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Study of some fractal and pathwise properties of continuous branching processes

Jean-Pierre Duhalde

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Université Pierre et Marie Curie

École Doctorale de Sciences Mathématiques de Paris Centre

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

Xan DUHALDE

SUR DES PROPRIÉTÉS FRACTALES ET TRAJECTORIELLES DE PROCESSUS DE BRANCHEMENT CONTINUS

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LE CYCLE DES SAISONS

Las de s'être contractés tout l'hiver les arbres tout à coup se flattent d'être dupes. Ils ne peuvent plus y tenir : ils lâchent leurs paroles, un flot, un vomissement de vert. Ils tâchent d'aboutir à une feuillaison complète de paroles. Tant pis ! Cela s'ordonnera comme cela pourra ! Mais en réalité, cela s'ordonne ! Aucune liberté dans la feuillaison... Ils lancent, du moins le croient-ils, n'importe quelles paroles, lancent des tiges pour y suspendre encore des paroles : nos troncs, pensent-ils, sont là pour tout assumer. Ils s'efforcent à se cacher, à se confondre les uns dans les autres. Ils croient pouvoir dire tout, recouvrir entièrement le monde de paroles variées : ils ne disent que « les arbres ». Incapables même de retenir les oiseaux qui repartent d'eux, alors qu'ils se réjouissaient d'avoir produit de si étranges fleurs. Toujours la même feuille, toujours le même mode de dépliement, et la même limite, toujours des feuilles symétriques à elles-mêmes, symétriquement suspendues ! Tente encore une feuille !-La même ! Encore une autre ! La même ! Rien en somme ne saurait les arrêter que soudain cette remarque : « L'on ne sort pas des arbres par des moyens d'arbres. » Une nouvelle lassitude, et un nouveau retournement moral. « Laissons tout ça jaunir, et tomber. Vienne le taciturne état, le dépouillement, l'AUTOMNE. »

Francis Ponge
Le parti pris des choses

RESUME

Cette thèse étudie certaines propriétés fractales et trajectorielles de processus de branchement en temps et espace continus. De façon informelle, ce type de processus est obtenu en considérant l'évolution d'une population où les individus se reproduisent et meurent au cours du temps, et ce de manière aléatoire. Le premier chapitre concerne la classe des processus de branchement avec immigration. On donne une formule semi-explicite pour la transformée de Laplace des temps d'atteinte ainsi qu'une condition nécessaire et suffisante de récurrence-transience. Ces deux résultats illustrent la compétition branchement/immigration. Le second chapitre considère l'arbre Brownien et ses mesures de temps local, dites mesures de niveau. On montre que celles-ci s'obtiennent comme restriction, à une constante près explicitée, d'une certaine mesure de Hausdorff sur l'arbre. Le résultat est montré simultanément pour tous niveaux. Le troisième chapitre étudie le Super-mouvement Brownien associé à un mécanisme de branchement général. Sa mesure d'occupation totale est obtenue comme restriction d'une certaine mesure de packing dans l'espace euclidien. Le résultat est valable en grande dimension. La condition sur la dimension de l'espace ambiant est discutée à travers le calcul, sous des hypothèses de régularité faibles pour le mécanisme de branchement, de la dimension de packing du range total du processus.

ABSTRACT

This thesis investigates some fractal and pathwise properties of branching processes with continuous time and state-space. Informally, this kind of process can be described by considering the evolution of a population where individuals reproduce and die over time, randomly. The first chapter deals with the class of continuous branching processes with immigration. We provide a semi-explicit formula for the hitting times and a necessary and sufficient condition for the process to be recurrent or transient. Those two results illustrate the competition between branching and immigration. The second chapter deals with the Brownian tree and its local time measures : the level-sets measures. We show that they can be obtained as the restriction, with an explicit multiplicative constant, of a Hausdorff measure on the tree. The result holds uniformly for all levels. The third chapter study the Super-Brownian motion associated with a general branching mechanism. Its total occupation measure is obtained as the restriction to the total range, of a given packing measure on the euclidean space. The result is valid for large dimensions. The condition on the dimension is discussed by computing the packing dimension of the total range. This is done under a weak assumption on the regularity of the branching mechanism.

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1 Processus et arbres de Galton-Watson.

Le présent travail de thèse étudie certains aspects des processus de branchement continus en temps et espace. Ceux-ci s'obtiennent notamment comme limite d'échelle de processus de branchement discrets dont nous rappelons les définitions dans cette section. Dans une première partie, nous rappelons la définition des processus de Galton-Watson et de leur généalogie : les arbres de Galton-Watson. Les notions de fonctions des hauteurs et de fonctions de contour sont ensuite introduites. Ces définitions dans le cadre discret permettent d'expliquer les définitions du processus des hauteurs en temps et espace continus. Pour plus de détails sur ces processus discrets, ainsi que pour les preuves des faits énoncés ci-dessous, on renvoie le lecteur au livre d'Athreya et Ney [8], au livre de Lyons et Peres [92], et à l'article de Neveu [94]. Pour les fonctions associées, voir l'article de Le Gall et Le Jan [86] et l'introduction de la monographie de Duquesne et Le Gall [40]. Sauf mention explicite du contraire, toutes les variables aléatoires considérées dans ce qui suit sont définies sur un même espace $(\Omega, \mathcal{F}, \mathbf{P})$.

1.1 Processus de Galton-Watson.

Soit $(V_i^{(n)}, i \in \mathbb{N}^*, n \in \mathbb{N})$, un tableau de variables aléatoires i.i.d. à valeurs dans \mathbb{N} de loi commune $\mu = (\mu(k), k \in \mathbb{N})$, qui est la *loi de reproduction*. On note $f_\mu(r) = \sum_{k=0}^{\infty} r^k \mu(k)$ la fonction génératrice de μ . Un processus de Galton-Watson $(Z_n)_{n \in \mathbb{N}}$ partant de $a \in \mathbb{N}$ et de loi de reproduction μ peut être défini récursivement par

$$\begin{cases} Z_0 = a \\ Z_{n+1} = \sum_{1 \leq i \leq Z_n} V_i^{(n)} \end{cases} \quad (1)$$

Le processus $(Z_n)_{n \in \mathbb{N}}$ est clairement une chaîne de Markov dont les probabilités de transition sont caractérisées par

$$\forall r \in [0, 1], \forall n \in \mathbb{N}, \quad \mathbf{E}[r^{Z_{n+1}} | Z_n] = f_\mu(r)^{Z_n} \quad (2)$$

La chaîne de Markov $(Z_n)_{n \in \mathbb{N}}$ décrit l'évolution au cours des générations d'une population constituée initialement de a ancêtres à la génération 0. À la génération $n \geq 0$, chaque individu donne naissance, indépendamment des autres, à un nombre aléatoire d'enfants suivant la loi μ . Les enfants des individus à la génération n constituent l'ensemble des individus à la génération $n + 1$.

Considérons deux populations indépendantes, l'une descendant de a ancêtres, l'autre descendant de b ancêtres. La réunion de ces deux populations suit clairement la même loi

qu'une population descendant de $a + b$ ancêtres. Ce fait est appelé *propriété de branchement*. Plus formellement, notons $P_n(a, \cdot)$ la loi de Z_n lorsque $Z_0 = a$ comme dans la définition (1); la propriété de branchement s'énonce par l'égalité suivante :

$$\forall n, a, b \in \mathbb{N} \quad P_n(a, \cdot) * P_n(b, \cdot) = P_n(a + b, \cdot), \quad (3)$$

où $*$ désigne le produit de convolution des lois sur \mathbb{N} . Cette propriété caractérise les processus de Galton-Watson parmi les chaînes de Markov d'espace d'états \mathbb{N} .

Soit $q = \sup \{r \in [0, 1] : f_\mu(r) \geq r\}$ qui est le plus petit point fixe de f_μ sur $[0, 1]$. Il est facile de montrer d'une part que

$$\mathbf{P}(\exists n \in \mathbb{N} : Z_n = 0) = \mathbf{E}[q^{Z_0}]. \quad (4)$$

D'autre part, si $\sum_{k=0}^{\infty} k\mu(k) \leq 1$, alors $q = 1$ et, par (4), la population s'éteint presque sûrement; si $\sum_{k=0}^{\infty} k\mu(k) > 1$, la population a une probabilité strictement positive de survivre. Pour ces raisons, μ est dite sous-critique si $\sum_{k=0}^{\infty} k\mu(k) < 1$, critique si $\sum_{k=0}^{\infty} k\mu(k) = 1$ et sur-critique si $\sum_{k=0}^{\infty} k\mu(k) > 1$.

1.2 Arbres de Galton-Watson.

Dans cette section, nous décrivons l'arbre généalogique d'un processus de Galton-Watson issu d'un seul ancêtre. De manière informelle, à un individu vivant à la génération n est associé un noeud à la hauteur n dans l'arbre généalogique. Si cet individu donne naissance à k enfants, autant de branches partiront de ce noeud dans l'arbre. Formellement, l'arbre généalogique est un arbre *étiqueté ordonné et enraciné* (un arbre *planaire enraciné*). Nous rappelons ici le formalisme d'arbres proposé par Neveu [94]; on pose tout d'abord

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n, \quad (5)$$

avec la convention $(\mathbb{N}^*)^0 = \emptyset$; \mathbb{U} est l'ensemble des mots finis écrits avec des entiers. Si $u = (u_1, \dots, u_n) \in \mathbb{U}$, on note $|u| = n$ sa longueur et on dira que u est un noeud à la hauteur (ou à la génération) n . Si $v = (v_1, \dots, v_m) \in \mathbb{U}$, on note $uv = (u_1, \dots, u_n, v_1, \dots, v_m)$ la concaténation des mots u et v . Un sous-ensemble $T \subset \mathbb{U}$ est un *arbre* (étiqueté, enraciné, ordonné) si

- (a) $\emptyset \in T$. On dit que \emptyset est la *racine* de T .
- (b) Si $u \in \mathbb{U}$ et si $j \in \mathbb{N}^*$ sont tels que $uj \in T$, alors $u \in T$.
- (c) Pour tout $u \in T$, il existe $k_u(T) \geq 0$ tel que $uj \in T$ si et seulement si $1 \leq j \leq k_u(T)$.

Notons $\mathbb{T} \subset \mathbb{U}$ l'ensemble des arbres étiquetés. Si $T \in \mathbb{T}$ et $u \in T$, on note $\theta_u(T)$ le sous-arbre de T constitué des descendants de u , c'est-à-dire $\theta_u(T) = \{v \in \mathbb{U} : uv \in T\}$. On munit \mathbb{T} de la tribu \mathcal{G} engendrée par les ensembles $\{T \in \mathbb{T} : u \in T\}$, $u \in \mathbb{U}$. Soit alors μ une probabilité sur \mathbb{N} . Il est montré dans Neveu [94] qu'il existe une unique probabilité Q_μ sur $(\mathbb{T}, \mathcal{G})$ telle que

- (a) $Q_\mu(k_\emptyset = j) = \mu(j)$, $j \in \mathbb{N}$.
- (b) Pour tout $j \in \mathbb{N}^*$ tel que $\mu(j) > 0$, sous $Q_\mu(\cdot \mid k_\emptyset = j)$, les sous-arbres $\theta_1(T), \dots, \theta_j(T)$ sont indépendants et de loi Q_μ .

La loi Q_μ est la loi de l'arbre de Galton-Watson de loi de reproduction μ . On vérifie facilement que si on pose $Z_n(T) := \#\{u \in T : |u| = n\}$, alors sous Q_μ , $(Z_n(T), n \in \mathbb{N})$ est un processus de GW de loi de reproduction μ issu d'un unique ancêtre ($Z_0 = 1$).

Soit \mathcal{G}_n la tribu sur \mathbb{T} engendrée par les ensembles $\{T \in \mathbb{T}, u \in T\}$, avec u variant dans \mathbb{U} tel que $|u| \leq n$: cette tribu contient l'information décrivant l'arbre jusqu'à la génération n . On note u_1, \dots, u_{Z_n} les sommets de l'arbre à la hauteur n . Sous Q_μ , conditionnellement à \mathcal{G}_n , la suite

$$\left(\theta_{u_1}(T), \dots, \theta_{u_{Z_n}}(T)\right) \quad (6)$$

est i.i.d de loi Q_μ . Cette propriété est appelée *propriété de branchement pour les arbres de Galton-Watson*.

1.3 Codages des arbres de Galton-Watson.

À un arbre $T \in \mathbb{T}$ fini, on associe deux fonctions de codage : sa fonction *de contour*, et sa fonction *des hauteurs*. La fonction de contour de T (que l'on voit comme un graphe dans le demi-plan supérieur enraciné sur l'horizon, et dont les arêtes sont des segments de longueur 1) se définit informellement comme suit. Soit une particule, à la racine au temps 0, parcourant l'arbre à vitesse unité, de la gauche vers la droite en revenant aussi peu que possible sur ses pas. On note $C_s(T)$ la distance de graphe dans T de la position de la particule au temps s à la racine \emptyset . Le temps mis par la particule pour revenir à la racine est égal à deux fois le nombre d'arêtes de T , quantité notée $\sigma(T) = 2(\#T - 1)$. Cette procédure définit une fonction $(C_s(T)), s \in [0, \sigma(T)]$ appelée *fonction de contour* de T (voir figure 1). Pour tout $k \in \{0, \dots, \sigma(T)\}$, on note v_k le sommet visité par la particule au temps k et la distance de graphe de T , notée d , s'exprime à l'aide de la fonction de contour comme suit :

$$\forall 0 \leq k_1 \leq k_2 \leq \sigma(T) \quad d(v_{k_1}, v_{k_2}) = C_{k_1}(T) + C_{k_2}(T) - 2 \min_{s \in [k_1, k_2]} C_s(T), \quad (7)$$

formule qui se généralise au cas continu.

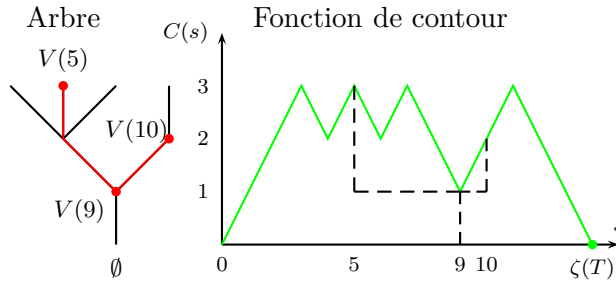


FIGURE 1 – La distance entre $V(5)$ et $V(10)$ peut s'exprimer uniquement à l'aide de distances à la racine : $d(V(5), V(10)) = d(V(5), \emptyset) + d(V(10), \emptyset) - 2d(V(9), \emptyset)$. Ici, $V(9)$ est le "plus récent ancêtre commun" de $V(5)$ et $V(10)$; il a une hauteur minimale parmi tous les sommets visités par la particule entre les temps $s = 5$ et $s = 10$.

Définissons maintenant la fonction des hauteurs associée à T . Pour cela, on observe d'abord que \mathbb{U} est totalement ordonné par l'ordre lexicographique ; on se donne une indexation des sommets de T pour l'ordre lexicographique : $u(0) = \emptyset, u(1), \dots, u(\#T - 1)$. La fonction des hauteurs $(H_n(T), 0 \leq n \leq \#T - 1)$ est alors définie par

$$\forall n \in \{0, \dots, \#T - 1\}, \quad H_n(T) := |u(n)|.$$

On définit ensuite les fonctions des hauteurs et de contour associées à une forêt d'arbres finis, c'est à dire une suite $(T_k)_{k \in \mathbb{N}^*}$ où $T_k \in \mathbb{T}$ et T_k est fini. Pour des raisons techniques

expliquées ci-dessous, on prolonge les fonctions de contour $(C_s(T_k), s \in [0, \sigma(T_k)])$ en posant pour tout $s \in (\sigma(T_k), 2\#T_k]$, $C_s(T_k) = 0$. La fonction de contour $(C_s, s \in \mathbb{R}_+)$ associée à la forêt est alors obtenue en concaténant les fonctions prolongées $(C_s(T_k), s \in [0, 2\#T_k])$, c'est-à-dire que pour tout $s \in \mathbb{R}_+$, on a

$$C_s = C_{s - (2\#(T_1 + \dots + \#T_{k-1}))}(T_k),$$

$$\text{si } 2\#(T_1 + \dots + \#T_{k-1}) \leq s < 2\#(T_1 + \dots + \#T_k).$$

On constate facilement que la donnée de la fonction de contour C permet de déterminer la fonction de chaque arbre de la forêt (c'est la raison du prolongement) et donc détermine entièrement la forêt.

On peut ordonner totalement les sommets de la forêt $(T_k)_{k \in \mathbb{N}^*}$, en parcourant dans l'ordre lexicographique les sommets de T_1 , puis à la suite ceux de T_2 , ainsi de suite. On note $(u_n)_{n \geq 0}$ la suite de ces sommets ainsi ordonnés et on pose ensuite

$$\forall n \geq 0, \quad H_n = |u_n|.$$

La fonction des hauteurs associée à la forêt est également obtenue en concaténant les fonctions des hauteurs associées à chaque arbre. Plus précisément, pour $k \geq 1$, on a

$$H_n = H_{n - (\#T_1 + \dots + \#T_{k-1})}(T_k), \quad \text{si } \#T_1 + \dots + \#T_{k-1} \leq n < \#T_1 + \dots + \#T_k.$$

La donnée du processus des hauteurs de la forêt détermine entièrement cette forêt. Le processus de contour peut s'écrire explicitement comme une certaine fonction du processus des hauteurs que nous ne détaillons pas ici (voir section 2.4 p. 61 de Duquesne et Le Gall [40]).

On peut également associer à la forêt $(T_k)_{k \in \mathbb{N}^*}$ une trajectoire issue de 0 et à valeurs dans \mathbb{Z} , appelé *chemin de Lukasiewicz*, que l'on note $(X_n)_{n \geq 0}$ et qui est définie récursivement par

$$X_0 = 0 \quad \text{et} \quad \forall n \geq 0, \quad X_{n+1} = X_n + k_{u_n}(T_k) - 1, \quad (8)$$

où $(u_n)_{n \geq 0}$ est la suite des sommets de la forêt $(T_k)_{k \in \mathbb{N}^*}$ parcourus dans l'ordre lexicographique, comme expliqué ci-dessus, et où $u_n \in T_k$ et $k_{u_n}(T_k)$ est le nombre d'enfants de u_n dans T_k . On observe que $u_n \in T_k$ si et seulement si $k = 1 - \inf_{0 \leq m \leq n} X_m$ et que X permet de complètement déterminer la forêt $(T_k)_{k \in \mathbb{N}^*}$.

La fonction de contour d'une forêt d'arbres de Galton-Watson i.i.d. de loi de reproduction géométrique de paramètre 1/2 est très proche d'une marche aléatoire simple symétrique réfléchie (c'est la concaténation d'excursions positives indépendantes de la marche prolongées par 0 sur deux unités de temps). Le cas des lois de reproduction géométriques sont les seuls cas où les fonctions de contour et des hauteurs ont des lois de description simple. Néanmoins, le lemme suivant montre que la fonction des hauteurs d'une forêt d'arbres de Galton-Watson i.i.d. s'obtient comme fonctionnelle adaptée d'une marche aléatoire.

Lemme 1.1 *On désigne par H et X , comme précédemment, le processus des hauteurs et le chemin de Lukasiewicz associés à une forêt d'arbres de Galton-Watson i.i.d. de loi de reproduction μ . Alors X est une marche aléatoire issue de 0 de loi de saut ν donnée par $\nu(k) := \mu(k+1)$ pour $k \in \{-1, 0, 1, \dots\}$. De plus H s'exprime en termes de X par*

$$\forall n \geq 0, \quad H_n := \# \left\{ k \in \{0, 1, \dots, n-1\} : X_k = \inf_{k \leq j \leq n} X_j \right\}. \quad (9)$$

Ce résultat, dû à Le Gall et Le Jan [86], est à l'origine de leur définition du processus des hauteurs dans le cadre continu (voir [86]) que nous rappelons plus loin en (46).

2 Processus de branchement continu.

2.1 Définition et caractérisation.

Les processus de branchement continu sont les analogues continus (en temps et en espace) des processus de Galton-Watson. Ils ont été introduits par Jirina [69] et Lamperti [78, 79] et étudiés également dans Bingham [12]. Pour plus de détails et les preuves des résultats rappelés ci-dessous, nous renvoyons à l'article de Bingham [12] et aussi à l'ouvrage de Kyprianou [73].

Soit un processus fellerien à valeurs dans $[0, \infty]$: ici l'état ∞ n'est pas vu comme un point cimetièrre ; ce processus est noté $(Z_t, t \in \mathbb{R}_+; \mathbf{P}_x, x \in \mathbb{R}_+)$; c'est un *processus de branchement continu* si 0 et ∞ sont des états absorbants et si ses noyaux de transition notés $(P_t(x, dy), x \in [0, \infty], t \in \mathbb{R}_+)$ sont des probabilités sur $[0, \infty]$ qui satisfont la *propriété de branchement* suivante :

$$\forall t \in \mathbb{R}_+, \forall x, x' \in [0, \infty], \quad P_t(x, \cdot) * P_t(x', \cdot) = P_t(x + x', \cdot), \quad (10)$$

qui est l'exacte analogue de la propriété de branchement (3) des processus de Galton-Watson.

Les noyaux de transition du processus de branchement Z sont caractérisés par leur transformée de Laplace. Plus précisément, pour tous $x, s, t, \lambda \in \mathbb{R}_+$, on a

$$\mathbf{E}_x \left[e^{-\lambda Z_{s+t}} \mid Z_s \right] = \exp(-Z_s u_t(\lambda)), \quad (11)$$

où l'application $t \mapsto u_t(\lambda)$ est dérivable, à valeurs positives, et satisfait l'équation

$$u_0(\lambda) = \lambda, \quad \text{et} \quad \frac{\partial}{\partial t} u_t(\lambda) + \psi(u_t(\lambda)) = 0, \quad t \in [0, \infty) \quad (12)$$

où la fonction ψ , appelée *mécanisme de branchement* du processus, est de la forme

$$\psi(\lambda) = \alpha_0 \lambda + \beta \lambda^2 + \int_{(0, \infty)} (e^{-\lambda r} - 1 + \lambda r \mathbf{1}_{(0,1)}(r)) \pi(dr), \quad \lambda \geq 0, \quad (13)$$

avec $\alpha_0 \in \mathbb{R}, \beta \geq 0$, et π , mesure sur $(0, \infty)$ telle que $\int_0^\infty (1 \wedge r^2) \pi(dr) < \infty$ et appelée *mesure de Lévy*.

Pour $\lambda > 0$, l'équation (12) admet une unique solution, ce qui implique que le mécanisme de branchement ψ caractérise entièrement la loi du processus de branchement continu Z . On parle donc d'un processus de branchement continu de mécanisme ψ ce que l'on abrège en $\text{CB}(\psi)$. On montre également qu'à tout mécanisme de branchement ψ de la forme (13), on peut associer un $\text{CB}(\psi)$.

Citons trois exemples de mécanismes de branchement.

- $\psi(\lambda) = \alpha_0 \lambda$, où $\alpha_0 \in \mathbb{R}$. Le $\text{CB}(\psi)$ associé est le processus déterministe $Z_t = Z_0 e^{-\alpha_0 t}$.
- $\psi(\lambda) = \lambda^2$. On a alors $u_t(\lambda) = \frac{\lambda}{1+t\lambda}$. Le $\text{CB}(\psi)$ associé est une diffusion, appelée *diffusion de Feller*, solution de l'EDS suivante : $dZ_t = \sqrt{2Z_t} dB_t$, où $(B_t, t \geq 0)$ est un mouvement Brownien standard.
- $\psi(\lambda) = \lambda^\gamma$, avec $\gamma \in (1, 2)$. Il s'agit du mécanisme de branchement stable non Brownien d'indice γ . Il correspond au cas où $\alpha_0 = \beta = 0$ et $\pi(dr) = \frac{\gamma(\gamma-1)}{\Gamma(2-\gamma)} r^{-\gamma-1} dr$. Dans ce cas, $u_t(\lambda)$ est explicitement calculable :

$$\forall t, \lambda \in \mathbb{R}_+, \quad u_t(\lambda) = \left((\gamma-1)t + \lambda^{-(\gamma-1)} \right)^{-\frac{1}{\gamma-1}}.$$

Le mécanisme ψ est une fonction convexe, analytique sur $(0, \infty)$. On la compare aux fonctions puissance grâce aux deux exposants suivants.

$$\gamma = \sup \{c \geq 0 : \lim_{\lambda \rightarrow \infty} \psi(\lambda)\lambda^{-c} = \infty\}, \quad \eta = \inf \{c \geq 0 : \lim_{\lambda \rightarrow \infty} \psi(\lambda)\lambda^{-c} = 0\}. \quad (14)$$

Les exposants γ et η ont été introduits par Blumenthal et Gettoor [17]. On a clairement $1 \leq \gamma \leq \eta \leq 2$; ces exposants sont en général distincts mais si ψ est à variation régulière en ∞ d'exposant γ , alors on a $\gamma = \eta = \gamma$.

Un $\text{CB}(\psi)$ peut être construit comme limite d'échelle de processus de branchement discrets voir Grimvall [62] (voir aussi Lamperti [79] et le livre de Le Gall [84]); nous rappelons plus loin en section 3.1 ces résultats.

Comme montré par Lamperti [78], un $\text{CB}(\psi)$ peut également se voir comme un processus de Lévy changé de temps, le changement de temps étant ici appelé *transformation de Lamperti*. En effet, un mécanisme de branchement ψ de la forme (13) est l'exposant de Laplace d'un processus de Lévy sans sauts négatifs $(X_t, t \geq 0)$ issu de 0. Plus précisément, pour tout $t \in (0, \infty)$, on a $\mathbf{E}[e^{-\lambda X_t}] = \exp(t\psi(\lambda))$: voir Bertoin [9], chapitre VII. Pour $x \geq 0$, on pose $T_x = \inf\{t \geq 0 : X_t = -x\}$, qui est le temps d'atteinte de $-x$; pour tout $t \in \mathbb{R}_+$, on pose ensuite

$$J_t = \inf \left\{ s \in \mathbb{R}_+ : \int_0^s \frac{du}{(x + X_{u \wedge T_x})} > t \right\},$$

avec la convention $\inf \emptyset = \infty$. On pose enfin

$$\forall t \in \mathbb{R}_+, \quad Z_t = x + X_{J_t \wedge T_x}, \quad (15)$$

avec la convention $Z_t = \infty$ si J_t et T_x sont infinis. Le processus $(Z_t, t \geq 0)$ est un $\text{CB}(\psi)$ issu de x . Ce résultat implique notamment que les trajectoires d'un $\text{CB}(\psi)$ n'ont pas de sauts négatifs. Par ailleurs, on peut montrer également que les trajectoires d'un $\text{CB}(\psi)$ sont à variations bornées si et seulement si c'est aussi le cas des trajectoires d'un processus de Lévy sans sauts négatifs d'exposant de Laplace ψ , ce qui est équivalent à la condition analytique

$$\beta = 0 \quad \text{et} \quad \int_0^1 r\pi(dr) < \infty. \quad (16)$$

On introduit ici le *drift effectif* de ψ noté \mathbf{d} :

$$\mathbf{d} := \begin{cases} \alpha_0 + \int_0^1 r\pi(dr) & \text{si le processus est à variations bornées} \\ \infty & \text{sinon.} \end{cases} \quad (17)$$

Dans tous les cas (variations bornées ou non), on a

$$\lim_{\lambda \rightarrow \infty} \psi(\lambda)/\lambda = \mathbf{d}. \quad (18)$$

Le drift effectif intervient dans les processus de branchement continus de la manière suivante : la propriété de branchement (10) permet de montrer qu'à t fixé, $\lambda \mapsto u_t(\lambda)$ est l'exposant de Laplace d'un subordonateur : on peut montrer (voir par exemple Labbé [75]) que le drift de $u_t(\lambda)$ vaut $e^{-\mathbf{d}t}$, avec la convention que ce dernier vaut 0 si $\mathbf{d} = \infty$.

Effectuons de bref rappels sur les comportements asymptotiques des $\text{CB}(\psi)$. Pour cela, on introduit

$$q = \sup \{ \lambda \in [0, \infty) : \psi(\lambda) \leq 0 \}, \quad (19)$$

qui est la plus grande racine de ψ . La propriété de Markov des $\text{CB}(\psi)$ (ou l'équation différentielle (12)) permet de montrer que pour tous $t, s, \lambda \in \mathbb{R}_+$, on a $u_{t+s}(\lambda) = u_t(u_s(\lambda))$. On observe que pour tout t , $u_t(q) = q$, et on peut montrer que si $\lambda < q$, la fonction $t \mapsto u_t(\lambda)$ croît strictement et tend vers q ; de même si $\lambda > q$, la fonction $t \mapsto u_t(\lambda)$ décroît strictement et tend vers q . Ainsi, l'équation différentielle (12) est équivalente à l'équation intégrale suivante.

$$\forall t \in [0, \infty), \forall \lambda \in (0, \infty) \setminus \{q\}, \quad \int_{u_t(\lambda)}^{\lambda} \frac{du}{\psi(u)} = t. \quad (20)$$

On déduit tout d'abord de cette équation que :

$$e^{-xq} = \mathbf{P}_x \left(\lim_{t \rightarrow \infty} Z_t = 0 \right) = 1 - \mathbf{P}_x \left(\lim_{t \rightarrow \infty} Z_t = \infty \right). \quad (21)$$

La connaissance de q permet donc de connaître les probabilités d'extinction et de survie du $\text{CB}(\psi)$ et, comme dans le cas discret, cela est relié aux valeurs moyennes du processus de la manière suivante : on rappelle d'abord que par convexité, ψ admet en 0 une dérivée à droite $\psi'(0+) \in [-\infty, \infty)$. On a notamment, pour tous $x, t \in (0, \infty)$

$$\mathbf{E}_x [Z_t] = x e^{-\psi'(0+)t}, \quad (22)$$

où les deux membres dans l'inégalité ci-dessus peuvent être infinis. Ainsi, on dit qu'un $\text{CB}(\psi)$ est *sous-critique* si $\psi'(0+) > 0$ ou *critique* si $\psi'(0+) = 0$ car, dans ces cas, il est clair que $q = 0$ et par (21), pour tout $x \in \mathbb{R}_+$, \mathbf{P}_x -p.s. $\lim_{t \rightarrow \infty} Z_t = 0$. Le processus de branchement est dit *sur-critique* si $\psi'(0+) < 0$ car, dans ce cas, il est clair que $q > 0$, et par (21), pour tout $x \in \mathbb{R}_+^*$, $\mathbf{P}_x(\lim_{t \rightarrow \infty} Z_t = \infty) > 0$.

On déduit facilement de (20) que le $\text{CB}(\psi)$ est *conservatif*, c'est-à-dire reste à valeurs dans \mathbb{R}_+ , ssi la condition suivante est satisfaite :

$$\int_{0+}^{\infty} \frac{d\lambda}{|\psi(\lambda)|} = \infty \iff \mathbf{P}_x\text{-p.s. } \forall t \in \mathbb{R}_+, \quad Z_t < \infty. \quad (23)$$

Si le $\text{CB}(\psi)$ n'est pas conservatif (ce qui ne peut se produire que si $\psi'(0+) = -\infty$), on déduit alors que $\mathbf{P}_x(Z_t = \infty) = e^{-xu_t(0+)}$, où $u_t(0+) = \lim_{\lambda \rightarrow 0+} u_t(\lambda)$ satisfait l'équation intégrale $\int_0^{u_t(0+)} d\lambda/|\psi(\lambda)| = t$. Dans ce cas, on note le temps d'explosion par $\zeta = \inf\{t \in \mathbb{R}_+ : Z_t \text{ ou } Z_{t-} = \infty\}$. On a alors, pour tout $t > 0$, $\mathbf{P}_x(\zeta > t) = \exp(-xu_t(0+))$ puis $\mathbf{P}(\zeta = \infty) = \exp(-xq)$. On a donc \mathbf{P}_x -p.s. $Z_{\zeta-} = Z_t = \infty$ pour tout $t \geq \zeta$ et $\mathbf{1}_{\{\zeta < \infty\}} = \mathbf{1}_{\{\lim_{t \rightarrow \infty} Z_t = \infty\}}$. Autrement dit, le processus *ou bien* tend vers 0 *ou bien* explose en temps fini et dans ce cas il atteint l'infini continûment.

Contrairement aux processus de Galton-Watson, l'extinction pour les $\text{CB}(\psi)$ peut s'effectuer de deux manières distinctes. Plus précisément, on déduit facilement de (20) qu'un $\text{CB}(\psi)$ est absorbé en 0 en temps fini ssi la condition analytique suivante est vérifiée :

$$\int_{0+}^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty \iff \mathbf{P}_x(\exists t \in \mathbb{R}_+ : Z_t = 0) = \mathbf{P}_x\left(\lim_{t \rightarrow \infty} Z_t = 0\right) = e^{-xq}. \quad (24)$$

Cette condition est parfois appelée *condition de Grey*. Si elle est satisfaite, on pose

$$\forall t \in \mathbb{R}_+, \quad v(t) = \lim_{\lambda \rightarrow \infty} u_t(\lambda) \quad (25)$$

et (20) implique que $v : (0, \infty) \rightarrow (q, \infty)$ est une bijection continue strictement décroissante telle que

$$\int_{v(t)}^{\infty} d\lambda/\psi(\lambda) = t \quad (26)$$

et $\mathbf{P}_x(Z_t = 0) = e^{-xv(t)}$, pour tous $t, x \in (0, \infty)$. Lorsque la condition de Grey (24) n'est pas satisfaite, on dit alors que le CB(ψ) est *persistant* car pour tout $x \in (0, \infty)$, \mathbf{P}_x -p.s. pour tout $t \in \mathbb{R}_+$, $Z_t > 0$.

2.2 Super-mouvement Brownien.

Les super-mouvements Browniens sont obtenus comme limites d'échelle de marches aléatoires branchantes qui combinent une population de Galton-Watson où chaque individu partant du lieu de sa naissance qui est le lieu de mort de son géniteur, se déplace selon une marche aléatoire dont les accroissements sont indépendants des déplacements spatiaux des autres individus. Les super-mouvements Browniens représentent donc l'évolution au cours du temps des déplacement spatiaux Browniens d'une population dont la taille est un CB(ψ). Les cas où ψ est un mécanisme de branchement quadratique $\psi(\lambda) = \alpha_0 + \beta\lambda^2$ ont été particulièrement étudiés et notamment le cas où $\psi(\lambda) = \lambda^2$, qui est appelé le *super-processus de Dawson-Watanabe*, d'après les travaux pionniers de Watanabe, Dawson et Hochberg [114, 23, 25]. Les super-diffusions de mécanisme de branchement plus général ψ ont été introduits et étudiés par Dynkin [47, 48, 50, 49]. Dans cette introduction, nous nous limitons à rappeler quelques définitions et les résultats strictement nécessaires à la compréhension des contributions de la thèse : pour une introduction plus complète, nous renvoyons le lecteur aux ouvrages de Dawson [24], Dynkin [51, 52], Le Gall [84], Perkins [19], Etheridge [54] et Li [89].

On note $M_f(\mathbb{R}^d)$ l'ensemble des mesures boréliennes finies sur \mathbb{R}^d , muni de la topologie de la convergence faible. Pour toute mesure $\mu \in M_f(\mathbb{R}^d)$ et toute fonction Borel-mesurable $f : \mathbb{R}^d \rightarrow \mathbb{R}$ bornée ou positive, on note $\langle \mu, f \rangle$ l'intégrale $\int_{\mathbb{R}^d} f(x)\mu(dx)$. On note également $\langle \mu \rangle = \mu(\mathbb{R}^d)$, la masse totale de μ et $\text{supp}(\mu)$, son *support topologique*, c'est à dire le complémentaire du plus grand ouvert de μ -mesure nulle.

Soit ψ un mécanisme de branchement de la forme (13) que l'on suppose critique ou sous-critique, c'est-à-dire que $\psi'(0+) \geq 0$. On suppose que ψ satisfait également la condition de Grey (24). Un super-mouvement Brownien de mécanisme de branchement ψ (un SBM(ψ) en abrégé) est un processus à valeurs dans $M_f(\mathbb{R}^d)$ markovien homogène càdlàg ($Z_t, t \in \mathbb{R}_+, \mathbb{P}_\mu, \mu \in M_f(\mathbb{R}^d)$) dont les noyaux de transition sont caractérisés de la manière suivante : pour toute mesure $\mu \in M_f(\mathbb{R}^d)$, pour toute fonction mesurable $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ et pour tous $s, t \in \mathbb{R}_+$,

$$\mathbb{P}_\mu(Z_0 = \mu) = 1 \quad \text{et} \quad \mathbb{E}_\mu [\exp(-\langle Z_{s+t}, f \rangle) | Z_s] = \exp(-\langle Z_s, u_t \rangle), \quad (27)$$

où la fonction $(u_t(x))_{t \geq 0, x \in \mathbb{R}^d}$ est mesurable, positive et satisfait l'équation

$$u_t(x) + \mathbf{E}_x \left[\int_0^t \psi(u_{t-s}(\xi_s)) ds \right] = \mathbf{E}_x [f(\xi_t)], \quad x \in \mathbb{R}^d, t \in [0, \infty). \quad (28)$$

Ici $\xi = (\xi_t)_{t \geq 0}$ est un processus continu défini sur (Ω, \mathcal{F}) et pour tout $x \in \mathbb{R}^d$, \mathbf{P}_x est une probabilité sur (Ω, \mathcal{F}) telle que sous \mathbf{P}_x , ξ soit distribué comme un mouvement brownien d -dimensionnel standard issu de x . Lorsque $f \in C_c^0(\mathbb{R}^d)$, c'est-à-dire est continue à support compact alors $u \in C^0(\mathbb{R}_+ \times \mathbb{R}^d) \cap C^{1,2}((0, \infty) \times \mathbb{R}^d)$ satisfait l'équation aux dérivées partielles semi-linéaire

$$u_0(x) = f(x) \quad \text{et} \quad \frac{\partial}{\partial t} u_t(x) + \psi(u_t(x)) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u_t(x), \quad x \in \mathbb{R}^d, t \in (0, \infty). \quad (29)$$

On note $(Q_t(\mu, d\nu); t \in \mathbb{R}_+, \mu \in M_f(\mathbb{R}^d))$ les noyaux de transition du SBM(ψ) Z , ils vérifient la *propriété de branchement spatiale* suivante

$$\forall \mu, \mu' \in M_f(\mathbb{R}^d), \forall t \in \mathbb{R}_+, \quad Q_t(\mu, \cdot) * Q_t(\mu', \cdot) = Q_t(\mu + \mu', \cdot). \quad (30)$$

Par conséquent, sous \mathbb{P}_μ , le processus $(\langle Z_t \rangle, t \geq 0)$, de la masse totale de Z satisfait la propriété la propriété de branchement (10) et il est facile de vérifier que c'est un $\text{CB}(\psi)$ issu de la valeur $\langle \mu \rangle$. Comme ψ est critique ou sous-critique et qu'il satisfait la condition de Grey (24), le $\text{CB}(\psi)$ $(\langle Z_t \rangle, t \geq 0)$ s'éteint en temps fini et donc

$$\forall \mu \in M_f(\mathbb{R}^d), \mathbb{P}_\mu\text{-p.s.} \quad \exists t_0 \in \mathbb{R}_+ : \forall t \geq 0, \quad Z_t = 0.$$

Cela permet de montrer que la définition suivante de la *mesure d'occupation totale* du $\text{SBM}(\psi)$ est bien définie par

$$\mathbf{M} = \int_0^\infty Z_t dt \quad (31)$$

comme une variable aléatoire à valeurs dans $M_f(\mathbb{R}^d)$. On introduit également la *trace totale* du $\text{SBM}(\psi)$ Z (en anglais *range*) par

$$\mathbf{R} = \bigcup_{\varepsilon > 0} \overline{\bigcup_{t \geq \varepsilon} \text{supp}(Z_t)}, \quad (32)$$

où pour toute partie B de \mathbb{R}^d , on note \overline{B} l'adhérence de B .

Bien que sous la condition de Grey (24) le super-processus s'éteigne en temps fini, une hypothèse plus restrictive est nécessaire pour obtenir la compacité du support. Plus précisément, dans [110] (voir aussi Hesse et Kyprianou [67] pour une preuve probabiliste), Sheu montre que si la mesure initiale μ est à *support compact*, alors

$$\int \frac{d\lambda}{\sqrt{\int_1^\lambda \psi(a) da}} < \infty \iff \mathbb{P}_\mu\text{-p.s.} \quad \mathbf{R} \text{ est borné.} \quad (33)$$

Cette dernière condition est plus forte que (24). Mais on observera que si $\gamma > 1$, où γ est l'exposant défini à (14), alors (33) et (24) sont vérifiées.

2.3 Processus de branchement avec immigration.

Revenons brièvement au cadre discret et rappelons la définition (1) des processus de Galton-Watson. Soit $(V_i^{(n)}, i \in \mathbb{N}^*, n \in \mathbb{N})$, un tableau de variables aléatoires i.i.d. de loi μ sur \mathbb{N} , appelée loi de reproduction. Soit $(I_n, n \geq 1)$, une suite de variables aléatoires i.i.d. valeurs dans \mathbb{N} de loi commune notée μ^* . Le *processus de Galton-Watson avec immigration de lois* (μ, μ^*) issu de a , noté $(Z_n^*)_{n \in \mathbb{N}}$, peut être récursivement défini par

$$\begin{cases} Z_0^* &= a \\ Z_{n+1}^* &= \sum_{1 \leq i \leq Z_n^*} V_i^{(n)} + I_n \end{cases} \quad (34)$$

A chaque temps $n \geq 1$, un nombre aléatoire I_n d'immigrés (de loi μ^*) rejoint la population. Au temps suivant, ces individus se reproduisent, comme tous les autres, selon la loi μ . Il est clair que Z^* est une chaîne de Markov. La classe des processus de Galton-Watson avec immigration a été introduite introduite par Heathcote [66] et étudiée notamment dans Seneta [109] et Foster et Williamson [59].

Dans [70], Kawazu et Watanabe étudient les limites d'échelle des processus de Galton-Watson avec immigration et introduisent les *processus de branchement avec immigration continus en temps et espace*, désignés dans la suite par l'acronyme CBI pour *Continuous Branching process with Immigration*. La loi d'un CBI est caractérisée par un couple de

fonctions (ψ, ϕ) où ψ est de la forme de Lévy-Khintchine (13) et où ϕ est l'exposant de Laplace d'un subordonateur conservatif, c'est-à-dire que

$$\phi(\lambda) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda r}) \nu(dr), \quad (35)$$

avec $b \in \mathbb{R}_+$ et ν , mesure sur $(0, \infty)$ telle que $\int_0^\infty (1 \wedge r) \nu(dr) < \infty$. La fonction ψ est un mécanisme de branchement produisant "les naissances et les morts" dans la population, et les accroissements d'un subordonateur d'exposant de Laplace ϕ entre t et $t + dt$ mesurent la taille des populations immigrées arrivant entre les "générations" t et $t + dt$. Un CBI(ψ, ϕ) est alors défini comme un processus de Feller à valeurs dans \mathbb{R}_+ dont le générateur sur l'espace des fonctions de classe C^2 et tendant vers 0 s'exprime par

$$Lf(x) := \beta x f''(x) + (b - \alpha_0 x) f'(x) + x \int_0^\infty (f(x+z) - f(x) - z \mathbf{1}_{[0,1]}(z) f'(x)) \pi(dz) + \int_0^\infty (f(x+z) - f(x)) \nu(dz). \quad (36)$$

De manière alternative, un CBI(ψ, ϕ), noté $(Z_t, t \geq 0; \mathbf{P}_x^*, x \in \mathbb{R}_+)$, voit ses noyaux caractérisés par leur transformée de Laplace comme suit : pour tous $x, s, t, \lambda \in \mathbb{R}_+$,

$$\mathbf{P}_x^*(Z_0 = x) = 1 \quad \text{et} \quad \mathbf{E}_x^* \left[e^{-\lambda Z_{s+t}} \mid Z_s \right] = \exp \left(-Z_s u_t(\lambda) - \int_0^t \phi(u_s(\lambda)) ds \right), \quad (37)$$

où pour tout $\lambda \in \mathbb{R}_+$, la fonction $t \mapsto u_t(\lambda)$ satisfait l'équation (12).

Pour les travaux concernant les CBI (chapitre 1 de la thèse) nous ne faisons qu'une hypothèse faible sur les mécanismes de branchement ψ que l'on considère : on suppose que

$$\exists \lambda_0 \in \mathbb{R}_+ : \quad \psi(\lambda_0) > 0. \quad (38)$$

Autrement dit, $-\psi$ n'est pas l'exposant de Laplace d'un subordonateur : sous cette hypothèse un CB(ψ) n'est pas monotone, c'est-à-dire que des morts se produisent dans la population. Cette hypothèse (38) est équivalente à ce que le drift effectif \mathbf{d} , qui est défini par (17), soit strictement positif, ce qui découle immédiatement de (18). Sous l'hypothèse (38), on pose $\underline{\mathbf{a}} := b/\mathbf{d}$, avec la convention que $\underline{\mathbf{a}} = 0$ si $\mathbf{d} = \infty$ (dans les cas à variations non-bornées). On montre alors pour tout $x \in \mathbb{R}^d$, que \mathbf{P}_x -p.s. pour tout $t > 0$,

$$Z_t \geq x e^{-\mathbf{d}t} + \underline{\mathbf{a}} (1 - e^{-\mathbf{d}t}), \quad (39)$$

ce qui implique en particulier \mathbf{P}_x -p.s. $\liminf_{t \rightarrow \infty} Z_t \geq \underline{\mathbf{a}}$. Notons que cette inégalité ne présente un intérêt que dans les cas à variations bornées.

Notons $P_t(x, dy)$, les noyaux de transition d'un CB(ψ) et notons $P_t^*(x, dy)$ ceux d'un CBI(ψ, ϕ). On a la propriété de branchement suivante :

$$\forall t \in \mathbb{R}_+, \forall x, x' \in [0, \infty], \quad P_t^*(x + x', dy) = P_t^*(x, dy) * P_t(x', dy). \quad (40)$$

Autrement dit, si on considère un CBI(ψ, ϕ) issu de x et un CB(ψ) issu de x' indépendant du précédent, leur somme est un CBI(ψ, ϕ) issu de $x + x'$. Comme dans le cas du CB(ψ), une transformée de Lamperti (voir Caballero, Pérez-Garmendia et Uribe-Bravo [20]) permet de construire un CBI(ψ, ϕ) comme solution d'une EDS mettant en jeu un processus de Lévy $(X_t, t \geq 0)$ sans sauts négatifs de cumulants ψ et un subordonateur $(Y_t, t \geq 0)$ d'exposant ϕ . En effet, on peut montrer que l'unique solution forte de l'EDS (écrite sous une forme intégrée et implicite) :

$$Z_t = x + X_{\int_0^t Z_s ds} + Y_t, \quad (41)$$

est un $\text{CBI}(\psi, \phi)$ partant de x . On notera cependant que l'équation implicite (41) ne peut pas être inversée à la manière de (15).

Donnons quelques exemples de CBI.

- Si $\psi(\lambda) = 2\lambda^2$, et si $\phi(\lambda) = d\lambda$, alors le $\text{CBI}(\psi, \phi)$ est le carré de Bessel de dimension d . Pour $d \in \mathbb{N}^*$, ce processus s'obtient comme le carré de la norme d'un mouvement Brownien standard en dimension d (voir Revuz et Yor [105], chapitre XI).
- Si $\psi(\lambda) = \alpha\lambda$, et si ϕ est quelconque, alors le CBI correspondant est à branchement déterministe et il appartient à la classe des processus de Ornstein-Uhlenbeck généralisés, introduits par Sato et Yamazato [108].
- Si ψ est critique ou sous-critique, s'il satisfaisant (24) et si $\phi(\lambda) = \psi'(\lambda) - \psi'(0)$, alors, comme dans le cas discret, le CBI correspondant s'interprète comme le processus $\text{CB}(\psi)$ conditionné à ne pas s'éteindre, comme montré dans Li [91] (voir aussi Lambert [77] et Duquesne [35]).
- $\psi(\lambda) = d\lambda^\gamma, \phi(\lambda) = d'\lambda^{\gamma-1}$, $\gamma \in (1, 2)$. Le $\text{CBI}(\psi, \phi)$ associé est un processus auto-similaire, étudié notamment dans [98]. Dans le cas $d' = d\gamma$, il s'agit du CB stable d'indice γ , conditionné à la non-extinction (étudié dans Kyprianou et Pardo [74]), et cas particulier de l'exemple précédent).

Les premiers travaux sur le comportement asymptotique des CBI sont dus à Pinsky [103] qui montre notamment qu'un $\text{CBI}(\psi, \phi)$ de mécanisme de branchement ψ critique ou sous-critique possède une loi invariante si et seulement si

$$\int_0^1 \frac{\phi(u)}{\psi(u)} du < \infty. \quad (42)$$

Dans le cas contraire, pour tous $x, b \in \mathbb{R}_+$, on a $\lim_{t \rightarrow \infty} \mathbf{P}_x(X_t \leq b) = 0$. De plus, Keller-Ressel et Mijatović montrent dans [72] que lorsque la condition (42) est remplie, le $\text{CBI}(\psi, \phi)$ converge en loi lorsque $t \rightarrow \infty$ vers sa loi invariante, qui a pour support $[\underline{a}, \infty)$, où $\underline{a} = b/d$.

Dans la suite, nous nous intéresserons aux possibles *retours à l'état nul* des CBI. Afin d'éviter les cas triviaux concernant cette question, on suppose que ψ satisfait (24). Le $\text{CB}(\psi)$ correspondant est absorbé avec probabilité positive et on se préoccupe de savoir si un $\text{CBI}(\psi, \phi)$ peut atteindre 0. Plus précisément, on dit que l'état 0 est *polaire* si et seulement si pour tout $x \in (0, \infty)$

$$\mathbf{P}_x^*(\exists t \in (0, \infty) : Z_t = 0) = 0. \quad (43)$$

On notera ici que lorsque $\phi \neq 0$, l'état 0 n'est plus absorbant. Rappelons la définition (19) de q , la plus grande racine de ψ . On fixe une constante $\theta \in (q, \infty)$ (ne jouant qu'un rôle arbitraire). Foucart et Uribe Bravo [60] montrent que 0 est polaire si et seulement si

$$\int_\theta^\infty \frac{dz}{\psi(z)} \exp \left[\int_\theta^z \frac{\phi(u)}{\psi(u)} du \right] = \infty. \quad (44)$$

La preuve utilise de façon cruciale un critère de recouvrement de \mathbb{R}_+ par des intervalles aléatoires.

3 Généalogie des processus de branchement.

3.1 Processus des hauteurs.

On présente ici brièvement la construction du processus des hauteurs de Le Gall et Le Jan [86] qui permet de définir rigoureusement la généalogie des CB. Pour cela on fixe

un mécanisme de branchement ψ de forme (13). On suppose que ψ est critique ou sous-critique, c'est-à-dire que $\psi'(0+) \geq 0$. Cela montre notamment que $\psi'(0+) \neq -\infty$, et il est facile de montrer que cette condition entraîne que $\int_{[1,\infty)} r\pi(dr) < \infty$. On pose alors $\alpha := \psi'(0+) = \alpha_0 - \int_{[1,\infty)} r\pi(dr)$ et (13) se réécrit alors de la manière suivante

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\pi(dr), \quad \lambda \geq 0, \quad (45)$$

avec $\alpha, \beta \geq 0, \int_0^\infty (r \wedge r^2)\pi(dr) < \infty$.

On note $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ l'espace des fonctions càdlàg muni de la topologie de Skorokhod et on note $X = (X_t, t \geq 0)$ le processus canonique. On note \mathbb{P} la loi du processus de Lévy sans sauts négatifs issu de 0 d'exposant de Laplace ψ , c'est-à-dire que

$$\forall t, \lambda \in \mathbb{R}_+, \quad \mathbb{E}[\exp(-\lambda X_t)] = \exp(t\psi(\lambda)).$$

Le processus des hauteurs en temps et espace continus se définit d'une manière analogue à la formule du lemme 1.1 de la section 1.3. Plus précisément, on suppose que ψ satisfait la condition de Grey (24) : sous cette hypothèse, Le Gall et Le Jan [86] (voir également Duquesne et Le Gall [40], chapitre 1) ont montré l'existence d'un processus $H = (H_t, t \geq 0)$ continu tel que pour tout $t \in \mathbb{R}_+$ la limite suivante ait lieu en \mathbb{P} -probabilité

$$H_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s \leq I_t^s + \varepsilon\}} ds, \quad (46)$$

avec les notations

$$\forall t \geq s \geq 0, \quad I_t^s = \inf_{s \leq r \leq t} X_r,$$

pour l'infimum de X entre les temps s et t . Le processus H est appelé *processus des hauteurs de mécanisme de branchement* ψ . Le processus des hauteurs est donc une fonctionnelle de type temps local de X , que l'on notera parfois $H(X)$: le processus de Lévy $(X_t, t \geq 0)$ joue le rôle de la marche aléatoire de Lukasiewicz $(X_n, n \geq 0)$ du lemme 1.1 de la section 1.3.

On rappelle la définition (14) de l'exposant γ . Si on suppose $\gamma > 1$, alors le théorème 1.4.4. [40] permet d'affirmer le résultat de régularité suivant.

$$\text{Pour tout } c \in (0, \frac{\gamma-1}{\gamma}), \text{ } \mathbb{P}\text{-p.s. } H \text{ est } c\text{-localement höldérien.} \quad (47)$$

Théorème de Ray-Knight généralisé. Lorsque $\psi(\lambda) = \lambda^2$, le processus de Lévy sous-jacent $(X_t, t \geq 0)$ a même loi que $(\sqrt{2}B_t, t \geq 0)$, où $(B_t, t \geq 0)$ est un mouvement Brownien linéaire standard issu de 0 et on montre que $(H_t, t \geq 0)$ a même loi que le processus $\sqrt{2}B$, réfléchi en son infimum, qui a même loi qu'un mouvement Brownien réfléchi. Dans ce cas le théorème de Ray-Knight montre que les processus temps locaux ont même loi qu'une diffusion de Feller, correspondant aux $\text{CB}(\psi)$.

Ce résultat se généralise aux processus de hauteurs de mécanisme de branchement plus général de la manière suivante. Pour cela, on introduit d'abord les processus temps locaux de H : la proposition 1.3.3. de [40] montre qu'il existe un processus $(L_t^a, a, t \in \mathbb{R}_+)$ tel que

- (a) Pour tout $t \in \mathbb{R}_+$, \mathbb{P} -p.s. $a \mapsto L_t^a$, est càdlàg.
- (b) Pour tout $a \in \mathbb{R}_+$, \mathbb{P} -p.s. $t \mapsto L_t^a$ est croissant.

(c) Pour tout $a, t \in \mathbb{R}_+$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \frac{1}{\varepsilon} \int_0^s dr \mathbf{1}_{\{a < H_r \leq a + \varepsilon\}} - L_s^a \right| \right] = 0. \quad (48)$$

Le théorème de Ray-Knight généralisé, dû à Le Gall et Le Jan [86] (voir également le théorème 1.4.1 [40]), s'énonce alors comme suit. On rappelle la notation

$$x \in \mathbb{R}_+, \quad T_x = \inf\{t \geq 0 : X_t = -x\}, \quad (49)$$

qui est bien définie car puisque ψ est critique ou sous-critique X ne dérive pas vers $+\infty$. Alors,

$$\text{sous } \mathbb{P}, (L_{T_x}^a, a \geq 0) \text{ a même loi qu'un CB}(\psi) \text{ issu de } x. \quad (50)$$

Théorème-limite pour le processus des hauteurs. Le théorème de Ray-Knight précédent justifie l'introduction du processus des hauteurs comme généalogie des CB. Le fait que le processus H défini par (46) est l'analogue continu des processus des hauteurs discrets est également justifié par les théorèmes limites que nous rappelons maintenant du chapitre 2 de la monographie de Duquesne et Le Gall [40].

Pour tout entier $p \geq 1$, on fixe une loi de reproduction $\mu_p = (\mu_p(k), k \in \mathbb{N})$ que l'on suppose critique ou sous-critique :

$$\sum_{k \geq 0} \mu_p(k) \leq 1.$$

À μ_p , on associe une loi de saut ν_p sur \mathbb{Z} définie par

$$\forall k \in \{-1, 0, 1, \dots\}, \quad \nu_p(k) = \mu_p(k+1).$$

Pour tout $p \geq 1$, on note $(\mathcal{T}_k^p, k \geq 1)$, une forêt (une suite) d'arbres de Galton-Watson i.i.d. de loi de reproduction μ_p . On note $H^p = (H_n^p, n \geq 0)$ et $C^p = (C_t^p, t \geq 0)$ le processus des hauteurs discrets et le processus de contour associés à la forêt $(\mathcal{T}_k^p, k \geq 1)$; on note $X^p = (X_n^p, n \geq 0)$ la marche aléatoire qui est le chemin de Lukasiewicz associé à la forêt $(\mathcal{T}_k^p, k \geq 1)$, comme définis à la section 1.3. Le lemme 1.1 affirme que X^p est une marche aléatoire de loi de saut ν_p . On fixe un réel $x > 0$ et on note également

$$\forall n \geq 0, \quad Z_n^p = \sum_{1 \leq k \leq \lfloor px \rfloor} \#\{v \in \mathcal{T}_k^p : |v| = n\}$$

qui est le processus de Galton-Watson comptant le nombre d'individus par génération dans les $\lfloor px \rfloor$ premiers arbres de la forêt.

Soit $(u_p)_{p \in \mathbb{N}^*}$ une suite croissante d'entiers tendant vers l'infini. Une variante d'un théorème de Grimvall [62] (énoncée comme le Théorème 2.1.1 dans [40]) montre que les convergences en loi ci-dessous sont équivalentes :

$$\left(\frac{1}{p} Z_{\lfloor u_p t \rfloor}^p \right)_{t \geq 0} \xrightarrow[p \rightarrow \infty]{\text{(fd)}} (Z_t)_{t \geq 0} \quad \text{ssi} \quad \left(\frac{1}{p} X_{\lfloor p u_p t \rfloor}^p \right)_{t \geq 0} \xrightarrow[p \rightarrow \infty]{\text{(fd)}} (X_t)_{t \geq 0}, \quad (51)$$

où X est un processus de Lévy sans sauts négatifs issu de 0 et d'exposant de Laplace ψ critique ou sous-critique qui satisfait (24) et où Z est un CB(ψ) issu de x . Ici la convergence est une convergence en distribution pour les lois marginales de dimension finie. Rappelons que si l'une de ces deux convergences a lieu en distribution pour les lois marginales de dimension finie, alors les deux convergences ont lieu en loi sur l'espace $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

En plus de l'hypothèse (51) on suppose que pour tout $\delta > 0$,

$$\liminf_{p \rightarrow \infty} f_{\mu_p}^{\circ[\delta u_p]}(0)^p > 0, \quad (52)$$

où pour tout $n \in \mathbb{N}$, $f_{\mu_p}^{\circ n}$ est la $n^{\text{ème}}$ itérée de la fonction génératrice f_{μ_p} de μ_p . Cette hypothèse est équivalente à la tension des temps d'extinction des processus de Galton-Watson. Si on suppose (51) et (52) alors la convergence suivante à lieu en loi sur l'espace $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$

$$\left(\frac{1}{p} X_{[pu_p t]}^p, \frac{1}{u_p} H_{[pu_p t]}^p, \frac{1}{u_p} C_{2pu_p t}^p \right)_{t \geq 0} \xrightarrow[p \rightarrow \infty]{(d)} (X_t, H_t, H_t)_{t \geq 0} \quad (53)$$

où $H_t = H(X)_t$ est le processus des hauteurs associé à X par (46). De plus, et conjointement à (53), on a la convergence en loi suivante dans $\mathbb{D}(\mathbb{R}, \mathbb{R})$:

$$\left(\frac{1}{p} Z_{[u_p a]}^p \right)_{a \geq 0} \xrightarrow[p \rightarrow \infty]{(d)} (L_{T_x}^a)_{a \geq 0}, \quad (54)$$

où $(L_t^a, t \geq 0, a \geq 0)$ est la famille de temps locaux de H donnés par (48) et où T_x est le premier temps d'atteinte de $-x$ par le processus X comme défini par (49). On observe que le processus de contour et le processus des hauteurs convergent vers le même processus limite et tendent donc à se confondre.

Lorsque les μ_p sont toutes égales à une même mesure μ alors (51) implique que μ est dans le domaine d'attraction d'une loi stable d'indice $\gamma \in (1, 2]$ et (52) est automatiquement vérifiée et dans ce cas $\psi(\lambda) = \lambda^\gamma$. Le cas Brownien a été prouvé par Aldous dans [7] ; les convergences jointes (53) et (54) sont prouvées dans la monographie de Duquesne et Le Gall [40], au théorème 2.3.2 et au corollaire 2.5.1.

Excursions du processus des hauteurs. Informellement, comme pour le processus de contour dans le cadre discret, le processus des hauteurs H en temps continu code une forêt d'arbres continus où chaque excursion de H au dessus de 0 correspondant à un arbre de cette forêt. Nous rappelons ici comment définir la loi d'un seul arbre en rappelant la décomposition en excursion du processus des hauteurs.

On commence par brièvement rappeler la décomposition de la trajectoire du processus de Lévy X en excursions au dessus de son infimum. Pour cela on pose

$$\forall t \in \mathbb{R}_+, \quad I_t = \inf_{0 \leq s \leq t} X_s,$$

qui est le processus infimum de X . Le processus $-I$ est croissant et, puisque X est supposé sans sauts négatifs, il est également continu. On rappelle que l'on suppose (24) qui implique que X est à variations non-bornées, ce qui est équivalent à

$$\beta > 0 \quad \text{ou bien} \quad \int_{(0,1)} r \pi(dr) = \infty. \quad (55)$$

Par un résultat standard de théorie des fluctuations (voir Bertoin [9] chapitre VI), le processus réfléchi $(X_t - I_t, t \geq 0)$ est un processus de Markov fort pour lequel le point 0 est régulier et instantané ; par ailleurs, le processus infimum $(-I_t, t \geq 0)$ est un temps local en 0 de $X - I$ (voir Bertoin [9], chapitre VII).

On note alors N , la mesure d'excursion associée : c'est une mesure sigma-finie sur $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$: notons (g_i, d_i) , $i \in \mathcal{I}$, les intervalles d'excursion de X au dessus de son infimum, on pose alors $X_s^i = X_{(g_i+s) \wedge d_i} - X_{g_i}$, $s \in \mathbb{R}_+$. Alors la mesure ponctuelle $\sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, X^i)}$ est une mesure de Poisson sur $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ d'intensité $dtN(dX)$.

Par (46), on voit que la valeur de H_t ne dépend en fait de X qu'à travers l'excursion de $X - I$ qui enjambe t . Si on note $H_s^i = H_{(g_i+s)\wedge d_i}$, la mesure ponctuelle

$$\sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, H^i)} \quad (56)$$

est une mesure de Poisson sur $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R})$ d'intensité $dtN(dH)$, où $N(dH)$ désigne la "loi" de $H(X)$ sous $N(dX)$.

La durée de vie de X et de H sous N sont les mêmes et cette durée est notée σ ; on vérifie que :

$$N\text{-p.p. } H_0 = H_t = 0, t \geq \sigma \quad \text{et} \quad H_t > 0, t \in (0, \sigma). \quad (57)$$

Un résultat standard de théorie des fluctuations montre que

$$\forall \lambda \in (0, \infty), \quad \psi^{-1}(\lambda) = N(1 - e^{-\lambda\sigma}), \quad (58)$$

où ψ^{-1} la fonction réciproque de ψ .

Les temps locaux de H peuvent se définir sous N de la manière suivante : pour tout $b \in (0, \infty)$, la continuité de H implique que la quantité $N(\sup_{t \geq 0} H_t > b) \in (0, \infty)$; il existe un processus $(L_t^a, t \geq 0, a > 0)$ tel que $a \mapsto L_t^a$ est càdlàg, $t \mapsto L_t^a$ est croissant et pour tout $a \in (0, \infty)$,

$$\lim_{\varepsilon \rightarrow 0} N\left(\mathbf{1}_{\{\sup H > b\}} \sup_{0 \leq s \leq t \wedge \sigma} \left| \frac{1}{\varepsilon} \int_0^s dr \mathbf{1}_{\{a - \varepsilon < H_r \leq a\}} - L_s^a \right| \right) = 0. \quad (59)$$

Cette approximation implique également que

$$N\text{-a.e. } \forall g \in C(\mathbb{R}_+, \mathbb{R}_+), \forall t \in \mathbb{R}_+, \quad \int_0^t g(H_s) ds = \int_0^\infty g(a) L_t^a da. \quad (60)$$

Nous renvoyons à [40], section 1.3, pour plus de détails. On observe que $N\text{-a.e. } L_t^a = L_\sigma^a$ pour tout $t \geq \sigma$. Une conséquence du théorème de Ray-Knight rappelé au (50) est que

$$\forall a, \lambda \in (0, \infty), \quad N(1 - e^{-\lambda L_\sigma^a}) = u_a(\lambda), \quad (61)$$

où $a \mapsto u_a(\lambda)$ est la fonction définie par l'équation (12), qui se formule par l'équation intégrale (20). On montre également que

$$\forall a \in (0, \infty), \quad N(L_\sigma^a \neq 0) = N(\sup_{t \in [0, \sigma]} H_t > a) = v(a), \quad (62)$$

où $v(a) = \lim_{\lambda \rightarrow \infty} u_a(\lambda)$ satisfait (26).

Lorsque $\psi(\lambda) = \lambda^\gamma$, $\gamma \in (1, 2]$, le processus de Lévy X sous \mathbb{P} satisfait la propriété de scaling suivante : pour tout $c \in (0, \infty)$, $(c^{-\frac{1}{\gamma}} X_{ct}, t \geq 0)$ a même loi que X . Par (46), cela entraîne sous \mathbb{P} la propriété de scaling pour H suivante :

$$(c^{-\frac{\gamma-1}{\gamma}} H_{ct})_{t \geq 0} \stackrel{(\text{loi})}{=} (H_t)_{t \geq 0}. \quad (63)$$

Cette propriété de scaling entraîne que

$$(c^{-\frac{\gamma-1}{\gamma}} H_{ct})_{t \geq 0} \text{ sous } c^{-\frac{1}{\gamma}} N \stackrel{(\text{loi})}{=} (H_t)_{t \geq 0} \text{ sous } N. \quad (64)$$

Par (58), $N(\sigma \in dc) = Kc^{-\frac{1}{\gamma}-1} dc$, où $1/K = \gamma\Gamma(1 - \frac{1}{\gamma})$. La propriété de scaling (64) entraîne l'existence d'une famille de lois, $N(\cdot | \sigma = c)$, $c \in (0, \infty)$, telle que $c \mapsto N(\cdot | \sigma = c)$

est continue pour la convergence en loi sur $C(\mathbb{R}_+, \mathbb{R})$, telle que $N(\cdot | \sigma = c)$ -p.s. $\sigma = c$ et telle que

$$N = \int_0^\infty N(\cdot | \sigma = c) N(\sigma \in dc) .$$

D'autre part $(c^{\frac{\gamma-1}{\gamma}} H_{t/c})_{t \in [0, c]}$ sous $N(\cdot | \sigma = c)$ a la même loi que $(H_t)_{t \in [0, 1]}$ sous $N(\cdot | \sigma = 1)$.

La loi $N(\cdot | \sigma = 1)$ est la loi du processus des hauteurs γ -stable normalisée. Lorsque $\gamma = 2$, cette loi est celle de l'excursion Brownienne normalisée, qui est la limite de la fonction de contour des arbres uniformes à n sommets comme démontré par Aldous [7] (voir aussi Le Gall [80]). Plus précisément, notons \mathcal{T}_n , un arbre (enraciné ordonné et étiqueté) aléatoire à n sommets, de loi uniforme; la fonction de contour $(C_t(\mathcal{T}_n), t \in [0, 2(n-1)])$ de cet arbre a même loi qu'une marche aléatoire symétrique simple conditionnée à atteindre la valeur -1 pour la première fois au temps $2n-1$ (ce résultat est dû à Dwass [46]); on en déduit que

$$\left(\frac{1}{\sqrt{n}} C_{2(n-1)t}(\mathcal{T}_n)\right)_{t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{} (H_t)_{t \in [0, 1]} \text{ sous } N(\cdot | \sigma = 1), \quad (65)$$

la convergence ayant lieu en loi sur $C(\mathbb{R}_+, \mathbb{R})$. Ce résultat s'étend aux fonctions de contour (et des hauteurs) d'arbres de Galton-Watson conditionnés à avoir n sommets et dont la loi de reproduction admet un moment d'ordre 2 (voir Aldous [7]). Le cas de lois de reproduction dans le domaine d'attraction d'une loi γ -stable a été traité par Duquesne [34], le processus limite étant dans ces cas, le processus des hauteurs γ -stable normalisé.

3.2 Arbres de Lévy.

Une façon de généraliser la notion de graphe est de considérer des espaces métriques, la distance de l'espace généralisant la distance de graphe. Les espaces métriques correspondant aux arbres sont appelés les *arbres réels* qui se définissent comme suit.

Définition 3.1 Un espace métrique (T, d) est un arbre réel s'il satisfait les deux conditions suivantes.

- (i) Pour tous $\sigma_1, \sigma_2 \in T$, il existe une unique isométrie f de $[0, d(\sigma_1, \sigma_2)]$ dans T telle que $f(0) = \sigma_1$ et $f(d(\sigma_1, \sigma_2)) = \sigma_2$. La fonction f est donc une application géodésique et on note son image comme suit :

$$\llbracket \sigma_1, \sigma_2 \rrbracket = f([0, d(\sigma_1, \sigma_2)]) .$$

- (ii) Si $q : [0, 1] \rightarrow T$ est injective et continue, alors $q([0, 1]) = \llbracket q(0), q(1) \rrbracket$.

Si on distingue un point $\varrho \in T$, le triplet (T, d, ϱ) est appelé *arbre réel enraciné* et ϱ est la *racine* de T .

Informellement les arbres réels sont obtenus en collant sans créer de cycle des segments de la droite réelle. Parmi les espaces métriques, les arbres réels sont caractérisés par la *propriété des quatre points* qui s'énonce comme suit : soit un (T, d) un espace métrique connexe; alors (T, d) est un arbre réel si et seulement si pour tous $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in T$,

$$d(\sigma_1, \sigma_2) + d(\sigma_3, \sigma_4) \leq \max(d(\sigma_1, \sigma_3) + d(\sigma_2, \sigma_4); d(\sigma_1, \sigma_4) + d(\sigma_2, \sigma_3)) . \quad (66)$$

On renvoie à Evans [55] ou à Dress, Moulton et Terhalle [30] pour plus de détails sur cette caractérisation métrique.

On se restreint dans ce qui suit aux *arbres réels compacts enracinés*. L'ensemble de tous les arbres réels compacts enracinés peut être muni de la *distance de Gromov-Hausdorff*

pointée qui se définit de la manière suivante : soient (T_1, d_1, ϱ_1) et (T_2, d_2, ϱ_2) , deux arbres réels compacts enracinés ; on pose

$$d_{\text{GH}}(T_1, T_2) = \inf \max (d_{\text{Haus}}(\phi_1(T_1), \phi_2(T_2)) ; d(\phi_1(\varrho_1), \phi_2(\varrho_2))) , \quad (67)$$

où l'infimum est pris sur tous les $(\phi_1, \phi_2, (E, d))$ où (E, d) est un espace métrique, où $\phi_1 : T_1 \rightarrow E$ et $\phi_2 : T_2 \rightarrow E$ sont des isométries et où d_{Haus} est la distance de Hausdorff sur les sous-ensembles compacts de E . On dit que deux arbres réels enracinés (T, d, ϱ) et (T', d', ϱ') sont isométriques s'il existe une isométrie bijective envoyant T sur T' et ϱ sur ϱ' . Clairement la quantité $d_{\text{GH}}(T_1, T_2)$ ne dépend que des classes d'isométrie de T_1 et de T_2 . On note \mathbb{T} l'ensemble des classe d'isométries des arbres réels compacts enracinés. Gromov [63], Evans, Pitman et Winter [57] ont montré que d_{GH} est une distance sur \mathbb{T} et que $(\mathbb{T}, d_{\text{GH}})$ est un espace polonais, que l'on appelle l'*espace des arbres réels compacts enracinés*.

Comme expliqué dans la section 1.3, les arbres discrets sont caractérisés par leur fonction de contour. Ce point de vue s'étend aux arbres réels compacts de la manière suivante : soit $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, à support compact, non-identiquement nulle et telle que $h(0) = 0$. On pose $\sigma_h = \sup\{t \in \mathbb{R}_+ : h(t) > 0\}$ qui est un réel strictement positif bien défini. Pour tous $s, t \in [0, \sigma_h]$, on pose

$$b_h(s, t) = \inf_{r \in [s \wedge t, s \vee t]} h(r) \quad \text{et} \quad d_h(s, t) = h(s) + h(t) - 2b_h(s, t) . \quad (68)$$

Si on interprète h comme la fonction de contour d'un arbre, en s'inspirant de la formule discrète (7), la quantité $b_h(s, t)$ représente la distance à la racine de l'ancêtre commun des sommets visités aux instants s et t et $d_h(s, t)$ est la distance séparant ces sommets. On vérifie que d_h vérifie la propriété des quatre points dans le sens suivant : pour tous $s_1, s_2, s_3, s_4 \in [0, \sigma_h]$,

$$d_h(s_1, s_2) + d_h(s_3, s_4) \leq \max (d_h(s_1, s_3) + d_h(s_2, s_4) ; d_h(s_1, s_4) + d_h(s_2, s_3)) . \quad (69)$$

En prenant $s_3 = s_4$, cela montre tout d'abord que d_h satisfait l'inégalité triangulaire. Cela montre que d_h est une pseudo-distance sur $[0, \sigma_h]$. On note alors $s \sim_h t$ la relation $d_h(s, t) = 0$ et on remarque que \sim_h est une relation d'équivalence. Des résultats élémentaires sur les espaces métriques montre que que d_h induit une distance sur l'espace quotient

$$\mathcal{T}_h = [0, \sigma_h] / \sim_h,$$

distance que l'on continue de noter d_h par souci de simplicité. On note

$$p_h : [0, \sigma_h] \rightarrow \mathcal{T}_h$$

la *projection canonique* du passage au quotient. Comme h est une fonction continue, on vérifie que p_h également et l'espace métrique (\mathcal{T}_h, d_h) est donc un espace métrique compact connexe. Il est clair que (69) implique que (\mathcal{T}_h, d_h) satisfait la propriétés des quatre points (66) et donc que (\mathcal{T}_h, d_h) est un arbre réel. Enfin on pose $\varrho_h = p_h(0) = p_h(\sigma_h)$, que l'on prend comme la racine de l'arbre réel (\mathcal{T}_h, d_h) . L'arbre réel compact enraciné $(\mathcal{T}_h, d_h, \varrho_h)$ est appelé l'*arbre codé par h*.

On rappelle ensuite le lemme 2.3 dans Duquesne et Le Gall [41], qui montre que pour toutes fonctions $h_1, h_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ à support compact telles que $h_1(0) = h_2(0) = 0$, on a

$$d_{\text{GH}}(\mathcal{T}_{h_1}, \mathcal{T}_{h_2}) \leq 4 \|h_1 - h_2\|_{\infty} \quad (70)$$

où $\|h_1 - h_2\|_\infty$ est la norme uniforme sur \mathbb{R}_+ de $h_1 - h_2$. Cela permet de définir le ψ -arbre de Lévy comme suit. On suppose que ψ est un mécanisme de branchement critique ou sous-critique de la forme (45) satisfaisant (24); on considère le ψ -processus des hauteurs H sous sa mesure d'excursion N , comme défini à la section 3.1. On rappelle que σ est la durée de l'excursion qui est telle que l'on ait (57). On définit alors le ψ -arbre de Lévy comme l'arbre compact enraciné

$$(\mathcal{T}, d, \varrho) = (\mathcal{T}_H, d_H, \varrho_H), \text{ avec } H \text{ sous } N. \quad (71)$$

Pour simplifier on note également $p := p_H$, la projection canonique de $[0, \sigma]$ sur \mathcal{T} . Le ψ -arbre de Lévy est donc l'arbre codé par le processus des hauteurs sous sa mesure d'excursion. L'inégalité (70) montre que la classe d'isométrie de $(\mathcal{T}, d, \varrho)$ est une variable aléatoire mesurable de $(\mathbb{T}, d_{\text{GH}})$. Lorsque $\psi(\lambda) = \lambda^\gamma, \gamma \in (1, 2]$, on parle d'*arbre stable* d'indice γ . Dans le cas particulier $\gamma = 2$, l'arbre de Lévy est appelé *arbre Brownien* (voir les commentaires relatifs à (65)).

Si la fonction de codage est le processus $H^x := (H_{t \wedge T_x}, t \geq 0)$, où T_x est le premier temps d'atteinte de la valeur $-x$ par le processus de Lévy X , comme défini dans la section 3.1, alors (53) implique la convergence suivante en loi sur $(\mathbb{T}, d_{\text{GH}})$

$$(\mathcal{T}_x^p, \frac{1}{u_p} d_p, \varrho_p) \xrightarrow{p \rightarrow \infty} (\mathcal{T}_{H^x}, d_{H^x}, \varrho_{H^x})$$

où $(\mathcal{T}_x^p, \frac{1}{u_p} d_p, \varrho_p)$ est l'espace métrique obtenu en identifiant les racines des $[px]$ arbres de Galton-Watson de loi de reproduction μ_p , en munissant le graphe obtenu de la distance de graphe d_p multipliée par le facteur de normalisation $1/u_p$. Pour plus de détails sur la construction des arbres de Lévy et leurs propriétés en tant que variables à valeurs dans l'espace des arbres $(\mathbb{T}, d_{\text{GH}})$ nous renvoyons aux articles de Evans, Pitman, Winter [57, 56], Duquesne et Le Gall [41], Duquesne et Winkel [45], Weill [115], et Abraham et Delmas [3].

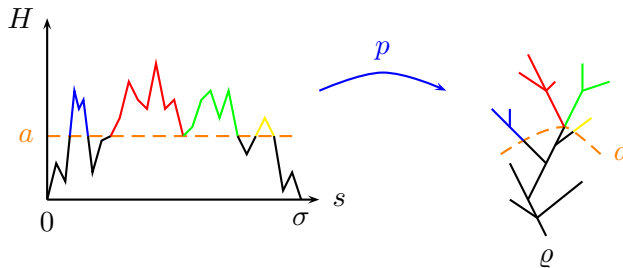


FIGURE 2 – Codage de l'arbre de Lévy par le processus des hauteurs H .

Mesures temps local et mesures de masse de l'arbre de Lévy. On rappelle que $p : [0, \sigma] \rightarrow \mathcal{T}$ est la projection canonique. La mesure temps local dL_t^a induite par p est une mesure ℓ^a portée par l'ensemble de niveau a . Plus précisément on pose

$$\mathcal{T}(a) = \{\sigma \in \mathcal{T} : d(\varrho, \sigma) = a\} \quad (72)$$

et pour toute fonction mesurable $f : \mathcal{T} \rightarrow \mathbb{R}_+$, on a

$$\langle \ell^a, f \rangle = \int_0^\sigma f(p(t)) dL_t^a. \quad (73)$$

La mesure ℓ^a , qui est finie et de masse totale $\ell^a(\mathcal{T}) = \ell^a(\mathcal{T}(a)) = L_\sigma^a$, est appelée *mesure de temps local au niveau a* . On prend $\ell^0 = 0$ et il est montré dans l'article de Duquesne et

Le Gall [41] au théorème 4.3, qu'il est possible de trouver une modification des mesures $(\ell^a, a \geq 0)$ telles que

$$N\text{-p.p. } a \in \mathbb{R}_+ \mapsto \ell^a \text{ est càdlàg pour la convergence faible sur } (\mathcal{T}, d). \quad (74)$$

Nous rappelons ici des propriétés de régularité du support topologique des mesures ℓ^a qui sont prouvées dans le théorème 4.4 de l'article de Duquesne et Le Gall [41]. On dit que $\sigma \in \mathcal{T}$ est un *point d'extinction* s'il existe $\varepsilon > 0$ tel que

$$d(\rho, \sigma) = \sup_{\sigma' \in B(\sigma, \varepsilon)} d(\rho, \sigma')$$

où $B(\sigma, \varepsilon)$ désigne la boule ouverte de centre σ et de rayon ε . On note \mathcal{E} l'ensemble des points d'extinctions de \mathcal{T} . Il est facile de vérifier qu'il y a une bijection entre les points d'extinction de \mathcal{T} et les maxima locaux de H : si $t \in \mathbb{R}_+$, tel que H admet un maximum local en t , alors $p(t) \in \mathcal{E}$ et réciproquement, à tout $\sigma \in \mathcal{E}$, il existe un unique temps $t \in \mathbb{R}_+$ tel que $p(t) = \sigma$ et H admet un maximum local en t . Si on note $\Gamma = \max_{\sigma \in \mathcal{T}} d(\rho, \sigma)$, la *hauteur totale de \mathcal{T}* , alors le théorème 4.4 de [41] affirme que N -p.p. les assertions suivantes sont vérifiées.

- (i) \mathcal{E} est infini dénombrable et pour tout $a \in (0, \Gamma)$, $\mathcal{E} \cap \mathcal{T}(a)$ a au plus un seul point.
- (ii) Si $a \in (0, \Gamma)$ est tel que $\mathcal{E} \cap \mathcal{T}(a) = \emptyset$, alors le support topologique de ℓ^a est $\mathcal{T}(a)$.
- (iii) Si $a \in (0, \Gamma)$ est tel que $\mathcal{E} \cap \mathcal{T}(a) = \{\sigma_a\}$ alors le support topologique de ℓ^a est $\mathcal{T}(a) \setminus \{\sigma_a\}$.

On introduit également la *mesure de masse de \mathcal{T}* , notée \mathbf{m} , qui est définie comme la mesure image de la mesure de Lebesgue sur $[0, \sigma]$ par la projection canonique p : pour toute fonction mesurable $f : \mathcal{T} \rightarrow \mathbb{R}_+$, on a

$$\langle \mathbf{m}, f \rangle = \int_0^\sigma f(p(t)) dt. \quad (75)$$

L'identité (60) implique alors facilement que

$$\mathbf{m} = \int_0^\infty \ell^a da. \quad (76)$$

La mesure de masse \mathbf{m} a une masse totale égale à σ et joue un rôle important dans l'étude des propriétés géométriques des arbres de Lévy car elle est en un certain sens la mesure la mieux répartie sur l'arbre. La construction que nous avons rappelée ici utilise le codage de \mathcal{T} par H mais il est possible de donner une définition des mesures de temps local ℓ^a qui soit intrinsèque, c'est-à-dire qui ne dépende que de la distance d sur \mathcal{T} et (76) permet alors d'obtenir également une définition de \mathbf{m} qui ne dépende que de la métrique de \mathcal{T} et non du codage par H : voir l'article de Duquesne et Le Gall [41], théorème 4.2.

Rappelons quelques propriétés géométriques des arbres de Lévy. Pour tout $\sigma \in \mathcal{T}$, le *degré* σ , noté $n(\sigma)$, est le nombre (peut-être infini) de composantes connexes de $\mathcal{T} \setminus \{\sigma\}$. Le point σ est une *feuille* de \mathcal{T} si $n(\sigma) = 1$, c'est-à-dire si $\mathcal{T} \setminus \{\sigma\}$ est connexe. C'est un point de branchement si $n(\sigma) \geq 3$: le point de branchement est dit binaire si $n(\sigma) = 3$ et infini si $n(\sigma) = \infty$. On note

$$\mathcal{L} = \{\sigma \in \mathcal{T} : n(\sigma) = 1\},$$

l'ensemble des feuilles de \mathcal{T} . Alors on peut montrer que

$$N\text{-p.p. } \mathbf{m} \text{ est diffuse, } \mathbf{m}(\mathcal{T} \setminus \mathcal{L}) = 0 \text{ et } \mathcal{L} \text{ est dense dans } \mathcal{T}. \quad (77)$$

Cela montre que l'arbre mesuré $(\mathcal{T}, d, \rho, \mathbf{m})$ est un *continuum tree* au sens d'Aldous [5]. Le théorème 4.6 de [41] démontre également les faits suivants.

- (i) N -p.p. pour tout $\sigma \in \mathcal{T}$, on a $n(\sigma) \in \{1, 2, 3, \infty\}$.
- (ii) L'ensemble $\{\sigma \in \mathcal{T} : n(\sigma) = 3\}$ des points de branchement binaires est vide N -p.p. si β dans (45) est nul et si $\beta > 0$, l'ensemble des points de branchement binaires est N -p.p. dénombrable dense dans \mathcal{T} .
- (iii) L'ensemble $\{\sigma \in \mathcal{T} : n(\sigma) = \infty\}$ des points de branchement infinis a une N -mesure strictement positive d'être non-vide si et seulement si la mesure de Lévy π de ψ est non-nulle. De plus si $\langle \pi, \mathbf{1} \rangle = \infty$, alors l'ensemble des points de branchement infini est un ensemble N -p.p. dénombrable dense dans \mathcal{T} . Si $\langle \pi, \mathbf{1} \rangle < \infty$, alors cet ensemble est N -p.p. fini.

On voit que les points de branchement infinis de \mathcal{T} sont liés aux sauts éventuels du processus de Lévy X servant à définir H et donc \mathcal{T} . Ce lien est explicité dans le théorème 4.7 de l'article de Duquesne et Le Gall [41], en lien avec les discontinuité des mesures ℓ^a . Plus précisément, on rappelle que Γ désigne la hauteur totale. Le théorème 4.7 de [41] montre que N -p.p. les assertions suivantes sont vérifiées.

- (i) $\{p(t); t \in \mathbb{R}_+ : \Delta X_t > 0\} = \{\sigma \in \mathcal{T} : n(\sigma) = \infty\}$.
- (ii) Pour tout $a \in (0, \Gamma)$, $\{\sigma \in \mathcal{T} : n(\sigma) = \infty\} \cap \mathcal{T}(a)$ compte au plus un point.
- (iii) Si $a \in (0, \Gamma)$ est tel que $\{\sigma \in \mathcal{T} : n(\sigma) = \infty\} \cap \mathcal{T}(a) = \emptyset$, alors $a \mapsto \ell^a$ est continu en a .
- (iv) Si $a \in (0, \Gamma)$ est tel que $\{\sigma \in \mathcal{T} : n(\sigma) = \infty\} \cap \mathcal{T}(a) = \{\gamma_a\}$, alors il existe $t \in \mathbb{R}_+$ tel que

$$p(t) = \gamma_a, \quad \Delta X_t > 0 \quad \text{et} \quad \ell^a = \ell^{a-} + \Delta X_t \delta_{\gamma_a}. \quad (78)$$

Le poids ΔX_t dans (78) utilise le codage de \mathcal{T} par H et X : une définition intrinsèque en est donnée dans [41] et la quantité ΔX_t , notée λ_{γ_a} est appelée le *temps local au point de branchement infini* γ_a .

Ces temps locaux aux points de branchement infinis interviennent dans plusieurs processus de fragmentation et d'élagage des arbres de Lévy : lorsque $\psi(\lambda) = \lambda^\gamma$, avec $\gamma \in (1, 2)$ et lorsque \mathcal{T} est l'arbre codé par H sous $N(\cdot | \sigma = 1)$, Miermont dans [93] a considéré une fragmentation aux points de branchement infini qui produit une fragmentation auto-similaire dont il a calculé les paramètres. Ce résultat a été généralisé aux arbres de Lévy dans plusieurs articles d'Abraham, Delmas et Voisin [1, 2, 4] traitant également de procédures générales d'élagage et de fragmentation impliquant ces temps locaux aux points de branchement infini.

L'arbre stable ($\psi(\lambda) = \lambda^\gamma, \gamma \in (1, 2]$) possède une propriété d'invariance par réenracinement uniforme (selon la mesure de masse \mathbf{m}); voir Aldous [6] pour le cas du CRT, et pour les cas stables Duquesne et Le Gall [41, 43], ou Haas, Pitman et Winkel [64].

3.3 Serpent de Lévy.

Le serpent Brownien est un processus à valeurs dans les chemins continus introduit par Le Gall [80, 82] permettant de faciliter l'étude du super-processus de Dawson-Watanabe : il s'agit d'une exploration markovienne des trajectoires historiques des individus de la population évoluant comme une diffusion de Feller. Cette construction a été généralisée dans la monographie de Duquesne et Le Gall [40] pour des mécanismes de branchement ψ plus généraux que le cas quadratique, le processus s'appelant dans le cas général *serpent de Lévy*. Dans cette section nous rappelons brièvement la définition du serpent de Lévy lorsque le déplacement spatial est un mouvement brownien et lorsque un mécanisme de branchement ψ de la forme (45) satisfait $\gamma > 1$. Rappelons que cette condition amène le

résultat de régularité (47) pour le processus des hauteurs associé à ψ et qu'elle implique (33), c'est à dire que la trace du SBM associé à ψ est bornée.

Dans ce qui suit, (Ω, \mathcal{F}) désigne un espace mesurable auxiliaire sur lequel est défini un processus continu $\xi = (\xi_t, t \geq 0)$ à valeurs dans \mathbb{R}^d ; pour tout $x \in \mathbb{R}^d$, on note \mathbf{P}_x une mesure de probabilité sur (Ω, \mathcal{F}) telle que sous \mathbf{P}_x , ξ est distribué comme un mouvement brownien standard en dimension d issu de x .

On note \mathcal{W} l'ensemble des trajectoires continues stoppées à valeurs dans \mathbb{R}^d , c'est-à-dire que

$$\mathcal{W} = \left\{ (w, \zeta_w); \zeta_w \in [0, \infty), w \in C([0, \zeta_w], \mathbb{R}^d) \right\}. \quad (79)$$

Ici, ζ_w désigne le *temps de vie de la trajectoire* w . Plutôt que de noter $(w, \zeta_w) \in \mathcal{W}$, on écrit simplement $w \in \mathcal{W}$. D'autre part, on plonge \mathbb{R}^d dans \mathcal{W} en considérant que x désigne la trajectoire $w \in \mathcal{W}$ telle que $\zeta_w = x$ et $w(0) = x$. On munit \mathcal{W} de la distance suivante

$$\forall w, w' \in \mathcal{W}, \quad \mathbf{d}(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{t \geq 0} \|w(t \wedge \zeta_w) - w'(t \wedge \zeta_{w'})\|,$$

où $\|\cdot\|$ désigne la norme euclidienne sur \mathbb{R}^d . Alors, $(\mathcal{W}, \mathbf{d})$ est un espace polonais.

On introduit ensuite le noyau de transition du serpent. Pour cela on fixe $w \in \mathcal{W}$, $a \in [0, \zeta_w]$ et $b \in [a, \infty)$. On définit sur \mathcal{W} la loi de probabilité $R_{a,b}(w, \cdot)$ comme suit.

- (i) $R_{a,b}(w, dw')$ -p.s., $w'(t) = w(t)$, $\forall t \in [0, a]$;
- (ii) $R_{a,b}(w, dw')$ -p.s., $\zeta_{w'} = b$.
- (iii) La loi de $(w'(a+t), 0 \leq t \leq b-a)$ sous $R_{a,b}(w, dw')$ est la loi de $(\xi_t, 0 \leq t \leq b-a)$ sous $\mathbf{P}_{w(a)}$.

En particulier, pour tout $w \in \mathcal{W}$, $R_{0,b}(w, \cdot)$ est la loi de $(\xi_t, 0 \leq t \leq b)$ sous $\mathbf{P}_{w(0)}$ et $R_{0,0}(w, \cdot) = \delta_{w(0)}(\cdot)$.

On munit $\mathcal{W}^{\mathbb{R}^+}$ de la tribu produit et on note $(W_t, t \geq 0)$ le processus canonique. On fixe $x \in \mathbb{R}^d$ et $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ telle que $h(0) = 0$, qui est appelé le *processus des temps de vie*. Pour tous $t \geq s \geq 0$, on note $b_h(s, t) = \min\{h(r), r \in [s, t]\}$. Le théorème d'extension de Kolmogorov implique qu'il existe une unique loi de probabilité Q_x^h sur $\mathcal{W}^{\mathbb{R}^+}$ telle que pour tous $0 \leq s_1 < \dots < s_n$,

$$\begin{aligned} Q_x^h[W_{s_1} \in A_1, \dots, W_{s_n} \in A_n] \\ = \int_{A_1 \times \dots \times A_n} R_{b_h(0, s_1), h(s_1)}(x, dw_1) \dots R_{b_h(s_{n-1}, s_n), h(s_n)}(w_{n-1}, dw_n), \end{aligned} \quad (80)$$

Pour tout $t \in \mathbb{R}_+$, Q_x^h -p.s. $\zeta_{W_t} = h(t)$. La figure 3 illustre la construction de Q_x^h lorsque h est élémentaire.

Rappelons ensuite quelques propriétés de régularité de W sous Q_x^h . Pour cela, on fait l'hypothèse suivante.

Le processus des temps de vie h est localement höldérien d'exposant $r \in (0, 1]$.

Fixons $p \in (1, \infty)$ and $t_0 \in (0, \infty)$. La dernière inégalité de la preuve de la proposition 4.4.1 de [40] p. 120 implique que

$$\forall s, t \in [0, t_0], \quad Q_x^h[\mathbf{d}(W_s, W_t)^p] \leq C |t-s|^{pr/2}. \quad (81)$$

où C est une constante dépendant de t_0 , de p et de la constante de Hölder de h sur $[0, t_0]$. Si $p > 2/r$, le critère de continuité de Kolmogorov montre qu'il existe une modification

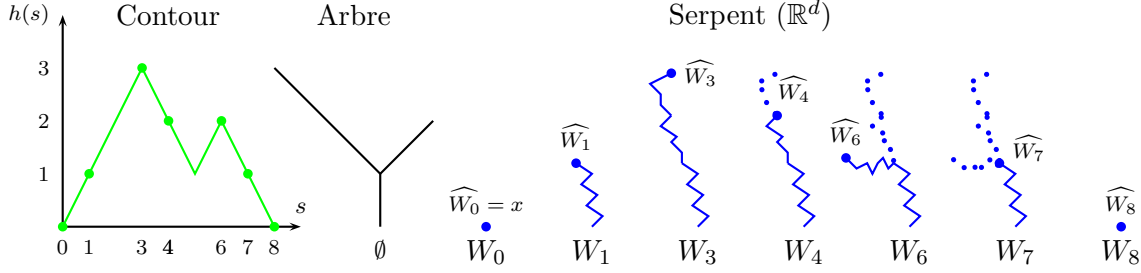


FIGURE 3 – La fonction h croît sur $[0, 3]$: au temps 3, le chemin W_3 est une trajectoire Brownienne de temps de vie $h(3) = 3$. Entre les temps 3 et 5, la fonction h décroît.

continue de W . On commet un léger abus de notation en continuant de noter Q_x^h la loi de cette modification sur $C(\mathbb{R}_+, \mathcal{W})$ et de noter $(W_t)_{t \geq 0}$ le processus canonique sur $C(\mathbb{R}_+, \mathcal{W})$. La loi Q_x^h est appelée loi du *serpent brownien issu de x et de processus de temps de vie déterministe h* .

On rappelle ensuite quelques résultats concernant l'extrémité du processus W sous Q_x^h . La continuité entraîne que Q_x^h -p.s. pour tout $t \in \mathbb{R}_+$, $\zeta_{W_t} = h(t)$. On pose alors

$$\forall t \in \mathbb{R}_+, \quad \widehat{W}_t = W_t(h(t)) .$$

Le processus $(\widehat{W}_t, t \geq 0)$ est appelé *le processus des extrémités du serpent*. C'est un processus continu à valeurs dans \mathbb{R}^d . On constate aussi que sous Q_x^h , le processus des extrémités du serpent est un processus Gaussien dont la covariance est caractérisée par

$$\forall s, t \in \mathbb{R}_+, \quad Q_x^h \left(\|\widehat{W}_t - \widehat{W}_s\|^2 \right) = h(t) + h(s) - 2 \inf_{s \wedge t \leq u \leq s \vee t} h(u) = d_h(t, s) . \quad (82)$$

On rappelle (68) définissant la distance d_h associée de l'arbre codé par la fonction h . L'égalité (82) permet alors de voir le processus des extrémités du serpent comme un mouvement Brownien indexé par l'arbre codé par h .

Le serpent de Lévy de déplacement spatial Brownien se définit comme le serpent dont le processus des temps de vie est le ψ -processus des hauteurs H . On rappelle qu'ici on suppose que $\gamma > 1$, ce qui implique que H est N -p.p. localement höldérien, ce qui permet d'appliquer à $h = H$ la construction du serpent. On définit alors la *mesure d'excursion du serpent de Lévy issue de x* comme

$$\mathbb{N}_x = \int_{C(\mathbb{R}_+, \mathbb{R})} N(H \in dh) Q_x^h , \quad (83)$$

qui est une mesure sigma finie sur l'espace polonais \mathcal{W} . On rappelle que σ désigne la durée de l'excursion de H sous N . La construction précédente implique que \mathbb{N}_x -a.e. $t \in [0, \sigma]$, $\zeta_{W_t} = H_t$. De plus, sous \mathbb{N}_x , la loi conditionnelle de W sachant H est Q_x^H . Nous renvoyons au chapitre 4 de [40] pour plus de détails. Enfin, si $\gamma > 1$, (47) combiné à (81) et au théorème de continuité de Kolmogorov, impliquent que

$$\text{pour tout } c \in (0, \frac{\gamma-1}{2\gamma}), \quad (\widehat{W}_t, t \geq 0) \text{ est localement } c\text{-höldérien.} \quad (84)$$

Les temps locaux de H , notés $(L_t^a, t \geq 0, a > 0)$ sous N sont bien définis sous \mathbb{N}_x également par (59) où \mathbb{N}_x se substitue à N . Pour tout $a \in (0, \infty)$ on définit la mesure aléatoire sur \mathbb{R}^d , notée $\mathcal{Z}_a = \mathcal{Z}_a(W)$, de la manière suivante : pour toute fonction mesurable $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$,

$$\langle \mathcal{Z}_a, f \rangle = \int_0^\sigma dL_s^a f(\widehat{W}_s) . \quad (85)$$

On introduit également la *mesure d'occupation de W* , notée $\mathcal{M} = \mathcal{M}_W$. L'identité (60) implique facilement que pour toute fonction mesurable $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$,

$$\langle \mathcal{M}, f \rangle = \int_0^\sigma f(\widehat{W}_t) dt = \int_0^\infty \langle Z_a, f \rangle da. \quad (86)$$

La *trace totale de W* , notée \mathcal{R}_W est quant à elle définie par

$$\mathcal{R}_W = \left\{ \widehat{W}_t; t \in [0, \sigma] \right\} \quad (87)$$

Le lien entre les ψ -super mouvements Browniens et le ψ -serpent de Lévy est donné par le théorème 4.2.1 de la monographie [40], que l'on rappelle dans l'énoncé suivant. Soit $\mu \in M_f(\mathbb{R}^d)$ et soit

$$\sum_{j \in \mathcal{J}} \delta_{(x_j, W^j)}$$

une mesure ponctuelle de Poisson d'intensité $\mu(dx)\mathbb{N}_x(dW)$. On pose $Z_0 = \mu$ et pour tout $a \in (0, \infty)$, on pose

$$Z_a = \sum_{j \in \mathcal{J}} Z_a(W^j).$$

Alors, \mathbf{P} -p.s. le processus $(Z_a, a \geq 0)$ est un super-mouvement Brownien càdlàg de mécanisme de branchement ψ issu de μ . De plus, si \mathbf{R} et \mathbf{M} sont déduits de Z par (32) et (31), on a

$$\mathbf{R} \cup \{x_j, j \in \mathcal{J}\} = \bigcup_{j \in \mathcal{J}} \mathcal{R}_{W^j} \quad \text{et} \quad \mathbf{M} = \sum_{j \in \mathcal{J}} \mathcal{M}_{W^j}. \quad (88)$$

Le dernier point (88) ne fait pas partie de l'énoncé du théorème 4.2.1 de [40] mais en est une conséquence simple.

4 Géométrie fractale des processus de branchement.

4.1 Dimensions fractales et mesures géométriques.

Dans cette section, on rappelle les définitions des mesures de Hausdorff et de packing sur un espace métrique (E, d) compact.

Les mesures de Hausdorff et les mesures de packing sont construites à partir de la métrique d et d'une fonction de jauge $g : [0, r_0) \rightarrow \mathbb{R}_+$ qui satisfait les propriétés de régularité suivantes

- (a) g est continue, strictement croissante et $g(0) = 0$.
- (b) g satisfait une condition de doublement :

$$\exists C \in (1, \infty) \quad \forall r \in (0, r_0/2) \quad g(2r) \leq Cg(r). \quad (89)$$

Mesures de Hausdorff. Pour toute partie $A \subset E$, et tout $\varepsilon \in (0, r_0)$, on pose

$$\mathcal{H}_g^{(\varepsilon)}(A) = \inf \left\{ \sum_{n \in \mathbb{N}} g(\text{diam}(A_n)), A_n \subset E, \text{diam}(A_n) \leq \varepsilon, A \subset \bigcup_{n \in \mathbb{N}} A_n \right\}, \quad (90)$$

puis

$$\mathcal{H}_g(A) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_g^{(\varepsilon)}(A).$$

\mathcal{H}_g est la g -mesure de Hausdorff sur E . C'est une mesure extérieure Borel régulière, diffuse. Nous renvoyons à Rogers [106] pour des détails. Dans les preuves du chapitre 2, nous utiliserons le lemme suivant permettant de comparer une mesure μ sur E à la g -mesure de Hausdorff. Pour $x \in E$, $r \in (0, \infty)$, on note $B(x, r)$ la boule ouverte de rayon r centrée en x .

Lemme 4.1 ([107]) *Soit E un espace métrique compact et g une jauge régulière satisfaisant la condition de doublement (89) pour une constante $C \in (1, \infty)$. Soit μ , une mesure borélienne finie sur E et soit A , un borélien de E .*

- (i) *Si pour tout $x \in A$, on a $\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{g(r)} < \kappa$, alors $\mu(A) \leq \kappa \mathcal{H}_g(A)$.*
- (ii) *Si pour tout $x \in A$, on a $\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{g(r)} > \kappa$, alors $\mu(A) \geq C^{-2} \kappa \mathcal{H}_g(A)$.*

Ce lemme a été démontré par Rogers et Taylor [107] dans le cas des espaces euclidiens. Nous renvoyons à Edgar [53] pour le cas des espaces métriques généraux. La fonction $x \mapsto \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{g(r)}$ est appelée *densité locale supérieure de μ par rapport à g* . On notera que dans la majoration (ii) du Lemme 4.1 apparaît la constante de doublement C . Mentionnons que même lorsque la densité locale supérieure d'une mesure μ est constante sur A égale à κ , on n'en déduit pas nécessairement que $\mu(A) = \kappa \mathcal{H}_g(A)$. Dans [31] (chapitre 2 de cette thèse) la géométrie particulière des niveaux d'un arbre réel est mise à profit pour énoncer un lemme de comparaison où n'apparaît plus la constante de doublement.

Mesures de packing. Les mesures de packing ont été introduites par Taylor et Tricot [112] pour étudier la courbe Brownienne en dimension plus grande que 3. Rappelons brièvement leur définition.

Soit $A \subset E$, non-vide. Soit $\varepsilon \in (0, \infty)$. Un ε -packing de A est une collection finie de boules fermées $\overline{B}(x_k, r_k)$, $1 \leq k \leq n$ deux à deux disjointes telles que les centres x_k appartiennent à A et les rayons r_k sont tous inférieurs à ε .

Soit une jauge g régulière satisfaisant la condition de doublement (89). On définit alors

$$\mathcal{P}_g^{(\varepsilon)}(A) = \sup \left\{ \sum_{1 \leq k \leq n} g(r_k) ; \overline{B}(x_k, r_k), 1 \leq k \leq n, \varepsilon\text{-packing de } A \right\},$$

et

$$\mathcal{P}_g^*(A) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{P}_g^{(\varepsilon)}(A) \in [0, \infty],$$

qui est la g pré-mesure de packing de A . La g -mesure de packing de A est alors donnée par

$$\mathcal{P}_g(A) = \inf \left\{ \sum_{n \geq 0} \mathcal{P}_g^*(A_n) ; A \subset \bigcup_{n \geq 0} A_n \right\}.$$

\mathcal{P}_g est une mesure extérieure Borel-régulière, diffuse. Le lemme ci-dessous permet de comparer une mesure μ sur E à la g -mesure de packing.

Lemme 4.2 ([112]) *Soit E un espace métrique compact et g une jauge régulière satisfaisant la condition de doublement (89) pour une constante $C \in (1, \infty)$. Soit μ , une mesure borélienne finie sur E et soit A , un borélien de E .*

- (i) *Si pour tout $x \in A$, on a $\liminf_{r \rightarrow 0} \frac{\mu(B(x, r))}{g(r)} < \kappa$, alors $\mu(A) \leq C^2 \kappa \mathcal{P}_g(A)$.*
- (ii) *Si pour tout $x \in A$, on a $\liminf_{r \rightarrow 0} \frac{\mu(B(x, r))}{g(r)} > \kappa$, alors $\mu(A) \geq \kappa \mathcal{P}_g(A)$.*

Ce lemme a été prouvé par Taylor et Tricot [112] dans le cas des espaces euclidiens. Nous renvoyons à Edgar [53] pour le cas des espaces métriques généraux. La fonction $x \mapsto \liminf_{r \rightarrow 0} \frac{\mu(B(x,r))}{g(r)}$ est appelée *densité locale inférieure de μ par rapport à g* . Mentionnons que même lorsque la densité locale inférieure d'une mesure μ est constante sur A égale à κ , on n'en déduit pas nécessairement que $\mu(A) = \kappa \mathcal{P}_g(A)$, sauf dans le cas où l'espace métrique considéré est l'espace euclidien \mathbb{R}^d (cf [53], chapitre 3).

Dimensions. Rappelons d'abord la définition des *dimensions de boîte* qui constituent l'exemple le plus élémentaire de dimension fractale. Soit $A \subset E$. Pour $\varepsilon \in (0, \infty)$, on note $N_\varepsilon(A)$ le nombre minimal de boules ouvertes de rayon ε nécessaires pour recouvrir A . Les dimensions de boîte inférieures et supérieures de A , notées respectivement $\underline{\dim}(A)$ et $\overline{\dim}(A)$, sont définies par

$$\underline{\dim}(A) = \liminf_{\varepsilon \rightarrow 0} \frac{\log N_\varepsilon(A)}{\log 1/\varepsilon}, \quad \overline{\dim}(A) = \limsup_{\varepsilon \rightarrow 0} \frac{\log N_\varepsilon(A)}{\log 1/\varepsilon}. \quad (91)$$

On définit les dimensions de Hausdorff et de packing de la manière suivante. Soit $A \subset E$.

$$\dim_H(A) = \sup\{\alpha \in (0, \infty) : \mathcal{H}_\alpha(A) = \infty\}, \quad \dim_p(A) = \sup\{\alpha \in (0, \infty) : \mathcal{P}_\alpha(A) = \infty\}, \quad (92)$$

où pour tout $\alpha \in (0, \infty)$, \mathcal{H}_α et \mathcal{P}_α désignent les mesures de Hausdorff et de packing associées à la jauge $g(r) = r^\alpha$. Ces dimensions possèdent notamment la propriété d'additivité dénombrable (ce qui n'est pas le cas des dimensions de boîte). Les inégalités suivantes sont vraies :

$$\dim_H(A) \leq \underline{\dim}(A) \quad \text{et} \quad \dim_H(A) \leq \dim_p(A) \leq \overline{\dim}(A) \quad (93)$$

On renvoie à Falconer [58] pour plus de détails.

4.2 Propriétés fractales des arbres de Lévy.

Dimensions. Les dimensions fractales d'un arbre de Lévy sont calculées par Duquesne et Le Gall dans [41] : soit un mécanisme de branchement ψ de la forme (45) ; soit l'arbre de Lévy \mathcal{T} associé par (71), sous la mesure N . On rappelle la définition (14) des exposants γ et η associés à ψ . Pour tout $a \in (0, \infty)$, on rappelle (72) définissant $\mathcal{T}(a)$, le niveau a de l'arbre. Sous l'hypothèse $\gamma > 1$, le théorème 5.5 dans [41] affirme que N -p.p.

$$\begin{aligned} \dim_H(\mathcal{T}(a)) &= \underline{\dim}(\mathcal{T}(a)) = \frac{1}{\eta-1}, & \dim_p(\mathcal{T}(a)) &= \overline{\dim}(\mathcal{T}(a)) = \frac{1}{\gamma-1} \\ \dim_H(\mathcal{T}) &= \underline{\dim}(\mathcal{T}) = \frac{\eta}{\eta-1}, & \dim_p(\mathcal{T}) &= \overline{\dim}(\mathcal{T}) = \frac{\gamma}{\gamma-1}. \end{aligned} \quad (94)$$

Mesures de Hausdorff exactes. Dans le cas Brownien où $\psi(\lambda) = \lambda^2$, Duquesne et Le Gall [42] ont montré que l'arbre Brownien ainsi que ses niveaux, admettent des mesures de Hausdorff exactes. Plus précisément, le théorème 1.3 de [42] affirme qu'il existe une constante universelle $c \in (0, \infty)$ telle que

$$\forall a \in (0, \infty), \quad N\text{-p.p.} \quad \ell^a(\cdot) = c \mathcal{H}_{g_1}(\cdot \cap \mathcal{T}(a)) \quad (95)$$

où \mathcal{H}_{g_1} est la g_1 -mesure de Hausdorff sur (\mathcal{T}, d) avec $g_1(r) = r \log \log 1/r$. Un résultat similaire est vérifié pour l'arbre Brownien tout entier : le théorème 1.1 dans [42] affirme qu'il existe une constante universelle $c \in (0, \infty)$ telle que N -p.p.

$$N\text{-p.p.} \quad \mathbf{m} = c \mathcal{H}_{g_2}, \quad (96)$$

où \mathcal{H}_{g_2} est la g_2 -mesure de Hausdorff sur (\mathcal{T}, d) avec $g_2(r) = r^2 \log \log 1/r$. Mentionnons que la Proposition 1.3. de Duquesne et Wang [44] précise que la densité locale supérieure de \mathbf{m} par rapport à g_2 vaut $4/\pi^2$.

Dans les cas stables non-Browniens où $\psi(\lambda) = \lambda^\gamma$, avec $\gamma \in (1, 2)$, Duquesne [37] établit qu'aucun des niveaux $\mathcal{T}(a)$ n'a de mesure de Hausdorff exacte, et également que \mathcal{T} n'a pas de mesure de Hausdorff exacte : pour toute jauge g à variation régulière, la g -mesure de Hausdorff de ces ensembles vaut N -p.p. 0 ou ∞ .

Mesures de packing exactes. Les arbres de Lévy se comportent de manière plus régulière vis-à-vis des mesures de packing. Duquesne a en effet démontré dans [38] que les arbres de Lévy admettent une mesure de packing exacte. Plus précisément, notons

$$\varphi = \psi' \circ \psi^{-1}.$$

On associe alors à ψ la jauge g définie par :

$$g(r) = \frac{\log \log \frac{1}{r}}{\varphi^{-1}\left(\frac{1}{r} \log \log \frac{1}{r}\right)}. \quad (97)$$

Dans l'article Duquesne [38], est introduit l'exposant suivant

$$\delta = \sup \{c \geq 0 : \exists C \in (0, \infty) \text{ tel que } C\psi(\mu)\mu^{-c} \leq \psi(\lambda)\lambda^{-c}, 1 \leq \mu \leq \lambda\}, \quad (98)$$

et il est montré au lemme 2.2. de [38] que la fonction de jauge g satisfait une condition de doublement (89) si et seulement si $\delta > 1$. On vérifie facilement que $1 \leq \delta \leq \gamma \leq \eta \leq 2$. En général ces exposants sont distincts : mentionnons qu'il est démontré dans [38] que pour tout $c \in (1, 2]$, il existe des mécanismes de branchement ψ de la forme (45) tels que $\delta = 1 < \gamma = \eta = c$. Mais lorsque ψ est à variation régulière en l'infini, $\delta = \gamma = \eta$.

Rappelons que \mathcal{T} est le ψ -arbre de Lévy sous N et que \mathbf{m} est sa mesure de masse définie par (75). Sous l'hypothèse $\delta > 1$, le théorème 1.1 de [38] affirme qu'il existe une constante $c_\psi \in (0, \infty)$ dépendant uniquement de ψ telle que l'égalité de mesures suivante est vraie :

$$N\text{-p.p. } \mathbf{m} = c_\psi \mathcal{P}_g. \quad (99)$$

4.3 Propriétés fractales du super-mouvement Brownien.

Dans cette section nous rappelons des propriétés fractales connues du super-processus de Dawson-Watanabe, ainsi qu'un résultat sur la dimension de Hausdorff des super-mouvements Browniens de mécanismes plus généraux. Soit $(Z_t, t \in \mathbb{R}_+, \mathbb{P}_\mu, \mu \in M_f(\mathbb{R}^d))$, processus de Dawson-Watanabe, c'est-à-dire un SBM(ψ) sur \mathbb{R}^d , où $\psi(\lambda) = \lambda^2$. Le processus de Dawson-Watanabe a fait l'objet de nombreux travaux concernant ses propriétés fractales.

Un premier résultat, dû à Dawson et Hochberg [25], précise la dimension de son support ainsi que celle de sa trace totale, définie par (32) : pour toute mesure $\mu \in M_f(\mathbb{R}^d)$

(i) Si $d \geq 2$, alors, \mathbb{P}_μ -p.s., $\forall t \in (0, \infty)$, $\dim_H(\text{supp}(Z_t)) = 2$.

(ii) Si $d \geq 4$, alors, \mathbb{P}_μ -p.s., $\dim_H(\mathbf{R}) = 4$.

Dans les résultats ci-dessus, la dimension de Hausdorff \dim_H peut être remplacée par les dimensions de packing ou de boîtes. Des résultats plus fins ont été établis par Dawson et Perkins [28] qui ont montré qu'il existe une constante $c_d \in (0, \infty)$ dépendant uniquement de d telle que

$$\forall d \geq 3, \forall t \in (0, \infty) \quad \mathbb{P}_\mu\text{-p.s. } Z_t = c_d \mathcal{H}_{g_3}(\cdot \cap \text{supp}(Z_t)), \quad (100)$$

où \mathcal{H}_{g_3} est la g_3 -mesure de Hausdorff sur \mathbb{R}^d avec $g_3(r) = r^2 \log \log 1/r$. Un résultat uniforme en temps a été obtenu par Perkins [100, 101] : il est rappelé plus loin dans cette introduction. En dimension critique $d = 2$, la fonction de jauge n'est pas la même comme démontré par Le Gall et Perkins dans [87] où il est prouvé qu'il existe une constante $c_2 \in (0, \infty)$ telle que

$$\forall t \in (0, \infty) \quad \mathbb{P}_{\mu\text{-p.s.}} \quad Z_t = c_2 \mathcal{H}_{g_4}(\cdot \cap \text{supp}(Z_t)), \quad (101)$$

où \mathcal{H}_{g_4} est la g_4 -mesure de Hausdorff sur \mathbb{R}^d avec $g_4(r) = r^2 \log 1/r \log \log \log 1/r$ (la preuve utilise le serpent Brownien). Le Gall, Perkins et Taylor ont également montré dans [88] que $\text{supp}(Z_t)$ n'a pas de fonction de packing exacte en dimension surcritique $d \geq 3$.

Concernant la trace totale \mathbf{R} du processus de Dawson-Watanabe, Dawson, Iscoe et Perkins [26] montrent qu'en dimension surcritique $d \geq 5$, il existe $c_d \in (0, \infty)$ ne dépendant que de d telle que

$$\mathbb{P}_{\mu\text{-p.s.}} \quad \mathbf{M} = c_d \mathcal{H}_{g_5}(\cdot \cap \mathbf{R}) \quad (102)$$

où \mathbf{M} est la mesure d'occupation totale du Dawson Watanabe définie par (31) et où \mathcal{H}_{g_5} est la g_5 -mesure de Hausdorff sur \mathbb{R}^d avec $g_5(r) = r^4 \log \log(1/r)$. En dimension critique $d = 4$, Le Gall a montré dans [83] un résultat similaire avec la fonction de jauge $g_6(r) = r^4 \log 1/r \log \log 1/r$.

La mesure d'occupation totale du Dawson Watanabe s'écrit aussi comme une mesure de packing, comme démontré par Duquesne dans [36] où il est prouvé l'existence, en dimension surcritique $d \geq 5$, l'existence d'une constant $c_d \in (0, \infty)$ ne dépendant que de d telle que

$$\mathbb{P}_{\mu\text{-p.s.}} \quad \mathbf{M} = c_d \mathcal{P}_{g_7}(\cdot \cap \mathbf{R}). \quad (103)$$

où \mathcal{P}_{g_7} est la g_7 -mesure de packing sur \mathbb{R}^d avec $g_7(r) = r^4 (\log \log(1/r))^{-3}$. C'est ce dernier résultat qui est généralisé dans le chapitre 3 de cette thèse.

Dans le cas d'un mécanisme de branchement ψ quelconque, de la forme (45), il y a peu de résultats connus concernant les propriétés fractales : par une méthode de subordination du serpent Brownien, Delmas dans [29] a calculé la dimension de Hausdorff de la trace des SBM de mécanisme de branchement stable, c'est-à-dire le cas des SBM(ψ) où $\psi(\lambda) = \lambda^\gamma$, $\gamma \in (1, 2]$. Ce résultat a été étendu par la suite par Duquesne et Le Gall dans [41] par des techniques différentes de celles de Delmas [29] : les arguments de Duquesne et Le Gall utilisent des propriétés des arbres de Lévy et la régularité Höldérienne du serpent de Lévy. Dans [41], il est montré que, sous l'hypothèse $\gamma > 1$, pour $t \in (0, \infty)$, $\mathbb{P}_{\mu\text{-p.s.}}$,

$$\dim_H(\text{supp } Z_t) = \underline{\dim}(\text{supp } Z_t) = \frac{2}{\eta-1} \wedge d, \quad \dim_H(\mathbf{R}) = \underline{\dim}(\mathbf{R}) = \frac{2\eta}{\eta-1} \wedge d. \quad (104)$$

La preuve de ce qui précède utilise le serpent de Lévy pour transporter sur \mathbb{R}^d , le résultat sur la dimension de Hausdorff de l'arbre énoncé en (94). Un résultat similaire concernant la dimension de parking est démontré au chapitre 3.

5 Contributions de cette thèse.

5.1 Temps d'atteinte pour les CBI et applications.

Les résultats présentés dans ce paragraphe sont tirés de l'article [33], écrit avec Clément Foucart et Chunhua Ma (chapitre 1 de ce manuscrit).

On a vu en section 2.3 que pour un CBI(ψ, ϕ) général, on dispose du critère (44) de polarité pour l'état 0 ainsi que de la condition nécessaire et suffisante (42) pour la convergence du processus vers sa probabilité invariante. Dans [33], on fait une étude des premiers temps d'atteinte pour un CBI(ψ, ϕ). Comme première application, on obtient une nouvelle preuve de la CNS (44) pour la polarité. Enfin, une condition nécessaire et suffisante pour la récurrence ou transience est proposée. Dans tout ce qui suit, le processus (ψ, ϕ) est un couple de la forme (35) et ($Z_t, t \geq 0, \mathbf{P}_x^*, x \in \mathbb{R}^d$) un CBI(ψ, ϕ), sur un espace (Ω, \mathcal{F}) adéquat.

Définition 5.1 (i). Pour $x \in (0, \infty)$ et $a \in [0, x]$, le temps d'entrée dans $[0, a]$ est noté σ_a et défini par

$$\sigma_a = \inf \{t \in (0, \infty) : Z_t \in [0, a]\}. \quad (105)$$

L'absence de sauts négatifs en fait aussi le premier temps d'atteinte de a .

(ii). On dit que $a \in (0, \infty)$ est *polaire* si pour tout $x \in (a, \infty)$, $\mathbf{P}_x^*(\sigma_a < \infty) = 0$.

(iii). On dit que le processus ($Z_t, t \geq 0$) est *récurrent* s'il existe $x \in \mathbb{R}_+$ tel que

$$\mathbf{P}_x^* \left(\liminf_{t \rightarrow \infty} |Z_t - x| = 0 \right) = 1. \quad (106)$$

Dans le cas contraire, le processus est dit *transient* et pour tout $x \in \mathbb{R}_+$, $\mathbf{P}_x^*(\lim_{t \rightarrow \infty} Z_t = \infty) = 1$.

(iv). On appelle progéniture totale du processus au temps t la quantité $\int_0^t Z_s ds$.

Dans le cas du carré de Bessel de dimension d , on sait (voir e.g. Revuz et Yor [105], chapitre XI) que 0 est polaire si et seulement si $d \geq 2$, et que le processus est transient si et seulement si $d > 2$. Ainsi, dans le cas $d = 2$, on a p.s. 0 polaire et $\liminf_{t \rightarrow \infty} Z_t = 0$. Dans le cas autosimilaire $\psi(\lambda) = d\lambda^\gamma, \phi(\lambda) = d'\lambda^{\gamma-1}, \gamma \in (1, 2)$, Patie [98] montre que 0 est polaire pour le CBI(ψ, ϕ) si et seulement si $d' \geq d(\gamma-1)$.

Dans le cas des processus de Ornstein-Uhlenbeck généralisés (ψ linéaire, ϕ quelconque), le CB(ψ) ne s'éteint pas, avec probabilité 1, ce qui, a fortiori, garantit la polarité de 0 pour le CBI(ψ, ϕ). La loi des temps d'entrée σ_a , sous \mathbf{P}_x^* , avec $x \geq a$, a été calculée par Hadjiev [65], étude complétée par Patie dans [96] avec la description de la loi du couple $(\sigma_a, \int_0^{\sigma_a} Z_s ds)$. Shiga [111] traite la question de la récurrence ou transience. Dans [33], ce critère est généralisé pour des CBI généraux. On notera notamment que dans ce qui suit, ni la condition de Grey (24), ni la condition $\int_{0+} d\lambda/|\psi(\lambda)| = \infty$ ne sont supposées ici a priori.

On rappelle la définition de \underline{a} donnée en (39). Le théorème principal de [33] donne, pour $x \geq a \geq \underline{a}$, une expression de la transformée de Laplace du couple $(\sigma_a, \int_0^{\sigma_a} Z_s ds)$ sous \mathbf{P}_x^* .

Théorème 5.1 ([33]) Soit $x > a \geq \underline{a}$. Pour tout $\lambda > 0$ et $\mu \geq 0$, on a

$$\mathbf{E}_x^* \left[\exp \left\{ -\lambda \sigma_a - \mu \int_0^{\sigma_a} Z_t dt \right\} \right] = \frac{f_{\lambda, \mu}(x)}{f_{\lambda, \mu}(a)}, \quad (107)$$

où, pour tout $x > \underline{a}$,

$$f_{\lambda, \mu}(x) = \int_{q(\mu)}^{\infty} \frac{dz}{\psi(z) - \mu} \exp \left(-xz + \int_{\theta}^z \frac{\phi(u) + \lambda}{\psi(u) - \mu} du \right), \quad (108)$$

avec $q(\mu) := \sup\{q \geq 0 : \psi(q) = \mu\}$, et où θ est une constante arbitraire, choisie plus grande que $q(\mu)$.

La preuve du théorème repose principalement sur le fait que la fonction d'échelle $f_{\lambda,\mu}$ est une fonction propre du générateur d'un CBI auxiliaire associé à (ψ, ϕ) . Plus précisément, pour $\mu \geq 0$ fixé, on définit $\bar{\psi} = \psi - \mu$, de sorte que si L est le générateur associé au CBI (ψ, ϕ) par (36), celui associé à un CBI $(\bar{\psi}, \phi)$ est donné, pour toute fonction $f \in \mathcal{C}^2(\mathbb{R}_+)$, tendant vers 0 en l'infini,

$$\bar{L}f(x) = Lf(x) - \mu x f(x).$$

On a alors que pour tout $\lambda > 0$, la fonction $f_{\lambda,\mu}$ est une fonction de classe \mathcal{C}^1 , strictement décroissante sur (\underline{a}, ∞) telle que $\bar{L}f_{\lambda,\mu} = \lambda f_{\lambda,\mu}$. Lorsque $\phi \equiv 0$, le CBI (ψ, ϕ) est un CB (ψ) , et il nous semble que même dans ce cas, les fonctions d'échelles définies en (108) n'apparaissent pas dans la littérature. Les deux applications principales du résultat ci-dessus sont obtenues en étudiant la loi de σ_a pour $a \geq 0$ ($\mu = 0$). On retrouve le critère de [60] pour la polarité de 0 et on obtient une CNS pour la récurrence du processus.

Théorème 5.2 ([60]) *Le seul point pouvant être polaire est \underline{a} . Si $\mathbf{d} < \infty$, alors \underline{a} est polaire. Dans le cas des variations non-bornées, $\underline{a} = 0$ et 0 est polaire si et seulement si*

$$\int_{\theta}^{\infty} \frac{dz}{\psi(z)} \exp \left[\int_{\theta}^z \frac{\phi(u)}{\psi(u)} du \right] = \infty. \quad (109)$$

Théorème 5.3 ([33]) *Supposons $\phi \not\equiv 0$.*

(a) *Dans le cas critique ou sous-critique ($\psi'(0+) = 0$ ou $\psi'(0+) > 0$), le CBI (ψ, ϕ) est récurrent si et seulement si*

$$\int_0^1 \frac{dz}{\psi(z)} \exp \left[- \int_z^1 \frac{\phi(x)}{\psi(x)} dx \right] = \infty. \quad (110)$$

Dans le cas récurrent, pour $x \geq \underline{a}$, on a $\liminf_{t \rightarrow \infty} Z_t = \underline{a}$, \mathbf{P}_x^ p.s.*

(b) *Dans le cas sur-critique $\psi'(0+) < 0$, le CBI (ψ, ϕ) est transient.*

Les deux critères ci-dessus précisent les rôles respectifs du branchement et de l'immigration : notons que les intégrales mettent en jeu le quotient ϕ/ψ , au voisinage de ∞ pour le théorème 5.2 et en 0 pour le théorème 5.3. On mentionne une application importante du théorème 5.3 au CB (ψ) conditionné à la non-extinction. Comme signalé plus haut, celui-ci a la même loi qu'un CBI (ψ, ϕ) , où $\phi(\lambda) = \psi'(\lambda) - \psi'(0)$. Lorsque $\psi'(0+) = 0$ (cas critique), l'intégrale dans (110) vaut 1 : on retrouve le résultat de Lambert [77] qui établit que le CB (ψ) conditionné à la non-extinction est transient.

5.2 Mesure de Hausdorff uniforme des niveaux de l'arbre Brownien.

Le théorème présenté ici est prouvé dans l'article [31] (chapitre 2 de ce manuscrit).

Le résultat principal de [31] concerne les mesures de temps local de l'arbre Brownien, définies en (73). Le résultat (95) y est revisité : on montre que la constante c est égale à $1/2$ et que le résultat est vrai N presque partout, *uniformément* pour tous niveaux. Le problème de l'uniformité dans la comparaison d'un temps local avec une mesure géométrique apparaît chez Perkins [99] dans l'étude des temps locaux du mouvement Brownien standard. Rappelons brièvement ce résultat pour le processus $(X_t, t \geq 0)$ défini sur $(\Omega, \mathcal{F}, \mathbb{P})$ et tel que $(X_t/\sqrt{2}, t \geq 0)$ est un mouvement Brownien standard, auquel est associée la

famille de temps locaux bi-continue $(L_t^a, t \geq 0, a \in \mathbb{R})$ définie par une approximation du type (48). Perkins dans [99] montre que

$$\mathbb{P}\text{-p.s.}, \quad \forall a \in \mathbb{R}, \quad L_t^a = \mathcal{H}_{g_8}(\{s \in [0, t] : X_s = x\}), \quad (111)$$

où \mathcal{H}_{g_8} est la mesure de Hausdorff sur \mathbb{R} associée à la jauge $g_8(r) = \sqrt{r \log \log 1/r}$. Le résultat fait suite aux travaux de Taylor et Wendel [113].

Un phénomène de régularité similaire est également vérifié pour le processus de Dawson-Watanabe $(Z_t, t \geq 0)$: en effet, Perkins [100, 101] montre que presque sûrement, uniformément pour tout temps t , la mesure Z_t est bornée entre deux multiples d'une même mesure de Hausdorff. Plus précisément, Perkins montre que si $d \geq 3$, il existe deux constantes c_d et C_d dans $(0, \infty)$, ne dépendant que de d , telles que

$$\mathbb{P}_{\mu}\text{-p.s.} \quad \forall t \in (0, \infty), \\ c_d \mathcal{H}_{g_3}(\cdot \cap \text{supp}(Z_t)) \leq Z_t \leq C_d \mathcal{H}_{g_3}(\cdot \cap \text{supp}(Z_t)). \quad (112)$$

où \mathcal{H}_{g_3} est la g_3 -mesure de Hausdorff sur \mathbb{R}^d avec $g_3(r) = r^2 \log \log 1/r$. Perkins conjecture $c_d = C_d$.

En s'inspirant des idées de la preuve de Perkins [100, 101] de (112), et dans le contexte plus simple et plus régulier de l'arbre Brownien, nous montrons dans le chapitre 2, le théorème suivant qui précise les résultats de Duquesne et Le Gall [42] rappelés au (95).

Théorème 5.4 ([31]) *On considère l'arbre Brownien \mathcal{T} , défini sous la mesure N . Pour tout $a \in (0, \infty)$, on rappelle que $\mathcal{T}(a) = \{\sigma \in \mathcal{T} : d(\varrho, \sigma) = a\}$ et que ℓ^a est la mesure de temps local au niveau a . Soit la fonction de jauge $g_1(r) = r \log \log 1/r$. Alors,*

$$N\text{-p.p.} \quad \forall a \in (0, \infty), \quad \ell^a = \frac{1}{2} \mathcal{H}_{g_1}(\cdot \cap \mathcal{T}(a)).$$

Commentaire 5.1 Bien qu'apparemment proches, il n'est pas clair que l'on puisse déduire ce théorème de (111), qui concerne pourtant les temps locaux du mouvement Brownien. \square

Commentaire 5.2 La stratégie de preuve du théorème 5.4 est empruntée à Perkins [100, 101] : l'uniformité en a est démontrée en utilisant, comme dans [100, 101], une grille discrète pour l'ensemble des niveaux et on est amené à étudier les boules à un niveau donné ayant un temps local anormalement grand, puis celles ayant un temps local qui est anormalement petit. \square

Commentaire 5.3 Un théorème de comparaison semblable au lemme 4.1 est démontré dans le cadre des niveaux d'un arbre réel : il s'agit du lemme 2.3 du chapitre 2, nouveau à notre connaissance ; la géométrie spécifique des niveaux d'un arbre réel permet de s'affranchir de la présence de la constante de doublement, ce qui donne la constante multiplicative exacte $1/2$ dans le théorème 5.4. \square

5.3 Mesure d'occupation totale et trace du super-mouvement Brownien.

Les deux théorèmes présentés ici sont prouvés dans l'article [32] écrit avec Thomas Duquesne (chapitre 3 de ce manuscrit).

On considère un SBM(ψ) noté Z , dont la trace totale \mathbf{R} est donnée par (32). Le mécanisme de branchement ψ est de la forme (45). On rappelle la définition (98) de l'exposant δ . On suppose

$$\delta > 1.$$

Cela implique notamment que $\gamma > 1$ et que (33) est satisfaite : si $\mu \in M_f(\mathbb{R}^d)$ est à support compact alors la trace totale \mathbf{R} est \mathbb{P}_μ -p.s. bornée.

Dans le prolongement des résultats de dimension (94) concernant les arbres de Lévy et de (104) concernant la trace d'un SBM(ψ), il est montré dans le chapitre 3 de cette thèse le théorème suivant.

Théorème 5.5 ([32]) *Soit $\mu \in M_f(\mathbb{R}^d)$, une mesure non-nulle. Soit ψ de la forme (45). Soit Z , un SBM(ψ) considéré sous \mathbb{P}_μ . Soit \mathbf{R} sa trace. On suppose*

$$\delta > 1 \quad \text{et} \quad d > \frac{2\delta}{\delta-1}.$$

Alors,

$$\mathbb{P}_\mu\text{-p.s.} \quad \dim_p(\mathbf{R}) = \frac{2\gamma}{\gamma-1}. \quad (113)$$

Si de plus $\text{supp}(\mu)$ est compact, alors $\mathbb{P}_\mu\text{-p.s.} \quad \overline{\dim}(\mathbf{R}) = \frac{2\gamma}{\gamma-1}$.

Commentaire 5.4 La dimension de packing $\frac{2\gamma}{\gamma-1}$ s'explique naturellement par le fait qu'elle vaut deux fois la dimension de packing d'un ψ -arbre de Lévy : en effet, le serpent de Lévy est processus Gaussien sur l'arbre qui est $(\frac{1}{2} - \varepsilon)$ -Hölder pour tout $\varepsilon > 0$ et on peut attendre qu'il double les dimensions (voir commentaires après (104)). \square

Commentaire 5.5 Le caractère hölderien du serpent (étudié dans [41] et rappelé dans (84)) implique facilement que si $\text{supp}(\mu)$ est compact et si $\gamma > 1$, alors $\overline{\dim}(\mathbf{R}) \leq d \wedge \frac{2\gamma}{\gamma-1}$. Puisque $\dim_p(A) \leq \overline{\dim}(A)$ pour tout sous-ensemble borné $A \subset \mathbb{R}^d$ (voir par exemple [58]), un argument simple entraîne que si $\mu \in M_f(\mathbb{R}^d)$ est non-nulle, et si $\gamma > 1$, alors $\mathbb{P}_\mu\text{-p.s.} \quad \dim_p(\mathbf{R}) \leq d \wedge \frac{2\gamma}{\gamma-1}$. Ceci combiné avec (104) entraîne que

$$\text{si } \eta = \gamma > 1, \text{ alors } \mathbb{P}_\mu\text{-p.s.} \quad \dim_H(\mathbf{R}) = \dim_p(\mathbf{R}) = d \wedge \frac{2\gamma}{\gamma-1}, \quad (114)$$

avec une égalité analogue pour les dimensions de boîte si $\text{supp}(\mu)$ est compact. L'égalité $\gamma = \eta$ est vérifiée si, par exemple, ψ est à variations régulières en ∞ . Par conséquent le caractère nouveau du théorème 5.5 ne concerne que les cas où $\gamma \neq \eta$. \square

Commentaire 5.6 Observons que l'hypothèse $\delta > 1$ n'est sans doute pas optimale car la valeur de $\dim_p(\mathbf{R})$ ne dépend que de γ . Nous conjecturons que si $\gamma > 1$, alors $\mathbb{P}_\mu\text{-p.s.} \quad \dim_p(\mathbf{R}) = d \wedge \frac{2\gamma}{\gamma-1}$. \square

Le résultat principal de [32] montre que la mesure d'occupation totale d'un SBM(ψ), notée \mathbf{M} et définie par (31), est une mesure de packing exacte, ce qui généralise (103) à des mécanismes de branchement ψ généraux de la forme (45). Plus précisément, on rappelle la notation $\varphi = \psi' \circ \psi^{-1}$ et on définit la jauge

$$g(r) = \frac{\log \log \frac{1}{r}}{\varphi^{-1}\left(\left(\frac{1}{r} \log \log \frac{1}{r}\right)^2\right)}. \quad (115)$$

On montre dans le chapitre 3 que $\delta > 1$ si et seulement si la fonction de jauge g satisfait une condition de doublement (89). Le théorème principal du chapitre 3 s'énonce alors comme suit.

Théorème 5.6 ([32]) *Soit $\mu \in M_f(\mathbb{R}^d)$, non-nulle. Soit ψ de la forme (45). Soit Z , un SBM(ψ) considéré sous \mathbb{P}_μ , de mesure d'occupation totale \mathbf{M} et de trace \mathbf{R} . On suppose :*

$$\delta > 1 \quad \text{et} \quad d > \frac{2\gamma}{\gamma - 1} .$$

Alors, il existe une constante $c_{d,\psi} \in (0, \infty)$ ne dépendant que de d et ψ telle que

$$\mathbb{P}_\mu\text{-p.s.} \quad \mathbf{M} = c_{d,\psi} \mathcal{P}_g(\cdot \cap \mathbf{R}) .$$

Commentaire 5.7 Nous pensons que les hypothèses du théorème 5.6 sont optimales dans le sens suivant : tout d'abord, puisque nous avons de fortes raisons de penser que $d \wedge \frac{2\gamma}{\gamma-1}$ est la dimension de packing de \mathbf{R} , le théorème 5.6 ne s'étend sans doute pas tel quel aux dimensions $d \leq \frac{2\gamma}{\gamma-1}$, en tous cas pas sans (au moins) changer la fonction de jauge g (il se peut également que \mathbf{M} ne se représente pas comme une mesure de packing en dimension sous-critique).

De plus, comme déjà mentionné, l'hypothèse $\delta > 1$ équivaut à ce que g satisfasse une condition de doublement ; bien qu'il soit possible de définir des mesures de packing dont la fonction de jauge est irrégulière (voir Edgar [53]), la condition de doublement est en quelque sorte l'hypothèse de régularité minimale sur une fonction de jauge pour que la mesure de packing associée ait des propriétés satisfaisantes (régularité, lemmes de comparaison, etc.) Dans ce sens, l'hypothèse $\delta > 1$ est nécessaire si l'on veut rester dans le cadre de mesures géométriques standard. \square

Commentaire 5.8 Lorsque $\psi(\lambda) = \lambda^\gamma$, la fonction de jauge g devient

$$g(r) = r^{\frac{2\gamma}{\gamma-1}} (\log \log 1/r)^{-\frac{\gamma+1}{\gamma-1}} .$$

Lorsque $\gamma = 2$, on retrouve la fonction de jauge g_7 et le résultat (103) précédemment mentionné.

Dans les cas stables le théorème 5.5 se déduit facilement du théorème 5.6. Cependant, cela ne semble pas être le cas lorsque $\gamma \neq \eta$. En effet un argument important de la preuve du théorème 5.6 consiste à calculer la densité inférieure locale de \mathbf{M} par rapport à g : cette limite inférieure est réalisée le long d'une suite de rayons dont l'image par g semble difficile à comparer à des fonctions puissance. Les théorèmes 5.5 et 5.6 ont donc deux preuves séparées dans le chapitre 3. \square

ON THE HITTING TIMES OF CONTINUOUS-STATE
BRANCHING PROCESSES WITH IMMIGRATION.

This article [33] has been written in collaboration with Clément Foucart and Chunhua Ma.

Abstract. We study a two-dimensional joint distribution related to the first passage time below a level for a continuous-state branching process with immigration. We provide an explicit expression of its Laplace transform and obtain a necessary and sufficient criterion for transience or recurrence. We follow the approach of Shiga, T. (1990) [A recurrence criterion for Markov processes of Ornstein-Uhlenbeck type. *Probability Theory and Related Fields*, 85(4), 425-447], by finding some λ -invariant functions for the generator.

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1 Introduction.

The continuous-state branching processes with immigration (CBI for short) are a class of time-homogeneous Markov processes with values in \mathbb{R}_+ . They have been introduced by Kawazu and Watanabe in 1971, see [70], as limits of rescaled Galton-Watson processes with immigration. Continuous-state branching processes with immigration form a class of Markov processes which has received significant attention in the literature. Any CBI process is characterized in law by the Laplace exponents Ψ and Φ of two independent Lévy processes : a spectrally positive process (which describes the reproduction) and a subordinator (which describes the immigration). The purpose of this paper is to give and exploit an explicit formula for the Laplace transform of the first passage time below a level. In particular we provide a necessary and sufficient condition for a CBI process to be recurrent or transient. We adopt the following definition of recurrence and transience.

Definition 1.1 We say that the process $(X_t, t \geq 0)$ is recurrent if there exists $x \in \mathbb{R}_+$ such that

$$\mathbb{P}_x(\liminf_{t \rightarrow \infty} |X_t - x| = 0) = 1. \quad (\text{I.1})$$

On the other hand, we say that the process is transient if

$$\mathbb{P}_x(\lim_{t \rightarrow \infty} X_t = \infty) = 1 \text{ for every } x \in \mathbb{R}_+. \quad (\text{I.2})$$

When the mechanisms of reproduction and immigration reduce to $\Psi(q) = \frac{\sigma^2}{2}q^2$ and $\Phi(q) = bq$, the process is the Feller diffusion, also called Cox-Ingersoll-Ross model in the financial setting. This is the unique solution to the stochastic equation :

$$X_t = x + \sigma \int_0^t \sqrt{X_s} dB_s + bt,$$

where $(B_t, t \geq 0)$ is a Brownian motion. A standard method to study the hitting times, as well as the transience and recurrence of a general diffusion, is to use potential theory and scale functions (see for instance pages 128-129 of Itô and McKean [68]). This theory yields the following classic result concerning the Feller diffusion $(X_t, t \geq 0)$: if $2b \geq \sigma^2$, then the point 0 is polar. If $2b > \sigma^2$, the process is transient, otherwise the process is recurrent. In particular, if $2b = \sigma^2$, then 0 is polar and the process is recurrent (we refer, for instance, to Chapter XI of Revuz-Yor [105] for a proof).

We shall study these path-properties for the general CBI processes. The polarity of zero has been studied in Foucart and Uribe Bravo [60]. However, this latter work focuses on the zero-set and does not provide a criterion for transience or recurrence of the process. Moreover, as we shall see, zero may be polar and recurrent (in sense of (I.1)).

On the one hand, when the mechanism Ψ reduces to $\Psi(q) = \gamma q$ with $\gamma > 0$, the class of CBI processes corresponds to positive Ornstein-Uhlenbeck processes. This class of processes has been intensively studied. Hadjiev [65] get a formula for the hitting times of generalized Ornstein-Uhlenbeck processes. Novikov [95], Patie [96], [97] apply potential theory to get some identities for the joint law of $(\sigma_a, \int_0^{\sigma_a} X_s ds)$, and for the first exit times. On the other hand, when no immigration is taken into account (namely, with $\Phi \equiv 0$), the corresponding CBI process is simply a continuous-state branching process (CB process) for which many results have been obtained using the Lamperti transform (which relates any CB process to a spectrally positive Lévy process). We refer, for instance, to Chapter 10 of [73]. We mention that a Lamperti-type representation for the CBI processes has been obtained by Caballero et al. in [20]. However, our methods do not rely on this representation.

Our aim is to generalize some of these results when immigration is taken into account for a general reproduction mechanism Ψ . In this framework, the integral from 0 to σ_a of the process can be interpreted as the total population up to time σ_a . The results reveal the interplay between Φ and Ψ in some path properties of CBI processes.

The paper is organized as follows. In Section 2, we recall the definition as well as some known properties of continuous-state branching processes. Section 3 discusses our main results. We study the state space of a CBI in Section 4. In Section 5 we prove a key lemma providing some λ -invariant functions, and apply it to establish our first theorem. In Section 6, we show a formula for the Laplace transform of the hitting times and study the polarity of 0. In Section 7, we establish our criterion for recurrence or transience and we study the law of the infimum of a transient CBI. We end the section by showing how to construct null-recurrent CBIs. In Section 8, we study the integral of the CBI process up to time σ_a .

Throughout the article, we take the convention that for any finite real number C , $C/\infty = 0$.

2 Preliminaries.

For an introduction to continuous-state branching processes with immigration, we refer to Li [90], [89] and Kyprianou [73]. We recall here some of their fundamental properties.

2.1 Continuous-state branching processes.

Let $x \in \mathbb{R}_+$. Notation \mathbb{P}_x denotes the law of the process started at $x \in \mathbb{R}_+$, and \mathbb{E}_x the corresponding expectation operator. A continuous-state branching process $(Z_t, t \geq 0)$ is a Markov process which satisfies the following property : for any x and $y \in \mathbb{R}_+$, the process $(Z_t, t \geq 0)$ starting from $x + y$ has the same law as

$$(Z_t^1 + Z_t^2, t \geq 0),$$

where $(Z_t^1, t \geq 0)$ and $(Z_t^2, t \geq 0)$ are two independent copies of $(Z_t, t \geq 0)$ starting respectively at x and y . There exists a unique function Ψ of the form :

$$\Psi(q) = \gamma q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu1_{\{u \in (0,1)\}})\pi(du)$$

such that the Laplace transform of the one-dimensional distribution of $(Z_t, t \geq 0)$ is given by

$$\mathbb{E}_x[e^{-\lambda Z_t}] = \exp(-xv_t(\lambda))$$

where the map $t \mapsto v_t(\lambda)$ is the solution to the differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\Psi(v_t(\lambda)), \quad v_0(\lambda) = \lambda.$$

Recall the following classification (see Chapter 10 of [73] for details) : the branching mechanism Ψ is said to be subcritical if $\Psi'(0+) > 0$, critical if $\Psi'(0+) = 0$ and supercritical if $\Psi'(0+) < 0$. For any $x \geq 0$, define the extinction time of $(Z_t, t \geq 0)$ by

$$\zeta := \inf\{t \geq 0; Z_t = 0\} \in [0, \infty].$$

The extinction time has the following law (see for instance Theorem 3.5 and 3.8 pages 59-60 of [89]):

$$\mathbb{P}_x[\zeta \leq t] = \exp(-xv_t)$$

with $v_t = \lim_{\lambda \rightarrow \infty} v_t(\lambda)$.

The asymptotic behavior of a $\text{CB}(\Psi)$ process is well understood. Indeed, the $\text{CB}(\Psi)$ process $(Z_t, t \geq 0)$ is absorbed in 0 with positive probability if and only if there exists $\theta > 0$ such that $\Psi(z) > 0$ for $z \geq \theta$ and

$$\int_\theta^\infty \frac{dq}{\Psi(q)} < \infty. \tag{I.3}$$

Under this condition, $v_t < \infty$ for all t and $v := \lim_{t \rightarrow \infty} v_t \in [0, \infty[$ is the largest root of the equation $\Psi(q) = 0$ and

$$\mathbb{P}_x[\zeta < \infty] = \exp(-xv).$$

In the (sub)critical case $v = 0$ and the process is absorbed in zero almost surely. In the supercritical case, we have $v > 0$ and the process goes to infinity with positive probability. Lastly, the process is absorbed in ∞ in finite time if and only if

$$\int_{0+} \frac{dq}{|\Psi(q)|} < \infty. \tag{I.4}$$

2.2 Continuous-state branching processes with immigration.

A continuous-state branching processes with immigration is a Markov process $(X_t, t \geq 0)$ whose law is characterized by two functions of the variable $q \geq 0$:

$$\Psi(q) = \gamma q + \frac{1}{2} \sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu 1_{\{u \in (0,1)\}}) \pi(du), \quad (\text{I.5})$$

$$\Phi(q) = bq + \int_0^\infty (1 - e^{-qu}) \nu(du), \quad (\text{I.6})$$

where $\sigma, b \geq 0$, $\gamma \in \mathbb{R}$ and ν, π are two Lévy measures such that $\int_0^\infty (1 \wedge u) \nu(du) < \infty$ and $\int_0^\infty (1 \wedge u^2) \pi(du) < \infty$. The measure ν characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The non-negative constants σ and b correspond respectively to the continuous reproduction and the continuous immigration. To simplify our notation, a continuous-state branching process with reproduction mechanism Ψ and immigration mechanism Φ is called CBI(Ψ, Φ) process.

Let $(X_t, t \geq 0)$ be a CBI(Ψ, Φ) process, its one-dimensional marginal law satisfies :

$$\mathbb{E}_x[e^{-qX_t}] = \exp\left(-xv_t(q) - \int_0^t \Phi(v_s(q)) ds\right), \quad (\text{I.7})$$

with $\frac{\partial v_t(q)}{\partial t} = -\Psi(v_t(q))$ and $v_0(q) = q$.

A CBI(Ψ, Φ) process $(X_t, t \geq 0)$ starting from $x + y$ has the same law as

$$(X_t^1 + X_t^2, t \geq 0), \quad (\text{I.8})$$

where $(X_t^1, t \geq 0)$ is a CBI(Ψ, Φ_1) process starting at x and $(X_t^2, t \geq 0)$ is an independent CBI(Ψ, Φ_2) process starting at y , with $\Phi_1 + \Phi_2 = \Phi$.

Denote by $C_0^2(\mathbb{R}_+)$ the space of twice-differentiable functions that tend to zero as x goes to ∞ . Kawazu and Watanabe [70] establish that a CBI(Ψ, Φ) process is a Feller process with generator the operator L acting on $C_0^2(\mathbb{R}_+)$ as follows

$$Lf(x) := \frac{\sigma^2}{2} x f''(x) + (b - \gamma x) f'(x) + x \int_0^\infty (f(x+z) - f(x) - z 1_{[0,1]}(z) f'(x)) \pi(dz) + \int_0^\infty (f(x+z) - f(x)) \nu(dz). \quad (\text{I.9})$$

Let us define the *effective drift* \mathbf{d} by

$$\mathbf{d} := \begin{cases} \gamma + \int_0^1 z \pi(dz) & \text{if the process has bounded variation paths} \\ \infty & \text{if the process has unbounded variation paths,} \end{cases} \quad (\text{I.10})$$

One can plainly check that in both cases, one has

$$\lim_{q \rightarrow \infty} \frac{\Psi(q)}{q} = \mathbf{d}.$$

Apart if explicitly mentioned, we always assume that the CBI process is non-deterministic. This is equivalent to assume that one of the three conditions holds : $\sigma \neq 0$, $\pi \neq 0$, or $\nu \neq 0$. Moreover, we assume that there exists $q \in \mathbb{R}_+$ such that $\Psi(q) > 0$ (i.e. $-\Psi$ is

not the Laplace exponent of a subordinator). This is equivalent to assume that \mathbf{d} belongs to $(0, \infty]$. Otherwise, the corresponding CBI process would be non-decreasing, and the problems studied in the present work are trivial.

We are now ready to state our main results.

3 Main results.

Denote the first entrance time in $[0, a]$ by σ_a :

$$\sigma_a := \inf\{t > 0; X_t \leq a\}. \quad (\text{I.11})$$

Remark that the process has no downward jumps, therefore $X_{\sigma_a} = a$ almost surely. We will discuss the law of σ_a when the process starts from a state x greater than a . The first theorem is the following. Set $\underline{a} := \frac{b}{\mathbf{d}}$ with \mathbf{d} defined by (I.10).

Theorem 3.1 *Let $x > a \geq \underline{a}$. For every $\lambda > 0$ and $\mu \geq 0$, we have*

$$\mathbb{E}_x \left[\exp \left\{ -\lambda \sigma_a - \mu \int_0^{\sigma_a} X_t dt \right\} \right] = \frac{\int_{q(\mu)}^{\infty} \frac{dz}{\Psi(z) - \mu} \exp \left(-xz + \int_{\theta}^z \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \right)}{\int_{q(\mu)}^{\infty} \frac{dz}{\Psi(z) - \mu} \exp \left(-az + \int_{\theta}^z \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \right)}, \quad (\text{I.12})$$

where $q(\mu) := \sup\{q \geq 0 : \Psi(q) = \mu\}$, and θ is an arbitrary constant larger than $q(\mu)$.

When Φ is null or taken of the specific form Ψ' , some formulas are simplified and we recover certain results on continuous-state branching processes.

We recover and complete some results of [60] through more classic techniques.

Theorem 3.2 [Foucart and Uribe-Bravo [60]] *The only point that may be polar is \underline{a} . If $\mathbf{d} < \infty$ then \underline{a} is polar. In the unbounded variation case, $\underline{a} = 0$ and 0 is polar or hit with positive probability accordingly as*

$$\int_{\theta}^{\infty} \frac{dz}{\Psi(z)} \exp \left[\int_{\theta}^z \frac{\Phi(x)}{\Psi(x)} dx \right] = \infty \text{ or } < \infty, \quad (\text{I.13})$$

where θ is an arbitrary constant larger than $q(\mu) = \sup\{q \geq 0 : \Psi(q) = \mu\}$.

The third theorem yields a necessary and sufficient criterion for the recurrence or transience property of a CBI(Ψ, Φ) process when $\Phi \not\equiv 0$. We mention that results in the same vein have been obtained by Chen et al. [22] in the setting of discrete branching Markov chains with immigration.

Theorem 3.3 *Let us assume $\Phi \not\equiv 0$.*

(a) *In the critical or subcritical case, the process is recurrent or transient accordingly as*

$$\int_0^1 \frac{dz}{\Psi(z)} \exp \left[- \int_z^1 \frac{\Phi(x)}{\Psi(x)} dx \right] = \infty \text{ or } < \infty. \quad (\text{I.14})$$

(b) *In the supercritical case, the CBI(Ψ, Φ) process is transient.*

For a critical CBI process with finite variance, we have a simpler criterion for its recurrence and transience, which somehow is reminiscent of the Feller diffusion.

Corollary 3.4 Assume $\int_1^\infty u^2 \log u \pi(du) < \infty$ and $\int_1^\infty u^2 \nu(du) < \infty$ and consider

$$\begin{aligned}\Psi(q) &= \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du), \\ \Phi(q) &= bq + \int_0^\infty (1 - e^{-qu})\nu(du).\end{aligned}$$

Let $\tilde{\sigma}^2 := \Psi''(0)$ and $\tilde{b} := \Phi'(0)$. The process is transient if and only if $2\tilde{b} > \tilde{\sigma}^2$.

Remark 3.1 – The integrability condition $\int_\theta^\infty \frac{1}{\Psi(z)} dz < \infty$ implies that $\mathbf{d} = \infty$, which entails that $\underline{\mathbf{a}} = 0$. However, it is worth mentioning that none of these implications are equivalences.

- In the criterion of Theorem 3.3, when the mechanism Ψ is subcritical, one can replace Φ by the map $q \mapsto \int_1^\infty (1 - e^{-qx})\nu(dx)$ in (I.14). In other words, neither the continuous immigration nor its small jumps play a role for the process to be transient. Moreover, we should mention that when $\Psi(q) = \gamma q$, the criterion coincides with that of Shiga [111]. Note that a subcritical CBI with $\Phi(q) = bq$ is always recurrent.
- If the state 0 is not polar, then one has the same necessary and sufficient conditions for both neighborhood-recurrence and point-recurrence (studied in [60]) of the state 0.

Example 3.1 Consider $\Psi(q) = dq^\alpha$, $\Phi(q) = d'q^\beta$ with $\alpha \in (1, 2]$ and $\beta \in (0, 1]$.

- If $\beta > \alpha - 1$, the process is recurrent and 0 is polar.
- If $\beta < \alpha - 1$, the process is transient and 0 is not polar.
- If $\beta = \alpha - 1$ and $\alpha \in (1, 2]$, the process is recurrent if $d'/d \leq \alpha - 1$ and transient if $d'/d > \alpha - 1$. The point 0 is polar if and only if $d'/d \geq \alpha - 1$. We observe that if $d'/d = \alpha - 1$, then 0 is polar but $\liminf_{t \rightarrow \infty} X_t = 0$. We point out that in this case, the CBI process is selfsimilar. Patie in [98] obtained the condition for 0 to be polar via other arguments.

4 State space of CBI processes.

We study here the state space of a general CBI process. A trivial example of CBI process which is not irreducible in \mathbb{R}_+ is the deterministic one. Namely, if $\Phi(q) = bq$ and $\Psi(q) = \gamma q$ with $\gamma > 0$, the associated CBI is $X_t = X_0 e^{-\gamma t} + \frac{b}{\gamma}(1 - e^{-\gamma t})$. The path of this process is above $\frac{b}{\gamma}$ as soon as $X_0 > \frac{b}{\gamma}$. As already mentioned the case when $-\Psi$ is the Laplace exponent of a subordinator is excluded. Recall $\mathbf{d} > 0$ and $\underline{\mathbf{a}} = \frac{b}{\mathbf{d}}$. We state a lower bound for any CBI process.

Proposition 4.1 Let X be a CBI(Ψ, Φ) process started at $x \in (0, \infty)$. Then, \mathbf{P}_x almost surely, for all $t > 0$,

$$X_t \geq e^{-\mathbf{d}t}x + \underline{\mathbf{a}}(1 - e^{-\mathbf{d}t}). \quad (\text{I.15})$$

In particular, this implies $\liminf_{t \rightarrow \infty} X_t \geq \underline{\mathbf{a}}$.

Proof. Firstly, one can notice that when X has unbounded variation paths, then $\mathbf{d} = \infty$ and $\underline{\mathbf{a}} = 0$. The lower bound in the lemma is then null and the statement is clear. We then focus on the case of bounded variation and denote, for all $t > 0$, $x_t := e^{-\mathbf{d}t}x + \underline{\mathbf{a}}(1 - e^{-\mathbf{d}t})$. Using the càdlàg regularity, it will be sufficient to prove that for a fixed $t \in (0, \infty)$,

$$\mathbf{P}_x(X_t < x_t) = 0. \quad (\text{I.16})$$

Let $(\tilde{X}_t, t \geq 0)$ be a CBI $(\Psi, \tilde{\Phi})$, where $\tilde{\Phi}(\lambda) = b\lambda$. We have then for all λ ,

$$\mathbb{E}_x[e^{-\lambda\tilde{X}_t}] = \exp\left(-xv_t(\lambda) - b \int_0^t v_s(\lambda) ds\right),$$

The function $\Phi - \tilde{\Phi}$ is an immigration mechanism. Thus, one can plainly derive from (I.8) the bound $\mathbf{P}_x(X_t < x_t) \leq \mathbf{P}_x(\tilde{X}_t < x_t)$. We will show that the latter probability is 0.

It is well-known that for a fixed $t > 0$, the map $\lambda \mapsto v_t(\lambda)$ is the Laplace exponent of a subordinator (see for instance Bertoin-Le Gall [10]). More precisely the underlying subordinator has for drift e^{-dt} (see Duquesne and Labbé in [39] Section 2.1 for details). Consider the Laplace exponent of the driftless subordinator :

$$w_t(\lambda) := v_t(\lambda) - e^{-dt}\lambda.$$

One can write

$$\mathbb{E}_x[e^{-\lambda\tilde{X}_t}] = \exp\left(-\lambda x_t - xw_t(\lambda) - b \int_0^t w_s(\lambda) ds\right), \quad (\text{I.17})$$

and

$$\mathbb{E}_x[\exp(-\lambda(\tilde{X}_t - x_t))] = \exp\left(-xw_t(\lambda) - b \int_0^t w_s(\lambda) ds\right). \quad (\text{I.18})$$

One can plainly check that the map $\lambda \mapsto xw_t(\lambda) + b \int_0^t w_s(\lambda) ds$ is the Laplace exponent of a non-negative random variable. We deduce $\tilde{X}_t \geq x_t$, \mathbf{P}_x -a.s., and thus (I.16).

Remark 4.1 Alternatively, one can use stochastic calculus. Consider the case of bounded variation paths, for which $\sigma = 0$ and $\int_0^1 x\pi(dx) < \infty$ (the result is vacuous in the unbounded variation case). Let $N_0(ds, du)$ and $N_1(ds, dz, du)$ be two independent Poisson random measures on $(0, \infty)^2$ and $(0, \infty)^3$ with intensity $ds\nu(dz)$ and $ds\pi(dz)du$, respectively. For each $x \geq 0$ there is a pathwise unique positive strong solution to the following stochastic equation :

$$X_t = x + \int_0^t (b - \mathbf{d}X_s) ds + \int_0^t \int_0^\infty z N_0(ds, dz) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z N_1(ds, dz, du).$$

By Itô's formula, the solution $(X_t, t \geq 0)$ is a CBI (Ψ, Φ) with $\sigma = 0$; See Theorem 3.1 of Dawson and Li [27]. On the other hand, let $(x_t, t \geq 0)$ be the solution to the ordinary differential equation :

$$x_t = x + \int_0^t (b - \mathbf{d}x_s) ds.$$

It follows from Theorem 2.2 of Dawson and Li [27] that $\mathbb{P}_x(X_t \geq x_t \text{ for all } t \geq 0) = 1$.

In the (sub)critical case, a necessary and sufficient condition for the existence of a stationary distribution was announced by Pinsky [103] and obtained by Li :

Theorem 4.2 (Theorem 3.20 in Li [89]) *i) If $\int_0^1 \frac{\Phi(u)}{\Psi(u)} du < \infty$, then the CBI (Ψ, Φ) process, $(X_t, t \geq 0)$, has an invariant probability distribution. In the subcritical case ($\Psi'(0+) > 0$), this integral condition is equivalent to*

$$\int_1^\infty \log(u)\nu(du) < \infty.$$

ii) If $\int_0^1 \frac{\Phi(u)}{\Psi(u)} du = \infty$, then for all $x, b \in \mathbb{R}_+$,

$$\lim_{t \rightarrow \infty} \mathbf{P}_x(X_t \leq b) = 0.$$

Remark 4.2 – Following the usual terminology, a CBI process with a non-degenerate limit distribution is said to be positive recurrent. One can easily check that any positive recurrent CBI process is indeed recurrent in the sense of (I.1) by using Theorem 3.3.

– The second statement of Theorem 4.2 is not plainly stated in [89]. Nevertheless, one can observe in the proof of Theorem 3.20 in [89] that if $\int_0^1 \frac{\Phi(u)}{\Psi(u)} du = \infty$, then $\mathbf{E}_x \left[e^{-\lambda X_t} \right] \xrightarrow{t \rightarrow \infty} 0$. We refer also to the Appendix A of Keller-Ressel and Mijatović [71].

It follows from Theorem 4.2 and Proposition 4.4 of [71] that either $(X_t, t \geq 0)$ has the non-degenerate limit distribution with support $[\underline{a}, \infty)$ or $X_t \rightarrow \infty$ in probability as $t \rightarrow \infty$. Thus, applying Fatou's lemma, it is not hard to see that

$$\mathbf{P}_x \left(\limsup_{t \rightarrow \infty} X_t = \infty \right) = 1, \text{ for any } x \in \mathbb{R}_+, \quad (\text{I.19})$$

when $\Phi \not\equiv 0$. Starting from a point in $\mathcal{S} = [\underline{a}, \infty)$, the process stays in \mathcal{S} , so we shall work with \mathcal{S} as the state space.

5 Proof of Theorem 3.1.

Recall that L is the infinitesimal generator of a CBI(Ψ, Φ) stated in (I.9). Let $\mu \geq 0$ and set $\bar{\Psi}(q) = \Psi(q) - \mu$. Denote \bar{L} the generator of $(\bar{X}_t, t \geq 0)$, a CBI($\bar{\Psi}, \Phi$). For all $f \in C_0^2(\mathbb{R}_+)$

$$\bar{L}f(x) = Lf(x) - \mu x f(x).$$

Recall $q(\mu) = \sup\{q \geq 0 : \Psi(q) = \mu\}$. Note that $q(\mu) < \infty$ since by assumption there exists q such that $\Psi(q) > 0$. We fix a constant $\theta = \theta(\mu) \in (q(\mu), \infty)$. The next Lemma provides some invariant functions for the generator \bar{L} .

Lemma 5.1 *Let $\lambda, \mu \geq 0$. Define, for $x \in (q(\mu), \infty)$,*

$$g_{\lambda, \mu}(x) := \frac{1}{\Psi(x) - \mu} \exp \left[\int_{\theta}^x \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \right], \quad (\text{I.20})$$

and

$$f_{\lambda, \mu}(x) := \int_{q(\mu)}^{\infty} e^{-xz} g_{\lambda, \mu}(z) dz.$$

If $\lambda > 0$, the function $f_{\lambda, \mu}$ is a C_0^2 -function decreasing on (\underline{a}, ∞) such that

$$\bar{L}f_{\lambda, \mu} = \lambda f_{\lambda, \mu}.$$

Proof.[Proof of Lemma 5.1.] Let $\lambda > 0, \mu \geq 0$. Firstly, we check that $f_{\lambda, \mu}(x)$ is well-defined for $x > \underline{a}$. We have

$$\frac{\Phi(u)}{\Psi(u) - \mu} = \frac{\Phi(u)}{u} \frac{u}{\Psi(u) - \mu} \xrightarrow{u \rightarrow \infty} \frac{b}{\mathbf{d}} =: \underline{a},$$

therefore

$$\frac{1}{z} \int_{\theta}^z \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \xrightarrow{z \rightarrow \infty} \underline{a}.$$

Since $x > \underline{a}$ and $\Psi(z) - \mu \geq Cz$ with large enough z , and a constant $C > 0$, we get for all $\lambda \geq 0$

$$\int_{\theta}^{\infty} \frac{dz}{\Psi(z)} \exp \left[-xz + \int_{\theta}^z \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \right] < \infty. \quad (\text{I.21})$$

It remains to verify the integrability at $q(\mu)$. We have

$$\int_{q(\mu)}^{\theta} \frac{dz}{\Psi(z) - \mu} \exp \left[-xz - \int_z^{\theta} \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \right] \leq \int_{q(\mu)}^{\theta} \frac{dz}{\Psi(z) - \mu} \exp \left[- \int_z^{\theta} \frac{\lambda}{\Psi(u) - \mu} du \right].$$

Consider $\lambda > 0$, an antiderivative of the integrand in the right hand side is

$$z \mapsto \frac{1}{\lambda} \exp \left[-\lambda \int_z^{\theta} \frac{1}{\Psi(u) - \mu} du \right]. \quad (\text{I.22})$$

This takes a finite value at $q(\mu)$ and yields the wished integrability. Similar arguments entails that $f_{\lambda, \mu}$ is twice-differentiable and tends to 0 when x goes to ∞ .

Remark that $g_{\lambda, \mu}$ solves the ordinary differential equation

$$\forall z \in (q(\mu), \infty) \quad \Psi'(z)g_{\lambda, \mu}(z) + (\Psi(z) - \mu)g'_{\lambda, \mu}(z) = (\Phi(z) + \lambda)g_{\lambda, \mu}(z). \quad (\text{I.23})$$

For all z , define $h_z(x) = e^{-xz}$, one can easily check that

$$\bar{L}h_z(x) = [x(\Psi(z) - \mu) - \Phi(z)]h_z(x).$$

We compute

$$\begin{aligned} \bar{L}f_{\lambda, \mu}(x) - \lambda f_{\lambda, \mu}(x) &= \int_{q(\mu)}^{\infty} \left(\bar{L}h_z(x) - \lambda h_z(x) \right) g_{\lambda, \mu}(z) dz \\ &= \int_{q(\mu)}^{\infty} e^{-xz} (x(\Psi(z) - \mu) - \Phi(z) - \lambda) g_{\lambda, \mu}(z) dz \\ &= \int_{q(\mu)}^{\infty} e^{-xz} \left(\Psi'(z)g_{\lambda, \mu}(z) + (\Psi(z) - \mu)g'_{\lambda, \mu}(z) - (\Phi(z) + \lambda)g_{\lambda, \mu}(z) \right) dz \\ &= 0. \end{aligned}$$

The third equality follows from integration by parts. Indeed, we have

$$(\Psi(x) - \mu)g_{\lambda, \mu}(x) = \exp \left(\int_{\theta}^x \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} du \right) \xrightarrow{x \rightarrow q(\mu)} 0$$

because $\int_{q(\mu)+} \frac{du}{\Psi(u) - \mu} = \infty$, since $\Psi(u) - \mu$ is always sub-linear near $q(\mu)$. The last equality holds true because of the ODE (I.23).

We establish now Theorem 3.1.

Proof.[Proof of Theorem 3.1] Consider a CBI(Ψ, Φ) process $(X_t, t \geq 0)$ and define $I_t := \int_0^t X_s ds$. The family $(e^{-\mu I_t}, t \geq 0)$ is a continuous multiplicative functional of $(X_t, t \geq 0)$. Denote the subordinate semi-group (in the terminology of Blumenthal and Gettoor [18]) by Q_t , and the subprocess by $(\bar{X}_t, t \geq 0)$. We have for all $f \in C^2(\mathbb{R}_+)$

$$Q_t f(x) = \bar{\mathbb{E}}[f(\bar{X}_t)] := \mathbb{E}_x[f(X_t)e^{-\mu I_t}].$$

We refer the reader to Theorem 3.3 and 3.12 pages 106 and 110 of Blumenthal and Gettoor [18]. The bivariate process $((X_t, I_t); t \geq 0)$ is a Markov process. Similarly as Patie [98] (see Lemma 7), one can see by Itô's formula that for any function $f \in C_c^2(\mathbb{R}_+)$,

$$f(X_t)e^{-\mu \int_0^t X_s ds} - f(x) - \int_0^t e^{-\mu \int_0^s X_u du} (Lf(X_s) - \mu X_s f(X_s)) ds$$

is a local martingale. Theorem 4.1.2 in [90] applies and ensures that $(\bar{X}_t, t \geq 0)$ is a CBI $(\bar{\Psi}, \Phi)$ process. Firstly we consider $a > \underline{a}$, and recall $\sigma_a = \inf\{t \geq 0, X_t = a\}$. From Lemma 5.1, one can apply Dynkin's formula to the Markov process $(\bar{X}_t, t \geq 0)$ killed at time σ_a , we get

$$\bar{\mathbb{E}}_x[e^{-\lambda \sigma_a \wedge t} f_{\lambda, \mu}(\bar{X}_{\sigma_a \wedge t})] = f_{\lambda, \mu}(x),$$

and thus

$$\mathbb{E}_x[e^{-\mu I_{\sigma_a \wedge t}} e^{-\lambda \sigma_a \wedge t} f_{\lambda, \mu}(X_{\sigma_a \wedge t})] = f_{\lambda, \mu}(x).$$

If we start from a point $x > a$, since the process has no downward jumps, $X_t > a$ for all time $t < \sigma_a$, and $f_{\lambda, \mu}(X_{t \wedge \sigma_a}) \leq f_{\lambda, \mu}(a)$. Therefore the left hand side of the above equality is bounded and when $t \rightarrow \infty$, we get

$$\mathbb{E}_x \left[\exp \left\{ -\mu \int_0^{\sigma_a} X_t dt - \lambda \sigma_a \right\} \right] = \frac{f_{\lambda, \mu}(x)}{f_{\lambda, \mu}(a)},$$

with the convention $e^{-\infty} = 0$. To prove the formula in the case $a = \underline{a}$, we notice that σ_a is increasing towards $\sigma_{\underline{a}}$, when $a \downarrow \underline{a}$, by quasi-left continuity of the CBI. The result follows by monotonicity.

6 Hitting times and polarity of the boundary point.

6.1 Hitting times.

By a slight abuse of notation, define $f_\lambda := f_{\lambda, 0}$ and $g_\lambda := g_{\lambda, 0}$, that is to say

$$g_\lambda(x) = \frac{1}{\Psi(x)} \exp \left[\int_\theta^x \frac{\Phi(u) + \lambda}{\Psi(u)} du \right] \quad (\text{I.24})$$

and $f_\lambda(x) = \int_{q(0)}^\infty e^{-xz} g_\lambda(z) dz$. As a direct consequence of Theorem 1.2, when $\mu = 0$ we get the following corollary.

Corollary 6.1 *For all $\lambda \in (0, \infty)$, and $x > a \geq \underline{a}$*

$$\mathbf{E}_x \left[e^{-\lambda \sigma_a} \right] = \frac{f_\lambda(x)}{f_\lambda(a)}. \quad (\text{I.25})$$

Remark 6.1 The process $(e^{-\lambda t} f_\lambda(X_t), t \geq 0)$ is not a martingale. This issue comes from the fact that f_λ is not in the domain of the generator associated to the CBI (Ψ, Φ) process. Indeed, we can plainly check that for any mechanisms Ψ, Φ : $|f'_\lambda(0)| = \infty$. In the same vein as scale functions for Lévy processes, one has to stop the process to get a martingale.

To the best of our knowledge these functions do not appear in the literature even when no immigration is taken into account. Consider that particular case and assume here that $\Phi \equiv 0$. The CBI is then a CB (Ψ) process. In the supercritical case, an easy calculation of the limit when λ goes to 0 yields

$$\mathbb{P}_x(\sigma_a < \infty) = \exp(-(x-a)q(0)), \quad \forall a \in]0, x].$$

Note that this equality holds for $a = 0$ under Grey's condition (I.3) (see for instance Theorem 3.8 in [89]). Furthermore, the function f_λ has a simpler expression. Indeed, since $\frac{z}{\Psi(z)} \rightarrow 1/\mathbf{d} \in [0, \infty)$ as $z \rightarrow \infty$, there exists $k > 0$ such that for $x > 0$,

$$e^{-xz} \exp\left(\lambda \int_\theta^z \frac{du}{\Psi(u)}\right) \leq e^{-xz} \exp\left(\lambda k \int_\theta^z \frac{du}{u}\right) = e^{-xz} (z/\theta)^{\lambda k} \xrightarrow{z \rightarrow \infty} 0.$$

Since $\Psi'(q(0)) < \infty$, we also have $e^{-xz} \exp\left(\lambda \int_\theta^z \frac{du}{\Psi(u)}\right) \rightarrow 0$ as $z \rightarrow q(0)$. Performing integration by parts, a notable cancellation occurs, we get :

$$f_\lambda(x) = \frac{x}{\lambda} \int_{q(0)}^\infty e^{-xz} \exp\left(\lambda \int_\theta^z \frac{du}{\Psi(u)}\right) dz,$$

and then for $x > a > \underline{a}$,

$$\mathbf{E}_x[e^{-\lambda\sigma_a}] = \frac{x \int_{q(0)}^\infty e^{-xz} \exp\left(\lambda \int_\theta^z \frac{du}{\Psi(u)}\right) dz}{a \int_{q(0)}^\infty e^{-az} \exp\left(\lambda \int_\theta^z \frac{du}{\Psi(u)}\right) dz}.$$

We return to the general case for which $\Phi \neq 0$. We end the section with a formula for the mean of the hitting time. When the branching mechanism is supercritical, we have $\mathbf{P}_x(\sigma_a = \infty) > 0$ for $x > a \geq \underline{a}$. This follows readily by taking $\lambda \rightarrow 0$ in (6.1). Hence, we focus on the critical or subcritical case.

Corollary 6.2 *In the critical or sub-critical case, for all $x > a \geq \underline{a}$,*

$$\mathbf{E}_x[\sigma_a] = \int_0^\infty \frac{dz}{\Psi(z)} (e^{-az} - e^{-xz}) \exp\left(\int_0^z \frac{\Phi(u)}{\Psi(u)} du\right). \quad (\text{I.26})$$

Proof. Consider firstly $a > \underline{a}$; the case $a = \underline{a}$ will follow by monotonicity. One has

$$\mathbf{E}_x[\sigma_a] = \lim_{\lambda \rightarrow 0} \lambda^{-1} \left(1 - \mathbf{E}_x[e^{-\lambda\sigma_a}]\right) = \lim_{\lambda \rightarrow 0} \frac{f_\lambda(a) - f_\lambda(x)}{\lambda f_\lambda(a)}.$$

By integration by parts, we compute

$$\begin{aligned} \lambda f_\lambda(a) &= \int_0^\infty dz \left(a - \frac{\Phi(z)}{\Psi(z)}\right) \exp\left(-az + \int_1^z \frac{\Phi(u)}{\Psi(u)} du + \lambda \int_1^z \frac{du}{\Psi(u)}\right) \\ &\xrightarrow{\lambda \rightarrow 0} \exp\left(-\int_0^1 \frac{\Phi(u)}{\Psi(u)} du\right). \end{aligned} \quad (\text{I.27})$$

Moreover, by monotonicity,

$$f_\lambda(a) - f_\lambda(x) \xrightarrow{\lambda \rightarrow 0} \int_0^\infty \frac{dz}{\Psi(z)} (e^{-az} - e^{-xz}) \exp\left(\int_1^z \frac{\Phi(u)}{\Psi(u)} du\right). \quad (\text{I.28})$$

The above integral is finite if and only if $\int_{0+} \frac{z}{\Psi(z)} \exp\left(\int_1^z \frac{\Phi(u)}{\Psi(u)} du\right) dz < \infty$. Since $z \leq C\Phi(z)$ at the neighbourhood of 0, for some constant $C \in (0, \infty)$, we have plainly the integrability by anti-differentiation. When $\int_1^z \frac{\Phi(u)}{\Psi(u)} du < \infty$ one can combine (I.27) and (I.28) to get the desired formula. When $\int_0^1 \frac{\Phi(u)}{\Psi(u)} du = \infty$, we have $\mathbf{E}_x[\sigma_a] = \infty$.

Remark 6.2 Let us recall Theorem 4.2. Note that for a CBI(Ψ, Φ) with $\Phi \neq 0$ one has for all $a > 0$, $\mathbf{E}_x[\sigma_a] < \infty$ if and only if $\int_0^1 \frac{\Phi(u)}{\Psi(u)} du < \infty$ (this is nothing but the positive recurrence of the process). The first moment of a CB(Ψ) process is finite if and only if $\int_0 \frac{z}{\Psi(z)} dz < \infty$. One can plainly check those assertions for $a = 0$ in the non polar case.

6.2 Proof of Theorem 3.2 : polarity of zero.

When $\underline{a} = 0$, Corollary 6.1 provides the Laplace transform of σ_0 , the hitting time of 0. We study now the polarity of the boundary. Recall that a point $a \in \mathcal{S}$ is said to be polar if for all $x \in \mathcal{S}$ such as $x \neq a$,

$$\mathbb{P}_x(\sigma_a < \infty) = 0.$$

Let $\lambda > 0$. From Corollary 6.1, the point a is polar if and only if $f_\lambda(a) = \infty$. We have seen that $f_\lambda(x) \in (0, \infty)$ for any $x \in (\underline{a}, \infty)$. Thus only \underline{a} may be polar. Firstly, if $\mathbf{d} < \infty$, note that

$$\frac{\Phi(x)}{\Psi(x)} - \underline{a} = \frac{1}{b\Psi(x)} \left[\mathbf{d} \int_0^\infty (1 - e^{-xu})\nu(du) + b \int_0^\infty (1 - e^{-xu})\pi(du) \right] \geq 0.$$

Then

$$\int_\theta^\infty \frac{dz}{\Psi(z)} \exp \left[-\underline{a}z + \int_\theta^z \frac{\Phi(x)}{\Psi(x)} dx \right] \geq e^{-\underline{a}\theta} \int_\theta^\infty \frac{dz}{\Psi(z)} = \infty,$$

and therefore we have $f_\lambda(\underline{a}) = \infty$.

Assume now $\mathbf{d} = \infty$ (thus $\underline{a} = 0$) and $\int_\theta^\infty \frac{dz}{\Psi(z)} \exp \left[\int_\theta^z \frac{\Phi(x)}{\Psi(x)} dx \right] = \infty$. We have $f_\lambda(0) = \infty$ and the same arguments hold.

We show now that if $\int_\theta^\infty \frac{dz}{\Psi(z)} \exp \left[\int_\theta^z \frac{\Phi(x)}{\Psi(x)} dx \right] < \infty$, then

$$\mathbb{P}_x[\sigma_0 < \infty] > 0. \tag{I.29}$$

Recall g_λ defined in (I.24). Using Corollary 6.1 for $a = 0$ and $\lambda \in (0, \infty)$, one gets

$$\mathbb{E}_x[e^{-\lambda\sigma_0}] = \frac{\int_{q(0)}^\theta e^{-xz} g_\lambda(z) dz + \int_\theta^\infty e^{-xz} g_\lambda(z) dz}{\int_{q(0)}^\theta g_\lambda(z) dz + \int_\theta^\infty g_\lambda(z) dz} \tag{I.30}$$

$$\geq \frac{e^{-x\theta} \int_{q(0)}^\theta g_\lambda(z) dz}{\int_{q(0)}^\theta g_\lambda(z) dz + \int_\theta^\infty g_\lambda(z) dz} \tag{I.31}$$

$$\geq \frac{e^{-x\theta}}{1 + \int_\theta^\infty g_\lambda(z) dz / \int_{q(0)}^\theta g_\lambda(z) dz} \tag{I.32}$$

The assumption entails that $\int_\theta^\infty g_\lambda(z) dz \xrightarrow{\lambda \rightarrow 0} \int_\theta^\infty g_0(z) dz < \infty$. On the one hand, if $\int_{q(0)}^\theta g_0(z) dz < \infty$, then, in the right hand-side of (I.32), the integrals tends to a positive and finite limit when λ goes to 0. On the other hand, if $\int_{q(0)}^\theta g_0(z) dz = \infty$, the right-hand-side tends to $e^{-x\theta}$. In both cases, we deduce that $\mathbb{E}_x[e^{-\lambda\sigma_0}]$ decreases to a positive limit when λ decreases to 0, which entails (I.29).

7 Recurrence and transience.

The next paragraph is devoted to some corollaries of Theorem 3.3, which is proved in paragraph 7.2. In paragraph 7.3, we provide examples of null-recurrent CBI processes.

7.1 Infimum of a transient CBI and long-term behavior of a CB process conditioned to be non-extinct.

Assume that the process $(X_t, t \geq 0)$ is transient. One can plainly check that the function

$$f_0(x) = \int_{q(0)}^{\infty} \frac{dz}{\Psi(z)} \exp\left(-xz + \int_{\theta}^z \frac{\Phi(u)}{\Psi(u)} du\right)$$

takes finite values for all $x > \underline{a}$. Applying Corollary 6.1 and Theorem 3.3, we obtain the following proposition.

Proposition 7.1 *Denote the overall infimum of the transient process $(X_t, t \geq 0)$ by I . We have*

$$\mathbb{P}_x(I \leq a) = \mathbb{P}_x(\sigma_a < \infty) = \frac{f_0(x)}{f_0(a)}.$$

If $f_0(0) = \int_{q(0)}^{\infty} \frac{dz}{\Psi(z)} \exp\left(\int_{\theta}^z \frac{\Phi(u)}{\Psi(u)} du\right) < \infty$ (i.e. 0 is not polar and the process is transient) then the law of I has an atom at 0.

Proof. Firstly, note that $\mathbb{P}_x[I \leq a] = \mathbb{P}_x[\sigma_a < \infty]$. By Theorem 3.3, the integrability condition needed to define f_0 is satisfied. Taking $\lambda = 0$, in the formula for the Laplace transform of σ_a , yields $\mathbb{P}_x[\sigma_a < \infty] = \frac{f_0(x)}{f_0(a)}$.

The $\text{CB}(\Psi)$ process conditioned to be non extinct corresponds to a $\text{CBI}(\Psi, \Phi)$ process with $\Phi(q) = \Psi'(q) - \Psi'(0+)$ (see e.g. [91]). As a direct corollary of Theorem 3.3, we recover and complete some results due to Lambert (see Theorem 4.2-i in [77]). In the (sub)critical case, $q(0) = 0$ and we choose $\theta = 1$.

Corollary 7.2 *The critical CB process conditioned to be non extinct is transient. Moreover, if the process starts at x , its minimum is uniformly distributed over $[0, x]$. The subcritical $\text{CB}(\Psi)$ process conditioned to be non extinct is recurrent or transient according to*

$$\int_0^1 \frac{dz}{z} \exp\left(-\int_z^1 \left(\frac{1}{u} - \frac{\Psi'(0+)}{\Psi(u)}\right) du\right) = \infty \text{ or } < \infty.$$

Proof. Let Ψ be a critical reproduction mechanism. Consider the case $\Phi = \Psi'$, the $\text{CBI}(\Psi, \Phi)$ process has the same law as the $\text{CB}(\Psi)$ process conditioned to the non extinction. In that case, we have clearly $\int_1^z \frac{\Psi'(u)}{\Psi(u)} du = \log(\Psi(z)) - \log(\Psi(1))$, and therefore

$$\int_0^1 \frac{dz}{\Psi(z)} \exp\left(\int_1^z \frac{\Psi'(u)}{\Psi(u)} du\right) = \frac{1}{\Psi(1)} < \infty.$$

In order to deal with the minimum, one can readily check that $f_0(x) = 1/x$. Thus, the random variable I is uniformly distributed over $[0, x]$. For the subcritical case, plugging $\Phi = \Psi' - \Psi'(0+)$ in the integral of Theorem 3.3, yields easily the statement.

Remark 7.1 The fact that the minimum of a critical $\text{CBI}(\Psi, \Psi')$ is uniformly distributed can be obtained alternatively from Proposition 3 in Chaumont [21], which states the corresponding result for Lévy processes conditioned to stay positive. Indeed, Lambert, in [76], shows that the CB process conditioned to be non-extinct has the same law as a time-changed Lévy process conditioned to stay positive.

7.2 Proofs of Theorem 3.3 and Corollary 3.4.

7.2.1 Proof of Theorem 3.3.

Firstly, we establish statement (a) of Theorem 3.3. The proof relies on the study of the Laplace transform of the hitting times provided by Corollary 6.1. Recall

$$g_\lambda(x) = \frac{1}{\Psi(x)} \exp \left[\int_1^x \frac{\Phi(u) + \lambda}{\Psi(u)} du \right].$$

Recurrence. Assume that

$$\int_0^1 \frac{1}{\Psi(x)} \exp \left[- \int_x^1 \frac{\Phi(u)}{\Psi(u)} du \right] dx = \infty. \quad (\text{I.33})$$

For every $x \geq a$,

$$\mathbb{P}_x[\sigma_a < \infty] = \lim_{\lambda \rightarrow 0} \mathbb{E}_x[e^{-\lambda\sigma_a}].$$

By Corollary 6.1, we have for all $a > \underline{a}$, for all $\varepsilon \in (0, \infty)$,

$$\mathbb{E}_x[e^{-\lambda\sigma_a}] = \frac{\int_0^\varepsilon e^{-xz} g_\lambda(z) dz + \int_\varepsilon^\infty e^{-xz} g_\lambda(z) dz}{\int_0^\varepsilon e^{-az} g_\lambda(z) dz + \int_\varepsilon^\infty e^{-az} g_\lambda(z) dz} \quad (\text{I.34})$$

$$\geq \frac{e^{-x\varepsilon} \int_0^\varepsilon g_\lambda(z) dz}{\int_0^\varepsilon g_\lambda(z) dz + \int_\varepsilon^\infty e^{-az} g_\lambda(z) dz} \quad (\text{I.35})$$

$$\geq \frac{e^{-x\varepsilon}}{1 + \int_\varepsilon^\infty e^{-az} g_\lambda(z) dz / \int_0^\varepsilon g_\lambda(z) dz}. \quad (\text{I.36})$$

The assumption (I.33) entails that $\int_0^\varepsilon g_\lambda(z) dz \rightarrow \infty$, as λ goes to 0. On the other hand, equation (I.21) ensures that $\int_\varepsilon^\infty e^{-az} g_\lambda(z) dz < \infty$, because $a > \underline{a}$. Therefore, the right-hand-side in (I.36) tends to $e^{-x\varepsilon}$, as $\lambda \rightarrow 0$. The real number ε can be chosen arbitrarily small, so we get

$$\mathbb{E}_x[e^{-\lambda\sigma_a}] \xrightarrow{\lambda \rightarrow 0} 1.$$

We deduce that $\mathbf{P}_x(\sigma_a < \infty) = 1$ for any $x \geq a > \underline{a}$, which implies $\mathbf{P}_x \left(\liminf_{t \rightarrow \infty} X_t \leq \underline{a} \right) = 1$. The lower bound provided in Proposition 4.1 then entails $\mathbf{P}_x \left(\liminf_{t \rightarrow \infty} X_t = \underline{a} \right) = 1$, so the process is recurrent in sense of (I.1).

Transience. We now work under the assumption

$$\int_0^1 \frac{dz}{\Psi(z)} \exp \left(- \int_z^1 \frac{\Phi(u)}{\Psi(u)} du \right) < \infty. \quad (\text{I.37})$$

Let $a > \underline{a}$. We show that $\mathbf{P}_x \left(\liminf_{t \rightarrow \infty} X_t < a \right) = 0$. One has

$$\begin{aligned} \mathbf{P}_x \left(\liminf_{t \rightarrow \infty} X_t < a \right) &\leq \lim_{t \rightarrow \infty} \mathbf{P}_x(\sigma_a \circ \theta_t < \infty) \\ &= \lim_{t \rightarrow \infty} \mathbf{E}_x[\mathbf{P}_{X_t}(\sigma_a < \infty)]. \end{aligned} \quad (\text{I.38})$$

Moreover one can write that,

$$\mathbf{E}_x[\mathbf{P}_{X_t}(\sigma_a < \infty)] \leq \mathbf{P}_x(X_t \leq a) + \mathbf{E}_x[\mathbf{1}_{\{X_t > a\}} \mathbf{P}_{X_t}(\sigma_a < \infty)]. \quad (\text{I.39})$$

Firstly, under (I.37), one has $\int_0^1 \frac{\Phi(u)}{\Psi(u)} du = \infty$. According to ii) in Theorem 4.2, it implies that

$$\lim_{t \rightarrow \infty} \mathbf{P}_x(X_t \leq a) = 0.$$

Thus, the first term in (I.39) goes to 0 when $t \rightarrow \infty$. We focus now on the second term. Under (I.37), one can take $\lambda = 0$ in Corollary 6.1. For $x > a > \underline{a}$,

$$\mathbf{P}_x(\sigma_a < \infty) = \frac{\int_0^\infty \frac{dz}{\Psi(z)} \exp\left(-xz + \int_1^z \frac{\Phi(u)}{\Psi(u)} du\right)}{\int_0^\infty \frac{dz}{\Psi(z)} \exp\left(-az + \int_1^z \frac{\Phi(u)}{\Psi(u)} du\right)} = c_a \int_0^\infty g_0(z) e^{-xz} dz \quad (\text{I.40})$$

with $c_a = \left(\int_0^\infty g_0(z) e^{-az} dz\right)^{-1}$. Hence,

$$\begin{aligned} \mathbf{E}_x \left[\mathbf{1}_{\{X_t > a\}} \mathbf{P}_{X_t}(\sigma_a < \infty) \right] &= c_a \mathbf{E}_x \left[\mathbf{1}_{\{X_t > a\}} \int_0^\infty g_0(z) e^{-zX_t} dz \right] \\ &= c_a \int_0^\infty g_0(z) \mathbf{E}_x \left[\mathbf{1}_{\{X_t > a\}} e^{-zX_t} \right] dz. \end{aligned} \quad (\text{I.41})$$

Moreover, $\mathbf{E}_x \left[\mathbf{1}_{\{X_t > a\}} e^{-zX_t} \right] \leq e^{-za}$ and by (I.37) and (I.21), $\int_0^\infty g_0(z) e^{-za} dz < \infty$. Furthermore, since $\int_{0+} \frac{\Phi(u)}{\Psi(u)} du = \infty$,

$$\mathbf{E}_x \left[\mathbf{1}_{\{X_t > a\}} e^{-zX_t} \right] \leq \mathbf{E}_x \left[e^{-zX_t} \right] = \exp\left(-xv_t(z) - \int_{v_t(z)}^z \frac{\Phi(u)}{\Psi(u)} du\right) \xrightarrow{t \rightarrow \infty} 0. \quad (\text{I.42})$$

Thus, by dominated convergence, the integral (I.41) tends to 0, which entails the desired result. Therefore the process is transient in the sense of Definition (I.2).

In order to prove statement (b) (transience in the supercritical case), one has just to adapt the proof above. Indeed, we have $\int_{q(0)}^\theta \frac{dz}{\Psi(z)} \exp\left(-\int_z^\theta \frac{\Phi(u)}{\Psi(u)} du\right) < \infty$, so we can write (I.40). Moreover, one can use that $\int_{v_t(z)}^z \frac{\Phi(u)}{\Psi(u)} du \xrightarrow{t \rightarrow \infty} \int_{q(0)}^z \frac{\Phi(u)}{\Psi(u)} du = \infty$ in (I.42). \square

7.2.2 Proof of Corollary 3.4

Recall Corollary 3.4. We assume $\int_1^\infty u^2 \nu(du) < \infty$, we have then

$$\begin{aligned} \Phi(q) &= \left(b + \int_0^\infty u \nu(du)\right) q + \int_0^\infty (1 - e^{-qu} - qu) \nu(du) \\ &= \tilde{b}q + O(q^2) \text{ as } q \rightarrow 0. \end{aligned}$$

On the other hand

$$\Psi(q) = \frac{1}{2} \tilde{\sigma}^2 q^2 - \int_0^\infty (1 - e^{-qu} - qu + \frac{1}{2} q^2 u^2) \pi(du). \quad (\text{I.43})$$

Set

$$h(q) := \int_0^\infty (1 - e^{-qu} - qu + \frac{1}{2} q^2 u^2) \pi(du).$$

The function h is non-negative and increasing. Consider

$$\begin{aligned} h_1(q) &= \int_0^1 (1 - e^{-qu} - qu + \frac{1}{2} q^2 u^2) \pi(du) \\ h_2(q) &= \int_1^\infty (1 - e^{-qu} - qu + \frac{1}{2} q^2 u^2) \pi(du). \end{aligned}$$

It is easy to see that $h_1(q) = O(q^3)$ as $q \rightarrow 0$. Moreover, by assumption $\int_1^\infty u^2 \log u \pi(du) < \infty$ and it follows from Bingham et al. [14, Theorem 8.1.8, page 335] (with $n = 1$ and $\beta = 2$) that $\int_0^1 h_2(q)q^{-3}dq < \infty$. Then we have

$$\int_0^1 h(q)q^{-3}dq < \infty, \quad (\text{I.44})$$

which implies that $h(q) = o(q^2)$ as $q \rightarrow 0$. The following limit is finite and positive :

$$\lim_{q \rightarrow 0} \frac{\frac{1}{\Psi(q)} \exp \left[- \int_q^1 \frac{\Phi(x)}{\Psi(x)} dx \right]}{\frac{1}{q^2} \exp \left[\int_q^1 \frac{\tilde{b}x}{\frac{1}{2}\tilde{\sigma}^2 x^2} dx \right]} = \frac{2}{\tilde{\sigma}^2} \exp \left[- \int_0^1 \left(\frac{\Phi(x)}{\Psi(x)} - \frac{2\tilde{b}x}{\tilde{\sigma}^2 x^2} \right) dx \right] \in]0, \infty[. \quad (\text{I.45})$$

Indeed a simple calculation yields

$$\begin{aligned} \int_0^1 \left| \frac{\Phi(x)}{\Psi(x)} - \frac{2\tilde{b}x}{\tilde{\sigma}^2 x^2} \right| dx &\leq \int_0^1 \left| \frac{\Phi(x) - \tilde{b}x}{\Psi(x)} \right| dx + \frac{2\tilde{b}}{\tilde{\sigma}^2} \int_0^1 \frac{h(x)}{\Psi(x)x} dx \\ &= J_1 + J_2, \end{aligned}$$

and since $|\Phi(q) - \tilde{b}q| = O(\Psi(q))$ as $q \rightarrow 0$, we have $J_1 < \infty$, and (I.44) ensures that $J_2 < \infty$. Thus, it follows from (I.45) that

$$\int_0^1 \frac{dz}{\Psi(z)} \exp \left[- \int_z^1 \frac{\Phi(x)}{\Psi(x)} dx \right] < \infty \iff \int_0^1 \frac{1}{z^2} \exp \left[- \int_z^1 \frac{2\tilde{b}}{\tilde{\sigma}^2 x} dx \right] < \infty \iff 2\tilde{b} > \tilde{\sigma}^2.$$

7.3 Construction of subcritical null-recurrent CBI processes.

We look here for examples of null recurrent CBI processes. Assume that $\Psi(q) = \gamma q$, with $\gamma > 0$. The following computations remains valid for any general subcritical mechanism Ψ , because only the behaviour of Ψ at 0 matters. To avoid positive recurrence, we need to choose Φ such that $\int_0^{\frac{\Phi(q)}{q}} dq = \infty$, which is equivalent to

$$\int^\infty \log(u)\nu(du) = \infty. \quad (\text{I.46})$$

Moreover, to get a recurrent process, we know from Theorem 3.3 that Φ has to satisfy

$$\int_0^1 \frac{dz}{z} \exp \left(- \int_z^1 \frac{\Phi(u)}{\gamma u} du \right) = \infty. \quad (\text{I.47})$$

From condition (I.46), we know that the example of Φ we are looking for is not a deterministic drift. Moreover, when ν is not null, the value of the drift coefficient b has no influence for (I.47) to be fulfilled. Therefore, we will take $b = 0$ and we will exhibit a sufficient condition involving the Lévy measure ν to get (I.47). Denote $\bar{\nu}(u) := \nu([u, \infty))$ and recall from Chapter III of Bertoin [9] that there exists a universal constant κ such that

$$\Phi(q)/q \leq \kappa J_\Phi(1/q), \forall q > 0, \quad \text{where } J_\Phi(x) := \int_0^x \bar{\nu}(u) du, x > 0.$$

Thus, we have

$$\begin{aligned} \int_z^1 \frac{\Phi(u)}{\gamma u} du &\leq \frac{\kappa}{\gamma} \int_z^1 J_\Phi(1/u) du = \frac{\kappa}{\gamma} \int_1^{1/z} J_\Phi(u)/u^2 du \\ &= \frac{\kappa}{\gamma} \left(J_\Phi(1) - z J_\Phi(1/z) + \int_1^{1/z} \bar{\nu}(u)/u du \right), \end{aligned}$$

by integration by parts. Hence, a sufficient condition to get (I.47) is

$$\int_0^1 \frac{dz}{z} \exp\left(-\frac{\kappa}{\gamma} \int_1^{1/z} \bar{\nu}(u)/u du\right) = \infty. \quad (\text{I.48})$$

Example 2. We consider $\alpha \in \mathbb{R}$ and define ν such that

$$\int_1^{1/z} \bar{\nu}(u)/u du = \alpha \log \log 1/z \quad \text{up to an add. constant.}, \quad (\text{I.49})$$

so that the integral in (I.48) is of the same nature as $\int_0 \frac{dz}{z \log \log(1/z)^{\kappa\alpha/\gamma}}$. The integral will be infinite if α is chosen such that $\kappa\alpha/\gamma \leq 1$. We can get (I.49) taking $\bar{\nu}(u) := \alpha u \frac{d}{du} \log \log u = \frac{\alpha}{\log u}$ on $[100, \infty]$, that is $\nu(du) = \frac{\alpha}{u \log^2 u} \mathbf{1}_{[100, \infty]} du$. We can easily check that ν is a Lévy measure and that the condition (I.46) is satisfied. This example is related to that given by Sato and Yamazato in Section 7 of [108], in which the authors observe that the null recurrence or transience of this process is a function of κ/γ . The form of the criterion (I.47) and the role played by Bertrand's integrals provide a better understanding of the criterion. In the next example, the value of γ has no influence.

Example 3. We choose ν such that

$$\int_1^{1/z} \bar{\nu}(u)/u du = \log \log \log 1/z \quad \text{up to an add. constant.}, \quad (\text{I.50})$$

so that the integral in (I.47) is $\int_0 \frac{dz}{z \log \log(1/z)^{\kappa/\gamma}} = \infty$. We can get (I.50) taking

$$\bar{\nu}(u) := u \frac{d}{du} \log \log \log u = \frac{1}{\log u \log \log u}, \quad \text{on } [100, \infty[,$$

that is $\nu(du) = \frac{\log \log(u)+1}{u \log^2 u \log^2(\log u)} \mathbf{1}_{[100, \infty]} du$. The density of the last Lévy measure is equivalent at ∞ to $\frac{1}{u \log^2 u \log \log u}$. Hence, we can check that ν is indeed a Lévy measure and that it satisfies (I.46).

8 Total population.

As already said, one can see the integral $\int_0^{\sigma_a} X_s ds$ as the total population up to time σ_a . In the case of the CB(Ψ) ($\Phi \equiv 0$), and $a = 0$, this is known as the total progeny. Lamperti [78] observed that a CB(Ψ) process is connected to a spectrally positive Lévy process with Laplace exponent Ψ by a simple time-change $\int_0^t X_s ds$. This allows ones to transfer the study of $\int_0^{\sigma_a} X_s ds$ to that of the hitting time of a Lévy process. See Bingham [12] and Corollary 10.9 in Kyprianou [73]. In what follows, we recover the latter corollary and obtain its analogue with immigration.

Proposition 8.1 *Let $x > a \geq \underline{a}$, and assume that $\Phi \equiv 0$. For all $\mu > 0$,*

$$\mathbb{E}_x \left[\exp \left\{ -\mu \int_0^{\sigma_a} X_t dt \right\} \right] = \exp \left(-(x-a)q(\mu) \right). \quad (\text{I.51})$$

Proof. Firstly, let $\lambda > 0$. By integration by parts, we have

$$\int_{q(\mu)}^{\infty} \frac{dz}{\Psi(z) - \mu} \exp \left(-xz + \int_{\theta}^z \frac{\lambda}{\Psi(u) - \mu} du \right) = \int_{q(\mu)}^{\infty} dz x e^{-xz} \exp \left(-xz + \int_{\theta}^z \frac{\lambda}{\Psi(u) - \mu} du \right),$$

which tends to $\int_{q(\mu)}^{\infty} dz xe^{-xz} = \exp(-xq(\mu))$, as λ goes to 0. Thus, let $a > \underline{a} \geq 0$. The desired result follows from Theorem 1.2, with $\Phi \equiv 0$, letting $\lambda \rightarrow 0$. One can obtain now the case $a = \underline{a}$ by monotonicity and quasi-left continuity. More generally, we have the following corollary of Theorem 1.2.

Corollary 8.2 *Let $x > a \geq \underline{a}$, and assume that $\Phi \not\equiv 0$. For all $\mu > 0$,*

$$\mathbb{E}_x \left[\exp \left\{ -\mu \int_0^{\sigma_a} X_t dt \right\} \right] = \frac{\int_{q(\mu)}^{\infty} \frac{dz}{\Psi(z)-\mu} \exp \left(-xz + \int_{\theta}^z \frac{\Phi(u)}{\Psi(u)-\mu} du \right)}{\int_{q(\mu)}^{\infty} \frac{dz}{\Psi(z)-\mu} \exp \left(-az + \int_{\theta}^z \frac{\Phi(u)}{\Psi(u)-\mu} du \right)}. \quad (\text{I.52})$$

In the particular case of the CBI(Ψ, Ψ') with $\Psi'(0+) = 0$ (this is the CB(Ψ) conditioned to be non extinct), we have

$$\mathbb{E}_x \left[\exp \left\{ -\mu \int_0^{\sigma_a} X_t dt \right\} \right] = \frac{a}{x} \exp(-(x-a)q(\mu)), \quad \forall \mu > 0, x > a \geq \underline{a}.$$

Proof. It follows readily from Theorem 1.2 by letting $\lambda \rightarrow 0$. We only have to check that the integral in the numerator of (I.52) is finite. At infinity, this follows from (I.21). At $q(\mu)$, one can use that

$$\Psi'(z)-\mu \underset{z \rightarrow q(\mu)}{\sim} \Psi'(q(\mu))(z - q(\mu)), \quad \text{and} \quad \int_{\theta}^z \frac{\Phi(u)}{\Psi(u)-\mu} du \underset{z \rightarrow q(\mu)}{\sim} \frac{\Phi(q(\mu))}{\Psi'(q(\mu))} \log \left(\frac{z - q(\mu)}{\theta - q(\mu)} \right),$$

where $\Phi(q(\mu))$ and $\Psi'(q(\mu)) \in (0, \infty)$ because $\mu \in (0, \infty)$.

UNIFORM HAUSDORFF MEASURE OF THE LEVEL SETS OF
THE BROWNIAN TREE.

This chapter is taken from the submitted article [31].

Abstract. Let (\mathcal{T}, d) be the random real tree with root ρ coded by a Brownian excursion. So (\mathcal{T}, d) is (up to normalisation) Aldous CRT [5] (see Le Gall [80]). The a -level set of \mathcal{T} is the set $\mathcal{T}(a)$ of all points in \mathcal{T} that are at distance a from the root. We know from Duquesne and Le Gall [42] that for any fixed $a \in (0, \infty)$, the measure ℓ^a that is induced on $\mathcal{T}(a)$ by the local time at a of the Brownian excursion, is equal, up to a multiplicative constant, to the Hausdorff measure in \mathcal{T} with gauge function $g(r) = r \log \log 1/r$, restricted to $\mathcal{T}(a)$. As suggested by a result due to Perkins [100, 101] for super-Brownian motion, we prove in this paper a more precise statement that holds almost surely uniformly in a , and we specify the multiplicative constant. Namely, we prove that almost surely for any $a \in (0, \infty)$, $\ell^a(\cdot) = \frac{1}{2} \mathcal{H}_g(\cdot \cap \mathcal{T}(a))$, where \mathcal{H}_g stands for the g -Hausdorff measure.

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1 Introduction.

The Continuum Random Tree was introduced by Aldous [5] as a random compact metric space $(\mathcal{T}_1, d, \mathbf{m}_1)$, endowed with a mass measure \mathbf{m}_1 such that almost surely $\mathbf{m}_1(\mathcal{T}_1) = 1$. It appears as the scaling limit of a large class of discrete models of random trees, and can be alternatively encoded by a normalised Brownian excursion (see Le Gall [80]). This encoding procedure will be the viewpoint of the present paper, but for the sake of simplicity, we will *not* ask the total mass to be equal to one. Instead, we work on the tree encoded by a Brownian excursion $(e_t, t \geq 0)$, under its excursion measure \mathbf{N} . Let us mention that our result remains true for the CRT.

The Brownian tree has a distinguished vertex ρ called the root, so it makes sense to define, for all $a \in (0, \infty)$ the a -level set $\mathcal{T}(a) = \{\sigma \in \mathcal{T} : d(\rho, \sigma) = a\}$. Moreover, one can define the collection of measures $(\ell^a(d\sigma), \sigma \in \mathcal{T}, a \in (0, \infty))$, as the image of the local times on the levels of the excursion. Those measures are called local time measures. Indeed, \mathbf{N} -a.e. for all $a \in (0, \infty)$, the topological support of ℓ^a is included in $\mathcal{T}(a)$. Duquesne and Le Gall [42] showed that for a fixed level a , one has

$$\mathbf{N}\text{-a.e.} \quad \ell^a(\cdot) = c \mathcal{H}_g(\cdot \cap \mathcal{T}(a)), \tag{II.1}$$

where \mathcal{H}_g stands for the Hausdorff measure associated with the gauge function $g(r) = r \log \log 1/r$ and $c \in (0, \infty)$ is a multiplicative constant. In this paper, we prove that $c = \frac{1}{2}$ and that the result holds \mathbf{N} -a.e. simultaneously for all levels a . Let us mention that the value $\frac{1}{2}$ depends on the normalisation chosen for the excursion measure \mathbf{N} . The later leads to an underlying branching process with branching mechanism $\psi(\lambda) = \lambda^2$ (see II.12). A result similar to (II.1) has been obtained by Perkins [100, 101] for Super Brownian Motion. Briefly, let $(Z_a, a \geq 0)$ a version of this measure-valued process on \mathbb{R}^d , defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Perkins proves that if the dimension d of the space is such that $d \geq 3$ (which corresponds to the supercritical dimension case), there exists two constants c_d, C_d in $(0, \infty)$, only depending on d such that the following holds

$$\mathbf{P}\text{-a.s.} \quad \forall a \in (0, \infty) \quad c_d \mathcal{H}_g(\cdot \cap \text{supp}(Z_a)) \leq Z_a(\cdot) \leq C_d \mathcal{H}_g(\cdot \cap \text{supp}(Z_a)), \quad (\text{II.2})$$

where $\text{supp}(Z_a)$ is the topological support of the measure Z_a and \mathcal{H}_g is the Hausdorff measure associated to the gauge function $g(r) = r^2 \log \log 1/r$. In this paper, we use the ideas and techniques of [100, 101] to get a result similar to (II.2), an equality being accessible in the setting of trees.

Before stating formally our result, let us recall precisely basic facts. A metric space (T, d) is a real tree if and only if the following two properties hold for any σ_1, σ_2 in T :

- (i) There is a unique isometric map f_{σ_1, σ_2} from $[0, d(\sigma_1, \sigma_2)]$ into T such that $f_{\sigma_1, \sigma_2}(0) = \sigma_1$ and $f_{\sigma_1, \sigma_2}(d(\sigma_1, \sigma_2)) = \sigma_2$. We set $[[\sigma_1, \sigma_2]] = f_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)])$ that is the geodesic path joining σ_1 and σ_2 .
- (ii) If q is a continuous injective map from $[0, 1]$ into T , such that $q(0) = \sigma_1$ and $q(1) = \sigma_2$, we have

$$q([0, 1]) = f_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)]).$$

If $\sigma_1 \in [[\rho, \sigma_2]]$, we will say that σ_1 is an *ancestor* of σ_2 (σ_2 is a *descendant* of σ_1).

Real trees can be derived from continuous functions that represent their contour functions. Namely, let us consider a (deterministic) excursion e , that is to say a continuous function for which there exists $\zeta \in (0, \infty)$ such that : $\forall t \geq \zeta, e(0) = e(t) = 0$, and $\forall t \in (0, \zeta), e(t) > 0$. A real tree T can be associated with e in the following way. For $s, t \in [0, \zeta]$, we set

$$d(s, t) = e(s) + e(t) - 2 \inf_{r \in [s \wedge t, s \vee t]} e(r).$$

It is easy to see that d is a pseudo-distance on $[0, \zeta]$. Defining the equivalence relation $s \sim t$ iff $d(s, t) = 0$, one can set

$$T = [0, \zeta] / \sim. \quad (\text{II.3})$$

The function d induces a distance on the quotient set T . For a fixed excursion e , let

$$p : [0, \zeta] \longrightarrow (T, d) \quad (\text{II.4})$$

be the canonical projection. Clearly p is continuous, which implies that (T, d) is a compact metric space. Moreover, it can be shown (see [41] for a proof) that (T, d) is a real tree. We take $\rho = p(0)$ as the root of T . For all $a \in (0, \infty)$, the a -level set $T(a) = \{\sigma \in T : d(\rho, \sigma) = a\}$ is the image by p of the set $\{t \in [0, \zeta] : e(t) = a\}$. The total *height* of the tree is defined by

$$h(T) = \sup \{d(\rho, \sigma); \sigma \in T\}. \quad (\text{II.5})$$

We define the *Brownian tree* as the metric space (\mathcal{T}, d) coded by the Brownian excursion. More precisely, let $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space, large enough to carry all the random variables we need. We consider on that space a process $(X_t, t \in [0, \infty))$ such that $(\frac{1}{\sqrt{2}}X_t, t \in [0, \infty))$ is a standard real-valued Brownian motion (the choice of the normalizing constant $\sqrt{2}$ is explained below). Let us set $\underline{X}_t = \inf_{s \in [0, t]} X_s$. Then, the reflected process $X - \underline{X}$ is a strong Markov process, and the state 0 is instantaneous in $(0, \infty)$ and recurrent (see [9], chapter VI). We denote by \mathbf{N} the excursion measure associated with the local time $-\underline{X}$; \mathbf{N} is a sigma-finite measure on the space of continuous functions on $[0, \infty)$, denoted \mathbf{C}^0 in this work. More precisely, let $\bigcup_{j \in \mathcal{J}} (l_j, r_j) = \{t > 0 : X_t - \underline{X}_t > 0\}$ be the excursion intervals of the reflected process, and for all $j \in \mathcal{J}$, we set $e_j(s) = X_{(l_j+s) \wedge d_j} - \underline{X}_{l_j}$, $s \in [0, \infty)$. Then,

$$\mathcal{M}(dt, de) = \sum_{j \in \mathcal{J}} \delta_{(-\underline{X}_{l_j}, e_j)}$$

is a Poisson point measure on $[0, \infty) \times \mathbf{C}^0$ of intensity $dt \mathbf{N}(de)$. Let us recall that the two processes $(|X_t|, 2L_t)_{t \geq 0}$ and $(X_t - \underline{X}_t, -\underline{X}_t)_{t \geq 0}$ have the same law under \mathbf{P} by a celebrated result of Lévy (see Blumenthal [16], Th. II 2.2) where the process $(L_t, t \geq 0)$ is defined by the approximation $L_t = \lim_{\varepsilon \rightarrow 0} (2\varepsilon)^{-1} \int_0^t \mathbf{1}_{\{|X_s| \leq \varepsilon\}} ds$ that holds uniformly in t on compact subsets of $[0, \infty)$.

We shall denote by $(e_t, t \geq 0)$ the canonical process on \mathbf{C}^0 . Under \mathbf{N} , it is a strong Markov process, with transition kernel of the original process X killed when it hits 0 (see [16] III 3(f)). The following properties hold for the process \mathbf{N} -a.e. : there exists a unique real $\zeta \in (0, \infty)$ such that $\forall t \in (0, \zeta), e(t) > 0$, and $\forall t \in [\zeta, \infty), e(t) = e(0) = 0$. Moreover, with our normalization, one has (see [16] IV 1.1)

$$\forall \lambda \in [0, \infty), \mathbf{N}(1 - e^{-\lambda \zeta}) = \sqrt{\lambda} \quad \text{and} \quad \mathbf{N}(\zeta \in dr) = \frac{r^{-3/2}}{2\sqrt{\pi}} dr. \quad (\text{II.6})$$

One can show that $\mathbf{N}(\cdot \mid \zeta \in [1 - \varepsilon, 1 + \varepsilon])$ converges when ε goes to 0, towards a probability measure that is denoted by $\mathbf{N}(\cdot \mid \zeta = 1)$. It can be seen as the law of the excursion of $X - \underline{X}$ conditioned to have length one. The tree encoded by e under $\mathbf{N}(\cdot \mid \zeta = 1)$ is the CRT defined in [5]. The choice of the normalising constant $\sqrt{2}$ is explained by the following. Let τ_n be uniformly distributed as the set of rooted planar trees with n vertices. We view it as a real tree, the edges of τ_n being intervals of length one, and we denote by (τ_n, d_n) the resulting metric space. Denote by $(C_t^{(n)}, t \in [0, 2(n-1)])$ its contour function that is (informally) defined as follows. We let a particle explore the planar tree at speed one, from the left to the right, beginning at the root. We set $C_t^{(n)}$ as the distance from the root of the particle at time t . It can be shown (see [85] Th. 1.17) that $(C_t^{(n)}, t \in [0, 2(n-1)])$ has the law of a simple random walk conditioned to be positive on $[1, 2(n-1) - 1]$ and null at $2(n-1)$. Using Donsker invariance principle, the rescaled contour function $(n^{-1/2} C_{2(n-1)t}^{(n)}, t \in [0, 1])$ converges in law towards the law of $(e_t, t \in [0, 1])$ under $\mathbf{N}(\cdot \mid \zeta = 1)$. In terms of trees, $(\tau_n, n^{-1/2} d_n)$ converges towards the CRT, that is the tree (\mathcal{T}_1, d) coded by e under $\mathbf{N}(\cdot \mid \zeta = 1)$. The latter convergence can be stated using the distance of Gromov-Hausdorff (see Evans, Pitman, Winter [57]).

Recalling definition (II.5), we get from [16] IV 1.1 that with our normalization,

$$\forall a \in (0, \infty) \quad \mathbf{N}\left(\sup_{t \in [0, \zeta]} e_t > a\right) = \mathbf{N}(h(\mathcal{T}) > a) = \frac{1}{a}. \quad (\text{II.7})$$

In the paper, for $a \in (0, \infty)$ we shall use the probability measure,

$$\mathbf{N}_a = \mathbf{N}(\cdot \mid h(\mathcal{T}) > a) = a\mathbf{N}(\cdot \mid \mathbf{1}_{\{h(\mathcal{T}) > a\}}). \quad (\text{II.8})$$

Recall that the a -level set of the Brownian tree is defined by

$$\mathcal{T}(a) = \{\sigma \in \mathcal{T} : d(\rho, \sigma) = a\}. \quad (\text{II.9})$$

As a consequence of Trotter's theorem on the regularity of Brownian local time ([16] sec VI.3) there exists a $[0, \infty)$ -valued process $(L_t^a)_{a,t \in [0, \infty)}$ such that \mathbf{N} -a.e. the following holds true:

- $(a, t) \mapsto L_t^a$ is continuous,
- for all $a \in [0, \infty)$, $t \mapsto L_t^a$ is non-decreasing,
- for all $a \in [0, \infty)$, for all $t \in [0, \infty)$ and for all $b \in (0, \infty)$,

$$\lim_{\varepsilon \rightarrow 0} \mathbf{N}\left(\mathbf{1}_{\{\sup e > b\}} \sup_{0 \leq s \leq t \wedge \zeta} \left| \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{a - \varepsilon < \ell(u) \leq a\}} du - L_s^a \right| \right) = 0. \quad (\text{II.10})$$

We refer to [40], Proposition 1.3.3. for details in a more general setting.

The image by the projection $p : [0, \zeta] \rightarrow \mathcal{T}$ of those local times defines the collection of local time measures on the tree, $(\ell^a(d\sigma), \sigma \in \mathcal{T}, a \in (0, \infty))$. More precisely,

$$\mathbf{N}\text{-a.e. for all } f : \mathcal{T} \xrightarrow{\text{meas.}} [0, \infty) \quad \forall a \in (0, \infty) \quad \int_{\mathcal{T}} f(\sigma) \ell^a(d\sigma) = \int_0^\zeta f(p(t)) dL_t^a. \quad (\text{II.11})$$

See [41], Th. 4.2 for an intrinsic definition of the measure ℓ^a (for fixed a). Let \mathcal{G}_a the σ -field generated by the excursion below level a (formal definitions and details on what follows are given in section 3.1). The approximation (II.10) entails that for fixed a , $\ell^a(\mathcal{T}) = L_\zeta^a$ is \mathcal{G}_a measurable. Moreover, the Ray-Knight theorem ([16] VI 2.10) entails that under $\mathbf{N}_a(\cdot)$ conditionally on \mathcal{G}_a , the process $(\ell^{a+a'}(\mathcal{T}), a' \geq 0)$ is a Feller diffusion started at $\ell^a(\mathcal{T})$. In particular, one has

$$\forall a, \lambda \in (0, \infty) \quad \mathbf{N} \left[1 - e^{-\lambda \ell^a(\mathcal{T})} \right] = \frac{\lambda}{1 + a\lambda}, \quad (\text{II.12})$$

which implies that under \mathbf{N}_a , $\ell^a(\mathcal{T})$ is exponentially distributed with mean a . The regularity of $a \mapsto \ell^a(\mathcal{T})$ is extended by Duquesne and Le Gall [41] : they prove that \mathbf{N} -a.e. the process $a \mapsto \ell^a$ is continuous for the weak topology of measures. In the same work, the topological support of the level set measures is precised as follows. A vertex $\sigma \in \mathcal{T}$ is called an extinction point if there exists $\varepsilon \in (0, \infty)$ such that $d(\rho, \sigma) = \sup\{d(\rho, \tau), \tau \in B(\sigma, \varepsilon)\}$, where $B(\sigma, \varepsilon)$ is the open ball in \mathcal{T} with center σ and radius ε . For $s \in [0, \zeta]$, the vertex $p(s) \in \mathcal{T}$ is an extinction time iff $s \in [0, \zeta]$ is a local maximum of e . As a consequence, the set of all extinction points, denoted \mathcal{E} , is countable. Let us denote $\text{supp}(\mu)$ for the topological support of the measure μ . The result states that

$$\mathbf{N}\text{-a.e. } \forall a \in (0, \infty) \setminus \mathcal{E}, \text{supp}(\ell^a) = \mathcal{T}(a), \quad \text{and} \quad \forall a \in \mathcal{E}, \text{supp}(\ell^a) = \mathcal{T}(a) \setminus \{\sigma_a\}, \quad (\text{II.13})$$

where σ_a is the (unique) extinction point at level a (see Perkins [102] for previous results on Super-Brownian motion).

Let us briefly introduce the construction of the Hausdorff measure. We set the *gauge* function g as

$$g(r) = r \log \log 1/r, \quad r \in (0, e^{-1}). \quad (\text{II.14})$$

In all the paper it will be assumed implicitly that $g(r)$ is considered only for $r \in (0, e^{-1})$. On that interval, g is an increasing continuous function. For any subset A of \mathcal{T} , one can define

$$\mathcal{H}_g(A) = \lim_{\varepsilon \rightarrow 0} \inf \left\{ \sum_{i \in \mathbb{N}} g(\text{diam}(E_i)) ; A \subset \bigcup_{i \in \mathbb{N}} E_i, \text{diam}(E_i) < \varepsilon \right\}. \quad (\text{II.15})$$

Standard results on Hausdorff measures (see e.g. [106]) ensure that \mathcal{H}_g defines a Borel-regular outer measure on \mathcal{T} called the g -Hausdorff measure on \mathcal{T} . The main result of the paper is the following.

Theorem 1.1 *Let \mathcal{T} be the Brownian tree, that is the tree encoded by the excursion e under \mathbf{N} . Let $(\ell^a(d\sigma), \sigma \in \mathcal{T}, a \in (0, \infty))$ the collection of local time measures and \mathcal{H}_g the g -Hausdorff measure on \mathcal{T} , where $g(r) = r \log \log 1/r$. Then, the following holds :*

$$\mathbf{N}\text{-a.e.} \quad \forall a \in (0, \infty) \quad \ell^a(\cdot) = \frac{1}{2} \mathcal{H}_g(\cdot \cap \mathcal{T}(a)). \quad (\text{II.16})$$

Comment 1.1 Thanks to the scaling properties of the Brownian excursion, one can derive from Theorem 1.2 a similar statement for the tree coded by e under $\mathbf{N}(\cdot \mid \zeta = 1)$, that is Aldous CRT.

Comment 1.2 Our result seems close to a theorem of Perkins [99] on linear Brownian motion. Let $(L_t^a, t \geq 0, a \in \mathbb{R})$ be the bi-continuous version of the local times for the process $(X_t, t \geq 0)$ defined above. Those local times are given by an approximation of the type of (II.10). Perkins proves that almost surely, uniformly in a , one has $L_t^a = \mathcal{H}_g(\{s \in [0, t] : X_s = x\})$, where $\overline{\mathcal{H}_g}$ stands for the Hausdorff measure on the line associated with the gauge $g(r) = \sqrt{r \log \log 1/r}$ (the result for fixed a had been obtain by Taylor and Wendel in [113]). The Brownian tree being coded by the Brownian excursion, everything happens as if the projection mapping $p : [0, \zeta] \rightarrow \mathcal{T}$ is 1/2-Hölder and induces a strong "doubling", such that the entire gauge function is squared. Nevertheless, we don't see how to derive our result from [99].

The paper is organised as follows. In section 2, we state some deterministic facts on the geometry of the level sets for a real tree. In particular, we provide two comparison lemmas with respect to Hausdorff measure on real trees. The second one, that is specific to our setting, seems new to us. In section 3, we recall basic facts on the Brownian tree and we establish some technical estimates. Section 4 is devoted to the proof of Theorem 1.2. As a first step, we prove Theorem 4.1, which gives an upper bound for the local time measures. To that end, we need to control the total mass of the balls that are "too large". The second step is the proof of Theorem 4.2, which requires a control of the number of balls that are "too small". Let us mention again that our strategy and many ideas in this work were borrowed from [100, 101].

Acknowledgments. I would like to thank my advisor Thomas Duquesne for introducing this problem, as well as for his help and the many improvements he suggested.

2 Geometric properties of the level sets of real trees.

2.1 The balls of the level sets of real trees.

Let (T, d, ρ) be a compact rooted real tree as defined in the introduction. Recall that for any $\sigma, \sigma' \in T$, $[[\sigma, \sigma']]$ stands for the unique geodesic path joining σ to σ' . We shall view T as a family tree whose ancestor is the root ρ and we then denote by $\sigma \wedge \sigma'$ the most recent common ancestor of σ and σ' that is formally defined by

$$[[\rho, \sigma \wedge \sigma']] = [[\rho, \sigma]] \cap [[\rho, \sigma']] .$$

Observe that

$$\forall \sigma, \sigma' \in T, \quad d(\sigma, \sigma') = d(\rho, \sigma) + d(\rho, \sigma') - 2d(\rho, \sigma \wedge \sigma') . \quad (\text{II.17})$$

Let $a \in [0, \infty)$. Recall that the a -level set of T is given by

$$T(a) = \{\sigma \in T : d(\rho, \sigma) = a\} .$$

Subtrees above level b . Let $b \in [0, \infty)$ and denote by $(T_j^{\sigma, b})_{j \in \mathcal{J}_b}$ the connected components of the open set $\{\sigma \in T : d(\rho, \sigma) > b\}$:

$$\bigcup_{j \in \mathcal{J}_b} T_j^{\sigma, b} = \{\sigma \in T : d(\rho, \sigma) > b\} .$$

Then for any $j \in \mathcal{J}_b$, there exists a unique point $\sigma_j \in T(b)$ such that $T_j^b := T_j^{\sigma, b} \cup \{\sigma_j\}$ is the closure of $T_j^{\sigma, b}$ in T . Note that (T_j^b, d, σ_j) is a compact rooted real tree and that

$$\forall j \in \mathcal{J}_b, \quad \forall \sigma \in T_j^b, \quad \sigma_j \in [[\rho, \sigma]] .$$

Open balls in $T(a)$. Recall that $B(\sigma, r)$ stands for the open ball in T with center σ and radius r . We shall also denote by $\Gamma(\sigma, r)$ the open ball with center σ and radius r in the level set of σ , namely

$$\Gamma(\sigma, r) = B(\sigma, r) \cap T(a), \quad \text{where } a = d(\rho, \sigma) . \quad (\text{II.18})$$

If $\sigma \in T(a)$, then we call $\Gamma(\sigma, r)$ a $T(a)$ -ball with radius r ; we denote by $\mathcal{B}_{a,r}$ the set of all the $T(a)$ -balls with radius r :

$$\mathcal{B}_{a,r} = \{\Gamma(\sigma, r); \sigma \in T(a)\} . \quad (\text{II.19})$$

The following proposition provides the geometric properties of $T(a)$ -balls that we shall use. The last point could be proved by noticing that restricted to a level-set $T(a)$, the distance d in the tree is ultrametric.

Proposition 2.1 *Let (T, d, ρ) be a compact rooted real tree. Let $a, r \in (0, \infty)$ be such that $a \geq r/2$. Then, the number of $T(a)$ -balls with radius r is finite. We set*

$$Z_{a,r} = \#\mathcal{B}_{a,r} \quad \text{and} \quad \{\Gamma_i, 1 \leq i \leq Z_{a,r}\} = \mathcal{B}_{a,r} . \quad (\text{II.20})$$

Then, the following holds true.

(i) Set $b = a - \frac{1}{2}r$. Then, there are $Z_{a,r}$ distinct subtrees above b denoted by $(T_{j_i}^b, d, \sigma_{j_i})$, $j_i \in \mathcal{J}_b$, $1 \leq i \leq Z_{a,r}$ such that

$$\Gamma_i = T(a) \cap T_{j_i}^b = \{\sigma' \in T_{j_i}^b : d(\sigma_{j_i}, \sigma') = r/2\}.$$

Thus, the $T(a)$ -balls with radius r are pairwise disjoint.

(ii) For all $\sigma \in T(a)$, one has $\text{diam}(\Gamma(\sigma, r)) \leq r$. If furthermore $r \in (0, 2a)$, then $\text{diam}(\Gamma(\sigma, r)) < r$ and

$$\forall r' \in (\text{diam}(\Gamma(\sigma, r)), r) \quad \Gamma(\sigma, r') = \Gamma(\sigma, r). \quad (\text{II.21})$$

Therefore, the set of all $T(a)$ -balls is countable.

(iii) Two $T(a)$ -balls are either contained one in the other or disjoint. Namely, for all $r' < r$ and all $\sigma, \sigma' \in T(a)$, either $\Gamma(\sigma', r') \subset \Gamma(\sigma, r)$ or $\Gamma(\sigma', r') \cap \Gamma(\sigma, r) = \emptyset$.

Proof. Let us prove (i). Let $\sigma, \sigma' \in T(a)$ and set $b = a - \frac{1}{2}r$. By (II.17), $d(\sigma, \sigma') = 2a - 2d(\rho, \sigma \wedge \sigma')$. Thus, $d(\sigma, \sigma') < r$ iff $d(\rho, \sigma \wedge \sigma') > b$. Let $j \in \mathcal{J}_b$ be such that $\sigma \in T_j^b$; namely, T_j^b is the unique subtree above b containing σ and σ_j is the unique point $\gamma \in \llbracket \rho, \sigma \rrbracket$ such that $d(\rho, \gamma) = b$. Now observe that for all $\sigma' \in T(a)$,

$$d(\rho, \sigma \wedge \sigma') > b \iff \sigma \wedge \sigma' \in \llbracket \sigma_j, \sigma \rrbracket \iff \sigma' \in T_j^b.$$

This proves that

$$\Gamma(\sigma, r) = T(a) \cap T_j^b. \quad (\text{II.22})$$

Conversely, let $j \in \mathcal{J}_b$ be such that $h(T_j^b) := \max\{d(\sigma_j, \gamma); \gamma \in T_j^b\} \geq r/2$. Let $\sigma \in T(a) \cap T_j^b$; then the previous arguments imply (II.22). Since T is compact, the set $\{j \in \mathcal{J}_b : h(T_j^b) \geq r/2\}$ is finite, which completes the proof of (i).

Let us prove (ii): let $\sigma \in T(a)$, let $r \in (0, 2a)$ and set $\delta = \text{diam}(\Gamma(\sigma, r))$. Then (II.22) implies that $\Gamma(\sigma, r)$ is compact and there are $\sigma_1, \sigma_2 \in \Gamma(\sigma, r)$ such that $d(\sigma_1, \sigma_2) = \delta$. Observe that it implies

$$\Gamma(\sigma, r) = \{\sigma' \in T(a) : \sigma_1 \wedge \sigma_2 \in \llbracket \rho, \sigma' \rrbracket\}.$$

Thus, $\Gamma(\sigma, r) = \overline{\Gamma(\sigma, \delta)}$, that is the closure of $\Gamma(\sigma, \delta)$, and it implies (II.21). The set of all $T(a)$ -balls is therefore $\bigcup_{q \in \mathbb{Q} \cap [0, \infty)} \mathcal{B}_{a,q}$, which is a countable set.

Let us prove (iii): $r' < r$ and $\sigma, \sigma' \in T(a)$ and suppose that $\Gamma(\sigma', r') \cap \Gamma(\sigma, r) \neq \emptyset$. Then (i) and (ii) implies that $\Gamma(\sigma, r) = \Gamma(\sigma', r)$, which implies that $\Gamma(\sigma', r') \subset \Gamma(\sigma, r)$. ■

2.2 Comparison lemmas for Hausdorff measures on real trees.

Let (T, d, ρ) be a compact real tree. We briefly recall the definition of Hausdorff measures on T and we state two comparison lemmas that are used in the proofs. Let $r_0 \in (0, \infty)$ and let $g : [0, r_0) \rightarrow [0, \infty)$ be a function that is assumed to be increasing, continuous and such that $g(0) = 0$. For all $\varepsilon \in (0, r_0)$ and all $A \subset T$, we set

$$\mathcal{H}_g^{(\varepsilon)}(A) = \inf \left\{ \sum_{n \in \mathbb{N}} g(\text{diam}(E_n)) ; A \subset \bigcup_{n \in \mathbb{N}} E_n, \text{diam}(E_n) < \varepsilon \right\}$$

and

$$\mathcal{H}_g(A) = \lim_{\varepsilon \downarrow 0} \uparrow \mathcal{H}_g^{(\varepsilon)}(A).$$

Under our assumptions, \mathcal{H}_g is a Borel-regular outer measure : this is the g -Hausdorff measure on T (see Rogers [106]). The following comparison lemma was first stated for Euclidean spaces by Rogers and Taylor [107]. The proof can be easily adapted to general metric spaces (see Edgar [53]). We include a brief proof of it in order to make the paper self-contained.

Lemma 2.2 *Let (T, d, ρ) be a compact rooted real tree. Let μ be a Borel measure on T . Let A be a Borel subset of T and let $c \in (0, \infty)$. Assume that*

$$\forall \sigma \in A \quad \limsup_{r \rightarrow 0} \frac{\mu(B(\sigma, r))}{g(r)} < c .$$

Then, $\mu(A) \leq c\mathcal{H}_g(A)$.

Proof. For any $\varepsilon \in (0, r_0)$, set

$$A_\varepsilon = \left\{ \sigma \in A : \sup_{r \in (0, \varepsilon)} \frac{\mu(B(\sigma, r))}{g(r)} < c \right\} .$$

Observe that for all $\varepsilon' < \varepsilon$, $A_\varepsilon \subset A_{\varepsilon'} \subset A$ and $A = \bigcup_{\varepsilon \in (0, r_0)} A_\varepsilon$. Let $(E_n)_{n \in \mathbb{N}}$ be a ε -covering of A_ε : namely $A_\varepsilon \subset \bigcup_{n \in \mathbb{N}} E_n$ and $\text{diam}(E_n) < \varepsilon$, for all $n \in \mathbb{N}$. Set $I = \{n \in \mathbb{N} : E_n \cap A_\varepsilon \neq \emptyset\}$ and for all $n \in I$, fix $\sigma_n \in E_n \cap A_\varepsilon$. Since g is continuous, for all $n \in I$ there exists $r_n \in (\text{diam}(E_n), \varepsilon)$ such that

$$E_n \subset B(\sigma_n, r_n) \quad \text{and} \quad g(r_n) \leq 2^{-n-1}\varepsilon + g(\text{diam}(E_n)) .$$

Observe that $\mu(B(\sigma_n, r_n)) < cg(r_n)$ and that $A_\varepsilon \subset \bigcup_{n \in I} B(\sigma_n, r_n)$. Thus,

$$\begin{aligned} \mu(A_\varepsilon) &\leq \mu\left(\bigcup_{n \in I} B(\sigma_n, r_n)\right) \leq \sum_{n \in I} \mu(B(\sigma_n, r_n)) \\ &\leq \sum_{n \in I} cg(r_n) \leq c\varepsilon + \sum_{n \in \mathbb{N}} cg(\text{diam}(E_n)) . \end{aligned}$$

Taking the infimum over all the possible ε -coverings of A_ε yields

$$\mu(A_\varepsilon) \leq c\varepsilon + c\mathcal{H}_g^{(\varepsilon)}(A_\varepsilon) \leq c\varepsilon + c\mathcal{H}_g(A_\varepsilon) \leq c\varepsilon + \mathcal{H}_g(A) ,$$

which implies the desired result since $\mu(A) = \lim_{\varepsilon \downarrow 0} \mu(A_\varepsilon)$. ■

In the next comparison lemma, that seems new to us, we restrict our attention to the level sets of real trees. A more general variant of this result involves a multiplicative constant depending on the gauge function. It has been first stated in Euclidian spaces by Rogers and Taylor [107] (see also Perkins [100]) and in general metric spaces (see Edgar [53]).

Lemma 2.3 *Let (T, d, ρ) be a compact rooted real tree. Let $a \in (0, \infty)$ be such that the a -level set $T(a)$ is not empty. Let μ be a finite Borel measure on T such that $\mu(T \setminus T(a)) = 0$. Let $A \subset T(a)$ be a Borel subset and let $c \in (0, \infty)$. Assume that*

$$\forall \sigma \in A \quad \limsup_{r \rightarrow 0} \frac{\mu(B(\sigma, r))}{g(r)} > c .$$

Then, $\mu(A) \geq c\mathcal{H}_g(A)$.

Proof. Let $\varepsilon \in (0, (2a) \wedge r_0)$. Let U be an open set of T such that $A \subset U$. For all $\sigma \in A$, there exists $r_\sigma \in (0, \varepsilon)$ such that

$$\mu(\Gamma(\sigma, r_\sigma)) = \mu(B(\sigma, r_\sigma)) > cg(r_\sigma) \quad \text{and} \quad \Gamma(\sigma, r_\sigma) \subset U.$$

Thus, $A \subset \bigcup_{\sigma \in A} \Gamma(\sigma, r_\sigma) \subset U$. Then, Proposition 2.1 (ii) asserts that the set of all $T(a)$ -balls is countable and Proposition 2.1 (iii) asserts that two $T(a)$ -balls are either contained one in the other or disjoint. Therefore, there exists $I \subset \mathbb{N}$ and $\sigma_n \in A$, $n \in I$, such that the $\Gamma(\sigma_n, r_{\sigma_n})$, $n \in I$, are pairwise disjoint and $A \subset \bigcup_{n \in I} \Gamma(\sigma_n, r_{\sigma_n}) \subset U$. Moreover, by Proposition 2.1 (ii), $\text{diam}(\Gamma(\sigma_n, r_{\sigma_n})) \leq r_{\sigma_n}$. Thus, we get

$$\begin{aligned} c\mathcal{H}_g^{(\varepsilon)}(A) &\leq \sum_{n \in I} cg(\text{diam}(\Gamma(\sigma_n, r_{\sigma_n}))) \leq \sum_{n \in I} cg(r_{\sigma_n}) \\ &\leq \sum_{n \in I} \mu(\Gamma(\sigma_n, r_{\sigma_n})) = \mu\left(\bigcup_{n \in I} \Gamma(\sigma_n, r_{\sigma_n})\right) \leq \mu(U). \end{aligned}$$

As $\varepsilon \rightarrow 0$, it entails $c\mathcal{H}_g(A) \leq \mu(U)$, for all open set U containing A . Since μ is a finite Borel measure, it is outer-regular for the open subsets, which implies the desired result. \blacksquare

3 Preliminary results on the Brownian tree.

3.1 Basic facts on the Brownian excursion.

We work under the excursion measure \mathbf{N} defined in the introduction and e denote the canonical excursion whose duration is denoted by ζ (see (II.6)). We shall denote by (\mathcal{T}, d, ρ) the compact rooted real tree coded by e .

The branching property. Fix $b \in (0, \infty)$. We discuss here a decomposition of e in terms of its excursions above level b ; this yields a decomposition of the Brownian tree called the branching property. To that end we first introduce the following time change: for all $t \in [0, \infty)$, we set

$$\tau_b(t) = \inf \left\{ s \in [0, \infty) : \int_0^s \mathbf{1}_{\{e_u \leq b\}} du > t \right\} \quad \text{and} \quad \tilde{e}_b(t) = e(\tau_b(t)). \quad (\text{II.23})$$

Note that $(\tilde{e}_b(t))_{t \in [0, \infty)}$ codes the tree below b namely $\{\sigma \in \mathcal{T} : d(\rho, \sigma) \leq b\}$ that is the closed ball with center ρ and radius b . We denote by \mathcal{G}_b , the sigma-field generated by $(\tilde{e}_b(t))_{t \in [0, \infty)}$ and completed with the \mathbf{N} -negligible sets. The approximation (II.10) implies that L_ζ^b is \mathcal{G}_b -measurable. Then denote by (α_j, β_j) , $j \in \mathcal{J}_b$, the connected components of the time-set $\{s \in [0, \infty) : e(s) > b\}$. Namely,

$$\bigcup_{j \in \mathcal{J}_b} (\alpha_j, \beta_j) = \{s \in [0, \infty) : e(s) > b\},$$

and we call (α_j, β_j) the excursion intervals of e above level b . For all $j \in \mathcal{J}_b$, we next set

$$l_j^b = L_{\tau_b(\alpha_j)}^b \quad \text{and} \quad \forall s \in [0, \infty), \quad e_j^b(s) = e_{(\alpha_j + s) \wedge \beta_j} - b.$$

Then, the $(e_j^b)_{j \in \mathcal{J}_b}$ are the excursions of e above level b . Recall from (II.7) and (II.8) the notation $\mathbf{N}_b = \mathbf{N}(\cdot | \sup e > b)$, that is a probability measure. The *branching property* asserts the following: under \mathbf{N}_b and conditionally on \mathcal{G}_b , the measure

$$\mathcal{M}_b(dl, de) = \sum_{j \in \mathcal{J}_b} \delta_{(l_j^b, e_j^b)} \quad (\text{II.24})$$

is a Poisson point measure on $[0, L_\zeta^b] \times \mathbf{C}^0$ with intensity $\mathbf{1}_{[0, L_\zeta^b]}(l) dl \mathbf{N}(de)$.

The following decomposition of e is interpreted in terms of the Brownian tree \mathcal{T} as follows. Recall that $p : [0, \zeta] \rightarrow \mathcal{T}$ stands for the canonical projection. Then for all $j \in \mathcal{J}_b$, we set

$$\sigma_j = p(\alpha_j) = p(\beta_j), \quad \mathcal{T}_j^{o,b} = p((\alpha_j, \beta_j)) \quad \text{and} \quad \mathcal{T}_j^b = p([\alpha_j, \beta_j]).$$

Then, we easily check that the $\mathcal{T}_j^{o,b}$, $j \in \mathcal{J}_b$, are the connected components of the open subset $\{\sigma \in \mathcal{T} : d(\rho, \sigma) > b\}$ and that $\mathcal{T}_j^{o,b} = \mathcal{T}_j^b \setminus \{\sigma_j\}$. Namely, the $(\mathcal{T}_j, d, \sigma_j)$, $j \in \mathcal{J}_b$ are the subtrees above level b of \mathcal{T} as introduced in Section 2.1. Moreover note that for all $j \in \mathcal{J}_b$, the rooted compact real tree $(\mathcal{T}_j, d, \sigma_j)$ is isometric to the tree coded by the excursion e_j^b . We next use this and Proposition 2.1 to discuss the balls in a fixed level of \mathcal{T} .

To that end, we fix $a, r \in (0, \infty)$ such that $a > r/2$ and we conveniently set $b = a - r/2$. Recall that $\mathcal{T}(a) = \{\sigma \in \mathcal{T} : d(\rho, \sigma) = a\}$ and that for all $\sigma \in \mathcal{T}(a)$, we have set $\Gamma(\sigma, r) = \mathcal{T}(a) \cap B(\sigma, r)$ that is the ball in $\mathcal{T}(a)$ with center σ and radius r . We also recall that $\mathcal{B}_{a,r} = \{\Gamma(\sigma, r); \sigma \in \mathcal{T}(a)\}$ stands for the set of all $\mathcal{T}(a)$ -balls with radius r . By Proposition 2.1, $\mathcal{B}_{a,r}$ is a finite set and that

$$\mathcal{B}_{a,r} = \{\mathcal{T}(a) \cap \mathcal{T}_j^b; j \in \mathcal{J}_b : h(\mathcal{T}_j^b) \geq r/2\},$$

where the trees $(\mathcal{T}_j^b, d, \sigma_j)$, $j \in \mathcal{J}_b$, are the subtrees of \mathcal{T} above level b as previously defined; here $h(\mathcal{T}_j^b) = \sup_{\sigma \in \mathcal{T}_j^b} d(\sigma_j, \sigma)$ stands for the total height of \mathcal{T}_j^b . Note that $h(\mathcal{T}_j^b) = \sup e_j^b$ that is maximum of the excursion corresponding to \mathcal{T}_j^b , as explained above.

Then, we set $Z_{a,r} = \#\mathcal{B}_{a,r}$, that is the number of $\mathcal{T}(a)$ -ball with radius r . Assume that $Z_{a,r} \geq 1$. We then define the indices $j_1, \dots, j_{Z_{a,r}} \in \mathcal{J}_b$ by

$$\{j_1, \dots, j_{Z_{a,r}}\} = \{j \in \mathcal{J}_b : h(\mathcal{T}_j^b) \geq r/2\} \quad \text{and} \quad \alpha_{j_1} < \dots < \alpha_{j_{Z_{a,r}}}.$$

and we set

$$\forall i \in \{1, \dots, Z_{a,r}\}, \quad \Gamma_i := \mathcal{T}(a) \cap \mathcal{T}_{j_i}^b. \quad (\text{II.25})$$

Namely $\mathcal{B}_{a,r} = \{\Gamma_i; 1 \leq i \leq Z_{a,r}\}$ is the set of the $\mathcal{T}(a)$ -balls with radius r listed in their order of visit by the excursion e coding \mathcal{T} .

Lemma 3.1 *Let $a, r \in (0, \infty)$ such that $a > r/2$. Let $\{\Gamma_i; 1 \leq i \leq Z_{a,r}\}$ is the set of the $\mathcal{T}(a)$ -balls with radius r listed in their order of visit as explained above. Then the following holds true.*

- (i) *Under $\mathbf{N}_a = \mathbf{N}(\cdot \mid \sup e > a)$, $Z_{a,r}$ has a geometric law with parameter $2a/r$. Namely,*

$$\forall k \geq 1, \quad \mathbf{N}_a[Z_{a,r} = k] = \left(1 - \frac{r}{2a}\right)^{k-1} \frac{r}{2a}.$$

- (ii) *For all $k \geq 1$, under $\mathbf{N}_a(\cdot \mid Z_{a,r} = k)$, the r.v. $(\ell^a(\Gamma_i))_{1 \leq i \leq k}$ are independent and exponentially distributed with mean $r/2$.*

Proof. Let $a \in (0, \infty)$ and denote $b = a - r/2$. Let $k \geq 1$ and $F_1, \dots, F_k : \mathbf{C}^0 \rightarrow [0, \infty)$ be measurable functionals. Recall from (II.7) that $\mathbf{N}(\sup e \geq r/2) = 2/r$. Then, the definition of the j_i combined with the branching property and basic results on Poisson point measures entail

$$\mathbf{N}_b \left[\mathbf{1}_{\{Z_{a,r}=k\}} \prod_{1 \leq i \leq k} F_i(e_{j_i}^b) \mid \mathcal{G}_b \right] = \frac{\left(\frac{2}{r} L_\zeta^b\right)^k}{k!} e^{-\frac{2}{r} L_\zeta^b} \prod_{1 \leq i \leq k} \mathbf{N}_{r/2}[F_i(e)]. \quad (\text{II.26})$$

Then recall (II.12) that implies that L_ζ^b under \mathbf{N}_b is exponentially distributed with mean b . Thus,

$$\frac{1}{k!} \mathbf{N}_b \left[\left(\frac{2}{r} L_\zeta^b \right)^k e^{-\frac{2}{r} L_\zeta^b} \right] = \frac{\left(\frac{2}{r} b \right)^k}{\left(1 + \frac{2}{r} b \right)^{k+1}} = \frac{r}{2a} \left(1 - \frac{r}{2a} \right)^k,$$

because $b = a - r/2$ and $\left(1 + \frac{2}{r} b \right)^{-1} = r/(2a)$. It implies

$$\mathbf{N}_b \left[\mathbf{1}_{\{Z_{a,r}=k\}} \prod_{1 \leq i \leq k} F_i(e_{j_i}^b) \right] = \frac{r}{2a} \left(1 - \frac{r}{2a} \right)^k \prod_{1 \leq i \leq k} \mathbf{N}_{r/2} [F_i(e)].$$

Next observe that \mathbf{N}_b -a.s. $\mathbf{1}_{\{\sup e > a\}} = \mathbf{1}_{\{Z_{a,r} \geq 1\}}$. Thus, we get

$$\begin{aligned} \mathbf{N}_a \left[\mathbf{1}_{\{Z_{a,r}=k\}} \prod_{1 \leq i \leq k} F_i(e_{j_i}^b) \right] &= \frac{a}{b} \mathbf{N}_b \left[\mathbf{1}_{\{Z_{a,r}=k\}} \prod_{1 \leq i \leq k} F_i(e_{j_i}^b) \right] \\ &= \frac{r}{2a} \left(1 - \frac{r}{2a} \right)^{k-1} \prod_{1 \leq i \leq k} \mathbf{N}_{r/2} (F_i(e)) \end{aligned} \quad (\text{II.27})$$

because $a/b = \left(1 - \frac{r}{2a} \right)^{-1}$. Recall that (II.12) implies that under $\mathbf{N}_{r/2}$, $\ell^{r/2}(\mathcal{T}) = L_\zeta^{r/2}$ is exponentially distributed with mean $r/2$. By taking $F_i(e) = f_i(L_\zeta^{r/2})$ in (II.27) we then get

$$\mathbf{N}_a \left[\mathbf{1}_{\{Z_{a,r}=k\}} \prod_{1 \leq i \leq k} f_i(\ell^a(\Gamma_i)) \right] = \frac{r}{2a} \left(1 - \frac{r}{2a} \right)^{k-1} \prod_{1 \leq i \leq k} \int_0^\infty f_i(s) \frac{2}{r} e^{-\frac{2}{r}s} ds,$$

with entails the desired result. ■

Ray-Knight theorem under \mathbf{N} . We first recall the definition of Feller diffusion, namely a Continuous States space Branching Process (CSBP) with branching mechanism $\psi(\lambda) = \lambda^2$. Let $x \in [0, \infty)$ and let $(Y_a^x)_{a \in [0, \infty)}$ be a $[0, \infty)$ -valued continuous process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is a Feller diffusion with branching mechanism $\psi(\lambda) = \lambda^2$ and initial value $Y_0^x = x$ if it is a Markov process such that

$$\mathbf{E} \left[\exp(-\lambda Y_{a+a'}^x) | Y_a^x \right] = \exp \left(-\frac{Y_a^x \lambda}{1 + a' \lambda} \right), \quad a, a', \lambda \in [0, \infty).$$

Recall notation $\mathbf{N}_a = \mathbf{N}(\cdot | \sup e > a)$ and \mathcal{G}_a for the sigma-field generated by the excursion \tilde{e}_a defined in (II.23). Recall that $\ell^a(\mathcal{T}) = L_\zeta^a$, the total mass of the local-time measure at level a , is \mathcal{G}_a -measurable.

We shall use the following statement of Ray-Knight theorem. Let $a \in (0, \infty)$.

(i) $\mathbf{N}_a[\exp(-\lambda \ell^a(\mathcal{T}))] = \frac{1}{1+a\lambda}$.

(ii) Under \mathbf{N}_a and conditionally given \mathcal{G}_a , the process $(\ell^{a+a'}(\mathcal{T}))_{a' \in [0, \infty)}$ is a Feller diffusion with branching mechanism $\psi(\lambda) = \lambda^2$ and initial value $\ell^a(\mathcal{T})$.

This is an immediate consequence of the Ray-Knight theorem for standard Brownian motion and of the Markov property under \mathbf{N} : see [16] III 3 and VI 2.10.

Combined with the branching property, the above Ray-Knight theorem, has the following consequence. Let us recall that we enumerate the $\mathcal{T}(a)$ -balls of $\mathcal{B}_{a,r}$ as $\{\Gamma_i, 1 \leq i \leq Z_{a,r}\}$ (see (II.25)). Let Γ such a $\mathcal{T}(a)$ -ball. For $a' \geq 0$, we define

$$\Gamma^{a+a'} = \{ \sigma \in \mathcal{T}(a+a') \exists \sigma' \in \Gamma : \sigma' \in \llbracket \rho, \sigma \rrbracket \}, \quad (\text{II.28})$$

the set of vertices at level $a+a'$ that have an ancestor in Γ (notice that $\Gamma^a = \Gamma$). The following lemma is a straightforward consequence of Ray-Knight theorem.

Lemma 3.2 *Let $a \in (0, \infty)$, $r \in [0, 2a]$. Let $\{\Gamma_i, 1 \leq i \leq Z_{a,r}\}$ the set of $\mathcal{T}(a)$ -balls of radius r . Under \mathbf{N}_a conditionally on \mathcal{G}_a , the processes $(\ell^{a+a'}(\Gamma_i^{a+a'}), a' \geq 0)$, $1 \leq i \leq Z_{a,r}$, are independent Feller diffusions started at $(\ell^a(\Gamma_i))$, $1 \leq i \leq Z_{a,r}$.*

Proof. Recalling for $b = a - r/2$ the decomposition (II.25), we see that

$$\forall i \in \{1, \dots, Z_{a,r}\}, \quad \Gamma_i^{a+a'} := \mathcal{T}(a+a') \cap \mathcal{T}_{j_i}^b. \quad (\text{II.29})$$

Hence, one can use (II.26), and the Ray-Knight theorem (see (ii) above) to get the desired result. ■

Spinal decomposition. We recall another decomposition of the Brownian tree called *spinal decomposition*. This is a consequence of Bismut's decomposition of the Brownian excursion that we recall here.

Let X be a real valued process defined on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $(\frac{1}{\sqrt{2}}X_t)_{t \in [0, \infty)}$ is distributed as a standard Brownian motion with initial value 0. Let X' be an independent copy of X on $(\Omega, \mathcal{F}, \mathbf{P})$. We fix $a \in (0, \infty)$ and we set

$$T_a = \inf\{t \in [0, \infty) : X_t = -a\} \quad \text{and} \quad T'_a = \inf\{t \in [0, \infty) : X'_t = -a\}.$$

We next set for any $s \in [0, \infty)$,

$$\check{e}_s^t = e_{(t-s)_+} \quad \text{and} \quad \hat{e}_s^t = e_{t+s}.$$

Then the Bismut's identity (see [15] or [81]) states that for any non-negative measurable functional F on $(\mathbf{C}^0)^2$,

$$\mathbf{N} \left[\int_0^\zeta dL_t^a F(\check{e}^t; \hat{e}^t) \right] = \mathbf{E} [F(a + X_{\cdot \wedge T_a}; a + X'_{\cdot \wedge T'_a})]. \quad (\text{II.30})$$

We derive from (II.30) an identity involving the excursions above the infimum of \hat{e}^t and \check{e}^t . To that end, we introduce the following. Let $g : [0, \infty) \rightarrow [0, \infty)$ with compact support. We define a point measure $\mathcal{N}(g)$ as follows: set $\underline{g}(t) = \inf_{[0,t]} g$ and denote by (l_j, r_j) , $j \in \mathcal{I}(g)$ the excursion intervals of $g - \underline{g}$ away from 0 that are the connected component of the open set $\{t \geq 0 : g(t) - \underline{g}(t) > 0\}$. For any $j \in \mathcal{I}(g)$, set $g^j(s) = ((g - \underline{g})(l_j + s) \wedge r_j)$, $s \geq 0$ and denote $h_j := g^j(l_j)$ the height where the excursion g^j starts. We then define $\mathcal{N}(g)$ as the point measure on $[0, \infty) \times \mathbf{C}^0$ given by

$$\mathcal{N}(g) = \sum_{j \in \mathcal{I}(g)} \delta_{(h_j, g^j)}.$$

Then, for any $t, a \in (0, \infty)$,

$$\mathcal{N}_t := \mathcal{N}(\check{e}^t) + \mathcal{N}(\hat{e}^t) \quad (\text{II.31})$$

and

$$\mathcal{N}_a^* := \mathcal{N}(a + X_{\cdot \wedge T_a}) + \mathcal{N}(a + X'_{\cdot \wedge T'_a}) \quad (\text{II.32})$$

We deduce from (II.30) that for all any a and for all nonnegative measurable function F on the set of positive measures on $[0, \infty) \times \mathbf{C}^0$, one has

$$\mathbf{N} \left[\int_0^\zeta dL_t^a F(\mathcal{N}_t) \right] = \mathbf{E} [F(\mathcal{N}_a^*)] \quad (\text{II.33})$$

and as consequence of Itô's decomposition of Brownian motion above its infimum, \mathcal{N}_a^* is a Poisson point measure on $[0, \infty) \times \mathbf{C}^0$ with intensity $2\mathbf{1}_{[0,a]}(h)dh\mathbf{N}(de)$.

Let us interpret this decomposition in terms of the Brownian tree. Choose $t \in (0, \zeta)$ such that $e_t = a$ and set $\sigma = p(t) \in \mathcal{T}$ (namely $\sigma \in \mathcal{T}(a)$). Then, the geodesic $[\rho, \sigma]$ is interpreted as the ancestral line of σ . Let us denote by \mathcal{T}_j^o , $j \in \mathcal{J}$, the connected components of the open set $\mathcal{T} \setminus [\rho, \sigma]$ and denote by \mathcal{T}_j the closure of \mathcal{T}_j^o . Then, there exists a point $\sigma_j \in [\rho, \sigma]$ such that $\mathcal{T}_j = \{\sigma_j\} \cup \mathcal{T}_j^o$. Recall notation \mathcal{N}_t from (II.31) and let us denote $\mathcal{N}_t = \sum_{j \in \mathcal{I}_t} \delta_{(h_j^t, e^{t,j})}$. Recall also the definition (II.32) of \mathcal{N}_a^* and denote $\mathcal{N}_a^* = \sum_{j \in \mathcal{I}_a^*} \delta_{(h_j^*, e^{*,j})}$. The specific coding of \mathcal{T} by e entails that for any $j \in \mathcal{J}$ there exists a unique $j' \in \mathcal{I}_t$ such that $d(\rho, \sigma_j) = h_j^t$, and such that the rooted compact real tree $(\mathcal{T}_j, d, \sigma_j)$ is isometric to the tree coded by $e^{t,j'}$.

Recall that $p(t) = \sigma$. We fix $r, r' \in [0, 2a)$ such that $r' \leq r$. We now compute the mass of the ring $B(\sigma, r) \setminus B(\sigma, r')$ in terms of \mathcal{N}_t . First, observe that for any $s \in [0, \zeta]$ such that $e_s = a$, we have

$$r' \leq d(s, t) < r \iff a - (r'/2) \geq \inf_{u \in [s \wedge t, s \vee t]} e_u > a - (r/2).$$

We then get

$$\ell^a(B(\sigma, r) \setminus B(\sigma, r')) = \sum_{j \in \mathcal{I}_t} \mathbf{1}_{(a-\frac{r}{2}, a-\frac{r'}{2}]}(h_j^t) L_{\zeta_j^t}^{a-h_j^t}(t, j), \quad (\text{II.34})$$

where $L_{\zeta_j^t}^{a-h_j^t}(t, j)$ stands for the local time at level $a - h_j^t$ of the excursion $e^{t,j}$.

Then, for any $a \in (0, \infty)$ and any $r, r' \in (0, 2a)$ such that $r' \leq r$, we also set

$$\Lambda_{r',r}^a = \sum_{j \in \mathcal{I}_a^*} \mathbf{1}_{(a-\frac{r}{2}, a-\frac{r'}{2}]}(h_j^*) L_{\zeta_j^*}^{a-h_j^*}, \quad (\text{II.35})$$

where, $L_{\zeta_j^*}^{a-h_j^*}$ stands for the local time at level $a - h_j^*$ of the excursion $e^{*,j}$ defined in (II.32). Let us consider $a \in (0, \infty)$, $n \in \mathbb{N}^*$ and (r_1, r_2, \dots, r_n) such that the $0 < r_n < \dots < r_2 < r_1 < 2a$. Then, (II.33) implies that for all non-negative measurable function F on \mathbb{R}^{n-1}

$$\mathbf{N} \left[\int_{\mathcal{T}} \ell^a(d\sigma) F(\ell^a(B(\sigma, r_k) \setminus B(\sigma, r_{k+1})); 1 \leq k \leq n-1) \right] = \mathbf{E} \left[F \left(\Lambda_{r_{k+1}, r_k}^a; 1 \leq k \leq n-1 \right) \right] \quad (\text{II.36})$$

On the right-hand-side, the dependency with respect to the level a is a bit artificial. Indeed, for $a \in (0, \infty)$, the Poisson point measure $\mathcal{N}_a^*(dhde)$ has its law invariant under the transformation $(h, e) \mapsto (a - h, e)$. Thus, let us consider on $(\Omega, \mathcal{F}, \mathbf{P})$ a new Poisson point measure $\mathcal{M}^* = \sum_{j \in \mathcal{I}^*} \delta_{(h_j^*, e_j^*)}$ with intensity $2dh\mathbf{N}(de)$ (we abuse notations and keep the notation (h_j^*, e_j^*) for the atoms). We set

$$\Lambda_{r',r}^* = \sum_{j \in \mathcal{I}^*} \mathbf{1}_{[\frac{r'}{2}, \frac{r}{2})}(h_j^*) L_{\zeta_j^*}^{h_j^*}, \quad (\text{II.37})$$

where $L_{\zeta_j^*}^{h_j^*}$ stands for the local time at height h_j^* for the excursion e_j^* . One can now rewrite (II.36) as

$$\begin{aligned} \mathbf{N} \left[\int_{\mathcal{T}} \ell^a(d\sigma) F(\ell^a(B(\sigma, r_k) \setminus B(\sigma, r_{k+1})); 1 \leq k \leq n-1) \right] &= \\ &= \mathbf{E} \left[F \left(\Lambda_{r_{k+1}, r_k}^*; 1 \leq k \leq n-1 \right) \right] \end{aligned} \quad (\text{II.38})$$

The law of the $\Lambda_{r',r}^*$ is quite explicit as shown by the following lemma.

Lemma 3.3 *Let $0 \leq r_n \leq r_{n-1} \leq \dots \leq r_1 \leq 2a$. Then,*

$$\Lambda_{r_n, r_{n-1}}^*, \Lambda_{r_{n-1}, r_{n-2}}^*, \dots, \Lambda_{r_2, r_1}^*$$

are independent. Moreover, for any $0 \leq r' \leq r \leq 2a$,

$$\forall y \in (0, \infty) \quad \mathbf{P}(\Lambda_{r',r}^* > y) = \left(1 - \frac{r'}{r}\right)^2 \frac{2y}{r} e^{-2y/r} + \left(1 - \left(\frac{r'}{r}\right)^2\right) e^{-2y/r},$$

and $\mathbf{P}(\Lambda_{r',r}^* = 0) = (r'/r)^2$.

Proof. The intervals $[r_{k+1}/2, r_k/2)$ being pairwise disjoint, the independence of the increments is a straightforward consequence of the properties the Poisson point measure \mathcal{M}^* . Using Campbell formula and (II.12), we compute, for all $\lambda \geq 0$,

$$\begin{aligned} \mathbf{E} \left[e^{-\lambda \Lambda_{2r',2r}^*} \right] &= \exp \left(- \int_{r'}^r 2dh \mathbf{N} \left[1 - e^{-\lambda e^{h(2r)}} \right] \right) \\ &= \exp \left(- \int_{r'}^r 2dh \frac{\lambda}{1+h\lambda} \right) = \left(\frac{1+r'\lambda}{1+r\lambda} \right)^2. \end{aligned}$$

Thus, $\Lambda_{2r',2r}^* \stackrel{(law)}{=} X_1 + X_2$, where X_1 and X_2 are i.i.d random variables where

$$\mathbf{E} \left[e^{-\lambda X_1} \right] = \frac{r'}{r} + \left(1 - \frac{r'}{r}\right) \frac{1}{1+r\lambda}.$$

Thus, $X_1 = 0$ with probability r'/r and conditionally on being non-zero, it is exponentially distributed with mean r . Thus, for $y > 0$,

$$\begin{aligned} \mathbf{P} \left(\Lambda_{2r',2r}^* > y \right) &= 2\mathbf{P} \left(X_1 = 0; X_2 > y \right) + \mathbf{P} \left(X_1 > 0; X_2 > 0; X_1 + X_2 > y \right) \\ &= 2\frac{r'}{r}\mathbf{P}(X_1 > y) + \left(1 - \frac{r'}{r}\right)^2 \mathbf{P}(Z > y), \end{aligned}$$

where Z has law Gamma(2, 1/r). The result proceeds now from elementary computations. ■

3.2 Estimates.

The following elementary computation is needed twice in our proofs.

Lemma 3.4 *Let $(X_n)_{n \geq 1}$ a sequence of i.i.d real valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, with mean 0 and a moment of order 4. Let Z be a random variable taking its values in \mathbb{N} , independent of the sequence (X_n) . Then*

$$\mathbf{E} \left[(X_1 + X_2 + \dots + X_Z)^4 \right] \leq 3\mathbf{E}[X_1^4] \mathbf{E} \left[Z^2 \right].$$

Moreover, the following holds : $\mathbf{E} \left[(X_1 - \mathbf{E}[X_1])^4 \right] \leq 2\mathbf{E}[X_1^4]$.

Proof. One has

$$\mathbf{E} \left[(X_1 + X_2 + \cdots + X_Z)^4 \mid Z \right] = \sum_{1 \leq i_1, i_2, i_3, i_4 \leq Z} \mathbf{E} [X_{i_1} X_{i_2} X_{i_3} X_{i_4}].$$

When (i_1, i_2, i_3, i_4) contains an index that is distinct of the three others, then the contribution of the corresponding term will be null. Thus the latter mean equals $Z\mathbf{E}[X_1^4] + 3Z(Z-1)\mathbf{E}[X_1^2]^2 \leq 3Z^2\mathbf{E}[X_1^4]$ (using Jensen's inequality). The second statement follows from

$$\begin{aligned} \mathbf{E} \left[(X_1 - \mathbf{E}[X_1])^4 \right] &= \mathbf{E} \left[(X_1 - \mathbf{E}[X_1])^4 \mathbf{1}_{\{X_1 \geq \mathbf{E}[X_1]\}} \right] + \mathbf{E} \left[(X_1 - \mathbf{E}[X_1])^4 \mathbf{1}_{\{X_1 < \mathbf{E}[X_1]\}} \right] \\ &\leq \mathbf{E} \left[X_1^4 \right] + \mathbf{E}[X_1]^4, \end{aligned}$$

and using Jensen's inequality. \blacksquare

We explained in Section 3.1 the link between the process $(\ell^a(\mathcal{T}), a \in (0, \infty))$ and the Feller diffusion, for which we provide here some basic estimates.

Lemma 3.5 *Let $(Y_a^x)_{a \geq 0}$ be a Feller diffusion starting at $x \geq 0$, defined on $(\Omega, \mathcal{F}, \mathbf{P})$. For all $x, y \in [0, \infty)$, for all $a \in (0, \infty)$, the following inequalities hold :*

- (i) *If $y \leq x$, then $\mathbf{P} \left(\inf_{b \in [0, a]} Y_b^x \leq y \right) \leq \exp \left(-\frac{1}{a} (\sqrt{x} - \sqrt{y})^2 \right)$.*
- (ii) *If $y \geq x$, then $\mathbf{P} \left(\sup_{b \in [0, a]} Y_b^x \geq y \right) \leq \exp \left(-\frac{1}{a} (\sqrt{y} - \sqrt{x})^2 \right)$.*

Proof. Let us prove (i). Recall that for all $x, b, \lambda \in [0, \infty)$, $\mathbf{E} \left[e^{-\lambda Y_b^x} \right] = \exp \left(-\frac{\lambda x}{1+b\lambda} \right)$. Thus, for fixed $a \in (0, \infty)$, and for $\lambda \in [0, \frac{1}{a})$, we set

$$\forall b \in [0, a], \quad M_b^{(\lambda, x)} := \exp \left(-\frac{\lambda Y_b^x}{1-b\lambda} \right). \quad (\text{II.39})$$

We stress that for $b \in [0, a]$, one has $1-b\lambda \geq 1-a\lambda > 0$, and one can compute

$$\mathbf{E}[M_b^{(\lambda, x)}] = \exp \left(-\frac{\lambda}{1-b\lambda} x / \left(1 + \frac{b\lambda}{1-b\lambda} \right) \right) = e^{-\lambda x}.$$

Combined with the Markov property, this entails that $(M_b^{(\lambda, x)}, b \in [0, a])$ is a martingale. Moreover, on $\left\{ \inf_{b \in [0, a]} Y_b^x \leq y \right\}$, one has $\inf_{b \in [0, a]} \frac{\lambda Y_b^x}{1-b\lambda} \leq \inf_{b \in [0, a]} \frac{\lambda Y_b^x}{1-a\lambda} \leq \frac{\lambda y}{1-a\lambda}$. Hence, the maximal inequality for sub-martingales entails

$$\begin{aligned} \mathbf{P} \left(\inf_{b \in [0, a]} Y_b^x \leq y \right) &\leq \mathbf{P} \left(\sup_{b \in [0, a]} M_b^{(\lambda, x)} \geq e^{-\frac{\lambda y}{1-a\lambda}} \right) \\ &\leq e^{\frac{\lambda y}{1-a\lambda}} \mathbf{E} \left[M_a^{(\lambda, x)} \right] = \exp \left(\frac{\lambda y}{1-a\lambda} - \lambda x \right). \end{aligned}$$

The reader can check using elementary computations that the function $\lambda \mapsto \frac{\lambda y}{1-a\lambda} - \lambda x$ has a negative minimum on $(0, 1/a)$ at the value $\lambda = \frac{1}{a} \left(1 - \sqrt{\frac{y}{x}} \right)$, and this minimum is $-\frac{1}{a} (\sqrt{x} - \sqrt{y})^2$, which completes the proof.

In order to prove (ii), one could extend the definition of $(M_b^{(\lambda,x)}, b \in [0, a])$ for $\lambda \in (-1/a, 0)$. In what follows, we use a simpler argument. Let us begin with the following remark: let $b \in (0, \infty)$, let \mathcal{E} be a r.v. on $(\Omega, \mathcal{F}, \mathbf{P})$ that is exponentially distributed with mean b , then for all $\lambda \geq 0$, $\mathbf{E}[e^{-\lambda\mathcal{E}}] = \frac{1}{1+b\lambda}$, and this Laplace transform remains finite for $\lambda \in (-1/b, 0)$. Moreover, one can plainly check that for $x, b \in (0, \infty)$, Y_b^x has the same law as $\sum_{i=1}^N \mathcal{E}_i$, where the \mathcal{E}_i are independent copies of \mathcal{E} and N is an independent Poisson r.v. with mean x/b . Thus, one has

$$\forall \mu \in (0, 1/b), \quad \mathbf{E} \left[e^{\mu Y_b^x} \right] = \exp \left(\frac{\mu x}{1 - \mu b} \right). \quad (\text{II.40})$$

The Feller diffusion $(Y_b^x, b \geq 0)$ is a martingale, so by convexity $(e^{\mu Y_b^x}, b \geq 0)$ is a submartingale. Thus, for all $\mu \in (0, 1/a)$, and $y \geq x \geq 0$, one has

$$\begin{aligned} \mathbf{P} \left(\sup_{b \in [0, a]} Y_b^x \geq y \right) &\leq \mathbf{P} \left(\sup_{b \in [0, a]} e^{\mu Y_b^x} \geq e^{\mu y} \right) \\ &\leq e^{-\mu y} \mathbf{E} \left[e^{\mu Y_a^x} \right] = \exp \left(\frac{\mu x}{1 - a\mu} - \mu y \right), \end{aligned}$$

and the result follows by optimizing the same function as before. ■

The next result is a corollary of Lemma 3.5 (ii).

Lemma 3.6 *Let $m \in (0, 1/2)$. For all $y \in (0, \infty)$,*

$$\mathbf{N} \left(\sup_{b \in [m, m^{-1}]} \ell^b(\mathcal{T}) > y \right) \leq (2/m) \exp(-my/2).$$

Proof. Let $m \in (0, 1/2)$ and recall from (II.23) the definition of \mathcal{G}_m . As recalled in Section 3.1, under \mathbf{N}_m , conditionally on \mathcal{G}_m , the process $(\ell^b(\mathcal{T}), b \geq m)$ is a Feller diffusion started at $\ell^m(\mathcal{T})$. Hence, conditioning with respect to \mathcal{G}_m and using Lemma 3.5 (ii), we get

$$\mathbf{N}_m \left(\sup_{b \in [m, m^{-1}]} \ell^b(\mathcal{T}) > y \right) \leq \mathbf{N}_m \left[\exp \left(-m \left(\sqrt{y} - \sqrt{\ell^m(\mathcal{T})} \right)^2 \right) \right].$$

Expanding $(\sqrt{u/2} - \sqrt{2v})^2$, one shows that for all $u, v \geq 0$, $(\sqrt{u} - \sqrt{v})^2 \geq u/2 - v$. Thus, $\mathbf{N}_m \left(\sup_{b \in [m, m^{-1}]} \ell^m(\mathcal{T}) > y \right) \leq \exp(-m\frac{y}{2}) \mathbf{N}_m \left[e^{m\ell^m(\mathcal{T})} \right]$. Recalling from (II.12) that under \mathbf{N}_m , $\ell^m(\mathcal{T})$ is exponentially distributed with mean m , we get $\mathbf{N}_m \left[e^{m\ell^m(\mathcal{T})} \right] = (1 - m^2)^{-1} \leq 2$, because $m < 1/2$. This entails the desired result, recalling that $\mathbf{N}_m(\cdot) = m\mathbf{N}(\cdot \mathbf{1}_{\{h(\mathcal{T}) > m\}})$ and that the events $\{h(\mathcal{T}) > m\}$ and $\{\ell^m(\mathcal{T}) > 0\}$ are equal, up to a \mathbf{N} negligible set. ■

Estimates for small balls. We consider here a level $a \in (0, \infty)$ and recall that $\mathcal{T}(a)$ is the a -level set of the Brownian tree \mathcal{T} . If $r \in [0, 2a]$, we recall from (II.18) the notation $\Gamma(\sigma, r)$ for the $\mathcal{T}(a)$ -ball of radius r and center $\sigma \in \mathcal{T}(a)$, the set of $\mathcal{T}(a)$ -balls of radius r being denoted $\mathcal{B}_{a,r}$. Let Γ be a $\mathcal{T}(a)$ -ball of radius r' , where $r' \in [0, 2a]$. From Proposition 2.1 (iii), we know that if $r \in [r', 2a]$, there exists a unique $\mathcal{T}(a)$ -ball of radius r that contains Γ , and we shall denote this "enlarged" ball by

$$\Gamma[r] := \Upsilon \quad \text{where } \Upsilon \in \mathcal{B}_{a,r} \text{ and } \Gamma \subset \Upsilon. \quad (\text{II.41})$$

We consider positive real numbers $r_1 > r_2 > \dots > r_n > 0$, and $\varepsilon_1 > \dots > \varepsilon_{n-1} > 0$, where $n \in \mathbb{N}^*$. We set $\mathbf{r} = \{r_1, \dots, r_n\}$ and $\boldsymbol{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_{n-1}\}$. We shall say that Γ , a $\mathcal{T}(a)$ -ball of radius r_n , is $(\mathbf{r}, \boldsymbol{\varepsilon})$ -small if and only if for all $1 \leq k \leq n-1$, the enlarged ball of radius r_k has a local time smaller than ε_k , namely

$$\forall k \in \{1, \dots, n-1\} \quad \ell^a(\Gamma[r_k]) \leq \varepsilon_k. \quad (\text{II.42})$$

We denote by $S_{a,\mathbf{r},\boldsymbol{\varepsilon}}$ the total number of such $(\mathbf{r}, \boldsymbol{\varepsilon})$ -small balls at level a :

$$S_{a,\mathbf{r},\boldsymbol{\varepsilon}} := \sum_{\Gamma \in \mathcal{B}_{a,r_n}} \mathbf{1}_{\{\Gamma \text{ is } (\mathbf{r}, \boldsymbol{\varepsilon})\text{-small}\}}. \quad (\text{II.43})$$

To control that number, we introduce

$$\mu(\mathbf{r}, \boldsymbol{\varepsilon}) := \mathbf{N} \left[S_{r_1/2, \mathbf{r}, \boldsymbol{\varepsilon}} \right]. \quad (\text{II.44})$$

Let us stress that its definition does not depend on a .

Lemma 3.7 *Let $a \in (0, \infty)$, $\mathbf{r} = \{r_1, \dots, r_n\}$, and $\boldsymbol{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_{n-1}\}$, where $r_1 > \dots > r_n > 0$, and $\varepsilon_1 > \dots > \varepsilon_{n-1} > 0$. There exists a constant $c_0 \in (0, 10^4]$ such that if $a/r_1 > 1$ and $r_1/r_n > 2$,*

$$\mathbf{N} \left[(S_{a,\mathbf{r},\boldsymbol{\varepsilon}} - \mu(\mathbf{r}, \boldsymbol{\varepsilon}) \ell^a(\mathcal{T}))^4 \right] \leq c_0 a \frac{r_1^2}{r_n^4}.$$

Proof. Let $a, \mathbf{r}, \boldsymbol{\varepsilon}$ as above. From Proposition 2.1 (iii), we know that the $\mathcal{T}(a)$ -balls of radius r_n are disjoint and that for all $\Upsilon \in \mathcal{B}_{a,r_n}$, there exists a unique $\mathcal{T}(a)$ -ball $\Gamma \in \mathcal{B}_{a,r_1}$ such that $\Upsilon \subset \Gamma$. Let us enumerate \mathcal{B}_{a,r_1} as $\{\Gamma_i, 1 \leq i \leq Z_{a,r_1}\}$, and set

$$\begin{aligned} \forall i \in \{1 \dots Z_{a,r_1}\}, \quad \mathcal{B}_{a,r_n}^{(i)} &= \{\Upsilon \in \mathcal{B}_{a,r_n} : \Upsilon \subset \Gamma_i\} \\ \text{and } S_{a,\mathbf{r},\boldsymbol{\varepsilon}}^{(i)} &= \# \left\{ \Upsilon \in \mathcal{B}_{a,r_n}^{(i)} : \Upsilon \text{ is } (\mathbf{r}, \boldsymbol{\varepsilon})\text{-small} \right\}. \end{aligned}$$

One has

$$S_{a,\mathbf{r},\boldsymbol{\varepsilon}} - \mu(\mathbf{r}, \boldsymbol{\varepsilon}) \ell^a(\mathcal{T}) = \sum_{i=1}^{Z_{a,r_1}} \left(S_{a,\mathbf{r},\boldsymbol{\varepsilon}}^{(i)} - \mu(\mathbf{r}, \boldsymbol{\varepsilon}) \ell^a(\Gamma_i) \right) =: \sum_{i=1}^{Z_{a,r_1}} X_i. \quad (\text{II.45})$$

Let us denote $b = a - r_1/2$ and recall from (II.23) the definition of the sigma-field \mathcal{G}_b . Adapting the proof of Lemma 3.1, it is not difficult to see that under \mathbf{N}_b , conditionally on \mathcal{G}_b , and conditionally on $\{Z_{a,r_1} = k\}$, the r.v. X_1, \dots, X_k are independent and have the same law as $S_{r_1/2, \mathbf{r}, \boldsymbol{\varepsilon}} - \mu(\mathbf{r}, \boldsymbol{\varepsilon}) \ell^{r_1/2}(\mathcal{T})$ under $\mathbf{N}_{r_1/2}$. Recalling from (II.12) that $\mathbf{N} \left[\ell^{r_1/2}(\mathcal{T}) \right] = (2/r_1) \mathbf{N}_{r_1/2} \left[\ell^{r_1/2}(\mathcal{T}) \right] = 1$, we see that

$$\mathbf{N}_b[X_1 \mid \mathcal{G}_b] = \mathbf{N}_{r_1/2} \left[S_{r_1/2, \mathbf{r}, \boldsymbol{\varepsilon}} - \mu(\mathbf{r}, \boldsymbol{\varepsilon}) \ell^{r_1/2}(\mathcal{T}) \right] = 0,$$

which explains the definition (II.44). We thus apply Lemma 3.4 to get from (II.45):

$$\mathbf{N}_b \left[(S_{a,\mathbf{r},\boldsymbol{\varepsilon}} - \mu(\mathbf{r}, \boldsymbol{\varepsilon}) \ell^a(\mathcal{T}))^4 \mid \mathcal{G}_b \right] \leq 3 \mathbf{N}_{r_1/2} \left[X_1^4 \right] \mathbf{N}_b \left[Z_{a,r_1}^2 \mid \mathcal{G}_b \right]. \quad (\text{II.46})$$

The second assertion in Lemma 3.4 entails $\mathbf{N}_{r_1/2}[X_1^4] \leq 2 \mathbf{N}_{r_1/2} \left[S_{r_1/2, \mathbf{r}, \boldsymbol{\varepsilon}}^4 \right]$. Moreover, we can use that $S_{r_1/2, \mathbf{r}, \boldsymbol{\varepsilon}}$ is smaller than $Z_{r_1/2, r_n}$, the total number of $\mathcal{T}(r_1/2)$ -balls of radius

r_n which has under $\mathbf{N}_{r_1/2}$ a geometric distribution with success probability $r_n/r_1 < 1/2$. Thus,

$$\mathbf{N}_{r_1/2} [X_1^4] \leq 2\mathbf{N}_{r_1/2} [S_{r_1/2, \mathbf{r}, \varepsilon}^4] \leq 2\mathbf{N}_{r_1/2} [Z_{r_1/2, r_n}^4] \leq \frac{48}{1 - r_n/r_1} \left(\frac{r_1}{r_n}\right)^4 \leq 96 \left(\frac{r_1}{r_n}\right)^4. \quad (\text{II.47})$$

In addition, according to the branching property, under \mathbf{N}_b , conditionally on \mathcal{G}_b , Z_{a, r_1} is a Poisson variable with mean $\mathbf{N}(h(\mathcal{T}) > r_1/2) \ell^b(\mathcal{T}) = (2/r_1)\ell^b(\mathcal{T})$. Thus,

$$\mathbf{N}_b [Z_{a, r_1}^2] = (2/r_1)\mathbf{N}_b [\ell^b(\mathcal{T})] + (2/r_1)^2\mathbf{N}_b [\ell^b(\mathcal{T})^2]. \quad (\text{II.48})$$

We know from (II.12) under \mathbf{N}_b , $\ell^b(\mathcal{T})$ has exponential law with mean b . Thus, $\mathbf{N}_b [\ell^b(\mathcal{T})] = b$ and $\mathbf{N}_b [\ell^b(\mathcal{T})^2] = 2b^2$. Recall that $b = a - r_1/2 \leq a$, so we get $(2/r_1)\mathbf{N}_b [\ell^b(\mathcal{T})] \leq \frac{2a}{r_1} \leq 2\frac{a^2}{r_1^2}$ because we assumed that $a/r_1 > 1$. Moreover $(2/r_1)^2\mathbf{N}_b [\ell^b(\mathcal{T})^2] = 8b^2/r_1^2 \leq 8b^2/r_1^2$. Thus $\mathbf{N}_b [Z_{a, r_1}^2] \leq 10a^2/r_1^2$. Combined with (II.46) and (II.47) it entails

$$\mathbf{N}_b [(S_{a, \mathbf{r}, \varepsilon} - \mu(\mathbf{r}, \varepsilon)\ell^a(\mathcal{T}))^4] \leq c_0 a^2 \frac{r_1^2}{r_n^4},$$

with c_0 a positive constant smaller than $(1/2)10^4$. This implies the desired result, using that $\mathbf{N}(h(\mathcal{T}) > b) = 1/b \leq 2/a$. ■

We state now the main technical Lemma of the paper. Let us recall from (II.44) the definition of $\mu(\mathbf{r}, \varepsilon)$. The proof of the lemma makes use of the spinal decomposition described in Section 3.1. In particular, a geometric argument allows to relate the problem to the variables introduced in (II.37).

Lemma 3.8 *Let $\mathbf{r} = \{r_1, \dots, r_n\}$, where $r_1 > \dots > r_n > 0$, and $\varepsilon = \{\varepsilon_1, \dots, \varepsilon_{n-1}\}$, where $\varepsilon_1 > \dots > \varepsilon_{n-1} > 0$. The following inequality holds :*

$$\mu(\mathbf{r}, \varepsilon) \leq \frac{5}{r_n} \sqrt{\prod_{k=1}^{n-1} \mathbf{P}(\Lambda_{r_{k+1}, r_k}^* \leq \varepsilon_k)}. \quad (\text{II.49})$$

Proof. Let $\mathbf{r} = \{r_1, \dots, r_n\}$ and $\{\varepsilon_1, \dots, \varepsilon_{n-1}\}$ as above. In that proof, we denote, for convenience, $b = r_1/2$; hence, a dependency with respect to b is actually a dependency with respect to \mathbf{r} . Let us consider Γ a $\mathcal{T}(b)$ -ball of radius r_n and recall the notation (II.41). The ball Γ is $(\mathbf{r}, \varepsilon)$ -small iff (II.42) holds. But, for all $\sigma \in \Gamma, k \in \llbracket 1, n-1 \rrbracket$,

$$\Gamma[r_k] = \Gamma(\sigma, r_k) \supset \Gamma(\sigma, r_k) \setminus \Gamma(\sigma, r_{k+1}).$$

Thus, if Γ is $(\mathbf{r}, \varepsilon)$ -small, then all the vertices in Γ belong to the set

$$\mathcal{S}(\mathbf{r}, \varepsilon) := \left\{ \sigma \in \mathcal{T}(b) : \forall k \in \{1 \dots n-1\} \quad \ell^b(\Gamma(\sigma, r_k) \setminus \Gamma(\sigma, r_{k+1})) \leq \varepsilon_k \right\}. \quad (\text{II.50})$$

The last set is easy to handle using the spinal decomposition. Indeed, according to (II.38) and the independence stated in Lemma 3.3, one has

$$\nu(\mathbf{r}, \varepsilon) := \mathbf{N} \left[\int \ell^b(d\sigma) \mathbf{1}_{\{\sigma \in \mathcal{S}(\mathbf{r}, \varepsilon)\}} \right] = \prod_{k=1}^{n-1} \mathbf{P}(\Lambda_{r_{k+1}, r_k}^* \leq \varepsilon_k). \quad (\text{II.51})$$

To relate $\mu(\mathbf{r}, \varepsilon)$ and $\nu(\mathbf{r}, \varepsilon)$, one can write

$$\mathbf{1}_{\{\Gamma \text{ is } (\mathbf{r}, \varepsilon)\text{-small}\}} \leq \mathbf{1}_{\{\ell^b(\Gamma) \leq r_n \sqrt{\nu(\mathbf{r}, \varepsilon)}\}} + \frac{\ell^b(\Gamma)}{r_n \sqrt{\nu(\mathbf{r}, \varepsilon)}} \mathbf{1}_{\{\Gamma \text{ is } (\mathbf{r}, \varepsilon)\text{-small}\}}. \quad (\text{II.52})$$

Moreover, (II.50) entails that $\ell^b(\Gamma) \mathbf{1}_{\{\Gamma \text{ is } (\mathbf{r}, \varepsilon)\text{-small}\}} \leq \int_{\Gamma} \ell^b(d\sigma) \mathbf{1}_{\{\sigma \in \mathcal{S}(\mathbf{r}, \varepsilon)\}}$. Recall now from Proposition 2.1 (i) that the balls of the set \mathcal{B}_{b, r_n} are pairwise disjoint. Summing in (II.52) over this set entails

$$S_{b, \mathbf{r}, \varepsilon} \leq \sum_{\Gamma \in \mathcal{B}_{b, r_n}} \mathbf{1}_{\{\ell^b(\Gamma) \leq r_n \sqrt{\nu(\mathbf{r}, \varepsilon)}\}} + \frac{\int \ell^b(d\sigma) \mathbf{1}_{\{\sigma \in \mathcal{S}(\mathbf{r}, \varepsilon)\}}}{r_n \sqrt{\nu(\mathbf{r}, \varepsilon)}}. \quad (\text{II.53})$$

Now, recalling Lemma 3.1, we compute

$$\begin{aligned} \mathbf{N}_b \left[\sum_{\Gamma \in \mathcal{B}_{b, r_n}} \mathbf{1}_{\{\ell^b(\Gamma) \leq r_n \sqrt{\nu(\mathbf{r}, \varepsilon)}\}} \right] &= \mathbf{N}_b [Z_{b, r_n}] \left(1 - \exp \left(-(2/r_n) r_n \sqrt{\nu(\mathbf{r}, \varepsilon)} \right) \right) \\ &\leq \frac{r_1}{r_n} 2 \sqrt{\nu(\mathbf{r}, \varepsilon)}, \end{aligned}$$

so the \mathbf{N} -measure of the first term in (II.53) is smaller than $\frac{2b^{-1}r_1}{r_n} \sqrt{\nu(\mathbf{r}, \varepsilon)}$. Recalling that $b = r_1/2$, we get that the latter equals $\frac{4}{r_n} \sqrt{\nu(\mathbf{r}, \varepsilon)}$. Moreover, by the mere definition (II.51), the \mathbf{N} -measure of the second term in (II.53) equals $\frac{1}{r_n} \sqrt{\nu(\mathbf{r}, \varepsilon)}$, so the first inequality is checked. \blacksquare

4 Proof of Theorem 1.1.

The proof of Theorem 1.1 will combine the following two theorems.

Theorem 4.1 *Let $\kappa \in (\frac{1}{2}, \infty)$ and $m \in (0, \frac{1}{2})$. Then, there exists a Borel subset $\mathbf{V} = \mathbf{V}(\kappa, m) \subset \mathbf{C}^0$ such that $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}) = 0$ and such that*

$$\text{on } \mathbf{V}, \quad \text{for all Borel subset } \mathcal{A} \subset \mathcal{T}, \quad \forall a \in [m, m^{-1}], \quad \ell^a(\mathcal{A}) \leq \kappa \mathcal{H}_g(\mathcal{A} \cap \mathcal{T}(a)).$$

For all $a, \alpha \in (0, \infty)$, let us set

$$\Delta_a^\alpha := \left\{ \sigma \in \mathcal{T}(a) : \limsup_{r \rightarrow 0} \frac{\ell^a(B(\sigma, r))}{g(r)} < \alpha \right\}. \quad (\text{II.54})$$

Theorem 4.2 *Let $\alpha \in (0, \frac{1}{2})$ and $m \in (0, 1/2)$. Then, there exists a Borel subset $\mathbf{V}' = \mathbf{V}'(\alpha, m) \subset \mathbf{C}^0$ such that $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}') = 0$ and such that*

$$\text{on } \mathbf{V}', \quad \forall a \in [m, m^{-1}], \quad \mathcal{H}_g(\Delta_a^\alpha) = 0.$$

The proofs of Theorem 4.1 and 4.2 share a common strategy, taken from Perkins [100, 101]. We need to control the mass, or the number of "bad" $\mathcal{T}(a)$ -balls where "bad" means too large or too small. And we want to do it uniformly for all levels a . This problem will be linked with a discrete one using a finite grid, and the measure or the number of bad $\mathcal{T}(a)$ -balls will be compared with a convenient multiple of $\ell^a(\mathcal{T})$, the total mass at level a .

4.1 Proof of Theorem 4.1.

4.1.1 Large balls.

Let us fix a level $a \in (0, \infty)$, and recall from section 3.1 the definition of the sigma-field \mathcal{G}_a , generated by the excursion below level a . We also recall the definition of $\mathcal{T}(a)$ -balls (II.18). We fix a threshold $y \in (0, \infty)$ and we consider the following set of "large" points on $\mathcal{T}(a)$:

$$\mathcal{L}_{a,r,y} = \{\sigma \in \mathcal{T}(a) : \ell^a(\Gamma(\sigma, r)) > y\}. \quad (\text{II.55})$$

According to Lemma 3.1, the "total large mass" $\ell^a(\mathcal{L}_{a,r,y}) = \sum_{\Gamma \in \mathcal{B}_{a,r}} \ell^a(\Gamma) \mathbf{1}_{\{\ell^a(\Gamma) > y\}}$ is \mathcal{G}_a -measurable.

Lemma 4.3 *For all $a, l, y, r, \delta \in (0, \infty)$, for all $c \in (1, \infty)$,*

$$\mathbf{N}\left(\ell^a(\mathcal{L}_{a,r,y/c}) \leq l ; \sup_{b \in [a, a+\delta]} \ell^b(\mathcal{L}_{b,r,y}) > 4l\right) \leq \frac{1}{a} \exp(-l/\delta) + \frac{2}{r} \exp\left(-(1-c^{-1/2})^2 y/\delta\right).$$

Proof. Let $a, l, y, r, \delta \in (0, \infty)$, where $r \leq 2a$, and $c \in (1, \infty)$. We define A_0 , a Borel subset of \mathbf{C}^0 , as the event

$$A_0 = \left\{ \ell^a(\mathcal{L}_{a,r,y/c}) \leq l ; \sup_{b \in [a, a+\delta]} \ell^b(\mathcal{L}_{b,r,y}) > 4l \right\}. \quad (\text{II.56})$$

We recall from Proposition 2.1 that $\mathcal{B}_{a,r} = \{\Gamma_i, 1 \leq i \leq Z_{a,r}\}$ is the collection of $\mathcal{T}(a)$ -balls of radius r at level a . For Γ a $\mathcal{T}(a)$ -ball and $b \in [a, \infty)$, we defined $\Gamma^b = \{\sigma \in \mathcal{T}(b) : \exists \sigma' \in \Gamma, \sigma' \in \llbracket \rho, \sigma \rrbracket\}$ as the set of vertices at level b having an ancestor in Γ (see (II.29) for details). Next we define A_1 a Borel subset of \mathbf{C}^0 as the event

$$A_1 := \left\{ \exists i \in \{1, \dots, Z_{a,r}\}, \ell^a(\Gamma_i) \leq y/c \text{ and } \sup_{b \in [a, a+\delta]} \ell^b(\Gamma_i^b) > y \right\}, \quad (\text{II.57})$$

and set

$$\mathcal{L}_{a,r,y/c}^b := \bigcup_{\substack{i \in \{1, \dots, Z_{a,r}\} \\ \ell^a(\Gamma_i) \geq y/c}} \Gamma_i^b \subset \mathcal{T}(b), \quad (\text{II.58})$$

which is the set of all vertices at level b having a "large" ancestor at level a . We prove the following :

$$\text{on } \mathbf{C}^0 \setminus A_1, \quad \forall b \in [a, a+\delta] \quad \mathcal{L}_{b,r,y} \subset \mathcal{L}_{a,r,y/c}^b. \quad (\text{II.59})$$

Proof of (II.59). Let $b \in [a, a+\delta]$ and let $\sigma \in \mathcal{L}_{b,r,y}$. Thus, the ball $\Gamma := \Gamma(\sigma, r) \in \mathcal{B}_{b,r}$ is such that $\ell^b(\Gamma) \geq y$. Let σ_a the unique ancestor of σ at level a , namely $d(\rho, \sigma_a) = a$ and $\sigma_a \in \llbracket \rho, \sigma \rrbracket$. We set $\Upsilon := \Gamma(\sigma_a, r) \in \mathcal{B}_{a,r}$ and we first claim that $\Gamma \subset \Upsilon^b$. Indeed, let $\sigma' \in \Gamma$ (so $d(\sigma, \sigma') < r$) and let σ'_a the unique ancestor of σ' at level a . Recalling that $\sigma \wedge \sigma'$ stands for the most recent common ancestor of σ and σ' , one get $d(\rho, \sigma \wedge \sigma') = \frac{1}{2}(2b - d(\sigma, \sigma'))$. Then two cases may occur. First, if $d(\sigma, \sigma') \leq 2(b-a)$, then $d(\rho, \sigma \wedge \sigma') \geq a$, thus $\sigma_a = \sigma'_a$ and $\sigma' \in \Upsilon^b$. If $d(\sigma, \sigma') \in (2(b-a), r)$. Then, one has $d(\rho, \sigma \wedge \sigma') < a$. We deduce from that inequality that $\sigma_a \neq \sigma'_a$ and that $\sigma_a \wedge \sigma'_a = \sigma \wedge \sigma'$. Hence,

$$\begin{aligned} d(\sigma_a, \sigma'_a) &= 2a - 2d(\rho, \sigma_a \wedge \sigma'_a) \\ &= 2b - 2d(\rho, \sigma \wedge \sigma') + 2a - 2b \\ &= d(\sigma, \sigma') - 2(b-a) < r. \end{aligned}$$

Thus $\sigma'_a \in \Upsilon$ and $\sigma' \in \Upsilon^b$, which ends the proof of the inclusion $\Gamma \subset \Upsilon^b$. We get that for all $b \in [a, a + \delta]$, $\ell^b(\Upsilon^b) \geq \ell^b(\Gamma) > y$. On $\mathbf{C}^0 \setminus A_1$, one cannot have both $\ell^a(\Upsilon) \leq y/c$ and $\sup_{b \in [a, a + \delta]} \ell^b(\Upsilon^b) > y$, which entails that here, $\ell^a(\Upsilon) > y/c$.

To sum up, on $\mathbf{C}^0 \setminus A_1$, a vertex σ , taken in $\mathcal{L}_{b,r,y}$, has an ancestor in a ball Υ , such that $\ell^a(\Upsilon) \geq y/c$. Thus, this ancestor belongs to $\mathcal{L}_{a,r,y/c}$ and $\sigma \in \mathcal{L}_{a,r,y/c}^b$.

End of the proof of (II.59).

Let us finish the proof of the lemma. From (II.59), we see that

$$\mathbf{N}(A_0) \leq \mathbf{N}(A_1) + \mathbf{N}(A_0 \cap (\mathbf{C}^0 \setminus A_1)) \leq \mathbf{N}(A_1) + \mathbf{N}(A_2), \quad (\text{II.60})$$

where A_2 is defined by

$$A_2 := \left\{ \ell^a(\mathcal{L}_{a,r,y/c}) \leq l ; \sup_{b \in [a, a + \delta]} \ell^b(\mathcal{L}_{a,r,y/c}^b) \geq 4l \right\}. \quad (\text{II.61})$$

We control $\mathbf{N}(A_1)$ and $\mathbf{N}(A_2)$ thanks to Lemma 3.5 (ii). Indeed, Lemma 3.2 states that under \mathbf{N}_a , conditionally on \mathcal{G}_a , the processes $(\ell^{a+a'}(\Gamma_i^{a+a'}), a' \geq 0), 1 \leq i \leq Z_{a,r}$ are independent Feller diffusions started at $\ell^a(\Gamma_i), 1 \leq i \leq Z_{a,r}$. Sub-additivity and Lemma 3.5 (ii) entails

$$\begin{aligned} \mathbf{N}(A_1) &\leq \frac{1}{a} \mathbf{N}_a \left[\sum_{i=1}^{Z_{a,r}} \mathbf{1}_{\{\ell^a(\Gamma_i) \leq y/c\}} \exp \left(-\delta^{-1} \left(\sqrt{y} - \sqrt{\ell^a(\Gamma_i)} \right)^2 \right) \right] \\ &\leq \frac{1}{a} \exp \left(-(1-c^{-1/2})^2 \delta^{-1} y \right) \mathbf{N}_a[Z_{a,r}] = \frac{2}{r} \exp \left(-(1-c^{-1/2})^2 \delta^{-1} y \right). \end{aligned} \quad (\text{II.62})$$

Since $\ell^b(\mathcal{L}_{a,r,y/c}^b) = \sum_{i=1}^{Z_{a,r}} \mathbf{1}_{\{\ell^a(\Gamma_i) > y/c\}} \ell^b(\Gamma_i^b)$, it implies that $(\ell^{a+a'}(\mathcal{L}_{a,r,y/c}^{a+a'}), a' \geq 0)$ is a Feller diffusion started at $\ell^a(\mathcal{L}_{a,r,y/c})$. Thus, we use Lemma 3.5 (ii) again to get

$$\begin{aligned} \mathbf{N}(A_2) &\leq \frac{1}{a} \mathbf{N}_a \left[\mathbf{1}_{\{\ell^a(\mathcal{L}_{a,r,y/c}) \leq l\}} \exp \left(-\delta^{-1} \left(2\sqrt{l} - \sqrt{\ell^a(\mathcal{L}_{a,r,y/c})} \right)^2 \right) \right] \\ &\leq \frac{1}{a} \exp(-l/\delta). \end{aligned} \quad (\text{II.63})$$

Hence, the desired result follows from (II.60), (II.62), and (II.63). \blacksquare

Recall that $g(r) = r \log \log(1/r)$. We fix $\kappa \in (\frac{1}{2}, \infty)$, and we shall apply the previous lemma with $y = \kappa g(r)$. The next lemma allows to control $\ell^a(\mathcal{L}_{a,r,\kappa g(r)})$ uniformly for all levels a . Its proof involves a discrete grid : for $m < 1/2$ and $r \in (0, \infty)$, we set

$$G(r, m) := \left\{ m + k\delta_r, k \in \mathbb{N}^* \right\} \cap [m, m^{-1}], \quad (\text{II.64})$$

where δ_r is the mesh of the grid, defined by

$$\delta_r = r^{3/2}. \quad (\text{II.65})$$

Note that $G(r, m)$ contains less than $(m\delta_r)^{-1}$ points.

Lemma 4.4 *Let $m \in (0, 1/2)$. Let $\kappa \in (\frac{1}{2}, \infty)$ and $\beta \in (1, \infty)$ such that $2\kappa - \beta > 0$. There exists a constant $r_1 \in (0, \infty)$ only depending on κ, β, m , such that*

$$\forall r \in (0, r_1), \quad \mathbf{N} \left(\sup_{b \in [m, m^{-1}]} \ell^b \left(\mathcal{L}_{b,r,\kappa g(r)} \right) > 4 \log(1/r)^{-\beta} \right) \leq \log(1/r)^{-2}. \quad (\text{II.66})$$

Proof. In what follows, we denote T_0 the left-hand-side of (II.66). Let us consider $c \in (1, \infty)$ such that $2\kappa/c - \beta > 0$. Recall that $G(r, m)$ stands for the grid defined by (II.64). Then we have $T_0 \leq T_1 + T_2$, where :

$$T_1 = \mathbf{N} \left(\sup_{a \in G(r, m)} \ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) \leq \log(1/r)^{-\beta} ; \sup_{b \in [m, m^{-1}]} \ell^b \left(\mathcal{L}_{b,r,\kappa g(r)} \right) \geq 4 \log(1/r)^{-\beta} \right),$$

$$T_2 = \mathbf{N} \left(\sup_{a \in G(r, m)} \ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) > \log(1/r)^{-\beta} \right).$$

Using sub-additivity and Lemma 4.3, one get

$$\begin{aligned} T_1 &= \mathbf{N} \left(\bigcup_{a \in G(r, m)} \left\{ \ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) \leq \log(1/r)^{-\beta} ; \sup_{b \in [a, a+\delta_r]} \ell^b \left(\mathcal{L}_{b,r,\kappa g(r)} \right) \geq 4 \log(1/r)^{-\beta} \right\} \right) \\ &\leq (m\delta_r)^{-1} \sup_{a \in G(r, m)} \mathbf{N} \left(\ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) \leq \log(1/r)^{-\beta} ; \sup_{b \in [a, a+\delta_r]} \ell^b \left(\mathcal{L}_{b,r,\kappa g(r)} \right) \geq 4 \log(1/r)^{-\beta} \right) \\ &\leq (m\delta_r)^{-1} \left(m^{-1} \exp \left(-\delta_r^{-1} \log(1/r)^{-\beta} \right) + \frac{2}{r} \exp \left(-(1-c^{-1/2})^2 \kappa \delta_r^{-1} g(r) \right) \right). \end{aligned}$$

One has $\delta_r^{-1} \log(1/r)^{-\beta} \geq r^{-1}$ and $\delta_r^{-1} g(r) \geq r^{-1/2}$ for all r sufficiently small. Thus, for example, $T_1 \leq \exp(-r^{-1/4}) \leq (1/2) \log(1/r)^{-2}$ for all r sufficiently small.

Let us bound T_2 . To that end, we set

$$\lambda(r, \kappa, c) := (2/r) \mathbf{E} \left[\mathcal{E} \mathbf{1}_{\{\mathcal{E} > \kappa g(r)/c\}} \right], \quad (\text{II.67})$$

where \mathcal{E} is a r.v. on $(\Omega, \mathcal{F}, \mathbf{P})$ exponentially distributed with mean $r/2$. For fixed κ and c , elementary computations entail

$$\begin{aligned} \lambda(r, \kappa, c) &= (2/r) \mathbf{P} \left(\mathcal{E} > \kappa g(r)/c \right) \mathbf{E}[\kappa g(r)/c + \mathcal{E}] \\ &= (2/r) \exp \left(-2(\kappa/c) \log \log 1/r \right) \left((\kappa/c)r \log \log 1/r + r/2 \right) \\ &\underset{r \rightarrow 0}{\sim} (2\kappa/c) \log(1/r)^{-2\kappa/c} \log \log 1/r. \end{aligned} \quad (\text{II.68})$$

We set $T_2 \leq T_3 + T_4$, where

$$T_3 = \mathbf{N} \left(\sup_{a \in G(r, m)} \left| \ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) - \lambda(r, \kappa, c) \ell^a(\mathcal{T}) \right| > \frac{1}{2} \log(1/r)^{-\beta} \right),$$

$$T_4 = \mathbf{N} \left(\sup_{a \in G(r, m)} \lambda(r, \kappa, c) \ell^a(\mathcal{T}) > \frac{1}{2} \log(1/r)^{-\beta} \right).$$

By sub-additivity and a Markov inequality involving a moment of order 4, we get

$$\begin{aligned} T_3 &\leq (m\delta_r)^{-1} \sup_{a \in G(r, m)} \mathbf{N} \left(\left| \ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) - \lambda(r, \kappa, c) \ell^a(\mathcal{T}) \right| > \frac{1}{2} \log(1/r)^{-\beta} \right) \\ &\leq (m\delta_r)^{-1} 2^4 \log(1/r)^{4\beta} \sup_{a \in G(r, m)} a^{-1} \mathbf{N}_a \left[\left(\ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) - \lambda(r, \kappa, c) \ell^a(\mathcal{T}) \right)^4 \right]. \end{aligned} \quad (\text{II.69})$$

Recall notation $\mathcal{B}_{a,r} = \{\Gamma_i, 1 \leq i \leq Z_{a,r}\}$ for the set of $\mathcal{T}(a)$ -balls with radius r . Then, consider the decomposition

$$\ell^a \left(\mathcal{L}_{a,r,\kappa g(r)/c} \right) - \lambda(r, \kappa, c) \ell^a(\mathcal{T}) = \sum_{i=1}^{Z_{a,r}} X_i,$$

where $X_i := \ell^a(\Gamma_i) \left(\mathbf{1}_{\{\ell^a(\Gamma_i) \geq \kappa g(r)/c\}} - \lambda(r, \kappa, c) \right)$. Using Lemma 3.1, we see that under \mathbf{N}_a , conditionally on $Z_{a,r}$, the random variables $\ell^a(\Gamma_1), \dots, \ell^a(\Gamma_{Z_{a,r}})$ are independent and exponentially distributed with mean $r/2$. Thus, the definition (II.67) of $\lambda(r, \kappa, c)$ entails that under \mathbf{N}_a , conditionally on $Z_{a,r}$, the r.v. $X_1, \dots, X_{Z_{a,r}}$ are i.i.d., with mean 0 and a moment of order 4. Then, by Lemma 3.4,

$$\mathbf{N}_a \left[\left(\sum_{i=1}^{Z_{a,r}} X_i \right)^4 \right] \leq 3 \mathbf{N}_a(X_1^4) \mathbf{N}_a \left[Z_{a,r}^2 \right]. \quad (\text{II.70})$$

From (II.68), we know that $\lambda(r, \kappa, c) \xrightarrow{r \rightarrow 0} 0$, so for all sufficiently small r , $\lambda(r, \kappa, c) \leq 1/2$ and $|X_1| \leq \ell^a(\Gamma_1)$, which implies $\mathbf{N}_a[X_1^4] \leq \mathbf{N}_a[\ell^a(\Gamma_1)^4] = \frac{3}{2}r^4$ for all sufficiently small r . Moreover, $Z_{a,r}$ is under \mathbf{N}_a a geometric r.v. with "success" probability $p = r/2a$ (see Lemma 3.1), thus $\mathbf{N}_a \left[Z_{a,r}^2 \right] = (2-p)/p^2 \leq 8a^2/r^2$. Combining (II.69) and (II.70), we get, for all sufficiently small r ,

$$T_3 \leq 3.2^4 \cdot (m\delta_r)^{-1} \log(1/r)^{4\beta} \sup_{a \in G(r,m)} a^{-1} \frac{3r^4}{2} \frac{8a^2}{r^2} \leq 10^3 m^{-2} \log(1/r)^{4\beta} r^{1/2}, \quad (\text{II.71})$$

recalling that $\delta_r = r^{3/2}$. Observe now that the right hand side is smaller than $(1/4) \log(1/r)^{-2}$ for all sufficiently small r .

For the term T_4 , Lemma 3.6 entails

$$\begin{aligned} T_4 &\leq \mathbf{N} \left(\sup_{b \in [m, m^{-1}]} \ell^b(\mathcal{T}) > \frac{1}{2} \lambda(r, \kappa, c)^{-1} \log(1/r)^{-\beta} \right) \\ &\leq (2/m) \exp \left(-(m/4) \lambda(r, \kappa, c)^{-1} \log(1/r)^{-\beta} \right). \end{aligned} \quad (\text{II.72})$$

By (II.68),

$$\lambda(r, \kappa, c)^{-1} \log(1/r)^{-\beta} \underset{r \rightarrow 0}{\sim} \frac{c}{2\kappa} \log(1/r)^{2\kappa/c - \beta} \log \log(1/r)^{-1}.$$

Recall that $2\kappa/c - \beta > 0$ and take $\varepsilon \in (0, 2\kappa/c - \beta)$. Thus, for all sufficiently small r ,

$$T_4 \leq (2/m) \exp \left(-\log(1/r)^\varepsilon \right),$$

which is smaller than $(1/4) \log(1/r)^{-2}$ for all sufficiently small r . ■

4.1.2 Proof of Theorem 4.1.

Let $\kappa \in (1/2, \infty)$, and let $m \in (0, 1/2)$. Let $\beta \in (1, \infty)$ such that $2\kappa - \beta > 0$. For all $a \in (0, \infty)$, $y \in (1, \infty)$ recall from (II.54) the definition :

$$\Delta_a^{y\kappa} = \left\{ \sigma \in \mathcal{T}(a) : \limsup_{r \rightarrow 0} \frac{\ell^a(B(\sigma, r))}{g(r)} < y\kappa \right\}. \quad (\text{II.73})$$

For any $p \in \mathbb{N}$, set $r_p := y^{-p}$. By Lemma 4.4, for all sufficiently large p ,

$$\mathbf{N} \left(\sup_{a \in [m, m^{-1}]} \ell^a \left(\mathcal{L}_{a, r_p, \kappa g(r_p)} \right) > 4 \log(1/r_p)^{-\beta} \right) \leq \log(1/r_p)^{-2} = \log(y)^{-2} p^{-2}, \quad (\text{II.74})$$

whose sum over p is finite. By Borel Cantelli lemma,

$$\mathbf{N}\text{-a.e.}, \text{ for all sufficiently large } p, \quad \sup_{a \in [m, m^{-1}]} \ell^a \left(\mathcal{L}_{a, r_p, \kappa g(r_p)} \right) \leq 4 \log(1/r_p)^{-\beta}. \quad (\text{II.75})$$

Moreover, $\log(1/r_p)^{-\beta} = \log(y)^{-\beta} p^{-\beta}$, and recall that $\beta > 1$. Thus, (II.75) entails that there exists a Borel subset $\mathbf{V}_y \subset \mathbf{C}^0$, such that $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}_y) = 0$, and on \mathbf{V}_y :

$$\forall a \in [m, m^{-1}], \quad \sum_{p=1}^{\infty} \ell^a \left(\mathcal{L}_{a, r_p, \kappa g(r_p)} \right) = \sum_{p=1}^{\infty} \ell^a \left(\{ \sigma : \ell^a(B(\sigma, r_p)) > \kappa g(r_p) \} \right) < \infty.$$

We can apply again the Borel-Cantelli Lemma, to the finite measures ℓ^a to get that,

$$\text{on } \mathbf{V}_y \quad \forall a \in [m, m^{-1}], \quad \ell^a(\text{d}\sigma)\text{-a.e.} \quad \exists p_0(a, \sigma), \quad \forall p \geq p_0(a, \sigma), \quad \frac{\ell^a(B(\sigma, r_p))}{g(r_p)} \leq \kappa. \quad (\text{II.76})$$

If $u \in (r_{p+1}, r_p]$, one has $\frac{\ell^a(B(\sigma, u))}{g(u)} < \frac{\ell^a(B(\sigma, r_p))}{g(r_{p+1})} \leq y \frac{\ell^a(B(\sigma, r_p))}{g(r_p)}$. Combined with (II.76), this entails that on \mathbf{V}_y , for all a in $[m, m^{-1}]$, for ℓ^a -almost every σ in $\mathcal{T}(a)$, $\limsup_{r \rightarrow 0} \ell^a(B(\sigma, r))/g(r) < y\kappa$. This can be rewritten in

$$\text{on } \mathbf{V}_y, \quad \forall a \in [m, m^{-1}], \quad \ell^a(\mathcal{T}(a) \setminus \Delta_a^{y\kappa}) = 0. \quad (\text{II.77})$$

Now set $\mathbf{V} = \bigcap \{ \mathbf{V}_y; y > 1; y \in \mathbb{Q} \}$. Clearly, $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}) = 0$ and by monotonicity, for all $\kappa' \in (\kappa, \infty)$, $\mathcal{T}(a) \setminus \Delta_a^{\kappa'} \subset \bigcup_{y > 1; y \in \mathbb{Q}} \{ \mathcal{T}(a) \setminus \Delta_a^{y\kappa} \}$. It follows easily from (II.77) that

$$\text{on } \mathbf{V}, \quad \forall a \in [m, m^{-1}], \quad \forall \kappa' \in (\kappa, \infty) \quad \ell^a(\mathcal{T}(a) \setminus \Delta_a^{\kappa'}) = 0. \quad (\text{II.78})$$

Thus, using Lemma 2.2, we get :

$$\begin{aligned} \text{on } \mathbf{V} \quad \forall \mathcal{A} \text{ Borel subset of } \mathcal{T} \quad \forall a \in [m, m^{-1}] \quad \forall \kappa' \in (\kappa, \infty) \\ \ell^a(\mathcal{A}) = \ell^a(\mathcal{A} \cap \Delta_a^{\kappa'}) \leq \kappa' \mathcal{H}_g(\mathcal{A} \cap \Delta_a^{\kappa'}) \leq \kappa' \mathcal{H}_g(\mathcal{A} \cap \mathcal{T}(a)). \end{aligned}$$

This ends the proof of Theorem 4.1 letting $\kappa' \searrow \kappa$.

4.2 Proof of Theorem 4.2.

4.2.1 Small balls.

For given level $a \in (0, \infty)$ and $r \in (0, \infty)$ we recall the notation $\mathcal{B}_{a,r}$ for the set of $\mathcal{T}(a)$ -balls of radius r . We recall from (II.41) that for $r \geq r' > 0$, a ball $\Gamma \in \mathcal{B}_{a,r'}$ is contained in a unique ball in $\mathcal{B}_{a,r}$, denoted $\Gamma[r]$. Let $\mathbf{r} = (r_1, \dots, r_n)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n-1})$, where the r_i and the ε_i are strictly decreasing. Recall from (II.42) that Γ , a $\mathcal{T}(a)$ -ball of radius r_n is $(\mathbf{r}, \boldsymbol{\varepsilon})$ -small iff

$$\forall k \in [1, n-1] \quad \ell^a(\Gamma[r_k]) \leq \varepsilon_k.$$

The total number of $(\mathbf{r}, \boldsymbol{\varepsilon})$ -small balls at level a is denoted by $S_{a,\mathbf{r},\boldsymbol{\varepsilon}}$ (see (II.43)). For $u \in (0, \infty)$, we write $u\mathbf{r} = (ur_1, \dots, ur_n)$. We recall from (II.28) the following notation : if Γ is a $\mathcal{T}(a)$ -ball, then, for all $b \geq a$, Γ^b is the subset of all the vertices in $\mathcal{T}(b)$ having an ancestor in Γ . Namely, $\Gamma^b = \{ \sigma \in \mathcal{T}(b), \exists \sigma' \in \Gamma : \sigma' \in \llbracket \rho, \sigma \rrbracket \}$.

Lemma 4.5 *Let $a, \delta \in (0, \infty)$, and $n \geq 2$. Let $\mathbf{r} = (r_1, \dots, r_n)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n-1})$, where the r_i and the ε_i are strictly decreasing. Let $c \in (1, 2)$, $\alpha \in (0, 1/2)$ and $\tilde{\alpha} \in (\alpha, 1/2)$. If $\delta < \frac{c-1}{2c}r_n$, then*

$$\mathbf{N} \left(\sup_{b \in [a, a+\delta]} S_{b, \mathbf{r}, \alpha \boldsymbol{\varepsilon}} > S_{a, c^{-1} \mathbf{r}, \tilde{\alpha} \boldsymbol{\varepsilon}} \right) \leq \frac{4n}{r_n} \exp \left(- \left(\sqrt{\tilde{\alpha}} - \sqrt{\alpha} \right)^2 \varepsilon_{n-1} / \delta \right).$$

Proof. Let us denote $B_0 = \{ \sup_{b \in [a, a+\delta]} S_{b, \mathbf{r}, \alpha \boldsymbol{\varepsilon}} > S_{a, c^{-1} \mathbf{r}, \tilde{\alpha} \boldsymbol{\varepsilon}} \}$. Next, we define the event B_1 by

$$B_1 = \{ \exists k \in \{1, \dots, n-1\}, \exists \Gamma \in \mathcal{B}_{a, r_k/c} : \ell^a(\Gamma) \geq \tilde{\alpha} \varepsilon_k \text{ and } \inf_{b \in [a, a+\delta]} \ell^b(\Gamma^b) < \alpha \varepsilon_k \}. \quad (\text{II.79})$$

We will prove that $B_0 \subset B_1$, that is to say

$$\text{on } \mathbf{C}^0 \setminus B_1, \quad \sup_{b \in [a, a+\delta]} S_{b, \mathbf{r}, \alpha \boldsymbol{\varepsilon}} \leq S_{a, c^{-1} \mathbf{r}, \tilde{\alpha} \boldsymbol{\varepsilon}}. \quad (\text{II.80})$$

Proof of (II.80). We work deterministically on $\mathbf{C}^0 \setminus B_1$. The inequality (II.80) follows from the following claim.

*For every $b \in [a, a+\delta]$, for every Γ a $\mathcal{T}(b)$ -ball of radius r_n which is $(\mathbf{r}, \alpha \boldsymbol{\varepsilon})$ -small, there exists Υ a $\mathcal{T}(a)$ -ball of radius r_n/c such that Υ is $(c^{-1} \mathbf{r}, \tilde{\alpha} \boldsymbol{\varepsilon})$ -small **and** $\Upsilon^b \subset \Gamma$.*

Assume that the latter is true. Then, to any $(\mathbf{r}, \alpha \boldsymbol{\varepsilon})$ -small ball at level b corresponds a $(c^{-1} \mathbf{r}, \tilde{\alpha} \boldsymbol{\varepsilon})$ -small ball at level a and the correspondence is injective. Summing over all $\mathcal{T}(b)$ -ball, we obtain (II.80).

Now let $b \in [a, a+\delta]$ and $\Gamma \in \mathcal{B}_{b, r_n}$ such that Γ is $(\mathbf{r}, \alpha \boldsymbol{\varepsilon})$ -small. Let $\sigma \in \Gamma$ and let σ_a its unique ancestor at level a . Namely $\sigma_a \in \mathcal{T}(a)$ and $\sigma_a \in \llbracket \rho, \sigma \rrbracket$. We denote $\Upsilon = \Gamma(\sigma_a, r_n/c) \in \mathcal{B}_{a, r_n/c}$ the $\mathcal{T}(a)$ -ball of radius r_n/c that contains σ_a . We claim that Υ is $(c^{-1} \mathbf{r}, \tilde{\alpha} \boldsymbol{\varepsilon})$ -small and that $\Upsilon^b \subset \Gamma$. To prove this, we show

$$\forall k \in \{1, \dots, n\} \quad (\Upsilon[r_k/c])^b \subset \Gamma[r_k]. \quad (\text{II.81})$$

Let $k \in \{1, \dots, n\}$ and let $\gamma \in (\Upsilon[r_k/c])^b$. Its unique ancestor at level a , denoted γ_a , is such that $\gamma_a \in \Upsilon[r_k/c]$. Two cases may occur. First, if $d(\sigma, \gamma) \leq 2(b-a)$, then we have $2(b-a) \leq 2\delta < \frac{c-1}{c}r_n < r_n \leq r_k$. The other case corresponds to $d(\sigma, \gamma) > 2(b-a)$. Then $d(\rho, \sigma \wedge \gamma) = \frac{1}{2}(2b - d(\sigma, \gamma)) < a$. Thus, $\sigma \wedge \gamma = \sigma_a \wedge \gamma_a$ and we have

$$\begin{aligned} d(\sigma, \gamma) &= 2b - 2d(\rho, \sigma \wedge \gamma) \\ &= 2a - 2d(\rho, \sigma_a \wedge \gamma_a) + 2b - 2a \\ &\leq d(\sigma_a, \gamma_a) + 2\delta \\ &< \frac{r_k}{c} + \frac{c-1}{c}r_n \leq r_k, \end{aligned}$$

where we used in the last line that $\sigma_a \in \Gamma \subset \Gamma[r_k/c]$. In both cases, $d(\sigma, \gamma) < r_k$ so $\gamma \in \Gamma(\sigma, r_k) = \Gamma[r_k]$, the last equality being a consequence of Proposition 2.1 (ii), and the definition of $\Gamma = \Gamma(\sigma, r_n)$. Thus, (II.81) is proved and it implies

$$\forall k \in \{1, \dots, n-1\} \quad \ell^b \left((\Upsilon[r_k/c])^b \right) \leq \ell^a(\Gamma[r_k]) \leq \alpha \varepsilon_k,$$

which, on $\mathbf{C}^0 \setminus B_1$, implies

$$\forall k \in \{1, \dots, n-1\} \quad \ell^a(\Upsilon[r_k/c]) \leq \tilde{\alpha} \varepsilon_k.$$

This entails that Υ is $(c^{-1}\mathbf{r}, \tilde{\alpha}\varepsilon)$ -small. The inclusion $\Upsilon^b \subset \Gamma$ was proved at line (II.81) with $k = n$ because $\Upsilon = \Upsilon[r_n/c] \subset \Gamma[r_n] = \Gamma$.

End of the proof of (II.80)

As in the proof of Lemma 4.3, we can use the fact that under \mathbf{N}_a , conditionally on \mathcal{G}_a , if Γ is a $\mathcal{T}(a)$ -ball, then the process $\{\ell^{a+a'}(\Gamma^{a+a'}), a' \geq 0\}$ is a Feller diffusion started at $\ell^a(\Gamma)$. Using sub-additivity and Lemma 3.5 (i), we get

$$\mathbf{N}(B_1) \leq \sum_{k=1}^{n-1} \frac{1}{a} \mathbf{N}_a \left[\sum_{i=1}^{Z_{a,r_k/c}} \mathbf{1}_{\{\ell^a(\Gamma_i) \geq \tilde{\alpha}\varepsilon_k\}} \exp \left(-\delta^{-1} \left(\sqrt{\ell^a(\Gamma_i)} - \sqrt{\alpha\varepsilon_k} \right)^2 \right) \right] \quad (\text{II.82})$$

$$\leq \frac{1}{a} \exp \left(-\delta^{-1} \left(\sqrt{\tilde{\alpha}\varepsilon_k} - \sqrt{\alpha\varepsilon_k} \right)^2 \right) \sum_{k=1}^{n-1} \mathbf{E} [Z_{a,r_k/c}] \quad (\text{II.83})$$

The proof is completed recalling that for all $k \in \{1 \dots n-1\}$, $\varepsilon_k \leq \varepsilon_{n-1}$, and that, by Lemma 3.1, $\mathbf{N}_a [Z_{a,r_k}] = \frac{2a}{r_k/c} \leq \frac{2ac}{r_n} \leq \frac{4a}{r_n}$. ■

Let us introduce

$$\forall j \in \mathbb{N}, \quad r_j = 2^{-j} \quad \text{and} \quad \varepsilon_j = g(r_j) \quad (\text{II.84})$$

and then

$$\forall p \in \mathbb{N}, \quad j_p = \lfloor (4/3)^p \rfloor, \quad \mathbf{r}^{(p)} = (r_j; j_p \leq j \leq j_{p+1}-1) \quad \text{and} \quad \varepsilon^{(p)} = (\varepsilon_j; j_p \leq j < j_{p+1}-1). \quad (\text{II.85})$$

Let $m \in (0, 1/2)$, we also introduce the following discrete grid

$$G'(p, m) := \left\{ m^{-1} + k\delta_p, k \in \mathbb{N}^* \right\} \cap [m, m^{-1}], \quad (\text{II.86})$$

where δ_p is the mesh of the grid, given by

$$\delta_p = r_{j_{p+1}}^{5/4}. \quad (\text{II.87})$$

Note that $G'(p, m)$ contains less than $(m\delta_p)^{-1}$ points.

Lemma 4.6 *Let $\alpha \in (0, 1/2)$, $m \in (0, 1/2)$. For $p \in \mathbb{N}$, denote $u_p := g(r_{j_{p+1}})^{-1} p^{-2}$. Then there exists $p_0 \in \mathbb{N}$ only depending on α, m such that for all $p \geq p_0$,*

$$\mathbf{N} \left(\sup_{b \in [m, m^{-1}]} S_{b, \mathbf{r}^{(p)}, \alpha\varepsilon^{(p)}} > u_p \right) \leq p^{-2}. \quad (\text{II.88})$$

Proof. Let $\tilde{\alpha} \in (\alpha, 1/2)$ and c in $(1, \infty)$ such that $2c\tilde{\alpha} \in (0, 1)$. In what follows, we denote T'_0 the left-hand-side of (II.88). Observe that $T'_0 \leq T'_1 + T'_2$, where we have set

$$T'_1 = \mathbf{N} \left(\sup_{a \in G'(p, m)} S_{a, c^{-1}\mathbf{r}^{(p)}, \tilde{\alpha}\varepsilon^{(p)}} \leq u_p ; \sup_{b \in [m, m^{-1}]} S_{b, \mathbf{r}^{(p)}, \alpha\varepsilon^{(p)}} > u_p \right),$$

$$T'_2 = \mathbf{N} \left(\sup_{a \in G'(p, m)} S_{a, c^{-1}\mathbf{r}^{(p)}, \tilde{\alpha}\varepsilon^{(p)}} > u_p \right).$$

Using sub-additivity and Lemma 4.5, we get

$$\begin{aligned} T'_1 &\leq \mathbf{N} \left(\bigcup_{a \in G'(p,m)} \left\{ \sup_{b \in [a, a+\delta_p]} S_{b, \mathbf{r}^{(p)}, \alpha \varepsilon^{(p)}} > S_{a, c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}} \right\} \right) \\ &\leq (m\delta_p)^{-1} \sup_{a \in G'(p,m)} \mathbf{N} \left(\sup_{b \in [a, a+\delta_p]} S_{b, \mathbf{r}^{(p)}, \alpha \varepsilon^{(p)}} > S_{a, c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}} \right) \\ &\leq (m\delta_p)^{-1} \frac{4(j_{p+1} - j_p)}{r(j_{p+1})} \exp \left(-(\sqrt{\tilde{\alpha}} - \sqrt{\alpha})^2 \delta_p^{-1} g(r(j_{p+1} - 2)) \right). \end{aligned}$$

One has $\delta_p^{-1} g(r(j_{p+1} - 2)) \geq \delta_p^{-1} g(r(j_{p+1})) = r(j_{p+1})^{-1/4} \log \log 1/r(j_{p+1})$, which implies that T'_1 is smaller than $(1/2)p^{-2}$, for all p sufficiently large (it is obviously not a sharp bound).

Recalling the definitions (II.44), we set

$$\mu_p = \mu(c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}) = \mathbf{N} \left(S_{r(j_p)/(2c), c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}} \right). \quad (\text{II.89})$$

We will prove that $T'_2 \leq T'_3 + T'_4$, where

$$\begin{aligned} T'_3 &= \mathbf{N} \left(\sup_{a \in G'(p,m)} |S_{a, c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}} - \mu_p \ell^a(\mathcal{T})| > u_p/2 \right), \\ T'_4 &= \mathbf{N} \left(\sup_{a \in G'(p,m)} \mu_p \ell^a(\mathcal{T}) > u_p/2 \right). \end{aligned}$$

By sub-additivity and a Markov inequality involving a moment of order 4, we get

$$\begin{aligned} T'_3 &\leq (m\delta_p)^{-1} \sup_{a \in G'(p,m)} \mathbf{N} \left(|S_{a, c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}} - \mu_p \ell^a(\mathcal{T})| > u_p/2 \right) \\ &\leq (m\delta_p)^{-1} 2^4 u_p^{-4} \sup_{a \in G'(p,m)} \mathbf{N} \left[\left(S_{a, c^{-1} \mathbf{r}^{(p)}, \tilde{\alpha} \varepsilon^{(p)}} - \mu_p \ell^a(\mathcal{T}) \right)^4 \right]. \quad (\text{II.90}) \end{aligned}$$

We want to apply Lemma 3.7 with $\mathbf{r} = c^{-1} \mathbf{r}^{(p)}$ and $\varepsilon = \tilde{\alpha} \varepsilon^{(p)}$. Thus, recalling (II.84) and (II.85), we check that for all sufficiently large p , $m/r(j_p) > 1$ and $r(j_p)/r(j_{p+1} - 1) > 2$. Recalling that $c_0 \in (0, 10^4]$ is the universal constant given by Lemma 3.7, we get from (II.90)

$$T'_3 \leq (m\delta_p)^{-1} 2^4 u_p^{-4} \sup_{a \in G'(p,m)} c_0 a \frac{r(j_p)^2}{r(j_{p+1} - 1)^4} \leq 2^4 c_0 m^{-2} \frac{r(j_p)^2}{\delta_p u_p^4 r(j_{p+1})^4}. \quad (\text{II.91})$$

Recall that $u_p = g(r(j_{p+1}))^{-1} p^{-2}$, and by (II.84) and (II.85), we get $\log \log(1/r(j_p)) \underset{p \rightarrow \infty}{\sim} p \log(4/3)$. Hence, $u_p \geq p^{-3} r(j_{p+1})^{-1}$ and (II.91) implies

$$T'_3 \leq 2^4 c_0 m^{-2} p^{12} \frac{r(j_p)^2 r(j_{p+1})^4}{r(j_{p+1})^{5/4} r(j_{p+1})^4}. \quad (\text{II.92})$$

Now, one can plainly check that $\frac{r(j_p)^2}{r(j_{p+1})^{5/4}}$ is smaller than $r(j_p)^{1/3}$. Thus, T'_3 is smaller than $(1/4)p^{-2}$ for all p sufficiently large.

For the term T'_4 , we use Lemma 3.6 to obtain

$$T'_4 \leq (2/m) \exp\left(- (m/4) u_p \mu_p^{-1}\right). \quad (\text{II.93})$$

Recalling (II.89) and Lemma 3.8, we get that for all p ,

$$\mu_p \leq \frac{5}{r(j_{p+1})} \left(\prod_{j=j_p}^{j_{p+1}-2} \mathbf{P}\left(\Lambda_{r_{j+1}/c, r_j/c}^* \leq \tilde{\alpha} r_j \log \log(1/r_j)\right) \right)^{1/2} \quad (\text{II.94})$$

We want to get an lower bound of $u_p \mu_p^{-1}$, so we compute an upper bound for $u_p^{-1} \mu_p$. Recalling that $u_p \geq p^{-3} r(j_{p+1})^{-1}$, one has

$$u_p^{-1} \mu_p \leq 5p^3 \exp\left(\frac{1}{2} \sum_{j=j_p}^{j_{p+1}-2} \log(1 - q_j)\right) \leq 5p^3 \exp\left(-\frac{1}{2} \sum_{j=j_p}^{j_{p+1}-2} q_j\right), \quad (\text{II.95})$$

where $q_j = \mathbf{P}(\Lambda_{r_{j+1}/c, r_j/c}^* > \tilde{\alpha} r_j \log \log(1/r_j))$. Recalling that $r_j = 2^{-j}$, it follows from Lemma 3.3 that

$$\begin{aligned} q_j &= \left(1 - \frac{1}{2}\right)^2 \frac{2\tilde{\alpha} r_j \log \log 1/r_j}{r_j/c} \exp\left(-\frac{2\tilde{\alpha} r_j \log \log 1/r_j}{r_j/c}\right) + \left(1 - \frac{1}{4}\right) \exp\left(-\frac{2\tilde{\alpha} r_j \log \log 1/r_j}{r_j/c}\right) \\ &\underset{j \rightarrow \infty}{\sim} \frac{\tilde{\alpha} c}{2} \log \log(1/r_j) e^{-2\tilde{\alpha} c \log \log(1/r_j)} \\ &\underset{j \rightarrow \infty}{\sim} c' \log(j) j^{-2\tilde{\alpha} c}, \end{aligned}$$

where c' is a positive constant depending on $\alpha, \tilde{\alpha}, c$. We stress that the particular choice of c was made to ensure that $\chi := 1 - 2\tilde{\alpha} c$ is strictly positive, so that the following is true for all large p :

$$\sum_{j_p}^{j_{p+1}-2} q_j \geq \sum_{j_p}^{j_{p+1}-2} j^{-2\tilde{\alpha} c} \geq \int_{j_p}^{j_{p+1}-1} x^{-2\tilde{\alpha} c} dx \underset{p \rightarrow \infty}{\sim} \chi^{-1} \left(\left(\frac{4}{3}\right)^\chi - 1\right) \left(\frac{4}{3}\right)^{p\chi}.$$

Thus, for all p sufficiently large, $\sum_{j_p}^{j_{p+1}-1} q_j \geq 2p$ which, combined with (II.95), entails that $u_p^{-1} \mu_p \leq 5p^3 \exp(-p)$. Thus, $u_p \mu_p^{-1} \geq 5^{-1} p^{-3} e^p$. Finally, we see from (II.93) that T'_3 is smaller than $(1/4)p^{-2}$ for all p sufficiently large, which ends the proof. \blacksquare

4.2.2 Proof of Theorem 4.2.

Let $\alpha \in (0, 1/2)$. For a level $a \in (0, \infty)$, we recall the definition (II.54) of Δ_a^α . To show that the g -Hausdorff measure of Δ_a^α is null, we need an efficient covering of this set. Let us recall the integer sequence $j_p = \lfloor (4/3)^p \rfloor$ and the radii $r_j = 2^{-j}$. For $p \in \mathbb{N}$, we recall the definition of the finite subsets $\mathbf{r}^{(p)} = \{r_j, j_p \leq j \leq j_{p+1} - 1\}$, and $\boldsymbol{\varepsilon}^{(p)} = \{\varepsilon_j, j_p \leq j < j_{p+1} - 1\}$ where $\varepsilon_j = g(r_j)$. Recalling the definition (II.42) for small balls, we set

$$\mathcal{C}_n := \bigcup_{p=n}^{\infty} \left\{ \Gamma \in \mathcal{B}_{a, r(j_{p+1})} : \Gamma \text{ is } (\mathbf{r}^{(p)}, \alpha \boldsymbol{\varepsilon}^{(p)})\text{-small} \right\}.$$

Observe that if $\sigma \in \Delta_a^\alpha$, then the $\mathcal{T}(a)$ -ball $\Gamma(\sigma, r(j_{p+1}))$ is $(\mathbf{r}^{(p)}, \alpha \boldsymbol{\varepsilon}^{(p)})$ -small for all large p , thus for all $n \in \mathbb{N}$, we have $\Delta_a^\alpha \subset \mathcal{C}_n$. Let us recall the definition (II.15) of Hausdorff measures, and the fact that the diameter of a $\mathcal{T}(a)$ -ball is smaller than its radius. We get

$$\forall a \in [m, m^{-1}] \quad \mathcal{H}_g^{(r(j_{n+1}))}(\Delta_a^\alpha) \leq \sum_{p=n}^{\infty} S_{a, \mathbf{r}^{(p)}, \alpha \boldsymbol{\varepsilon}^{(p)}} \cdot g(r(j_{n+1})), \quad (\text{II.96})$$

because $\Delta_a^\alpha \subset \mathcal{C}_n$. Thus,

$$\forall a \in [m, m^{-1}] \quad \mathcal{H}_g(\Delta_a^\alpha) \leq \limsup_{n \rightarrow \infty} \sum_{p=n}^{\infty} S_{a, \mathbf{r}^{(p)}, \alpha \varepsilon^{(p)}} \cdot g(r(j_{p+1})). \quad (\text{II.97})$$

Now, let $m \in (0, 1/2)$. Applying Lemma 4.6, we easily get that

$$\sum_{p=1}^{\infty} \mathbf{N} \left(\sup_{a \in [m, m^{-1}]} S_{a, \mathbf{r}^{(p)}, \alpha \varepsilon^{(p)}} > u_p \right) < \infty,$$

where we recall the notation $u_p = g(r(j_{p+1}))^{-1} p^{-2}$. By Borel-Cantelli lemma there exists a subset $\mathbf{V}' \subset \mathbf{C}^0$ such that $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}') = 0$ and such that

$$\text{on } \mathbf{V}', \quad g(r(j_{p+1})) \sup_{a \in [m, m^{-1}]} S_{a, \mathbf{r}^{(p)}, \alpha \varepsilon^{(p)}} \leq p^{-2}, \quad \text{for all suff. large } p.$$

Combined with (II.96), we deduce on \mathbf{V}' , for $a \in [m, m^{-1}]$, one has

$$\mathcal{H}_g(\Delta_a^\alpha) \leq \lim_{n \rightarrow \infty} \sum_{p=n}^{\infty} p^{-2} = 0,$$

which is the desired result.

4.3 Proof of Theorem 1.2.

Let $\kappa \in (\frac{1}{2}, \infty)$, $\alpha \in (0, \frac{1}{2})$, and $m \in (0, 1/2)$. Theorem 4.1 entails that there exists a Borel subset $\mathbf{V} = \mathbf{V}(\kappa, m) \subset \mathbf{C}^0$ such that $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}) = 0$ and

$$\text{on } \mathbf{V}(\kappa, m), \quad \text{for all Borel subset } \mathcal{A} \subset \mathcal{T}, \quad \forall a \in [m, m^{-1}], \quad \ell^a(\mathcal{A}) \leq \kappa \mathcal{H}_g(\mathcal{A} \cap \mathcal{T}(a)). \quad (\text{II.98})$$

Now, let us rewrite the definition (II.54)

$$\Delta_a^\alpha = \left\{ \sigma \in \mathcal{T}(a) : \limsup_{r \rightarrow 0} \frac{\ell^a(B(\sigma, r))}{g(r)} < \alpha \right\}. \quad (\text{II.99})$$

According to Theorem 4.2, there exists a Borel subset $\mathbf{V}' = \mathbf{V}'(\alpha, m) \subset \mathbf{C}^0$ such that $\mathbf{N}(\mathbf{C}^0 \setminus \mathbf{V}') = 0$ and

$$\text{on } \mathbf{V}'(\alpha, m) \quad \forall a \in [m, m^{-1}] \quad \mathcal{H}_g(\Delta_a^\alpha) = 0. \quad (\text{II.100})$$

Let $\alpha' < \alpha$ and notice that $\mathcal{T}(a) \setminus \Delta_a^\alpha \subset \left\{ \sigma : \limsup_{r \rightarrow 0} \frac{\ell^a(B(\sigma, r))}{g(r)} > \alpha' \right\}$. Moreover, from (II.13), we know that \mathbf{N} -a.e. for all $a \in (0, \infty)$, $\ell^a(\mathcal{T} \setminus \mathcal{T}(a)) = 0$. Thus, on \mathbf{V}' , for all Borel subset $\mathcal{A} \subset \mathcal{T}$, and for all $a \in [m, m^{-1}]$ and all $\tilde{\alpha} < \alpha$, Lemma 2.3 entails

$$\ell^a(\mathcal{A}) \geq \ell^a(\mathcal{A} \cap (\mathcal{T}(a) \setminus \Delta_a^\alpha)) \geq \alpha' \mathcal{H}_g(\mathcal{A} \cap (\mathcal{T}(a) \setminus \Delta_a^\alpha)) = \alpha' \mathcal{H}_g(\mathcal{A} \cap \mathcal{T}(a)), \quad (\text{II.101})$$

where we used (II.100) for the last equality. Letting $\alpha' \rightarrow \alpha$, we get

$$\text{on } \mathbf{V}'(\alpha, m) \quad \text{for all Borel subset } \mathcal{A} \subset \mathcal{T}, \quad \forall a \in [m, m^{-1}] \quad \ell^a(\mathcal{A}) \geq \alpha \mathcal{H}_g(\mathcal{A} \cap \mathcal{T}(a)). \quad (\text{II.102})$$

Now, let us set

$$\tilde{\mathbf{V}} := \left(\bigcap_{\substack{\kappa \in (1/2, \infty) \cap \mathbb{Q} \\ m \in (0, 1/2) \cap \mathbb{Q}}} \mathbf{V}(\kappa, m) \right) \cap \left(\bigcap_{\substack{\alpha \in (0, 1/2) \cap \mathbb{Q} \\ m \in (0, 1/2) \cap \mathbb{Q}}} \mathbf{V}'(\alpha, m) \right). \quad (\text{II.103})$$

Clearly, $\tilde{\mathbf{V}}$ is a Borel subset of \mathbf{C}^0 such that $\mathbf{N}(\mathbf{C}^0 \setminus \tilde{\mathbf{V}}) = 0$. Moreover, combining (II.98) and (II.102), we get that on $\tilde{\mathbf{V}}$, for all Borel subset $\mathcal{A} \subset \mathcal{T}$, and for all level $a \in (0, \infty)$, one has $\ell^a(\mathcal{A}) = \frac{1}{2} \mathcal{H}_g(\mathcal{A} \cap \mathcal{T}(a))$.

EXACT PACKING MEASURE OF THE RANGE OF
 ψ -SUPER-BROWNIAN MOTIONS.

This article [32] has been written in collaboration with Thomas Duquesne.

Abstract. We consider super processes whose spatial motion is the d -dimensional Brownian motion and whose branching mechanism ψ is critical or subcritical; such processes are called ψ -super Brownian motions. If $d > 2\gamma/(\gamma-1)$, where $\gamma \in (1, 2]$ is the lower index of ψ at ∞ , then the total range of the ψ -super Brownian motion has an exact packing measure whose gauge function is $g(r) = (\log \log 1/r)/\varphi^{-1}((1/r \log \log 1/r)^2)$, where $\varphi = \psi' \circ \psi^{-1}$. More precisely, we show that the occupation measure of the ψ -super Brownian motion is the g -packing measure restricted to its total range, up to a deterministic multiplicative constant only depending on d and ψ . This generalizes the main result of [36] that treats the quadratic branching case. For a wide class of ψ , the constant $2\gamma/(\gamma-1)$ is shown to be equal to the packing dimension of the total range.

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1 Introduction.

The main result of this paper provides an exact packing gauge function for the total range of super processes whose spatial motion is the d -dimensional Brownian motion and whose branching mechanism ψ is critical or subcritical. We call such super processes *ψ -super Brownian motions* (or *ψ -SBM*, for short). This generalizes the main result of [36] that concerns the Dawson-Watanabe super process corresponding to the quadratic branching mechanism $\psi(\lambda) = \lambda^2$.

Before stating precisely our main results, let us briefly recall previous works related to the fine geometric properties of super processes. Most of these results concern the Dawson-Watanabe super process $(Z_t)_{t \geq 0}$. Dawson and Hochberg [25] have proved that a.s. for all $t > 0$, the Hausdorff dimension of the topological support of Z_t is equal to $2 \wedge d$. In [28], Dawson and Perkins prove that in supercritical dimensions $d \geq 3$, the Dawson-Watanabe super process Z_t is a.s. equal to the h_1 -Hausdorff measure restricted to the topological support of Z_t , where $h_1(r) = r^2 \log \log 1/r$ (see also Perkins [100, 101] for a close result holding a.s. for all times t). By use of the Brownian snake, Le Gall and Perkins [87] prove a similar result in the critical dimension $d=2$ with the gauge function

$h_2(r) = r^2 \log 1/r \log \log 1/r$. In [88], Le Gall, Perkins and Taylor have proved that in dimension $d \geq 3$, the topological support of Z_t has no exact packing measure.

Dawson and Hochberg in [25] also proved that the Hausdorff dimension of the total range of the Dawson Watanabe super process is a.s. equal to $4 \wedge d$. In [26], Dawson Iscoe and Perkins investigate the fine geometric properties of the total occupation measure $\mathbf{M} = \int_0^\infty Z_t dt$ of the Dawson-Watanabe super process: they prove that in supercritical dimensions $d \geq 5$, \mathbf{M} is a.s. equal to the h_3 -Hausdorff measure restricted on the total range of the super process, where $h_3(r) = r^4 \log \log 1/r$. In [83], Le Gall considers the critical dimension $d=4$ and he proves a similar result with respect to the gauge function $h_4(r) = r^4 \log 1/r \log \log 1/r$. In [36], the occupation measure \mathbf{M} is also shown to coincide a.s. with the g_1 -packing measure restricted to the total range of the super process, where $g_1(r) = r^4 (\log \log 1/r)^{-3}$.

For super Brownian motions whose branching mechanism is general, less results are available: in [29], Delmas computes the Hausdorff dimension of super Brownian motions whose branching mechanism is stable; this result is eventually extended in [41] to general branching mechanism ψ thanks to geometric considerations on ψ -Lévy trees. The ψ -Lévy trees are the actual genealogical structures of the ψ -SBM; they are compact random real trees coded by the height process (introduced by Le Gall and Le Jan [86] and further studied in [40]) and they appear as the scaling limits of Galton-Watson trees; their geometric properties are discussed in [41, 42, 37, 38]. In particular, it is proved in [38] that Lévy trees have an exact packing measure, which is closely related to the main result of our article.

Let us introduce precisely the main results of our paper. We first fix a *branching mechanism* ψ that is critical or subcritical: namely, $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the Laplace exponent of a spectrally positive Lévy process that is of the following Lévy-Khintchine form:

$$\forall \lambda \in \mathbb{R}_+, \quad \psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r) \pi(dr), \quad (\text{III.1})$$

where $\alpha, \beta \in \mathbb{R}_+$, and π is the *Lévy measure* that satisfies $\int_{(0,\infty)} (r \wedge r^2) \pi(dr) < \infty$. The branching mechanism ψ is the main parameter that governs the law of the processes that are considered in this paper. We introduce two exponents that compare ψ with power functions at infinity:

$$\gamma = \sup \{c \in \mathbb{R}_+ : \lim_{\lambda \rightarrow \infty} \psi(\lambda)\lambda^{-c} = \infty\}, \quad \eta = \inf \{c \in \mathbb{R}_+ : \lim_{\lambda \rightarrow \infty} \psi(\lambda)\lambda^{-c} = 0\}. \quad (\text{III.2})$$

The *lower index* γ and the *upper index* η have been introduced by Blumenthal and Gettoor [17]: they appear in the fractal dimensions and the regularity of the processes that we consider. The statements of the paper also involve a third exponent:

$$\delta = \sup \{c \in \mathbb{R}_+ : \exists C \in (0, \infty) \text{ such that } C\psi(\mu)\mu^{-c} \leq \psi(\lambda)\lambda^{-c}, 1 \leq \mu \leq \lambda\} \quad (\text{III.3})$$

that has been introduced in [38]. It is easy to check that $1 \leq \delta \leq \gamma \leq \eta \leq 2$. If ψ is regularly varying at ∞ , all these exponents coincide. In general, they are however distinct and we mention that there exist branching mechanisms ψ of the form (III.1) such that $\delta = 1 < \gamma = \eta$ (see Lemma 2.3 and Lemma 2.4 in [38] for more details). In our paper we shall often assume that $\delta > 1$ which is a mild regularity assumption on ψ (see Comment 1.4 below).

The space \mathbb{R}^d stands for the usual d -dimensional Euclidian space. We denote by $M_f(\mathbb{R}^d)$ the space of finite Borel measures equipped with the topology of weak convergence.

For all $\mu \in M_f(\mathbb{R}^d)$ and for all Borel measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$, we use the following notation:

$$\langle f, \mu \rangle = \int_{\mathbb{R}^d} f(x) \mu(dx) \quad \text{and} \quad \langle \mu \rangle = \mu(\mathbb{R}^d).$$

Then, $\langle \mu \rangle$ is the *total mass* of μ . We shall also denote by $\text{supp}(\mu)$ the *topological support* of μ that is the smallest closed subset supporting μ .

Unless the contrary is explicitly mentioned, all the random variables that we consider are defined on the same measurable space (Ω, \mathcal{F}) . We first introduce a \mathbb{R}^d -valued continuous process $\xi = (\xi_t)_{t \geq 0}$; for all $x \in \mathbb{R}^d$, we let \mathbf{P}_x be a probability measure on (Ω, \mathcal{F}) such that ξ under \mathbf{P}_x is distributed as a standard d -dimensional Brownian motion starting from x . We also introduce $Z = (Z_t)_{t \in \mathbb{R}_+}$ that is a $M_f(\mathbb{R}^d)$ -valued càglàd process defined on (Ω, \mathcal{F}) , and for all $\mu \in M_f(\mathbb{R}^d)$, we let \mathbb{P}_μ be a probability measure on (Ω, \mathcal{F}) such that Z under \mathbb{P}_μ is distributed as *super Brownian motion with branching mechanism ψ* . Namely, under \mathbb{P}_μ , Z is a Markov process whose transitions are characterized as follows: for all bounded Borel measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and for all $s, t \in \mathbb{R}_+$,

$$\mathbb{P}_\mu\text{-a.s.} \quad \mathbb{E}_\mu[\exp(-\langle Z_{t+s}, f \rangle) \mid Z_s] = \exp(-\langle Z_s, v_t \rangle), \quad (\text{III.4})$$

where the function $(v_t(x))_{t \in \mathbb{R}_+, x \in \mathbb{R}^d}$ is the unique nonnegative solution of the integral equation

$$v_t(x) + \mathbf{E}_x \left[\int_0^t \psi(v_{t-s}(\xi_s)) ds \right] = \mathbf{E}_x [f(\xi_t)], \quad x \in \mathbb{R}^d, t \in [0, \infty).$$

Dawson-Watanabe super processes correspond, up to scaling in time and space, to the branching mechanism $\psi(\lambda) = \lambda^2$. Super diffusions with general branching mechanisms of the form (III.1) have been introduced by Dynkin [50]; for a detailed account on super processes, we refer to the books of Dynkin [51, 52], Le Gall [84], Perkins [19], Etheridge [54] and Li [89].

We easily check that, under \mathbb{P}_μ , the process $(\langle Z_t \rangle)_{t \geq 0}$ of the total mass of the ψ -SBM is a *continuous states branching process* with branching mechanism ψ . Continuous states branching processes have been introduced by Jirina [69] and Lamperti [78, 79], and further studied by Bingham [13]. The assumption $\delta > 1$, implies $\gamma > 1$, which easily entails $\int^\infty d\lambda/\psi(\lambda) < \infty$ that is called the *Grey condition*. Under this condition, standard results on continuous states branching processes (see Bingham [13]) imply that $\langle Z \rangle$ is absorbed in 0 in finite time: namely, $\mathbb{P}_\mu(\exists t \in \mathbb{R}_+ : Z_t = 0) = 1$. Thus the following definition makes sense:

$$\mathbf{M} = \int_0^\infty Z_t dt \quad (\text{III.5})$$

and \mathbf{M} is therefore a random finite Borel measure on \mathbb{R}^d : it is the *occupation* measure of the ψ -SBM Z . We also define the *total range* of Z by

$$\mathbf{R} = \bigcup_{\varepsilon > 0} \overline{\bigcup_{t \geq \varepsilon} \text{supp}(Z_t)}, \quad (\text{III.6})$$

where for any subset B in \mathbb{R}^d , \overline{B} stands for its closure. We recall here a result due to Sheu [110] that gives a condition on ψ for \mathbf{R} to be bounded:

$$\mathbb{P}_\mu\text{-a.s. } \mathbf{R} \text{ is bounded} \iff \int_1^\infty \frac{db}{\sqrt{\int_1^b \psi(a) da}} < \infty \quad \text{and } \text{supp}(\mu) \text{ is compact.} \quad (\text{III.7})$$

See also Hesse and Kyprianou [67] for a simple probabilistic proof. Note that if $\gamma > 1$, then (III.7) holds true.

We next denote by \dim_H and \dim_p respectively the *Hausdorff and the packing dimensions* on \mathbb{R}^d . We also denote by $\underline{\dim}$ and $\overline{\dim}$ the *lower and the upper box dimensions*. We refer to Falconer [58] for precise definitions. We next recall Theorem 6.3 [41] that asserts that for all $\mu \in M_f(\mathbb{R}^d)$ distinct from the null measure, the following holds true.

$$\text{If } \gamma > 1, \text{ then } \mathbb{P}_\mu\text{-a.s.} \quad \dim_H(\mathbf{R}) = d \wedge \frac{2\eta}{\eta - 1}. \quad (\text{III.8})$$

If furthermore $\text{supp}(\mu)$ is compact, then \mathbf{R} is bounded (by (III.7)) and Theorem 6.3 [41] also asserts that $\mathbb{P}_\mu\text{-a.s. } \underline{\dim}(\mathbf{R}) = d \wedge \frac{2\eta}{\eta - 1}$. As already mentioned (III.8) generalizes the work of Delmas [29] that treats SBMs whose branching mechanism is stable. Note that the assumption $\gamma > 1$ is not completely satisfactory for $\dim_H(\mathbf{R})$ only depends on d and η (see Proposition 5.7 [41] and the discussion in Section 5.3 of this article). The first result of our paper computes the packing dimension of \mathbf{R} under more restrictive assumptions.

Theorem 1.1 *Let $\mu \in M_f(\mathbb{R}^d)$ be distinct from the null measure. Let ψ be of the form (III.1). Let \mathbf{R} be the total range of the ψ -SBM with initial measure μ , as defined in (III.6). Assume that $\delta > 1$ and that $d > \frac{2\delta}{\delta - 1}$. Then,*

$$\mathbb{P}_\mu\text{-a.s.} \quad \dim_p(\mathbf{R}) = \frac{2\gamma}{\gamma - 1}. \quad (\text{III.9})$$

If furthermore $\text{supp}(\mu)$ is compact, then $\mathbb{P}_\mu\text{-a.s. } \overline{\dim}(\mathbf{R}) = \frac{2\gamma}{\gamma - 1}$.

Comment 1.1 Theorem 5.5 [41] shows that if $\gamma > 1$, the Hausdorff and the packing dimensions of the ψ -Lévy tree are resp. $\frac{\eta}{\eta - 1}$ and $\frac{\gamma}{\gamma - 1}$. Thus, for all sufficiently large d , (III.8) and (III.9) imply that the Hausdorff and the packing dimensions of \mathbf{R} are *twice* that of the ψ -Lévy tree. This can be informally explained by the fact that \mathbf{R} appears as the range of the ψ -Lévy snake that can be viewed as a Gaussian process indexed by the ψ -Lévy tree that is $(\frac{1}{2} - \varepsilon)$ -Hölder regular for any $\varepsilon \in (0, 1/2)$ (see Lemma 6.4 [41] and see also (III.35) in Section 2.4 of the present paper for more details). \square

Comment 1.2 The above mentioned Hölder-regularity of the Lévy snake studied in [41] also entails that if $\text{supp}(\mu)$ is compact and if $\gamma > 1$, then $\overline{\dim}(\mathbf{R}) \leq d \wedge \frac{2\gamma}{\gamma - 1}$. Since $\dim_p(A) \leq \overline{\dim}(A)$ for any bounded $A \subset \mathbb{R}^d$ (see e.g. [58]), an easy argument entails that for all non-null finite measure μ , if $\gamma > 1$, then $\mathbb{P}_\mu\text{-a.s. } \dim_p(\mathbf{R}) \leq d \wedge \frac{2\gamma}{\gamma - 1}$. This, combined with (III.8) entails that

$$\text{if } \eta = \gamma > 1, \text{ then } \mathbb{P}_\mu\text{-a.s.} \quad \dim_H(\mathbf{R}) = \dim_p(\mathbf{R}) = d \wedge \frac{2\gamma}{\gamma - 1}, \quad (\text{III.10})$$

with the same equality for the lower and the upper box dimensions if $\text{supp}(\mu)$ is compact. The equality $\gamma = \eta$ holds true for instance if ψ is regularly varying at ∞ . Therefore, the novelty of Theorem 1.1 only concerns the cases where $\gamma \neq \eta$. \square

Comment 1.3 In Theorem 1.1, note that Assumption $\delta > 1$ is not optimal since the value of $\dim_p(\mathbf{R})$ only depends on γ . Our arguments fail to prove (III.9) when $d \in (\frac{2\gamma}{\gamma - 1}, \frac{2\delta}{\delta - 1})$. We conjecture that *if $\gamma > 1$, then $\mathbb{P}_\mu\text{-a.s. } \dim_p(\mathbf{R}) = d \wedge \frac{2\gamma}{\gamma - 1}$.* \square

Let us set $\varphi = \psi' \circ \psi^{-1}$. The main properties of that increasing function are stated in Section 2.2. Here, we just notice that the reciprocal function of φ , that is denoted by φ^{-1} , is defined from $[\alpha, \infty)$ to $[0, \infty)$. Then, we set

$$g(r) = \frac{\log \log \frac{1}{r}}{\varphi^{-1}\left(\left(\frac{1}{r} \log \log \frac{1}{r}\right)^2\right)}, \quad r \in (0, r_0) \quad (\text{III.11})$$

where $r_0 = \min(\alpha^{-1/2}, e^{-e})$, with the convention $\alpha^{-1/2} = \infty$ if $\alpha = 0$. We check (see Section 2.2) that g is a continuous increasing function such that $\lim_{0+} g = 0$.

We next denote by \mathcal{P}_g the g -packing measure on \mathbb{R}^d , whose definition is recalled in Section 2.1. The following theorem is the main result of the paper.

Theorem 1.2 *Let $\mu \in M_f(\mathbb{R}^d)$ be distinct from the null measure. Let ψ be of the form (III.1). Let Z be a ψ -SBM starting from μ ; let \mathbf{R} be its total range, as defined by (III.6), and let \mathbf{M} be its occupation measure, as defined by (III.5). Let g be defined by (III.11). Assume that*

$$\delta > 1 \quad \text{and} \quad d > \frac{2\gamma}{\gamma - 1}.$$

Then, there exists a positive constant $\kappa_{d,\psi}$ that only depends on d and ψ such that

$$\mathbb{P}_\mu\text{-a.s.} \quad \mathbf{M} = \kappa_{d,\psi} \mathcal{P}_g(\cdot \cap \mathbf{R}). \quad (\text{III.12})$$

Comment 1.4 Unlike Theorem 1.1, we think that the assumptions of Theorem 1.2 are optimal in the following sense. Indeed, since $d \wedge \frac{2\gamma}{\gamma-1}$ is thought to be the packing dimension of \mathbf{R} , (III.12) probably does not hold true when $d \leq \frac{2\gamma}{\gamma-1}$.

Moreover, arguing as in Lemma 2.3 [38], we easily check that $\delta > 1$ if and only if g satisfies a *doubling condition* (see (III.13) Section 2.1). Although, it is possible to define packing measures with respect to irregular gauge functions (see Edgar [53]), the doubling condition is the minimal assumption on a gauge function which implies that the corresponding packing measure has nice properties (regularity, comparison lemmas). In this sense, Assumption $\delta > 1$ is required to stay within the framework of standard geometric measures. \square

Comment 1.5 In the stable cases where $\psi(\lambda) = \lambda^\gamma$ with $\gamma \in (1, 2]$, then

$$\forall r \in (0, \infty), \quad g(r) = r^{\frac{2\gamma}{\gamma-1}} (\log \log 1/r)^{-\frac{\gamma+1}{\gamma-1}}.$$

If $\gamma = 2$, then $g(r) = r^4 (\log \log 1/r)^{-3}$ and we recover the result from [36]. Moreover, note that in the stable cases, Theorem 1.1 is a straightforward consequence of Theorem 1.2. However, when $\gamma \neq \eta$, it turns out that Theorem 1.1 cannot be simply derived from Theorem 1.2: indeed, one important ingredient of the proof of Theorem 1.2 consists in computing the lower local density of \mathbf{M} with respect to g ; this lower limit is achieved along a deterministic sequence of radii whose images by g are hard to compare with a power function. \square

The paper is organised as follows. In Section 2, we recall definitions and basic results. Section 2.1 is devoted to packing measures and to two comparison lemmas that are standard technical tools used to compute packing measures. Section 2.2 gather elementary facts on the power exponents δ, γ and η associated with the branching mechanism ψ . In Section 2.3 and in Section 2.4, we recall the definitions of – and various results on – the

ψ -height process, the corresponding ψ -Lévy tree and the associated ψ -Lévy snake. In Section 3, we prove estimates on a specific subordinator (Sections 3.1 and Section 3.3) and on functionals of the snake involving the hitting time of a given ball (Section 3.2 and Section 3.4). Section 4 is devoted to the proof of the two main theorems: we prove Theorem 1.2 first and Theorem 1.1 next.

2 Notations, definitions and preliminary results.

2.1 Packing measures.

In this section, we briefly recall basic results on packing measures on the Euclidian space \mathbb{R}^d that have been introduced by Taylor and Tricot in [112].

A *gauge function* is an increasing continuous function $g : (0, r_0) \rightarrow (0, \infty)$, where $r_0 \in (0, \infty)$, such that $\lim_{0+} g = 0$ and that satisfies a *doubling condition*: namely, there exists $C \in (0, \infty)$ such that

$$\forall r \in (0, r_0/2), \quad g(2r) \leq Cg(r). \quad (\text{III.13})$$

Let $B \subset \mathbb{R}^d$ and let $\varepsilon \in (0, \infty)$. Recall that a (closed) ε -*packing* of B is a finite collection of pairwise disjoint closed ball $(\overline{B}(x_m, r_m))_{1 \leq m \leq n}$ whose centers x_m belong to B and whose radii r_m are not greater than ε . We then set

$$\mathcal{P}_g^{(\varepsilon)}(B) = \sup \left\{ \sum_{1 \leq m \leq n} g(r_m) ; \left(\overline{B}(x_m, r_m) \right)_{1 \leq m \leq n}, \varepsilon\text{-packing of } B \right\} \quad (\text{III.14})$$

and

$$\mathcal{P}_g^*(B) = \lim_{\varepsilon \rightarrow 0+} \mathcal{P}_g^{(\varepsilon)}(B) \in [0, \infty], \quad (\text{III.15})$$

that is called the g -*packing pre-measure* of B . The g -*packing measure* of B is then given by

$$\mathcal{P}_g(B) = \inf \left\{ \sum_{n \geq 0} \mathcal{P}_g^*(B_n) ; B \subset \bigcup_{n \geq 0} B_n \right\}. \quad (\text{III.16})$$

Remark 2.1 The definition (III.14) of $\mathcal{P}_g^{(\varepsilon)}$ that we adopt here is slightly different from the one introduced by Taylor and Tricot [112] who take the infimum of $\sum_{m=1}^n g(2r_m)$ over ε -packings with open balls. However, since g is increasing, continuous and satisfies a doubling condition (III.13), the resulting measure is quite close to Taylor and Tricot's definition: the difference is irrelevant to our purpose and their main results on packing measures immediately apply to the g -packing measures defined by (III.15). \square

Next recall from Lemma 5.1 [112] that \mathcal{P}_g is a Borel-regular outer measure. Moreover, it is obvious from the definition that for any subset $B \subset \mathbb{R}^d$,

$$\mathcal{P}_g(B) \leq \mathcal{P}_g^*(B). \quad (\text{III.17})$$

Next, if B is a \mathcal{P}_g -measurable set such that $\mathcal{P}_g(B) < \infty$, then for any $\varepsilon > 0$, there exists a closed subset $F_\varepsilon \subset B$ such that

$$\mathcal{P}_g(B) \leq \mathcal{P}_g(F_\varepsilon) + \varepsilon. \quad (\text{III.18})$$

We recall here Theorem 5.4 [112] that is a standard comparison result for packing measures.

Theorem 2.1 (Theorem 5.4 [112]) *Let μ be a finite Borel measure on \mathbb{R}^d . Let B be a Borel subset of \mathbb{R}^d . Let g be a gauge function satisfying a doubling condition (III.13) with a constant $C > 0$. Then, the following holds true.*

- (i) *If $\liminf_{r \rightarrow 0} \frac{\mu(B(x,r))}{g(r)} > 1$ for any $x \in B$, then $\mu(B) \geq \mathcal{P}_g(B)$.*
- (ii) *If $\liminf_{r \rightarrow 0} \frac{\mu(B(x,r))}{g(r)} < 1$ for any $x \in B$, then $\mu(B) \leq C^2 \mathcal{P}_g(B)$.*

We also recall the following specific result due to Edgar in [53], Corollary 5.10.

Lemma 2.2 (Corollary 5.10 [53]) *Let μ be a finite Borel measure on \mathbb{R}^d . Let $\kappa \in (0, \infty)$ and let B be a Borel subset of \mathbb{R}^d such that*

$$\forall x \in B, \quad \liminf_{r \rightarrow 0^+} \frac{\mu(B(x,r))}{g(r)} = \kappa.$$

Then, $\mu(B) = \kappa \mathcal{P}_g(B)$.

Remark 2.2 The main purpose of Edgar's article [53] is to deal with fractal measures in metric spaces with respect to possibly irregular gauge functions. Corollary 5.10 [53] (stated here as Lemma 2.2) holds true in this general setting if μ satisfies the so called *Strong Vitali Property* (see [53] p.43 for a definition and a discussion of this topic). A result due to Besicovitch [11] ensures that any finite measure on \mathbb{R}^d enjoys the Strong Vitali Property. Therefore, Lemma 2.2 is an immediate consequence of Edgar's Corollary 5.10 [53]. \square

We finally recall the definition of the *packing dimension*: let $\alpha \in (0, \infty)$; we simply write \mathcal{P}_α instead of \mathcal{P}_g when $g(r) = r^\alpha$, $r \in (0, \infty)$. Let $B \subset \mathbb{R}^d$. Then, the *packing dimension* of B , denoted by $\dim_p(B)$ is the unique real number in $[0, d]$ such that

$$\mathcal{P}_\alpha(B) = \infty \text{ if } \alpha < \dim_p(B) \quad \text{and} \quad \mathcal{P}_\alpha(B) = 0 \text{ if } \alpha > \dim_p(B). \quad (\text{III.19})$$

2.2 Exponents.

In this section we briefly recall from [38] several results relating power exponents associated with ψ to properties of the gauge function g introduced in (III.11). Recall that the branching mechanism ψ has the Lévy-Khintchine form (III.1). It is well-known that ψ' is the Laplace exponent of a subordinator, just like ψ^{-1} , the reciprocal of ψ . Thus, $\varphi = \psi' \circ \psi^{-1}$ is also the Laplace exponent of a subordinator. Note that $\psi'(0) = \alpha$. As already mentioned, the reciprocal function of φ , denoted by φ^{-1} , is then defined from $[\alpha, \infty)$ to $[0, \infty)$. We also introduce the function $\tilde{\psi}(\lambda) = \psi(\lambda)/\lambda$ that easily shown to be also the Laplace exponent of subordinator. Next observe that $1/\varphi$ is the derivative of ψ^{-1} and recall that ψ is convex and that ψ' , $\tilde{\psi}$, ψ^{-1} and φ are concave. In particular, this implies $\tilde{\psi}(2\lambda) \leq 2\tilde{\psi}(\lambda)$ and the following

$$\psi(2\lambda) \leq 4\psi(\lambda), \quad \tilde{\psi}(\lambda) \leq \psi'(\lambda) \leq \frac{\psi(2\lambda) - \psi(\lambda)}{\lambda} \leq 4\tilde{\psi}(\lambda) \quad \text{and} \quad \frac{\lambda}{\psi^{-1}(\lambda)} \leq \varphi(\lambda) \leq \frac{4\lambda}{\psi^{-1}(\lambda)}. \quad (\text{III.20})$$

Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function. We agree on $\sup \emptyset = 0$ and $\inf \emptyset = \infty$ and we define the following exponents that compare ϕ with power functions at infinity:

$$\gamma_\phi = \sup \{c \in \mathbb{R}_+ : \lim_{\lambda \rightarrow \infty} \phi(\lambda)\lambda^{-c} = \infty\}, \quad \eta_\phi = \inf \{c \in \mathbb{R}_+ : \lim_{\lambda \rightarrow \infty} \phi(\lambda)\lambda^{-c} = 0\}. \quad (\text{III.21})$$

Then, γ_ϕ (resp. η_ϕ) is the lower exponent (resp. the upper) of ϕ at ∞ . We also introduce the following exponent

$$\delta_\phi = \sup \{c \in \mathbb{R}_+ : \exists C \in (0, \infty) \text{ such that } C\phi(\mu)\mu^{-c} \leq \phi(\lambda)\lambda^{-c}, 1 \leq \mu \leq \lambda\} \quad (\text{III.22})$$

that plays an important role for the regularity of the gauge function. Thus by (III.2) and (III.3)

$$\gamma = \gamma_\psi, \quad \eta = \eta_\psi, \quad \delta = \delta_\psi.$$

It is easy to check that $1 \leq \delta \leq \gamma \leq \eta \leq 2$. If ψ is regularly varying at ∞ , all these exponents coincide. In general, they are however distinct and we mention that there exist branching mechanisms ψ of the form (III.1) such that $\delta = 1 < \gamma = \eta$: see Lemma 2.3 and Lemma 2.4 in [38] for more detail. As a direct consequence of (III.20) we have $\delta_{\tilde{\psi}} = \delta_{\psi'} = \delta - 1$, $\gamma_{\tilde{\psi}} = \gamma_{\psi'} = \gamma - 1$ and $\eta_{\tilde{\psi}} = \eta_{\psi'} = \eta - 1$. Moreover, we get $\delta_\varphi = (\delta - 1)/\delta$, $\gamma_\varphi = (\gamma - 1)/\gamma$ and $\eta_\varphi = (\eta - 1)/\eta$.

Recall from (III.11) the definition of the function g . The arguments of the proof of Lemma 2.3 (i) in [38] can be immediately adapted to prove that $g : (0, r_0) \rightarrow (0, \infty)$ is an increasing continuous function such that $\lim_{0+} g = 0$ and such that it satisfies the following.

(i) The function g satisfies the doubling condition (III.13) iff $\delta > 1$.

(ii) If ψ is regularly varying at ∞ with exponent $c > 1$, then $\delta = \gamma = \eta = c$ and g is regularly varying at ∞ with exponent $c/(c - 1)$.

We shall further need the following bound that is a consequence of (III.20).

Lemma 2.3 *Let g the gauge function defined by (III.11). Let $c \in (0, \infty)$. Then there exists $r(c) \in (0, r_0)$ that only depends on c such that*

$$\forall r \in (0, r(c)), \quad g(r) \psi'^{-1}(c/r^2) \leq 4r^2.$$

Proof. Take $r(c) \in (0, r_0)$ such that $\log \log 1/r(c) \geq 1 \vee \sqrt{c}$. Thus,

$$r \in (0, r(c)), \quad \psi'^{-1}(cr^{-2}) \leq \psi'^{-1}((r^{-1} \log \log r^{-1})^2).$$

Recall that $\varphi^{-1} = \psi \circ \psi'^{-1}$. By comparing $\tilde{\psi}$ and ψ' thanks to (III.20), we get for all $r \in (0, r(c))$

$$\begin{aligned} g(r) \psi'^{-1}(cr^{-2}) &= \frac{\psi'^{-1}(cr^{-2}) \log \log \frac{1}{r}}{\psi(\psi'^{-1}((\frac{1}{r} \log \log \frac{1}{r})^2))} \leq \frac{\psi'^{-1}((\frac{1}{r} \log \log \frac{1}{r})^2) \log \log \frac{1}{r}}{\psi(\psi'^{-1}((\frac{1}{r} \log \log \frac{1}{r})^2))} \\ &\leq \frac{\log \log \frac{1}{r}}{\tilde{\psi}(\psi'^{-1}((\frac{1}{r} \log \log \frac{1}{r})^2))} \leq \frac{4r^2}{\log \log \frac{1}{r}} \leq 4r^2, \end{aligned}$$

which is the desired result. ■

2.3 Height process and Lévy trees.

In this section we recall the definition of the height process that encodes Lévy trees. The Lévy trees are the scaling limit of Galton-Watson trees and they are the genealogy of super-Brownian motion.

The height process. Recall that ψ stands for a branching mechanism of the form (III.1). We always assume that $\gamma > 1$. It is convenient to work on the canonical space $\mathbb{D}([0, \infty), \mathbb{R})$ of càdlàg paths equipped with Skorohod topology and the corresponding Borel sigma-field. We denote by $X = (X_t)_{t \geq 0}$ the canonical process and by \mathbb{P} the distribution of the spectrally positive Lévy processes starting from 0 whose Laplace exponent is ψ . Namely,

$$\forall t, \lambda \in [0, \infty), \quad \mathbb{E}[\exp(-\lambda X_t)] = \exp(t\psi(\lambda)).$$

Note that the specific form (III.1) of ψ implies that X_t is integrable and that $\mathbb{E}[X_t] = -\alpha t$. This easily entails that X does not drift to ∞ . Conversely, if a spectrally positive Lévy process does not drift to ∞ , then its Laplace exponent is necessarily of the form (III.1). We shall assume that $\gamma > 1$ which easily implies that either $\beta > 0$ (and $\gamma = 2$) or $\int_{(0,1)} r\pi(dr) = \infty$. It entails that \mathbb{P} -a.s. X has unbounded variation paths: see Bertoin [9], Chapter VII, Corollary 5 (iii).

Note that $\gamma > 1$ entails $\int_0^\infty d\lambda/\psi(\lambda) < \infty$ and, as shown by Le Gall and Le Jan [86] (see also [40], Chapter 1), there exists a *continuous process* $H = (H_t)_{t \geq 0}$ such that for any $t \in [0, \infty)$, the following limit holds true in \mathbb{P} -probability:

$$H_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{I_t^s < X_s < I_t^s + \varepsilon\}} ds, \quad (\text{III.23})$$

where I_t^s stands for $\inf_{s \leq r \leq t} X_r$. The process $H = (H_t)_{t \geq 0}$ is called the ψ -height process; it turns out to encode the genealogy of super-Brownian motion with branching mechanism ψ as explained below. We shall need the following result that is proved in [40].

Lemma 2.4 ([40] Theorem 1.4.4) *Assume that $\gamma > 1$. Then for every $c \in (0, \frac{\gamma-1}{\gamma})$, H is \mathbb{P} -a.s. locally c -Hölder continuous.*

Excursions of the height process. We denote by I the infimum process of X :

$$\forall t \in \mathbb{R}_+, \quad I_t = \inf_{0 \leq r \leq t} X_r.$$

When ψ is of the form $\psi(\lambda) = \beta\lambda^2$, X is distributed as a Brownian motion and (III.23) easily implies that H is proportional to $X - I$, which is distributed as a reflected Brownian motion. In the general cases, H is neither a Markov process nor a martingale. However it is possible to develop an excursion theory for H as follows.

Since X has unbounded variation sample paths, basic results of fluctuation theory (see Bertoin [9], Sections VI.1 and VII.1) entail that $X - I$ is a strong Markov process in $[0, \infty)$ and that 0 is regular for $(0, \infty)$ and recurrent with respect to this Markov process. Moreover, $-I$ is a local time at 0 for $X - I$ (see Bertoin [9], Theorem VII.1). We denote by N the corresponding excursion measure of $X - I$ above 0. More precisely, we denote by (a_j, b_j) , $j \in \mathcal{I}$, the excursion intervals of $X - I$ above 0 and we define the corresponding excursions by $X^j = X_{(a_j+.) \wedge b_j} - I_{a_j}$, $j \in \mathcal{I}$. Then, $\sum_{j \in \mathcal{I}} \delta_{(-I_{a_j}, X^j)}$ is a Poisson point measure on $[0, \infty) \times \mathbb{D}([0, \infty), \mathbb{R})$ with intensity $dx N(dx)$.

Next observe that under \mathbb{P} , the value of H_t only depends on the excursion of $X - I$ straddling time t and we easily check that

$$\bigcup_{j \in \mathcal{I}} (a_j, b_j) = \{t \geq 0 : H_t > 0\}.$$

This allows to define the height process under N as a certain measurable function $H(X)$ of X . We denote by $C(\mathbb{R}_+, \mathbb{R})$ the space of the continuous functions from $[0, \infty)$ to

\mathbb{R} equipped with the topology of the uniform convergence on every compact subsets of $[0, \infty)$; by convenience, we shall slightly abuse notation by denoting by $H = (H_t)_{t \geq 0}$ the *canonical process on $C(\mathbb{R}_+, \mathbb{R})$* and by denoting by $N(dH)$ the "distribution" of the height process $H(X)$ associated with X under the excursion measure $N(dX)$. Then we derive from the previous results the following Poisson decomposition of the height process $H(X)$ associated with X under \mathbb{P} : for any $j \in \mathcal{I}$, set $H^j = H_{(a_j + \cdot) \wedge b_j}$; then, under \mathbb{P} , the point measure

$$\sum_{j \in \mathcal{I}} \delta_{(-I_{a_j}, H^j)} \tag{III.24}$$

is distributed as a Poisson point measure on $[0, \infty) \times C(\mathbb{R}_+, \mathbb{R})$ with intensity $dx N(dH)$. For more details, we refer to [40], Chapter 1.

We denote by σ the *duration* of X under its excursion measure N (with an obvious definition). It is easy to check that H and X under N have the same duration and that the following holds true.

$$N\text{-a.e. } \sigma < \infty, \quad H_0 = H_\sigma = 0 \quad \text{and} \quad H_t > 0 \iff t \in (0, \sigma) .$$

Basic results of fluctuation theory (see Bertoin [9], Chapter VII) also entail:

$$\forall \lambda \in (0, \infty) \quad N[1 - e^{-\lambda\sigma}] = \psi^{-1}(\lambda) . \tag{III.25}$$

Local times of the height process. We recall from [40], Chapter 1, Section 1.3, the following result: there exists a jointly measurable process $(L_s^a)_{a, s \in [0, \infty)}$ such that \mathbb{P} -a.s. for any $a \in [0, \infty)$, $s \mapsto L_s^a$ is continuous, non-decreasing and such that

$$\forall t, a \geq 0, \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \frac{1}{\varepsilon} \int_0^s dr \mathbf{1}_{\{a < H_r \leq a + \varepsilon\}} - L_s^a \right| \right] = 0 . \tag{III.26}$$

The process $(L_s^a)_{s \in [0, \infty)}$ is called the *a-local time of H*. Recall that I stands for the infimum process of X . One can show (see [41], Lemma 1.3.2) that for fixed t , $L_t^0 = -I_t$. Moreover, one can observe that the support of the random Stieltjes measure dL^a is contained in the closed set $\{t \geq 0 : H_t = a\}$.

A general version of the *Ray-Knight theorem for H* asserts the following. For any $x \geq 0$, set $T_x = \inf\{t \geq 0 : X_t = -x\}$; then, *the process $(L_{T_x}^a ; a \geq 0)$ is distributed as a continuous-states branching process CSBP with branching mechanism ψ and initial state x* (see Le Gall and Le Jan [86], Theorem 4.2, and also [40], Theorem 1.4.1).

It is possible to define the local times of H under the excursion measure N as follows. For any $b > 0$, let us set $v(b) = N(\sup_{t \in [0, \sigma]} H_t > b)$. Since H is continuous, the Poisson decomposition (III.24) implies that $v(b) < \infty$, for any $b > 0$. It is moreover clear that v is non-increasing and that $\lim_{\infty} v = 0$. Then, for every $a \in (0, \infty)$, we define a continuous increasing process $(L_t^a)_{t \in [0, \infty)}$, such that for every $b \in (0, \infty)$ and for any $t \in [0, \infty)$, one has

$$\lim_{\varepsilon \rightarrow 0} N \left[\mathbf{1}_{\{\sup H > b\}} \sup_{0 \leq s \leq t \wedge \sigma} \left| \frac{1}{\varepsilon} \int_0^s dr \mathbf{1}_{\{a - \varepsilon < H_r \leq a\}} - L_s^a \right| \right] = 0 . \tag{III.27}$$

We refer to [40] Section 1.3 for more details. Note that N -a.e. $L_t^a = L_\sigma^a$ for all $t \geq \sigma$. The process $(L_t^a)_{t \in [0, \infty)}$ is the *a-local time of the excursion of the height process*.

Lévy trees. We briefly explain how the height process H under its excursion measure N can be viewed as the contour process of a tree called the Lévy tree. Recall that σ is the duration of H under N . For any $s, t \in [0, \sigma]$, we set

$$d(s, t) = H_t + H_s - 2 \inf_{u \in [s \wedge t, s \vee t]} H_u . \quad (\text{III.28})$$

The quantity $d(s, t)$ represents the distance between the points corresponding to s and t in the Lévy tree. Indeed d is obviously symmetric in s and t and we easily check that d satisfies the triangle inequality. Two real numbers $s, t \in [0, \sigma]$ correspond to the same point in the Lévy tree iff $d(s, t) = 0$, which is denoted by $s \sim t$. Observe that \sim is an equivalence relation. The Lévy tree is then given by the quotient set

$$\mathcal{T} = [0, \sigma] / \sim .$$

Then, d induces a true metric on \mathcal{T} that we keep denoting d . Denote by $p : [0, \sigma] \rightarrow \mathcal{T}$ the *canonical projection*. Since H is continuous N -a.e., so is p , which implies that (\mathcal{T}, d) is a random compact connected metric space. More specifically, (\mathcal{T}, d) is N -a.e a compact *real-tree*, namely a compact metric space such that any two points are connected by a unique self-avoiding path, that turns out to be geodesic: see [41] for more details on Lévy trees viewed as real-trees.

The *mass measure* of the Lévy tree \mathcal{T} , denoted by \mathbf{m} , is the pushforward measure of the Lebesgue measure ℓ on $[0, \sigma]$ via the canonical projection p . Namely, N -a.e. for all Borel measurable function $f : \mathcal{T} \rightarrow \mathbb{R}_+$,

$$\langle \mathbf{m}, f \rangle = \int_0^\sigma \ell(dt) f(p(t)) .$$

One can show that N -a.e. the mass measure is diffuse; obviously its topological support is \mathcal{T} and $\mathbf{m}(\mathcal{T}) = \sigma$. We refer to [41] for more details.

Let $r \in (0, \infty)$ and let $t \in [0, \sigma]$. Let $B(p(t), r)$ denote the open ball in (\mathcal{T}, d) with center $p(t)$ and radius r . Then the mass measure of $B(p(t), r)$ in (\mathcal{T}, d) is then given by

$$\mathbf{a}(t, r) := \mathbf{m}(B(p(t), r)) = \int_0^\sigma \ell(ds) \mathbf{1}_{\{d(s, t) \leq r\}} . \quad (\text{III.29})$$

We shall need the following result on the lower density of \mathbf{m} at typical points that is proved in [38], Theorem 1.2. To that end, we set

$$\forall r \in (0, \alpha \wedge e^{-e}), \quad k(r) := \frac{\log \log \frac{1}{r}}{\varphi^{-1}(\frac{1}{r} \log \log \frac{1}{r})} . \quad (\text{III.30})$$

Lemma 2.5 ([38] Theorem 1.2) *Let ψ be a branching mechanism of the form (III.1). Let σ be the duration of the height process H under its excursion measure N . Let k be as in (III.30). Assume that $\delta > 1$. Then, there exists a constant $c_1 \in (0, \infty)$ that only depends on ψ such that*

$$N\text{-a.e. for } \ell\text{-almost all } t \in [0, \sigma] \quad \liminf_{r \rightarrow 0} \frac{\mathbf{a}(t, r)}{k(r)} \geq c_1 ,$$

where $\mathbf{a}(t, r)$ is defined by (III.29) for all $r \in (0, \infty)$ and for all $t \in [0, \sigma]$.

The exploration process. As already mentioned, the height process is not a Markov process. To explore in a Markovian way the Lévy tree, Le Gall and Le Jan in [86] introduce a measure valued process $\rho = (\rho_t)_{t \geq 0}$ that is called *the exploration process* whose definition is the following. Denote by $M_f(\mathbb{R}_+)$ the set of finite measures on $[0, \infty)$ equipped with the *total variation distance*. Recall that X under \mathbb{P} is a spectrally positive Lévy process starting from 0 whose Laplace exponent is ψ that satisfies $\gamma > 1$. We denote by \mathcal{F}_t^X the sigma-field generated by $X_{\cdot \wedge t}$ augmented with the \mathbb{P} -negligible events. Recall that for all $s, t \in [0, \infty)$ such that $s \leq t$, I_t^s stands for $\inf_{u \in [s, t]} X_u$. Then, for all $t \in [0, \infty)$, the following definition makes sense under \mathbb{P} or N :

$$\rho_t(dr) = \beta \mathbf{1}_{[0, H_t]}(r) dr + \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(dr). \quad (\text{III.31})$$

Note that the $M_f(\mathbb{R}_+)$ -valued process ρ is $(\mathcal{F}_t^X)_{t \geq 0}$ -adapted. The height process H can be deduced from ρ as follows: for any $\mu \in M_f(\mathbb{R}_+)$, we denote by $\text{supp}(\mu)$ its topological support and we define

$$H(\mu) = \sup(\text{supp}(\mu)),$$

that is possibly infinite. We can show that

$$\mathbb{P}\text{-a.s. (or } N\text{-a.e.) } \forall t \in [0, \infty), \quad \text{supp}(\rho_t) = [0, H_t].$$

As proved in [40], the exploration process ρ admits a càdlàg modification under \mathbb{P} and N . By Proposition 1.2.3 [40], under N , ρ is a càdlàg strong Markov process with respect to $(\mathcal{F}_{t+}^X)_{t \geq 0}$.

2.4 The Lévy Snake.

The ψ -Lévy snake is a generalization of Le Gall's Brownian snake that greatly facilitates the study of super processes: it provides a Markovian parametrisation of the genealogy and the spatial positions of the underlying continuous population that gives rise to the super process. We recall from [40], Chapter 4, the following definition of the ψ -Lévy snake. To that end, recall that $\xi = (\xi_t)_{t \geq 0}$ is a \mathbb{R}^d -valued continuous process defined on (Ω, \mathcal{F}) and that for any $x \in \mathbb{R}^d$, \mathbf{P}_x is a probability measure on (Ω, \mathcal{F}) such that under \mathbf{P}_x , ξ is distributed as a standard d -dimensional Brownian motion starting from x .

Snake with a deterministic Hölder-regular lifetime process. We denote \mathcal{W} the set of *continuous stopped paths*, namely the set of pairs (w, ζ_w) where $\zeta_w \in [0, \infty)$ and $w : [0, \zeta] \rightarrow \mathbb{R}^d$ is a *continuous* function. Here ζ_w is the *lifetime* of w . We shall slightly abuse notation by simply denoting w instead of (w, ζ) in the sequel. The set \mathcal{W} is equipped with the metric \mathbf{d} defined for $w, w' \in \mathcal{W}$ by :

$$\mathbf{d}(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{t \geq 0} \|w(t \wedge \zeta_w) - w'(t \wedge \zeta_{w'})\|.$$

Here $\|\cdot\|$ stands for the Euclidian norm on \mathbb{R}^d . It can be shown that $(\mathcal{W}, \mathbf{d})$ is a Polish space.

To define the finite dimensional marginal distributions of the snake, we first need to introduce its transition kernels. Let $w \in \mathcal{W}$, let $a \in [0, \zeta_w]$ and let $b \in [a, \infty)$. We plainly define a law $R_{a,b}(w, dw')$ on \mathcal{W} by requiring the following.

- (i) $R_{a,b}(w, dw')$ -a.s. $w'(t) = w(t), \forall t \in [0, a]$.

(ii) $R_{a,b}(w, dw')$ -a.s. $\zeta_{w'} = b$.

(iii) The law of $(w'(a+t))_{0 \leq t \leq b-a}$ under $R_{a,b}(w, dw')$ is the law of $(\xi_t)_{0 \leq t \leq b-a}$ under $\mathbf{P}_{w(a)}$. In particular, $R_{0,b}(w, dw')$ is the law of $(\xi_t)_{0 \leq t \leq b}$ under $\mathbf{P}_{w(0)}$.

We denote by $(W_s)_{s \geq 0}$ the canonical process on the space $\mathcal{W}^{\mathbb{R}^+}$ of the \mathcal{W} -valued functions on \mathbb{R}_+ equipped with the product sigma-field. We next fix $x \in \mathbb{R}^d$. We slightly abuse notation by denoting x the stopped path with null lifetime starting from x (and therefore ending at x). Let $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $h(0) = 0$. We call h the *lifetime process*. For all $s, s' \in \mathbb{R}_+$ such that $s' \geq s$, we use the notation $b_h(s, s') = \inf_{s \leq r \leq s'} h(r)$. From the definition of the laws $R_{a,b}$ and Kolmogorov extension theorem there is a unique probability measure Q_x^h on $\mathcal{W}^{\mathbb{R}^+}$ such that for all $0 = s_0 < s_1 < \dots < s_n$,

$$\begin{aligned} & Q_x^h[W_{s_0} \in A_0, \dots, W_{s_n} \in A_n] \\ &= \mathbf{1}_{A_0}(x) \int_{A_1 \times \dots \times A_n} R_{b_h(s_0, s_1), h(s_1)}(w_0, dw_1) \dots R_{b_h(s_{n-1}, s_n), h(s_n)}(w_{n-1}, dw_n). \end{aligned} \quad (\text{III.32})$$

Note that $(h, x) \mapsto Q_x^h$ is measurable and for all $t \in \mathbb{R}_+$, Q_x^h -a.s. $\zeta_{W_t} = h(t)$.

We next discuss the regularity of the process W under Q_x^h . To that end, we assume the following.

The lifetime process h is locally Hölder continuous with exponent $r \in (0, 1]$.

Fix $p \in (1, \infty)$ and $t_0 \in (0, \infty)$. The last inequality of the proof of Proposition 4.4.1[40] p. 120, entails that there exists a constant C that only depends on t_0 , on p and on the Hölder constant of h on $[0, t_0]$, such that

$$\forall s, t \in [0, t_0], \quad Q_x^h[\mathbf{d}(W_s, W_t)^p] \leq C |t-s|^{pr/2}. \quad (\text{III.33})$$

If $p > 2/r$, then the Kolmogorov continuity criterion applies and asserts that there exists a continuous modification of the process W . We slightly abuse notation by keeping notation Q_x^h for law on $C(\mathbb{R}_+, \mathcal{W})$ of such a modification; likewise, we also keep denoting by $(W_t)_{t \geq 0}$ the canonical process on $C(\mathbb{R}_+, \mathcal{W})$. We then call Q_x^h the law of the *snake with lifetime process h starting from x* . Working on $C(\mathbb{R}_+, \mathcal{W})$, we see that Q_x^h -a.s. for all $t \in \mathbb{R}_+$, $\zeta_{W_t} = h(t)$. We then set

$$\widehat{W}_t = W_t(h(t)). \quad (\text{III.34})$$

The process \widehat{W} is called the snake's *endpoint process* that is Q_x^h -a.s. continuous. From (III.32), we easily get that under Q_x^h , the endpoint process is Gaussian whose covariance is characterized by

$$\forall s, t \in \mathbb{R}_+, \quad Q_x^h(\|\widehat{W}_t - \widehat{W}_s\|^2) = h(t) + h(s) - 2 \inf_{s \wedge t \leq u \leq s \vee t} h(u). \quad (\text{III.35})$$

Moreover since (III.33) holds for any $p > 1$, the Kolmogorov criterion implies that for any $q \in (0, r/2)$, \widehat{W} is Q_x^h -a.s. locally q -Hölder continuous.

The definition of Lévy snake. The Lévy snake is the snake whose lifetime process is the height process H introduced in Section 2.3. Recall that we assume that $\gamma > 1$ and recall from Lemma 2.4 that H is \mathbb{P} -a.s. (or N -a.e.) Hölder regular and that the previous construction of the snake applies. We then define the excursion measure of the ψ -Brownian snake starting from $x \in \mathbb{R}^d$ by

$$\mathbf{N}_x = \int_{C(\mathbb{R}_+, \mathbb{R})} N(H \in dh) Q_x^h. \quad (\text{III.36})$$

Then H is the lifetime process of W . Namely \mathbf{N}_x -a.e. for all $t \in [0, \sigma]$, $\zeta_{W_t} = H_t$ and thus, $\widehat{W}_t = W_t(H_t)$. Moreover, under \mathbf{N}_x , the conditional law of W given H is Q_x^H : we refer to [40], Chapter 4, for more details. Lemma 2.4 and the results discussed right after (III.35) entail the following lemma.

Lemma 2.6 *Assume that $\gamma > 1$. Then, for any $q \in (0, \frac{\gamma-1}{2\gamma})$, \widehat{W} is \mathbf{N}_x -a.e. locally q -Hölder.*

The range of the endpoint process \widehat{W} is a connected compact subset of \mathbb{R}^d and we use the following notation

$$\mathcal{R}_W = \{\widehat{W}_t; t \in [0, \sigma]\}. \quad (\text{III.37})$$

Recall that for any $a \in (0, \infty)$, $(L_s^a)_{s \geq 0}$ stands for the local time of H at level a . We then denote by $\mathcal{Z}_a(W)$ the random measure on \mathbb{R}^d defined by

$$\langle \mathcal{Z}_a(W), f \rangle = \int_0^\sigma dL_s^a f(\widehat{W}_s),$$

for any Borel measurable $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$. We also set $\mathcal{Z}_0(W) = 0$, the null measure. Recall that $M_f(\mathbb{R}^d)$ stands for the space of finite Borel measures on \mathbb{R}^d equipped with the topology of weak convergence. We can prove that under \mathbf{N}_x , the $M_f(\mathbb{R}^d)$ -valued process $(\mathcal{Z}_a)_{a \geq 0}$ has a càdlàg modification that is denoted in the same way by convenience. The *occupation measure of the snake* \mathcal{M}_W is then defined by

$$\langle \mathcal{M}_W, f \rangle = \int_0^\infty \langle \mathcal{Z}_a(W), f \rangle da = \int_0^\sigma f(\widehat{W}_s) ds, \quad (\text{III.38})$$

for any Borel measurable $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$.

We then recall Theorem 4.2.1 [40] that connects the ψ -Lévy snake to super Brownian motions.

Theorem 2.7 ([40] Theorem 4.2.1) *We keep notation from above. Assume that $\gamma > 1$. Let $\mu \in M_f(\mathbb{R}^d)$. Let*

$$\sum_{i \in \mathcal{J}} \delta_{(x_i, W^i)}$$

be a Poisson point measure on $\mathbb{R}^d \times \mathcal{W}$ with intensity $\mu(dx) \mathbf{N}_x(dW)$. For every $a \in (0, \infty)$ set

$$Z_a = \sum_{i \in \mathcal{J}} \mathcal{Z}_a(W^i).$$

and also set $Z_0 = \mu$. Then, the process $(Z_a)_{a \geq 0}$ is a ψ -super Brownian motion starting from μ (as defined in the introduction section). Moreover, if \mathbf{R} and \mathbf{M} are defined in terms of Z by (III.6) and (III.5), then,

$$\mathbf{R} \cup \{x_j; j \in \mathcal{J}\} = \bigcup_{j \in \mathcal{J}} \mathcal{R}_{W^j} \quad \text{and} \quad \mathbf{M} = \sum_{j \in \mathcal{J}} \mathcal{M}_{W^j}. \quad (\text{III.39})$$

The last point (III.39) is not part of Theorem 4.2.1 [40] but it is an easy consequence of that result. To simplify notation, when there is no risk of confusion we shall simply write

$$\mathcal{Z}_a := \mathcal{Z}_a(W), \quad \mathcal{R} := \mathcal{R}_W \quad \text{and} \quad \mathcal{M} := \mathcal{M}_W.$$

Consequences of Markov property. As the height process, the ψ -Lévy snake $(W_t)_{t \geq 0}$ defined above is not a Markov process. However $\overline{W} := (\rho_t, W_t)_{t \geq 0}$ is a strong Markov process under \mathbf{N}_x : see Theorem 4.1.2 [40] for more details. Instead of fully discussing the Markov property of \overline{W} , we only state here the various results we need, that are consequences of the strong Markov property.

Denote by $(\mathcal{F}_t^W)_{t \geq 0}$ the filtration generated by $(\overline{W}_t)_{t \geq 0}$. As a consequence of the strong Markov property for \overline{W} (see [40], Theorem 4.1.2) and a specific decomposition of the snake into excursions above the infimum of its lifetime proved in Lemma 4.2.4 [40], we get the following result that is used in Section 3.4.

Lemma 2.8 ([40] Theorem 4.1.2 and Lemma 4.2.4) *Let T be a $(\mathcal{F}_{t+}^W)_{t \geq 0}$ -stopping time. Let Y be a nonnegative \mathcal{F}_{T+}^W -measurable random variable. Let $G : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be Borel measurable. Then,*

$$\begin{aligned} \mathbf{N}_0 \left[\mathbf{1}_{\{0 < T < \infty\}} Y \exp \left(- \int_T^\sigma G(\widehat{W}_s) ds \right) \right] \\ = \mathbf{N}_0 \left[\mathbf{1}_{\{0 < T < \infty\}} Y \exp \left(- \int_{[0, H_T]} \rho_T(dh) \mathbf{N}_{W_T(h)} (1 - e^{-\int_0^\sigma G(\widehat{W}_s) ds}) \right) \right]. \end{aligned}$$

We shall apply the strong Markov property of \overline{W} at specific hitting times of the snake. More precisely, let us introduce several notations. Let $r \in (0, \infty)$. We define the first hitting time of W of the closed ball $\overline{B}(0, r)$ in \mathbb{R}^d by

$$\tau_r := \inf \{ t \in \mathbb{R}_+ : \widehat{W}_t \in \overline{B}(0, r) \}, \quad (\text{III.40})$$

with the convention $\inf \emptyset = \infty$. We also introduce the following function

$$\forall x \in \overline{B}(0, r)^c, \quad u_r(x) := \mathbf{N}_x(\tau_r < \infty). \quad (\text{III.41})$$

Since $t \mapsto \widehat{W}_t$ is continuous, we also get

$$\forall x \in \overline{B}(0, r)^c, \quad u_r(x) = \mathbf{N}_x(\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset). \quad (\text{III.42})$$

From [40] p. 121 and p. 131, we know that $u_r(x) \in (0, \infty)$, for all $r \in (0, \infty)$ and all $x \in \overline{B}(0, r)^c$, and that u_r is moreover radial. We then denote by \tilde{u}_r the function from (r, ∞) to $(0, \infty)$ such that

$$\forall x \in \overline{B}(0, r)^c, \quad \tilde{u}_r(\|x\|) = u_r(x).$$

Let $r' \in (0, \infty)$. For all stopped path $w \in \mathcal{W}$, we next set

$$T_{r'}(w) = \inf \{ s \in [0, \zeta_w] : w(s) \in \overline{B}(0, r') \}, \quad (\text{III.43})$$

with the convention $\inf \emptyset = \infty$. We then define a function $\varpi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ by

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}_+, \quad \varpi(\lambda_1, \lambda_2) = \begin{cases} (\psi(\lambda_1) - \psi(\lambda_2)) / (\lambda_1 - \lambda_2) & \text{if } \lambda_1 \neq \lambda_2, \\ \psi'(\lambda_1) & \text{if } \lambda_1 = \lambda_2. \end{cases} \quad (\text{III.44})$$

Recall that $\xi = (\xi_t)_{t \geq 0}$ is a \mathbb{R}^d -valued continuous process defined on (Ω, \mathcal{F}) and that for any $x \in \mathbb{R}^d$, \mathbf{P}_x is a probability measure on (Ω, \mathcal{F}) such that under \mathbf{P}_x , ξ is distributed as a standard d -dimensional Brownian motion starting from x .

The following proposition is a specific application of Theorem 4.6.2 [40] that we use in the proof of Lemma 3.15 in Section 3.4.

Proposition 2.9 ([40] **Theorem 4.6.2**) *Let $x \in \mathbb{R}^d$. Let $r, r' \in (0, \infty)$ be such that $r' > r$ and $x \in \overline{B}(0, r')^c$. We keep the previous notation. Let $F, G : \mathcal{W} \rightarrow \mathbb{R}_+$ be Borel-measurable. Then,*

$$\begin{aligned} & \mathbf{N}_x \left[\mathbf{1}_{\{\tau_r < \infty\}} F((W_{\tau_r}(s))_{0 \leq s \leq T_{r'}(W_{\tau_r})}) \exp \left(- \int_{[0, T_{r'}(W_{\tau_r})]} \rho_{\tau_r}(dh) G((W_{\tau_r}(s))_{0 \leq s \leq h}) \right) \right] \\ &= \tilde{u}_r(r') \mathbf{E}_x \left[\mathbf{1}_{\{T_{r'}(\xi) < \infty\}} F((\xi_s)_{0 \leq s \leq T_{r'}(\xi)}) \exp \left(- \int_{[0, T_{r'}(\xi)]} dh \varpi(u_r(\xi_h), G((\xi_s)_{0 \leq s \leq h})) \right) \right]. \end{aligned}$$

Palm formula We introduce the following notation

$$\forall \lambda \in \mathbb{R}_+, \quad \psi^*(\lambda) = \psi(\lambda) - \alpha \lambda, \tag{III.45}$$

that is clearly the Laplace exponent of a spectrally Lévy process. We then fix $x \in \mathbb{R}^d$. Again, recall that $\xi = (\xi_t)_{t \geq 0}$ under \mathbf{P}_x is distributed as a standard d -dimensional Brownian motion starting from x . Let $U = (U_a)_{a \geq 0}$ be a subordinator defined on $(\Omega, \mathcal{F}, \mathbf{P}_x)$ that is assumed to be independent of ξ and whose Laplace exponent is

$$\forall \lambda \in \mathbb{R}_+, \quad \tilde{\psi}^*(\lambda) := \frac{\psi(\lambda)}{\lambda} - \alpha.$$

For any $a \in \mathbb{R}_+$, we denote by $R_a(db)$ the random measure $\mathbf{1}_{[0, a]}(b) dU_b$. We first recall from [40], formula (106) p. 105, that for any measurable function $F : M_f(\mathbb{R}_+) \times \mathcal{W} \rightarrow \mathbb{R}_+$, the following holds true:

$$\mathbf{N}_x \left(\int_0^\sigma ds F(\rho_s, W_s) \right) = \int_0^\infty da e^{-\alpha a} \mathbf{E}_x [F(R_a, (\xi_s)_{0 \leq s \leq a})]. \tag{III.46}$$

We next provide a Palm decomposition for the occupation measure \mathcal{M} of the snake that is used to estimate its lower local density at "typical" points. To that end we need to introduce the following auxiliary random variables. Let $(V_t)_{t \geq 0}$ be a subordinator defined on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ that is independent of ξ and whose Laplace exponent is $\psi^*(\lambda) := \psi'(\lambda) - \alpha$. We then introduce the following point measure on $[0, \infty) \times C(\mathbb{R}_+, \mathcal{W})$:

$$\mathcal{N}^*(dt dW) = \sum_{j \in \mathcal{J}^*} \delta_{(t_j, W^j)}, \tag{III.47}$$

such that under \mathbf{P}_0 and conditionally given (ξ, V) , \mathcal{N}^* is distributed as a Poisson point process with intensity $dV_t \mathbf{N}_{\xi_t}(dW)$.

For all $j \in \mathcal{J}^*$, we denote by \mathcal{M}_j the occupation measure of the snake W^j . Then for all $a \in \mathbb{R}_+$, we define the following random measure on \mathbb{R}^d ;

$$\mathcal{M}_a^* = \sum_{j \in \mathcal{J}^*} \mathbf{1}_{[0, a]}(t_j) \mathcal{M}_j. \tag{III.48}$$

Note that \mathcal{M}_a^* is \mathbf{P}_0 -a.s. a random finite Borel measure on \mathbb{R}^d . Informally \mathcal{M}_a^* is the sum of the the occupation measure of the snakes grafted at a rate given by V on the *spatial spine* ξ between time 0 and a . As a by-product of Formula (113) p.113 in [40], we get the following Palm decomposition of \mathcal{M} under \mathbf{N}_0 .

Proposition 2.10 ([40] (113)) *Let $F : \mathbb{R}^d \times M_f(\mathbb{R}^d) \rightarrow \mathbb{R}_+$ be measurable. Then,*

$$\mathbf{N}_0 \left[\int_{\mathbb{R}^d} \mathcal{M}(dy) F(y, \mathcal{M}) \right] = \int_0^\infty da e^{-\alpha a} \mathbf{E}_0 [F(\xi_a, \mathcal{M}_a^*)]. \tag{III.49}$$

We shall mostly use the Palm formula in this way: for any measurable functional $G : \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}_+$, we get

$$\mathbf{N}_0 \left[\int_{\mathbb{R}^d} \mathcal{M}(dy) G((\mathcal{M}(B(y, r)))_{r \geq 0}) \right] = \int_0^\infty da e^{-\alpha a} \mathbf{E}_0 [G((\mathcal{M}_a^*(B(0, r)))_{r \geq 0})]. \quad (\text{III.50})$$

3 Estimates.

3.1 Tail of a subordinator.

Recall from (III.45) that $\psi^*(\lambda) = \psi(\lambda) - \alpha\lambda$, is the Laplace exponent of a spectrally positive Lévy process. Therefore, $\psi^{*'}$ is the Laplace exponent of a subordinator that is conservative for $\psi^{*'}(0) = 0$. By subordination, $\sqrt{\psi^{*'} \circ \psi^{-1}}$ is also the Laplace exponent of a conservative subordinator. The main idea of the proof of Theorem 4.1 consists in comparing the mass of a typical ball with a subordinator whose Laplace exponent is $\sqrt{\psi^{*'} \circ \psi^{-1}}$. To that end, we first need the following result.

Lemma 3.1 *Assume $\delta > 1$. Recall from (III.11) the definition of the gauge function $g : (0, r_0) \rightarrow (0, \infty)$. Let $(S_r)_{r \in [0, \infty)}$ be a subordinator defined on the auxiliary probability space $(\Omega, \mathcal{F}, \mathbf{P}_0)$. We assume that the Laplace exponent of S is $\sqrt{\psi^{*'} \circ \psi^{-1}}$. Let $\rho_n \in (0, r_0)$, $n \in \mathbf{N}$, be such that*

$$\rho_{n+1} \leq e^{-n} \rho_n \quad \text{and} \quad \sup_{n \geq 0} n^{-2} \log 1/\rho_n < \infty. \quad (\text{III.51})$$

Then, $\sum_{n \geq 0} \mathbf{P}_0(S_{\rho_n} \leq g(4\rho_n)) = \infty$. Moreover, we get

$$\mathbf{P}_0\text{-a.s.} \quad \limsup_{n \rightarrow \infty} \frac{S_{\rho_{n+1}}}{g(4\rho_n)} < \infty.$$

Proof. To simplify notation, we set $\Phi = \sqrt{\psi^{*'} \circ \psi^{-1}}$. Thus, $\mathbf{E}_0[\exp(-\lambda S_r)] = \exp(-r\Phi(\lambda))$. Denote by Φ^{-1} the reciprocal function of Φ . For any $r \in (0, e^{-1})$, we set

$$g_*(r) = \frac{\log \log \frac{1}{r}}{\Phi^{-1}(\frac{1}{r} \log \log \frac{1}{r})}.$$

An easy computation implies that $\Phi^{-1}(y) = \varphi^{-1}(y^2 + \alpha)$. Since $\alpha \in [0, \infty)$, we easily get $g_*(r) \leq g(r)$, $r \in (0, r_0)$. For any $n \in \mathbf{N}$, we then set $\lambda_n = \Phi^{-1}((4\rho_n)^{-1} \log \log 1/4\rho_n)$. Then, observe that

$$\lambda_n g(4\rho_n) \geq \lambda_n g_*(4\rho_n) = \log \log(1/4\rho_n).$$

Next note that for all $a, x \in [0, \infty)$, $(1 - e^{-a}) \mathbf{1}_{\{0 \leq x \leq a\}} \geq e^{-x} - e^{-a}$, which easily entails

$$\mathbf{P}_0(S_{\rho_n} \leq g(4\rho_n)) \geq \frac{\exp(-\rho_n \Phi(\lambda_n)) - \exp(-\lambda_n g(4\rho_n))}{1 - \exp(-\lambda_n g(4\rho_n))} \underset{n \rightarrow \infty}{\sim} (\log 1/(4\rho_n))^{-\frac{1}{4}}.$$

By the second assumption in (III.51), $\sum_{n \geq 0} (\log 1/(4\rho_n))^{-\frac{1}{4}} = \infty$, which proves the first point of the lemma.

Let us prove the second point. By a standard Markov inequality, we get

$$\mathbf{P}_0(S_{\rho_{n+1}} \geq g_*(4\rho_n)) \leq \frac{1 - \exp(-\rho_{n+1} \Phi(\lambda_n))}{1 - \exp(-\lambda_n g_*(4\rho_n))} \leq \frac{\rho_{n+1} \Phi(\lambda_n)}{1 - \exp(-\lambda_n g_*(4\rho_n))}.$$

First observe that $1 - \exp(-\lambda_n g_*(4\rho_n)) = 1 - (\log 1/4\rho_n)^{-1} \rightarrow 1$, as $n \rightarrow \infty$. By (III.51), there exists a constant $c_2 \in (0, \infty)$ such that

$$\rho_{n+1} \Phi(\lambda_n) = \frac{\rho_{n+1}}{4\rho_n} \log \log 1/(4\rho_n) \leq c_2 e^{-n} \log n ,$$

Thus, $\sum_{n \geq 0} \mathbf{P}_0(S_{\rho_{n+1}} \geq g(4\rho_n)) \leq \sum_{n \geq 0} \mathbf{P}_0(S_{\rho_{n+1}} \geq g_*(4\rho_n)) < \infty$, which completes the proof by the Borel-Cantelli lemma. \blacksquare

3.2 Estimates on hitting probabilities.

As already mentioned in (III.7), the total range \mathbf{R} of a ψ -super Brownian motion is bounded if the starting measure μ has compact support and if

$$\int_1^\infty \frac{db}{\sqrt{\int_1^b \psi(a) da}} < \infty. \tag{III.52}$$

Observe that if $\delta > 1$, then $\gamma > 1$ and (III.52) holds true, which allows to define the following function

$$\forall v \in (0, \infty), \quad I(v) = \int_v^\infty \frac{db}{\sqrt{\int_v^b \psi(a) da}} = \int_0^\infty \frac{db}{\sqrt{\int_0^b \psi(v+a) da}}. \tag{III.53}$$

This function is clearly decreasing and continuous and it plays a role in the proof of an upper bound of the hitting probabilities of the ψ -Lévy snake. We first need the following elementary lemma.

Lemma 3.2 *Assume $\delta > 1$. Then, there exists $c_3 \in (0, \infty)$ that only depends on ψ such that for all $v \in (1, \infty)$ and all $r \in (0, \infty)$ satisfying $r \leq I(v)$, we have*

$$v \leq \psi'^{-1}(4c_3 r^{-2}).$$

Proof. By an elementary change of variable, we get

$$I(v) = \int_1^\infty \frac{\sqrt{v} db}{\sqrt{\int_1^b \psi(va) da}}.$$

Fix $c \in (1, \delta)$. By the definition (III.3) of δ , there exists $C \in (0, \infty)$ such that $\psi(va) \geq C\psi(v)a^c$, for any $a, v \in (1, \infty)$. Let $v \in (1, \infty)$ and $r \in (0, \infty)$ be such that $r \leq I(v)$. Then,

$$r \leq I(v) \leq C^{-1/2}(v/\psi(v))^{1/2} \int_1^\infty \frac{db}{\sqrt{\int_1^b a^c da}} =: (c_3 v/\psi(v))^{1/2},$$

which implies the desired result since $\psi'(v) \leq 4\psi(v)/v$ by (III.20). \blacksquare

Recall from (III.37) the definition of the range \mathcal{R} of the snake. Recall from (III.36) the notation \mathbf{N}_x for the excursion measure of the snake starting from x . Let $r \in (0, \infty)$. Recall that $B(0, r)$ stands for the open ball in \mathbb{R}^d with radius r and center 0. Then, we set for all $x \in B(0, r)$

$$v_r(x) = \mathbf{N}_x(\mathcal{R} \cap B(0, r)^c \neq \emptyset). \tag{III.54}$$

From [40] p. 121 and p. 131, we know that

$$\forall x \in B(0, r), \quad v_r(x) \in (0, \infty) \quad \text{and} \quad \lim_{\|x\| \rightarrow r^-} v_r(x) = \infty.$$

Moreover, v_r is C^2 on $B(0, r)$ and it satisfies $\frac{1}{2}\Delta v_r = \psi(v_r)$. As an easy consequence of Brownian motion isotropy, v_r is a radial function: namely, $v_r(x)$ only depends on $\|x\|$ (and r). Therefore, one can derive estimates on v_r by studying the associated ordinary differential equation corresponding to the radial function, as done by Keller in [71], p. 507 inequality (25), who proves the following:

$$\forall r \in (0, \infty), \quad \frac{2}{\sqrt{d}}r \leq I(v_r(0)) \leq 2r. \quad (\text{III.55})$$

We use this bound as follows. For any $r \in (0, \infty)$ and any $x \in B(0, r)^c$, recall that

$$u_r(x) = \mathbf{N}_x(\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset). \quad (\text{III.56})$$

From [40] p. 121 and p. 131, we know that $u_r(x) \in (0, \infty)$ for all $x \in B(0, r)^c$, that

$$u_r \text{ is radial, } \quad \lim_{\|x\| \rightarrow \infty} u_r(x) = 0 \quad \text{and} \quad \lim_{\|x\| \rightarrow r+} u_r(x) = \infty. \quad (\text{III.57})$$

Recall that we denote by \tilde{u}_r the radial function yielded by u_r , namely:

$$\forall x \in \overline{B}(0, r)^c, \quad \tilde{u}_r(\|x\|) = u_r(x). \quad (\text{III.58})$$

Moreover, u_r is C^2 in $B(0, r)^c$ and it satisfies

$$\frac{1}{2}\Delta u_r = \psi(u_r) \quad \text{in } B(0, r)^c. \quad (\text{III.59})$$

We shall use several times the following upper bound of u_r .

Lemma 3.3 *Assume that $\delta > 1$ and that $d \geq 3$. Let $\varrho \in (0, \infty)$. Then there exist $r_1, c_4 \in (0, \infty)$, that only depend on ψ, d and ϱ , such that*

$$\forall r \in (0, r_1), \quad \forall x \in B(0, (1+\varrho)r)^c, \quad u_r(x) \leq ((1+\varrho)r/\|x\|)^{d-2} \psi'^{-1}(c_4 r^{-2}).$$

Proof. Let $y \in \mathbb{R}^d$ be such that $\|y\| = (1+\varrho)r$. First note that

$$\mathbf{N}_y(\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset) \leq \mathbf{N}_y(\mathcal{R} \cap B(y, \varrho r)^c \neq \emptyset).$$

By translation invariance of Brownian motion, the right member of the previous inequality does not depend on y and we get $u_r(y) \leq v_{\varrho r}(0)$, where $v_{\varrho r}$ is defined by (III.54). For any $x \in B(0, (1+\varrho)r)^c$, we next set $w(x) = u_r(x) - v_{\varrho r}(0)((1+\varrho)r/\|x\|)^{d-2}$ that is clearly subharmonic. The previous upper bound implies that $w(y) \leq 0$, if $\|y\| = (1+\varrho)r$. By (III.57), $\lim_{\|x\| \rightarrow \infty} w(x) = 0$ and by the maximum principle, we get that $w \leq 0$ on $B(0, (1+\varrho)r)^c$. Namely,

$$\forall r \in (0, \infty), \quad \forall x \in B(0, (1+\varrho)r)^c, \quad u_r(x) \leq v_{\varrho r}(0)((1+\varrho)r/\|x\|)^{d-2}. \quad (\text{III.60})$$

Since $\delta > 1$, (III.52) is satisfied and the function I given by (III.53) is well-defined; we easily check that $I(v) \rightarrow 0$ iff $v \rightarrow \infty$. Then, (III.55) implies that $\lim_{r \rightarrow 0} v_{\varrho r}(0) = \infty$, so we can find $r_1 \in (0, \infty)$ such that for all $r \in (0, r_1)$, $v_{\varrho r}(0) \geq 1$ and by the left member of (III.55) we also have $2r\varrho/\sqrt{d} \leq I(v_{\varrho r}(0))$. Thus, Lemma 3.2 applies and asserts that $v_{\varrho r}(0) \leq \psi'^{-1}(c_4 r^{-2})$, where $c_4 := c_3 d \varrho^{-2}$, which completes the proof thanks to (III.60). \blacksquare

We use the previous lemma to get an upper bound of the expectation of a specific additive functional of the Brownian motion that involves u_r . More precisely, for any $r \in (0, \infty)$, we define

$$q_r = \psi'^{-1}(c_4 r^{-2}). \quad (\text{III.61})$$

Recall that c_4 is the constant appearing in Lemma 3.3. Note that for any $0 < r < (c_4/\psi'(1))^{1/2}$, we have $q_r \geq 1$. We then define

$$\forall r \in (0, (c_4/\psi'(1))^{1/2}), \quad J(r) = r^2 q_r^{\frac{2}{d-2}} \int_1^{q_r} \psi'(v) v^{-\frac{d}{d-2}} dv. \quad (\text{III.62})$$

Recall that $\xi = (\xi_t)_{t \geq 0}$ stands for a standard Brownian motion starting from 0 that is defined on the auxiliary probability space $(\Omega, \mathcal{F}, \mathbf{P}_0)$. We next prove the following lemma.

Lemma 3.4 *Assume that $\delta > 1$ and that $d \geq 3$. Let $a \in (0, \infty)$. Then, there exist $c_5, c_6, r_2 \in (0, \infty)$ that only depend on ψ, d and a , such that*

$$\forall r \in (0, r_2) \quad \mathbf{E}_0 \left[\int_0^{2a} ds \mathbf{1}_{\{\|\xi_s\| \geq 2r\}} \psi'(u_r(\xi_s)) \right] \leq c_5 + c_6 J(r).$$

Proof. To simplify notation, we denote by L the expectation in the left member of the previous inequality. Recall from (III.58) the notation \tilde{u}_r for the radial function yielded by u_r . By Fubini and easy changes of variable, we have the following.

$$\begin{aligned} L &= \int_{B(0, 2r)^c} dx \int_0^{2a} ds (2\pi s)^{-d/2} e^{-\|x\|^2/2s} \psi'(\tilde{u}_r(\|x\|)) \\ &= c_7 \int_{2r}^{\infty} dy y^{d-1} \psi'(\tilde{u}_r(y)) \int_0^{2a} ds s^{-d/2} e^{-y^2/2s} \\ &= c_8 \int_{2r}^{\infty} dy y f(y) \psi'(\tilde{u}_r(y)), \end{aligned}$$

where for any $y \in (0, \infty)$, we have set $f(y) = \int_{y^2/(4a)}^{\infty} du u^{d/2-2} e^{-u}$ and where c_7 and c_8 are constants that only depend on d . Since we assume that $d \geq 3$, $f(0)$ is well-defined and finite, and it is easy to check that $\int_0^{\infty} y f(y) dy < \infty$.

We next use Lemma 3.3 with $\varrho = 1$ to get for all $r \in (0, r_1)$ and all $y \in (2r, \infty)$ that $\tilde{u}_r(y) \leq (2r/y)^{d-2} q_r$. We then set $\alpha_r = 2r q_r^{1/(d-2)}$. Thus, $\tilde{u}_r(y) \leq (\alpha_r/y)^{d-2}$, for all $r \in (0, r_1)$ and all $y \in (2r, \infty)$. We next set

$$r_2 := r_1 \wedge (c_4/\psi'(1))^{1/2}.$$

Observe that for any $r \in (0, r_2)$, $q_r \geq 1$, which implies that $\alpha_r \geq 2r$. Next, observe that for all $r \in (0, r_2)$ and all $y \in (\alpha_r, \infty)$, $(\alpha_r/y)^{d-2} \leq 1$. Thus, $\psi'(\tilde{u}_r(y)) \leq \psi'(1)$. It implies

$$\begin{aligned} L &\leq c_8 \psi'(1) \int_{\alpha_r}^{\infty} y f(y) dy + c_8 \int_{2r}^{\alpha_r} y f(y) \psi'((\alpha_r/y)^{d-2}) dy \\ &\leq c_5 + c_8 f(0) \int_{2r}^{\alpha_r} y \psi'((\alpha_r/y)^{d-2}) dy, \end{aligned} \quad (\text{III.63})$$

where $c_5 := c_8 \psi'(1) \int_0^{\infty} y f(y) dy$. By using the change of variable $v = (\alpha_r/y)^{d-2}$ we get

$$c_8 f(0) \int_{2r}^{\alpha_r} y \psi'((\alpha_r/y)^{d-2}) dy = \frac{1}{d-2} c_8 f(0) \alpha_r^2 \int_1^{(\alpha_r/2r)^{d-2}} \psi'(v) v^{-\frac{d}{d-2}} dv = c_6 J(r),$$

where we have set $c_6 := \frac{4}{d-2}c_8f(0)$. Then, the desired result follows from (III.63). \blacksquare

When d is greater than $\frac{2\delta}{\delta-1}$, the function J is bounded for all small values of r as proved in the following lemma.

Lemma 3.5 *Assume that $\delta > 1$ and that $d > \frac{2\delta}{\delta-1}$. Then, there exists a constant $c_9 \in (0, \infty)$ that depends on d and ψ such that $J(r) \leq c_9$ for all $r \in (0, r_2)$.*

Proof. Observe that $\frac{2}{d-2} < \delta - 1$. Recall that $\delta_{\psi'} = \delta - 1$. Let us fix $u \in (\frac{2}{d-2}, \delta - 1)$. By the definition (III.3) of the exponent $\delta_{\psi'}$, there exists $K \in (0, \infty)$ depending on ψ and u such that $\forall 1 \leq \lambda \leq \mu$, $\psi'(\lambda) \leq K \psi'(\mu)(\lambda/\mu)^u$. Recall from (III.61) that $\psi'(q_r) = c_4 r^{-2}$, where c_4 is the constant appearing in Lemma 3.3. Then we get the following.

$$\begin{aligned} J(r) &= r^2 q_r^{\frac{2}{d-2}} \int_1^{q_r} \psi'(v) v^{-\frac{d}{d-2}} dv \leq r^2 q_r^{\frac{2}{d-2}} \psi'(q_r) \int_1^{q_r} (v/q_r)^u v^{-\frac{d}{d-2}} dv \\ &\leq c_4 q_r^{\frac{2}{d-2}-u} \int_1^{q_r} v^{u-\frac{2}{d-2}-1} dv \\ &\leq c_4 q_r^{\frac{2}{d-2}-u} \int_0^{q_r} v^{u-\frac{2}{d-2}-1} dv = \frac{c_4}{u-\frac{2}{d-2}}, \end{aligned}$$

which implies the desired result with $c_9 := \frac{c_4}{u-\frac{2}{d-2}}$. \blacksquare

When $d \in (\frac{2\gamma}{\gamma-1}, \frac{2\delta}{\delta-1}]$, we are only able to prove that $\liminf_{r \rightarrow 0} J(r) < \infty$. More precisely, we prove the following lemma.

Lemma 3.6 *Assume that $\gamma > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Recall that c_4 appears in Lemma 3.3. Then, there exists a decreasing function*

$$\theta \in (\psi'(1), \infty) \mapsto r_\theta \in (0, (c_4/\psi'(1))^{\frac{1}{2}})$$

such that $\lim_{\theta \rightarrow \infty} r_\theta = 0$ and such that there exists $c_{10} \in (0, \infty)$, that only depends on d and ψ , and that satisfies

$$\forall \theta \in (\psi'(1), \infty), \quad J(r_\theta) \leq c_{10}. \quad (\text{III.64})$$

Moreover, for any $\theta', \theta \in (\psi'(1), \infty)$ such that $\theta' \geq \theta$, we also have

$$r_{\theta'}/r_\theta \leq (\theta/\theta')^{1/2} \quad \text{and} \quad r_\theta \geq c_{11} \theta^{-c_{12}}, \quad (\text{III.65})$$

where $c_{11}, c_{12} \in (0, \infty)$ only depend on d and ψ .

Proof. Note that $\frac{2}{d-2} < \gamma - 1 = \gamma_{\psi'}$. Let us fix $c \in (\frac{2}{d-2}, \gamma_{\psi'})$. Thus, $\lambda^{-c} \psi'(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$, which allows to define the following for any $\theta \in (\psi'(1), \infty)$:

$$r_\theta = (c_4/\psi'(\lambda_\theta))^{\frac{1}{2}} \quad \text{where} \quad \lambda_\theta = \inf \{ \lambda \in [1, \infty) : \lambda^{-c} \psi'(\lambda) = \theta \}. \quad (\text{III.66})$$

Note that if $\theta > \psi'(1)$, then $r_\theta < (c_4/\psi'(1))^{\frac{1}{2}}$ and $J(r_\theta)$ is well-defined by (III.62). Clearly, $\theta \mapsto \lambda_\theta$ increases to ∞ as $\theta \rightarrow \infty$. Consequently $\theta \mapsto r_\theta$ decreases to 0 as $\theta \rightarrow \infty$. Recall from (III.61) and (III.62) the definitions of q_r and $J(r)$ and note that $q_{r_\theta} = \lambda_\theta$. By definition, $\psi'(v) \leq \theta v^c$, for any $v \in [1, \lambda_\theta]$, which implies

$$\begin{aligned} J(r_\theta) &= r_\theta^2 \lambda_\theta^{\frac{2}{d-2}} \int_1^{\lambda_\theta} \psi'(v) v^{-\frac{d}{d-2}} dv \leq r_\theta^2 \lambda_\theta^{\frac{2}{d-2}} \theta \int_1^{\lambda_\theta} v^{c-\frac{2}{d-2}-1} dv \\ &\leq \frac{1}{c-\frac{2}{d-2}} r_\theta^2 \lambda_\theta^{\frac{2}{d-2}} \theta \lambda_\theta^{c-\frac{2}{d-2}} = c_{13} \frac{\theta \lambda_\theta^c}{\psi'(\lambda_\theta)} = c_{13}, \end{aligned}$$

where $c_{13} = c_4 / (c - \frac{2}{d-2})$ and since $\psi'(\lambda_\theta) = \theta \lambda_\theta^c$, by definition. Next, observe that

$$\theta r_\theta^2 = \frac{c_4 \theta}{\psi'(\lambda_\theta)} = c_4 \lambda_\theta^{-c}.$$

Thus, $\theta \mapsto \theta r_\theta^2$ decreases, which proves the first inequality in (III.65). To prove the second inequality, we fix $\varepsilon \in (0, \infty)$ such that $c + \varepsilon < \gamma_{\psi'}$. By definition of $\gamma_{\psi'}$, there exists $K \in (0, \infty)$ such that $\lambda^{-c} \psi'(\lambda) \geq K \lambda^\varepsilon$, for any $\lambda \in [1, \infty)$. It entails that $\theta = \lambda_\theta^{-c} \psi'(\lambda_\theta) \geq K \lambda_\theta^\varepsilon$. Thus,

$$r_\theta = (c_4 / \psi'(\lambda_\theta))^{\frac{1}{2}} = (c_4 / (\theta \lambda_\theta^c))^{\frac{1}{2}} \geq (c_4 K^{\frac{c}{\varepsilon}})^{\frac{1}{2}} \theta^{-\frac{1}{2}(1 + \frac{c}{\varepsilon})},$$

which implies the desired result with $c_{11} = (c_4 K^{\frac{c}{\varepsilon}})^{\frac{1}{2}}$ and $c_{12} = \frac{1}{2}(1 + \frac{c}{\varepsilon})$. \blacksquare

By combing the previous lemmas we obtain the following result.

Lemma 3.7 *Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then $d \geq 4$. Let $a \in (0, \infty)$. For any $n \in \mathbf{N}$, set $\rho_n = r_{e^{n^2}}$, where $(r_\theta)_{\theta \in [\psi'(1), \infty)}$ is defined as in Lemma 3.6. Then, the sequence $(\rho_n)_{n \geq 0}$ satisfies (III.51) in Lemma 3.1. Moreover, there exists a constant $c_{14} \in (0, \infty)$, that only depends on d, ψ and a , such that for all sufficiently large $n \in \mathbf{N}$,*

$$\mathbf{E}_0 \left[\int_0^{2a} ds \mathbf{1}_{\{\|\xi_s\| \geq 2\rho_n\}} \psi'(u_{\rho_n}(\xi_s)) \right] \leq c_{14}.$$

Proof. Note that $\gamma \leq 2$, which easily entails that $d \geq 4$. By (III.65) in Lemma 3.6,

$$\rho_{n+1} / \rho_n \leq (e^{n^2} / e^{(n+1)^2})^{1/2} = e^{-n - \frac{1}{2}} \leq e^{-n} \quad \text{and} \quad n^{-2} \log 1/\rho_n \leq c_{12} - n^{-2} \log c_{11},$$

which proves that $(\rho_n)_{n \geq 0}$ satisfies (III.51) in Lemma 3.1. Moreover, for all $n \in \mathbf{N}$ such that $\rho_n \in (0, r_2)$, Lemma 3.4 and (III.64) in Lemma 3.6 imply

$$\mathbf{E}_0 \left[\int_0^{2a} ds \mathbf{1}_{\{\|\xi_s\| \geq 2\rho_n\}} \psi'(u_{\rho_n}(\xi_s)) \right] \leq c_5 + c_6 J(r_{e^{n^2}}) \leq c_5 + c_6 c_{10} =: c_{14},$$

which completes the proof of the lemma. \blacksquare

3.3 The spine and the associated subordinator.

Recall from Section 2.4, the Palm formula for the occupation measure of the snake. Recall that $\xi = (\xi_t)_{t \geq 0}$ is d -dimensional Brownian motion starting from 0 that is defined on $(\Omega, \mathcal{F}, \mathbf{P}_0)$. Recall that $(V_t)_{t \geq 0}$ is a subordinator defined on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ that is independent of ξ and whose Laplace exponent is $\psi^{*\prime}(\lambda) = \psi'(\lambda) - \alpha$. Recall from (III.47) that under \mathbf{P}_0 , conditionally given (ξ, V) , $\mathcal{N}^*(dtdW) = \sum_{j \in \mathcal{J}^*} \delta_{(t_j, W^j)}$ is a Poisson point process on $[0, \infty) \times C(\mathbb{R}_+, \mathcal{W})$ with intensity $dV_t \mathbf{N}_{\xi_t}(dW)$. Then recall from (III.48) that for all $a \in \mathbb{R}_+$, we have set $\mathcal{M}_a^* = \sum_{j \in \mathcal{J}^*} \mathbf{1}_{[0, a]}(t_j) \mathcal{M}_j$ where for all $j \in \mathcal{J}^*$, \mathcal{M}_j stands for the occupation measure of the snake W^j as defined in (III.38).

For any $a \in \mathbb{R}_+$, we then introduce

$$T_a := \langle \mathcal{M}_a^*, \mathbf{1} \rangle = \sum_{j \in \mathcal{J}^*} \mathbf{1}_{[0, a]}(t_j) \sigma_j, \tag{III.67}$$

where σ_j is the total duration of the excursion of the snake W^j . By construction of the snake excursion measure,

$$\mathbf{N}_x[1 - e^{-\lambda \sigma}] = N[1 - e^{-\lambda \sigma}] = \psi^{-1}(\lambda). \tag{III.68}$$

We shall assume throughout the paper that $d \geq 4$. We then introduce the following two last exit times: for all $r \in \mathbb{R}_+$, we set

$$\vartheta(r) = \sup\{t \geq 0 : \|\xi_t\| \leq r\} \quad (\text{III.69})$$

$$\gamma(r) = \sup\left\{t \geq 0 : \sqrt{(\xi_t^{(1)})^2 + (\xi_t^{(2)})^2 + (\xi_t^{(3)})^2} \leq r\right\} \quad (\text{III.70})$$

where $(\xi_t^{(i)})_{t \geq 0}$ stands for the i -th coordinate of ξ . We then recall the two basic facts on the processes γ and ϑ .

(a) The increments of $(\vartheta(r))_{r \geq 0}$ are independent. Moreover, $(\|\xi_{\vartheta(r)+t}\|)_{t \geq 0}$ is independent of the two processes $(\vartheta(r'))_{0 \leq r' \leq r}$ and $(\|\xi_{t \wedge \vartheta(r)}\|)_{t \geq 0}$.

(b) The process $(\gamma(r))_{r \geq 0}$ has independent and stationary increments: it is a subordinator with Laplace exponent $\lambda \mapsto \sqrt{2\lambda}$.

The first point is proved in Gettoor [61]. The second is a celebrated result of Pitman [104].

Before stating our lemma, we introduce the following random variables:

$$\forall t \geq s \geq 0, \quad N_r(s, t) = \#\left\{j \in \mathcal{J}^* : s < t_j < t \text{ and } \mathcal{R}_j \cap \overline{B}(0, r) \neq \emptyset\right\}, \quad (\text{III.71})$$

that counts the snakes that are grafted on the spatial spine ξ between times s and t , and that hit the ball $\overline{B}(0, r)$.

Lemma 3.8 *Assume that $d \geq 4$. We keep the previous notation. Then, the following holds true.*

(i) *For all real numbers $r > r' > \rho > \rho' > 0$ and all $a \in (0, \infty)$, the random variables*

$$T_{\vartheta(2\rho)} - T_{\vartheta(2\rho')}, \quad T_{\vartheta(2r)} - T_{\vartheta(2r')} \quad \text{and} \quad N_r(\vartheta(2r), a + \vartheta(2r))$$

are independent.

(ii) *The process $(T_{\gamma(r)})_{r \geq 0}$ is a subordinator with Laplace exponent $\sqrt{2\psi^{*'} \circ \psi^{-1}}$.*

Remark 3.1 We take the opportunity to mention that it is stated incorrectly in [36] (60), that $T_{\gamma(2\rho)} - T_{\gamma(2\rho')}$, $T_{\gamma(2r)} - T_{\gamma(2r')}$ and $N(\vartheta(2r), a)$ are independent. More precisely, the statement (49) in [36] is incorrect. We provide here a correct statement and a correct proof.

Proof. Let us prove (i). Let $\lambda, \mu, \theta \in \mathbb{R}_+$. To simplify notation we set

$$Y = \exp\left(-\lambda(T_{\vartheta(2r)} - T_{\vartheta(2r')}) - \mu(T_{\vartheta(2\rho)} - T_{\vartheta(2\rho')}) - \theta N_r(\vartheta(2r), a + \vartheta(2r))\right).$$

Then recall (III.68) and recall from (III.58) notation $\tilde{u}_r(\|x\|) = \mathbf{N}_x(\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset)$. Then, basic results on Poisson processes imply

$$\mathbf{E}_0[Y | (\xi, V)] = \exp\left(-\psi^{-1}(\lambda)(V_{\vartheta(2r)} - V_{\vartheta(2r')}) - \psi^{-1}(\mu)(V_{\vartheta(2\rho)} - V_{\vartheta(2\rho')}) - (1 - e^{-\theta}) \int_{(0, a)} dV_t \tilde{u}_r(\|\xi_{\vartheta(2r)+t}\|)\right).$$

Since V is a subordinator with Laplace exponent $\psi^{*'}$, we then get

$$\begin{aligned} \mathbf{E}_0[Y | \xi] &= \exp\left(-(\vartheta(2r) - \vartheta(2r'))\psi^{*'}(\psi^{-1}(\lambda))\right. \\ &\quad \left.- (\vartheta(2\rho) - \vartheta(2\rho'))\psi^{*'}(\psi^{-1}(\mu)) - \int_0^a dt \psi^{*'}((1 - e^{-\theta})\tilde{u}_r(\|\xi_{\vartheta(2r)+t}\|))\right). \end{aligned}$$

The above mentioned property (a) for last exit times then implies that

$$\vartheta(2r) - \vartheta(2r'), \quad \vartheta(2\rho) - \vartheta(2\rho') \quad \text{and} \quad \int_0^a dt \psi^{*'}((1 - e^{-\theta})\tilde{u}_r(\|\xi_{\vartheta(2r)+t}\|))$$

are independent which easily implies (i).

The second point is proved in a similar way: let $0 = r_0 < r_1 < \dots < r_n$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$. We set

$$Y' = \exp\left(-\sum_{1 \leq k \leq n} \lambda_k (T_{\gamma(r_k)} - T_{\gamma(r_{k-1})})\right)$$

Thus,

$$\begin{aligned} \mathbf{E}_0[Y'] &= \mathbf{E}_0\left[\exp\left(-\sum_{1 \leq k \leq n} \psi^{-1}(\lambda_k) (V_{\gamma(r_k)} - V_{\gamma(r_{k-1})})\right)\right] \\ &= \mathbf{E}_0\left[\exp\left(-\sum_{1 \leq k \leq n} \psi^{*'}(\psi^{-1}(\lambda_k)) (\gamma(r_k) - \gamma(r_{k-1}))\right)\right] \\ &= \exp\left(-\sum_{1 \leq k \leq n} (r_k - r_{k-1}) \sqrt{2\psi^{*'}(\psi^{-1}(\lambda_k))}\right). \end{aligned}$$

Indeed, the first equality comes from basic results on Poisson point measures, the second equality comes from the fact that V is a subordinator with Laplace exponent $\psi^{*'}$ and the last equality is a consequence of the above mentioned Property (b) of the last exit times $\gamma(r)$. This completes the proof of (ii). \blacksquare

Recall from (III.71) the notation $N_r(s, t)$ that counts the snakes that are grafted on the spatial spine ξ between times s and t , and that hit the ball $\overline{B}(0, r)$. We state the following lemma that actually means, in some sense, that in supercritical dimension there is no snake W^j grafted far away that hit $\overline{B}(0, r)$.

Lemma 3.9 *Assume that $\gamma > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then, for all $t > s > 0$,*

$$\forall t > s > 0, \quad \lim_{r \rightarrow 0} \mathbf{P}_0(N_r(s, t) = 0) = 1.$$

Proof. Recall from (III.56) the definition of u_r and from (III.58) that of \tilde{u}_r . By the definition (III.71), conditionally given (ξ, V) , $N_r(s, t)$ is a Poisson random variable with parameter $\int_s^t dV_w u_r(\xi_w)$. Thus

$$\mathbf{P}_0(N_r(s, t) = 0) = \mathbf{E}_0\left[\exp\left(-\int_s^t dw \psi^{*'}(u_r(\xi_w))\right)\right]. \quad (\text{III.72})$$

Next, note that $d > \frac{2\gamma}{\gamma-1}$ implies that $d - 2 > \frac{2}{\gamma-1}$. Then, there exists $b \in (0, 1)$ and $a \in (0, \gamma - 1)$ such that

$$(d - 2)(1 - b) > \frac{2}{a}. \quad (\text{III.73})$$

By the definition (III.69) of the last exit time process ϑ ,

$$\mathbf{E}_0\left[\exp\left(-\int_s^t dw \psi^{*'}(u_r(\xi_w))\right)\right] \geq \exp(-t\psi^{*'}(\tilde{u}_r(r^b))) \mathbf{P}_0(s > \vartheta(r^b)). \quad (\text{III.74})$$

Clearly $\lim_{r \rightarrow 0} \mathbf{P}_0(s > \vartheta(r^b)) = 1$. Thus, we only have to choose b such that $\lim_{r \rightarrow 0} \tilde{u}_r(r^b) = 0$. To that end, we apply Lemma 3.3 with $\varrho = 1$: for all $r \in (0, r_1 \wedge 2^{-\frac{1}{1-b}})$, we get

$$\tilde{u}_r(r^b) \leq (2r/r^b)^{d-2} \psi'^{-1}(c_4 r^{-2}) = 2^{d-2} r^{(d-2)(1-b)} \psi'^{-1}(c_4 r^{-2}). \quad (\text{III.75})$$

Since $a \in (0, \gamma - 1)$ and since $\gamma_{\psi'} = \gamma - 1$, the definition (III.21) of $\gamma_{\psi'}$ entails that for all sufficiently large λ , $\psi'(\lambda) \geq \lambda^a$ and thus $\psi'^{-1}(\lambda) \leq \lambda^{1/a}$. Then (III.75) implies that there exists a constant $c \in (0, \infty)$ such that $\tilde{u}_r(r^b) \leq c r^{(d-2)(1-b)-2/a}$ for all sufficiently small r . By (III.73), this entails $\lim_{r \rightarrow 0} \tilde{u}_r(r^b) = 0$, which completes the proof by (III.72) and (III.74). \blacksquare

We next prove a similar estimate for $N_r(\vartheta(2r), a + \vartheta(2r))$.

Lemma 3.10 *Assume that $\delta > 1$ and that $d > \frac{2\delta}{\delta-1}$. Then, there exist two constants $c_{15}, r_3 \in (0, \infty)$ that only depends on d, ψ and a , such that*

$$\forall r \in (0, r_3) \quad \mathbf{P}_0(N_r(\vartheta(2r), a + \vartheta(2r)) = 0) \geq c_{15}.$$

Proof. Recall from (III.56) the definition of u_r and from (III.58) that of \tilde{u}_r . By the definition (III.71), conditionally given (ξ, V) , $N_r(\vartheta(2r), a + \vartheta(2r))$ is a Poisson random variable with parameter $\int_{\vartheta(2r)}^{a+\vartheta(2r)} dV_t u_r(\xi_t)$. Thus

$$\mathbf{P}_0(N_r(\vartheta(2r), a + \vartheta(2r)) = 0) = \mathbf{E}_0 \left[\exp \left(- \int_{\vartheta(2r)}^{a+\vartheta(2r)} dt \psi^{*'}(u_r(\xi_t)) \right) \right]. \quad (\text{III.76})$$

Next note that on $\{\vartheta(2r) \leq a\}$, $a + \vartheta(2r) \leq 2a$ and that $t \geq \vartheta(2r)$ implies that $\|\xi_t\| \geq 2r$. Thus, by (III.76)

$$\begin{aligned} \mathbf{P}_0(N_r(\vartheta(2r), a + \vartheta(2r)) = 0) &\geq \mathbf{E}_0 \left[\mathbf{1}_{\{\vartheta(2r) \leq a\}} \exp \left(- \int_0^{2a} dt \mathbf{1}_{\{\|\xi_t\| \geq 2r\}} \psi^{*'}(u_r(\xi_t)) \right) \right] \\ &\geq \mathbf{E}_0 \left[\exp \left(- \int_0^{2a} dt \mathbf{1}_{\{\|\xi_t\| \geq 2r\}} \psi^{*'}(u_r(\xi_t)) \right) \right] - \mathbf{P}_0(\vartheta(2r) > a) \\ &\geq \exp \left(- \mathbf{E}_0 \left[\int_0^{2a} dt \mathbf{1}_{\{\|\xi_t\| \geq 2r\}} \psi^{*'}(u_r(\xi_t)) \right] \right) - \mathbf{P}_0(\vartheta(2r) > a) \end{aligned} \quad (\text{III.77})$$

where we use Jensen inequality in the last line. By Lemma 3.4 and Lemma 3.5, for any $r \in (0, r_2)$,

$$\exp \left(- \mathbf{E}_0 \left[\int_0^{2a} dt \mathbf{1}_{\{\|\xi_t\| \geq 2r\}} \psi^{*'}(u_r(\xi_t)) \right] \right) \geq \exp(-c_5 - c_6 c_9).$$

Since $\mathbf{P}_0(\vartheta(2r) > a) \rightarrow 0$ as $r \rightarrow 0$, there exists $r_3 \in (0, r_2)$ such that $\mathbf{P}_0(\vartheta(2r_3) > a) \leq \frac{1}{2} e^{-c_5 - c_6 c_9}$. Thus by (III.77) and the previous inequality, for any $r \in (0, r_3)$,

$$\mathbf{P}_0(N_r(\vartheta(2r), a + \vartheta(2r)) = 0) \geq \frac{1}{2} \exp(-c_5 - c_6 c_9) =: c_{15},$$

which completes the proof of lemma. \blacksquare

Under the less restrictive condition $d > \frac{2\gamma}{\gamma-1}$, we get a similar lower bound but only for the family of radii $(\rho_n)_{n \geq 1}$ introduced in Lemma 3.7.

Lemma 3.11 *Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Let $(\rho_n)_{n \geq 1}$ be the sequence introduced in Lemma 3.7. Then, there exists a constant $c_{16} \in (0, \infty)$, that only depend on d, ψ and a , such that*

$$\forall n \geq 1 \quad \mathbf{P}_0(N_r(\vartheta(2\rho_n), a + \vartheta(2\rho_n)) = 0) \geq c_{16}.$$

Proof. The lower bound (III.77) applies for $r = \rho_n$. Then, Lemma 3.7, entails that for all sufficiently large n ,

$$\exp\left(-\mathbf{E}_0\left[\int_0^{2a} dt \mathbf{1}_{\{\|\xi_t\| \geq 2\rho_n\}} \psi^{*'}(u_{\rho_n}(\xi_t))\right]\right) \geq \exp(-c_{14}),$$

and we complete the proof arguing as in the proof of Lemma 3.10 with $c_{16} := \frac{1}{2}e^{-c_{14}}$. ■

We then shall need the following result that is used in the proof of Lemma 4.4.

Lemma 3.12 *Assume that $\delta > 1$ and that $d \geq 4$. Let $(\rho_n)_{n \geq 1}$ be the sequence of radii introduced in Lemma 3.7. Then,*

$$(i) : \sum_{n \geq 1} \mathbf{P}_0\left(T_{\gamma(2\rho_n)} \leq g(8\rho_n)\right) = \infty \quad \text{and} \quad (ii) : \mathbf{P}_0\text{-a.s.} \limsup_{n \rightarrow \infty} \frac{T_{\gamma(2\rho_{n+1})}}{g(8\rho_n)} < \infty.$$

Proof. Lemma 3.8 shows that $T_{\gamma(\cdot)}$ is a subordinator with Laplace exponent $\sqrt{2\psi^{*'} \circ \psi^{-1}}$. Then, Lemma 3.7 asserts that the sequence $(\rho_n)_{n \geq 0}$ satisfies the conditions (III.51) in Lemma 3.1, which immediately entails (i) and (ii). ■

We end this section with the following result that is close to the previous one and that is used in the proof of Theorem 1.1.

Lemma 3.13 *Assume that $\gamma > 1$. Let $u \in (0, \frac{2\gamma}{\gamma-1})$. Then, there exists a decreasing sequence $(s_n)_{n \geq 1}$ that tends to 0, that only depends on ψ and u , and that satisfies the following.*

$$(i) : \sum_{n \geq 1} \mathbf{P}_0\left(T_{\gamma(2s_n)} \leq s_n^u\right) = \infty \quad \text{and} \quad (ii) : \mathbf{P}_0\text{-a.s.} \limsup_{n \rightarrow \infty} \frac{T_{\gamma(2s_{n+1})}}{s_n^u} < \infty.$$

Proof. Let $u' \in (u, \frac{2\gamma}{\gamma-1})$. We set $\varphi^* = \psi^{*'} \circ \psi^{-1}$, where $\psi^*(\lambda) = \psi(\lambda) - \alpha$. Recall from (III.21) that $\gamma_\varphi = \gamma_{\varphi^*} = \frac{\gamma-1}{\gamma}$. We fix $a \in (0, \frac{\gamma-1}{\gamma})$. By the definition (III.21) of γ_{φ^*} , there exists $\lambda_0 \in (0, \infty)$ such that $\varphi^*(\lambda) \geq \lambda^a$ for any $\lambda \in [\lambda_0, \infty)$. Next observe that $2/u' > \frac{\gamma-1}{\gamma} = \gamma_{\varphi^*}$. As an easy consequence of definition (III.21) of γ_{φ^*} , we get $\liminf_{\lambda \rightarrow \infty} \varphi^*(\lambda)\lambda^{-2/u'} = 0$. Consequently, there exists an increasing sequence $(\lambda_n)_{n \geq 0}$ such that

$$\forall n \in \mathbf{N}, \quad 2^n \leq \lambda_n \quad \text{and} \quad \lambda_n^a \leq \varphi^*(\lambda_n) \leq \lambda_n^{2/u'}. \quad (\text{III.78})$$

We next fix $\varepsilon \in (0, \infty)$ such that $(1 + \varepsilon)u/u' < 1$, which is possible since $u' > u$. Then, we set

$$\forall n \in \mathbf{N}, \quad s_n = \varphi^*(\lambda_n)^{-\frac{1+\varepsilon}{2}}. \quad (\text{III.79})$$

Since, by Lemma 3.8, $(T_{\gamma(r)})_{r \geq 0}$ is a subordinator with Laplace exponent $\sqrt{2\varphi^*}$, a Markov inequality entails

$$\mathbf{P}_0\left(T_{\gamma(2s_n)} > s_n^u\right) \leq \frac{1 - \exp\left(-2\sqrt{2}s_n\sqrt{\varphi^*(\lambda_n)}\right)}{1 - \exp(-\lambda_n s_n^u)} \leq 2\sqrt{2} \frac{s_n\sqrt{\varphi^*(\lambda_n)}}{1 - \exp(-\lambda_n s_n^u)}. \quad (\text{III.80})$$

By (III.79) and the last inequality of (III.78), we get

$$\lambda_n s_n^u = \lambda_n \varphi^*(\lambda_n)^{-(1+\varepsilon)u/2} \geq \lambda_n^{1-(1+\varepsilon)u/u'} \xrightarrow[n \rightarrow \infty]{} \infty. \quad (\text{III.81})$$

Moreover, the first two inequalities in (III.78) and (III.79) imply

$$s_n \sqrt{\varphi^*(\lambda_n)} = \varphi^*(\lambda_n)^{-\varepsilon/2} \leq \lambda_n^{-a\varepsilon/2} \leq 2^{-na\varepsilon/2}. \quad (\text{III.82})$$

Then (III.80), (III.81) and (III.82) imply that there exists $c \in (0, \infty)$ such that

$$\mathbf{P}_0 \left(T_{\gamma(2s_n)} > s_n^u \right) \leq c 2^{-na\varepsilon/2}.$$

It immediately implies (i) and (ii) follows from $\mathbf{P}_0 \left(T_{\gamma(2s_{n+1})} > s_n^u \right) \leq \mathbf{P}_0 \left(T_{\gamma(2s_n)} > s_n^u \right)$ and from the Borel Cantelli lemma. \blacksquare

3.4 Estimates for bad points.

Recall from (III.36) the definition of the excursion measure \mathbf{N}_0 of the ψ -Lévy snake W . Recall that the lifetime process of W is the height process H . Namely, \mathbf{N}_0 -a.e. for all $t \in \mathbb{R}_+$, $\zeta_{W_t} = H_t$. Therefore, the duration σ of W under \mathbf{N}_0 is the duration of the excursion of H . Recall that \widehat{W} is the endpoint process of the snake. Recall from (III.38) that \mathcal{M} stands for the occupation measure of the snake, namely the random measure on \mathbb{R}^d that is the image via \widehat{W} of the Lebesgue measure on $[0, \sigma]$. Recall from (III.37) the definition of \mathcal{R} , the range of the snake.

Let $\lambda, r \in (0, \infty)$. Note that $\langle \mathcal{M} \rangle = \sigma$, therefore we get

$$\mathbf{N}_0 [1 - e^{-\lambda \mathcal{M}(B(0,r))}] \leq \mathbf{N}_0 [1 - e^{-\lambda \sigma}] = N [1 - e^{-\lambda \sigma}] = \psi^{-1}(\lambda), \quad (\text{III.83})$$

where the last two equalities come from (III.25) and (III.68). The next lemma states a lower bound of the same kind.

Lemma 3.14 *There exists $c_{17} \in (0, 1)$, that only on depends d , such that*

$$\forall \mu, r \in (0, \infty), \quad \frac{r^2 \mu}{\psi^{-1}(\mu)} \geq 1 \implies \mathbf{N}_0 [1 - e^{-\mu \mathcal{M}(B(0,r))}] \geq c_{17} \psi^{-1}(\mu). \quad (\text{III.84})$$

Proof. To simplify, set $q(\mu, r) := \mathbf{N}_0 [1 - e^{-\mu \mathcal{M}(B(0,r))}]$. Note that $\mathcal{M}(B(0,r)) = \int_0^\sigma dt \mathbf{1}_{\{\|\widehat{W}_t\| < r\}}$. An easy argument combined with Fubini first entails the following:

$$q(\mu, r) = \mu \int_0^\infty dt \mathbf{N}_0 \left[\mathbf{1}_{\{t \leq \sigma; \|\widehat{W}_t\| < r\}} e^{-\mu \int_t^\sigma ds \mathbf{1}_{\{\|\widehat{W}_s\| < r\}}} \right]. \quad (\text{III.85})$$

We next apply Lemma 2.8 at the (deterministic) time t (recall that this lemma is a specific form of the Markov property for $\overline{W} = (\rho, W)$). Then we get for any $t \in \mathbb{R}_+$:

$$\begin{aligned} & \mathbf{N}_0 \left[\mathbf{1}_{\{t \leq \sigma; \|\widehat{W}_t\| < r\}} e^{-\mu \int_t^\sigma ds \mathbf{1}_{\{\|\widehat{W}_s\| < r\}}} \right] \\ &= \mathbf{N}_0 \left[\mathbf{1}_{\{t \leq \sigma; \|\widehat{W}_t\| < r\}} \exp \left(- \int_{[0, H_t]} \rho_t(dh) \mathbf{N}_{W_t(h)} [1 - e^{-\mu \mathcal{M}(B(0,r))}] \right) \right]. \end{aligned} \quad (\text{III.86})$$

From (III.83), we get that for all $t, h \geq 0$, $\mathbf{N}_{W_t(h)} [1 - e^{-\mu \mathcal{M}(B(0,r))}] \leq \psi^{-1}(\mu)$. Then,

$$\begin{aligned} & \mathbf{N}_0 \left[\mathbf{1}_{\{t \leq \sigma; \|\widehat{W}_t\| < r\}} \exp \left(- \int_{[0, H_t]} \rho_t(dh) \mathbf{N}_{W_t(h)} [1 - e^{-\mu \mathcal{M}(B(0,r))}] \right) \right] \\ & \geq \mathbf{N}_0 \left[\mathbf{1}_{\{t \leq \sigma; \|\widehat{W}_t\| < r\}} e^{-\psi^{-1}(\mu) \int_{[0, H_t]} \rho_t(dh)} \right] \end{aligned} \quad (\text{III.87})$$

Then by (III.85), (III.86), (III.87) and Fubini we get

$$q(\mu, r) \geq \mu \mathbf{N}_0 \left[\int_0^\sigma dt \mathbf{1}_{\{\|\widehat{W}_t\| < r\}} e^{-\psi^{-1}(\mu) \int_{[0, H_t]} \rho_t(dh)} \right]. \quad (\text{III.88})$$

We next apply (III.46) to the right member of the previous inequality: to that end, recall that $U = (U_a)_{a \geq 0}$ stands for a subordinator defined on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ that is independent of ξ and whose Laplace exponent is $\widetilde{\psi}^*(\lambda) = \psi(\lambda)/\lambda - \alpha$. Then, (III.46) to the right member of (III.88) entails the following:

$$\begin{aligned} \mu \mathbf{N}_0 \left[\int_0^\sigma dt \mathbf{1}_{\{\|\widehat{W}_t\| < r\}} e^{-\psi^{-1}(\mu) \int_{[0, H_t]} \rho_t(dh)} \right] &= \mu \int_0^\infty da e^{-\alpha a} \mathbf{E}_0 \left[\mathbf{1}_{\{\|\xi_a\| < r\}} e^{-\psi^{-1}(\mu) U_a} \right] \\ &= \mu \int_0^\infty da e^{-a\mu/\psi^{-1}(\mu)} \mathbf{P}_0(\|\xi_a\| < r), \end{aligned}$$

since ξ and U are independent. Thus, (III.88) and a simple change of variable using the scaling property of Brownian motion, entail

$$q(\mu, r) \geq \psi^{-1}(\mu) \int_0^\infty \mathbf{P}_0 \left(\|\xi_c\| \leq r \sqrt{\mu/\psi^{-1}(\mu)} \right) e^{-c} dc. \quad (\text{III.89})$$

If $r^2\mu/\psi^{-1}(\mu) \geq 1$, then (III.89) implies (III.84) with $c_{17} := \int_0^\infty \mathbf{P}_0(\|\xi_c\| \leq 1) e^{-c} dc$. Clearly, c_{17} only depends on d and $c_{17} \in (0, 1)$. \blacksquare

Before stating the next lemma, we recall a result about the first exit time from a ball for a d -dimensional Brownian motion. First set

$$\chi_{d,r} := \inf \{t \in \mathbb{R}_+ : \|\xi_t\| = r\}. \quad (\text{III.90})$$

In dimension 1, one get

$$\forall r, \lambda \in \mathbb{R}_+, \quad \mathbf{E}_0[\exp(-\lambda \chi_{1,r})] = \left(\cosh(r\sqrt{2\lambda}) \right)^{-1} \leq 2 \exp(-r\sqrt{2\lambda}). \quad (\text{III.91})$$

Indeed note that $t \mapsto \exp(\sqrt{2\lambda}\xi_t - \lambda t)$ is a martingale (see e.g. Revuz and Yor [105], Chapter II, (3.7)). In dimension d , observe that

$$\mathbf{P}_0\text{-a.s.} \quad \chi_{d,r} \geq \min_{1 \leq j \leq d} \inf \{t \in \mathbb{R}_+ : \sqrt{d} |\xi_t^{(j)}| = r\},$$

where $\xi_t^{(j)}$ stands for the j -th coordinate of ξ_t . The previous inequality combined with (III.91) then entails

$$\forall \lambda, r \in \mathbb{R}_+, \quad \mathbf{E}_0[\exp(-\lambda \chi_{d,r}(\xi))] \leq 2d \exp(-r\sqrt{2\lambda/d}). \quad (\text{III.92})$$

Lemma 3.15 *Assume that $\delta > 1$ and that $d \geq 3$. Recall that r_1 and c_4 are the constants appearing in Lemma 3.3. Then, there exist $c_{18}, c_{19} \in (0, \infty)$, that only depend on d and ψ , such that*

$$\begin{aligned} \forall \mu \in (0, \infty), \forall r \in (0, r_1) \text{ such that } \frac{r^2\mu}{\psi^{-1}(\mu)} \geq 16, \quad \forall x \in \mathbb{R}^d \setminus \overline{B}(0, 2r), \\ \mathbf{N}_x \left[\mathbf{1}_{\{\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset\}} e^{-\mu \mathcal{M}(B(0, 2r))} \right] \leq c_{18} (r/\|x\|)^{d-2} \psi'^{-1}(c_4 r^{-2}) \exp(-c_{19} r \sqrt{\varphi(\mu)}). \end{aligned}$$

Proof. We fix $\mu \in (0, \infty)$ and $x \in \mathbb{R}^d \setminus \overline{B}(0, 2r)$. To simplify notation we set

$$p(x, \mu, r) = \mathbf{N}_x \left[\mathbf{1}_{\{\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset\}} e^{-\mu \mathcal{M}(B(0, 2r))} \right].$$

Recall from (III.40) the definition of τ_r that is the hitting time in $\overline{B}(0, r)$ of the snake W . First recall that $\{\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset\} = \{\tau_r < \infty\}$ and observe that

$$\mathbf{N}_x\text{-a.e. on the event } \{\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset\}, \quad \int_{\tau_r}^{\sigma} \mathbf{1}_{\{\|\widehat{W}_s\| < 2r\}} ds \leq \mathcal{M}(B(0, 2r)).$$

Thus, we get

$$p(x, \mu, r) \leq \mathbf{N}_x \left[\mathbf{1}_{\{\tau_r < \infty\}} e^{-\mu \int_{\tau_r}^{\sigma} \mathbf{1}_{\{\|\widehat{W}_s\| < 2r\}} ds} \right]. \quad (\text{III.93})$$

We next apply Proposition 2.8 to the right member of (III.93) and to the stopping time τ_r ; thus we get we get

$$p(x, \mu, r) \leq \mathbf{N}_x \left[\mathbf{1}_{\{\tau_r < \infty\}} \exp \left(- \int_{[0, H_{\tau_r}]} \rho_{\tau_r}(dh) \mathbf{N}_{W_{\tau_r}(h)} [1 - e^{-\mu \mathcal{M}(B(0, 2r))}] \right) \right]. \quad (\text{III.94})$$

Let $w \in \mathcal{W}$ be a continuous stopped path starting from $x \in \overline{B}(0, 2r)$; we denote by ζ_w its lifetime. We define $T_1(w)$, $T_2(w)$ and $T_3(w)$ by

$$\begin{aligned} T_1(w) &= \inf \{t \in [0, \zeta_w] : \|w(t)\| \leq 3r/2\} \\ T_2(w) &= \inf \{t \in [0, \zeta_w - T_1(w)] : \|w(t + T_1(w)) - w(T_1(w))\| > r/4\} \\ T_3(w) &= \inf \{t \in [0, \zeta_w] : \|w(t)\| \leq 5r/4\} \end{aligned} \quad (\text{III.95})$$

with the convention that $\inf \emptyset = \infty$. Observe that if $T_3(w) < \infty$, then $T_1(w) + T_2(w) \leq T_3(w)$. Moreover, since $x \in \overline{B}(0, 2r)^c$, \mathbf{N}_x -a.e. on the event $\{\tau_r < \infty\}$, we have $T_1(W_{\tau_r}) + T_2(W_{\tau_r}) \leq T_3(W_{\tau_r}) < \tau_r < \infty$ and for any $t \in [T_1(W_{\tau_r}), T_1(W_{\tau_r}) + T_2(W_{\tau_r})]$, the following inequality holds true:

$$\mathbf{N}_{W_{\tau_r}(t)} [1 - e^{-\mu \mathcal{M}(B(0, 2r))}] \geq \mathbf{N}_{W_{\tau_r}(t)} [1 - e^{-\mu \mathcal{M}(B(W_{\tau_r}(t), r/4))}] = \mathbf{N}_0 [1 - e^{-\mu \mathcal{M}(B(0, r/4))}] =: \Lambda_{\mu, r},$$

the last equality being a consequence of the invariance of the snake under translation; here $\Lambda_{\mu, r}$ only depends on μ and r . To simplify notation we set $T_1 = T_1(W_{\tau_r})$ and $T_2 = T_2(W_{\tau_r})$. An elementary inequality combined with (III.94) entails

$$\begin{aligned} p(x, \mu, r) &\leq \mathbf{N}_x \left[\mathbf{1}_{\{\tau_r < \infty\}} \exp \left(- \int_{[T_1, T_1 + T_2]} \rho_{\tau_r}(dh) \mathbf{N}_{W_{\tau_r}(h)} [1 - e^{-\mu \mathcal{M}(B(0, 2r))}] \right) \right] \\ &\leq \mathbf{N}_x \left[\mathbf{1}_{\{\tau_r < \infty\}} \exp \left(- \Lambda_{\mu, r} \int_{[T_1, T_1 + T_2]} \rho_{\tau_r}(dh) \right) \right]. \end{aligned} \quad (\text{III.96})$$

Recall from (III.95) the definition of T_3 . Recall from (III.44) the definition of the function ϖ . We next apply Proposition 2.9 with $r' = \frac{5}{4}r$ to the right member of (III.96). Then we get

$$\begin{aligned} &\mathbf{N}_x \left[\mathbf{1}_{\{\tau_r < \infty\}} \exp \left(- \Lambda_{\mu, r} \int_{[T_1, T_1 + T_2]} \rho_{\tau_r}(dh) \right) \right] \\ &= \tilde{u}_r(5r/4) \mathbf{E}_x \left[\mathbf{1}_{\{T_3(\xi) < \infty\}} \exp \left(- \int_0^{T_3(\xi)} dt \varpi(u_r(\xi_t), \Lambda_{\mu, r} \mathbf{1}_{[T_1(\xi), T_1(\xi) + T_2(\xi)]}(t)) \right) \right] \\ &\leq \tilde{u}_r(5r/4) \mathbf{E}_x \left[\mathbf{1}_{\{T_1(\xi) < \infty\}} \exp \left(- \int_{T_1(\xi)}^{T_1(\xi) + T_2(\xi)} dt \varpi(u_r(\xi_t), \Lambda_{\mu, r}) \right) \right]. \end{aligned} \quad (\text{III.97})$$

Here recall that ξ under \mathbf{P}_x is distributed as a standard d -dimensional Brownian motion starting from x . The convexity of ψ provides the following lower bounds for $\varpi(u_r(\xi_t), \Lambda_{\mu, r})$:

- if $u_r(\xi_t) \geq \Lambda_{\mu,r}$, then $\varphi(u_r(\xi_t), \Lambda_{\mu,r}) \geq \psi'(\Lambda_{\mu,r})$;
- if $u_r(\xi_t) < \Lambda_{\mu,r}$, then $\varpi(u_r(\xi_t), \Lambda_{\mu,r}) \geq \psi(\Lambda_{\mu,r})/\Lambda_{\mu,r} \geq \frac{1}{4}\psi'(\Lambda_{\mu,r})$, by (III.20) for the last inequality.

These inequalities combined with (III.96) and (III.97), entail

$$p(x, \mu, r) \leq \tilde{u}_r(5r/4) \mathbf{E}_x \left[\mathbf{1}_{\{T_1(\xi) < \infty\}} \exp\left(-\frac{1}{4}\psi'(\Lambda_{\mu,r})T_2(\xi)\right) \right]. \quad (\text{III.98})$$

We now assume that

$$\frac{r^2\mu}{16\psi^{-1}(\mu)} \geq 1.$$

Recall that $\Lambda_{\mu,r} = \mathbf{N}_0[1 - e^{-\mu\mathcal{M}(B(0,r/4))}]$. By Lemma 3.14, $\Lambda_{\mu,r} \geq c_{17}\psi^{-1}(\mu)$. We next use the concavity of ψ' and the fact that $c_{17} \in (0, 1)$, to get

$$\psi'(\Lambda_{\mu,r}) \geq \psi'(c_{17}\psi^{-1}(\mu)) \geq c_{17}\psi'(\psi^{-1}(\mu)) = c_{17}\varphi(\mu). \quad (\text{III.99})$$

Recall that r_1 and c_4 are the constants appearing in Lemma 3.3. We assume that $r \in (0, r_1)$. Then, Lemma 3.3 with $\varrho = 1/4$ implies that $\tilde{u}_r(5r/4) \leq \psi'^{-1}(c_4r^{-2})$. Thus, by (III.98) and (III.99) we get

$$p(x, \mu, r) \leq \psi'^{-1}(c_4r^{-2}) \mathbf{E}_x \left[\mathbf{1}_{\{T_1(\xi) < \infty\}} \exp\left(-\frac{1}{4}c_{17}\varphi(\mu)T_2(\xi)\right) \right]. \quad (\text{III.100})$$

Recall from (III.95) the definition of $T_1(\xi)$ and $T_2(\xi)$. By the Markov property at time $T_1(\xi)$, we get

$$\mathbf{E}_x \left[\mathbf{1}_{\{T_1(\xi) < \infty\}} \exp\left(-\frac{1}{4}c_{17}\varphi(\mu)T_2(\xi)\right) \right] = \mathbf{P}_x(T_1(\xi) < \infty) \mathbf{E}_0 \left[\exp\left(-\frac{1}{4}c_{17}\varphi(\mu)\chi_{d,r/4}(\xi)\right) \right].$$

Since $d \geq 3$, we get $\mathbf{P}_x(T_1(\xi) < \infty) = (3r/2\|x\|)^{d-2}$. Moreover by (III.92), we get

$$\mathbf{E}_0 \left[\exp\left(-\frac{1}{4}c_{17}\varphi(\mu)\chi_{d,r/4}(\xi)\right) \right] \leq 2d \exp\left(-\frac{1}{8}r\sqrt{2c_{17}\varphi(\mu)/d}\right).$$

Then we set $c_{18} = 2d(3/2)^{d-2}$ and $c_{19} = \sqrt{c_{17}/(32d)}$ and we get

$$\mathbf{E}_x \left[\mathbf{1}_{\{T_1(\xi) < \infty\}} \exp\left(-\frac{1}{4}c_{17}\varphi(\mu)T_2(\xi)\right) \right] \leq c_{18}(r/\|x\|)^{d-2} \exp\left(-c_{19}r\sqrt{\varphi(\mu)}\right),$$

which implies the desired result by (III.100). ■

Recall from (III.11) the definition of the gauge function g :

$$g(r) = \frac{\log \log \frac{1}{r}}{\varphi^{-1}\left(\left(\frac{1}{r} \log \log \frac{1}{r}\right)^2\right)}.$$

Lemma 3.16 *Assume they $\delta > 1$ and that $d \geq 3$. Recall that c_{18} is the constant appearing in Lemma 3.15 and that c_4 is the constant appearing in Lemma 3.3. There exists $c_{20}, c_{21}, r_4, \kappa_0 \in (0, \infty)$, that only depend d and ψ , such that for all $r \in (0, r_4)$, for all $\kappa \in (0, \kappa_0)$ and for all $x \in \overline{B}(0, 2r)^c$,*

$$\begin{aligned} \mathbf{N}_x(\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset; \mathcal{M}(B(0, 2r))) &\leq \kappa g(r) \\ &\leq c_{18} \psi'^{-1}(c_4r^{-2}) (r/\|x\|)^{d-2} (\log 1/r)^{-c_{20} \kappa^{-c_{21}}}. \end{aligned}$$

Proof. Let $\kappa, \mu \in (0, \infty)$. Let $r \in (0, r_1)$ where r_1 is the constant appearing in Lemma 3.3. Assume that such that $\frac{r^2 \mu}{\psi^{-1}(\mu)} \geq 16$. A Markov inequality, combined with Lemma 3.15 entails :

$$\begin{aligned} \mathbf{N}_x \left(\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset; \mathcal{M}(B(0, 2r)) \leq \kappa g(r) \right) &\leq e^{\kappa \mu g(r)} \mathbf{N}_x \left[\mathbf{1}_{\{\mathcal{R} \cap \overline{B}(0, r) \neq \emptyset\}} e^{-\mu \mathcal{M}(B(0, 2r))} \right] \\ &\leq c_{18} (r/\|x\|)^{d-2} \psi'^{-1}(c_4 r^{-2}) \exp(F(\mu, r, \kappa)), \end{aligned} \quad (\text{III.101})$$

where we have set

$$F(\mu, r, \kappa) := \kappa \mu g(r) - c_{19} r \sqrt{\varphi(\mu)}.$$

For any $q \in [1, \infty)$ and any $r \in (0, r_0)$, we set

$$\mu_{r,q} = \varphi^{-1} \left(q \left(\frac{1}{r} \log \log \frac{1}{r} \right)^2 \right).$$

We first get an estimate for $\mu_{q,r}$. To that end fix a such that $1/a \in (0, \delta_\varphi)$. By the definition (III.22) of δ_φ , there exists $C \in (0, \infty)$ such that for any $p, \lambda \in [1, \infty)$, $C p^{1/a} \varphi(\lambda) \leq \varphi(p\lambda)$. Thus, for any $z \geq \varphi(1)$ and any $q \geq C$, we get $\varphi^{-1}(qz) \leq (q/C)^a \varphi^{-1}(z)$. This easily entails that there exists $r_5 \in (0, r_0 \wedge r_1)$ and $c_{22} \in (0, \infty)$ such that

$$\forall r \in (0, r_5), \forall q \in [1, \infty), \quad \mu_{r,1} \leq \mu_{r,q} \leq c_{22} q^a \mu_{r,1}. \quad (\text{III.102})$$

Recall that $\tilde{\psi}(\lambda) = \psi(\lambda)/\lambda$. We next observe that for all $r \in (0, e^{-1})$ and all $q \in [1, \infty)$, (III.20) implies

$$r^2 \frac{\mu_{r,q}}{\psi^{-1}(\mu_{r,q})} = r^2 \tilde{\psi}(\psi^{-1}(\mu_{r,q})) \geq \frac{1}{4} r^2 \psi'(\psi^{-1}(\mu_{r,q})) \geq \frac{1}{4} r^2 \psi'(\psi^{-1}(\mu_{r,1})) = \frac{1}{4} (\log \log 1/r)^2.$$

Then we set $r_4 = \exp(-e^8) \wedge r_5$ and we get

$$\forall r \in (0, r_4), \forall q \in [1, \infty), \quad \frac{r^2 \mu_{r,q}}{\psi^{-1}(\mu_{r,q})} \geq 16. \quad (\text{III.103})$$

Let $r \in (0, r_4)$, $q \in [1, \infty)$ and $\kappa \in (0, \infty)$. Observe that $g(r) = (\log \log 1/r)/\mu_{r,1}$ and that $r \sqrt{\varphi(\mu_{r,q})} = \sqrt{q} \log \log 1/r$. Thus, we get

$$F(\mu_{r,q}, r, \kappa) = \left(\kappa \frac{\mu_{r,q}}{\mu_{r,1}} - c_{19} \sqrt{q} \right) \log \log \frac{1}{r} \leq \sqrt{q} (c_{22} \kappa q^{a-\frac{1}{2}} - c_{19}) \log \log \frac{1}{r}.$$

Since $\delta_\varphi \leq 1$, we get $a > 1$ and thus $a - \frac{1}{2} > \frac{1}{2}$. We then set $\kappa_0 = c_{19}/(2c_{22})$ and for all $\kappa \in (0, \kappa_0)$ we also set $q_\kappa = (\kappa_0/\kappa)^{\frac{1}{a-\frac{1}{2}}}$. Then $q_\kappa \geq 1$ and

$$\sqrt{q_\kappa} (c_{22} \kappa q_\kappa^{a-\frac{1}{2}} - c_{19}) = -\frac{1}{2} c_{19} (\kappa_0/\kappa)^{\frac{1}{2a-1}}.$$

We then set $c_{20} = \frac{1}{2} c_{19} \kappa_0^{\frac{1}{2a-1}}$ and $c_{21} = \frac{1}{2a-1}$. Then

$$F(\mu_{r,q_\kappa}, r, \kappa) \leq -c_{20} \kappa^{-c_{21}} \log \log \frac{1}{r}$$

and we complete the proof thanks to (III.103) that allows to apply (III.101) for any $r \in (0, r_4)$. \blacksquare

4 Proof of the results.

Recall from (III.38) the definition of the total occupation measure of the snake \mathcal{M} . We prove in Section 4.1 the following results on the lower density of \mathcal{M} .

Theorem 4.1 *Let ψ be a branching mechanism the form (III.1). Let $(W_t)_{t \geq 0}$ be the associated snake. Let g be defined by (III.11). Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then, there exists a constant $\kappa_{d,\psi} \in (0, \infty)$, that only depends on d and ψ , such that*

$$\mathbf{N}_0\text{-a.e. for } \mathcal{M}\text{-almost all } x, \quad \liminf_{r \rightarrow 0_+} \frac{\mathcal{M}(B(x, r))}{g(r)} = \kappa_{d,\psi}. \quad (\text{III.104})$$

This result is then used to prove the following theorem in Section 4.2.

Theorem 4.2 *Let ψ be a branching mechanism the form (III.1). Let $(W_t)_{t \geq 0}$ be the associated snake. Let g be defined by (III.11). Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then, there exists a constant $\kappa_{d,\psi} \in (0, \infty)$, that only depends on d and ψ , such that*

$$\mathbf{N}_0\text{-a.e. for any Borel set } B, \quad \mathcal{M}(B) = \kappa_{d,\psi} \mathcal{P}_g(B \cap \mathcal{R}).$$

4.1 Proof of theorem 4.1.

Recall from Section 2.4, the Palm formula for the occupation measure of the snake. To that end, recall that $\xi = (\xi_t)_{t \geq 0}$ is a continuous process defined on the auxiliary measurable space (Ω, \mathcal{F}) and recall that \mathbf{P}_0 is a probability measure on (Ω, \mathcal{F}) under which ξ is distributed as a standard d -dimensional Brownian motion starting from the origin 0. Recall that $(V_t)_{t \geq 0}$ be a subordinator defined on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ that is independent of ξ and whose Laplace exponent is $\psi^{*\prime}(\lambda) = \psi'(\lambda) - \alpha$. Recall from (III.47) that under \mathbf{P}_0 , conditionally given (ξ, V) , $\mathcal{N}^*(dtdW) = \sum_{j \in \mathcal{J}^*} \delta_{(t_j, W^j)}$ is a Poisson point process on $[0, \infty) \times C(\mathbb{R}_+, \mathcal{W})$ with intensity $dV_t \mathbf{N}_{\xi_t}(dW)$. Recall from (III.48) that for all $a \in \mathbb{R}_+$, we have set $\mathcal{M}_a^* = \sum_{j \in \mathcal{J}^*} \mathbf{1}_{[0, a]}(t_j) \mathcal{M}_j$ where for all $j \in \mathcal{J}^*$, \mathcal{M}_j stands for the occupation measure of the snake W^j as defined in (III.38). Also recall from (III.71) the definition of the following random variables:

$$\forall t \geq s \geq 0 \quad N_r(s, t) = \# \left\{ j \in \mathcal{J}^* : s < t_j < t \text{ and } \mathcal{R}_j \cap \overline{B}(0, r) \neq \emptyset \right\},$$

that counts the snakes that are grafted on the spatial spine ξ between times s and t , and that hit the ball $\overline{B}(0, r)$. We first prove the following lemma that is a consequence of the Blumenthal 0-1 law.

Lemma 4.3 *Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Recall from (III.11) the definition of the gauge function g . There exists a constant $\kappa_{d,\psi} \in [0, \infty]$ that only depend on d and ψ , such that*

$$\forall a \in (0, \infty), \quad \mathbf{P}_0\text{-a.s.} \quad \liminf_{r \rightarrow 0_+} \frac{\mathcal{M}_a^*(B(0, r))}{g(r)} = \kappa_{d,\psi}. \quad (\text{III.105})$$

Proof. Let $a \in (0, \infty)$. Let $s \in (0, a)$. Observe that if $N_r(s, a) = 0$, then $\mathcal{M}_a^*(B(0, r)) = \mathcal{M}_s^*(B(0, r))$. By Lemma 3.9, there exists a deterministic sequence $(r_n)_{n \geq 0}$ decreasing to 0 such that

$$\sum_{n \geq 0} \mathbf{P}_0(N_{r_n}(s, a) \neq 0) < \infty.$$

By the Borel Cantelli Lemma, \mathbf{P}_0 -a.s. for all sufficiently large n , $N_{r_n}(s, a) = 0$. Since $r \mapsto N_r(s, a)$ is non-decreasing, we get that \mathbf{P}_0 -a.s. for all sufficiently small r , $N_r(s, a) = 0$. Consequently,

$$\forall s \in (0, a), \quad \mathbf{P}_0\text{-a.s.} \quad \liminf_{r \rightarrow 0^+} \frac{\mathcal{M}_a^*(B(0, r))}{g(r)} = \liminf_{r \rightarrow 0^+} \frac{\mathcal{M}_s^*(B(0, r))}{g(r)}. \quad (\text{III.106})$$

Let \mathcal{G}_s be the sigma-field generated by $\mathbf{1}_{[0, s]}(t) \mathcal{N}^*(dt dW)$ and completed by the \mathbf{P}_0 -negligible sets. Using properties of Poisson random measures and the Blumenthal zero-one law for ξ , we easily check that $\mathcal{G}_{0^+} := \bigcap_{s > 0} \mathcal{G}_s$ is \mathbf{P}_0 -trivial: namely, for all $A \in \mathcal{G}_{0^+}$, either $\mathbf{P}_0(A) = 0$ or $\mathbf{P}_0(A) = 1$. Then observe that (III.106) implies that the random variable $\liminf_{r \rightarrow 0} \mathcal{M}_a^*(B(0, r))/g(r)$ is \mathcal{G}_{0^+} -measurable. It is therefore \mathbf{P}_0 -a.s. equal to a deterministic constant $\kappa_{d, \psi} \in [0, \infty]$ that does not depend on a . \blacksquare

Remark 4.1 We point out that Lemma 4.3 holds true for all gauge function g . \square

By (III.50), the previous lemma entails that

$$\mathbf{N}_0\text{-a.e. for } \mathcal{M}\text{-almost all } x, \quad \liminf_{r \rightarrow 0^+} \frac{\mathcal{M}(B(x, r))}{g(r)} = \kappa_{d, \psi} \in [0, \infty]. \quad (\text{III.107})$$

Now, we need to prove that $0 < \kappa_{d, \psi} < \infty$, which is done in two steps.

Lemma 4.4 *Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then, $\kappa_{d, \psi} < \infty$.*

Proof. Fix $a \in (0, \infty)$ and recall from (III.67) the definition of T_a that is the sum of the durations of the snakes that are grafted on the spine $(\xi_s)_{0 \leq s \leq a}$. Recall from (III.71) the definition of $N_r(s, t)$. Observe that if $\vartheta(2r) \geq a$, then $\mathcal{M}_a^*(B(0, r)) \leq T_a \leq T_{\vartheta(2r)}$. Next note that if $\vartheta(2r) < a$ and if $N_r(\vartheta(2r), a + \vartheta(2r)) = 0$, then, $\mathcal{M}_a^*(B(0, r)) \leq T_{\vartheta(2r)}$. Therefore,

$$\mathbf{P}_0\text{-a.s. on } \{N_r(\vartheta(2r), a + \vartheta(2r)) = 0\}, \quad \mathcal{M}_a^*(B(0, r)) \leq T_{\vartheta(2r)}.$$

Then, for all $r > r' \geq 0$, and all $A > 0$,

$$\mathbf{P}_0\text{-a.s. on } \{N_r(\vartheta(2r), a + \vartheta(2r)) = 0\} \cap \{T_{\vartheta(2r)} - T_{\vartheta(2r')} \leq A\}, \quad \mathcal{M}_a^*(B(0, r)) \leq A + T_{\vartheta(2r')}. \quad (\text{III.108})$$

Recall from (III.69) the definition of $\gamma(r)$ and $\vartheta(r)$ and observe that $\vartheta(r) \leq \gamma(r)$, which implies

$$\mathbf{P}_0\text{-a.s.} \quad \forall r \in (0, \infty), \quad T_{\vartheta(r)} \leq T_{\gamma(r)}.$$

Thus, by (III.108), for all $r > r' \geq 0$, and all $A > 0$,

$$\mathbf{P}_0\text{-a.s. on } \{N_r(\vartheta(2r), a + \vartheta(2r)) = 0\} \cap \{T_{\vartheta(2r)} - T_{\vartheta(2r')} \leq A\}, \quad \mathcal{M}_a^*(B(0, r)) \leq A + T_{\gamma(2r')}, \quad (\text{III.109})$$

which allows us to bound $\mathcal{M}_a^*(B(0, r))$ by the subordinator $T_{\gamma(\cdot)}$ studied in section 3.3.

Recall Lemma 3.7, where the sequence $(\rho_n)_{n \geq 1}$ is introduced; recall Lemma 3.11 and Lemma 3.12 that provide a control respectively on $\mathbf{P}_0(N_{\rho_n}(\vartheta(2\rho_n), a + \vartheta(2\rho_n)) = 0)$ and on $T_{\gamma(2\rho_n)}$. For any $n \geq 1$, we next introduce the following random variable:

$$Y_n = \mathbf{1}_{\{N_{\rho_n}(\vartheta(2\rho_n), a + \vartheta(2\rho_n)) = 0\} \cap \{T_{\vartheta(2\rho_n)} - T_{\vartheta(2\rho_{n+1})} \leq g(8\rho_n)\}}. \quad (\text{III.110})$$

By (III.109), we get

$$\forall n \geq 1, \quad \mathbf{P}_0\text{-a.s. on } \{Y_n = 1\}, \quad \mathcal{M}_a^*(B(0, \rho_n)) \leq g(8\rho_n) + T_{\gamma(2\rho_{n+1})}.$$

We then define the following event:

$$E = \left\{ \sum_{n \geq 1} Y_n = \infty \right\}.$$

We claim that

$$\mathbf{P}_0(E) > 0. \quad (\text{III.111})$$

Then the proof of the lemma is completed as follows: the previous arguments first entail

$$\mathbf{P}_0\text{-a.s. on } E, \quad \liminf_{n \rightarrow \infty} \frac{\mathcal{M}_a^*(B(0, \rho_n))}{g(8\rho_n)} \leq 1 + \limsup_{n \rightarrow \infty} \frac{T_{\gamma(2\rho_{n+1})}}{g(8\rho_n)}. \quad (\text{III.112})$$

Recall that the assumption $\delta > 1$ implies that g satisfies a C -doubling condition (III.13), which entails $g(8r) \leq C^3 g(r)$. Then by (III.112),

$$\mathbf{P}_0\text{-a.s. on } E, \quad \kappa_{d,\psi} = \liminf_{r \rightarrow 0^+} \frac{\mathcal{M}_a^*(B(0, r))}{g(r)} \leq C^3 \left(1 + \limsup_{n \rightarrow \infty} \frac{T_{\gamma(2\rho_{n+1})}}{g(8\rho_n)} \right). \quad (\text{III.113})$$

By Lemma 3.12 (ii) the right member of (III.113) is \mathbf{P}_0 -a.s. finite, which completes the proof of the lemma. It only remain to prove our claim (III.111).

Proof of (III.111). We use a second moment method. By the independence property of Lemma 3.8 (i), we first get

$$\mathbf{E}_0[Y_n] = \mathbf{P}_0(N_{\rho_n}(\vartheta(2\rho_n), a + \vartheta(2\rho_n)) = 0) \mathbf{P}_0(T_{\vartheta(2\rho_n)} - T_{\vartheta(2\rho_{n+1})} \leq g(8\rho_n)).$$

Then, the lower bound of Lemma 3.11 and the fact that $T_{\vartheta(r)} \leq T_{\gamma(r)}$ entail

$$\begin{aligned} \mathbf{E}_0[Y_n] &\geq c_{16} \mathbf{P}_0(T_{\vartheta(2\rho_n)} - T_{\vartheta(2\rho_{n+1})} \leq g(8\rho_n)) \\ &\geq c_{16} \mathbf{P}_0(T_{\vartheta(2\rho_n)} \leq g(8\rho_n)) \\ &\geq c_{16} \mathbf{P}_0(T_{\gamma(2\rho_n)} \leq g(8\rho_n)). \end{aligned} \quad (\text{III.114})$$

So by Lemma 3.12 (i), we get

$$\sum_{n \geq 1} \mathbf{E}_0[Y_n] = \infty. \quad (\text{III.115})$$

Besides, for $n > m \geq 1$, by the independence property of Lemma 3.8 (i)

$$\begin{aligned} \mathbf{E}_0[Y_n Y_m] &\leq \mathbf{P}_0(T_{\vartheta(2\rho_n)} - T_{\vartheta(2\rho_{n+1})} \leq g(8\rho_n); T_{\vartheta(2\rho_m)} - T_{\vartheta(2\rho_{m+1})} \leq g(8\rho_m); N_{\rho_m}(\vartheta(2\rho_m), a + \vartheta(2\rho_m)) = 0) \\ &\leq \mathbf{P}_0(T_{\vartheta(2\rho_n)} - T_{\vartheta(2\rho_{n+1})} \leq g(8\rho_n)) \mathbf{E}_0[Y_m] \\ &\leq \frac{1}{c_{16}} \mathbf{E}_0[Y_n] \mathbf{E}_0[Y_m], \end{aligned} \quad (\text{III.116})$$

where the last inequality follows from (III.114). Therefore, if we denote $L_n = \sum_{1 \leq k \leq n} Y_k$, then (III.115) and (III.116) entail

$$\limsup_{n \rightarrow \infty} \frac{\mathbf{E}[L_n^2]}{\mathbf{E}[L_n]^2} < \infty.$$

and the desired result (III.111) follows from the Kochen Stone Lemma. \blacksquare

The following lemma completes the proof of Theorem 4.1. Its proof relies on a density result on the ψ -Lévy tree that is proved in [38] and that has been recalled here as Lemma 2.5.

Lemma 4.5 *Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then $\kappa_{d,\psi} > 0$.*

Proof. We work with W under \mathbf{N}_0 . The proof consists in lifting to \widehat{W} the estimate of Lemma 2.5 by using the fact that conditionally given H , \widehat{W} is a Gaussian process (see (III.35) in Section 2.4). More precisely, recall that the height process H is the lifetime process of the snake W and recall from (III.36) the definition of \mathbf{N}_0 that shows that conditionally given H , the law of W is Q_0^H (see Section 2.4). Recall from (III.28) the following notation d for the pseudo-distance in the Lévy tree:

$$\forall s, t \in [0, \sigma], \quad d(s, t) = H_t + H_s - 2 \inf_{u \in [s \wedge t, s \vee t]} H_u.$$

Let $r, R \in (0, \infty)$ and let $t \in [0, \sigma]$. We set

$$\mathbf{a}(t, r) = \int_0^\sigma \mathbf{1}_{\{d_H(s,t) \leq r\}} ds \quad \text{and} \quad \mathbf{b}(t, r, R) = \int_0^\sigma \mathbf{1}_{\{d_H(s,t) \leq r\} \cap \{\|\widehat{W}_s - \widehat{W}_t\| \geq R\}} ds.$$

The quantity $\mathbf{a}(t, r)$ has been already introduced in Lemma 2.5. First note that

$$\forall t \in [0, \sigma], \quad \mathbf{a}(t, r) \leq \mathbf{b}(t, r, R) + \mathcal{M}(B(\widehat{W}_t, R)). \quad (\text{III.117})$$

By (III.35), we then get

$$N(dH)\text{-a.e. } \forall t \in [0, \sigma], \quad Q_0^H[\mathbf{b}(t, r, R)] \leq \mathbf{a}(t, r) \int_{\mathbb{R}^d \setminus B(0, R/\sqrt{r})} (2\pi)^{-d/2} e^{-\|x\|^2/2} dx. \quad (\text{III.118})$$

For any integer $n \geq 2$, we next set $R_n = 2^{-n}$ and $r_n = \frac{1}{4} R_n^2 (\log \log 1/R_n)^{-1}$. By elementary computation,

$$\forall n \geq 2, \quad \int_{\mathbb{R}^d \setminus B(0, R_n/\sqrt{r_n})} (2\pi)^{-d/2} e^{-\|x\|^2/2} dx \leq c_{23} n^{-3/2},$$

where $c_{23} \in (0, \infty)$ only depends on d (note that the power $3/2$ is not optimal). By (III.118), we get

$$N(dH)\text{-a.e. } \forall t \in [0, \sigma], \quad Q_0^H \left[\sum_{n \geq 2} \frac{\mathbf{b}(t, r_n, R_n)}{\mathbf{a}(t, r_n)} \right] < \infty.$$

Thus, $N(dH)$ -a.e. for all $t \in [0, \sigma]$, $Q_0^H(\limsup_{n \rightarrow \infty} \mathbf{b}(t, r_n, R_n)/\mathbf{a}(t, r_n) > 0) = 0$. Then, by Fubini,

$$N(dH)\text{-a.e.} \quad Q_0^H \left[\int_0^\sigma \mathbf{1}_{\left\{ \limsup_{n \rightarrow \infty} \frac{\mathbf{b}(t, r_n, R_n)}{\mathbf{a}(t, r_n)} > 0 \right\}} dt \right] = 0,$$

which implies that

$$\mathbf{N}_0\text{-a.e. for } \ell\text{-almost all } t \in [0, \sigma], \quad \lim_{n \rightarrow \infty} \frac{\mathbf{b}(t, r_n, R_n)}{\mathbf{a}(t, r_n)} = 0 \quad (\text{III.119})$$

(ℓ stands here for the Lebesgue measure on the real line). Recall from (III.30), the notation $k(r)$:

$$\forall r \in (0, \alpha \wedge e^{-e}), \quad k(r) := \frac{\log \log \frac{1}{r}}{\varphi^{-1}\left(\frac{1}{r} \log \log \frac{1}{r}\right)}.$$

Then, (III.119) combined with (III.117), entails

$$\mathbf{N}_0\text{-a.e. for } \ell \text{ almost all } t \in [0, \sigma], \quad \liminf_{n \rightarrow \infty} \frac{\mathbf{a}(t, r_n)}{k(r_n)} \leq \liminf_{n \rightarrow \infty} \frac{\mathcal{M}(B(\widehat{W}_t, R_n))}{k(r_n)}.$$

The definition of $\delta_\varphi \in (0, 1)$ easily implies that there exists a constant $c_{24} \in (0, \infty)$ such that for all n sufficiently large $c_{24}k(r_n) \geq g(R_n)$. Then, Lemma 2.5 entails that

$$\mathbf{N}_0\text{-a.e. for almost all } t \in [0, \sigma], \quad \liminf_{n \rightarrow \infty} \frac{\mathcal{M}(B(\widehat{W}_t, 2^{-n}))}{g(2^{-n})} \geq \frac{c_1}{c_{24}}.$$

An easy argument involving the doubling property (III.13) for g completes the proof thanks to (III.107). ■

4.2 Proof of Theorem 4.2.

We first introduce a specific decomposition of \mathbb{R}^d into dyadic cubes. We adopt the following notation: we denote by $\lfloor \cdot \rfloor$ the integer part application and we write \log_2 for the logarithm in base 2; we fix $d > \frac{2\gamma}{\gamma-1}$ and we set

$$p := \lfloor \log_2(4\sqrt{d}) \rfloor,$$

so that $2^p > 2\sqrt{d}$. To simplify notation, we set $\mathcal{D}_n = 2^{-n-p}\mathbb{Z}^d$, for any $n \geq 0$. For any $y = (y_1, \dots, y_d)$ in \mathcal{D}_n , we also set

$$D_n(y) = \prod_{j=1}^d [y_j - \frac{1}{2}2^{-n}; y_j + \frac{1}{2}2^{-n}) \quad \text{and} \quad D_n^\bullet(y) = \prod_{j=1}^d [y_j - \frac{1}{2}2^{-n-p}; y_j + \frac{1}{2}2^{-n-p}).$$

It is easy to check the following properties.

- Prop(1) If y, y' are distinct points in \mathcal{D}_n , then $D_n^\bullet(y) \cap D_n^\bullet(y') = \emptyset$.
- Prop(2) Let $y \in \mathcal{D}_n$. Then, we have

$$D_n^\bullet(y) \subset \overline{B}(y, \frac{1}{2}2^{-n-p}\sqrt{d}) \subset \overline{B}(y, 2^{-n-p}\sqrt{d}) \subset D_n(y).$$

For any $r < (2d)^{-1}$, we set $n(r) = \lfloor \log_2(r^{-1}(1+2^{-p})\sqrt{d}) \rfloor$, so that the following inequalities hold:

$$\frac{1}{2}(1+2^{-p})\sqrt{d}2^{-n(r)} < r \leq (1+2^{-p})\sqrt{d}2^{-n(r)}. \quad (\text{III.120})$$

Next, for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and for $j \in \{1, \dots, d\}$, we set

$$y_j = 2^{-n(r)-p} \lfloor x_j 2^{n(r)+p} + \frac{1}{2} \rfloor.$$

Therefore, $y = (y_1, \dots, y_d) \in \mathcal{D}_{n(r)}$ and we easily check the following:

- Prop(3) The point x belongs to $D_{n(r)}^\bullet(y)$ and $D_{n(r)}(y) \subset B(x, r)$.

We work under \mathbf{N}_0 . Recall Lemma 3.16: we fix

$$\kappa_1 \in (0, \kappa_0) \quad \text{such that} \quad c_{20}\kappa_1^{-c_{21}} > 2. \quad (\text{III.121})$$

Since we assume $\delta > 1$, g satisfies the doubling condition (III.13), which implies that there exists $\kappa_2 \in (0, \infty)$, that only depends on d, ψ and κ_1 , such that for all sufficiently large n ,

$$\kappa_2 g(2^{-n}) \leq \kappa_1 g(\frac{1}{2}2^{-n-p}\sqrt{d}). \quad (\text{III.122})$$

We then fix $A > 100$ and for any n such that $2^{-n} \leq 1/(2A)$, we set

$$U_n(A) = \sum_{\substack{y \in \mathcal{D}_n \\ 1/A \leq \|y\| \leq A}} g(\sqrt{d}(1 + 2^{-p})2^{-n}) \mathbf{1}_{\{\mathcal{M}(D_n(y)) \leq \kappa_2 g(2^{-n})\} \cap \{\mathcal{R} \cap D_n^\bullet(y) \neq \emptyset\}}.$$

We first prove the following lemma.

Lemma 4.6 *Assume that $\delta > 1$ (so that (III.122) holds) and that $d \geq 3$. Then, for all $A > 100$,*

$$\mathbf{N}_0\text{-a.e.} \quad \lim_{N \rightarrow \infty} \sum_{n \geq N} U_n(A) = 0. \quad (\text{III.123})$$

Proof. We fix n such that $2^{-n} \leq 1/(2A)$ and we fix $y \in \mathcal{D}_n$ such that $1/A \leq \|y\| \leq A$. By Prop(2) and (III.122), we get

$$\begin{aligned} \mathbf{N}_0(\mathcal{M}(D_n(y)) \leq \kappa_2 g(2^{-n}); \mathcal{R} \cap D_n^\bullet(y) \neq \emptyset) \\ \leq \mathbf{N}_0\left(\mathcal{M}(B(y, 2^{-n-p}\sqrt{d})) \leq \kappa_1 g(\tfrac{1}{2}2^{-n-p}\sqrt{d}); \mathcal{R} \cap \overline{B}(y, \tfrac{1}{2}2^{-n-p}\sqrt{d}) \neq \emptyset\right) \\ = \mathbf{N}_{-y}\left(\mathcal{M}(B(0, 2^{-n-p}\sqrt{d})) \leq \kappa_1 g(\tfrac{1}{2}2^{-n-p}\sqrt{d}); \mathcal{R} \cap \overline{B}(0, \tfrac{1}{2}2^{-n-p}\sqrt{d}) \neq \emptyset\right), \end{aligned}$$

the last equality being an immediate consequence of the invariance of the snake by translation. We next apply Lemma 3.16 with $x = -y$ and $r = \frac{1}{2}2^{-n-p}\sqrt{d}$ and $\kappa = \kappa_1$ (that satisfies (III.121); thus, there exists $c_{25}, c_{26} \in (0, \infty)$, that only depends on d and ψ , such that

$$\mathbf{N}_0(\mathcal{M}(D_n(y)) \leq \kappa_2 g(2^{-n}); \mathcal{R} \cap D_n^\bullet(y) \neq \emptyset) \leq c_{25}(2^{-n-p})^{d-2} \|y\|^{2-d} n^{-2} \psi'^{-1}(c_{26} 2^{2n+2p}).$$

By Lemma 2.3 and the doubling property (III.13) of g , there exists $c_{27} \in (0, \infty)$, that only depends on d and ψ , such that for all sufficiently large n ,

$$g(\sqrt{d}(1 + 2^{-p})2^{-n}) \psi'^{-1}(c_{26} 2^{2n+2p}) \leq c_{27} 2^{-2n-2p},$$

which entails the following:

$$g(\sqrt{d}(1 + 2^{-p})2^{-n}) \mathbf{N}_0(\mathcal{M}(D_n(y)) \leq \kappa_2 g(2^{-n}); \mathcal{R} \cap D_n^\bullet(y) \neq \emptyset) \leq c_{28} (2^{-n-p})^d \|y\|^{2-d} n^{-2}, \quad (\text{III.124})$$

where $c_{28} = c_{25} c_{27}$. Elementary arguments entail the following inequalities:

$$\begin{aligned} \mathbf{N}_0(U_n(A)) &\leq c_{28} n^{-2} \sum_{\substack{y \in \mathcal{D}_n \\ 1/A \leq \|y\| \leq A}} (2^{-n-p})^d \|y\|^{2-d} \\ &\leq c_{29} n^{-2} \int \mathbf{1}_{\{1/A \leq \|x\| \leq A\}} \|x\|^{2-d} dx \\ &\leq c_{30} n^{-2} \int_{1/A}^A \rho d\rho \\ &\leq c_{31} A^2 n^{-2}, \end{aligned}$$

where $c_{29}, c_{30}, c_{31} \in (0, \infty)$ only depend on d and ψ . Therefore, $\mathbf{N}_0(\sum_{n \geq N} U_n(A)) < \infty$, which easily implies the lemma. \blacksquare

We next prove the following lemma.

Lemma 4.7 *Assume that $\delta > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Then,*

$$\mathbf{N}_0\text{-a.e. } \mathcal{P}_g \left(\left\{ x \in \mathcal{R} : \liminf_{r \rightarrow 0^+} g(r)^{-1} \mathcal{M}(B(x, r)) \neq \kappa_{d, \psi} \right\} \right) = 0, \quad (\text{III.125})$$

where $\kappa_{d, \psi}$ is the constant appearing in Theorem 4.1.

Proof. We fix $A > 100$. By Theorem 4.1 and Lemma 4.6 there exists a Borel subset \mathcal{W}_A of \mathcal{W} such that $\mathbf{N}_0(\mathcal{W} \setminus \mathcal{W}_A) = 0$ and such that on \mathcal{W}_A , (III.104) and (III.123) hold true. We shall work deterministically on $W \in \mathcal{W}_A$.

Let B be any Borel subset of $\{x \in \mathbb{R}^d : 1/A \leq \|x\| \leq A\}$. Let $\varepsilon \in (0, \infty)$ and let $\overline{B}(x_1, r_1), \dots, \overline{B}(x_k, r_k)$ be any closed ε -packing of $B \cap \mathcal{R}$. Namely, the later balls are disjoint, their centres belong to $B \cap \mathcal{R}$ and their radii are smaller than ε . Let $c_{32} \in (0, \infty)$ be a constant to be specified later. First observe that

$$\begin{aligned} \sum_{i=1}^k g(r_i) &= \sum_{i=1}^k g(r_i) \mathbf{1}_{\{\mathcal{M}(B(x_i, r_i)) > c_{32} g(r_i)\}} + \sum_{i=1}^k g(r_i) \mathbf{1}_{\{\mathcal{M}(B(x_i, r_i)) \leq c_{32} g(r_i)\}} \\ &\leq c_{32}^{-1} \mathcal{M}(B(\varepsilon)) + \sum_{i=1}^k g(r_i) \mathbf{1}_{\{\mathcal{M}(B(x_i, r_i)) \leq c_{32} g(r_i)\}}, \end{aligned} \quad (\text{III.126})$$

where we have set $B(\varepsilon) = \{x \in \mathbb{R}^d : \text{dist}(x, B) \leq \varepsilon\}$. Next, fix $1 \leq i \leq k$; recall notation $n(r_i)$ from (III.120) and denote by y_i the point of $\mathcal{D}_{n(r_i)}$ corresponding to x_i such that Prop(3) holds true. Therefore, by (III.120), we have

$$\begin{aligned} \mathcal{M}(B(x_i, r_i)) &\leq c_{32} g(r_i) \quad \text{and} \quad x_i \in B \cap \mathcal{R} \implies \\ \mathcal{M}(D_{n(r_i)}(y_i)) &\leq c_{32} g((1 + 2^{-p})\sqrt{d}2^{-n(r_i)}) \quad \text{and} \quad \mathcal{R} \cap D_{n(r_i)}^\bullet(y_i) \neq \emptyset. \end{aligned}$$

We use the doubling condition (III.13) to choose c_{32} such that $c_{32} g((1 + 2^{-p})\sqrt{d}2^{-n(r)}) \leq \kappa_2 g(2^{-n(r)})$ for all sufficiently small $r \in (0, 1)$. Thus, we get

$$\sum_{i=1}^k g(r_i) \mathbf{1}_{\{\mathcal{M}(B(x_i, r_i)) \leq c_{32} g(r_i)\}} \leq \sum_{n: 2^{-n} \leq c_{33} \varepsilon} U_n(A),$$

where $c_{33} = 2((1 + 2^{-p})\sqrt{d})^{-1}$. Since W belongs to \mathcal{W}_A where (III.123) holds, this inequality combined with (III.126) implies the following.

$$\mathcal{P}_g(B \cap \mathcal{R}) \leq \mathcal{P}_g^*(B \cap \mathcal{R}) \leq c_{32}^{-1} \mathcal{M}\left(\bigcap_{\varepsilon > 0} B(\varepsilon)\right). \quad (\text{III.127})$$

We next apply (III.127) with $B = B_A$ given by

$$B_A = \left\{ x \in \mathcal{R} : 1/A \leq \|x\| \leq A \text{ and } \liminf_{r \rightarrow 0^+} g(r)^{-1} \mathcal{M}(B(x, r)) \neq \kappa_{d, \psi} \right\}.$$

Therefore $\mathcal{P}_g(B_A) < \infty$. Suppose now that $\mathcal{P}_g(B_A) > 0$. Then, as a consequence of (III.18), there exists a closed subset F , with $F \subset B_A$, such that $\mathcal{P}_g(F) > 0$. Since F is closed then $F = \bigcap_{\varepsilon > 0} F(\varepsilon)$; since F is a subset of B_A and since $W \in \mathcal{W}_A$ (where (III.104) holds true), we get $\mathcal{M}(F) \leq \mathcal{M}(B_A) = 0$ and by (III.127) applied to $B = F$, we obtain $\mathcal{P}_g(F) = 0$, which rises a contradiction. Thus, we have proved that $\mathbf{N}_0\text{-a.e. } \mathcal{P}_g(B_A) = 0$, which easily entails the lemma by letting A go to ∞ , since $\mathcal{P}_g(\{0\}) = 0$. \blacksquare

We now complete the proof of Theorem 4.2: by Theorem 4.1 and Lemma 4.7 there exists a Borel subset \mathcal{W}^* of \mathcal{W} such that $\mathbf{N}_0(\mathcal{W} \setminus \mathcal{W}^*) = 0$ and such that (III.104) and (III.125) hold true on \mathcal{W}^* . We fix $W \in \mathcal{W}^*$ and we set

$$\text{Good} = \left\{ x \in \mathcal{R} : \liminf_{r \rightarrow 0^+} g(r)^{-1} \mathcal{M}(B(x, r)) = \kappa_{d, \psi} \right\} \quad \text{and} \quad \text{Bad} = \mathcal{R} \setminus \text{Good}.$$

Let B be any Borel subset of \mathbb{R}^d . By (III.104) and (III.125), we have

$$\mathcal{M}(B \cap \text{Bad}) = \mathcal{P}_g(B \cap \mathcal{R} \cap \text{Bad}) = 0.$$

Then, we apply Lemma 2.2 to $\text{Good} \cap B$ and we get

$$\mathcal{M}(B \cap \text{Good}) = \kappa_{d, \psi} \mathcal{P}_g(B \cap \mathcal{R} \cap \text{Good}).$$

Therefore, on \mathcal{W}^* , for Borel subset B of \mathbb{R}^d , $\mathcal{M}(B) = \kappa_{d, \psi} \mathcal{P}_g(B \cap \mathcal{R})$, which completes the proof of Theorem 4.2.

4.3 Proof of theorem 1.2

We derive Theorem 1.2 from Theorem 4.2. To that end, we first need an upper bound of the upper box-counting dimension of \mathcal{R} under \mathbf{N}_x . Let us briefly recall the definition of the box-counting dimensions of a bounded subset $K \subset \mathbb{R}^d$: let $\varepsilon \in (0, \infty)$ and let $n_\varepsilon(K)$ stands for the minimal number of open balls of radius ε that are necessary to cover K . Then,

$$\underline{\dim}(K) = \liminf_{\varepsilon \rightarrow 0} \frac{\log n_\varepsilon(K)}{\log 1/\varepsilon} \quad \text{and} \quad \overline{\dim}(K) = \limsup_{\varepsilon \rightarrow 0} \frac{\log n_\varepsilon(K)}{\log 1/\varepsilon}. \quad (\text{III.128})$$

Fix $x \in \mathbb{R}^d$ and recall Lemma 2.6 that asserts that for any $q \in (0, \frac{\gamma-1}{2\gamma})$, \mathbf{N}_x -a.e. $(\widehat{W}_s)_{s \in [0, \sigma]}$ is q -Hölder continuous. As already mentioned in Comment 1.2, it easily implies the following:

$$\mathbf{N}_x\text{-a.e.} \quad \overline{\dim}(\mathcal{R}) \leq \frac{2\gamma}{\gamma-1}, \quad (\text{III.129})$$

where \mathcal{R} is the range of the Lévy snake as defined (III.37). We next prove the following lemma.

Lemma 4.8 *Assume that $\gamma > 1$ and that $d > \frac{2\gamma}{\gamma-1}$. Let $x \in \mathbb{R}^d$. For any compact subset K such that $\overline{\dim}(K) \leq \frac{2\gamma}{\gamma-1}$, we have \mathbf{N}_x -a.e. $\mathcal{M}(K) = 0$.*

Proof. Let us first assume that $x \notin K$ and set $k := \inf_{y \in K} \|x - y\| > 0$. For any $\varepsilon \in (0, k/2)$, denote by n_ε the minimal number of balls with radius ε that are necessary to cover K , and denote by $B(x_1^\varepsilon, \varepsilon), \dots, B(x_{n_\varepsilon}^\varepsilon, \varepsilon)$ such balls. Then, (III.46) combined with standard estimates of d -dimensional Green function entail the following inequalities.

$$\begin{aligned} \mathbf{N}_x(\mathcal{M}(K)) &\leq \sum_{i=1}^{n_\varepsilon} \mathbf{N}_x(\mathcal{M}(B(x_i^\varepsilon, \varepsilon))) \\ &\leq \sum_{i=1}^{n_\varepsilon} \int_0^\infty e^{-\alpha a} \mathbf{P}_x(\xi_a \in B(x_i^\varepsilon, \varepsilon)) da \\ &\leq c_{34} \sum_{i=1}^{n_\varepsilon} \int_{B(x_i^\varepsilon, \varepsilon)} \|x - y\|^{2-d} dy \\ &\leq c_{35} k^{2-d} \varepsilon^d n_\varepsilon, \end{aligned}$$

where $c_{34}, c_{35} \in (0, \infty)$ only depend on d . Since $d > \frac{2\gamma}{\gamma-1} \geq \overline{\dim}(K)$, the previous inequality implies that $\mathbf{N}_x(\mathcal{M}(K)) = 0$ as $\varepsilon \rightarrow 0$.

Let us now consider the general case: for any $r > 0$, the previous case applies to the compact set $K' = K \setminus B(x, r)$ and we get

$$\mathbf{N}_x\text{-a.e.} \quad \mathcal{M}(K) = \mathcal{M}(K \cap B(x, r)) + \mathcal{M}(K \setminus B(x, r)) \leq \mathcal{M}(B(x, r)),$$

which implies the desired result as $r \rightarrow 0$ since \mathcal{M} is diffuse. \blacksquare

The end of the proof of Theorem 1.2 follows an argument due to Le Gall in [83] pp. 312-313. Theorem 4.2 and Lemma 4.8 imply that for any compact set K such that $\overline{\dim}(K) \leq \frac{2\gamma}{\gamma-1}$, and for any $x \in \mathbb{R}^d$,

$$\mathbf{N}_x\text{-a.e.} \quad \mathcal{P}_g(K \cap \mathcal{R}) = 0. \tag{III.130}$$

Recall the connection (III.39) in Theorem 2.7 between \mathbf{R} , \mathbf{M} and the excursions W^j , $j \in \mathcal{J}$, of the Brownian snake. An easy argument on Poisson point processes combined with (III.129) and (III.130) implies that almost surely $\mathcal{P}_g(\mathcal{R}_{W^j} \cap \mathcal{R}_{W^i}) = 0$ for any $i \neq j$ in \mathcal{J} . Then, (III.39) entails

$$\mathcal{P}_g(\cdot \cap \mathbf{R}) = \sum_{j \in \mathcal{J}} \mathcal{P}_g(\cdot \cap \mathcal{R}_{W^j}).$$

Theorem 2.7 and (III.39) thus imply

$$\kappa_{d,\psi} \mathcal{P}_g(\cdot \cap \mathbf{R}) = \sum_{j \in \mathcal{J}} \kappa_{d,\psi} \mathcal{P}_g(\cdot \cap \mathcal{R}_{W^j}) = \sum_{j \in \mathcal{J}} \mathcal{M}_{W^j} = \mathbf{M},$$

which is the desired result.

4.4 Dimension of the range of the ψ -SBM.

We now prove Theorem 1.1. To that end, recall that $\xi = (\xi_t)_{t \geq 0}$ is a continuous process defined on the auxiliary measurable space (Ω, \mathcal{F}) and recall that \mathbf{P}_0 is a probability measure on (Ω, \mathcal{F}) under which ξ is distributed as a standard d -dimensional Brownian motion starting from the origin 0. Recall that $(V_t)_{t \geq 0}$ be a subordinator defined on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ that is independent of ξ and whose Laplace exponent is $\psi^{*\prime}(\lambda) = \psi'(\lambda) - \alpha$. Recall from (III.47) that under \mathbf{P}_0 , conditionally given (ξ, V) , $\mathcal{N}^*(dtdW) = \sum_{j \in \mathcal{J}^*} \delta_{(t_j, W^j)}$ is a Poisson point process on $[0, \infty) \times C(\mathbb{R}_+, \mathcal{W})$ with intensity $dV_t \mathbf{N}_{\xi_t}(dW)$. Then recall from (III.48) that for all $a \in \mathbb{R}_+$, we have set $\mathcal{M}_a^* = \sum_{j \in \mathcal{J}^*} \mathbf{1}_{[0,a]}(t_j) \mathcal{M}_j$ where for all $j \in \mathcal{J}^*$, \mathcal{M}_j stands for the occupation measure of the snake W^j as defined in (III.38). Also recall from (III.71) the definition of the following random variables:

$$\forall t \geq s \geq 0 \quad N_r(s, t) = \# \left\{ j \in \mathcal{J}^* : s < t_j < t \text{ and } \mathcal{R}_j \cap \overline{B}(0, r) \neq \emptyset \right\},$$

that counts the snakes that are grafted on the spatial spine ξ between times s and t , and that hit the ball $\overline{B}(0, r)$.

Lemma 4.9 *Assume that $\delta > 1$ and that $d > \frac{2\delta}{\delta-1}$. Then, for all $a \in (0, \infty)$, and for all $u \in (0, \frac{2\gamma}{\gamma-1})$,*

$$\mathbf{P}_0\text{-a.s.} \quad \liminf_{r \rightarrow 0^+} r^{-u} \mathcal{M}_a^*(B(0, r)) < \infty. \tag{III.131}$$

Proof. The present proof is very similar to that of Lemma 3.10. We present only the main steps. Recall from Lemma 3.8 the definition of the processes $(T_{\vartheta(r)})_{r \geq 0}$ and $(T_{\gamma(r)})_{r \geq 0}$. Recall from (III.71) the definition of $N_r(s, t)$. Recall from Lemma 3.13 the definition of the sequence $(s_n)_{n \geq 0}$. Then, for all $n \geq 0$, we set

$$Y'_n = \mathbf{1}_{\{N_{s_n}(\vartheta(2s_n), a + \vartheta(2s_n)) = 0\} \cap \{T_{\vartheta(2s_n)} - T_{\vartheta(2s_{n+1})} \leq s_n^u\}}. \quad (\text{III.132})$$

Reasoning as in the proof of (III.109), we get

$$\mathbf{P}_0\text{-a.s. on } \{Y'_n = 1\}, \quad \mathcal{M}_a^*(B(0, s_n)) \leq 1 + T_{\gamma(2s_{n+1})}. \quad (\text{III.133})$$

Thus, if we show that

$$\mathbf{P}_0\left(\sum_{n \geq 1} Y'_n = \infty\right) > 0, \quad (\text{III.134})$$

we get (III.131), by use of Lemma 3.13 (ii).

The inequality (III.134) is obtained using the Kochen Stone Lemma, as in the proof of (4.4). Indeed, under the assumption $d > \frac{2\delta}{\delta-1}$, Lemma 3.10 entails that for all $n \in \mathbf{N}$, $\mathbf{P}_0(N_{s_n}(\vartheta(2s_n), a + \vartheta(2s_n)) = 0) \geq c_{15} > 0$; we then argue exactly as in the proof of Lemma 4.4 to obtain (III.134). We leave the details to the reader. \blacksquare

Proof of Theorem 1.1. Assume that $\delta > 1$ and that $d > \frac{2\delta}{\delta-1}$. Recall from (III.37) the definition of \mathcal{R} , the range of the Lévy snake. By Theorem 2.7 and by spatial invariance of the snake, it is sufficient to prove

$$\mathbf{N}_0\text{-a.e.} \quad \dim_p(\mathcal{R}) = \overline{\dim}(\mathcal{R}) = \frac{2\gamma}{\gamma-1}.$$

Recall that for every bounded subset $K \subset \mathbb{R}^d$, $\dim_p(A) \leq \overline{\dim}(A)$ (see e.g. Falconer [58]). By (III.129), it then only remains to prove

$$\mathbf{N}_0\text{-a.e.} \quad \dim_p(\mathcal{R}) \geq \frac{2\gamma}{\gamma-1}. \quad (\text{III.135})$$

Lemma 4.9 combined with (III.50) implies that for all $u \in (0, \frac{2\gamma}{\gamma-1})$,

$$\mathbf{N}_0\text{-a.e. for } \mathcal{M}\text{-almost all } x, \quad \liminf_{r \rightarrow 0^+} r^{-u} \mathcal{M}(B(x, r)) < \infty, \quad (\text{III.136})$$

which implies that $\mathbf{N}_0\text{-a.e. } \dim_p(\mathcal{R}) \geq u$ by the comparison results stated here as Theorem 2.1. This entails (III.135) and the proof of Theorem 1.1 is completed. \blacksquare

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