/q_\Upsilon(n)) = \ln(q_\alpha(0)/q_\Upsilon(0)) + \sum_j \ln(p(j|\alpha)/p(j|\Upsilon)) N_j(n) \quad (4.31)$$

with  $N_j(n)$  the number of time the outcome  $j$  was obtained up to time  $n$ . In section 4.2.1 we saw that  $\lim_n N_j(n)/n = \text{tr}[M_j^* M_j \rho(\infty)]$  with  $\rho(\infty) = \lim_n \frac{1}{n} \sum_{k=0}^n \rho(k)$ . Of course here  $\rho(\infty)$  is known. We have  $\rho(\infty) = |\Upsilon\rangle\langle\Upsilon|$ . Thus  $\lim_n N_j(n)/n = p(j|\Upsilon)$ . We immediately obtain

$$\lim_n \ln(q_\alpha(n)/q_\Upsilon(n))/n = -S(\Upsilon|\alpha) \quad (4.32)$$

with

$$S(\Upsilon|\alpha) = \sum_j p(j|\Upsilon) \ln(p(j|\Upsilon)/p(j|\alpha)) \quad (4.33)$$

the relative entropy of the distribution of the outcomes knowing the limit pointer with respect to the same distribution knowing the pointer we are interested in. This relative entropy is always strictly positive unless the two distributions are equal. Hence it is strictly positive when the two pointers are different. But if  $\alpha = \Upsilon$ ,  $S(\Upsilon|\alpha) = 0$ .

Going back to the probability of  $\alpha$ ,

$$q_\alpha(n) = e^{-nS(\Upsilon|\alpha)+o(n)}, \alpha \neq \Upsilon. \quad (4.34)$$

Hence the system state converges exponentially to  $|\Upsilon\rangle\langle\Upsilon|$  with a rate given by the relative entropy  $S(\Upsilon|\alpha)$ . Let  $S_{\min}(\gamma)$  be the minimum of these rates when the limit state is  $\gamma$ :  $S_{\min}(\gamma) = \min_{\alpha \neq \gamma} S(\gamma|\alpha)$ . Then using Cauchy-Schwarz inequality for semi definite matrices, almost surely,

$$\|\rho(n) - |\Upsilon\rangle\langle\Upsilon|\|_{\max} = e^{-nS_{\min}(\Upsilon)/2+o(n)}. \quad (4.35)$$

The norm  $\|X\|_{\max}$  is equal to the maximum of the norm of the elements of  $X$ .

#### 4.3.2. Non demolition measurement and martingale change of measure

##### Generalized non demolition measurements

In S. Haroche's group experiments [61, 83], they tried to optimize the rate of convergence. They used different axes for the atom measurements. They even tried to choose the probe measurement with respect to previous measurement outcomes [83]. In this section we introduce the possibility of such a change of indirect measurement method depending on the whole history of the measurement. This subsection refers to results obtained in [20].

We assume we have a finite number of different methods of indirect measurements available. A measurement method is a set of Kraus operators  $\{M_{o,j}|j \in \{j\}\}$  with  $o$  fixed. For any  $o$  we

### 4.3. Non demolition indirect measurements and wave function collapse

have  $\sum_j M_{o,j}^* M_{o,j} = I$ . The index  $o$  correspond to the measurement method. The different methods can differ either by the initial probe state, the system–probe interaction or the measurement performed on the probe. In the case where the same measurement method is used every time,  $o$  takes only one value and then can be forgotten. Each method of indirect measurement is a well define indirect measurement on its own.

Since we can use any of the indirect measurement methods, they a priori all need to fulfill the non demolition condition. Actually this time we are not sure we will be using all the methods. So we have only an implication.

$$\forall j, o, M_{o,j} = \sum_{\alpha \in \mathcal{P}} m(j|o, \alpha) |\alpha\rangle \langle \alpha| \Rightarrow \text{non demolition condition for } \mathcal{P}. \quad (4.36)$$

Of course this is true if for all methods the system–probe interaction is bloc diagonal in the basis  $\mathcal{P}$ . From now on we assume this non demolition condition is fulfilled.

We need to implement the way we choose the measurement method at each time. Remember that the way we choose it can depend on the whole history of the measurement. Hence at time  $n$  the measurement method is chosen with the distribution  $c_n(o|o_1, j_1, \dots, o_{n-1}, j_{n-1})$ . Hence for any  $n \geq 0$ , for any finite sequence  $(o_1, j_1, \dots, o_{n-1}, j_{n-1})$ , for any  $o$  we have  $c_n(o|o_1, j_1, \dots, o_{n-1}, j_{n-1}) \geq 0$  and  $\sum_o c_n(o|o_1, j_1, \dots, o_{n-1}, j_{n-1}) = 1$ . The distribution of the measurement method can depend both on time and the history of the measurement.

The update rule for  $q_\alpha(n+1)$  knowing  $o_1, j_1, \dots, o_n, j_n$  is now

$$q_\alpha(n+1) = q_\alpha(n) \frac{p(j|o, \alpha)}{\sum_\beta q_\beta(n) p(j|o, \beta)} \text{ with probability } c_{n+1}(o|\dots) \sum_\beta q_\beta(n) p(j|o, \beta) \quad (4.37)$$

with  $p(j|o, \alpha) = |m(j|o, \alpha)|^2$ . A straightforward computation confirms that  $q_\alpha(n)$  is still a martingale. It is still bounded, thus as before it converges almost surely to  $q_\alpha(\infty)$ .

The non degeneracy condition is here a bit trickier that when we only used one method. To find the most general one, let us step back a little. Let us define a family of probability measure  $(\mathbb{P}_\alpha)$ . Each measure is indexed by a pointer state. The measures  $\mathbb{P}_\alpha$  are probability measures on the set of measurement realizations  $\Omega = \{o_1, j_1, \dots, o_n, j_n, \dots\}$ .  $\mathbb{P}_\alpha$  corresponds to the probability measure if the system state was initially in the pointer state  $\alpha$ . We have an explicit expression for each of these measures.

$$\mathbb{P}_\alpha(o_1, j_1, \dots, o_n, j_n) = c_1(o_1) p(j_1|o_1, \alpha) \dots c_n(o_n|o_1, j_1, \dots, o_{n-1}, j_{n-1}) p(j_n|o_n, \alpha). \quad (4.38)$$

The probability measure when the initial system state is general is simply a weighted sum of these measures.

$$\mathbb{P} = \sum_\alpha q_\alpha(0) \mathbb{P}_\alpha \quad (4.39)$$

Thus we have absolute continuity of all the  $\mathbb{P}_\alpha$  such that  $q_\alpha(0) \neq 0$  with respect to  $\mathbb{P}$ :  $\mathbb{P}(A) = 0 \Rightarrow \mathbb{P}_\alpha(A) = 0$ .

As shown in [20],  $q_\alpha(\infty)/q_\alpha(0)$  is actually the Radon-Nikodym derivative of  $\mathbb{P}_\alpha$  with respect to  $\mathbb{P}$ . Hence for any integrable random variable  $X$ ,

$$q_\alpha(0) \mathbb{E}_\alpha(X) = \mathbb{E}(q_\alpha(\infty) X). \quad (4.40)$$

#### 4. Repeated indirect measurements

This definition of  $q_\alpha(\infty)$  allows us to find an equivalence between the wave function collapse and the mutual singularity of the measures  $\mathbb{P}_\alpha$ .

The mutual singularity of the measures means it exists a partition  $\{\Omega_\alpha\}_{\alpha \in \mathcal{P}}$  of our measurement realization space  $\Omega$  such that  $\mathbb{P}_\beta(\Omega_\alpha) = \delta_{\alpha,\beta}$  with  $\delta$  the Kronecker delta.

Let us illustrate this property with the case where only one method is used. We can define parts of  $\Omega$ :  $\Omega_\alpha = \{\omega \in \Omega \text{ s.t. } \lim_n N_j(n)/n = p(j|\alpha)\}$ . The part corresponding to  $\alpha$  is the set of all realizations whose outcome frequencies are  $p(j|\alpha)$ . Let  $\tilde{\Omega}$  be the part of  $\Omega$  not covered by the union of these  $\Omega_\alpha$ :  $\tilde{\Omega} = \Omega \setminus \cup_\alpha \Omega_\alpha$ . Let us redefine  $\Omega_0$  as  $\Omega_0 = \{\omega \in \Omega \text{ s.t. } \lim_n N_j(n)/n = p(j|0)\} \cup \tilde{\Omega}$ . Let us assume the measurement is non degenerate, then  $\cup_\alpha \Omega_\alpha = \Omega$  and  $\Omega_\alpha \cap \Omega_\beta = \emptyset$  for  $\alpha \neq \beta$ , so  $\mathbb{P}_\beta(\Omega_\alpha) = \delta_{\alpha,\beta}$ . We have explicitly build a partition of  $\Omega$  such that  $\mathbb{P}_\alpha(\Omega_\beta) = \delta_{\alpha,\beta}$ . Hence the measures  $\mathbb{P}_\alpha$  are all mutually singular. This construction is possible if and only if the measurement is non degenerate. If not there is at least two different pointers corresponding to a same  $\Omega$  part.

In general we have the following equivalence

$$\mathbb{P}_\alpha \text{'s are mutually singular} \Leftrightarrow \text{It exists a partition } (\Omega_\alpha) \text{ of } \Omega \text{ s.t. } q_\alpha(\infty) = 1_{\Omega_\alpha} \quad (4.41)$$

where  $1_{\Omega_\alpha}(\omega)$  is equal to 1 if  $\omega \in \Omega_\alpha$  and to 0 otherwise. Hence either  $q_\alpha(\infty) = 0$  or 1. But only one  $q_\alpha(\infty)$  can be equal to 1 for each realization. We note  $\Upsilon$  the corresponding pointer. Since  $q_\Upsilon(\cdot) = 1$  imply  $\rho(\cdot) = |\Upsilon\rangle\langle\Upsilon|$ , we have  $\rho(\infty) = |\Upsilon\rangle\langle\Upsilon|$ . We can write explicitly

$$\rho(\infty)(\omega) = |\Upsilon(\omega)\rangle\langle\Upsilon(\omega)| = \sum_\alpha |\alpha\rangle\langle\alpha| 1_{\Omega_\alpha}(\omega). \quad (4.42)$$

From the martingale property of  $q_\alpha(n)$  we also have  $\mathbb{P}(\alpha = \Upsilon) = \mathbb{E}(1_{\Omega_\alpha}) = q_\alpha(0)$ .

As before we have an almost sure collapse of the wave function onto one of the pointer with a distribution of the limit pointer given by the von Neumann postulate applied to the initial state.

Let us give the equivalence proof. Assume the  $\mathbb{P}_\alpha$  are all mutually singular. Then it exists a partition  $(\Omega_\alpha)$  of  $\Omega$  such that  $\mathbb{P}_\alpha(\Omega_\beta) = \delta_{\alpha,\beta}$ . Since  $q_\beta(\infty)/q_\beta(0)$  is the Radon-Nikodym derivative of  $\mathbb{P}_\beta$  with respect to  $\mathbb{P}$ , we have  $\mathbb{E}(q_\beta(\infty)1_{\Omega_\alpha}) = q_\beta(0)\mathbb{E}_\beta(1_{\Omega_\alpha}) = q_\beta(0)\delta_{\alpha,\beta}$ . Moreover  $q_\beta(\infty)1_{\Omega_\alpha} \geq 0$  so,  $q_\beta(\infty)1_{\Omega_\alpha} = q_\beta(\infty)\delta_{\alpha,\beta}$  almost surely. Then summing over  $\beta$  on both sides, we obtain  $q_\alpha(\infty) = 1_{\Omega_\alpha}$ . For the converse, assume it exists a partition  $(\Omega_\alpha)$  of  $\Omega$  such that  $q_\alpha(\infty) = 1_{\Omega_\alpha}$ . Then  $q_\alpha(0)\mathbb{E}_\alpha(1_{\Omega_\beta}) = q_\alpha(0)\mathbb{P}_\alpha(\Omega_\beta) = \mathbb{E}(1_{\Omega_\alpha}1_{\Omega_\beta}) = \mathbb{E}(q_\alpha(\infty))\delta_{\alpha,\beta}$ . Since  $\mathbb{E}(q_\alpha(\infty)) = q_\alpha(0)$ , we obtain  $\mathbb{P}_\alpha(\Omega_\beta) = \delta_{\alpha,\beta}$ .

A general condition to assure mutual singularity is given in [20]. Essentially, the experimenter must, for any two different pointers, use infinitely many times a method which fulfill the previous non demolition condition for these two pointers. For example if 4 pointers are involved, if the experimenter use alternatively every time two methods such that  $p(j|0,0) = p(j|0,2)$  for any  $j$  and  $p(0|0,1) \neq p(0|0,3)$  for the first and  $p(0|1,0) \neq p(0|1,2)$  and  $p(j|1,1) = p(j|1,3)$  for any  $j$ , for the second, the state will collapse even if the first method does not fulfill non demolition condition for 0 and 2 and the second method does not for 1 and 3.

We can also apply this condition of mutual singularity to direct projective measurements. The probability space is, in this case, the set of eigenvalues of an observable, hence  $\mathcal{P}$  directly. The pointers are the corresponding eigenvectors. Each part of the probability space corresponds to the singlet  $\Omega_\alpha = \{\alpha\}$  and the measure with respect to the pointers are  $\mathbb{P}_\alpha(\{\beta\}) = \delta_{\alpha,\beta}$ . There is no actual martingale since the measurement process is assumed

### 4.3. Non demolition indirect measurements and wave function collapse

instantaneous. We only have the Radon-Nikodym derivatives of  $\mathbb{P}_\alpha$  with respect to  $\mathbb{P}$  which are  $1_{\{\alpha\}}/q_\alpha(0)$ . We only have a two step process for the probabilities of the pointers. Before the measurement  $q_\alpha(0)$  and after it  $q_\alpha(\text{after}) = 1_{\{\alpha\}}$ .

#### Estimated state stability

From this mutual singularity condition, we can see that we only need the measurement outcomes to know in which state the system collapse. We do not need to compute the updated state. Each realization belong only in one of the parts  $\Omega_\alpha$  and since we have a bijection between these parts and the pointer states, the realization will correspond to one and only one part and thus to one and only one limit pointer state. Thus when we use only one method, looking only at the limit of the outcome frequencies  $N_j(n)/n$  is sufficient to know the limit pointer.

With no surprise we can thus show that if we start the computation of the state evolution with a estimated state which is not the physical initial state we will nevertheless find the right pointer state on which the system has collapsed. Assume we start with an estimated state  $\hat{\rho}(0)$  such that  $\hat{q}_\alpha(0) > 0$  whenever  $q_\alpha(0) > 0$ . For example  $\hat{\rho}(0) = I/\dim(\mathcal{H}_{\text{sys.}})$  is a good estimate. We update this state using the same formula as for the physical state. Knowing the  $n$  first methods and outcomes,

$$\hat{q}_\alpha(n+1) = \hat{q}_\alpha(n) \frac{p(j|o, \alpha)}{\sum_\beta \hat{q}_\beta(n) p(j|o, \beta)}. \quad (4.43)$$

The distribution of the method and outcome is still the physical one. For example, the probability to have  $j_{n+1} = j$  knowing all measurement results up to time  $n$  and all method choices up to time  $n+1$  is  $\sum_\beta q_\beta(n) p(j|o_{n+1}, \alpha)$ . Hence  $\hat{q}_\alpha(n)$  is not a martingale under the measure  $\mathbb{P}$  so its convergence is not guarantied.

Instead we can define a new probability measure, an estimated probability measure  $\hat{\mathbb{P}} = \sum_\beta \hat{q}_\beta(0) \mathbb{P}_\beta$ . It is the measure *as if* the estimated state was the physical state. Under this measure  $\hat{q}_\alpha(n)$  is a martingale. Repeating our previous discussion we have  $\hat{q}_\alpha(\infty) = 1_{\Omega_\alpha}$   $\hat{\mathbb{P}}$  almost surely. But we want an almost sure collapse under  $\mathbb{P}$ , the physical measure not under  $\hat{\mathbb{P}}$ . Conveniently, the definition of our estimated state guaranties we have absolute continuity of  $\mathbb{P}$  with respect to  $\hat{\mathbb{P}}$ , hence  $\hat{\mathbb{P}}(A) = 1 \Rightarrow \mathbb{P}(A) = 1$ . So  $\hat{\mathbb{P}}$  almost sure convergence implies  $\mathbb{P}$  almost sure convergence. Since the definition of the sets  $\Omega_\alpha$  depends only on the measures  $\mathbb{P}_\alpha$ , we have

$$\lim_n \hat{q}_\alpha(n) = \lim_n q_\alpha(n) = 1_{\Omega_\alpha} \quad (4.44)$$

$\mathbb{P}$  almost surely. Hence

$$\lim_n \hat{\rho}(n) = \lim_n \rho(n) = |\Upsilon\rangle\langle\Upsilon| \quad (4.45)$$

$\mathbb{P}$  almost surely.

#### Exponential rate improvement

Another interesting property is that imposing the limit pointer state of the system is equivalent to project initially the system in this pointer state. Formally we have for any measurable

#### 4. Repeated indirect measurements

subset  $A$  of  $\Omega$ ,

$$\mathbb{P}(A|\Upsilon = \alpha) = \mathbb{P}_\alpha(A). \quad (4.46)$$

This follows directly from the expression of  $\mathbb{P}$  and the conditional probability and from the mutual singularity of the  $\mathbb{P}_\alpha$ 's. It allows us to study any process under the law conditioned on the limit state and then go back to the law  $\mathbb{P}$ . We use this to compute the convergence rate of the system state.

We cannot use an ergodic argument as before since the result does not hold a priori when we change the method at each time. Instead we will use the usual law of large numbers under each measure  $\mathbb{P}_\alpha$ .

Assume the way we chose the measurement method only depends on the preceding method used and measurement outcome obtained. Especially it does not depend on time. We have

$$c_{n+1}(o|o_1, j_1, \dots, o_n, j_n) = c(o|o_n, j_n). \quad (4.47)$$

Under each measure  $\mathbb{P}_\alpha$ , the process corresponding to the couple measurement method and outcome,  $((o_n, j_n))_{n \in \mathbb{N}}$  is a Markov chain with kernel

$$K_\alpha(o', j'|o, j) = p(j|o, \alpha)c(o|o', j'). \quad (4.48)$$

Note that  $((o_n, j_n))$  is not a Markov chain under  $\mathbb{P}$  since the distribution of  $(o_{n+1}, j_{n+1})$  depends not only on  $(o_n, j_n)$  but also on  $(q_n(\alpha))$ . The expression of the kernel invites us to define a reduced kernel only for the measurement method process.

$$K_\alpha^{red}(o'|o) = \sum_{j'} p(j'|o', \alpha)c(o|o', j'). \quad (4.49)$$

If the Markov chain corresponding to this kernel is irreducible and aperiodic, it admits only one invariant distribution of the methods  $\mu_\alpha^{red}(o)$ . It follows that  $K_\alpha(o', j'|o, j)$  has also a unique invariant distribution which is

$$\mu_\alpha(o, j) = \mu_\alpha^{red}(o)p(j|o, \alpha). \quad (4.50)$$

Hence the law of large numbers for Markov chains implies

$$\lim_n N_{o,j}(n)/n = \mu_\alpha^{red}(o)p(j|o, \alpha), \quad \mathbb{P}_\alpha \text{ almost surely}, \quad (4.51)$$

with  $N_{j,o}(n)$  the number of time  $o$  was used and  $j$  obtained as an outcome up to time  $n$ . Since under  $\mathbb{P}_\alpha$ ,  $\Upsilon = \alpha$  almost surely, or equivalently  $\Upsilon = \alpha$  on  $\Omega_\alpha$ , we can replace  $\alpha$  by  $\Upsilon$  in the limit. Hence for any pointer  $\alpha$ ,

$$\lim_n N_{o,j}(n)/n = \mu_\Upsilon^{red}(o)p(j|o, \Upsilon), \quad \mathbb{P}_\alpha \text{ almost surely}. \quad (4.52)$$

Since  $\mathbb{P} = \sum_\alpha q_\alpha(o)\mathbb{1}_\alpha$  and  $\sum_\alpha q_\alpha(0) = 1$  we immediately conclude

$$\lim_n N_{o,j}(n)/n = \mu_\Upsilon^{red}(o)p(j|o, \Upsilon), \quad \mathbb{P} \text{ almost surely}. \quad (4.53)$$

Hence we have a law of large numbers where the limit frequencies of the methods and outcomes depend on the limit pointer state as in the one method case.

### 4.3. Non demolition indirect measurements and wave function collapse

Following the same path as before we obtain again an exponential convergence rate for  $q_\alpha(n)$ .

$$q_\alpha(n) = e^{-n\bar{S}(\Upsilon|\alpha)+o(n)}. \quad (4.54)$$

The new rate corresponds to a mean of the one method rates corresponding to each method used. The distribution of the methods is given by the invariant distribution of the reduced chain  $K_\Upsilon^{red}$ .

$$\bar{S}(\Upsilon|\alpha) = \sum_o \mu_\Upsilon(o) S_o(\Upsilon|\alpha), \quad (4.55)$$

$$S_o(\Upsilon|\alpha) = \sum_j p(j|o, \Upsilon) \ln(p(j|o, \Upsilon)/p(j|o, \alpha)). \quad (4.56)$$

This freedom in the rate of convergence introduced by the choice of measurement method allows for some tuning of the convergence rate. For instance, the mean convergence rate will always be greater or equal to the minimum of the convergence rate corresponding to the different methods taken alone. Let  $\bar{S}_{\min}(\gamma) = \min_{\alpha \neq \gamma} \bar{S}(\gamma|\alpha)$  and  $S_{\min,o}(\gamma) = \min_{\alpha \neq \gamma} S_o(\gamma|\alpha)$ . We have

$$\bar{S}_{\min}(\gamma) \geq \min_o S_{\min,o}(\gamma). \quad (4.57)$$

Let us give an example of such rate improvement. A four dimensional system is subject to a non degenerate, non demolition measurement. The pointers are indexed 0 to 3. At each step we choose independently with an equiprobability between two measurement methods  $a$  and  $b$ . They both have two possible outcomes 0 or 1. It is typically the setup of Serge Haroche's group experiment where they used 4 methods with two possible outcomes. The distribution of the outcomes given the pointers are

$p(j o, \alpha)$		pointers			
	outcomes	0	1	2	3
method $a$	0	3/4	1/3	2/3	1/4
	1	1/4	2/3	1/3	3/4
method $b$	0	1/3	1/4	2/3	3/4
	1	2/3	3/4	1/3	1/4

and  $c(a) = c(b) = 1/2$ .

We compute  $S_{\min,a}(\gamma)$ ,  $S_{\min,b}(\gamma)$  and  $\bar{S}_{\min}(\gamma)$ :

$\gamma$	0	1	2	3
$S_{\min,a}(\gamma)$	0.016	0.017	0.016	0.017
$S_{\min,b}(\gamma)$	0.017	0.017	0.016	0.017
$\bar{S}_{\min}(\gamma)$	0.12	0.12	0.12	0.28

The minimal mean rate is almost always ten times greater than each of the minimal rates corresponding to the two methods. From this data we are tempted to say that the state will take almost 10 times less time to converge. We can make this statement more accurate by studying mean convergence times.

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Let us define a stopping time for the first time we have a much greater probability to end up in the state  $\Upsilon$  instead of the state  $\alpha$ :

$$\tau(\alpha|\Upsilon) = \inf\{n \geq 0 \text{ such that } q_\alpha(n)/q_\Upsilon(n) \leq \epsilon\}. \quad (4.58)$$

At time  $\tau(\alpha|\Upsilon)$ , the probability to project onto  $\alpha$  is at most equal to  $\epsilon$  times the probability to project onto  $\Upsilon$ . Hence, if  $\epsilon$  is small enough, we have a much greater chance to project onto  $\Upsilon$  than onto  $\alpha$ . Thus we call this stopping time a convergence time. It is the first time the probability to project onto  $\alpha$  will be negligible. The notation  $\tau$  is sometimes used as a shorthand for  $\tau(\alpha|\Upsilon)$ .

When we choose the methods independently with the same distribution at each time (*i.e.*  $c_n(o|\dots) = c(o)$ ), we can compute the mean of this convergence time conditioned on the limit pointer state through Wald's identity. Under the law conditioned on the limit pointer state, the methods and measurement outcomes are i.i.d. random variables with  $\mathbb{P}_\Upsilon((o_n, j_n) = (o, j)) = c(o)p(j|o, \Upsilon)$ . In this context  $\ln(q_\alpha(n)/q_\Upsilon(n))$  is a sum of i.i.d. random variables.

$$\ln(q_\alpha(n)/q_\Upsilon(n)) = \ln(q_\alpha(0)/q_\Upsilon(0)) + \sum_{k=1}^n \ln(p(j_k|o_k, \alpha)/p(j_k|o_k, \Upsilon)). \quad (4.59)$$

Using Markov's inequality we show the expectation of  $\tau$  under  $\mathbb{P}_\Upsilon$  is finite<sup>2</sup>. Thus we can use Wald's equation. It gives us

$$\mathbb{E}_\Upsilon(\tau) = \frac{\ln(q_\alpha(0)/q_\Upsilon(0)) - \mathbb{E}_\Upsilon(\ln(q_\alpha(\tau)/q_\Upsilon(\tau)))}{\bar{S}(\Upsilon|\alpha)}. \quad (4.60)$$

Of course this is well defined only when  $\alpha \neq \Upsilon$ . When  $\alpha = \Upsilon$  we set it formally to  $\infty$  since we have  $q_\alpha(n)/q_\Upsilon(n) = 1$  at any time.

This result may appear unnatural at first sight. Usually using Doob's optional stopping theorem, one can show that since  $\mathbb{E}_\Upsilon(\tau) < \infty$ ,  $\mathbb{E}_\Upsilon(q_\alpha(\tau)/q_\Upsilon(\tau)) = q_\alpha(0)/q_\Upsilon(0)$ . In our situation the almost sure upper bound  $\epsilon$  on  $q_\alpha(\tau)/q_\Upsilon(\tau)$  contradicts this last equality. But it is not the condition  $\mathbb{E}(\tau) < \infty$  that fails. It is the condition that it exists a  $C > 0$  such that  $\mathbb{E}_\Upsilon\left(\left|\frac{q_\alpha(n+1)}{q_\Upsilon(n+1)} - \frac{q_\alpha(n)}{q_\Upsilon(n)}\right| \middle| j_1, \dots, j_n\right) < C$  on the event  $n < \tau$ . For  $n < \tau$ ,  $\frac{q_\alpha(n)}{q_\Upsilon(n)}$  is finite but can be as large as possible.

Let  $\Delta_{\min}l(\alpha|\gamma) = \min_j \ln\left[\frac{p(j|\alpha)}{p(j|\gamma)}\right] > -\infty$ . It is the minimal (negative) increment of  $\ln(q_\alpha(n)/q_\Upsilon(n))$  possible. Hence  $\ln(q_\alpha(n)/q_\Upsilon(n)) - \ln(q_\alpha(n-1)/q_\Upsilon(n-1)) \geq \Delta_{\min}l(\alpha|\gamma)$  almost surely for any time  $n$ . Since  $\ln(q_\alpha(\tau-1)/q_\Upsilon(\tau-1)) > \epsilon$ , we have the following almost sure bounds  $\ln(\epsilon) + \Delta_{\min}l(\alpha|\Upsilon) \leq \ln(q_\alpha(\tau)/q_\Upsilon(\tau)) \leq \ln(\epsilon)$ . We obtain

$$\frac{\ln(q_\Upsilon(0)/q_\alpha(0)) - \ln(\epsilon)}{\bar{S}(\Upsilon|\alpha)} \leq \mathbb{E}_\Upsilon(\tau(\alpha|\Upsilon)) \leq \frac{\ln(q_\Upsilon(0)/q_\alpha(0)) - \ln(\epsilon) - \Delta_{\min}l(\alpha|\gamma)}{\bar{S}(\Upsilon|\alpha)}. \quad (4.61)$$

The introduction of  $\Delta_{\min}l(\alpha|\gamma)$  allows us to deal with cases where  $\ln(q_\alpha(\tau)/q_\Upsilon(\tau))$  overshoots  $\ln(\epsilon)$ . Then the maximal value of the overshoot is  $\Delta_{\min}l(\alpha|\Upsilon)$ .

We see that the greater the convergence rate is, the shorter the mean convergence time is. The simulations of figure 4.2 using our previous example show this influence. When the convergence rate increases we expect that the convergence will happen sooner.

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2. See proof in Appendix A.1.

### 4.3. Non demolition indirect measurements and wave function collapse

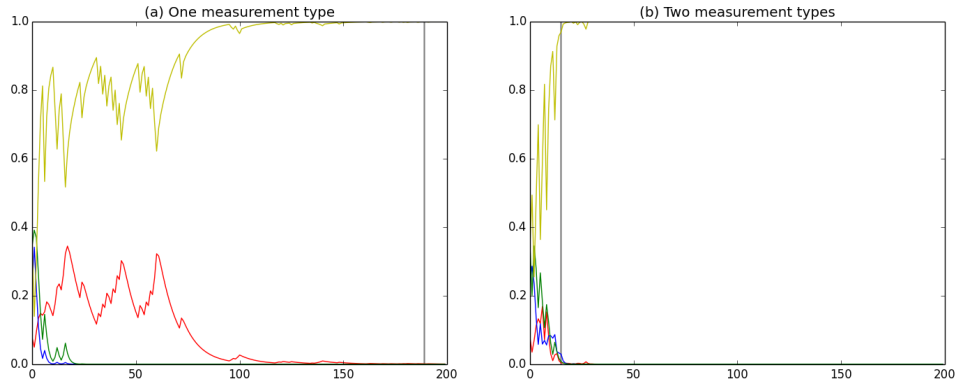


Figure 4.2.: Simulations of the pointer probability evolution for one realization of the example in the text. On the left (a) only the measurement method  $a$  is used. On the right (b) both methods are used. They are picked up at each time independently with equiprobability. The colored curves correspond to different pointers (red=0, blue=1, green=2, yellow=3). The black line in both graphs corresponds to the maximum over  $\alpha$  of the mean convergence time lower bound corresponding to the limit pointer of the realization:  $\max_{\alpha \neq 3} \mathbb{E}_3(\tau(\alpha|3))$ , when  $\epsilon = 0.01$ . It corresponds to the first time 3 has at least 100 more chance to be the limit pointer than any other pointer. We explicitly see that we can expect a larger number of realizations leading to an earlier collapse when using two measurement methods.

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Actually, in next section we will see that the rate and the mean time of convergence correspond, in an other context, to two different characterizations of the convergence. While the mean convergence time indicates when we can expect the state to converge, the rate corresponds to how sharply, or how fast, it will converge once it starts converging. This is enlightening when looking at the graphs of figure 4.2. In the QND case the two improvements of the measurement seem to happen together when the rate increase. We have both a sharper and an earlier convergence.

To conclude our discussion on the convergence rate, let us recall that since we have an exponential convergence for the pointer probabilities, we have an exponential convergence for the state

$$\|\rho(n) - |\Upsilon\rangle\langle\Upsilon|\|_{\max} = e^{-n\bar{S}_{\min}(\Upsilon)/2 + o(n)}. \quad (4.62)$$

##### 4.3.3. A note on degenerate measurements

Up to now we assumed the non demolition measurement was non degenerate. Of course measurements are not always non degenerate and several pointers might share the same value for the physical quantity we are interested in. In this subsection we discuss the possibility of degenerate non demolition measurements when only one measurement method is used. The generalization to different methods is straight forward and does not lead to radically different results.

Let us recall that the almost sure convergence of  $q_\alpha(n)$  does not depend on the degeneracy status of the measurement. The degeneracy will only change the limit value  $q_\alpha(\infty)$  which may no longer be 0 or 1.

In order to have degeneracy we need two different pointers such that  $p(j|\alpha) = p(j|\beta)$  for any outcome  $j$ . We define sectors which are sets of pointers sharing the same conditional outcome distribution. The sector in which  $\alpha$  belong is denoted  $\mathcal{P}_\alpha$ . We have an equivalence relation between the pointers of a same sector.

$$\alpha \equiv \beta \iff \forall j, p(j|\beta) = p(j|\alpha) \iff \beta \in \mathcal{P}_\alpha. \quad (4.63)$$

$\mathcal{P}_\alpha$  is the equivalence class of  $\alpha$ . We can define the probabilities of theses sectors.

$$q_{\mathcal{P}_\alpha}(n) = \sum_{\beta \in \mathcal{P}_\alpha} q_\beta(n). \quad (4.64)$$

The update rule for  $q_{\mathcal{P}_\alpha}(n)$  is actually the same as for  $q_\alpha(n)$ .

$$q_{\mathcal{P}_\alpha}(n+1) = q_{\mathcal{P}_\alpha}(n) \frac{p(j|\alpha)}{\sum_{\mathcal{P}_\beta} q_{\mathcal{P}_\beta}(n) p(j|\beta)} \text{ with probability } \sum_{\mathcal{P}_\beta} q_{\mathcal{P}_\beta}(n) p(j|\beta). \quad (4.65)$$

Between the sectors, the non degeneracy criteria is fulfilled. For two different sectors  $\mathcal{P}_\alpha$ ,  $\mathcal{P}_\beta$ , it exists a  $j$  such that  $p(j|\alpha) \neq p(j|\beta)$ . If it was not the case, we would have  $\mathcal{P}_\alpha = \mathcal{P}_\beta$ . Therefore we have almost surely

$$\lim_n q_{\mathcal{P}_\alpha}(n) = 1_{\alpha \in \mathcal{P}_\Upsilon} \quad (4.66)$$

with  $\mathcal{P}_\Upsilon$  one of the sectors. As before the probability for the limit sector to be  $\mathcal{P}_\alpha$  is  $q_{\mathcal{P}_\alpha}(0)$ . Hence we recover, in the long time limit, the same behavior as for an initial degenerate

#### 4.4. Invariant subspace stabilization

von Neumann projective measurement. This is true at least for the probabilities of the sectors. But as showed in [20], it is also true for the probabilities of the pointers. We have  $q_\alpha(\infty) = q_\alpha(0)/q_{\mathcal{P}_\alpha}(0)$  if  $\mathcal{P}_\alpha = \mathcal{P}_\Upsilon$  and  $q_\alpha(\infty) = 0$  else.

The convergence of the state is on the contrary not always obtained. We are in a typical situation where the state will not purify in the long time limit. Let us decompose the eigenvalues of the  $M_j$ 's in their modulus and argument:  $m(j|\alpha) = e^{i\theta(j|\alpha)}\sqrt{p(j|\alpha)}$ . The Kraus operators  $M_j$  can be written as a weighted sum of unitaries.

$$U_j(\alpha) = \sum_{\beta \in \mathcal{P}_\alpha} e^{i\theta(j|\beta)} |\beta\rangle\langle\beta| \quad (4.67)$$

$$M_j = \sum_{\mathcal{P}_\alpha} \sqrt{p(j|\alpha)} U_j(\alpha). \quad (4.68)$$

We recover here the action of the measurement as equivalent to a unitary action when the system state is supported on a subspace of the system Hilbert space. If  $q_\beta(n) = 0$  for any  $\beta \notin \mathcal{P}_\alpha$ , then  $\rho(n+1) = U_{j_{n+1}}(\alpha)\rho(n)U_{j_{n+1}}(\alpha)^*$ .

Let us define a unitary operator process

$$U'(n) = \sum_{\beta} e^{i\sum_{k=1}^n \theta(j_k|\beta)} |\beta\rangle\langle\beta|. \quad (4.69)$$

The limit of the system state is obtained when the rotation induced by this unitary evolution  $U'(n)$  is compensated. This is necessary since, contrary to the projective measurement, the measurement is not instantaneous.

$$\lim_n U'(n)^* \rho(n) U'(n) = \rho(\infty). \quad (4.70)$$

The limit state is the initial state projected on the system Hilbert space subspace corresponding to the limit sector. Let  $P_\alpha = \sum_{\beta \in \mathcal{P}_\alpha} |\beta\rangle\langle\beta|$ .

$$\rho(\infty) = \frac{P_\Upsilon \rho(0) P_\Upsilon}{\text{tr}[P_\Upsilon \rho(0)]}. \quad (4.71)$$

Of course this limit state can be a mixed state. That contradicts the almost sure purification as expected.

For the estimated state we also have a convergence towards the same sector. But the limit state is the estimated initial one projected onto the limit sector.

$$\lim_n U'(n)^* \hat{\rho}(n) U'(n) = \frac{P_\Upsilon \hat{\rho}(0) P_\Upsilon}{\text{tr}[P_\Upsilon \hat{\rho}(0)]} \quad (4.72)$$

almost surely.

#### 4.4. Invariant subspace stabilization

One of the current aim of quantum optics experiments is not to measure a set of pointer state but the preparation of specific pure system states, a Bell state for example. Maybe the most common technique used for state preparation is feedback control. The system state is continuously, or repeatedly indirectly measured and given the result a feedback action

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is performed on the system, driving it to the desired state [102]. Serge Haroche's group experiment [89] is a good example of such a setup in the discrete time case. In the late nineties and early two thousands the idea of engineering the coupling of the system to its environment such that it is autonomously driven to the desired state emerged [41, 47, 71, 87, 88, 99]. This *reservoir engineering* can also be implemented by specific types of feedback. It is the case for example of the Markovian feedback unitary control models [74, 95]. The feedback is minimal since only the last measurement result is used to determine the feedback action. Some experiments have already implemented this idea of reservoir engineering [19, 68, 72, 77, 90].

In this section we study discrete time quantum trajectories driving the system state toward an invariant subspace of the system Hilbert space. In the case where this subspace is one dimensional, it corresponds to a pure state preparation. If the evolution given the state is supported on the subspace is unitary then the limit subspace is a decoherence free subspace.

This section presents preliminary results of an article in preparation in collaboration with B. Cloez, C. Pellegrini and F. Ticozzi. This work was inspired by [94].

First we define precisely what we mean by invariant subspace and "driven towards". Then we recall the results of [33] on the invariance and global asymptotic convergence of the system state when the evolution is given by the repeated action of a CP map corresponding to the average evolution of an indirect measurement procedure. From these results we show that the convergence still holds when the measurements are taken into account. Moreover, recording the measurement results leads to a greater exponential convergence rate compared to the one obtained for the average evolution. Hence the convergence towards the invariant subspace is characterized by two time scales. We conclude with a discussion about these time scales.

##### 4.4.1. Invariant and globally asymptotically stable subspaces

Recall that  $\mathcal{H}_{\text{sys.}}$  is our finite dimensional system Hilbert space. We define an orthogonal decomposition of  $\mathcal{H}_{\text{sys.}}$ . We define  $\mathcal{H}_S$  and  $\mathcal{H}_R$  two Hilbert spaces such that  $\mathcal{H}_{\text{sys.}} = \mathcal{H}_S \oplus \mathcal{H}_R$  is an orthogonal decomposition of  $\mathcal{H}_{\text{sys.}}$ .  $\mathcal{H}_S$  is the Hilbert space towards which we want the state to converge. We can decompose any linear operator  $X$  on  $\mathcal{H}_{\text{sys.}}$  in four blocs:

$$X = \left( \begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array} \right) \quad (4.73)$$

$X_S$  is a linear operator on  $\mathcal{H}_S$ ,  $X_R$  a linear operator on  $\mathcal{H}_R$ ,  $X_P$  a linear operator from  $\mathcal{H}_R$  to  $\mathcal{H}_S$  and  $X_Q$  a linear operator from  $\mathcal{H}_S$  to  $\mathcal{H}_R$ .

Let  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  be the set of states such that  $\rho_R = 0$ , so  $\rho_P = 0$  too.

$$\rho \in \mathcal{I}_S(\mathcal{H}_{\text{sys.}}) \iff \rho = \begin{pmatrix} \rho_S & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.74)$$

Let also  $\mathcal{I}_R(\mathcal{H}_{\text{sys.}})$  be the set of states such that  $\rho_S = 0$ , so  $\rho_P = 0$  too.

$$\rho \in \mathcal{I}_R(\mathcal{H}_{\text{sys.}}) \iff \rho = \begin{pmatrix} 0 & 0 \\ 0 & \rho_R \end{pmatrix}. \quad (4.75)$$

We say that  $\mathcal{H}_S$  is an almost surely invariant subspace of the indirect measurement process if almost surely,

$$\rho(0) = \begin{pmatrix} \rho_S(0) & 0 \\ 0 & 0 \end{pmatrix} \text{ implies } \rho(n) = \begin{pmatrix} \rho_S(n) & 0 \\ 0 & 0 \end{pmatrix} \quad (4.76)$$

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for any time  $n > 0$ . Hence, if we started with a state in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ , the state will stay in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ .

For example, take a  $\mathcal{H}_{\text{sys.}} = \mathbb{C}^2$  system. We define  $\mathcal{H}_S = \text{linspan}(\{|0\rangle\})$  and  $\mathcal{H}_R = \text{linspan}(\{|1\rangle\})$ .  $\mathcal{H}_S$  almost surely invariant means that if our initial state is the ground state, then with probability one, at any later time the system state is the ground state.

Equivalently we say  $\mathcal{H}_S$  is an invariant subspace in mean if the average system state stays in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  for any time  $n > 0$ . In other words:

$$\rho(0) = \begin{pmatrix} \rho_S(0) & 0 \\ 0 & 0 \end{pmatrix} \text{ implies } \mathbb{E}(\rho(n)) = \begin{pmatrix} \mathbb{E}(\rho_S(n)) & 0 \\ 0 & 0 \end{pmatrix} \quad (4.77)$$

for anytime  $n > 0$ . We will see later that the almost sure invariance and invariance in mean are in fact equivalent.

We say that  $\mathcal{H}_S$  is an almost surely globally asymptotically stable (GAS) subspace if with probability one, for any initial system state  $\rho(0)$ ,

$$\lim_{n \rightarrow \infty} \left\| \rho(n) - \begin{pmatrix} \rho_S(n) & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0. \quad (4.78)$$

Hence, in the long time limit the system state is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ . In our  $\mathbb{C}^2$  example it means that for any initial state, the system converges with probability one towards its ground state.

As before we also say that  $\mathcal{H}_S$  is GAS in mean if the convergence property is fulfilled by the state average evolution whatever is the initial state  $\rho(0)$ :

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E}(\rho(n)) - \begin{pmatrix} \mathbb{E}(\rho_S(n)) & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0. \quad (4.79)$$

If  $\mathcal{H}_S$  is simply a one dimensional space corresponding to an entangled state, we just define the invariance of this state with probability one or in mean and the convergence towards this state with probability one or in mean.

The question we might ask is whether the conditions are the same if we want these properties in mean or almost surely. In next subsection we show that they are effectively the same.

##### 4.4.2. Invariance and asymptotic stability in mean

In [33] necessary and sufficient conditions were found such that a subspace  $\mathcal{H}_S$  is invariant and GAS in mean when the system undergoes repeated indirect measurements. Here we recall the results we are interested in that can be found in this article.

Let us recall that the mean evolution of the state is given by the repeated action of a trace preserving CP map. Let us write  $\bar{\rho}(n) = \mathbb{E}(\rho(n))$ . For any time  $n$  we have

$$\bar{\rho}(n+1) = \Phi(\bar{\rho}(n)) \text{ with } \Phi(\rho) = \sum_j M_j \rho M_j^*, \sum_j M_j^* M_j = I. \quad (4.80)$$

##### Invariance

A simple computation of  $\Phi(\rho)$  leads to the equivalence

$$\mathcal{H}_S \text{ is invariant in mean if and only if } M_{j,Q} = 0 \forall j. \quad (4.81)$$

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Hence  $M_j = \begin{pmatrix} M_{j,S} & M_{j,P} \\ 0 & M_{j,R} \end{pmatrix}$ . If moreover we consider the trace preserving condition  $\sum_j M_j^* M_j = I$ , we have  $\sum_j M_{j,P}^* M_{j,S} = 0$ ,  $\sum_j M_{j,S}^* M_{j,S} = I_{\mathcal{H}_S}$  and  $\sum_j M_{j,P}^* M_{j,P} + M_{j,R}^* M_{j,R} = I_{\mathcal{H}_R}$ . As we could have expected,  $\mathcal{H}_S$  is invariant in mean if and only if no transition from states of  $\mathcal{H}_S$  to states of  $\mathcal{H}_R$  is induced by the measurement process. This is actually a well known condition in the study of reducible CP maps.

Using this decomposition we can define two new CP maps which will be useful afterwards. First  $\Phi_S(\rho_S) = \sum_j M_{j,S} \rho_S M_{j,S}^*$  is a trace preserving CP map on the states of  $\mathcal{H}_S$ ,  $\mathcal{S}(\mathcal{H}_S)$ . If  $\rho$  is in  $\mathcal{I}_S(\mathcal{H})$ , we have

$$\Phi(\rho) = \begin{pmatrix} \Phi_S(\rho_S) & 0 \\ 0 & 0 \end{pmatrix} \quad (4.82)$$

with  $\text{tr}[\Phi_S(\rho_S)] = \text{tr}[\rho_S] = 1$ . Hence  $\Phi_S$  corresponds to the effective evolution when  $\rho$  is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ .

Then we define  $\Phi_R(\rho_R) = \sum_j M_{j,R} \rho_R M_{j,R}^*$ . Since  $\sum_j M_{j,P}^* M_{j,P} + M_{j,R}^* M_{j,R} = I_{\mathcal{H}_R}$ ,  $\text{tr}[\Phi_R(\rho_R)] = \text{tr}[\rho_R] - \text{tr}[\sum_j M_{j,P}^* M_{j,P} \rho_R] \leq \text{tr}[\rho_R]$ . Hence  $\Phi_R$  is trace non increasing. It corresponds to the evolution of the  $R$  bloc of the system state. We have

$$\Phi(\rho) = \begin{pmatrix} \star & \star \\ \star & \Phi_R(\rho_R) \end{pmatrix} \quad (4.83)$$

with  $\text{tr}[\Phi_R(\rho_R)] \leq \text{tr}[\rho_R]$ . The next state  $R$  bloc only depends on the previous one and not on the other blocs of the state. This property is in fact a manifestation of the invariance in mean of  $\mathcal{H}_R$  under the dual map  $\Phi^*$ . Let  $X$  be a linear operator on  $\mathcal{H}_{\text{sys.}}$  such that  $X_S = 0$ ,  $X_P = 0$  and  $X_Q = 0$ , then

$$\Phi^*(X) = \Phi^* \left( \begin{pmatrix} 0 & 0 \\ 0 & X_R \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_R^*(X_R) \end{pmatrix}. \quad (4.84)$$

#### Asymptotic stability

Let us now turn to the conditions under which  $\mathcal{H}_S$  is GAS in mean. Hence when the mean effect of the measurement is to drive the state into  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ . It is obvious that  $\mathcal{H}_S$  is GAS when  $\Phi_R$  is strictly trace decreasing. Let  $\lambda_0$  be the largest eigenvalue of  $\sum_j M_{j,R}^* M_{j,R}$ . Since  $\mathcal{H}_R$  is finite dimensional and  $\Phi_R$  is strictly trace decreasing, we have  $\lambda_0 < 1$  and  $\text{tr}[\Phi_R(\rho_R)] \leq \lambda_0 \text{tr}[\rho_R]$  for any  $\rho_R$ . Thus  $\text{tr}[\bar{\rho}_R(n)] \leq \lambda_0^n \text{tr}[\rho_R(0)]$  and  $\text{tr}[\rho_R(n)]$  converges to 0 when  $n$  tends to infinity. Thanks to Cauchy–Schwarz inequality on positive semi definite matrices it implies  $\bar{\rho}(n)$  converges to the subset  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  in the long time limit. Moreover we can characterize the convergence as exponential with a rate  $\sqrt{\lambda_0}$  for the  $\bar{\rho}_P(n)$  bloc and  $\lambda_0$  for the  $\bar{\rho}_R(n)$  bloc.

Actually it is not the only case where  $\mathcal{H}_S$  is GAS. In [33] the authors proved using La Salle's invariance principle that  $\mathcal{H}_S$  is GAS if and only if there is no invariant subspace included in the kernel of  $\sum_j M_{j,P}^* M_{j,P}$ . If such a subspace existed the system could be stuck in it. Let us make this necessary and sufficient condition more explicit. Let  $\mathcal{H}_{R_0}$  be a subspace of  $\sum_j M_{j,P}^* M_{j,P}$  kernel. We write  $\mathcal{H}_{R_1}$  its orthogonal complement in  $\mathcal{H}_R$ . We can decompose the operators  $M_{j,R}$  and  $M_{j,P}$  with respect to the direct sum decomposition  $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_{R_1} \oplus \mathcal{H}_{R_0}$ .

We have

$$M_{j,R} = \begin{pmatrix} M_{j,R_1} & M_{j,R_{10}} \\ M_{j,R_{01}} & M_{j,R_0} \end{pmatrix} \quad (4.85)$$

$$M_{j,P} = \begin{pmatrix} M_{j,P_1} & 0 \end{pmatrix}. \quad (4.86)$$

Then  $\mathcal{H}_S$  is invariant if and only if we cannot find any  $\mathcal{H}_{R_0}$  such that  $\forall j, M_{j,R_{10}} = 0$ .

This criteria is interesting since we can know if  $\mathcal{H}_S$  is GAS directly from the shape of the Kraus operators  $M_j$ . But the exponential convergence can be a bit trickier to find since we can have  $\text{tr}[\sum_j M_{j,R}^* M_{j,R} \rho_R] = 1$ . May be the most simple example of such situation is when we have a  $\mathbb{C}^3$  system with two measurement outcomes  $j = 0, 1$  such that

$$M_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.87)$$

In the long time limit the state will be  $\bar{\rho}(\infty) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . If  $\rho_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , then  $\text{tr}[\Phi_R(\rho_R)] = \text{tr}[\rho_R] = 1$ . Actually in this example we have  $\Phi^2(\rho(0)) = \rho(\infty)$  but it nevertheless shows that our previous approach cannot be generalized. We will come back on this issue later.

#### 4.4.3. Almost sure invariance and asymptotic stability

Let us now turn to the equivalence between the almost sure and in mean invariance and asymptotic stability. We show that the invariance and GAS of  $\mathcal{H}_S$  do not depend on the tracking of the outcomes.

Let  $P_S$  and  $P_R$  be the projectors onto respectively  $\mathcal{H}_S$  and  $\mathcal{H}_R$ .

$$P_S = \begin{pmatrix} I_S & 0 \\ 0 & 0 \end{pmatrix}, P_R = \begin{pmatrix} 0 & 0 \\ 0 & I_R \end{pmatrix}. \quad (4.88)$$

First remark that  $\rho \in \mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  is equivalent to  $\text{tr}[P_R \rho] = 0$  or  $\text{tr}[P_S \rho] = 1$ . We define  $V(\rho) = \text{tr}[\rho P_R]$ .

$$\rho \in \mathcal{I}_S(\mathcal{H}_{\text{sys.}}) \iff V(\rho) = 0. \quad (4.89)$$

The function  $V$  is strictly positive outside of  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  and null on  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ . Thus  $V$  is a good Lyapunov candidate function for our convergence problem. Since  $V(\Phi(\rho)) - V(\rho) \leq 0$ , it is a good Lyapunov function for the mean evolution.

#### Invariance

Suppose  $\mathcal{H}_S$  is invariant in mean. From the result of the previous subsection we have that if initially the state is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ , then  $\bar{\rho}(n)$  is in this same set of states. Hence if  $V(\rho(0)) = 0$ , we have for any time  $n \geq 0$ ,  $V(\bar{\rho}(n)) = 0$ . Since  $\mathcal{H}_{\text{sys.}}$  is finite dimensional and  $V$  linear, for any time  $n$   $\mathbb{E}(V(\rho(n))) = 0$ . Moreover we have  $V(\rho) \geq 0$  for any state, thus if  $\rho(0) \in \mathcal{I}_S(\mathcal{H})$ , almost surely for any time  $n$   $V(\rho(n)) = 0$ . Hence  $\rho(n) \in \mathcal{I}_S(\mathcal{H})$  almost surely which proves the almost sure invariance of  $\mathcal{H}_S$ . The converse implication is trivial.

$$\mathcal{H}_S \text{ invariant in mean} \iff \mathcal{H}_S \text{ almost surely invariant.} \quad (4.90)$$

#### 4. Repeated indirect measurements

##### Global asymptotic stability

We now discuss the global asymptotic stability of  $\mathcal{H}_S$ . We first show the GAS of  $\mathcal{H}_S$  in  $L^1$  norm and then turn to the almost sure GAS.

Suppose  $\mathcal{H}_S$  is GAS in mean. Then  $\lim_{n \rightarrow \infty} V(\bar{\rho}(n)) = 0$ . Once again we equivalently have  $\lim_{n \rightarrow \infty} \mathbb{E}(V(\rho(n))) = 0$ . Since  $V(\rho) \geq 0$  we immediately have the  $L^1$  norm convergence of  $V(\rho(n))$  towards 0.

For the almost sure convergence we use the supermartingale convergence theorem [80]. We have  $\mathbb{E}(V(\rho(n+1)) | j_1, \dots, j_n) = \text{tr}[\Phi_R(\rho_R(n))]$ . Recall that  $\Phi_R$  is trace non increasing, hence  $\text{tr}[\Phi_R(\rho_R(n))] \leq V(\rho_R(n))$  and  $(V(\rho(n)))_{n \in \mathbb{N}}$  is a supermartingale. Moreover  $V(\rho(n))$  is bounded, so from the supermartingale convergence theorem, we have that  $V_\infty = \lim_{n \rightarrow \infty} V(\rho(n))$  exists almost surely. Since  $V(\rho(n))$  converges in  $L^1$  to zero, using the existence of an almost sure convergent sub sequence and the uniqueness of the limit, we obtain  $V_\infty = 0$ . Hence  $\mathcal{H}_S$  is almost surely GAS. Using dominated convergence, the converse is once again trivial and we finally have.

$$\mathcal{H}_S \text{ GAS in mean} \Leftrightarrow \mathcal{H}_S \text{ almost surely GAS.} \quad (4.91)$$

From this equivalence between the properties in mean and almost surely one could think that taking into account the measurement outcomes instead of averaging them would not change much the behavior of the system state. But as we will see in next section, taking into account the measurement outcome recording can lead to a peculiar evolution.

##### 4.4.4. Exponential convergence

###### Mean rate

We can prove that the mean convergence is exponential with a rate corresponding to the spectral radius of  $\Phi_R$ . To show this, we need to modify our Lyapunov candidate function. One can remark that any function  $V_K(\rho) = \text{tr}[K\rho]$  with  $K$  a positive semi definite operator with  $\mathcal{H}_R$  as its support is suitable ( $\mathcal{H}_S$  is its kernel). We have

$$\rho \in \mathcal{I}_S(\mathcal{H}) \iff V_K(\rho) = 0. \quad (4.92)$$

For simplicity we write  $K$  either for the positive semi definite operator on  $\mathcal{H}_{\text{sys.}}$  or the positive definite operator on  $\mathcal{H}_R$ . Let  $\lambda_0$  be  $\Phi_R$  spectral radius. Our idea is to find  $K$  such that  $V_K(\Phi(\rho)) \leq \lambda_0 V_K(\rho)$ .

If  $\Phi_R$  is irreducible the existence of such  $K$  follows directly from Perron–Frobenius theorem for irreducible positive maps [54]. If  $\Phi_R$  is irreducible, it exists a unique definite positive  $K$  such that  $\Phi_R^*(K) = \lambda_0 K$ . Hence  $V_K(\Phi_R(\rho)) = \lambda_0 V_K(\rho)$ .

If  $\Phi_R$  is reducible it exists a positive semi definite operator  $K'$  on  $\mathcal{H}_R$  such that  $\Phi_R^*(K') = \lambda_0 K'$  but  $K'$  can have a non trivial kernel on  $\mathcal{H}_R$ . So  $V_{K'}$  is not always a good Lyapunov candidate function. Helpfully, following the demonstration of the existence of  $K'$  in [54], we can always find a  $K$  corresponding to a convergence rate as close as we want from  $\lambda_0$ .

$$\forall \epsilon > 0, \exists K_\epsilon \in \mathcal{B}(\mathcal{H}_R), K_\epsilon > 0 \text{ s.t. } \Phi_R(K_\epsilon) \leq (\lambda_0 + \epsilon) K_\epsilon. \quad (4.93)$$

Let  $\Phi_\eta = \Phi_R + \eta\Psi$  with  $\eta > 0$  and  $\Psi$  an irreducible CP map.  $\Phi_\eta$  is an irreducible CP map and its spectral radius  $\lambda_\eta$  converges continuously to the one of  $\Phi_R$  (*i.e.*  $\lambda_0$ ) when  $\eta$

#### 4.4. Invariant subspace stabilization

goes continuously to 0. Hence for any  $\epsilon > 0$  we can find a  $\eta_\epsilon > 0$  such that  $\lambda_{\eta_\epsilon} \leq \lambda_0 + \epsilon$ . Then it exists a  $K_\epsilon > 0$  such that  $\Phi_{\eta_\epsilon}^*(K_\epsilon) = \lambda_{\eta_\epsilon} K_\epsilon \leq (\lambda_0 + \epsilon) K_\epsilon$ . It follows  $\Phi_R^*(K_\epsilon) \leq (\lambda_0 + \epsilon) K_\epsilon - \eta_\epsilon \Psi^*(K_\epsilon) \leq (\lambda_0 + \epsilon) K_\epsilon$  which is what we wanted. Of course we need  $\lambda_0 + \epsilon < 1$ . It is always possible to find such a  $\epsilon$ . For example  $\epsilon = (1 - \lambda_0)/2$  match the requirements.

Finally whatever is the reducibility status of  $\Phi_R$ , for any strictly positive  $\epsilon$  it always exists a  $K > 0$  such that  $V_K(\Phi(\rho)) = \text{tr}[\Phi^*(K)\rho] = \text{tr}[\Phi_R^*(K)\rho_R] \leq (\lambda_0 + \epsilon)V_K(\rho)$ .

Hence in any case we have that for all  $\epsilon > 0$ , it exists  $K > 0$  on  $\mathcal{H}_R$  such that

$$V_K(\bar{\rho}(n)) \leq (\lambda_0 + \epsilon)^n V_K(\rho(0)). \quad (4.94)$$

This inequality implies that  $V_K(\bar{\rho}(n)) = O((\lambda_0 + \epsilon)^n)$  with  $O$  the usual asymptotic comparison when  $n$  goes to infinity.

We translate this exponential convergence to an exponential convergence of  $\bar{\rho}_R(n)$  norm at the end of this section.

#### Almost sure rate

Let us first find back the equivalent of the mean convergence rate when measurement outcomes are recorded.

The process  $(V_K(\rho_n)(\lambda_0 + \epsilon)^{-n})_{n \in \mathbb{N}}$  is a positive supermartingale.

$$\mathbb{E}(V_K(\rho(n+1))(\lambda_0 + \epsilon)^{-n-1} | j_1, \dots, j_n) = \quad (4.95)$$

$$V_K(\Phi(\rho(n)))(\lambda_0 + \epsilon)^{-n-1} \leq V_K(\rho(n))(\lambda_0 + \epsilon)^{-n}. \quad (4.96)$$

Hence  $V_K(\rho(n))(\lambda_0 + \epsilon)^{-n}$  converges almost surely by Doob supermartingale convergence theorem. Thus  $V_K(\rho(n)) = O((\lambda_0 + \epsilon)^n)$  with  $O$  an almost sure asymptotic comparison when  $n$  tends to infinity.

We recovered the mean convergence rate in an almost sure sense. This bound may seem the best we can find but actually, in some cases, we can obtain a higher convergence rate.

#### Rate improvement

From now on we fix  $\epsilon$  and the corresponding  $K$ . Accordingly we redefine  $\lambda_0$  as  $\lambda_0 = \text{spectral radius}(\Phi_R) + \epsilon$ . Hence  $\Phi_R^*(K) \leq \lambda_0 K$ .

In this section we use the knowledge we have on the convergence rate in the non demolition case to try to improve the convergence rate toward  $\mathcal{H}_S$ . As in the QND case, our aim is to find a bound for  $\ln(V_K(\rho(n)))/n$  when  $n$  goes to infinity.

To avoid infinities, let us assume  $M_j^* M_j$  and  $M_{j,R}^* M_{j,R}$  are definite positive for any  $j$  and  $V_K(\rho(n)) > 0$  almost surely at any time  $n$ . Hence the limit is reached only in infinite time. Thus at any time  $\ln(V_K(\rho(n)))$  is well defined. These assumptions are the equivalent of the one we made in the non demolition case for the derivation of the convergence rate.

We define two new processes  $\hat{\rho}_S(n) = \rho_S(n)/\text{tr}[\rho_S(n)]$  which is well defined at least after a fixed bounded time  $n_0$  and  $\hat{\rho}_K(n) = K^{1/2} \rho_R(n) K^{1/2} / \text{tr}[K \rho_R(n)]$  which is always well defined since we assumed  $V_K(\rho(n)) > 0$  almost surely.  $(\hat{\rho}_S(n))_{n \in \mathbb{N}}$  and  $(\hat{\rho}_K(n))_{n \in \mathbb{N}}$  are processes of states on respectively  $\mathcal{H}_S$  and  $\mathcal{H}_R$ . We also define new operators  $M_{j,K} = K^{1/2} M_{j,R} K^{-1/2}$  and the corresponding  $\Phi_K(\rho_K) = \sum_j M_{j,K} \rho_K M_{j,K}^*$ . These definitions may seem artificial but they lead to great improvements in the notations. We have a simpler bound for  $\Phi_K^*$ :  $\Phi_K^*(I_R) \leq \lambda_0 I_R$ .  $K$  is replaced by  $I_R$  for  $\Phi_K$  and  $\hat{\rho}_K$  is normalized so  $V_K(\Phi_R(\rho))/V_K(\rho) = \text{tr}[\Phi_K(\hat{\rho}_K)] \leq \lambda_0$ .

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We write  $\ln(V_K(\rho(n)))$  as a sum of a predictable process and a martingale using Doob decomposition,

$$\ln(V_K(\rho(n))) = \ln(V_K(\rho(0))) + \sum_{k=0}^{n-1} \sum_j \text{tr}[M_j^* M_j \rho(k)] \ln \left( \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K(k)]}{\text{tr}[M_j^* M_j \rho(k)]} \right) + \mathcal{M}(n) \quad (4.97)$$

with  $\mathcal{M}(n)$  a martingale. Using martingale law of large numbers, under the assumptions we gave at the beginning of this section we have  $\lim_{n \rightarrow \infty} \mathcal{M}(n)/n = 0$ . We also have  $\text{tr}[M_j^* M_j \rho(n)] = \text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S(n)] + O(\sqrt{V_K(\rho(n))})$  as  $n$  tends to infinity.

Since  $V_K(\rho(n))$  is almost surely bounded and  $V_K(\rho(n)) = O(\lambda_0^n)$ ,  $\sum_{k=0}^{\infty} \sqrt{V_K(\rho(k))} < \infty$  almost surely. Thus we have,

$$\ln(V_K(\rho(n)))/n = \frac{1}{n} \sum_{k=0}^{n-1} \sum_j \text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S(k)] \ln \left( \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K(k)]}{\text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S(k)]} \right) + o(1) \quad (4.98)$$

almost surely as  $n$  tends to infinity.

This rate seems to have an expression equivalent to the relative entropy we obtained in the previous section for the non demolition measurement. But this time  $(\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K(k)])_j$  is not a probability distribution since we have  $\sum_j \text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K(k)] < 1$ . Actually here the rate can never be equal to zero. It will always be strictly negative.

We have  $\text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S] > 0$  for any  $j$  and  $\sum_j \text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S] = 1$ . So using Jensen inequality and the concavity of  $\ln$ , we obtain

$$\sum_j \text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S] \ln \left( \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K]}{\text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S]} \right) \leq \ln(\text{tr}[\Phi_K(\hat{\rho}_K)]) \leq \ln(\lambda_0). \quad (4.99)$$

For any  $\hat{\rho}_K$  and any  $\hat{\rho}_S$ .

Hence

$$\limsup_n \ln(V_K(\rho(n)))/n \leq \ln(\lambda_0). \quad (4.100)$$

Thus  $V_K(\rho(n)) \leq \lambda_0^n e^{o(n)}$ . We seem to have used a complicated derivation for a weaker version of the bound we already had. As a matter of fact, as announced earlier, in some cases we can find a better bound for  $\sum_j \text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S] \ln \left( \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K]}{\text{tr}[M_{j,S}^* M_{j,S} \hat{\rho}_S]} \right)$ . We write this bound  $\ln(\lambda_1)$ . The improvement origin can be understood as the presence of a non demolition measurement between  $\mathcal{H}_S$  and  $\mathcal{H}_R$  added to the convergence towards  $\mathcal{H}_S$ . For example let us take a  $\mathbb{C}^2$  system with two measurement outcomes possible  $j = \pm$ . Let the Kraus operators be

$$M_{\pm} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \pm \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (4.101)$$

Then  $M_{j,K}^* M_{j,K}$  and  $M_{j,S}^* M_{j,S}$  are just positive numbers. The rate  $\lambda_1$  is equal to  $\lambda_0$  and is  $\lambda_1 = \lambda_0 = \frac{1}{2}$ .

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Now assume just after each of these measurement we perform a non demolition measurement with outcomes  $j' = 0, 1$  and Kraus operators

$$M_0 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (4.102)$$

We have four outcomes possibles  $j \in \{(+, 0), (+, 1), (-, 0), (-, 1)\}$ . The corresponding Kraus operators are

$$M_{0,\pm} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \pm \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{2}} \end{pmatrix} \text{ and } M_{1,\pm} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \pm \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{2\sqrt{2}} \end{pmatrix}. \quad (4.103)$$

We have  $\lambda_0 = 1/2$  and  $\lambda_1 \simeq 0.47 < \lambda_0$ . Here the improvement is not great but it can be increased. We are left with two different rates. We need to understand their meanings better.

#### Mean convergence time

The difference between the mean rate  $\lambda_0$  and the almost sure rate  $\lambda_1$  is best understood when studying the mean convergence time.

Let us define a stopping time

$$\tau_\epsilon = \inf\{n \geq 0 \text{ s.t. } V_K(\rho(n)) \leq \epsilon\}. \quad (4.104)$$

It corresponds to the first time  $V_K(\rho(n))$  will be smaller or equal to  $\epsilon$ . Following the similar discussion we had in the non demolition case, we expect that if  $\epsilon$  is small enough,  $V_K(\rho(n))$  will stay “close” to zero afterwards. With this limit in mind we nevertheless call  $\tau_\epsilon$  the convergence time and  $\mathbb{E}(\tau_\epsilon)$  the mean convergence time.

Remark that  $\epsilon$  here is different from the  $\epsilon$  used to approach the spectral radius of  $\Phi_R$ .

Let  $\epsilon'$  be such that  $\ln(\epsilon') = \ln(\epsilon) + \min_{j,\rho,\hat{\rho}_K} \ln \left[ \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K]}{\text{tr}[M_j^* M_j \rho]} \right]$ .  $\epsilon'$  is strictly positive and smaller than  $\epsilon$ . This is assured by the boundness and positive definiteness of the operators  $M_j^* M_j$  and  $M_{j,K}^* M_{j,K}$ .  $\ln(\epsilon') - \ln(\epsilon)$  is the minimal (negative) increment of  $\ln(V_K(\rho(n)))$  possible between two times. For any time  $n$ ,  $\ln(V_K(\rho(n+1))) - \ln(V_K(\rho(n))) \geq \ln(\epsilon') - \ln(\epsilon)$  almost surely.

We define the stopped supermartingale  $(\ln(V_K(\rho(\tau_\epsilon \wedge n))))$  where  $k \wedge n = \min(k, n)$ .

$$\begin{aligned} \mathbb{E}(\ln(V_K(\rho(\tau_\epsilon \wedge n)))) &= \\ \ln(V_K(\rho(0))) + \mathbb{E} \left( \sum_{k=0}^{\tau_\epsilon \wedge n - 1} \sum_j \text{tr}[M_j^* M_j \rho(k)] \ln \left( \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K(k)]}{\text{tr}[M_j^* M_j \rho(k)]} \right) \right). \end{aligned} \quad (4.105)$$

As before we can use Jensen's inequality to bound each term of the sum over time.

$$\sum_j \text{tr}[M_j^* M_j \rho(k)] \ln \left( \frac{\text{tr}[M_{j,K}^* M_{j,K} \hat{\rho}_K(k)]}{\text{tr}[M_j^* M_j \rho(k)]} \right) \leq \ln(\lambda_0), \forall k. \quad (4.106)$$

We cannot find a better bound since we need a bound at any time and the  $\ln(\lambda_1)$  bound is valid only in the large  $n$  limit. We have

$$\mathbb{E}(\ln(V_K(\rho(\tau_\epsilon \wedge n)))) \leq \ln(V_K(\rho(0))) + \ln(\lambda_0) \mathbb{E}(\tau_\epsilon \wedge n). \quad (4.107)$$

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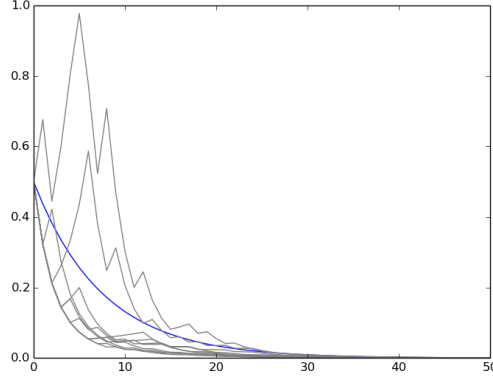


Figure 4.3.: Simulation of the evolution of  $V_{IR}$  for a 2 level system with  $\lambda_0 = \lambda_1 = \frac{7}{8}$ . The blue curve corresponds to the mean evolution ( $V_{IR}(\bar{\rho}(n))$ ) and the grey lines to ten different realizations of the evolution when taking into account the measurement results ( $V_{IR}(\rho(n))$ ).

Since for  $n < \tau_\epsilon$ ,  $\ln(V_K(\rho(n))) > \ln(\epsilon)$  we have  $0 \geq \ln(V_K(\rho(\tau_\epsilon \wedge n))) \geq \ln(\epsilon')$  for any time  $n$ . Thus  $\ln(V_K(\rho(\tau_\epsilon \wedge n)))$  is almost surely bounded and thus converge almost surely and in  $L^1$ . Using Markov's inequality, one can show  $\mathbb{E}(\tau_\epsilon)$  is finite<sup>3</sup>. The sequence  $\tau \wedge n$  is monotone in  $n$  so  $\lim_n \mathbb{E}(\tau_\epsilon \wedge n) = \mathbb{E}(\tau_\epsilon)$  using Lebesgue's monotone convergence theorem.

Taking the limit  $n$  to infinity in our last inequality, we have

$$\mathbb{E}(\ln(V_K(\rho(\tau_\epsilon)))) \leq \ln(\lambda_0)\mathbb{E}(\tau_\epsilon) + \ln(V_K(\rho(0))). \quad (4.108)$$

Recall that  $\ln(V_K(\rho(\tau_\epsilon))) \geq \ln(\epsilon')$  almost surely. Since  $\ln(\lambda_0) < 0$ , we have

$$\mathbb{E}(\tau_\epsilon) \leq \frac{\ln(\epsilon') - \ln(V_K(\rho(0)))}{\ln(\lambda_0)}. \quad (4.109)$$

Of course this is valid only for  $\epsilon < V_K(\rho(0))$ . For  $\epsilon \geq V_K(\rho(0))$ ,  $\tau_\epsilon$  is trivially 0 almost surely.

As in the non demolition case the introduction of  $\epsilon'$  allowed us to deal with cases where  $V_K(\rho(\tau_\epsilon))$  overshoots  $\epsilon$ .

We remark that the mean convergence time depends on  $\lambda_0$  and not  $\lambda_1$ . Looking at figures 4.3 and 4.4, we can see that when taking into account the measurement results, for some realizations the time after which the state converges is much shorter. For some other they stay chaotic and then converge rapidly. They can even get close to being completely supported in  $\mathcal{H}_R$  ( $V_K(\rho) = 1$ ). Recall that the result for  $\lambda_1$  is true only in large time, when  $n \gg 1$ , whereas  $\lambda_0$  is a bound in mean for any time. Hence  $\lambda_1$  is more an asymptotic stability rate, when  $\lambda_0$  is characteristic of the time it takes to converge. The improvement of  $\lambda_1$  is in the convergence speed when the convergence happen but it can happen after a long time. The distribution of this time depends on  $\lambda_0$ . The smaller is  $\lambda_0$  the more chance the convergence will happen after short time.

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3. The proof is given in Appendix A.2.

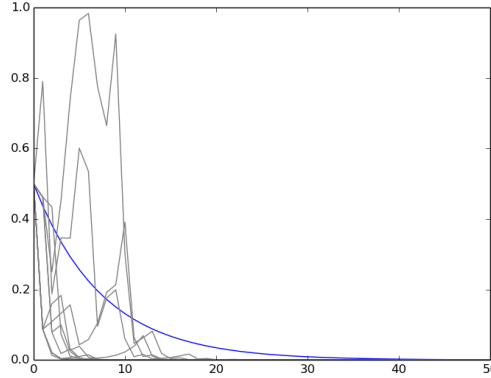


Figure 4.4.: Simulation of the evolution of  $V_{IR}$  for a 2 level system with  $\lambda_0 = \frac{7}{8}$  and  $\lambda_1 = \frac{21}{40}$ . The blue curve corresponds to the mean evolution ( $V_{IR}(\bar{\rho}(n))$ ) and the grey lines to ten different realizations of the evolution when taking into account the measurement results ( $V_{IR}(\rho(n))$ ).

This observation is similar to the behavior of the zero temperature case of [23]. In this article, the authors showed that a two level system connected to a thermal bath and continuously monitored by a diffusive non demolition measurement will, when the non demolition measurement have a large convergence rate, jump from one energy state to the other with time between the jumps distributed exponentially with a parameter depending on the temperature.

Here we have the same behavior, but once the state reaches the equivalent of the ground state ( $\mathcal{H}_S$ ) it stays in it.

In next chapter we will discuss the same type of processes for continuous quantum trajectories. The influence of the added non demolition measurement on the convergence rate will be much clearer.

### Exponential convergence for the state

As conclusion for this section, we translate the convergence rate for  $V_K(\rho(n))$  to a convergence rate for our system state. For now we only have a convergence rate for  $V_K(\rho(n))$ . The generalization to the state is quite straightforward.

We have an exponential convergence rate for our Lyapunov candidate function  $V_K(\rho(n)) \leq \lambda_1^n e^{o(n)}$ . We need to get back to a characterization of the exponential convergence for the state. Since  $K$  is definite positive, its smallest eigenvalue  $k_0$  is such that  $K \geq k_0 I_R$  with  $k_0 > 0$ . Hence  $\text{tr}[\rho_R(n)] \leq \frac{1}{k_0} \lambda_1^n e^{o(n)} = \lambda_1^n e^{o(n)}$ . Using the Cauchy-Schwarz inequality for semi definite positive matrix,

$$\|\rho_R(n)\|_{\max} \leq \lambda_1^n e^{o(n)} \quad (4.110)$$

$$\text{and } \|\rho_P(n)\|_{\max} \leq \lambda_1^{\frac{n}{2}} e^{o(n)}. \quad (4.111)$$

Hence, all the matrix blocs different from  $\rho_S(n)$  converge almost surely to zero. The convergence is exponential with a rate  $\lambda_1$  for  $\rho_R(n)$  and  $\lambda_1^{1/2}$  for  $\rho_P(n)$ .



## 5. Continuous indirect measurements

Continuous time quantum trajectories are meant to describe the evolution of a quantum system undergoing some continuous indirect measurements. As in the discrete case, one consider the system and its environment. By their interaction they become entangled. One then measures the environment and tries to evaluate the influence of this measurement on the system state. Contrary to discrete quantum trajectories, the environment is not considered to be a sequence of probes but is a large other system continuously in interaction with the system. Thus it is not a priori renewed between each indirect measurement. Moreover the environment is continuously monitored. Thus at every time a measurement is being performed not only at discrete fixed times. Hence the measurement outcomes are described by classical stochastic

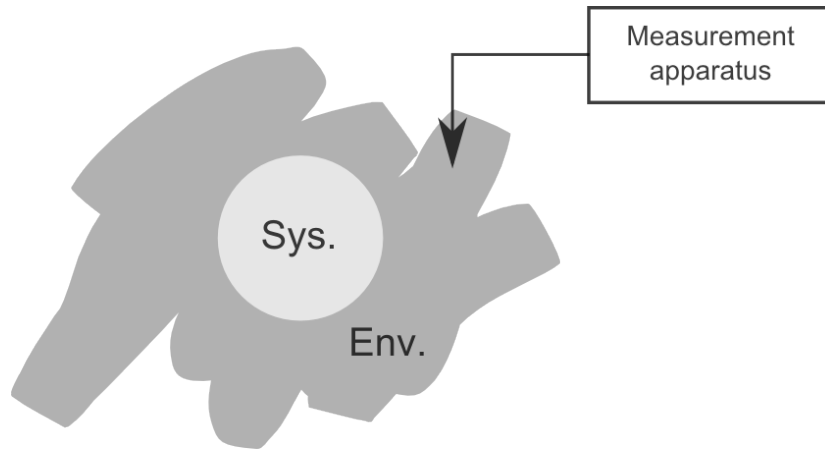


Figure 5.1.: Principle of continuous indirect measurement.

processes defined in continuous time. Thus the system evolution is described by a continuous time stochastic process.

Maybe the first attempts at describing continuous time measurements were made, using the instrument formalism, by E. B. Davies for the counting case [45, 46] and A. Barchielli, L. Lanz and G. M. Prosperi for the diffusive case [14, 15]. A. Barchielli and G. Lupieri made the connections with quantum stochastic calculus in [7, 16].

Not long after this first approach, V. P. Belavkin formulated and used quantum filtering theory to derive quantum trajectories SDE from quantum stochastic differential equations (QSDE) [26–28]. Sometimes, quantum trajectories SDEs are called Belavkin's equations. Referring to the technique used in the derivation, they are also called quantum filters.

The link between the two approaches was made by A. Barchielli and V. P. Belavkin in [10]. They derived both counting and diffusive SDE's found by V. P. Belavkin using instrument formalism. They also give the connection with parallel works of N. Gisin and L. Diosi. In [12], A. Barchielli and M. Gregoratti also showed the equivalence of quantum filtering and instrument formalism derivations starting from a quantum stochastic evolution.

## 5. Continuous indirect measurements

These two authors postulated stochastic generalizations of the Schrödinger equation reproducing wave function collapse [49, 58]. Quantum trajectories equations, in their pure state formalization, are therefore sometimes called stochastic Schrödinger equations. In section 5.4, we will see that these postulated stochastic differential equations (SDE) are merely special case of indirect measurements. This was first noticed in [10].

Independently, J. Dalibard *et al.* proposed a similar stochastic Schrödinger equation with jumps as a mean of master equations simulation [44]. Their idea was to consider that the master equation corresponds to the mean evolution of a system undergoing successive indirect “gedanken” measurements at random times. Their work has been an important contribution to numerical computation of master equations. Although, they only considered these measurements as merely computational tools and did not linked them to actual time continuous measurements. Their idea was further pursued through the study of what may now be called master equation unraveling [8, 10, 13]. The link with Davies’ instrument formalism was done by A. Barchielli [8]. H. Carmichael used a similar approach. He added physical motivations justifying the distribution of the measurement outcomes [40].

In 2010, C. Pellegrini proved what can be seen as a consistency between discrete and continuous quantum trajectories. The latter can be found as a continuous time approximation of the former [84–86].

Beside this work on the physical situations where quantum trajectories are applicable, a lot of work has been done on the properties of these quantum trajectories SDE. They are now a widely used tool in quantum optics [36, 102].

Note that here we refer only to Markovian quantum trajectories. Non Markovian physical extensions deal with a classical source of memory (*i.e.* feedback) which is a subject beyond the scope of this thesis. The interpretation of non Markovian versions of quantum trajectories as the evolution of a continuously monitored system interacting in a non Markovian manner with its environment is yet not clear [18, 56, 93].

The main purpose of this chapter is to present results obtained during this thesis on the long time behavior of continuous quantum trajectories in two situations. Either the quantum trajectory reproduces effectively a direct projective measurement of the system through the measurement of its environment. Or the quantum trajectory drives the system state towards a subspace of the state space. For example, towards one specific pure state. In both cases we prove convergence, compute an exponential rate of convergence and bound the mean convergence time. Before the discussion of these results, we give a general presentation of quantum trajectories.

The chapter is structured as follows. In a first section we define the stochastic differential equations (SDE) corresponding to quantum trajectories and fix our notations. In a second section we review the different derivations of quantum trajectories from physical situations. For instance we will not present the postulated equation of N. Gisin [58] and L. Diosi [49] as they are specific cases of the SDEs obtained by other means. In a third section we discuss general long time behavior of quantum trajectories. We have the same ergodic and purification properties as in the discrete case. In a fourth section we present the convergence results for non demolition quantum trajectories. We recover results equivalent as the ones obtained in the discrete case. In the long time limit the system wave function collapse with a distribution of the limit state given by von Neumann projection postulate. The convergence is exponential. We give its rate and bound the mean convergence time. In a fifth and last section we present results on globally attractive subspaces. As in the discrete case, we show that a convergence in mean is equivalent to a convergence when measurement outcomes are recorded. We show

that the asymptotic exponential convergence rate can be increased by the measurements and we compute a bound on the mean convergence time. We conclude by a discussion on the different significations of the rates in mean and when measurement outcomes are recorded.

## 5.1. Continuous time quantum trajectories

Let us consider a finite dimensional system. Hence its Hilbert space is  $\mathcal{H}_{\text{sys.}} = \mathbb{C}^{d_{\text{sys.}}}$  with  $d_{\text{sys.}} < \infty$ . We recall that the set of system state is  $\mathcal{S}(\mathcal{H}_{\text{sys.}}) = \{\rho \in \mathcal{B}(\mathcal{H}_{\text{sys.}}) \text{ s.t. } \rho \geq 0, \text{tr}[\rho] = 1\}$ . Let us define useful functions on  $\mathcal{S}(\mathcal{H}_{\text{sys.}})$ .

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{j=1}^n L_j \rho L_j^* - \frac{1}{2} \{L_j^* L_j, \rho\} \quad (5.1)$$

$$D_j(\rho) = L_j \rho + \rho L_j^* - \text{tr}[(L_j + L_j^*)\rho] \rho, \quad j = 1, \dots, p-1 \quad (5.2)$$

$$J_j(\rho) = L_j \rho L_j^*, \quad v_j(\rho) = \text{tr}[J_j(\rho)], \quad j = p, \dots, n. \quad (5.3)$$

The system Hamiltonian  $H$  and the  $L_j$ 's are operators on  $\mathcal{H}_{\text{sys.}}$ . Since we are in finite dimension they are automatically bounded. In the rest of the text we may omit the lower bound 1 and upper bound  $n$  for the sums over  $j$  but they are always assumed. This lighten the notations.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  be a filtered probability space with usual conditions. Let  $(W_j(t)), j < p$  be  $p-1$  independent Wiener processes and let  $(\tilde{N}_j(dx, dt)), j \geq p$  be  $n-p+1$  independent adapted Poisson point processes of intensity  $dx dt$ ; they are independent of the Wiener processes. We assume  $(\mathcal{F}_t)$  is the natural filtration of the processes  $W, \tilde{N}$  and we assume  $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t = \mathcal{F}$ .

If the system undergo Markovian continuous indirect measurement, then its evolution is given by a SDE on  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  of the following form.

$$\begin{aligned} d\rho(t) = & \mathcal{L}(\rho(t))dt \\ & + \sum_{j < p} D_j(\rho(t)) dW_j(t) \\ & + \sum_{j \geq p} \left( \frac{J_j(\rho(t))}{v_j(\rho(t))} - \rho(t) \right) [dN_j(t) - v_j(\rho(t))dt] \end{aligned} \quad (5.4)$$

with the initial system state  $\rho(0)$  fixed. The increments  $dN_j(t)$  are for now shorthand notations for the increments of  $\int_0^t \int_{\mathbb{R}} 1_{0 < x < v_j(\rho(s))} \tilde{N}_j(dx, ds)$ . This shorthand notation being understood we have the definition of the quantum trajectory SDE. Once this definition is set we have that the processes  $N_j(t) = \int_0^t \int_{\mathbb{R}} 1_{0 < x < v_j(\rho(s))} \tilde{N}_j(dx, ds)$  are counting processes with stochastic intensities  $(v_j(\rho(t)))$ . Hence, we have  $\mathbb{E}(dN_j(t) | \mathcal{F}_t) = v_j(\rho(t))dt$  with  $\mathbb{E}$  the expectation under  $\mathbb{P}$  [29, 86]. This construction of the counting processes  $(N_j(t))$  is not the only one possible. One can also build, using Girsanov transformation, a probability measure under which initially Poisson processes with unit intensities become counting processes with stochastic intensities  $(v_j(\rho(t)))$ . We will use this technique when we will present quantum filtering in section 5.2.3 and master equation unraveling in section 5.2.4.

The solutions  $(\rho(t))$  of this type of SDE are called quantum trajectories. And the SDE itself is the quantum trajectory SDE. The existence and uniqueness of the solution for these SDE in the set of state were proven in [11, 13, 84–86].

## 5. Continuous indirect measurements

Quantum trajectory SDE's are non linear SDE's. As we will see in the different derivations and particularly in section 5.2.4 on master equation unraveling, the non linearity is not an ad-hoc assumption. It is the result of the system state normalization. Hence it is the result of the conditioning on the measurement outcomes. The underlying quantum theory on which quantum trajectories are based is not modified from the standard one, it stays linear.

The equation (5.4) contains three parts. Each corresponds to different contributions to the system state evolution. On the first line appear the usual Lindblad operator. When we look at the average evolution implied by the SDE we recover a master equation with Lindbladian  $\mathcal{L}$ . This is not a surprise since averaging over the measurements should be equivalent to a trace over the environment degrees of freedom. As expected we recover a usual reduced Markovian evolution.

$$d\mathbb{E}(\rho(t)) = \mathcal{L}(\mathbb{E}(\rho(t)))dt. \quad (5.5)$$

The second line corresponds to the effect of measurements whose outcomes are continuous signals, namely currents. The third line corresponds to the effect of measurements whose outcome signals are discontinuous, namely particle counting.

For discontinuous signal measurements it is not hard to see that the actual outcome measured is  $N_j(t)$  for each  $j$ . When an additional +1 count is recorded at time  $t$  a POVM is performed on the system with the corresponding operator  $L_j$ . The probability to have such POVM performed on the system between time  $t$  and  $t + dt$  is  $v_j(\rho(t))dt$ . Our first derivation of quantum trajectories will be based on this interpretation.

For the continuous signal measurements, it is not that obvious. From the different derivations we will perform we know that actually the outcome recorded is

$$Y_j(t) = W_j(t) + \int_0^t \text{tr}[(L_j + L_j^*)\rho(s)]ds \quad (5.6)$$

The instantaneous mean current is then  $\mathbb{E}(dY_j(t)|\mathcal{F}_t)/dt = \text{tr}[(L_j + L_j^*)\rho(t)]$ .

The trajectories involving only continuous signals are often called diffusive and the one involving only discontinuous signal are called jump trajectories. We also talk about diffusive part and jump part of a quantum trajectory SDE when referring to the continuous and discontinuous part respectively.

The processes  $(N_j(t))$  and  $(Y_j(t))$  are what an experimenter has access to. In the section on non demolition we will use this to compute the evolution of an estimation of the system state.

One can generalize (5.4) by introducing time dependent random coefficients [13]. Here we will not explore this possibility. So, when we will present quantum filtering theory in section 5.2.3, we will not discuss heterodyne detection schemes. The derivation is a straight forward generalization of the homodyne case that we treat. Identically here we did not take into account a possible faulty detection or the presence of an additional unmeasured environment. Actually the results on non demolition and subspace stabilization still hold in the presence of an additional unmeasured environment, as long as this environment fulfills the non demolition or invariance conditions respectively. In the rest of the text we warn the reader whenever the results might not hold with an additional unmeasured environment.

## 5.2. Physical models of continuous indirect measurement

### 5.2.1. Phenomenological approach

Let us start with an enlightening phenomenological derivation. We follow the idea of J. Dalibard *et al.* [44] to introduce jump quantum trajectories. From these we obtain the diffusive one. The interested reader will find a similar derivation in [102]. H. Carmichael also investigated this approach to continuous measurement [40]. A more rigorous treatment of this approach is made in [10]. It is actually in this last article that the limit from a jump trajectory to a diffusive one is performed.

An experimenter counts the number of photon emitted by a system. Whenever a photon is emitted it is detected by some macroscopic apparatus. There is no faulty measurement. At each time the result on the detector is either 0 (no detection) or 1 (detection). If at a given time the system state is  $\rho$ , then after a measurement the system state is either

$$\frac{M_0 \rho M_0^*}{\text{tr}[M_0^* M_0 \rho]} \text{ if no photon is detected, or } \frac{M_1 \rho M_1^*}{\text{tr}[M_1^* M_1 \rho]} \text{ if a photon is detected,} \quad (5.7)$$

with  $M_j$  the Kraus operators corresponding to each outcome. The probability to detect a photon is  $p(1) = \text{tr}[M_1^* M_1 \rho]$ . The conservation of the trace for the mean state evolution imposes  $M_0^* M_0 + M_1^* M_1 = I_{\text{sys.}}$ . If  $\rho'$  is the state after a measurement, then  $\rho' = \frac{M_0 \rho M_0^*}{\text{tr}[M_0^* M_0 \rho]}(1 - \epsilon) + \frac{M_1 \rho M_1^*}{\text{tr}[M_1^* M_1 \rho]} \epsilon$  where  $\epsilon$  is a random variable equal to 1 with probability  $p(1)$  and 0 otherwise. Hence  $\rho'$  is a random variable.

Assume we know from the interaction between the system and the field that if a photon is detected, the system state becomes  $\frac{L \rho L^*}{\text{tr}[L^* L \rho]}$ . The operator  $L$  corresponds to the energy level jump the system undergoes when a photon is emitted. For example  $L \propto \sigma_-$  for a two level atom<sup>1</sup>. The probability of a photon detection during an infinitesimal time  $dt$  is  $\text{tr}[L^* L \rho] dt$ . This jump probability corresponds to  $M_1 := L \sqrt{dt}$ . The conservation of the trace imposes that  $M_0$  is such that  $M_0^* M_0 + L^* L dt = I_{\text{sys.}}$ . Using the shorthand  $dt^2 = 0$  to obtain a development up to order  $dt$ , one finds that  $M_0 := I_{\text{sys.}} - i(H - i/2 L^* L) dt$ , with  $H$  Hermitian, fits.

Let  $N(t)$  be the process counting the number of photons detected. At time  $t$ ,  $dN(t) \equiv \epsilon$ ,  $\rho \equiv \rho(t)$  and  $\rho' \equiv \rho(t + dt)$ . Using the definition of  $\rho(t + dt)$ , we get

$$\rho(t+dt) = \frac{(\rho(t) - i[H, \rho(t)]dt - 1/2\{L^* L, \rho(t)\}dt)}{1 - \text{tr}[L^* L \rho(t)]dt} \times (1 - dN(t)) + \frac{L \rho(t) L^*}{\text{tr}[L^* L \rho(t)]} dN(t). \quad (5.8)$$

Following Ito rules we have  $dt dN(t) = 0$ . Thus

$$d\rho(t) = \mathcal{L}(\rho(t))dt + \left( \frac{L \rho(t) L^*}{\text{tr}[L^* L \rho(t)]} - \rho(t) \right) (dN(t) - \text{tr}[L^* L \rho(t)]dt) \quad (5.9)$$

---

1. In this case  $\sigma_-$  is the usual lowering operator  $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  or  $\sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2)$  in terms of Pauli matrices.

## 5. Continuous indirect measurements

with

$$\mathcal{L}(\rho) = -i[H, \rho] + L\rho L^* - \frac{1}{2}\{L^*L, \rho\}. \quad (5.10)$$

This is our quantum trajectory SDE with one discontinuous signal measurement. We infer this equation should describe photon counting measurement.

This counting equation is well suited for spontaneous emission experiments. In an interferometry experiment, the system is stimulated by an external strong coherent field. The photon emission is not spontaneous anymore but driven by the external field. The measurement result on the system is given by the perturbation of the external field induced by the presence of the system. In optics, a canonical situation of interferometry is the homodyne detection scheme (figure 5.2). A laser beam is split. One part of the beam interacts with the system, the other evolves freely. The two parts are then compared. Information on the system is given by the phase difference of the two halves of the laser light beam. In these experiments one detects not single photons but a continuous current [102]. So we expect to find a diffusive quantum trajectory SDE in this case.

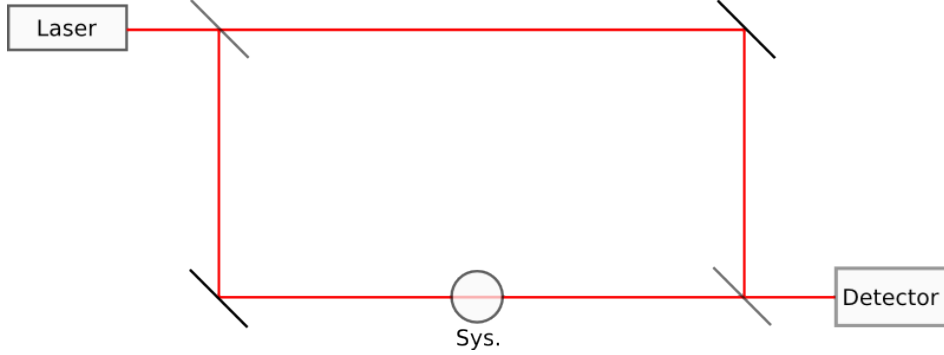


Figure 5.2.: Schematics of a typical homodyne measurement. A local oscillator (laser) is added to stimulate photo emission by the system. The phase difference between the free field (upper arm) and the field perturbed by the system (lower arm) is then evaluated by the detector.

The passage from a counting quantum trajectory to a diffusive one has been done by A. Barchielli and V. P. Belavkin in [10]. We present here a derivation inspired by this work. The laser electromagnetic field state  $|\alpha\rangle$  is assumed highly coherent and powerful. We can treat it classically. The presence of this new field  $\alpha = |\alpha|e^{i\theta}$  induce a rotation of the system in time. The system is driven by the field. This translates to a modification of the operators describing the evolution. The Hamiltonian in the Lindblad is changed in  $H \rightarrow H + i(\alpha L^* - \bar{\alpha}L)$  and the  $L$  operator modeling the creation of a photon is modified in  $L' = L - \alpha$ .

The Hamiltonian modification can be motivated by a simple model. Let the system be a two level one. The usual Jaynes–Cummings interaction Hamiltonian between the coherent laser field and the atom is  $H_{\text{int.}} = -i(\sigma_- \otimes a^* - \sigma_+ \otimes a)$  with  $a^*, a$  the usual creation, annihilation operators of the laser field. Remark that we implicitly rescaled time such that no constant appear in the interaction Hamiltonian. Since we consider the laser field to be classical we can directly replace  $a$  by  $\alpha e^{-i\omega_0 t}$ , with  $H = \frac{1}{2}\omega_0\sigma_3$  in our time scale. We can get rid of the time  $t$  dependency by going into the rotating wave frame. We infer that the Hamiltonian in the

## 5.2. Physical models of continuous indirect measurement

Lindbadian must be modified accordingly. The interaction between the laser coherent light and the background electromagnetic field leads to the shift  $L \rightarrow L - \alpha$ . The justification of such modification is similar using the dipolar interaction Hamiltonian between the background electromagnetic field and the laser field. The justification for these modifications will be much stronger and clearer in subsection 5.2.3 dedicated to quantum filtering.

The previously found jump equation is now:

$$\begin{aligned} d\rho(t) = & \mathcal{L}^\alpha(\rho(t))dt \\ & + \left( \frac{L\rho(t)L^* - |\alpha|(e^{i\theta}L\rho(t) + e^{-i\theta}\rho(t)L^*) + |\alpha|^2\rho(t)}{\text{tr}[L^*L\rho(t)] - |\alpha|\text{tr}[(e^{i\theta}L + e^{-i\theta}L^*)\rho(t)] + |\alpha|^2} - \rho(t) \right) \\ & \times (dN^\alpha(t) - \text{tr}[L^*L\rho(t)] - |\alpha|\text{tr}[(e^{i\theta}L + e^{-i\theta}L^*)\rho(t)] + |\alpha|^2)dt. \end{aligned}$$

The superscript  $\alpha$  on  $dN^\alpha(t)$  reminds of the counting process intensity dependency on the classical field. The introduction of the classical field only affects the unitary part of the mean evolution.

$$\begin{aligned} d\mathbb{E}(\rho(t)) &= \mathcal{L}^\alpha(\mathbb{E}(\rho(t)))dt \\ &= -i[H + i(\alpha L^* - \bar{\alpha}L), \mathbb{E}(\rho(t))] + L\mathbb{E}(\rho(t))L^* - \frac{1}{2}\{L^*L, \mathbb{E}(\rho(t))\}. \end{aligned} \quad (5.11)$$

We are interested in the limit where the photons cannot be distinguished individually. Namely when the laser total energy sent per second is large compared with the typical energy of the system. This assumption is far from being restrictive. For example a common red laser pointer, namely a 3mW power 635nm wave length laser, emits about  $10^{16}$  photons per second when the typical energy of the system corresponds approximately to the energy of one photon. Hence we can take the limit  $|\alpha| \rightarrow \infty$  with no guilt. One gets, for the jump part of the equation,

$$\begin{aligned} \lim_{|\alpha| \rightarrow \infty} |\alpha| \left( \frac{L\rho(t)L^* - |\alpha|(e^{i\theta}L\rho(t) + e^{-i\theta}\rho(t)L^*) + |\alpha|^2\rho(t)}{\text{tr}[L^*L\rho(t)] - |\alpha|\text{tr}[(e^{i\theta}L + e^{-i\theta}L^*)\rho(t)] + |\alpha|^2} - \rho(t) \right) \\ = -e^{i\theta}L\rho(t) - e^{-i\theta}\rho(t)L^* + \text{tr}[(e^{i\theta}L + e^{-i\theta}L^*)\rho(t)]\rho(t) \end{aligned}$$

The increased intensity of the counting process make it converge to a continuous process. Let  $W^\alpha(t)$  be such that  $dW^\alpha(t) = \frac{1}{|\alpha|}(dN^\alpha(t) - (\text{tr}[L^*L\rho(t)] - |\alpha|\text{tr}[(e^{i\theta}L + e^{-i\theta}L^*)\rho(t)] + |\alpha|^2)dt)$ . Has discussed in [10], in the limit  $|\alpha| \rightarrow \infty$ , we have  $dW^\alpha(t)dW^\alpha(s) \rightarrow \delta_{s,t}dt$ . Thus the process  $W^\alpha(t)$  converges in distribution to a Wiener process  $W(t)$ . And in this limit, the SDE satisfied by the system state is

$$d\rho(t) = \mathcal{L}(\rho(t))dt \quad (5.12)$$

$$+ (L\rho(t) + \rho(t)L^* - \text{tr}[(L + L^*)\rho(t)]\rho(t))dW(t) \quad (5.13)$$

The phase  $e^{i\theta}$  has been absorbed in the definition of  $L$ . We obtained the diffusive quantum trajectory SDE as a limit of the photon counting one when a powerful driving field is added.

It is interesting to see how the measurement signal  $N(t)$  is modified in this limit. With the introduction of the field, the measurement outcome is inside  $W^\alpha(t)$ . If we want to “extract” it we can define

$$Y^\alpha(t) = W^\alpha(t) + \int_0^t \frac{1}{|\alpha|}(\text{tr}[L^*L\rho(s)] + |\alpha|\text{tr}[(L + L^*)\rho(s)])ds. \quad (5.14)$$

## 5. Continuous indirect measurements

We added back the non diverging parts of the counting process intensity. Taking the limit  $|\alpha| \rightarrow \infty$  we obtain as expected a measurement outcome process

$$Y(t) = W(t) + \int_0^t \text{tr}[(L + L^*)\rho(s)]ds. \quad (5.15)$$

It is the increased intensity of the jump process which leads to the apparition of a continuous signal.

The diffusive quantum trajectory SDE can also be directly derived with well chosen  $M_0$  and  $M_1$ . The suitable choice is not really physically meaningful for now. It makes sense in the light of repeated indirect measurement continuous limit subsection results. Therefore we do not try to give physical arguments for the choice of  $M_0$  and  $M_1$ .

As before we consider that an experimenter continuously measures the environment of a system. At each time he can obtain two different outcomes, 0 or 1. Now we suppose that over an infinitesimal time  $dt$  the measurement effect is given by Kraus operators  $M_0^{dt}$  and  $M_1^{dt}$  defined by

$$M_0^{dt} = \frac{1}{\sqrt{2}}(I_{\text{sys.}} + \sqrt{dt}L - \frac{1}{2}dtL^*L) \quad (5.16)$$

$$M_1^{dt} = \frac{1}{\sqrt{2}}(I_{\text{sys.}} - \sqrt{dt}L - \frac{1}{2}dtL^*L) \quad (5.17)$$

This choice fulfills the trace conservation condition  $M_0^{dt*}M_0^{dt} + M_1^{dt*}M_1^{dt} = I_{\text{sys.}}$ . Of course the notation  $\sqrt{dt}$  is a liberty we take.

From these definitions and the convention  $dt^{1+\epsilon} = dN(t)dt = 0$  for any  $\epsilon > 0$  we obtain

$$\rho(t+dt) = \frac{\rho(dt) + \sqrt{dt}(L\rho(t) + \rho(t)L^*) + \mathcal{L}(\rho(t))dt}{1 + \sqrt{dt}\text{tr}[(L + L^*)\rho(t)]}(1 - dN(t)) \quad (5.18)$$

$$+ \frac{\rho(dt) - \sqrt{dt}(L\rho(t) + \rho(t)L^*) + \mathcal{L}(\rho(t))dt}{1 - i\sqrt{dt}\text{tr}[(L + L^*)\rho(t)]}dN(t) \quad (5.19)$$

with  $\mathbb{E}[dN(t)|\rho(t)] = \frac{1}{2}(1 - \sqrt{dt}\text{tr}[(L + L^*)\rho(t)])$ . From this expression we get

$$d\rho(t) = \mathcal{L}(\rho(t))dt \quad (5.20)$$

$$+ D(\rho(t))\sqrt{dt}(1 - \sqrt{dt}\text{tr}[(L + L^*)\rho(t)] - 2dN(t)) \quad (5.21)$$

with  $D(\rho) = L\rho + \rho L^* - \text{tr}[(L + L^*)\rho]\rho$ . If we define  $dW^{dt}(t) = \sqrt{dt}(1 - i\sqrt{dt}\text{tr}[(L - L^*)\rho(t)] - 2dN(t))$ , then we get  $\mathbb{E}[dW^{dt}(t)|\rho(t)] = 0$  and  $dW^{dt}(t)^2 = dt$  thus we can infer that  $W(t) = \int_0^t dW^{dt}(t)$  is a Wiener process. We recover the diffusive quantum trajectory SDE

$$d\rho(t) = \mathcal{L}(\rho(t))dt + D(\rho(t))dW(t). \quad (5.22)$$

This phenomenological approach motivates the use of quantum trajectories to describe continuous indirect measurement. In next subsections we present more rigorous approach for the physical justification of continuous time quantum trajectories.

### 5.2.2. Repeated indirect measurement continuous approximation

In a series of articles [84–86] C. Pellegrini, showed that continuous time quantum trajectories can be seen as continuous time approximations of discrete quantum trajectories. With M. Bauer and D. Bernard we proved an equivalent result in the non demolition case using a different method [20,24]. The idea of discrete time indirect measurement was also the starting point in A. Barchielli, L. Lanz and G. M. Prosperi article [14].

We recall the basic definition of discrete quantum trajectories. A system repeatedly interacts with probes. After each interaction the probe that just interacted is measured. Recording these measurement outcomes implies the system state evolution is a Markov chain  $(\rho(n))$  defined as follow. Knowing all first  $n$  outcomes,

$$\rho(n+1) = \frac{M_j \rho(n) M_j^*}{\text{tr}[M_j^* M_j \rho(n)]} \text{ with probability } \text{tr}[M_j^* M_j \rho(n)]. \quad (5.23)$$

The operators  $M_j$  are Kraus operators acting on the system. They are defined by  $M_j = \langle j|U|\Psi\rangle$  with  $|\Psi\rangle$  the initial state of the probe,  $U$  the interaction and  $|j\rangle$  the probe state corresponding to the outcome  $j$ .

The continuous approximation is obtained when a large number of indirect measurements are performed in a finite fixed time. We have a time  $t$  fixed and  $n$  measurements performed during this time interval. We assume, without loss of generality that the time interval is 1. Hence, each measurement happens during a time  $1/n$ . Since the measurement on the probe after the interaction is assumed to be instantaneous,  $1/n$  is the interaction time. We take the continuous limit by sending  $n$  to infinity. Of course as the interaction time decreases, the interaction strength must increase. If not the effect of the interaction becomes irrelevant. So, as in the repeated interaction continuous limit, let us take a Hamiltonian for the unitary interaction with the following  $n$  scaling.

$$H_{\text{tot.}} = H_{\text{sys.}} \otimes I_{\text{probe}} + I_{\text{sys.}} \otimes H_{\text{probe}} + \sqrt{n} H_{\text{int.}}. \quad (5.24)$$

The interaction unitary operator is

$$U^{(n)} = \exp(-i \frac{t}{n} H_{\text{tot.}}) \quad (5.25)$$

$$= \exp(-it [\frac{1}{n} (H_{\text{sys.}} \otimes I_{\text{probe}} + I_{\text{sys.}} \otimes H_{\text{probe}}) + \frac{1}{\sqrt{n}} H_{\text{int.}}]). \quad (5.26)$$

The subscript  $(n)$  is a reminder of the  $n$  dependency of this operator. The scaling of the Kraus operators  $M_j^{(n)}$  follows from the unitary operator scaling. The purpose of the continuous time limit is to make sense of the limit  $n \rightarrow \infty$  for the discrete process  $(\rho^{(n)}(t))$  defined by

$$\rho^{(n)}(t) = \rho([nt]) \quad (5.27)$$

where  $[nt]$  is the integer part of  $nt$ . If  $t = k/n$  then  $\rho^{(n)}(t) = \rho(k)$ . Thus  $\rho^{(n)}(t)$  is a process in continuous time which is equal to the discrete time process for any time  $k/n$  with  $k$  an integer which can be greater than  $n$ . Note that we use the convention that whenever  $\rho(\cdot)$  as an explicitly integer argument, it corresponds to the discrete time process. If the argument is not explicitly integer, it corresponds to a continuous time process. Hence here  $\rho([nt])$  is the discrete time quantum trajectory at time  $[nt]$ .

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The only other assumption we will need is that  $\langle \Psi | H_{\text{int.}} | \Psi \rangle = 0$  with  $|\Psi\rangle$  the initial probe state. This assumption is needed to avoid infinities. During the derivation, this quantity appears as a mean Hamiltonian acting on the system added to  $H_{\text{sys.}}$ . But it is scaled as  $\sqrt{n}$  so it explodes in the limit  $n \rightarrow \infty$ . Making sure it is null assures the unitary part of the limit evolution does not include an exploding rotation of the system state. In the non demolition case, this assumption is only needed for the convergence of the process  $\rho^{(n)}(t)$  not for the diagonal elements  $q_\alpha^{(n)}(t)$ . This is due to the bloc diagonal property of  $H_{\text{int.}}$  in the pointer basis.

Before we present the general result, let us present the method, restricted to non demolition measurements, we formulated with M. Bauer and D. Bernard [20, 24]. The key tool is martingale change of measure. On this aspect it makes the connection with the quantum filtering approach on next subsection. The convergence of  $(\rho^{(n)}(t))$  and the measurement outcome processes are obtained using the convergence of finite dimension distributions.

### Non demolition continuous limit

We recall that in the non demolition case we have a specific system Hilbert space orthonormal basis  $\mathcal{P}$  in which  $U^{(n)}$  is bloc diagonal. Hence  $H_{\text{sys.}} = \sum_{\alpha \in \mathcal{P}} E_\alpha |\alpha\rangle\langle\alpha|$  and  $H_{\text{int.}} = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes H_{\text{int.}}(\alpha)$ . The operators  $H_{\text{int.}}(\alpha)$  are hermitian operators on the probe Hilbert space.

The technique we used in [20, 24], goes as follow. From the measurement outcome counting processes we define processes which will have a continuous limit. We recall that the measures  $\mathbb{P}_\alpha^{(n)}$  are the measures on the outcome space when the system is initially in the pointer state  $\alpha$ . Under each law  $\mathbb{P}_\alpha^{(n)}$  we use the i.i.d. properties of the outcome counting processes  $N_j(n)$  to compute their limit finite dimensional characteristic functions. Using the expression of the measure for any initial system state  $\mathbb{P}^{(n)}$  as a weighted sum of the measures  $\mathbb{P}_\alpha^{(n)}$  we obtain the limit finite dimensional characteristic functions under the measure  $\mathbb{P}^{(n)}$ .

On the continuous side, we define a standard filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \nu)$  which support standard independent Wiener and Poisson processes. Using a push forward measure and martingale change of measure we build a family of measure  $(\mu_\alpha)$  such that, under them, the Wiener and Poisson processes finite dimensional characteristic functions are equal to the continuous limit of the discrete time ones under each  $\mathbb{P}_\alpha^{(n)}$ . We write the martingale performing this change of measure  $(M_\alpha(t))$ . Using these martingales we define a new one  $M(t)$ . This martingale is a weighted sum of the previous ones. The change of measure implied by this martingale leads to a new measure  $\mu$ . Under this measure the Wiener and Poisson processes finite dimensional characteristic functions are equal to the continuous limit of the discrete time ones under  $\mathbb{P}^{(n)}$ .

Thus we constructed a probability space and processes such that their finite dimensional distributions are equal to the continuous limit of the outcome process ones. Here it is only in this sense we consider the continuous time limit. This is a weaker convergence than the usual weak convergence. The interested reader can find an extensive discussion of the meaning of this convergence and a comparison with weak convergence in Appendix B of [20].

Once the outcome processes convergence to continuous time processes is obtained, we recover more readable SDE for those using Girsanov theorem. The continuous time limit of the system state process should then formally be obtained through the limit of its finite dimensional characteristic functions and its identification to an equivalent finite dimensional

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characteristic function on the continuous time side. Though, we do not do it explicitly. From the expression of the natural limit of the Doob–Meyer decomposition of the system state process, the result we would obtain is obvious. And we infer this is the actual limit. In [20] the interested reader can find a proof of the convergence for the pointer probabilities  $q_\alpha(t)$  using Mellin transforms.

Let us now explain in details the continuous time limit.

First we find the continuous time limit of the outcome processes  $N_j(n)$ . These processes count the number of times the outcome  $j$  is obtained up to time  $n$ . If the initial system state is one of the pointer states  $\alpha$ , then the  $N_j(n)$ 's are just sum of i.i.d. random variables. Their joint characteristic function at time  $n$  under the measure  $\mathbb{P}_\alpha^{(n)}$  is then

$$\mathbb{E}_\alpha^{(n)}(e^{\lambda_j N_j(n)}) = \left( \sum_j p^{(n)}(j|\alpha) e^{\lambda_j} \right)^n. \quad (5.28)$$

Note that the measures themselves depend on  $n$ . This is why we are not able to use standard central limit theorems for the continuous limit derivation.

Let us define processes from the counting processes  $N_j(n)$ . Let us organize the outcomes such that for  $j = 1, \dots, p-1$ ,  $\langle \Psi | j \rangle \neq 0$  and for  $j = p, \dots$ ,  $\langle \Psi | j \rangle = 0$ . Hence for  $j \geq p$  we have a null probability to obtain the outcome  $j$  in absence of interaction. Let  $p_0(j) = |\langle \Psi | j \rangle|^2$  be the probability to obtain the outcome  $j$  in absence of interaction. Then we define

$$Y_j^{(n)}(t) = \frac{1}{\sqrt{n}}(N_j([nt]) - p_0(j)nt) \quad (5.29)$$

$$N_j^{(n)}(t) = N_j([nt]). \quad (5.30)$$

The outcome probabilities can be expanded in series of  $\frac{1}{\sqrt{n}}$  powers.

$$p^{(n)}(j|\alpha) = p_0(j) \left( 1 + \frac{1}{\sqrt{n}} r(j|\alpha) + \frac{1}{n} \vartheta(j|\alpha) + o\left(\frac{1}{n}\right) \right), \quad j < p \quad (5.31)$$

$$p^{(n)}(j|\alpha) = \frac{1}{n} \theta(j|\alpha) + o\left(\frac{1}{n}\right), \quad j \geq p. \quad (5.32)$$

From the identity  $\sum_j p^{(n)}(j|\alpha) = 1$ , we have  $\sum_j p_0(j) r(j|\alpha) = 0$  and  $\sum_{j < p} p_0(j) \vartheta(j|\alpha) + \sum_{j \geq p} \theta(j|\alpha) = 0$ . The different constants introduced are defined from the probe state, the interaction Hamiltonian and the measurement outcomes.

$$r(j|\alpha) = 2\text{Im} \left( \frac{\langle j | H_{\text{int.}}(\alpha) \Psi \rangle}{\langle j | \Psi \rangle} \right), \quad j < p \quad (5.33)$$

$$\theta(j|\alpha) = |\langle j | H_{\text{int.}}(\alpha) \Psi \rangle|^2, \quad j \geq p. \quad (5.34)$$

We recall that  $H_{\text{int.}}(\alpha)$  is the  $\alpha$  bloc of  $H_{\text{int.}}$ . We do not give the expression of the  $\vartheta(j|\alpha)$ 's. From now on they disappear from our notations. One just need to care about their relation with the  $\theta(j|\alpha)$  we just gave. Namely  $\sum_{j < p} p_0(j) \vartheta(j|\alpha) + \sum_{j \geq p} \theta(j|\alpha) = 0$ .

Using these expansions, a straightforward computation leads to the limit

$$\lim_n \mathbb{E}_\alpha^{(n)}(e^{\sum_{j < p} \lambda_j Y_j^{(n)}(t) + \sum_{j \geq p} \lambda_j N_j^{(n)}(t)}) = \quad (5.35)$$

$$\exp \left[ t \left( \sum_{j < p} p_0(j) r(j|\alpha) + \frac{1}{2} \sum_{i,j < p} \lambda_i \sigma_{ij} \lambda_j + \sum_{j \geq p} \theta(j|\alpha) (e^{\lambda_j} - 1) \right) \right] \quad (5.36)$$

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with  $\sigma_{ji} = \delta_{i,j}p_0(j) - p_0(i)p_0(j)$ . Note that  $\sum_i \sigma_{ji} = 0$ . Since the processes  $N_n(j)$  have independent increments under each  $\mathbb{P}_\alpha^{(n)}$ , the generalization of this formula to any finite dimensional characteristic function of the processes  $Y_j^{(n)}(t)$  and  $N_j^{(n)}(t)$  under  $\mathbb{P}_\alpha^{(n)}$  is direct.

Let us now build a filtered probability space supporting continuous time process with same finite dimensional characteristic functions. Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \nu)$  be a filtered probability space. Let  $(W_j(t))$ ,  $j = 1, \dots, p-1$  be independent Wiener processes and  $(N_j(t))$ ,  $j = p, \dots$  be independent Poisson point process with unit intensity under  $\nu$ . The Poisson and Wiener processes are also independent. We assume the filtration  $(\mathcal{F}_t)$  is the natural filtration of the processes  $W, N$ . Let  $\mu^0$  be the image measure of  $\nu$  under the map  $x(j) \mapsto \sqrt{p_0(j)}(x(j) - \sqrt{p_0(j)} \sum_i \sqrt{p_0(i)}x(i))1_{j < p} + x(j)1_{j \geq p}$ . Hence under  $\mu^0$ , the processes  $(W_j(t))$  are centered Gaussian processes with covariance  $\mathbb{E}_{\mu^0}(W_j(t)W_j(s)) = \min(s, t)\sigma_{ji}$ . The processes  $(N_j(t))$  are still Poisson point processes with unit intensity.

Let us define a family of non negative martingales  $M_\alpha(t)$  with respect to  $\mu^0$  and the filtration  $(\mathcal{F}_t)_{t \leq T}$ .

$$M_\alpha(0) = 1 \quad (5.37)$$

$$dM_\alpha(t) = M_\alpha(t) \left( \sum_{j < p} r(j|\alpha) dW_j(t) + \sum_{j \geq p} (\theta(j|\alpha) - 1) [dN_j(t) - dt] \right). \quad (5.38)$$

Using Radon-Nikodym change of measure we can define a family of measures  $(\mu_\alpha)$  defined on  $\mathcal{F}_T$  by

$$d\mu_{\alpha, T} = M_\alpha(T) d\mu^0. \quad (5.39)$$

In the rest of the text we may omit the index  $T$ . But the definition is still limited to  $\mathcal{F}_T$  with a  $T$  as large as we want. Note that it means the convergence is true only up to a arbitrary large finite time  $T$ .

Using Girsanov theorem we deduce that under  $\mu_\alpha$  the processes  $N_j(t)$  are Poisson point processes with intensity  $\theta(j|\alpha)$  and the processes  $(W_j(t))$  Gaussian processes with independent increments such that

$$\mathbb{E}_{\mu_\alpha}(W_j(t)) = t \sum_i \sigma_{ji} r(i|\alpha) = t p_0(j) r(j|\alpha) \quad (5.40)$$

$$\text{Cov}_{\mu_\alpha}(W_j(t), W_i(s)) = \min(s, t) \sigma_{ji}. \quad (5.41)$$

In the first line we used  $\sum_j p_0(j) r(j|\alpha) = 0$ . We have the characteristic function

$$\begin{aligned} \mathbb{E}_{\mu_\alpha}(e^{\sum_{j < p} \lambda_j W_j(t) + \sum_{j \geq p} \lambda_j N_j(t)}) = \\ \exp[t(\sum_{j < p} p_0(j) r(j|\alpha) + \frac{1}{2} \sum_{i, j < p} \lambda_i \sigma_{ij} \lambda_j + \sum_{j \geq p} \theta(j|\alpha) (e^{\lambda_j} - 1))]. \end{aligned} \quad (5.42)$$

Once again, since the processes  $(W_j(t))$  and  $(N_j(t))$  have independent increments the generalization to any finite dimensional characteristic function is straightforward. Hence in the meaning we gave earlier the processes  $(Y_j^{(n)}(t))$  and  $(N_j^{(n)}(t))$  under  $\mathbb{P}_\alpha^{(n)}$  converge respectively to the processes  $(W_j(t))$  and  $(N_j(t))$  under  $\mu_\alpha$ .

Note that, since  $\sum_i \sigma_{ij} = \sum_j p_0(j) r(j|\alpha) = 0$ , we have  $\sum_j W_j(t) = 0$  at least in distribution under each  $\mu_\alpha$ . This is a reminder that the measurement outcomes are not independent since

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at least one must be obtained for each indirect measurement. Hence we defined one too many measurement outcome process. Also if the initial probe state is exactly one of the outcome eigenvectors, we only have Poisson processes as outcome continuous time processes. Namely, if  $|\Psi\rangle = |1\rangle$  then  $p = 2$ , only the counting processes remain relevant since we have  $\sigma_{ij} = \sigma_{11} = p_0(j)r(j|\alpha) = p_0(1)r(1|\alpha) = 0$ , thus  $W_1(t) = 0$  at least in distribution under each  $\mu_\alpha$ . Actually  $W_1(t) = 0$   $\mu_0$  almost surely, thus  $\mu_\alpha$  almost surely since the map  $f(x)$  defining the push forward measure is identically 0. Hence for any element  $A$  of  $\mathcal{F}$ ,  $\mu_0(A) = \nu(f^{-1}(A)) = \nu(\Omega)1_{0 \in A} = 1_{0 \in A}$ .

Let us define a new measure  $\mu = \sum_\beta q_\beta(0)\mu_\alpha$ . We recall that  $q_\alpha(0)$  is the initial “probability” of the pointer state  $\alpha$ ,  $q_\alpha(0) = \langle \alpha | \rho(0) | \alpha \rangle$ . Since the convergence of the processes is true for each  $\mu_\alpha$  it will also be for  $\mu$  starting from the initial discrete time measure  $\mathbb{P}^{(n)} = \sum_\alpha q_\alpha(0)\mathbb{P}_\alpha^{(n)}$ . We need to unravel a clearer expression for the law of the processes under this new measure.

Let us define a new martingale  $M(t) = \sum_\beta q_\beta(0)M_\beta(t)$ . From the definition of  $\mu_\beta$ , we obtain that  $M(t)$  is the Radon-Nikodym derivative of  $\mu$  with respect to  $\mu^0$ .

$$\left. \frac{d\mu}{d\mu^0} \right|_{\mathcal{F}_t} = M(t), \quad t \leq T. \quad (5.43)$$

We remark that, since  $M(t) = 0$  implies  $M_\alpha(t) = 0$  for any pointers such that  $q_\alpha(0) \neq 0$ , the corresponding measures  $\mu_\alpha$  are all absolutely continuous with respect to  $\mu$ . Since  $M(t)$  is  $\mu$  almost surely strictly positive, using the chain rule, we obtain

$$d\mu_\alpha = \frac{M_\alpha(T)}{M(T)} d\mu \quad (5.44)$$

for any  $\alpha$  such that  $q_\alpha(0) \neq 0$ .

This martingale change of measure remind us of the one we used in the non demolition discrete study. So let us define a process

$$q_\alpha(t) = q_\alpha(0) \frac{M_\alpha(t)}{M(t)}. \quad (5.45)$$

This process is a  $\mu$  martingale with  $\mathbb{E}_\mu(q_\alpha(t)) = q_\alpha(0)$ . In terms of these martingales the SDE verified by  $M(t)$  is

$$dM(t) = M(t) \left( \sum_{j < p} \sum_\beta q_\beta(t) r(j|\beta) dW_j(t) + \sum_{j \geq p} \left( \sum_\beta q_\beta(t) \theta(j|\beta) - 1 \right) [dN_j(t) - dt] \right). \quad (5.46)$$

Hence Girsanov theorem tells us that under  $\mu$ ,

$$X_j(t) = W_j(t) - \int_0^t \sum_\beta q_\beta(s) p_0(j) r(j|\beta) ds \quad (5.47)$$

is a centered Gaussian process with covariance  $\mathbb{E}(X_j(t)X_i(s)) = \min(s, t)\sigma_{ji}$ . Under the same measure, the processes  $N_j(t)$  are counting processes and

$$N_j(t) - \int_0^t \sum_\beta q_\beta(s) \theta(j|\beta) ds \quad (5.48)$$

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is a martingale. Thus  $N_j(t)$  are counting processes with stochastic intensity  $\sum_\beta q_\beta(t)\theta(j|\beta)$ .

From their similar status of Radon-Nikodym derivative the question naturally arising is: can we show that the processes  $q_\alpha(t)$  are the continuous limit of the processes  $q_\alpha^{(n)}(t)$ . Let us use the “naive scaling” approach used in [24]. We assume the state processes defined from the outcome processes should well behave under the limit as long as the limit obtained is coherent with the initial discrete process. We recall that if one seek rigor, we refer to the computations of the Mellin transforms done in [20].

Let us consider the Doob–Meyer decomposition of the evolution of one density matrix element in the basis  $\mathcal{P}$ .

$$\begin{aligned} \rho_{\alpha\beta}^{(n)}(t+1/n) - \rho_{\alpha\beta}^{(n)}(t) = & \\ & \rho_{\alpha\beta}^{(n)}(t) \left( \sum_j m^{(n)}(j|\alpha) \overline{m^{(n)}(j|\beta)} - 1 \right) \\ & + \rho_{\alpha\beta}^{(n)}(t) \sum_j \frac{m^{(n)}(j|\alpha) \overline{m^{(n)}(j|\beta)}}{\sum_\gamma q_\gamma^{(n)}(t) p^{(n)}(j|\gamma)} [\Delta N_j([nt]) - \sum_\gamma q_\gamma^{(n)}(t) p^{(n)}(j|\gamma)] \end{aligned} \quad (5.49)$$

with  $M_j^{(n)} = \sum_\alpha m^{(n)}(j|\alpha) |\alpha\rangle \langle \alpha|$ . The expansion in  $1/\sqrt{n}$  gives

$$\begin{aligned} \rho_{\alpha\beta}^{(n)}(t+1/n) - \rho_{\alpha\beta}^{(n)}(t) = & \\ & \rho_{\alpha\beta}^{(n)}(t) \times (-i(E_\alpha - E_\beta) + \sum_j l(j|\alpha) \overline{l(j|\beta)} - \frac{1}{2}(|l(j|\alpha)|^2 + |c(j|\beta)|^2)1/n \\ & + \rho_{\alpha\beta}^{(n)}(t) \sum_{j < p} (l'(j|\alpha) + \overline{l'(j|\beta)} - \sum_\gamma r(j|\gamma) q_\gamma^{(n)}(t)) [\Delta Y_j^{(n)}(t) - 1/n \sum_\gamma p_0(j) r(j|\gamma) q_\gamma^{(n)}(t)] \\ & + \rho_{\alpha\beta}^{(n)}(t) \sum_{j \geq p} \left( \frac{l(j|\alpha) \overline{l(j|\beta)}}{\sum_\gamma q_\gamma^{(n)}(t) \theta(j|\gamma)} - 1 \right) [\Delta N_j^{(n)}(t) - 1/n \sum_\gamma q_\gamma^{(n)}(t) \theta(j|\gamma)] \\ & + o(1/n) \end{aligned} \quad (5.50)$$

with  $E_\alpha |\alpha\rangle = H_{\text{sys.}} |\alpha\rangle$ ,  $l(j|\alpha) = -i \langle j | H_{\text{int.}}(\alpha) \Psi \rangle$  and  $l'(j|\alpha) = l(j|\alpha) / \langle j | \Psi \rangle$  for  $j < p$ . Remark that  $r(j|\alpha) = 2\text{Re}(l'(j|\alpha))$ . Hence, considering the limit of  $1/n$  to be an infinitesimal time increment  $dt$ , the natural limit of this discrete difference equation is the differential equation

$$\begin{aligned} d\rho_{\alpha\beta}^{(0)}(t) = & \rho_{\alpha\beta}^{(0)}(t) \times (-i(E_\alpha - E_\beta) + \sum_j l(j|\alpha) \overline{l(j|\beta)} - \frac{1}{2}(|l(j|\alpha)|^2 + |c(j|\beta)|^2)dt \\ & + \rho_{\alpha\beta}^{(0)}(t) \sum_{j < p} (l'(j|\alpha) + \overline{l'(j|\beta)} - \sum_\gamma r(j|\gamma) q_\gamma^{(0)}(t)) [dY_j^{(0)}(t) - dt \sum_\gamma p_0(j) r(j|\gamma) q_\gamma^{(0)}(t)] \\ & + \rho_{\alpha\beta}^{(0)}(t) \sum_{j \geq p} \left( \frac{l(j|\alpha) \overline{l(j|\beta)}}{\sum_\gamma q_\gamma^{(0)}(t) \theta(j|\gamma)} - 1 \right) [dN_j^{(0)}(t) - dt \sum_\gamma q_\gamma^{(0)}(t) \theta(j|\gamma)]. \end{aligned} \quad (5.51)$$

Let us define operators  $L_j = \langle j | H_{\text{int.}} \Psi \rangle$  and  $L'_j = L_j / \langle j | \Psi \rangle$  for  $j < p$ . We also define a new process  $X_j^{(0)} = Y_j^{(0)}(t) - \int_0^t \sum_\gamma p_0(j) r(j|\gamma) q_\gamma^{(0)}(s) ds$ .  $X_j^{(0)}(t)$  has the same finite dimensional distribution under  $\mathbb{P}^{(0)}$  as  $X_j(t)$  under  $\mu$ . Similarly  $N_j^{(0)}(t)$  under  $\mathbb{P}^{(0)}$  has the same finite

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dimensional distribution as  $N_j(t)$  under  $\mu$ . Then using our newly defined operators  $L_j, L'_j$ , we expect  $\rho^{(0)}$  under  $\mathbb{P}^{(0)}$  to have the same finite dimensional distribution as  $\rho(t)$ , the solution of the following SDE, under  $\mu$ .

$$\begin{aligned} d\rho(t) = & \mathcal{L}(\rho(t))dt \\ & + \sum_{j < p} (L'_j \rho(t) + \rho(t) L'^*_j - \text{tr}[(L'_j + L'^*_j)\rho(t)]\rho(t))dX_j(t) \\ & + \sum_{j \geq p} \left( \frac{L_j \rho(t) L_j^*}{\text{tr}[L_j^* L_j \rho(t)]} - \rho(t) \right) [dN_j(t) - \text{tr}[L_j^* L_j \rho(t)]dt] \end{aligned} \quad (5.52)$$

where

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_j L_j \rho L_j^* - \frac{1}{2} \{L_j^* L_j, \rho\} \quad (5.53)$$

is the usual Lindbladian. Of course we assume identical initial values  $\rho^{(0)}(0) = \rho(0)$ .

We can deduce the expression of the limit SDE for the diagonal elements of  $\rho^{(0)}(t)$  in the basis  $\mathcal{P}$ . These pointer probabilities are solution of the following SDE

$$\begin{aligned} dq_\alpha^{(0)}(t) = & q_\alpha^{(0)}(t) \sum_{j < p} (r(j|\alpha) - \sum_\beta q_\beta^{(0)}(t) r(j|\beta)) dX_j^{(0)}(t) \\ & + q_\alpha^{(0)}(t) \sum_{j \geq p} \left( \frac{\theta(j|\alpha)}{\sum_\beta q_\beta^{(0)}(t) \theta(j|\beta)} - 1 \right) [dN_j^{(0)}(t) - dt \sum_\beta q_\beta^{(0)}(t) \theta(j|\beta)]. \end{aligned} \quad (5.54)$$

We can compare this expression with the SDE of  $q_\alpha(t)$  obtained using the usual Itô formula.

$$\begin{aligned} dq_\alpha(t) = & q_\alpha(t) \sum_{j < p} (r(j|\alpha) - \sum_\beta q_\beta(t) r(j|\beta)) dX_j(t) \\ & + q_\alpha(t) \sum_{j \geq p} \left( \frac{\theta(j|\alpha)}{\sum_\beta q_\beta(t) \theta(j|\beta)} - 1 \right) [dN_j(t) - dt \sum_\beta q_\beta(t) \theta(j|\beta)]. \end{aligned} \quad (5.55)$$

Hence we infer that the  $q_\alpha^{(0)}$  under  $\mathbb{P}^{(0)}$  has the same finite dimensional distribution as  $q_\alpha(t)$  under  $\mu$ . Thus we infer that the finite dimensional distribution of  $(\rho^{(n)}(t))$  converges to the one of  $(\rho(t))$ .

The SDE defining  $\rho(t)$  has not exactly the same expression as usual quantum trajectories. The processes  $X_j(t)$  are not Wiener processes under  $\mu$ . But a simple computation shows that the processes  $X_j(t)$  can be expressed in terms of  $p-1$  independent Brownian motions  $B_j(t)$  under  $\mu$ :  $X_j(t) = \sum_i \sqrt{p_0(j)}(\delta_{i,j} - \sqrt{p_0(j)}\sqrt{p_0(i)})B_i(t)$ . Then the quantum trajectory SDE becomes

$$\begin{aligned} d\rho(t) = & \mathcal{L}(\rho(t))dt \\ & + \sum_{j < p} (L_j \rho(t) + \rho(t) L_j^* - \text{tr}[(L_j + L_j^*)\rho(t)]\rho(t))dB_j(t) \\ & + \sum_{j \geq p} \left( \frac{L_j \rho(t) L_j^*}{\text{tr}[L_j^* L_j \rho(t)]} - \rho(t) \right) [dN_j(t) - \text{tr}[L_j^* L_j \rho(t)]dt] \end{aligned} \quad (5.56)$$

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where the  $L_j$  are redefined by  $L_j = \sqrt{p_0(j)} \frac{\langle j | H_{\text{int.}} | \Psi \rangle}{\langle j | \Psi \rangle}$  for  $j < p$ . We used  $\sum_j p_0(j) L_j' = 0$ .

We recovered the usual quantum trajectory SDE. Each outcome  $j$  with  $p_0(j) \neq 0$  leads to a diffusive part. The outcomes  $j$  such that  $p_0(j) = 0$  lead to the counting processes. This is not a surprise since we expect to frequently measure outcomes which had originally a strictly positive probability to be measured, and to measure just few times outcomes which had a null probability to be measured without interaction with the system.

### General continuous limit

In [84–86], C. Pellegrini proved a more general and stronger result. For instance, he did not assume that the initial discrete quantum trajectory corresponds to a non demolition indirect measurement. We briefly sketch the method he used in [86]. Note that the existence and uniqueness in distribution of the solution of the quantum trajectory SDE is needed to obtain the convergence. This result is also proven in [86].

The derivation technique is based on the convergence of Markov generators. We recall that  $(\rho^{(n)}(t))$  is a continuous time step process version of the discrete quantum trajectory  $(\rho(n))$ . We have  $\rho^{(n)}(t) = \rho(k)$  for  $k/n \leq t < (k+1)/n$ . We define the following operator acting on  $C_c^2$  function  $f$  of operators acting on the system Hilbert space. The index  $c$  means  $f$  has compact support. We define

$$\mathcal{A}_n f(\rho) = n \sum_j \left( f \left( \frac{M_j \rho M_j^*}{\text{tr}[M_j^* M_j \rho]} \right) - f(\rho) \right) \text{tr}[M_j^* M_j \rho]. \quad (5.57)$$

This operator associates to  $f$  the increment of the predictable process  $\sum_{k=1}^n \mathbb{E}(f(\rho(k)) | \rho(k-1)) - f(\rho(k-1)) = \sum_{k=1}^n \frac{1}{n} \mathcal{A}_n f(\rho(k-1))$ . The operator  $\mathcal{A}_n$  is called the Markov generator of the Markov chain  $(\rho(k))$  (or of the process  $(\rho^{(n)}(t))$ ).

Equivalently, we can compute the generator of the continuous time quantum trajectory for any  $C_c^2$  function  $f$  on the set of operators acting on the system.

$$\mathcal{A} f(\rho) = \lim_{t \rightarrow 0} \frac{\mathbb{E}(f(\rho(t)) | \rho) - f(\rho)}{t}. \quad (5.58)$$

The operator  $\mathcal{A}$  is a generalization of the dual of the more known Fokker-Planck operator for Markov diffusive processes. It also describes Markov processes including jumps. Using Itô formula it is easy to compute the generator corresponding to a given SDE. For example, for the quantum trajectory SDE, we have

$$\begin{aligned} \mathcal{A} f(\rho) = & D_\rho f(\mathcal{L}(\rho)) \\ & + \sum_{j < p} D_\rho^2 f(D_j(\rho), D_j(\rho)) \\ & + \sum_{j \geq p} [f(\rho_j) - f(\rho) - D_\rho f(\rho_j)] v_j(\rho) \end{aligned} \quad (5.59)$$

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where  $\rho_j = \frac{J_j(\rho)}{v_j(\rho)}$ .  $D_\rho$  and  $D_\rho^2$  are shorthand notations for

$$D_\rho f(X) = \sum_{k=1}^{2 \dim_{\text{sys.}}^2} X_k \frac{\partial f}{\partial \rho_k}(\rho) \quad (5.60)$$

$$D_\rho^2 f(X, Y) = \sum_{k, l=1}^{2 \dim_{\text{sys.}}^2} X_k Y_l \frac{\partial^2 f}{\partial \rho_k \partial \rho_l}(\rho) \quad (5.61)$$

where  $X, Y$  are operators on the system Hilbert space and  $X_k$  are the real and imaginary parts of their elements of matrix. The first line of (5.59) corresponds to the mean evolution which is given by the Lindbladian  $\mathcal{L}$ . The second line corresponds to the continuous (diffusive) outcome processes. Considering only these two lines we are in presence of the usual dual of a Fokker-Planck operator. The third and last line corresponds to the discontinuous (jump) outcome processes.

A rapid computation using Taylor expansion shows that  $\lim_n \mathcal{A}_n f(\rho) = \mathcal{A}f(\rho)$  on the set of states. The convergence is uniform on  $\mathcal{S}(\mathcal{H}_{\text{sys.}})$  since this set is compact and the functions  $f$  are  $C_c^2$ . Note that if  $|\Psi\rangle = |1\rangle$  then  $p = 2$  and  $D_1(\rho) = 0$  for any  $\rho$ . Thus, in the continuous limit, we will not have any continuous (diffusive) outcome process.

The continuous time process  $\rho(t)$  is actually the unique solution (in distribution) such that

$$f(\rho(t)) - f(\rho(0)) - \int_0^t \mathcal{A}f(\rho(s)) ds \quad (5.62)$$

is a martingale with respect to its natural filtration  $\mathcal{F}_t^\rho = \sigma(\rho(s), s \leq t)$ . Hence any process such that the last expression is a martingale has the same distribution as  $\rho(t)$ . We say  $(\rho(t))$  is the solution of the martingale problem  $(\mathcal{A}, \rho(0))$ . Note that we expressed the solution of the martingale problem in terms of process and not measure. We found a process solving the martingale problem on an already defined probability space.

We reduced the characterization of the distribution of  $\rho(t)$  to the expression of its generator. Hence it will be the object we will be mostly interested in. We expect that computing the limit of  $\mathcal{A}_n$  when  $n \rightarrow \infty$ , will lead to  $\mathcal{A}$  and that it will imply that  $\rho^{(n)}(t)$  converges in distribution (*i.e.* weakly) to  $\rho(t)$ . Instead of computing the limit of all finite dimensional distributions, we will formally need to compute only the limit of  $\mathcal{A}_n$  and not the whole history of the state.

One down side compared to the martingale change of measure of the QND case is that we do not obtain explicitly the limit distribution of the measurement processes. Moreover we lose the explanation of the change of measurement distribution through a martingale change of measure induced by the interaction with the system.

A theorem of Ethier and Kurtz cited and adapted to quantum trajectories in [86] (Theorem 5) gives us the conditions under which the weak convergence is obtained from the generator convergence. We do not state the full theorem here and refer the interested reader to the cited article. Let us just indicate that in addition to the quantum trajectories properties we already gave (mainly uniqueness of the solution of the martingale problem  $(\mathcal{A}, \rho(0))$  in distribution), the family of processes  $(\rho^{(n)}(t))$  need to be tight (equivalently the family of measures implied by these processes) and we need that for any increasing sequence of times  $(t_k)_{1 \leq k \leq m}$  such that  $t_m \leq t < t + s$ , for any family of  $C_c^2$  functions  $(\theta_k)_{1 \leq k \leq m}$  and for any  $C_c^2$  function  $f$ ,

$$\lim_n \mathbb{E} \left( \left( f(\rho^{(n)}(t+s)) - f(\rho^{(n)}(t)) - \int_t^{t+s} \mathcal{A}f(\rho^{(n)}(u)) du \right) \prod_{k=1}^m \theta_k(\rho^{(n)}(t_k)) \right) = 0. \quad (5.63)$$

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The convergence of the expectation assure finite dimensional distribution convergence up to a finite, arbitrary large time  $T$ . The tightness condition assure that the convergence is actually weak.

The tightness condition on the family of processes  $(\rho^{(n)}(t))_{n \in \mathbb{N}}$  guaranties that they are mostly concentrated on a same compact subset of probability space. More precisely,  $(\rho^{(n)}(t))$  is a thigh family of processes if the family of corresponding measure  $(\mu_n)_{n \in \mathbb{N}}$  is such that for any  $\epsilon > 0$  it exists a compact subset  $K_\epsilon$  of the probability space such that for all  $\mu_n$

$$\mu_n(K_\epsilon) > 1 - \epsilon. \quad (5.64)$$

If the tightness criteria is not fulfilled, the processes could escape indefinitely to a previously unexplored part of the probability space.

A good example of a non tight family of measures is a family of Gaussian probability measures on the real line. Let  $(\nu_n)$  be this family of measures. Their means are  $\mathbb{E}_{\nu_n}(X) = n$ . Their variances are  $\mathbb{E}_{\nu_n}((X - n)^2) = 1$ . Assume it exists a compact subset of  $\mathbb{R}$  such that  $\nu_n(K) > \frac{1}{2}$  for any  $n$ . From Heine–Borel theorem,  $K$  is bounded. Let  $x_0 = \max(K)$ . For  $n$  large enough we have  $\nu_n(K) \leq \frac{1}{\sqrt{2\pi}} \lambda(K) e^{-\frac{1}{2}(x_0 - n)^2} < \frac{1}{2}$  with  $\lambda$  the Lebesgue's measure on  $\mathbb{R}$ . This contradicts  $\nu_n(K) > \frac{1}{2}$  for any  $n$ . Thus the family  $(\nu_n)$  is not tight. We see that  $(\nu_n)$  is not tight because when  $n \rightarrow \infty$ , the pointwise limit of the density corresponding to  $\nu_n$  vanishes. Hence as  $n$  increase, the random variable  $X$  will always be concentrated farther towards infinity.

The tightness of  $(\rho^{(n)}(t))$  is proved in [86]. We do not reproduce the proof here. The convergence of the expectation is, as expected, a direct consequence of  $\mathcal{A}_n$  convergence to  $\mathcal{A}$  and the fact that  $f(\rho^{(n)}(t+s)) - f(\rho^{(n)}(t)) - \sum_{k=1}^n \frac{1}{n} \mathcal{A}_n f(\rho^{(n)}(t))$  is a martingale with respect to the natural filtration of  $(\rho^{(n)}(t))$ . We refer the interested reader to the cited article for the complete proof.

We have weak convergence of general discrete quantum trajectories to general continuous quantum trajectories. The discussion of the extension of this result to mixed and random i.i.d. initial probe states is made in [78].

In order to better understand the meaning of this limit and in particular to obtain the distribution of the measurement outcome processes, one may like to actually extend the technique used in the QND case. This might be possible considering the process  $(\text{tr}[M_{j_n}^* M_{j_n} \rho(n-1)])_n$  as a martingale with respect to a measure with equiprobability for the outcomes  $j$ . Then one could see the actual distribution of the measurements as a martingale change of measure using  $\text{tr}[M_{j_n}^* M_{j_n} \rho(n-1)]$ . If we find a continuous limit for the initially equiprobable law and for the probability outcome martingale we might be able to find the continuous limit using Girsanov theorem. The introduction of a ad-hoc i.i.d. distribution for the outcomes should allow avoiding the difficulty introduced by the state dependency of the counting processes intensities.

In the next subsection we turn to a derivation of quantum trajectories directly from continuous time Markovian open system dynamics. We recall that in section 3.2.3, we saw that the limit of repeated interactions (without measurement) leads to quantum stochastic differential equation (QSDE). Hence a natural question would be, can we derive continuous quantum trajectories directly from the QSDE ? Is the result coherent with the one we just obtained ? One of the purpose of next subsection is to answer these questions.

### 5.2.3. Quantum filtering

As we saw in chapter 3 in some limits, the extended evolution of an open system is described through quantum stochastic (QS) processes. Compared to the reduced Markovian limit, the noises of the QS evolution hold some information on the environment. This will allow us to describe continuous monitoring of these environment. We expect to derive quantum trajectories SDE for the system state when the QS noises are measured and taken into account.

This derivation is here obtained using quantum filtering techniques. V.P. Belavkin was the first to define and use quantum filtering to obtain quantum trajectories SDE [26–28]. As far as the author knows, it was the first derivation of quantum trajectories with no added hypothesis on the behavior of the system under measurement. Only the QS unitary evolution and the projection postulate were used. Since then this derivation has been mostly applied to quantum optics situations, namely interferometry and photo counting measurements. Quantum filtering is not the only method one can use to derive quantum trajectories from QSDE's. Alternative derivations can be found in [9, 12, 25, 102].

In this subsection we present a sketch of derivation inspired by the more recent work of L. Bouten *et al.* [34]. We begin with an application of quantum filtering to one simple indirect measurement. We then generalize this approach to measurements on quantum noises. We give elements of proof for the derivation of the quantum trajectory SDE, first for a simple interferometry measurement and then for a spontaneous photon emission. The generalization by combination of these cases is straightforward.

Note that we will not discuss the general meaning and purpose of quantum filtering theory. We just use the technique to derive quantum trajectories.

#### Discrete introduction to quantum filtering

Let us first consider a more simple situation. We apply the ideas of quantum filtering to a simple discrete indirect measurement.

The setup is the following. A finite dimensional Hilbert space probe in a definite pure state interacts with a system and is then measured. The probe plays the role of the environment. The total Hilbert space we are interested in is a tensor product of the system Hilbert space and the environment Hilbert space.

$$\mathcal{H}_{tot.} = \mathcal{H}_{sys.} \otimes \mathcal{H}_{probe} \quad (5.65)$$

The system and the probe are entangled through a unitary operator  $U$  acting on the total Hilbert space  $\mathcal{H}_{tot.}$ . Starting with a disentangled state  $\rho = \rho_{sys.} \otimes \rho_{probe}$ , the state becomes  $U\rho U^*$  after the interaction.

Let  $O$  be the observable one measures on the environment (*i.e.*  $O \equiv I_{sys.} \otimes O$ ). Its spectrum is  $\text{spec}(O)$ .  $O$  can be written as a weighted sum of projectors  $O = \sum_{o \in \text{spec}(O)} o P_o$ . The projectors only act on the environment part of the Hilbert space:  $P_o \equiv I_{sys.} \otimes P_o$ . After a measurement of  $O$ , if the result of the measurement is  $o$ , the total state is

$$\frac{P_o U \rho U^* P_o}{\text{tr}[P_o U \rho U^*]} \quad (5.66)$$

The probability of this event is  $p(o) = \text{tr}[P_o U \rho U^*]$ .

This measurement is described by the probability space  $(\text{spec}(O), \mathcal{F}_O, p)$  with  $\mathcal{F}_O$  the smallest  $\sigma$ -algebra making any singlet  $\{o\}$  measurable.

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We can compute the expectation of any observable conditioned on the measurement. The expectation of any operator  $Y$  acting on the total Hilbert space is  $\text{tr}[Y P_o U \rho U^* P_o]/p(o)$ . Then the expectation of  $Y$  conditioned on the measurement of  $O$  is the random variable

$$\sum_o \text{tr}[Y P_o U \rho U^* P_o]/p(o) \times 1_{\{o\}}(\omega) \quad (5.67)$$

where  $\omega \in \text{spec}(O)$  and  $\mathbb{E}_p(1_o) = p(o)$ . We would like to be able to interpret this conditioning as an actual conditional expectation as defined in the context of probability theory. First we need to have some sort of equivalence between the quantum physics objects we use and the probability theory ones.

For now we drop the interaction  $U$  and just try to describe a measurement. Let  $\mathbb{P}$  be the probability measure defined by  $\mathbb{P}(A) = \sum_{o \in A} p(o)$  for any  $A \in \mathcal{F}_O$ . We have  $\mathbb{P}(A) = \text{tr}[P_A \rho]$  with  $P_A = \sum_{o \in A} P_o$ . The triplet  $(\text{spec}(O), \mathcal{F}_O, \mathbb{P})$  defines a probability space. It is the same as before but with a new measure  $\mathbb{P}$ . Let  $L^\infty(\text{spec}(O), \mathcal{F}, \mathbb{P})$  be the algebra of bounded  $\mathbb{P}$  measurable complex functions of  $\text{spec}(O)$ . Hence  $L^\infty(\text{spec}(O), \mathcal{F}, \mathbb{P})$  is the set of all complex linear combinations of singlet indicator functions  $1_{\{o\}} : \text{spec}(O) \rightarrow \{0, 1\}$ . Let  $\mathcal{O}$  be the set of all complex linear combination of projectors  $(P_o)_{o \in \text{spec}(O)}$ . There is obviously a \*-isomorphism  $\iota$  between  $\mathcal{O}$  and  $L^\infty(\text{spec}(O), \mathcal{F}, \mathbb{P})$ . Just replace the projectors by the indicator functions in the expression of the operator and conversely. Moreover  $\mathbb{E}(X) = \text{tr}[X \rho]$  for any element  $X$  of  $\mathcal{O}$  or equivalently  $L^\infty(\text{spec}(O), \mathcal{F}, \mathbb{P})$ . We have a \*-isomorphism between the two spaces we want to use.

Using spectral theorem we can build such \*-isomorphism for any commutative von Neumann algebra (*i.e.* a weakly closed commutative sub algebra of the bounded operators acting on the total Hilbert space) and any well defined state. For any commutative von Neumann algebra  $\mathcal{O}$ , any well defined state  $\rho$ , it exists a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a \*-isomorphism  $\iota$  between  $\mathcal{O}$  and  $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  such that for any element  $O$  of  $\mathcal{O}$ ,  $\rho(O) \equiv \text{tr}[O \rho] = \mathbb{E}(\iota(O))$  (see [34] Theorem 3.3). So, since they are equivalent through the \*-isomorphism, from now on we do not distinguish the notations between the probability and the operator contexts. Actually through this mapping we interpret some quantum noises as classical processes as we explained in section 3.1.

We want to compute the expectation of some operator conditioned on the measurement outcome. This quantity is obviously a random variable. A probability theory tool which would correspond to such random variable is a conditional expectation. Let us give an example of conditional expectation we would like to use expressed in operator terms.

$$\mathbb{E}_0(Y|\mathcal{O}) = \sum_{o \in \text{spec}(O)} \text{tr}[Y P_o \rho P_o]/p(o) \times P_o. \quad (5.68)$$

This definition is a quite straightforward application of the postulates of quantum mechanics. We actually defined an observable with eigenvalues corresponding to some operator  $Y$  expectation with respect to a state depending on the measurement outcome. Alas it is actually well defined only for  $Y$  commuting with every element of  $\mathcal{O}$ . Hence for  $Y$  in  $\mathcal{O}'$  the commutant of  $\mathcal{O}$ .

Let us show that with this limitation, it is actually a conditional expectation. We need to show that for any  $K$  in  $\mathcal{O}$  ( $K = \sum_\lambda k(o) P_o$ ) and any  $Y$  in  $\mathcal{O}'$ ,  $\mathbb{E}_0(\mathbb{E}_0(Y|\mathcal{O})K) = \mathbb{E}_0(YK) = \mathbb{E}_0(KY)$ . The last equality is easy to prove from the commutation of  $Y$  with  $K$ . The proof of the first equality needs more attention. From the definition of the conditional expectation,

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the decomposition of  $K$  in a sum of the projectors  $P_o$  and the commutation of  $Y$  with any projector  $P_o$ , we have

$$\mathbb{E}_0(\mathbb{E}_0(Y|\mathcal{O})K) = \sum_o \text{tr}[Y P_o \rho] / p(o) \times \text{tr}[k(o) P_o \rho]. \quad (5.69)$$

Since  $p(o) = \text{tr}[P_o \rho]$ , we obtain

$$\mathbb{E}_0(\mathbb{E}_0(Y|\mathcal{O})K) = \sum_o \text{tr}[Y k(o) P_o \rho] = \mathbb{E}_0(YK). \quad (5.70)$$

This achieves the proof that  $\mathbb{E}_0(\cdot|\mathcal{O})$  is a conditional expectation defined for elements of  $\mathcal{O}'$ . For  $Y$  not commuting with every element in  $\mathcal{O}$ , the last equality is not true. We are stuck with a sum  $\sum_o \text{tr}[P_o Y k(o) P_o \rho]$  which cannot be simplified in general. This limitation comes down to fundamental quantum mechanics. If we have two non commuting observables  $A$  and  $B$ , they cannot be measured simultaneously because the order in which we need to compute the probabilities is indefinite. Either  $p(a, b) = \text{tr}[P_a P_b \rho]$  or  $p(a, b) = \text{tr}[P_b P_a \rho]$  with  $a, b$  eigenvalues of  $A$  and  $B$  respectively. Hence we cannot define a probability space corresponding to such measurement unless  $[P_a, P_b] = 0$  for any  $a, b$ . It is actually a manifestation of Heisenberg uncertainty principle. We cannot exactly know the value of two non commuting observables at the same time.

Let us stress that the conditional expectation is only well defined for operator of the form  $Y = \sum_{o \in \text{spec}(O)} X_o \otimes P_o$  with  $X_o$  operators on the system Hilbert space. This is not surprising. These type of observables can actually be mapped to random variables defined on  $\text{spec}(O)$  and taking value in the space of system operators. we have

$$\iota(Y) = \sum_{o \in \text{spec}(O)} X_o 1_o \quad (5.71)$$

Hence as we would have expected, following [34], the definition of the conditional expectation can be extended to any commuting von Neumann algebra, its commutant and any physical state. One just need to go back and forth between the operators and the probability space using the  $*$ -isomorphism  $\iota$ . The definition of the conditional expectation in the probability space is then translated in the operator space. The elements of the commutant of the von Neumann algebra which are not themselves in the von Neumann algebra are then mapped to random variables which take value in the set of system operators.

We are interested in indirect measurements. Thus let us introduce back an interaction taking place before the measurement. Hence we have two probability measures.

$$\mathbb{P}_0(A) = \text{tr}[P_A \rho] \quad (5.72)$$

$$\mathbb{P}(A) = \text{tr}[P_A U \rho U^*]. \quad (5.73)$$

Under  $\mathbb{P}$ , we have

$$\mathbb{E}(Y|\mathcal{O}) = \sum_{o \in \text{spec}(O)} \text{tr}[Y P_o U \rho U^* P_o] / p_U(o) \times P_o \quad (5.74)$$

with  $p_U(o) = \mathbb{P}(\{o\})$ .

We would like to find a non negative random variable allowing us to pass from  $\mathbb{P}_0$  to  $\mathbb{P}$ . Hence we would like to find a Radon–Nikodym derivative of  $\mathbb{P}$  with respect to  $\mathbb{P}_0$ . Let us

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formulate this in operator terms. Assume it exists an operator  $V$  in  $\mathcal{O}'$  such that, for any operator  $Y$  in  $\mathcal{O}'$ ,  $\mathbb{E}[Y] = \mathbb{E}_0(V^*YV)$ . Remark that  $\mathbb{E}_0(V^*V) = \mathbb{E}(I_{\text{tot.}}) = \mathbb{P}(\text{spec}(O)) = 1$ . The existence of such  $V$  will allow us to find the law of the random variables of  $\mathcal{O}'$  under  $\mathbb{P}$  knowing their law under  $\mathbb{P}_0$ . In particular for any  $Y \in \mathcal{O}$ , following the rules for conditional expectations,

$$\mathbb{E}(Y) = \mathbb{E}_0(V^*YV) \quad (5.75)$$

$$= \mathbb{E}_0(\mathbb{E}_0(V^*V|\mathcal{O})Y). \quad (5.76)$$

Therefore if such  $V$  exists,  $\mathbb{P}$  is absolutely continuous with respect to  $\mathbb{P}_0$  and  $\mathbb{E}_0(V^*V|\mathcal{O})$  is the Radon-Nikodym derivative, restricted to  $\mathcal{O}$  of  $\mathbb{P}$  with respect to  $\mathbb{P}_0$ . Finding this  $V$  is a key point of the derivation of quantum trajectories.

We finish this discrete measurement introduction by showing the following Bayes' rule equivalent. It will turn useful for the derivation of quantum trajectories. For any  $Y \in \mathcal{O}'$ , we have

$$\mathbb{E}(Y|\mathcal{O}) = \frac{\mathbb{E}_0(V^*YV|\mathcal{O})}{\mathbb{E}_0(V^*V|\mathcal{O})} \quad (5.77)$$

$\mathbb{P}$  almost surely.

First let us show that for any  $K$  in  $\mathcal{O}$  and any  $Y$  in  $\mathcal{O}'$ ,

$$\mathbb{E}_0(\mathbb{E}_0(V^*YV|\mathcal{O})K) = \mathbb{E}_0(\mathbb{E}_0(V^*V|\mathcal{O})\mathbb{E}(Y|\mathcal{O})K) \quad (5.78)$$

The left hand side is obviously equal to  $\mathbb{E}[YK]$ . The right hand side is equal to

$$\mathbb{E}_0(\mathbb{E}_0(V^*V\mathbb{E}(YK|\mathcal{O})|\mathcal{O})) \quad (5.79)$$

where we used that  $K$  is in  $\mathcal{O}$  as well as  $\mathbb{E}(YK|\mathcal{O})$ . Since  $V$  commutes with any element of  $\mathcal{O}$ , we have

$$\mathbb{E}_0(\mathbb{E}_0(V^*V\mathbb{E}(YK|\mathcal{O})|\mathcal{O})) = \mathbb{E}_0(V^*\mathbb{E}(YK|\mathcal{O})V). \quad (5.80)$$

Then, since  $\mathbb{E}_0(V^*YV|\mathcal{O}) - \mathbb{E}_0(V^*V|\mathcal{O})\mathbb{E}(Y|\mathcal{O})$  is an element of  $\mathcal{O}$ , we have

$$\mathbb{E}_0((\mathbb{E}_0(V^*YV|\mathcal{O}) - \mathbb{E}_0(V^*V|\mathcal{O})\mathbb{E}(Y|\mathcal{O}))^2) = 0 \quad (5.81)$$

Thus

$$\mathbb{E}_0(V^*YV|\mathcal{O}) = \mathbb{E}_0(V^*V|\mathcal{O})\mathbb{E}(Y|\mathcal{O}) \quad (5.82)$$

$\mathbb{P}_0$  almost surely and therefore  $\mathbb{P}$  almost surely.

Finally  $\mathbb{E}_0(V^*V|\mathcal{O})$  is  $\mathbb{P}$  almost surely strictly greater than zero. So, we can divide both sides of the inequality by  $\mathbb{E}_0(V^*V|\mathcal{O})$  and we obtain the expected result.

The benefit of this Bayes' rule formula is that we extracted the interaction from the definition of the measure. The price is that we need to find a suitable  $V$ .

### QSDE and continuous measurement

Before we derive the quantum trajectories let us review some useful properties of QSDE's. We either recall some results of chapter 3 or show properties based on results of this chapter. The evolution unitary operator process is the solution of the following QSDE

$$\begin{aligned} U_0 &= I \\ dU_t &= -i(H - i\frac{1}{2}L^*L)U_t dt + L^*U_t dA_t - LU_t dA_t^*. \end{aligned} \quad (5.83)$$

The evolution of a system operator  $X_t = U_t^* X U_t$  is given by

$$dX_t = \mathcal{L}(X)_t + [L^*, X]_t dA_t - [L, X]_t dA_t^*. \quad (5.84)$$

We do not take into account the possible presence of  $\Lambda_t$  noises in the QSDE. The generalization to the presence of several noises is straight forward, thus we simplify the explanations by considering only one noise and thus one operator  $L$ .

We have the Itô computation rules

$dX_t dY_t$	$d\Lambda_t$	$dA_t$	$dA_t^*$	$dt$
$d\Lambda_t$	$d\Lambda_t$	0	$dA_t^*$	0
$dA_t$	$dA_t$	0	$dt$	0
$dA_t^*$	0	0	0	0
$dt$	0	0	0	0

A continuous measurement up to time  $t$  corresponds to a measurement of an observable  $Z_t$  which is a Hermitian linear combination of  $\Lambda_t$ ,  $A_t$ ,  $A_t^*$  and  $t$ . As the time of the measurement is increasing, so is  $t$  in the index of  $Z_t$ . Thus along a continuous time measurement we do not have a commutative algebra generated by only one operators but a commutative algebra  $\mathcal{Z}_t$  generated by a family of operator  $(Z_s)_{s \leq t}$ . Thankfully the family is commutative. We have  $[Z_t, Z_s] = 0$  for any time  $s, t$ . If  $Y_s \in \mathcal{Z}_s$  then  $[Y_s, Z_t] = 0$  for any  $t$ . Thus the commutative algebra increases when a new member is added to the generating family. The family of commutative algebras  $(\mathcal{Z}_t)_t$  has a filtration behavior. For any time  $s < t$ ,  $\mathcal{Z}_s \subset \mathcal{Z}_t$ .

We would now like to map these algebras to a probability space. As we explained in chapter 3, when the environment is in its vacuum state,  $B_t = A_t + A_t^*$  and  $N_t = \Lambda_t + A_t + A_t^* + t$  can be mapped to, respectively, a Brownian motion and a Poisson process with unit intensity on definite probability spaces. Since  $[N_t, B_t] \neq 0$ , only one of the two mapping can be made at once. Note that since  $N_t$  and  $B_t$  are unbounded, the spectral theorem we stated earlier in this subsection can not be applied directly to the corresponding commutative algebra  $\mathcal{B}_t$  and  $\mathcal{N}_t$ . Nevertheless, since  $N_t$  and  $B_t$  are Hermitian we can still define a \*-isomorphism between the commutative algebra and a probability space [34]. Moreover, in the case of quantum noise, we already explained in chapter 3, that S. Attal proved that both  $B_t$  and  $N_t$  can be interpreted as multiplication operators by respectively a Brownian motion and a Poisson process of unit intensity [3]. Hence through the mapping to the probability space, we can set  $\mathcal{Z} = \mathcal{Z}_\infty = \bigvee_t \mathcal{Z}_t$  and we have that  $(\mathcal{Z}_t)_t$  is actually a filtration.

It remains only to check that we can define a probability measure  $\mathbb{P}_t$  for evolved observables  $U_t^* Z_t U_t$ . Hence a measure after the interaction. From the quantum physics rules, the expectation of any operator  $Y$  in  $\mathcal{Z}'_t$  after time  $t$  should be  $\mathbb{E}_0(U_t^* Y U_t)$ . But we can have a  $Y$  acting trivially on  $\Gamma(L^2([s, \infty[))$  with  $s < t$ . Hence a  $Y$  corresponding to a measurement only up to

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time  $s$  only. Thankfully, since  $U_s^* Y U_s = U_t^* Y U_t$ , its expectation is still the one given. So even if we stopped the measurement at time  $s$ , we can compute the expectation using  $U_t$ . This is actually necessary for the definition of the measure  $\mathbb{P}_t$ . This one of the difficulty encountered when one tries to derive non Markovian quantum trajectories using quantum filtering.

Assume it exists a QS process  $V_t$  in  $\mathcal{Z}'_t$  such that for any  $Y \in \mathcal{Z}'_t$ ,  $\mathbb{E}_0(U_t^* Y U_t) = \mathbb{E}_0(V_t^* Y V_t)$ . Let  $Y$  be  $\mathcal{Z}_t$  measurable. Then  $\mathbb{E}_0(U_t^* Y U_t) = \mathbb{E}_0(\mathbb{E}_0(V_t^* V_t | \mathcal{Z}_t) Y)$ . Assume  $\mathbb{E}_0(V_t^* V_t | \mathcal{Z}_t)$  is a  $(\mathcal{Z}_t)$  martingale under  $\mathbb{P}_0$  defined at least up to a finite fixed time  $T$  as large as we want. Then  $\mathbb{P}_t$  is defined as the measure whose Radon–Nykodim derivative with respect to  $\mathbb{P}_0$  restricted to  $\mathcal{Z}_t$  is  $\mathbb{E}_0(V_t^* V_t | \mathcal{Z}_t)$ . The expectation of any  $Y \in \mathcal{Z}'_t$  under  $\mathbb{P}_t$  is given from its definition on  $\mathcal{Z}_t$  through

$$\mathbb{E}_t(Y) = \mathbb{E}_t \left( \frac{\mathbb{E}_0(V_t^* Y V_t | \mathcal{Z}_t)}{\mathbb{E}_0(V_t^* V_t | \mathcal{Z}_t)} \right). \quad (5.85)$$

In the two following subsections we use these techniques to study two measurement situation. First an interferometry and then a photon counting.

### Homodyne measurement scheme

In a coherent interferometry (Homodyne) measurement, the phase shift between two parts of a laser beam is measured. The laser frequency is tuned with the system energy level difference. One of the parts evolves freely when the other one interact with the system before the measurement. This measurement corresponds to a continuous measurement of a quadrature of the field. Namely of linear combinations of the position operator,  $A_t^* + A_t$ , and the impulsions operator  $-i(A_t - A_t^*)$  of the boson field of the coherent laser light.

We restrict our selves to the continuous measurement of  $B_t = A_t + A_t^*$ . The generalization is straightforward. Similarly we suppose the field is in its ground state  $|e(0)\rangle = |0\rangle$ . It seems to be in complete contradiction with the presence of an intense laser. Actually, the state of the laser field is assumed to be coherent. If we limit our selves to measurement up to time  $t$  the state is  $|\alpha 1_{[0,t]}\rangle = e^{-\bar{\alpha} A_t + \alpha A_t^*} |0\rangle$ . Fortunately, a simple modification of the unitary evolution brings us back to a vacuum initial laser field. It would be sufficient to redefine  $U_t$  as  $U_t^\alpha = U_t e^{-\bar{\alpha} A_t + \alpha A_t^*}$  and we would have  $U_t(|\phi\rangle \otimes |\alpha\rangle) = U_t^\alpha(|\phi\rangle \otimes |0\rangle)$  on  $\mathcal{H}_{\text{sys.}} \otimes \Gamma(L^2([0, t]))$  with  $|\phi\rangle$  the initial system state. For now we assume that  $U_t$  already takes into account this modification. Thus we consider the bosonic field in its ground state. Hence the total initial state is

$$\rho_{\text{tot.}}(0) = \rho(0) \otimes |0\rangle\langle 0|. \quad (5.86)$$

Let us look at the measurement outcome process. We already know that under  $\mathbb{P}_0$ ,  $B_t$  is a Wiener process. Let us investigate its distribution under  $\mathbb{P}_T = \mathbb{E}_0(V_T^* V_T | \mathcal{B}_T) \mathbb{P}_0$ . Hence we need to find a suitable process  $V_t$ .

Let  $V_t$  be the solution of

$$dV_t = -i(H - i\frac{1}{2}L^*L)V_t dt - LV_t(dA_t^* + dA_t). \quad (5.87)$$

We have  $\mathbb{E}_0((U_t^* - V_t^*)(U_t - V_t)) = 0$  thus this choice is a good one.

For any system operator  $X$ , let

$$\pi_t(X) = \frac{\mathbb{E}_0(V_t^* X V_t | \mathcal{B}_t)}{\mathbb{E}_0(V_t^* V_t | \mathcal{B}_t)}. \quad (5.88)$$

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We have

$$d\mathbb{E}_0(V_t^* X V_t | \mathcal{B}_t) = \mathbb{E}_0(V_t^* V_t | \mathcal{B}_t) \pi_t(\mathcal{L}^*(X)) dt - \mathbb{E}_0(V_t^* V_t | \mathcal{B}_t) \pi_t(L^* X + X L) dB_t. \quad (5.89)$$

Particularly

$$d\mathbb{E}_0(V_t^* V_t | \mathcal{B}_t) = -\mathbb{E}_0(V_t^* V_t | \mathcal{B}_t) \pi_t(L^* + L) dB_t. \quad (5.90)$$

Thus  $\mathbb{E}_0(V_t^* V_t | \mathcal{B}_t)$  is a  $(\mathcal{B}_t)$  exponential martingale under  $\mathbb{P}'_0$ .

Hence using Girsanov theorem we find that under  $\mathbb{P}_T$ ,  $W_t = B_t + \int_0^t \pi_s(L^* + L) ds$  is a Wiener process. Hence the measurement process  $B_t$  under  $\mathbb{P}_T$  is such that

$$B_t = W_t - \int_0^t \pi_s(L^* + L) ds \quad (5.91)$$

with  $W_t$  a Wiener process.

We turn to the computation of the quantum trajectory itself. The expectation of any system observable conditioned on the measurement of  $B_t$  is given by

$$\mathbb{E}_t(X | \mathcal{B}_t) = \frac{\mathbb{E}_0(V_t^* X V_t | \mathcal{B}_t)}{\mathbb{E}_0(V_t^* V_t | \mathcal{B}_t)} \quad (5.92)$$

$$= \pi_t(X) \quad (5.93)$$

under  $\mathbb{P}_t$ .

Using Itô rules we easily compute

$$d\pi_t(X) = \pi_t(\mathcal{L}^*(X)) dt - (\pi_t(L^* X + X L) - \pi_t(X) \pi_t(L^* + L)) dW_t \quad (5.94)$$

with  $\pi_0(X) = \mathbb{E}_0[X]$  and  $W_t$  the previously defined Wiener process under  $\mathbb{P}_t$ .

Let us now come back to the state. Let  $\rho(t)$  be the process of system state such that  $\text{tr}[X\rho(t)] = \mathbb{E}_t(X | \mathcal{B}_t)$  for any operator  $X$  acting on the system Hilbert space. The distribution of  $\text{tr}[X\rho(t)]$  is equal to the distribution of  $\pi_t(X)$ . Thus the SDE giving the distribution of  $\pi_t$  gives the distribution of  $\text{tr}[X\rho(t)]$ . We just have to replace  $\pi_t(X)$  in the previous equation by  $\text{tr}[X\rho(t)]$ . The SDE defining the distribution of  $\rho(t)$  under  $\mathbb{P}_t$  is then

$$d\rho(t) = \mathcal{L}(\rho(t)) dt - (L\rho(t) + \rho(t)L^* - \text{tr}[(L + L^*)\rho(t)]\rho(t)) dW_t. \quad (5.95)$$

Note that if we had used  $U_t^\alpha$  instead of  $U_t$  we would have found the same stochastic equation with just a different Hamiltonian. We have  $dU_t^\alpha = -i(H - i\frac{1}{2}(L^*L + |\alpha|^2 - 2\alpha L^*))U_t^\alpha dt + (L^* - \bar{\alpha})U_t^\alpha dA_t - (L - \alpha)U_t^\alpha dA_t^*$ . Thus a suitable  $V_t^\alpha$  is the solution of  $dV_t^\alpha = -i(H - i\frac{1}{2}L^*L + |\alpha|^2 - 2\alpha L^*)V_t^\alpha dt + (L - \alpha)V_t^\alpha(dA_t^* + dA_t)$ . Repeating the same procedure with these operators, we obtain the same equation with a new Hamiltonian  $H \rightarrow H + i(\alpha L^* - \bar{\alpha}L)$ . The outcome process depends also on  $\alpha$ . We have

$$dB_t^\alpha = dW_t - (\pi_t^\alpha(L + L^*) - (\bar{\alpha} + \alpha)) dt. \quad (5.96)$$

Hence the drift is shifted by the non vacuum coherent state.

This derivation can also be done when  $L$  depends on time. For example for an heterodyne measurement. But has announced at the beginning of the chapter we do not present in details such time dependency.

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### Photon counting

We now deal with the case where photons are individually counted. As we already saw, in the limit of a large number of photons detected per second we recover a diffusive quantum trajectory. Here we assume we can distinguish every photon individually. The term photon can of course be changed to any bosonic particle. The measured observable is the particle number operator  $\Lambda_t$ .

As before we assume the environment bosonic field is in its vacuum state. It will be easier to work not with  $\Lambda_t$  but with a unitarily equivalent observable  $N_t = \Lambda_t + A_t + A_t^* + t = R_t \Lambda_t R_t^*$  with  $R_t = e^{A_t - A_t^*}$ . In chapter 3 we saw that  $N_t = \Lambda_t + A_t^* + A_t + t$  has the distribution of a classical Poisson process with unit intensity. We even saw that actually  $N_t$  is a multiplication operator by a Poisson process. Let  $\mathcal{I}_t$  and  $\mathcal{N}_t$  be respectively the commutative algebras generated by the  $\Lambda_s$ 's and the  $N_s$ 's up to time  $t$ . Let us remark that  $\mathcal{N}_t = R_t \mathcal{I}_t R_t^*$ . The law  $\mathbb{P}_t$  is defined through  $\mathbb{E}_t(Y) = \mathbb{E}_0(U_t^* Y U_t)$  for any  $Y \in \mathcal{I}_t$ . Since we will work with  $\mathcal{N}_t$  and not  $\mathcal{I}_t$ , let us preferably define  $\mathbb{P}'_t$  through  $\mathbb{E}'_t(Y') = \mathbb{E}_0(U_t^* R_t^* Y' R_t U_t)$  for any  $Y' \in \mathcal{N}_t$ . From the unitary equivalence between the two algebras, we have  $\mathbb{E}_t(Y) = \mathbb{E}'_t(R_t Y R_t^*)$  for any operator  $Y$  in  $\mathcal{I}_t$ . Thus any random variable  $Y$  has the same law under  $\mathbb{P}_t$  as  $R_t Y R_t^*$  under  $\mathbb{P}'_t$ . Hence  $\Lambda_t$  has the same law under  $\mathbb{P}_t$  as  $N_t$  under  $\mathbb{P}'_t$ . Hence we can do all the study with  $N_t$  under the law  $\mathbb{P}'_t$  instead of  $\Lambda_t$  under  $\mathbb{P}_t$ .

First let us give the QSDE for  $U'_t = R_t U_t$ .

$$dU'_t = -i(H - i/2(L^* L + I_{\text{sys.}} - 2L))U'_t dt + (L - I_{\text{sys.}})U'_t dA_t^* - (L^* - I_{\text{sys.}})U'_t dA_t \quad (5.97)$$

$U_0 = I$ . A suitable  $V'_t$  is the solution of

$$dV'_t = -i(H - i/2(L^* L - I_{\text{sys.}}))V'_t dt + (L - I_{\text{sys.}})V'_t dN_t \quad (5.98)$$

For any system operator  $X$  let

$$\pi'_t(X) = \frac{\mathbb{E}_0(V_t'^* X V'_t | \mathcal{N}_t)}{\mathbb{E}_0(V_t'^* V'_t | \mathcal{N}_t)}. \quad (5.99)$$

Then

$$d\mathbb{E}_0(V_t'^* X V'_t | \mathcal{N}_t) = \mathbb{E}_0(V_t'^* V'_t | \mathcal{N}_t) \left[ \pi'_t(\mathcal{L}^*(X))dt + (\pi'_t(L^* X L) - \pi'_t(X))[dN_t - dt] \right]. \quad (5.100)$$

Particularly

$$d\mathbb{E}_0(V_t'^* V'_t | \mathcal{N}_t) = \mathbb{E}_0(V_t'^* V'_t | \mathcal{N}_t) (\pi'_t(L^* L) - 1)[dN_t - dt]. \quad (5.101)$$

Thus  $\mathbb{E}_0(V_t'^* V'_t | \mathcal{N}_t)$  is a  $(\mathcal{N}_t)$  exponential martingale under  $\mathbb{P}'_0$ .

Hence using Girsanov theorem once again, we have that

$$N_t - \int_0^t \pi'_s(L^* L) ds \quad (5.102)$$

is a  $(\mathcal{N}_t)$  martingale under  $\mathbb{P}'_T$ . Thus under  $\mathbb{P}'_t$ ,  $N_t$  is a counting process with stochastic intensity  $\pi'_t(L^* L)$ .

We turn now to the evolution of the expectation of any system observable conditioned on the measurement outcomes. Hence to  $\pi'_t(X)$ . Using Itô rules we find it is the solution of

$$d\pi'_t(X) = \pi'_t(\mathcal{L}^*(X))dt + \left( \frac{\pi'_t(L^* X L)}{\pi'_t(L^* L)} - \pi'_t(X) \right) (dN_t - \pi'_t(L^* L)dt) \quad (5.103)$$

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with  $\pi'_0(X) = \mathbb{E}'_0(X)$ . We have  $\pi'_t(L^*L) = \text{tr}[L^*L\rho(t)]$ . Thus the equation for the system state is,

$$d\rho(t) = \mathcal{L}(\rho(t)) + \left( \frac{L^*\rho(t)L}{\text{tr}[L^*L\rho(t)]} - \rho(t) \right) (dN_t - \text{tr}[L^*L\rho(t)]dt). \quad (5.104)$$

We have recovered the jump quantum trajectory.

Note that the martingale change of measure we used here is not guarantied to work when we consider not  $N_t$  but  $\Lambda_t$ . The latter can be seen as a counting process with zero intensity when the environment is in the vacuum. Then the event  $\Lambda_t > 0$  as zero probability 0 under  $\mathbb{P}_0$  whereas it would have a strictly positive probability under the probability measure after the interaction  $\mathbb{P}_t$ . So  $\mathbb{P}_t$  is not absolutely continuous with respect to  $\mathbb{P}_0$ , thus we should not be able to find a martingale to pass from  $\mathbb{P}_0$  to  $\mathbb{P}_t$ . If the environment is not in a vacuum state but in a coherent state with parameter  $\alpha$  this is not an issue. Under the induced probability measure  $\mathbb{P}_\alpha$ ,  $\Lambda_t$  is a Poisson process with intensity  $|\alpha|^2$ . And we have absolute continuity of  $\mathbb{P}_t$  with respect to  $\mathbb{P}_\alpha$ . The impossibility to obtain a positive intensity starting with a null one is a limit of the approach through martingale change of measure. But as we saw we can always find a way around this problem. Here we used  $N_t$  instead of  $\Lambda_t$ .

Both the results we just obtained can be combined adding more noises in the QSDE. Then the general form of quantum trajectories SDE is recovered. One can also check that using quantum filtering on the QSDE corresponding to the repeated interactions continuous limit leads to the quantum trajectory SDE we found in last subsection.

For the present derivation, we assumed the environment was in a pure state. This is not a limitation since we can always find a representation in which the environment state is a pure one. Though, the identification of the observables corresponding to the actual physical measurement might be harder. The only true limitation is that the unitary evolution must correspond to a Markovian reduced one. The derivation and interpretation of non Markovian quantum trajectories describing continuous measurements of a system whose reduced evolution would be non Markovian is still an open question [18, 56, 93].

### Continuous and two time measurements

Before we end this subsection let us stress the importance of the fact that  $U_t^*Z_sU_t = U_s^*Z_sU_s$  for  $s < t$ . Usually in quantum mechanics, the joint distribution of two measurements with observable  $A$  and  $B$  at two different times, respectively  $s$  and  $t$ , is given by  $G(\lambda_A, \lambda_B) = \text{tr}[U_t^*e^{\lambda_A A}U_tU_s^*e^{\lambda_B B}U_s\rho]$ . The sometime used expression  $\text{tr}[e^{\lambda_A U_t^*AU_t + \lambda_B U_s^*BU_s}\rho]$  has no clear physical meaning. But in the Markovian context these two quantities are actually equal. Let  $A$  be the measurement of  $Z$  at time  $t$  and  $B$  the measurement of  $Z$  at time  $u$ . Their joint characteristic distribution is then  $G(\lambda_t, \lambda_s) = \text{tr}[U_t^*e^{\lambda_t Z_t}U_tU_s^*e^{\lambda_s Z_s}U_s\rho]$ . Now since  $U_s^*Z_sU_s = U_t^*Z_sU_t$  and  $[Z_t, Z_s] = 0$ , we have  $G(\lambda_t, \lambda_s) = \text{tr}[e^{\lambda_t U_t^*Z_tU_t + \lambda_s U_s^*Z_sU_s}\rho]$ . Moreover, since  $\text{tr}[e^{\lambda U_t^*Z_tU_t - \lambda U_s^*Z_sU_s}\rho] = \text{tr}[U_t^*e^{\lambda(Z_t - Z_s)}U_t\rho] = \mathbb{E}_t(e^{\lambda(Z_t - Z_s)})$ , the statistics of the two times measurement is given by the statistics of the continuous measurement outcome process increment between the two times.

This is interesting for particle counting measurements.  $\Lambda_t$  count the number of particles detected up to time  $t$ . Formally  $\Lambda_\infty$  is the measurement of the total number of particles. Before any interaction happen we have  $\Lambda_\infty = 0$  since the environment is considered in its ground state. Let us consider a two time measurement of  $\Lambda_\infty$ . At time  $t$  we measure  $U_t^*\Lambda_\infty U_t$

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and at time  $s$ ,  $U_s^* \Lambda_\infty U_s$ . Now, from the factorization property of  $\Lambda_\infty$  and the fact that the interaction up to time  $t$  only acts on  $\mathcal{H}_{\text{sys.}} \otimes \Gamma(L^2([0, t]))$ , we have  $U_t^* \Lambda_\infty U_t = U_t^* \Lambda_t U_t$ . Thus  $\text{tr}[U_t^* e^{\lambda \Lambda_\infty} U_t U_s^* e^{-\lambda \Lambda_\infty} U_s \rho] = \mathbb{E}_t(e^{\lambda(\Lambda_t - \Lambda_s)})$  which is the characteristic function of the increment of the measurement outcome process  $(N_t)$  between time  $s$  and  $t$ . Thus the continuous measurement is equivalent to a two time measurement. As pointed out by D. Bernard and B. Doyon in [30] it should be in this continuous measurement context that quantum fluctuation relations should be envisioned. In experiments we cannot measure the entire environment at once. Experimenters use continuous measurement of currents.

### 5.2.4. Unraveling of master equation

Quantum trajectory SDE's can be also be obtained through the unraveling of master equations [8, 13, 17]. The unraveling of the master equation consists in finding a classical stochastic process preserving the purity of the states such that the mean evolution of this state corresponds to a given master equation. The idea is to say that the master equation found by other means is actually the mean evolution of a natural indirect measurement on the environment. The link between this approach and the dilation of the master equation is clear. On one side we try to find a measurement which would lead in average to the master equation. On the other side we try to find an environment which, once traced out, would lead to the master equation. Using quantum filtering we should be able to go from this environment to the unraveling [13].

This approach can be understood as a more rigorous generalization of the phenomenological approach we presented at the beginning of this section.

In this subsection we just present the unraveling of the master equation, without involving instrument formalism. We neither do the unraveling in the most general context where the operators  $L_j$  and  $H$  only have to be bounded adapted processes with respect to the natural filtration of the system state process. This generality is actually the strength of this approach. Here we only consider operators which are independent of time and of the system state.

#### The linear process

First let us recall the expression of the master equation.

$$\begin{aligned} d\hat{\rho}(t) &= \mathcal{L}(\hat{\rho}(t))dt \\ &= (-i[H, \hat{\rho}(t)] + \sum_j L_j \hat{\rho}(t) L_j^* - \frac{1}{2} \{L_j^* L_j, \hat{\rho}(t)\})dt. \end{aligned} \quad (5.105)$$

Suppose the initial state is pure,  $\rho(0) = |\phi(0)\rangle\langle\phi(0)|$ . We want to find a process  $|\phi(t)\rangle$  on  $\mathcal{H}_{\text{sys.}}$  such that  $\hat{\rho}(t) = \mathbb{E}(|\phi(t)\rangle\langle\phi(t)|)$ . Suppose only two operators  $L_j$  are involved in the Lindbladian  $\mathcal{L}$ ,  $j \in \{1, 2\}$ . Suppose we expect one continuous in time measurement outcome (a current) and one discrete (a counting). Thus let us define a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$  supporting one Wiener process  $(W(t))$  and one Poisson process with unit intensity  $(N(t))$ . These two processes are independent. Let  $(\mathcal{F}_t)$  be their joint natural filtration. We define a process

$$d|\phi(t)\rangle = K|\phi(t)\rangle + D_1|\phi(t)\rangle dW(t) + D_2|\phi(t)\rangle [dN(t) - dt]. \quad (5.106)$$

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with  $K, D_1, D_2$  some linear operators on  $\mathcal{H}_{\text{sys.}}$ . It is the most general linear SDE with time independent coefficients. If one computes the expectation of  $\hat{\rho}(t) = \mathbb{E}_{\mathbb{Q}}(|\phi(t)\rangle\langle\phi(t)|)$ , one gets

$$d\hat{\rho}(t)/dt = K\hat{\rho}(t) + \hat{\rho}(t)K^* + D_1\hat{\rho}(t)D_1^* + D_2\hat{\rho}(t)D_2^*. \quad (5.107)$$

Since the right hand side must be equal to  $\mathcal{L}(\hat{\rho}(t))$ , we shall choose

$$\begin{aligned} K &= -i(H - i\frac{1}{2}(L_1^*L_1 + L_2^*L_2 - 1)) \\ D_1 &= L_1 \\ D_2 &= L_2 - 1. \end{aligned} \quad (5.108)$$

For any  $2 \times 2$  unitary matrix  $u$ ,  $D_1 = u_{11}L_1 + u_{12}L_2$ ,  $D_2 = u_{21}L_1 + u_{22}L_2 - 1$  are suitable  $D_j$  operators. But let us stick with our initial definition.

We have a process with values in  $\mathcal{H}_{\text{sys.}}$  such that the average evolution of the corresponding density matrix is given by the master equation. Still there is an issue. The norm of  $|\phi(t)\rangle$  is not almost surely conserved in time. Thankfully, a change of measure will allow us to define a non linear SDE such that the normalized state stay pure and its average leads to the master equation.

### The non linear SDE

Let  $\sigma(t) = |\phi(t)\rangle\langle\phi(t)|$ . The trace of  $\sigma(t)$  is the solution of the SDE

$$d\text{tr}(\sigma(t)) = \text{tr}[(L_1 + L_1^*)\sigma(t)]dW(t) + \text{tr}[(L_2^*L_2 - 1)\sigma(t)][dN(t) - dt] \quad (5.109)$$

Let  $\rho(t) = \sigma(t)/\text{tr}[\sigma(t)]$  with  $\rho(t) = \tilde{\rho}$  a fixed state whenever  $\text{tr}[\sigma(t)] = 0$ . Then  $\text{tr}[\sigma(t)]$  is the solution of a Doleans–Dade SDE

$$d\text{tr}[\sigma(t)] = \text{tr}[\sigma(t)]dZ(t) \quad (5.110)$$

with

$$dZ(t) = \text{tr}[(L_1 + L_1^*)\rho(t)]dW(t) + (\text{tr}[(L_2^*L_2\rho(t)] - 1)[dN(t) - dt]. \quad (5.111)$$

$Z(t)$  is a  $\mathcal{F}_t$  local martingale under  $\mathbb{Q}$ . The operators  $L_1$  and  $L_2$  are bounded and  $\rho(t)$  is  $\mathbb{Q}$  almost surely bounded too. Thus, up to a time  $T$  arbitrary large,  $\text{tr}[\sigma(t)]$  is a martingale. Therefore we can define  $\mathbb{P}$  the measure such that its Radon-Nikodym derivative with respect to  $\mathbb{Q}$  limited to  $\mathcal{F}_T$  is  $\text{tr}[\sigma(T)]$ . Using Girsanov theorem, we find that under  $\mathbb{P}$ ,  $X_t = W_t - \int_0^t \text{tr}[(L_1 + L_1^*)\rho(s)]ds$  is a Wiener process and  $N_t$  is a counting process with intensity  $\text{tr}[L_2^*L_2\rho(t)]$ .

Let us explicitly write the expectation of  $\sigma(t)$  under  $\mathbb{Q}$ .

$$\mathbb{E}_{\mathbb{Q}}(\sigma(t)) = \int_{\Omega} \sigma(t)(\omega) \mathbb{Q}(d\omega). \quad (5.112)$$

This expectation is equal to the expectation of  $\rho(t)$  under  $\mathbb{P}$ .  $\rho(t)$  is the solution of the SDE

$$\begin{aligned} d\rho(t) &= \mathcal{L}(\rho(t))dt \\ &+ (L_1\rho(t) + \rho(t)L_1^* - \text{tr}[(L_1 + L_1^*)\rho(t)]\rho(t))dX(t) \\ &+ \left( \frac{L_2\rho(t)L_2^*}{\text{tr}[L_2^*L_2\rho(t)]} - \rho(t) \right) [dN(t) - \text{tr}[L_2^*L_2\rho(t)]dt]. \end{aligned} \quad (5.113)$$

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Note that  $\text{tr}[\sigma(t)]$  is almost surely never equal to 0 under  $\mathbb{P}$ . Thus  $\rho(t)$  is almost always the normalized state. We have recovered the quantum trajectory SDE since under  $\mathbb{P}$ ,  $X(t)$  is a Wiener process and  $N(t)$  has intensity  $\text{tr}[L_2^* L_2 \rho(t)]$ .

The differential equation for  $\mathbb{E}_{\mathbb{P}}(\rho(t))$  is the master equation and  $\text{tr}[\rho(t)] = 1$   $\mathbb{P}$  almost surely at any time. Moreover, as we will see in next section, if the initial state is a pure one, then  $\rho(t)$  corresponds also to a pure state  $\mathbb{P}$  almost surely.

Note that we can relax the assumption of an initial pure system state to any system mixed state. Then  $\sigma(0)$  is a convex linear combination of rank one projectors. And following the same steps we obtain the same quantum trajectory SDE whatever is the initial system state.

The liberty in the choice of  $D_1$  and  $D_2$  shows that there is not a unique unraveling for a given master equation. This is coherent with the fact that there is not a unique dilation. This is actually a result of the non unicity of the form of the Lindbladian we discussed in section 2.2. But moreover, once the dilation is done, we can perform different measurements on the environment obtained. Hence we could have chosen only continuous in time measurement. We would have obtained a purely diffusive quantum trajectory. Equivalently we could have chosen a purely jump type quantum trajectory. Thus the suitable unraveling depends on the context. Either one tries to find the best match to an actual continuous measurement experiment or one search for the best quantum trajectory suited for a numerical simulation of the master equation.

We chose to organize the presentation of the quantum trajectory derivations with the underlying physical microscopic model in mind. Other point of views are adopted in the literature [8, 11].

### 5.3. Asymptotic behavior

We have covered all known derivations of quantum trajectories as a mean to model continuous indirect measurements. Now we study the behavior of these quantum trajectories. We will recover the different results we obtained for discrete time quantum trajectories in the continuous case.

#### 5.3.1. Ergodicity

Continuous quantum trajectories are an important tool for numerical simulations of master equations [44]. As we will see in next section they preserve the purity of states. In the context of numerical simulations, dealing with pure states is much more efficient than dealing with density matrices. For a system of dimension  $d$ , one needs to compute  $d$  coordinates for a pure state where as, for a density matrix, one need to compute  $d^2$  matrix elements. The complexity of the algorithm is thus reduced using pure states. The algorithm of computation for the reduced Markovian evolution goes as follows. One simulates not the evolution given by the master equation, but a quantum trajectory unraveling the reduced evolution. One then average over a lot of realizations of such quantum trajectories. Usually since we have the choice in the unraveling, one prefers to use jump quantum trajectories. They are much simpler to compute. This is this idea that lead J. Dalibard *et al.* to the introduction of quantum trajectories [44].

These numerical simulations raise one question. Do we need to simulate several realizations of quantum trajectories or can we just average a single one over time? Hence we are faced

with a question on the equality between the ensemble average and the time average. Do we have

$$\lim_{t \rightarrow \infty} \mathbb{E}(\rho(t)) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \rho(s) ds ? \quad (5.114)$$

We face the same question as in the discrete case. We will answer it not only for jump quantum trajectories but for any continuous time quantum trajectory. Indeed this question is not only interesting for numerical situations but also to understand asymptotic behavior of quantum systems subject to indirect measurements. For example, as in the discrete case, we will use the ergodic properties of quantum trajectories to find an exponential rate of convergence toward a collapse of the wave function in the context of non demolition measurements.

The results of this section were obtained by B. Kümmerer and H. Maassen in [69].

Again, the first remark we need to make is that  $\lim_t \mathbb{E}(\rho(t))$  is not always well defined. For example if all the  $L_j$  operators of the Lindbladian are proportional to the identity, we are left with a unitary evolution and  $\mathbb{E}(\rho(t))$  has a limit if and only if  $\rho(0)$  is an eigenvector of the Hamiltonian. Hence when  $\mathbb{E}(\rho(t))$  is a constant. As in the discrete case we avoid this issue since the ergodic mean of the trajectory will converge with probability one towards one of the invariant states of the mean evolution.

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \rho(s) ds = \rho(\infty) \quad (5.115)$$

almost surely with  $\mathcal{L}(\rho(\infty)) = 0$ . Moreover

$$\mathbb{E}(\rho(\infty)) = P_\Phi(\rho(0)) \quad (5.116)$$

with  $P_\Phi$  the projector on the invariant states of the semigroup  $\Phi_t$  corresponding to the average evolution.

$$P_\Phi(\rho) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{s\mathcal{L}} \rho ds. \quad (5.117)$$

The proof of this convergence is once again obtained through the convergence of two specific martingales. First if we apply  $P_\Phi$  to the integral expression for the trajectory, we remark that the predictable part disappears.

$$P_\Phi \left( \int_0^t \mathcal{L}(\rho(s)) ds \right) = 0. \quad (5.118)$$

One can either compute  $\frac{1}{u} \int_0^u dv \int_0^t ds e^{v\mathcal{L}} \mathcal{L}(\rho(s)) = \int_0^t ds \frac{1}{u} (e^{u\mathcal{L}} - I) \rho(s)$  and take the limit  $u \rightarrow \infty$ . It is 0 since  $e^{u\mathcal{L}}$  preserves the set of states. Or one can express  $P_\Phi$  as a sum of non Hermitian projectors onto the invariant states of  $e^{t\mathcal{L}}$ . We have  $P_\Phi = \sum_k |\rho_k\rangle \langle M_k|$  with  $|\mathcal{L}\rho_k\rangle = 0$  and  $\langle \mathcal{L}^* M_k| = 0$  [103].  $|\rho_k\rangle$  is the vector representation of the invariant state  $\rho_k$ .  $k$  runs from 1 to the number of linearly independent invariant states. The cancellation is then obvious.

Since its predictable part is null,  $P_\Phi(\rho(t))$  is a martingale. It is bounded, so it converges almost surely. Actually its limit is  $\rho(\infty)$ . It remains to prove that  $\frac{1}{t} \int_0^t \rho(s) - P_\Phi(\rho(s)) ds$  converges almost surely to 0. It is a consequence of the second martingale convergence. It is actually a law of large numbers for a specific martingale.

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The martingale part of the integral expression for the trajectory is

$$\sum_{j < p} \int_0^t D_j(\rho(s)) dW_j(s) + \sum_{j \geq p} \int_0^t \left( \frac{J_j(\rho(s))}{v_j(\rho(s))} - \rho(s) \right) [dN_j(s) - v_j(\rho(s)) ds]. \quad (5.119)$$

Let us write this martingale  $M_t$  and its quadratic variation  $[M]_t$ ,

$$[M]_t = \sum_{j < p} \int_0^t D_j(\rho(s))^2 ds + \sum_{j \geq p} \int_0^t \left( \frac{J_j(\rho(s))}{v_j(\rho(s))} - \rho(s) \right)^2 dN_j(s). \quad (5.120)$$

All the integrands in this integral expression of  $[M]_t$  are almost surely bounded and the intensity of the  $N_j(t)$ 's are also almost surely bounded. Hence, we have

$$\int_1^\infty \text{tr} \left[ \frac{1}{t^2} d\mathbb{E}([M]_t) \right] < \infty. \quad (5.121)$$

Thus the martingale law of large numbers [67] implies

$$\lim_{t \rightarrow \infty} M_t/t = 0 \quad (5.122)$$

almost surely.

Since  $\rho(t)$  and  $\rho(0)$  are bounded we have  $\lim_t \frac{1}{t} \int_0^t \mathcal{L}(\rho(s)) ds = 0$ . This result is sufficient to prove  $\lim_t \frac{1}{t} \int_0^t \rho(s) - P_\Phi(\rho(s)) ds = 0$ . For the details we refer the reader to [69]. This conclude the proof of B. Kümmerer and H. Maassen ergodic theorem for continuous quantum trajectories.

Using this result we can prove a law of large numbers on the measurement record processes. Let us recall that the continuous measurement record processes are

$$Y_j(t) = W_j(t) + \int_0^t \text{tr}[(L_j + L_j^*)\rho(s)] ds. \quad (5.123)$$

The discontinuous ones are just the counting processes  $N_j(t)$ . From the Brownian law of large numbers, we have  $\lim_t W_j(t)/t = 0$ . From the ergodic theorem we just presented, we have  $\lim_t \frac{1}{t} \int_0^t \text{tr}[(L_j + L_j^*)\rho(s)] ds = \text{tr}[(L_j + L_j^*)\rho(\infty)]$ . Thus

$$\lim_t \frac{1}{t} \int_0^t Y_j(s) ds = \text{tr}[(L_j + L_j^*)\rho(\infty)]. \quad (5.124)$$

Hence, the limit of the time average drift of the continuous measurement record corresponds to the one expected with a system in an invariant state of  $\Phi_t$ .

Has stated before, the intensities of the processes  $N_j(t)$  are almost surely bounded. It follows that the related compensated Poisson process, which is a martingale, obey to the martingale law of large numbers. So  $\frac{1}{t} \left( N_j(t) - \int_0^t v_j(\rho(s)) ds \right)$  converges almost surely to 0. Thus

$$\lim_t N_j(t)/t = v_j(\rho(\infty)). \quad (5.125)$$

Hence, the limit jump frequencies correspond to the one expected with a system in an invariant state of  $\Phi_t$ .

We will use these laws of large numbers to find the exponential rate of convergence of the system state in the context of non demolition measurements.

### 5.3.2. State purification

As announced in previous subsections a useful property of quantum trajectories is that they preserve almost surely the purity of the system state. Of course if an additional dissipation process takes place, this is no longer true. We concentrate on quantum trajectories which conserve the purity, hence when no part of the environment is left unmeasured.

Maybe the most simple way to see that pure states stay pure almost surely along the trajectory is that we can actually write a stochastic differential equation (SDE) for the pure state  $|\phi_t\rangle$  corresponding to the density matrix  $\rho(t) = |\phi_t\rangle\langle\phi_t|$ . Then we recover the usual quantum trajectory SDE when we compute  $d(|\phi_t\rangle\langle\phi_t|)$  using Itô rules.

A better proof of purity conservation was obtained by A. Barchielli and A. M. Paganoni in [17]. A good indicator of purity is the linear entropy:  $G(\rho) = \text{tr}[\rho(I - \rho)]$ . It is positive and bounded from above by 1. It is equal to 0 if and only if  $\rho$  corresponds to a pure state.

$$G(\rho) = 0 \iff \exists |\phi\rangle \text{ s.t. } \rho = |\phi\rangle\langle\phi|. \quad (5.126)$$

The conservation of purity follows directly from the expression of the mean evolution of  $G(\rho(t))$ ,  $\mathbb{E}(G(\rho(t)))$ .

$$\mathbb{E}(G(t)) = G(0) - \int_0^t \mathbb{E}(y(\rho(s)))ds. \quad (5.127)$$

with

$$y(\rho) = \sum_{j < p} \text{tr} \left[ \left( \sqrt{\rho}(L_j + L_j^* - \text{tr}[(L_j + L_j^*)\rho])\sqrt{\rho} \right)^2 \right] \quad (5.128)$$

$$+ \sum_{j \geq p} v_j(\rho) \text{tr} \left[ \left( \sqrt{\rho} \left( \frac{L_j^* L_j}{v_j(\rho)} - I \right) \sqrt{\rho} \right)^2 \right]. \quad (5.129)$$

Since  $y(\rho)$  is a sum of two positive terms,  $y(\rho(s))$  is almost surely positive. Hence  $\mathbb{E}(G(\rho(t))) \leq G(0)$ . Actually using the conditional expectation with respect to  $\mathcal{F}_s$ , we have  $\mathbb{E}(G(\rho(t))|\mathcal{F}_s) \leq G(\rho(s))$ . If the state is pure at time  $s$ , we have  $G(\rho(s)) = 0$ . Thus  $\mathbb{E}(G(\rho(t))) \leq 0$ . But remember that  $G(\rho)$  is always non negative, so  $G(\rho(s)) = 0 \Rightarrow \mathbb{E}(G(\rho(t))|\mathcal{F}_s) = 0$ . It follows that for any  $t > s$ ,  $G(\rho(s)) = 0 \Rightarrow G(\rho(t)) = 0$  almost surely. Hence once the state is pure it stays pure.

As in the discrete case, we do not only have conservation of the purity but also a tendency to purify mixed states along the quantum trajectory.

The inequality  $\mathbb{E}(G(\rho(t))|\mathcal{F}_s) \leq G(\rho(s))$  informs us that  $G(\rho(t))$  is actually a supermartingale. It is bounded thus it converges. Now the question is whether or not  $G(\rho(t))$  converges to 0. In [17] the authors gave a sufficient criteria for the almost sure purification, hence for the almost sure convergence to zero of the linear entropy.

If it does not exists a, at least, two dimensional projector  $P$  and scalars  $r_j$  and  $\theta_j$  such that

$$\forall j < p, P(L_j + L_j^*)P = r_j P \text{ and } \forall j \geq p, PL_j^* L_j P = \theta_j P, \quad (5.130)$$

then any initial state tends to purify along the trajectory when  $t$  goes to infinity. In other words, if no such projector exists, then  $\lim_t G(\rho(t)) = 0$  almost surely.

The proof is based on the convergence to zero of  $\mathbb{E}(y(\rho(t)))$ . Remember that  $\mathbb{E}(G(\rho(t))) = G(\rho(0)) - \int_0^t \mathbb{E}(y(\rho(s)))ds$ . Since  $G(\rho(t))$  converges almost surely and is bounded, the integral

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over  $s$  in last expression converges too. So  $\lim_t \mathbb{E}(y(\rho(t))) = 0$ . Since  $y(\rho) \geq 0$  for any state, this last convergence means that  $y(\rho(t))$  converges in  $L^1$  norm to 0. But  $y(\rho) = 0$  if and only if

$$\sqrt{\rho}(L_j + L_j^* - \text{tr}[(L_j + L_j^*)\rho])\sqrt{\rho} = 0, \forall j < p \quad (5.131)$$

$$\text{and } \sqrt{\rho} \left( \frac{L_j^* L_j}{v_j(\rho)} - I \right) \sqrt{\rho} = 0, \forall j \geq p. \quad (5.132)$$

This is possible only if it exists a projector  $P$  and reals  $r_j, \theta_j$  such that

$$P(L_j + L_j^*)P = r_j P, \forall j < p \quad (5.133)$$

$$\text{and } PL_j^* L_j P = \theta_j P, \forall j \geq p. \quad (5.134)$$

If the condition on the projectors is true, this is true only for rank one projectors. Hence  $y(\rho) = 0$  only for pure states  $\rho$ . Thus we have a convergence in  $L^1$  norm towards pure states. We would like an almost sure convergence.

Since  $y(\rho(t))$  converges to zero in  $L^1$  norm, it exists a divergent subsequence of times  $(t_n)$  such that  $y(\rho(t_n))$  converges almost surely to 0. In [17], the authors show an implication between the almost sure convergence of  $y(\rho(t_n))$  to zero and the almost sure convergence of  $G(\rho(t_n))$  to zero. For the details we refer the reader to this article. From this almost sure convergence and the unicity of the limit we obtain  $\lim_t G(\rho(t)) = 0$  almost surely.

In next section on non demolition measurements we will discuss a typical case where the condition on the non existence of a two dimensional projectors is violated. The non purity in the limit is then obvious.

### 5.4. Non demolition indirect measurements and wave function collapse

In this section and the following section we translate the results of sections 4.3 and 4.4. In this first one we discuss continuous quantum trajectories corresponding to non demolition measurements. As explain in the section on discrete time non demolition measurements, their aim is to measure a system with minimal perturbation. Avoiding the direct measurement of the system can also protect its integrity. A good example of this protection is of course the photon number measurement performed by S. Haroche's group [61]. As in the discrete case we will give the conditions under which the quantum trajectory equation respects the preservation of the pointer states. It is not a surprise that this condition imposes that  $H$  and all the  $L_j$  operators need to be diagonal in the same basis. Under a non degeneracy condition, we show that the state converges almost surely to one of the pointer states with a distribution of the limit pointer states equal to the one expected for a direct measurement of the pointers. We, once again, reproduce in the long time limit the behavior corresponding to a projective von Neumann measurement. This result is not completely new. Actually, the stochastic Schrödinger equation emerged in attempts to find a stochastic equation leading to wave function collapse [49, 58]. In [2, 98], wave function collapse for different quantum diffusive trajectories was shown.

During this thesis, in [20] with M. Bauer and D. Bernard and then using similar techniques in [29] with C. Pellegrini we generalized these results to any kind of quantum trajectories. The fact that these same evolution equations arise as a special case of quantum trajectories

#### 5.4. Non demolition indirect measurements and wave function collapse

shows that they indeed explain wave function collapse. But we still need to postulate Born's rule. It is applied to the environment in the derivation of quantum trajectories. We recall that in the discrete indirect measurement setting, we saw in chapter 4 that only Born's rule is necessary to obtain the state update knowing the result of the indirect measurement. The same is true for our derivations of continuous quantum trajectories. Hence the collapse we obtain is a direct consequence of Born's rule, not von Neumann postulate.

In these articles we also showed that the convergence is exponential and gave the explicit rate of convergence. We showed stability of the estimated state. If one starts the computation of the state evolution with a state different from the physical state and use the actual outcomes to update it, the estimated state will collapse on the same state as the physical one almost surely. In this section we present these new results.

Even if this might seem redundant with the discrete case, the continuous case applies to different physical situations and is often used as an approximation of repeated indirect measurements when the computation of the unitary  $U$  becomes too difficult. Hence knowing the non degeneracy criteria and the convergence rate for continuous non demolition measurement is of interest. We should stress that we do not aim at the same kind of generality as in the discrete case. For example we do not discuss the introduction of a feedback. Hence we do not treat the continuous approximation of a non i.i.d. choice of measurement methods. This would need a general description of non Markovian quantum trajectory. A lot of work is done nowadays on this subject [18, 56, 93] but the question is beyond the subject of this thesis.

As in the discrete case, even if this results may seem redundant with the results of previous section, they concern actually a much more specific physically interesting case for which much stronger results can be obtained. Once again we will introduce the concept of pointer states which are directly related to experiments. Moreover the exponential convergence and the stability of the estimated state are both new results.

The section is structured as follow. In a first subsection we give a necessary and sufficient condition for a quantum trajectory SDE to be non demolition. We also prove that this is equivalent to the pointer probability processes being martingales. In a second subsection we present the non degeneracy condition and show the collapse. In a third subsection we prove the exponential collapse and give the expression of the rate. In a fourth subsection we evaluate the mean convergence time. In a fifth section we discuss the possibility of a collapse in a finite time. Finally in a last subsection we show the stability of the estimated state.

##### 5.4.1. Non demolition condition

A quantum trajectory SDE corresponds to a non demolition measurement if and only if it preserves a set of states, the pointer states  $\mathcal{P}$ . This set forms an orthonormal basis of  $\mathcal{H}_{\text{sys.}}$ . The stability means that if at time  $s$  the system state is one of the pointers, then, with probability one, it will be in this pointer state at any later time. This non demolition condition is fulfilled if and only if the Hamiltonian  $H$  and all the operators  $L_j$  are diagonal in the basis of the pointers.

$$\begin{aligned} \forall \alpha \in \mathcal{P}, \forall t \geq s, \rho(s) = |\alpha\rangle\langle\alpha| \Rightarrow \rho(t) = |\alpha\rangle\langle\alpha| \text{ a.s.} \\ \Updownarrow \\ H = \sum_{\alpha \in \mathcal{P}} E(\alpha) |\alpha\rangle\langle\alpha|, \forall j, L_j = \sum_{\alpha \in \mathcal{P}} l(j|\alpha) |\alpha\rangle\langle\alpha|. \end{aligned} \tag{5.135}$$

If the former is true, then the invariance of the pointer state is also true in mean:  $\mathbb{E}(\rho(t)) = |\alpha\rangle\langle\alpha|$  which implies  $\mathcal{L}(|\alpha\rangle\langle\alpha|) = 0$ . From this equality looking at the different matrix elements

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of  $\mathcal{L}(|\alpha\rangle\langle\alpha|)$  we obtain that  $H$  and the  $L_j$  are diagonal in  $\mathcal{P}$ .

If the latter is true, then the pointers are invariant under the mean (linear) evolution. Thus if at time  $s$   $\rho(s) = |\alpha\rangle\langle\alpha|$ , at any later time  $t$ ,  $\mathbb{E}(1 - \text{tr}[\rho(t)|\alpha\rangle\langle\alpha|]|\mathcal{F}_s) = 0$  almost surely. Since  $1 - \text{tr}[\rho(t)|\alpha\rangle\langle\alpha|] \geq 0$ , it means  $\text{tr}[\rho(t)|\alpha\rangle\langle\alpha|] = 1$  almost surely. Hence  $\rho(t) = |\alpha\rangle\langle\alpha|$  almost surely.

The fact that the pointer states are eigenvectors of  $H$  restrict the possibility of non demolition measurements to the measurement of effective energy state. It is the duty of the experimenter to built a context in which the states he wants to measure are eigenstates of the effective Hamiltonian. This effective Hamiltonian is usually different from the free one since the presence of an environment often modifies the effective unitary evolution (Lamb shift for example).

The non demolition condition is also equivalent to the processes  $q_\alpha(t) = \text{tr}[\rho(t)|\alpha\rangle\langle\alpha|]$  being martingales.

Assume our quantum trajectory SDE fulfills the non demolition condition for  $\mathcal{P}$ . Then the pointers are invariant states of the average evolution:  $\mathcal{L}(|\alpha\rangle\langle\alpha|) = 0$ . The rank one projector corresponding to the pointer state is also invariant under the dual average evolution:  $\mathcal{L}^*(|\alpha\rangle\langle\alpha|) = 0$ . Thus the drift part of the SDE for  $q_\alpha(t)$  is null and  $q_\alpha(t)$  is a martingale.

The converse is based on the same argument. Since  $q_\alpha(t)$  is a martingale, the drift part of its SDE must be null for any state. Hence  $\mathcal{L}^*(|\alpha\rangle\langle\alpha|) = 0$ . It follows that  $H$  and all the  $L_j$  must be diagonal. So the quantum trajectory SDE is non demolition for  $\mathcal{P}$ .

As one can guess this martingale property will turn useful when showing wave function collapse and studying the mean convergence time.

Now that we know that  $q_\alpha(t)$  is a martingale, let us define new quantities which simplify its study.

$$r(j|\alpha) = l(j|\alpha) + \overline{l(j|\alpha)}, \forall j < p \quad (5.136)$$

$$\theta(j|\alpha) = |l(j|\alpha)|^2, \forall j \geq p. \quad (5.137)$$

Then  $q_\alpha(t)$  SDE is

$$dq_\alpha(t) = q_\alpha(t) \left[ \sum_{j < p} (r(j|\alpha) - \langle r(j) \rangle_t) dW_j(t) + \sum_{j \geq p} \left( \frac{\theta(j|\alpha)}{\langle \theta(j) \rangle_t} - 1 \right) [dN_j(t) - \langle \theta(j) \rangle_t dt] \right] \quad (5.138)$$

with  $\langle f(j) \rangle_t = \sum_{\beta \in \mathcal{P}} q_\beta(t) f(j|\beta)$ . These  $r(j|\alpha)$  and  $\theta(j|\alpha)$  are the eigenvalues of respectively the operators  $L_j + L_j^*$  and  $L_j^* L_j$ . Hence  $\langle r(j) \rangle_t = \text{tr}[(L_j + L_j^*)\rho(t)]$  and  $\langle \theta(j) \rangle_t = \text{tr}[L_j^* L_j \rho(t)]$ . We have obtained a SDE similar to the one we found when computing the continuous limit of the discrete time equivalents of  $q_\alpha(t)$ .

### 5.4.2. Wave function collapse

Since the  $(q_\alpha(t))$  are bounded martingales, they converge almost surely. We have a non degenerate measurement when one and only one  $q_\alpha(\infty)$  is equal to one and the others are null.

Let us assume that for any couple of different pointer  $\alpha, \beta$ , there is at least one  $j$  such that

$$\begin{aligned} r(j|\alpha) &\neq r(j|\beta) \text{ if } j < p \\ \theta(j|\alpha) &\neq \theta(j|\beta) \text{ if } j \geq p. \end{aligned} \quad (5.139)$$

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This is our non degeneracy condition. If this condition is fulfilled, then the system state converges with probability one towards one of the pointer states, and the distribution of this limit state correspond to that of the projection postulate.

If the non degeneracy condition is fulfilled,

$$\begin{aligned} \lim_{t \rightarrow \infty} \rho(t) &= |\Upsilon\rangle\langle\Upsilon| \text{ almost surely,} \\ \mathbb{P}(\Upsilon = \alpha) &= q_\alpha(0). \end{aligned} \quad (5.140)$$

$\Upsilon$  is a random variable on the set of pointer states. This result was obtained in [20] for diffusive trajectories and extended to any trajectory in [29]. It corresponds to a generalization of previous results [2, 98].

The convergence of the state is once again obtained through the convergence of the pointer state probabilities  $q_\alpha(t)$ . These probabilities converge almost surely and are bounded. From Lebesgue convergence theorem, we have that  $\mathbb{E}(q_\alpha(t)^2)$  converges. Hence using Itô-Levy isometry, we have

$$\begin{aligned} \lim_t \mathbb{E}(q_\alpha(t)^2 (r(j|\alpha) - \langle r(j) \rangle_t)^2) &= 0, j < p \\ \lim_t \mathbb{E}(q_\alpha(t)^2 (\theta(j|\alpha) - \langle \theta(j) \rangle_t)^2) &= 0, j \geq p. \end{aligned} \quad (5.141)$$

We have a  $L^1$  convergence to zero of the quantities inside the expectations since they are positive. Using the almost sure convergence of an extracted subsequence and basic algebra operations we obtain

$$\begin{aligned} q_\alpha(\infty)q_\beta(\infty)(r(j|\alpha) - r(j|\beta)) &= 0, j < p \\ q_\alpha(\infty)q_\beta(\infty)(\theta(j|\alpha) - \theta(j|\beta)) &= 0, j \geq p. \end{aligned} \quad (5.142)$$

The non degeneracy condition implies  $q_\alpha(\infty)q_\beta(\infty) = 0$  hence only one  $q_\alpha(\infty)$  can be strictly greater than zero. Since the sum of these probabilities is one, we have

$$q_\alpha(\infty) = 1_{\Upsilon=\alpha}. \quad (5.143)$$

Moreover  $q_\alpha(t)$  is a martingale, so  $\mathbb{E}(1_{\Upsilon=\alpha}) = q_\alpha(0)$ . Since  $q_\alpha(\cdot) = 1 \iff \rho(\cdot) = |\alpha\rangle\langle\alpha|$  we obtain the expected result.

This convergence is not a completely new result. In next section we present new results on the convergence rate towards the limit pointer state.

##### 5.4.3. Exponential convergence rate

As in the discrete case we can find an exponential convergence rate towards the limit pointer state  $\Upsilon$ . We even have an explicit expression for the convergence rate. The results of this section for diffusive quantum trajectories were proven in [20]. The generalization to any quantum trajectory was made in [29]. The discussion of the mean convergence time is new.

The exponential convergence is found through the study of  $\lim_{t \rightarrow \infty} \ln(q_\alpha(t))/t$ . Of course for this quantity to be well defined we need  $q_\alpha(t)$  to be strictly positive. So for now we assume that for any  $j$  and any  $\alpha$ ,  $\theta(j|\alpha) > 0$ . This guaranties that  $q_\alpha(t) > 0$  almost surely for any time  $t$ . One can check by studying the formal solution of  $q_\alpha(t)$  as a Doleans-Dade exponential

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martingale:

$$\begin{aligned}
q_\alpha(t) &= q_\alpha(0) \\
&\times \exp \left[ \sum_{j < p} \int_0^t (r(j|\alpha) - \langle r(j) \rangle_s) dW_j(s) - \frac{1}{2} \int_0^t (r(j|\alpha) - \langle r(j) \rangle_s)^2 ds \right] \\
&\times \prod_{j \geq p} \prod_{s \leq t} \left( 1 + \left( \frac{\theta(j|\alpha)}{\langle \theta(j) \rangle_s} - 1 \right) \Delta N_j(s) \right) \times \exp \left[ - \int_0^t (\theta(j|\alpha) - \langle \theta(j) \rangle_s) ds \right].
\end{aligned} \tag{5.144}$$

$\Delta N_j(s)$  is equal to 1 at each time where a jump  $j$  occurs and is equal to zero otherwise. With our strict positive assumption on the  $\theta(j|\alpha)$ 's, we never have  $\frac{\theta(j|\alpha)}{\langle \theta(j) \rangle_t} = 0$  hence we can write  $q_\alpha(t)$  as an exponential of a supermartingale.

$$\begin{aligned}
q_\alpha(t) &= q_\alpha(0) \\
&\times \exp \left[ \sum_{j < p} \int_0^t (r(j|\alpha) - \langle r(j) \rangle_s) dW_j(s) - \frac{1}{2} \int_0^t (r(j|\alpha) - \langle r(j) \rangle_s)^2 ds \right. \\
&\times \left. \sum_{j \geq p} \int_0^t \ln \left( \frac{\theta(j|\alpha)}{\langle \theta(j) \rangle_s} \right) dN_j(s) - \int_0^t (\theta(j|\alpha) - \langle \theta(j) \rangle_s) ds \right].
\end{aligned} \tag{5.145}$$

From this last expression of  $q_\alpha(t)$  we can infer what will be our convergence rate. Let us define a rate

$$\sigma(\gamma|\alpha) = \frac{1}{2} \sum_{j < p} (r(j|\alpha) - r(j|\gamma))^2 - \sum_{j \geq p} \theta(j|\gamma) \left( \ln \left[ \frac{\theta(j|\alpha)}{\theta(j|\gamma)} \right] - \frac{\theta(j|\alpha)}{\theta(j|\gamma)} + 1 \right). \tag{5.146}$$

We have  $\sigma(\gamma|\alpha) \geq 0$  with equality if and only if  $\alpha = \gamma$ .

In [20] and [29] we showed that

$$q_\alpha(t) = e^{-t\sigma(\Upsilon|\alpha) + o(t)} \tag{5.147}$$

almost surely.

Here we do not use the same derivation of the rate. We prefer using the ergodic property of quantum trajectory already discussed instead. We will use the derivation of [29] for the study of the mean convergence time.

First let us remark that  $\lim_{t \rightarrow \infty} \ln(q_\alpha(t))/t = \lim_{t \rightarrow \infty} \ln(q_\alpha(t)/q_\Upsilon(t))/t$  since  $\lim_t q_\Upsilon(t)$  is 1 almost surely. From the expression (5.145) of  $q_\alpha(t)$  we immediately find that

$$\begin{aligned}
q_\alpha(t)/q_\Upsilon(t) &= q_\alpha(0)/q_\Upsilon(0) \exp[-t\sigma(\Upsilon|\alpha)] \\
&\times \exp \left[ \sum_{j < p} (r(j|\alpha) - r(j|\Upsilon)) X_{j,\Upsilon}(t) + \sum_{j \geq p} \ln \left( \frac{\theta(j|\alpha)}{\theta(j|\Upsilon)} \right) M_{j,\Upsilon}(t) \right]
\end{aligned} \tag{5.148}$$

with

$$X_{j,\gamma}(t) = W_j(t) - t r(j|\gamma) + \int_0^t \langle r(j) \rangle_s ds \tag{5.149}$$

$$M_{j,\gamma}(t) = N_j(t) - t \theta(j|\gamma). \tag{5.150}$$

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From the ergodic properties of  $N_j(t)$ ,  $W_j(t)$  and  $\rho(t)$  we obtain

$$\begin{aligned}\lim_{t \rightarrow \infty} X_{j,\Upsilon}(t)/t &= 0 \\ \lim_{t \rightarrow \infty} M_{j,\Upsilon}(t)/t &= 0\end{aligned}\tag{5.151}$$

almost surely.

It follows directly that

$$\lim_{t \rightarrow \infty} \ln(q_\alpha(t))/t = \lim_{t \rightarrow \infty} \ln(q_\alpha(t)/q_\Upsilon(t))/t = -\sigma(\Upsilon|\alpha)\tag{5.152}$$

almost surely. Hence we have the the result we expected.

Let us define  $\sigma_{\min}(\gamma) = \min_{\alpha \neq \gamma} \sigma(\gamma|\alpha)$ . Using Cauchy Schwarz inequality we have

$$\|\rho(t) - |\Upsilon\rangle\langle\Upsilon|\|_{\max} = e^{-t\sigma_{\min}(\Upsilon)+o(t)}\tag{5.153}$$

almost surely.

The exponential convergence rate we just obtained is an asymptotic one. As in the discrete case we can expect an early chaotic behavior. Hence, once again, we would like to interpret this exponential convergence in terms of mean convergence time.

##### 5.4.4. Mean convergence time

As for the discrete case, let us define a convergence stopping time  $\tau(\alpha|\Upsilon)$ .

$$\tau(\alpha|\Upsilon) = \inf\{t \geq 0 \text{ s.t. } q_\alpha(t)/q_\Upsilon(t) \leq \epsilon\}.\tag{5.154}$$

Since the convergence rate  $\sigma(\Upsilon|\alpha)$  is valid only in asymptotic time, we cannot evaluate the mean of  $\tau(\alpha|\Upsilon)$  directly under  $\mathbb{P}$ . In the discrete case the mean convergence time depended directly on the limit state. The mean was evaluated under the probability measure conditioned on the limit pointer state. We need to build such conditioned probability measure in the continuous case for the evaluation of the mean convergence time.

The following martingale change of measure is the key point of [29] and was inspired by the similar technique used in [20].

The pointers probabilities are positive bounded martingales. So, using martingale change of measure we can define a family of probability measures

$$d\mathbb{P}_\gamma^t(\omega) = \frac{q_\gamma(t)(\omega)}{q_\gamma(0)} d\mathbb{P}(\omega), t \geq 0\tag{5.155}$$

for all  $\gamma \in \mathcal{P}$  such that  $q_\gamma(0) \neq 0$ . Since  $q_\gamma(t) = \mathbb{E}(q_\gamma(\infty)|\mathcal{F}_t)$ , we can extend these measures to unique measures  $\mathbb{P}_\gamma$  such that

$$d\mathbb{P}_\gamma(\omega) = \frac{q_\gamma(\infty)(\omega)}{q_\gamma(0)} d\mathbb{P}(\omega).\tag{5.156}$$

Using Girsanov transformation we obtain that under  $\mathbb{P}_\gamma$ , the  $X_{j,\gamma}(t)$  and  $M_{j,\gamma}(t)$  are  $\mathcal{F}_t$  martingales. The effect of this change of measure is to change the drift in the continuous measurement processes  $Y_j(t)$  from  $\int_0^t \langle r(j) \rangle_s ds$  to  $t r(j|\gamma)$  and the intensity of the discrete measurement processes  $N_j(t)$  from  $\langle \theta(j) \rangle_t$  to  $\theta(j|\gamma)$ . Hence the measurement processes under  $\mathbb{P}_\gamma$  have laws corresponding to a system state in the pointer state  $\gamma$ .

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Let us remark that

$$d\mathbb{P}(\omega) = \sum_{\beta \in \mathcal{P}, q_\beta(0) > 0} q_\beta(0) d\mathbb{P}(\omega | \Upsilon = \beta) \quad (5.157)$$

since  $\mathbb{P}(\Upsilon = \beta) = q_\beta(0)$ . Multiplying by  $\frac{q_\gamma(\infty)}{q_\gamma(0)}$  on the left and the right, we see that it implies

$$d\mathbb{P}_\gamma(\omega) = d\mathbb{P}(\omega | \Upsilon = \gamma). \quad (5.158)$$

Hence the family of measures we just defined are the measures conditioned on the limit pointer state.

These measure being defined, we turn to the computation of the mean convergence time.

Let us first remark that  $(\ln(q_\alpha(t)/q_\gamma(t)) - \ln(q_\alpha(0)/q_\gamma(0)))$  given from expression (5.148) is a Levy process<sup>2</sup> under  $\mathbb{P}_\gamma$ . We can prove, using Markov inequality, that the expectation of  $\tau(\alpha | \Upsilon)$  under  $\mathbb{P}_\gamma$  is finite<sup>3</sup>. Under these conditions we have (see [50] p.380)

$$\mathbb{E}_\Upsilon(\ln(q_\alpha(\tau)/q_\Upsilon(\tau))) = -\sigma(\Upsilon | \alpha) \mathbb{E}(\tau). \quad (5.159)$$

This equality is a continuous time equivalent of Wald's equality.

Now we need to bound  $\left( \ln \left[ \frac{q_\alpha(\tau)}{q_\Upsilon(\tau)} \right] \right)$ . If the quantum trajectory was a purely diffusive one, we would be almost sure that  $\epsilon$  was reached, so we would have the equality  $\ln \left[ \frac{q_\alpha(\tau)}{q_\Upsilon(\tau)} \right] = \ln(\epsilon)$ . The possible introduction of jumps breaks this equality. But the discontinuities due to the jumps are bounded so we are able to find the desired bound. Let us define  $\Delta_{\min} l(\alpha | \Upsilon) = \min(0, \min_j \ln(\theta(j|\alpha)/\theta(j|\Upsilon))) > -\infty$ . Let us adopt the notation  $\lim_{h \rightarrow 0} f(t-h) = f(t-)$ . This is always well defined for càdlàg processes like  $(\rho(t))$ . We have almost surely for any time  $t$ ,  $\ln \left[ \frac{q_\alpha(t)}{q_\Upsilon(t)} \right] - \ln \left[ \frac{q_\alpha(t-)}{q_\Upsilon(t-)} \right] \geq \Delta_{\min} l(\alpha | \Upsilon)$ . Since  $\ln \left[ \frac{q_\alpha(t-)}{q_\Upsilon(t-)} \right] \geq \ln(\epsilon)$  for an time  $t \leq \tau$ , we have the almost sure bounds  $\ln(\epsilon) \geq \ln \left[ \frac{q_\alpha(\tau)}{q_\Upsilon(\tau)} \right] \geq \ln(\epsilon) + \Delta_{\min} l(\alpha | \Upsilon)$ .

Hence

$$\frac{\ln(q_\alpha(0)/q_\Upsilon(0)) - \ln(\epsilon)}{\sigma(\Upsilon | \alpha)} \leq \mathbb{E}_\Upsilon(\tau(\alpha | \Upsilon)) \leq \frac{\ln(q_\alpha(0)/q_\Upsilon(0)) - \ln(\epsilon) - \Delta_{\min} l(\alpha | \Upsilon)}{\sigma(\Upsilon | \alpha)}. \quad (5.160)$$

The mean convergence time is smaller when the convergence rate increases. Thus we can expect more realizations leading to an early collapse. Of course the smaller the  $\epsilon$  the greater is the mean convergence time. We recover exactly the same behavior as in the discrete case. We do not repeat here the same discussion.

Up until now we have never included the possibility of the equivalent of a i.i.d. choice of probe. Actually, as shown in [20] for diffusive processes and in [78] the introduction of a i.i.d choice of probe increases the number of measurement processes. One measurement process corresponds to one probing method and one related measurement outcome. The effect on the convergence rate is the same as in the discrete case, thus we do not repeat the discussion.

The discussion on the non degenerate cases is also equivalent to the discrete time one. Let us just remark that the non degeneracy condition that there exist two different pointers  $\alpha$  and  $\beta$  such that for any  $j$  either  $r(j|\alpha) = r(j|\beta)$  or  $\theta(j|\alpha) = \theta(j|\beta)$  depending on the value of  $j$  corresponds to cases where the almost sure purification criteria for the trajectory is not matched. Hence we can have  $\lim_t G(\rho(t)) > 0$ . In the non demolition case, any initial state  $p|\alpha\rangle\langle\alpha| + (1-p)|\beta\rangle\langle\beta|$ ,  $0 < p < 1$  is invariant along the trajectory and thus it stays mixed in the long time limit.

2. Its increments are independent, stationary, and it is right continuous.

3. See proof in Appendix A.3.

### 5.4.5. Collapse stopping time

In this subsection we discuss the cases where at least one  $\theta(j|\alpha)$  is null. In this case all our previous discussion fail since infinities arise in the logarithms. This indicates that the collapse can happen in a finite time. Here we make explicit the distribution of this collapse time.

Let us define two family of stopping times.

$$T_j = \min\{t \in \mathbb{R}_+ \text{ s.t. } N_j(t) > 0\}, j \geq p \quad (5.161)$$

$$T_\alpha = \min\{T_j \text{ s.t. } \theta(j|\alpha) = 0\} \quad (5.162)$$

The first family corresponds to the first time jumps  $j$  are recorded. The second corresponds to the first times the corresponding  $q_\alpha(t)$  will be equal to 0. Of course they stay almost surely equal to 0 after this time. We have

$$\mathbb{P}_\Upsilon(T_\alpha \geq t) = 1 - e^{-\lambda(\alpha|\Upsilon)t} \quad (5.163)$$

with  $\lambda(\alpha|\Upsilon) = \sum_{j \text{ s.t. } \theta(j|\alpha)=0} \theta(j|\Upsilon)$ . Hence, under the measure conditioned on the limit pointer, the times at which the other pointers become null are distributed exponentially with a parameter  $\lambda(\alpha|\Upsilon)$ . If ever  $\lambda(\alpha|\Upsilon) = 0$ , then  $T_\alpha = \infty$   $\mathbb{P}_\Upsilon$  almost surely.

Under the unconditioned measure  $\mathbb{P}$ , we have

$$\mathbb{P}(T_\alpha \geq t) = 1 - \sum_{\beta \in \mathcal{P}} q_\beta(0) e^{-\lambda(\alpha|\beta)t}. \quad (5.164)$$

Let us remark that

$$\mathbb{P}(T_\alpha = \infty) \geq \mathbb{P}(\Upsilon = \alpha) \quad (5.165)$$

thus, it can take an infinite time for  $q_\alpha(t)$  to reach zero even if it is not the limit pointer.

### 5.4.6. Estimated state stability

We finish this section with a discussion on the stability of state estimates. Remember that in the discrete case we used the measures  $\mathbb{P}_\alpha$  to find that the estimated state reaches the same limit state as the physical state. Here we do not have access to such  $\mathbb{P}_\alpha$  defined without reference to any initial state besides the pointer states. Our  $\mathbb{P}_\alpha$  are defined relatively to the physical initial state. If the continuous quantum trajectory is not the continuous approximation of a discrete quantum trajectory, we do not have such initial state independent measures. We use the exponential convergence rate we found earlier to prove the stability of the estimated state.

Let us recall that the measurement processes are

$$Y_j(t) = W_j(t) + \int_0^t \text{tr}[(L_j + L_j^*)\rho(s)]ds \quad (5.166)$$

$$N_j(t) \quad (5.167)$$

The evolution of the state  $\rho(t)$  given these processes is

$$d\rho(t) = \mathcal{L}(\rho(t))dt + \sum_{j < p} D_j(\rho(t)) [dY_j(t) - \text{tr}[(L_j + L_j^*)\rho(t)]dt] \quad (5.168)$$

$$+ \sum_{j \geq p} \left( \frac{J_j(\rho(t))}{v_j(\rho(t))} - \rho(t) \right) [dN_j(t) - v_j(\rho(t))dt]. \quad (5.169)$$

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If we start with a estimated state  $\hat{\rho}(0)$ , we will update it according to the SDE

$$d\hat{\rho}(t) = \mathcal{L}(\hat{\rho}(t))dt + \sum_{j < p} D_j(\hat{\rho}(t)) [dY_j(t) - \text{tr}[(L_j + L_j^*)\hat{\rho}(t)]dt] \quad (5.170)$$

$$+ \sum_{j \geq p} \left( \frac{J_j(\hat{\rho}(t))}{v_j(\hat{\rho}(t))} - \hat{\rho}(t) \right) [dN_j(t) - v_j(\hat{\rho}(t))dt]. \quad (5.171)$$

As shown in [29], we have

$$\frac{\hat{q}_\alpha(t)}{\hat{q}_\Upsilon(t)} = \frac{\hat{q}_\alpha(0)}{\hat{q}_\Upsilon(0)} \times \exp \left[ -\sigma(\alpha|\Upsilon)t + \sum_{j < p} (r(j|\alpha) - r(j|\Upsilon))X_{j,\Upsilon}(t) + \sum_{j \geq p} \ln \left( \frac{\theta(j|\alpha)}{\theta(j|\Upsilon)} \right) M_{j,\Upsilon}(t) \right]. \quad (5.172)$$

This is well defined as soon as  $q_\alpha(0) > 0 \Rightarrow \hat{q}_\alpha(0) > 0$ .

Hence, as before, from exponential convergence, we have that for any  $\alpha \neq \Upsilon$ ,  $\hat{q}_\alpha(t)$  converges almost surely to 0.

$$\hat{q}_\alpha(t) = \hat{q}_\Upsilon(t) e^{-t\sigma(\Upsilon|\alpha) + o(t)}. \quad (5.173)$$

Only one pointer,  $\Upsilon$ , is such that  $\hat{q}_\Upsilon(\infty)$  can be greater than zero. From the conservation of the trace of  $\hat{\rho}(t)$ , we have  $\sum_{\alpha \in \mathcal{P}} q_\alpha(\infty) = 1$ . Thus  $\hat{q}_\Upsilon(\infty) = 1$  almost surely and

$$\lim_{t \rightarrow \infty} \hat{\rho}(t) = \lim_{t \rightarrow \infty} \rho(t) = |\Upsilon\rangle\langle\Upsilon| \quad (5.174)$$

almost surely. We have proved the stability of the estimate.

## 5.5. Invariant subspace stabilization

As presented in section 4.4, in the late nineties and early two thousands the idea of engineering the coupling of the system to its environment such that it is autonomously driven to a desired state emerged [41, 47, 71, 87, 88, 99]. Some experimenters have already implemented this idea of reservoir engineering [19, 68, 72, 77, 90]. Actually reservoir engineering is mostly formulated and applied in a continuous time context. So in this section we study continuous time quantum trajectories driving the system state toward an invariant subspace of the system Hilbert space. Once again, the results will be equivalent to the ones obtained for discrete quantum trajectories but they apply in a different context. We recall that in the case where this subspace is one dimensional, it corresponds to a pure state preparation. If the evolution given the state is supported on the subspace is unitary then the limit subspace is a decoherence free subspace.

This section presents preliminary results of an article in preparation in collaboration with B. Cloez, C. Pellegrini and F. Ticozzi. This work was inspired by [94]

The section is structured as follows. First we recall and adapt the definitions of invariant subspace and “driven towards”. Then we recall the results of [96, 97] on the invariance and global asymptotic convergence of a system state undergoing an evolution given by a CP semi-group. From these results we show that the convergence still holds when measurements on the

environment are performed. We obtain a bound on the exponential convergence rate which is always greater or equal to the one obtained for the CP semi-group. The convergence is characterized by two time scales. We conclude with a discussion about these time scales.

### 5.5.1. Invariant and globally asymptotically stable subspaces

Recall that  $\mathcal{H}_{\text{sys.}}$  is our finite dimensional system Hilbert space. We define  $\mathcal{H}_S$  and  $\mathcal{H}_R$  two Hilbert spaces such that  $\mathcal{H}_{\text{sys.}} = \mathcal{H}_S \oplus \mathcal{H}_R$  is an orthogonal decomposition of  $\mathcal{H}_{\text{sys.}}$ .  $\mathcal{H}_S$  is the Hilbert space where we want the state to converge. We can decompose any linear operator  $X$  on  $\mathcal{H}_{\text{sys.}}$  in four blocs:

$$X = \left( \begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array} \right) \quad (5.175)$$

$X_S$  is a linear operator on  $\mathcal{H}_S$ ,  $X_R$  a linear operator on  $\mathcal{H}_R$ ,  $X_P$  a linear operator from  $\mathcal{H}_R$  to  $\mathcal{H}_S$  and  $X_Q$  a linear operator from  $\mathcal{H}_S$  to  $\mathcal{H}_R$ .

Let  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  be the set of states such that  $\rho_R = 0$ , so  $\rho_P = 0$  too.

$$\rho \in \mathcal{I}_S(\mathcal{H}_{\text{sys.}}) \iff \rho = \begin{pmatrix} \rho_S & 0 \\ 0 & 0 \end{pmatrix}. \quad (5.176)$$

Let also  $\mathcal{I}_R(\mathcal{H}_{\text{sys.}})$  be the set of states such that  $\rho_S = 0$ , so  $\rho_P = 0$  too.

$$\rho \in \mathcal{I}_R(\mathcal{H}_{\text{sys.}}) \iff \rho = \begin{pmatrix} 0 & 0 \\ 0 & \rho_R \end{pmatrix}. \quad (5.177)$$

We say that  $\mathcal{H}_S$  is an almost surely invariant subspace of the quantum trajectory process if almost surely,

$$\rho(0) = \begin{pmatrix} \rho_S(0) & 0 \\ 0 & 0 \end{pmatrix} \text{ implies } \rho(t) = \begin{pmatrix} \rho_S(t) & 0 \\ 0 & 0 \end{pmatrix} \quad (5.178)$$

for any time  $t > 0$ . Hence, if we started with a state in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ , the state will stay in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ .

Equivalently we say  $\mathcal{H}_S$  is an invariant subspace in mean if the average system state stays in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  for any time  $t > 0$ . In other words:

$$\rho(0) = \begin{pmatrix} \rho_S(0) & 0 \\ 0 & 0 \end{pmatrix} \text{ implies } \mathbb{E}(\rho(t)) = \begin{pmatrix} \mathbb{E}(\rho_S(t)) & 0 \\ 0 & 0 \end{pmatrix} \quad (5.179)$$

for any time  $t > 0$ . We will see later that, as in the discrete case, the almost sure invariance and invariance in mean are equivalent.

We say that  $\mathcal{H}_S$  is an almost surely globally asymptotically stable (GAS) subspace if with probability one, for any initial system state  $\rho(0)$ ,

$$\lim_{t \rightarrow \infty} \left\| \rho(t) - \begin{pmatrix} \rho_S(t) & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0. \quad (5.180)$$

Hence, in the long time limit the system state is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ .

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As before we also say that  $\mathcal{H}_S$  is GAS in mean if the convergence property is fulfilled by the state average evolution whatever is the initial state  $\rho(0)$ :

$$\lim_{t \rightarrow \infty} \left\| \mathbb{E}(\rho(t)) - \begin{pmatrix} \mathbb{E}(\rho_S(t)) & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0. \quad (5.181)$$

If  $\mathcal{H}_S$  is simply a one dimensional space corresponding to an entangled state, we have just defined the invariance of this state with probability one and in mean. And the convergence towards this state with probability one and in mean.

As in the discrete case, in next subsection, we show that the conditions under which the above properties are true in mean and almost surely are actually the same.

### 5.5.2. Invariance and asymptotic stability in mean

In [96, 97] necessary and sufficient conditions were found such that a subspace  $\mathcal{H}_S$  is invariant and GAS in mean when the system evolution is given by a CP semi-group. Here we recall these results.

Let us write  $\bar{\rho}(t) = \Phi_t(\rho(0)) = e^{t\mathcal{L}}\rho(0)$  with  $\mathcal{L}$  a Lindblad operator such that for any  $\rho$ ,  $\text{tr}[\mathcal{L}(\rho)] = 0$ .  $\bar{\rho}(t)$  is equal to the average evolution when measurements are performed on the environment:  $\bar{\rho}(t) = \mathbb{E}(\rho(t))$ .

Explicitly we recall we have

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_j L_j \rho L_j^* - \frac{1}{2} \{L_j^* L_j, \rho\}. \quad (5.182)$$

#### Invariance

In [96, 97] the authors showed, using the semi-group structure of  $\Phi_t$ , that

$$\mathcal{H}_S \text{ is invariant in mean} \iff L_{j,Q} = 0 \forall j \text{ and } iH_P - \frac{1}{2} \sum_j L_{j,S}^* L_{j,P} = 0. \quad (5.183)$$

As we could have expected,  $\mathcal{H}_S$  is invariant in mean if and only if no transition from states of  $\mathcal{H}_S$  to states of  $\mathcal{H}_R$  is induced by the environment and the Hamiltonian. This is actually a well known condition in the study of reducible CP semi-groups.

Using this decomposition we define two new CP semi-groups which will be useful afterwards. First  $\Phi_{S,t}(\rho_S) = e^{t\mathcal{L}_S}\rho_S$  is a trace preserving CP semi-group acting on the states of  $\mathcal{S}(\mathcal{H}_S)$ . The Lindbladian  $\mathcal{L}_S$  is

$$\mathcal{L}_S(\rho_S) = -i[H_S, \rho_S] + \sum_j L_{j,S} \rho_S L_{j,S}^* - \frac{1}{2} \{L_{j,S}^* L_{j,S}, \rho_S\}. \quad (5.184)$$

If  $\rho$  is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys}})$ , we have

$$\Phi_t(\rho) = \begin{pmatrix} \Phi_{S,t}(\rho_S) & 0 \\ 0 & 0 \end{pmatrix} \quad (5.185)$$

with  $\text{tr}[\Phi_{S,t}(\rho_S)] = \text{tr}[\rho_S] = 1$ . Hence  $\Phi_{S,t}$  corresponds to the effective evolution when  $\rho$  is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys}})$ .

Then we define  $\Phi_{R,t}(\rho_R) = e^{t\mathcal{L}_R}\rho_R$ . The generator  $\mathcal{L}_R$  is

$$\mathcal{L}_R(\rho_R) = -i[H_R, \rho_R] + \sum_j L_{j,R}\rho_R L_{j,R}^* - \frac{1}{2}\{L_{j,P}^* L_{j,P} + L_{j,R}^* L_{j,R}, \rho_R\}. \quad (5.186)$$

Thus  $\text{tr}[\mathcal{L}_R(\rho_R)] = -\text{tr}[\sum_j L_{j,P}^* L_{j,P} \rho_R] \leq 0$ . Hence  $\Phi_{R,t}$  is trace non increasing. It corresponds to the evolution of the  $R$  bloc of the system state. We have

$$\Phi_t(\rho) = \begin{pmatrix} \star & \star \\ \star & \Phi_{R,t}(\rho_R) \end{pmatrix} \quad (5.187)$$

with  $\text{tr}[\Phi_{R,t}(\rho_R)] \leq \text{tr}[\rho_R]$ .

### Asymptotic stability

Let us now turn to the conditions under which  $\mathcal{H}_S$  is GAS in mean. Hence when the mean effect of the measurement is to drive the state into  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ . It is obvious that  $\mathcal{H}_S$  is GAS when  $\Phi_{R,t}$  is strictly trace decreasing. Hence when  $\sum_j L_{j,P}^* L_{j,P}$  is positive definite. Let  $\alpha_0$  be the smallest eigenvalue of  $\sum_j L_{j,P}^* L_{j,P}$ . Since  $\mathcal{H}_R$  is finite dimensional and  $\sum_j L_{j,P}^* L_{j,P}$  is positive definite, we have  $\alpha_0 > 0$  and  $\text{tr}[\mathcal{L}_R(\rho_R)] \leq -\alpha_0 \text{tr}[\rho_R]$  for any  $\rho_R$ . Thus  $\text{tr}[\bar{\rho}_R(t)] \leq e^{-\alpha_0 t} \text{tr}[\rho_R(0)]$  and  $\text{tr}[\bar{\rho}_R(t)]$  converges to 0 when  $t$  tends to infinity. Thanks to Cauchy Schwarz inequality on positive semi definite matrices it implies  $\bar{\rho}(t)$  converges to the subset  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  in the long time limit. Moreover we can characterize the convergence as exponential with a rate  $\alpha_0/2$  for the  $\bar{\rho}_P(t)$  bloc and  $\alpha_0$  for the  $\bar{\rho}_R(t)$  bloc.

As in the discrete case, it is not the only case where  $\mathcal{H}_S$  is GAS. In [96, 97] the authors proved using La Salle's invariance principle that  $\mathcal{H}_S$  is GAS if and only if there is no invariant subspace included in the kernel of  $\sum_j L_{j,P}^* L_{j,P}$ . If such a subspace existed the system could be stuck in it. Given its importance, let us make this necessary and sufficient condition more explicit again. Let  $\mathcal{H}_{R_0}$  be a subspace of  $\sum_j L_{j,P}^* L_{j,P}$  kernel. We write  $\mathcal{H}_{R_1}$  its orthogonal complement in  $\mathcal{H}_R$ . We can decompose the operators  $L_{j,R}$  and  $L_{j,P}$  with respect to the direct sum decomposition  $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_{R_1} \oplus \mathcal{H}_{R_0}$ . We have

$$H_R = \begin{pmatrix} H_{R_1} & H_{R_{01}}^* \\ H_{R_{01}} & H_{R_0} \end{pmatrix} \quad (5.188)$$

$$L_{j,R} = \begin{pmatrix} L_{j,R_1} & L_{j,R_{10}} \\ L_{j,R_{01}} & L_{j,R_0} \end{pmatrix} \quad (5.189)$$

$$L_{j,P} = \begin{pmatrix} L_{j,P_1} & 0 \end{pmatrix}. \quad (5.190)$$

Then  $\mathcal{H}_S$  is invariant if and only if we cannot find any  $\mathcal{H}_{R_0}$  such that  $\forall j, L_{j,R_{10}} = 0$  and  $-iH_{R_{01}} - \frac{1}{2}\sum_j L_{j,R_{01}}^* L_{j,R_1} = 0$ .

As in the discrete case we can check the convergence to  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  directly from the shape of the operators  $L_j$  and  $H$ . But once again, the exponential convergence is not obvious since we can have  $\text{tr}[\sum_j L_{j,P}^* L_{j,P} \rho_R] = 0$ .

### 5.5.3. Almost sure invariance and asymptotic stability

Let us now turn to the equivalence between the almost sure and in mean invariance and asymptotic stability. We show that the invariance and GAS of  $\mathcal{H}_S$  do not depend on the measurements performed on the environment.

## 5. Continuous indirect measurements

Let  $P_S$  and  $P_R$  be the projectors onto respectively  $\mathcal{H}_S$  and  $\mathcal{H}_R$ .

$$P_S = \begin{pmatrix} I_S & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & I_R \end{pmatrix}. \quad (5.191)$$

We still have that  $\rho \in \mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  is equivalent to  $\text{tr}[P_R \rho] = 0$  or  $\text{tr}[P_S \rho] = 1$ . We define  $V(\rho) = \text{tr}[\rho P_R]$ .

$$\rho \in \mathcal{I}_S(\mathcal{H}_{\text{sys.}}) \iff V(\rho) = 0. \quad (5.192)$$

The function  $V$  is strictly positive outside of  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$  and null on  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ . Thus  $V$  is a good Lyapunov candidate function for our convergence problem. Since  $dV(\bar{\rho}(t))/dt \leq 0$ , it is a good Lyapunov function for the mean evolution.

### Invariance

Suppose  $\mathcal{H}_S$  is invariant in mean. From the result of the previous subsection we have that if initially the state is in  $\mathcal{I}_S(\mathcal{H}_{\text{sys.}})$ , then  $\bar{\rho}(t)$  is in this same set of states. Hence if  $V(\rho(0)) = 0$ , we have for any time  $t \geq 0$ ,  $V(\bar{\rho}(t)) = 0$ . Since  $\mathcal{H}_{\text{sys.}}$  is finite dimensional and  $V$  linear, for any time  $t$ ,  $\mathbb{E}(V(\rho(t))) = 0$ . Moreover we have  $V(\rho) \geq 0$  for any state, thus if  $\rho(0) \in \mathcal{I}_S(\mathcal{H})$ , almost surely for any time  $t$   $V(\rho(t)) = 0$ . Hence  $\rho(t) \in \mathcal{I}_S(\mathcal{H})$  almost surely which proves the almost sure invariance of  $\mathcal{H}_S$ . The converse implication is trivial.

$$\mathcal{H}_S \text{ invariant in mean} \Leftrightarrow \mathcal{H}_S \text{ almost surely invariant.} \quad (5.193)$$

### Global asymptotic stability

We now discuss the global asymptotic stability of  $\mathcal{H}_S$ . As for the discrete case, we first show the GAS of  $\mathcal{H}_S$  in  $L^1$  norm and then turn to the almost sure GAS.

Suppose  $\mathcal{H}_S$  is GAS in mean. Then  $\lim_{t \rightarrow \infty} V(\bar{\rho}(t)) = 0$ . Once again we equivalently have  $\lim_{t \rightarrow \infty} \mathbb{E}(V(\rho(t))) = 0$ . Since  $V(\rho) \geq 0$  we immediately have the  $L^1$  norm convergence of  $V(\rho(t))$  towards 0.

For the almost sure convergence we use the supermartingale convergence theorem [80]. We then follow the same path as in the discrete case. Using dominated convergence theorem, the converse is trivial and we finally have,

$$\mathcal{H}_S \text{ GAS in mean} \Leftrightarrow \mathcal{H}_S \text{ almost surely GAS.} \quad (5.194)$$

As for the discrete case, in next section we show that even if we have this equivalence, the measurements on the environment can greatly alter the behavior of the system state. This time the interpretation in terms of an added non demolition will be more evident.

#### 5.5.4. Exponential convergence

##### Mean rate

As in the discrete case, we need a new Lyapunov candidate function. As before any  $V_K(\rho) = \text{tr}[K\rho]$  with  $K$  a positive semi definite operator with  $\mathcal{H}_R$  as its support is a suitable Lyapunov candidate function ( $\mathcal{H}_S$  is the kernel of  $K$ ). We have

$$\rho \in \mathcal{I}_S(\mathcal{H}) \iff V_K(\rho) = 0. \quad (5.195)$$

### 5.5. Invariant subspace stabilization

We write  $K$  either for the positive semi definite operator on  $\mathcal{H}_{\text{sys}}$  or the positive definite operator on  $\mathcal{H}_R$ .

Let  $-\alpha_0 = \max\{\text{Re}(\alpha)/\alpha \in \text{spec}(\mathcal{L})\}$ . Then the spectral radius of  $\Phi_{R,t}$  is  $e^{-\alpha_0 t}$  for any  $t \geq 0$ .

If  $\Phi_{R,t}$  is irreducible the existence of a  $K$  such that  $V_K(\Phi_t(\rho)) \leq e^{-\alpha_0 t} V_K(\rho)$  follows directly from Perron Frobenius theorem for irreducible positive maps [54].

If  $\Phi_{R,t}$  is reducible we can, once again, follow a demonstration of the existence of a semi definite eigenoperator  $K'$  of  $\mathcal{L}_R^*$  with eigenvalue equal to  $-\alpha_0$ . We can always find a  $K$  corresponding to a convergence rate as close as we want from  $\alpha_0$ .

$$\forall \epsilon > 0, \exists K_\epsilon \in \mathcal{B}(\mathcal{H}_R), K_\epsilon > 0 \text{ s.t. } \mathcal{L}_R(K_\epsilon) \leq -(\alpha_0 - \epsilon)K_\epsilon. \quad (5.196)$$

We reproduce here the proof for convenience. It is slightly different but follows the same idea as in the discrete case.

Let  $\mathcal{L}_\eta = \mathcal{L}_R + \eta\Psi$  with  $\eta > 0$  and  $\Psi$  an irreducible CP map.  $e^{t\mathcal{L}_\eta}$  is a semi-group of irreducible CP maps. The spectrum of  $\mathcal{L}_\eta$  converges continuously to the one of  $\mathcal{L}_R$  when  $\eta$  goes continuously to 0. Let  $-\alpha_\eta = \max\{\text{Re}(\lambda)/\lambda \in \text{spec}(\mathcal{L}_\eta)\}$ . For any  $\epsilon > 0$  we can find a  $\eta_\epsilon > 0$  such that  $\alpha_{\eta_\epsilon} \geq \alpha_0 - \epsilon$ . We apply Perron-Frobenius theorem for irreducible CP maps. It exists a  $K_\epsilon > 0$  such that  $\mathcal{L}_{\eta_\epsilon}^*(K_\epsilon) = -\alpha_{\eta_\epsilon} K_\epsilon \leq -(\alpha_0 - \epsilon)K_\epsilon$ . It follows  $\mathcal{L}_R^*(K_\epsilon) \leq -(\alpha_0 - \epsilon)K_\epsilon - \eta_\epsilon \Psi^*(K_\epsilon)$ . Since  $\Psi^*$  is a CP map,  $\Psi^*(K_\epsilon) > 0$  and we have  $\mathcal{L}_R^*(K_\epsilon) \leq -(\alpha_0 - \epsilon)K_\epsilon$  which is what we wanted. Of course we need  $\alpha_0 - \epsilon > 0$ . It is always possible to find such a  $\epsilon$ . For example  $\epsilon = \alpha_0/2$  match the requirements.

Finally which ever is the reducibility status of  $\Phi_{R,t}$ , for any strictly positive  $\epsilon$  it always exists a  $K > 0$  such that  $V_K(\Phi_t(\rho)) = \text{tr}[\Phi_t^*(K)\rho] = \text{tr}[\Phi_{R,t}^*(K)\rho_R] \leq e^{-(\alpha_0 - \epsilon)t} V_K(\rho)$ .

Hence in any case, for all  $\epsilon > 0$  it exists  $K$  such that

$$V_K(\bar{\rho}(t)) \leq e^{-(\alpha_0 - \epsilon)t} V_K(\rho(0)). \quad (5.197)$$

It implies that  $V_K(\bar{\rho}(t)) = O(e^{-(\alpha_0 - \epsilon)t})$  with  $O$  the usual asymptotic comparison when  $t$  goes to infinity.

We translate this exponential convergence to an exponential convergence of  $\bar{\rho}_R(t)$  norm at the end of this section.

#### Almost sure rate

First we recover the mean rate. Again we remark that the process  $(V_K(\rho(t))e^{(\alpha_0 - \epsilon)t})_{t \in \mathbb{R}_+}$  is a positive supermartingale. For any  $t \geq s$ ,

$$\mathbb{E}(V_K(\rho(t))e^{(\alpha_0 - \epsilon)t} | \mathcal{F}_s) = V_K(\Phi_{t-s}\rho(s))e^{(\alpha_0 - \epsilon)t} \leq V_K(\rho(s))e^{(\alpha_0 - \epsilon)s}. \quad (5.198)$$

Hence  $V_K(\rho(t))e^{(\alpha_0 - \epsilon)t}$  converges almost surely by Doob supermartingale convergence theorem. Thus  $V_K(\rho(t)) = O(e^{-(\alpha_0 - \epsilon)t})$  with  $O$  an almost sure asymptotic comparison when  $t$  tends to infinity.

We recovered the mean convergence rate but in the almost sure sense.

As in the discrete case we now improve this convergence rate.

## 5. Continuous indirect measurements

### Rate improvement

From now on we fix  $\epsilon$  and the corresponding  $K$ . Accordingly we redefine  $\alpha_0$  as  $-\alpha_0 = \max\{\operatorname{Re}(\lambda)/\lambda \in \operatorname{spec}(\mathcal{L}_R)\} + \epsilon$ . Hence  $\mathcal{L}_R^*(K) \leq -\alpha_0 K$ .

Once again we search for an upper bound for  $\ln(V_K(\rho(t)))/t$  when  $t$  goes to infinity.

To avoid infinities, let us assume  $L_j^* L_j$  and  $L_{j,R}^* L_{j,R}$  are definite positive for any  $j$  and  $V_K(\rho(t)) > 0$  almost surely at any time  $t$ . Hence the limit is reached only in infinite time. Thus at any time  $\ln(V_K(\rho(t)))$  is well defined. These assumptions are the equivalent of the ones we made in the non demolition case for the derivation of the convergence rate.

We define two new processes  $\hat{\rho}_S(t) = \rho_S(t)/\operatorname{tr}[\rho_S(t)]$  which is well defined at least after a fixed bounded time  $t_0$  and  $\hat{\rho}_K(t) = K^{1/2} \rho_R(t) K^{1/2} / \operatorname{tr}[K \rho_R(t)]$  which is always well defined since we assumed  $V_K(\rho(t)) > 0$  almost surely.  $(\hat{\rho}_S(t))_{t \in \mathbb{R}_+}$  and  $(\hat{\rho}_K(t))_{t \in \mathbb{R}_+}$  are processes of states on respectively  $\mathcal{H}_S$  and  $\mathcal{H}_R$ . We also define new operators  $L_{j,K} = K^{1/2} L_{j,R} K^{-1/2}$  and the corresponding  $\mathcal{L}_K(\rho_K) = K^{\frac{1}{2}} \mathcal{L}_R(K^{-\frac{1}{2}} \rho_K K^{-\frac{1}{2}}) K^{\frac{1}{2}}$ . So we have  $\mathcal{L}_K^*(I_R) \leq -\alpha_0 I_R$  and  $\operatorname{tr}[\mathcal{L}_K(\hat{\rho}_K)] \leq -\alpha_0$ .

We write  $\ln(V_K(\rho(t)))$  as a sum of a predictable process and a martingale using Doob–Meyer decomposition,

$$\begin{aligned} \ln(V_K(\rho(t))) &= \ln(V_K(\rho(0))) + \int_0^t \operatorname{tr}[\mathcal{L}_K(\hat{\rho}_K(s))] ds \\ &\quad - \frac{1}{2} \sum_{j < p} \int_0^t (\operatorname{tr}[(L_{j,K} + L_{j,K}^*) \hat{\rho}_K(s)] - \operatorname{tr}[(L_j + L_j^*) \rho(s)])^2 ds \\ &\quad + \sum_{j \geq p} \int_0^t \operatorname{tr}[L_j^* L_j \rho(s)] \left( \ln \left[ \frac{\operatorname{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K(s)]}{\operatorname{tr}[L_j^* L_j \rho(s)]} \right] + 1 - \frac{\operatorname{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K(s)]}{\operatorname{tr}[L_j^* L_j \rho(s)]} \right) ds \\ &\quad + \mathcal{M}(t) \end{aligned} \tag{5.199}$$

with  $\mathcal{M}(t)$  a martingale. Using martingale law of large numbers, under the assumptions we gave at the beginning of this section, we once again have  $\lim_{t \rightarrow \infty} \mathcal{M}(t)/t = 0$ . We also have  $\operatorname{tr}[(L_j + L_j^*) \rho(t)] = \operatorname{tr}[(L_{j,S} + L_{j,S}^*) \hat{\rho}_S(t)] + O(\sqrt{V_K(\rho(t))})$  and  $\operatorname{tr}[L_j^* L_j \rho(t)] = \operatorname{tr}[L_{j,S}^* L_{j,S} \hat{\rho}_S(t)] + O(\sqrt{V_K(\rho(t))})$  as  $t$  tends to infinity.

Since  $V_K(\rho(t))$  is bounded and  $V_K(\rho(t)) = O(e^{-\alpha_0 t})$ ,  $\int_0^\infty \sqrt{V_K(\rho(t))} dt < \infty$  almost surely. Thus we have,

$$\begin{aligned} \ln(V_K(\rho(t)))/t &\leq -\alpha_0 \\ &\quad - \frac{1}{2} \sum_{j < p} \frac{1}{t} \int_0^t (\operatorname{tr}[(L_{j,K} + L_{j,K}^*) \hat{\rho}_K(s)] - \operatorname{tr}[(L_{j,S} + L_{j,S}^*) \hat{\rho}_S(s)])^2 ds \\ &\quad + \sum_{j \geq p} \frac{1}{t} \int_0^t \operatorname{tr}[L_{j,S}^* L_{j,S} \hat{\rho}_S(s)] \left( \ln \left[ \frac{\operatorname{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K(s)]}{\operatorname{tr}[L_{j,S}^* L_{j,S} \hat{\rho}_S(s)]} \right] + 1 - \frac{\operatorname{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K(s)]}{\operatorname{tr}[L_{j,S}^* L_{j,S} \hat{\rho}_S(s)]} \right) ds \\ &\quad + o(1) \end{aligned} \tag{5.200}$$

almost surely as  $t$  tends to infinity.

### 5.5. Invariant subspace stabilization

We have recovered (painfully) our previous rate  $\alpha_0$ . But we also have two more negative terms. Under some conditions similar to the QND measurements non degeneracy ones we obtain an improved rate.

For each  $j$ , let us define two intervals of  $\mathbb{R}$ ,  $\mathcal{R}_{j,R/S} = [\min \text{spec}(L_{j,R/S} + L_{j,R/S}^*), \max \text{spec}(L_{j,R/S} + L_{j,R/S}^*)]$  and two intervals of  $\mathbb{R}_+$ ,  $\Theta_{j,R/S} = [\min \text{spec}(L_{j,R/S}^* L_{j,R/S}), \max \text{spec}(L_{j,R/S}^* L_{j,R/S})]$ . These intervals are such that

$$\text{tr}[(L_{j,K/S} + L_{j,K/S}^*)\hat{\rho}_{K/S}] \in \mathcal{R}_{j,R/S} \quad (5.201)$$

$$\text{tr}[L_{j,K/S}^* L_{j,K/S} \hat{\rho}_{K/S}] \in \Theta_{j,R/S} \quad (5.202)$$

for any states  $\hat{\rho}_{K/S}$ . Let us define

$$\alpha(r, \theta) = \frac{1}{2} \sum_{j>p} (r_{j,R} - r_{j,S})^2 - \theta_{j,S} \left( \ln \left[ \frac{\theta_{j,R}}{\theta_{j,S}} \right] + 1 - \frac{\theta_{j,R}}{\theta_{j,S}} \right) \quad (5.203)$$

and

$$\alpha_1 = \min \left\{ \alpha(r, \theta) \middle/ (r_{j,R/S}, \theta_{j,R/S}) \in \mathcal{R}_{j,R/S} \otimes \Theta_{j,R/S} \right\}. \quad (5.204)$$

$\alpha_1$  is always positive or null. And if for at least one  $j$ ,  $d(\mathcal{R}_{j,R}, \mathcal{R}_{j,S}) \neq 0$  or  $d(\Theta_{j,R}, \Theta_{j,S}) \neq 0$  with  $d(x, y) = |x - y|$  the usual distance on  $\mathbb{R}$ , then  $\alpha_1 > 0$ . Thus we can have an improvement of the convergence rate  $\alpha_0$ . We have

$$\limsup_t \ln(V_K(\rho(t)))/t \leq -(\alpha_0 + \alpha_1). \quad (5.205)$$

The criteria for  $\alpha_1 > 0$  is equivalent to the non degeneracy one we introduced for QND measurements. If the system is a two level one. Then the  $L_{j,R/S}$  are just elements of  $\mathbb{C}$  and the criteria for  $\alpha_1 > 0$  is that it exists  $j$  such that  $r_{j,R} \neq r_{j,S}$  or  $\theta_{j,R} \neq \theta_{j,S}$  depending on the value of  $j$ . It corresponds exactly to the non degeneracy condition for a QND measurement discriminating between  $\mathcal{H}_R$  and  $\mathcal{H}_S$ .  $\alpha_1$  is the rate we obtained in the QND case. Hence, the convergence rate improvement can be understood as an added non demolition measurement effect.

### Mean convergence time

As in the discrete case we study the meaning of the two convergence rates through mean convergence times.

Let us define a stopping time

$$\tau_\epsilon = \inf\{t \geq 0 \text{ s.t. } V_K(\rho(t)) \leq \epsilon\}. \quad (5.206)$$

It corresponds to the first time  $V_K(\rho(t))$  will be smaller or equal to  $\epsilon$ . We call  $\tau_\epsilon$  the convergence time and  $\mathbb{E}(\tau_\epsilon)$  the mean convergence time.

## 5. Continuous indirect measurements

Let us define a stopped time  $\tau_\epsilon \wedge t$ . As in the discrete case from (5.199) we have

$$\begin{aligned} \mathbb{E}(\ln(V_K(\rho(\tau_\epsilon \wedge t)))) &\leq \ln(V_K(\rho(0))) - \alpha_0 \mathbb{E}(\tau_\epsilon \wedge t) \\ &\quad - \frac{1}{2} \mathbb{E} \left( \sum_{j < p} \int_0^{\tau_\epsilon \wedge t} (\text{tr}[(L_{j,K} + L_{j,K}^*) \hat{\rho}_K(s)] - \text{tr}[(L_j + L_j^*) \rho(s)])^2 ds \right) \\ &\quad + \mathbb{E} \left( \sum_{j \geq p} \int_0^{\tau_\epsilon \wedge t} \text{tr}[L_j^* L_j \rho(s)] \left( \ln \left[ \frac{\text{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K(s)]}{\text{tr}[L_j^* L_j \rho(s)]} \right] + 1 - \frac{\text{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K(s)]}{\text{tr}[L_j^* L_j \rho(s)]} \right) ds \right) \end{aligned} \quad (5.207)$$

This time we do not have a better upper bound than 0 for the second and third line since the bound  $\alpha_1$  is true only for large times. We have

$$\mathbb{E}(\ln(V_K(\rho(\tau_\epsilon \wedge t)))) \leq -\alpha_0 \mathbb{E}(\tau_\epsilon \wedge t) + \ln(V_K(\rho(0))). \quad (5.208)$$

We need a lower bound on  $\mathbb{E}(\ln(V_K(\rho(\tau_\epsilon \wedge t))))$ . We have a bound on the maximal discontinuity of our trajectory. Let  $\ln(\epsilon') = \ln(\epsilon) + \min_{j, \hat{\rho}_K, \rho}(0, \ln \left[ \frac{\text{tr}[L_{j,K}^* L_{j,K} \hat{\rho}_K]}{\text{tr}[L_j^* L_j \rho]} \right])$ .  $\ln(\epsilon') - \ln(\epsilon)$  is the minimal (negative) value of  $\ln(V_K(\rho(t))) - \ln(V_K(\rho(t-)))$  for any time  $t$ . Since  $\ln(V_K(\rho(\tau_\epsilon \wedge t-))) \geq \epsilon$  almost surely, we have  $\ln(V_K(\rho(\tau_\epsilon \wedge t))) \geq \ln(\epsilon')$ . Hence  $0 \geq \ln(V_K(\rho(\tau_\epsilon \wedge t))) \geq \ln(\epsilon')$ , so  $\ln(V_K(\rho(\tau_\epsilon \wedge t)))$  is almost surely bounded. It converges almost surely and in  $L^1$  when  $t$  goes to infinity. Then  $\lim_t \mathbb{E}(\ln(V_K(\rho(\tau_\epsilon \wedge t)))) = \mathbb{E}(\ln(V_K(\rho(\tau_\epsilon))))$ . Moreover  $\tau_\epsilon \wedge t$  is monotone in  $t$  and the expectation of its point wise limit  $\tau_\epsilon$  is finite<sup>4</sup>. Taking the limit  $t$  to infinity on both sides of (5.208) gives us

$$\mathbb{E}(\ln(V_K(\rho(\tau_\epsilon)))) \leq -\alpha_0 \mathbb{E}(\tau_\epsilon) + \ln(V_K(\rho(0))). \quad (5.209)$$

Since  $\ln(V_K(\rho(\tau_\epsilon))) \geq \ln(\epsilon')$  almost surely, we have

$$\mathbb{E}(\tau_\epsilon) \leq \frac{\ln(V_K(\rho(0))) - \ln(\epsilon')}{\alpha_0}. \quad (5.210)$$

The mean convergence time depends on  $\alpha_0$  and not  $\alpha_1$ . Looking at figures 5.3 and 5.4, we see that when measurements are performed with a non degeneracy like condition ( $\alpha_1 > 0$ ), for some realizations the time after which the state converges is much shorter. For some other they stay chaotic and then converge rapidly. They can even get close to being completely supported in  $\mathcal{H}_R$  ( $V_\rho = 1$ ). These observations are equivalent to the ones made in the discrete case. Hence we can interpret  $\alpha_0$  and  $\alpha_1$  in the same manner.  $\alpha_1 + \alpha_0$  is more an asymptotic convergence (or stability) rate, when  $\alpha_0$  is characteristic of the time it takes to converge. The improvement of  $\alpha_0 + \alpha_1$  is in the convergence speed when the convergence happen but it can happen after a long time. The distribution of this time depends on  $\alpha_0$ . The smaller is  $\alpha_0$  the more chance the convergence will happen after short time.

The simulation presented corresponds exactly to the zero temperature case of [23]. In this article, M. Bauer and D. Bernard showed that a two level system connected to a thermal bath and continuously monitored by a diffusive non demolition measurement will, when the non demolition measurement has a large convergence rate, jump from one energy state to the other with time between the jumps distributed exponentially with a parameter depending on the temperature. Hence we find a similar behavior here but with a null temperature and for a general GAS subspace.

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4. See proof in Appendix A.4.

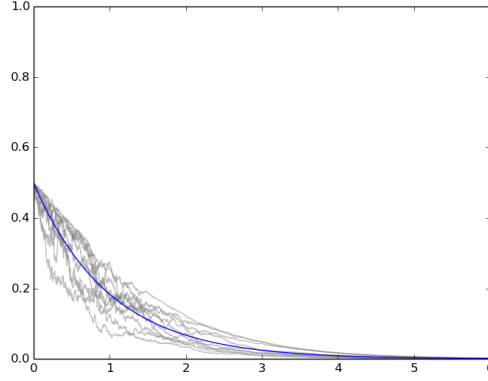


Figure 5.3.: Simulation of the evolution of  $V_{I_R}$  for a diffusive quantum trajectory of a 2 level system with  $\alpha_0 = 1$  and  $\alpha_1 = 0$ . The blue curve corresponds to the mean evolution ( $V_{I_R}(\bar{\rho}(t))$ ) and the grey lines to ten different realizations of the evolution when taking into account the measurement results ( $V_{I_R}(\rho(t))$ ).

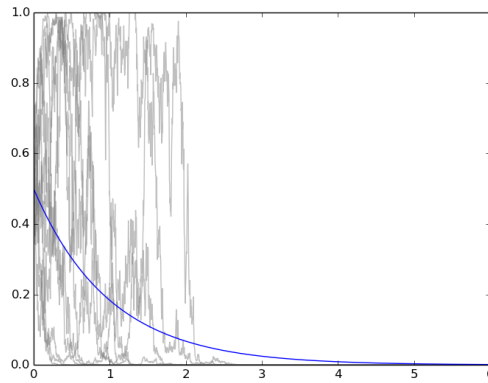


Figure 5.4.: Simulation of the evolution of  $V_{I_R}$  for a diffusive quantum trajectory of a 2 level system with  $\alpha_0 = 1$  and  $\alpha_1 = 1$ . The blue curve corresponds to the mean evolution ( $V_{I_R}(\bar{\rho}(t))$ ) and the grey lines to ten different realizations of the evolution when taking into account the measurement results ( $V_{I_R}(\rho(t))$ ).

## 5. Continuous indirect measurements

### Exponential convergence for the state

In conclusion of this section, we translate the convergence rate for  $V_K(\rho(t))$  to a convergence rate for the system state.

Since  $K$  is definite positive, its smallest eigenvalue  $k_0$  is such that  $K \geq k_0 I_R$  with  $k_0 > 0$ . Hence  $\text{tr}[\rho_R(t)] \leq \frac{1}{k_0} e^{-(\alpha_0 + \alpha_1)t + o(t)} = e^{-(\alpha_0 + \alpha_1)t + o(t)}$ . Using the Cauchy–Schwarz inequality for semi definite positive matrix,

$$\|\rho_R(t)\|_{\max} \leq e^{-(\alpha_0 + \alpha_1)t + o(t)} \quad (5.211)$$

$$\text{and } \|\rho_P(t)\|_{\max} \leq e^{-(\alpha_0 + \alpha_1)t/2 + o(t)}. \quad (5.212)$$

Hence, all the matrix blocs different from  $\rho_S(t)$  converge almost surely to zero. The convergence is exponential with a rate  $\alpha_0 + \alpha_1$  for  $\rho_R(t)$  and  $(\alpha_0 + \alpha_1)/2$  for  $\rho_P(t)$ .

## 6. Conclusion

In this thesis, after a general presentation and a discussion of the motivations of the formalism we used, we investigated the large time behavior of systems undergoing indirect measurements. The preliminary presentation included particularly a derivation of the path integral formalization of quantum stochastic processes. We then studied non demolition measurements and stabilization through reservoir engineering. In both cases we found new results implied by the measurement recording.

For quantum non demolition measurements, indirect measurement recording leads to the collapse of the wave function whereas the mean evolution would have lead only to decoherence. Moreover we showed that the convergence rate is exponential with a fixed rate. This rate allowed us to bound the mean time of convergence towards the collapse.

For discrete time quantum trajectories we did this analysis for the most general non demolition measurement possible. Hence we allowed for a change of probing method at each time with an arbitrary method choice procedure. This generality was made possible by the introduction of martingale change of measure techniques. Since the pointer probabilities were bounded martingales, we pictured the measurement as a change of measure implied by these martingales. This change of measure allowed us to study the quantum trajectories under laws conditioned on the limit state. Under these laws the measurement outcome processes become much more simple to study. For instance, if only one probing method is used, the measurement outcomes become i.i.d. random variables.

We used this i.i.d. property to derive the continuous limit of non demolition quantum trajectories. We again used martingale change of measure in the process and in particular Girsanov's theorem.

Inspired by these results we adapted the martingale techniques to continuous time non demolition trajectories. Through martingale change of measure we showed that the measurement outcome processes conditioned on the limit pointer state are Levy processes. It is thanks to this property that we have been able to compute the bounds on the mean convergence time.

Concerning reservoir engineering, from the results of convergence of the system state for the mean evolution, we showed an equivalence with convergence when measurement outcomes are recorded. We found two convergence rates. The first one corresponds to the mean evolution convergence rate whereas the second one corresponds to an asymptotic stability rate. This last rate is interpreted as the consequence of an added non demolition measurement. It leads to a peculiar evolution of the system state. When it becomes large enough, two phases appear in the behavior of the system state. First during a random time the evolution does not show any convergence towards the limit state. The state can even come close to have almost no support on the limit Hilbert subspace. This random time distribution seems to depend only on the mean convergence rate. Just after this first phase, the system state converges rapidly towards the limit state and stays in this state afterwards. Hence the convergence is rapid but can happen after a long time.

To conclude this thesis let us mention some future directions of research around the subjects we presented that might be interesting.

## 6. Conclusion

- *Minimal feedback in the repeated indirect measurements scheme.* As we presented in this thesis, repeated indirect measurement continuous time limit leads to the usual quantum trajectories (section 5.2.2). For discrete non demolition quantum trajectories we saw that conditioning the next indirect measurement on the result of the last one could lead to an increased wave function collapse convergence rate. The dependency on the last predecessor can be seen as a minimal feedback. The next action on the system is partly determined by the previous measurement. Hence it is a feedback without state estimation. Besides the consequence of such dependency on the convergence rate in the non demolition case, it could lead to more general results on convergence towards specific states. Hence it could be useful to derive the continuous time approximation of such minimal feedback repeated indirect measurements. We started to work on this subject with C. Pellegrini.
- *Error due to the continuous time approximation.* As explained in last paragraph, continuous time approximation of repeated indirect measurement leads to continuous time quantum trajectories. In order to use this approximation for the estimation of the system state, one need to estimate the error it induces. The idea is to compare the error due to the discretization of a quantum trajectory with the one due to a discretization of the interaction between the system and the probe used for indirect measurement. Estimating these errors could lead to better system state estimations. With C. Pellegrini, we plan to investigate this subject.
- *QSDE path integral applications.* The path integral formulation for QSDE we derived connect QSDE with a widely used tool in condensed matter. Hence it would be interesting to adapt condensed matter related techniques to study these equations. For example, these techniques should be useful for the study of systems including non integrable interactions or disorder. A related subject of interest is the competition between tunneling and dissipation for transport across potential barriers. This question has already been studied by Caldeira and Leggett [39] and others after them using their model. But the introduction of QSDE formalism may lead to a general Arrhenius like law. This idea was first formulated by D. Bernard.
- *Extended limits for scattering interactions.* As we saw in section 3.2.3, scattering like interactions between the system and the reservoir lead in the quantum stochastic limit to unusual Lindbladian expressions. These limits have not been completely investigated for now. In particular the quantum stochastic limits for out of vacuum reservoirs have not been all fully studied. Yet this type of interaction, including number like operators ( $a_i^* a_j$ ) on the environment in the interaction Hamiltonian, appear in interesting physics (Kondo model, Coulomb Blockade ...). Thus one should be interested in studying the extended limit for such interactions. This limits obtained, the study of their behavior under continuous measurement through quantum filtering could lead to new results in quantum transport.

Let us conclude with a remark on the Markovian approximations we used through out this report. They imply that the interaction between the system and its environment is extremely localized in time. To be more precise, as we can infer from the repeated interactions model, it is as if at each time a new environment interacts with the system. Some physical phenomena cannot, at least up to now, be understood under such approximation. The coherence between the system and the environment is at the heart of the phenomenon. As we explained in last paragraph, their might still be problems which can be mapped to Markovian evolution in some limits. But in general, as pointed out by D. Bernard and B. Doyon [30], in order to match the

experiments, one would like to describe continuous measurement when the interaction between the environment and the system is non Markovian. One expect it would lead to some non Markovian quantum trajectories [18, 56, 93]. Yet, the interpretation of these non Markovian quantum trajectories as the result of continuous measurements of some open system with non Markovian system–environment interaction is not established. Even the possibility of such interpretation is not assured. Hence up to now only continuous measurement of Markovian open systems model are really understood. Non Markovian behavior can only come from the action of the experimenter. For example through the choice of probes in the discrete non demolition setup. Hence the question of whether it is possible to define quantum trajectories modeling intrinsically non Markovian open quantum systems is also of interest.



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## A. Mean convergence time proofs

### A.1. QND finite mean convergence time

Our aim is to prove that  $\mathbb{E}_\gamma(\tau(\alpha|\gamma)) = \sum_n \mathbb{P}_\gamma(\tau > n) < \infty$ . First let us remark that  $\mathbb{P}_\gamma(\tau > n) \geq \mathbb{P}_\gamma(q_\alpha(n)/q_\gamma(n) \geq \epsilon)$ . Hence, if we prove  $\sum_n \mathbb{P}_\gamma(q_\alpha(n)/q_\gamma(n) \geq \epsilon) < \infty$  we prove that the mean convergence time is finite. We actually prove by Markov inequality that  $\mathbb{P}_\gamma(q_\alpha(n)/q_\gamma(n) \geq \epsilon)$  is bounded by the  $n^{\text{th}}$  term of a geometric suite whose ratio is smaller than 1. We have  $\mathbb{P}_\gamma(q_\alpha(n)/q_\gamma(n) \geq \epsilon) \leq \sqrt{\frac{q_\alpha(0)}{\epsilon q_\gamma(0)}} p^n$  with  $p < 1$ .

Let us remark that  $\mathbb{P}_\gamma(\frac{q_\alpha(n)}{q_\gamma(n)} \geq \epsilon) = \mathbb{P}_\gamma(\sqrt{\frac{q_\alpha(n)}{q_\gamma(n)}} \geq \sqrt{\epsilon})$ . By Markov inequality we have

$$\mathbb{P}_\gamma \left( \sqrt{\frac{q_\alpha(n)}{q_\gamma(n)}} \geq \sqrt{\epsilon} \right) \leq \sqrt{\frac{1}{\epsilon}} \mathbb{E}_\gamma \left( \sqrt{\frac{q_\alpha(n)}{q_\gamma(n)}} \right). \quad (\text{A.1})$$

Since the outcomes  $j_k$  are i.i.d. random variables under  $\mathbb{P}_\gamma$ , the expectation on the right hand side is easily computed.

$$\mathbb{E}_\gamma \left( \sqrt{\frac{q_\alpha(n)}{q_\gamma(n)}} \right) = \sqrt{\frac{q_\alpha(0)}{q_\gamma(0)}} \left( \sum_j p(j|\gamma) \left( \frac{p(j|\alpha)}{p(j|\gamma)} \right)^{\frac{1}{2}} \right)^n. \quad (\text{A.2})$$

Since, from the non degeneracy condition, there is at least two outcomes  $j$  and  $j'$  such that  $\frac{p(j|\alpha)}{p(j|\gamma)} \neq \frac{p(j'|\alpha)}{p(j'|\gamma)}$ , the strict concavity of  $x \mapsto \sqrt{x}$ , gives

$$p = \sum_j p(j|\gamma) \left( \frac{p(j|\alpha)}{p(j|\gamma)} \right)^{\frac{1}{2}} < 1. \quad (\text{A.3})$$

Thus we have

$$\mathbb{E}_\gamma(\tau) \leq \sqrt{\frac{q_\alpha(0)}{\epsilon q_\gamma(0)}} \frac{1}{1-p}. \quad (\text{A.4})$$

Hence  $\mathbb{E}_\gamma(\tau)$  is finite.

Since  $\frac{1}{2}S(\gamma|\alpha) \geq 1-p$  and  $\ln(q_\alpha(0)/q_\gamma(0)) - \ln(\epsilon) - \Delta_{\min}(\alpha|\gamma) \leq 2\sqrt{\frac{q_\alpha(0)}{\epsilon q_\gamma(0)}}$ , at least for  $\epsilon$  small enough, the bound we find in the text is better than the one we just found.

### A.2. Asymptotic subspace finite mean converge time

As for the QND case, we want to show  $\mathbb{E}(\tau_\epsilon) = \sum_n \mathbb{P}(\tau_\epsilon > n) < \infty$ . We will bound this sum by the following:  $\mathbb{E}(\tau_\epsilon) \leq \sum_n \mathbb{P}(V_K(\rho(n)) \geq \epsilon)$ .

### A. Mean convergence time proofs

Recall we have  $\mathbb{E}(V_K(\rho(n))) \leq V_K(\rho(0))\lambda_0^n$  with  $\lambda_0 < 1$ . Using Markov inequality, we have

$$\mathbb{P}(V_K(\rho(n)) \geq \epsilon) \leq \frac{V_K(\rho(0))}{\epsilon} \lambda_0^n. \quad (\text{A.5})$$

The convergence of  $\sum_n \mathbb{P}(V_K(\rho(n)) \geq \epsilon)$  follows directly from that of  $\sum_n \lambda_0^n$ .

From this convergence we immediately have the bound  $\mathbb{E}(\tau_\epsilon) \leq \frac{V_K(\rho(0))}{\epsilon} \frac{1}{1-\lambda_0}$ . Since for a  $\epsilon$  small enough,  $\ln(V_K(\rho(0))/\epsilon) \leq V_K(\rho(0))/\epsilon$  and  $1 - \lambda_0 \leq \ln(\lambda_0)$ , this is a worse bound than the one of the text. At least when  $\epsilon$  is small enough.

### A.3. Continuous QND finite mean convergence time

We want to prove  $\mathbb{E}_\gamma(\tau(\alpha|\gamma)) < \infty$ .

First let us show that  $\mathbb{E}(\tau) = \int_0^\infty \mathbb{P}(\tau > t) dt$  when  $\lim_t t\mathbb{P}(\tau > t) = 0$ .

We have  $\mathbb{E}(\tau_\epsilon) = \lim_t \int_0^t s \mathbb{P}(\tau_\epsilon = s) ds$ .  $\mathbb{P}(\tau = s) = -\frac{d}{ds} \mathbb{P}(\tau_\epsilon > s)$ . The equality  $\mathbb{E}(\tau_\epsilon) = \int_0^\infty \mathbb{P}(\tau_\epsilon > t) dt$  follows then directly from integration by part formula.

Here we use Markov inequality to bound  $\mathbb{P}_\gamma(\frac{q_\alpha(t)}{q_\gamma(t)} \geq \epsilon)$ . Let us first remark that this probability is an upper bound on  $\mathbb{P}(\tau > t)$ . Then  $\mathbb{P}_\gamma(\frac{q_\alpha(t)}{q_\gamma(t)} \geq \epsilon) = \mathbb{P}_\gamma(\sqrt{\frac{q_\alpha(t)}{q_\gamma(t)}} \geq \epsilon^{1/2})$ . From Markov inequality,

$$\mathbb{P}_\gamma(\sqrt{\frac{q_\alpha(t)}{q_\gamma(t)}} \geq \epsilon^{1/2}) \leq \frac{1}{\sqrt{\epsilon}} \mathbb{E}(\sqrt{\frac{q_\alpha(t)}{q_\gamma(t)}}). \quad (\text{A.6})$$

The processes  $X_{j,\gamma}(t)$  and  $N_j(t)$  are independent Brownian motions and Poisson processes with intensity  $\theta(j|\gamma)$  respectively under  $\mathbb{P}_\gamma$ . Using expression (5.148) for  $\sqrt{\frac{q_\alpha(t)}{q_\gamma(t)}}$ , we can compute explicitly  $\mathbb{E}_\gamma(\sqrt{\frac{q_\alpha(t)}{q_\gamma(t)}})$ .

$$\begin{aligned} \mathbb{E}_\gamma\left(\sqrt{\frac{q_\alpha(t)}{q_\gamma(t)}}\right) = \\ \sqrt{\frac{q_\alpha(0)}{q_\gamma(0)}} \exp \left[ -\frac{t}{2} \left( \sum_{j < p} \frac{1}{4} (r(j|\alpha) - r(j|\gamma))^2 + \sum_{j \geq p} \theta(j|\gamma) \left( \sqrt{\frac{\theta(j|\alpha)}{\theta(j|\gamma)}} - 1 \right)^2 \right) \right]. \end{aligned} \quad (\text{A.7})$$

Let us write

$$\sigma'(\gamma|\alpha) = \sum_{j < p} \frac{1}{8} (r(j|\alpha) - r(j|\gamma))^2 + \frac{1}{2} \sum_{j \geq p} \theta(j|\gamma) \left( \sqrt{\frac{\theta(j|\alpha)}{\theta(j|\gamma)}} - 1 \right)^2. \quad (\text{A.8})$$

We have

$$\mathbb{P}_\gamma(\tau > t) \leq \sqrt{\frac{q_\alpha(0)}{\epsilon q_\gamma(0)}} e^{-t\sigma'(\gamma|\alpha)}. \quad (\text{A.9})$$

Thus

$$\mathbb{E}_\gamma(\tau(\alpha|\gamma)) \leq \sqrt{\frac{q_\alpha(0)}{\epsilon q_\gamma(0)}} \frac{1}{\sigma'(\gamma|\alpha)}. \quad (\text{A.10})$$

#### A.4. Continuous asymptotic subspace finite mean convergence time

The rate involved in the bound is smaller than  $\sigma(\gamma|\alpha)$  and for a  $\epsilon$  small enough we have  $\ln(q_\alpha(0)/q_\gamma(0)) - \ln(\epsilon) - \Delta_{\min} l(\alpha|\gamma) \leq \sqrt{q_\alpha(0)/(\epsilon q_\gamma(0))}$ . Hence the bound of the text is better, at least for a small enough  $\epsilon$ .

#### A.4. Continuous asymptotic subspace finite mean convergence time

We want to prove  $\mathbb{E}(\tau_\epsilon) < \infty$ .

As in the discrete case we bound  $\mathbb{P}(\tau_\epsilon > t)$  by an exponentially decreasing quantity using Markov inequality.

$\mathbb{P}(\tau_\epsilon > t) \leq \mathbb{P}(V_K(\rho(t)) \geq \epsilon)$ . By Markov inequality  $\mathbb{P}(V_K(\rho(t)) \geq \epsilon) \leq \frac{V_K(\rho(0))}{\epsilon} e^{-\alpha_0 t}$ .

From  $\mathbb{E}(\tau_\epsilon) = \int_0^\infty \mathbb{P}(\tau > t)$ , it follows

$$\mathbb{E}(\tau_\epsilon) \leq \frac{V_K(\rho(0))}{\epsilon \alpha_0}. \quad (\text{A.11})$$

Again, this bound is worse than the text one when  $\epsilon$  is small enough.



## B. Path integral derivations

### B.1. Derivation of the path integral for an environment in its vacuum

As explained in the text, the derivation of the path integral can be reduced to finding a path integral for  $U_{t,s}^f = {}_{t,s}\langle e(f) | U_{t,s}^f | 0 \rangle_{t,s}$ . Indeed,

$$\mathbb{E}(B(s)C(t)) = \int df e^{-|f|^2} \text{tr}_{sys.} [C U_{t,s}^f U_{s,0}^f \rho_{sys.} U_{s,0}^{f*} B U_{t,s}^{f*}]$$

If we insert the closure relation for the system at well chosen places, we obtain

$$\begin{aligned} \mathbb{E}(B(s)C(t)) &= \int df e^{-|f|^2} \int dx_B dx_C dx_0 dx'_0 \\ &\quad B(x_B) T_{s,t}(x_B, x_C; f) C(x_C) T_{0,t}(x_C, x_0; f) \\ &\quad \rho(x_0, x'_0) \overline{T_{s,0}(x'_0, x_B; f)} \end{aligned}$$

with transition amplitudes  $T_{t,s}(x_f, x_0; f) = \langle x_f | U_{t,s}^f | x_0 \rangle$  and  $\rho(x_0, x'_0) = \langle x_0 | \rho | x'_0 \rangle$ .

Our last task is to compute the path integral formulation of  $T_{t,s}(x_f, x_0; f)$ . First, using the definition of  $A_t$ , we find that  $U_{t,s}^f$  is the solution of the differential equation

$$dU_{t,s}^f/dt = -i(H - i/2L^*L)U_{t,s}^f - LU_{t,s}^f \overline{f(t)}, \quad U_{s,s}^f = I_{sys.}$$

Using this expression, we can derive a path integral using standard techniques. It leads us to the following expression.

$$T_{t,s}(x_f, x_0; f) = \int_{x(s)=x_0}^{x(t)=x_f} dx dp e^{i \int_s^t du p(u) \dot{x}(u) - H(u) + i/2 L^*(u) L(u) + i L(u) \overline{f(u)}}.$$

Inserting this expression in  $\mathbb{E}(\dots)$ , we straight forwardly obtain the expression expected.

The generalization to multi time correlation functions and more general time ordering, is straight forward too. It is sufficient to add a time slicing for each time an observable is inserted. One can also extend the time over which the path integral is defined by inserting  $U_{\infty,s}$  on the left just before  $B$  and  $U_{\infty,s}^*$  on the right after  $U_{s,0}^*$ . Using the cyclic property of the trace and the unitarity of  $U_{\infty,s}$  one immediately find that these insertions do not modify the result for the correlation. This property is useful if one wants to compute correlation functions with a characteristic functional.

### B.2. Path integral derivation for $n_t \neq 0$

As in the previous proof we divide the environment exponential states in different time parts and include closure relations on the system wherever needed and obtain the following

### B. Path integral derivations

expression:

$$\begin{aligned}\mathbb{E}(B(s)C(t)) &= \int df dg_{\pm} dx_B dx_C dx_0 dx'_0 \\ &\quad e^{-|f|^2 - |g_-|^2 - |g_+|^2 + (f, \mathbf{1}_{[t, \infty[} g_+) + (g_-, \mathbf{1}_{[t, \infty[} f)} \\ &\quad B(x_B) T_{t,s}(x_B, x_C; f, g_+) C(x_C) T_{s,0}(x_C, x_0; f, g_+) \\ &\quad \rho_{sys.}(x_0, x'_0) \rho_{env.}(g_+, g_-) \overline{T_{t,0}(x_B, x'_0; f, g_-)},\end{aligned}$$

with  $T_{t,s}(x_f, x_0; f, g) = \langle x_f | U_{t,s}^{f,g} | x_0 \rangle$ ,  $U_{t,s}^{f,g} = {}_{t,s} \langle e(f) | U_{t,s} | e(g) \rangle_{t,s}$  and  $\rho_{env.}(g_+, g_-)$  given in the text. It remains to compute  $T_{t,s}(x_f, x_0; f, g)$  in order to conclude.  $U_{t,s}^{f,g}$  is the solution of the differential equation

$$dU_{t,s}^{f,g}/dt = [-i(H - i/2L^*L) - L\bar{f}(s) + L^*g(s)]U_{t,s}^{f,g} \text{ and } U_{s,s}^{f,g} = I_{sys.} e^{(f, \mathbf{1}_{[s, t[} g)}.$$

This expression does not seem to correspond to the QSDE of  $U_{t,s}$ . Though if we look at the formal solution of this equation, we have

$$U_{t,s} = \mathcal{T}_> e^{\int_s^t -iH du - L dA_u^* + L^* \int_s^t dA_u}$$

whatever is the value of  $n_t$ . Hence, the solution does not depend on the state of the environment.  $U_{t,s}$  is an operator on  $\mathcal{H}_{sys.} \otimes \Gamma(L^2(\mathbb{R}_+))$ . It is defined independently of the environment state. Thus the differential equation for  $U_{t,s}^{f,g}$  should only depend on  $f$  and  $g$  and not on  $n_t$ . This explains the expression for the differential equation of  $U_{t,s}^{f,g}$ .

As earlier, given the expression of  $U_{t,s}^{f,g}$  we use standard path integral derivation techniques and obtain

$$\begin{aligned}T_{t,s}(x_f, x_0; f, g) &= \\ &\int_{x(s)=x_0}^{x(t)=x_f} dx dp \exp \left[ i \int_s^t du \left( \dot{x}(u) p(u) - H(u) + i/2 L^* L(u) \right. \right. \\ &\quad \left. \left. + i L(u) \bar{f}(u) - i L^*(u) g(u) - i \bar{f}(u) g(u) \right) \right].\end{aligned}\tag{B.1}$$

Inserting this expression in  $\mathbb{E}(\dots)$  we get

$$\begin{aligned}\mathbb{E}(B(s)C(t)) &= \\ &\int_{x_+(\infty)=x_-(\infty)} df dg_{\pm} dx_{\pm} dp_{\pm} \\ &\quad B_+(s) C_+(t) \rho_{sys.}(x_+(0), x_-(0)) e^{iG[x_{\pm}, p_{\pm}; f, g_{\pm}]}\end{aligned}\tag{B.2}$$

with  $G$  given in the text.

### B.3. Classical limit derivation

The first step in our demonstration is a change of variable on the paths  $x_{\pm}, p_{\pm}$ . We consider that the forward  $(x_+, p_+)$  and backward  $(x_-, p_-)$  differ only because of quantum non commutativity behavior. Hence their difference must be of the order of the quantum

### B.3. Classical limit derivation

effects. More precisely their difference must be of order  $\hbar$ . Hence we make the change of variables :  $x_{\pm} = x \pm \hbar q, p_{\pm} = p \pm \hbar \pi$ . We can then write a development of the action  $iG$  in  $\hbar$ :

$$iG = - \int_0^\infty ds \left( |f(s)|^2 + |g_+(s)|^2 + |g_-(s)|^2 - \overline{g_+}(s) e^{-\lambda(s)} g_-(s) + \overline{f}(s) g_+(s) + \overline{g_-}(s) f(s) \right) \quad (\text{B.3})$$

$$+ 2i \int_0^\infty ds \left( \pi(\dot{x}(s) - \frac{\partial H}{\partial p}(s)) - q(\dot{p}(s) + \frac{\partial H}{\partial x}(s)) \right) \quad (\text{B.4})$$

$$- \int_0^\infty ds \frac{1}{\hbar} L^* L(x, p; s) \quad (\text{B.5})$$

$$- \frac{1}{\sqrt{\hbar}} \int_0^\infty ds \left( \overline{L(x, p; s)} (f(s) - g_+(s)) + L(x, p; s) (\overline{f}(s) - \overline{g_-}(s)) \right) \quad (\text{B.6})$$

$$+ \sqrt{\hbar} \int_0^\infty ds \left( \frac{\partial \overline{L}}{\partial x} q(s) + \frac{\partial \overline{L}}{\partial p} \pi(s) \right) (f(s) + g_+(s)) \quad (\text{B.7})$$

$$- \sqrt{\hbar} \int_0^\infty ds \left( \frac{\partial L}{\partial x} q(s) + \frac{\partial L}{\partial p} \pi(s) \right) (\overline{f}(s) + \overline{g_-}(s)) \quad (\text{B.8})$$

$$+ O(\hbar).$$

Let us simplify the notations introducing:

$$S_{cl}^0 = \int_0^\infty ds \left( \pi(\dot{x}(s) - \frac{\partial H}{\partial p}(s)) - q(\dot{p}(s) + \frac{\partial H}{\partial x}(s)) \right) \quad (\text{B.9})$$

$$D(s) = \left( \frac{\partial L}{\partial x} q(s) + \frac{\partial L}{\partial p} \pi(s) \right) \quad (\text{B.10})$$

$$L(s) = L(x, p; s) \quad (\text{B.11})$$

At first glance on the dependency on  $\hbar$  of the different parts of  $iG$ , we need to get rid of lines (B.5) and (B.6) which would lead to infinities as  $\hbar$  goes to 0. We suppress these infinities through an integration over specific linear combinations of  $f$  and  $g_{\pm}$ .

Let  $h_{\pm} = f - g_{\pm}$ . Then (B.3) becomes

$$- \int_0^\infty ds \left( |h_+(s)|^2 + |h_-(s)|^2 - \overline{h_+}(s) h_-(s) + \overline{g_+}(s) (1 - e^{-\lambda(s)}) g_-(s) \right)$$

and (B.6) becomes

$$- \frac{1}{\sqrt{\hbar}} \int_0^\infty ds \left( \overline{L(x, p; s)} h_+(s) + L(x, p; s) \overline{h_-}(s) \right).$$

We do not take into account the change in the measure of the path integration since it is a simple multiplication by a constant. After integration over  $h_+$  and  $h_-$  we obtain for (B.3)

$$- \int_0^\infty ds \overline{g_+}(s) (1 - e^{-\lambda(s)}) g_+(s).$$

Remark that  $g_- = h_+ - h_- + g_+$ . This point is important in the computation. After integration, (B.6) disappears and two new terms appear. The first one depends only on  $L(s)$  and  $D(s)$ :

$$+ \frac{1}{\hbar} \int_0^\infty ds (\overline{L}(s) - \hbar \overline{D}(s)) (L(s) + \hbar D(s)).$$

### B. Path integral derivations

The second one depends on  $D(s)$  and  $g_+$ :

$$+2\sqrt{\hbar} \int_0^\infty ds \bar{g}_+(s)(1 - e^{-\lambda(s)})D(s).$$

Hence we obtain

$$iG = - \int_0^\infty ds |g_+(s)|^2 (1 - e^{-\lambda(s)}) \quad (\text{B.12})$$

$$+ 2iS_{cl.}^0 \quad (\text{B.13})$$

$$- \frac{1}{\hbar} \int_0^\infty ds \left( L^* L(x, p, s) - \bar{L}(s) L(s) \right) \quad (\text{B.14})$$

$$+ \int_0^\infty ds \left( \bar{L}(s) D(s) - \bar{D}(s) L(s) \right) \quad (\text{B.15})$$

$$+ 2\sqrt{\hbar} \int_0^\infty ds \left( \bar{D}(s) g_+(s) - \bar{g}_+(s) D(s) \right) \quad (\text{B.16})$$

$$+ 2\sqrt{\hbar} \int_0^\infty ds \bar{g}_+(s)(1 - e^{-\lambda(s)})D(s) \quad (\text{B.17})$$

$$+ O(\hbar). \quad (\text{B.18})$$

The next step would be to take the limit  $\hbar \rightarrow 0$  but it remains an exploding term in this limit. Thus we need to check under which conditions  $L^* L(x, p; s) - \bar{L}(s) L(s) = 0$ . Of course this is true if  $[L^*, L] = 0$ , hence if  $L = F(X)$  or  $L = F(P)$  with  $F$  any complex function of reals. It is also true if  $L = \alpha X + \beta P$ . In this case  $L^* L(x, p; s) = |\alpha x(s) + \beta p(s)|^2 + i\hbar\alpha\beta$  and  $\bar{L}(s) L(s) = |\alpha x(s) + \beta p(s)|^2$ . Hence  $L^* L(x, p; s) - \bar{L}(s) L(s) = i\hbar\alpha\beta$ . This additional constant add the same exact weigh to any path. It could have been suppressed in the initial definition of the path integral. Thus we can get rid of it. So from now on let us assume  $L^* L(x, p; s) - \bar{L}(s) L(s) = 0$ .

We are now equipped to take the limit  $\hbar \rightarrow 0$ . First let us make a simple change of variables on  $g_+$  such that all  $\hbar$  dependency remains in front of  $|g_+(s)|^2(1 - e^{-\lambda(s)})$ :  $g(s) = \sqrt{\hbar}g_+(s)$ . Remember that  $\lambda$  is such that  $e^{-\lambda(s)} = 1 - \hbar\tilde{\lambda}(s) + o(\hbar)$ . Hence  $(1 - e^{-\lambda(s)})/\hbar = \tilde{\lambda}(s) + O(\hbar)$ . It follows we have  $|g(s)|^2(1 - e^{-\lambda(s)})/\hbar = |g(s)|^2\tilde{\lambda}(s) + O(\hbar)$  and  $\bar{g}(s)(1 - e^{-\lambda(s)})D(s) = O(\hbar)$ .

We obtain the action

$$iG_{cl.} = \lim_{\hbar \rightarrow 0} iG = - \int_0^\infty ds |g(s)|^2 \tilde{\lambda}(s) \quad (\text{B.19})$$

$$+ 2iS_{cl.}^0 + \int_0^\infty ds \left( \bar{L}(s) D(s) - L(s) \bar{D}(s) \right) \quad (\text{B.20})$$

$$+ 2 \int_0^\infty ds \left( \bar{D}(s) g(s) - D(s) \bar{g}(s) \right). \quad (\text{B.21})$$

We know modify this expression to match the classical MSR action one.

$$iG_{cl.} = \quad (\text{B.22})$$

$$- \int_0^\infty ds \frac{\tilde{\lambda}(s)}{4} (\eta_1(s)^2 + \eta_2(s)^2) \quad (\text{B.23})$$

$$- 2i \int_0^\infty ds q(s) \left( \dot{p}(s) + \frac{\partial H}{\partial x}(s) + \gamma_x(s) + I_x(s)\eta_1(s) - R_x(s)\eta_2(s) \right) \quad (\text{B.24})$$

$$+ 2i \int_0^\infty ds \pi(s) \left( \dot{x}(s) - \frac{\partial H}{\partial p}(s) - \gamma_p(s) - I_p(s)\eta_1(s) + R_p(s)\eta_2(s) \right) \quad (\text{B.25})$$

with

$$\gamma_y(s) = \text{Im}\left(L(s)\frac{\partial \bar{L}}{\partial y}(s)\right) \quad (\text{B.26})$$

$$I_y(s) = \text{Im}\left(\frac{\partial L}{\partial y}(s)\right) \quad (\text{B.27})$$

$$R_y(s) = \text{Re}\left(\frac{\partial L}{\partial y}(s)\right). \quad (\text{B.28})$$

This MSR action correspond to the Langevin equations

$$\dot{x}(s) = \frac{\partial H}{\partial p}(s) + \gamma_p(s) - R_p(s)\eta_2(s) + I_p(x)\eta_1(s) \quad (\text{B.29})$$

$$\dot{p}(s) = -\frac{\partial H}{\partial x}(s) - \gamma_x(s) + R_x(s)\eta_2(s) - I_x(x)\eta_1(s). \quad (\text{B.30})$$

$\eta_1$  and  $\eta_2$  are two independent white noises whose correlation function is  $\langle \eta_i(s)\eta_j(t) \rangle = \delta_{i,j}\delta(t-s)2\tilde{\lambda}(t)^{-1}$ .



# C. Article: Repeated quantum non-demolition measurements: convergence and continuous-time limit

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Michel Bauer<sup>♣</sup><sup>1</sup>, Tristan Benoist<sup>♠</sup><sup>2</sup> and Denis Bernard<sup>♠</sup><sup>3</sup>

<sup>♣</sup> Institut de Physique Théorique<sup>4</sup> de Saclay, CEA-Saclay, France.

<sup>♠</sup> Laboratoire de Physique Théorique de l'ENS, CNRS & Ecole Normale Supérieure de Paris, France.

## Abstract

We analyze general enough models of repeated indirect measurements in which a quantum system interacts repeatedly with randomly chosen probes on which Von Neumann direct measurements are performed. We prove, under suitable hypotheses, that the system state probability distribution converges after a large number of repeated indirect measurements, in a way compatible with quantum wave function collapse. Similarly a modified version of the system density matrix converges. We show that the convergence is exponential with a rate given by some relevant mean relative entropies. We also prove that, under appropriate rescaling of the system and probe interactions, the state probability distribution and the system density matrix are solutions of stochastic differential equations modeling continuous-time quantum measurements. We analyze the large time convergence of these continuous-time processes and prove convergence.

## C.1. Introduction

Repeated indirect quantum measurements aim at getting (partial) information on quantum systems with minimal impact on it. A possibility consists in repeating non-demolition

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1. michel.bauer@cea.fr

2. tristan.benoist@ens.fr

3. Member of CNRS; denis.bernard@ens.fr

4. CEA/DSM/IPhT, Unité de recherche associée au CNRS

measurements (QND). At each step, one lets the quantum system under study interact with another quantum system, called the probe, and performs a Von Neumann measurement [4] on this probe. Information on the quantum system is gained through intrication between the probe and the quantum system. If one is aiming at progressively measuring a quantum observable, one has to make sure that a system prepared in one eigenstate of this observable remains in it after a cycle of intrication and direct measurement on the probe, and that the set of stable states, called pointer states, forms an orthonormal basis of the system Hilbert space. The experiment of ref. [2], in which repeated QND measurements is used to fix and measure the number of photons in a cavity without destroying them, illustrates this strategy.

Repeated indirect measurements were studied in ref. [1]. There, the discussion was limited to QND measurements consisting of identical probes, interactions and measurements on the probe, and assuming a non-degeneracy condition. In the present article, we extend these results to cases where different indirect measurements (probes, interactions and direct measurements on probes) are used. We also study the degenerate case.

Let  $Q_n(\cdot)$  be the pointer state distribution after the  $n^{\text{th}}$  indirect measurement, that is  $Q_n(\alpha)$  is the probability to find the system in the state  $\alpha$  after  $n$  steps ( $\alpha$  labels the pointer states),  $\sum_{\alpha} Q_n(\alpha) = 1$ . As explain in section C.3, each cycle of indirect measurement updates the distribution through Bayes' law. The analysis of the distribution is reformulated as a problem in classical probability theory (with no quantum interference).

We shall prove that this sequence of distributions converges at large  $n$ , that is after a large – strictly speaking, infinite – number of QND measurements. If a non-degeneracy assumption is verified, the limit distribution is  $Q(\alpha) = \delta_{\Upsilon, \alpha}$  for some random limit pointer state  $\Upsilon$ . This reflects the collapse of the system wave function. The convergence is exponential,

$$Q_n(\alpha) \simeq \text{const.} \exp(-n\bar{S}(\Upsilon|\alpha)), \quad \text{for large } n, \alpha \neq \Upsilon,$$

with rate given by an appropriate relative entropy  $\bar{S}(\Upsilon|\alpha)$  defined in eq.(C.9). In probabilistic terms, the limit  $Q(\alpha)$  possesses a natural interpretation as a Radon-Nikodym derivative and,  $Q_n(\alpha) = \mathbb{E}(Q(\alpha)|\mathcal{F}_n)$  is a closed martingale with respect to an appropriate filtration, see section C.5. As a consequence, we show that the expectations conditioned on the limit pointer state are identical to expectations starting from this same pointer state. That is:  $\mathbb{E}_{\Upsilon}(\cdot) = \mathbb{E}(\cdot|\mathcal{A})$  where  $\mathcal{A}$  is the tail  $\sigma$ -algebra, the smallest one making the limit distribution measurable. See below for a precise definition of  $\mathbb{E}_{\Upsilon}(\cdot)$ .

Convergence of  $Q_n(\alpha)$  is also studied when the non-degeneracy hypothesis is not fulfilled. In this degenerate case, the limit pointer state distribution vanishes outside a random finite set of pointer states. The quantum system density matrix, when properly modified, also converges in the limit of infinite number of QND measurements. The limit density matrix then coincides with that predicted for degenerate Von Neumann measurements [4], see section C.5.

Of course repeated indirect measurements have already been studied in the physics literature, mostly through time continuous measurement formalisms – as far as we know, little was done on the discrete setting as we do in the present paper. E. B. Davies [6] probably made the first rigorous approach to time continuous quantum measurement. This was later studied by N. Gisin [7] and L. Diosi [8] using the non linear Schrödinger equation. Simultaneously, A. Barchielli and V. P. Belavkin derived the equations governing continuous measurements in terms of instruments [9]. They derived jump equations which, when properly rescaled, are equivalent to diffusive equations for continuous measurements. Another approach uses quantum stochastic differential equations and quantum filtering theory to obtain the so-called

Belavkin equations [10–14] and [15]. More recently, C. Pellegrini derived Belavkin equations for continuous time measurements [20–22] using discrete repeated indirect measurement models. The problem of convergence of quantum density matrix has also been analyzed within the time continuous measurement framework. In refs. [16, 17], V. P. Belavkin showed the convergence of mixed states toward pure states. A derivation of wave function collapse from the non-linear stochastic Schrödinger equation has been presented in refs. [18, 19]. It makes use of martingale theory as we do in the present paper.

In the following we also connect our discrete model to the time continuous measurement formalism. Taking the time continuous limit requires rescaling appropriately the interaction between the quantum system under study and the probes. In that sense, the time continuous model we consider is close to that of ref. [21] but our proofs are different and slightly more general. Our derivation is based on the convergence of some discrete counting processes – related to the number of occurrences of outputs in the successive indirect measurements – toward a time continuous Gaussian process. Under appropriate hypotheses, spelled out in section C.6, the pointer state distribution satisfies a random diffusive stochastic equation driven by Gaussian processes. Suppose that at each step the probe system is randomly selected (independently of the past history and with time independent probability, for simplicity) among a finite set  $\mathcal{O}$  whose elements are indexed by  $o \in \mathcal{O}$  and that the output measurements on the probe can take finite number of values indexed by  $i \in \text{spec}(o)$ . Then, the pointer state probabilities  $Q_t(\alpha)$  are time continuous martingales (with respect to an appropriate filtration) whose evolutions are governed by the non linear stochastic equations:

$$dQ_t(\alpha) = Q_t(\alpha) \sum_{(o,i)} (\Gamma^{(o)}(i|\alpha) - \langle \Gamma^{(o)}(i) \rangle_t) dX_t(o, i)$$

where  $X_t(o, i)$  are some centered Gaussian processes,  $\Gamma^{(o)}(i|\alpha)$  are coding for the probability of output probe measurement  $i$  within the probe system  $o$  conditional on the quantum system be prepared in the state  $\alpha$  and  $\langle \Gamma^{(o)}(i) \rangle_t = \sum_{\alpha} \Gamma^{(o)}(i|\alpha) Q_t(\alpha)$ . The pointer state distribution again converges as a finite-dimensionnal bounded vector martingale. Under non-degeneracy assumptions, the limit distribution is again  $Q(\alpha) = \delta_{\alpha, \alpha_0}$  and the convergence is still exponential with a rate given by the scaling limit of the mean relative entropy.

These results extend to the system density matrix. In the time continuous scaling limit, the system density matrix is a solution of a diffusive Belavkin equation (C.23), as expected. Although not a martingale, properly modified, it converges to the density matrix predicted by Von Neumann measurement theory.

The article is organized as follow : In section C.2 we define the repeated QND measurement process we study. In section C.3 we establish the link with a classical random process in which the pointer state distribution is repeatedly updated through Bayes' law. In section C.4 we prove the convergence of the pointer state distribution under some assumptions and we determine the convergence rate in general cases. In section C.5 we extend these results to the degenerate case. Finally in section C.6 we study the time continuous scaling limit of our model. Some technical details appearing along the article are postponed to appendices.

## C.2. QND measurements as stochastic processes

The aim of this section is to describe the relation between repeated non-demolition measurements, positive operator valued measurements (POVM's) and classical stochastic processes.

### C.2.1. Repeated indirect quantum measurements

Let us consider a quantum system with initial density matrix  $\rho$ . Repeated non-demolition measurements aim at getting indirectly information on the system (without demolishing it as a projective quantum measurement à la Von Neumann or a direct connection to a macroscopic apparatus might do).

To gain information, we let the system interact with another quantum system called *the probe*, and then perform a Von Neumann measurement on the probe. Assume the probe is initially in the pure state  $|\Psi\rangle\langle\Psi|$ . Let  $U$  be the unitary operator, acting on the tensor product Hilbert space  $\mathcal{H}_{\text{sys.}} \otimes \mathcal{H}_{\text{probe}}$ , coding for the interaction between the system and the probe. After interaction, the system and the probe are entangled. Their joint state is

$$U(\rho_0 \otimes |\Psi\rangle\langle\Psi|)U^\dagger$$

A perfect non-degenerate projective measurement is then performed on the probe. That is, one is measuring an observable with a non-degenerate spectrum  $i \in \mathcal{I}$ . Let  $\{|i\rangle\}$  be the corresponding eigenbasis of  $\mathcal{H}_{\text{probe}}$ . If the output of the probe measurement is  $i$ , the system state is projected into

$$\rho'(i) = \frac{1}{\pi(i)} \langle i|U|\Psi\rangle \rho \langle\Psi|U^\dagger|i\rangle$$

because the probe and the system have been entangled. This projection occurs with probability

$$\pi(i) := \text{Tr}[\langle i|U|\Psi\rangle \rho \langle\Psi|U^\dagger|i\rangle].$$

We do not have to worry about cases where  $\pi(i) = 0$ , because these cases, almost surely, never happen.

The process of “interaction plus probe measurement” is an example of a positive operator valued measurement (POVM). Let us define operators  $M_i$ , acting on the system Hilbert space, by

$$M_i := \langle i|U|\Psi\rangle.$$

They satisfy  $\sum_i M_i^\dagger M_i = \mathbb{I}_{\text{sys.}}$  as a consequence of the unitarity of  $U$ . After measurement with output  $i$ , the system density matrix  $\rho'(i)$  can be written as

$$\rho'(i) = \frac{1}{\pi(i)} M_i \rho M_i^\dagger,$$

with  $\pi(i) = \text{Tr}[M_i \rho M_i^\dagger]$ . This characterizes a POVM.

Let us now assume that we repeat the process of “interaction plus probe measurement” ad libitum. As we shall see below, even for purely practical reasons, it is useful to keep the freedom of changing some or all features of the process. For instance, the experimenter might tune (or let fluctuate randomly, or tune but leaving a certain amount of randomness or ...) the initial state  $|\Psi\rangle$  of the probe at each step. Or he/she might tune (or let fluctuate randomly, or ...) the interaction operator  $U$  at each step, for instance by playing on the time lapse that the probe spends close enough to the system to interact significantly with it. He/She might even tune (or ...) the type of probe (in particular the dimension of its Hilbert space) at each step. Finally, he/she might tune (or ...) the non-degenerate probe measurement (equivalently the  $\mathcal{H}_{\text{probe}}$  basis made of its eigenvectors  $\{|i\rangle\}$ ).

## C.2. QND measurements as stochastic processes

We let  $|\Psi_n\rangle$ ,  $U_n$ ,  $\mathcal{I}_n$  denote the initial state, interaction operator and set of possible outcomes of the  $n^{\text{th}}$  step. Setting  $\rho_0 := \rho$ ,  $\rho_1 := \rho'$  and so on, we get a random recursion equation, namely that for  $i \in \mathcal{I}_n$ :

$$\rho_n = \frac{1}{\pi_n(i)} M_i^{(n)} \rho_{n-1} M_i^{(n)\dagger} \quad (\text{C.1})$$

with probability  $\pi_n(i) = \text{Tr}[M_i^{(n)} \rho_{n-1} M_i^{(n)\dagger}]$  where  $M_i^{(n)} = \langle i|U_n|\Psi_n\rangle$  (note that the meaning of the expectation itself, i.e. the Hilbert space with respect to which it is taken, might depend on  $n$ , however we may arrange to choose the  $\mathcal{I}_n$ 's so that  $i$  determines  $\langle i|$  completely).

It is worth noticing that, under such an evolution, a pure state remains a pure state, that is:  $|\phi_n\rangle = M_i^{(n)}|\phi_{n-1}\rangle/\|M_i^{(n)}|\phi_{n-1}\rangle\|$  with probability  $\langle\phi_{n-1}|M_i^{(n)\dagger}M_i^{(n)}|\phi_{n-1}\rangle$ . This case is included in that of density matrices.

Let us now specialize this scheme in such a way that it preserves a preferred basis of the system Hilbert space. That is, we assume there exists a fixed basis  $\{|\alpha\rangle\}$  of  $\mathcal{H}_{\text{sys.}}$  such that all interactions can be decomposed as

$$U_n := \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U_n(\alpha) \quad (\text{C.2})$$

where the  $U_n(\alpha)$ 's are unitary operators on  $\mathcal{H}_{\text{probe}_n}$ . The states  $|\alpha\rangle$  are called *pointer states*. The density matrices  $|\alpha\rangle\langle\alpha|$  with  $|\alpha\rangle$  a pointer state are fixed points of the recursion relation (C.1).

The pointer states have to be eigenstates of the system Hamiltonian  $H_s$  for these indirect measurements to be quantum non-demolition (QND) measurements, since there is a waiting time between two successive indirect measurements during which the quantum system evolves freely. That is,  $H_s = \sum_{\alpha} E_{\alpha} |\alpha\rangle\langle\alpha|$  where  $E_{\alpha}$  is the energy of the pointer state  $|\alpha\rangle$  for the free system.

After each indirect measurement one gains information on the system state. Repeating the process (infinitely) many times amounts (as we shall explain) to perform a measurement of a system observable whose eigenstates are the pointer states. This observable commutes with the system free evolution. A system in one of the pointer states remains unchanged by the successive indirect measurements.

It has been shown in ref. [1] that a system subject to repeated QND measurements as described above converges toward one of the pointer states. This convergence was only proved in the case where the probes, interactions and observables on the probes are all the same. A non-degeneracy hypothesis was also used. One of the present article aim is to generalize the convergence statements without those assumptions.

A word on terminology: we are going to name *partial measurement* one iteration of “interaction plus probe measurement” and *complete measurement* an infinite sequence of successive partial measurements.

### C.2.2. A toy model

We shall illustrate this framework and our results with a simple toy model inspired by experiments done on quantum electrodynamics in cavities [2]. The present work is actually inspired by these experiments. There, the system is a monochromatic photon field and the

probes are modeled by two level systems. The observable we aim at measuring is the photon number. This is a non-demolishing measurement.

The system-probe interaction is well described by the unitary operator

$$U = \exp[-i(\epsilon\Delta t \hat{p} \otimes \mathbb{I}_2 + \frac{\pi}{4} \hat{p} \otimes \sigma_3)]$$

where  $\hat{p}$  is the photon number operator,  $\epsilon$  the energy of a photon and  $\Delta t$  the interaction duration.

This interaction amounts to the rotation of the two level system effective spin half if the cavity happened to be in a photon number operator eigenstate. The probes are assumed to be initially in a state  $|\Psi\rangle = e^{-i\theta\sigma_3}|+1\rangle$  where  $|+1\rangle$  is an eigenvector of  $\sigma_1$  corresponding to the eigenvalue  $+1$ . The probe observables, which are measured after the interaction between the system and the probe has taken place, are  $O_{\theta'} = e^{-i\theta'\sigma_3}\sigma_2e^{i\theta'\sigma_3}$ . Their eigenvectors are  $|\pm_{\theta'}\rangle = e^{-i\theta'\sigma_3}|\pm_2\rangle$ .

The resulting POVM operators for the process of “interaction plus probe measurement” only depend on the difference between the two angles  $\theta$  and  $\theta'$ . They are  $M_{\pm}^{\theta-\theta'}$  with

$$\begin{aligned} M_{\pm}^{\theta-\theta'} &= \langle \pm_2 | e^{i\theta'\sigma_3} e^{-i(\epsilon\Delta t \hat{p} \otimes \mathbb{I}_2 + \frac{\pi}{4} \hat{p} \otimes \sigma_3)} e^{-i\theta\sigma_3} | +1 \rangle \\ &= e^{-i\epsilon\Delta t \hat{p}} \langle \pm_2 | e^{i(\theta'-\theta)\sigma_3 - i\frac{\pi}{4} \hat{p} \otimes \sigma_3} | +1 \rangle \end{aligned}$$

Using the identity  $e^{-i\theta\sigma_3} = \cos(\theta)\mathbb{I}_2 - i\sin(\theta)\sigma_3$  and  $\sigma_3|+1\rangle = |-1\rangle$ , one gets

$$M_{\pm}^{\theta-\theta'} = \frac{1}{\sqrt{2}} e^{-i(\epsilon\Delta t \hat{p} \pm \frac{\pi}{4})} \left[ \cos(\theta - \theta' + \frac{\pi}{4}) \hat{p} \pm \sin(\theta - \theta' + \frac{\pi}{4}) \right]$$

One may verify that  $M_{+}^{\theta-\theta'} M_{+}^{\theta-\theta'} + M_{-}^{\theta-\theta'} M_{-}^{\theta-\theta'} = \mathbb{I}$ . Remark that if  $|p\rangle, p \in \mathbb{N}$  is a fixed photon number state, then  $|\langle p | M_{\pm}^{\theta} | p \rangle|$  is identical to  $|\langle p + 4k | M_{\pm}^{\theta} | p + 4k \rangle|$ , with  $k \in \mathbb{N}$ . This property leads to degeneracies in iterative QND measurement methods. Two states  $|p\rangle$  and  $|p + 4k\rangle$  can not be distinguished by this method. These degeneracies are discussed in section C.5.1.

### C.3. Measurement apparatus and Bayes' law

The aim of this section is to reformulate (part of) of the iterative QND measurement method in terms of classical probability theory. We are interested in the eigenstate probability distribution  $q_n(\alpha)$ , with

$$q_n(\alpha) := \langle \alpha | \rho_n | \alpha \rangle,$$

and its evolution during the iterative procedure. At each step, the system density matrix is updated via the relation (C.1), and as a consequence of the factorization property (C.2), this distribution is updated through the random recursion relation:

$$q_n(\alpha) = q_{n-1}(\alpha) \frac{|M^{(n)}(i|\alpha)|^2}{\sum_{\beta} q_{n-1}(\beta) |M^{(n)}(i|\beta)|^2}$$

with probability  $\sum_{\beta} q_{n-1}(\beta) |M^{(n)}(i|\beta)|^2$ . We have defined  $M^{(n)}(i|\alpha) := \langle i | U_n(\alpha) | \Psi_n \rangle$ .

### C.3. Measurement apparatus and Bayes' law

Since  $|M^{(n)}(i|\alpha)|^2$  is the probability to get a probe measurement result  $i$  conditioned on the system state being  $|\alpha\rangle$ , we introduce a (hopefully) suggestive notation

$$p_n(i|\alpha) := |M^{(n)}(i|\alpha)|^2$$

We have  $\sum_i p_n(i|\alpha) = 1$  since  $\sum_i M^{(n)}(i|\alpha)^\dagger M^{(n)}(i|\alpha) = \mathbb{I}$ . The recursion relation on the distribution reads

$$q_n(\alpha) = q_{n-1}(\alpha) \frac{p_n(i|\alpha)}{\sum_\beta q_{n-1}(\beta) p_n(i|\beta)} \quad (\text{C.3})$$

with probability  $\pi_n(i) = \sum_\beta q_{n-1}(\beta) p_n(i|\beta)$ . This update rule corresponds to Bayes' law. The study of the eigenstate distribution convergence is thus a question of classical probability theory.

Let us now put on stage the classical probability theory framework we shall be using.

We imagine building a measurement apparatus which performs a sequence of partial measurements. As we have stressed above, we allow for a protocol where the characteristics of partial measurements may vary at each step. However, we shall assume that these characteristics are chosen within some finite set  $\mathcal{O}$  called the set of measurement methods. In a quantum setting, one measurement method  $o$  is a triplet (probe state  $|\Psi\rangle$ , interaction  $U$ , probe eigen-basis  $\{|i\rangle\}$  of direct measurement). Each measurement method  $o \in \mathcal{O}$  defines a set, called the spectrum of  $o$  and denoted by  $\text{spec}(o)$ , of possible outcomes. For each  $o$  we have a family of probability measures  $p^o(\cdot|\alpha)$  on  $\text{spec}(o)$  indexed by  $\alpha \in \mathcal{S}$ , where  $\mathcal{S}$  is the index set of pointer states.

As time goes by, the experimenter records the sequence  $o_1, i_1, o_2, i_2, \dots$  where  $o_1$  is the first measurement method,  $i_1$  the outcome of the first partial measurement, performed using method  $o_1$ , and so on. So it is natural to take as the space of events the space  $\Omega$  of infinite sequences  $(o_1, i_1, o_2, i_2, \dots)$  where each  $o_n$  belongs to  $\mathcal{O}$  and each  $i_n$  to  $\text{spec}(o_n)$ . To be even more formal, set  $E := \cup_{o \in \mathcal{O}} \{o\} \times \text{spec}(o)$ , so that  $E$  is the set of couples  $(o, i)$  with  $o \in \mathcal{O}$  and  $i \in \text{spec}(o)$ . Then  $\Omega := E^{\mathbb{N}^*}$ .

For a finite sequence  $(o_1, i_1, o_2, \dots, i_n, o_{n+1}) \in E^n \times \mathcal{O}$ ,  $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$  is defined as the subset of  $\Omega$  made of all those  $\omega$ 's whose first  $2n+1$  components are  $o_1, i_1, o_2, \dots, i_n, o_{n+1}$ . We define  $B_{o_1, i_1, o_2, \dots, o_n, i_n}$  analogously. We let  $\mathcal{F}_n$  be the  $\sigma$ -algebra generated by all the  $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ . Note that  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all the  $B_{o_1}$ , i.e.  $\mathcal{F}_0$  codes for the first measurement method choice. For convenience we define  $\mathcal{F}_{-1} \equiv \{\emptyset, \Omega\}$ . Then  $\mathbb{F} := (\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, \dots)$  is an increasing sequence of  $\sigma$ -algebras. We take  $\mathcal{F}$  to be the smallest  $\sigma$ -algebra on  $\Omega$  containing all the  $\mathcal{F}_n$ , making  $(\Omega, \mathcal{F}, \mathbb{F})$  a filtered measurable space. We could define another filtration by taking  $\mathcal{F}'_n$  to be the  $\sigma$ -algebra generated by all the  $B_{o_1, i_1, o_2, \dots, o_n, i_n}$ . While this may seem superficially a more natural choice of filtration, we shall see below that  $\mathbb{F}$  is slightly more convenient. There is a natural collection of measurable functions on  $(\Omega, \mathcal{F})$ , namely the projections : for  $\omega = (o_1, i_1, o_2, \dots, o_n, i_n, \dots)$  we set  $O_n(\omega) = o_n$ ,  $I_n(\omega) = i_n$ . These can be used to define counting functions that play an important role in the following. We set  $\epsilon_n(o, i) := \mathbf{1}_{O_n=o, I_n=i}$  and  $N_n(o, i) := \sum_{1 \leq m \leq n} \epsilon_m(o, i)$  (with the usual empty sum convention  $N_0(o, i) := 0$ ).

The first task is to put a probability measure on  $\Omega$ . The next one will be to define a sequence of random variables on  $\Omega$  solving the recursion relation (C.3).

If the measurement methods are given, the distributions of partial measurements are described by the  $p^o(\cdot|\alpha)$ . So what remains to be discussed is how the measurement methods are chosen, and we put the condition that this does not involve precognition. We

suppose that a collection of non-negative numbers  $d_{-1} = 1$ ,  $d_0(o_1)$  (for all  $o_1 \in \mathcal{O}$ ),  $\dots$ ,  $d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1})$  (for all  $o_1 \in \mathcal{O}$ ,  $i_1 \in \text{spec}(o_1)$ ,  $\dots$ ,  $o_{n+1} \in \mathcal{O}$ ) is given in such a way that

$$\sum_{o_{n+1} \in \mathcal{O}} d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1}) = d_{n-1}(o_1, i_1, o_2, \dots, i_{n-1}, o_n). \quad (\text{C.4})$$

It is not difficult to produce such a collection. For instance if  $c_0(\cdot)$  is a probability measure on  $\mathcal{O}$  and for each  $n \geq 1$   $c_n(\cdot|o, i)$  is a probability measure on  $\mathcal{O}$  indexed by  $(o, i) \in E$  then

$$d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1}) \equiv c_0(o_1) c_1(o_2|o_1, i_1) \cdots c_n(o_{n+1}|o_n, i_n)$$

does the job. We call this special choice the Markovian feedback protocol. Some special cases are of interest. If we assume that for  $n \geq 1$   $c_n(\cdot|o, i) = c_n(\cdot)$  does not depend on  $o, i$ , we arrive at something we call the random protocol. On the other hand, if  $b_0$  is an element of  $\mathcal{O}$  and  $b_n$ ,  $n \geq 1$  a family of maps from  $E$  to  $\mathcal{O}$ , then taking  $c_0(\cdot) := \delta_{\cdot, b_0}$  and, for  $n \geq 1$ ,  $c_n(\cdot|o, i) := \delta_{\cdot, b_n(o, i)}$  we arrive at the description of an experimenter deciding of the next measurement method by taking into account the previous measure outcome.

Given a collection of such non negative numbers  $d_n(o_1, i_1, \dots, i_n, o_{n+1})$ , and using the Kolmogorov extension theorem, it is easy to see that there is a unique probability measure  $\mathbb{P}_\alpha$  on  $(\Omega, \mathcal{F})$  such that

$$\mathbb{P}_\alpha(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}) = p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha) d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1}).$$

Indeed, we see that the mandatory consistency condition is fulfilled : if the left-hand side is summed over  $o_{n+1}$  (using (C.4)) and then over  $i_n$  the formula for  $\mathbb{P}_\alpha(B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n})$  is recovered, which is needed since  $B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n}$  is the disjoint union of the  $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$  over the possible  $o_{n+1}$  and  $i_n$ . The normalization condition  $\mathbb{P}_\alpha(\Omega) = 1$  and the positivity condition are obvious.

Note that in general, conditional on the sequence of measurement methods  $o_1, \dots, o_n$ , one has  $\mathbb{P}_\alpha(i_1, \dots, i_n|o_1, \dots, o_n) \neq p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha)$ . This is due to the feedback. For the cases when the  $d_n$ 's do not depend on the outcomes, in particular for the independent random protocol, equality is recovered.

We define also,

$$\mathbb{P} \equiv \sum_{\alpha \in \mathcal{S}} q_0(\alpha) \mathbb{P}_\alpha. \quad (\text{C.5})$$

We use  $\mathbb{E}_\alpha$  and  $\mathbb{E}$  to denote expectations with respect to  $\mathbb{P}_\alpha$  and  $\mathbb{P}$  respectively.

A simple computation shows that, for each  $\alpha$ , the conditional probability

$$\mathbb{P}_\alpha(O_{n+1} = o_{n+1}|O_1, I_1, \dots, O_n, I_n) = \frac{d_n(O_1, I_1, O_2, \dots, I_n, o_{n+1})}{d_{n-1}(O_1, I_1, O_2, \dots, I_{n-1}, O_n)}$$

whenever the denominator is nonzero. The same formula holds for  $\mathbb{P}$ . The right-hand side is simply  $c_n(o_{n+1}|O_n, I_n)$  for the Markovian feedback protocol. So indeed, the functions  $d_{-1}, d_0, d_1, \dots$  embody the probabilistic description of the choice of measurement methods.

These definitions may seem arbitrary at that point, but now we can make contact with the initial problem. Define a sequence of random variables  $Q_n(\alpha)$  by the initial condition  $Q_0(\alpha) = q_0(\alpha)$  and the recursion relation

$$Q_n(\alpha) = \frac{Q_{n-1}(\alpha) p^{O_n}(I_n|\alpha)}{\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{O_n}(I_n|\beta)}.$$

To show that the recursion relation (C.3) is verified, we need to show that the transition probabilities are correct. A simple way to do that is to solve this random recursion relation. For  $\omega \in B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$  one checks that

$$Q_n(\alpha, \omega) := \frac{q_0(\alpha) p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha)}{\sum_{\beta \in \mathcal{S}} q_0(\beta) p^{o_1}(i_1|\beta) \cdots p^{o_n}(i_n|\beta)} \quad (\text{C.6})$$

whenever  $\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}) \neq 0$ . Note that this condition ensures that the denominator of  $Q_n(\alpha, \omega)$  is nonzero. We observe that whenever defined,  $Q_n(\alpha, \omega) \geq 0$  and  $\sum_{\alpha} Q_n(\alpha, \omega) = 1$ . If  $\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}) = 0$  the value of  $Q_n(\alpha, \omega)$  is mostly immaterial from a probabilistic viewpoint, because in any case the full sequence  $Q_n(\alpha, \omega)$  is well-defined on a set  $\tilde{\Omega}$  of  $\mathbb{P}$ -measure 1 (note that the collection of all  $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$  is countable, so the collection of those with  $\mathbb{P}$ -measure 0 is countable as well, or empty).

Since there is no dependence on  $o_{n+1}$  on the right-hand side we observe that for  $\omega \in \tilde{\Omega}$ , and conditional on  $\mathcal{F}_{n-1}$ ,

$$Q_n(\alpha) = \frac{Q_{n-1}(\alpha) p^{o_n}(i_n|\alpha)}{\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n|\beta)}$$

with probability

$$\sum_{o_{n+1} \in \mathcal{O}} \frac{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}})}{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n})} = \frac{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n})}{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n})} = \sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n|\beta).$$

So the recursion relation (C.3) is recovered with the identifications  $q_n(\alpha) \rightarrow Q_n(\alpha)$  and  $p_n \rightarrow p^{o_n}$ . To summarize, we have accomplished our goal : find a probability space on which (C.3) has a solution, which we have even written explicitly.

In the following sections we study the convergence of  $Q_n(\cdot)$  and its dependence with respect to the initial pointer state distribution. On our way, we shall understand the probabilistic meaning of the recursion relation (C.3).

## C.4. Convergence

In [1] the convergence of  $q_n(\cdot)$  has been shown under the hypothesis that only one partial measurement method  $o$  is used. The properties of the limit were elucidated under the further assumption that for every couple of pointer states  $(\alpha, \beta)$  there exists at least one partial measurement result  $i$  such that  $p^o(i|\alpha) \neq p^o(i|\beta)$ . This last assumption can be understood as a non-degeneracy hypothesis because two different pointer states  $\alpha, \beta$  do not induce identical partial measurement results distribution  $p^o(i|\alpha)$ . Our aim is to generalize the convergence of  $Q_n(\cdot)$  while weakening the hypotheses made in [1]. We discuss the convergence when different partial measurement methods are used. We focus on the influence of this extension on the rate of convergence. The degenerate case will be studied in section C.5.1. In the case of one measurement method, a convergence result similar to that of [1] has been obtained by H. Amini, P. Rouchon and M. Mirrahimi through sub-martingale convergence in [3].

#### C.4.1. Convergence with different partial measurement methods

The extension of the convergence result of [1] to cases with different measurement methods is straightforward. From the fact that, conditional on  $\mathcal{F}_{n-1}$ ,

$$Q_n(\alpha) = \frac{Q_{n-1}(\alpha)p^{o_n}(i_n|\alpha)}{\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta)p^{o_n}(i_n|\beta)}$$

with probability  $\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta)p^{o_n}(i_n|\beta)$ , the average of  $Q_n(\alpha)$ , again conditioned on  $\mathcal{F}_{n-1}$ , is

$$\sum_{i_n \in \text{spec}_{o_n}} Q_n(\alpha) \sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta)p^{o_n}(i_n|\beta) = Q_{n-1}(\alpha).$$

So  $Q_{n-1}(\alpha)$  is conserved in average. Though the computation involved to prove it is essentially the same, a mathematically cleaner statement is that

$$\mathbb{E}(Q_n(\alpha)|\mathcal{F}_{n-1}) = Q_{n-1}(\alpha),$$

i.e. each  $Q_n(\alpha)$  is an  $\mathbb{F}$ -martingale.

In fact,  $Q_n(\alpha)$  has a deeper probabilistic meaning, which makes the martingale property obvious.

For a while, forget the previous definition of  $Q_n(\alpha)$ . Observe that, under the assumption that  $q_0(\alpha) > 0$  for every  $\alpha \in \mathcal{S}$ , any set of  $\mathbb{P}$ -measure 0 has also  $\mathbb{P}_\alpha$ -measure 0. Then the Radon-Nikodym theorem states that for each  $\alpha \in \mathcal{S}$ , there is a  $\mathbb{P}$ -integrable non-negative random variable  $Q(\alpha)$  on  $(\Omega, \mathcal{F})$  such that

$$q_0(\alpha)\mathbb{E}_\alpha(X) = \mathbb{E}(Q(\alpha)X)$$

for every  $\mathbb{P}_\alpha$ -integrable random variable  $X$  on  $(\Omega, \mathcal{F})$ . The random variable  $Q(\alpha)$  is a Radon-Nikodym derivative of  $q_0(\alpha)\mathbb{P}_\alpha$  with respect to  $\mathbb{P}$ . It is obvious that two Radon-Nikodym derivatives can differ only on a set of  $\mathbb{P}$ -measure 0: in that sense the Radon-Nikodym derivative is unique if it exists. We have also that,  $\mathbb{P}$ -almost surely,  $\sum_\alpha Q(\alpha) = 1$ , so that,  $\mathbb{P}$ -almost surely, each  $Q(\alpha) \leq 1$ . This existence theorem is a bit abstract but if one replaces  $\mathcal{F}$  by  $\mathcal{F}_n$  one can get a concrete formula. The same argument ensures the existence of a  $\mathbb{P}$ -integrable non-negative random variable  $Q_n(\alpha)$  on  $(\Omega, \mathcal{F}_n)$  such that  $q_0(\alpha)\mathbb{E}_\alpha(X) = \mathbb{E}(Q_n(\alpha)X)$  for every  $\mathbb{P}_\alpha$ -integrable random variable  $X$  on  $(\Omega, \mathcal{F}_n)$ . As  $\mathcal{F}_n$  is finite, it suffices to let  $X$  run over the indicator functions for the  $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ . This implies that

$$Q_n(\alpha, \omega) = q_0(\alpha) \frac{\mathbb{P}_\alpha(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}})}{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}})}$$

for every  $\omega \in B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$  (such that the denominator is nonzero, else the value of  $Q_n(\alpha, \omega)$  is immaterial). Explicitly one finds

$$Q_n(\alpha, \omega) = \frac{q_0(\alpha)p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha)}{\sum_{\beta \in \mathcal{S}} q_0(\beta)p^{o_1}(i_1|\beta) \cdots p^{o_n}(i_n|\beta)}$$

on  $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ .

This is exactly our previous definition of  $Q_n(\alpha)$ , which is probabilistically a Radon-Nikodym derivative. This makes the martingale property obvious without any computation, just by

the definition and general properties of conditional expectations. In fact,  $Q_n(\alpha)$  is a closed martingale:

$$Q_n(\alpha) = \mathbb{E}(Q(\alpha)|\mathcal{F}_n).$$

As  $Q_n(\alpha)$  is also bounded, the martingale convergence theorem [5] ensures that  $Q_n(\alpha) \rightarrow Q(\alpha)$   $\mathbb{P}$ -almost surely and in  $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ . Since by assumption  $\mathcal{S}$  is a finite set, all  $Q_n(\alpha)$ 's converge simultaneously  $\mathbb{P}$ -almost surely.

Let us now observe that the following two statements are equivalent:

- the measures  $\mathbb{P}_\alpha$  are all mutually singular,
- there is a collection  $(\Omega_\alpha)_{\alpha \in \mathcal{S}}$  of disjoint measurable subsets of  $\Omega$  such that,  $\mathbb{P}$ -almost surely  $\mathbf{1}_{\Omega_\alpha} = Q(\alpha)$ .

The proof is simple. The statement that the measures  $\mathbb{P}_\alpha$  are all mutually singular is equivalent to the existence of a collection  $(\Omega_\alpha)_{\alpha \in \mathcal{S}}$  of disjoint measurable subsets of  $\Omega$  such that  $\mathbb{P}_\beta(\Omega_\alpha) = \delta_{\alpha,\beta}$  for all  $\alpha, \beta \in \mathcal{S}$ . From the defining property of Radon-Nikodym derivatives,  $\mathbf{1}_{\Omega_\alpha} Q(\alpha)$  is also a Radon-Nikodym derivative of  $q_0(\alpha)\mathbb{P}_\alpha$  with respect to  $\mathbb{P}$ , and  $\mathbb{E}(Q(\alpha)\mathbf{1}_{\Omega_\beta}) = q_0(\alpha)\delta_{\alpha,\beta}$  which by positivity implies that, for  $\alpha \neq \beta$ ,  $Q(\alpha)\mathbf{1}_{\Omega_\beta} = 0$  except maybe on a set of  $\mathbb{P}$ -measure 0. Hence  $\mathbb{P}$ -almost surely  $\mathbf{1}_{\Omega_\alpha} Q(\beta) = Q(\beta)\delta_{\alpha,\beta}$ . Summing over  $\beta$  gives  $\mathbf{1}_{\Omega_\alpha} = Q(\alpha)$   $\mathbb{P}$ -almost surely. The converse is also true: if there is a collection  $(\Omega_\alpha)_{\alpha \in \mathcal{S}}$  of disjoint measurable subsets of  $\Omega$  such that  $\mathbb{P}$ -almost surely  $\mathbf{1}_{\Omega_\alpha} = Q(\alpha)$ , then the measures  $\mathbb{P}_\alpha$  are all mutually singular and concentrated on the  $\Omega_\alpha$ 's.

A striking consequence is that, if the measures  $\mathbb{P}_\alpha$  are all mutually singular, for each  $\omega$  in a set of  $\mathbb{P}$ -measure 1,  $Q_n(\alpha)$  converges to either 0 or 1, and it converges to 1 with probability  $\mathbb{P}(\Omega_\alpha) = q_0(\alpha)$ .

Hence when the measures  $\mathbb{P}_\alpha$  are all mutually singular *there is a full experimental equivalence between an infinite sequence of partial measurements and a direct projective measurement on the system*. We further study this equivalence in section C.4.2.

We shall now give a criterion, that the experimenter may enforce on the protocol, ensuring that the measures  $\mathbb{P}_\alpha$  are all mutually singular. This involves a non-degeneracy hypothesis, similar but weaker than that made in [1].

We say that  $o \in \mathcal{O}$  is recurrent in  $\omega \in \Omega$  if  $O_n(\omega) = o$  for infinitely many  $n$ 's. Our (sufficient) criterion for all the measures  $\mathbb{P}_\alpha$  to be mutually singular is:

There is a subset  $\mathcal{O}_s$  of  $\mathcal{O}$  such that

- Each  $o \in \mathcal{O}_s$  is recurrent with probability 1 under each  $\mathbb{P}_\alpha$
- For every  $\alpha, \beta \in \mathcal{S}$ ,  $\alpha \neq \beta$  there is some  $o \in \mathcal{O}_s$  and  $i \in \text{spec}(o)$  such that  $p^o(i|\alpha) \neq p^o(i|\beta)$ .

This condition says that with probability one, infinitely many partial measurements that distinguish between any two states of the system will occur.

Consider the event  $A_o$  made of the  $\omega$ 's such that  $Q_n(\alpha)$  converges for each  $\alpha$  and  $o$  is recurrent. Note that by our assumptions  $\mathbb{P}(A_o) = 1$ . We show that for any  $i \in \text{spec}(o)$

$$Q(\alpha) \sum_{\gamma} Q(\gamma) p^o(i|\gamma) = Q(\alpha) p^o(i|\alpha)$$

on  $A_o$ . There are two cases to consider. Either  $(O_n(\omega), I_n(\omega)) = (o, i)$  for infinitely many  $n$ 's: then the announced relation follows by taking the limit of the basic recursion relation along a subsequence. Or  $(O_n(\omega), I_n(\omega)) = (o, i)$  for only finitely many  $n$ 's: then, as shown in

Appendix C.A,  $\sum_{\gamma} Q(\gamma)p^o(i|\gamma) = 0$  so that in particular  $Q(\alpha)p^o(i|\alpha) = 0$  and the announced relation still holds.

This implies that

$$\forall \alpha, \beta \in \mathcal{S}, Q(\alpha)Q(\beta)(p^o(i|\alpha) - p^o(i|\beta)) = 0 \text{ on } A_o \text{ for every } i \in \text{spec}(o). \quad (\text{C.7})$$

Then, by (C.7),  $\forall \alpha, \beta \in \mathcal{S}, \alpha \neq \beta$  one has  $Q(\alpha)Q(\beta) = 0$  on  $A_{\mathcal{O}_s} := \cap_{o \in \mathcal{O}_s} A_o$ , which has  $\mathbb{P}$ -measure 1. As the sum of the  $Q(\alpha)$ 's is 1, this means that on  $A_{\mathcal{O}_s}$  one has  $Q(\alpha) = \delta_{\alpha, \Upsilon}$  where  $\Upsilon(\omega)$  is some  $\omega$ -dependent element of  $\mathcal{S}$ . So there is a family of disjoint subsets  $\Omega_{\alpha}$  of  $A_{\mathcal{O}_s}$  such that  $\cup_{\alpha} \Omega_{\alpha}$  has  $\mathbb{P}$ -measure 1, and  $Q(\alpha) = \mathbf{1}_{\Omega_{\alpha}}$  except maybe on a set of  $\mathbb{P}$ -measure 0.

We shall give two examples.

For the first one, the task to ensure that enough measurement methods  $o$  are recurrent is left to the experimenter.

The second example is the Markovian feedback protocol. For this protocol, under  $\mathbb{P}_{\alpha}$ , the process  $(o_n, i_n)$  is Markovian with transition kernel  $K_{\alpha, n}(o, i; o', i') = p^{o'}(i'|\alpha)c_n(o'|o, i)$ , with initial distribution  $p^o(i|\alpha)c_0(o)$ . Recurrence questions are well under control at least when the kernels do not depend on  $n$ . So we assume that  $c_n(o'|o, i) = c(o'|o, i)$  is time independent<sup>5</sup> and set  $K_{\alpha}(o, i; o', i') = p^{o'}(i'|\alpha)c(o'|o, i)$ . The product structure of  $K_{\alpha}$  leads to introduce the reduced transition kernel  $K_{\alpha}^{red}(o; o') := \sum_{i \in \text{spec}(o)} p^o(i|\alpha)c(o'|o, i)$ . One can rely on classical Markov chain computations to make sure that the measures  $\mathbb{P}_{\alpha}$  are all mutually singular. Assuming that the reduced Markov chain  $K_{\alpha}^{red}$  is irreducible and aperiodic, it admits a unique invariant probability  $\mu_{\alpha}^{red}$  on  $\mathcal{O}$ , which is strictly positive. Then all partial measurement methods with  $\mu_{\alpha}^{red}(o) > 0$  will be recurrent on a set of  $\mathbb{P}_{\alpha}$ -measure 1. Moreover the full Markov chain has a unique invariant probability  $\mu_{\alpha}(o, i) = p^o(i|\alpha)\mu_{\alpha}^{red}(o)$ . Then the strong law of large numbers for Markov chains states that  $N_n(o, i)$ , the number of occurrence of  $(o, i)$  up to the  $n^{\text{th}}$  experiment, satisfies

$$N_n(o, i) \sim n\mu_{\alpha}(o, i) \text{ for large } n$$

on a set of  $\mathbb{P}_{\alpha}$ -measure 1. This ergodicity result will be put in use in section C.4.3.

To summarize, we have proved that if there are enough  $\mathbb{P}_{\alpha}$  recurrent partial measurement methods then the measures  $\mathbb{P}_{\alpha}$  are all mutually singular, so that there is a full experimental equivalence between an infinite sequence of partial measurements and a direct projective measurement on the system.

### C.4.2. Conditioning or projecting

In the previous section we pointed out the connection between complete measurements and direct projective measurements. This holds whenever the measures  $\mathbb{P}_{\alpha}$  are mutually singular.

Under this hypothesis, we show in this section that we can solve the random recursion relation (C.3) on a space where the final pointer state is determined before the measurement process starts: for the class of experiments we are dealing with, it is consistent to assess that the total measurement outcome can be decided in advance and by a classical probabilistic choice.

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5. Were  $c_n$  periodic in  $n$  we could look at a Markov chain with a larger state space to reduce to that case.

We assume that the measures  $\mathbb{P}_\alpha$  are mutually singular, and, as usual, that all  $q_0(\alpha)$  are  $> 0$ . To avoid clumsy statements, we remove from  $\Omega$  the ( $\mathbb{P}$ -negligible) set of events for which either  $Q_n(\alpha)$  is not defined for all  $n$ 's, or  $Q_n(\alpha)$  does not converge to 0 or 1. So we assume that the sets  $\Omega_\alpha$  form a partition of  $\Omega$ , and the random variable  $\Upsilon$  defined by  $\lim_{n \rightarrow \infty} Q_n(\alpha, \omega) = \mathbf{1}_{\omega \in \Omega_\alpha} = \delta_{\alpha, \Upsilon(\omega)}$  is defined everywhere on  $\Omega$ . We let  $\mathcal{A}$  be the smallest  $\sigma$ -algebra making any  $\Omega_\alpha$  measurable. We claim that if  $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$  is any integrable random variable,

$$\mathbb{E}(X|\mathcal{A}) = \mathbb{E}_\Upsilon(X).$$

This can be rephrased as : conditioning  $\mathbb{P}$  on the limit of  $Q_n(\alpha)$  being 1 leads to  $\mathbb{P}_\alpha$ . This is essentially obvious from the Radon-Nikodym viewpoint and the fact that the measures  $\mathbb{P}_\alpha$  are mutually singular. But a direct computation is easy. The fact that  $\mathcal{S}$  is finite (countable would do the job as well) has two consequences. First, any  $\mathcal{A}$ -measurable random variable  $Y$  can be written as a linear combination  $Y = \sum_{\alpha \in \mathcal{S}} y_\alpha \mathbf{1}_{\Omega_\alpha}$ . Second, to test that  $\mathbb{E}(X|\mathcal{A}) = Y$  it suffices to check that  $\mathbb{E}(X \mathbf{1}_{\Omega_\alpha}) = \mathbb{E}(Y \mathbf{1}_{\Omega_\alpha})$  for every  $\alpha$ . Now, by definition,  $\mathbb{E}(X \mathbf{1}_{\Omega_\alpha}) = q_0(\alpha) \mathbb{E}_\alpha(X)$ , whereas  $\mathbb{E}(Y \mathbf{1}_{\Omega_\alpha}) = q_0(\alpha) y_\alpha$ , so, if  $\mathbb{E}(X|\mathcal{A}) = Y$ ,  $y_\alpha = \mathbb{E}_\alpha(X)$ , i.e.  $Y = \sum_{\alpha \in \mathcal{S}} \mathbb{E}_\alpha(X) \mathbf{1}_{\Omega_\alpha} = \mathbb{E}_\Upsilon(X)$ . Hence  $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}_\Upsilon(X)$  as announced.

This proves the equivalence between projecting first on a given state  $\alpha$  and conditioning on the limit state being  $\alpha$ .

### C.4.3. Convergence rates and trial distribution independence

Experimentally, the initial distribution  $q_0(\cdot)$  may not be known. One would then use the sequence of partial measurements to gain information and reconstruct it from these measurements. This may be done using Bayes' law starting from a trial distribution  $\hat{q}_0(\cdot)$  (supposed to be nowhere vanishing) and recursively improving it using the relation

$$\hat{q}_n(\alpha) = \hat{q}_{n-1}(\alpha) \frac{p^{o_n}(i_n|\alpha)}{\sum_{\beta} \hat{q}_{n-1}(\beta) p^{o_n}(i_n|\beta)}$$

if the outcome is  $i$ , which happens with probability  $\sum_{\beta} \hat{q}_{n-1}(\beta) p^{o_n}(i_n|\beta)$ . The difference with eq.(C.3) is that the recursion involves  $\hat{q}_n(\cdot)$  and not  $q_n(\cdot)$ . However, the probability is the one given by the  $q_n(\cdot)$ . If the initial trial distribution  $\hat{q}_0(\cdot)$  coincides with the initial system distribution  $q_0(\cdot)$ , then  $\hat{q}_n(\cdot) = q_n(\cdot)$  for all  $n$ . Both  $\hat{q}_n(\cdot)$  and  $q_n(\cdot)$  are realization dependent. We shall define the random process  $\hat{Q}_n(\cdot)$  as  $Q_n(\cdot)$  in (C.6) but with a different initial distribution

$$\hat{Q}_n(\alpha) := \hat{q}_0(\alpha) \frac{p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha)}{\sum_{\beta} \hat{q}_0(\beta) p^{o_1}(i_1|\beta) \cdots p^{o_n}(i_n|\beta)}$$

The probability law still depends on the true initial distribution. Notice that  $\hat{Q}_n(\cdot)$  is not an  $\mathbb{F}$ -martingale under this law, contrary to  $Q_n(\cdot)$ . As we shall show, they nevertheless have identical limit, that is:  $\lim_{n \rightarrow \infty} \hat{Q}_n(\alpha)$  exists and is equal to  $\mathbf{1}_{\Omega_\alpha}$  with  $\mathbb{P}$ -probability 1.

Moreover, if a time independent Markovian feedback protocol is used, the convergence of the state probability distribution is exponential. Its convergence rate is the mean relative entropy of the partial measurement result distribution conditioned on the system be in the state  $\Upsilon$  with respect to the one conditioned on the system be in the state  $\alpha$ . This means that for  $n$  large enough,

$$\hat{Q}_n(\alpha) \simeq e^{-n\bar{S}(\Upsilon|\alpha)}, \quad \text{for } \alpha \neq \Upsilon \quad (\text{C.8})$$

with

$$\bar{S}(\beta|\alpha) := \sum_{o \in \mathcal{O}} \mu_{\Upsilon}^{red}(o) \sum_{i \in \text{spec}(o)} p^o(i|\beta) \ln \left[ \frac{p^o(i|\beta)}{p^o(i|\alpha)} \right] \quad (\text{C.9})$$

Here, all the  $p^o(i|\alpha)$  are assumed to be strictly positive, thus any  $(o, i)$  with  $o \in \mathcal{O}_s$  is recurrent. In the case of time independent random protocols, the rate is the same with  $\mu_{\Upsilon}^{red}(\cdot)$  replaced by  $c(\cdot)$  the distribution of measurement methods. This coincides with the result of [1] if  $\mathcal{O}$  contains only one partial measurement method.

The independence of the limiting distribution with respect to the initial trial distribution is obtained whenever one starts with a trial distribution such that  $\hat{q}_0(\alpha) > 0$  wherever  $q_0(\alpha) > 0$ . This happens for example if we start with  $\hat{q}_0(\alpha) > 0$  for any  $\alpha \in \mathcal{S}$ .

To see this, we analyse the behavior of  $\hat{Q}_n$  under the probability measure  $\hat{\mathbb{P}} := \sum_{\alpha \in \mathcal{S}} \hat{q}_0(\alpha) \mathbb{P}_{\alpha}$ , which can be seen as a trial probability measure on  $\Omega$  as it corresponds to a system initially in the trial state. Under  $\hat{\mathbb{P}}$ ,  $\hat{Q}_n(\alpha)$  is a martingale, so by the above arguments, it converges  $\hat{\mathbb{P}}$  almost surely to  $\mathbf{1}_{\Omega_{\alpha}}$ . As by hypothesis  $\hat{q}_0(\alpha) > 0$  whenever  $q_0(\alpha) > 0$  (which can be rephrased as:  $\mathbb{P}$  is absolutely continuous with respect to  $\hat{\mathbb{P}}$ ), a subset of  $\Omega$  of  $\hat{\mathbb{P}}$  probability 1 has also  $\mathbb{P}$  probability 1. So  $\lim_{n \rightarrow \infty} \hat{Q}_n(\alpha) = \mathbf{1}_{\Omega_{\alpha}} = \lim_{n \rightarrow \infty} Q_n(\alpha)$  with  $\mathbb{P}$  probability 1.

What is less direct is the determination of the convergence rate. This requires controlling the behavior of the counting processes  $N_n(o, i)$ . As recalled at the end of section C.4.1,  $N_n(o, i)/n \rightarrow \mu_{\alpha}(o, i)$  on a set<sup>6</sup> of  $\mathbb{P}_{\alpha}$ -measure 1. We want to infer that  $\mathbb{P}$ -almost surely,

$$\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_{\Upsilon}(o, i).$$

To prove it, we set  $\mathcal{L}(o, i) := \{\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_{\Upsilon}(o, i)\}$  and write

$$\mathbb{P}(\mathcal{L}(o, i)) = \sum_{\alpha} q_0(\alpha) \mathbb{P}(\mathcal{L}(o, i) | \Omega_{\alpha}) = \sum_{\alpha} q_0(\alpha) = 1,$$

where we used

$$\begin{aligned} \mathbb{P}(\mathcal{L}(o, i) | \Omega_{\alpha}) &= \mathbb{P}(\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_{\alpha}(o, i) | \Omega_{\alpha}) \\ &= \mathbb{P}_{\alpha}(\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_{\alpha}(o, i)) = 1. \end{aligned}$$

Observe now that  $\hat{Q}_n(\alpha)$  can be expressed as a function of the counting processes:

$$\hat{Q}_n(\alpha) = \frac{\hat{q}_0(\alpha) \prod_{(o, i) \in E} p^o(i|\alpha)^{N_n(o, i)}}{\sum_{\beta} \hat{q}_0(\beta) \prod_{(o, i) \in E} p^o(i|\beta)^{N_n(o, i)}}$$

Under the hypothesis that all  $p^o(i|\alpha)$ 's are  $> 0$ , we have that  $\hat{Q}_n(\alpha) > 0$  for every  $n$ , and the logarithm of the ratio between  $\hat{Q}_n(\alpha)$  and  $\hat{Q}_n(\beta)$  is well defined. Using the previous result, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left[ \frac{\hat{Q}_n(\beta)}{\hat{Q}_n(\alpha)} \right] &= \sum_{(o, i) \in E} \mu_{\Upsilon}(o, i) \ln \left[ \frac{p^o(i|\beta)}{p^o(i|\alpha)} \right] \\ &= \sum_{(o, i) \in E} \mu_{\Upsilon}^{red}(o) p^o(i|\Upsilon) \left( \ln \left[ \frac{p^o(i|\beta)}{p^o(i|\Upsilon)} \right] + \ln \left[ \frac{p^o(i|\Upsilon)}{p^o(i|\alpha)} \right] \right) \end{aligned}$$

6. For the record, the set  $\{N_n(o, i)/n \rightarrow \mu_{\alpha}(o, i)\}$  is measurable, as it can be written  $\bigcap_{m \in \mathbb{N}^*} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n > n_0} \{\omega \in \Omega, |N_n(o, i; \omega)/n - \mu_{\alpha}(o, i)| < 1/m\}$ .

Then, for a large enough  $n$ ,

$$\frac{\hat{Q}_n(\beta)}{\hat{Q}_n(\alpha)} \simeq e^{-n\bar{S}(\Upsilon|\beta)} e^{n\bar{S}(\Upsilon|\alpha)}$$

with  $\bar{S}(\Upsilon|\alpha)$  the mean relative entropy,  $\bar{S}(\Upsilon|\alpha) = \sum_{o \in \mathcal{O}} \mu_{\Upsilon}^{red}(o) S^o(\Upsilon|\alpha)$  where

$$S^o(\Upsilon|\alpha) = \sum_{i \in \text{spec}(o)} p^o(i|\Upsilon) (\ln[p^o(i|\Upsilon)] - \ln[p^o(i|\alpha)])$$

The relative entropy is always non negative, subsequently, the mean relative entropy is non negative too. Moreover, the mean relative entropy is null if and only if  $\Upsilon = \beta$  (all relative entropies null).

Using this property and  $\sum_{\beta} \hat{Q}_n(\beta) = 1$ , we obtain for  $\alpha \neq \Upsilon$ ,

$$\hat{Q}_n(\alpha)^{-1} = \sum_{\beta} \frac{\hat{Q}_n(\beta)}{\hat{Q}_n(\alpha)} \simeq e^{n\bar{S}(\Upsilon|\alpha)} (1 + \sum_{\beta \neq \Upsilon} e^{-n\bar{S}(\Upsilon|\beta)})$$

Then, to leading exponential order

$$\hat{Q}_n(\alpha) \simeq e^{-n\bar{S}(\Upsilon|\alpha)}$$

Hence, for  $n$  large enough, we proved that

$$\hat{Q}_n(\alpha) \simeq \begin{cases} 1 & \text{if } \alpha = \Upsilon \\ \text{const.} e^{-n\bar{S}(\Upsilon|\alpha)} & \text{else} \end{cases}$$

The limit distribution does not depend on the trial initial distribution but only on the complete measurement realization. The probability to have  $\hat{Q}_{\infty}(\alpha) = \delta_{\alpha, \gamma}$  equals  $q_0(\gamma)$ . With time independent Markovian feedback protocol, the convergence is exponential with a leading rate  $\bar{S}(\Upsilon|\alpha)$ .

#### C.4.4. Convergence rate tuning

Most of the time, when performing a measurement, one prefers it to take as little time as possible. The use of different partial measurement methods allows us to tune the convergence rate. Let us take an example. Suppose we want to discriminate between three possible pointer states of a system, and suppose that the partial measurement methods give only **True/False** as possible outputs. We denote  $T, F$  the partial measurement results and  $1, 2, 3$  the pointer states. Each partial measurement method can be tuned to maximize, up to a measurement error  $\varepsilon \ll 1$ , the probability of one of its outcome knowing the system is in one of the three pointer states. We shall show that this is not enough to maximize all convergence rates for arbitrary limit pointer state. It is the use of different measurement methods picked randomly that allows us to overcome this convergence rate problem.

Let us consider for instance two measurement methods. The first one, denoted  $a$ , has conditioned probabilities

$$p^a(T|1) = \varepsilon, p^a(T|2) = q, p^a(T|3) = 1 - \varepsilon$$

with  $q = O(1)$ . The second one, denoted  $b$ , is obtained by switching the probability conditioned on 1 and 2, that is

$$p^b(T|1) = q, p^b(T|2) = \varepsilon, p^b(T|3) = 1 - \varepsilon$$

Let us now look at the convergence rate conditioned on the limit pointer state to be 1. These are coded in the relative entropies. If only the measurement method  $a$  is used, one has:

$$\begin{aligned} S^a(1|2) &= \varepsilon \ln \left[ \frac{\varepsilon}{q} \right] + (1 - \varepsilon) \ln \left[ \frac{1 - \varepsilon}{1 - q} \right] \sim -\ln[1 - q] = O(1) \\ S^a(1|3) &= \varepsilon \ln \left[ \frac{\varepsilon}{1 - \varepsilon} \right] + (1 - \varepsilon) \ln \left[ \frac{1 - \varepsilon}{\varepsilon} \right] \sim -\ln[\varepsilon] \gg 1 \end{aligned}$$

The convergence of  $Q_n(3)$  toward 0 is quick but the one of  $Q_n(2)$  is rather slow. If measurement method  $b$  is used the interesting relative entropies are now

$$\begin{aligned} S^b(1|2) &= q \ln \left[ \frac{q}{\varepsilon} \right] + (1 - q) \ln \left[ \frac{1 - q}{1 - \varepsilon} \right] \sim -\ln[\varepsilon] \\ S^b(1|3) &= q \ln \left[ \frac{q}{1 - \varepsilon} \right] + (1 - q) \ln \left[ \frac{1 - q}{\varepsilon} \right] \sim -\ln[\varepsilon] \end{aligned}$$

All the convergences rates are then high if the limit pointer state is 1. But if the limit pointer state is not 1 but 2 then, using only the measurement method  $b$ , the relative entropy  $S^b(2|1)$  is

$$S^b(2|1) = \varepsilon \ln \left[ \frac{\varepsilon}{q} \right] + (1 - \varepsilon) \ln \left[ \frac{1 - \varepsilon}{1 - q} \right] \sim -\ln[q] = O(1)$$

and the convergence rate toward 2 is slow.

Now, if at each time one of the two measurement methods is used with equal probability  $\frac{1}{2}$ . The convergence rate for any  $i, j$  with  $i \neq j$  is

$$\bar{S}(i|j) = \frac{1}{2}(S^a(i|j) + S^b(i|j)) \sim -\ln[\varepsilon] \gg 1$$

As a consequence, the convergence rate is always high, whichever the limit pointer state is.

In the toy model, if the first partial measurement method correspond to  $\theta - \theta' = \frac{\pi}{3}$ , then

$$S^{\frac{\pi}{3}}(0|3) \sim 0.116$$

This is the slowest of all convergence rates. If the partial measurement method with  $\theta - \theta' = \frac{\pi}{6}$  is introduced and the partial measurement methods are chosen with equal probabilities each time, then

$$\bar{S}(0|3) \sim 1.18$$

and the slowest of all convergence rates is

$$\bar{S}(1|3) \sim 1.10$$

If only the  $\frac{\pi}{3}$  measure is used and the limit pointer state is 0, then a theoretical 99% confidence level is reached after about 50 measures. With the use of the two different partial measurement methods the same confidence level for the same limit state is reached in 5 measures. The same number of measurements is needed if the limit pointer state is 1, 2 or 3.

## C.5. Degeneracy and limit quantum state

Often the quantity we measure is a property common to several pointer states. In the quantum case, this corresponds to a degenerate projective Von Neumann measurement. There, at least two different eigenstates share the same eigenvalue. For our measurement process, degeneracies happen when several distributions  $p^o(\cdot|\cdot)$  are equal for different pointer states, so that some states cannot be distinguished. For example, in our toy model, whatever  $\theta - \theta'$  is, we have  $p^{\theta-\theta'}(\pm|p) = p^{\theta-\theta'}(\pm|p+4k)$  with  $k$  an integer. The pointer state with  $p$  photons cannot be distinguished from the one with  $p+4k$  photons.

In this section we study the system state evolution when degenerate repeated partial measurements are performed. In a first part we show the convergence of the system pointer state distribution. In a second part we focus on the quantum case and the influence of phases introduced between degenerate states by the repeated partial measurement process.

We shall partition the set of configurations into sectors. Let us define an equivalence relation among pointers by identifying two pointers whose partial measurement distributions are identical. That is: two pointers  $\alpha$  and  $\beta$  are said to be equivalent (denoted  $\alpha \sim \beta$ ) if, for any partial measurement method  $o$  and result  $i$ ,

$$p^o(i|\alpha) = p^o(i|\beta).$$

By definition the sector  $\underline{\alpha}$  is the equivalence class of  $\alpha$ . In the toy model the sectors are the sets  $\underline{p} = \{p+4k, k \in \mathbb{N}\}$  with  $p = 0, 1, 2, 3$ .

### C.5.1. State distribution convergence

We first look at the convergence of the pointer state distribution  $Q_n(\cdot)$  in case of degeneracy. The system distributions  $Q_n(\cdot)$  induce probability distributions  $\bar{Q}_n(\cdot)$  on sectors by

$$\bar{Q}_n(\underline{\alpha}) := \sum_{\alpha' \in \underline{\alpha}} Q_n(\alpha').$$

Since sectors form a partition of the set of pointer states, we have  $\sum_{\underline{\alpha}} \bar{Q}_n(\underline{\alpha}) = 1$ . The initial probability of a sector is  $\bar{q}_0(\underline{\alpha}) = \sum_{\alpha' \in \underline{\alpha}} q_0(\alpha')$ . The recursion relation (C.3) can obviously be lifted to a recursion relation for the sector distributions,

$$\bar{Q}_{n+1}(\underline{\alpha}) = \bar{Q}_n(\underline{\alpha}) \frac{p^{o_n}(i_n|\alpha)}{\sum_{\underline{\beta}} \bar{Q}_n(\underline{\beta}) p^{o_n}(i_n|\beta)}.$$

It is identical in structure to eq.(C.3) but with the bonus that it now is non degenerate. Lets assume that for two different sectors, it exists at least one  $\mathbb{P}$ -recurrent partial measurement method distinguishing between the two sectors: if  $\alpha \not\sim \beta$ , it exists  $o \in \mathcal{O}_s$  and  $i \in \text{spec}(o)$  such that  $p^o(i|\alpha) \neq p^o(i|\beta)$ . Thus we can use the non-degenerate case results but applied to the sector distribution. Hence,  $\bar{Q}_n(\cdot)$  almost surely converge and

$$\bar{Q}(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\gamma}}$$

with  $\underline{\gamma}$  the realization dependent limit sector. The probability that the limit sector be  $\underline{\gamma}$  is equal to  $\bar{q}_0(\underline{\gamma})$ .

From the martingale property, the state distribution converge (not only the sector distribution), and the point which remains to be discussed is what is this limit. Thanks to the relation  $q_0(\alpha') Q_n(\alpha) = q_0(\alpha) Q_n(\alpha')$  valid for any  $n$  if  $\alpha \sim \alpha'$ , we shall show that this limit is

$$Q(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \underline{\Upsilon} \\ q_0(\alpha)/\bar{q}_0(\underline{\Upsilon}) & \text{if } \alpha \in \underline{\Upsilon} \end{cases} \quad (\text{C.10})$$

with  $\underline{\Upsilon}$  the limit sector.

Indeed, the state distribution satisfies the recursion relation (C.3). Thus, if  $(i_k)_{k=0, \dots, n-1}$  are the  $n$  first partial measurements results, one has

$$\frac{Q_n(\alpha')}{q_0(\alpha')} = \frac{\prod_{k=1}^n p^{o_k}(i_k|\alpha)}{\sum_{\beta} q_0(\beta) \prod_{k=1}^n p^{o_k}(i_k|\beta)}$$

for any  $\alpha'$  in the sector  $\underline{\alpha}$ . The right hand side only depends on the sector  $\underline{\alpha}$ , and thus  $\frac{Q_n(\alpha)}{q_0(\alpha)} = \frac{Q_n(\alpha')}{q_0(\alpha')}$  if  $\alpha' \in \underline{\alpha}$ . From this equality, it follows that

$$Q(\alpha') = Q(\alpha) \frac{q_0(\alpha')}{q_0(\alpha)}$$

Since,  $\bar{Q}(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$ , we have  $Q(\alpha) = 0$  if  $\alpha \notin \underline{\Upsilon}$  and  $1 = \sum_{\alpha' \in \underline{\Upsilon}} Q(\alpha') = \frac{Q(\alpha)}{q_0(\alpha)} \bar{q}_0(\underline{\Upsilon})$ , for any  $\alpha \in \underline{\Upsilon}$ . Hence,

$$Q(\alpha) = \frac{q_0(\alpha)}{\bar{q}_0(\underline{\Upsilon})}, \quad \text{for } \alpha \in \underline{\Upsilon}$$

The probability of convergence to a sector as well as the limit state distribution (C.10) coincide, in quantum mechanics, with what would have been predicted by Von Neumann rules for degenerate projective measurements. The approach we have been following so far, based on tools from classical probability theory, gives no information on the convergence of the density matrix off-diagonal elements. It is the next section's purpose to discuss the evolution of the density matrix  $\rho_n$  and not only the evolution of the probabilities  $Q_n(\alpha) = \langle \alpha | \rho_n | \alpha \rangle$ .

### C.5.2. Density matrix convergence

We are now interested in the convergence of the system density matrix. In the basis of pointer states we may write:

$$\rho_n = \sum_{\alpha, \beta} A_n(\alpha, \beta) |\alpha\rangle \langle \beta|$$

with  $A_n(\alpha, \alpha) = Q_n(\alpha)$ . It evolves according to the recursion relation (C.1). The processes  $A_n(\alpha, \beta)$ , are not martingales, and their convergence can not be obtain through the martingale convergence theorem. Actually, they do not always converge. To obtain convergence a unitary evolution process has to be subtracted.

For each POVM, a phase between the pointer states is introduced by the operators  $M_i^{(o)}$ . Even inside a sector this phase can be nonzero. This possibility comes from the degeneracy criteria we unraveled previously. Two pointer states,  $\alpha, \beta$  can have a nonzero limit probability if they are in the same sector :  $\alpha \sim \beta$ . This criterion implies a norm equality  $|M^{(o)}(i|\alpha)| = |M^{(o)}(i|\beta)|$  for any  $i$  in the spectrum of any partial measurement, but not a full equality. So  $M^{(o)}(i|\alpha)$  and  $M^{(o)}(i|\beta)$  can differ by a phase. The density matrix converges either if this phase can be set to zero or if we absorb it through a transformation of the evolution.

Let us write the operators  $M^{(o)}(i|\alpha)$  in a phase times norm form

$$M^{(o)}(i|\alpha) = e^{-i\Delta t(E_\alpha + \theta^{(o)}(i|\alpha))} \sqrt{p^{(o)}(i|\alpha)}$$

The specific form of the phase is inspired by the Hamiltonian  $H^{(o)} = \sum_\alpha (E_\alpha \mathbb{I} + H_p^{(o)} + H_\alpha^{(o)})|\alpha\rangle\langle\alpha|$ . This is the most general Hamiltonian if one want  $U$  to fulfill the non demolition condition (C.2). In the above formula,  $\Delta t$  is the interaction time between the probe and the system.

Let us define a unitary operator process, diagonal in the pointer state basis,

$$\tilde{U}_n = \sum_\alpha e^{-i\Delta t(nE_\alpha + \sum_{(o,i) \in E} \theta^{(o)}(i|\alpha) N_n(o,i))} |\alpha\rangle\langle\alpha|$$

and the unitary equivalent conjugate density matrix process

$$\tilde{\rho}_n = \tilde{U}_n^\dagger \rho_n \tilde{U}_n \quad (\text{C.11})$$

The diagonal elements of  $\rho_n$  in the basis  $\{|\alpha\rangle\}$ , are not affected by this transformation. Their limits stay the same. Thus if  $\alpha$  or  $\beta$  are not in the limit sector  $\underline{\Upsilon}$ , according to the Cauchy-Schwartz theorem,  $\tilde{A}_\infty(\alpha, \beta) = 0$ . We are then interested in the limit of the elements  $\tilde{A}_n(\alpha, \beta)$  with  $\alpha, \beta \in \underline{\Upsilon}$ .

If  $\beta \in \underline{\alpha}$ , then  $q_0(\alpha) \tilde{A}_n(\alpha, \beta) = a_0(\alpha, \beta) Q_n(\alpha)$ . Repeating the discussion made in the section C.5.1, we get

$$\tilde{A}_\infty(\alpha, \beta) = \begin{cases} a_0(\alpha, \beta) / \bar{q}_0(\underline{\Upsilon}) & \text{if } \alpha, \beta \in \underline{\Upsilon} \\ 0 & \text{else} \end{cases}$$

Hence,  $\tilde{\rho}_n$  has an almost sure limit which coincides with the result of a Von Neumann measurement:  $\tilde{\rho}_\infty$  is equal to  $\rho_0$  projected on the system subspace corresponding to the sector  $\underline{\Upsilon}$ .

$$\lim_{n \rightarrow \infty} \tilde{\rho}_n = \frac{1}{\bar{q}_0(\underline{\Upsilon})} P_{\underline{\Upsilon}} \rho_0 P_{\underline{\Upsilon}} \quad (\text{C.12})$$

where  $P_{\underline{\Upsilon}} := \sum_{\gamma \in \underline{\Upsilon}} |\gamma\rangle\langle\gamma|$  is the projector on the subspace corresponding to the sector  $\underline{\Upsilon}$ .

In some cases the unitary operator process can be reduced to a deterministic one. For example if  $\theta^{(o)}(i|\alpha) = \theta^{(o)}(i|\beta)$  for every  $(o, i) \in E$  and every  $\alpha \sim \beta$ , then we can chose  $\tilde{U}_n = e^{-i\Delta t H_s}$ . This is the case in the toy model, from the periodicity of trigonometric functions, in a given sector all phases introduced by partial measurements are equal. If the limit sector is  $\underline{p} = \{p + 4k, k \in \mathbb{N}\}$ , then the photon field limit state will be

$$\lim_{n \rightarrow \infty} e^{in\Delta t \hat{p}} \rho_n e^{-in\Delta t \hat{p}} = \sum_{k, k' \in \mathbb{N}} \frac{a_0(p + 4k, p + 4k')}{\bar{q}_0(\underline{p})} |p + 4k\rangle\langle p + 4k'|$$

One other example corresponds to the case where the phases  $\theta^{(o)}(i|\alpha)$  do not depend on  $(o, i)$  but depend on  $\alpha$ . Then one can define  $H_{eff.} = H_s + \sum_\alpha \theta(\alpha) |\alpha\rangle\langle\alpha|$  and  $\tilde{U}_n = e^{-in\Delta t H_{eff.}}$ .

In most of the cases the unitary evolution  $\tilde{U}_n$  is a stochastic process and then in the limit  $n \rightarrow \infty$ , it remains a stochastic rotation inside the limit sector. When  $\tilde{U}_n$  is deterministic, the remaining rotation is deterministic too.

## C.6. Continuous diffusive limit

We shall now prove the convergence of the discrete processes we consider toward processes driven by time continuous Belavkin diffusive equations. Our proof, different from that used in [21], allows us to derive the continuous measurement diffusive equation not only for the quantum repeated indirect measurement process but also for the macroscopic Bayesian apparatus we defined. The quantum case is a peculiar realization of it.

The time continuous equation is found as a scaling limit of the discrete evolution when  $n$  goes to infinity with  $t = n\delta$  fixed ( $\delta = \Delta t$ ). We first study the pointer state distribution scaling limit,  $Q_t(\alpha) := \lim_{\delta \rightarrow 0} Q_{[t/\delta]}(\alpha)$ . The evolution equation for  $Q_t(\alpha)$  is given in eq.(C.15) below. We then look at the time continuous limit of the density matrix evolution and get the Belavkin diffusive equation eq.(C.23). In the quantum case, the continuous limit requires rescaling appropriately the system-probe interaction Hamiltonian as

$$H = H_s \otimes \mathbb{I}_p + \mathbb{I}_s \otimes H_p + \frac{1}{\sqrt{\delta}} H_I \quad (\text{C.13})$$

We present in some details the case with a unique partial measurement method. The results are then easily extended to cases with different measurement methods.

### C.6.1. Continuous time limit of the pointer state distribution

We are here interested in the state distribution continuous time limit. The results presented in this section apply to the general Bayesian recursion relation (C.3) — which in particular includes the case of repeated QND measurements. To begin with, we assume that there is only one partial measurement method<sup>7</sup>. Henceforth we suppress  $o$  from all the notations and let  $\mathcal{I}$  stand for the index set of outcomes. Note that the two filtrations  $\mathcal{F}_n$  and  $\mathcal{F}'_n$  coincide and carry the information on the first  $n$  partial measurements.

We assume that the conditional probabilities  $p(i|\alpha)$  depend on a further small parameter  $\delta$ , and are of the form

$$p(i|\alpha) = p_0(i)(1 + \sqrt{\delta} \Gamma_\delta(i|\alpha)) \quad (\text{C.14})$$

with  $p_0(i) > 0$  for all  $i$ 's and that  $\Gamma(i|\alpha) := \lim_{\delta \rightarrow 0+} \Gamma_\delta(i|\alpha)$  exists. Then  $\sum_i p_0(i) = 1$ , so that the  $p_0(i)$ 's specify a probability measure, and for every  $\delta$ ,  $\sum_i p_0(i) \Gamma_\delta(i|\alpha) = 0$ . The important point is that  $p_0(i)$  is independent of  $\alpha$ .

These hypothesis are of course satisfied in the quantum case with QND interaction Hamiltonian  $H_I = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes H_\alpha$  and rescaling  $H_I \rightarrow \frac{1}{\sqrt{\delta}} H_I$ . Then  $p_0(i) = |\langle i|\Psi\rangle|^2$  is the probability measure in absence of interaction, and

$$\Gamma(i|\alpha) := 2 \operatorname{Im} \left( \frac{\langle i|H_\alpha|\Psi\rangle}{\langle i|\Psi\rangle} \right)$$

We assume that  $\langle i|\Psi\rangle \neq 0$  for all  $i$ .

We first need to make precise the sense in which a limit on  $Q_n(\alpha)$  is to be taken.

For a fixed  $\omega \in \Omega$  the limit  $\lim_{\delta \rightarrow 0} Q_{[t/\delta]}(\alpha)$  is not expected to exist<sup>8</sup>. But there is some hope that, properly defined, a limit for the law of the process  $Q_{[t/\delta]}(\alpha)$ ,  $t \in \mathbb{R}^+$  exists. We

7. This condition will be relaxed in section C.6.1.

8. Think of the simple random walk: the convergence to Brownian motion is not sample by sample because  $\frac{S_{2n}}{\sqrt{2n}}$  has no reason to be close to  $\frac{S_n}{\sqrt{n}}$ .

shall prove in Appendix C.B that this is the case, specifying a bit the kind of convergence that is involved. Under the limiting law, the process  $Q_t(\cdot)$  satisfies the stochastic equation

$$dQ_t(\alpha) = Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \langle \Gamma_i \rangle_t) dX_t(i) \quad (\text{C.15})$$

where

$$\langle \Gamma_i \rangle_t := \sum_{\beta} Q_t(\beta) \Gamma(i|\beta).$$

Here  $X_t(i)$ , with  $\sum_i X_t(i) = 0$ , are continuous martingales with quadratic covariation

$$dX_t(i)dX_t(j) = dt (\delta_{i,j} p_0(i) - p_0(i)p_0(j)). \quad (\text{C.16})$$

We shall show that a vector solving this equation is a bounded martingale, to which the martingale convergence theorem can be applied with results similar to those in the discrete case:

$$Q_{\infty}(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \underline{\Upsilon} \\ q_0(\alpha)/\bar{q}_0(\underline{\alpha}) & \text{if } \alpha \in \underline{\Upsilon} \end{cases}$$

with  $\underline{\Upsilon}$  the limit sector. However, the sector definition is not the same as in the discrete case. In time continuous,  $\alpha$  and  $\beta$  are in the same sector if and only if  $\Gamma(i|\alpha) = \Gamma(i|\beta)$  for all partial measurement result  $i$ . The probability for the system to be in the sector  $\underline{\alpha}$  in the limit  $t$  goes to infinity is  $\bar{q}_0(\underline{\alpha}) = \sum_{\alpha' \in \underline{\alpha}} q_0(\alpha')$ .

The convergence is still exponential

$$Q_t(\alpha) = \exp(-t/\tau_{\gamma\alpha}), \quad \text{if } \alpha \notin \underline{\Upsilon}$$

with characteristic convergence time  $\tau_{\gamma\alpha}$ ,

$$2/\tau_{\gamma\alpha} = \sum_i p_0(i) (\Gamma(i|\alpha) - \Gamma(i|\gamma))^2 \quad (\text{C.17})$$

This coincides with the convergence rate we would have found by taking the relative entropy  $S(\gamma|\alpha)$  scaling limit. However, it is somewhat difficult to decipher that it originates from a relative entropy by only knowing its expression in the continuous-time limit.

## Preparation

We work with the model  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Our derivation is based on the use of the counting processes  $N_n(i)$ . Recall that  $N_0(i) = 0$  and that  $N_n(i) := \sum_{1 \leq m \leq n} \epsilon_m(i)$  for  $n \geq 1$ , where  $\epsilon_n(i) := \mathbf{1}_{I_n=i}$  is 1 if the  $n^{\text{th}}$  partial measurement outcome is  $i$  and 0 otherwise.

We start by listing some properties of these counting processes and their relationship to the solution of (C.3). Then we shall formulate and prove the analogous statements for the continuous time limit.

It is obvious that the filtration  $\mathcal{F}_0, \mathcal{F}_1, \dots$  is the natural filtration of the vector counting processes  $N_n$ .

Also recall that the random recursion relation (C.3) can be solved in terms of the counting processes as

$$Q_n(\alpha) = q_0(\alpha) \frac{\prod_i p(i|\alpha)^{N_n(i)}}{\sum_{\beta} q_0(\beta) \prod_i p(i|\beta)^{N_n(i)}}.$$

A trivial but crucial observation is that under each  $\mathbb{P}_\alpha$ ,  $N_n$  is the sum of independent identically distributed (i.i.d) random vectors.

As a first consequence, a simple computation leads to

$$\mathbb{E} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (N_{n_l}(i) - N_{n_{l-1}}(i))} \right) = \sum_{\alpha} q_0(\alpha) \prod_{l=1}^k \left( \sum_i e^{\lambda_l(i)} p(i|\alpha) \right)^{n_l - n_{l-1}} \quad (\text{C.18})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences of integers  $0 = n_0 \leq n_1 \leq \dots \leq n_k$  of length  $k$ , and arbitrary (complex)  $\lambda_l(i)$ 's. A second consequence is that under  $\mathbb{P}_\alpha$  each  $N_n(i)$  is a sub-martingale and  $N_n(i) = (N_n(i) - np(i|\alpha)) + np(i|\alpha)$  is its Doob decomposition as a martingale plus a predictable (in that case deterministic) increasing process. Moreover, if  $n \geq 1$ , and if  $X$  is an  $\mathcal{F}_{n-1}$  measurable random variable, we compute

$$\begin{aligned} \mathbb{E}(X \epsilon_n(i)) &= \sum_{\alpha} q_0(\alpha) \mathbb{E}_{\alpha}(X \epsilon_n(i)) \\ &= \sum_{\alpha} q_0(\alpha) \mathbb{E}_{\alpha}(X) p(i|\alpha) = \sum_{\alpha} \mathbb{E}(X Q_{n-1}(\alpha)) p(i|\alpha). \end{aligned}$$

For the last equality we used the  $Q$ 's characterization as Radon-Nikodym derivatives. This proves that  $\mathbb{E}(\epsilon_n(i) | \mathcal{F}_{n-1}) = \sum_{\alpha} Q_{n-1}(\alpha) p(i|\alpha) = \pi_n(i)$ . Hence, setting

$$A_n(i) := \sum_{m=1}^n \pi_m(i),$$

an increasing predictable process, we find that  $X_n(i) := N_n(i) - A_n(i)$  is an  $\mathcal{F}_n$ -martingale under  $\mathbb{P}$ , so each  $N_n(i)$  is again a sub-martingale with Doob decomposition

$$N_n(i) = X_n(i) + A_n(i) \quad (\text{C.19})$$

under  $\mathbb{P}$ .

Finally, by some simple algebra we may rephrase the random recursion relation satisfied by the  $Q$ 's as a stochastic difference equation

$$Q_n(\alpha) - Q_{n-1}(\alpha) = Q_{n-1}(\alpha) \sum_i \frac{p(i|\alpha)}{\pi_{n-1}(i)} (X_n(i) - X_{n-1}(i)). \quad (\text{C.20})$$

### Derivation of the pointer state distribution evolution

Equation (C.20) admits eq.(C.15) as a naive continuous time limit when  $\delta$ , the scaling parameter, goes to  $0^+$ . To put the validity of this formal approach on a firmer ground, one needs to prove the existence of a continuous time limit. This is a classical topic, but the presence of the scaling parameter  $\delta$  in various places prevents us from applying standard theorems straightforwardly. So we rely on a down-to-earth approach, which is rather technical. For this reason we relegated the argument to appendix C.B. This is where the interested reader should look for some background, precise definitions, etc. We give here a brief summary:

– By an appropriate interpolation procedure, one defines a  $\delta$ -dependent push-forward  $\mu_\alpha(\delta)$  of each  $\mathbb{P}_\alpha$  and  $\mu(\delta)$  of  $\mathbb{P}$  in  $C_0(\mathbb{R}^+, \mathbb{R}^I)$ , the space of continuous functions from  $\mathbb{R}^+$  to  $\mathbb{R}^I$  vanishing at 0.

- We are not able to prove the convergence in law of the  $\mu_\alpha(\delta)$  or of  $\mu(\delta)$  when  $\delta \rightarrow 0^+$ .
- However, the finite dimensional distributions of the joint processes  $N_n(i)$  and  $Q_n(\alpha)$  under each  $\mathbb{P}_\bullet$  (where  $\bullet$  stands either for an element of  $\mathcal{S}$  or for nothing) admit, after appropriate time dependent centering and scaling, continuous time limits which are the joint finite dimensional distributions, under a probability measure  $\mu_\bullet$  on  $C_0(\mathbb{R}^+, \mathbb{R}^I)$ , for processes  $W_t(i)$ , to be thought of as

$$\lim_{\delta \rightarrow 0^+} \sqrt{\delta}(N_{t/\delta}(i) - p_0(i)t/\delta),$$

and  $Q_t(\alpha)$ , to be thought of as  $\lim_{\delta \rightarrow 0^+} Q_{t/\delta}(\alpha)$ .

- The process  $W_t$  is the canonical coordinate process on  $C(\mathbb{R}^+, \mathbb{R}^I)$ , and its natural filtration  $\mathcal{G}_t$  is to be thought of as the continuous time limit of the natural filtration for  $N_n$ , i.e. as the information collected by indirect measurements up to time  $t$ .

- The identity  $\mu = \sum_\alpha q_0(\alpha)\mu_\alpha$  holds. The Radon-Nikodym derivative of  $\mu(\alpha)$  with respect to  $\mu$  on  $\mathcal{G}_t$  is  $M_t(\alpha)/M_t$  where

$$M_t(\alpha) := e^{\sum_i \Gamma(i|\alpha)W_t(i) - \frac{t}{2} \sum_i p_0(i)\Gamma(i|\alpha)^2}, \quad M_t := \sum_\alpha q_0(\alpha)M_t(\alpha)$$

For each  $\alpha$ ,  $M_t^{-1}(\alpha)$  is a  $\mathcal{G}_t$ -martingale under  $\mu_\alpha$ , and  $M_t^{-1}$  is a  $\mathcal{G}_t$ -martingale under  $\mu$ .

- For each  $T > 0$ , under the measure  $M_T^{-1}d\mu$  (which coincides with  $M_T^{-1}(\alpha)d\mu_\alpha$  for every  $\alpha$ ), the process  $(W_t)_{t \in [0, T]}$  is a continuous time-homogeneous centered Gaussian process with covariance  $\min(t, s)(\delta_{i,j}p_0(i) - p_0(i)p_0(j))$ . Thus, by Girsanov's theorem, under each  $\mu_\alpha$ ,  $W_t$  is a continuous time-homogeneous Gaussian process with independent increments, characterized by

$$\begin{aligned} \mathbb{E}^{\mu_\alpha}(W_t(i)) &= tp_0(i)\Gamma(i|\alpha) \\ \text{Cov}^{\mu_\alpha}(W_t(i), W_s(j)) &= \min(t, s)(\delta_{i,j}p_0(i) - p_0(i)p_0(j)). \end{aligned}$$

- There is an explicit formula for the  $Q_t$ 's in terms of the  $W_t$ 's, namely:

$$Q_t(\alpha) = q_0(\alpha) \frac{M_t(\alpha)}{M_t} = q_0(\alpha) \frac{e^{\sum_i \Gamma(i|\alpha)W_t(i) - \frac{t}{2} \sum_i p_0(i)\Gamma(i|\alpha)^2}}{\sum_\beta q_0(\beta) e^{\sum_i \Gamma(i|\beta)W_t(i) - \frac{t}{2} \sum_i p_0(i)\Gamma(i|\beta)^2}}.$$

We are now in position to check that all the properties established in the discrete setting, as listed in section C.6.1, have a direct naive counterpart in the continuous time setting.

The construction of the filtration  $\mathcal{G}_t$  as the natural filtration for the canonical process was already explained. We have also already mentioned that there is an explicit formula for the  $Q_t$ 's. The counterpart of (C.18), the Laplace transform of the counting processes joint distributions is given for the canonical process in eq.(C.29), Appendix C.B.

The counterpart of the counting process Doob decomposition under  $\mathbb{P}_\alpha$  is  $W_t(i) = (W_t(i) - tp_0(i)\Gamma(i|\alpha)) + tp_0(i)\Gamma(i|\alpha)$  under  $\mu_\alpha$ .

To get the counterpart of the counting process Doob-Meyer decomposition under  $\mathbb{P}$ , i.e. the Doob-Meyer decomposition of  $W_t(i)$  under  $\mu$ , we use Girsanov's theorem. As recalled above, for every  $T > 0$   $(W_t(i))_{t \in [0, T]}$  is a continuous martingale under  $M_T^{-1}d\mu$ . From

$$dM_t/M_t = \sum_\alpha q_0(\alpha) \frac{M_t(\alpha)}{M_t} \sum_i \Gamma(i|\alpha) dW_t(i) = \sum_\alpha Q_t(\alpha) \sum_i \Gamma(i|\alpha) dW_t(i),$$

we infer that the increasing process

$$A_t(i) := \int_0^t ds \sum_{\alpha} Q_s(\alpha) p_0(i) \Gamma(i|\alpha)$$

is the compensator of  $W_t(i)$ , i.e.

$$X_t(i) := W_t(i) - A_t(i)$$

is a  $\mathcal{G}_t$  martingale under  $\mu$ , with quadratic variation given by (C.16). It is easily seen that  $A_t(i), X_t(i)$  are the obvious continuous time limits of  $A_n(i), X_n(i)$ .

It remains to write down the stochastic evolution equations for the  $Q_t$ 's. By Itô's formula for a ratio, we find

$$\frac{dQ_t(\alpha)}{Q_t(\alpha)} = \left( \frac{dM_t(\alpha)}{M_t(\alpha)} - \frac{dM_t}{M_t} \right) \left( 1 - \frac{dM_t}{M_t} \right).$$

leading immediately to (C.15) which we reproduce for convenience:

$$dQ_t(\alpha) = Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \langle \Gamma_i \rangle_t) dX_t(i)$$

where  $\langle \Gamma_i \rangle_t := \sum_{\beta} Q_t(\beta) \Gamma(i|\beta)$ . Note again that this equation is also the naive continuous time limit of the discrete equation (C.20).

To summarize, one makes no mistakes if one works naively and forgets about the lengthy rigorous construction of the continuous time limit. This gives us confidence in what follows to proceed straightforwardly in the derivation of continuous time equations in more complicated situations.

### Convergence of the continuous time evolution

We now prove the convergence of  $Q_t(\alpha)$  when  $t$  goes to infinity. Its almost sure convergence is a direct consequence of its martingale property. We need to prove that the final distribution is

$$Q_{\infty}(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \underline{\Upsilon} \\ q_0(\alpha)/\bar{q}_0(\underline{\alpha}) & \text{if } \alpha \in \underline{\Upsilon} \end{cases} \quad (\text{C.21})$$

and that the convergence is exponential with the characteristic time  $\tau_{\Upsilon\alpha}$ .

First we prove that the limit of the sector distribution  $\bar{Q}_t(\underline{\alpha}) := \sum_{\alpha \in \underline{\alpha}} Q_t(\alpha)$  is

$$\bar{Q}_{\infty}(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$$

The equation of evolution for the sector distribution is

$$d\bar{Q}_t(\underline{\alpha}) = \sum_{\alpha' \in \underline{\alpha}} dQ_t(\alpha') = \bar{Q}_t(\underline{\alpha}) \sum_i (\Gamma(i|\alpha) - \langle \Gamma_i \rangle_t) dX_t(i)$$

In the limit  $t \rightarrow \infty$ , we have  $Q_{\infty}(\underline{\alpha})(\Gamma(i|\alpha) - \langle \Gamma_i \rangle_{\infty}) = 0$  for all  $i$ ,  $\mu$ -almost surely. Then either  $\bar{Q}_{\infty}(\underline{\alpha}) = 0$  or  $\Gamma(i|\alpha) = \sum_i \bar{Q}_{\infty}(\underline{\beta}) \Gamma(i|\beta)$ . Since  $\Gamma(i|\alpha) \neq \Gamma(i|\beta)$  if  $\underline{\alpha} \neq \underline{\beta}$ , the solution to the limit equation is  $\bar{Q}_{\infty}(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$ .

Second we show that  $\frac{Q_t(\alpha')}{Q_t(\alpha)} = \frac{q_0(\alpha')}{q_0(\alpha)}$  if  $\alpha', \alpha$  are in the same sector. As in the discrete case, this relation implies eq.(C.21). We compute:

$$\frac{Q_t(\alpha)}{Q_t(\alpha')} d\frac{Q_t(\alpha')}{Q_t(\alpha)} = \frac{dQ_t(\alpha')}{Q_t(\alpha')} - \frac{dQ_t(\alpha)}{Q_t(\alpha)} + \left(\frac{dQ_t(\alpha)}{Q_t(\alpha)}\right)^2 - \frac{dQ_t(\alpha')}{Q_t(\alpha')} \frac{dQ_t(\alpha)}{Q_t(\alpha)}$$

Since,  $\frac{dQ_t(\alpha')}{Q_t(\alpha')} = \frac{dQ_t(\alpha)}{Q_t(\alpha)}$  for  $\alpha$  and  $\alpha'$  in the same sector, we obtain  $d\frac{Q_t(\alpha')}{Q_t(\alpha)} = 0$  if  $\alpha, \alpha' \in \underline{\alpha}$ . For all time  $t$ ,

$$\frac{Q_t(\alpha')}{Q_t(\alpha)} = \frac{q_0(\alpha')}{q_0(\alpha)} \text{ if } \alpha \sim \alpha'$$

which achieves the proof for the limit pointer state distribution.

Finally we prove the exponential convergence. The tools we use are the convergence of the pointer state distribution and the Itô calculus. From the distribution convergence, we have  $\langle \Gamma_i \rangle_t \simeq \Gamma(i|\Upsilon)$  for  $t$  large enough. The evolution equation (C.15) for  $\alpha \notin \underline{\Upsilon}$  becomes  $dQ_t(\alpha) \simeq Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \Gamma(i|\Upsilon)) dX_t(i)$ . This equation is a well known stochastic exponential equation. Thus, at large time  $t$ , with good approximation

$$Q_t(\alpha) \simeq \text{const.} e^{-\frac{t}{2} \sum_i p_0(i) (\Gamma(i|\alpha) - \Gamma(i|\Upsilon))^2 + \sum_i X_t(i) (\Gamma(i|\alpha) - \Gamma(i|\Upsilon))}$$

for  $\alpha \notin \underline{\Upsilon}$ . Keeping only the leading term in the exponential we obtain the exponential decrease,  $Q_t(\alpha) \simeq \exp(-t/\tau_{\Upsilon\alpha})$ , with  $\tau_{\Upsilon\alpha}$  given in eq.(C.17).

### Different partial measurement methods

The previous results can easily be extended to cases where different measurement methods are randomly used. Since proofs are similar to those of previous sections, here we only present a general outline of the approach. We limit ourselves to a time and realization independent partial measurement method distribution. In this case  $d_n(o_1, i_1, \dots, i_n, o_{n+1}) = \prod_{k=1}^{n+1} c(o_k)$  with  $\sum_o c(o) = 1$ .

To stay in the scope of the diffusive limit we assume that for any  $o$ ,  $\langle i|\Psi^{(o)} \rangle \neq 0$ .

Following previous sections, we define linear interpolations  $W_t^{(\delta)}(o, i)$  of the counting processes which naively read

$$\sqrt{\delta}(N_{t/\delta}(o, i) - c(o)p_0^o(i)t/\delta)$$

See Appendix C.C for precise definitions. As shown in this appendix, all finite dimensional distribution functions of  $W_t^{(\delta)}(o, i)$  under (a push-forward of)  $\mathbb{P}_\alpha$  (resp.  $\mathbb{P}$ ) have a finite limit as  $\delta \rightarrow 0^+$  which coincide with those of continuous random processes, denoted  $W_t(o, i)$ , under appropriate measures denoted  $\mu_\alpha$  (resp.  $\mu$ ). Under  $\mu_\alpha$ ,  $W_t(o, i)$  is a Gaussian process with

$$\begin{aligned} \mathbb{E}^{\mu_\alpha}(W_t(o, i)) &= t c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha) \\ \text{Cov}^{\mu_\alpha}(W_t(o, i), W_s(o', j)) &= \min(t, s) (c(o)p_0^o(i)\delta_{(o,i),(o',j)} - c(o)p_0^o(i)c(o')p_0^{o'}(j)). \end{aligned}$$

with  $p_0^o(i) = |\langle i|\Psi^{(o)} \rangle|^2$  and  $\Gamma^{(o)}(i|\alpha) = 2\text{Im} \left( \frac{\langle i|H_\alpha^{(o)}|\Psi^{(o)} \rangle}{\langle i|\Psi^{(o)} \rangle} \right)$ .

The measure  $\mu$  is the sum  $\mu = \sum_\alpha q_0(\alpha)\mu_\alpha$ . The Radon-Nikodym derivative of  $\mu(\alpha)$  with respect to  $\mu$  is  $M_t(\alpha)/M_t$  where  $M_t = \sum_\alpha q_0(\alpha)M_t(\alpha)$  with

$$M_t(\alpha) = e^{\sum_{(o,i) \in E} \Gamma^{(o)}(i|\alpha) W_t(o,i) - \frac{t}{2} \sum_{(o,i) \in E} c(o)p_0^o(i) \Gamma^{(o)}(i|\alpha)^2}$$

As in the section C.6.1 we define

$$X_t(o, i) = W_t(o, i) - \int_0^t \sum_{\alpha} Q_s(\alpha) c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha) ds$$

The  $X_t(o, i)$  are martingales under  $\mu$ . From this definition we obtain straightforwardly

$$dQ_t(\alpha) = Q_t(\alpha) \sum_{(o,i) \in E} (\Gamma^{(o)}(i|\alpha) - \langle \Gamma^{(o)}(i) \rangle_t) dX_t(o, i)$$

with  $\langle \Gamma^{(o)}(i) \rangle_t = \sum_{\alpha} \Gamma^{(o)}(i|\alpha) Q_t(\alpha)$  and

$$dX_t(o, i) dX_t(o', j) = dt (c(o) p_0^o(i) \delta_{(o,i), (o',j)} - c(o) p_0^o(i) c(o') p_0^{o'}(j))$$

The limit of  $Q_t(\alpha)$  is the same but the sectors are now the sets of basis states such that  $\Gamma^{(o)}(i|\alpha') = \Gamma^{(o)}(i|\alpha)$  for all partial measurement methods and all partial measurement results. The convergence toward the limit distribution is exponential

$$Q_t(\alpha) \simeq \exp \left[ -\frac{t}{2} \sum_{(o,i) \in E} c(o) p_0^o(i) (\Gamma^{(o)}(i|\alpha) - \Gamma^{(o)}(i|\Upsilon))^2 \right]$$

The approximation hold if  $t$  is large enough. The convergence is exponential with a characteristic time

$$\frac{2}{\tau_{\Upsilon\alpha}} = \sum_{(o,i) \in E} c(o) p_0^o(i) (\Gamma^{(o)}(i|\alpha) - \Gamma^{(o)}(i|\Upsilon))^2$$

We find a convergence rate which is a mean convergence rate as in the discrete case. The same result is found by taking the scaling limit of the discrete case mean relative entropy.

### C.6.2. Density matrix evolution

We are now interested in the density matrix evolution.

As in section C.5.2, the density matrix at time  $n$  can be decomposed in the basis of pointer states:

$$\rho_n = \sum_{\alpha, \beta} A_n(\alpha, \beta) |\alpha\rangle \langle \beta|$$

The same decomposition applies to the time continuous density matrix we will define. The recurrence relation (C.1) translates for  $A_n(\alpha, \beta)$  in

$$A_n(\alpha, \beta) = \frac{A_{n-1}(\alpha, \beta) M^{(o_n)}(i_n|\alpha) M^{(o_n)}(i_n|\beta)^{\star}}{\sum_{\gamma} q_{n-1}(\gamma) p^{o_n}(i_n|\gamma)}$$

Where  $M^{(o)}(i|\alpha) = \langle i|U^{(o)}(\alpha)|\Psi^{(o)}\rangle$ . For  $\alpha = \beta$ , this reproduces the pointer state distribution recurrence relation (C.3), as expected.

We first limit ourselves to the case where only one partial measurement method is used and we omit the index  $o$ . The results will then be generalized to different partial measurement methods. We used a few hypotheses to get the continuous-time limit:

– The first two assumptions are related to the development in  $\sqrt{\delta}$  of the conditional probabilities  $p(i|\alpha)$ . As stated before, the interaction Hamiltonian must be rescaled  $H_I \rightarrow \frac{1}{\sqrt{\delta}}H_I$  and for any partial measurement result  $i$ ,  $\langle i|\Psi \rangle \neq 0$ . Then

$$p(i|\alpha) = p_0(i)(1 + \sqrt{\delta}\Gamma_\delta(i|\alpha))$$

with  $p_0(i) = |\langle i|\psi \rangle|^2$ . The assumption  $\langle i|\Psi \rangle \neq 0$  leads to the diffusive limit. If this condition is not fulfilled for every  $i$ , then a jump-diffusion limit is found as shown in [23].

– A third assumption is needed to obtain a convergence of the evolution of the phases between different pointer states. The interaction Hamiltonian expectation must be zero :

$$\langle \Psi | H_I | \Psi \rangle = 0$$

Under these assumptions, we show in Appendix C.D that the time continuous evolution derived from the discrete time case is

$$A_t(\alpha, \beta) = A_0(\alpha, \beta) \frac{e^{l(\alpha, \beta)t - i \sum_i (c(i|\alpha) - c(i|\beta)^*) W_t(i)}}{\sum_\gamma q_0(\gamma) e^{\sum_i -i \Gamma(i|\gamma) W_t(i) - \frac{t}{2} p_0(i) \Gamma(i|\gamma)^2}} \quad (\text{C.22})$$

with  $c(i|\alpha) = \frac{\langle i|H_I|\Psi \rangle}{\langle i|\Psi \rangle}$  and

$$l(\alpha, \beta) := -i(E_\alpha - E_\beta) - \frac{1}{2} \sum_i p_0(i) (|c(i|\alpha)|^2 + |c(i|\beta)|^2 - c(i|\alpha)^2 - c(i|\beta)^2)$$

If we set  $\alpha = \beta$  we recover the result on the pointer state distribution.

A simple computation using Itô rules shows that this process is solution of a Belavkin diffusive equation:

$$d\rho_t = L(\rho_t) - i \sum_i (C_i \rho_t - \rho_t C_i^\dagger - \rho_t \text{Tr}[(C_i - C_i^\dagger) \rho_t]) dX_t(i) \quad (\text{C.23})$$

with the Lindbladian

$$L(\rho) = -i[H_s, \rho] + \sum_i p_0(i) (C_i \rho C_i^\dagger - \frac{1}{2} \{C_i^\dagger C_i, \rho\})$$

and  $C_i := \sum_\alpha c(i|\alpha) |\alpha\rangle \langle \alpha| = \frac{\langle i|H_I|\Psi \rangle}{\langle i|\Psi \rangle}$ .

As shown in [20], this equation corresponds to the time continuous limit of repeated POVM processes (C.1) even if the non destruction assumption (C.2) is not fulfilled.

In the next section we study the long time behavior of such evolution in the non destructive case.

### Long time convergence of the density matrix

The pointer state distribution convergence indicates that, in the long time limit, the system is in a subspace of basis  $\underline{\Upsilon}$ . This information only tells us what is the limit of the elements  $A_t(\alpha, \beta)$  when  $\alpha$  or  $\beta$  are not in the limit sector  $\underline{\Upsilon}$ . From the Cauchy-Schwarz theorem,  $\lim_{t \rightarrow \infty} Q_t(\alpha) Q_t(\beta) = 0$  implies  $\lim_{t \rightarrow \infty} A_t(\alpha, \beta) = 0$ . For the elements  $A_t(\alpha, \beta)$  with  $\alpha, \beta \in \underline{\Upsilon}$ , the limit  $t \rightarrow \infty$  is yet unknown.

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We decompose the operators  $C_i$  in a sum of two hermitian operators

$$C_i = R_i + iS_i$$

with  $R_i = \sum_{\alpha} \text{Re}(c(i|\alpha)) |\alpha\rangle\langle\alpha|$  and  $S_i = \sum_{\alpha} \frac{1}{2} \Gamma(i|\alpha) |\alpha\rangle\langle\alpha|$ .

As in the discrete time case, the density matrix evolution has to be modified by a unitary process in order to get convergence when  $t$  goes to infinity. Let  $\tilde{U}_t$  be the unitary diagonal operator defined via

$$\tilde{U}_t^{-1} d\tilde{U}_t = -i(H_s - \sum_i p_0(i)[R_i(S_i - 2\langle S_i \rangle_t) - \frac{i}{2} R_i^2])dt - i \sum_i R_i dX_t(i)$$

and let  $\tilde{\rho}_t$  be the modified density matrix

$$\tilde{\rho}_t = \tilde{U}_t^\dagger \rho_t \tilde{U}_t$$

As we show below it has an almost sure limit

$$\lim_{t \rightarrow \infty} \tilde{\rho}_t = \frac{1}{q_0(\underline{\Upsilon})} P_{\underline{\Upsilon}} \rho_0 P_{\underline{\Upsilon}} \quad (\text{C.24})$$

where  $P_{\underline{\Upsilon}} := \sum_{\gamma \in \underline{\Upsilon}} |\gamma\rangle\langle\gamma|$  is the projector on the subspace corresponding to the sector  $\underline{\Upsilon}$ . Therefore,  $\tilde{\rho}_\infty$  is equivalent to the density matrix we would have found if an initial Von Neumann measurement had been performed on the system. The unitary operator  $\tilde{U}_t$  only induces a rotation inside the limit subspace.

Recall that we only need to prove the convergence of the  $\tilde{\rho}_t$  matrix elements corresponding to two pointer states in the same sector. From the Belavkin equation (C.23) and using Itô rules, we find the evolution equation for  $\tilde{\rho}_t$ :

$$d\tilde{\rho}_t = \sum_i p_0(i)(S_i \tilde{\rho}_t S_i - \frac{1}{2} \{S_i S_i, \tilde{\rho}_t\})dt - i \sum_i (\{S_i, \tilde{\rho}_t\} - 2\text{Tr}[S_i \tilde{\rho}_t])dX_t(i)$$

Thus, the time evolution of matrix elements  $\tilde{A}_t(\alpha, \beta)$  of  $\tilde{\rho}_t$  with  $\beta$  and  $\alpha$  in the same sector is,

$$d\tilde{A}_t(\alpha, \beta) = \tilde{A}_t(\alpha, \beta) \sum_i (\Gamma(i|\alpha) - \langle \Gamma(i) \rangle_t) dX_t(i)$$

Noticing that  $Q_t(\alpha) d\tilde{A}_t(\alpha, \beta) = \tilde{A}_t(\alpha, \beta) dQ_t(\alpha)$  and repeating the discussion of section C.6.1, we get

$$\tilde{A}_\infty(\alpha, \beta) = \begin{cases} \frac{A_0(\alpha, \beta)}{q_0(\underline{\Upsilon})} & \text{if } \alpha \text{ and } \beta \in \underline{\Upsilon} \\ 0 & \text{else} \end{cases}$$

This proves the limit (C.24).

### Extension to different partial measurement methods

We can extend our results to cases where different partial measurement methods are used. Once again we limit ourselves to time independent random protocols. The density matrix evolution is modified as follows:

$$d\rho_t = L(\rho_t) dt + \sum_{(o,i) \in E} \mathcal{D}_{(o,i)}(\rho_t) dX_t(o, i)$$

with

$$\mathcal{D}_{(o,i)}(\rho_t) = -i(C_i^{(o)}\rho_t - \rho_t C_i^{(o)\dagger} - \rho_t \text{Tr}[C_i^{(o)}\rho_t - \rho_t C_i^{(o)\dagger}])$$

where  $C_i^{(o)} = \frac{\langle i|H_f^{(o)}|\Psi_o\rangle}{\langle i|\Psi_o\rangle}$  and

$$L(\rho_t) = -i[H_s, \rho_t] + \sum_{(o,i) \in E} c(o) p_0^o(i) (C_i^{(o)}\rho_t C_i^{(o)\dagger} - \frac{1}{2}\{C_i^{(o)\dagger} C_i^{(o)}, \rho_t\})$$

As before  $c(o)$  is the probability of using measurement method  $o$ . The limit density matrix can be analyzed as above: we obtain identical convergence statements once the density matrix has been rotated using an appropriate unitary  $\tilde{U}_t$ .

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### C.A. Details for mutual singularity

We prove that if  $o$  is recurrent but  $(o, i)$  is not then  $\sum_{\gamma} Q(\gamma) p^o(i|\gamma) = 0$ .

Observe that under  $\mathbb{P}_{\alpha}$  we have the Markov property

$$\mathbb{E}_{\alpha}(\mathbf{1}_{I_n=i} | \mathcal{F}_{n-1}) = p^{O_n}(i|\alpha) \quad (\text{C.25})$$

It says that, under  $\mathbb{P}_{\alpha}$ ,  $O_n = o$  the next measurement outcome is  $i \in \text{spec}(o)$  with probability  $p^o(i|\alpha)$  independently of what happened before.

Now assume that under  $\mathbb{P}_{\alpha}$  measurement method  $o$  is recurrent with probability 1. Take  $0 \leq T_1 < T_2 < \dots < T_k \dots$  to be the times when the measurement method is  $o$ . We show, using the strong Markov property, that  $I_{T_1}, I_{T_2}, \dots$  are independent identically distributed random variables with distribution  $p^o(\cdot|\alpha)$ . This is quite natural: the functions  $d_n$  help choosing the measurement method, but they do not influence the measurement result.

Indeed, note first that the above statement is trivial when there is only one measurement method, because then there is no need to invoke stopping times and the strong Markov property. In the general case, note the slight mismatch with usual notations:  $\{T_k \leq n\}$  is in fact  $\mathcal{F}_{n-1}$  measurable, so it is natural to write  $\mathcal{F}_{T_k-1}$  for the algebra associated to the stopping time  $T_k$ . Then write

$$\begin{aligned} \mathbb{E}_{\alpha}(\mathbf{1}_{I_{T_1}=i_1} \dots \mathbf{1}_{I_{T_k}=i_k} | \mathcal{F}_{T_k-1}) &= \mathbf{1}_{I_{T_1}=i_1} \dots \mathbb{E}_{\alpha}(\mathbf{1}_{I_{T_k}=i_k} | \mathcal{F}_{T_k-1}) \\ &= \mathbf{1}_{I_{T_1}=i_1} \dots p^o(i_k|\alpha) \end{aligned}$$

One can go on to condition with respect to  $\mathcal{F}_{T_{k-1}-1}, \dots$  until one finds the plain expectation

$$\mathbb{E}_{\alpha}(\mathbf{1}_{I_{T_1}=i_1} \dots \mathbf{1}_{I_{T_k}=i_k}) = p^o(i_1|\alpha) \dots p^o(i_k|\alpha).$$

As a consequence, for any  $\alpha$  such that  $o$  is recurrent under  $\mathbb{P}_{\alpha}$ :

- either  $p^o(i|\alpha) > 0$  and with  $\mathbb{P}_{\alpha}$ -probability 1 the outcome  $i$  appears infinitely many times in the sequence  $I_{T_1}, I_{T_2}, \dots$ , i.e.  $(o, i)$  is recurrent with probability 1,
- or  $p^o(i|\alpha) = 0$  and  $i$  never appears in the sequence  $I_{T_1}, I_{T_2}, \dots$ , i.e.  $(o, i)$  never appears.

Now assume that  $o$  is recurrent under all  $\mathbb{P}_{\alpha}$ 's. The above implies immediately that the probability under  $\mathbb{P}$  that  $(o, i)$  is non-recurrent (this event is denoted by  $\tilde{A}_{(o,i)}$ ) is given by  $\sum_{\gamma, p^o(i|\gamma)=0} q_0(\gamma)$ .

If  $p^o(i|\beta) = 0$  then  $Q(\beta) = Q(\beta) \mathbf{1}_{\tilde{A}_{(o,i)}}$  because by the recursion relation  $Q_n(\beta) = 0$  whenever  $(o, i)$  has shown up before time  $n$ . So

$$\mathbb{E}(Q(\beta) | \tilde{A}_{(o,i)}) = \frac{\mathbb{E}(Q(\beta))}{\mathbb{E}(\mathbf{1}_{\tilde{A}_{(o,i)}})} = \frac{q_0(\beta)}{\sum_{\gamma, p^o(i|\gamma)=0} q_0(\gamma)}$$

which implies that

$$\mathbb{E}\left(\sum_{\gamma, p^o(i|\gamma)=0} Q(\gamma) | \tilde{A}_{(o,i)}\right) = 1$$

Hence, conditional on  $\tilde{A}_{(o,i)}$ , the  $Q(\alpha)$ 's for which  $p^o(i|\alpha) > 0$  have to vanish. Equivalently,  $Q(\gamma) p^o(i|\gamma) = 0$  for each  $\gamma$  and  $\sum_{\gamma} Q(\gamma) p^o(i|\gamma) = 0$ , which was to be proved.

## C.B. Proof of existence of a continuous time limit

We first put the notion of continuous time limit in context.

Let  $V$  be the vector space  $C_0(\mathbb{R}^+, \mathbb{R}^I)$  of continuous functions  $f$  from  $\mathbb{R}^+$  to  $\mathbb{R}^I$  such that  $f_0 = 0 \in \mathbb{R}^I$ . For each  $\delta > 0$ , and each  $\omega \in \Omega$  we define a function  $W_t^{(\delta)}(i)$  on  $\mathbb{R}^+$  by linear interpolation of  $W_t^{(\delta)}(i) := \sqrt{\delta}(N_{t/\delta}(i) - p_0(i)t/\delta)$  if  $t/\delta$  is an integer. Explicitly, for  $t \in [\delta n, \delta(n+1)]$

$$W_t^{(\delta)}(i) = \sqrt{\delta}((n+1 - t/\delta)N_n(i) + (t/\delta - n)N_{n+1}(i) - p_0(i)t/\delta).$$

For every  $\omega \in \Omega$  the function  $W_t^{(\delta)}(i)$  is continuous for  $t \in \mathbb{R}^+$ . So we have a map  $W^{(\delta)} : \Omega \rightarrow V$ . But, as already pointed out before, there is no hope that, for a fixed  $\omega \in \Omega$ ,  $W_t^{(\delta)}(i)$  has a limit when  $\delta \rightarrow 0^+$ . The only clear fact is that for a fixed  $t$ , the central limit theorem ensures that the distribution of  $W_t^{(\delta)}(i)$  under each  $\mathbb{P}_\alpha$  has a Gaussian limit when  $\delta \rightarrow 0^+$ . Note that this observation fixes the scaling  $\sqrt{\delta}$  as the only one possible.

But if we are interested in convergence as a process, a deeper approach is needed. If we endow  $V$  with the topology of uniform convergence on compact sets  $\mathcal{T}(V)$  and with the corresponding Borel  $\sigma$ -algebra  $\mathcal{B}(V)$ , we can show that the map  $W^{(\delta)}$  is measurable from  $(\Omega, \mathcal{F})$  to  $(V, \mathcal{B}(V))$ . This is not difficult, because by a classical result,  $\mathcal{B}(V)$  is the smallest  $\sigma$ -algebra on  $V$  containing the family of sets

$$B^{t,i,a} := \{f \in C_0(\mathbb{R}^+, \mathbb{R}^I), a < f_t(i)\}$$

indexed by  $t \in ]0, +\infty[$ ,  $i \in I$  and  $a \in \mathbb{R}$ . It is plain that the inverse image of  $B^{t,i,a}$  under  $W^{(\delta)}$  is in  $\mathcal{F}_n$  whenever  $n > t/\delta$ . As the filtration on  $\Omega$  is exactly the one making the  $N_n$   $\mathcal{F}_n$ -measurable, the appropriate filtration on  $C(\mathbb{R}^+, \mathbb{R}^I)$  should be the natural one, the smallest making the canonical process adapted<sup>9</sup>. We denote it by  $\mathcal{G}_t$ .

Then any probability measure on  $(\Omega, \mathcal{F})$  induces via  $W^{(\delta)}$  a probability measure on  $(V, \mathcal{B}(V))$ . Note that, via (C.14), the measures we defined previously on  $(\Omega)$  depend on  $\delta$ , and to be explicit we write  $\mathbb{P}^{(\delta)}$ ,  $\mathbb{E}^{(\delta)}$ , etc. to stress this fact. Let  $\mu_\alpha(\delta)$  be the image measure of  $\mathbb{P}_\alpha^{(\delta)}$  pushed forward by  $W^{(\delta)}$  on  $(V, \mathcal{B}(V))$ . As  $(V, \mathcal{T}(V))$  is a so-called Polish space, there is a nice notion of convergence for measures on it, called “weak convergence” of measures, and we could ask if the  $\mu_\alpha(\delta)$ ’s converge weakly to some probability measure on  $(V, \mathcal{B}(V))$  (and then, so would  $\mu(\delta) = \sum_\alpha q_0(\alpha)\mu_\alpha(\delta)$ ). Note that despite its name, weak convergence is strong enough to ensure the convergence of the expectations of rather general functionals, so we could hope to control the  $Q$ ’s continuous time limit as well, because they are nice functionals of the counting process.

It is usually in this context that continuous time limits have a meaning. In this setting, there are a number of theorems, called functional central limit theorems, or Donsker invariance principles, that express the continuous time limit of random walks (with independent increments) in terms of Brownian motions in fine details. Alas, though under each  $\mathbb{P}_\alpha$ ,  $N_n(i)$  is a random walk with independent increments, the theorems we are aware do not apply immediately. The problem is that the  $\delta$  dependence is not only in  $W$ , but also in the  $\mathbb{P}_\alpha$ ’s. This problems would show up even more dramatically to deal with the  $Q$ ’s convergence and there relationships with  $W$ . While we think these are purely technical details in our case, we shall

9. Of course, as long as  $\delta > 0$ ,  $W_t$  looks a bit forward in the future, as it involves  $N_{n+1}$  for  $t \in [\delta n, \delta(n+1)]$ , but for continuous process this does not matter.

not try to deal with them: we shall instead rely on a weaker and slightly less natural notion of continuous time limit that will suffice for our purposes. To say things in more mathematical terms: we shall content ourselves with proving that the joint finite dimensional distributions of  $W$ 's and  $Q$ 's converge to joint finite dimensional distributions of continuous processes we can identify explicitly, but we do not embark on the more technical task of proving tightness.

We now turn to our explicit approach of the continuous time limit.

From the characteristic function (C.18) we obtain easily that

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \mathbb{E}^{(\delta)} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}^{(\delta)}(i) - W_{t_{l-1}}^{(\delta)}(i))} \right) = \\ \left( \sum_{\alpha} q_0(\alpha) e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_i \lambda_l(i) p_0(i) \Gamma(i|\alpha)} \right) \times \\ e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \end{aligned} \quad (\text{C.26})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences  $0 = t_0 \leq t_1 \leq \dots \leq t_k$  of length  $k$  and arbitrary (complex)  $\lambda_l(i)$ 's.

By a standard theorem on characteristic functions, we have thus proved that the  $W_t^{(\delta)}(i)$  finite marginals (under each  $\mathbb{P}_{\alpha}^{(\delta)}$  and under  $\mathbb{P}^{(\delta)}$ ) have a limit for  $\delta \rightarrow 0^+$ . This is much weaker than what weak convergence of measures would ensure. It is enough to ensure that the limit marginals satisfy the Kolmogorov consistency criterion, but it does not guarantee that it is possible to concentrate the corresponding process on  $C_0(\mathbb{R}^+, \mathbb{R}^I)$ . However in the case at hand, we can bypass this problem because of the simple form of the result, which is Gaussian for each  $\alpha$ .

Let  $\nu$  be the Wiener measure of a standard Brownian motion on  $C_0(\mathbb{R}^+, \mathbb{R}^I)$ . The linear map from  $\mathbb{R}^I$  to itself defined by

$$y(i) := \sqrt{p_0(i)}(x(i) - \sqrt{p_0(i)} \sum_j \sqrt{p_0(j)} x(j))$$

induces a map from  $C_0(\mathbb{R}^+, \mathbb{R}^I)$  to itself. Let  $\mu^0$  be the image measure of  $\nu$  under this map. It is easily seen that under this law, the canonical process  $W$  on  $C_0(\mathbb{R}^+, \mathbb{R}^I)$  satisfies

$$\begin{aligned} \mathbb{E}^{\mu^0} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}(i) - W_{t_{l-1}}(i))} \right) = \\ e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \end{aligned} \quad (\text{C.27})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences  $0 = t_0 \leq t_1 \leq \dots \leq t_k$  of length  $k$  and arbitrary (complex)  $\lambda_l(i)$ 's.

As a consequence, Itô's formula holds with

$$dW_t(i) dW_t(j) = dt (\delta_{i,j} p_0(i) - p_0(i) p_0(j)).$$

Our aim is to use Girsanov's theorem to deform the measure and go from the right-hand side in (C.27) to the right-hand in (C.26). If the process  $U_t$  with values in  $\mathbb{R}^I$  is adapted and satisfies some further technical integrability conditions,

$$M_t^U := e^{\int_0^t \sum_i U_s(i) dW_s(i) - \frac{1}{2} \int_0^t (\sum_i p_0(i) U_s(i)^2 - (\sum_i p_0(i) U_s(i))^2) ds}$$

is a martingale. Moreover, by Itô's formula,  $dM_t^U = M_t^U \sum_i U_s(i) dW_s(i)$ . Then, by Girsanov's theorem, for any  $T > 0$ , under the measure  $d\mu_T^U := M_T^U d\mu^0$  on  $C_0([0, T], \mathbb{R}^I)$ , the process  $W_t(i) - p_0(i) \int_0^t (U_s(i) - \sum_j p_0(j) U_s(j)) ds$ ,  $t \in [0, T]$ , has the same law as  $W_t(i)$ ,  $t \in [0, T]$ , under  $d\mu^0$ .

Note that for  $t \leq T$ ,  $M_t^U$  is a Radon-Nikodym derivative.

$$M_t^U := \left[ \frac{d\mu_T^U}{d\mu^0} \right]_t = \mathbb{E}^{\mu^0} \left( \frac{d\mu_T^U}{d\mu^0} \middle| \mathcal{G}_t \right),$$

In general this construction cannot work for infinite  $T$ , because  $\mu_T^U$  and  $\mu^0$  become singular. However,  $T$  plays only a dummy role : for  $T' \leq T$ ,  $d\mu_T^U$  and  $d\mu_{T'}^U$  coincide on  $\mathcal{G}_{T'}$ . So it is only a slight abuse, which lightens notations a bit, to write  $d\mu^U$  for  $d\mu_T^U$  and

$$M_t^U = \left[ \frac{d\mu^U}{d\mu^0} \right]_t = \mathbb{E}^{\mu^0} \left( \frac{d\mu^U}{d\mu^0} \middle| \mathcal{G}_t \right).$$

For the special choice  $U_t(i) := \Gamma(i|\alpha)$ , using  $\sum_i p_0(i) \Gamma(i|\alpha) = 0$ , we compute

$$M_t(\alpha) := e^{\sum_i \Gamma(i|\alpha) W_t(i) - \frac{t}{2} \sum_i p_0(i) \Gamma(i|\alpha)^2}$$

which is certainly a martingale, and we obtain for every  $T$  a measure  $d\mu_\alpha$  on  $C_0([0, T], \mathbb{R}^I)$  such that

$$\begin{aligned} \mathbb{E}^{\mu_\alpha} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}(i) - W_{t_{l-1}}(i))} \right) &= \\ e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_i \lambda_l(i) p_0(i) \Gamma(i|\alpha)} e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \end{aligned} \quad (\text{C.28})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences  $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq T$  of length  $k$  and arbitrary (complex)  $\lambda_l(i)$ 's.

Finally, setting  $M_t := \sum_\alpha q_0(\alpha) M_t(\alpha)$  (trivially a martingale again) and  $d\mu := M_t d\mu^0$  we obtain for every  $T$  a measure  $d\mu$  on  $C_0([0, T], \mathbb{R}^I)$  such that

$$\begin{aligned} \mathbb{E}^\mu \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}(i) - W_{t_{l-1}}(i))} \right) &= \\ \left( \sum_\alpha q_0(\alpha) e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_i \lambda_l(i) p_0(i) \Gamma(i|\alpha)} \right) \times \\ e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \end{aligned} \quad (\text{C.29})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences  $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq T$  of length  $k$  and arbitrary (complex)  $\lambda_l(i)$ 's. So we have found the continuous time limit of the counting process as the canonical process on  $C_0([0, T], \mathbb{R}^I)$  with the measure  $\mu$ .

It remains to deal with the continuous time limit of the  $Q_n$ 's.

We note that by the chain rule  $\frac{M_t(\alpha)}{M_t} = \left[ \frac{d\mu(\alpha)}{d\mu} \right]_t = \mathbb{E}^\mu \left( \frac{d\mu(\alpha)}{d\mu} \middle| \mathcal{G}_t \right)$  is the Radon-Nikodym derivative of  $\mu(\alpha)$  with respect to  $\mu$ . So with some memory of what happened in the discrete case, it is natural to define  $Q_t(\alpha) := q_0(\alpha) \frac{M_t(\alpha)}{M_t}$ .

Using the explicit formula for the  $Q_n$ 's in terms of the counting processes, we define  $Q_t^{(\delta)}$  by an interpolation procedure:

$$Q_t^{(\delta)}(\alpha) := q_0(\alpha) \frac{\prod_i p(i|\alpha) W_t^{(\delta)}(i) / \sqrt{\delta} + p_0(i)t/\delta}{\sum_\beta q_0(\beta) \prod_i p(i|\beta) W_t^{(\delta)}(i) / \sqrt{\delta} + p_0(i)t/\delta},$$

so that if  $t/\delta = n$ , an integer,  $Q_t^{(\delta)}(\alpha) = Q_n(\alpha)$ . Note that in this formula the  $p(i|\beta)$ 's depend implicitly on  $\delta$  via (C.14).

One can prove that the joint finite dimensional distributions of the processes  $(W_t^{(\delta)}, Q_t^{(\delta)})$  under  $\mathbb{P}^{(\delta)}$  have limits when  $\delta \rightarrow 0^+$  and that these limit are nothing but the joint finite dimensional distributions of the processes  $(W_t, Q_t)$  under  $\mu$ . In this precise sense, the  $Q_n$ 's continuous time limit is deciphered.

This result is really no big surprise, but to prove it we have to rely on an *ad hoc* trick and an explicit elementary but tedious computation. The details are neither illuminating nor elegant so we omit them.

When  $\delta$  is small enough, all non-empty sets in  $\mathcal{F}_n$  have strictly positive measure, so that if  $q_0(\alpha) > 0$  the same is true for  $Q_n(\alpha)$  for all  $n$ 's. As furthermore  $\sum_{\alpha} Q_n(\alpha) = 1$  for all  $n$ , all the information on the  $Q_n$ 's (joint) laws is embodied in the joint laws of ratios of  $Q_n$ 's.

As these ratios have a simple product structure in terms of the counting processes, the explicit computation of

$$\mathbb{E} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (N_{n_l}(i) - N_{n_{l-1}}(i))} \prod_{l=1}^k \prod_{\alpha, \beta} \left( \frac{Q_{n_l}(\alpha)}{Q_{n_l}(\beta)} \right)^{\eta_l(\alpha, \beta)} \right)$$

for  $k \geq 1$ , arbitrary non-decreasing sequences of integers  $0 = n_0 \leq n_1 \leq \dots \leq n_k$  of length  $k$ , and arbitrary (complex)  $\lambda_l(i)$ 's and  $\eta_l(\alpha, \beta)$ 's is in some sense a special case of (C.18).

The same remark applies to computations involving ratios of  $Q_t$ 's.

This allows to compute explicitly that

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \mathbb{E}^{(\delta)} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}^{(\delta)}(i) - W_{t_{l-1}}^{(\delta)}(i))} \prod_{l=1}^k \prod_{\alpha, \beta} \left( \frac{Q_{t_l}^{(\delta)}(\alpha)}{Q_{t_l}^{(\delta)}(\beta)} \right)^{\eta_l(\alpha, \beta)} \right) = \\ \mathbb{E}^{\mu} \left( e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}(i) - W_{t_{l-1}}(i))} \prod_{l=1}^k \prod_{\alpha, \beta} \left( \frac{Q_{t_l}(\alpha)}{Q_{t_l}(\beta)} \right)^{\eta_l(\alpha, \beta)} \right) \end{aligned} \quad (\text{C.30})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences  $0 = t_0 \leq t_1 \leq \dots \leq t_k$  of length  $k$ , and arbitrary (complex)  $\lambda_l(i)$ 's and  $\eta_l(\alpha, \beta)$ 's.

As such a mixture of Laplace and Mellin transforms characterizes the distributions completely, this concludes the existence of a natural continuous time limit.

## C.C. Details on the continuous time limit with different partial measurement methods

We use the linear interpolation of appendix C.B on  $W_t^{(\delta)}(o, i) := \sqrt{\delta} (N_{t/\delta}(o, i) - c(o)p_0^o(i)t/\delta)$  if  $t/\delta$  is an integer. Explicitly for  $t \in [\delta n, \delta(n+1)]$ ,

$$W_t^{(\delta)}(o, i) = \sqrt{\delta} ((n+1 - t/\delta)N_n(o, i) + (t/\delta - n)N_{n+1}(o, i) - c(o)p_0^o(i)t/\delta).$$

We remind that  $E = \bigcup_{o \in \mathcal{O}} \{o\} \otimes \text{spec}(o)$  is the set of all possible measurement methods and outcomes. We expect the limit time continuous process to live on the vector space of continuous function from  $\mathbb{R}^+$  to  $\mathbb{R}^E$ .

### C.D. Derivation of the density matrix evolution

Compare to previous sections, the main changes are in the correlation functions calculations. Thanks to the measurement method distribution time and realization independency, we find:

$$\mathbb{E} \left( e^{\sum_{l=1}^k \sum_{(o,i) \in E} \lambda_l(o,i) (N_{n_l}(o,i) - N_{n_{l-1}}(o,i))} \right) = \sum_{\alpha} q_0(\alpha) \prod_{l=1}^k \left( \sum_{(o,i) \in E} e^{\lambda_l(o,i) c(o) p^o(i|\alpha)} \right)^{n_l - n_{l-1}} \quad (\text{C.31})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences of integers  $0 = n_0 \leq n_1 \leq \dots \leq n_k$  of length  $k$ , and arbitrary (complex)  $\lambda_l(o, i)$ 's.

As a consequence, in the limit  $\delta \rightarrow 0^+$ ,

$$\lim_{\delta \rightarrow 0^+} \mathbb{E}^{(\delta)} \left( e^{\sum_{l=1}^k \sum_{(o,i) \in E} \lambda_l(o,i) (W_{t_l}^{(\delta)}(o,i) - W_{t_{l-1}}^{(\delta)}(o,i))} \right) = \left( \sum_{\alpha} q_0(\alpha) e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_{(o,i) \in E} \lambda_l(o,i) c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha)} \right) \times e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_{(o,i) \in E} c(o) p_0^o(i) \lambda_l(o,i)^2 - (\sum_{(o,i) \in E} c(o) p_0^o(i) \lambda_l(o,i))^2)} \quad (\text{C.32})$$

for  $k \geq 1$ , arbitrary non-decreasing sequences  $0 = t_0 \leq t_1 \leq \dots \leq t_k$  of length  $k$  and arbitrary (complex)  $\lambda_l(o, i)$ 's and with  $\Gamma^{(o)}(i|\alpha) = 2\text{Im} \left( \frac{\langle i|H_{\alpha}^{(o)}|\Psi^{(o)}\rangle}{\langle i|\Psi^{(o)}\rangle} \right)$ . Then each  $W_t^{(\delta)}(o, i)$  under  $\mu^{(\delta)}$  converges toward a process  $W_t(o, i)$  under  $\mu$ .

The demonstration is then as in previous section except for notational differences which keep track of the  $o$ -dependency of  $W_t(o, i)$ . As in the previous section, the measure  $\mu^0$  is defined as the push-forward measure of  $\nu$  on  $\mathbb{R}^E$  under the linear map

$$y(o, i) := \sqrt{c(o) p_0^o(i)} (x(o, i) - \sqrt{c(o) p_0^o(i)} \sum_{(o', j) \in E} \sqrt{c(o') p_0^{o'}(j)} x(o', j))$$

The martingale  $M_t$  is defined by

$$M_t = \sum_{\alpha} q_0(\alpha) e^{\sum_{(o,i) \in E} \Gamma^{(o)}(i|\alpha) W_t(o,i) - \frac{t}{2} \sum_{(o,i) \in E} c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha)^2}$$

The measure  $\mu$  is defined via Girsanov's transformation:  $\mathbb{E}^{\mu}(\cdot) = \mathbb{E}^{\mu^0}(M_t \cdot)$ .

### C.D. Derivation of the density matrix evolution

Let us derive the density matrix continuous time limit. Recall that at time  $n$  its elements are functions of the counting processes

$$A_n(\alpha, \beta) = \frac{A_0(\alpha, \beta) \prod_i (M(i|\alpha) M(i|\beta)^*)^{N_n(i)}}{\sum_{\gamma} q_0(\gamma) \prod_i p(i|\gamma)^{N_n(i)}}$$

from this expression we define time continuous processes

$$A_t^{(\delta)}(\alpha, \beta) = \frac{A_0(\alpha, \beta) \prod_i (M(i|\alpha) M(i|\beta)^*)^{W_t^{(\delta)}(i)/\sqrt{\delta} + p_0(i) \frac{t}{\delta}}}{\sum_{\gamma} q_0(\gamma) \prod_i p(i|\gamma)^{W_t^{(\delta)}(i)/\sqrt{\delta} + p_0(i) \frac{t}{\delta}}}$$

equal to  $A_n(\alpha, \beta)$  if  $t/\delta = n$  is an integer. The  $M(i|\alpha)$ 's depend explicitly on  $\delta$  via

$$M(i|\alpha) = \langle i|e^{-i\delta(E_\alpha \mathbb{I} + H_p + \frac{1}{\sqrt{\delta}}H_\alpha)}|\Psi\rangle$$

We rewrite the products over the partial measurement results as exponentials of sums

$$A_t^{(\delta)}(\alpha, \beta) = \frac{A_0(\alpha, \beta)e^{\sum_i \ln(M(i|\alpha)M(i|\beta)^*/p_0(i))(W_t^{(\delta)}(i)/\sqrt{\delta} + p_0(i)\frac{t}{\delta})}}{\sum_\gamma q_0(\gamma)e^{\sum_i \ln(p(i|\gamma)/p_0(i))(W_t^{(\delta)}(i)/\sqrt{\delta} + p_0(i)\frac{t}{\delta})}}}$$

A detailed analysis of the limit  $\delta \rightarrow 0^+$  would require to perform the same study as in section C.B. However, at this stage we are confident enough to state that we can safely shortcut a few steps and use directly that  $W_t^{(\delta)}(i)$  converge. Using  $\langle \Psi|H_I|\Psi\rangle = 0$ , and the identity  $\sum_i p_0(i) \frac{\langle i|H_\alpha^2|\Psi\rangle}{\langle \Psi|i\rangle} = \sum_i p_0(i)|c(i|\alpha)|^2$ , we obtain

$$\begin{aligned} \lim_{\delta \rightarrow 0} \sum_i \ln(M(i|\alpha)M(i|\beta)^*/p_0(i))(W_t^{(\delta)}(i)/\sqrt{\delta} + p_0(i)\frac{t}{\delta}) \\ = l(\alpha, \beta)t - i \sum_i (c(i|\alpha) - c(i|\beta)^*)W_t(i) \end{aligned}$$

where the limit as to be understood as the limit of any finite dimensional correlation functions. Therefore

$$\lim_{\delta \rightarrow 0} A_t^{(\delta)}(\alpha, \beta) = A_t(\alpha, \beta)$$

with  $A_t(\alpha, \beta)$  defined in (C.22). It is then a simple matter, using Itô rules for  $W_t(i)$ , to derive the Belavkin equation (C.23) for the density matrix  $\rho_t = \sum_{\alpha, \beta} A_t(\alpha, \beta)|\alpha\rangle\langle\beta|$ .

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## D. Article: Iterated Stochastic Measurements

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Michel Bauer<sup>♣</sup><sup>1</sup>, Denis Bernard<sup>♠</sup><sup>2</sup> and Tristan Benoist<sup>♠</sup><sup>3</sup>

<sup>♣</sup> Institut de Physique Théorique<sup>4</sup> de Saclay, CEA-Saclay, France.

<sup>♠</sup> Laboratoire de Physique Théorique de l'ENS, CNRS & Ecole Normale Supérieure de Paris, France.

### Abstract

We describe a measurement device principle based on discrete iterations of Bayesian updating of system state probability distributions. Although purely classical by nature, these measurements are accompanied with a progressive collapse of the system state probability distribution during each complete system measurement. This measurement scheme finds applications in analysing repeated non-demolition indirect quantum measurements. We also analyse the continuous time limit of these processes, either in the Brownian diffusive limit or in the Poissonian jumpy limit. In the quantum mechanical framework, this continuous time limit leads to Belavkin equations which describe quantum systems under continuous measurements.

### D.1. Introduction

Informal and formal similarities between Bayesian inference [1] and quantum mechanics have been noted quite some time ago, see e.g. [2]. Bayesian inference may be seen as a way to update trial probability distributions by taking into account the partial information one has gained on the system under study. Indirect quantum measurement consists in obtaining partial information on a quantum system by letting it interact with another quantum system, called a probe, and performing a direct Von Neumann measurement on this probe. Iterating the process of system-probe interaction and probe measurement increases the information on the system because of system-probe entanglements.

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1. [michel.bauer@cea.fr](mailto:michel.bauer@cea.fr)

2. Member of CNRS; [denis.bernard@ens.fr](mailto:denis.bernard@ens.fr)

3. [tristan.benoist@ens.fr](mailto:tristan.benoist@ens.fr)

4. CEA/DSM/IPhT, Unité de recherche associée au CNRS

This has been experimentally implemented in electrodynamics in cavities [3], but also in superconductor circuits [4]. As shown by these experiments, repeating a large number of times (formally, infinitely many times) indirect non-demolition measurements [5] reproduces macroscopic direct measurements with collapse of the system quantum wave function. Each collapse is stochastic and progressive, becoming sharper and sharper as the number of indirect measurements increases.

Controlling quantum systems [7] by repeating measurements is, in some way, as old as quantum mechanics, but it has recently been further developed aiming at quantum state manipulations and quantum information processing [8]. At a theoretical level, the concept of quantum trajectories [9,10] emerges from the need to describe quantum jumps and randomness inherent to repeated measurements. In parallel, studies of open quantum systems [11] led to the theory of quantum feedback [12] and quantum continual measurements [13]. Belavkin equations [14] are stochastic non-linear generalizations of the Schrödinger equation adapted to quantum systems under continual measurements.

Contact between experiments of the type described in ref. [3] and classical stochastic processes was made in ref. [6], showing in particular that the approach to the collapse is controlled by a relevant relative entropy. The aim of this note is to follow and complement the study of ref. [6], by, in some way, reversing the logic. We start by forgetting quantum mechanics for a while and we study a random process obtained by discretely and randomly updating a system state probability distribution using Bayes' rules. Iterated stochastic measurements refer to this random recursive updating. We describe why and how this leads to a stochastic measurement principle allowing to measure the initial system state probability distribution but which implements a random collapse of the system state distribution at each individual complete system measurement. The initial system state distribution is nevertheless reconstructed by repeating the complete system measurements. We point out a connection between De Finetti's theorem on exchangeable random variables, see e.g. ref. [15], and iterated stochastic measurements. We also show that these discrete measurement devices admit continuous formulations with continual updating. There are two limits: a Brownian diffusive limit in which the random data used to update the system state distribution are coded into Brownian motions, this case was studied in ref. [16], and a Poissonian jumpy limit in which these random data are coded in point processes. The construction of the continuous time process relies on deforming an a priori probability measure on the updating data. The key tool is Girsanov's theorem. Then we transport these results, in an almost automatic way, to quantum mechanics, and we show that quantum mechanical systems under repeated non-demolition indirect measurements admit a continuous time limit described by Belavkin equations (D.18,D.19). This completes results proved in ref. [17] and makes contact with those described in ref. [18].

## D.2. Iterated indirect stochastic measurements

Let  $\mathcal{S}$  be the system under study and  $A$  be a chosen countable set of system states  $\alpha \in A$  that we shall call *pointer states*<sup>5</sup>. The model apparatus is going to measure the probability distribution  $Q_0(\alpha)$ , with  $\sum_{\alpha} Q_0(\alpha) = 1$ , for the system  $\mathcal{S}$  to be in one of the pointer state.

The model apparatus is made of an infinite series of indirect partial measurements. Let  $I$  denote the set of possible results of one partial measurements, which we assume to be

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5. According to the quantum terminology, but the concept of states is here more general as it simply refers to a complete list of labels characterizing the system behavior.

finite or countable. For each system complete measurement, the output datum is thus an infinite sequence of data  $(i_1, i_2, \dots)$ ,  $i_k \in I$ , associated to the series of successive partial measurements. The output data are random. The probability distribution  $Q_0(\alpha)$  is to be reconstructed from the sequences  $(i_1, i_2, \dots)$ .

To be concrete one may keep in mind that the indirect partial measurements arise from direct measurements on probes which have been coupled to the system. The model apparatus is then made of an infinite set of in-going probes – which, for simplicity, are supposed to be all identical – passing through the system  $\mathcal{S}$  and interacting with it one after the other. Measurements are done on the out-going probes.

Specifications of the model apparatus depend on the chosen set of pointer states. One of its manufacturing characteristics is a collection of probability distributions  $p(i|\alpha)$ ,  $\sum_i p(i|\alpha) = 1$ , for the output partial measurement to be  $i \in I$  conditioned on the system  $\mathcal{S}$  be in the state  $\alpha \in A$ . For simplicity, we shall assume a non-degeneracy hypothesis which amounts to suppose that all probability distributions  $p(\cdot|\alpha)$  are distinct, i.e. for any pair of distinct pointer states  $\alpha$  and  $\beta$  there exists  $i \in I$  such that  $p(i|\alpha) \neq p(i|\beta)$ .

### D.2.1. Discrete time description

In the model apparatus, a *complete measurement* is made of an infinite series of *partial measurements* such that each output of these partial measurements provides a gain of information on the system. Our first aim is to decipher which informations one is gaining from the  $n^{\text{th}}$  first partial measurements. This will allow us to spell out the way the model apparatus is working as a measurement device.

- *Series of partial measurements and specification of the model apparatus.* Suppose that the first partial measurement gives result  $i_1 \in I$ . Bayes' law then tells us that the probability for the system  $\mathcal{S}$  to be in the state  $\alpha$  conditioned on the first measurement be  $i_1$  is  $Q_1(\alpha|i_1) = Q_0(\alpha)p(i_1|\alpha)/\pi_0(i_1)$ , with  $\pi_0(i) := \sum_{\alpha} Q_0(\alpha)p(i|\alpha)$ , if  $Q_0(\alpha)$  is the initial probability for the system  $\mathcal{S}$  to be in the state  $\alpha$  (this probability is yet unknown but shall be recovered from the series of partial measurements making a complete measurement). Let us now ask ourselves what is the probability to get  $i_2$  as second output partial measurement? By the law of conditioned probabilities,  $\pi_1(i_2|i_1) = \sum_{\alpha} p(i_1, i_2|\alpha) Q_0(\alpha)/\pi_0(i_1)$  with  $p(i_1, i_2|\alpha)$  the probability to measure  $i_1$  and  $i_2$  on the two first partial measurements conditioned on the system to be in the state  $\alpha$ . At this point we need to make an assumption: we assume that the output partial measurements are independent and identically distributed (i.i.d.) provided that the system  $\mathcal{S}$  is in one of the pointer state  $\alpha \in A$ . This translates into the relation

$$p(i_1, i_2|\alpha) = p(i_2|\alpha)p(i_1|\alpha),$$

which implies that  $\pi_1(i_2|i_1) = \sum_{\alpha} p(i_2|\alpha) Q_1(\alpha|i_1)$ . That is the probability  $\pi_1(i_2|i_1)$  is identical to the probability to get  $i_2$  as output partial measurement assuming that the system distribution is  $Q_1(\alpha|i_1)$ .

Hence, as a defining characteristic property of our model apparatus, we assume that the output of the  $n^{\text{th}}$  partial measurements is independent of those of the  $(n-1)$ -first outputs provided the system  $\mathcal{S}$  is in one of the pointer state  $\alpha \in A$ , that is:

$$p(i_1, \dots, i_{n-1}, i_n|\alpha) = p(i_n|\alpha)p(i_1, \dots, i_{n-1}|\alpha) = \prod_{k=1}^n p(i_k|\alpha), \quad \forall \alpha. \quad (\text{D.1})$$

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This specifies our model apparatus. This specification is clearly attached to the chosen set of pointer states.

Conversely, the pointer states associated to this device are those system states for which the values of the output partial measurements are independent, i.e. conditioned on the system to be in a pointer state, the output variables  $i_1, i_2, \dots$  are independent and identically distributed. If the system is initially in a pointer state  $\alpha$ , that is its probability distribution is peaked,  $Q_0(\cdot) = \delta_{\cdot;\alpha}$ , the occurrence frequency  $\nu(i)$  of the value  $i$  in the output sequence  $(i_1, i_2, \dots)$  is  $p(i|\alpha)$ . As we shall see later, one may then identify the pointer states as the system states for which independent infinite series of partial measurements (i.e. independent complete measurements) provide identical occurrence frequencies  $\nu(\cdot)$ , and this gives a way to calibrate the device and to determine the conditioned probabilities  $p(\cdot|\alpha)$ .

If the system is not in a pointer state, its initial distribution  $Q_0(\alpha)$  – to be determined – is un-peaked. Let  $Q_n(\alpha|i_1, \dots, i_n)$  be the probability for the system to be in the state  $\alpha$  conditioned on the  $n$ -first output partial measurements be  $i_1, i_2, \dots, i_n$ . From our hypothesis (D.1), the probability to get  $i$  as the  $n^{\text{th}}$  output conditioned on the  $(n-1)^{\text{th}}$  first outputs be  $(i_1, \dots, i_{n-1})$  is

$$\pi_{n-1}(i|i_1, \dots, i_{n-1}) = \sum_{\alpha} p(i|\alpha) Q_{n-1}(\alpha|i_1, \dots, i_{n-1}). \quad (\text{D.2})$$

By Bayes' law, the probability for the system to be in the state  $\alpha$  conditioned on the  $n$ -first measurements be  $i_1, i_2, \dots, i_n$  is then recursively computed by

$$Q_n(\alpha|i_1, \dots, i_n) = \frac{p(i_n|\alpha) Q_{n-1}(\alpha|i_1, \dots, i_{n-1})}{\pi_{n-1}(i_n|i_1, \dots, i_{n-1})}, \quad (\text{D.3})$$

where  $\pi_{n-1}$  is the probability to get  $i_n$  as the  $n^{\text{th}}$  output. To simplify notations we denote  $Q_n(\alpha|i_1, \dots, i_n)$  by  $Q_n(\alpha)$  and  $\pi_{n-1}(i|i_1, \dots, i_{n-1})$  by  $\pi_{n-1}(i)$ . Eq.(D.3) can be solved explicitly:

$$Q_n(\alpha) = Q_0(\alpha) \frac{\prod_i p(i|\alpha)^{N_n(i)}}{\sum_{\beta} Q_0(\beta) \prod_i p(i|\beta)^{N_n(i)}}, \quad (\text{D.4})$$

with  $N_n(i)$  the number of times the value  $i$  appears in the  $n^{\text{th}}$  first outputs.

Let us point out an interesting reformulation of the above conditions on the outputs of the model apparatus. A sequence  $(i_1, i_2, \dots)$  of random variables is called exchangeable if the distribution of  $(i_1, i_2, \dots, i_n)$  is the same as the distribution of  $(i_{\sigma_1}, i_{\sigma_2}, \dots, i_{\sigma_n})$  for each  $n$  and each permutation  $\sigma$  of  $[1, 2, \dots, n]$ . A remarkable theorem due to De Finetti (see e.g. ref. [15] or the last two items of ref. [19]) asserts that an infinite sequence  $(i_1, i_2, \dots)$  of random variables is exchangeable if and only if there is a random variable  $A$  such that, conditionally on  $A$ ,  $(i_1, i_2, \dots)$  is a sequence of independent identically distributed random variables. In our construction, the values taken by  $A$  are nothing but the pointer states and the measure on  $A$  is  $Q_0$ . So the hypotheses on the model apparatus can be rephrased as the fact that the order of partial measurements is immaterial.

More concretely, let  $\Omega$  be the data set of all complete measurements. This is made of all infinite series  $\omega := (i_1, i_2, \dots)$ ,  $i_k \in I$ , of output partial measurements. We may endow  $\Omega$  with the filtration  $\mathcal{F}_n$  of  $\sigma$ -algebras generated by the sets  $B_{i_1, \dots, i_n} := \{\omega = (i_1, \dots, i_n, \text{anything else}) \in \Omega\}$ , i.e.  $\mathcal{F}_n$  codes for the knowledge of the  $n^{\text{th}}$  first partial measurements. This filtered space is

## D.2. Iterated indirect stochastic measurements

equipped with a probability measure recursively defined by  $\mathbb{P}[i_n = i | \mathcal{F}_{n-1}] = \pi_{n-1}(i)$ . Notice that, given  $Q_0(\alpha)$ , this probability measure decomposes as a sum

$$\mathbb{P} = \sum_{\alpha} Q_0(\alpha) \mathbb{P}_{\alpha}$$

where  $\mathbb{P}_{\alpha}$  will be the probability measure induced on  $\Omega$  if the system happened to be initially in the pointer state  $\alpha$ , i.e. if  $Q_0(\cdot)$  is peaked at  $\alpha$ . Under  $\mathbb{P}_{\alpha}$  the partial outputs are independent random variables so that

$$\mathbb{P}_{\alpha}[B_{i_1, \dots, i_n}] = \prod_{k=1}^n p(i_k | \alpha).$$

Let us then quote properties of the random probability distribution  $Q_n(\cdot)$ , which will be keys for specifying the model measurement device:

(i) Peaked distributions are stable under the recursion relation (D.3). That is, if  $Q_0(\cdot) = \delta_{\cdot; \alpha}$  then  $Q_n(\cdot) = \delta_{\cdot; \alpha}$  for any  $n$ .

(ii) Given  $Q_0(\cdot)$  generic, the random variables  $Q_n(\alpha)$  converge as  $n$  goes to infinity almost surely and in  $\mathbb{L}^1$ . The limiting distribution  $Q_{\infty}(\cdot)$  is peaked at a random target pointer state. That is:

$$Q_{\infty}(\cdot) = \delta_{\cdot; \gamma_{\omega}}$$

with target pointer state  $\gamma_{\omega}$  depending on the event  $\omega$ . The probability for the target to be a given pointer state  $\alpha$  is the initial probability distribution:

$$\mathbb{P}[\gamma_{\omega} = \alpha] = Q_0(\alpha).$$

(iii) The asymptotic occurrence frequencies  $\nu(i) := \lim_n N_n(i)/n$ , with  $N_n(i)$  the number of times the value  $i$  appears in the  $n^{\text{th}}$  first outputs, are those of the target pointer state. That is:

$$\lim_{n \rightarrow \infty} N_n(i)/n = p(i | \gamma_{\omega}).$$

(iv) The convergence is exponentially fast:

$$Q_n(\alpha) \simeq \exp(-nS(\gamma_{\omega} | \alpha)), \quad \alpha \neq \gamma_{\omega},$$

for  $n$  large enough, with  $S(\gamma_{\omega} | \alpha)$  the relative entropy of  $p(\cdot | \gamma_{\omega})$  relative to  $p(\cdot | \alpha)$ .

These facts have been proved in ref. [6]. They are based on the fact that the random variables  $Q_n(\alpha)$  are bounded  $\mathbb{P}$ -martingales with respect to the filtration  $\mathcal{F}_n$ . That is  $\mathbb{E}[Q_n(\alpha) | \mathcal{F}_{n-1}] = Q_{n-1}(\alpha)$ . A classical theorem of probability theory [19] says that a bounded martingale converges almost surely and in  $\mathbb{L}^1$ , so that  $Q_{\infty}(\alpha) := \lim_n Q_n(\alpha)$  exists and  $Q_n(\alpha) = \mathbb{E}[Q_{\infty}(\alpha) | \mathcal{F}_n]$ . More general results, involving for instance extra randomness on the partial measurements or relaxing the non-degeneracy hypothesis on the conditioned probability  $p(\cdot | \alpha)$ , have been obtained in ref. [16].

• *How to read-off a complete measurement and consequences.* Let us summarize how the model apparatus is (concretely) working and how data are analysed, see Fig.D.1. For a given system measurement, the data is an infinite sequence  $\omega = (i_1, i_2, \dots)$  of output partial measurements. From its asymptotic behaviour, the apparatus computes the asymptotic frequencies  $\nu(i)$  of occurrences of the values  $i$  in the sequence  $\omega$ , and it compares it to one

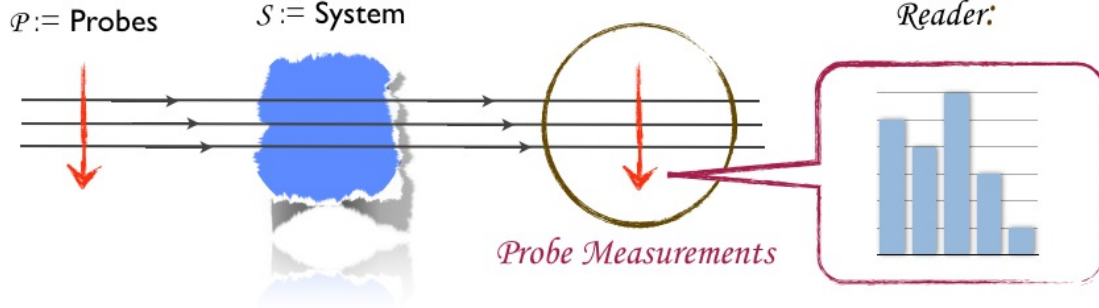


Figure D.1.: A schematic view of iterated stochastic measurements: probes are send one after the other to interact with the system for a while. After the interaction, a measurement is performed on each probe. The information gained is summarized in the occurrence frequencies, which allow to identify the limiting state.

of the apparatus data-base distributions  $p(i|\alpha)$ . By the non-degeneracy hypothesis and the above convergence theorem [6], each of the asymptotic frequencies coincide with one of the data-base distributions, so that the comparison identifies uniquely the target pointer state and that identified state is by definition the result of a complete system measurement. Since by the above theorem the distribution of the target pointer states is the initial distribution  $Q_0(\cdot)$ , the histogram of repeated independent complete system measurements yields the initial distribution.

Notice that by the end of a complete measurement the system state distribution has collapsed into one of the pointer states. The need for an infinite series of partial measurement reflects the need for a macroscopic apparatus to implement the collapse. If the system measurement is stopped after a finite number of partial measurements the collapse is only partial, i.e. the probability distribution  $Q_n(\cdot)$  is still smeared around the target pointer state. The target pointer state may nevertheless be identified with high fidelity if the differences between the data-base probability distributions  $p(\cdot|\alpha)$  are bigger than the fluctuations of the frequencies  $\nu_n(\cdot)$  which generically scale like  $n^{-1/2}$ .

### D.2.2. Continuous time limit

We now describe continuous time limits of the previous model apparatus in which the partial measurements are done continuously in time. There are different continuous time limits, depending on the behaviour of the data-base conditioned probability distributions  $p(\cdot|\alpha)$ : a Brownian diffusive limit, a Poissonian jumpy limit, or a mixture of them.

These limits may be understood by looking at properties of the counting process  $N_n(i) := \sum_{k=1}^n \mathbb{I}_{i_k=i}$  which is the number of times the value  $i$  appears in the  $n^{\text{th}}$  first partial measurements. Recall that  $\pi_{m-1}(i) = \mathbb{E}[\mathbb{I}_{i_m=i}|\mathcal{F}_{m-1}]$  is the probability to get  $i$  as the  $m^{\text{th}}$  partial output conditioned on the  $(m-1)^{\text{th}}$  first partial outputs. We may tautologically decompose

$N_n(i)$  as

$$N_n(i) = X_n(i) + A_n(i), \quad \text{with } A_n(i) := \sum_{m=0}^{n-1} \pi_m(i), \quad (\text{D.5})$$

where this equation serves as the definition of  $X_n(i)$ , i.e.  $X_n(i) := N_n(i) - A_n(i)$ , with  $\sum_i X_n(i) = 0$  as both  $N_n(i)$  and  $A_n(i)$  add up to  $n$ . Then by construction  $\mathbb{E}[X_n(i)|\mathcal{F}_{n-1}] = X_{n-1}(i)$ , so that the processes  $X_n(i)$  are  $\mathbb{P}$ -martingales with respect to the filtration  $\mathcal{F}_n$ . Equation (D.5) is the so-called Doob decomposition of  $N_n(i)$  as the processes  $A_n(i)$  are predictable, i.e.  $A_n(i)$  is  $\mathcal{F}_{n-1}$ -mesurable, see ref. [19]. The martingale property in particular implies that  $\mathbb{E}[X_n(i)] = 0$ .

Recall now the recursion relation (D.3) that we may rewrite as

$$Q_n(\alpha) - Q_{n-1}(\alpha) = Q_{n-1}(\alpha) \sum_i \frac{p(i|\alpha)}{\pi_{n-1}(i)} (\mathbb{I}_{i_n=i} - \pi_{n-1}(i))$$

which holds true because  $\sum_i p(i|\alpha) = 1$ . By construction  $\mathbb{I}_{i_n=i} - \pi_{n-1}(i) = X_n(i) - X_{n-1}(i)$ , so that

$$(\Delta Q)_n(\alpha) = Q_{n-1}(\alpha) \sum_i \frac{p(i|\alpha)}{\pi_{n-1}(i)} (\Delta X)_n(i). \quad (\text{D.6})$$

with  $(\Delta Q)_n(\alpha) := Q_n(\alpha) - Q_{n-1}(\alpha)$  and  $(\Delta X)_n(i) := X_n(i) - X_{n-1}(i)$ . We thus have rewritten the recursion relation (D.3) as a discrete non-linear difference equation for the probability distribution  $Q_n(\cdot)$  driven by discrete differences of the martingales  $X_n(i)$ . This will be the starting point of the continuous time limits.

Before going on let us point out a geometrical interpretation of  $Q_n(\alpha)$  which will be useful later. On the set of complete measurements, we have defined a global measure  $\mathbb{P}$  and a series of measures  $\mathbb{P}_\alpha$  associated to each of the pointer states with  $\mathbb{P} = \sum_\alpha Q_0(\alpha) \mathbb{P}_\alpha$ . It is a simple matter to check that  $\mathbb{P}_\alpha$  is non singular with respect to  $\mathbb{P}$ , so that there exists a Radon-Nikodym derivative of  $\mathbb{P}_\alpha$  with respect to  $\mathbb{P}$ , see ref. [19]. This derivative is  $Q_\infty(\alpha)/Q_0(\alpha)$ . More concretely, for any  $\mathcal{F}_n$ -measurable integrable function  $X$ ,

$$Q_0(\alpha) \mathbb{E}_\alpha[X] = \mathbb{E}[Q_n(\alpha) X],$$

with  $Q_n(\alpha) = \mathbb{E}[Q_\infty(\alpha)|\mathcal{F}_n]$ , as may be checked directly.

We may tautologically refine this geometrical construction. Let us start from an arbitrary probability measure  $\mathbb{P}^0$  on  $\Omega$ , and let us set  $\mathfrak{z}_{i_1, \dots, i_n} := \mathbb{P}^0[B_{i_1, \dots, i_n}]$ , assuming that none of these probabilities vanish. Let  $Z_n$  and  $Z_n(\alpha)$  be  $\mathcal{F}_n$ -measurable functions defined by

$$Z_n(\alpha)(i_1, \dots, i_n) := \mathfrak{z}_{i_1, \dots, i_n}^{-1} \prod_k p(i_k|\alpha), \quad Z_n := \sum_\alpha Q_0(\alpha) Z_n(\alpha),$$

so that  $Q_n(\alpha)/Q_0(\alpha) = Z_n(\alpha)/Z_n$ . It is easy to check that each  $Z_n(\alpha)$  is a  $\mathbb{P}^0$ -martingale,  $\mathbb{E}^0[Z_n(\alpha)|\mathcal{F}_{n-1}] = Z_{n-1}(\alpha)$ . As it is clear from their definition, the  $Z_n(\alpha)$ 's are the Radon-Nikodym derivative of the measures  $\mathbb{P}_\alpha$ 's with respect to  $\mathbb{P}^0$  on  $\mathcal{F}_n$ -measurable functions, that is

$$\mathbb{E}_\alpha[X] = \mathbb{E}^0[Z_n(\alpha) X],$$

for any  $\mathcal{F}_n$ -measurable function  $X$ . Of course,  $Z_n$  is the Radon-Nikodym derivatives of  $\mathbb{P}$  with respect to  $\mathbb{P}^0$ , i.e.  $\mathbb{E}[X] = \mathbb{E}^0[Z_n X]$  for any  $\mathcal{F}_n$ -measurable function  $X$ . Choosing adequately  $\mathbb{P}^0$  helps taking the continuous time limit, a fact that we shall illustrate below.

### Brownian diffusive limit

The Brownian diffusive limit occurs when the conditioned probability  $p(\cdot|\alpha)$  depends on an extra small parameter  $\delta$  such that

$$p(i|\alpha) \simeq_{\delta \rightarrow 0} p_0(i) (1 + \sqrt{\delta} \Gamma(i|\alpha) + \dots),$$

with all  $p_0(i)$ 's non vanishing and  $\alpha$ -independent. Since  $\sum_i p_0(i) = 1$ , the  $p_0(\cdot)$ 's define an  $\alpha$ -independent probability measure on  $I$ . Note that  $\sum_i p_0(i) \Gamma(i|\alpha) = 0$  for all  $\alpha$  since  $\sum_i p(i|\alpha) = 1$  for all  $\delta$ . By the non-degeneracy assumption, the functions  $\Gamma(\cdot|\alpha)$  on  $I$  are all different.

The continuous time limit is then obtained by performing the scaling limit  $\delta \rightarrow 0$ ,  $n \rightarrow \infty$  with  $t := n/\delta$  fixed.

To understand the scaling limit of the counting processes  $N_n(i)$ , let us look at its behaviour under  $\mathbb{P}_\alpha$ , i.e. for a system in the pointer state  $\alpha$  with initial distribution  $Q_0(\cdot) = \delta_{\cdot;\alpha}$ . Then by hypothesis the output partial measurements are random independent variables, so that  $N_n(i) = \sum_{k=1}^n \mathbb{I}_{i_k=i}$  is the sum of  $n$  independent identically distributed (i.i.d.) variables  $\epsilon_k(i)$  with value 1 (if the output of  $k^{\text{th}}$  partial measurement is  $i$ ) with probability  $p(i|\alpha)$  and zero (if the output of  $k^{\text{th}}$  partial measurement is different from  $i$ ) with complementary probability. By the law of large numbers, the  $N_n(i)$ 's at large  $n$  become Gaussian processes with mean  $np(i|\alpha)$  and covariance  $\min(n, m) (p(i|\alpha)\delta_{i;j} - p(i|\alpha)p(j|\alpha))$ . Under these hypotheses, the probability  $\pi_{m-1}(i)$  for the  $m^{\text{th}}$  output partial measurement to be  $i$  is  $p(i|\alpha)$  for all  $m$ , so that  $A_n(i) = np(i|\alpha)$ . Hence, under  $\mathbb{P}_\alpha$  and for such peaked initial distribution, the law of the processes  $X_n(i)$  at large  $n$  is that of Gaussian processes with zero mean and covariance

$$\mathbb{E}_\alpha[X_n(i)X_m(j)] = \min(n, m) (p(i|\alpha)\delta_{i;j} - p(i|\alpha)p(j|\alpha)), \quad \text{for } Q_0(\cdot) = \delta_{\cdot;\alpha}.$$

After appropriate rescaling, this clearly admits a finite limit as  $\delta \rightarrow 0$  which is  $\alpha$ -independent. Hence under this hypothesis,  $X_n(i)$  admits a continuous time limit  $X_t(i)$ , formally to be thought of as  $\lim_{\delta \rightarrow 0} \sqrt{\delta} X_{[t/\delta]}(i)$ . However, the previous equation is not enough to describe this limit under the law  $\mathbb{P}$  and some care has to be taken, see ref. [16].

So, let us define the scaling diffusive limit of the state distribution and the Doob martingales:

$$Q_t(\delta) := \lim_{\delta \rightarrow 0} Q_{[t/\delta]}(\alpha), \quad X_t(i) := \lim_{\delta \rightarrow 0} \sqrt{\delta} X_{[t/\delta]}(i),$$

and of the counting processes,

$$W_t(i) := \lim_{\delta \rightarrow 0} \sqrt{\delta} (N_{[t/\delta]}(i) - p_0(i)t/\delta).$$

These equalities have to be thought in law, but we shall still denote by  $\mathbb{P} = \sum_\alpha Q_0(\alpha) \mathbb{P}_\alpha$  the probability measure for the continuous time processes. By construction,  $X_t(i)$  are  $\mathbb{P}$ -martingales.

The discrete difference equation (D.6) naively translates into the non-linear stochastic equation for  $Q_t(\alpha)$ . Recall that in the diffusive limit,  $p(i|\alpha) \simeq p_0(i)[1 + \sqrt{\delta} \Gamma(i|\alpha) + \dots]$  as  $\delta$  goes to zero, so that  $\pi_{n-1}(i) \simeq p_0(i)[1 + \sqrt{\delta} \langle \Gamma(i) \rangle_t + \dots]$  with  $\langle \Gamma(i) \rangle_t := \sum_\alpha \Gamma(i|\alpha) Q_t(\alpha)$ . In the continuous time limit, eq.(D.6) then becomes:

$$dQ_t(\alpha) = Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \langle \Gamma(i) \rangle_t) dX_t(i) \tag{D.7}$$

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with Itô's convention. We used  $\sum_i X_t(i) = 0$  and  $\sum_i p_0(i)\Gamma(i|\alpha) = 0$  to take this limit. Remark that this equation preserves the normalisation condition  $\sum_\alpha Q_t(\alpha) = 1$ . This equation is that which governs the evolution of the system probability distribution under continuous Bayes' updating in the diffusive limit. The random fields  $X_t(i)$  code for the information of the continuous time series of partial measurements. Not all of these fields are independent since  $\sum_i X_t(i) = 0$ . As we shall see, the main feature of the Brownian diffusive limit is that the  $X_t(i)$ 's are Gaussian processes with zero mean and covariance

$$\mathbb{E}[X_t(i)X_s(j)] = \min(t, s) (p_0(i)\delta_{i;j} - p_0(i)p_0(j)), \quad (\text{D.8})$$

Alternatively, the fields  $X_t(i)$  are zero mean Gaussian martingales with quadratic variation

$$dX_t(i)dX_t(j) = (p_0(i)\delta_{i;j} - p_0(i)p_0(j)) dt,$$

which is of course compatible with the relation  $\sum_i X_t(i) = 0$ . Actually the proofs of equation (D.7) and of the correctness of (D.8) are a bit tricky, see [16] for details. We shall here present an alternative less rigorous but quicker and simpler argument.

Let us now argue for eq.(D.8). Recall the Doob decomposition of the counting process  $N_n(i) = X_n(i) + A_n(i)$ . Its naive scaling limit reads

$$W_t(i) = X_t(i) + \int_0^t ds p_0(i) \langle \Gamma(i) \rangle_s,$$

whose infinitesimal differential form is

$$dW_t(i) = dX_t(i) + p_0(i) \langle \Gamma(i) \rangle_t dt \quad (\text{D.9})$$

Contrary to the  $X_t(i)$ 's, the  $W_t(i)$ 's are not  $\mathbb{P}$ -martingales but they are globally defined and independent of the initial distribution  $Q_0(\cdot)$  because they are defined as limit of the counting process. We however know that, under  $\mathbb{P}_\alpha$ , the  $W_t(i)$ 's are Gaussian processes with mean and covariance

$$\begin{aligned} \mathbb{E}_\alpha[W_t(i)] &= tp_0(i)\Gamma(i|\alpha), \\ \text{Cov}_\alpha[W_t(i)W_s(j)] &= \min(t, s) (p_0(i)\delta_{i;j} - p_0(i)p_0(j)). \end{aligned}$$

We now would like to use this information to read off properties of the martingales  $X_t(i)$ 's. The key point consists in using Girsanov's theorem [19]. Recall that  $Q_\infty(\alpha)$  may be thought as the Radon-Nikodym derivative of  $\mathbb{P}_\alpha$  with respect to  $\mathbb{P}$  and that  $Q_t(\alpha) = \mathbb{E}[Q_\infty(\alpha)|\mathcal{F}_t]$ . Assume for a while that the  $X_t(i)$ 's are Gaussian processes under  $\mathbb{P}$  with zero mean and quadratic variation  $G(i, j)dt := \langle dX_t(i), dX_t(j) \rangle$  to be determined. Girsanov's theorem tells us that modifying the measure  $\mathbb{P}$  by multiplication by the martingale  $Q_t(\alpha)$  adds a supplementary drift in the stochastic differential equation (D.9), given by the logarithmic derivative of the martingales. In the present case, using eq.(D.7) Girsanov's theorem implies that

$$dW_t(i) = d\hat{X}_t(i) + p_0(i) \langle \Gamma(i) \rangle_t dt + \sum_j G(i, j)(\Gamma(j|\alpha) - \langle \Gamma(j) \rangle_t)dt,$$

with  $\hat{X}_t(i)$ 's Gaussian processes under  $\mathbb{P}_\alpha$  with zero mean and identical quadratic variation  $G(i, j)dt$ . Comparing now with the known properties of  $W_t(i)$  under  $\mathbb{P}_\alpha$ , spelled out above,

we deduce that  $G(i, j) = (p_0(i)\delta_{i,j} - p_0(i)p_0(j))$ , as claimed<sup>6</sup>, so that the previous equation reduces to

$$dW_t(i) = d\hat{X}_t(i) + p_0(i)\Gamma(i|\alpha)dt,$$

under  $\mathbb{P}_\alpha$ , as required. This ends our argument for equations (D.7, D.8).

A way to rigorously construct processes with all the above properties is to deform a suitable a priori measure  $\mathbb{P}^0$ . Details have been given in ref. [16]. In this note, we shall illustrate this strategy in the Poissonian case.

Eq.(D.7) may actually be integrated explicitly, see ref. [16]. Furthermore, as bounded martingales the  $Q_t(\alpha)$ 's again converge almost surely and in  $\mathbb{L}^1$ . Under the non-degeneracy assumption that all  $\Gamma(\cdot|\alpha)$  are different, the limit distribution is peaked,  $Q_\infty(\cdot) = \delta_{\cdot, \gamma_\omega}$ , at a random target pointer state. The convergence is still exponential.

### Poissonian jumpy limit

The Poissonian limit occurs when the conditioned probabilities  $p(i|\alpha)$  vanish as a small parameter  $\delta$  vanishes. Not all  $p(i|\alpha)$ 's may vanish simultaneously as they sum up to 1. So let us single out one value  $i^*$  for which  $p(i|\alpha)$  goes to 1 as  $\delta \rightarrow 0$  for all  $\alpha$  and assume that all other  $p(i|\alpha)$  vanish in this limit:

$$p(i^*|\alpha) \simeq_{\delta \rightarrow 0} 1, \quad p(i|\alpha) \simeq_{\delta \rightarrow 0} \delta \theta(i|\alpha) \quad \text{for } i \neq i^*, \forall \alpha.$$

By consistency  $p(i^*|\alpha) = 1 - \delta(\sum_{i \neq i^*} \theta(i|\alpha)) + o(\delta)$  and all  $\theta(i|\alpha)$  are positive and assumed to be non-vanishing. In the limit  $\delta \rightarrow 0$ , the output of the partial measurements is most frequently  $i^*$  with sporadic jumps to another value  $i$  different from  $i^*$ <sup>7</sup>.

The continuous time limit is obtained by performing the scaling limit  $\delta \rightarrow 0$ ,  $n \rightarrow \infty$  with  $t = n/\delta$  fixed.

To understand the continuous time limit of the counting processes  $N_n(i)$ , let us again look at its behaviour under  $\mathbb{P}_\alpha$ . As before, the output partial measurements are then random independent variables, so that  $N_n(i) = \sum_{k=1}^n \mathbb{I}_{i_k=i}$  is the sum of  $n$  independent identically distributed variables  $\epsilon_k(i)$  with value 1 with probability  $p(i|\alpha)$  and zero with complementary probability. Let us first consider  $i \neq i^*$  and compute  $\log \mathbb{E}_\alpha[e^{zN_n(i)}] = n \log[(1 - p(i|\alpha) + e^z p(i|\alpha))]$ . In the scaling limit with  $n = t/\delta$  and  $p(i|\alpha) \simeq \delta \theta(i|\alpha)$  we get

$$\lim_{\delta \rightarrow 0} \log \mathbb{E}_\alpha[e^{zN_{[t/\delta]}(i)}] = t\theta(i|\alpha)(e^z - 1), \quad i \neq i^*.$$

Similar computations, based on the general formula

$$\mathbb{E}_\alpha \left[ e^{\sum_{l=1}^k \sum_i z_l(i)(N_{n_l}(i) - N_{n_{l-1}}(i))} \right] = \prod_{l=1}^k \left( \sum_i e^{z_l(i)} p(i|\alpha) \right)^{n_l - n_{l-1}},$$

for  $k \geq 1$ , arbitrary non-decreasing sequences of integers  $0 = n_0 \leq n_1 \leq \dots \leq n_k$  of length  $k$ , and arbitrary (complex)  $z_l(i)$ 's, show that, in the limit  $\delta \rightarrow 0$ , the  $\mathbb{P}_\alpha$ -distributions of the counting processes  $N_{[t/\delta]}(i)$ ,  $i \neq i^*$ , converge to those of independent Poisson point processes

6. This expression for the quadratic variation of the  $X_t(i)$ 's is compatible with the relation  $\sum_i X_t(i) = 0$ , since  $\sum_i G(i, j) = 0$  as it should.

7. We may generalize this by assuming that more than one conditioned probabilities remain finite as  $\delta$  goes to zero. In these cases, the continuous time limit will be a mixture between the Brownian and Poissonian limits.

## D.2. Iterated indirect stochastic measurements

with intensities  $\theta(i|\alpha) dt$ . Note that this statement is true under  $\mathbb{P}_\alpha$  but not under  $\mathbb{P}$ . However, we can compute their  $\mathbb{P}$ -generating functions using the decomposition of the measure  $\mathbb{P} = \sum_\alpha Q_0(\alpha) \mathbb{P}_\alpha$ . For instance

$$\mathbb{E}[e^{zN_{[t/\delta]}(i)}] \simeq_{\delta \rightarrow 0} \sum_\alpha Q_0(\alpha) e^{t\theta(i|\alpha)(e^z-1)}, \quad i \neq i^*.$$

The properties of  $N_n(i^*)$  and their limits are reconstructed using the sum rule,  $\sum_i N_n(i) = n$ . In particular, for small  $\delta$ ,  $N_{[t/\delta]}(i^*) \simeq t/\delta$  up to order 1 random corrections.

So, let us define the scaling Poisson limits of the state distribution and of the Doob martingales  $X_n(i)$ 's,

$$Q_t(\alpha) := \lim_{\delta \rightarrow 0} Q_{[t/\delta]}(\alpha), \quad Y_t(i) := \lim_{\delta \rightarrow 0} X_{[t/\delta]}(i)$$

and of the jump counting processes

$$N_t(i) := \lim_{\delta \rightarrow 0} N_{[t/\delta]}(i), \quad \text{for } i \neq i^*,$$

and  $M_t(i^*) := \lim_{\delta \rightarrow 0} (N_{[t/\delta]}(i^*) - t/\delta)$ . Again, these equalities have to be thought in law, but we still denote by  $\mathbb{P} = \sum_\alpha Q_0(\alpha) \mathbb{P}_\alpha$  the probability measure for the time continuous processes. By construction, the martingales  $Y_t(i)$  sum up to zero,  $\sum_i Y_t(i) = 0$ , and have zero mean,  $\mathbb{E}[Y_t(i)] = 0$ . Similarly,  $M_t(i^*) + \sum_{i \neq i^*} N_t(i) = 0$ .

Again, the naive scaling limit of the difference equation (D.6) yields a stochastic equation for the system state distribution. In the Poissonian limit, one has  $p(i|\alpha) \simeq \delta\theta(i|\alpha) + \dots$  for  $i \neq i^*$  as  $\delta \rightarrow 0$ , so that  $\pi_{n-1}(i) \simeq \delta \langle \theta(i) \rangle_t + \dots$  with  $\langle \theta(i) \rangle_t := \sum_\alpha \theta(i|\alpha) Q_t(\alpha)$ , for  $i \neq i^*$ , whereas both  $p(i^*|\alpha)$  and  $\pi_{n-1}(i^*)$  approach 1 as  $\delta$  goes to zero. The continuous time limit of eq.(D.6) is then

$$dQ_t(\alpha) = Q_t(\alpha) \sum_{i \neq i^*} \left( \frac{\theta(i|\alpha)}{\langle \theta(i) \rangle_t} - 1 \right) dY_t(i). \quad (\text{D.10})$$

where we used  $dY_t(i^*) = -\sum_{i \neq i^*} dY_t(i)$  to deal with the term associated to  $i^*$  in eq.(D.6). As we shall show just below, the  $Y_t(i)$ 's are related to the counting processes by

$$dN_t(i) = dY_t(i) + \langle \theta(i) \rangle_t dt, \quad i \neq i^*, \quad (\text{D.11})$$

We shall furthermore argue that the processes  $dN_t(i)$ ,  $i \neq i^*$ , are point processes with intensities  $\langle \theta(i) \rangle_t dt$ . This intensity is sample dependent – a point that we shall explain –, but predictable. Equations (D.10,D.11) are those which governs the evolution of the system probability distribution under continuous Bayes' updating in the Poissonian limit. The random counting processes  $N_t(i)$  code for the informations on the continuous time series of partial measurements.

Let us now argue for eq.(D.11). Consider again the Doob decomposition  $N_n(i) = X_n(i) + A_n(i)$ . Because  $\pi_{[t/\delta]}(i) \simeq \delta \langle \theta(i) \rangle_t$  for small  $\delta$ , its naive scaling reads

$$N_t(i) = Y_t(i) + \int_0^t ds \langle \theta(i) \rangle_s, \quad i \neq i^*.$$

Its infinitesimal version is eq.(D.11), as announced. Since  $p(i^*|\alpha) \simeq 1 - \delta\sigma(i^*|\alpha)$  with  $\sigma(i^*|\alpha) := \sum_{i \neq i^*} \theta(i|\alpha)$ , the counting function  $N_n(i^*)$  slightly deviates from  $n$ , and  $M_t(i^*) = Y_t(i^*) - \int_0^t ds \langle \sigma(i^*) \rangle_s$  with  $\langle \sigma(i^*) \rangle_s := \sum_\alpha \sigma(i^*|\alpha) Q_s(\alpha)$ .

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By the martingale property,  $\mathbb{E}[dY_t(i)|\mathcal{F}_t] = 0$  so that

$$\mathbb{E}[dN_t(i)|\mathcal{F}_t] = \langle \theta(i) \rangle_t dt, \quad i \neq i^*.$$

That is the number of jumps in the direction  $i$  in the time interval  $[t, t + dt)$  depends on the past of the process and is equal to  $\langle \theta(i) \rangle_t dt$  in mean. We may go a little further and compute the generating function of those jumps. Indeed, since the conditional measure  $\mathbb{E}[\cdot|\mathcal{F}_t]$  decomposes as a sum,  $\mathbb{E}[\cdot|\mathcal{F}_t] = \sum_{\alpha} Q_t(\alpha) \mathbb{E}_{\alpha}[\cdot|\mathcal{F}_t]$ , and since under  $\mathbb{E}_{\alpha}[\cdot|\mathcal{F}_t]$  the  $dN_t(i)$ 's are Poisson point processes with intensity  $\theta(i|\alpha)dt$ , we have

$$\log \mathbb{E}[e^{z dN_t(i)}|\mathcal{F}_t] = dt \langle \theta(i) \rangle_t (e^z - 1), \quad (\text{D.12})$$

with  $\langle \theta(i) \rangle_t = \sum_{\alpha} Q_t(\alpha) \theta(i|\alpha)$ . That is, under  $\mathbb{P}$ , the  $dN_t(i)$ 's are point processes with intensities  $\langle \theta(i) \rangle_t dt$ , as announced. As above, a similar computation shows that the  $dN_t(i)$ 's, for fixed  $t$ , are independent variables for  $i \neq j$  under  $\mathbb{P}_{\alpha}$  but not under  $\mathbb{P}$ . An alternative description of this limit is given in ref. [17], see also the forthcoming ref. [21].

Up to now, our arguments have been only in law. A rigorous construction<sup>8</sup> of processes, living on a well-defined probability space, and having all the required properties, is to deform a suitable a priori measure  $\mathbb{P}^0$ . The hint that this is possible is the formula for  $Q_t(\alpha)$  obtained by taking the continuous time limit of eq.(D.4). There are some cancellations of powers of  $\delta$  between numerator and denominator yielding

$$Q_t(\alpha) = Q_0(\alpha) \frac{Z_t(\alpha)}{Z_t}, \quad \text{with } Z_t := \sum_{\beta} Q_0(\beta) Z_t(\beta),$$

where

$$Z_t(\alpha) := \prod_{i \neq i^*} \theta(i|\alpha)^{N_t(i)} e^{-t(\theta(i|\alpha)-1)}. \quad (\text{D.13})$$

One recognizes  $Z_t(\alpha)$  as the standard exponential Poisson martingale. So, let us start from an a priori probability measure  $\mathbb{P}^0$  accommodating for independent Poisson processes  $N_t(i)$ ,  $i \neq i^*$ , of intensity  $dt$ . Define  $\mathbb{P}_{\alpha} := Z_t(\alpha) \mathbb{P}^0$  on  $\mathcal{F}_t$ . Then, under  $\mathbb{P}_{\alpha}$ , the  $N_t(i)$ 's are independent Poisson processes with intensity  $\theta(i|\alpha)dt$ . Defining

$$\mathbb{P} := \left( \sum_{\alpha} Q_0(\alpha) Z_t(\alpha) \right) \mathbb{P}^0,$$

it is plain that the  $Q_t(\alpha)$ 's are  $\mathbb{P}$ -martingales and the  $N_t(i)$ 's have the law we were after. For instance, since  $Q_t(\alpha) = Q_0(\alpha) Z_t(\alpha)/Z_t$ , we have  $\mathbb{E}[\cdot|\mathcal{F}_t] = \sum_{\alpha} Q_t(\alpha) \mathbb{E}_{\alpha}[\cdot|\mathcal{F}_t]$  so that  $dN_t := N_{t+dt}(i) - N_t(i)$  is at most 1 and  $\mathbb{P}[dN_t(i) = 1|\mathcal{F}_t] = dt \langle \theta(i) \rangle_t$ .

### D.3. Iterated indirect quantum measurements

Although purely probabilistic – involving classical probability only – the previous description of iterated stochastic measurements finds applications in the quantum world, in particular in the framework of repeated indirect non-demolition measurements [5]. Recall that an

<sup>8</sup>. Which is an alternative to ref. [17] in that case.

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indirect quantum measurement consists in letting a quantum system interact with another quantum system, called the probe, and implementing a direct Von Neumann measurement on the probe. One then gains information on the system because the probe and the system have been entangled. Repeating the cycle of entanglement and measurement progressively increases the information on the system as in the model apparatus we described above.

Let  $\mathcal{S}$  be the quantum system and  $\mathcal{H}_s$  be its Hilbert space of states. Pick a basis of states  $\{|\alpha\rangle\}$  in  $\mathcal{H}_s$ , which are going to play the role of pointer states. Let  $\mathcal{P}$  be the probe and  $\mathcal{H}_p$  be its Hilbert space. We assume that the probe-system interaction preserves the pointer states: a system initially prepared in one of the pointer state remains in this state after having interacted with the probes. This requires a peculiar form for the unitary operator  $U$  of the probe-system interaction:

$$U = \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U_{\alpha}, \quad (\text{D.14})$$

with  $U_{\alpha}$  an unitary operators on  $\mathcal{H}_p$ . Alternatively,  $U|\alpha\rangle \otimes |\nu\rangle = |\alpha\rangle \otimes U_{\alpha}|\nu\rangle$  for any  $|\nu\rangle \in \mathcal{H}_p$ , a property coding for the fact that the pointer states  $|\alpha\rangle$  are preserved by this interaction.

We imagine sending identical copies of the probe, denoted  $\mathcal{P}_1, \mathcal{P}_2, \dots$ , one after the other through the system and measuring an observable on each probe after the interaction. We assume that the in-going probes have all been prepared in the same state  $|\psi\rangle \in \mathcal{H}_p$ , and that the observables measured in the out-going channel are all identical with non-degenerate spectrum  $I$ . Let  $\{|i\rangle\} \in \mathcal{H}_p$ ,  $i \in I$ , be the basis of eigenstates of the measured observable. We denote by  $i_k$  the output of the measurement on the  $k^{\text{th}}$  out-going probe. In analogy with previous section, we call the cycle entanglement and measurement on a probe a partial measurement. The results of repetitions of theses cycles of partial measurements are random sequences  $(i_1, i_2, \dots)$ ,  $i_k \in I$ . As before, such infinite series of partial measurements will be called a complete measurement. The unitary operator  $U$  codes for the probability of measuring a given value  $i$  on the out-going probe. Suppose that the in-going probe has been prepared in the state  $|\psi\rangle$  and the system  $\mathcal{S}$  in the state  $|\alpha\rangle$ . After interaction, the system-probe state is  $|\alpha\rangle \otimes U_{\alpha}|\psi\rangle$  and the probability to measure the value  $i$  of the probe observable is

$$p(i|\alpha) := |\langle i|U_{\alpha}|\psi\rangle|^2,$$

by the rule of quantum mechanics. So  $|\langle i|U_{\alpha}|\psi\rangle|^2$  is the probability to measure  $i$  in the out-going channel conditioned on the system state be  $|\alpha\rangle$ . The analogy with the previous section should start to become clear.

#### D.3.1. Discrete time description

Let  $\rho$  be the system density matrix. The system state probability distribution is  $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$ . The aim of this section is to describe how the system state distribution and the density matrix evolve when the cycles of entanglement and measurement are repeated, and to make explicit contact with previous sections.

Assume that the system is initially prepared in a density matrix state  $\rho_0$ , and let us look at what happens during a cycle of entanglement and interaction. Recall that the probe is assumed to be prepared in the density matrix state  $|\psi\rangle\langle\psi|$ . After interaction, the joint system-probe density matrix is  $U\rho_0 \otimes |\psi\rangle\langle\psi|U^{\dagger}$ . The observable, with spectrum  $I$ , is then measured on the probe. If  $i_1$  is the output value of this measurement, the joint system-probe state is

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projected into  $\rho_1 \otimes |i_1\rangle\langle i_1|$  with

$$\rho_1 := \frac{1}{\pi_0(i_1)} \langle i_1|U|\psi\rangle \rho_0 \langle \psi|U^\dagger|i_1\rangle,$$

This occurs with probability  $\pi_0(i_1) = \text{Tr}(\langle i_1|U|\psi\rangle \rho_0 \langle \psi|U^\dagger|i_1\rangle)$ . Using the assumed property of  $U$ , eq.(D.14), this can be rewritten as

$$\pi_0(i) := \text{Tr}(\langle i|U|\psi\rangle \rho_0 \langle \psi|U^\dagger|i\rangle) = \sum_{\alpha} p(i|\alpha) Q_0(\alpha).$$

How this cycle is to be repeated is clear. Let  $\rho_{n-1}$  be the system density matrix after the  $n-1$  first partial measurements – this density matrix depends on the random values of these measurements, so that  $\rho_{n-1} = \rho_{n-1}(i_1, \dots, i_{n-1})$ , but we simplify the notation by not writing explicitly the values of the measurements. We let the system interact with the  $n^{\text{th}}$  probe and do a measurement on this probe. If  $i_n$  is the output value of this  $n^{\text{th}}$  partial measurement, the system state is projected into

$$\rho_n = \frac{1}{\pi_{n-1}(i_n)} \langle i_n|U|\psi\rangle \rho_{n-1} \langle \psi|U^\dagger|i_n\rangle, \quad (\text{D.15})$$

where again we simplified the notation by not writing the values of the partial measurements –  $\rho_n$  should have been written as  $\rho_n(i_n|i_1, \dots, i_{n-1})$  and similarly for  $\pi_{n-1}$ . This projection occurs with probability  $\pi_{n-1}(i_n)$ , with

$$\pi_{n-1}(i) := \text{Tr}(\langle i|U|\psi\rangle \rho_{n-1} \langle \psi|U^\dagger|i\rangle) = \sum_{\alpha} p(i|\alpha) Q_{n-1}(\alpha).$$

The diagonal matrix elements of the density matrix are the probabilities for the system to be in a pointer state, that is  $Q_n(\alpha) = \langle \alpha|\rho_n|\alpha\rangle$ . From eq.(D.15) we read that

$$Q_n(\alpha) = \frac{p(i_n|\alpha) Q_{n-1}(\alpha)}{\pi_{n-1}(i_n)}.$$

The two above equations exactly coincide with eqs.(D.2,D.3) defining iterated stochastic measurements. So everything we wrote in the previous sections applies. In particular the collapse of the system probability distribution is a discrete implementation of the wave function collapse in Von Neumann measurement. The quantum system observable measured by the iteration of cycles of entanglement and indirect measurement is that with eigenstate basis  $\{|\alpha\rangle\}$ . The collapse happens only for an infinite sequence of partial measurements reflecting the fact that the iterated stochastic measurement apparatus is macroscopic only if an infinite sequence of partial measurements is implemented, see ref. [6].

#### D.3.2. Continuous time limit

The aim of this section is to take the continuous time limit of the discrete recurrence equation (D.15) for the quantum density matrix using the results of the previous section. Doing this we will make contact with the so-called Belavkin equations [14], describing continuous time measurements in quantum mechanics and which are non-linear stochastic Schrödinger equations [20].

### D.3. Iterated indirect quantum measurements

The small parameter  $\delta$  is the time duration of the system-probe interaction, so that the unitary operator is  $U = \exp(-\imath\delta H)$  with  $H$  the system-probe hamiltonian<sup>9</sup>. As is well known, the dynamics of a quantum system under continuous measurements is frozen by continuous wave packet reductions, a fact named the quantum Zeno effect. To avoid it, we have to rescale the system-probe interaction at the same time we decrease the interaction time duration. So we assume the following form of the hamiltonian  $H$ ,

$$H = H_s \otimes 1 + 1 \otimes H_p + \frac{1}{\sqrt{\delta}} H_I, \quad (\text{D.16})$$

where  $H_s$  is the system hamiltonian,  $H_p$  the probe hamiltonian and  $H_I$  the interaction hamiltonian.

For the pointer state to be stable under the action of  $U = e^{-\imath\delta H}$ , eq.(D.14), we should assume that  $H_s$  is diagonal in the pointer basis,  $H_s = \sum_{\alpha} |\alpha\rangle E_{\alpha} \langle \alpha|$  for some energies  $E_{\alpha}$ , – this is linked to the non-demolition character of the measurement – and that

$$H_I = \sum_{\alpha} |\alpha\rangle \langle \alpha| \otimes H_{\alpha},$$

with  $H_{\alpha}$  acting on  $\mathcal{H}_p$  but  $\alpha$  dependent. The conditioned probabilities  $p(i|\alpha)$  are then

$$p(i|\alpha) = |\langle i|\psi\rangle - \imath\sqrt{\delta} \langle i|H_{\alpha}|\psi\rangle + \dots|^2,$$

so that the Brownian diffusive case corresponds  $\langle i|\psi\rangle \neq 0$  and the Poissonian jumpy case to  $\langle i|\psi\rangle = 0$ .

In both cases, the continuous time limit is then obtained by performing the scaling limit  $\delta \rightarrow 0$ ,  $n \rightarrow \infty$  with  $t := n/\delta$  fixed as above.

It is useful to recast the quantum recursion relation (D.15) into a difference equation. This simplifies matter when taking the continuous time limit. Let us write  $\rho_n = \sum_i \rho_n(i_n) \mathbb{I}_{i_n=i}$  with  $\rho_n(i_n)$  defined in eq.(D.15). Recall that  $\mathbb{E}[\mathbb{I}_{i_n=i} | \mathcal{F}_{n-1}] = \pi_{n-1}(i)$  and write  $\mathbb{I}_{i_n=i} = (\mathbb{I}_{i_n=i} - \pi_{n-1}(i)) + \pi_{n-1}(i)$ . This leads to the Doob decomposition of the difference  $\rho_n - \rho_{n-1}$  as,

$$\rho_n - \rho_{n-1} = (D\rho)_{n-1} + (\Delta\rho)_n, \quad (\text{D.17})$$

with  $(D\rho)_{n-1} := \mathbb{E}[\rho_n | \mathcal{F}_{n-1}] - \rho_{n-1}$ , which is  $\mathcal{F}_{n-1}$ -measurable, and  $(\Delta\rho)_n := \rho_n - \mathbb{E}[\rho_n | \mathcal{F}_{n-1}]$ , which satisfies  $\mathbb{E}[(\Delta\rho)_n | \mathcal{F}_{n-1}] = 0$ . Explicitely,

$$\begin{aligned} (D\rho)_{n-1} &= \sum_i \langle i|U|\psi\rangle \rho_{n-1} \langle \psi|U^{\dagger}|i\rangle - \rho_{n-1}, \\ (\Delta\rho)_n &= \sum_i \frac{\langle i|U|\psi\rangle \rho_{n-1} \langle \psi|U^{\dagger}|i\rangle}{\pi_{n-1}(i)} (X_n(i) - X_{n-1}(i)), \end{aligned}$$

where we used  $\mathbb{I}_{i_n=i} - \pi_{n-1}(i) = X_n(i) - X_{n-1}(i)$ , as in previous section. In the continuous time limit, the first term  $(D\rho)_{n-1}$  is going to converge towards the drift term and the second one  $(\Delta\rho)_n$  to the noisy source of the stochastic differential equation.

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9. We use the notation  $\imath$ , without a dot, to code for the square root of  $-1$ .

### Brownian diffusive limit

The Brownian diffusive limit occurs when  $\langle i|\psi\rangle \neq 0$  for all  $i$ . Then  $p(i|\alpha) \simeq p_0(i)(1 + \sqrt{\delta}\Gamma(i|\alpha) + \dots)$  for  $\delta$  small with,

$$p_0(i) = |\langle i|\psi\rangle|^2, \quad \Gamma(i|\alpha) = 2\text{Im}\left(\frac{\langle i|H_\alpha|\psi\rangle}{\langle i|\psi\rangle}\right).$$

This is the situation we encountered in the previous section on the classical diffusive limit, so that we can borrow all results obtained there.

It is then a simple matter to naively take the continuous time limit of the difference equations (D.17). This limit exists only if  $\langle \psi|H_I|\psi\rangle = 0$ , which is equivalent to

$$\langle \psi|H_\alpha|\psi\rangle = 0 \text{ for all } \alpha,$$

a criteria which we assume to hold true. Recall that this scaling limit consists in  $\delta \rightarrow 0$  with  $t = n\delta$  fixed. Let us first expand the term  $(D\rho)_{n-1}$  in power of  $\sqrt{\delta}$ . The term of order  $\sqrt{\delta}$  vanishes due to the condition  $\langle \psi|H_I|\psi\rangle = 0$ , and for the term of order  $\delta$  we get:  $(D\rho)_{[t/\delta]} \simeq_{\delta \rightarrow 0} L_d(\rho_t)\delta$ , with Linbladian

$$L_d(\rho_t) := -i[H_s, \rho_t] + \sum_i p_0(i)(C_i \rho_t C_i^\dagger - \frac{1}{2}\{C_i^\dagger C_i, \rho_t\}),$$

where we defined the operators  $C_i$ 's acting on  $\mathcal{H}_s$  by  $C_i := -i\frac{\langle i|H_I|\psi\rangle}{\langle i|\psi\rangle}$ , or equivalently

$$C_i := -i \sum_\alpha |\alpha\rangle \frac{\langle i|H_\alpha|\psi\rangle}{\langle i|\psi\rangle} \langle \alpha|,$$

using the decomposition of  $H_I$  on pointer states. Remark that  $\sum_i p_0(i)C_i = 0$  thanks to the assumed condition  $\langle \psi|H_\alpha|\psi\rangle = 0$ . Similarly, expanding the term  $(\Delta\rho)_n$  using  $\pi_{n-1}(i) \simeq p_0(i)[1 + \delta\text{Tr}[(C_j + C_j^\dagger)\rho_t] + \dots]$ , we get  $\lim_{\delta \rightarrow 0}(\Delta\rho)_{[t/\delta]} = \sum_j \mathcal{D}_j(\rho_t) dX_t(j)$ , with

$$\mathcal{D}_j(\rho_t) := C_j \rho_t + \rho_t C_j^\dagger - \rho_t \text{Tr}[(C_j + C_j^\dagger)\rho_t].$$

Note that computing these limits only uses the decomposition (D.16) of the hamiltonian  $H$  and not the existence of a preferred pointer state basis<sup>10</sup>. Gathering shows that the Brownian limit of the difference equation (D.15) is

$$d\rho_t = L_d(\rho_t) dt + \sum_j \mathcal{D}_j(\rho_t) dX_t(j) \quad (\text{D.18})$$

where the  $X_t(j)$ 's are the Gaussian centred processes, with quadratic variation

$$dX_t(i)dX_t(j) = (p_0(i)\delta_{i,j} - p_0(i)p_0(j)) dt,$$

defined in eq.(D.8) and in the discussion around this equation. This is an example of the diffusive Belavkin equation [14,18]. It is important to recall that  $\sum_i p_0(i)C_i = 0$  since without this condition, but with  $\sum_i X_t(i) = 0$  as we do have, eq.(D.18) would not be positivity preserving [18]. Contact with previous sections can be made explicit by recalling that the state probability distribution is  $Q_t(\alpha) = \langle \alpha|\rho_t|\alpha\rangle$  and by noticing that  $\text{Tr}[(C_j + C_j^\dagger)\rho_t] = \langle \Gamma(i)\rangle_t$ . We only took a naive limit of the difference equation (D.15), to mathematically prove the diffusive Belavkin equation for the system density matrix in the scaling limit would require more delicate arguments.

<sup>10</sup>. The existence of the pointer state basis was however used in determining the statistical properties of the fields  $X_t(i)$ .

### Poissonian jumpy limit

The Poissonian limit occurs when  $\langle i|\psi\rangle = 0$ . This cannot happen for all  $i$  as  $\{|i\rangle\}$  forms an orthonormal basis of  $\mathcal{H}_p$  and  $|\psi\rangle$  is non zero. So, we assume, for simplicity, that one element of this basis is  $|\psi\rangle$ , say  $|i^*\rangle = |\psi\rangle$ , and all others are orthogonal to  $|\psi\rangle$ , i.e.  $\langle i|\psi\rangle = 0$  for all  $i \neq i^*$ . Then,  $p_0(i^*|\alpha) \simeq_{\delta \rightarrow 0} 1$  and  $p_0(i|\alpha) \simeq_{\delta \rightarrow 0} \delta \theta(i|\alpha)$ , for  $i \neq i^*$ , with

$$\theta(i|\alpha) = |\langle i|H_\alpha|\psi\rangle|^2.$$

This is the situation we encountered in the previous section on the classical Poisson jumpy limit, so that we can borrow all results obtained there.

As in the diffusive case, it is a simple matter to naively take the continuous time limit of the difference equation (D.17). This only uses the decomposition the hamiltonian  $H$  but the limit exists only if  $\langle \psi|H_I|\psi\rangle = 0$ , and we assume this to be true. Expanding the first term  $(D\rho)_{n-1}$  to second order in  $\sqrt{\delta}$ , we get  $(D\rho)_{[t/\delta]} \simeq_{\delta \rightarrow 0} L_p(\rho_t)\delta$ , with Linbladian

$$L_p(\rho_t) := -i[H_s, \rho_t] + \sum_{i \neq i^*} (D_i \rho_t D_i^\dagger - \frac{1}{2}\{D_i^\dagger D_i, \rho_t\}),$$

where we defined the operators  $D_i := -i\langle i|H_I|\psi\rangle$  acting on  $\mathcal{H}_s$ , that is

$$D_i := -i \sum_{\alpha} |\alpha\rangle \langle i|H_\alpha|\psi\rangle \langle \alpha|.$$

To compute the limit of the second term  $(\Delta\rho)_n$ , we notice that, to leading order in  $\delta$ ,  $\langle i|U|\psi\rangle\rho\langle\psi|U|i\rangle \simeq \delta D_i \rho D_i^\dagger$  and  $\pi_{n-1}(i) \simeq \delta \text{Tr}[D_i \rho_t D_i^\dagger]$  for  $i \neq i^*$ , and we get  $\lim_{\delta \rightarrow 0} (\Delta\rho)_{[t/\delta]} = \sum_{i \neq i^*} \widehat{\mathcal{D}}_i(\rho_t) dY_t(i)$ , with

$$\widehat{\mathcal{D}}_i(\rho_t) := \frac{D_i \rho_t D_i^\dagger}{\text{Tr}[D_i \rho_t D_i^\dagger]} - \rho_t,$$

where the last term  $-\rho_t$  comes from using  $dY_t(i^*) = -\sum_{i \neq i^*} dY_t(i)$  when computing the contribution of the  $i^*$ -term in  $(\Delta\rho)_n$ , as in previous section. Gathering shows that the Poissonian limit of the difference equation (D.15) is

$$d\rho_t = L_p(\rho_t) dt + \sum_{i \neq i^*} \widehat{\mathcal{D}}_i(\rho_t) dY_t(i) \quad (\text{D.19})$$

where the  $Y_t(j)$ 's are the Poisson-like compensated martingales defined in eq.(D.11) above. That is,

$$dN_t(i) = dY_t(i) + \text{Tr}[D_i \rho_t D_i^\dagger] dt, \quad i \neq i^*,$$

where  $dN_t(i)$ 's are the point processes with intensities  $\langle \theta(i) \rangle_t dt$  defined in previous section, see e.g. eq.(D.12,D.13). Note that, using the decomposition of the interaction hamiltonian  $H_I$  on the pointer state basis,  $H_I = \sum_{\alpha} |\alpha\rangle \langle \alpha| \otimes H_\alpha$ , we have

$$\text{Tr}[D_i \rho_t D_i^\dagger] = \sum_{\alpha} \theta(i|\alpha) Q_t(\alpha) = \langle \theta(i) \rangle_t,$$

so that the previous equation indeed coincides with eq.(D.11). Equation (D.19) is an example of a jumpy Belavkin equation [14]. Let us finally point out that the stochastic processes

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(D.18,D.19) are not the most general one because we assumed that they preserve the pointer state basis<sup>11</sup> so that the operator  $H_s$ ,  $C_i$  or  $D_j$  are diagonal in the pointer basis. This is of course related to the non-demolition property. Eqs.(D.18,D.19) are also peculiar examples of more general class of models for continuous quantum measurements whose long time behavior leads to purification of mixed states, see e.g. [18,22].

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11. We made this assumption when computing the probability distributions of the processes  $X_t(i)$  and  $Y_t(i)$  but not when formally taking the continuous time limit of the discrete eq.(D.15).

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# E. Article: Large Time Behaviour and Convergence Rate for Non Demolition Quantum Trajectories

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Tristan Benoist<sup>♠1</sup> and Clément Pellegrini<sup>♣2</sup>

<sup>♠</sup>Laboratoire de Physique Théorique de l'ENS, CNRS & École Normale Supérieure de Paris, 24 rue Lhomond 75005 Paris, France.

<sup>♣</sup>Institut de Mathématiques de Toulouse, Equipe de Statistique et de Probabilité, Université Paul Sabatier, 31062 Toulouse Cedex 9, France.

## Abstract

A quantum system  $\mathcal{S}$  undergoing continuous time measurement is usually described by a jump-diffusion stochastic differential equation. Such an equation is called a stochastic master equation and its solution is called a quantum trajectory. This solution describes actually the evolution of the state of  $\mathcal{S}$ . In the context of Quantum Non Demolition measurement, we investigate the large time behavior of this solution. It is rigorously shown that, for large time, this solution behaves as if a direct Von Neumann measurement has been performed at time 0. In particular the solution converges to a random pure state which is related to the wave packet reduction postulate. Using theory of Girsanov transformation, we determine precisely the exponential rate of convergence towards this random state. The important problem of state estimation (used in experiment) is also investigated.

## Introduction

In quantum optics, indirect measurements are often used [28, 29]. Usually a system is probed by light beams (direct photodetection, homodyne and heterodyne detection schemes) or conversely, atoms probe a photon field trapped in a cavity. Such experiments are promising

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1. tristan.benoist@ens.fr

2. clement.pellegrini@math.univ-toulouse.fr

towards the manipulation of quantum states [3, 4, 29, 32, 37, 39]. They are designed to extract information from a quantum system on site and without destroying it. The idea is to avoid direct interaction of the quantum system with a macroscopic apparatus (photo detector, screen ...). Instead the physical setup is the following: a quantum system  $\mathcal{S}$  (from which we want to extract information) is put in interaction with an auxiliary quantum system  $\mathcal{E}$ . After interaction, a measurement on  $\mathcal{E}$  is performed. Due to the entanglement between  $\mathcal{S}$  and  $\mathcal{E}$ , the result of the measurement holds some information on  $\mathcal{S}$ . Conditionally to this result one can compute the evolution of  $\mathcal{S}$ .

One of the best example of such setups is Serge Haroche's group experiment at LKB [27]. They have successfully implemented a scheme of repeated interactions and measurements which allows to measure the number of photons in a cavity (without destroying the photons). The principle consist in putting the system  $\mathcal{S}$  (the cavity photon field) in contact with a sequence of auxiliary systems (Rydberg atoms)  $\mathcal{E}_k$  which interact one after the other with  $\mathcal{S}$ . After each interaction a measurement on the atom  $\mathcal{E}_k$  which has just finish to interact is performed. Such a procedure, called *repeated quantum indirect measurements*, allows to monitor the system  $\mathcal{S}$  and to have an estimation of the number of photons inside the cavity.

A particular feature in the Serge Haroche's group experiment is that only Quantum Non Demolition (QND) measurement are performed. Such a scheme is at the cornerstone of the mathematical study of the long time behavior of  $\mathcal{S}$ . In [13, 14], the authors show that the state of the system  $\mathcal{S}$  converges when the number of interactions tends to infinity. More precisely they show a convergence which is compatible with the wave function collapse postulate. In particular it is shown that the state of  $\mathcal{S}$  behaves in infinite time as if a direct Von Neumann measurement on  $\mathcal{S}$  would have been performed at time 0. Essentially these results concern discrete time model where the time of interaction  $\tau$  between  $\mathcal{S}$  and a piece  $\mathcal{E}_k$  is fixed. They apply to general nondemolition measurement scheme of which Serge Haroche's group experiment is an example (see [22] and references therein).

When the time of interaction goes to zero, this yields to continuous time models. In [5] it has been shown that quantum repeated interactions model are a powerful approximation of the so called *Quantum Langevin equation*. In [33–35], it is shown that the continuous time approximation ( $\tau$  goes to zero) of repeated quantum indirect measurements lead to jump-diffusion stochastic differential equations (see also [15]). Such equations are namely the equations which describe the evolution of a quantum system undergoing indirect continuous measurements [7–12, 18, 19, 25, 39]. They are called *stochastic master equations* and their solutions *quantum trajectories*.

In this article, we focus on the stochastic master equations describing general continuous time quantum nondemolition measurement. Our main purpose is to describe the long time behavior of the state of  $\mathcal{S}$  when the time  $t$  goes to infinity. In particular if  $(\rho(t))$ <sup>3</sup> describes the stochastic evolution of  $\mathcal{S}$  undergoing indirect QND measurement, we show that  $(\rho(t))$  converges to a pure state  $|\Upsilon\rangle\langle\Upsilon|$ . This convergence is obtained by studying in detail the quantities  $(q_\alpha(t))$  defined by

$$q_\alpha(t) = \text{Tr}[\rho(t)|\alpha\rangle\langle\alpha|], \alpha \in \mathcal{P}, t \geq 0$$

where  $\mathcal{P}$  is a preferred basis of the Hilbert space describing  $\mathcal{S}$ . In particular the quantity  $q_\alpha(t)$  gives the probability for  $\mathcal{S}$  to be in the state  $|\alpha\rangle$  if a direct measurement on  $\mathcal{S}$  would have

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3. The process  $(\rho(t))$  is actually the quantum trajectory describing the evolution of the state of  $\mathcal{S}$  which undergoes indirect continuous measurement

ben performed at time  $t$ . The elements  $|\alpha\rangle$  are often called pointer states. We show that  $\Upsilon$  is a random variable on the set of pointer states. By studying the distribution of  $\Upsilon$  we are able to connect the convergence towards  $|\Upsilon\rangle\langle\Upsilon|$  with the wave function collapse postulate at time 0. From this convergence we study the behavior of the system conditioned on the limit state  $\Upsilon$  being  $\gamma$ . This conditioning corresponds to a particular martingale change of measure. Using a standard Girsanov transformation, we show rigorously that the convergence towards  $|\Upsilon\rangle\langle\Upsilon|$  is exponentially fast and we give its explicit rate. The problem of estimation is also investigated when we are in the situation where the initial state  $\rho(0)$  is unknown. In this context, we only have access to the result of the measurements. Since  $\rho(0)$  is assumed to be unknown, this is not sufficient for describing totally  $(\rho(t))$ . In particular we are only able to describe the evolution of an estimate state of  $(\rho(t))$  denoted by  $(\tilde{\rho}(t))$ . We show that this estimation is efficient since  $(\tilde{\rho}(t))$  converges in long time to the same limit  $|\Upsilon\rangle\langle\Upsilon|$ . Such a property often refers to the notion of stability of quantum filter [1, 2, 32, 36, 38].

The article is structured as follows. In Section 1, we introduce the stochastic models describing the generic stochastic master equations. Next we present the particular case of nondemolition stochastic master equations. This allows us to define the processes  $(q_\alpha(t))$ . We then study the properties of these processes and show that they are bounded martingales. Section 2 is devoted to the main convergence theorem. From the martingale and boundedness property of  $(q_\alpha(t))$ , we conclude that these processes converge almost surely. This allows us to present the main convergence result and to define precisely the random variable  $\Upsilon$ . Next, using appropriate Girsanov change of measure, we show that this convergence is exponentially fast. Finally we investigate the problem of estimation.

## E.1. Non destructive quantum trajectories

### E.1.1. System state evolution

This section is devoted to present the continuous time stochastic processes which describe quantum trajectories. As announced these stochastic processes are solutions of particular type of jump-diffusion stochastic differential equations.

Before presenting the SDEs, let us introduce some notations. The quantum system is represented by a finite dimensional Hilbert space denoted by  $\mathcal{H}$ . We denote the set of density matrices by  $\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}), \rho \geq 0, \text{Tr}[\rho] = 1\}$ . A density matrix represents a general system mixed state. A system in a pure state  $|\phi\rangle \in \mathcal{H}$  corresponds to a special case where the density matrix is the projector onto  $|\phi\rangle$ . In this situation the corresponding density matrix is  $\rho_{|\phi\rangle} = |\phi\rangle\langle\phi|$ . In the rest of the article, if not specified, the term state refers to a density matrix.

Let us consider a family  $C_i, i = 0, \dots, n$  of operators in  $\mathcal{B}(\mathcal{H})$  and let  $H \in \mathcal{B}(\mathcal{H})$  such that  $H = H^*$  i.e.  $H$  is a Hermitian operator. On  $\mathcal{S}(\mathcal{H})$ , we introduce the following functions:

$$\begin{aligned} L(\rho) &= -i[H, \rho] + \sum_{i=0}^n \left( C_i \rho C_i^* - \frac{1}{2} (C_i^* C_i \rho + \rho C_i^* C_i) \right) \\ J_i(\rho) &= C_i \rho C_i^*, \quad i = 0, \dots, n \\ v_i(\rho) &= \text{Tr}[J_i(\rho)], \quad i = 0, \dots, n \\ H_i(\rho) &= C_i \rho + \rho C_i^* - \text{Tr}[(C_i + C_i^*)\rho]\rho, \quad i = 0, \dots, n, \end{aligned} \tag{E.1}$$

for all states  $\rho \in \mathcal{S}(\mathcal{H})$ .

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  be a filtered probability space with usual conditions. Let  $(W_j(t)), j = 0, \dots, p$  be standard independent Wiener processes and let  $(N_j(dx, dt)), j = p+1, \dots, n$  be independent adapted Poisson point processes of intensity  $dxdt$ ; the  $N_j$ 's are independent of the Wiener processes. We assume that  $(\mathcal{F}_t)$  is the natural filtration of the processes  $W, N$  and we assume also that  $\mathcal{F}_\infty = \bigvee_{t>0} \mathcal{F}_t = \mathcal{F}$ .

On  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ , we consider the following SDE

$$\begin{aligned} \rho(t) = & \rho_0 + \int_0^t L(\rho(s-))ds \\ & + \sum_{i=0}^p \int_0^t H_i(\rho(s-))dW_i(s) \\ & + \sum_{i=p+1}^n \int_0^t \int_{\mathbb{R}} \left( \frac{J_i(\rho(s-))}{v_i(\rho(s-))} - \rho(s-) \right) \mathbf{1}_{0 < x < v_i(\rho(s-))} [N_i(dx, ds) - dxds], \end{aligned} \quad (\text{E.2})$$

where  $\rho_0 \in \mathcal{S}(\mathcal{H})$ .

**Definition 1.** *The equation (E.2) is called a stochastic master equation and its solution is called a quantum trajectory.*

Equation (E.2) is a "generic"<sup>4</sup> SDE describing the evolution of a system undergoing continuous indirect measurements. Results of existence and uniqueness of the solution of (E.2) can be found in [8, 12, 33–35].

In Eq. (E.2), the operator  $L$  is a usual Lindblad operator [26, 30]. These operators appear in the definition of the master equation in the Markovian approach of Open Quantum Systems.

From Eq. (E.2), one can introduce the measurement record counting processes:

$$\hat{N}_i(t) = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0 < x < v_i(\rho(s-))} N_i(dx, ds), i = p+1, \dots, n.$$

These processes are counting processes with stochastic intensity given by

$$\int_0^t v_i(\rho(s-))ds, i = p+1, \dots, n.$$

In particular, for any  $i \in \{p+1, \dots, n\}$ , the process  $(\hat{N}_i(t) - \int_0^t v_i(\rho(s-))ds)$  is a  $(\mathcal{F}_t)$  martingale under the probability  $\mathbb{P}$ .

During an experiment, these processes would correspond to the counting measurement records an experimenter would obtain. For example  $\hat{N}_i(t)$  could correspond to the total number of photons arrived on a detector up to time  $t$ . In section E.2.3 we discuss in more details what would be the equivalent for a continuous measurement record.

In terms of  $\hat{N}_i(t)$ , Eq. (E.2) can be written as

$$\begin{aligned} d\rho(t) = & L(\rho(t-))dt + \sum_{i=0}^p H_i(\rho(t-))dW_i(t) \\ & + \sum_{i=p+1}^n \left( \frac{J_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (d\hat{N}_i(t) - v_i(\rho(t-))dt). \end{aligned} \quad (\text{E.3})$$

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4. One can generalize these equations by introducing time dependent and random coefficients [12]

In the next section we introduce a nondemolition condition on this evolution and study the large time behaviour of  $(\rho(t))$ .

### E.1.2. Non demolition condition

A measurement process is called nondemolition if one can find a basis  $\mathcal{P}$  of  $\mathcal{H}$  such that any element of  $\mathcal{P}$  is unmodified by the measurement process. If, at a given time, a system is in one of the basis states, it will remain in it at any future time with probability one.

**Definition 2.** *Let  $\mathcal{P}$  be a basis of  $\mathcal{H}$ . A measurement process fulfills a nondemolition condition for  $\mathcal{P}$  if any state of  $\mathcal{P}$  is stable under the measurement process: for any  $|\alpha\rangle \in \mathcal{P}$  if at time  $s$ ,  $\rho(s) = |\alpha\rangle\langle\alpha|$  then for any time  $t > s$ ,  $\rho(t) = \rho(s)$ , almost surely.*

The stable states  $|\alpha\rangle\langle\alpha|, \alpha \in \mathcal{P}$ , are called *pointer states*.

We assume from now on that  $H$  and the  $C_i$ 's are diagonal in the basis  $\mathcal{P}$ . The main result we prove in this section is the equivalence between this diagonal assumption and a nondemolition condition for  $\mathcal{P}$ .

The diagonal assumption can be expressed as follows. There exist  $\epsilon(\alpha) \in \mathbb{R}$ ,  $c(i|\alpha) \in \mathbb{C}$  such that

$$H = \sum_{\alpha \in \mathcal{P}} \epsilon(\alpha) |\alpha\rangle\langle\alpha|$$

$$C_i = \sum_{\alpha \in \mathcal{P}} c(i|\alpha) |\alpha\rangle\langle\alpha|, i = 0, \dots, n.$$

Attached to these decompositions we introduce the following quantities which will be used further

$$r(i|\alpha) = c(i|\alpha) + \overline{c(i|\alpha)}, i = 0, \dots, n,$$

$$\theta(i|\alpha) = |c(i|\alpha)|^2, i = 0, \dots, n.$$

Here  $\bar{z}$  is  $z$  complex conjugate. In the basis  $\mathcal{P}$ , we denote a matrix  $A = (A_{\alpha\beta})_{\alpha,\beta}$ .

Our study of  $(\rho(t))$  is mainly based on the study of the diagonal elements of  $(\rho(t))$  in the basis  $\mathcal{P}$ . If a direct measurement identifying all the pointer states would have been performed at time  $t = 0$ , then the system after this measurement would have been in the pointer state  $|\alpha\rangle\langle\alpha|$  with probability  $\rho_{\alpha\alpha}(0) = \text{Tr}[\rho(0)|\alpha\rangle\langle\alpha|]$ . If the same direct measurement is performed at time  $t > 0$  the probability to obtain the same system pointer state is  $\rho_{\alpha\alpha}(t) = \text{Tr}[\rho(t)|\alpha\rangle\langle\alpha|]$ . So, the evolution of the diagonal elements of  $(\rho(t))$  in  $\mathcal{P}$  gives us information on the distribution of such direct measurement outcomes.

In the sequel, for all  $\alpha \in \mathcal{P}$ , we use the notations

$$q_\alpha(t) = \rho_{\alpha\alpha}(t) = \text{Tr}[\rho(t)|\alpha\rangle\langle\alpha|], t \geq 0. \quad (\text{E.4})$$

As a preliminary, let us prove that the  $(q_\alpha(t))$ ,  $\alpha \in \mathcal{P}$  are  $(\mathcal{F}_t)$  martingales solutions of Dade-Doleans type of SDEs.

**Theorem 1.** On  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ , the stochastic processes  $(q_\alpha(t)), \alpha \in \mathcal{P}$  satisfy the following system of stochastic differential equations

$$dq_\alpha(t) = q_\alpha(t-) \left[ \sum_{i=0}^p (r(i|\alpha) - \langle r_i(t-) \rangle) dW_i(t) + \sum_{i=p+1}^n \left( \frac{\theta(i|\alpha)}{\langle \theta_i(t-) \rangle} - 1 \right) (d\hat{N}_i(t) - \langle \theta_i(t-) \rangle dt) \right], \quad \alpha \in \mathcal{P}, \quad (\text{E.5})$$

where  $\langle r_i(t) \rangle = \sum_{\gamma} r(i|\gamma) q_\gamma(t)$  and  $\langle \theta_i(t) \rangle = \sum_{\gamma} \theta(i|\gamma) q_\gamma(t)$ , for all  $t \geq 0$ .

In particular, the stochastic processes  $(q_\alpha(t)), \alpha \in \mathcal{P}$  are  $(\mathcal{F}_t)$  martingales. As solution of Dade-Doleans type of SDEs, they can be expressed in the following form

$$\begin{aligned} q_\alpha(t) &= q_\alpha(0) \times \exp \left[ \sum_{i=0}^p \left( \int_0^t (r(i|\alpha) - \langle r_i(s-) \rangle) dW_i(s) - \frac{1}{2} \int_0^t (r(i|\alpha) - \langle r_i(s-) \rangle)^2 ds \right) \right] \\ &\quad \times \prod_{i=p+1}^n \prod_{s \leq t} \left( 1 + \left( \frac{\theta(i|\alpha)}{\langle \theta_j(s-) \rangle} - 1 \right) \Delta \hat{N}_i(t) \right) \times \exp \left[ - \int_0^t (\theta(i|\alpha) - \langle \theta_j(s-) \rangle) ds \right] \\ &= q_\alpha(0) \times \exp \left[ \sum_{i=0}^p \left( \int_0^t (r(i|\alpha) - \langle r_i(s-) \rangle) dW_i(s) - \frac{1}{2} \int_0^t (r(i|\alpha) - \langle r_i(s-) \rangle)^2 ds \right) \right. \\ &\quad \left. + \sum_{i=p+1}^n \left( \int_0^t \ln \left( \frac{\theta(i|\alpha)}{\langle \theta_j(s-) \rangle} \right) d\hat{N}_i(s) - \int_0^t (\theta(i|\alpha) - \langle \theta_j(s-) \rangle) ds \right) \right]. \quad (\text{E.6}) \end{aligned}$$

In the case where  $\theta(i|\alpha) = 0$ , for some  $i = p+1, \dots, n$ , if a jump of the corresponding  $\hat{N}_i$  occurs at some time  $t$ , one can see that  $q_\alpha(t)$  vanishes ( $q_\alpha(u) = 0$ , for all  $u \geq t$ ). In this situation in order to give a sense to the second expression one can consider that  $\ln(0) = -\infty$  and  $\exp(-\infty) = 0$ . Nevertheless the second expression will be used only when  $\theta(i|\alpha) > 0$  for all  $\alpha \in \mathcal{P}$  and for all  $i = p+1, \dots, n$ . Let us stress that in this situation, if  $q_\alpha(0) \neq 0$ , we have  $q_\alpha(t) > 0$ , for all  $t \geq 0$ . Although, in section E.2.2, we discuss some interesting properties of  $(q_\alpha(t))$  when  $\theta(i|\alpha) = 0$  for some  $i$ .

*Proof.* In order to obtain the expression (E.5), we have to compute  $d\rho_{\alpha\alpha}(t)$  by using (E.3). To this end we have to plug the diagonal condition into the expression of  $L$ ,  $H_i$ ,  $J_i$  and  $v_i$ . This way we can compute the following expression.

$$\begin{aligned} v_i(\rho) &= \sum_{\alpha \in \mathcal{P}} |c(i|\alpha)|^2 \rho_{\alpha\alpha}, \quad i = p+1, \dots, n \\ (J_i(\rho))_{\alpha\beta} &= \rho_{\alpha\beta} \times c(i|\alpha) \overline{c(i|\beta)}, \quad i = p+1, \dots, n \\ (H_i(\rho))_{\alpha\beta} &= \rho_{\alpha\beta} \times \left( c(i|\alpha) + \overline{c(i|\beta)} - \sum_{\gamma \in \mathcal{P}} (c(i|\gamma) + \overline{c(i|\gamma)}) \rho_{\gamma\gamma} \right), \quad i = 0, \dots, p \\ (L(\rho))_{\alpha\beta} &= \rho_{\alpha\beta} \times \left( -i(\epsilon_\alpha - \epsilon_\beta) + \sum_{i=0}^n c(i|\alpha) \overline{c(i|\beta)} - \frac{1}{2} (|c(i|\alpha)|^2 + |c(i|\beta)|^2) \right). \end{aligned} \quad (\text{E.7})$$

### E.1. Non destructive quantum trajectories

From now on we only need the expression of the diagonal elements  $\alpha = \beta$  in the pointer basis. In other words the stochastic differential equations for  $(q_\alpha(t))$  do not depend on the off diagonal elements of the system state. This way remarking also that  $(L(\rho))_{\alpha\alpha} = 0$  for any  $\alpha \in \mathcal{P}$ , it is easy to derive Equation (E.5).

The second part follows from the fact that the processes  $(W_i(t))$  and  $(\hat{N}_i(t) - \int_0^t v_i(\rho(s-))ds)$  are  $(\mathcal{F}_t)$  martingales. This way the stochastic processes  $(q_\alpha(t))$  are local martingales but since they are bounded they are true martingales. The solution (E.6) is the usual expression of the solution of a Dade-Doleans SDE.  $\square$

We are now equipped to prove the equivalence between the diagonal assumption and the nondemolition condition.

**Theorem 2.** *The quantum stochastic master equation (E.3) fulfills a nondemolition condition for  $\mathcal{P}$  if and only if  $H$  and all the operators  $C_i$  are diagonal in the basis  $\mathcal{P}$ .*

*Proof.* Let us first prove that the diagonal condition imply the nondemolition condition. We need to prove that if at time  $s$ ,  $\rho(s) = |\alpha\rangle\langle\alpha|$ , then at any time  $t > s$ ,  $\rho(t) = |\alpha\rangle\langle\alpha|$  almost surely. Since  $\rho(t)$  is a Markovian process, we can, without loss of generality, set  $s = 0$ .

Put  $\rho(0) = |\alpha\rangle\langle\alpha|$  for some  $\alpha \in \mathcal{P}$ . We have  $q_\beta(0) = 0$  for any  $\beta \neq \alpha$ ,  $\beta \in \mathcal{P}$  and  $\text{Tr}[\rho(t)] = 1$  for any time  $t$ . Then, looking at (E.6), we have  $q_\beta(t) = 0$  and  $q_\alpha(t) = 1$  for all  $t \geq 0$  almost surely. It implies

$$\rho(t) = |\alpha\rangle\langle\alpha|, \quad \forall t \geq 0, \quad \text{a.s.}$$

We now prove that the nondemolition condition implies the diagonal assumption. If at time  $s$ ,  $\rho(s) = |\alpha\rangle\langle\alpha|$ , then for any time  $t > s$ ,  $\rho(t) = |\alpha\rangle\langle\alpha|$ , almost surely. The expectation of  $\rho(t)$  conditioned on  $\rho(s) = |\alpha\rangle\langle\alpha|$  with  $\alpha \in \mathcal{P}$  is:

$$\begin{aligned} \mathbb{E}[\rho(t) | \rho(s) = |\alpha\rangle\langle\alpha|] - \rho(s) &= \mathbb{E} \left[ \int_s^t L(\rho(u-)) du \middle| \rho(s) = |\alpha\rangle\langle\alpha| \right] \\ &+ \mathbb{E} \left[ \sum_{i=0}^p \int_s^t H_i(\rho(u-)) dW_i(u) \middle| \rho(s) = |\alpha\rangle\langle\alpha| \right] \\ &+ \mathbb{E} \left[ \sum_{i=p+1}^n \int_s^t \left( \frac{J_i(\rho(u-))}{v_i(\rho(u-))} - \rho(u-) \right) (d\hat{N}_i(u) - v_i(\rho(u-))du) \middle| \rho(s) = |\alpha\rangle\langle\alpha| \right]. \end{aligned}$$

Since  $(W_i(t))$  and  $(\hat{N}_i(t) - \int_0^t v_i(\rho(u-))du)$  are martingales,

$$\mathbb{E}[\rho(t) | \rho(s) = |\alpha\rangle\langle\alpha|] - \rho(s) = \int_s^t \mathbb{E}[L(\rho(u)) | \rho(s) = |\alpha\rangle\langle\alpha|] du.$$

At this stage, since  $\rho(u) = \rho(s)$  for all  $u \geq s$  almost surely, we get

$$\begin{aligned} 0 &= \mathbb{E}[\rho(t) | \rho(s) = |\alpha\rangle\langle\alpha|] - \rho(s) \\ &= L(\rho(s))(t-s), \quad \forall t \geq s \quad \text{a.s.} \end{aligned}$$

Let  $\beta \in \mathcal{P}, \beta \neq \alpha$ . The condition  $L(|\alpha\rangle\langle\alpha|)_{\beta\beta} = 0$  implies  $(C_i)_{\beta\alpha} = 0$  for all  $i = 0, \dots, n$ . Using this result, the condition  $L(|\alpha\rangle\langle\alpha|)_{\beta\alpha} = 0$  implies  $H_{\beta\alpha} = 0$ . Hence,  $H$  and all the  $C_i$ 's must be diagonal in the basis  $\mathcal{P}$ .  $\square$

In the next section we use the martingale property of  $(q_\alpha(t))$  to study its long time behavior. Before let us prove that this martingale property is equivalent to the nondemolition condition.

**Proposition 1.** *The processes  $(q_\alpha(t))$  are  $(\mathcal{F}_t)$  martingales if and only if (E.3) fulfills a nondemolition condition for  $\mathcal{P}$ .*

*Proof.* We already proved that the nondemolition condition implies the martingale property of  $(q_\alpha(t))$ . Let us prove the converse.

We suppose that for any  $\alpha \in \mathcal{P}$ , the process  $(\rho_{\alpha\alpha}(t))$  is a martingale. This assumption is true only if the drift part of (E.3) for  $\rho_{\alpha\alpha}(t)$  is null whatever is the initial state. Hence, for any arbitrary  $\rho \in \mathcal{S}(\mathcal{H})$ , we must have  $L(\rho)_{\alpha\alpha} = 0$ . Take  $\rho = |\beta\rangle\langle\beta|$  with  $\alpha \neq \beta$ . As seen earlier, the condition  $L(\rho)_{\alpha\alpha} = 0$ , implies that the  $C_i$ 's must be diagonal in the basis  $\mathcal{P}$ . Now put  $|\alpha + \beta\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle)$ . Take  $\rho = |\alpha + \beta\rangle\langle\alpha + \beta|$ . The condition  $L(\rho)_{\alpha\alpha} = 0$  implies  $H_{\alpha\beta} - H_{\beta\alpha} = 0$ . Put  $|\alpha + i\beta\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + i|\beta\rangle)$ . Take  $\rho = |\alpha + i\beta\rangle\langle\alpha + i\beta|$ . The condition  $L(\rho)_{\alpha\alpha} = 0$  implies  $H_{\alpha\beta} + H_{\beta\alpha} = 0$ . We can thus conclude that  $H$  must also be diagonal in the basis  $\mathcal{P}$ .

As proved earlier this diagonal property is equivalent to the nondemolition condition.  $\square$

The next section is devoted to the large time behavior of  $(q_\alpha(t))$  and to interpretations of the obtained convergence in terms of wave function collapse.

## E.2. Convergence and wave function collapse

### E.2.1. Wave function collapse

In this section we show the almost sure convergence of the processes  $(q_\alpha(t))$  when  $t$  goes to infinity. Under some non degeneracy conditions, we can identify the limit random variables  $(q_\alpha(\infty))$ . More precisely, in this context we show that  $q_\alpha(\infty)$  is equal to 1 for a pointer  $\Upsilon \in \mathcal{P}$  and 0 for the others. The pointer state  $\Upsilon$  is a random variable and we find its distribution. We next show that this imply that  $(\rho(t))$  converges almost surely to one of the pointer states  $|\alpha\rangle\langle\alpha|$ . In particular, we show that the probability for the limit pointer state to be  $|\alpha\rangle\langle\alpha|$  is  $q_\alpha(0) = \rho_{\alpha\alpha}(0)$ . This is what is predicted by the von Neumann projection postulate if a direct measurement would have been performed at time 0. Thinking of the limit state in terms of random variable, in the limit  $t \rightarrow \infty$ , the system state is a random variable with the same law as the one predicted by von Neumann projection postulate for a direct measurement at time 0.

In the following subsections we present some useful properties implied by this convergence.

Let us express our non degeneracy condition

**Assumption (ND):** For any  $(\alpha, \beta)$  with  $\alpha \neq \beta$  there exists  $i \in \{0, 1, \dots, n\}$  such that

- either  $r(i|\alpha) \neq r(i|\beta)$  if  $i \leq p$
- or  $\theta(i|\alpha) \neq \theta(i|\beta)$  if  $i > p$ .

**Theorem 3.** *Under Assumption (ND), there exist random variables  $q_\alpha(\infty), \alpha \in \mathcal{P}$  which take values in  $\{0, 1\}$  such that*

$$\lim_{t \rightarrow \infty} q_\alpha(t) = q_\alpha(\infty), \forall \alpha \in \mathcal{P}, \quad (\text{E.8})$$

*almost surely and in  $L^1$  norm. Moreover, we have  $\mathbb{E}[q_\alpha(\infty)|\mathcal{F}_t] = q_\alpha(t)$ , for all  $\alpha \in \mathcal{P}$  and for all  $t \geq 0$ .*

## E.2. Convergence and wave function collapse

The random variables  $q_\alpha(\infty), \alpha \in \mathcal{P}$ , satisfy

$$\mathbb{P}(q_\alpha(\infty) = 1) = q_0(\alpha), \forall \alpha \in \mathcal{P}, \quad (\text{E.9})$$

$$q_\alpha(\infty)q_\beta(\infty) = 0, \forall \alpha \neq \beta, \text{ a.s.} \quad (\text{E.10})$$

As a consequence there exists a random variable  $\Upsilon$  with values in  $\mathcal{P}$  such that

$$\mathbb{P}(\Upsilon = \alpha) = q_\alpha(0), \forall \alpha \in \mathcal{P}$$

and such that

$$\lim_{t \rightarrow \infty} \rho(t) = |\Upsilon\rangle\langle\Upsilon|, \text{ a.s.}$$

*Proof.* Let  $\alpha \in \mathcal{P}$  be fixed. The almost sure convergence of  $(q_\alpha(t))$  follows from the fact that  $(q_\alpha(t))$  are bounded  $(\mathcal{F}_t)$  martingales. More precisely, the family  $(q_\alpha(t))$  is uniformly integrable. Therefore there exists  $q_\alpha(\infty)$  such that  $\lim_{t \rightarrow \infty} q_\alpha(t) = q_\alpha(\infty)$ , almost surely and in  $L^1$  norm and we have  $\mathbb{E}[q_\alpha(\infty)|\mathcal{F}_t] = q_\alpha(t)$ , for all  $t \geq 0$ . It remains to show that these random variables take values in  $\{0, 1\}$ . To this end, for  $\alpha$  being fixed, using Itô-Lévy isometry, we have for all  $t > 0$

$$\mathbb{E}[q_\alpha(t)^2] = \mathbb{E}[[q_\alpha(t), q_\alpha(t)]],$$

where  $[q_\alpha(t), q_\alpha(t)]$  corresponds to the stochastic bracket of  $(q_\alpha(t))$ . We then have

$$\begin{aligned} \mathbb{E}[q_\alpha(t)^2] &= \sum_{i=0}^p \int_0^t \mathbb{E} [q_\alpha(s)^2 (r(i|\alpha) - \langle r_i(s) \rangle)^2] ds \\ &\quad + \sum_{i=p+1}^n \int_0^t \mathbb{E} \left[ q_\alpha(s)^2 \left( \frac{\theta(i|\alpha)}{\langle \theta_i(s) \rangle} - 1 \right)^2 \langle \theta_i(s) \rangle \right] ds \end{aligned} \quad (\text{E.11})$$

Since the processes  $(q_\alpha(t))$  converge almost surely and are bounded, using Lebesgue dominated convergence Theorem, we have that the quantity  $\mathbb{E}[q_\alpha(t)^2]$  converges when  $t$  goes to infinity. This implies that

$$\begin{aligned} \int_0^\infty \mathbb{E} [q_\alpha(s)^2 (r(i|\alpha) - \langle r_i(s) \rangle)^2] ds &< \infty, \quad i = 0, \dots, p, \\ \int_0^\infty \mathbb{E} \left[ q_\alpha(s)^2 \left( \frac{\theta(i|\alpha)}{\langle \theta_i(s) \rangle} - 1 \right)^2 \langle \theta_i(s) \rangle \right] ds &< \infty, \quad i = p+1, \dots, n. \end{aligned} \quad (\text{E.12})$$

Again by the dominated convergence Theorem, the quantities

$$\begin{aligned} &\mathbb{E} [q_\alpha(t)^2 (r(i|\alpha) - \langle r_i(t) \rangle)^2], \quad i = 0, \dots, p, \\ &\mathbb{E} \left[ \left( \frac{\theta(i|\alpha)q_\alpha(t)}{\langle \theta_i(t) \rangle} - q_\alpha(t) \right)^2 \langle \theta_i(t) \rangle \right], \quad i = p+1, \dots, n \end{aligned} \quad (\text{E.13})$$

converge when  $t$  goes to infinity. Then from (E.12) it follows that necessarily

$$\lim_{t \rightarrow \infty} \mathbb{E}[q_\alpha(t)^2 (r(i|\alpha) - \langle r_i(t) \rangle)^2] = 0, \quad i = 0, \dots, p \quad (\text{E.14})$$

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[ q_\alpha(t)^2 \left( \frac{\theta(i|\alpha)}{\langle \theta_i(t) \rangle} - 1 \right)^2 \langle \theta_i(t) \rangle \right] = 0, \quad i = p+1, \dots, n. \quad (\text{E.15})$$

The above  $L^2$  convergences imply the almost sure convergences up to an extraction. More precisely, there exist subsequences  $(t_n^i)$  such that for all  $i = 0, \dots, n$ ,  $\lim_{n \rightarrow \infty} t_n^i = \infty$  and such that almost surely

$$\lim_{n \rightarrow \infty} q_\alpha(t_n^i)^2 (r(i|\alpha) - \langle r_i(t_n^i) \rangle)^2 = 0, \quad i = 0, \dots, p \quad (\text{E.16})$$

$$\lim_{n \rightarrow \infty} q_\alpha(t_n^i)^2 \left( \frac{\theta(i|\alpha)}{\langle \theta_i(t_n^i) \rangle} - 1 \right)^2 \langle \theta_i(t_n^i) \rangle = 0, \quad i = p+1, \dots, n. \quad (\text{E.17})$$

Since the processes  $(q_\alpha(t))$  converge almost surely, by uniqueness of the almost sure limits and using the boundedness of  $\langle \theta_i(t) \rangle$ , we can conclude that almost surely, for all  $\alpha \in \mathcal{P}$

$$\lim_{t \rightarrow \infty} q_\alpha(t)^2 (r(i|\alpha) - \langle r_i(t) \rangle)^2 = 0, \quad i = 0, \dots, p \quad (\text{E.18})$$

$$\lim_{t \rightarrow \infty} q_\alpha(t)^2 (\theta(i|\alpha) - \langle \theta_i(t) \rangle)^2 = 0, \quad i = p+1, \dots, n. \quad (\text{E.19})$$

Then it follows that, almost surely, for all  $\alpha \in \mathcal{P}$

$$q_\alpha(\infty) (r(i|\alpha) - \langle r_i(\infty) \rangle) = 0, \quad i = 0, \dots, p \quad (\text{E.20})$$

$$q_\alpha(\infty) (\theta(i|\alpha) - \langle \theta_i(\infty) \rangle) = 0, \quad i = p+1, \dots, n. \quad (\text{E.21})$$

This way, almost surely, for all  $\alpha \neq \beta$

$$q_\alpha(\infty) (r(i|\alpha) - \langle r_i(\infty) \rangle) = q_\beta(\infty) (r(i|\beta) - \langle r_i(\infty) \rangle) = 0, \quad i = 0, \dots, p \quad (\text{E.22})$$

$$q_\alpha(\infty) (\theta(i|\alpha) - \langle \theta_i(\infty) \rangle) = q_\beta(\infty) (\theta(i|\beta) - \langle \theta_i(\infty) \rangle) = 0, \quad i = p+1, \dots, n. \quad (\text{E.23})$$

It follows that, almost surely, for all  $\alpha \neq \beta$

$$q_\alpha(\infty) q_\beta(\infty) (r(i|\alpha) - r(i|\beta)) = 0, \quad i = 0, \dots, p \quad (\text{E.24})$$

$$q_\alpha(\infty) q_\beta(\infty) (\theta(i|\alpha) - \theta(i|\beta)) = 0, \quad i = p+1, \dots, n. \quad (\text{E.25})$$

Finally using Assumption (ND) one can conclude that

$$q_\alpha(\infty) q_\beta(\infty) = 0, \quad \forall \alpha \neq \beta \text{ a.s.}$$

This way, there exists a set  $\Omega'$  such that  $\mathbb{P}(\Omega') = 1$  and such that for all  $\omega \in \Omega'$  there exists a unique  $\Upsilon \in \mathcal{P}$  such that  $q_\Upsilon(\infty)(\omega) \neq 0$  and for all  $\beta \neq \Upsilon$ ,  $q_\beta(\infty)(\omega) = 0$ . Moreover, since  $\text{Tr}[\rho(t)] = 1$ , we have that  $\sum_{\alpha \in \mathcal{P}} q_\alpha(\infty)(\omega) = 1$ . Therefore  $q_\Upsilon(\infty)(\omega) = 1$ . Then we have proved that for all  $\alpha$ ,  $q_\alpha(\infty) \in \{0, 1\}$ . Using now the martingale property, we have  $\mathbb{E}[q_\alpha(\infty)] = q_\alpha(0)$ , which implies that  $\mathbb{P}(q_\alpha(\infty) = 1) = q_\alpha(0)$  and the first part of the theorem is proved.

For the second part, let us come back to the definition of  $\Upsilon$ . This defines a random variable taking values in the set of pointer states  $\mathcal{P}$  (for  $\omega \in \Omega \setminus \Omega'$ , we can put  $\Upsilon(\omega) = \psi$ , where  $\psi \notin \mathcal{P}$ , this will appear with probability 0). It is then clear that

$$\mathbb{P}(\Upsilon = \alpha) = \mathbb{P}(q_\alpha(\infty) = 1) = q_\alpha(0), \quad \forall \alpha \in \mathcal{P}.$$

One can note in particular that  $q_\alpha(\infty) = \mathbb{1}_{\Upsilon=\alpha}$ . Now, we are in the position to conclude the proof. Indeed, since  $q_\Upsilon(\infty) = 1$ , we get

$$\lim_{t \rightarrow \infty} \text{Tr}[\rho(t)|\Upsilon\rangle\langle\Upsilon|] = 1, \quad \text{a.s.}$$

## E.2. Convergence and wave function collapse

Now let  $\omega \in \Omega'$  and let  $\mu$  be a limit of a convergent subsequence  $(\rho(t_n)(\omega))$ , we have that  $\text{Tr}[\mu|\Upsilon(\omega)\rangle\langle\Upsilon(\omega)|] = 1$  which implies that  $\mu = |\Upsilon(\omega)\rangle\langle\Upsilon(\omega)|$ . Therefore all convergent subsequences of  $(\rho(t)(\omega))$  converge to  $|\Upsilon(\omega)\rangle\langle\Upsilon(\omega)|$  which implies

$$\lim_{t \rightarrow \infty} \rho(t) = |\Upsilon\rangle\langle\Upsilon|, \text{ a.s.}$$

□

This result is crucial in the following. In particular it will allow us to use a Girsanov transformation "in infinite time horizon".

### E.2.2. Exponential rate of convergence

In this section, we study the convergence speed of the processes  $(q_\alpha(t))$ . In particular, we shall show an exponential convergence. To this end, we study the following quantities

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\gamma(t)} \right), \quad \alpha, \gamma \in \mathcal{P}.$$

Since  $(q_t(\alpha))$  can vanish in the case where  $\theta(i|\alpha) = 0$  for some  $i$ , this quantity can be finite or infinite. Furthermore, we limit our study to pointer states  $\alpha, \gamma \in \mathcal{P}$  such that  $q_\alpha(0) \neq 0$  and  $q_\gamma(0) \neq 0$ . Remark, if  $q_\beta(0) = 0$  for some pointer state  $\beta$  we have  $q_\beta(t) = 0$  for any time  $t$ .

First, let us start by studying the case where  $q_\alpha(t) > 0$  and  $q_\gamma(t) > 0$ , for all  $t \geq 0$ , almost surely. As already discussed this is ensured when for any  $i \in \{p+1, \dots, n\}$ ,  $\theta(i|\alpha) > 0$  and  $\theta(i|\gamma) > 0$ . In this case, using (E.5), we have almost surely, for all  $t \geq 0$ ,

$$\begin{aligned} \frac{q_\alpha(t)}{q_\gamma(t)} &= \frac{q_\alpha(0)}{q_\gamma(0)} \times \exp \left[ \sum_{i=0}^p \left( \int_0^t (r(i|\alpha) - r(i|\gamma)) dW_i(s) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \int_0^t \left[ (r(i|\alpha) - \langle r_i(s-) \rangle)^2 - (r(i|\gamma) - \langle r_i(s-) \rangle)^2 \right] ds \right) \right. \\ &\quad \left. + \sum_{i=p+1}^n \left( \int_0^t \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) d\hat{N}_i(s) - \int_0^t (\theta(i|\alpha) - \theta(i|\gamma)) ds \right) \right]. \quad (\text{E.26}) \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \frac{q_\alpha(t)}{q_\gamma(t)} &= \frac{q_\alpha(0)}{q_\gamma(0)} \times \exp \left[ \sum_{i=0}^p \left( \int_0^t (r(i|\alpha) - r(i|\gamma)) dX_i^\gamma(s) - \frac{1}{2} \int_0^t (r(i|\alpha) - r(i|\gamma))^2 ds \right) \right. \\ &\quad \left. + \sum_{i=p+1}^n \left( \int_0^t \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) d\hat{M}_i^\gamma(s) - \int_0^t (\theta(i|\alpha) - \theta(i|\gamma)) - \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \theta(i|\gamma) ds \right) \right], \quad (\text{E.27}) \end{aligned}$$

for all  $t \geq 0$ , where

$$\begin{aligned} X_i^\gamma(t) &= W_j(t) - \int_0^t \left[ r(i|\gamma) - \langle r_i(s-) \rangle \right] ds, \quad i = 0, \dots, p \\ \hat{M}_i^\gamma(t) &= \hat{N}_i(t) - \int_0^t \theta(i|\gamma) ds = \hat{N}_i(t) - \theta(i|\gamma)t, \quad i = p+1, \dots, n \end{aligned} \quad (\text{E.28})$$

for all  $t \geq 0$ . Now we shall see that these processes are martingales under a suitable change of measure. To this end we consider the family of probability measures  $(\mathbb{Q}_\gamma^t)$  defined by

$$d\mathbb{Q}_\gamma^t(\omega) = \frac{q_\gamma(t)(\omega)}{q_\gamma(0)} d\mathbb{P}(\omega), \quad t \geq 0,$$

for all  $\gamma \in \mathcal{P}$  such that  $q_\gamma(0) \neq 0$ .

This family of probability measure is consistent. Moreover, since  $\mathbb{E}[q_\gamma(\infty)|\mathcal{F}_t] = q_\gamma(t)$ , for all  $t \geq 0$ , any element of this family can be extended to a unique probability measure  $\mathbb{Q}_\gamma$  on  $\mathcal{F}_\infty = \mathcal{F}$ . In particular, we have

$$d\mathbb{Q}_\gamma(\omega) = \frac{q_\gamma(\infty)(\omega)}{q_\gamma(0)} d\mathbb{P}(\omega) \quad (\text{E.29})$$

and in terms of filtration we get the following Radon Nykodim formula

$$\mathbb{E} \left[ \frac{d\mathbb{Q}_\gamma(\omega)}{d\mathbb{P}(\omega)} \middle| \mathcal{F}_t \right] = \frac{q_\gamma(t)}{q_\gamma(0)} = \frac{d\mathbb{Q}_\gamma^t(\omega)}{d\mathbb{P}(\omega)}, \quad t \geq 0. \quad (\text{E.30})$$

This way we can consider the quantity  $\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\gamma(t)} \right)$  under  $\mathbb{Q}_\gamma$ . We shall need the following lemma which relies on Girsanov transformation.

**Lemma 1.** *Let  $\gamma \in \mathcal{P}$  such that  $q_\gamma(0) \neq 0$ . Under  $\mathbb{Q}_\gamma$ , the processes  $(X_j^\gamma(t)), j = 0, \dots, p$  and  $(\hat{M}_j^\gamma(t)), j = p+1, \dots, n$  are  $(\mathcal{F}_t)$  martingales. More precisely  $(X_j^\gamma(t)), j = 0, \dots, p$  are standard  $\mathbb{Q}_\gamma$  Brownian motions and  $(\hat{N}_j^\gamma(t)), j = p+1, \dots, n$  are usual Poisson processes with deterministic intensities  $\theta(i|\gamma)$ .*

The following theorem expresses the exponential convergence speed towards  $\Upsilon$ .

**Theorem 4.** *Assume Assumption (ND) is satisfied. Assume that  $\alpha, \gamma \in \mathcal{P}$  are such that  $q_\alpha(0) \neq 0$ ,  $q_\gamma(0) \neq 0$ , and such that  $\theta(i|\alpha) > 0$ ,  $\theta(i|\gamma) > 0$  for all  $i = p+1, \dots, n$ . Then, we have*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\gamma(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\gamma))^2 + \sum_{i=p+1}^n \theta(i|\gamma) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\gamma)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \right], \quad (\text{E.31})$$

$\mathbb{Q}_\gamma$  almost surely.

More generally, in terms of the random variable  $\Upsilon$ , we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\Upsilon(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right], \quad (\text{E.32})$$

$\mathbb{P}$  almost surely.

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*Proof.* From (E.26), we have

$$\begin{aligned} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\gamma(t)} \right) &= \frac{1}{t} \ln \left( \frac{q_\alpha(0)}{q_\gamma(0)} \right) \\ &+ \sum_{i=0}^p (r(i|\alpha) - r(i|\gamma)) \frac{X_j^\gamma(t)}{t} - \frac{1}{2} (r(i|\alpha) - r(i|\gamma))^2 \\ &+ \sum_{i=p+1}^n \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \frac{\hat{M}_i^\gamma(t)}{t} + (\theta(i|\gamma) - \theta(i|\alpha)) + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \theta(i|\gamma). \end{aligned} \quad (\text{E.33})$$

Now from Lemma 1, we know that under  $\mathbb{Q}_\gamma$ , the processes  $(X_j^\gamma(t))$  are standard Brownian motions and  $(\hat{N}_i(t))$  are a usual Poisson processes of intensities  $\theta(i|\gamma)$ . This way, using law of large number for Brownian motion and Poisson processes, we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\gamma(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\gamma))^2 + \sum_{i=p+1}^n \theta(i|\gamma) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\gamma)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \right], \quad (\text{E.34})$$

$\mathbb{Q}_\gamma$  almost surely.

Now let us remark that

$$d\mathbb{P}(\omega) = \sum_{\gamma \text{ s.t. } q_\gamma(0) > 0} q_\gamma(0) d\mathbb{P}(\omega | \Upsilon(\omega) = \gamma)$$

which yields

$$d\mathbb{Q}_\gamma(\omega) = d\mathbb{P}(\omega | \Upsilon(\omega) = \gamma).$$

As a consequence

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{q_\alpha(t)}{q_\Upsilon(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right], \quad (\text{E.35})$$

$\mathbb{P}$  almost surely and the result follows.  $\square$

From the above Theorem since  $q_\Upsilon(\infty) = 1$ ,  $\mathbb{P}$  almost surely, we get the following asymptotic expansion. For all  $\alpha \in \mathcal{P}$  such that  $q_\alpha(0) \neq 0$

$$q_\alpha(t) = e^{-t \left[ \frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 - \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right] \right]} \times (1 + o(1)), \quad (\text{E.36})$$

$\mathbb{P}$  almost surely. From the inequality  $\ln(x) \leq x - 1$ , we see that the rate

$$\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 - \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right] \geq 0, \text{ a.s.} \quad (\text{E.37})$$

More precisely each term of the sums is nonnegative. Now from Assumption (ND), the quantity (E.37) is equal to zero if and only if  $\alpha = \Upsilon$ . This underlines the exponential rate convergence towards  $\Upsilon$ .

**Remark.** Let us assume the interaction of our measurement apparatus with the system involves only one hermitian operator  $C_i = C$ . In other words  $n = 0$ . Let us also assume we can choose either to have a continuous process as our measurement record or a counting process. In other words, either the quantum stochastic master equation is

$$d\rho(t) = L(\rho(t))dt + \left( C\rho(t) + \rho(t)C^* - \text{Tr}[(C + C^*)\rho(t)]\rho(t) \right) dW_t \quad (\text{E.38})$$

or

$$d\rho(t) = L(\rho(t-))dt + \left( \frac{C\rho(t-)C^*}{\text{Tr}[C^*C\rho(t-)]} - \rho(t-) \right) [d\hat{N}(t) - \text{Tr}[C^*C\rho(t-)]dt]. \quad (\text{E.39})$$

In the diffusive case (E.38), the convergence rate (E.37) is equal to  $(c(\alpha) - c(\Upsilon))^2$ . In the counting case (E.39) it is equal to  $-c(\Upsilon)^2[\ln(c(\alpha)^2/c(\Upsilon)^2) + 1 - c(\alpha)^2/c(\Upsilon)^2]$ . A simple study shows that

$$(c(\alpha) - c(\Upsilon))^2 \leq -c(\Upsilon)^2[\ln(c(\alpha)^2/c(\Upsilon)^2) + 1 - c(\alpha)^2/c(\Upsilon)^2].$$

So the choice of a counting process may lead to a higher convergence rate. But it comes at a price. Suppose  $C$  has two different eigenvalues of equal norm:  $c(\alpha) \neq c(\beta)$ ,  $|c(\alpha)| = |c(\beta)|$ . The non degeneracy assumption (ND) is not fulfilled for the jump equation (E.39) whereas it is fulfilled for the diffusive equation (E.38).

We now study the situation where it exists  $i \in \{p+1, \dots, n\}$  and  $\alpha \in \mathcal{P}$ , such that  $\theta(i|\alpha) = 0$ . In this case we shall study the following stopping times

$$T_i = \min\{t \in \mathbb{R}_+ \text{ s.t. } \hat{N}_i(t) > 0\}, i = p+1, \dots, n \quad (\text{E.40})$$

$$T(\alpha) = \min\{T_i \text{ s.t. } \theta(i|\alpha) = 0\}, \alpha \in \mathcal{P}. \quad (\text{E.41})$$

Assume that  $q_\alpha(0) \neq 0$ , we have  $q_\alpha(T(\alpha)) = 0$  and  $q_\alpha(t) > 0$ , for all  $t < T(\alpha)$  as well as  $q_\alpha(t) = 0$ , for all  $t \geq T(\alpha)$ . This way, if  $T(\alpha) < \infty$ , the process  $(q_\alpha(t))$  converges to zero in finite time.

We have the following proposition which describes the distribution of  $T(\alpha), \alpha \in \mathcal{P}$ .

**Proposition 2.** Let  $\alpha \in \mathcal{P}$  such that there exists  $i \in \{p+1, \dots, n\}$  such that  $\theta(i|\alpha) = 0$ . Then,

$$\mathbb{P}(T(\alpha) \leq t | \Upsilon = \gamma) = 1 - e^{-\lambda(\alpha|\gamma)t},$$

where  $\lambda(\alpha|\gamma) = \sum_{i, \text{ s.t. } \theta(i|\alpha)=0} \theta(i|\gamma)$ .

Finally, we have

$$\mathbb{P}(T(\alpha) \leq t) = 1 - \sum_{\beta \in \mathcal{P}} q_\beta(0) e^{-\lambda(\alpha|\beta)t}.$$

*Proof.* Remember that, under  $\mathbb{Q}_\gamma$ ,  $(\hat{N}_i(t))$  are usual Poisson processes of intensities  $\theta(i|\gamma)$ . So, under  $\mathbb{Q}_\gamma$ , the law of  $T_i$  is exponential with parameter equal to  $\theta(i|\gamma)$ , that is,

$$\mathbb{Q}_\gamma(T_i \leq t) = 1 - e^{-\theta(i|\gamma)t}, i = p+1, \dots, n.$$

Under  $\mathbb{Q}_\gamma$ ,  $T(\alpha)$  is thus the minimum of a finite set of random variables obeying exponential laws. Since we have assumed that the Poisson point processes  $N_i(.,.)$  are independent, under

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$\mathbb{Q}_\gamma$  the processes  $(\hat{N}_i(t))$  are also independent as well as the stopping times  $T_i$ . From the properties of exponential law, we have

$$\mathbb{Q}_\gamma(T(\alpha) \leq t) = 1 - e^{-\lambda(\alpha|\gamma)t}.$$

Since  $\mathbb{P} = \sum_{\beta \in \mathcal{P}} q_\beta(0) \mathbb{Q}_\beta$ , we have

$$\mathbb{P}(T(\alpha) \leq t) = 1 - \sum_{\beta \in \mathcal{P}} q_\beta(0) e^{-\lambda(\alpha|\beta)t},$$

which concludes the proof. □

Let us note that taking  $t$  goes to infinity we get

$$\mathbb{P}(T(\alpha) = \infty) = \sum_{\beta, \text{ s.t. } \lambda(\alpha, \beta)=0} q_\beta(0) \geq q_\alpha(0) = \mathbb{P}(q_\alpha(\infty) = 1).$$

In the next section, we address the problem of convergence when one does not have access to the process  $(\rho(t))$  but only to the measurement records.

### E.2.3. Stability

Usually, in experiments the initial state  $\rho_0$  of the system  $\mathcal{H}$  is unknown (this is sometimes that we want to estimate). This way we cannot have access to the quantum trajectory  $(\rho(t))$ . Nevertheless we have still access to the results given by the measurement apparatus. These results are directly connected to the quantum trajectory (E.2). In terms of processes, the results of the measurement are described by output processes in the following way. The observed processes are given by

$$dy_i(t) = dW_i(t) + \text{Tr}[(C_i + C_i^*)\rho(t-)]dt, i = 0, \dots, p$$

for the diffusive part of the evolution and by

$$d\hat{N}_i(t), i = p+1, \dots, n$$

for the counting processes. The quantities  $y_i(t)$  and  $\hat{N}_i(t)$  are the measurements recorded by the apparatus. In an homodyne or heterodyne detection scheme,  $y_i(t)$  would represent the detected photo current integrated up to time  $t$  whereas, in a direct photodetection scheme,  $\hat{N}_i(t)$  would be the number of photons detected up to time  $t$  [19,39]. The quantum trajectory can be expressed as follows

$$\begin{aligned} \rho(t) &= L(\rho(t-))dt + \sum_{i=0}^p H_i(\rho(t-))(dy_i(t) - \text{Tr}[(C_i + C_i^*)\rho(t-)]dt) \\ &\quad + \sum_{i=p+1}^n \left( \frac{J_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (d\hat{N}_i(t) - v_i(\rho(t-))dt), \end{aligned} \tag{E.42}$$

$$\rho(0) = \rho_0. \tag{E.43}$$

In the case where we do not know the initial state  $\rho_0$ , we can estimate the quantum trajectory  $(\rho(t))$  by using an estimate quantum trajectory  $(\tilde{\rho}(t))$  satisfying the following stochastic differential equation

$$\begin{aligned} \tilde{\rho}(t) = & L(\tilde{\rho}(t-))dt + \sum_{i=0}^p H_i(\tilde{\rho}(t-))(dy_i(t) - \text{Tr}[(C_i + C_i^*)\tilde{\rho}(t-)]dt) \\ & + \sum_{i=p+1}^n \left( \frac{J_i(\tilde{\rho}(t-))}{v_i(\tilde{\rho}(t-))} - \tilde{\rho}(t-) \right) (d\hat{N}_i(t) - v_i(\tilde{\rho}(t-))dt), \end{aligned} \quad (\text{E.44})$$

$$\tilde{\rho}(0) = \tilde{\rho}_0, \quad (\text{E.45})$$

where  $\tilde{\rho}_0$  is an arbitrary state. Let us stress that  $(y_i(t)), i = 0, \dots, p$  and  $(\hat{N}_i(t)), i = p+1, \dots, n$  are the output processes attached to the true quantum trajectory  $(\rho(t))$ . This way, if  $\tilde{\rho}_0 = \rho_0$  we get  $(\tilde{\rho}(t)) = (\rho(t))$ . In particular the estimate quantum trajectory  $(\tilde{\rho}(t))$  is governed by the measurement record as if it was the true quantum trajectory. This can allow to simulate an estimation of the true quantum trajectory. Such an estimate is often called a *quantum filter*. An important question is to know if the estimate become closer and closer to the true quantum trajectory when  $t$  goes to infinity. In particular does the distance between the estimate and the true quantum trajectory converges to zero? Such a property is related to the so called *stability of quantum filter*. For general schemes of indirect measurement, partial results in this direction has been developed in [2] (it is shown that the fidelity between  $(\rho(t))$  and  $(\tilde{\rho}(t))$  increases at least in average when  $t$  increases).

Here, in the context of QND we show that the estimate quantum trajectory converges to the same state than the one of the true quantum trajectory. We shall show namely that  $(\tilde{\rho}(t))$  converges to  $|\Upsilon\rangle\langle\Upsilon|$  when  $t$  goes to infinity. This is achieved by a direct analysis of the quantities

$$\tilde{q}_\alpha(t) = \text{Tr}[\tilde{\rho}(t)|\alpha\rangle\langle\alpha|],$$

for all  $\alpha \in \mathcal{P}$  and for all  $t \geq 0$ . The processes  $(\tilde{q}_\alpha(t))$  represent actually the estimate of the true  $(q_\alpha(t))$ .

From Equation (E.44), we can see that the processes  $(\tilde{q}_\alpha(t))$  are solutions of

$$\begin{aligned} d\tilde{q}_\alpha(t) = \tilde{q}_\alpha(t-) \left[ \sum_{i=0}^p \left( r(i|\alpha) - \langle \tilde{r}_i(t-) \rangle \right) \left( dW_i(t) + (\langle r_i(t-) \rangle - \langle \tilde{r}_i(t-) \rangle) \right) \right. \\ \left. + \sum_{i=p+1}^n \left( \frac{\theta(i|\alpha)}{\langle \tilde{\theta}_i(t-) \rangle} - 1 \right) \left( d\hat{N}_i(t) - \langle \tilde{\theta}_i(t-) \rangle dt \right) \right]. \end{aligned} \quad (\text{E.46})$$

Let us treat the case where  $\theta(i|\alpha) \neq 0$ , for all  $\alpha \in \mathcal{P}$  and for all  $i = p+1, \dots, n$ . Eq. (E.46) are still Dade-Doleans exponential and the solution of (E.46) are given by

$$\begin{aligned} \tilde{q}_t(\alpha) = \tilde{q}_0(\alpha) \times \exp & \left[ \sum_{i=0}^p \left( \int_0^t (r(i|\alpha) - \langle \tilde{r}_i(s-) \rangle) (dW_i(s) + (\langle r_i(s-) \rangle - \langle \tilde{r}_i(s-) \rangle) ds) \right. \right. \\ & \left. \left. - \frac{1}{2} \int_0^t (r(i|\alpha) - \langle \tilde{r}_i(s-) \rangle)^2 ds \right) \right. \\ & \left. + \sum_{i=p+1}^n \left( \int_0^t \ln \left( \frac{\theta(i|\alpha)}{\langle \tilde{\theta}_i(s-) \rangle} \right) d\tilde{N}_i(s) - \int_0^t (\theta(i|\alpha) - \langle \tilde{\theta}_i(s-) \rangle) ds \right) \right], \quad (\text{E.47}) \end{aligned}$$

for all  $\alpha \in \mathcal{P}$  and for all  $t \geq 0$ .

Let us stress that, in general, these processes are no more  $(\mathcal{F}_t)$  martingales. We cannot then conclude to their convergence by using martingale convergence results. Nevertheless the quantity  $\tilde{q}_t(\alpha)/\tilde{q}_t(\gamma)$  takes exactly the same form than the one of the true quantum trajectory. More precisely, assume that  $\tilde{q}_\gamma(0) \neq 0$ , after computations we get that

$$\begin{aligned} \frac{\tilde{q}_\alpha(t)}{\tilde{q}_\gamma(t)} = \frac{\tilde{q}_\alpha(0)}{\tilde{q}_\gamma(0)} \times \exp & \left[ \sum_{i=0}^p \left( \int_0^t (r(i|\alpha) - r(i|\gamma)) dX_i^\gamma(s) - \frac{1}{2} \int_0^t (r(i|\alpha) - r(i|\gamma))^2 ds \right) \right. \\ & \left. + \sum_{i=p+1}^n \left( \int_0^t \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) d\hat{M}_i^\gamma(s) - \int_0^t (\theta(i|\alpha) - \theta(i|\gamma)) - \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \theta(i|\gamma) ds \right) \right], \quad (\text{E.48}) \end{aligned}$$

for all  $\alpha, \gamma \in \mathcal{P}$ , where the processes  $(X_i^\gamma(t)), i = 0, \dots, p$  and  $(M_i^\gamma(t)), i = p+1, \dots, n$  have been defined in (E.28). Now we would like to consider the limit of this quantity under  $\mathbb{Q}_\gamma$ . To this end, we first need to consider a  $\gamma$  such that  $q_\gamma(0) \neq 0$ . In this case, if  $\tilde{q}_\alpha(0) \neq 0$ , we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{\tilde{q}_\alpha(t)}{\tilde{q}_\gamma(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\gamma))^2 + \sum_{i=p+1}^n \theta(i|\gamma) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\gamma)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\gamma)} \right) \right], \quad (\text{E.49})$$

$\mathbb{Q}_\gamma$  almost surely. As a consequence, we get again

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{\tilde{q}_\alpha(t)}{\tilde{q}_\Upsilon(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right], \quad (\text{E.50})$$

$\mathbb{P}$  almost surely. This yields that  $\mathbb{P}$  almost surely

$$\tilde{q}_\alpha(t) = \tilde{q}_\Upsilon(t) e^{-t \left[ \frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 - \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right] + o(1) \right]}. \quad (\text{E.51})$$

Now we consider  $\Omega''$  with  $\mathbb{P}(\Omega'') = 1$  and such that (E.51) is fulfilled for all  $\omega \in \Omega''$ . Let  $\omega \in \Omega''$  be fixed. For any  $\alpha$  such that  $\alpha \neq \Upsilon(\omega)$ , under Assumption (ND) there exists  $i$  such that either  $r(i|\alpha) \neq r(i|\Upsilon(\omega))$  or  $\theta(i|\alpha) \neq \theta(i|\Upsilon(\omega))$ . Hence, since  $0 \leq \tilde{q}_\Upsilon(t) \leq 1$ , we can conclude that

$$\lim_{t \rightarrow \infty} \tilde{q}_\alpha(t)(\omega) = 0.$$

Now, since  $\sum \tilde{q}_\alpha(t) = 1$  still holds, we deduce that

$$\lim_{t \rightarrow \infty} \tilde{q}_{\Upsilon(\omega)}(t)(\omega) = 1.$$

Note that the result (E.50) requires only that  $\tilde{q}_\alpha(0) \neq 0$  as soon as  $q_\alpha(0) \neq 0$ . Then, we can express the following result.

**Proposition 3.** *Assume that  $\theta(i|\alpha) > 0$ , for all  $\alpha \in \mathcal{P}$  and for all  $i = p+1, \dots, n$ . Assume Assumption (ND) is satisfied. Assume that  $\tilde{q}_\alpha(0) \neq 0$  for any  $\alpha \in \mathcal{P}$  such that  $q_\alpha(0) \neq 0$ .*

*Let  $(\tilde{q}_\alpha(t))$  be the stochastic processes defined by (E.47) with  $\tilde{q}_\alpha(0) \neq 0$ . We have*

$$\begin{aligned} \lim_{t \rightarrow \infty} \tilde{q}_{\Upsilon}(t) &= 1, \\ \lim_{t \rightarrow \infty} \tilde{q}_\alpha(t) \mathbf{1}_{\Upsilon \neq \alpha} &= 0, \end{aligned} \tag{E.52}$$

$\mathbb{P}$  almost surely. Moreover, the convergence is exponentially fast, that is

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{\tilde{q}_\alpha(t)}{\tilde{q}_{\Upsilon}(t)} \right) = -\frac{1}{2} \sum_{i=0}^p (r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{i=p+1}^n \theta(i|\Upsilon) \left[ 1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln \left( \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} \right) \right], \tag{E.53}$$

$\mathbb{P}$  almost surely.

Finally

$$\lim_{t \rightarrow \infty} \tilde{\rho}(t) = |\Upsilon\rangle\langle\Upsilon|,$$

$\mathbb{P}$  almost surely.

In particular, it appears clearly that

$$\lim_{t \rightarrow \infty} \tilde{\rho}(t) = \lim_{t \rightarrow \infty} \rho(t) = |\Upsilon\rangle\langle\Upsilon|$$

which shows the stability of the estimation and the convergence rate is the same.

Remark that if  $\tilde{q}_\alpha(0) = 0$ , then the conditions of our proposition impose  $q_\alpha(0) = 0$ . Otherwise the estimation would be irrelevant. In the case  $q_\alpha(0) = 0$ , this does not necessarily impose that  $\tilde{q}_\alpha(0) = 0$ . In the case  $\tilde{q}_\alpha(0) \neq 0$ , we still have that  $\lim_{t \rightarrow \infty} \tilde{q}_\alpha(t) = 0$ , almost surely (exponentially fast). In particular, in experiment a safe choice is to choose  $\tilde{\rho}(0)$  such that  $\tilde{q}_\alpha(0) \neq 0$ , for all  $\alpha \in \mathcal{P}$ .

In the situation where it exists  $i \in \{p+1, \dots, n\}$ , such that  $\theta(i|\alpha) = 0$ , it is interesting to note that when a  $q_\alpha(t)$  vanishes, the same happens for the estimate  $\tilde{q}_\alpha(t)$ . This follows from the fact that we have access to the jumping times of the processes  $\hat{N}_i(\cdot)$  through the measurement records.

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