Some contributions to financial market modelling with transaction costs

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Résumé

Cette thèse traite plusieurs problèmes qui se posent pour les marchés financiers avec coûts de transactions et se compose de quatre parties.

On commence, dans la première partie, par une étude du problème de couverture asymptotique d'une option Européenne pour un marché soumis à des coûts de transaction proportionnels. Pour cela, on utilise la méthode d'approximation des portefeuilles suggérée par Hayne Leland (1994). Plus précisément, on adapte la méthode modifiée proposée par Emmanuel Lépinette (2008) et généralise les résultats à des modèles de volatilité locale largement utilisés en pratique. En utilisant les techniques de PDEs pour estimer les Greeks, on montre la convergence en probabilité des portefeuilles discrétisés vers le pay-off lorsque le nombre de dates de révision converge vers l'infini.

Dans la seconde partie, on considère le problème de Merton sur l'optimisation de consommation et d'investissement dans le modèle de Kabanov, modèle abstrait de marchés avec coûts de transaction proportionnel, lorsque les prix sont conduits par un processus de Lévy. Le modèle que l'on considère est générique au sens que le risque de défaut est autorisé, et que les stratégies d'investissement sont supposées làdlàg au lieu de càdlàg. Dans ce contexte, on prouve que la fonction valeur est solution d'une équation HJB integro-differentielle au sens des solutions de viscosité. De plus, la stratégie optimale est construite à partir de la solution d'une équation différentielle stochastique avec réflexions obliques.

Dans la troisième partie, on propose un modèle générique permettant de couvrir les modèles avec la présence de coûts fixes et coûts de transaction proportionnels sur le marché. En général, le modèle que l'on considère n'est pas nécessairement convexe, donc la technique de dualité classique ne marche plus. Néanmoins, la méthode de Maximum Essentiel Multidimensionel proposée par Kabanov et Lépinette (2013) est encore valable grâce à laquelle on arrive à établir un théorème de sur-réplication d'une option de type Européen ou Américaine. En suite, avec le même modèle, on s'intéresse à l'absence d'opportunité arbitrage. En introduisant la notion de fonction liquidative, on réduit le problème au cas unidimensionnel et étudie plusieurs types d'opportunités d'arbitrage.

La dernière partie est consacrée à l'étude du problème de maximisation d'utilité de la richesse terminale d'un portefeuille sous contraintes de risques. Dans ce cas, le risque est représenté par une fonction de perte qui est typiquement concave et aléatoire. Pour cela, les techniques de dualité convexe et de BSDEs sont appliquées pour obtenir la richesse optimale.

Abstract

This thesis deals with different problems related to markets with transaction costs and is composed of four parts.

In the first part, we begin with the study of asymptotic hedging a European option in markets with proportional transaction costs. To do so, we make use of the approximating method suggested by Hayne Leland (1994). More precisely, we adapt the modified strategy proposed by Emmanuel Lépinette (2008) and generalize the results to the models of local volatility. By using the PDE technics for estimating the Greeks, we obtain the convergence in probability of discrete portfolios to the given pay-off function when the number of revision dates tends to infinity.

In the second part, we consider the portfolio optimization problem of Merton in Kabanov's model, an abstract model of markets with transaction costs, when the prices are driven by a Lévy process. Our model is quite generic in the sense that the default risk is allowed, and that the strategies are làdlàg instead of càdlàg as usually supposed in the literature. In this setting, we prove that the value function is solution to a integro-differential HJB equation in the viscosity sense. Moreover, the optimal policy is constructed thanks to the solution to a differential stochastic equation with oblique reflections.

In the third part, we suggest a general model allowing to cover the case of markets with both fixed and proportional transaction costs. In general, the model we consider is not necessarily convex. Therefore, the duality technic is not valid anymore. However, the Multidimensional Essential Supremum method suggested by Kabanov and Lépinette (2013) still works in our setting thanks to which we can characterize the super-hedging prices of an option of European or American type. In the same model, we are also interested in arbitrage theory. By introducing the notion of liquidation function, we reduce the problem to the unidimensional case and study several types of arbitrage opportunities.

The last part of the thesis is devoted to the study of the utility maximization problem under expected shortfall risk constraints. In our setting, the risk is represented by some kind of loss function which is typically random and increasing concave. To do so, the convex duality and BSDE technic are used for obtaining the optimal wealth.

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Chapitre 1

Introduction générale

1.1 Motivation : Marchés avec coûts de transaction

Parmi les hypothèses fondamentales des modèles mathématiques pour les marchés financiers est l'hypothèse que les marchés sont sans friction. En particulier, les investisseurs peuvent faire leurs transactions sans frais de courtage, taxes ou écarts acheteur-vendeur ... Cela conduit au fait que trading peut se faire en temps continu, ce qui est une hypothèse irréaliste car trading en temps continu, en réalité, signifie que les investisseurs peuvent subir une perte infinie en raison des coûts de transaction. Les articles fondamentaux qui étudient des modèles de marché avec coûts de transaction appartiennent à Jouini et Kallal (1995) qui ont étudié le cas de marchés avec deux actifs et le trading est soumis à des écarts acheteur-vendeur. Par la suite, un grand nombre d'articles en économie et finance ont été proposées pour relâcher l'hypothèse de l'absence des coûts de transaction. De nombreuses études revisitent les problèmes classiques avec des contraintes de coûts de transaction. Une attention particulière a également été consacrée pour la modélisation des structures de coûts de transaction

Les structures de coûts de transaction peuvent être très sophistiquées. Nous renvoyons le lecteur à Kissell et Glantz (2003) pour une classification complète des coûts de transaction. Dans le sens le plus général, les coûts de transaction se composent de coûts fixes et variables, ainsi que les coûts visibles et cachés. Les composants fixes sont les coûts qui sont indépendants des prix courants du marché ou de la stratégie mise en œuvre et communément connue à l'avance tels que les commissions, les frais. Les coûts variables sont les coûts qui sont déterminés par les prix courants du marché et qui dépendent de la stratégie de trading. Coûts visibles sont connues exactement à l'avance ou facilement mesurables à partir des données actuelles du marché. Ils se composent des coûts fixes et une partie des coûts variables tels que les écarts de cours acheteur-vendeur ou les taxes. Les coûts cachés sont les composants qui ne sont pas facilement connus ou observables à partir des données actuelles du marché (par exemple, l'impact sur le marché ou les autres effets de liquidité tels que le coût de retard, l'appréciation des prix, le risque de timing et le coût d'opportunité). Dans cette thèse, nous limitons le concept de coûts de transaction dans les coûts visibles. Dans la pratique, les coûts visibles sont généralement représentés par une fonction affine par morceaux du volume des transactions.

Beaucoup d'études simplifient la notion de coûts de transaction en supposant qu'ils sont proportionnels au volume de trading. Plus réaliste est de modéliser les coûts de transaction en incluant les coûts fixes et proportionnels (coûts linéaires). Jouini, Kallal et Napp (2001-2006) ont d'abord tenté d'étudier la théorie de l'arbitrage en présence de coûts fixes. En fait, les coûts fixes sont difficiles à traiter parce que la technique classique de dualité convexe n'est plus valide. Par conséquent, il existe très peu d'études sur les marchés avec à la fois des coûts

fixes et proportionnels. Au contraire, beaucoup a été fait pour les marchés avec seulement les coûts de transaction proportionnels. Dans le cadre multidimensionnel où les échanges directes entre les actifs sont possibles, ce qui est généralement le cas pour les marchés de change, une approche géométrique a été introduite par Kabanov (1999). Dans ce modèle, le portefeuille est représenté par un processus multidimensionnel (V_t), dont chaque processus composant (V_t^i) reflète la position sur l'actif i exprimé en termes de quantité physique. Cette approche permet d'éviter la question de numéraire et, tout aussi important, il est plus réaliste pour la description des stratégies admissibles. Le cône de solvabilité K_t au temps t est défini comme l'ensemble des positions qui peuvent être transformées en positions dont les composants sont non-négatifs après des échanges appropriées, et la condition auto-finançant devient $\dot{V}_t \in -K_t$ pour tout t.

Bien que la linéarité des coûts de transaction est une forte simplification du monde réel, il peut être suffisant dans certains cas et conduit à des résultats mathématiques satisfaisants. Cependant, cette approche peut produire des résultats erronés lorsque le montant de l'investissement est faible et il est attribué à de nombreux actifs dans les petites fractions. La raison en est simple : lorsque le volume de transactions est faible, les coûts fixes ne sont pas négligeables, et ces coûts sont à la baisse lors de l' augmentation de volume de trading (coûts de transaction sont concaves). Sinon, la linéarité n'est pas une bonne idée pour modéliser les coûts de transaction lorsque le volume de trading est grand, en raison de l'effet d'illiquidité. Autrement dit, si pour un certain actif le volume de trading est élevé, alors il ne peut y avoir pas assez d'offre (ou demande) de cet actif, et donc les coûts de transaction vont augmenter (coûts de transaction sont convexes). Par conséquent, une fonction de coûts de transaction non-linéaire devrait être une approche pertinente : une fonction concave des coûts de transaction est bonne pour la modélisation de petits investisseurs, tandis qu'une fonction convexe des coûts de transaction est appropriée pour les modèles avec des grands investisseurs. Beaucoup de chercheurs supposent un mélange des deux structures concave et convexe pour modéliser les coûts de transaction : jusqu'à un certain niveau du volume de transaction, les coûts de transaction est une fonction concave du volume des transactions. Cependant, la fonction de coût de transaction est une fonction convexe lorsque le volume de transaction va au-delà de ce niveau, pour les références, voir Demchuk (2002), Kono et Wijayanayake (2001) et les références dedans.

Il existe une littérature considérable qui revoit les problèmes classiques (et nouvelles) lorsque les marchés sont soumis à la présence des coûts de transaction. Beaucoup d'articles analysent l'effet des coûts de transaction sur les opportunités d'arbitrage, à la fois en temps discret et continu (théorie de l'arbitrage avec coûts de transaction). Nous pouvons citer ici Jouini et Kallal (1995), Soner, Shreve et Cvitanic (1995), Guasoni (2006), Cherny (2007), Guasoni et al. (2010), Lépinette et Kabanov (2012). Une attention particulière a été portée sur le problème de couverture approximative avec les coûts de transaction, par exemple la méthode de Leland (1985) et certaines tentatives par la suite comme Kabanov et Safarian (1997), Bensaid et al. (1992), Soner et al. (1995), Hodges et Neuberger (1989), Pergamenshchikov et Nguyen (2012). Il y a aussi plusieurs recherches sur les problèmes d'optimisation de portefeuille ou de minimisation du risque sous les coûts de transaction, par exemple, Constantinides (1979) et Constantinides (1986), Davis et Norman (1990), Dumas et Luciano (1991), Amihud et Mendelson (1986), Framstad et al. (1999), Gerhold et al. (2013), et de nombreux autres documents.

1.2 Contributions de la thèse

La partie principale de cette thèse traite de divers problèmes sur les marchés avec coûts de transaction. En particulier, nous essayons de répondre aux questions suivantes :

Q.1 Comment peut-on modifier la stratégie de couverture de Leland pour les modèles à volatilité locale ?

Q.2 Comment peut-on généraliser le problème de la consommation optimale de Merton au modèle de Kabanov lorsque les prix sont conduits par un processus de Lévy, et que le risque de défaut est possible ?

Q.3 Comment peut-on utiliser la notion de "Essential maximum" introduite par Kabanov et Lépinette (2013) afin de caractériser le prix de couverture d'une option Européenne avec coûts de transaction fixes et proportionnels ?

Q.4 Dans un modèle avec à la fois des coûts de transaction fixes et proportionnels, quels sont les analogues de FTAP (The Fundamental Theorem of Asset Pricing) ?

Q.5 Est-ce que la technique classique de dualité fonctionne toujours dans le problème de la maximisation de l'utilité sous les contraintes de perte espérée ?

Chaque question ci-dessus touche à un problème spécifique en finance et sera présentée dans les différentes parties de la thèse. Les cinq parties peuvent alors être lues indépendamment. Sauf le dernier, toutes les autres parties sont liées aux modèles de marché avec coûts de transaction.

La thèse se compose de sept chapitres

- Chapitres $1, 2$. Dans ces deux premiers chapitres, nous proposons une brève introduction à la thèse : le contexte de marchés avec coûts de transaction, la contribution de cette thèse, un aperçu de chaque partie de la thèse.
- Chapitre 3. Dans ce chapitre, nous étudions le problème de la couverture approximative dans un modèle à volatilité locale en deux dimensions en présence de coûts de transaction proportionnels. Nos principales contributions présentées dans ce chapitre sont
  - Introduction d'une technique d'EDP qui permet d'obtenir des bornes supérieures des Grecs d'une option Européenne dans un modèle à volatilité locale.
  - Généralisation de la stratégie modifiée de Leland pour les modèles à volatilité locale.
- Chapitre 4. Dans ce chapitre, nous étudions le problème de la consommation optimale dans le modèle Kabanov lorsque les prix sont conduits par un processus de Lévy. Le problème Merton classique est généralisé dans trois directions :
  - Tout d'abord, on permet des chocs sur le marché en modélisant les prix par un processus de Lévy.
  - Deuxièmement, le risque de défaut est pris en compte : les transactions s'arrêtent à la première fois lorsque le portefeuille sort du cône de solvabilité. C'est plus réaliste de supposer que les investisseurs pourrait être en défaillance.
  - Troisièmement, les stratégies de trading sont làglàd afin de capturer des chocs accessibles et inaccessibles sur le marché. En outre, cette approche est également compatible avec la construction de la stratégie optimale en fonction des SDEs avec des réflexions obliques.
- Chapitre 5. Dans ce chapitre, nous étudions le problème de la sur-couverture dans un modèle général de marché, y compris le cas des marchés avec coûts fixes et proportionnels. Nos principales contributions dans ce chapitre sont
  - Introduction d'un nouveau modèle de marché défini par le processus de liquidation ce qui permet de réduire l'étude de portefeuilles multidimensionnelles aux processus

de liquidation en valeur réelle.
— Caractérisation des prix de sur-couverture d'une option Européenne dans ce modèle en utilisant la notion de maximum essentielle dans le cas multidimensionnel.

- Chapitre 6. Ce chapitre continue à étudier le modèle proposé dans le chapitre précédent dans le cadre de la théorie de l'arbitrage. Nos principales contributions présentées dans ce chapitre sont
  — L'introduction d'une condition équivalente à l'existence d'une mesure de probabilité risque neutre, c'est à dire l'absence de possibilité asymptotique-arbitrage.
  — Caractérisation des prix de sur-couverture pour une option Européenne (en termes d'argent, pas vecteur de quantités) dans le modèle Kabanov par la technique de dualité.
  — Etude des différentes notions des opportunités d'arbitrage pour ce modèle ainsi que les relations entre les marchés avec coûts fixes & coûts proportionnels et les marchés avec coûts proportionnels seulement.
- Chapitre 7. Ce chapitre traite du problème de maximisation de l'utilité sous contraintes de risque. Nous utilisons la technique de dualité dans diverses situations. Nos principales contributions présentées dans ce chapitre sont
  — Une preuve de relation duale et la construction de la richesse optimale via la fonction duale de Fenchel.
  — Une solution à ce problème dans le cas de marché complète basée sur les techniques d'EDSR et de dualité convexe.

Liste des publications ou des pré-publications

1. Approximate Hedging in a Local Volatility Model with Proportional Transaction Costs, with E. Lépinette. *Applied Mathematical Finance*. 2014.
2. Consumption-Investment Optimization Problem in a Levy Financial Model with Transaction Costs, with Y. Kabanov and E. Lépinette. Preprint.
3. General Financial Market Model Defined by a Liquidation Value Process, with E. Lépinette. Submitted to *Stochastics*. 2014.
4. Risk Neutral Probabilities and Absence of Arbitrage Opportunity in Non Convex Financial Market Models, with E. Lépinette. Preprint.
5. Utility Maximization Problem With Expected Loss Constraints. Preprint.

1.3 Couverture approximative dans un modèle à volatilité locale avec coûts de transaction

• Motivation.

Ce troisième chapitre de la thèse traite le problème de couverture asymptotique une option Européenne dans un modèle à volatilité locale avec deux actifs lorsque l'écart de cours acheteur-vendeur est proportionnel à la valeur trading. Ce problème était tout d'abord étudié dans le papier séminal de Leland (1985) pour le modèle de Black-Scholes. Dans son travail, Leland suppose que la transaction de ν actifs coûte une quantité de $\kappa\nu S$, soit l'achat ou la vente, où κ est un coefficient des coûts de

transaction et S_t est le prix d'une unité d'actif au t. Le portefeuille de couverture est révisé chaque δt où δt est un petit pas de temps fini et fixe, malgré cela est optimale ou non dans tous les sens. Dans ce modèle, la valeur actuelle de portefeuille au temps t est définie par

$$V_t^n = V_0^n + \int_0^t D_u^n dS_u - \sum_{t_i \leq t} k_n S_{t_i} |D_{i+1}^n - D_i^n|, \quad t < 1.$$

où $t_i = t_i^n, 0 \leq i \leq n$, $t_0 = 0, t_n = 1$, sont les dates de révision telles que $t_{i+1} - t_i = \delta t$; et D^n désigne la stratégie de trading satisfaisant $D^n = D_i^n$ sur l'intervalle $]t_{i-1}, t_i]$. En présence des coûts de transaction, Leland suggère une stratégie qui peut être considérée comme un delta modifié de la formule de Black-Scholes. L'idée est de substituer la volatilité σ par une "volatilité élargie" $\hat{\sigma}$ afin de compenser pour les coûts de transaction. La "volatilité élargie" est définie par

$$\widehat{\sigma}^2 = \sigma^2 + 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}} = \sigma^2 + 2\kappa\sigma n^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}.$$

La stratégie de trading est donnée par $\hat{D}_i^n = \hat{C}_x(t_i, S_{t_i})$ où $\widehat{C}(t,x) = \widehat{C}^n(t,x)$ est la solution à l'équation de chaleur avec un nouveau coefficient de diffusion $\hat{\sigma}$

$$\begin{cases} \widehat{C}_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2 x^2 \widehat{C}_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\times[0,1[\ , \\ \widehat{C}(1,x) & = & h(x), \quad x \in]0,\infty[\ . \end{cases}$$

D'après la stratégie de Leland, lorsque le nombre de révision n est grand, c'est à dire quand $\delta t = \frac{1}{n}$ converge vers 0, la valeur du portefeuille de couverture convergera au pay-off $h(S_T)$ en probabilité, autrement dit, cette stratégie de couverture va asymptotiquement couvrir l'option Européenne sans erreur. En outre, la valeur initiale de cette stratégie est donnée par $V_0^n = \widehat{C}(0, S_0)$ qui est supérieure au prix de couverture dans les formules de Black-Scholes en raison de l'augmentation de la volatilité de la stratégie de Leland. Ceci s'explique par l'intuition que le prix de l'option doit inclure un montant supplémentaire que le vendeur de l'option doit payer en raison des écarts de cours acheteurs-vendeurs quand il ou elle construit un portefeuille de couverture. Lorsque l'intervalle de révision tend vers zéro, ce prix va augmenter au prix bid S_0^a, soit le coût de la stratégie "buy-and-hold". Cela signifie que la stratégie de Leland n'est pas mieux que la stratégie sur-couverture lorsque les transactions se passent presque en temps continu.

Malheureusement, la conjecture de Leland n'est vraie que dans le cas où κ_n tend vers 0 avec la vitesse de $n^{-\alpha}, 0 < \alpha \leq \frac{1}{2}$ comme montrée par Kabanov et Safarian (1997), mais elle est fausse dans le cas des coûts constants $\kappa_n = const$. Dans ce cas, il est rapporté dans Kabanov et Safarian (1997) ou plus tard dans Pergamenshchikov (2003) que l'erreur de couverture converge avec une vitesse de convergence de $n^{-\frac{1}{4}}$ vers une variable non-nulle. L'interprétation économique de ce phénomène est claire : un trader ne dispose pas de motivation à la transaction "presque continuelle" s'il ne reçoit pas une réduction sur les coûts de transaction lors que les transactions sont

plus fréquentes (de manière équivalente, le trade avec des ordres nombreux), si non il subira une perte importante en raison des coûts de transaction.
Une tentative de traiter le cas des coûts constants appartient à Lépinette (2011). Il suggère une modification de la stratégie classique de Leland pour supprimer l'erreur de couverture. A savoir, il a construit une stratégie de la façon suivante

$$D^n_{t_i} = \widehat{C}_x(t_i, S_{t_i}) + \int_0^{t_i} \widehat{C}_{xt}(t, S_t)dt. \tag{1.3.1}$$

Il est important de souligner que le prix de couverture de la stratégie de Lépinette est exactement le même que celui de la stratégie de Leland. Puisque l'erreur de couverture asymptotiquement disparaît, il est claire que la stratégie modifiée surperforme l'ancienne. Une étude de la vitesse de convergence est également effectuée dans un article ultérieur de Lépinette et Darses (2012).

• **Nouveaux résultats.**

Le but de mon premier projet dans cette thèse est donc de généraliser les résultats de Lépinette pour le cas des modèles à volatilité locale. Nous montrons que la stratégie de couverture modifiée fonctionne bien dans ce nouveau contexte, i.e. la couverture est asymptotiquement sans erreur. Nous limitons notre attention à un modèle à volatilité locale pour deux raisons. Premièrement, les modèles à volatilité locale sont couramment utilisés dans la pratique car ils peuvent être tout simplement calibrés à des options "vanillas" par les formules de Dupire. Un des avantages des modèles à volatilité locale par rapport à des modèles à volatilité stochastique est que les premiers conservent la complétude du marché, ce qui est un facteur clé en pratique pour évaluer des produits dérivés. Deuxièmement, la stratégie de Lépinette ne pourrait pas être valable dans des modèles à volatilité stochastique parce que la couverture exacte échoue dans ces modèles. Il est donc sans intérêt pour étudier une stratégie de trading spécifique qui conduit à une erreur de couverture positif asymptotiquement (sauf que nous pouvons montrer que cette stratégie est optimale dans un certain sens). Notre principal résultat de cette partie est le théorème suivant :

Theorem 1.3.1. *Soit $\alpha \in [0, \frac{1}{2}]$ et supposons que h et σ sont tels que $\widehat{C}_{xx} \geq 0$. Si la stratégie $\widehat{D}^n_i$ est donnée par (2.3.1), alors sous quelques conditions techniques, la valeur terminale du portefeuille*

$$V^n_1 = \widehat{C}^n_0 + \int_0^1 \widehat{D}^n_u dS_u - k_n \sum_{i=1}^{n-1} S_{t_i} |\widehat{D}^n_{i+1} - \widehat{D}^n_i|$$

converges vers le pay-off $h(S_1)$ en probabilité.

La preuve de ce théorème est essentiellement similaire à celle de Lépinette (2011). La technique clé réside dans l'estimation des dérivées successives de $\widehat{C}(t, x)$. Lorsque la fonction de volatilité est constante ou déterministe, les choses sont simples car nous pouvons employer explicitement les expressions de ces dérivés. Dans le cas de volatilité locale, nous sommes confrontés à une difficulté considérable de ne pas savoir

de telles expressions. La seule chose que nous pouvons faire est de faire appel à des techniques classiques d'EDP afin d'obtenir des bornes précises pour les dérivés de $\widehat{C}(t,x)$. Une utilisation intelligente du changement de temps technique est nécessaire pour obtenir les résultats souhaités.

• **Perspectives.**

Les résultats obtenus peuvent être généralisés ou développées dans des directions différentes.
Tout d'abord, notons que la fonction des coûts de transaction considérée dans ce chapitre est de la forme linéaire, c'est à dire $G(t,S_t,x) = |x|S_t$ où x désigne le volume de transaction et S_t désigne le prix de l'action au temps t. La linéarité de G peut donc être relaxée afin que le modèle soit plus réaliste, par exemple, l'impact du marché ou contraintes réglementaires. Nous référons à Elie et Lépinette (2013) pour la même problématique dans le cas de modèle de Black-Scholes lorsque G ne dépend pas du prix de l'action, soit $G = G(x,t)$. Un problème similaire est également étudié dans Nguyen (2014) pour les modèles à volatilité stochastique et avec la fonction de transaction de la forme $G = G(t, xS_t)$ (courbe de l'offre).
Deuxièmement, on sait que l'erreur de couverture $V_1^n - h(S_1)$ converge vers zéro quand n tend vers l'infini. Il est naturel d'étudier la vitesse de cette convergence. Mais même dans le cas le plus simple de modèle de Black-Scholes, la preuve a déjà utilisé des techniques probabilistes sophistiquées. Par conséquent, la question de la vitesse de convergence reste ouverte dans notre modèle à volatilité locale. En outre, nous espère que quand n tend vers l'infini, le coût de couverture $C^n(0,S_0)$ converge vers le prix de sur-couverture du pay-off. Au moins c'est le cas pour les Calls Européens, quand $C^n(0,S_0)$ converge vers le prix de la stratégie buy-and-hold, c'est à dire $S_0^a := S_0(1+\kappa)$. Nous nous intéressons également à la vitesse de convergence $S_0^a - C^n(0,S_0)$. Cette question est d'une grande importance pour les praticiens, en particulier si la vitesse de convergence de $C^n(0,S_0)$ vers le prix de la stratégie buy-and-hold S_0^a est nettement inférieure à la vitesse de convergence de V_1^n vers $h(S1)$, alors l'on peut conclure que la stratégie modifiée de Leland est vraiment utile dans la pratique.

1.4 Optimisation de consommation dans le modèle de Kabanov avec sauts

• **Motivation.**

Dans ce troisième chapitre, nous étudions le problème classique de la consommation optimale dans le modèle de Kabanov avec des sauts, c'est à dire les marchés avec coûts de transaction proportionnels et des prix étant conduits par un processus de Lévy. Le problème de consommation-investissement optimale en temps continu a été initié par le papier séminal de Merton (1971). Il a considéré un modèle de marché sans friction où le processus de prix est un mouvement brownien géométrique. Etant

donné une fonction d'utilité qui représente la référence de risque de l'investisseur, son objectif est de maximiser l'utilité espérée actualisée de la consommation sur l'intervalle de temps infini. Pour la fonction d'utilité puissance, il a obtenu une solution explicite du problème de contrôle optimal. Merton a prouvé que, dans la stratégie d'investissement optimale, l'investisseur doit maintenir une proportion constante de la richesse attribuée à l'actif risqué qui est facilement calculée à partir des paramètres du modèle (Merton proportion). Ce résultat très important parmi les théories d'investissement et largement utilisé par les praticiens en gestion de fonds mutuels.

Le travail de Merton a été étendu par de nombreux auteurs dans de diverses directions, y compris les modèles avec coûts de transaction et avec sauts, qui sont les principaux objets de notre intérêt. Dans le cas des marchés avec deux actifs et coûts de transaction proportionnels (sans sauts), Davis et Norman (1990) ont étudié la structure de la fonction de valeur. Ils ont également fourni une construction rigoureuse de la solution optimale. Les auteurs ont montré que la politique optimale est de garder les proportions de la richesse en actifs risqués toujours dans une région contenant la ligne de Merton. Les limites supérieure et inférieure de cette région peuvent être évidemment calculées en fonction de paramètres du modèle. Lorsque les proportions de la richesse de l'investisseur investies dans les actifs risqués se trouvent dans cette région, l'investisseur ne fait aucune transaction. Lorsque les fluctuations du processus de prix conduisent les proportions de la richesse investies dans les actifs risqués à la frontière de la région de l'inaction, l'investisseur fait une transaction de montant minimal nécessaire pour maintenir la proportion dans la région de l'inaction. En outre, la quantité optimale à consommer du compte bancaire est déduite à partir de la solution à un EDP non-linéaire, connue sous le nom de l'équation de Hamilton-Jacobi-Bellman (HJB). La principale difficulté pour obtenir cette solution est que le domaine de l'équation HJB n'est pas spécifié de manière exogène. Au contraire, il est spécifié de façon endogène par les conditions qui déterminent la région de l'inaction. Pour un schéma de calcul afin de résoudre cette équation HJB dans le cas multidimensionnel, nous renvoyons le lecteur à l'article de Muthuraman et Kumar (2006), une solution complète pour le problème en termes de solutions de viscosité est donnée dans Shreve et Soner (1994), le problème d'optimisation de portefeuille sous des petites coûts de transaction est étudié par Touzi et Soner (2013).

Bien que le mouvement géométrique brownien gagne une popularité dans la modélisation financière, il est critiqué en raison de sa faible prévisibilité des mouvements de prix sur les marchés. Les modèles de diffusion à l'aide de processus de Lévy semblent être plus souple et permet de capturer les propriétés statistiques et économiques des données du marché, et sont encore mathématiquement tractable. Récemment, plusieurs articles ont étudié le problème de la consommation et de l'investissement optimal lorsque les prix sont modélisés par un processus de Lévy et les marchés sont soumis à des coûts de transaction. Par exemple, Framstad et al. (2001) ont étudié le cas de deux actifs, tandis que Kabanov et De Vallière (2009) ont généralisé le problème dans un cadre très abstrait de modèles de multi-actifs avec coûts de transaction. Le but de notre travail est de deux plis. Tout d'abord, il fournit des preuves mathématiques rigoureuse pour des résultats dans ces deux documents, tels que l'équation HJB et l'unicité de la solution, la construction d'une stratégie optimale. Deuxièmement, il unifie les approches des deux documents dans un modèle général

de Kabanov, où les prix sont conduits par un processus de Lévy et les stratégies de trading sont autorisés à être làdlàg, mais pas càdlàg comme on les suppose dans la littérature standard. Cela permet le travail, non seulement pour être cohérent avec la construction d'une stratégie optimale qui est généralement càglàd dans un modèle Lévy, mais aussi de capturer les chocs accessibles et inaccessibles du marché .

• **Nouveaux résultats.**

Le chapitre 4 est divisé en deux parties. Dans la première partie, nous étudions le problème de la consommation optimale dans un cadre abstrait basé sur le modèle de Kabanov. La deuxième partie est consacrée à un cas particulier des marchés avec deux actifs. Le modèle abstrait est décrit comme suit.
Soit $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ un espace probabilisé filtré complète vérifiant les hypothèses habituelles. Considérons un agent qui investit sa richesse dans un portefeuille de plusieurs actifs dont le processus de la rentabilité est conduit par un processus de Lévy $d-$ dimentionnel $(Y_t)_{t\geq 0}$

$$dY_t = \mu t + \Xi dW_t + \int_{\mathbf{R}^d} z\tilde{N}(dz, dt),$$

où $\mu \in \mathbf{R}^d$, W est un mouvement brownien standard m-dimensionnel et Ξ est un $d \times m$-matrice. De plus, $\tilde{N}(dz, dt)$ désigne la mesure aléatoire compensé de Poisson dont la compensation prendra la forme $dt\Pi(dz).\Pi(dz)$ est généralement appelé la mesure de Lévy qui est un mesure $\sigma-$finie sur les ensembles de Borel sur $\mathbf{R}^d \setminus \{0\}$ vérifiant la condition suivante

$$\int_{\mathbf{R}^d\setminus\{0\}} \left(|z|^2 \wedge |z|\right) \Pi(dz) < \infty.$$

On considère une version du modèle de Kabanov avec deux cônes constants K et $\mathcal{C}$ qui sont supposés être fermés et que $K \cap (-K) = \{0\}, \mathcal{C} \cap (-\mathcal{C}) = \{0\}$. Supposons que $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. En termes financiers, K représente le cône de solvabilité, par exemple, on peut choisir K pour l'ensemble des positions de sorte que lorsque l'on va convertir le portefeuille en n'importe quel actif et payer les coûts de transaction, les valeurs liquidatives sont non-négatives ; tandis que $\mathcal{C}$ se compose de toute possibilités de consommation, par exemple, si l'on consomme uniquement sur le compte bancaire, alors $\mathcal{C} = \mathbf{R}_+ e_1$. La dynamique d'un processus de portefeuille est défini pour chaque $i = 1, \ldots, d$ par :

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_0 = V_{0-} = x,$$

où le control $\pi = (B, C)$ est un processus làdlàgs, prévisibles et de variations bornées. On exige que $dB_t \in -Kdt$ et $dC_t = c_t dt \in \mathcal{C}dt$. Ces conditions seront décrite en détail dans la thèse. La dynamique signifie que un tel portefeuille V est auto-finançant, c'est à dire ses incréments sont uniquement dues aux incréments de Y, et les coûts de

transaction modélisés par B sont retirés tandis que C représente la somme cumulée de la richesse consommée.
Pour chaque control admissible $\pi \in \mathcal{A}_x$, on va introduire le temps d'arrêt

$$\theta^\pi = \inf \{t : V_t^\pi \notin \operatorname{int} K\}.$$

On définit le processus de l'utilité

$$J_t^\pi(x) := \int_0^{t\wedge\theta^\pi} e^{-\beta s} U(c_s) ds,$$

où β est un taux d'actualisation positif et U est une application non-négatif défini sur $\mathcal{C}$ ce qui représente une fonction d'utilité ; il est supposé être concave, $U(0) = 0$ et $U(x)/|x| \to 0$ quand $|x| \to \infty$. Le problème de la consommation optimale consiste à optimiser le processus de l'utilité au cours de l'ensemble des stratégies admisibles. Pour ce faire, on va définir la fonction de Bellman comme suit

$$W(x) := \sup_{\pi\in\mathcal{A}_x} \mathbb{E}[J_\infty^\pi(x)], \quad x \in \operatorname{int} K. \tag{1.4.2}$$

Le résultat principal de ce chapitre est de montrer que, sous certaines conditions faibles, la fonction de Bellman est la solution unique de l'équation HJB du formulaire ci-dessous

$$\begin{aligned} &F(W''(x), W'(x), H(W, x), W(x), x) = 0, \quad x \in \operatorname{int} K, \\ &W(x) = 0 \quad \text{on } \partial K. \end{aligned}$$

On laisse tous les détails sur l'équation et les conditions nécessaires au chapitre 4. A ce stade, on fait quelques remarques suivantes.

- Tout d'abord, comme les contrôles sont supposés être làdlàg, on doit redéfinir la notion des intégrales stochastiques par rapport à des intégrants prévisibles et làdlàgs. Ceci est présenté dans l'annexe du chapitre 4.
- Deuxièmement, on permet la possibilité que l'investisseur peut faire faillite si sa situation est insolvable. Cette contrainte rend le problème plus difficile dans la mesure où il n'est plus simple pour prouver la concavité de la fonction de Bellman comme dans les modèles des diffusions continues. En fait, la concavité est importante car il permet d'étudier la structure de la fonction de Bellman. Heureusement, on peut prouver que si l'équation HJB admet la solution unique, alors la contrainte de défaut n'est pas obligatoire, c'est à dire qu'il n'est pas optimal de délibérer le portefeuille hors de la cône de solvabilité.
- Enfin, notons que la seule différence entre notre modèle et un modèle conduit par un mouvement brownien est la présence de l'opérateur $H(f, x)$ qui est donnée par

  $$H(f, x) := \int_{\mathbf{R}^d} \left[f(x + \operatorname{diag}(x) z) 1_{x+\operatorname{diag}(x)z\in \mathrm{intK}} - f(x) - f'(x) \operatorname{diag}(x) z \right] \Pi(dz).$$

  Cet opérateur intégro-différentiel n'est pas défini de manière locale. Par conséquent, on doit ensuite définir des solutions de viscosité dans le sens global.

Dans la deuxième partie du chapitre 4, on revisite le problème de la consommation optimale dans le cas bidimensionnel. Cette affaire a été complètement étudiée par Soner et Shreve (1994) lorsque les prix sont conduits par un mouvement brownien géométrique. Comme l'unicité de la solution est vérifiée dans ce modèle, la fonction de Bellman est concave et donc l'on peut utiliser les techniques de l'analyse convexe pour étudier la solution de l'équation HJB. On retrouve donc la plupart de leurs résultats pour notre modèle, de la structure et de la régularité de la fonction de Bellman, à la construction de la stratégie optimale basée sur la notion des EDSs avec des réflexions obliques qui est à son tour, doit être reformuler de façon rigoureuse lorsque sauts sont tenus en compte.

• **Perspectives.**

Le modèle de Kabanov s'avère être un bon cadre pour étudier les marchés avec frictions. Il est non seulement assez général pour modéliser les coûts de transaction proportionnels, mais conduit également à des résultats mathématiques satisfaisants tels que les équations HJB. Dans ce nouveau contexte, on peut également généraliser le problème de consommation et d'investissement optimal dans des directions différentes. Par exemple, si on permet à l'investisseur de fournir une certain capital pour sauver l'investisseur lorsqu'il déclare de faillite, il n'est plus évident que l'investisseur reste toujours dans la zone de solvabilité à l'optimalité. Certaines conditions doivent être relaxées ou modifiées, par exemple la fonction d'utilité peut être dépendante de la richesse, les coefficients du modèle peuvent être stochastiques, le marché peut être confronté à des risques de liquidité, l'horizon peut être fini ou aléatoire, l'investisseur peut reçoit les revenus du travail, ect.. On peut également envisager un problème d'optimisation plus générale de l'utilité récursive, et ajouter plus de contraintes de risque sur les stratégies ou sur les richesses tels que contrainte sur la vente à découvert, ou contrainte de drawndown ... On croit que ces problèmes peuvent être résolus dans le cadre du modèle de Kabanov et conduisent à de nombreux résultats prometteurs dans l'avenir.

1.5 Un modèle général de marché défini par le processus de valeur liquidative

• **Motivation.**

Dans les cinquième et sixième chapitres, nous introduisons un modèle général qui capture les coûts de transaction fixes et proportionnels. Dans ce nouveau contexte, nous étudions la théorie de l'arbitrage. Deux questions fondamentales se posent. Premièrement, pour un modèle de marché financier, nous nous intéressons à la possibilité de faire des profits à partir de rien par les activités de transaction sur le marché, c'est à dire à partir d'une position nulle ou endettée, peut-on se retrouver avec une richesse non négative dans tous les scénarios, et un gain avec une probabilité strictement positive ? Si la réponse à cette question est oui, on dit que le marché admet une opportunité d'arbitrage. Deuxièmement, considérons un contingent-claim dans

un modèle avec l'absence d'opportunités d'arbitrage qui rapporte un flux financier dans l'avenir à son titulaire, quel est le plus petit montant initial x à partir de lequel le vendeur du contingent-claim peut commencer un portefeuille de sorte que les valeurs de ce portefeuille dominent les flux financiers du contingent-claim ? Nous appelons cela le problème de sur-couverture, et la valeur x est appelée le prix de sur-couverture du contingent-claim. Dans le cas d'un marché discrétisé sans coûts de transaction, la théorie de l'arbitrage est lancée à partir du fameux théorème de Dalang-Morton-Willinger qui indique que le marché est sans arbitrage si et seulement s'il existe une mesure de martingale équivalente. Ce résultat est généralement considéré comme le théorème fondamental de l'évaluation d'actif (FTAP). Une analogue de ce théorème dans des modèles en temps continu est fournie par Harrison, Kreps et Pliska (1981) ou par Delbaen Schachermayer (1994). Dans les deux cas, la théorie de l'arbitrage a été bien développée par un grand nombre d'auteurs en faisant appel au célèbre théorème de séparation d'Hahn-Banach dans l'analyse fonctionnelle, ou le théorème de Kreps-Yan par la suite. Les prix de sur-couverture des options Européennes ou Américaines sont également caractérisés en utilisant le théorème de décomposition optionnelle, voir Kramkov (1996).

Pour les coûts de transaction proportionnels, la théorie a été initiée par E. Jouini et H. Kallal (1995). Dans leur article pionnier, les auteurs ont considéré un modèle de deux actifs et établi l'équivalence entre la condition non-arbitrage avec l'existence d'un *consistent price system*, c'est à dire une martingale (sous une mesure probabilisé équivalente) évoluant dans les écarts acheteur-vendeur. Cette nouvelle notion est une généralisation naturelle des mesures de martingale équivalentes dans le cas sans coûts de transaction. Kabanov (1999) a introduit un modèle général en temps discret pour modéliser les marché des devises avec coûts de transaction et a donné une description des dotations initiales qui permettent de couvrir un produit dérivé écrit en diverses monnaies par un portefeuille autofinançant. Ce modèle possède une belle structure géométrique et a rapidement devenu un modèle standard de la théorie de l'arbitrage pour les marchés avec coûts de transaction proportionnels. Il existe plusieurs documents importants qui étudient le problème de la sur-couverture dans le modèle de Kabanov, par exemple Kabanov, Rasonyi, Stricker (2002), Schachermayer (2004), Campi et Schachermayer (2006), Kabanov et Lépinette (2013).

Dans la pratique, les coûts fixes représentent une partie importante des coûts de transaction. Rappelons que les coûts fixes sont bornés indépendamment de la taille de la transaction. Il y a beaucoup d'exemples de coûts fixes dans la réalité comme les frais fixes de courtage, les ententes de courtage où les frais marginaux vont à zéro au-delà d'un volume donné qui est périodiquement remis à zéro, les taxes fixes d'investissement pour avoir accès à un marché (comme un marché étranger), les coûts d'exploitation et de transformation, les coûts fixes liés à la mise en place d'un bureau ou pour obtenir l'accès à l'information, et le coût d'opportunité de la recherche d'un marché ou de faire un commerce particulier, etc ... Par conséquent, il est raisonnable d'étudier des modèles qui capturent de l'impact des coûts fixes. Toutefois, jusqu'à présent il y avait très peu de travaux académiques sur cette direction. Il y a plusieurs tentatives d'étudier formellement des modèles avec coûts fixes, par exemple dans Jouini, Kallal, et Napp (2001, 2006) où les auteurs ont établi l'équivalence entre l'absence des opportunités arbitrages et l'existence d'une mesure probabilisé

de martingale absolument continue.

• **Nouveaux résultats.**

Dans le chapitre 5, nous présentons un modèle mathématique qui est assez général pour capturer à la fois les coûts fixes et proportionnels. Rappelons que, dans le modèle de Kabanov, la dynamique de portefeuilles doivent nécessairement être écrite pour chaque actif à la place d'un processus de richesse unique, comme dans le cas sans coûts de transaction. Cela rend le modèle multidimensionnel. Notre idée est de considérer le processus des valeurs liquidatives au lieu du processus de portefeuille multidimensionnel afin de réduire le problème au cas unidimensionnel. Pour ce faire, nous formalisons le modèle comme suit.

Etant donné une base stochastique $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, P)$, on définit *processus des cônes de solvabilité* un processus adapté à valeurs ensembles $(G_t)_{0\leq t\leq T}$ vérifiant les conditions suivantes

Conditions sur (G_t) :

(i) G_t est un ensemble fermé et $\mathcal{F}_t$-mesurable, $0 \leq t \leq T$,

(ii) $G_t + G_t \subseteq G_t$, $0 \leq t \leq T$, p.p.,

(iii) $\lambda G_t \subseteq G_t$, $\forall \lambda \geq 1$, p.p.,

(iv) $G_t + \mathbf{R}^d_+ = G_t$, p.p.,

(v) Le cône $\overline{\mathbf{R}_+ G_t}$ est propre , i.e. $\overline{\mathbf{R}_+ G_t} \cap (-\overline{\mathbf{R}_+ G_t}) = \{0\}$ p.p.

Les conditions (i) et (iv) sont évidentes. Les conditions (ii) et (iii) sont satisfaites pour tout modèle avec coûts fixes, parce que les coûts fixes sont indépendants du volume de transaction, ce qui conduit au fait que les coûts fixes sont relativement décroissants quand on augmente la taille des transactions. La dernière condition signifie que les coûts de transaction sont efficaces, c'est à dire que l'on ne peut pas faire de transaction sans payer les coûts. Notons que les ensembles de solvabilité ne sont pas nécessairement convexe, par conséquent, on ne peut plus appliquer les techniques traditionnelles de l'analyse convexe. C'est le point clé qui rend le problème plus difficile que dans le cas des coûts de transaction proportionnels. Le processus de liquidation $(\mathbf{L}_t)$ est définie de telle sorte que

$$G_t = \{x \in \mathbf{R}^d : \mathbf{L}_t(x) \geq 0\}.$$

Plus précisément, on a

$$\mathbf{L}_t(z) := \sup\{\alpha \in \mathbf{R} : \ z - \alpha e_1 \in G_t\}. \tag{1.5.3}$$

Comme dans le cas de coûts proportionnels, un processus stochastique adapté $(V_u)_{t\leq u\leq T}$ est appelé un portefeuille si $V_u - V_{u-1} \in G_u$ p.p. On peut donc écrire $V_T = \sum_{t\leq u\leq T} \xi_u$ où $\xi_u \in L^0(-G_u, \mathcal{F}_u)$. Désignons R^t_T l'ensemble de tous les valeurs terminales V_T des portefeuilles V telles que $V_{t-1} = 0$. i.e.

$$\mathrm{R}^t_T := \mathrm{R}^t_T(G) := \sum_{t\leq u\leq T} L^0(-G_u, \mathcal{F}_u).$$

En utilisant la fonction de liquidation, on se concentre sur l'ensemble des valeurs liquidatives terminales

$$\mathrm{LV}_T^t := \{\mathbf{L}_T(V_T) : \ V_T \in \mathrm{R}_T^t\}.$$

Dans ce cadre, nous étudions les conditions non-arbitrages telles que **NA**, **NA**2 en termes de LV_T^t. La seule différence avec le cas classique est la caractérisation des prix de sur-couverture des contingent-claims. Pour cela, la technique de dualité n'est plus valide. Nous devons employer la notion de *Multidimensional Essential Supremum* proposée par Kabanov et Lépinette (2013). Cette notion est une généralisation de son analogue classique pour les variables aléatoires réelles au cas multidimensionnel, où la relation d'ordre linéaire sur $\mathbf{R}$ est remplacé par une nouvelle définie par le cône de solvabilité, c'est à dire, par définition, $x \succeq^t y \Leftrightarrow x - y \in G_t$. Notons que cette relation est bien définie en raison de la condition (ii). On rappelle ici la définition de Multidimensional Essential Supremum, où l'on désigne par $\succeq$ une relation d'ordre entraînée par certains ensembles de solvabilité défini comme ci-dessus.

Definition 1.5.1. *Soit Γ un sous-ensemble de $L^0(\mathbf{R}^d, \mathcal{F})$. On note par $\mathcal{H}$-Esssup Γ un sous-ensemble $\hat{\Gamma}$ de $L^0(\mathbf{R}^d, \mathcal{H})$ tel que la condition suivante est vérifiée :*

(a) $\hat{\Gamma} \succeq \Gamma$,

(b) Si $\gamma \in L^0(\mathbf{R}^d, \mathcal{H})$ et $\gamma \succeq \Gamma$, alors il existe $\hat{\gamma} \in \hat{\Gamma}$ tel que $\gamma \succeq \hat{\gamma}$,

(c) Si $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ entraîne $\hat{\gamma}_1 = \hat{\gamma}_2$.

Etant donné un pay-off Européen $Y_T \in L^0(\mathcal{F}_T)$, un processus de portefeuille (V_t) sur-couvre Y_T si $V_T \succeq^T Y_T$. De plus, il est appelé *minimal* si tout portefeuille $W \in \mathcal{V}$ tel que $W_T \succeq^T Y_T$ et $V \succeq W$ (i.e. $V \succeq^t W$ pour tout t) coïncide avec V. Notons $\mathcal{V}_{min}^E(Y_T)$ l'ensemble de tout processus sur-couvrant Y_T. Le théorème de sur-couverture dans ce nouveau contexte est le suivant.

Proposition 1.5.2. *Supposons que **NA**2 est vérifié et qu'il existe au moins une $V \in \mathcal{V}$ tel que $V_T \succeq^T Y_T$. Alors $\mathcal{V}_{\min}^E(Y_T) \neq \emptyset$ and it coincides with the set of solutions of backward inclusions et il coïncide avec l'ensemble des solutions à des inclusions rétrogrades*

$$V_t \in (\mathcal{F}_t, \succeq^{t+1})\text{-Esssup}\,\{V_{t+1}\}, \quad t \leq T-1, \quad V_T = Y_T. \tag{1.5.4}$$

De plus, tout $W \in \mathcal{V}$ tel que $W_T \succeq^T Y_T$ vérifie $W \succeq V$ pour certaine $V \in \mathcal{V}_{min}^E(Y_T)$.

Une définition du prix de sur-couverture nécessite une formulation plus mathématique et sera détaillée au chapitre 5. Le problème de sur-couverture d'une option Américaine se fait de la même manière que dans le cas des coûts proportionnels. Pour la démonstration du théorème ci-dessus, la partie la plus difficile est de prouver que $\mathcal{V}_{min}^E(Y_T) \neq \emptyset$. Pour ce faire, il est souligné que toutes les conditions nécessaires dans [57] ne sont pas vérifiées. En particulier, on ne sait pas s'il existe une représentation dénombrable d'utilité mesurable pour les relations d'ordre partiel ou non (pour plus de détails, voir [57], [35] et les références citées dedans). Par conséquent, on doit utiliser une autre approche pour résoudre le problème.

Chapitre 6 est un projet en cours, nous continuons à étudier le modèle non-convexe donné dans le chapitre 5. Notre objectif est d'examiner les différents critères de

non-arbitrage dans ce nouveau contexte. Plus précisément, nous nous intéressons à répondre aux questions suivantes.

Q.1 *Quelle est la condition d'absence de l'arbitrage qui est équivalent à l'existence d'un mesure risque-neutre équivalente ? Dans le cas où l'une de ces deux conditions est vérifiée, comment caractériser le prix de sur-couverture d'une option Européenne ?*

Pour cette question, nous introduisons une nouvelle notion des opportunités d'arbitrage appelé *Extended No Asymptotic Arbitrage opportunity condition* **(ENAA)**. Nous montrons que la condition **(ENAA)** est équivalente à l'existence d'un mesure risque-neutre équivalente $Q \sim P$ tel que $E_Q[L_T(V_T)] \leq 0$ pour toutes les valeurs liquidatives terminales du portefeuille V_T de notre modèle, où L_T désigne la fonction liquidative à l'échéance.

Soit $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ un pay-off qui est borné inférieurement. On définit le prix de sur-couverture de ξ comme suit

$$p(\xi) = \inf\{x \in \mathbf{R} | \, \exists V_T \in R_T^0 : \, x + L_T(V_T) \geq \xi\}.$$

Dans le modèle de Kabanov, on peut prouver que, sous la condition **(ENAA)**, ce prix est caractérisé par

$$V_0^\xi = \sup_{Q \in \mathcal{D}} E_Q \xi,$$

où $\mathcal{D}$ désigne l'ensemble des mesures probabilisé risque-neutre équivalente.

Q.2 *Quelle est la relation entre un modèle avec seulement les coûts proportionnels et un modèle avec à la fois coûts fixes et proportionnels ?*

Rappelons que si le marché avec les coûts fixes et proportionnels admet une opportunité d'arbitrage, c'est également le cas pour le marché avec seulement les coûts proportionnels (puisque les transactions dans le dernier sont moins chères). Réciproquement, si le marché avec seulement les coûts proportionnels admet une opportunité d'arbitrage (en quelque sorte), alors on peut multipier cette stratégie d'arbitrage par une grande constante suffisante pour réduire l'impact des coûts fixes aussi bas que possible. En conséquence, cette stratégie modifiée produira une nouvelle opportunité d'arbitrage dans le modèle avec des coûts fixes et proportionnels. Cette idée est formalisée dans la notion d'opportunités d'arbitrage faible **(WAO)**. Nous montrons que l'absence de **(WAO)** est équivalente dans les deux marchés.

Q.3 *Comment peut-on relier une condition non-arbitrage définie à l'aide de la fonction liquidative à celle définie en termes des vecteurs de portefeuille ?*

Pour cette question, on pourrait espérer que certaines conditions non-arbitrages sont équivalentes dans tous les deux contextes. Si c'est le cas, cela montre que notre théorie est compatible avec la théorie de l'arbitrage pour les modèles de Kabanov. Dans un modèle général avec plusieurs actifs, cette question reste encore ouverte. Mais au minimum, on peut montrer que c'est le cas pour le modèle avec deux actifs.

Q.4 *Dans le cas où les coûts fixes ne sont pas négligeables, c'est à dire lorsque les coûts fixes sont bornés inférieurement par une constante strictement positive, quel est l'impact des coûts fixes sur les conditions non-arbitrages ?*

Si les coûts fixes ne sont pas négligeables, une transaction sur le marché conduira à un coût strictement positif. Par conséquent, toute opportunité d'arbitrage de type

asymptotique (par exemple, **NFLVR**) aurait pu conduire à la présence d'une nouvelle opportunité d'arbitrage dans le sens classique (par exemple **NA**) s'il n'y avait pas des coûts fixes sur le marché. Cependant, par un argument similaire à celui mentionné dans le commentaire de **Q.2**, la présence d'une opportunité d'arbitrage (non asymptotique) sur le marché avec seulement les coûts proportionnels conduira à l'existence d'une nouvelle opportunité d'arbitrage (non asymptotique) sur le marché avec les coûts fixes et proportionnels. En conclusion, la présence des opportunités d'arbitrage asymptotiques et non asymptotiques sont équivalentes lorsque les coûts fixes sont non négligeables.

1.6 Maximisation d'utilité avec contraintes de risque

• **Motivation.**

Un des problèmes les plus classiques de la finance mathématique est la maximisation de l'utilité espérée de la richesse terminale. Mathématiquement, on s'intéresse au problème d'optimisation suivant :

$$u(x) = \sup_{\pi \in \mathcal{H}} E[U(X_T^{x,\pi})].$$

Ici, $(X_t^{x,\pi})_{0 \le t \le T}$ désigne le processus de la richesse produite par un capital initial x avec une stratégie admissible $\pi \in \mathcal{H}$, U est une fonction d'utilité qui peut être aléatoire. Un choix commun de U est $U(x) = u(x-B)$ où B désigne, par exemple, un passif aléatoire et u est une fonction non-décroissante, concave, déterministe comme fonction logarithmique, exponentielle ou de la puissance. Il existe une vaste littérature sur la maximisation de l'utilité espérée et on se contente à un très bref aperçu des approches suivantes.

Théorie de dualité convexe. La méthodologie de dualité convexe remonte à Bismut (1973). Dans la forme la plus simple des marchés complets, cette méthodologie est essentiellement basée sur l'application de la transformation de Legendre-Fenchel de la fonction d'utilité donnée par

$$\tilde{U}(y) := \sup_{x \ge 0} [U(x) - xy], y > 0.$$

La fonction duale permet d'associer le problème primal au problème duale formulé comme suit

$$v(x) = \inf_{y>0} E[\tilde{U}(yH)],$$

où H désigne la densité risque-neutre. Il a été montré que l'utilité marginale de la richesse terminale du portefeuille optimal est, à une constante, égale à la densité risque-neutre, c'est à dire $U'(X_T^{x,\pi^*}) = y^*H$. Le portefeuille optimal est construit en utilisant le théorème de représentation martingale (ou le théorème de décomposition optionnelle pour le cas des marchés incomplets). La méthode de dualité convexe a été développé par Pliska (1986), Cox et Huang (1989, 1991) et Karatzas, Lehoczky et Shreve (1987) pour les marchés financiers complets, et par He et Pearson (1991),

Karatzas, Lehoczky, Shreve et Xu (1991), Kramkov & Schachermayer (1999) pour le cas incomplète.

Programmation dynamique et les équations HJB. Bien que la théorie de dualité convexe est puissante pour prouver l'existence des portefeuilles optimaux dans le cas non-markovien, il ne montre pas une façon de caractériser ces stratégies optimales. L'approche de programmation dynamique devrait être considérée comme complémentaire à la dualité convexe et reste comme une méthode appropriée pour obtenir la caractérisation la stratégie optimale et la fonction de valeur. L'idée de cette méthode consiste à appliquer les outils de la théorie de contrôle stochastique pour obtenir un EDP (qui est appelée l'équation Hammilton-Jacobi-Bellman) pour la fonction valeur. Cependant, cette approche est basée sur l'hypothèse que les prix sont conduites par un processus de Markov. Pour les références, voir Merton (1971) Karatzas et Zitkovic (2003), Karatzas et Kardaras (2007).

EDS rétrograde. La théorie des EDSRs a été développée par Pardoux & Peng (1990). Il a rapidement devenu l'un des principaux courants de la théorie de contrôle stochastique avec une variété d'applications en finance mathématique. Pour le problème de maximisation de l'utilité, la méthode de EDSR s'avère être un bon remplacement pour les équations HJB dans un cadre non-markovien. Lorsque l'utilité prend certaines formes communes telles que la puissance, logarithmique ou exponentielle, il a été montré par Hu, Imkeller et Müller (2005) que la maximisation de l'utilité peut être essentiellement réduite à la résolution d'un EDSR qui caractérise la stratégie optimale et la fonction de valeur. Une approche EDS forward-backward est proposée par Horst et al. (2011) pour le cas des fonctions d'utilité générales.

L'optimisation statique et multiplicateurs de Lagrange. Cette méthode a été utilisée pour traiter le problème maximisation de l'utilité avec des contraintes de risque imposés sur la valeur terminale de portefeuilles. L'idée est de réduire le problème dynamique à un problème d'optimisation statique avec des contraintes. Le dernier est résolu en utilisant les multiplicateurs de Lagrange classiques. La stratégie d'investissement optimale sous contraintes de risque en termes de valeur à risque et un deuxième risque fonctionnelle ont été étudiés dans un cadre brownien par Basak & Shapiro (2001) et Gabih et al. (2005). Une solution complète dans une modèle général de semi-martingale avec les contraintes de type "utility-based shortfall risk" est donnée par Gundel et Weber (2005).

Comme mentionné ci-dessus, lorsque l'on impose une contrainte de risque sur la richesse terminale, le problème est résolu par des techniques d'optimisation statique. Ce travail vise à résoudre le problème par une approche différente. Au début, notre projet a visé à utiliser la méthode d'EDSR pour attaquer le problème. Malheureusement, une telle tentative n'a pas réussi et nous avons dû choisir une autre approche. D'un point de vue de contrôle stochastique, il n'est pas difficile de résoudre ce problème au moyen de la programmation dynamique et des équations de Bellman, car il est juste un cas particulier du problème de contrôle optimal sous contraintes des cibles (voir Bouchard-Elie-Touzi (2008) pour plus de détails sur la technique). Par conséquent, la dualité convexe reste comme une méthode de choix pour étudier notre problème de maximisation de l'utilité.

Comme discuté dans Gabih et al. (2005), la fonction de distribution de la richesse

optimale dans un problème de maximisation de l'utilité sans contrainte possède beaucoup de propriétés indésirables. D'un point de vue pratique, une mesure du risque devrait être envisagée pour une meilleure gestion de l'investissement. Par exemple, un problème commun est de limiter le déficit de la richesse finale au-dessous certain repère B sur le marché. Ici, B pourrait se présenter pour un niveau de défaut dans la gestion des risques, ou un indice boursier dans le cas où un investisseur veut vaincre le marché, ou tout simplement un passif ciblé dans la gestion des fonds ... Dans ces cas, un choix de mesure de risque pourrait être d'une grande importance. Quelques exemples courants de mesures de risque sont la Value at Risk (VaR) et Expected Shortfall (ES). Nous nous concentrons sur une autre catégorie de mesures de risque de la forme $\rho(X) := E[l(X-B)^-]$, où $l : \mathbf{R}_+ \to \mathbf{R}_+$ étant une fonction convexe non décroissante. Ce genre de mesures de risque doit être une bonne candidate pour la gestion des risques, car il possède les propriétés suivantes :

— Tout d'abord, il s'agit d'une mesure convexe, par conséquent, il soutient le principe de la diversification en investissement.
— Deuxièmement, en choisissant une fonction de poids l appropriée, la mesure du risque ρ est sensible aux pertes, ce qui n'est pas le cas pour Valeur à Risque (VaR).
— Troisièmement, cette mesure ne vérifie pas l'additivité ce qui est critiqué dans la gestion des risques (rappelons que la mesure de risque ρ est dit cash-additif si pour tout nombre réel c et position financière de $X, \rho(X+c) = \rho(X) - c$), il est donc pertinent pour une mesure interne du risque.
— Enfin, ρ ne dépend que de la part de perte, pas la partie de gain. Il s'agit d'une exigence naturelle parce que dans la gestion des risques, nous nous concentrons sur les pertes plutôt que des gains.

De arguments ci-dessus, nous sommes amenés à une forme générale de mesure du risque $\rho(X) := E[\ell(X)]$ où ℓ est une fonction de perte concave, non-décroissante et éventuellement aléatoire. Dans ce cas, la contrainte de risque est de la forme $\rho(X_T^{x,\pi}) \geq m$ pour certaine constante m et richesse terminale $X_T^{x,\pi}$.

• Nouveaux résultats.

Ce chapitre vise à appliquer la méthodologie de dualité convexe dans des situations différentes du problème de maximisation de l'utilité sous contraintes de risque.
Nous commençons par le cas d'un modèle de marché incomplète avec contrainte de risque quand le portefeuille est non-négative.

$$u(x) = \sup\{E[U(X_T^{x,\pi})] : \quad X_T^{x,\pi} \geq 0, E[\ell(X_T^{x,\pi})] \geq m\}.$$

Notons par $\mathcal{C}(x,m)$ l'ensemble des positions réplicables, c'est à dire l'ensemble des variables aléatoires qui sont bornées inférieurement et dominées par des richesses terminales $X_T^{x,\pi}$ vérifiant la contrainte de risque. Le domaine dual est défini par

$$\mathcal{Y} := \{(y,Y) \in \mathbf{R}_+ \times L_+^1 : EY \leq y, \delta(Y) := \sup_{X \in \mathcal{C}(x,m)-x} E[XY] < \infty\}.$$

Ici, δ est appelée la fonction de support. Notre premier résultat principal est la relation de dualité suivante. La preuve n'est pas très différente du cas sans contraintes.

La technique de la preuve est empruntée à Kramkov et Schachermayer (1999).

Theorem 1.6.1. *Considérons le problème dual*

$$w(x) = \inf_{(y,Y)\in\mathcal{Y}}[v(Y) + xy],$$

où $v(Y) := E[V(Y)] + \delta(Y)$ *et* V *est la fonction duale de Fenchel de* U*. Sous des conditions nécessairement techniques, on a les résultats suivants*

(i) *Le problème dual admet une solution optimale* $(y^*, Y^*) \in \mathcal{Y}$:

$$w(x) = v(Y^*) + xy^*.$$

(ii) *La dualité est vérifiée*

$$u(x) = \inf_{(y,Y)\in\mathcal{Y}}[v(Y) + xy] = v(Y^*) + xy^*.$$

De plus, la richesse optimale est donnée par $X^* = I(Y^*) := (U')^{-1}(Y^*)$.

Ensuite, dans le même modèle, on relaxe la condition de positivité sur le processus de richesse. L'idée de la preuve est de considérer le problème d'approximation (voir Schachermayer (2001)) :

$$u_n(x) = \sup_{X\in\mathcal{C}_n(x,m)} EU_n(X),$$

où $U_n(x) = U(x)$ if $x \geq -n$ et $U_n(x) = -\infty$ par ailleurs. Dans une même manière que le cas de richesses non-negatives, le problème primal approximative admet solution unique X_n donnée par $X_n = I(Y_n) \vee -n \in \mathcal{C}_n(x, m)$. En faisant n tendre vers l'infini, on obtient un résultat similaire comme dans le cas précédent.

Il est facile de voir que, dans la méthode de dualité que nous utilisons ci-dessus, la contrainte de la perte est déterminée implicitement par le domaine primal $\mathcal{C}(x, m)$ et par la fonction de support. Par conséquent, il peut être directement généralisé au cas de contraintes convexes générales. Nous proposons une autre approche qui est toujours basée sur la méthode de dualité. Pour des raisons de simplicité, nous ne considérons que le cas des marchés complets avec des contraintes de positivité sur la richesse. Cette approche permet de montrer la contrainte de risque explicitement dans le problème dual. L'idée est de remplacer la contrainte espérée par un nombre infini de contraintes presque-sure comme considérée dans Bouchard-Touzi-Elie (2009).

$$E[l(X_T^{x,\pi})] \geq m \Leftrightarrow \exists M \in \mathcal{M} :\ X_T^{x,\pi} \geq M,$$

où

$$\mathcal{M} = \mathcal{M}(x, m) := \{M \in L_+^0(\mathcal{F}_T) :\ E_Q M \leq x, E[l(M)] \geq m\}.$$

Il est alors naturel de remplacer la fonction duale de Fenchel sans contrainte par une fonction duale de Fenchel avec contrainte $V_p(y) := \sup_{r\geq p}[U(r) - ry]$. En l'utilisant, on est amené aux inégalités suivantes

$$u(x) \leq \sup_{M\in\mathcal{M}} w_M(x) \leq \inf_{y>0}[v(y) + xy], \tag{1.6.5}$$

où

$$w_M(x) := \inf_{y>0}\{E[V_M(yH)] + xy\}, \tag{1.6.6}$$

et

$$v(y) = v_{x,m}(y) := \sup_{M\in\mathcal{M}} E[V_M(yH)]. \tag{1.6.7}$$

Ici, Q désigne la mesure de martingale risque-neutre. Comme prévu, on peut montrer que sous certaines conditions techniques, la dualité est vérifiée

$$u(x) = \sup_{M\in\mathcal{M}} w_M(x) = \inf_{y>0}[v(y) + xy], \tag{1.6.8}$$

et la richesse optimale est donnée par

$$X^* = -V'_{M^*}(y^*\frac{dQ}{dP}) = \max\{M^*, I(y^*\frac{dQ}{dP})\},$$

où M^* est le maximizer du problème dual $\sup_{M\in\mathcal{M}} w_M(x)$ et $y^* = y^*(M^*)$ est le minimizer de (2.6.6).

D'un point de vue pratique, tous les résultats obtenus ci-dessus ne sont pas tout à fait satisfaisants, car ils ne présentent pas un moyen de calculer numériquement, au moins la fonction de valeur. Même en cas des marchés complets , la complexité du problème dual n'est pas significativement plus faible que le problème primal (puisque nous introduisons un nouveau ensemble $\mathcal{M}(x,m)$ de variables aléatoires supplémentaires). Nous présentons ensuite une nouvelle méthode basée sur la relation entre le problème de maximisation d'utilité et le problème de type de couverture (quantile). On travaille encore sur un marché avec contrainte de positivité sur la richesse et une contrainte de risque à l'échéance. Considérons le problème suivant :

$$v(y,m) := \inf\{x : \exists\pi : E[U(X_T^{x,\pi})] \geq y,\, E[l(X_T^{x,\pi})] \geq m\}.$$

Il est clair que $u(.,m) = v^{-1}(.,m)$. Par conséquent, le problème primal est réduit à étudier un nouveau problème de type de couverture avec des cibles stochastiques multiples, ou EDSR avec les conditions terminales faibles, comme la terminologie de Bouchard-Elie-Réveillac (2013). Pour ce nouveau problème, nous appliquons la technique de dualité pour caractériser la fonction de valeur et la richesse optimale. Plus précisément, on s'intéresse à la problématique suivante

$$v(\mathbf{m}) := \inf\{x : \exists\pi : E[\mathbf{l}(X_T^{x,\pi})] \succeq \mathbf{m}\},$$

où $\mathbf{l} := (\mathbf{l}_1, \ldots, \mathbf{l}_n)$ est une fonction de perte qui est déterministe et multidimensionnelle, $\mathbf{m} \in \operatorname{Im}\mathbf{l}$ et $\succeq$ est entendu au sens composant. En outre, $\mathbf{l_k}$ est supposé être strictement croissante pour tous $1 \leq k \leq n$. On a donc

$$E[\mathbf{l}(X_T^{x,\pi})] \succeq \mathbf{m} \Leftrightarrow \exists M \in \mathcal{M} : X_T^{x,\pi} \geq F(M),$$

où $F(M) = \max_{1\leq k\leq n}\{\mathbf{l}_k^{-1}(M_k)\}$ étant une fonction convexe et croissante en composant de M, et

$$\mathcal{M} := \{M = (M_0, \ldots, M_K) \in L^1(\operatorname{Im}\mathbf{l}) : \mathbf{l}_k^{-1}(M_k) \in L^2_+, E(M_k) = m_k\, \forall\, k\}.$$

Par le principe de comparaison, on obtient l'estimation suivante

$$v(\mathbf{m}) \geq \inf_{M \in \mathcal{M}} \{X_0^{F(M)}\},$$

où $X_0^{F(M)}$ est la solution à l'EDSR suivante avec la condition terminale $F(M)$.

$$X_0^{F(M)} = F(M) - \int_0^T (r_t X_t + \theta_t Z_t) dt - \int_0^T Z_t dW_t.$$

En utilisant la fonction duale de Fenchel de F : $\tilde{F}(p) := \sup_{\mathbf{r} \in \mathrm{Im}\, \mathbf{l}} [p.\mathbf{r} - F(\mathbf{r})]$, on obtient la relation duale suivante

$$v(\mathbf{m}) = \inf_{M \in \mathcal{M}} \{X_0^{F(M)}\} = \sup_{p \in \mathbf{R}_+^{K+1}} E[p.\mathbf{m} - \tilde{F}(L_T^{-1} p) L_T],$$

où (L_t) désigne le processus de prix d'état. Le dernier problème est facile à résoudre car il est considéré sur $\mathbf{R}_+^{K+1}$. On peut alors caractériser la richesse optimale du problème primal au moyen de la solution au problème dual.

• **Perspectives.**

On présente ici quelques perspectives du travail.

Tout d'abord, puisque l'on peut facilement simuler L_T à partir des paramètres du modèle, la dernière équation de dualité pourrait être exploitée pour calculer numériquement la double fonction de la valeur $v(\mathbf{m})$, donc la fonction valeur primale.

Deuxièmement, la technique de dualité utilisée dans les deux dernières sections pourraient être généralisés à des marchés incomplets. Dans ce cas, il n'est pas difficile de prouver le théorème de vérification en utilisant la fonction duale de Fenchel avec contraintes. Cependant, l'existence de solutions au problème dual reste encore comme une question ouverte en raison du fait que certaines conditions d'intégrabilité ne sont plus vérifiées lorsque nous introduisons des variables aléatoires M.

Troisièmement, les contraintes convexes de portefeuille peuvent être prises en compte. Notons que lorsque l'on utilise directement la méthode de dualité convexe, la question clé est la convexité de l'ensemble des portefeuilles. Lorsque le problème primal est lié au problème de type quantile-hedging par la méthode EDSR, nous sommes amenés à une EDSR avec les contraintes en maturité et des contraintes convexes dynamiques, ce qui pourrait faire objet intéressant pour les études futures.

Enfin, la contrainte terminale peut être remplacée par une contrainte de type américain, c'est à dire les pertes sont contrôlées à tout moment avant l'échéance. On peut prouver que ce type de contrainte peut être transformé en un nombre infini de contraintes presque-sûre, où une situation similaire de celui-ci a été étudiée, par exemple, dans Karoui & Jeanblanc (1998) en présence de revenus du travail.

Chapitre 2

General Introduction

2.1 Motivation : Market with transaction costs

Among the fundamental assumptions of mathematical models for financial markets is the assumption that markets are frictionless. In particular, investors can make their transactions without brokerage fees, taxes or bid-ask spreads... This leads to the fact that trading can be done in continuous time, which is an unrealistic assumption since continuous trading in reality implies that investors may suffer an infinite loss due to transaction costs. The seminal papers studying market models with transaction costs belong to Jouini and Kallal (1995) who studied the case of markets with two assets and trading is subject to bid-ask spreads. Thereafter, many articles in economics and finance relaxing the assumption of no transaction costs have been proposed. Numerous studies revisit classical problems with transaction cost constraints. A special attention has also been paid to framework with transaction cost structures.

The structures of transaction costs may be very sophisticated. We refer readers to Kissell and Glantz (2003) for a thorough classification of transaction costs. In the most general sense, transaction costs consist of both fixed and variable costs, as well as both visible and hidden costs. Fixed components are those costs that are independent of the current market prices or implementing strategy and commonly known in advance such as commissions, fees. Variable costs are costs that are determined by the current market prices and dependent upon the trading strategy. Visible costs are either known exactly in advance or easily measurable from current market data. They consists of fixed costs and part of variable costs such as bid-ask spreads or taxes. Hidden costs are those components that are not readily known or observable from the current market data (e.g. market impact and other liquidity effects such as delay cost, price appreciation, timing risk and opportunity costs). In this thesis, we restrict the concept of transaction costs to visible costs. In practice, visible costs are commonly represented by some piecewise function of the transaction volume.

Lots of studies simplify the notion of transaction costs by assuming that they are proportional to the trading volume. More realistic is to model transaction costs by including both fixed and proportional costs (i.e. linear costs). Jouini, Kallal and Napp (2001-2006) first attempted to study the theory of no arbitrage in the presence of fixed costs. As a matter of fact, fixed costs are difficult to deal with as the classical technic of convex duality is no longer valid. Therefore, there are very few studies

about markets with both fixed and proportional transaction costs. On the contrary, much has been done for markets with only proportional transaction costs. In the multivariate setting where direct exchanges between assets is possible, which is typically the case for currency markets, a geometric approach is introduced by Kabanov (1999). In this model, the portfolio is represented by a multidimensional process (V_t), whose each component process (V_t^i) reflects the dynamics of trading position in a special asset i expressed in terms of physical quantity. This approach allows to avoid the question of numéraire and, equally important, it is more realistic in describing the admissibility condition of trading strategies. The solvency cone K_t at time t is defined as the set of positions which can be transformed into non negative holdings after suitable exchanges, and the self-financing condition reads as $\dot{V}_t \in -K_t$ for all t.

Though linearity of transaction costs is a strong simplification of the real world, it may be adequate under certain circumstances and leads to satisfying mathematical results. However, this approach may produce an erroneous result when the amount of investment is small and it is allocated to many assets in smaller fractions. The reason is simple : when transaction volume is small, the fixed costs are not negligible, and this cost is decreasing when trading volume increase (transaction cost is concave). Otherwise, linearity is not a good idea to model transaction costs when trading volume is so large, due to the illiquidity effect. That is, if for a certain security the transaction volume is high, then there can be not enough supply (demand) of this security, and thus the unit transaction cost will increase (transaction cost is convex). Therefore, a non-linear transaction cost function should be a relevant approach : a concave transaction cost function is good for modelling small investors, while a convex transaction cost function is appropriate for models with large investors. Lots of researchers suppose a mixture of both concave and convex transaction cost structure : Up to a certain level of the transaction volume, the transaction cost is a concave function of the transaction volume. However, the transaction cost function becomes a convex function when transaction volume goes beyond this level, for references, see Demchuk (2002), Kono and Wijayanayake (2001) and references therein.

There is a considerable literature revisiting classical (and new) problems when markets are subject to the presence of transaction costs. Lots of papers analyse the effect of transaction costs on arbitrage opportunities, both in discrete and continuous time (arbitrage and pricing theory under transaction costs). We can cite here Jouini and Kallal (1995), Soner, Shreve and Cvitanic (1995), Guasoni (2006), Cherny (2007), Guasoni et al. (2010), Lépinette and Kabanov (2012). A special attention has been paid for the problem of approximating hedging under transaction costs, for instance the method of Leland (1985) and some further development thereafter such as Kabanov and Safarian (1997), Bensaid et al. (1992), Soner et al. (1995), and Hodges and Neuberger (1989), Pergamenshchikov and Nguyen (2012). There are also several researches on problems of portfolio optimisation or risk minimisation under transaction costs, for example, Constantinides (1979) and Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Amihud and Mendelson (1986), Framstad et al. (1999), Gerhold et al. (2013), and many other papers.

2.2 Contributions of the thesis

The main part of this dissertation deals with various problems for markets with transaction costs. In particular, we try to answer the following main questions :

Q.1 How can we modify the hedging strategy of Leland for models with local volatility ?

Q.2 How to generalise the optimal consumption problem of Merton to the Kabanov model when the prices are driven by a Lévy process, and default risk is possible ?

Q.3 How can we employ the notion of Essential Maximum introduced by Kabanov and Lépinette (2013) in order to characterise the hedging price of a European claim under both fixed and proportional transaction costs ?

Q.4 In a model with both fixed and proportional transaction costs, what are analogies of FTAP ?

Q.5 Does the classical duality technic still work in the problem of utility maximisation under expected loss constraints ?

Each question above touches on a specific problem in Finance and will be presented in separate parts of the thesis. The five parts can then be read independently. Except the last one, all other parts are related to market models with transaction costs.

The thesis is split into seven chapters

- Chapters 1, 2. In these two first chapters we provide an brief introduction to the thesis : the context of markets with transaction costs, the contribution of the thesis, an overview of every part of the thesis.
- Chapter 3. In this chapter we study the problem of approximate hedging in a two dimensional local volatility model in the presence of proportional transaction costs. Our principal contributions in this chapter are
  — Introduction of a PDE technic that allows to obtain upper bounds of Greeks of a European option in a local volatility model.
  — Generalisation of the modified Leland's strategy to models with local volatility.
- Chapter 4. In this chapter we study the problem of optimal consumption in the Kabanov model when prices are driven by a Lévy process. The classical Merton problem is generalised in three directions :
  — First, we allow shocks in markets by modelling prices by Lévy processes.
  — Second, default risk is taken to account : transactions stop at the first time when the portfolio goes out of the solvency cone. This is more realistic than assuming that investors never default.
  — Third, trading strategies are allowed to be làglàd in order to capture both accessible and inaccessible shocks in the market. Moreover, this approach is also consistent with the construction of optimal strategy based on SDEs with oblique reflections.
- Chapter 5. In this chapter we study the problem of super hedging in a general market model including the case of markets with both fixed and proportional transaction costs. Our principal contributions in this chapter are
  — Introduction of a new market model defined by the liquidation process that allows to reduce the study of multidimensional portfolios to real valued liquidation processes.

  - — Characterisation of super hedging prices of a European option in this model by using the notion of Essential maximum in the multidimensional case.
- Chapter 6. This chapter continues to study the model suggested in the previous chapter in the context of arbitrage theory. Our principal contributions in this chapter are
  - — Introduction of an equivalent condition for existence of a risk neutral probability measure, i.e. absence of asymptotic-arbitrage opportunity.
  - — Characterisation of super hedging prices for a European option (in terms of cash, not vector of quantities) in the Kabanov model via the duality technic.
  - — Study of different notions of arbitrage opportunity for this model as well as the relationships between model with both fixed & proportional transaction costs and markets with only proportional costs.
- Chapter 7. This chapter deals with the problem of utility maximisation under target risk constraints. We make use of the duality technic in various situations. Our principal contributions in this chapter are
  - — A proof of duality relation and construction of the optimal wealth via the constrained Fenchel dual function.
  - — A solution to the problem in the case of complete market based on a mixture use of both BSDE technic and duality technic.

List of publications or pre-publications

i. Approximate Hedging in a Local Volatility Model with Proportional Transaction Costs, with E. Lépinette. *Applied Mathematical Finance*. 2014.

ii. Consumption-Investment Optimisation Problem in a Levy Financial Model with Transaction Costs, with Y. Kabanov and E. Lépinette. Preprint.

iii. General Financial Market Model Defined by a Liquidation Value Process, with E. Lépinette. Submitted to *Stochastics*. 2014.

iv. Risk Neutral Probabilities and Absence of Arbitrage Opportunity in Non Convex Financial Market Models, with E. Lépinette. Preprint.

v. Utility Maximisation Problem With Expected Loss Constraints. Preprint.

2.3 Approximate hedging in a local volatility model with proportional transaction costs

• Motivation.

This first chapter of the thesis deals with the problem of asymptotic hedging a European option in a two dimensional local volatility market model when the bid-ask spread is proportional to the traded value. This problem is first studied in the seminal paper of Leland (1985) for the Black-Scholes model. In his work, Leland assumes that the cost of trading ν assets costs an amount $\kappa\nu S$ for both buying and selling, where κ is coefficient of transaction costs and S is price of one unit share. The hedging portfolio is revised every δt where δt is a finite and fixed, small time step, whether or not this is optimal

in any sense. In this model the current value of the portfolio process at time t is defined by

$$V_t^n = V_0^n + \int_0^t D_u^n dS_u - \sum_{t_i \leq t} k_n S_{t_i} |D_{i+1}^n - D_i^n|, \quad t < 1.$$

where $t_i = t_i^n, 0 \leq i \leq n$, $t_0 = 0, t_n = 1$, are the revision dates such that $t_{i+1} - t_i = \delta t$; and D^n is the trading strategy which is piece-wise, i.e. $D^n = D_i^n$ on the interval $]t_{i-1}, t_i]$. In the presence of transaction costs, Leland suggests a strategy which can be considered as a modified Delta of Black-Scholes replication formula. The idea is to substitute the volatility σ by an enlarged one $\hat{\sigma}$ in order to compensate for the transaction costs. The "enlarged volatility" is defined by

$$\widehat{\sigma}^2 = \sigma^2 + 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}} = \sigma^2 + 2\kappa\sigma n^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}.$$

The hedging strategy is given by $\hat{D}_i^n = \hat{C}_x(t_i, S_{t_i})$ where $\widehat{C}(t,x) = \widehat{C}^n(t,x)$ is the solution to the Black-Scholes heat equation with new diffusion $\hat{\sigma}$

$$\begin{cases} \widehat{C}_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2 x^2 \widehat{C}_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\times[0,1[\ , \\ \widehat{C}(1,x) & = & h(x), \quad x \in]0,\infty[\ . \end{cases}$$

Leland argued that when the number of revision dates n is large, i.e. when $\delta t = \frac{1}{n}$ converges to 0, the value of hedging portfolio will converge to the value of contingent claim h in probability, i.e. this hedging strategy will asymptotically hedge the European option without error. Moreover, the initial value of this hedging strategy is given by $V_0^n = \widehat{C}(0, S_0)$ which is greater than the hedging price in the Black-Scholes formulae due to the increasing of the volatility in the Leland's strategy. This is implied from the intuition that the option price should include an necessary extra cost that the writer of the option has to pay due to the bid-ask spreads when he or she constructs a hedging portfolio. When the revision interval tends to zero, this price will increase to S_0, i.e. the cost of the buy-and-hold strategy. This means that at the Leland strategy is not better than the trivial supper hedging when trading happens almost continuously.

Unfortunately, the claim of Leland is only true in the case when κ is decreasing to 0 with a speed of $n^{-\alpha}, 0 < \alpha \leq \frac{1}{2}$ as proved by Kabanov and Safarian (1997), But it fails to hold true in the case of constant cost coefficient $\kappa = const$. In this case, it is reported in Kabanov and Safarian (1997) or later in Pergamenshchikov (2003) that the error when using the approximate portfolio to hedge the option converges with a rate of convergence of $n^{-\frac{1}{4}}$ to a non-zero random variable. The economical interpretation of this phenomenon is clear : a trader does not have any incentive to trading "almost continuously" if he does not receive a reduction in transaction costs when trading more frequently (equivalently, trading with large orders), otherwise he will bear a significant loss due to transaction costs.

An attempt to deal with the case of constant cost belongs to Lépinette (2011). He suggests a modification of the conventional Leland's strategy in order to cancel out the hedging discrepancy. Namely, he construct the strategy as follows

$$D^n_{t_i} = \widehat{C}_x(t_i, S_{t_i}) + \int_0^{t_i} \widehat{C}_{xt}(t, S_t)dt. \tag{2.3.1}$$

It is important to stress that the hedging price of Lépinette's strategy is exactly the same as that of Leland's strategy. Since the asymptotic hedging error disappears, it shows that the modified strategy outperforms the older one. A study of convergence rate is also provided in a later paper of Lépinette and Darses (2012).

• **New results.**

The goal of my first project in the present thesis is then to generalize the result of Lépinette to the case of local volatility models. We prove that the modified hedging strategy still works in this new setting, i.e. asymptotic hedging without error. We restrict our attention to a local volatility model for two reasons. First, local volatility models are popularly used in practice as it can be simply calibrated to vanilla options by Dupire's formulae. One of the advantages of local volatility models over stochastic volatility models is that they conserve the completeness of the market, which is a key factor in derivative pricing practices. Second, Lépinette's strategy might not be valid in models with stochastic volatility due to the fact that the exact replication fails in these models. It is therefore of no interest to study a specific trading strategy that leads to an asymptotic positive hedging error (except that we can show that this strategy is optimal in some sense). Our main result in this part is the following theorem

Theorem 2.3.1. *Let $\alpha \in [0, \frac{1}{2}]$ and assume that h and σ are such that $\widehat{C}_{xx} \geq 0$. If the strategy $\widehat{D}^n_i$ is given by (2.3.1), then under some technical conditions, the terminal value of the portfolio*

$$V^n_1 = \widehat{C}^n_0 + \int_0^1 \widehat{D}^n_u dS_u - k_n \sum_{i=1}^{n-1} S_{t_i} |\widehat{D}^n_{i+1} - \widehat{D}^n_i|$$

converges to the contingent claim $h(S_1)$ in probability.

The proof of this theorem is basically similar to the one in Lépinette (2011). The key technic resides in the estimation of successive derivatives of $\widehat{C}(t, x)$. When the volatility function is constant or deterministic, things are simpler because we can employ explicitly expressions for these derivatives. In the case of local volatility, we face a considerable difficulty as we do not know such expressions. The only thing we can do is to appeal classical technics from the PDE theories to obtain accurate bounds for the derivatives of $\widehat{C}(t, x)$. A clever use of time change technic is necessary to obtain the desired results.

• **Future perspectives.**

The results obtained can be generalized or developed in different directions. First, notice that the transaction cost function that we consider in this chapter is of the linear form, i.e. $G(t, S_t, x) = |x|S_t$ where x denotes the trading volume and S_t denotes the spot at time t. The linearity of G therefore can be relaxed in order that the model be more realistic, e.g. trading impact or regulatory constraints. We refer to Elie and Lépinette (2013) for the same problematic in the case of Black-Scholes model and when G is not dependent on the spot price , i.e $G = G(t, x)$. A similar problem is also studied in Nguyen (2014) for stochastic volatility models and transaction function of the form $G = G(t, xS_t)$ (supply curve).
Second, since we know that the hedging error $V_1^n - h(S_1)$ converges to zero when n tends to infinity. It is natural to study the rate of this convergence. But even in the simplest case of Black-Scholes model, the proof has already used heavily probabilistic technics. Therefore, the question of convergence rate remains open in our local volatility model. Moreover, we expect that when n tends to infinity, the hedging cost $C^n(0, S_0)$ converges to the super hedging price of the pay-off. At least it is the case for Eropean Calls, when $C^n(0, S_0)$ converges to the price of the buy and hold strategy, i.e. $S_0^a := S_0(1 + \kappa)$. We are also concerned with the rate of convergence $S_0^a - C^n(0, S_0)$. This question is of great importance for practitioners, in particular if the rate of convergence of $C^n(0, S_0)$ towards the buy and hold price S_0^a is significantly lower than the rate of convergence of V_1^n towards $h(S1)$, then we can conclude that the modified strategy of Lépinette is really valuable in practice.

2.4 Optimal consumption problem in a Kabanov's model with jumps

• **Motivation.**

In this second chapter, we study the classical optimal consumption problem in a Kabanov's model with jumps, i.e. markets with proportional transaction costs and prices being driven by Lévy processes. The study of consumption–investment problems in continuous time was initiated by the seminal paper of Merton (1971). He considered a model of frictionless market where the price processes are geometric Brownian motions. Given a utility function that represents the risk reference of the investor, his goal is to maximize the expected discounted utility of consumption on the infinite time interval. For the power utility function, he obtained an explicit solution of the optimal control problem. Merton found that, in computing the optimal investment rules, the investor has to keep the proportions of the total wealth held in risky securities equal to a constant vector which is easily calculated from the model parameters (Merton proportion). This result is of great importance

in investment theories and largely used by practitioners in mutual funds management.

The work of Merton was extended by many authors in various directions including models with transaction costs and jumps, which are the main objects of our interest. In the case of markets with two assets and proportional transaction costs (without jumps), Davis and Norman (1990) studied the structure of the value function but also provided a rigourous construction of the optimal solution. The authors claimed that the optimal policy is to keep the proportions of the total wealth held in risky security always staying in a wedge region containing the Merton line. The upper and lower bounds of this wedge can be of course computed in terms of model parameters. When the proportions of investor's wealth invested in the risky asset lies within this region, the investor does not transact. He merely consumes from his bank account. When fluctuations in the price processes drive the proportions of wealth invested in the stock to the boundary of the region of inaction, the investor transacts the minimal amount required to keep the proportion in the region of inaction. Moreover, the optimal amount to consume from the bank account is deduced from the solution to a non-linear PDE, known as the Hamilton-Jacobi-Bellman (HJB) equation. The main difficulty to obtain this solution is that the domain of the HJB equation is not exogenously specified. Rather it is specified endogenously via conditions that determine the region of inaction. For a computational scheme to solve this HJB equation in multidimensional cases, we refer readers to the paper of Muthuraman and Kumar (2006), a complete solution to the problem in terms of viscosity solutions is given in Shreve and Soner (1994), the portfolio optimization problem under small transaction costs is studied in Touzi and Soner (2013).

Though geometric Brownian motion gains a popularity in financial modelling, it is criticized because of its weak predictability of price movements on markets. Diffusion models using Lévy processes seem to be more flexible and capture statistical and economical properties of market data and are still mathematically tractable. Recently, several papers have investigated the optimal consumption and investment problem when prices are modelled by Lévy processes and markets are subject to transaction costs. For example, Framstad et al. (2001) studied the two assets case, while Kabanov and De Vallière (2009) generalized the problem to a very abstract setting of models with transaction costs and multi-assets. The goal of our work is two-folds. First, it provide rigourously mathematical proofs for some results in these two papers such as the HJB equation and uniqueness of solutions, the construction of an optimal strategy. Second, it unifies the approaches of two papers in a general Kabanov's model where prices are driven by Lévy processes and trading strategies are allowed to be làdlàg, but not càdlàg as supposed in the standard literature. This allows the work not only to be consistent with the construction of an optimal strategy which is typically càglàd in a Lévy model, but also to capture both accessible and inaccessible market shocks.

• **New results.**

Chapter 3 is divided in two parts. In the first part we study the optimal consumption problem in a abstract setting based on Kabanov's model. The second part is devoted to a special case of markets with two assets. The abstract model is described as follows.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual hypotheses. We consider an agent who invests his wealth in a portfolio of multiple assets whose price return process is driven by a $d-$ dimensional Lévy process $(Y_t)_{t\geq 0}$

$$dY_t = \mu t + \Xi dW_t + \int_{\mathbf{R}^d} z\tilde{N}(dz, dt),$$

where $\mu \in \mathbf{R}^d$, W is an m-dimensional standard Brownian motion and Ξ is a $d \times m$-matrix. Moreover, $\tilde{N}(dz, dt)$ denotes the compensated Poisson random measure whose compensator take the form $dt\Pi(dz)$.$\Pi(dz)$ is usually called the Lévy measure which is a $\sigma-$ finite measure on the Borel sets of $\mathbf{R}^d \setminus \{0\}$ satisfying the following condition

$$\int_{\mathbf{R}^d\setminus\{0\}} \left(|z|^2 \wedge |z|\right) \Pi(dz) < \infty.$$

We consider a version of Kabanov's model which is slightly more general than that of the standard model of financial market under constant proportional transaction costs. We are given two constant cones K and $\mathcal{C}$ which are supposed to be closed and proper, i.e. $K \cap (-K) = \{0\}$ and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. We assume that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. In financial terms, K stands for the solvency cone, for instance we can chose K to be the set of positions such that when we convert the actual portfolio to any asset and pay transaction costs, the liquidation values are nonnegative ; while $\mathcal{C}$ consists of consumption possibilities, for example, if we only consume on the bank account, then $C = \mathbf{R}_+ e_1$. The dynamics of a portfolio process is defined for each $i = 1, \ldots, d$ by :

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_0 = V_{0-} = x,$$

where the controls $\pi = (B, C)$ are làdlàg and predictable processes of bounded variations. We require that $dB_t \in -Kdt$ and $dC_t = c_t dt \in \mathcal{C}dt$. These condition will be described in detail in the thesis. The dynamics means that such a portfolio V is self-financing, i.e. its increments are only due to the increments of Y, and transaction costs modelled by B are withdrawn while C represents the cumulated sum of consumed wealth.

Example 2.4.1. *We give here an example of the abstract Kabanov's model. We consider a market consisting of two asset : the first one is a risk-free asset with interest rate zero an the second one is a risky asset whose prices*

are driven by a Lévy process. Suppose that the dynamics of portfolio are given by

$$\begin{aligned} dV_t^1 &= dL_t^{21} - (1+\lambda^{12})dL^{12} - c_t dt, \\ dV_t^2 &= V_{t-}^2 \left(\mu dt + \sigma dW_t + \int_{\mathbf{R}} x(p(dy,dt) - q(dy,dt)) \right) + dL_t^{12} - (1+\lambda^{21})dL^{21}, \end{aligned}$$

where L^{ij}, $i,j = 1,2$, are the transfer processes we suppose to be làdlàg and λ^{ij}, $i,j = 1,2$, are the transaction costs coefficients. We rewrite the dynamics of a portfolio process under the standard form :

$$dV_t = \operatorname{diag} V_{t-} \left(\tilde{\mu} dt + \tilde{\sigma} dW_t + \int z(N(dz,dt) - \tilde{\Pi}(dz)dt) \right) + dB_t - dC_t.$$

In this framework, the controls are ${}^t dC_t = (c_t dt, 0)$, and

$${}^t dB_t = (dL_t^{21} - (1+\lambda^{12})dL^{12}, dL_t^{12} - (1+\lambda^{21})dL^{21}).$$

The solvency cone K is simply a sector generated by the two vectors

$$g_1 = (1+\lambda^{12})e_1 - e_2, g_2 = (1+\lambda^{21})e_2 - e_1.$$

The consumption region is $\mathcal{C} = \mathbf{R}_+ e_1$.

For every admissible control $\pi \in \mathcal{A}_x$, let us introduce the stopping time

$$\theta^\pi = \inf \{ t : V_t^\pi \notin \operatorname{int} K \}.$$

We define the utility process

$$J_t^\pi(x) := \int_0^{t \wedge \theta^\pi} e^{-\beta s} U(c_s) ds,$$

where β is a positive discount rate and U is a given non-negative mapping defined on $\mathcal{C}$ which represents a utility function ; it is assumed to be concave, $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. The optimal consumption problem consists in maximizing the utility process over the set of admissible strategies. To do so, we define the Bellman function as

$$W(x) := \sup_{\pi \in \mathcal{A}_x} \mathbb{E}[J_\infty^\pi(x)], \quad x \in \operatorname{int} K. \tag{2.4.2}$$

The main result of this chapter is to prove that, under some mild conditions, the Bellman function is a unique solution to the HJB equation of the following form

$$\begin{aligned} &F(W''(x), W'(x), H(W,x), W(x), x) = 0, \quad x \in \operatorname{int} K, \\ &W(x) = 0 \quad \text{on } \partial K. \end{aligned}$$

We leave all details about the equation and conditions necessary to Chapter 3. At this stage, we have several remarks as follows.

- First, as the controls are assumed to be làdlàg, we have to redefine the notion of stochastic integrals with respect to làdlàg predicable integrands. This is done and presented in the appendix of the chapter 3.
- Second, we allow the possibility that the investor may go bankrupt if his position is insolvent. This constraint makes the problem more difficult in the sense that it is no longer straightforward to conclude about the concavity of Bellman function as in continuous diffusion models. Actually, the concavity is important as it allows to study the structure of the Bellman function. Fortunately, we can prove that if the HJB equation admits unique solution, then the bankruptcy constraint is not binding, i.e. it is not optimal to deliberately get the portfolio out of the solvency cone.
- Last, note that the only difference between our model and a model driven by a multidimensional. Brownian motion is the presence of the operator $H(f,x)$ which is given by

$$H(f,x) := \int_{\mathbf{R}^d} \left[f(x + \mathrm{diag}\,(x)z) 1_{x+\mathrm{diag}\,(x)z \in \mathrm{intK}} - f(x) - f'(x)\mathrm{diag}\,(x)z \right] \Pi(dz).$$

  This integro-differential operator is not defined in a local way. Therefore, we then need to define viscosity solutions in the global sense.

In the second part of Chapter 3, we revisit the optimal consumption problem in the two dimensional case. This case has been fully studied by Soner and Shreve (1994) when the prices are driven by a geometric Brownian motion. As the uniqueness of solutions holds in this model, the Bellman function is concave and therefore we can make use of the machinery of convex analysis to study the solution to the HJB equation. We then recover most of their results in this new setting, from the structure and regularity of the Bellman function, to the construction of the optimal strategy based on the notion of SDEs with reflections which is in turn, has to be rigourously rewritten when jumps are involved.

• Future perspectives.

The Kabanov's model turns out to be a good framework for studying markets with friction. It is not only general enough to capture the case of proportional transaction costs, but also leads to satisfying mathematical results such as HJB equations. In this new setting, we can also generalize the optimal consumption-investment problem in different directions. For instance, if we allow an endowment to rescue the investor in case of default, it is no longer obvious that the investor always stays in solvency at optimality. Some conditions should be relaxed or modified, for example the utility function may be wealth-dependent, the market coefficients may be stochastic, the trading may be faced to liquidity risks, the horizon may be finite or random, the investor may receives labor income, so on and so forth. We can also consider a more general optimization problem of recursive utility, and add some more risk constraints on portfolios or wealths such as limit trading constraint or drawn-down constraints... We believe that these problems can be solved

within the context of Kabanov's model and lead to many promising results in the future.

2.5 A general market model defined by the liquidation processes

• **Motivation.**

In the third and fourth chapters, we introduce a general model that captures both fixed and proportional transaction costs. In this new setting, we study the pricing theory in the absence of arbitrage opportunities. Two fundamental questions rise. First, given a financial market model, we are concerned about the possibility of making profit out of nothing via trading activities on the market, i.e. starting from a null or debted position, can we end up with a nonnegative wealth and making a gain with strictly probability ? If the answer to this question is yes, we say that the market admits an arbitrage opportunity. Second, given a market model which is free of arbitrage opportunities and a contingent claim that pays off a financial flow in the future to its holder, what is the smallest initial amount x of money from which the writer of the claim can start a portfolio such that the values of this portfolio dominate the financial flow of the contingent claim ? We call it the super hedging problem, and the value x is called to be the super-hedging price of contingent claim. In the case of a discrete market model without transaction costs, the arbitrage theory is initiated from the famous theorem of Dalang-Morton-Willinger which states that the market is arbitrage-free if and only if there exists an equivalent martingale measure. This result is usually referred to as The Fundamental Theorem of Asset Pricing (FTAP). A analog of this theorem in continuous time models are provided in Harrison, Kreps and Pliska (1981) or by Delbaen Schachermayer (1994). In both cases, the arbitrage theory has been well developed by a lot of authors by appealing to the well-known Hahn-Banach separation theorem in convex functional analysis, or the Kreps-Yan theorem thereafter. The super-hedging prices of European or American options are also characterized by means of the optional decomposition theorem, see Kramkov (1996).

For proportional transaction costs, the theory was initiated by E. Jouini and H. Kallal (1995). In their pioneering paper, the authors considered a two-assets models and established the equivalence between the no-arbitrage opportunity condition with the existence of which is so-called *consistent price system*, i.e. a martingale evolving within the bid-ask spreads under some equivalent probabilistic measure. This new notion is a natural generalization of equivalent martingale measures in the frictionless case. Kabanov (1999) introduced a general semimartingale model in discrete time of a currency market with transaction costs and gave a description of the initial endowments which allow to hedge a contingent claim in various currencies by a self-financing portfolio. This model possesses a nice geometric structure

and rapidly became a standard model in the theory of no arbitrage with proportional transaction costs. There are several important papers studying the super hedging problem in Kabanov's model, for instance Kabanov, Rasonyi, Stricker (2002), Schachermayer (2004), Campi and Schachermayer (2006), Kabanov and Lépinette (2013).

In practice, fixed costs account for a significant amount of trading costs. Recall that fixed costs are bounded regardless of the transaction size. There are a lot of examples of fixed costs in reality such as fixed brokerage fees, brokerage arrangements where marginal fees go to zero beyond a given volume that is reset periodically, fixed investment taxes to gain access to a market (such as a foreign market), operational and processing costs that typically exhibit strong economies of scale (e.g. through automation), fixed costs involved in setting up an office and obtaining access to information, and the opportunity cost of looking at a market or of doing a special trade, so on and so forth. Therefore, it is reasonable to study models capturing the impact of fixed costs. However, to the date there is very few academic papers on this direction. There are several attempts to formally study models with fixed costs, for example in Jouini, Kallal, and Napp (2001, 2006) where the authors established the equivalence between the absence of arbitrage opportunity and the existence of an absolutely continuous martingale probability measure.

• **New results.**

In chapter 4, we introduce a mathematical model which is general enough to capture both fixed and proportional costs. Recall that, in "standard models" like Kabanov's model, the dynamics of portfolios necessarily need to be written separately for each asset instead of a unique wealth process as in the frictionless case. This makes the model to be multidimensional. Our idea is to consider the liquidation value process instead of multidimensional portfolio process in order to reduce the problem to the unidimensional case. In order to do so, let us formalize the setting as follows.

Given a stochastic basic $(\Omega, (\mathcal{F}_t)_{t=0,\dots,T}, P)$, we define *solvency process* an adapted set-valued process $(G_t)_{0\leq t\leq T}$ satisfying the following conditions

Conditions on (G_t) :

(i) G_t is a closed $\mathcal{F}_t$-adapted set, $0 \leq t \leq T$,

(ii) $G_t + G_t \subseteq G_t$, $0 \leq t \leq T$, a.s.,

(iii) $\lambda G_t \subseteq G_t$, $\forall \lambda \geq 1$, a.s.,

(iv) $G_t + \mathbf{R}^d_+ = G_t$, a.s.,

(v) The cone $\overline{\mathbf{R}_+ G_t}$ is proper , i.e. $\overline{\mathbf{R}_+ G_t} \cap (-\overline{\mathbf{R}_+ G_t}) = \{0\}$ a.s.

The condition (i) and (iv) are obvious. Conditions (ii) and (iii) are satisfied for any model with fixed costs, because fixed costs are independent of

transaction volume, which leads to the fact that fixed costs are relatively decreasing when we increase the size of transactions. The last condition means that transaction costs are efficient, i.e. we cannot transact without paying costs. Notice that the solvency sets are not necessarily convex, therefore we cannot apply the traditional technics of convex analysis anymore. This is the key point that makes the problem more difficult than the case of proportional transaction costs. The liquidation process $(\mathbf{L}_t)$ is defined in such a way that

$$G_t = \{x \in \mathbf{R}^d : \mathbf{L}_t(x) \geq 0\}.$$

More precisely, we have

$$\mathbf{L}_t(z) := \sup\{\alpha \in \mathbf{R} : \ z - \alpha e_1 \in G_t\}. \tag{2.5.3}$$

As in the case of proportional transaction costs, an adapted stochastic process $(V_u)_{t\leq u\leq T}$ is called to be a portfolio if $V_u - V_{u-1} \in G_u$ a.s. We can therefore write $V_T = \sum_{t\leq u\leq T} \xi_u$ where $\xi_u \in L^0(-G_u, \mathcal{F}_u)$ and denote by R_T^t the set of all terminal values V_T of portfolio processes V such that $V_{t-1} = 0$. i.e.

$$\mathrm{R}_T^t := \mathrm{R}_T^t(G) := \sum_{t\leq u\leq T} L^0(-G_u, \mathcal{F}_u).$$

By using the liquidation function, we focus on the set of all terminal liquidation values

$$\mathrm{LV}_T^t := \{\mathbf{L}_T(V_T) : \ V_T \in \mathrm{R}_T^t\}.$$

In this setting, we study the classical no arbitrage conditions such as **NA**, **NA**2 by means of LV_T^t. The only difference from the classical case is the characterization of super-hedging prices of contingent claims. For it, the duality technic is not valid anymore. We have to employ the notion of *Multidimensional Essential Supremum* proposed by Kabanov and Lépinette (2013). This notion is a generalization of its classical analog of real random variables to the multidimensional case, where the natural order relation on $\mathbf{R}$ is replaced by the new one defined by the solvency cone, i.e. by definition, $x \succeq^t y \Leftrightarrow x - y \in G_t$. Notice that this relation is well-defined because of Condition (ii). We recall here the definition of Multidimentional Essential Supremum, where we denote by $\succeq$ an order relation implied by certain solvency set as above.

Definition 2.5.1. *Let Γ be a subset of $L^0(\mathbf{R}^d, \mathcal{F})$. We denote by $\mathcal{H}$-*Esssup Γ *a subset $\hat{\Gamma}$ of $L^0(\mathbf{R}^d, \mathcal{H})$ such that the following conditions hold :*

(a) $\hat{\Gamma} \succeq \Gamma$,

(b) if $\gamma \in L^0(\mathbf{R}^d, \mathcal{H})$ and $\gamma \succeq \Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\gamma \succeq \hat{\gamma}$,

(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 = \hat{\gamma}_2$.

Given a European claim $Y_T \in L^0(\mathcal{F}_T)$, a portfolio process (V_t) super-replicates Y_T if $V_T \succeq^T Y_T$. Moreover, it is called *minimal* if any portfolio process $W \in \mathcal{V}$

such that $W_T \succeq^T Y_T$ and $V \succeq W$ (i.e. $V \succeq^t W$ for all t) coincides with V. We denote $\mathcal{V}^E_{min}(Y_T)$ the set of all minimal processes super replicating the European claim Y_T. The hedging theorem in this new setting is the following.

Proposition 2.5.2. *Suppose that* **NA**2 *holds and suppose there exits at least one $V \in \mathcal{V}$ such that $V_T \succeq^T Y_T$. Then $\mathcal{V}^E_{min}(Y_T) \neq \emptyset$ and it coincides with the set of solutions of backward inclusions*

$$V_t \in (\mathcal{F}_t, \succeq^{t+1})\text{-Esssup}\,\{V_{t+1}\}, \quad t \leq T-1, \quad V_T = Y_T. \tag{2.5.4}$$

Moreover, any $W \in \mathcal{V}$ with $W_T \succeq^T Y_T$ is such that $W \succeq V$ for some $V \in \mathcal{V}^E_{min}(Y_T)$.

A definition of minimal hedging price needs more mathematical formulation and will be detailed in Chapter 3. The hedging problem of an American option is done in a similar manner as in the case of proportional costs. For the proof of the above theorem, the most dificult part is to prove that $\mathcal{V}^E_{min}(Y_T) \neq \emptyset$. To do so, it is stressed that all necessary conditions as in [57] are not available. In particular, we do not know whether there exists a countable measurable utility representation for the order reference or not (for more detail, see [57], [35] and the references therein). Therefore, we have to use another approach to deal with the problem.

Chapter 4 is an ongoing project, we continue to study the non convex model given in chapter 3. Our goal is to review various no arbitrage criteria in this new setting. More precisely, we are interested in answering the following questions.

Q.1 *What is the no arbitrage condition that is equivalent to the existence of an Equivalent Risk Neutral Probability Measure ? In the case where one of these two conditions satisfies, how to characterize the super hedging price of a European option ?*

For this question, we introduce a new notion of arbitrage opportunity which is called *Extended* No Asymptotic Arbitrage opportunity condition **(ENAA)**. We show that Condition **(ENAA)** is equivalent to the existence of an Equivalent Risk Neutral Probability Measure $Q \sim P$ such that $E_Q[L_T(V_T)] \leq 0$ for all terminal liquidation values of the portfolio processes V_T of our model, where L_T denotes the liquidation function at maturity.

Let $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ be an arbitrary contingent claim which is bounded below. We define the super hedging price of ξ as follows

$$p(\xi) = \inf\{x \in \mathbf{R} |\, \exists V_T \in R^0_T : \, x + L_T(V_T) \geq \xi\}.$$

In Kabanov's model, we can prove that, under **(ENAA)** this price is characterized by

$$V^\xi_0 = \sup_{Q \in \mathcal{D}} E_Q \xi,$$

where $\mathcal{D}$ denotes the set of Equivalent Risk Neutral Probability Measures.

Q.2 *What is the relationship between a model with only proportional costs and a model with both fixed and proportional costs ?*

Recall that if the market with both fixed and proportional costs admits an arbitrage opportunity, then it is also the case for the market with only proportional costs (since trading in the latter is cheaper). Reciprocally, if the market with only proportional costs admits an arbitrage opportunity (in some sense), then we can leverage this arbitrage strategy by a sufficient large constant in order to reduce the impact of fixed costs as low as possible. As a consequence, this modified strategy will produce a new arbitrage opportunity in the model with both fixed and proportional costs. This idea is formalized in the notion of Weak Arbitrage Opportunity **(WAO)**. We prove that the absence of **(WAO)** are equivalent in both markets.

Q.3 *How can we link our no arbitrage conditions (defined by the liquidation function) to the ones defined in terms of portfolio vectors ?*

For this question, we would expect that most of no arbitrage conditions are equivalent in both settings. If this is the case, it shows that our theory is consistent with the no arbitrage theory for Kabanov's models. In a general model with multiple assets, this question still remains open. But at least, we can show that it is the case for models with two assets.

Q.4 *In the case where the fixed cost is not negligible, i.e. when fixed cost is bounded from below by some strictly positive constant, what is the impact of fixed costs on no arbitrage conditions ?*

If fixed costs are not negligible, any trading activity in the market will lead to a strictly positive cost. Therefore, any arbitrage opportunity of asymptotic kind (for example, **NFLVR**) could have led to the presence of a new arbitrage opportunity in classical sense (e.g.**NA**) if there were not fixed costs in the market. However, by a similar argument as mentioned in the comment of **Q.2**, the presence of an (non asymptotic) arbitrage opportunity in the market with only proportional costs will lead to the existence of a new (non asymptotic) arbitrage opportunity in the market with both fixed and proportional costs. In conclusion, we see that the presence of asymptotic and non asymptotic arbitrage opportunities are equivalent in the model where fixed costs are non negligible.

2.6 Utility maximization problem under target risk constraints

• **Motivation.**

One of the most classical problems of mathematical finance is the maximization of expected utility from terminal wealth. Mathematically, we are concerned with the following optimization problem

$$u(x) = \sup_{\pi \in \mathcal{H}} E[U(X_T^{x,\pi})].$$

Here, $(X_t^{x,\pi})_{0 \leq t \leq T}$ denotes the wealth process produced by an initial capital x together with an admissible trading strategy $\pi \in \mathcal{H}$, U is a utility function

which might be random. A common choice of U is $U(x) = u(x - B)$ where B denotes, for example, a given random liability and u is some deterministic concave nondecreasing function such as power, logarithmic or exponential functions. There is a vast literature on the maximization of expected utility and we confine ourselves to a very brief overview of approaches as follows.

Convex duality theory. The convex duality methodology originally dates back to Bismut (1973). In the simplest form (complete market), this methodology is essentially based on applying the Legendre-Fenchel transformation of the utility function given by

$$\tilde{U}(y) := \sup_{x \geq 0} [U(x) - xy], y > 0.$$

The dual function allows to associate the primal problem to the dual problem formulated as

$$v(x) = \inf_{y>0} E[\tilde{U}(yH)],$$

where H denotes the risk-neutral density of the unique martingale measure. It was shown that the marginal utility of the terminal wealth of the optimal portfolio is, up to a constant, equal to the risk-neutral density, i.e $U'(X_T^{x,\pi^*}) = y^*H$. The optimal portfolio is constructed by using the martingale representation theorem (or the optional decomposition theorem for the case of incomplete markets). The convex duality method was developed by Pliska (1986), Cox and Huang (1989, 1991) and Karatzas, Lehoczky and Shreve (1987) for complete financial markets, and by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Kramkov& Schachermayer (1999) for the incomplete case.

Dynamic programming and HJB equations. Though convex duality theory is powerful in proving the existence of optimal portfolios in a general non-Markovian framework, it does not show a way to characterize these optimal strategies. The dynamic programming approach should be seen as complementary to convex duality, which remains a relevant method to obtain the characterization of both optimal strategies and the value function. The idea of this method consists in applying tools from the stochastic control theory to derive a PDE (which is so-called Hammilton-Jacobi-Bellman equation) for the value function. However, this approach is based on the assumption that prices are driven by aQuartzes Markovian process. For references, see Merton (1971) Karatzas and Zitkovic (2003) , Karatzas and Kardaras (2007).

Backward SDEs. The theory of backward SDEs has been developed by Pardoux & Peng (1990). It has rapidly become one of the mainstreams of stochastic control theory with a variety of applications in mathematical finance. For the utility maximization problem, the BSDE method turns out to be a good replacement for HJB equations in a non-Markovian setting. When the utility takes some common forms such as power, logarithmic or exponential, it has been shown by Hu, Imkeller and Muller (2005) that the

utility maximization can essentially be reduced to solving a BSDE that characterizes the optimal strategy and the value function. A forward-backward SDE approach is given in Horst et al. (2011) for the case of general utility functions.

Static optimization and Lagrange multipliers. This method has been used to deal with the utility maximization problem with risk constraints imposed on the terminal value of portfolios. The idea is to reduce the problem from a dynamic framework to a static optimization problem with constraints and then solve it by means of classical Lagrange multipliers. Optimal investment policies under downside risk constraints in terms of value at risk and and a second risk functional have been studied in a Brownian setting by Basak & Shapiro (2001) and Gabih et al. (2005). A complete solution in a general semimartingale with utility-based shortfall risk constraints is given by Gundel and Weber (2005).

As mentioned above, when we impose a risk constraint on the terminal wealth, the problem is solved by static optimization technics. This work aims at solve the problem by a different approach. At the beginning, our project aimed at using BSDE method to tackle the problem. Unfortunately, such an attempt did not succeed and we had to choose another approach. From a stochastic control point of view, it is not difficult to solve this problem by means of dynamic programming and Bellman equations, as it is just a particular case of the optimal control problem under target constraints problem, see Bouchard-Elie-Touzi (2008) for more details about the technic. Therefore, convex duality remains as a method of choice to study our utility maximization problem.

As discussed in Gabih et al. (2005), the distribution function of optimal wealth in an unconstrained utility maximization problem possesses lots of undesired properties. From a practical point of view, a risk measure should be considered for a better investing management. For example, a common problem is to limit the shortfall of terminal wealth below some benchmark B on the market. Here B might stand for a default level in risk management, or a stock index in the case an investor wants to defeat the market, or just a target liability in mutual fund management... In this case, a choice of risk mesure could be of great importance. Some common examples of risk measures are Value at Risk (VaR) and Expected Shortfall (ES). We focus on another class of risk measures of the form $\rho(X) := E[l(X-B)^-]$, where $l : \mathbf{R}_+ \to \mathbf{R}_+$ being a non decreasing convex function. This kind of risk measures should be a good candidate for risk management purposes since it possesses the following properties :

— First, it is a convex measure, therefore it supports the diversification principle in investment.
— Second, by choosing an appropriate weight function l, the risk measure ρ is sensitive to losses, which is not the case for Value at Risk.
— Third, this measure is not cash-additive which is criticized in risk management (recall that a risk measure ρ is said to be cash-additive if for any

real number c and financial position $X, \rho(X+c) = \rho(X) - c$), therefore it is relevant for a internal measure of risk.

— Last, ρ only depends on the shortfall part, not the gain part. This is a natural requirement because in risk management, we focus on losses rather than in gains.

From above arguments, we are led to a general form of risk measure, namely, $\rho(X) := E[\ell(X)]$ where ℓ is a possibly random non-decreasing concave loss function. In this case, the risk constraint is of the form $\rho(X_T^{x,\pi}) \geq m$ for some constant m and terminal wealth $X_T^{x,\pi}$.

• **New results.**

This chapter aims at applying the convex duality methodology in different situations of the utility maximization problem under risk constraints.

We begin with the case of an incomplete market model with risk constraint when the wealth is supposed to be non negative.

$$u(x) = \sup\{E[U(X_T^{x,\pi})] : \quad X_T^{x,\pi} \geq 0, E[\ell(X_T^{x,\pi})] \geq m\}.$$

Let us denote $\mathcal{C}(x,m)$ the set of hedgeable positions, i.e. the set of random variables which are bounded from below and dominated by terminal wealths $X_T^{x,\pi}$ satisfying the risk constraint. The dual domain is defined by

$$\mathcal{Y} := \{(y,Y) \in \mathbf{R}_+ \times L_+^1 : EY \leq y, \delta(Y) := \sup_{X \in \mathcal{C}(x,m)-x} E[XY] < \infty\}.$$

Here, δ is called to be support function. Our first main result is the following duality relation. The proof is not much different from the non-constrained case. The technic of proof is borrowed from Kramkov and Schachermayer (1999).

Theorem 2.6.1. *Consider the dual problem*

$$w(x) = \inf_{(y,Y)\in\mathcal{Y}} [v(Y) + xy],$$

where $v(Y) := E[V(Y)] + \delta(Y)$ and V is the Fenchel dual function of U. Under necessarily technical conditions, we have the following

(i) *The dual problem admits an optimal solution $(y^*, Y^*) \in \mathcal{Y}$:*

$$w(x) = v(Y^*) + xy^*.$$

(ii) *The duality holds*

$$u(x) = \inf_{(y,Y)\in\mathcal{Y}} [v(Y) + xy] = v(Y^*) + xy^*.$$

Moreover, the optimal wealth is given by $X^* = I(Y^*) := (U')^{-1}(Y^*)$.

Next, in the same model, we relax the positivity condition on the wealth process. The idea of proof is to consider the approximating problem (see Schachermayer (2001)) :

$$u_n(x) = \sup_{X \in \mathcal{C}_n(x,m)} E[U_n(X)],$$

where $U_n(x) = U(x)$ if $x \geq -n$ and $U_n(x) = -\infty$ otherwise. In a same manner as the case of positive wealth, the approximating primal problem admits unique solution X_n given by $X_n = I(Y_n) \vee -n \in \mathcal{C}_n(x,m)$. By letting n tend to infinity, we obtain a similar result as in the previous case.

It is easy to see that in the duality method we use above, the loss constraint is implicitly determined by the primal domain $\mathcal{C}(x,m)$ and by the support function. Therefore, it can be directly generalized to the case of general convex constraints. We suggest another approach which is still based on duality method. For sake of simplicity, we only consider the case of complete markets with positive-wealth constraints. This approach allows to show the risk constraint explicitly from the dual problem. The idea is to replace the expected constraint by an infinite number of almost sure constraints as considered in Bouchard-Touzi-Elie (2009).

$$E[l(X_T^{x,\pi})] \geq m \Leftrightarrow \exists M \in \mathcal{M} : \ X_T^{x,\pi} \geq M,$$

where

$$\mathcal{M} = \mathcal{M}(x,m) := \{M \in L^0_+(\mathcal{F}_T) : \ E_Q M \leq x, E[l(M)] \geq m\}.$$

It is then natural to replace the Fenchel dual function by a constrained one $V_p(y) := \sup_{x \geq p}[U(x) - xy]$. By using it we are led to the following inequalities

$$u(x) \leq \sup_{M \in \mathcal{M}} w_M(x) \leq \inf_{y>0}[v(y) + xy], \tag{2.6.5}$$

where

$$w_M(x) := \inf_{y>0}\{E[V_M(yH)] + xy\}, \tag{2.6.6}$$

and

$$v(y) = v_{x,m}(y) := \sup_{M \in \mathcal{M}} E[V_M(yH)]. \tag{2.6.7}$$

Here, Q denotes the unique risk-neutral martingale measure. As expected, we can show that under some technical conditions, the duality holds

$$u(x) = \sup_{M \in \mathcal{M}} w_M(x) = \inf_{y>0}[v(y) + xy], \tag{2.6.8}$$

and the optimal wealth is given by

$$X^* = -V'_{M^*}(y^* \frac{dQ}{dP}) = \max\{M^*, I(y^* \frac{dQ}{dP})\},$$

where M^* is the maximizer of the dual problem $\sup_{M\in\mathcal{M}} w_M(x)$ and $y^* = y^*(M^*)$ is the minimizer of (2.6.6).

From a practical point of view, all results obtained above are not quite satisfactory because they do not show a way to calculate numerically, at least the value function. Even in complete market case, the complexity of the dual problem is not significantly lower than the primal problem (since we introduce a new set $\mathcal{M}(x,m)$ of extra random variables). We then present a new method based on the relationship between the utility maximization problem and the (quantile) hedging-type problem. We still work on a complete market with positivity constraint on the wealth and risk constraint at maturity. Let us consider the following problem

$$v(y,m) := \inf\{x : \exists\pi : E[U(X_T^{x,\pi})] \geq y,\, E[l(X_T^{x,\pi})] \geq m\}.$$

It is clear that $u(.,m) = v^{-1}(.,m)$. Therefore, the primal problem boils down to a new hedging-type problem under multiple targets, or backward SDEs with weak-terminal conditions as in the terminology of Bouchard-Elie-Réveillac (2013). For this new problem, we apply the duality technic to characterize the value function and the optimal wealth. More precisely, we are concerned with the following problem

$$v(\mathbf{m}) := \inf\{x : \exists\pi : E[\mathbf{l}(X_T^{x,\pi})] \succeq \mathbf{m}\},$$

where $\mathbf{l} := (\mathbf{l}_1, \ldots, \mathbf{l}_n)$ is a multidimensional deterministic loss function, $\mathbf{m} \in \operatorname{Im}\mathbf{l}$ and $\succeq$ is understood componentwisely. Moreover, $\mathbf{l_k}$ is supposed to be strictly increasing for all $1 \leq k \leq n$. We have

$$E[\mathbf{l}(X_T^{x,\pi})] \succeq \mathbf{m} \Leftrightarrow \exists M \in \mathcal{M} : X_T^{x,\pi} \geq F(M),$$

where $F(M) = \max_{1\leq k\leq n}\{\mathbf{l}_k^{-1}(M_k)\}$ being a convex function and componentwisely increasing of M, and

$$\mathcal{M} := \{M = (M_0, \ldots, M_K) \in L^1(\operatorname{Im}\mathbf{l}) : \mathbf{l}_k^{-1}(M_k) \in L^2_+, E[M_k] = m_k \,\forall\, k\}.$$

By the comparison principle, we have the following estimation

$$v(\mathbf{m}) \geq \inf_{M\in\mathcal{M}}\{X_0^{F(M)}\},$$

where $X_0^{F(M)}$ is solution to the following BSDE with terminal condition $F(M)$.

$$X_0^{F(M)} = F(M) - \int_0^T (r_t X_t + \theta_t Z_t)dt - \int_0^T Z_t dW_t.$$

By using the Fenchel duality of $F : \tilde{F}(p) := \sup_{\mathbf{r}\in\operatorname{Im}\mathbf{l}}[p.\mathbf{r} - F(\mathbf{r})]$, we obtain the following duality relation

$$v(\mathbf{m}) = \inf_{M\in\mathcal{M}}\{X_0^{F(M)}\} = \sup_{p\in\mathbf{R}_+^{K+1}} E[p.\mathbf{m} - \tilde{F}(L_T^{-1}p)L_T],$$

where (L_t) denotes the state price process. The last problem is easier to solve because it is considered on $\mathbf{R}_+^{K+1}$. We can then characterize the optimal wealth of the primal problem by means of the solution to the dual one.

• **New perspectives.**

We briefly present here several outlooks of the work.

First, since we can easily simulate L_T from model parameters, the last duality equation could be exploited to calculate numerically the dual value function $v(\mathbf{m})$, hence the primal value function.

Second, duality technics used in the two last sections could be generalized to incomplete markets. In this case, it is not hard to prove the verification theorem by using the constrained Fenchel dual function. However, the existence of solutions to the dual problem still remains as an open question due to the fact that some integrability conditions are no longer satisfied when we introduce extra random variables M.

Third, convex portfolio constraints could be taken into account. Note that when we directly use the convex duality method, the key issue is the convexity of portfolio set. When the primal problem is related to the hedging problem by BSDEs method, we are led to a BSDE with both terminal constraints and dynamic convex constraints, which might rise an interesting subject for future studies.

Last, terminal constraint can be replaced by an American-type constraint, i.e. losses are controlled at any time prior to maturity. We can prove that this type of constraint can be transformed to an infinite number of almost sure constraints, where a similar situation of the latter has been investigated in, for example, Kara & Jeanblanc (1998) in the presence of labor income.

Chapitre 3

Approximate hedging in local volatility models under transaction costs

Abstract

Local volatility models are popular because they can be simply calibrated to the market of European options. We extend the results of [76], [75] for such models, i.e. we propose a modified Leland method which allows us to approximately replicate a European contingent claim when the market is under proportional transaction costs.

Keywords : Black–Scholes formula, transaction costs, Leland strategy, approximate hedging.

Note. This chapter is based on the paper *Approximate hedging in local volatility models under transaction costs,* T. Tran, E. Lépinette, To appear in *Applied Mathematical Finance.*

3.1 Introduction

There are indications that the Black and Scholes model does not fit certain financial market phenomenons. In practice, the implied volatility is not constant, i.e. depends on the expiration date and the strike. This problem is known as the implied smile effect. More natural is then to consider local volatility models. Indeed, under mild assumptions, the strong Markov property and the continuity of the price process S implies that S is given by a local volatility function $\sigma(t, S_t)$. Bouchouev and Isakov [13] and later Egger, Hein and Hofmann [33] suggest to search for local volatility in the product form $\sigma(t, x) = \sigma(x)\rho(t)$. These local volatility models are very popular because they are simple to calibrate the call prices by the Dupire formula. In this chapter, we consider a standard two-asset local volatility

model in presence of transaction costs. Our goal is to provide a strategy which replicates approximately a European contingent claim $h(S_T)$ when the number of revision dates of the portfolio tends to ∞. We show that the Leland method can be adapted to the case of a local volatility model. This is a generalization of Papers [76] and [75] to the case where the volatility is not constant and depends on the underlying asset. The Leland strategy derives from the solution $\widehat{C}$ to the heat equation of the Black and Scholes model where we increase the volatility in order to compensate for the transaction costs. Contrarily to the case where the volatility is constant, we do not have explicit expressions of the solution to the heat equation. The main difficulty is to obtain appropriate estimates of the derivatives of $\widehat{C}$, which is necessary to show the convergence of our scheme. To do so, we use PDEs techniques and our main concern is the case where the proportional transaction costs coefficient is constant, i.e. does not depend on the number n of revision dates. For this model, the usual Leland method fails [76]. By modifying the Leland technique, we show that it is possible to obtain an approximate hedging of the pay-off $h(S_1)$ without limit error as $n \to \infty$ for a large class of pay-off functions h.

3.2 The model

We assume without loss of generality that the time horizon is $T = 1$. The non-risky asset is the *numéraire* $S^0 = 1$ and the risky asset is given under the martingale probability measure by the SDE :

$$dS_t = S_t \sigma(t, S_t) dW_t, \qquad 0 \leq t \leq 1.$$

We suppose that trading involves proportional transaction costs with the transaction cost coefficients depending on the number of revision dates n :

$$k_n = k_0 n^{-\alpha}, \qquad \alpha \in [0, \frac{1}{2}].$$

The case $\alpha > 0$ means that the transaction costs decreases as the number of times the agent rebalances his/her portfolio increases. This is the Leland model. If $\alpha = 0$, the transaction costs coefficient is constant, i.e. does not depend on n. This model may be considered as more realistic and represents our main concern. Although, a transaction costs rate, e.g. $r = 0.001$, can be also seen as $r = k_0 n^{-\alpha}$ with $\alpha > 0$ and n large enough. In this case, k_0 depends on α and we leave the qualitative analysis of such approach for future research.

In our model the current value of the portfolio process at time t is defined by

$$V_t^n = V_0^n + \int_0^t D_u^n dS_u - \sum_{t_i \leq t} k_n S_{t_i} |D_{i+1}^n - D_i^n|, \qquad t < 1. \tag{3.2.1}$$

where $t_i = t_i^n, 0 \leq i \leq n$, $t_0 = 0, t_n = 1$, are the revision dates ; and $D^n = D_i^n$ on the interval $]t_{i-1}, t_i]$ where D_i^n is $\mathcal{F}_{t_{i-1}}$-mesurable, i.e. D^n is the trading strategy. The number of the risky asset that the holder possesses in the period i is then D_i^n. The dynamics (3.2.1) means that the portfolio process V^n is self-financed and with transaction costs which are proportional to the traded volume. We suppose that the dates $t_i's$ are uniform i.e. $t_i = t_i^n = i/n$, $i = 0, \ldots, n$.

In the complete model without friction, a contingent claim $h(S_1)$ is exactly replicated by the terminal value of the self-financed portfolio :

$$V_t = E(h(S_1)|\mathcal{F}_t) = C(t, S_t) = V_0 + \int_0^t C_x(r, S_r)dS_r, \quad t \leq 1$$

where C is solution of the PDE :

$$(\mathbf{e_0}) = \left\{ \begin{array}{lcl} C_t(t,x) & + & \frac{1}{2}\sigma^2(t,x)x^2C_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\times[0,1[\\ C(1,x) & = & h(x), \quad x \in]0,\infty[\end{array} \right.$$

With transaction costs, we follow Leland's approach [77], i.e. we construct a strategy which can be considered as a modified Delta of Black-Scholes replication formula. The idea is to subtitute the volatility σ by an enlarged one $\widehat{\sigma}(t,x)$ in order to compensate for the transaction costs. The "enlarged volatility" is defined by :

$$\widehat{\sigma}^2(t,x) = \sigma^2(t,x) + \sigma(t,x)\gamma_n,$$

where

$$\gamma_n = k_n n^{\frac{1}{2}} \sqrt{\frac{8}{\pi}}.$$

So, we introduce the PDE :

$$(\mathbf{e}) = \left\{ \begin{array}{lcl} \widehat{C}_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2(t,x)x^2\widehat{C}_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\times[0,1[\, , \\ \widehat{C}(1,x) & = & h(x), \quad x \in]0,\infty[. \, , \end{array} \right.$$

The existence of a solution to the PDE $(\mathbf{e})$ is ensured by Lemma 3.5.2. Let us precise the intuition behind the Leland strategy. By the Ito Formula, assuming that the solution $\widehat{C}$ to $(\mathbf{e})$ is smooth, we have

$$\begin{aligned} \widehat{C}(t,S_t) &= \widehat{C}(0,S_0) + \int_0^t \widehat{C}_x(u,S_u)dS_u \\ &\quad + \frac{1}{2}\int_0^t \left[\sigma^2(u,S_u) - \widehat{\sigma}^2(u,S_u)\right] S_u^2 \widehat{C}_{xx}(u,S_u)du. \end{aligned}$$

Then, $\widehat{C}(t,S_t)$ can be seen as the continuous version of a portfolio process (3.2.1) provided that $D_i^n = \widehat{C}(t_{i-1}, S_{t_{i-1}})$ and the drift term in the formula above corresponds to the cumulated transaction costs, i.e. we want to make equal the two following increments :

$$\frac{1}{2}\left[\sigma^2(u,S_u) - \widehat{\sigma}^2(u,S_u)\right] S_u^2 \widehat{C}_{xx}(u,S_u)\Delta u$$

and

$$-k_0 n^{-\alpha} \left| \widehat{C}_x(u+\Delta u, S_{u+\Delta u}) - \widehat{C}_x(u, S_u) \right| S_{u+\Delta u}.$$

To do so, we use the Taylor approximation

$$\begin{aligned} \widehat{C}_x(u+\Delta u, S_{u+\Delta u}) - \widehat{C}_x(u, S_u) &\simeq \widehat{C}_{xt}(u, S_u)\Delta_u + \widehat{C}_{xx}(u, S_u)\left(S_{u+\Delta u} - S_u\right), \\ &\simeq \widehat{C}_{xx}(u, S_u)\left(S_{u+\Delta u} - S_u\right) \end{aligned}$$

where

$$S_{u+\Delta u} - S_u \simeq \sigma(u, S_u) S_u \left(W_{u+\Delta u} - W_u\right).$$

Assuming that $\widehat{C}_{xx} \geq 0$, this implies that we should look for $\widehat{\sigma}$ such that

$$\frac{1}{2}\left[\sigma^2(u, S_u) - \widehat{\sigma}^2(u, S_u)\right] \Delta u \simeq -k_0 n^{-\alpha} \sigma(u, S_u) \left|W_{u+\Delta u} - W_u\right| \frac{S_{u+\Delta u}}{S_u}.$$

Then, considering the conditional expectation knowing $\mathcal{F}_u$, and the equalities

$$E|W_{\Delta u}| = \sqrt{\Delta u}\sqrt{\frac{2}{\pi}}, \quad \frac{S_{u+\Delta u}}{S_u} = 1 + \sigma(u, S_u)\left(W_{u+\Delta u} - W_u\right).$$

we obtain, considering only the main terms, that

$$\frac{1}{2}\left[\sigma^2(u, S_u) - \widehat{\sigma}^2(u, S_u)\right] \Delta u = -k_0 n^{-\alpha} \sigma(u, S_u) \sqrt{\Delta_u}\sqrt{\frac{2}{\pi}}.$$

We deduce that

$$\widehat{\sigma}^2(u, S_u) = \sigma^2(u, S_u) + k_0 n^{1/2-\alpha} \sqrt{\frac{8}{\pi}} \sigma(u, S_u).$$

We recall that the Leland strategy fails in the case $\alpha = 0$, i.e. an approximation error appears. We propose a modified strategy as in [75] to treat the general case $\alpha \in [0, \frac{1}{2}]$:

$$\widehat{V}_t^n = \widehat{V}_0^n + \int_0^t \widehat{D}_u^n dS_u - \sum_{t_i \leq t} k_n S_{t_i} |\widehat{D}_{i+1}^n - \widehat{D}_i^n|, \qquad 0 \leq t < 1,$$

where $\widehat{V}_0^n = \widehat{C}_x^n(0, S_0)$, and $\widehat{D}_t^n = \widehat{D}_{t_i}^n$ on $]t_{i-1}, t_i]$, $\widehat{D}_{t_i}^n$ is given by

$$\widehat{D}_{t_i}^n = \widehat{C}_x^n(t_{i-1}, S_{t_{i-1},}) - \sum_{1 \leq j \leq i-1} \left[\widehat{C}_x^n(t_j, S_{t_{j-1},}) - \widehat{C}_x^n(t_{j-1}, S_{t_{j-1},})\right]. \quad (3.2.2)$$

We make use the abbreviations $\hat{H}_t = \widehat{C}_x(t, S_t) = \widehat{C}_x^n(t, S_t)$ where we often omit the index n, $\hat{h}_t = \widehat{C}_{xx}(t, S_t)$, $H_{t_i^n}^n = \widehat{C}_x(t_i, S_{t_i,})$, $t_i = t_i^n$, and

$$K_t^n = \sum_{t_i^n < t} \Delta K_{t_i^n}^n$$

where $\Delta K^n_{t^n_0} = 0$ and for $i \leq 1$:

$$\Delta K^n_{t^n_i} = -\left[\widehat{C}_x(t_j, S_{t_{j-1}},) - \widehat{C}_x(t_{j-1}, S_{t_{j-1}},)\right].$$

In this chapter, we impose hypothesis on the pay-off function :

Assumption (H) : h is a continuous convex function on $[0, \infty[$ which is differentiable except at the points $K_1 < \cdots < K_p < \ldots$ where the first derivative h' admit right and left limits. Morever, h has polynomial growth, i.e, $|h(x)| \leq M(1+|x|^a)$ for some positive constants M, a and h' is bounded.

In the case $\alpha = 0$, we only consider pay-off functions h behaving as affine functions at infinity. The pay-off functions in the European option contracts are generally of this form.

Assumption (H') $h(x) = h_1(x) + ax + b$ where h_1 is a bounded continuous function.

Assumption (E) : There exists some positive constant K such that :
(a) $m \leq \sigma \leq M$ for some positive constants M, m.
(b) $|\sigma(t,x) - \sigma(t',x')| \leq K(|t-t'| + |x-x'|)$.
(c) $\dfrac{\partial^k}{\partial x^k}\sigma(t,x)$ and $\dfrac{\partial^2}{\partial x \partial t}\sigma(t,x)$ are continuous, and

$$|x^k \frac{\partial^k}{\partial x^k}\sigma(t,x)| + |\frac{\partial^2}{\partial x \partial t}\sigma(t,x)| \leq K, \qquad \forall\, 0 \leq k \leq 3.$$

Remark 3.2.1. *Observe that* **Assumption (E)** *is satisfied by the local volatility functions* $\sigma(t,x) = \gamma(t)e^{-cx^2}$, $\sigma(t,x) = \gamma(t)(\text{atan}(x) + \pi)$, *where* $c > 0$ *and* γ *is a bounded, positive and Lipschitz function. In the second case, the local volatility of the model increases as the price of the risky asset increases. The following example is given in [33] :*

$$\sigma(t,x) = \sqrt{\frac{1}{10}\left(1 - \frac{1}{2}\exp\left(-4\ln^2(x)\right)\sin(2\pi x)\right)}\sqrt{1 + \frac{3}{5}\sin(2\pi t)}.$$

It is clear that Condition (a) holds. Since we have the inequalities

$$\exp\left(-2\ln^2(x)\right) \leq \exp\left(2\ln(x)\right) \leq x^2$$

if $x \leq e^{-1}$, *we easily deduce that the first derivative* $\sigma_x(t,x)$ *is uniformly bounded in* x, *hence Condition (b) holds.*
Using the bounds $\exp\left(-4\ln^2(x)\right) \leq \exp\left(-4\ln(x)\right) \leq x^{-4}$ *if* $x \geq e^1$, *and the fact that* $\ln^k(x)\exp\left(-2\ln^2(x)\right)$ *is bounded for all* $k > 0$,*we also deduce that Condition (c) holds.*

Remark 3.2.2. *Note that Condition (c) implies that* $(t,x) \mapsto \sigma(t,e^x)$ *is a Lipschitz function uniformly with respect to* $(t,x) \in \mathbf{R}$ *(since the first derivative of this function is bounded).*

Our main result in this chapter is the following convergence theorem.

Theorem 3.2.3. *Assume that $\alpha \in [0, \frac{1}{2}]$ and the conditions (H),(H'),(E) hold. Morever, suppose that h and σ are such that $\widehat{C}_{xx} \geq 0$. If the strategy $\widehat{D}_i^n$ is given by (3.2.2), then the terminal value of the portfolio*

$$V_1^n = \widehat{C}_0^n + \int_0^1 \widehat{D}_u^n dS_u - k_n \sum_{i=1}^{n-1} S_{t_i} |\widehat{D}_{i+1}^n - \widehat{D}_i^n|$$

converges to the contingent claim $h(S_1)$ in probability.

Remark 3.2.4. *We do not need the portfolio rebalancing at the horizon date ; that is why the cumulated sum of all transaction costs is taken only up to $n-1$. Notice that we need the condition $\widehat{C}_{xx} \geq 0$ to avoid the approximative error. In fact, this condition will not restrict so many our interesting cases. We refer to Lemma 5.6 for a sufficient condition ensuring the convexity of $C(t,x)$ in the variable x and more generally to the papers [78] and [80] on the convexity propagation.*

The chapter is structured as follows. In Section 3.3, we state the proof of our main result, i.e. the convergence of the suggested terminal portfolio value to the pay-off. To do so, we need the upper bounds of the function $\widehat{C}(t,x)$ and its derivatives given in Section 3.4. Appendix is devoted to some auxiliary results used in the proofs.

3.3 Proof of the Theorem 3.2.3

By the Ito formula, we represent the difference $V_1^n - h(S_1)$ in a convenient form.

Lemma 3.3.1. *We have $V_1^n - h(S_1) = F_1^n + F_2^n + F_3^n$ where*

$$F_1^n := \int_0^1 (H_t^n - \widehat{H}_t) dS_t, \tag{3.3.3}$$

$$F_2^n := \frac{1}{2} \int_0^1 \sigma(t, S_t) \gamma_n S_t^2 |\widehat{C}_{xx}(t, S_t)| dt - k_n \sum_{i=1}^{n-1} |\Delta H_{t_i}^n + \Delta K_{t_i}^n| S_{t_i} \tag{3.3.4}$$

$$F_3^n := k_n \int_0^1 K_t^n dS_t. \tag{3.3.5}$$

Our objective is to prove that F_1^n, F_2^n, F_3^n converge to zero in probability. We will only separate two cases $\alpha = 0$ and $\alpha \in]0, 1/2]$ in proving $F_3^n \to 0$ since $F_1^n, F_2^n \to 0$ without Condition **(H')**. Throughout the section, C will designate a strictly positive constant independent of n that could be changed from a line to another one. We put $\gamma_n = n^{1/2-\alpha}$.

Lemma 3.3.2. *We have $P - \lim F_1^n = 0$.*

Proof Let us first consider the case $\alpha < \frac{1}{2}$. As $\widehat{C}_x(t,x)$ and $E(\sup_{u\in[0,1]} S_u^2)$ are bounded, the integral

$$\int_{1-\frac{1}{\gamma_n}}^{1} (H_t^n - \widehat{H}_t) dS_t$$

tends to 0 in L^2, so in probability. For each $t \in [t_{i-1}, t_i[, t < 1 - \frac{1}{\gamma_n}$, let us consider the Taylor expansion

$$\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t) = \widehat{C}_{xt}(\theta_i, S_{t_{i-1}})(t_{i-1} - t) + \widehat{C}_{xx}(t, \widetilde{S}_t)(S_{t_{i-1}} - S_t)$$

where $\theta_i \in [t_{i-1}, t]$ and $\widetilde{S}_t \in [S_{t_{i-1}}, S_t]$. Using Theorem 3.4.4, we obtain that :

$$|\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)| \leq C \left| \frac{t - t_{i-1}}{1-t} \right| + C \frac{|S_{t_{i-1}} - S_t|}{\widetilde{S}_t \sqrt{\gamma_n (1-t)}}$$

As $|\Delta t_i| \leq \dfrac{C}{n}$ and $1 - t \geq \frac{1}{\gamma_n}$, the first term tends to 0. Moreover, since $(S_t)_{0\leq t\leq 1}$ is strictly positive and continuous we deduce that the second term tends to 0 a.s. Hence

$$\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t) \to 0, \quad \text{a.s.}$$

Keeping in mind that $E(\sup_{u\in[0,1]} S_u^2)$ and $\widehat{C}_x(t,x)$ are bounded, we get the convergence

$$E(\int_0^{1-\frac{1}{\gamma_n}} (H_t^n - \widehat{H}_t) dS_t)^2 = \int_0^{1-\frac{1}{\gamma_n}} E[S_t^2 (H_t^n - \widehat{H}_t)^2] dt \to 0 \qquad \text{as} \quad n \to \infty$$

by virtue of the Lebesgue dominated convergence theorem. Hence we have shown that $\lim E(F_1^n)^2 = 0$ with $\alpha < \frac{1}{2}$. The case $\alpha = \frac{1}{2}$ is obivious because $\widehat{C}$ does not depend on n. ■

Let us now show that $F_2^n \longrightarrow 0$. By the Ito Formula, we have

$$\widehat{C}_x(t, S_t) = \widehat{C}_x(0, S_0) + M_t^n + A_t^n$$

where

$$\begin{aligned} M_t^n &:= \int_0^t \sigma(u, S_u) S_u \widehat{C}_{xx}(u, S_u) dW_u, \\ A_t^n &:= \int_0^t \left[\widehat{C}_{xt}(u, S_u) + \frac{1}{2} \sigma^2(u, S_u) S_u^2 \widehat{C}_{xxx}(u, S_u) \right] du. \end{aligned}$$

We write $F_2^n = \sum_{i=1}^5 L_i$ where

$$
\begin{aligned}
L_1^n &:= \frac{1}{2}\int_0^1 \sigma(t,S_t)\gamma_n S_t^2 \widehat{h}_t dt - \frac{1}{2}\int_0^1 \sum_{i=1}^{n-1} \sigma(t_{i-1},S_{t_{i-1}})\gamma_n S_{t_{i-1}}^2 \widehat{h}_{t_{i-1}} I_{]t_{i-1},t_i]}(t)dt \\
L_2^n &:= \sum_{i=1}^{n-1} \sigma(t_{i-1},S_{t_{i-1}})\widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 \left(\frac{1}{2}\gamma_n \Delta t_i - k_n\, n^{1/2}\sqrt{\Delta t_i}|\Delta W_{t_i}|\right), \\
L_3^n &:= k_n \sum_{i=1}^{n-1} \sigma(t_{i-1},S_{t_{i-1}}) S_{t_{i-1}}^2 \widehat{h}_{t_{i-1}} n^{1/2}\sqrt{\Delta t_i}|\Delta W_{t_i}| - k_n \sum_{i=1}^{n-1} S_{t_{i-1}}|\Delta M_{t_i}|, \\
L_4^n &:= k_n \sum_{i=1}^{n-1} S_{t_{i-1}}|\Delta M_{t_i}| - k_n \sum_{i=1}^{n-1} S_{t_{i-1}}|\Delta H_{t_i} + \Delta K_{t_i}|, \\
L_5^n &:= -k_n \sum_{i=1}^{n-1} \Delta S_{t_i}|\Delta H_{t_i} + \Delta K_{t_i}|
\end{aligned}
$$

Lemma 3.3.3. *We have* $P - \lim L_1^n = 0$.

Proof. We rewrite

$$L_1^n = \frac{1}{2}\int_{t_{n-1}}^1 \sigma(t,S_t)\gamma_n S_t^2 \widehat{h}_t dt + \sum_{k=1}^{6} L_{1k}^n.$$

It is obvious that the first integral tends to 0 as n tends to ∞. We now show that $L_{1k}^n \to 0$, $\forall k$. These terms are defines as follows.

$$L_{11}^n := \frac{1}{2}\int_0^1 \sum_{i=1}^{n-1} \left[\sigma(t,S_t) - \sigma(t_{i-1},S_t)\right]\gamma_n S_t^2 \widehat{h}_t I_{]t_{i-1},t_i]}(t)dt.$$

Using the hypothesis on σ, and Theorem 3.4.4, we deduce a constant c_ω depending on $\omega \in \Omega$ such that :

$$L_{11}^n \le \frac{c_\omega}{n}\int_0^{t_{n-1}} \frac{\sqrt{\gamma_n}}{\sqrt{1-t}} \le c_\omega n^{-\frac{3}{4}-\frac{\alpha}{2}} \to 0.$$

$$L_{12}^n := \frac{1}{2}\int_0^1 \sum_{i=1}^{n-1} \left[\sigma(t_{i-1},S_t) - \sigma(t_{i-1},S_{t_{i-1}})\right]\gamma_n S_t^2 \widehat{h}_t I_{]t_{i-1},t_i]}(t)dt.$$

Similarly, there is a constant c_ω depending on $\omega \in \Omega$ such that $|L_{12}^n| \le c_\omega \widetilde{L}_{12}^n$, where

$$\widetilde{L}_{12}^n = \sqrt{\gamma_n}\int_0^{t_{n-1}} \sum_{i=1}^{n-1} \frac{|S_t - S_{i-1}|}{\sqrt{1-t}} I_{]t_{i-1},t_i]}(t)dt.$$

Since $E|S_t - S_{i-1}| \le C\sqrt{t-t_i}$ we obtain that

$$E(\widetilde{L}_{12}^n) \le C\, n^{-\frac{1}{4}-\frac{\alpha}{2}} \to 0.$$

$$L^n_{13} := \frac{1}{2}\int_0^1 \sum_{i=1}^{n-1} \left[S_t^2 - S^2_{t_{i-1}}\right] \gamma_n \sigma(t_{i-1}, S_{t_{i-1}}) \widehat{h}_t I_{]t_{i-1},t_i]}(t)dt.$$

A reasoning similar to the one used for L^n_{12} leads to $P - |L^n_{13}| \to 0$.

$$\begin{aligned} L^n_{14} &:= \frac{1}{2}\int_0^1 R^n_t dt, \\ R^n_t &:= \sum_{i=1}^{n-1} \left[|\widehat{C}_{xx}(t, S_t)| - |\widehat{C}_{xx}(t, S_{t_{i-1}})|\right] \gamma_n \sigma(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} I_{]t_{i-1},t_i]}(t)dt. \end{aligned}$$

As R^n_t is bounded by $\frac{c_\omega}{\sqrt{1-t}}$, we deduce that $|\int_{1-\frac{1}{\gamma_n}}^1 R^n_t dt| \le \frac{c_\omega}{\sqrt{\gamma_n}} \to 0$.

It remains to show that $\bar{L}^n_{15} := \frac{1}{2}\int_0^{1-\frac{1}{\gamma_n}} R^n_t dt \to 0$. Using the inequality $||a| - |b|| \le |a - b|$ and the Taylor expansion

$$\widehat{C}_{xx}(t, S_t) - \widehat{C}_{xx}(t, S_{t_{i-1}}) = \widehat{C}_{xxx}(t, \widetilde{S}_i)\left(S_t - S_{t_{i-1}}\right)$$

where $\widetilde{S}_i \in [S_t, S_{t_{i-1}}]$, we deduce a.s. a constant c_ω such that $\left|\bar{L}^n_{15}\right| \le c_\omega \widetilde{L}^n_{15}$ with

$$\widetilde{L}^n_{15} := \int_0^{1-\frac{1}{\gamma_n}} \sum_{i=1}^{n-1} \gamma_n |S_t - S_{t_{i-1}}| \left(\frac{1}{\sqrt{\gamma_n(1-t)}} + \frac{1}{\gamma_n(1-t)}\right) I_{]t_{i-1},t_i]}(t)dt.$$

Since $E|S_t - S_{t_{i-1}}| \le C\sqrt{\Delta t_i}$, we get that

$$\begin{aligned} E\widetilde{L}^n_{15} &\le \frac{C}{\sqrt{n}} \int_0^{1-\frac{1}{\gamma_n}} \left(\frac{\sqrt{\gamma_n}}{\sqrt{1-t}} + \frac{1}{1-t}\right) dt \\ &\le C\, n^{-\frac{1}{4}-\frac{\alpha}{2}} + C\frac{\ln n}{\sqrt{n}} \to 0. \end{aligned}$$

$$L^n_{16} := \frac{1}{2}\int_0^1 \sum_{i=1}^{n-1} \left[|\widehat{C}_{xx}(t, S_{t_{i-1}})| - |\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})|\right] \gamma_n \sigma(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} I_{]t_{i-1},t_i]}(t)dt.$$

By the same argument as for L^n_{15}, we show that $\bar{L}^n_{16} \to 0$. To do so, we write :

$$\widehat{C}_{xx}(t, S_{t_{i-1}}) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) = \widehat{C}_{xxt}(\widetilde{t}_i, S_{t_{i-1}})(t - t_{i-1})$$

where $\widetilde{t}_i \in [t_{i-1}, t_i]$. Using Theorem 3.4.4, we deduce that, for some constant c_ω, $|L^n_{16}| \le c_\omega \widetilde{L}^n_{16}$ where

$$\widetilde{L}^n_{16} = \int_0^{1-\frac{1}{\gamma_n}} \frac{\gamma_n^2}{n}\left(\frac{1}{\sqrt{\gamma_n^3(1-t)^3}}\right) dt.$$

This integral tends to 0 as $n \to \infty$, i.e. $P - \lim L^n_{16} = 0$. ■

Lemma 3.3.4. *We have* $P-\lim L_2^n = 0$.

Proof. By hypothesis there exists a.s. a constant c_ω such that

$$L_2^n \leq c_\omega \sum_{i=1}^{n-1} \left|\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\right| S_{t_{i-1}} \xi_i$$

where

$$\xi_i = \frac{1}{2}\gamma_n \Delta t_i - k_n\, n^{1/2}\sqrt{\Delta t_i}|\Delta W_{t_i}|$$

is independent of $\mathcal{F}_{t_{i-1}}$, verifies $E\xi_i = 0$, and $E\xi_i^2 = k_n^2 n(\Delta t_i)^2$. Using Theorem 3.4.4 we obtain that $L_2^n \leq c_\omega \widetilde{L}_2^n$ where

$$\widetilde{L}_2^n := \sum_{i=1}^{n-1} \frac{\xi_i}{\rho_{t_{i-1}}}.$$

By independence, we have

$$\begin{aligned} E(\widetilde{L}_2^n)^2 &\leq \sum_{i=1}^{n-1} \frac{E\xi_i^2}{\rho_{t_{i-1}}^2} \\ &\leq \frac{nk_n^2(\Delta t_i)^2}{\gamma_n(1-t_{i-1})} \\ &\leq Cn^{-\alpha-\frac{1}{2}} \sum_{i=1}^{n-1} \frac{\Delta t_i}{1-t_{i-1}}. \end{aligned}$$

The sum $\sum_{i=1}^{n-1} \dfrac{\Delta t_i}{1-t_{i-1}}$ can be approximated by $\displaystyle\int_0^{t_{n-1}} \frac{dt}{1-t}$ which is $O(\ln n)$. It then follows that $\widetilde{L}_2^n \to 0$ in L^2 and then $L_2^n \to 0$ in probability. ■

Lemma 3.3.5. *We have* $P-\lim L_3^n = 0$.

Proof. Let us write $|L_3^n| \leq D_1^n + D_2^n + D_3^n$ where D_i^n, $i = 1, 2, 3$, are defined below.

$$D_1^n = k_n \sum_{i=1}^{n-1} \left| \int_{t_{i-1}}^{t_i} \xi_i(t) dW_t \right|$$

with

$$\xi_i(t) = S_{t_{i-1}}^2 \left[\sigma(t_{i-1}, S_{t_{i-1}}) - \sigma(t_{i-1}, S_t)\right] \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}).$$

We immediately get that

$$\|D_1^n\|_2 \leq C\, n^{-\alpha} \sum_{i=1}^{n-1} \left(\int_{t_{i-1}}^{t_i} E\xi_i^2(t) dt \right)^{1/2}.$$

Using the hypothesis on σ and Theorem 3.4.4, we obtain that :

$$\|D_1^n\|_2 \leq c\, n^{-\alpha} \sum_{i=1}^{n-1} \left(\int_{t_{i-1}}^{t_i} \frac{\Delta t_i dt}{\gamma_n(1-t_{i-1})} \right)^{1/2}.$$

Hence

$$\|D_1^n\|_2 \leq c\, n^{-\frac{\alpha}{2}-\frac{1}{4}} \sum_{i=1}^{n-1} \frac{\Delta t_i}{\sqrt{1-t_{i-1}}} \to 0.$$

We have

$$D_2^n = k_n \sum_{i=1}^{n-1} \left| \int_{t_{i-1}}^{t_i} \widetilde{\xi}_i(t) dW_t \right|$$

where

$$\widetilde{\xi}_i(t) = S_{t_{i-1}}^2 \left[\sigma(t_{i-1}, S_t) - \sigma(t, S_t)\right] \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}).$$

Using he hypothesis on σ and Theorem 3.4.4, we also deduce that $\|D_2^n\|_2 \to 0$. The last term is

$$D_3^n = k_n \sum_{i=1}^{n-1} \left| \int_{t_{i-1}}^{t_i} \mathcal{X}_i(t) dW_t \right|$$

where

$$\mathcal{X}_i(t) = \left[S_{t_{i-1}} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - S_t \widehat{C}_{xx}(t, S_t)\right] \sigma(t, S_t).$$

We first observe that

$$\|D_3^n\|_2 \leq C\, n^{-\alpha} \sum_{i=1}^{n-1} \left(\int_{t_{i-1}}^{t_i} E\mathcal{X}_i^2(t) dt \right)^{1/2}.$$

Using the Ito formula

$$d[S_t \widehat{h}^t] = d[S_t \widehat{C}_{xx}(t, S_t)] = f_t dW_t + g_t dt$$

where

$$f_t := \sigma S_t \widehat{C}_{xx}(t, S_t) + \sigma S_t^2 \widehat{C}_{xxx}(t, S_t)$$

$$g_t := S_t \widehat{C}_{xxt}(t, S_t) + \frac{1}{2}\sigma^2 S_t^3 \widehat{C}_{xxxx}(t, S_t) + \sigma^2 S_t^2 \widehat{C}_{xxx}(t, S_t),$$

we deduce that

$$E(S_t \widehat{h}^t - S_{t_{i-1}} \widehat{h}_{t_{i-1}})^2 \leq 2 \int_{t_{i-1}}^{t_i} E(f_u^2) du + 2\Delta t_i \int_{t_{i-1}}^{t_i} E(g_u^2) du.$$

It follows that

$$\left(\int_{t_{i-1}}^{t_i} E\mathcal{X}_i^2(t) dt \right)^{1/2} \leq \sqrt{2\Delta t_i} \left(\int_{t_{i-1}}^{t_i} E(f_u^2) du \right)^{\frac{1}{2}} + \Delta t_i \left(2 \int_{t_{i-1}}^{t_i} E(g_u^2) du \right)^{\frac{1}{2}}.$$

By Theorem 3.4.4, we get the bounds

$$\begin{aligned} E(|D_3^n|) &\leq C\, n^{-\frac{1}{2}} \sum_{i=1}^{n-1} \frac{\Delta t_i}{1-t_i} + C\, n^{-1/4-\alpha/2} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/2}}{(1-t_i)^{3/2}} \\ &\leq C\, n^{-\frac{1}{2}} \ln n + C\, n^{-1/4-\alpha/2} \ln n. \end{aligned}$$

where C is a constant. To obtain the last inequality, we have used the fact that $\sqrt{\Delta t_i/(1-t_i)}$ is bounded for $i \leq n-1$. It follows that $E(|D_3^n|)$ converges to 0. ■

Lemma 3.3.6. *We have* $P-\lim L_4^n = 0$.

Proof. We first show that we may replace ΔK_{t_i} by

$$\Delta\tilde{K}_{t_i} := -\int_{t_{i-1}}^{t_i} C_{xt}(u,S_u)du.$$

To do so, it suffices to show that $\chi^n \to 0$ where

$$\chi^n := k_n \sum_{i\leq n-1} S_{t_{i-1}} \int_{t_{i-1}}^{t_i} \left(\widehat{C}_{xt}(u,S_u) - \widehat{C}_{xt}(u,S_{t_{i-1}})\right) du.$$

Using a Taylor expansion

$$\widehat{C}_{xt}(u,S_u) - \widehat{C}_{xt}(u,S_{t_{i-1}}) = \widehat{C}_{xxt}(u,\bar{S}_{t_{i-1}})\left(S_u - S_{t_{i-1}}\right)$$

we deduce that $\chi^n \leq c_\omega \bar{\chi}^n$ where, by Theorem 3.4.4,

$$\bar{\chi}^n := k_n\gamma_n \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \frac{S_u - S_{t_{i-1}}}{(1-t)^{3/2}\gamma_n^{3/2}} du.$$

As $E\left|S_u - S_{t_{i-1}}\right| \leq c\sqrt{\Delta t_i}$, we easily conclude that $E\bar{\chi}^n \to 0$.
Using the inequality $||a|-|b|| \leq |a-b|$ and assuming that $\Delta K_{t_i} = \Delta\tilde{K}_{t_i}$, we get that

$$|L_n^4| \quad \leq \quad k_n \sum_{i=1}^{n-1} S_{t_{i-1}} |\Delta A_{t_i} + \Delta K_{t_i}| \leq c(\omega)\, k_n \int_0^{t_{n-1}} \sigma^2 S_u^2 |\widehat{C}_{xxx}(u,S_u)| du.$$

By virtue of Theorem 3.4.4, we get that :

$$|L_n^4| \leq c(\omega)\, k_n \int_0^{t_{n-1}} \left(\frac{1}{\sqrt{\gamma_n(1-t)}} + \frac{1}{\gamma_n(1-t)}\right) dt \to 0.$$

■

Lemma 3.3.7. *We have* $P-\lim L_5^n = 0$.

Proof. We use similar arguments. First, using a Taylor expansion and Theorem 3.4.11, we deduce that

$$A^n := k_n \sum_{i=1}^{n-1} \Delta S_{t_i} |\Delta H_{t_i}| \leq c_\omega k_n \sum_{i=1}^{n-1} \left[\frac{(\Delta S_{t_i})^2}{\sqrt{\gamma_n(1-t_i)}} + \frac{\Delta t_i \Delta S_{t_i}}{(1-t_i)}\right].$$

As $E(\Delta S_{t_i})^2 \leq C\Delta t_i$, we deduce that $A^n \to 0$. Secondly, as in Lemma 3.3.6 , we may assume without loss of generality that $\Delta K_{t_i} = \Delta\tilde{K}_{t_i}$. Therefore, it remains to estimate, by virtue of Theorem 3.4.4

$$B^n := k_n \sum_{i=1}^{n-1} \Delta S_{t_i} |\Delta\tilde{K}_{t_i}| \leq c_\omega k_n \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \frac{\Delta S_{t_i}}{1-u} du.$$

We easily get that $B^n \to 0$. ■

Lemma 3.3.8. *We have* $P-\lim F_3^n = 0$.

Proof. Let us define $\bar{F}_3^n := \int_0^1 K_t^n dS_t$. Observe that $\bar{F}_3^n$ can be rewritten as

$$\bar{F}_3^n = \sum_{i=1}^{n} K_{t_i}^n \Delta S_{t_i} = -\sum_{i=1}^{n-1} S_{t_i}\Delta K_{t_i}^n + K_1^n S_1 - K_{t_1}^n S_0 = -\sum_{i=1}^{n-1} S_{t_i}\Delta K_{t_i}^n + K_1^n S_1$$

Since $F_3^n = k_n \bar{F}_3^n$, we assume without loss of generality that $\Delta K_{t_i} = \Delta \tilde{K}_{t_i}$ as shown in Lemma 3.3.6. Therefore,

$$\begin{aligned}\bar{F}_3^n &= \sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i} \widehat{C}_{tx}(u,S_u)S_{t_i}du - S_1\int_0^{t_{n-1}} \widehat{C}_{tx}(u,S_u)du,\\ &= \int_0^{t_{n-1}} \widehat{C}_{tx}(u,S_u)(S_u-S_1)du + \sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i} \widehat{C}_{tx}(u,S_u)(S_{t_i}-S_u)du\end{aligned}$$

Note that $E|S_u - S_t| \le C\sqrt{|t-u|} \le C\sqrt{|1-u|}, \quad \forall u,t$, and recall that $E(\sup_{0\le u\le 1} S_u) < \infty$. Let us now separate the analysis in two cases :

- $\alpha \in]0, 1/2]$. We use the bound $\left|\widehat{C}_{xt}(t,x)\right| \le \dfrac{C}{1-t}$ and we deduce that

$$E|F_3^n| \le cn^{-\alpha}\int_0^{t_{n-1}} \sqrt{1-u}\frac{1}{1-u}du \le cn^{-\alpha} \to 0.$$

- $\alpha = 0$. In this case we use Lemma 3.4.5. We get that

$$E|F_3^n| \le c\int_0^{t_{n-1}} \frac{1}{\sqrt{\gamma_n}(1-t)}du \le cn^{-1/4}\ln n \to 0.$$

Therefore, we have shown that $P-\lim F_3^n = 0$. ■

3.4 Estimation of the Derivatives of $\widehat{C}(t,x)$

Before analysing the function $\widehat{C}(t,x)$ and getting some upper bounds for its derivatives, we are going to use the time change technique to reduce our problem into the case where the diffusion coefficient in (**e**) is bounded independently of n. We then give an explicit expression of $\widehat{C}(t,x)$ and $\widehat{C}_x(t,x)$ and we deduce estimates of the derivatives. To do so, we utilize the analysis of the fundamental solution associated to a parabolic-type PDE given in [37]. Notice that Friedman imposes that the coefficients of the parabolic-type PDE are Holder continuous of exponent $0 < \beta < 1$. In fact, the uniform Lipschitz condition is sufficient as shown in [74]. In all this section, we suppose that the assumptions (**E**), (**H**) hold. We shall pay special attention to the case $\alpha = 0$ with Assumption **(H')**. This is the most difficult case of our analysis for which we need specific bounds of the derivatives of $\widehat{C}$.

Recall that $\widehat{C}(t,x)$ satisfies the equation

$$\left\{ \begin{array}{rcl} \widehat{C}_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2(x)x^2\widehat{C}_{xx}(t,x)=0, \quad (x,t)\in]0,\infty[\otimes[0,1[\,, \\ \widehat{C}(1,x) & = & h(x), \quad x\in]0,\infty[\,., \end{array} \right.$$

Consider the change of time $s := t\gamma_n$. Then, $u(s,x) := \widehat{C}(\frac{s}{\gamma_n},x)$ satisfies the following PDE :

$$(\mathbf{f}) = \left\{ \begin{array}{rcl} \widehat{u}_t(t,x) & + & \frac{1}{2}(\widehat{\sigma}^*)^2(t,x)x^2\widehat{u}_{xx}(t,x)=0, \quad (x,t)\in]0,\infty[\otimes[0,\gamma_n[\,, \\ \widehat{u}(\gamma_n,x) & = & h(x), \quad x\in]0,\infty[\,, \end{array} \right.$$

where $(\widehat{\sigma}^*)^2(t,x) = \dfrac{\widehat{\sigma}^2(t,x)}{\gamma_n}$. We easily check that $\widehat{\sigma}^*(s,x)$ satisfies Assumption (**E**) for some constants which do not dependent of n. Using the definition of u, it can be shown that

$$\frac{\partial^k}{\partial x^k}\frac{\partial^r}{\partial s^r}u(s,x) = \frac{1}{\gamma_n^r}\frac{\partial^k}{\partial x^k}\frac{\partial^r}{\partial t^r}C(t,x). \tag{3.4.6}$$

According to the proof of Lemma 3.5.2, the unique solution of (**f**) is given by

$$\widehat{u}(s,x) = Eh(\widehat{S}_{s,x}(\gamma_n)) \tag{3.4.7}$$

where $\widehat{S}_{s,x}$ is solution of the PDE

$$(\hat{\mathbf{S}}) = \left\{ \begin{array}{rcl} d\widehat{S}_{s,x}(u) & = & \widehat{\sigma}^*(u,\widehat{S}_{s,x}(u))\widehat{S}_{s,x}(u)dW_u, \quad u\in]s,\gamma_n] \\ \widehat{S}_{s,x}(s) & = & x \end{array} \right.$$

We need the probabilistic representation (3.4.7) of $\widehat{u}(s,x)$ in the case $\alpha = 0$. If $\alpha > 0$, we use the following representation of $\widehat{u}_x(s,x)$. Let us define

$$\widehat{\Lambda}^*(t,x) := \left(\widehat{\sigma}^*(t,x) + x\widehat{\sigma}^*_x(t,x)\right)\widehat{\sigma}^*(t,x)$$

and, by Lemma 3.5.3, we consider $\widetilde{S}_{x,t}$ the solution of the SDE :

$$\left\{ \begin{array}{rcl} d\widetilde{S}_{x,t}(u) & = & \widehat{\sigma}^*(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)dW_u + \widehat{\Lambda}^*(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)du, \quad u\in]s,\gamma_n] \\ \widetilde{S}_{x,t}(t) & = & x. \end{array} \right.$$

Then, we have :

Lemma 3.4.1.

$$\widehat{u}_x(s,x) = Eh'(\widetilde{S}_{s,x}(\gamma_n)). \tag{3.4.8}$$

Proof. We write :

$$\begin{array}{rcl} \widehat{u}(t,x) - \widehat{u}(t,x_0) & = & Eh(\widehat{S}_{x,t}(\gamma_n)) - Eh(\widehat{S}_{x_0,t}(\gamma_n)), \\ \widehat{u}(t,x) - \widehat{u}(t,x_0) & = & E\displaystyle\int_0^{\gamma_n} \frac{d}{d\mu}h\left(\widehat{S}_{x_0,t}(\gamma_n) + \mu(\widehat{S}_{x,t}(\gamma_n) - \widehat{S}_{x_0,t}(\gamma_n))\right)d\mu. \end{array}$$

Since h' exists out of a countable set, $\left(\widehat{C}(t,x)-\widehat{C}(t,x_0)\right)/(x-x_0)$ is equal to

$$E\int_0^{\gamma_n} h'\left(\widehat{S}_{x_0,t}(\gamma_n)+\mu(\widehat{S}_{x,t}(\gamma_n)-\widehat{S}_{x_0,t}(\gamma_n)\right)\frac{\widehat{S}_{x,t}(\gamma_n)-\widehat{S}_{x_0,t}(\gamma_n)}{x-x_0}d\mu.$$

As Assumption (**E**) holds, we apply Theorem 5.12 p120 [36] and deduce that $\frac{\partial\widehat{S}_{x,t}(\gamma_n)}{\partial x}$ exists in the L^2 sense, i.e. :

$$\frac{\widehat{S}_{x,t}(\gamma_n)-\widehat{S}_{x_0,t}(\gamma_n)}{x-x_0}\to\frac{\partial\widehat{S}_{x_0,t}(\gamma_n)}{\partial x}\text{ in }L^2. \tag{3.4.9}$$

Indeed, we check that Condition (A) page 108 [36] is satisfied by the SDE $(\hat{\mathbf{S}})$. First, we have $|\widehat{\sigma}^*(t,x)x|\le C|x|$ and secondly :

$$|\widehat{\sigma}^*(t,x)x-\widehat{\sigma}^*(t,\overline{x})\overline{x}|\le|\widehat{\sigma}^*(t,x)||x-\overline{x}|+|\overline{x}\,(\widehat{\sigma}^*(t,x)-\widehat{\sigma}^*(t,\overline{x}))|$$

where a Taylor expansion yields

$$\widehat{\sigma}^*(t,x)-\widehat{\sigma}^*(t,\overline{x})=\widehat{\sigma}^*_x(t,x_0)(x-\overline{x}),\quad x_0\in[x,\overline{x}].$$

Then,

$$|\overline{x}\,(\widehat{\sigma}^*(t,x)-\widehat{\sigma}^*(t,\overline{x}))|\le|\widehat{\sigma}^*(t,x)-\widehat{\sigma}^*(t,\overline{x})||\overline{x}-x_0|+|x_0\widehat{\sigma}^*_x(t,x_0)||x-\overline{x}|$$

where $|x_0\widehat{\sigma}^*_x(t,x_0)|$ is bounded by virtue of Assumption **(E)**. It follows that there exists a constant such that for all x

$$|\widehat{\sigma}^*(t,x)x-\widehat{\sigma}^*(t,\overline{x})\overline{x}|\le C|x-\overline{x}|.$$

As σ_x is continuous, we conclude that Condition (A) holds.
Furthermore, we have :

$$\frac{\partial\widehat{S}_{x,t}(u)}{\partial x}=1+\int_t^u\frac{\widehat{\Lambda}^*\left(s,\widehat{S}_{x,t}(s)\right)}{\widehat{\sigma}^*\left(s,\widehat{S}_{x,t}(s)\right)}\frac{\partial\widehat{S}_{x,t}(s)}{\partial x}dW_s$$

which is a strictly positive martingale (see Lemma 3.5.4). In the proof Lemma 3.4.2, it is shown that the distribution of $\widehat{S}_{x_0,t}(\gamma_n)$ is of density with respect to the Lebesgue measure. We deduce that, out of the null-set $\widehat{S}_{x_0,t}(\gamma_n)\in\{K_p:p\in N^*\}$, we have almost surely :

$$\int_0^{\gamma_n}h'\left(\widehat{S}_{x_0,t}(\gamma_n)+\mu(\widehat{S}_{x_n,t}(\gamma_n)-\widehat{S}_{x_0,t}(\gamma_n)\right)d\mu\to h'(\widehat{S}_{x_0,t}(\gamma_n))$$

provided that x_n is sufficiently near to x_0 and x_n is a subsequence such that (3.4.9) holds. Since h' is bounded, it follows that

$$\widehat{u}_x(t,x)=Eh'(\widehat{S}_{x,t}(\gamma_n))\frac{\partial\widehat{S}_{x,t}(\gamma_n)}{\partial x}.$$

Finally, we define $d\overline{P} = \frac{\partial \widehat{S}_{x,t}(\gamma_n)}{\partial x} dP$ so that $\widehat{u}_x(t,x) = \overline{E}h'(\widehat{S}_{x,t}(\gamma_n))$. By virtue of the Girsanov theorem ((5.1) p 190 [72]), the process

$$B_u = W_u - W_t - \int_t^u \frac{\widehat{\Lambda}^*\left(s, \widehat{S}_{x,t}(s)\right)}{\widehat{\sigma}^*\left(s, \widehat{S}_{x,t}(s)\right)} du$$

is a standard brownian motion under $\overline{P}$. Moreover, $\widehat{S}_{x,t}$ verifies the SDE

$$d\widehat{S}_{x,t}(u) = \widehat{\sigma}^*(u, \widehat{S}_{x,t}(u))\widehat{S}_{x,t}(u)dB_u + \widehat{\Lambda}(t, \widehat{S}_{x,t}(u))\widehat{S}_{x,t}(u)du. \quad (3.4.10)$$

Since $\widehat{\sigma}^*$ and $\widehat{\Lambda}$ are bounded, (3.4.10) admits a unique strong solution, hence a unique weak solution. We conclude that $\widehat{u}_x(t,x) = Eh'(\widetilde{S}_{x,t}(\gamma_n))$. ■

Let us now give an explicit representation of $\widehat{u}_x(t,x)$ using the notion of fundamental solution of a PDE of parabolic type.

Lemma 3.4.2. *We have :*

$$\widehat{u}_x(t,x) = \int_{-\infty}^{\infty} h'(e^z)\Gamma^*(x,t,z,\gamma_n)dz$$

where $\Gamma^*(x,t,z,\tau)$ *is the fundamental solution of the operator :*

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

and

$$\begin{aligned} \widehat{\sigma}_a(t,x) &= \widehat{\sigma}^*(t,e^x), \\ \widehat{\sigma}_b(t,x) &= \widehat{\Lambda}(t,e^x) - \frac{1}{2}(\widehat{\sigma}^*)^2(t,e^x). \end{aligned}$$

Proof. Let us define the process $\widehat{\eta}_{x,t}(u) = \ln \widetilde{S}_{e^x,t}(u)$. It satisfies the following SDE :

$$(\mathbf{f}') = \left\{ \begin{array}{rcl} d\widehat{\eta}_{x,t}(u) &=& \widehat{\sigma}_a(u, \widehat{\eta}_{x,t}(u))dW_u + \widehat{\sigma}_b(u, \widehat{\eta}_{x,t}(u))du \\ \widehat{\eta}_{x,t}(t) &=& x \end{array} \right.$$

To see it, it suffices to apply the Ito formula with the process $\exp(\widehat{\eta}_{x,t})$ where $\widehat{\eta}_{x,t}$ is the solution of (**f'**). By virtue of Lemma 3.5.5, $\widehat{\eta}_{x,t}$ is a Markov process of transition density function $\Gamma^*(x,t,z,\gamma_n)$, the fundamental solution of the operator :

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

This means that : $P(\widehat{\eta}_{x,t}(u) \in dz) = \Gamma^*(x,t,z,u)dz$ hence

$$\widehat{C}_x(t,x) = \int_{-\infty}^{\infty} h'(e^z)\Gamma^*(\ln x,t,z,\gamma_n)dz.$$

■

Let us now consider the special case $\alpha = 0$. With Assumption **(H')**, the function h is assumed to be of the form $h(x) = h_1(x) + ax + b$. As $\widehat{S}_{s,x}(u)$ is a martingale, Lemma 3.4.1 yields

$$\widehat{u}(s,x) = Eh_1(\widehat{S}_{s,x}(\gamma_n)) + ax + b.$$

Using the same arguments as in Lemma 3.4.2 , the function $Eh_1(\widehat{S}_{s,x}(\gamma_n))$ has also an explicit representation, and so is $\widehat{u}(s,x)$, which is given by the following lemma.

Lemma 3.4.3. *Assume that Assumption* **(H')** *holds. Then,*

$$\widehat{u}(t,x) = \int_{-\infty}^{\infty} h_1(e^z)\Gamma^*(x,t,z,\gamma_n)dz + ax + b$$

where $\Gamma^*(x,t,z,\tau)$ *is the fundamental solution of the operator :*

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

and

$$\widehat{\sigma}_a(t,x) \quad = \quad \widehat{\sigma}(t,e^x), \quad \widehat{\sigma}_b(t,x) = -\frac{1}{2}\widehat{\sigma}^2(t,e^x).$$

We now aim to estimate the fundamental solution $\Gamma^*(x,t,z,\gamma_n)$. To do so we use the bounds given by Theorem 8, p.263 [37]. Notice that we only need the differentiability of $\sigma(t,x)$ up to order $n = 3$. In particular, we recall that

$$\left|\frac{\partial^{k+r}}{\partial x^k \partial t^r}\Gamma^*(x,t,\xi,\gamma_n)\right| \leq \frac{C'}{\rho_n(t)^{\frac{k+2r}{2}}}\exp\left\{-C\frac{|x-\xi|^2}{\rho_n(t)}\right\} \tag{3.4.11}$$

where C, C' are two positive constants which, because of the change of time, do not depend on n and $\rho_n(t) = \gamma_n - t$.

We then obtain explicit expressions of the derivatives of $\widehat{C}(t,x)$. According to Lemma 3.4.3, we have

$$\widehat{u}_x(t,x) = \int_{-\infty}^{\infty} h'(e^y)\Gamma^*(\ln x,t,y,\gamma_n)dy$$

and by the change of variable $z = e^y$, we obtain :

$$\begin{aligned}
\widehat{u}_x(t,x) &= \int_0^{\infty} \frac{h'(z)}{z}\Gamma^*(\ln x,t,\ln y,\gamma_n)dz,\\
\widehat{u}_{xx}(t,x) &= \frac{1}{x}\int_0^{\infty} \frac{h'(z)}{z}\frac{\partial}{\partial x}\Gamma^*(\ln x,t,\ln y,\gamma_n)dz,\\
\widehat{u}_{xxx}(t,x) &= -\frac{1}{x}\widehat{u}_{xx}(t,x) + \frac{1}{x^2}\int_0^{\infty} \frac{h'(z)}{z}\frac{\partial^2}{\partial^2 x}\Gamma^*(\ln x,t,\ln y,\gamma_n)dz,\\
\widehat{u}_{xt}(t,x) &= \int_0^{\infty} \frac{h'(z)}{z}\frac{\partial}{\partial t}\Gamma^*(\ln x,t,\ln y,\gamma_n)dz,\\
\widehat{u}_{xxt}(t,x) &= \int_0^{\infty} \frac{h'(z)}{z}\frac{\partial^2}{\partial t\partial x}\Gamma^*(\ln x,t,\ln y,\gamma_n)dz.
\end{aligned}$$

Using the bounds given by (3.4.11) and the relations (3.4.6), we immediately deduce some bounds for the derivatives of $\widehat{C}(t,x)$ which are given in the following theorem.

Theorem 3.4.4. *There exists a constant $C>0$ independent of n such that :*

$$\left|\widehat{C}_{xx}(t,x)\right| \leq \frac{C}{x\rho_t}, \tag{3.4.12}$$

$$\left|\widehat{C}_{xxx}(t,x)\right| \leq \frac{C}{x^2\rho_t}+\frac{C}{x^2\rho_t^2}, \tag{3.4.13}$$

$$\left|\widehat{C}_{xt}(t,x)\right| \leq \frac{C}{1-t}, \tag{3.4.14}$$

$$\left|\widehat{C}_{xxt}(t,x)\right| \leq \frac{C\gamma_n}{x\rho_t^3}, \tag{3.4.15}$$

where $\rho_t := \gamma_n(1-t)$.

We conclude this section with a specific bound we need in the case $\alpha=0$ under Assumption **(H')**. As $u(s,x)$ has a probabilistic representation given by Lemma 3.4.3) where the function h_1 is bounded, we get a bound for $\widehat{C}_{xt}(t,x)$ which is comparable to that of $\widehat{C}_{xxt}(t,x)$ in the case $\alpha>0$. Indeed, recall that the bound we obtain for $\widehat{C}_{xxt}(t,x)$ is deduced from the probabilistic representation of $u_x(s,x)$. Therefore, we have

Lemma 3.4.5. *Suppose that Assumption* **(H')** *holds. Then,*

$$\left|\widehat{C}_{xt}(t,x)\right| \leq \frac{C\gamma_n}{x\rho_t^3} \tag{3.4.16}$$

where $\rho_t := \gamma_n(1-t)$.

3.5 Appendix

Lemma 3.5.1. *The stochastic equation defined on $[s,+\infty[$ for all $s>0$ by :*

$$(\hat{\mathbf{S}}) = \left\{ \begin{array}{rcl} d\widehat{S}_{x,s}(t) & = & \widehat{\sigma}(t,\widehat{S}_{x,s}(t))\widehat{S}_{x,s}(t)dW_t \\ \widehat{S}_{x,s}(s) & = & x \end{array} \right.$$

has a unique solution such that for some constant $C^*=C^*(n,T)$,

$$E\sup_{0\leq t\leq T}\widehat{S}^2_{x,s}(t) \leq C^*(1+x^2).$$

Proof. It suffices to apply Theorem 2.2 p104 [36]. ■

Lemma 3.5.2. *Assume that Condition* (**H**) *holds. Then, the PDE* (**e**) *has a unique solution.*

Proof. Note that it is not possible to conclude immediately on the existence of a solution of (**e**) because the operator is not uniformly parabolic on

$]0,\infty[\otimes[0,1[$. This is why, we transform the problem in such a way that the required uniform parabolic condition holds.
By virtue of Lemma 3.5.1, recall that $\widehat{S}_{x,s}(t)$ is the unique solution of the stochastic equation defined on $[s,1]$, $s\in[0,1]$ by :

$$\left\{\begin{array}{rcl} d\widehat{S}_{x,s}(t) &=& \widehat{\gamma}(t,\widehat{S}_{x,s}(t))dW_t \\ \widehat{S}_{x,s}(s) &=& x \end{array}\right.$$

where $\widehat{\gamma}(t,x)=\sqrt{\sigma^2(t,x)x^2+\sigma(t,x)x^2\gamma_n}$. Recall that

$$E\sup_{s\le t\le 1}\widehat{S}^2_{x,s}(t)\le C_n^*(1+x^2)$$

where C_n^* is a constant depending on n. Let us define $g(x,t):=Eh(\widehat{S}_{x,t}(1))$. It satisfies the inequality

$$\begin{array}{rcl} |g(x,t)| &\le& c\left(1+E|\widehat{S}_{x,t}(1)|\right)\le c\left(1+(E\widehat{S}^2_{x,t}(1))^{1/2}\right) \\ &\le& c\,(1+|x|). \end{array}$$

Since h' is bounded, we obtain by virtue of the Cauchy–Schwarz inequality, that

$$|g(x,t)-g(y,u)|\le c\sqrt{E\left(\widehat{S}_{x,t}(1)-\widehat{S}_{y,u}(1)\right)^2}.$$

Let $\widehat{S}^{(m)}_{x,s}$ be the solution of the SDE

$$\left\{\begin{array}{rcl} d\widehat{S}^{(m)}_{x,s}(t) &=& \widehat{\gamma}^{(m)}(t,\widehat{S}^{(m)}_{x,s}(t))dW_t \\ \widehat{S}^{(m)}_{x,s}(s) &=& x \end{array}\right.$$

where $\widehat{\gamma}^{(m)}(t,x):=\sqrt{\sigma^2(t,x)x^2+\sigma(t,x)\gamma_n x^2+m^{-1}}$. Observe than $\|\widehat{\gamma}^{(m)}-\widehat{\gamma}\|_\infty\le m^{-1/2}$ hence $\widehat{S}_{x,s}(1)^{(m)}\to\widehat{S}_{x,s}(1)$ in $L^2(\Omega,P)$ as $m\to\infty$ uniformly in x and s. Then, $g^{(m)}(x,t):=Eh(\widehat{S}^{(m)}_{x,t}(1))$ converges uniformly to $g(x,t)$. Applying Lemma 3.3 p 112 with Condition (A') p 113 [36], we deduce the existence of a constant $C_R^{(m)}$ such that

$$|g^{(m)}(x,t)-g^{(m)}(y,u)|\le C_R^{(m)}\sqrt{(x-y)^2+|t-u|}$$

if $|x|,|y|\le R$. We deduce that $g^{(m)}$ is continuous hence so is g.
We use the notations of page 138 [36] where we replace t by $1-t$. Let us consider the following sets with $m\in\mathbb{N}\backslash\{0\}$:

$$\begin{array}{rcl} Q_m &=&]\frac{1}{m},m[\times]0,1[, \\ B_m &=&]\frac{1}{m},m[\times\{1\}, \\ T_m &=&]\frac{1}{m},m[\times\{0\}, \\ S_m &=& \{\frac{1}{m},m\}\times[0,1[. \end{array}$$

For each $y \in \partial Q_m$, it is easy to observe that there exists a closed ball K_y^m such that $K_y^m \cap Q_m = \emptyset$ and $K_y^m \cap \overline{Q_m} = \{y\}$. It follows that the function W_y proposed p 134 [36] defines a *barrier* for each $y \in S_m \subseteq \partial Q_m$. Moreover, observe that $g(x,t) = Eh(\widehat{S}_{x,1}(1)) = h(x)$ if $(x,t) \in \overline{B_m} \cap \overline{S_m}$. By virtue of Theorem 3.6 p 138 [36], Assumptions (**G**) and (**P**), implie that the Dirichlet problem

$$(\mathbf{D_m}) = \left\{ \begin{array}{rcll} u_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2(t,x)x^2u_{xx}(t,x) = 0 & (x,t) \in Q_m \cup T_m \\ u(1,x) & = & h(x) & x \in B_m \\ u(t,x) & = & g(x,t) & (x,t) \in S_m \end{array} \right.$$

admits a unique solution u^m. Indeed, g and h are continuous. Since $\overline{Q_m}$ is bounded, there is a constant c_m depending on m such that

$$\left|\widehat{\gamma}^2(t,x) - \widehat{\gamma}^2(t,\overline{x})\right| \leq c_m|x - \overline{x}|.$$

Note that u^m is continuous on $\overline{Q_m}$ and the derivatives u_t^m, u_{xx}^m are continuous on $Q_m \cup T_m$ (see Theorem 3.6 p 138 [36]). By virtue of Theorem 5.2 p 147 [36], we deduce that that u_m has the following stochastic representation

$$u^m(x,t) = Eg(\widehat{S}_{x,t}(\tau),\tau)I_{\tau<1} + Eh(\widehat{S}_{x,t}(1))I_{\tau=1},$$

where τ is a stopping time. It follows that $u^m(x,t) = Eg(\widehat{S}_{x,t}(\tau),\tau)$.
On the other hand,

$$g(\widehat{S}_{x,t}(\tau),\tau) = Eh\left(\widehat{S}_{\widehat{S}_{x,t}(\tau),\tau}(1)\right)$$

where $\widehat{S}_{\widehat{S}_{x,t}(\tau),\tau}(1) = \widehat{S}_{x,t}(1)$. It follows that $u^m(x,t) = g(x,t)$ and, as $m \to \infty$, we deduce that g is a solution to the PDE (**e**). Indeed, it is easy to check that g verifies (**e**). Moreover, $v(t,y) = u(t,e^y)$ is a solution of the following uniformly parabolic PDE

$$(\mathbf{f}) = \left\{ \begin{array}{rcll} v_t(t,y) & + & \frac{1}{2}\widehat{\sigma}^2(e^y)v_{yy}(t,y) \quad -\frac{1}{2}\widehat{\sigma}^2(e^y)v_y(t,y) = 0, & (y,t) \in \mathbf{R} \times [0,1[\\ v(1,y) & = & h(e^y), \quad x \in \mathbf{R}. & \end{array} \right.$$

By virtue of Theorem 3.6 [36], v is also the unique solution of the same PDE restricted to an arbitrary smooth bounded domain. Moreover, by virtue of Theorem 5.2 p 147 [36], we deduce that the solution v has a probabilistic representation which is unique. We then conclude that u is also unique. ■

Lemma 3.5.3. *Assume that $t \in [0,\gamma_n]$. Then, the stochastic equation :*

$$\left\{ \begin{array}{rcl} d\widetilde{S}_{x,t}(u) & = & \widehat{\sigma}^*(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)dW_u + \widehat{\Lambda}^*(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)du \\ \widetilde{S}_{x,t}(t) & = & x \end{array} \right.$$

has a unique solution on $[t,\gamma_n]$.

Proof. It suffices to use Theorem 2.2 p104 [36]. For this, we verify the needed conditions with :

$$\begin{aligned} \widetilde{\sigma}(t,x) &= \widehat{\sigma}^*(t,x)x, \\ b(t,x) &= \widehat{\Lambda}^*(t,x)x. \end{aligned}$$

From

$$(\widehat{\sigma}^*)^2(t,x) = \frac{1}{\gamma_n}\sigma^2(t,x) + \sigma(t,x)$$

we deduce that

$$2\widehat{\sigma}_x^*(t,x)\widehat{\sigma}^*(t,x) = \frac{2}{\gamma_n}\sigma_x(t,x)\sigma(t,x) + \sigma_x(t,x).$$

Recall that

$$\widehat{\Lambda}^*(t,x) = (\widehat{\sigma}^*)^2(t,x) + \widehat{\sigma}_x^*(t,x)\widehat{\sigma}^*(t,x)x.$$

Then, from the hypothesis $|x\sigma_x(t,x)| \leq const$ it is easy to deduce that $|b(t,x)| \leq const\,|x|$. Otherwise, it is clear that $|\widetilde{\sigma}(t,x)| \leq const\,|x|$.
Finally, we suppose that $|x|, |\overline{x}| \leq N$. We have

$$|b(t,x) - b(t,\overline{x})| \leq \left|\widehat{\Lambda}^*(t,x)\right| |x-\overline{x}| + |\overline{x}| \left|\widehat{\Lambda}^*(t,x) - \widehat{\Lambda}^*(t,\overline{x})\right|$$

where $\left|\widehat{\Lambda}^*(t,x)\right| \leq const$, $|\overline{x}| \leq N$ and

$$\widehat{\Lambda}^*(t,x) - \widehat{\Lambda}^*(t,\overline{x}) = (\widehat{\sigma}^*)^2(t,x) - (\widehat{\sigma}^*)^2(t,\overline{x}) + x\widehat{\sigma}_x^*(t,x)\widehat{\sigma}^*(t,x) - \overline{x}\widehat{\sigma}_x^*(t,\overline{x})\widehat{\sigma}^*(t,\overline{x}).$$

But we have $\left|(\widehat{\sigma}^*)^2(t,x) - (\widehat{\sigma}^*)^2(t,\overline{x})\right| \leq const\,|x-\overline{x}|$ and

$$2x\widehat{\sigma}_x^*(t,x)\widehat{\sigma}^*(t,x) = x\frac{2}{\gamma_n}\sigma_x(t,x)\sigma(t,x) + x\sigma_x(t,x).$$

Since the next expression is bounded, we first write $x = (x-\overline{x}) + \overline{x}$ and finally, we have to estimate

$$\begin{aligned} \sigma_x(t,x)\sigma(t,x) - \sigma_x(t,\overline{x})\sigma(t,\overline{x}) &= \sigma(t,x)\left(\sigma_x(t,x) - \sigma_x(t,\overline{x})\right) \\ &\quad + \sigma_x(t,\overline{x})\left(\sigma(t,x) - \sigma(t,\overline{x})\right) \end{aligned}$$

where

$$\begin{aligned} |\sigma_x(t,x) - \sigma_x(t,\overline{x})| &\leq const\,|x-\overline{x}|, \\ |\sigma(t,x) - \sigma(t,\overline{x})| &\leq const\,|x-\overline{x}|. \end{aligned}$$

because $\sigma_{xx}(t,x)$ is bounded. Then, we can conclude that for $|x|, |\overline{x}| \leq N$,

$$|b(t,x) - b(t,\overline{x})| \leq const(N)\,|x-\overline{x}|.$$

In a similar way, it is easy to prove that

$$|\widetilde{\sigma}(t,x) - \widetilde{\sigma}(t,\overline{x})| \leq const(N)\,|x-\overline{x}|.$$

■

Lemma 3.5.4. *The local martingale*

$$\frac{\partial \widehat{S}_{x,t}(u)}{\partial x} = 1 + \int_t^u \frac{\widehat{\Lambda}^*\left(s, \widehat{S}_{x,t}(s)\right)}{\widehat{\sigma}\left(s, \widehat{S}_{x,t}(s)\right)} \frac{\partial \widehat{S}_{x,t}(s)}{\partial x} dW_s$$

is a strictly positive martingale on any interval $[t,T] \subseteq [0,\infty)$.

Proof. The Doleans–Dade formula give us

$$\frac{\partial \widehat{S}_{x,t}(u)}{\partial x} = \exp\left\{ \int_t^1 \widehat{\Lambda}^*\left(v, \widehat{S}_{x,t}(v)\right) dW_v - \frac{1}{2}\int_t^1 (\widehat{\Lambda}^*)^2\left(v, \widehat{S}_{x,t}(v)\right) dv \right\}.$$

Since $\widehat{\Lambda}^*$ is bounded, we deduce that there exists a constant c such that

$$\left(\frac{\partial \widehat{S}_{x,t}(u)}{\partial x}\right)^2 \leq c\, N_u$$

where

$$N_u = \exp\left\{ \int_t^u 2\widehat{\Lambda}^*\left(v, \widehat{S}_{x,t}(v)\right) dW_v - \frac{1}{2}\int_t^1 4(\widehat{\Lambda}^*)^2\left(v, \widehat{S}_{x,t}(v)\right) dv \right\}$$

is a strictly positive locale martingale, hence a supermartingale verifying

$$dN_u = 2N_u \widehat{\Lambda}^*\left(u, \widehat{S}_{x,t}(u)\right) dW_u.$$

In particular N is integrable and finally

$$\sup_{u\in[t,T]} E\left(\frac{\partial \widehat{S}_{x,t}(u)}{\partial x}\right)^2 < \infty.$$

So, we can conclude about the lemma. ■

Lemma 3.5.5. *The process* $\widehat{\eta}_{x,t}$ *is a Markov process of transition density function* $\Gamma^*(x,t,z,\tau)$*, the fundamental solution of the operator :*

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

Proof. According to Theorem 5.4 p 149 [36], it suffices to verify the needed conditions. Condition (A_1) is well verified since $\widehat{\sigma}_a^2(t,x) \geq const > 0$. Let us verify Condition $(B_1)(i)$: First, $\widehat{\sigma}_a^2(t,x) = (\widehat{\sigma}^*)^2(t,x)$ and $\widehat{\sigma}_b(t,x)$ are bounded. Secondly, suppose that $|x|, |\overline{x}| \leq N$. Then

$$\left|\widehat{\sigma}^2(t,e^x) - \widehat{\sigma}^2(t',e^{\overline{x}})\right| \leq \left|(\widehat{\sigma}^*)^2(t,e^x) - (\widehat{\sigma}^*)^2(t',e^x)\right| + \left|(\widehat{\sigma}^*)^2(t',e^x) - (\widehat{\sigma}^*)^2(t',e^{\overline{x}})\right|$$

where, as already shown, $\left|(\widehat{\sigma}^*)^2(t',e^x) - (\widehat{\sigma}^*)^2(t',e^{\overline{x}})\right| \leq c|x-\overline{x}|$. Moreover,

$$\left|(\widehat{\sigma}^*)^2(t,e^x) - (\widehat{\sigma}^*)^2(t',e^x)\right| \leq \frac{1}{\gamma_n}\left|\sigma^2(t,e^x) - \sigma^2(t',e^x)\right| + \left|\sigma(t,e^x) - \sigma(t',e^x)\right|.$$

It follows that if $|x|, |\overline{x}| \leq N$,

$$\left|\widehat{\sigma}_a^2(t, e^x) - \widehat{\sigma}_a^2(t', e^{\overline{x}})\right| \leq C(N)\ \left(|t - t'| + |x - \overline{x}|\right).$$

In a similar way, since we suppose that $\widehat{\sigma}_{x,t}$ is bounded, we have

$$\left|\widehat{\sigma}_b(t, e^x) - \widehat{\sigma}_b(t', e^{\overline{x}})\right| \leq const(N)\ \left(|t - t'| + |x - \overline{x}|\right).$$

Finally, since $x\widehat{\sigma}_x(t, x)$ is bounded, we deduce that Condition $(B_1)(ii)$ holds, i.e. for any $x, \overline{x}$,

$$\left|\widehat{\sigma}_a^2(t, e^x) - \widehat{\sigma}_a^2(t, e^{\overline{x}})\right| \leq C\ (|x - \overline{x}|).$$

■

Lemma 3.5.6. *Assume that h is a convex function verifying Condition* **H**. *If σ does not depend on t, then $\widehat{C}_{xx} \geq 0$.*

According to the Tanaka–Meyer formula, we have :

$$h\left(\widehat{S}_{x,t}(1)\right) = h(x) + \int_t^1 h'_-\left(\widehat{S}_{x,t}(u)\right) dW_u + \frac{1}{2}\int_{\mathbb{R}} L_1^u \mu(du)$$

where h'_- is the left derivative and

$$\mu = h''(u)du + \sum_i [h'_+(K_i) - h'_+(K_i)]\delta_{K_i},$$

δ_{K_i} is the Dirac measure. Moreover, $(L_s^u)_{s\in[t,1]}$ is a continuous and positive semi-martingale verifying

$$\int_{\mathbb{R}} g(u) L_s^u du = \int_t^s g\left(\widehat{S}_{x,t}(u)\right) d\langle \widehat{S}_{x,t}\rangle_u, \quad s \in [t, 1]$$

for any positive and bounded measurable function g. It follows that

$$\begin{aligned} h\left(\widehat{S}_{x,t}(1)\right) &= h(x) + \int_t^1 h'_-\left(\widehat{S}_{x,t}(u)\right) dW_u + \frac{1}{2}\sum_i [h'_+(K_i) - h'_-(K_i)] L_1^{K_i} \\ &+ \frac{1}{2}\int_t^1 h''\left(\widehat{S}_{x,t}(u)\right) \widehat{\sigma}^2\left(\widehat{S}_{x,t}(u)\right) \widehat{S}_{x,t}^2(u) du. \end{aligned} \tag{3.5.17}$$

Recall that

$$\left(\widehat{S}_{x,t}(u) - K\right)^+ = (x - K)^+ + \int_t^u I_{\widehat{S}_{x,t}(s)>K} d\widehat{S}_{x,t}(s) + \frac{1}{2} L_u^K.$$

Then,

$$\frac{1}{2} E L_1^{K_i} = \widehat{C}_i(t, x) - (x - K_i)^+$$

where $\widehat{C}^i(t, x)$ is the solution of **(e)** when $h(x) = (x - K_i)^+$. Taking the expectations, we deduce from (3.5.17) that

$$\widehat{C}_t(t, x) = \sum_i \alpha_i \widehat{C}_t^i(t, x) - \frac{1}{2} E\left(h''\left(\widehat{S}_{x,t}(1)\right) \widehat{\sigma}^2\left(\widehat{S}_{x,t}(1)\right) \widehat{S}_{x,t}^2(1)\right) \tag{3.5.18}$$

where $\alpha_i = h'_+(K_i) - h'_+(K_i) \geq 0$. Indeed, to differentiate (3.5.17) with respect to t, we note that $\widehat{S}_{x,t}(u) = \overline{S}_{x,0}(u-t)$ where $\overline{S}_{x,0}$ verifies

$$d\overline{S}_{x,0}(v) = \widehat{\sigma}\left(\overline{S}_{x,0}(v)\right)\overline{S}_{x,0}(v)dW_v, \quad v \in [0, 1-t]$$

and we use the change of variable $v = u - t$. Since $h'' \geq 0$ and

$$\widehat{C}_t(t,x) = -\frac{1}{2}\widehat{\sigma}^2(t,x)x^2\widehat{C}_{xx}(t,x),$$

it is then sufficient to show the lemma for $h(x) = (x-K)^+$. To do so, we define :

$$\begin{aligned} h_n(x) &:= 0 \quad x \in [0, K - 1/n] \\ &:= n\left(x - K + 1/n\right)^2/4 \quad x \in [K - 1/n, K + 1/n] \\ &:= x \quad x \in [K + 1/n, \infty[. \end{aligned}$$

The function h_n is a continuous and convex function which satisfies the inequalities

$$\begin{aligned} &0 \leq h_n(x) - h(x) \leq \frac{1}{4n}, \\ &|h'_n(x) - h'(x)| \leq I_{[K-1/n, K+1/n]}(x). \end{aligned}$$

It follows that $\widehat{C}^n_x(t,x) \to \widehat{C}_x(t,x)$ where $\widehat{C}^n, \widehat{C}$ are the solutions of **(e)** respectively when the terminal conditions are given by h_n and $h(x) = (x-K)^+$. Since h_n is a C^1-function, the coefficients $\alpha_i = 0$ and we deduce from (3.5.18) that $\widehat{C}^n_{xx}(t,x) \geq 0$ and $x \to \widehat{C}^n_x$ is increasing. Then, $x \to \widehat{C}_x$ is also increasing and finally $\widehat{C}_{xx} \geq 0$. ■

Chapitre 4

Consumption-investment optimization problem in a Lévy financial model under transaction costs

Abstract

We consider a consumption-investment optimization problem for the Kabanov model when the proportional transaction costs rate is constant and the prices are modeled by a Lévy process. We naturally extend the preliminary work of [26] to portfolio processes that are only supposed to be làdlàg. This allows to suitably rebalance portfolio processes which jumps induced by the Lévy process and identify an optimal strategy in the two dimensional case.

Keywords : Optimal consumption-investment, Merton problem, Transaction costs, Dynamic programming priciple, HJB equation.

Note. This chapter is based on the article *Consumption-investment optimization problem in a Lévy financial model under transaction costs,* T. Tran, E. Lépinette, Y. Kabanov. Preprint.

4.1 Introduction

In this chapter, we study a consumption-investment problem with infinite horizon in the setting of Kabanov's model with proportional transaction costs. We consider a multidimensional and continuous-time model whose price's dynamics is driven by a Lévy process. This problem originates from the seminal paper of [18]. Davis and Norman [25] rigorously solve the problem and provide the optimal consumption plan in a diffusion model with transaction costs. In the case where the value function is not smooth, Soner and Shreve [94] have shown that it can be considered as a weak solution, i.e. a viscosity

solution of the HJB equation. When the risky asset prices follow exponential Lévy processes, Framstad et al. [40] have obtained the same results as those in [94] under some mild conditions. An extension of the jumping diffusion case is proposed by Kabanov et al. [26] where they consider a general market model with conic constraints.

This present chapter aims at giving an unified framework for the optimal consumption-investment problem with transaction costs. It extends [26] to the case where the controls (the portfolio strategies and consumption plans) are only supposed to be làdlàg but we also provide the optimal strategy accordingly. It should be stressed that it is more realistic as confirmed when constructing the optimal strategy. To be more precise, we go back to the paper of Framstad et al. [40]. The strategies are only supposed to be càdlàg but the optimal strategy, which is constructed by reflected diffusions at local times, appears to be làdlàg, i.e. does not belong to the set of admissible controls the authors consider. Nevertheless, it is a good intuition to consider làdlàg strategies as the negative effects induced by the jumps of the price process on the portfolio processes may be corrected by an immediate re-balancement. The economic intuition behind this consideration is also straightforward. When asset prices move in response to unpredictable events, which is typically the case for jumping times of a Lévy process, trading will take place immediately after the event, i.e. the strategy is càglàd. However, if a market information is announced at a predictable time, e.g. models with the presence of a predictable labor income stream, it is perfectly feasible to trade immediately before the announcement, i.e. the strategy is càdlàg. Therefore, the làdlàg assumption is economically relevant.

The main results in this chapter extend those of [26] and [40]. Though the general line of arguments is common (but more technical), one needs to re-examine carefully each step of the proofs. One of the most difficult part is to show that the Bellman function is the viscosity solution of an HJB equation. In the case of only one riskless and one risky assets, we study the regularity of the Bellman function and give a rigorous construction of the optimal strategy as conjectured in [40].

The rest of the chapter is structured as follows. In section 3.2, we study the optimal consumption-investment problem in a general Kabanov's model. After the formulation of the problem, we present the main results of the chapter, including the derivation of the Hamilton-Jacobi-Bellman equation and the uniqueness of the solution. Some needed results for the proof of the main theorems are also presented in this section. Section 3.3 is devoted to the classical Merton problem. We study the structure and give some results on the regularity of the Bellman function. Furthermore, the construction of an optimal strategy to the problem is also provided. Auxiliary results are presented in Appendix, Section 3.4, including the construction of an integral with respect to làdlàg bounded-variation processes, reflected SDEs, and some important properties of the Bellman function.

Notations : We shall use the notations A^+ and A^- to designate the left (resp.

right) limit of a process A and we also denote by A_{t-} and A_{t+} the left and right limit, respectively, at time t. If A is a làdlàg and predictable process of bounded variations, the jump processes are denoted by

$$\Delta A := A - A^-,\ \Delta^+ A := A^+ - A,$$

and we introduce the following càdlàg processes :

$$A_t^d := \sum_{s\le t} \Delta A_s,\ A_t^{d+} := \sum_{s<t} \Delta^+ A_s, \quad t \ge 0.$$

The continuous part of A is defined as

$$A^c := A - A^d - A_-^{d,+}.$$

We denote $\dot{A}^c$ the optional version of the Radon-Nikodym derivative $dA^c/d\|A^c\|$ where $\|A^c\|$ is the total variation of A^c.

4.2 Optimal Consumption-Investment in the Kabanov Model

4.2.1 Formulation of the Problem

We consider the financial market model with jumps adopted in [26]. The price return process is modeled by a d-dimensional Lévy process $(Y_t)_{t\ge 0}$ defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ satisfying the usual conditions. We denote by $p(dz, dt)$ its jump measure and $q(dz, dt) = \Pi(dz)dt$ its compensator such that $\Pi(dz)$ is a positive measure concentrated on $(-1, \infty)^d$ and

$$\int_{\mathbf{R}^d} \left(|z|^2 \wedge |z|\right) \Pi(dz) < \infty. \tag{4.2.1}$$

The dynamics of Y is given by

$$dY_t = \mu t + \Xi dW_t + \int_{\mathbf{R}^d} z\left(p(dz, dt) - q(dz, dt)\right), \tag{4.2.2}$$

where $\mu \in \mathbf{R}^d$, W is an m-dimensional standard Brownian motion and Ξ is a $d \times m$-matrix. In the identification of an optimal strategy we shall only consider a pure jump Lévy process with finite activity, i.e, $\int_{\mathbf{R}^d} |z|\pi(dz) < \infty$.

We consider a conic constraint model as a generalization of the Kabanov's model. We are given two constant cones K and $\mathcal{C}$ which are supposed to be closed and proper, i.e. $K \cap (-K) = \{0\}$ and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. We assume that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. In finance K and $\mathcal{C}$ stand for the set of transaction constraints (solvency cone) and consumption constraints, respectively. The dynamics of a portfolio process is defined for each $i = 1, \ldots, d$ by :

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_0 = V_{0-} = x, \tag{4.2.3}$$

where the controls $\pi = (B, C)$ are làdlàg and predictable processes of bounded variations. The notion of integral with respect to such a process is given in Appendix. The dynamics means that such a portfolio V is self-financed, i.e. its increments are only due to the increments of Y, and transaction costs modeled by B are withdrawn while C represents the cumulated sums of consumed wealth. Precisely, we assume that $\pi = (B, C)$ belongs to the class (we denote by $\mathcal{A}_x$) of admissible controls satisfying the following properties :

1) $\dot{B}^c \in -K$, dP d$\|B^c\|$ a.e., $\dot{C}^c \in \mathcal{C}$ dP d$\|C^c\|$ a.e.,

2) $\Delta^+ B_\tau \in -K$, $\Delta^+ C_\tau \in \mathcal{C}$, for all stopping times τ,

3) $\Delta B_\tau \in -K$, $\Delta C_\tau \in \mathcal{C}$, for all predictable stopping times τ,

4) If $V := V^\pi$, $\pi = (B, C)$, is such that $V_\tau \in \operatorname{int} K$ for some stopping time τ, then $V_\tau^+ = V_\tau + \Delta^+ B_\tau - \Delta^+ C_\tau \in \operatorname{int} K$,

5) If $V := V^\pi$, $\pi = (B, C)$, is such that $V_{\tau-} \in \operatorname{int} K$ (or K) for some stopping time τ, then $V_{\tau-} + \Delta B_\tau - \Delta C_\tau \in \operatorname{int} K$(or K),

6) $x + \Delta^+ B_0 \in \operatorname{int} K$.

The three last conditions mean that the agent is rational enough. He does not deliberately take his position out of the solvency cone. It is also assumed that $\Delta B_0^+ = \Delta C_0 = \Delta^+ C_0 = 0$ and dC^c is absolute continuous with respect to the Lebesgue measure and we write $dC_t^c = c_t dt$. It can be deduced, from the monotonicity of the controls B, C with respect to the partial order induced by K (i.e. $\forall\, x, y \in \mathbf{R}^d, x \preceq y \Leftrightarrow y - x \in K$), that B and C are of finite variations. Indeed, since $\operatorname{int} K \neq \emptyset$, by an appropriate change of coordinates we may assume w.l.o.g. that all coordinates of B, C are monotonic, hence are of bounded variations. Without loss of generality, in the rest of the present chapter we may assume that C is continuous since the jumps of C are not taken into account in the maximization problem. For the sake of simplicity, we assume that $dC_t = c_t dt$ almost everywhere w.r.t the Lesbegue measure on $\mathbf{R}$.

For every control $\pi \in \mathcal{A}_x$, let us introduce the stopping time

$$\theta^\pi = \inf \{ t : V_t^\pi \notin \operatorname{int} K \}. \tag{4.2.4}$$

We suppose that the strategy $\pi = (B, C)$ is frozen after the exit time, i.e. $\Delta^+ B_\theta = 0$ and $dB_t = 0$ for $t > \theta$. Throughout the chapter, we fix a discount coefficient $\beta > 0$.

For every control $\pi = (B, C) \in \mathcal{A}_x$, $x \in \operatorname{int} K$, we define the utility process

$$J_t^\pi(x) \;:=\; \int_0^{t \wedge \theta^\pi} e^{-\beta s} U(c_s) ds,$$

where U is a fixed non-negative mapping defined on $\mathcal{C}$ which represents a utility function; it is assumed to be concave, $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. The optimal consumption problem consists in maximizing the utility process over the set of admisible strategies. To do so, we define the Bellman function as

$$W(x) := \sup_{\pi \in \mathcal{A}_x} \mathbb{E}[J_\infty^\pi(x)], \quad x \in \operatorname{int} K. \tag{4.2.5}$$

The finiteness of the Bellman function is not a trivial property in the classical optimal consumption problem. The conditions ensuring it is studied in lemma 4.4.15 by means of the existence of classical supersolutions to the HJB equation. From now on, we will always suppose that W is finite. The continuity of W is proven in theorem 4.4.12, given in Appendix.

4.2.2 Main Results

In the following, we denote by $C_p(K)$ the set of all continuous functions f on K such that $\sup_{x\in K} |f(x)|(1+|x|)^{-p} < \infty$; and by $C^2(\operatorname{int} K)$ the set of all functions f which are C^2 on $\operatorname{int} K$. Let us define for each $\pi = (B, C) \in \mathcal{A}_x$ and any function $f \in C_1(K) \cap C^2(\operatorname{int} K)$, increasing with respect to the order $\preceq_K$, the operator

$$H(f,x) := \int_{\mathbf{R}^d} \left[f(x + \operatorname{diag}(x)z) I(x,z) - f(x) - f'(x)\operatorname{diag}(x)z \right] \Pi(dz),$$

where $I(v,z) = I_{v+\operatorname{diag}(v)z \in \operatorname{intK}}$. Notice that the operator $H(f,x)$ is well-defined if we set $f(x + \operatorname{diag}(x)z)I(x,z) = 0$ when $I(x,z) = 0$. For convenience, we shall always assume that all the functions f defined on K we consider are extended to $\mathbf{R}^d$ by setting $f(x) = 0$ for $x \in \mathbf{R}^d \setminus K$. Using a Taylor expansion for $f \in C_1(K)$, we claim that

$$|f(x + \operatorname{diag}(x)z) I(x,z) - f(x) - f'(x)\operatorname{diag}(x)z| \le C_x(|z| \wedge |z|^2),$$

where C_x is a constant depending on x. Therefore, the operator H is well defined. Let us now define

$$\begin{aligned}
G : &= (-K) \cap \partial \mathcal{O}_1(0), \\
\Sigma_G(p) : &= \sup_{x \in G} px, \\
U^*(p) : &= \sup_{x \in \mathcal{C}} \left(U(x) - px \right), \\
Av : &= (\operatorname{diag} \mathrm{v}).\Xi \left((\operatorname{diag} \mathrm{v}).\Xi \right)^T, \quad v \in \mathbf{R}^d,
\end{aligned}$$

and the operators :

$$\begin{aligned} F_0(X,p,H,W,x): &= \frac{1}{2}\mathrm{Tr}\, A(x)X + \mu^T(\mathrm{diag}\, x)p + H - \beta W, \quad (4.2.6)\\ \mathcal{L}_0\phi: &= F_0(\phi''(x),\phi'(x),H(\phi,x),\phi(x),x), \quad (4.2.7)\\ F(X,p,H,W,x): &= \max\{F_0(X,p,H,W,x) + U^*(p), \Sigma_G(p)\}, \quad (4.2.8)\\ \mathcal{L}\phi: &= F(\phi''(x),\phi'(x),H(\phi,x),\phi(x),x). \quad (4.2.9) \end{aligned}$$

Let us consider the Dirichlet problem for the HJB equation

$$\begin{aligned} &F(W''(x),W'(x),H(W,x),W(x),x) = 0, \quad x \in \mathrm{int}\, K, \quad (4.2.10)\\ &W(x) = 0 \quad \text{on } \partial K. \quad (4.2.11) \end{aligned}$$

We recall the concept of viscosity solution to an HJB equation. Note that the integro-differential operator considered in this chapter is not defined in a local way. We then need to define viscosity solutions in the **global** sense :

A function $v \in C(K)$ is called a *viscosity supersolution* of (4.2.10) if for every $x \in \mathrm{int}\, K$ and every $f \in C_1(K) \cap C^2(\mathrm{int}\, K)$ such that $v(x) = f(x)$ and $v \geq f$ on K, the inequality $\mathcal{L}f(x) \leq 0$ holds.

A function $v \in C(K)$ is called a *viscosity subsolution* of (4.2.10) if for every $x \in \mathrm{int}\, K$ and every $f \in C_1(K) \cap C^2(\mathrm{int}\, K)$ such that $v(x) = f(x)$ and $v \leq f$ on K, the inequality $\mathcal{L}f(x) \geq 0$ holds.

A function $v \in C(K)$ is a *viscosity solution* of (4.2.10) if v is simultaneously a viscosity super- and subsolution.A function $v \in C(K)$ is a *viscosity solution* of (4.2.10) if v is simultaneously a viscosity super- and subsolution.

At last, a function $v \in C_1(K) \cap C^2(\mathrm{int}\, K)$ is called *classical supersolution* of (4.2.10) if $\mathcal{L}v \leq 0$ on $\mathrm{int}\, K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\mathrm{int}\, K$.

We denote by Φ the set of all continuous function $f : K \mapsto \mathbf{R}_+$ increasing with respect to the partial ordering $\succeq_K$ and such that for every $x \in \mathrm{int}\, K$ and $\pi = (B, C)$ the positive process X_t^f is a supermartingale. The following lemma is a consequence of lemma 4.2.5, and confirms the relevance of the HJB equation.

Lemma 4.2.1. *Let $f \in C^2(\mathbf{R}^d)$ be a positive classical supersolution of (4.2.10) which vanishes outside* $\mathrm{int}\, K$*, then $f \in \Phi$.*

Proof. See [26].

Let us now present the first main result of this chapter. The proof of the following theorem and of the next one are given in Subsection 4.2.4.

Theorem 4.2.2.

i) The Bellman function W is a viscosity supersolution to (4.2.10).

ii) If $W \in C_{\frac{1}{2}}(K)$, or $\int_{\mathbf{R}^d} |\eta| \Pi(d\eta) < \infty$, then W is a viscosity subsolution to (4.2.10).

Theorem 4.2.2 shows that the Bellman function is, in particular, a viscosity solution to (4.2.10) under some mild conditions. We now establish an uniqueness theorem for this equation. To do so, we need the following concept :

Definition. We say that a positive function $\ell \in C_1(K) \cap C^2(\operatorname{int} K)$ is a *Lyapunov function* if the following properties are satisfied :
1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0\ell(x) \leq 0$ for all $x \in \operatorname{int} K$,
2) $\ell(x) \to \infty$ as $|x| \to \infty$.

In other words, ℓ is a classical strict supersolution of the **truncated** equation (excluding the term U^*), continuous up to the boundary, and increasing to infinity at infinity.

Let us introduce the following condition on Π which guaranties the uniqueness of solutions to the HJB equation we consider, under the condition that there exists a Lyapunov function as stated in the next theorem.

Condition Π^0 : $\forall x \in \operatorname{int} K$, $\Pi(z : \ x + \operatorname{diag} xz \in \partial K) = 0$.

Remark 4.2.3. *This condition holds in the two dimensional case if the first component of the underlying asset is a bond $B = 1$ so that $\Pi = \delta_0 \otimes \pi$ where we suppose that π does not charge the singletons.*

Theorem 4.2.4. *Suppose that there exists a Lyapunov function ℓ and Π satisfies Condition Π^0. Then, the Dirichlet problem (4.2.10) has at most one viscosity solution in the class of continuous functions satisfying the growth condition*

$$|W(x)|/\ell(x) \to 0, \quad |x| \to \infty. \tag{4.2.12}$$

The proof is given in [26]. Notice that Proposition 4.4.47 ensures the existence of a Lyapunov function under mild assumptions.

4.2.3 Technical Results

Itô Expansion

Let us define for each n the compact set

$$K_n := \{x \in K : \ |x| \leq n\} \cap \{x \in K : \ d(x, \partial K) \geq \frac{1}{n}\}.$$

It is obvious that $\{K_n\}$ is an increasing sequence whose union is $\operatorname{int} K$. For each $\pi = (B, C) \in \mathcal{A}_x, x \in \operatorname{int} K$, we define $V^{\theta^n_+}$ as the stopped portfolio process.

$$V^{\theta^n_+} := V^{\pi,x,\theta^n_+} := V^{\pi,x} I_{[0,\theta^n[} + V^{\pi,x}_{\theta^n_+} I_{[\theta^n,\infty[}, \tag{4.2.13}$$

where θ^n is the first time the portfolio exits K_n. We also define $B^{\theta^n_+}$ in a similar manner. Note that the value of $V^{\pi,x}_{\theta^n}$ may not be in $\operatorname{int} K$ due to a

possible jump of the Lévy process at θ^n but, in this case, $V^{\pi,x}_{\theta^n_-} \notin \partial K$ by virtue of lemma 4.4.21. We deduce from the dynamics of $V^{\pi,x}$ that

$$\begin{aligned} V_t^{\theta^n_+} &= x + \int_0^t I_{[0,\theta^n]}(s)\mathrm{diag}\,(V_t^{\theta^n_+})(\mu_s ds + \Xi dW_s) + B_t^{\theta^n_+} - C_t^{\theta^n}, \\ &+ \int_0^t \int_{\mathbf{R}^d} \mathrm{diag}\,(V_t^{\theta^n_+}) z I_{[0,\theta^n]}(s)\,(p(dz,ds) - q(dz,ds))\,. \end{aligned}$$

Below, we study the quantity

$$\bar{X}_t^{f,n} := e^{-\beta t} f(V_{t+}^{\theta^n}) + J_t^\pi$$

by means of the Ito formula. Recall that we set $f(x) = 0$ for $x \notin K$, hence $\bar{X}_t^{f,n}$ is well-defined. If K_n is replaced by K we note $\bar{X}_t^f$ instead of $\bar{X}_t^{f,n}$. We also denote $\widetilde{V}_t^n := V_{t+} I_{[0,\theta^n[}(t) + V_{\theta^n_-} I_{[\theta^n,\infty[}(t)$. We have the following key result :

Lemma 4.2.5 (Itô expansion)**.** *For any function $f \in C_1(K) \cap C^2(\mathrm{int\,K})$, increasing with respect to the order $\preceq_K$, we have*

$$\begin{aligned} \bar{X}_t^{f,n} &= f(x) + N_{t\wedge\theta^n} + R_{t\wedge\theta^n} \\ &+ \int_0^{t\wedge\theta^n} e^{-\beta u}\Big[\mathcal{L}_0 f(\widetilde{V}_{u_-}^n) - f'(\widetilde{V}_{u_-}^n)c_u + U(c_u)\Big]du, \end{aligned}$$

where N is a local martingale and R is a decreasing process such that $R_0 = 0$.

Proof. Note that we do not assume any regularity of f across ∂K, so that we can not apply directly the Ito formula to $X_t^{f,n}$. To overcome this difficulty, instead of considering $V^{\theta^n_+}$, we study the process $V^{\theta^n_-}$ defined by

$$V^{\theta^n_-} := V I_{[0,\theta^n[} + V_{\theta^n_-} I_{[\theta^n,\infty[}. \tag{4.2.14}$$

This process involves in K_n. We also have a representation for $V^{\theta^n_-}$:

$$\begin{aligned} V_t^{\theta^n_-} &= x + \int_0^t I_{[0,\theta^n]}(s)\mathrm{diag}\,(V_t^{\theta^n_-})(\mu_s ds + \Xi dW_s) + B_t^{\theta^n_-} - C_t^{\theta^n} \\ &+ \int_0^t \int_{\mathbf{R}^d} \mathrm{diag}\,(V_t^{\theta^n_-}) z I_{[0,\theta^n)}(s)\,(p(dz,ds) - q(dz,ds))\,. \end{aligned}$$

For a sake of simplicity, we denote $\widetilde{V}_t := V_{t+}^{\theta^n_-}$. The Itô formula applied to the process $e^{-\beta t} f(\widetilde{V}_t) := e^{-\beta t} f(V_{t+}^{\theta^n_-})$ yields

$$\begin{aligned} e^{-\beta t} f(\widetilde{V}_t) &= f(x) + \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) d\widetilde{V}_u - \beta \int_0^t e^{-\beta u} f(\widetilde{V}_{u_-}) I_{[0,\theta^n]}(s) du \\ &+ \frac{1}{2}\int_0^t e^{-\beta s} \mathrm{Tr}\, A\widetilde{V}_{s_-} f''(\widetilde{V}_{s_-}) I_{[0,\theta^n]}(s) ds \\ &+ \sum_{s\le t} e^{-\beta s}\Big[f(\widetilde{V}_s) - f(\widetilde{V}_{s_-}) - f'(\widetilde{V}_{s_-})(\widetilde{V}_s - \widetilde{V}_{s_-})\Big]. \end{aligned}$$

Using (4.2.14), we have

$$\begin{aligned}\int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) d\widetilde{V}_u &= \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) I_{[0,\theta^n)}(s) \mathrm{diag}\,(\widetilde{V}_s)(\mu_s ds + \Xi dW_s) \\ &+ \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) dB_s^c + \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) c_u du \\ &+ \sum_{s\le t} e^{-\beta s} I_{[0,\theta^n)}(s) f'(\widetilde{V}_{s_-}) \left(\Delta^+ B_s + \Delta B_s\right) \\ &+ \int_0^t \int_{\mathbf{R}^d} e^{-\beta u} f'(\widetilde{V}_{u_-}) \mathrm{diag}\,(\widetilde{V}_{s_-}) z I_{[0,\theta^n)}(s) \left(p(dz,ds) - q(dz,ds)\right).\end{aligned}$$

Note that

$$\widetilde{V}_s = \widetilde{V}_{s_-} + \mathrm{diag}\,\widetilde{V}_{s_-} \Delta Y_s + \Delta^+ B_s + \Delta B_s.$$

We rewrite

$$\begin{aligned}&\sum_{s\le t} e^{-\beta s} \left[f(\widetilde{V}_s) - f(\widetilde{V}_{s_-}) - f'(\widetilde{V}_{s_-})(\widetilde{V}_s - \widetilde{V}_{s_-})\right] \\ =\ &\sum_{s\le t} e^{-\beta s} \left[f(\widetilde{V}_s) - f(\widetilde{V}_{s_-} + \mathrm{diag}\,\widetilde{V}_{s_-} \Delta Y_s)\right] I_{[0,\theta^n)}(s) \\ +\ &\sum_{s\le t} e^{-\beta s} \left[f\left(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-}) \Delta Y_s\right) - f(\widetilde{V}_{s_-}) - f'(\widetilde{V}_{s_-}) \mathrm{diag}\,(\widetilde{V}_{s_-}) \Delta Y_s\right] I_{[0,\theta^n)}(s) \\ -\ &\sum_{s\le t} e^{-\beta s} f'(\widetilde{V}_{s_-}) \left(\Delta^+ B_s + \Delta B_s\right) I_{[0,\theta^n)}(s).\end{aligned}$$

Moreover,

$$\begin{aligned}&\sum_{s\le t} e^{-\beta s} \left[f\left(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-}) \Delta Y_s\right) - f(\widetilde{V}_{s_-}) - f'(\widetilde{V}_{s_-}) \mathrm{diag}\,(\widetilde{V}_{s_-}) \Delta Y_s\right] I_{[0,\theta^n)}(s) \\ =\ &\int_0^t \int_{\mathbf{R}^d} e^{-\beta s} \left[f(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-}) z) - f(\widetilde{V}_{s_-}) - f'(\widetilde{V}_{s_-}) \mathrm{diag}\,(\widetilde{V}_{s_-}) z\right] I_{[0,\theta^n)}(s)\, p(dz,ds).\end{aligned}$$

Since $I(\widetilde{V}_{s_-}, \Delta Y_s) = 1$ for $s < \theta^n$, we may omit the indicator I within the operator H for $s < \theta^n$. We deduce that

$$\begin{aligned}e^{-\beta t} f(\widetilde{V}_t) &= f(x) + \tilde{N}_{t\wedge\theta^n} + \tilde{R}_{t\wedge\theta^n} && (4.2.15) \\ &+ \int_0^{t\wedge\theta^n} e^{-\beta u} \left[\mathcal{L}_0 f(\widetilde{V}_{u_-}) - f'(\widetilde{V}_{u_-}) c_u\right] du, && (4.2.16)\end{aligned}$$

where

$$\begin{aligned}\tilde{N}_t &= \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) \mathrm{diag}\,(\widetilde{V}_s) \Xi I_{[0,\theta^n]}(u) dW_u && (4.2.17) \\ &+ \int_0^t \int_{\mathbf{R}^d} e^{-\beta u} \left[f(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-}) z) - f(\widetilde{V}_{s_-})\right] I_{[0,\theta^n)}(s) \left(p(dz,du) - q(dz,du)\right).\end{aligned}$$

The residual term is

$$\begin{aligned}\tilde{R}_t : &= \sum_{s\leq t} e^{-\beta s} I_{[0,\theta^n)}(s)\big[f\left(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-})\Delta Y_s + \Delta B_s^+ + \Delta B_s\right) \\ &- f(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-})\Delta Y_s)\big] + \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) I_{[0,\theta^n)}(s) dB_s^c.\end{aligned}$$

The process $\tilde{R}$ is decreasing due to the monotonicity of f with respect to K. Finally, observe that

$$\begin{aligned}\bar{X}_t^{f,n} &- e^{-\beta t} f(\widetilde{V}_t) = J_t^\pi + e^{-\beta\theta^n}\left[f(V_{\theta^n+}) I(V_{\theta^n-}, \Delta Y_{\theta_n}) - f(V_{\theta^n-})\right] 1_{t=\theta^n} \\ &= J_t^\pi + e^{-\beta\theta^n}\left[f(V_{\theta^n+}) - f(V_{\theta^n-} + \mathrm{diag}\,(V_{\theta_-^n})\Delta Y_{\theta^n-})\right] I(V_{\theta^n-}, \Delta Y_{\theta_n}) 1_{t=\theta^n} \\ &+ e^{-\beta\theta^n}\left[f(V_{\theta^n-} + \mathrm{diag}\,(V_{\theta_-^n})\Delta Y_{\theta^n-}) I(V_{\theta^n-}, \Delta Y_{\theta_n}) - f(V_{\theta^n-})\right] 1_{t=\theta^n}.\end{aligned} \tag{4.2.18}$$

Plugging the r.h.s of (4.2.18) into (4.2.15), as the Lesbesgue measure dt does not charge any point, we conclude the theorem with

$$\tilde{N}_t = \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) \mathrm{diag}\,(\widetilde{V}_s) \Xi I_{[0,\theta^n]}(u) dW_u \tag{4.2.19}$$

$$\begin{aligned}&+ \int_0^t \int_{\mathbf{R}^d} e^{-\beta u}\left[f(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-})z) I(\widetilde{V}_{s_-}, \Delta Y_s) - f(\widetilde{V}_{s_-})\right] \times \\ &\times\ I_{[0,\theta^n]}(s)\left(p(dz,du) - q(dz,du)\right).\end{aligned} \tag{4.2.20}$$

and

$$\begin{aligned}\tilde{R}_t : &= \sum_{s\leq t} e^{-\beta s} I_{[0,\theta^n]}(s)\big[f\left(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-})\Delta Y_s + \Delta B_s^+ + \Delta B_s\right) \\ &- f(\widetilde{V}_{s_-} + \mathrm{diag}\,(\widetilde{V}_{s_-})\Delta Y_s)\big] I(\widetilde{V}_{s_-}, \Delta Y_s) + \int_0^t e^{-\beta u} f'(\widetilde{V}_{u_-}) I_{[0,\theta^n]}(s) dB_s^c.\end{aligned} \tag{4.2.21}$$

■

Remark 4.2.6. *If the function $f \in C^2(\mathbf{R}^d)$, we may apply directly the Ito formula to $X_t^f := e^{-\beta t} f(V_{t+}) + J_t^\pi$ and obtain the following result*

$$X_t^f = \bar{X}_t^f = f(x) + N_t + R_t + \int_0^{t\wedge\theta} e^{-\beta u}\left[\mathcal{L}_0 f(\widetilde{V}_{u_-}) - f'(\widetilde{V}_{u_-}) c_u + U(c_u)\right] du$$

where $\widetilde{V} := V^\theta, N$ is a local martingale and R is a decreasing process with $R_0 = 0$. Moreover, $X_t^f = \bar{X}_t^f$ if f vanishes outside $\mathrm{int}\, K$.

Jets

Let f and g be functions defined on a neighborhood of zero. We shall write $f(.) \lessapprox g(.)$ if $f(h) \leq g(h) + o(|h|^2)$ as $|h| \to 0$. The notations $f(.) \gtrapprox g(.)$ and $f(.) \approx g(.)$ have the obvious meaning.

For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$ we consider the quadratic function

$$Q_{p,X}(z) := pz + (1/2)\langle Xz, z\rangle\,, \quad z \in \mathbf{R}^d\,,$$

and define the *super-* and *subjets* of a function v at the point x :

$$\begin{aligned} J^+v(x) &:= \{(p,X):\ v(x+.) \lessapprox v(x) + Q_{p,X}(.)\},\\ J^-v(x) &:= \{(p,X):\ v(x+.) \gtrapprox v(x) + Q_{p,X}(.)\}. \end{aligned}$$

In other words, $J^+v(x)$ (resp. $J^-v(x)$) is the family of coefficients of quadratic functions $v(x)+Q_{p,X}(y-.)$ dominating the function $v(.)$ (resp., dominated by this function) in a neighborhood of the point x with precision up to the second order included and coinciding with $v(.)$ at this point.

In the classical theory developed for differential equations, the notion of viscosity solutions admits an equivalent formulation in terms of super- and subjets. But this is not the case in our formulation due to the non-local property of the integro-differential operator. The following results establish a link between the notion of viscosity solutions and super- and subjets.

Lemma 4.2.7. *Let $v \in C_1(K)\cap C(K)$ be positive and $(p,X) \in J^-v(x), x \in \operatorname{int} K$. There is a function $f \in C_1(K) \cap C^2(x)$ such that $f'(x) = p$, $f''(x) = X$, $f(x) = v(x)$, $f \le v$ on a neighborhood of x. Moreover, if $v \ge 0$, the function f can be chosen such that $f \le v$ on K, $f \in C^2(\mathbf{R}^d)$ and f vanishes outside an abitrary neighborhood of x.*

Proof. See [26].

Remark 4.2.8. *As a consequence of the lemma above, if we assume that v is a global viscosity supersolution of the HJB equation, then, by definition, we have*

$$F(X, p, H(f,x), v(x), x) \le 0.$$

Lemma 4.2.9. *Let $v \in C_1(K) \cap C^2(x)$ and $(p,X) \in J^+v(x), x \in \operatorname{int} K$. Then, there is a function $f \in C_1(K)\cap C^2(x)$ such that $f'(x) = p$, $f''(x) = X$, $f(x) = v(x)$, $f \ge v$ on a neighborhood of x. Moreover, the function f can be chosen such that $f \ge v$ on K.*

Proof. Take $r > 0$ arbitrarily small such that the ball $\mathcal{O}_{4r}(x) = \{y:\ |y-x| \le 4r\}$ lays in the interior of K. By similar arguments as in lemma 4.2.7 we find a function $\phi \in C_1(K)\cap C^2(x)$ such that $\phi'(x) = p$, $\phi''(x) = X$, $\phi(x) = v(x)$, $\phi \ge v$ on a neighborhood of x.

For the second statement, Since $v \in C_1(K)$, there exists a constant A such that $v(z) \le A(1 + |z|) =: \psi$. Choosing A large enough, we have $\psi \ge \phi$ on $\mathcal{O}_{4r}(x)$. By virtue of the unity partition theorem, there exist a $C^\infty-$functions $0 \le \zeta \le 1$ such that :

$$\zeta \equiv 1, \text{on}\, \mathcal{O}_{2r}(x) \,\text{and}\, \zeta \equiv 0, \text{on}\, \mathcal{O}^c_{3r}(x). \tag{4.2.22}$$

We set $f = 1 - \zeta$ and we define $\tilde{\phi} := \phi\zeta + \psi\chi$. This function satisfies all required conditions of the lemma, and, by the construction, $f \ge v$ on K. ■

Remark 4.2.10. *In addition, if we assume that v is a global viscosity subsolution of the HJB equation, then we have*

$$F(X, p, H(f,x), v(x), x) \geq 0.$$

Strict Local Supersolutions

In this subsection, we formulate a strict supersolution property which is the key point for deducing, from the Dynamic Programming Principle, that W is a global viscosity subsolution of the HJB equation.
We fix a closed ball $\overline{\mathcal{O}_r}(x) \subseteq \operatorname{int} K$ such that the larger ball $\mathcal{O}_{2r}(x) \subseteq \operatorname{int} K$ and we define the stopping time $\tau^\pi := \tau_r^\pi$ as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.

$$\tau^\pi := \inf\{t \geq 0 : |V_t^{\pi,x} - x| \geq r\}. \tag{4.2.23}$$

Lemma 4.2.11. *Let $f \in C^2(\operatorname{int} K) \cap C_1(K)$ be such that $\mathcal{L}f \leq -\varepsilon \leq 0$ on $\overline{\mathcal{O}_r}(x)$. Suppose that $\int_{\mathbf{R}^d} |\eta| \Pi(d\eta) < \infty$ or $f \in C_{\frac{1}{2}}(K)$. Then, there exists a constant $\eta := \eta(\varepsilon, r)$ and an interval $(0, t_0]$ such that*

$$\sup_{\pi \in \mathcal{A}_x} E\bar{X}^{f,n}_{t\wedge\tau^\pi} \leq f(x) - \eta t, \quad t \in (0, t_0].$$

Observe that $\tau^\pi \leq \theta^n$ if n is large enough hence $X^{f,n}_{t\wedge\tau^\pi}$ does not depend on n.

Proof. We fix a strategy π and omit its symbol in the notations below. In what follows, only the behavior of the processes we consider on $[0, \tau]$ does matter. For n large enough, we have $\mathcal{O}_{2r}(x) \subseteq \operatorname{int} K_n$. Hence $\tau^\pi \leq \theta^n$ and we may apply theorem 4.2.5 and get that

$$\begin{aligned}
\bar{X}^{f,n}_{t\wedge\tau} &= f(x) + N_{t\wedge\tau} + R_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{-\beta u} \left[\mathcal{L}_0 f(V_{u-}) + U^*(V_{u-})\right] du \\
&- \int_0^{t\wedge\tau} e^{-\beta u} \Big[U^*(V_{u-}) + f'(V_{u-})c_u - U(c_u)\Big] du, \\
&= f(x) + N_{t\wedge\tau} + R_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{-\beta u} \mathcal{L}f(V_{u-}) du \\
&- \int_0^{t\wedge\tau} e^{-\beta u} \Big[U^*(V_{u-}) + f'(V_{u-})c_u - U(c_u)\Big] du \\
&+ \int_0^{t\wedge\tau} e^{-\beta u} \left[\mathcal{L}_0 f(V_{u-}) + U^*(V_{u-}) - \mathcal{L}f(V_{u-})\right] du.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\bar{X}^{f,n}_{t\wedge\tau} &= f(x) + N_{t\wedge\tau} + \widetilde{R}_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{-\beta u} \mathcal{L}f(V_{u-}) du \\
&- \int_0^{t\wedge\tau} e^{-\beta u} \Big[U^*(u, V_{u-}) + f'(V_{u-})c_u - U(c_u)\Big] du, \quad (4.2.24)
\end{aligned}$$

where $\widetilde{R}$ is a decreasing process such that $\widetilde{R}_0 = 0$ and N is a local martingale. We shall prove that $N^{t\wedge\tau}$ is a martingale. First, suppose that $f \in C_{\frac{1}{2}}(K)$. By definition, $V_{u_-} \in \overline{\mathcal{O}_r}(x)$ on $[0,\tau]$. It follows that $f'(V_{u_-})$ and $\mathrm{diag}\,(V_{u_-})$ are bounded on $[0,\tau)$. Moreover, choosing z small enough, the finite increment formula yields

$$f(V_{s_-} + \mathrm{diag}\,(V_{s_-})z) - f(V_{s_-}) = f'(\tilde{\theta})\mathrm{diag}\,(V_{s_-})z$$

where $\tilde{\theta} \in \mathcal{O}_{2r}(x)$. From there and the fact that f is in $C_{\frac{1}{2}}(K)$, we deduce two constants k_1, $k_2 > 0$ and $\epsilon > 0$ (depending on x, r) such that

$$\begin{aligned}\left|f(V_{s_-} + \mathrm{diag}\,(V_{s_-})z) - f(V_{s_-})\right|^2 1_{s\leq\tau} &\leq k_1 z^2 I_{|z|\leq\epsilon} + k_2|z|I_{|z|>\epsilon},\\ &\leq c\,(|z^2| \wedge |z|).\end{aligned}$$

By virtue of theorem II.1.33 [51] and (4.2.19), we deduce that N^τ is a square integrable martingale. Hence $EN^{t\wedge\tau} = 0$.
Similarly, if $f \in C_1(K)$ and $\int_{\mathbf{R}^d} |\eta|\Pi(d\eta) < \infty$, we also have

$$\begin{aligned}\left|f(V_{s_-} + \mathrm{diag}\,(V_{s_-})z) - f(V_{s_-})\right| 1_{s\leq\tau} &\leq k_1 z I_{|z|\leq\epsilon} + k_2(1+|z|)I_{|z|>\epsilon},\\ &\leq c\,|z|,\end{aligned}$$

which implies that N^τ is also a martingale and then $EN^{t\wedge\tau} = 0$.
By assumption, $\mathcal{L}f(y) \leq -\varepsilon$ for all $y \in \overline{\mathcal{O}_r}(x)$ and so $\Sigma_G(f'(y)) \leq -\varepsilon$ on $[0,\tau]$. It follows that $f'(y)k \leq -\varepsilon|k|$ whatever $k \in -K$ so that $f'(\overline{\mathcal{O}_r}(x)) \subseteq \mathrm{int}\,K^*$ on $[0,\tau]$. In particular, for $s \in [0,\tau]$

$$f'(V_{s_-})\dot{B}^c_s \leq -\varepsilon|\dot{B}^c_s|$$

We deduce that the following term in the expression (4.2.21) of R is bounded as follows :

$$\int_0^{t\wedge\tau} e^{-\beta u} f'(V_{u_-}) I_{s\leq\theta}\dot{B}^c_s d\|B^c\|_s \leq -\varepsilon\int_0^{t\wedge\tau} e^{-\beta u} I_{s\leq\theta}|\dot{B}^c_s| d\|B^c\|_s.$$

On the other hand, the other terms defining R can be estimated as follows :

$$\begin{aligned}&f\left(V_{s_-} + \mathrm{diag}\,(V_{s_-})\Delta Y_s + \Delta B^+_s + \Delta B_s\right) - f(V_{s_-} + \mathrm{diag}\,(V_{s_-})\Delta Y_s)\\ =\ &f'(\gamma_s)\left(\Delta B^+_s + \Delta B_s\right)\end{aligned}$$

where $\gamma_s \in [V_s, V_{s+}]$. Observe that $\gamma_s \in \overline{\mathcal{O}_r}(x)$ if $s < \tau$. If $s = \tau$ we may assume without loss of generality that the control ΔB^+_τ equals 0 which is less expensive. Therefore, we have

$$\begin{aligned}&\left[f\left(V_{s_-} + \mathrm{diag}\,(V_{s_-})\Delta Y_s + \Delta B^+_s + \Delta B_s\right) - f(V_{s_-} + \mathrm{diag}\,(V_{s_-})\Delta Y_s)\right] I_{s\leq\tau}\\ \leq\ &-\epsilon\left(|\Delta B^+_s| + |\Delta B_s|\right) I_{s\leq\tau}\end{aligned}$$

Therefore, by Equality (4.2.24), we deduce that

$$E\bar{X}^{f,n}_{t\wedge\tau} \leq f(x) - \varepsilon e^{-\beta t}(t\wedge\tau) - e^{-\beta t}EZ_t$$

where

$$\begin{aligned} Z_t : &= \int_0^{t\wedge\tau} r(c_s, f'(V_{s_-}))ds + \epsilon\int_0^{t\wedge\tau} |\dot{B}^c_s| d\|B^c\|_s + \epsilon\sum_{s\leq t\wedge\tau}\left(|\Delta B^+_s| + |\Delta B_s|\right), \\ r(c,p) : &= U^*(p) + pc - U(c). \end{aligned}$$

Recall that $U^*(p) = \sup_{x\in C}(U(x) - px) \geq 0$ since $U(0) = 0$. Moreover, by assumption,

$$\inf_{p\in f'(\overline{\mathcal{O}_r}(x)),\, c\in\mathcal{C},\, |c|=1} pc \geq \varepsilon.$$

Since $U(c)/|c| \to 0$ as $|c| \to \infty$, we finally deduce that there exists a constant $\kappa > 1$ such that

$$\inf_{p\in f'(\overline{\mathcal{O}_r}(x))} r(c,p)) \geq \kappa^{-1}|c|, \quad \forall c\in\mathcal{C}, \quad |c| \geq \kappa.$$

Therefore,

$$\int_0^{t\wedge\tau} r(c_s, f'(V_{s_-}))ds \geq \kappa^{-1}\int_0^{\tau} I_{|c_s|\geq\kappa}|c_s|ds.$$

Moreover, the second integral defining Z dominates $\kappa_1\|B^c\|_{t\wedge\tau}$ for some $\kappa_1 > 0$. Indeed, recall that all norms in $\mathbf{R}^d$ are equivalent, in particular $c^{-1}|.| \leq |.|_1 \leq c|.|$ for some $c > 0$ where $|x|_1 := \sum_{i=1}^d |x_i|$ and $|.|$ is the Euclidean norm. It follows that

$$c^{-1}\|B^c\| \leq \operatorname{Var} B^c \leq c\|B^c\|,$$

where $\operatorname{Var} B^c$ is the total variation of B^c with respect to $|.|_1$. At last, we have :

$$|\dot{B}^c|_1 = \sum_{i=1}^d |\dot{B}^{c\,i}| = \sum_{i=1}^d \left|\frac{dB^{c\,i}}{d\|B^c\|}\right| = \sum_{i=1}^d \left|\frac{dB^{c\,i}}{d\operatorname{Var} B^{c\,i}}\right| \frac{d\operatorname{Var} B^{c\,i}}{d\|B^c\|} = \frac{d\operatorname{Var} B^c}{d\|B^c\|}.$$

The claimed property follows. We deduce some constant $\gamma > 0$ such that

$$EX^f_{t\wedge\tau} \leq f(x) - e^{-\beta t}\gamma^{-1}E\widetilde{Z}_t$$

where

$$\widetilde{Z}_t := t\wedge\tau + \int_0^{t\wedge\tau} I_{|c_s|\geq\kappa}|c_s|ds + \|B\|_{t\wedge\tau+}.$$

Then

$$\widetilde{Z}_t \geq t\wedge\tau + \|B\|_{t\wedge\tau+} - \int_0^{t\wedge\tau} I_{|c_s|\leq\kappa}|c_s|ds \geq (1-\kappa)t\wedge\tau + \|B\|_{t\wedge\tau+}.$$

Using the stochastic formula given by lemma 4.4.23, since $\mathcal{E}_0(Y) = \mathcal{E}_{0+}(Y) = 1$, we immediately get the existence of a number $t_0 > 0$ and a measurable set Γ with $P(\Gamma) > 0$ on which

$$|V^{\pi,x} - x| \leq r/2 + \delta\|B\|_+, \quad t \in [0, t_0],$$

whatever the control $\pi = (B, C)$. Diminishing t_0, we may assume without loss of generality that $\kappa t_0 \leq r/(4\delta)$. For any $t \leq t_0$, the inequality $\|B\|_{\tau+} \geq r/(2\delta)$ holds on the set $\Gamma \leq \{\tau \leq t\}$. Therefore,

$$\widetilde{Z}_t \geq (1-\kappa)\tau + 2\kappa t_0 \geq \kappa t_0 \geq t.$$

On the set $\Gamma \leq \{\tau > t\}$, the inequality $\widetilde{Z}_t \geq t$ obviously holds. Thus, $E\widetilde{Z}_t \geq tP(\Gamma)$ if $t \in [0, t_0]$ and the result is proven. ■

Dynamic Programming Principle

Theorem 4.2.12. *Let τ be a finite stopping time. Then*

$$W(x) = \sup_{\pi \in \mathcal{A}_x} E\left(J_\tau^\pi(x) + e^{-\beta\tau} W(V_\tau^{x,\pi}) I_{\tau<\theta}\right).$$

This theorem is derived from the two following lemmas :

Lemma 4.2.13. *Let $\mathcal{T}_f$ be the sets of finite stopping times. Then,*

$$W(x) \leq \sup_{\pi \in \mathcal{A}_x} \inf_{\tau \in \mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_\tau^{x,\pi}) I_{\tau<\theta}\right). \tag{4.2.25}$$

Proof. The proof is the same than in [26].

Lemma 4.2.14. *For any stopping time $\tau \in \mathcal{T}_f$, we have :*

$$W(x) \geq \sup_{\pi \in \mathcal{A}_x} E\left(J_\tau^\pi(x) + e^{-\beta\tau} W(V_\tau^{x,\pi}) I_{\tau<\theta}\right).$$

Proof. See the similar proof in [26].

Remark 4.2.15. *By similar arguments than in [26], we can prove that lemma 4.2.13 and lemma 4.2.14 also hold if we replace V by V_+. The version of lemma 4.2.14 for V_+ will serve to prove that the Bellman function is a global viscosity subsolution to the HJB equation (see Section 4.2.4).*

4.2.4 Proof of the Main Results

Proof of theorem 4.2.2

Proof i). Using rem 4.2.6, we repeat the proof given in [26].

Proof ii). Let $x \in \operatorname{int} K$ and $\phi \in C_1(K) \cap C^2(\operatorname{int} K)$ be a function such that $\phi(x) = W(x)$ and $W \leq \phi$ on K. Suppose that ϕ does not satisfy the subsolution inequality at x. By continuity, suppose there exists a neighborhood

$\overline{\mathcal{O}_{r'}(x)} \subseteq$ int K of x on which $\mathcal{L}\phi \leq -\varepsilon$ where $\varepsilon > 0$. By virtue of lemma 4.2.11, there exists a constant $\eta := \eta(\varepsilon)$ and an interval $(0, t_0]$ such that

$$\sup_{\pi \in \mathcal{A}_x} E\left(J^{x,\pi}_{t\wedge\tau^\pi_r} + e^{-\beta(t\wedge\tau^\pi_r)}\phi(V^{x,\pi}_{t\wedge\tau^\pi_r+})I(V^{x,\pi}_{t\wedge\tau^\pi_r-}, \Delta Y_{t\wedge\tau^\pi_r})\right) \leq \phi(x) - \eta t, \quad t \in (0, t_0]. \tag{4.2.26}$$

We may assume w.l.o.g. that $r = r'$. Fix an arbitrary $t \in (0, t_0]$. Using lemma (4.2.13) for V_+, we deduce that there exists $\pi \in \mathcal{A}_x$ such that

$$W(x) \leq \inf_{\tau \in \mathcal{T}_f} E\left(J^{x,\pi}_{t\wedge\tau} + e^{-\beta(t\wedge\tau)}W(V^{x,\pi}_{t\wedge\tau+})I_{t\wedge\tau<\theta}\right) + \frac{1}{2}\eta t. \tag{4.2.27}$$

As $W \leq \phi$ and $I(V^{x,\pi}_{t\wedge\tau^\pi_r-}, \Delta Y_{t\wedge\tau^\pi_r}) = 0$ imply that $I_{t\wedge\tau<\theta} = 0$, we obtain from the above relations that $W(x) \leq \phi(x) - \frac{1}{2}\eta t$, which yields a contradiction since $W(x) = \phi(x)$. ■

4.3 Two-asset Model : Structure of the Bellman Function and Optimal Policy

4.3.1 Assumptions of the Model and Main Results

We investigate the structure of the Bellman function and construct an optimal policy for the case $d = 2$. In the following, we use the notations $z = (x, y)$ to designate a generic element z of $\mathbf{R}^2$. The canonical basis (e_1, e_2) is considered, i.e. $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The risk-free asset is supposed to be a constant, and the risky asset follows a geometric Lévy process :

$$\begin{aligned} dS^1_t &= 0,\ S^1_0 = 1 \\ dS^2_t &= S^2_{t-}\left(\mu dt + \sigma dW_t + \int_{\mathbf{R}} x(p(dy, dt) - q(dy, dt))\right),\ S^2_0 = 1, \end{aligned}$$

where p is the jump measure of S^2 and $q(dy, dt) = \pi(dy)dt$ is its compensator such that $\pi(dy)$ is a positive measure concentrated in $(-1, \infty)$ which does not charge the singletons and satisfies the following condition

Condition (I) :

$$\int_{-1}^{\infty} \max(1, |t|)\pi(dt) < \infty. \tag{4.3.28}$$

The inequality (4.3.28) ensures that π is a finite measure, and that the associated Lévy process has a finite activity. This implies that the Lévy process $Y = (Y^1, Y^2)$ has the following representations :

$$Y^1 = 0,\ Y^2_t = \sigma W_t + \mu t + \sum_{i=1}^{N_t} \chi_i$$

where $N_t = \sum_{n=1}^{\infty} 1_{T_n \leq t}$ is a Poisson process of intensity $\lambda > 0$ and (χ_i) is a family of i.i.d. $\frac{\pi}{\lambda}$-distributed random variables independent of N.
A portfolio process in the two-dimensional Kabanov model we consider satisfies the following dynamics :

$$\begin{aligned} dV_t^1 &= dL_t^{21} - (1+\lambda^{12})dL^{12} - c_t dt, \\ dV_t^2 &= V_{t-}^2 \left(\mu dt + \sigma dW_t + \int_{\mathbf{R}} x(p(dy,dt) - q(dy,dt)) \right) + dL_t^{12} - (1+\lambda^{21})dL^{21}, \end{aligned}$$

where L^{ij}, $i,j = 1,2$, are the transfer processes we suppose to be làdlàg. At last, λ^{ij}, $i,j = 1,2$, are the transaction costs coefficients. We rewrite the dynamics of a portfolio process under the vector form :

$$dV_t = \operatorname{diag} V_{t-} \left(\tilde{\mu} dt + \tilde{\sigma} dW_t + \int z(\tilde{p}(dz,dt) - \tilde{\Pi}(dz)dt) \right) + dB_t - dC_t.$$

Here, we denote ${}^t\tilde{\mu} = (0,\mu), {}^t\tilde{\sigma} = (0,\sigma)$ and, with $z = (x,y)$,

$$\tilde{p}(dz,dt) = \delta_0(dx)dt \otimes \sum_{n=1}^{\infty} \delta_{T_n,\chi_n}(dt,dy),\ \tilde{\Pi}(dz) = \delta_0(dx) \otimes \pi(dy)$$

This means that $A = \operatorname{diag}(0,\sigma^2)$. Morever, we denote ${}^t dC_t = (c_t dt, 0)$ and ${}^t dB_t = (dL_t^{21} - (1+\lambda^{12})dL^{12}, dL_t^{12} - (1+\lambda^{21})dL^{21})$
The optimization problem reads as

$$E \int_0^{\theta} e^{-\beta s} u(c_s) ds \to \max,$$

where $u : \mathbf{R}_+ \to \mathbf{R}$ is a concave utility function. In the sequel, we consider the case of power utility functions, i.e. $u(r) = \frac{r^\gamma}{\gamma}, \gamma \in (0,1)$. Observe that, in this case, W is homogeneous of degree γ :

$$W(\nu x) = \nu^\gamma W(x), \quad \forall x \in K,\ \nu \geq 0. \tag{4.3.29}$$

In this framework, the solvency cone K is simply a sector generated by the vectors $g_1 = (1+\lambda^{12})e_1 - e_2, g_2 = (1+\lambda^{21})e_2 - e_1$. The dual cone of K is given by $K^* = \operatorname{cone}\{\tilde{g}_1, \tilde{g}_2\}$ with $\tilde{g}_1 = (1+\lambda^{12})e_2 + e_1, \tilde{g}_2 = (1+\lambda^{21})e_1 + e_2$. For the sake of simplicity, we suppose that $\lambda^{12} = \lambda^{21} = \lambda$. The consumption region is $C = \mathbf{R}_+ e_1$. The HJB equation is given by :

$$F(W''(z), W'(z), H(W,z), W(z), z) = 0, \tag{4.3.30}$$
$$W(z) = 0 \quad \text{on } \partial K, \tag{4.3.31}$$

where

$$F_0(X,p,H,W,z) : = \frac{1}{2}\sigma^2 y(X)_{22} + \mu y p_2 + H - \beta W, \tag{4.3.32}$$
$$F(X,p,H,W,z) : = \max\{F_0(X,p,H,W,z) + U^*(p), \Sigma_G(p)\} \tag{4.3.33}$$

Some simple calculation shows that $U^*(p) = \frac{1}{\gamma}(p_1)^{\frac{\gamma}{\gamma-1}}$, where $p = (p_1, p_2)$. Moreover

$$\Sigma_G(p) \leq 0 \Leftrightarrow \max\{-g_1 p, -g_2 p\} \leq 0,$$

and

$$H(u,z) = \int_{\mathbf{R}} \left[u(x, y(1+t))I(z,t) - u(x,y) - u'_y(x,y)yt\right] \pi(dt),$$

where $I(z,t) := 1_{\{(x,y+yt)\in \text{intK}\}}, z = (x,y)$. We provide some simple properties on the operator $H(u,z)$ as follows

Lemma 4.3.1. *i) If $u \in C_1(K) \cap C^1(K \setminus \mathbf{R}e_1)$ is a concave function such that $\Sigma_G(u') \leq 0$ on K (i.e. u is increasing with respect to the natural order on K), then the operator $H(u,z)$ is non positive.*

ii) If $u \in C^1(K \setminus \mathbf{R}e_1)$ is homogeneous of degree $\gamma < 1$ and $\pi(\mathbf{R}) < \infty$ then $H(u,.)$ is continuous on $K \cap \{(x,y) : y > 0\}$. Moreover, if $u_y(x,0^+)$ exists and is finite, then so is $H(u,x,0^+)$.

iii) If $u \in C^2(K \setminus \mathbf{R}e_1)$ is homogeneous of degree $\gamma < 1$ and $\pi(\mathbf{R}) < \infty$ then $H(u,.)$ is C^1 on $K \cap \{(x,y) : y \geq 0\}$. Moreover, if $u_y(x,0^+), u_{yy}(x,0^+)$ exist and are finite, then so is $H(u,x,0^+)$.

Proof. i) We aim to prove that

$$P(z,t) := u(x, y(1+t))I(z,t) - u(x,y) - u'_y(x,y)yt \leq 0.$$

It is clear that $P(z,t) = 0$ in the case $y = 0$. If $y > 0$ or if $y < 0$ and $t \in (-1, 0)$, then $I(z,t) = 1$ for all $t > -1$ (since $y(1+t) > y$) and we get, by the concavity of the function $y \to u(x,y)$ that

$$P(z,t) = u(x, y(1+t)) - u(x,y) - u'_y(x,y)yt \leq 0.$$

The case $y < 0$ and $t \geq 0$ is obvious because $u'_y(x,y) \geq 0$ since $t \to u(z+te_2)$ is a increasing funtion. We then have $P(z,t) \geq 0$.

ii) Assume that $(x,y) \in K$ and $y > 0$. Let $z_n := (x_n, y_n) \in K, y_n > 0$ be a sequence convergent to $z := (x,y)$. By assumption, we have $P(x_n, y_n) - P(x,y) = o(1)$ uniformly on $t \in (-1, 0)$. We now consider $t \geq 0$. Using the homogeneity of u, we obtain

$$u(x_n, y_n + y_n t) - u(x, y+yt) = (1+t)^\gamma (u(\frac{x_n}{1+t}, y_n) - u(\frac{x}{1+t}, y)) = (1+t)^\gamma o(1),$$

uniformly on $t \geq 0$. Finally,

$$u(x_n, y_n) - u(x,y) = o(1),$$

and

$$u_y(x_n, y_n)y_n t - u_y(x,y)yt = to(1).$$

We deduce that

$$P(x_n, y_n) - P(x,y) = o(1)\max(1, |t|),$$

which implies that $H(u, z_n) \to H(u, z)$ when $n \to \infty$. The case $y = 0$ is proved similarly, using the boundedness of $\{u_y(z_n)\}$ when $y_n \to 0$.

iii) The proof is similar than ii). ■

Note that, under Condition 4.3.28, the Bellman function is a global viscosity solution to the HJB equation without the condition $\gamma \leq \frac{1}{2}$. To have more insight on the Bellman function and the HJB equation, we consider the following assumption

Condition (II) :

$$\beta > \frac{\gamma\mu^2}{2\sigma^2(1-\gamma)}. \tag{4.3.34}$$

By using lemma 4.3.1, we deduce that Condition (4.3.34) implies the condition which is supposed in Corollary 4.4.10, i.e.

$$\beta > \frac{\gamma\mu^2}{2\sigma^2(1-\gamma)} + h^*(\gamma),$$

where funtion h^* is given in Appendix A1. As shown in Corollary 4.4.10, Condition (4.3.34) ensures the existence of a classical super solution to the HJB equation. Therefore, using Proposition 4.4.15, we deduce that the Bellman function is finite on K and continuous up to the boudary ∂K.

It is not difficult to verify that Condition (Π^0) is satisfied in the two-dimentional model we consider. Additionally, the existence of a Lyapunov function with higher growing order than the Bellman function is also confirmed in Corollary 4.4.10 under Condition (4.3.34). Therefore, we deduce from theorem 4.2.4 that the HJB equation has a unique solution in the class $C_1(K)$. By combining with theorem 4.4.16 in Appendix A2 we conclude that the Bellman function is concave.

We now summarize some well known results of the literature on the structure of the Bellman function. Recall that these results holds provided that W is continuous, concave and monotone with respect to K. But, we may easily adapt all the proofs given for continuous diffusion processes, see [94] and [61], to the case with jumps. Indeed, to do so, it suffices to use lemma 4.3.1. Moreover, the regularity of the Bellman function is studied in Subsection 4.3.2.

Theorem 4.3.2. *The cone K can be splitted into three non-empty open disjoint cones $K_i, i = 0, 1, 2 : K = \bar{K}_0 \cup \bar{K}_1 \cup \bar{K}_2$ such that, if we denote $K_0 = \text{cone}\,\{\tilde{g}_1, \tilde{g}_2\}, K_1 = \text{cone}\,\{g_1, \tilde{g}_1\}, K_2 = \text{cone}\,\{\tilde{g}_2, g_2\}$ for some vectors $\tilde{g}_1, \tilde{g}_2$, then $K_i, i = 0, 1, 2$ are determined by :*

$$K_1 = \{(x, y) \in \text{int}\, K;\ -g_1 W' = 0\}, \tag{4.3.35}$$

$$K_2 = \{(x, y) \in \text{int}\, K;\ -g_1 W' = 0\}, \tag{4.3.36}$$

$$K_0 = \{(x, y) \in \text{int}\, K;\ -g_1 W' < 0,\ -g_2 W' < 0\}. \tag{4.3.37}$$

Moreover, K_1 contains $\operatorname{cone}(g_1, e_1)$. *On $K_1 \cup K_2, W$ is C^∞ and is given by*

$$W(z) = a_1 u(p_1 z) \text{ on } K_1,$$

$$W(z) = a_2 u(p_2 z) \text{ on } K_2,$$

where a_1, a_2 are some constants. On K_0, the derivatives W', W_{yy} are well defined and continuous, the Bellman function W is then a unique solution to the following PDE in **classical sense** *:*

$$\frac{1}{2}\sigma^2 x_2^2 \varphi_{yy} + \mu y \varphi_y + H(\varphi, x) - \beta\varphi + u^*(\varphi_x) = 0.$$

Below is the second main result which provides an optimal policy for the optimization problem. We give a rigorous proof based on the notion of solution to a Skorohod problem. We refer the readers to Appendix A3 where details on the Skorokhod problem are given.

Theorem 4.3.3. *Suppose that the boundaries of K_0 are different from the x- and y-axes. Let $(x, y) \in \bar{K}_0$, then the Skorokhod problem (4.4.54) with $\sigma(V_t) := (-W_x(V_t)^{\frac{1}{\gamma-1}}, V_t)$ has a unique solution. Moreover, the portfolio process V participating in the solution of this problem is an optimal portfolio. The optimal strategy is given by the formula*

$$B_t = \int_0^t g(V_s) dk_s, \tag{4.3.38}$$

$$c_t = W_x^{\frac{1}{\gamma-1}}, \tag{4.3.39}$$

where W is the Bellman function.

The proof of the theorem is postponed to subsection 4.3.3.

Remark 4.3.4. *The situations where $x \in K_i, i = 1, 2$, are easily reduced to the one treated in the theorem above. Indeed, recall that the function W restricted on the set K_i is constant along the direction g_i, $i = 1, 2$. Instead of considering the initial position $x \in K_i, i = 1, 2$, we consider the point $\tilde{x}$ lying on the boundary of K_0 by projecting x onto K_0 parallel to g_i. This translation does not change the value of the Bellman function, meaning that $W(x) = W(\tilde{x})$. Therefore, the optimal strategy for x is constructed simply by adding the initial jump $\Delta B_0 := \tilde{x} - x$ to the optimal strategy given by the Skorokhod problem for $\tilde{x}$.*

In the next subsection, we recall some well known facts on the regularity of the Bellman function.

4.3.2 Reduction to One Variable and Regularity of the Value Function .

Using the homogeneity property of the Bellman function we reduce our problem to the case of one variable by considering the restriction of the Bellman function on the intersection of the line {(x,y) : x+y=1} with the interior of

K. Indeed, if we denote $\psi(z) := W(1-z,z), z \in \Delta := [-\frac{1}{\lambda}, 1+\frac{1}{\lambda}]$, then we may reconstruct W from ψ by the formula

$$W(x,y) = (x+y)^\gamma \psi(\frac{y}{x+y}), \quad (x,y) \in \operatorname{int} K.$$

As in [94] and [61], we may show that ψ is the viscosity solution of the new HJB equation obtained by the change of variables above :

$$\max_{i=0,1,2} l_i(\psi) = 0, \tag{4.3.40}$$

with the two first-order operators

$$l_1(\psi) := -\lambda\gamma\psi + (1+\lambda z)\psi', \quad l_2(\psi) := -\lambda\gamma\psi + (1+\lambda-\lambda z)\psi',$$

and the second-order operator

$$l_0(\psi) := f_2\psi'' + f_1\psi' + f_0\psi + \frac{1-\gamma}{\gamma}[\gamma\psi - z\psi']^{\frac{\gamma}{\gamma-1}} + H(z,\psi,\psi'),$$

where

$$\begin{aligned}
f_2(z) : &= \frac{1}{2}\sigma^2 z^2(1-z)^2, \\
f_1(z) : &= -\sigma^2(1-\gamma)z(1-z)(z-\theta), \\
f_0(z) : &= \frac{1}{2}\sigma^2\gamma(\gamma-1)z^2 + \gamma\mu z - \beta, \\
H(z,\psi,\psi') : &= \int_{-1}^{\infty} \left[(1+z\eta)^\gamma \psi(\frac{z+z\eta}{1+z\eta}) 1_{(1-z,z(1+\eta))\in K} - \psi(z) - z\eta(\gamma\psi + (1-z)\psi')\right] \\
&\times \ d\pi(\eta).
\end{aligned}$$

Recall that W is concave hence ψ is also concave on Δ and its derivatives ψ', ψ'' exist almost everywhere. Therefore, (4.3.40) holds in classical sense a.s. Moreover, ψ has left and right derivatives which are continuous from the left and right, respectively, and satisfy the inequality $D^+\psi \le D^-\psi$, which is strict only on a countable set.

In what follows, we briefly recall some results of the literature on the regularity of the Bellman function and focus on then extra term $H(z,\psi,\psi')$.

Proposition 4.3.5. *The function ψ is continuously differentiable on the interval Δ except, maybe, zero. If ψ has a discontinuity at 0, then*

$$\psi(0) = \frac{1}{\gamma}\left(\frac{1-\gamma}{\beta}\right)^{1-\gamma} = \frac{1}{\gamma}\kappa_*^{1-\gamma}. \tag{4.3.41}$$

We refer to [61], lemma 4.8.6 for the proof of this lemma. The only difference is that : for each $(p,X) \in J^+\psi(z), z \in \Delta$, we have to construct a concave test function $h \in C^2(\Delta)$ such that $h \ge \psi$ and $h - \psi$ atains local minimum at z. The existence of such a function is straightforward by noting that $p \in (D^+\psi(z), D^-\psi(z))$. A direct consequence of the proposition above is that the Bellman function is C^1 on $\operatorname{int} K \setminus \mathbf{R}_+ e_1$. More precisely we have :

Corollary 4.3.6. *The value function is C^1 on $\operatorname{int} K \setminus \mathbf{R}_+ e_1$. If ψ is not C^1 on $\mathbf{R}_+ e_1$, then (4.3.41) holds. Furthermore, even if ψ is not C^1 on $\mathbf{R}_+ e_1$, the partial derivative W_x is defined and continuous, and the one-sided derivatives $W_y(x, 0\pm)$ are also defined and satisfy the one-sided continuity conditions*

$$W_y(x, 0\pm) = \lim_{(\xi,\eta)\to(x,0\pm)} W_y(\xi, \eta), \quad x > 0.$$

Since $K_0 \neq \emptyset$ and K_1 contains $\operatorname{cone}(g_1, e_1)$, there exists numbers $\theta_1, \theta_2 \in \Delta$ such that

$$0 \leq \theta_1 < \theta_2 < 1 + \frac{1}{\lambda}$$

and

$$K_0 = \{(x, y) \in \operatorname{int} K : \theta_1 < \frac{y}{x+y} < \theta_2\}.$$

The following result is Proposition 8.5 in [94]. The proof is easily adapted to the case with jumps, using the continuity of the function $z \to H(z, \psi, \psi')$ given in lemma 4.3.1 i).

Proposition 4.3.7. *The function ψ is C^2 on $(\theta_1, \theta_2) \setminus \{1\}$ and satisfies the HJB equation $l_0\psi = 0$ on this set in the classical sense.*

The proposition above implies that the value function satisfies the HJB equation $\mathcal{L}_0(W) + U^*(W_x) = 0$ on $K_0 \setminus \mathbf{R}e_2$ in the classical sense and is C^2 on this set. It remains to study W on the set $\mathbf{R}e_2$. To do so, we follow the proof of theorem 9.1. in [94] :

Proposition 4.3.8. *The second derivative W_{yy} is well defined and continuous accross $\mathbf{R}_+ e_2$ and W satisfies on K_0 the equation*

$$\mathcal{L}_0(W) + U^*(W_x) = 0,$$

in the classical sense.

Theorem 4.3.2 shows that the sector $\operatorname{cone}(e_1, g_1)$ is contained in K_1 but we do not know whether $\operatorname{cone}(e_1, g_1) = K_1$. In this case, it seems to be difficult to prove the smoothness of the value function on $\mathbf{R}_+ e_1$.

Proposition 4.3.9. *Suppose that $\pi(\mathbf{R})$ and $\int_{\mathbf{R}} \eta\pi(\eta)$ exists and satisfies $\pi(\mathbf{R}) - \int_{\mathbf{R}} \eta\pi(\eta) \geq 0$. Then, $e_1 \in \operatorname{int} K_1$.*

Proof. The proof of this proposition is similar to that of Proposition 4.8.8, [61]. The only difference is that we need to calculate the right derivative of the function

$$H(z) := \int_{-1}^{\infty} \left[(1+z\eta)^\gamma \psi(\frac{z+z\eta}{1+z\eta}) 1_{(1-z, z(1+\eta)) \in K} - \psi(z) - z\eta(\gamma\psi + (1-z)\psi') \right] d\pi(\eta)$$

at zero. Note that, if $z \geq 0$, then $1_{(1-z,z(1+\eta))\in K} = 1$ hence H admits a right derivative at zero. Some elementary calculations show that its value coincides with $(\pi(\mathbf{R}) - \int_{\mathbf{R}} \eta\pi(\eta))\gamma\psi(0)$. We then follow the proof of Proposition 4.8.8, [61]. ■

In order to apply the Itô formula and construct an optimal policy, we need the value function W to be C^2 across the boundary of the cone K_0, except

at 0. The following result shows that this condition is satisfied provided that the boundary does not coincide with the positive $x-$axis or the positive $y-$axis. We omit the proof which is similar to that of theorem 10.1 in [94] (since the operator H does not affect the proof).

Proposition 4.3.10. *If the cone K_0 does not shares its boundary with that of the first quadrant, then the Bellman function is C^2 across the boundary of K_0.*

4.3.3 Proofs of theorem 4.3.3.

Proof of theorem 4.3.3. Note that $\theta^\pi = \infty$ since $V_t^+ \in K_0, \forall\, t$, and W_x is positive (hence (4.3.39) makes sense). We shall only consider the case where K_0 is included in the first quadrant. Otherwise, we refer the readers to rem 10.4 in [94]. The Bellman function is then C^2 on K_0 hence σ is locally lipschitz on this set. We then deduce that the Skorokhod problem admits a unique solution. We check the second assertion. Applying lemma 4.2.5, we get that

$$\begin{aligned} e^{-\beta t}W(V_t^+) + J_t^\pi &= W(x,y) + N_t + R_t, \\ &+ \int_0^t e^{-\beta u}\Big[\mathcal{L}_0 W(V_{u_-}) - W'(V_{u_-})c_u + U(c_u)\Big]du. \end{aligned}$$

From construction of the solution to the Skorokhod problem, we obtain that $R_t = 0$. Moreover, when c is defined by (4.3.39), we have $W_x(V_{u_-})c_u + U(c_u) = U^*(c_u)$. Therefore, the integral is zero, by virtue of theorem 4.3.2. It remains to prove that N_t is a martingale and

$$\lim_{n\to\infty} Ee^{-\beta t_n}W(V_{t_n}^+) \to 0, \tag{4.3.42}$$

for a sequence of real numbers $t_n \to \infty$.

To prove (4.3.42), we observe that $|W(z)| \le \kappa|z|^\gamma$ and $|W_x(z)| \le \kappa|z|^{\gamma-1}$ where κ is an upper bound of W and W_x on the intersection Δ_0 of the set K_0 with the line $x + y = 1$. This is deduced from $\psi'(0+) < \infty$ and the continuity of W on Δ_0. We deduce (for some varying constant κ from one line to another one) that,

$$E\int_0^\infty e^{-\beta t}W(V_t^+)dt \le \kappa E\int_0^\infty e^{-\beta t}|V_t^+|^\gamma dt \le \kappa E\int_0^\infty e^{-\beta t}u(c_t)dt \le \kappa W(x). \tag{4.3.43}$$

Since W is finite, this implies the existence of a sequence $t_n \uparrow \infty$ for which (4.3.42) holds. Details of this assertion are given in lemma 4.3.11.

We now prove that N_t is a true martingale. Indeed, by a similar argument, we have

$$|zW_y(z)| \le \kappa|z|^\gamma \le (1 + |z|), \quad z \in K_0.$$

Hence, we infer that the stochastic process $\int_0^\cdot e^{-\beta u}W_y(V_{u_-}^2)V_s^2\sigma dW_u$ is a martingale. Let us focus on the integral with respect to the Lévy measure.

Observe that, for each fixed s, we have $I(V_{s_-}, z) = 1$ (because $V_s^+ \in \bar{K}_0$, $\forall\, s$). Moreover, using a finite taylor expansion, we get

$$\left|W(\widetilde{V}_{s_-} + \text{diag}\,(\widetilde{V}_{s_-})z) - W(\widetilde{V}_{s_-})\right| \leq |W'(\eta)||\text{diag}\,(\widetilde{V}_{s_-})z|,$$

where $\eta \in [\widetilde{V}_{s_-}, \widetilde{V}_{s_-} + \text{diag}\,(\widetilde{V}_{s_-})z]$ satisfies $|\eta| \leq |\widetilde{V}_{s_-}|(1 + |z|)$. It follows that

$$\left|W(\widetilde{V}_{s_-} + \text{diag}\,(\widetilde{V}_{s_-})z) - W(\widetilde{V}_{s_-})\right| \leq \kappa|\eta|^{\gamma-1}|\text{diag}\,(\widetilde{V}_{s_-})z| \leq \kappa|\widetilde{V}_{s_-}|^{\gamma}(1+|z|)^{\gamma-1}|z|,$$

where the last inequality is deduced from the inequality $|W'(\eta)| \leq \kappa|\eta|^{\gamma-1}$. We then obtain

$$\left|W(\widetilde{V}_{s_-} + \text{diag}\,(\widetilde{V}_{s_-})z) - W(\widetilde{V}_{s_-})\right| \leq \kappa|\widetilde{V}_{s_-}|^{\gamma}|z|.$$

Therefore, as the Lévy process is of finite activity and (4.3.43) holds,

$$E\int_0^t \int_{\mathbf{R}} e^{-\beta u}\left|W(\widetilde{V}_{s_-} + \text{diag}\,(\widetilde{V}_{s_-})z)I(\widetilde{V}_{s_-}, z) - W(\widetilde{V}_{s_-})\right| \pi(dz)ds \leq \kappa W(x) < \infty.$$

By theorem I.1.33 b. page 73 [51], we deduce that the purely discontinuous local martingale N satisfies $E\,\text{var}(N)_\infty < \infty$ hence is a martingale. ■

To complete the proof of the theorem, we need the following lemma

Lemma 4.3.11. *Suppose that* $\int_0^\infty X_u du < \infty$ *where* $X_u = Ee^{-\beta u}W(V_{u+}) \geq 0$. *Then* $\lim_{t\to\infty} X_t \to 0$.

Proof. Observe that $e^{-\beta t}W(V_t^+)+J_t^\pi = W(x)+N_t$. With $Y_u = e^{-\beta u}W(V_{u+})$, we get that for $u \geq s$, $Y_u - Y_s = N_u - N_s - (J_u^\pi - J_s^\pi)$ where $J_u^\pi - J_s^\pi \geq 0$. Since N is a martingale, we deduce that $X_u - X_s \leq 0$, i.e. X is decreasing. Therefore, the integrablity of $\int_0^\infty X_u du$ ensures that $\lim_{t\to\infty} X_t \to 0$. Indeed, if $\lim_{t\to\infty} X_t \to c > 0$ then $\int_0^\infty X_u du \geq \int_0^\infty c du = \infty$, hence a contradiction. ■

4.4 Appendix

4.4.1 A0 : Stochastic Integral with Làdlàg Integrands

This subsection is devoted to define the integral $\int_0^t X_u dB_u$ for any adapted càdlàg process X and any predictable làdlàg process B of bounded variations. Assume that we are given a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Such an integral may not exist in the standard Stieltjes sense. Typically, this is the case where X and B share common discontinuity points. We refer to the book of Wheeden and Zygmund [100] for more details and [43] where such a construction is considered.

For any predictable process of bounded variations B starting from zero and having trajectories with left and right limits, we put $\Delta B := B - B_-$ as usual and $\Delta^+ B := B_+ - B_-$. We also define the right-continuous processes

$$B_t^d := \sum_{s\leq t} \Delta B_s, \quad B_t^{d,+} := \sum_{s<t} \Delta^+ B_s$$

and, at last, the continuous part of B :

$$B^c := B - B^d - B_-^{d,+}.$$

Definition 4.4.1. *Let X be an adapted càdlàg process and B a predictable làdlàg process of bounded variations. Define :*

$$I_T(X,B) := \int_{[0,T]} X_u dB_u := \int_{[0,T]} X_u dB_u^c + \sum_{0<t\leq T} X_{t-}\Delta B_t + \sum_{0\leq t<T} X_t \Delta^+ B_t. \tag{4.4.44}$$

Note that the integral part in the definition above is finite in the sense of Stieltjes since the integrator is continuous and of finite variations ; the discrete sum is absolutely convergent (bounded by $2\|B\|_T \max_{0\leq t\leq T}|X(t)|$). Therefore, the integral is well-defined.

From the economic viewpoint, we may think of X as the dynamics of a risky asset while B is the quantity of assets invested in the portfolio. The cost of this strategy is then given by $\int_{[0,T]} X_u dB_u$ which consists of three terms. The integral term reflects the cumulative cost with respect to all changes in the trend of the strategy. The two last terms are cumulative costs charged at discontinuity points of the strategy (right before and after economic shocks).

Let us establish some major properties of the predictable Stieltjes integral given in the following theorem :

Theorem 4.4.2. *The map I satisfies the following properties :*

i) $I_\cdot(X,B)$ is an adapted làdlàg process.

ii) I_T is linear both in X and in B and, with $X_T^* = \sup_{t\in[0,T]}|X_t|$, we have :

$$|I_T(X,T)| \leq \|B\|_T X_T^*.$$

iii) If B^n is a sequence of predictable làdlàg processes of bounded variations converging pointwise to B such that $\sup_n \|B^n\|_T < \infty$, then $(\int_0^t X_u dB_u^n)_{t\leq T}$ converges pointwise to $(\int_0^t X_u dB_u)_{t\leq T}$.

iv) If $(B_n)_{n\geq 1}$ satisfy iii) and $X \geq 0$, then $\liminf_n I(X,\|B_n\|) \geq I(X,\|B\|)$.

Proof. The proofs of the asertions $ii) - iv)$ are given in [43]. It suffices to prove the first asertion. We check that $I_T(X,B)$ is $\mathcal{F}_T$-measurable for all $T>0$. First, note that the second and the third terms in 4.4.44 are trivially $\mathcal{F}_T$-measurable since $\mathcal{F}$ is right-continuous. Thus, it suffices to check that the first term, which is a pathwise Stieltjes integral, is $\mathcal{F}_T$-measurable. But this is true because this integral is the limit of Riemann sums along a deterministic grid

$$\int_{[0,T]} X_u dB_u^c = \lim_{n\to\infty} \sum_{k=0}^{[nT]} X_{\frac{k}{n}} \left(B^c_{\frac{k+1}{n}} - B^c_{\frac{k}{n}} \right),$$

each term of which is $\mathcal{F}_T$-measurable.

To prove that the paths $t \to I_t(X,B)$ have right and left limits, we observe that the integral in 4.4.44 is continuous, while the discrete summations are làdlàg.

The following theorem shows that we can approximate the integral $\int_{[0,T]} X_u dB_u$ by integrals with respect to simple integrators. The proof is given in [43].

Theorem 4.4.3. *Let B be a predictable làdlàg process of bounded variations and X be a locally bounded càdlàg process. Then, for all $\varepsilon > 0$, there exists a strictly increasing sequence of stopping times τ^n such that $\sup_n \tau^n > T$, and a predictable process B^ε of the form :*

$$B^\varepsilon = \sum_n B_{\tau^n} I_{[[\tau^n]]} + B_{\tau^n_+} I_{]\tau^n, \tau^{n+1}[} \tag{4.4.45}$$

such that $|B^\varepsilon - B| \leq \varepsilon$, $\left|\int_0^t X_u dB^\varepsilon_u - \int_0^t X_u dB_u\right| \leq \varepsilon$, $\|B^\varepsilon\| \leq \|B\|$ pointwise on $[0,T]$ outside of a non null set.

Lemma 4.4.4. *Let (B^k) be a sequence of predictable làdlàg processes of bounded variations. Suppose that B^k converges pointwise to B such that for all k, $|\Delta^+ B^k| \leq |\Delta^+ B|$ and $\Delta^+ B^k \to \Delta^+ B$. Then, $(B^k)^{d+} \to B^{d+}$ pointwise.*

Proof. Let us consider for each k, a sequence of stopping times $(\tau^{n,k})$ exhausting the jumps of B^k and let (τ^n) be a sequence of stopping times exhausting the jumps of B. Let us define

$$\mathbb{T} := \{\tau^{n,k} : (n,k) \in \mathbb{N}\} \cup \{\tau^n : n \in \mathbb{N}\} := \{s_i : i \in \mathbb{N}\}.$$

Consider the positive measure $\mu := \sum_{i\in\mathbb{N}} \delta_{s_i}$ where δ_{s_i} are Dirac measures at points s_i. Then,

$$(B^k)^{d+}_t = \int_0^t \Delta^+ B^k_u d\mu(u).$$

As $|\Delta^+ B^k| \leq |\Delta^+ B|$ and $|\Delta^+ B|$ is μ-integrable as a process of finite variations, we deduce by the dominated convergence theorem that

$$(B^k)^{d+}_t = \int_0^t \Delta^+ B^k_u d\mu(u) \to \int_0^t \Delta^+ B_u d\mu(u) = B^{d,+}.$$

■

From there, we deduce an integration by parts formula :

Theorem 4.4.5. *Let X be a locally bounded càdlàg semimartingale and B be a predictable làdlàg process of bounded variations. Then,*

$$X_t B_t = X_0 B_0 + \int_0^t X_u dB_u + \int_0^t B_{u_-} dX_u + \sum_{s\leq t} \Delta X_s \Delta B_s.$$

Proof. We may assume without loss of generality that $B_0 = X_0 = 0$. We first assume that B has the form

$$B = \sum_n B_{\tau^n} I_{[[\tau^n]]} + B_{\tau^n_+} I_{]\tau^n, \tau^{n+1}[}.$$

We may assume without generality that there exists n_0 such that $\tau^{n_0} = t$. From the definition 4.4.44, recall that

$$\int_0^t X_u dB_u = \sum_{\tau^n \leq t} X_{\tau^n_-} \Delta B_{\tau^n} + \sum_{\tau^n < t} X_{\tau^n} \Delta^+ B_{\tau^n}.$$

On the other hand, notice that

$$B_- = \sum_n B_{\tau^n_+} I_{]\tau^n, \tau^{n+1}]}.$$

Then,

$$\int_0^t B_{u_-} dX_u = \sum_{\tau^n \leq t} B_{\tau^{n-1}_+} (X_{\tau^n} - X_{\tau^{n-1}}).$$

It follows that

$$\begin{aligned} \int_0^t X_u dB_u \quad + \quad \int_0^t B_{u_-} dX_u &= \sum_{\tau^n \leq t} B_{\tau^{n-1}_+} \left(X_{\tau^n} - X_{\tau^n_-} \right) + X_{t_-} B_t \\ &+ \sum_{\tau^n < t} B_{\tau^n} (X_{\tau^n_-} - X_{\tau^n}) + \sum_{\tau^n < t} X_{\tau^n} B_{\tau^n_+} - \sum_{\tau^n \leq t} X_{\tau^{n-1}} B_{\tau^{n-1}_+}, \end{aligned}$$

$$\int_0^t X_u dB_u + \int_0^t B_{u_-} dX_u \quad = \quad - \sum_{\tau^n \leq t} \left(B_{\tau^n} - B_{\tau^n_-} \right) \left(X_{\tau^n} - X_{\tau^n_-} \right) + X_t B_t.$$

In the general case, we use theorem 4.4.3 with a refining sequence of stopping times $(\tau^n_k)_k$. Note that $\sum_{s \leq t} \Delta X_s \Delta B_s = [X, B^{c+d}]_t$ where $B^{c+d} = B^c + B^d$. If B^k converges uniformly to B where B^k is of the form (4.4.45) as in theorem 4.4.3, then by lemma 4.4.4, $(B^k)^{c+d}$ converges pointwise to B^{c+d} as $k \to \infty$. Therefore, it suffices to show that : $\sum_{s \leq t} \Delta X_s \Delta B^k_s = [X, (B^k)^{c+d}]_t \to [X, B^{c+d}]_t = \sum_{s \leq t} \Delta X_s \Delta B_s$. This is obvious if X is of finite variations. Otherwise, it suffices to observe that we may uniformly approximate X by a process X^ϵ of finite variations such that $|X - X^\epsilon| \leq \epsilon$ as done in [43]. Since $\sup_k \|B^k\| \leq \|B\|$, we then conclude. ■

We need more results :

Lemma 4.4.6. *Let X be locally bounded càdlàg processes and B be a predictable làdlàg process of bounded variations. Then,*

$$\Delta(\int_0^\cdot X_u dB_u)_t = X_{t_-} \Delta B_t, \quad \Delta^+(\int_0^\cdot X_u dB_u)_t = X_t \Delta^+ B_t.$$

Proof. The equalities obviously hold if B is of the form (4.4.45). Otherwise, by theorem 4.4.3, we may approximate B by a sequence of elementary processes B^n of the form (4.4.45) such that B^n_- and B^n converge pointwise to B and B_- and $X \cdot B^n_\pm \to X \cdot B_\pm$. ■

Using theorem 4.4.5 and lemma 4.4.6, we obtain the following result :

Theorem 4.4.7. *Let X, Y be locally bounded càdlàg processes and B be a predictable làdlàg process of bounded variations. Then,*

$$X_t \int_0^t Y_u dB_u = \int_0^t X_u Y_u dB_u + \int_0^t \left(\int_0^{u-} Y_v dB_v \right) dX_u + \sum_{s \leq t} \Delta X_s Y_{s-} \Delta B_s.$$

4.4.2 A1 : Existence of Lyapunov Functions and Classical Supersolutions

In this subsection we study the existence of Lyapunov functions and classical supersolutions to the HJB equation. Our purpose is to seek for explicit conditions which guarantees the concavity and finiteness of the Bellman funciton of the two-dimentional Merton problem with power utility functions. We only focus on the case where the matrix $A = (a_{ij})$ is diagonal with $a_{ii} = \sigma^i$, such that $\sigma^0 = 0, \mu^0 = 0$ and $\sigma^i \neq 0, i = 1 \dots d$ (i.e. the first asset is a *numéraire* and the others are risky assets). We also suppose that the utility function has the power form $U(x) = u_\gamma(xe_0)$, where $u_\gamma(t) = \frac{t^\gamma}{\gamma}, \gamma \in (0,1)$; and that $\mathcal{C} = \mathbf{R}_+ e_0$. For the general case, readers should consult the paper [26] for more details.
In this subsection we work under the following condition

$$\int_{\mathbf{R}^d} |z| \Pi(dz) < \infty.$$

Lyapunov Functions

For $p \in \operatorname{int} K^*$, we find a Lyapunov function of the form $v(x) := u_\eta(px)$,where $u_\eta(x) = \frac{x^\eta}{\eta}$ and $\gamma < \eta < 1$. Note that the Bellman function is homogeneous of degree γ. Therefore, the parameter η is chosen bigger than γ to assure that the Lyapunov function is growing faster than the Bellman function. If such a Lyapunov exists, the HJB equation has a unique solution, and, by virtue of theorem 4.4.16, this unique solution is a concave function.
We have $u'_\eta(x) = (px)^{\eta-1} p \in \operatorname{int} K^*$ as required for v to be a Lyapunov function. Moreover,

$$\begin{aligned} \mathcal{L}_0 v(x) &= \frac{1}{2}\langle A(x)p, p\rangle u''_\eta(px) + \langle \mu(x), p\rangle u'_\eta(px) - \beta u_\eta(px) \\ &+ \int_{\mathbf{R}^d} \left[u_\eta(px + {}^t p \operatorname{diag}(x) z) I(x,z) - u_\eta(px) - u'_\eta(px)\, {}^t p \operatorname{diag}(x) z \right] \Pi(dz). \end{aligned}$$

Let us denote the integral expression above by $H_\eta(x)$. Our goal is to choose u so that on $\operatorname{int} K$ we have $\mathcal{L}_0 v(x) \leq 0$, or equivalently on $\operatorname{int} K$

$$\beta \geq \sup_{x \in K} \left[p \operatorname{diag}(x) \mu \frac{u'_\eta(px)}{u_\eta(px)} + \frac{1}{2}\langle A(x)p, p\rangle \frac{u''_\eta(px)}{u_\eta(px)} + \frac{H_\eta(x)}{u_\eta(px)} \right].$$

Denote

$$L_\eta(p,x) = p \operatorname{diag}(x) \mu \frac{u'_\eta(px)}{u_\eta(px)} + \frac{1}{2}\langle A(x)p, p\rangle \frac{u''_\eta(px)}{u_\eta(px)}.$$

We have

$$\begin{aligned} L_\eta(p,x) &= \frac{1}{2}\frac{\eta}{1-\eta}\sum_{i=1}^d \frac{\mu_i^2}{\sigma_i^2} - \frac{1}{2}\eta(1-\eta)\sum_{i=1}^d \left(\frac{\sigma_i p_i x_i}{px} - \frac{\mu_i}{\sigma_i(1-\eta)}\right)^2, \\ &\leq \frac{1}{2}\frac{\eta}{1-\eta}\sum_{i=1}^d \frac{\mu_i^2}{\sigma_i^2}. \end{aligned}$$

We then choose β such that

$$\beta \geq \frac{1}{2}\frac{\eta}{1-\eta}\sum_{i=1}^d \frac{\mu_i^2}{\sigma_i^2} + \sup_{x\in K}\frac{H_\eta(x)}{u_\eta(px)}.$$

Put $k(p,x) = \frac{1}{px} p\,\mathrm{diag}\,(x)$. It is clear that

$$\frac{H_\eta(x)}{u_\eta(px)} \leq \int_{\mathbf{R}^d} \left[(1+k(p,x)z)^\eta 1_{\{1+k(p,x)z>0\}} - 1 - \eta k(p,x)z\right]\,\Pi(dz).$$

Note that $k(p,tx) = k(p,x)$ for $t>0$. Therefore, instead of considering the r.h.s of the equality above on K, we may simply consider it on $\mathcal{B}_1 := K\cap\mathcal{O}_1$, where $\mathcal{O}_1$ is the unit ball $\{x : |x| = 1\}$. It is easy to prove that this expression is bounded on $\mathcal{B}_1$. We then define

$$h(\eta) := \inf_{p\in \mathrm{int} K^*}\sup_{x\in\mathcal{B}_1}\int_{\mathbf{R}^d} \left[(1+k(p,x)z)^\eta 1_{\{1+k(p,x)z>0\}} - 1 - \eta k(p,x)z\right]\,\Pi(dz),$$

and chose β such that

$$\beta > \frac{1}{2}\frac{\eta}{1-\eta}\sum_{i=1}^d \frac{\mu_i^2}{\sigma_i^2} + h(\eta). \tag{4.4.46}$$

We get the following result as a direct consequence of all the calculations above.

Proposition 4.4.8.

i) If the condition (4.4.46) holds, then $v(x) = \frac{(px)^\eta}{\eta}, 1>\eta>\gamma$ *is a Lyapunov function with respect to* $\mathcal{L}_0$ *for some* $p\in K^*$ *with* $p^1 = 1$.

ii) Set $\bar{h}(\gamma) := \liminf_{\eta\searrow\gamma} h(\eta)$ *and suppose that*

$$\beta > \frac{1}{2}\frac{\gamma}{1-\gamma}\sum_{i=1}^d \frac{\mu_i^2}{\sigma_i^2} + \bar{h}(\gamma). \tag{4.4.47}$$

Then, the HJB equation has unique solution in the class $C_1(K)$.

■

Classical Supersolution

Let us set $v(x) := ku_\gamma(p.x)$. By definition,

$$\begin{aligned} U^*(v'(x)) &= \sup_{y\in\mathcal{C}}\big(U(y)-v'(x)y\big)=\sup_{y\in\mathcal{C}}\big(U(y)-ku'_\gamma(px)py\big), \\ &= \sup_{y_1\geq 0}\left(\frac{(y_1)^\gamma}{\gamma}-k(px)^{\gamma-1}p_1y_1\right) \end{aligned}$$

Then, with $p_1=1$, $U^*(v'(x))=k^{\frac{\gamma}{\gamma-1}}(px)^\gamma(1/\gamma-1)$. We have $\Sigma_G(v'(x))=\sup_{x\in G}u'_\gamma(p.x)px\leq 0$ since $u'\geq 0$ and $G\subseteq -K$, $p\in K^*\backslash\{0\}$.

Our goal is to choose p,k so that on $\mathrm{int}K$, we have

$$\begin{aligned} &k^{\frac{\gamma}{\gamma-1}}(px)^\gamma(1/\gamma-1)+\frac{k}{2}\langle A(x)p,p\rangle u''_\gamma(px)+k\langle\mu(x),p\rangle u'_\gamma(px)-k\beta u_\gamma(px) \\ &+\int_{\mathbf{R}^d}\left[u_\gamma(px+{}^tp\mathrm{diag}\,(x)z)I(x,z)-u_\gamma(px)-u'_\gamma(px)^t p\mathrm{diag}\,(x)z\right]\Pi(dz)\leq 0. \end{aligned}$$

Following a similar reasoning as in the preceding subsection, we choose β such that

$$\beta>\frac{1}{2}\frac{\gamma}{1-\gamma}\sum_{i=1}^d\frac{\mu_i^2}{\sigma_i^2}+h(\gamma)+k^{\frac{1}{\gamma-1}}(1-\gamma). \tag{4.4.48}$$

Proposition 4.4.9.

i) Suppose that Condition (4.4.48) holds. Then, the function

$$v(x)=\frac{k}{\gamma}(px)^\gamma,$$

is a classical solution of the HJB equation for some $p\in K^*$ with $p^1=1$.

ii) In the two–dimensional model with the power utility function, assume that the Merton parameter

$$\kappa_M:=\frac{1}{1-\gamma}\left(\beta-\frac{\gamma\mu^2}{2\sigma^2(1-\gamma)}-h(\gamma)\right)>0.$$

Then, there exists $p\in K^*$ with $p^1=1$ such that the function $f(x)=m(px)^\gamma$ is a classical solution of the HJB equation and $m>(1/\gamma)\kappa_M^{\gamma-1}$.

Combining Propositions 4.4.8 and 4.4.9, we obtain the following result

Corollary 4.4.10. *Let us set $h^*(\gamma):=\max(h(\gamma),\bar{h}(\gamma))$. The following condition*

$$\beta>\frac{1}{2}\frac{\gamma}{1-\gamma}\sum_{i=1}^d\frac{\mu_i^2}{\sigma_i^2}+h^*(\gamma) \tag{4.4.49}$$

guarantees the existence of a classical super solution to the HJB equation and a Lyapunov function with higher growing order than that of the Bellman funtion (in the considered case).

4.4.3 A2 : Some Elementary Properties of The Bellman Function

We denote by $\succeq$ the partial order defined by K, i.e. if $x, y \in \mathbf{R}^d$, $x \succeq y \Leftrightarrow x - y \in K$.

Proposition 4.4.11. *The function W is increasing with respect to the partial order $\succeq$.*

Proof. Suppose that $x_2 \succeq x_1$. Let $\pi = (B, C) \in \mathcal{A}_{x_1}$ and $V^{(1)}$ be such that

$$\begin{aligned} V_{0-}^{(1)} &= x_1, \\ dV_t^{(1)i} &= V_{t-}^{(1)i} dY_t^i + dB_t^i - dC_t^i, \quad t \geq 0,\ i = 1, \cdots, d. \end{aligned}$$

Let us define $V_{0-}^{(2)} = x_2$ and $V_t^{(2)} = V_t^{(1)}$ if $t \geq 0$. From the dynamics of $V^{(1)}$ we deduce that

$$\begin{aligned} V_{0-}^{(2)} &= x_2, \\ dV_t^{(2)i} &= V_{t-}^{(2)i} dY_t^i + d\tilde{B}_t^i - dC_t^i, \quad t \geq 0,\ i = 1, \cdots, d, \end{aligned}$$

where $\tilde{B}_t^i = B_t^i + (x_1^i - x_2^i) I_{[0,\infty)}(t)$ is still a làdlàg and predictable process of bounded variations satisfying $\Delta \tilde{B}_t \in -K$ a.s. since $x_1 - x_2 \in -K$. Since $V_{0-}^{(2)}, V_{0-}^{(1)} \in \operatorname{int} K$ it is straightforward that $\theta^2 = \theta^1$ where θ^i, $i = 1, 2$, are the stopping times defined by (4.2.4) respectively for $V^{(2)}$ and $V^{(1)}$. It follows that $\widetilde{\pi} = (\widetilde{B}, C) \in \mathcal{A}_{x_2}$ and we deduce that $W(x_1) \leq W(x_2)$. ■

Theorem 4.4.12. *Assume that $W(x_0) < \infty$ where $x_0 \in \operatorname{int} K$. Then, W is continuous at x_0.*

We need some following technical results

Lemma 4.4.13. *Let us consider $x_0 \in \operatorname{int} \mathrm{K}$. Then, $\limsup_{\lambda \to 1} W(\lambda x_0) \leq W(x_0)$.*

Proof. We may find $\tilde{\pi}^n = (\tilde{B}^n, \tilde{C}^n) \in \mathcal{A}_{\lambda x_0}$ depending on λ such that

$$W(\lambda x_0) \leq E \int_0^{\theta^{\pi^n}} e^{-\beta s} U(\tilde{c}_s^n) ds + \frac{1}{n}. \tag{4.4.50}$$

Observe that if $\tilde{\pi}^n = (\tilde{B}^n, \tilde{C}^n) \in \mathcal{A}_{\lambda x_0}$, then $\tilde{\pi}^n/\lambda := (\tilde{B}^n/\lambda, \tilde{C}^n/\lambda) \in \mathcal{A}_{x_0}$ and $\theta^{\tilde{\pi}^n/\lambda} = \theta^{\tilde{\pi}^n}$. We rewrite $\tilde{\pi}^n$ as $\tilde{\pi}^n = \lambda \pi^n$ where $\pi^n := (B^n, C^n) \in \mathcal{A}_{x_0}$. By (4.4.50), we deduce that

$$W(\lambda x_0) \leq E \int_0^{\theta^{\pi^n}} e^{-\beta s} U(\lambda c_s^n) ds + \frac{1}{n}.$$

As $\lambda^{-1} \in]0, 1[$ and $x \mapsto U(x)$ is concave with $U(0) = 0$, we get that

$$U(c_s^n) = U\left(\lambda^{-1}(\lambda c_s^n) + (1 - \lambda^{-1}) \times 0\right) \geq \lambda^{-1} U(\lambda c_s^n).$$

We then get $W(\lambda x_0) \leq \lambda W(x_0) + n^{-1}$ and the conclusion follows as $n \to \infty$. ■

Lemma 4.4.14. *Let us consider a sequence $x_n \to x_0 \in \operatorname{int} K$ and $T \in (0,\infty)$. Then, for any $\pi \in \mathcal{A}_{x_0}$, the sequence of portfolios $V^{(n)} = V^{\pi,x_n}$ with initial values $V^{(n)}_{0_-} = x_n$ is such that*

$$T \wedge \theta \leq \liminf_n \theta_n \wedge T$$

where θ, θ_n are the stopping times defined by $V := V^{\pi,x}$, $V^{(n)}$ respectively in (4.2.4).

Proof. By virtue of lemma (4.4.23), $V^{(n)}$ and V are uniquely defined by (4.2.3). Moreover, $\widetilde{V}^n := V^{(n)} - V$ satisfies the dynamic

$$d\widetilde{V}^{ni}_t \;=\; \widetilde{V}^{ni}_{t_-} dY^i_t, \quad \widetilde{V}^n_t = x_n - x_0.$$

Hence $\widetilde{V}^{ni}_t = (x^i_n - x^i_0)S^i_t$ or equivalently $\widetilde{V}^n_t = \operatorname{diag}\left((x_n - x_0)/S_0\right) S_t$. We deduce that

$$\sup_{t\leq T} |V^{(n)}_t - V_t| \leq \frac{|x_n - x_0|}{|S_0|} S^*_T$$

where $S^*_T := \sup_{t\leq T} |S_t|$. On the other hand, if $\delta > 0$ is small enough,

$$\inf_{s\leq T\wedge\theta-\delta} d(V_s;\partial K) = \lim_n d(V_{s_n};\partial K)$$

where we may assume that $s_n \in [0, T\wedge\theta-\delta]$ converges to $s_0 \in [0, T\wedge\theta-\delta]$ by a compactness argument. First assume that $s_n \uparrow s_0$. Then,

$$\inf_{s\leq T\wedge\theta-\delta} d(V_s;\partial K) = d(V_{s_{0_-}};\partial K) > 0,$$

by virtue of Corollary 4.4.22. Otherwise, we get that

$$\inf_{s\leq T\wedge\theta-\delta} d(V_s;\partial K) = d(V_{s_{0_+}};\partial K) > 0.$$

Therefore, there exists $\epsilon > 0$ such that

$$d(V_s;\partial K) > \epsilon, \quad \forall s \in [0, T\wedge\theta-\delta].$$

Applying the triangular inequality $d(V_s;\partial K) \leq d(V_s;V^n_s) + d(V^n_s;\partial K)$ we then deduce that

$$d(V^n_s;\partial K) \;>\; \epsilon - d(V_s;V^n_s) \geq \epsilon - \frac{|x_n - x_0|}{|S_0|} S^*_T, \quad s \in [0, T\wedge\theta-\delta].$$

It follows that for n large enough

$$d(V^n_s;\partial K) > 0, \quad s \in [0, T\wedge\theta-\delta].$$

We deduce that $\theta^n \geq T\wedge\theta-\delta$ if n is large enough. Therefore, we have shown that whatever $\delta > 0$ is small enough, there exists a.s. n_0 such that $n \geq n_0$ implies that $T\wedge\theta-\delta \leq \theta^n$, hence $T\wedge\theta-\delta \leq \inf_{n\geq n_0}\theta^n$ and finally $T\wedge\theta \leq \liminf_n \theta_n \wedge T$. ■

Proof of theorem 4.4.12. It suffices to show that W is both upper semicontinuous and lower semicontinuous.

• Let us first show that $\limsup_{x\to x_0} W(x) \leq W(x_0)$. In the contrary case, we have $\limsup_{x\to x_0} W(x) > W(x_0)$. Note that $\limsup_{x\to x_0} W(x) = \lim_k W(x_k)$ where x_k is a subsequence converging to x_0. As $x_0 \in \operatorname{int} K$, we may assume that $x_k \in \operatorname{int} K$. We define $\widetilde{x}_k = (1+k^{-1})x_0 \in \operatorname{int} K$ such that $\widetilde{x}_k \in x_0 + \operatorname{int} K$ if k is large enough. As $x_k \to x_0$ and $\widetilde{x}_k - \operatorname{int} K$ is an open set containing x_0, there exists a subsequence x_{n_k} such that $x_{n_k} \in \widetilde{x}_k - \operatorname{int} K$ hence $x_{n_k} \preceq \widetilde{x}_k$. Since W is increasing with respect to $\succeq$, we obtain that $\limsup_k W(\widetilde{x}_k) \geq \lim_k W(x_{n_k}) > W(x_0)$. On the other hand, by virtue of lemma 4.4.13, $W(\widetilde{x}_k) \leq W(x_0)$ hence a contradiction.

• Let us show that $\liminf_{x_n\to x_0} W(x_n) \geq W(x_0)$. For an arbitrary $\epsilon > 0$, there exists $\pi \in \mathcal{A}_{x_0}$ such that

$$W(x_0) \leq \epsilon + E\int_0^{\theta^\pi} e^{-\beta s}U(c_s)ds.$$

We then deduce $T \in (0,\infty)$ such that

$$W(x_0) \leq 2\epsilon + E\int_0^{\theta^\pi\wedge T} e^{-\beta s}U(c_s)ds.$$

We introduce the stopping times θ^n associated to the portfolios defined by the strategy π and the initial values x_n. Observe the inequality

$$I_{s<\liminf_n \theta^n\wedge T} \leq \liminf_n I_{s\leq\theta^n\wedge T}.$$

So by virtue of Proposition 4.4.14 and Fatou's lemma, we then deduce that

$$\begin{aligned} W(x_0) &\leq 2\epsilon + \liminf_n E\int_0^{\theta^n\wedge T} e^{-\beta s}U(c_s)ds, \\ W(x_0) &\leq 2\epsilon + \liminf_n W(x_n). \end{aligned}$$

Since $\epsilon > 0$ is arbitrarily chosen, we deduce that $W(x_0) \leq \liminf_n W(x_n)$. ■

Recall that Φ is the set of all continuous function $f : K \mapsto \mathbf{R}_+$ increasing with respect to the partial ordering $\succeq_K$ and such that for every $x \in \operatorname{int} K$ and $\pi = (B,C)$ the positive process X_t^f is a supermartingale. The following proposition presents the finiteness and continuity up to boundary ∂K of the Bellman function in term of Φ.

Proposition 4.4.15.

a) If $f \in \Phi$, then $W \leq f$ on $\operatorname{int} K$. Hence, if Φ is non-empty, then W is finite.

b) If a point $x_0 \in \partial K$ is such that there exists $f \in \Phi$ with $f(x_0) = 0$, then W is continuous at x_0, i.e. $W(x_n) \to W(x_0) := 0$ as $x_n \in \operatorname{int} K$ converges to x_0.

Proof. If $f \in \Phi$, $(X_t^f)_{t\in[0,\infty)}$ is a positive supermartingale. We deduce that $W(x) \leq EX_\infty^f \leq f(x)$ by virtue of Fatou's lemma.
From a), the inequality $0 \leq W(x_n) \leq f(x_n)$ allows us to immediately conclude about the second statement. ■

Theorem 4.4.16. *If the HJB equation (4.2.10) has an unique global viscosity solution in $C_1(K)$, then the Bellman function is concave.*

Proof. If we define the Bellman function only on the class of admissible strategies generating portfolio processes evolving in K all the time, i.e

$$\widehat{\mathcal{A}}_x := \{\pi \in \mathcal{A} : V_t^\pi \in K \,\forall\, t > 0\,\} \neq \emptyset, \text{ [1]}$$

and

$$\widehat{W}(x) := \sup_{\pi\in\widehat{\mathcal{A}}_x} EJ_\infty^\pi(x), \quad x \in \operatorname{int} K,$$

then the Bellman function $\widehat{W}$ is a global viscosity solution to the same HJB equation as W. By assumption, the global viscosity solution of this HJB equation is unique, i.e $\widehat{W} = W$. It is well known that the function $\widehat{W}$ is concave (the proof of it is straightforward and given in Framstad et al. [40]). Therefore, W is also concave. ■

4.4.4 A3 : Skorokhod Problem

In order to provide a rigorous construction of the optimal policy to the two-dimentional optimal consumption problem, we present in this section the existence and uniqueness of solutions to a class of SDEs with reflection, also called the Skorokhod problem. We first recall some results for the continuous diffusion case, and then pass to the case of finite activity pure-jumps Lévy processes.

Skorokhod Problem for Continuous Diffusion Processes

Let $\gamma : \partial K_0 \mapsto \mathbf{R}^2$ be a vector-valued function with $g(x) = -g_i$ on the set $(\partial K_0 \cap \partial K_i)\backslash\{0\}$ and $\gamma(0) = 0$. Let Y be the process $Y_t = (Y_0^1, Y_0^2) + (t, W_t)$, $t \geq 0$, where W is a standard brownian motion. Let $\sigma = \mathbf{R}^2 \mapsto \mathbf{R}^2 \times \mathbf{R}^2$ be a matrix-valued function which is Lipschitz-continuous.
On the closed cone $\bar{K}_0$, we consider the Skorokhod problem formulated as follows : find a pair of adapted continuous processes, V, starting from $x \in \bar{K}_0$ and k, real-valued, starting at zero, and increasing such that

$$dV_t = \sigma(V_t)dY_t + \gamma(V_t)dk_t, \tag{4.4.51}$$
$$dk_t = I_{V_t\in\partial K_0}dk_t, \tag{4.4.52}$$
$$V_t \in K_0\,, \forall t \geq 0. \tag{4.4.53}$$

1. Observe that $\widehat{\mathcal{A}}_x \neq \emptyset$. Indeed, rebalance the portfolio starting from $x \in \operatorname{int} K$ so that $V_+ = 0$.

The aim of this section is to show that this r.s.d.e has a solution on the set K_0 which is trapped at zero. To do so, we shall prove several intermediate lemmas. The main proof is based on the existence of a solution to a r.s.d.e. on a bounded domain G when the direction of the reflection is given by a C^2-function γ satisfying the following condition (see [30]) :

C1 : $\gamma \in C^2(\mathbf{R}^2, \mathbf{R}^2)$ and there is $b \in (0,1)$ such that

$$\bigcup_{0\leq t\leq b} B(x - t\gamma(x), tb) \subseteq G^c, \quad \text{for } x \in \partial G.$$

Theorem 4.4.17. *The Skorokhod problem (4.4.51) has a solution which is trapped at zero.*

Proof.

Denote by D the bisector of the cone K_0^n and we put $d > 0$ such that $D := \{-dx + y = 0\}$. Let us introduce the polygons

$$K_0^n := K_0 \cap \{\epsilon_n^{-1} \leq x + dy \leq \epsilon_n\}$$

where $\epsilon_n \to \infty$. Let $x \in \partial K_0^n$ be a starting point. The case $x = 0$ being trivial, we assume that $x \neq 0$ hence $x \in K_0^n$ if n is large enough.

• Step 1. There exists closed regions $\tilde{K}_0^n$ such that $K_0^n \subseteq \tilde{K}_0^n \subseteq K_0^{n+1}$ verifying the condition **C1** for some reflection function γ_n satisfying $\gamma_n(x) \to \gamma(x)$ for all $x \in \partial K_0$. Indeed, we denote by a_n and b_n the two points of $\partial K_0 \cap \{\epsilon_n = x + dy\}$ such that $y_{a_n} > y_{b_n}$. Observe that b_n is the symmetric of a_n with respect to the bisector D. Similarly, c_n and d_n are the two symmetric points of $\partial K_0 \cap \{\epsilon_n^{-1} = x + dy\}$. Denote $\tilde{e}_n := D \cap \{(\epsilon_n + \epsilon_{n+1})/2 = x + dy\}$. We then define $\tilde{K}_0^n$ as the polygon

$$\tilde{K}_0^n := K_0 \cap \{(\epsilon_n^{-1} + \epsilon_{n+1}^{-1})/2 \leq x + dy \leq (\epsilon_n + \epsilon_{n+1})/2\},$$

and denote by $\tilde{a}_n$ and $\tilde{b}_n$ the two points of $\partial K_0 \cap \{(\epsilon_n + \epsilon_{n+1})/2 = x + dy\}$ such that $y_{\tilde{a}_n} > y_{\tilde{b}_n}$. Similarly, $\tilde{c}_n$ and $\tilde{d}_n$ are the two points of $\partial K_0 \cap \{(\epsilon_n^{-1} + \epsilon_{n+1}^{-1})/2 = x + dy\}$ such that $y_{\tilde{c}_n} > y_{\tilde{d}_n}$.

Let η_1 be the outward normal to $\partial K_0 \cap K_1$ and η_2 be the outward normal to $\partial K_0 \cap K_2$. We consider g_3 a unit vector such that $g_3\eta_1 > 0$ and $g_3(1,d) > 0$. Similarly, we define g_4 as a unit vector such that $g_4\eta_2 > 0$ and $g_4(1,d) > 0$, g_5 is a unit vector such that $g_5\eta_1 > 0$, $g_5\eta_2 > 0$.

Let us introduce the smooth function

$$\begin{aligned}\gamma_n(x) &:= -\Big(g_1\chi^1(x) + g_2\chi^2(x) + g_3(1-\chi^1(x))(1-\chi^4(x))\chi^3(x) \\ &+ g_4(1-\chi^2(x))(1-\chi^3(x))\chi^4(x) + g_5(1-\chi^1(x))(1-\chi^2(x))\chi^5(x)\Big),\end{aligned}$$

where, by lemma 4.4.18, the functions $\chi_i \in C^\infty(\mathbf{R}^2, [0,1])$, $i = 1, \cdots, 5$ and, with $\gamma_n \to 0$ small enough,

$\chi^1(x) = 1$ on $[d_n, b_n]_{\gamma_n}$ and $\chi^1(x) = 0$ on $\mathbf{R}^2 \backslash [d_n, b_n]_{2\gamma_n}$
$\chi^2(x) = 1$ on $[c_n, a_n]_{\gamma_n}$ and $\chi^2(x) = 0$ on $\mathbf{R}^2 \backslash [c_n, a_n]_{2\gamma_n}$,
$\chi^3(x) = 1$ on $(C_3^n)_{\gamma_n} := \Big([\tilde{e}_n, \tilde{b}_n] \cup [\tilde{b}_n, b_n]\Big)_{\gamma_n}$ and $\chi^3(x) = 0$ on $\mathbf{R}^2 \backslash (C_3^n)_{2\gamma_n}$,
$\chi^4(x) = 1$ on $(C_4^n)_{\gamma_n} := ([\tilde{e}_n, \tilde{a}_n] \cup [\tilde{a}_n, a_n])_{\gamma_n}$ and $\chi^4(x) = 0$ on $\mathbf{R}^2 \backslash (C_4^n)_{2\gamma_n}$.
$\chi^5(x) = 1$ on $(C_5^n)_{\gamma_n} := \Big([c_n, \tilde{c}_n] \cup [\tilde{c}_n, \tilde{d}_n] \cup [\tilde{d}_n, d_n]\Big)_{\gamma_n}$ and $\chi^5(x) = 0$ on $\mathbf{R}^2 \backslash (C_5^n)_{2\gamma_n}$.
Let us denote by $\eta(x)$ the outward normal at each point of $\partial \tilde{K}_0^n$. The mapping $\eta : \partial \tilde{K}_0^n \mapsto \mathbf{R}^2$ is continuous except at the points $\tilde{a}_n, \tilde{b}_n, \tilde{c}_n, \tilde{d}_n$ where it admits left and righ limits we denote by $\eta(x\pm)$. Moreover, by construction we have $\gamma_n(x)\eta(x-) > 0$ and $\gamma_n(x)\eta(x+) > 0$ for all $x \in \partial \tilde{K}_0^n$.
Observe that

$$d(x - t\gamma_n(x), \partial \tilde{K}_0^n) \geq \min_{x \in \partial \tilde{K}_0^n} d(x - t\gamma_n(x), \partial \tilde{K}_0^n) := m, \quad \forall x \in \partial \tilde{K}_0^n.$$

Indeed, by a compactness argument, $m = d(x_\infty - t\gamma_n(x_\infty), \partial \tilde{K}_0^n)$ for some $x_\infty \in \tilde{K}_0^n$. Since

$$d(x_\infty - t\gamma_n(x_\infty), \partial \tilde{K}_0^n) \geq -t\gamma_n(x_\infty)\eta(x_\infty \pm) = 2bt,$$

where $2b := -\gamma_n(x_\infty)\eta(x_\infty \pm) > 0$, we finally deduce that Condition **C1** holds.

• Step 2. By virtue of Corollary 5.2 [30], there exists a unique strong solution (V^n, k^n) starting from x to the reflected s.d.e. (4.4.51) on the domain $\tilde{K}_0^n$. Let

$$\begin{aligned} \tau^n &:= \inf\{t : V_t^n(1, d) = \epsilon_n^{-1}\}, \\ \rho^n &:= \inf\{t : V_t^n(1, d) = \epsilon_n\}, \end{aligned}$$

and $\mu^n := \tau^n \wedge \rho^n$. On the intervall $[0, \mu^n]$, the process (V^n, k^n) is solution to (4.4.51) on the domain $\tilde{K}_0^{n+1}$ with respect to γ^{n+1}. Indeed, on the intervall $[0, \mu^n]$, the reflection only occurs on the boundary ∂K_0 on which γ^{n+1} and γ^n coincides with γ. By the uniqueness property given by Corollary 5.2 [30], we deduce that $(V^n, k^n) = (V^{n+1}, k^{n+1})$ on $[0, \mu^n]$. It follows that $\mu^n \leq \mu^{n+1}$. The rest of the proof is done as in [61], page 229. ■
For any compact subset C of $\mathbf{R}^2$ and for all $\epsilon > 0$, let us define

$$C_\epsilon := \bigcup_{c \in C} B_c(\epsilon).$$

Recall the well known result :

Lemma 4.4.18. *For any compact subset C of $\mathbf{R}^2$ and for all $\epsilon > 0$, there exists $\chi^\epsilon \in C^\infty(\mathbf{R}^2, [0, 1])$ such that $\chi^\epsilon = 1$ on C_ϵ and χ^ϵ vanishes outside $C_{2\epsilon}$.*

Skorokhod Problem for Pure-Jumps Lévy Processes

Let $\gamma : \partial K_0 \mapsto \mathbf{R}^2$ be a vector-valued function with $g(x) = -g_i$ on the set $(\partial K_0 \cap \partial K_i)\backslash\{0\}$ and $\gamma(0) = 0$. Recall that a Lévy process has finite activity when $\nu(\mathbf{R}^2) < \infty$. In this case, it can be represented as the sum of a compound poison process and a scaled Wiener process with drift. So, consider Y a process such that $Y_t = (Y_0^1, Y_0^2, 0) + (t, W_t, N_t)$, $t \geq 0$, where W is a standard brownian motion and N is a pure jump process of finite activity. This means that

$$N_t = \sum_{k=1}^{\tilde{N}_t} \Delta N_{T_k},$$

where ΔN_{T_k} are i.i.d. random variables and $\tilde{N}_t = \sum_k 1_{T_k \leq t}$ is a Poisson process with jump stopping times $(T_k)_k$.
Let $\sigma = \mathbf{R}^2 \mapsto \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ be a matrix-valued function assumed to be Lipschitz-continuous.
We consider the Skorokhod problem on K_0 formulated as follows : find a pair of adapted làdlàg (resp. càglàd) processes, V, starting from $x \in K_0$ and k, real-valued, starting at zero, and increasing such that

$$\begin{aligned} dV_t &= \sigma(V_{t-})dY_t + \gamma(V_t)dk_t, \\ dk_t &= I_{V_t \in K \backslash \operatorname{int} K_0} dk_t, \\ V_t^+ &\in \bar{K}_0 \, \forall t \geq 0. \end{aligned} \tag{4.4.54}$$

The aim of this subsection is to show that this r.s.d.e has a solution on the set $\bar{K}_0$.
Theorem 4.4.19. *There exists a unique solution to the Skorokhod problem (4.4.54).*
Proof. Let $(T_k)_{k \in \mathbb{N}\backslash\{0\}}$ the jump stopping times of the process Y. Assume we have already constructed a solution (V, k) to (4.4.54) on the interval $[0, T_k)$. Define

$$V_{T^k} := V_{T^k -} + \sigma(V_{T^k -})\Delta Y_{T^k}.$$

Let us set

$$V_{T^k +} = \mathbb{P}_{K_0}^{-K}(V_{T^k}) \in L^0(\mathbf{R}^2, \mathcal{F}_{T^k}) \in \partial K_0$$

where the projection operator $\mathbb{P}_{K_0}$ is defined in lemma 4.4.20. We define $\Delta^+ k_{T_k}$ by the equality

$$\Delta^+ V_{T_k} := \gamma(V_{T_k})\Delta^+ k_{T_k}.$$

Applying theorem 4.4.17 and the strong markov property, there exists a solution $(\tilde{V}, \tilde{k})$ to (4.4.54) from the starting point $\tilde{V}_0 := V_{T^k +}$ with respect to $\tilde{N}_t := N_{T^k + t} - N_{T^k}, \tilde{W}_t := W_{T^k + t} - W_{T^k}$ on the interval $[0, T^{k+1} - T^k]$. We then set $V_t := \tilde{V}_{t - T^k}$ and $k_t := k_{T_k +} + \tilde{k}_{t - T^k}$ on (T^k, T^{k+1}). The uniqueness follows from the uniqueness on each interval $[T_k, T_{k+1})$. ■

In the following, we assume that $K \subseteq \mathbf{R}^2$ is a constant cone satisfying the hypothesis of the introduction and $K_0 \subseteq K$ is a closed cone with $\partial K_0 \subseteq \operatorname{int} K$ and $\operatorname{int} K_0 \neq \emptyset$.

Lemma 4.4.20. *[Projection onto K_0 parallel to $-K$] Given $x \in K$, there exists a unique $y := \mathbb{P}_{K_0}^{-K}(x) \in K_0$ such that*

$$\|x - y\| = \min_{k \in K}\{\|x - k\| : x - k \in K\}.$$

We omit the proof which is standard. It is easily observable that the direction of $x - \mathbb{P}_{K_0}^{-K}(x)$ is given by g_2 if $x \in K_2$ and g_1 if $x \in K_1$.

4.4.5 A4 : Auxiliary Results

In the following, we present some technical results.

Lemma 4.4.21. *For every portfolio process $V = V^\pi$, $\pi \in \mathcal{A}_x$, the process $I_{V_- \in \partial K}|\Delta Y| I_{[0,\theta^\pi]}$ is indistinguishable from zero.*

Proof. Consider an arbitrary $\varepsilon > 0$,

$$\mathbb{E}\sum_{s \geq 0} I_{V_{s-} \in \partial K}|\Delta Y_s| I_{|\Delta Y_s| \geq \varepsilon} I_{s \leq \theta} = \mathbb{E}\int_{[0,\infty)\times \mathbf{R}^d} I_{V_{s-} \in \partial K}|z| I_{|z| \geq \varepsilon} I_{s \leq \theta} p(dz, ds)$$

Since the process $s \mapsto I_{V_{s-} \in \partial K}|z| I_{|z| \geq \varepsilon} I_{s \leq \theta}$ is predictable and $q(dz, dt) = \Pi(dz)dt$, we deduce that

$$\begin{aligned}\mathbb{E}\sum_{s \geq 0} I_{V_{s-} \in \partial K}|\Delta Y_s| I_{|\Delta Y_s| \geq \varepsilon} I_{s \leq \theta} &= \mathbb{E}\int_{[0,\infty)\times \mathbf{R}^d} I_{V_{s-} \in \partial K}|z| I_{|z| \geq \varepsilon} I_{s \leq \theta} \Pi(dz)dt, \\ &= \mathbb{E}\int_{[0,\infty)\times \mathbf{R}^d} I_{V_{s-} \in \partial K}|z| I_{|z| \geq \varepsilon} I_{s \leq \theta} I_{\Delta V_s \neq 0} \Pi(dz)dt.\end{aligned}$$

We recall that for $s < \theta$, $V_s \in \operatorname{int} K$. Hence, $\Delta V_s = 0$ implies that $V_{s-} \in \operatorname{int} K$. Since the Lebesgue measure does not charge any countable set, we deduce that

$$\mathbb{E}\sum_{s \geq 0} I_{V_{s-} \in \partial K}|\Delta Y_s| I_{|\Delta Y_s| \geq \varepsilon} I_{s \leq \theta} = 0.$$

Therefore, $I_{V_- \in \partial K}|\Delta Y| I_{|\Delta Y| \geq \varepsilon} I_{[0,\theta^\pi]} = 0$ a.s. and, by letting $\varepsilon \to 0$, we then conclude that $I_{V_- \in \partial K}|\Delta Y| I_{[0,\theta^\pi]} = 0$ a.s. ■

Corollary 4.4.22. *We have $V_{t-} \in \operatorname{int} K$ if $t \in [0, \theta[$.*

Proof. Suppose that $V_{t-} \in \partial K$ for some $t < \theta$. Then, by virtue of lemmas 4.4.21 and Assumption 5) of the model, we have $\Delta Y_t = \Delta B_t = \Delta C_t = 0$. Therefore, $V_t = V_{t-} \in \partial K$ hence a contradiction. ■

Note that we also deduce from lemma 4.4.21 that the portfolio process moves out of K either in a continuous manner (in the case $V_{\theta-} \in \partial K$) or due to a jump (in the case $V_{\theta-} \in \operatorname{int} K$). ■

Let us denote by $S_t^i := \mathcal{E}_t(Y^i)$ the Doleans–Dade exponential defined as

$$\mathcal{E}_t(Y^i) := e^{Y_t^i - \frac{1}{2}\langle Y^{ci}\rangle} \prod_{s\leq t}(1+\Delta Y_s^i)e^{-\Delta Y_s^i}.$$

Observe that S^i is strictly positive by assumption and $(S_t)_{t\geq 0}$ is the price process whose dynamics is given by

$$\frac{dS_t^i}{S_{t-}^i} = dY_t^i.$$

If X is a càdlàg process and B is a predictable làdlàg process of bounded variations we may define the Stieltjes integral $\int_0^t X_u dB_u$ following the definition of [43] (with some modifications given in the appendix of this chapter). We then deduce a closed form of the portfolio process expression controlled by $\pi = (B, C)$.

Lemma 4.4.23. *For every control $\pi = (B, C)$,*

$$(V_t^\pi)^i = x^i \mathcal{E}_t(Y^i) + \mathcal{E}_t(Y^i)\int_0^t \mathcal{E}_u(Y^i)^{-1} d(B_u^i - C_u^i),$$

is the unique solution of the dynamics (4.2.3).

Proof. Let us set

$$W_t^i = S_t^i x_i + S_t^i \int_0^t (S_u^i)^{-1} d(B_u^i - C_u^i).$$

The integration by part formula of theorem 4.4.7 yields

$$\begin{aligned} W_t^i &= x_i + \int_0^t x_i S_{u-}^i dY_u^i + \int_0^t S_u^i (S_u^i)^{-1} d(B_u^i - C_u^i) \\ &+ \int_0^t \left(\int_0^{u-} (S_v^i)^{-1} d(B_v^i - C_v^i) \right) S_{u-}^i dY_u^i + \sum_{u\leq t} S_{u-}^i \Delta Y_u^i (S_{u-}^i)^{-1} \Delta(B^i - C^i)_u. \end{aligned}$$

We then deduce that

$$W_t^i = x_i + \int_0^t W_{u-}^i dY_u^i + B_u^i - C_u^i.$$

Let V be an other process verifying (4.2.3). Then, $U = W - V$ satisfies the s.d.e. $dU_t^i = U_{t-}^i dY_t^i$, $U_{0-} = 0$, hence $U = 0$. ■

Chapitre 5

General Financial Market Model Defined by a Liquidation Value Process

Abstract

We present a general financial market model defined by a liquidation value process. This approach generalizes the conic models of Schachermayer and Kabanov where the transaction costs are proportional to the exchanged volumes of traded assets. This allows to consider financial market models where proportional transaction costs and fixed costs coexist. In this case, the solvency set of all portfolio positions that can be liquidated without any debt is not necessary convex. Therefore, the usual duality principle based on the Hahn-Banach separation theorem is not appropriate to characterize the prices super hedging a contingent claim. We propose an alternative method to price European or American contingent claims under absence of arbitrage opportunity of the second kind.

Keywords : Financial markets, Liquidation value, Transaction costs, European and American options, Hedging, Partial order.

Note. This chapter is based on the article *General Financial Market Model Defined by a Liquidation Value Process,* T. Tran, E. Lépinette, Submitted in *Journal of Stochastics* .

5.1 Introduction

Since the pioneering work of Jouini and Kallal [52], a lot has been done in the direction of financial market models with proportional transaction costs. On the one hand, the Schachermayer model [92] focuses on markets where prices are defined by bid-ask spreads, which is a generalization of [52]. The main theorems in the literature [52, 92, 45] are formulated for models with only one risky asset. They show the equivalence between absence of

arbitrage opportunity and existence of a martingale evolving in the bid-ask spread. They may be interpreted as generalizations of the well-known Dalang-Morton-Willinger theorem for frictionless market models defined by a unique price process S on a time horizon $[0,T]$ [24]. It asserts that there is no arbitrage opportunity if and only if there exists a deflator (a strictly positive martingale) $(\rho_t)_{t\in[0,T]}$ such that ρS is a martingale or, equivalently, S is a martingale under the equivalent probability measure Q such that $dQ/dP = \rho_T$.

On the other hand, the Kabanov model is oriented towards currency markets where any asset can be exchanged directly to any other one. A first approach was suggested in [56] (inspired by the paper [19] where a two-asset diffusion model is considered) : one should think in terms of the martingale density rather than of the measure and the natural analog of the latter (called in [92] a consistent price system) is a martingale whose values are in the dual of the *solvency cone*. The random solvency cone is the set of all portfolio positions one can change, paying transaction costs, into new ones with nonnegative positions in every asset. Most of the results in the literature are obtained under the condition that the solvency cone G is proper, i.e. $G\cap(-G) = \{0\}$, which means that at any instant there exists at least one strictly positive transaction costs coefficient. This ensures that the interior of the positive dual of the solvency set is nonempty, see [72] for more details and the general case for discrete-time models.

Abstract models are also considered with general random cones. A strong form of absence of arbitrage opportunity is required to deduce a dual characterization of prices super hedging a European or American claim, see [55], [15],[7] and [31]. This is equivalent to the existence of a strictly consistent price system, i.e. a martingale evolving in the interior of the positive dual of the solvency cone. As shown in [45], this corresponds to a robust absence of arbitrage opportunity, i.e. there is still no arbitrage opportunity for smaller transaction cost coefficients. This concept is extensively studied by Schachermayer [46] for small transaction costs. Moreover, in Proposition 9 [15], a new condition which is called (B) in [72] is considered meaning that the set of consistent price systems is rich enough. As shown by Rásonyi for discrete-time models, this condition is equivalent to the absence of arbitrage opportunity of second kind [91] and is studied in further papers [10] and [57] for continuous-time settings. Under Condition (B), it is possible to characterize the set of all *minimal* portfolio processes super hedging a European claim with respect to the random preorder induced by the random solvency cone [58], [59].

For all the models mentioned above, studying absence of arbitrage opportunity allows to exhibit dual elements, precisely either deflators for frictionless models or consistent price systems for Kabanov's models, which characterize the set of all super hedging prices of European or American options. The technique we use to construct such dual elements is very standard. Indeed, since the models are convex in the sense that the terminal values R_T of portfolio processes is a convex set, it is sufficient to show that R_T is closed under

absence of arbitrage opportunity so that we may apply the Hahn-Banach separation theorem. This is the well known Kreps-Yan theorem in the case of conic market models, see for instance [73] and [11]. Even if the solvency set is only convex but not conic, it is also possible to formulate results on absence of arbitrage opportunity as well as a characterization of super hedging prices as proposed in [85].

In presence of both fixed and proportional transaction costs, for instance when the prices are defined by bid-ask spreads, the solvency set is not necessarily convex. A first attempt to formally study models with fixed costs is proposed in [53] and [54] where the absence of arbitrage opportunity appears to be equivalent to the existence of an absolutely continuous martingale probability measure, at least for a finite probability space. To do so, the authors equivalently reformulate the absence of arbitrage opportunity in the model with friction in a new type of absence of arbitrage opportunity but for the associated frictionless model so that convexity may be used. It is quite reasonable to study such models including fixed costs as observed a long time ago by Brennan [14] since such models capture the basic features of the commission structure of many stock exchanges which involve a stated amount plus a declining percentage of the value of the transaction [53]. We send readers to [53] for a rich list of references on the subject.

In this chapter, we unify these different approaches by proposing a model defined by a liquidation value process in a discrete-time setting. This is very natural. Indeed, suppose that a discrete-time stochastic basis $(\Omega, (\mathcal{F}_t)_{t=0,\cdots,T}, P)$ is given and a portfolio process is a predictable vector-valued strategy V expressed in physical units, i.e. the quantities of assets an agent holds. The very question is how can we change the portfolio position V_{t-1} into V_t at time t ? The answer is immediate : write $V_{t-1} = V_t + (V_t - V_{t-1})$ and liquidate without any debt the remaining part $V_t - V_{t-1}$. This means that $V_t - V_{t-1}$ belongs to the solvency set. Equivalently, $\mathbf{L}_t(V_t - V_{t-1}) \geq 0$ where $\mathbf{L}_t(x)$ is the liquidation value function at time t of the position x, i.e. the maximal amount of cash we obtain when we liquidate all the risky positions. In this chapter, we only consider discrete-time models for the sake of simplicity. In Section 4.2, we introduce a financial market model defined by a liquidation value process. We consider a two-dimensional example defined by a riskless bond and a risky asset given by bid and ask prices in presence of fixed costs. In Section 4.3, we study the absence of arbitrage opportunity of the second kind as introduced by [91] under which we formulate, in Section 4.4, a characterization of super hedging prices for European and American options, by means of the essential supremum and maximum notions. Most of the proofs are presented in Appendix.

Notations.

$e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^d$, $d \geq 1$.

For some subset $G \subseteq \mathbf{R}^d, x \succeq^G y \Leftrightarrow x - y \in G$.

For random sets $(G_t)_{0 \leq t \leq T}, G_t \subseteq \mathbf{R}^d, x \succeq^t y \Leftrightarrow x - y \in G_t$.

$\mathbf{R}^d_+$ (resp. $\mathbf{R}^d_{++}$) is the set of vectors in $\mathbf{R}^d$ having positive (resp. positive and non null) components.

$\mathbb{E}$ designates the expectation of a random variable.

$L^0(E,\mathcal{F})$ is the metric space of all E-valued random variables which are $\mathcal{F}$-measurable.

$L^\infty(E,\mathcal{F})$ is the normed space of all E-valued random variables which are $\mathcal{F}$-measurable and bounded.

5.2 Model and basic properties

Let us consider a probability space $(\Omega,\mathcal{F},\mathrm{P})$ and let $(\mathcal{F}_t)_{t=0,\cdots,T}$ be a discrete-time filtration. Let $(U_t(x))_{t=0,\cdots,T}$ be a Caratheodory function defined on the product $\Omega\times\{0,\cdots,T\}\times\mathbf{R}^d$, i.e. a function which satisfies the following properties :

a) For each ω, P-a.s., and every $t=0,\cdots,T$, $U_t(\omega,\cdot)$ is continuous on $\mathbf{R}^d$,
b) For each $(t,x)\in\{0,\cdots,T\}\times\mathbf{R}^d$, $U_t(\cdot,x)\in L^0(\mathbf{R},\mathcal{F}_t)$,
c) $U_t(0)\geq 0$ a.s. for all $t=0,\cdots,T$,
d) For all $t=0,\cdots,T$, the property $(U_t(x)\geq 0$ and $U_t(y)\geq 0)$ for some $x,y\in\mathbf{R}^d$ implies that $U_t(x+y)\geq 0$ holds a.s.

Remark 5.2.1. Observe that we may replace U by $\tilde{U}$ defined by

$$\tilde{U}_t(\omega,x):=\frac{U_t(\omega,x)}{1+|U_t(\omega,x)|},$$

which satisfies $|\tilde{U}|\leq 1$.

We suppose that the portfolio processes we consider are expressed in physical units, i.e. the number of assets an agent holds. Moreover, we suppose that the first component of such portfolios corresponds to a cash account. Precisely, we assume without loss of generality that the bond is $S^1=1$ so that the first component of a portfolio position is an amount of cash.

Definition 5.2.2. *A self-financing portfolio process $(V_t)_{t=0,\cdots,T}$ is an $(\mathcal{F}_t)_{t=0,\cdots,T}$-adapted process such that*

$$U_t(V_{t-1}-V_t)\geq 0,\quad \forall t=0,\cdots,T\quad a.s. \tag{5.2.1}$$

Let us comment this definition. At time t, when we re-balance a portfolio process $(V_t)_{t=0,\cdots,T}$, we change the position V_{t-1} into V_t. To do so, we split the position V_{t-1} into two portfolio positions as follows : $V_{t-1}=V_t+(V_{t-1}-V_t)$. We need to liquidate without any debt the part $V_{t-1}-V_t$. By definition of our model, this means that $U_t(V_{t-1}-V_t)\geq 0$. Such a portfolio is said self-financing since no extra wealth is added for the trading of V. Observe that the binary relation $x\succeq^t y$ if and only if $U_t(x-y)\geq 0$ is a preference relation so that (5.2.1) reads as $V_{t-1}\succeq^t V_t$.

Let $(U_t(x))_{t=0,\cdots,T}$ be a Caratheodory function and let us define the solvency set

$$G_t(\omega):=\{x\in\mathbf{R}^d:\,U_t(x)\geq 0\}.$$

Observe that G is an adapted random set in the sense that its graph at time t, i.e.

$$\operatorname{Gr} G_t := \{(\omega, x) : \ x \in G_t(\omega)\},$$

belongs to $\mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d)$ where $\mathcal{B}(\mathbf{R}^d)$ is the Borel σ-algebra on $\mathbf{R}^d$. This is a closed set containing 0 and such that $G_t + G_t \subseteq G_t$ a.s. This set corresponds to the set of all solvent positions at time t. Conversely, suppose that at each instant t we are given a random $\mathcal{F}_t$-adapted closed set G_t containing 0 and such that $G_t + G_t \subseteq G_t$ a.s. We interpret G_t as the solvency set at time t. Let us define $U_t(x) = -d(x, G_t)$ where d is the Euclidean distance on $\mathbf{R}^d$. We observe that $U_t(x) \geq 0$ if and only if $U_t(x) = 0$, i.e. if $x \in G_t$. Obviously, $(U_t(x))_{t=0,\cdots,T}$ is a Caratheodory function. It follows that Caratheodory functions and solvency sets equivalently define the financial market model we suggest. Many situations may be described by such a solvency set and we may restrict G_t to $\mathbf{N}^d$ so that the model is realistic, i.e. the number of stocks an agent holds is integer-valued provided that $V_{0-} \in \mathbf{N}^d$.

Note that the solvency sets $(G_t)_{t\geq 0}$ are not necessarily convex cones in general. Contrary to the classical financial market models with proportional transaction costs, the model is not convex hence it is not possible a priori to formulate absence of arbitrage opportunity thanks to the Hahn–Banach separation theorem as usual. Consequently, it is not possible to derive a dual characterization of all super hedging prices of European or American claims except if we consider the artificial enlarged market defined by the random cone $(K_t)_{t\geq 0}$ where K is given below by (6.1.2).

In the present chapter, we restrict our attention to models satisfying the following conditions. They are motivated by markets with both fixed and proportional transaction costs.

Condition $\mathbf{G}^0$:

(i) G_t is a closed $\mathcal{F}_t$-adapted set, $0 \leq t \leq T$,

(ii) $G_t + G_t \subseteq G_t$, $0 \leq t \leq T$, a.s.,

(iii) $\lambda G_t \subseteq G_t$, $\forall \lambda \geq 1$, a.s.,

(iv) $G_t + \mathbf{R}^d_+ = G_t$, a.s.,

(v) The cone $\overline{\mathbf{R}_+ G_t}$ is proper , i.e. $\overline{\mathbf{R}_+ G_t} \cap (-\overline{\mathbf{R}_+ G_t}) = \{0\}$ a.s.

Remark 5.2.3. Under the conditions above, we easily show the following equalities

$$\operatorname{conv} G_t = \operatorname{cone} G_t = \mathbf{R}_+ G_t =: K_t, \tag{5.2.2}$$

where $\operatorname{conv} G_t$ and $\operatorname{cone} G_t$ respectively designate the convex and the conic hull of G_t. In particular, if G_t is convex, then $G_t = K_t$. The enlarged market given by the solvency set $(K_t)_{t\geq 0}$ is therefore the minimal conic market model that dominates $(G_t)_{t\geq 0}$. For instance, if $(G_t)_{t\geq 0}$ models a financial market with both fixed and proportional transaction costs, then $(K_t)_{t\geq 0}$ defines the market model with only proportional transaction costs.

Since we interpret the closed random set G_t as the set of all solvent positions at time t, it is natural to introduce a liquidation value process $\mathbf{L}_t$ such that $G_t = \{x \in \mathbf{R}^d : \mathbf{L}_t(x) \geq 0\}$. Very naturally, if x is a portfolio position, then $\mathbf{L}_t(x)$ is the largest amount of cash we get at time t, when liquidating the risky assets contained in the portfolio x. The new portfolio position after liquidation is therefore $\mathbf{L}_t(x)e_1$. The liquidation value process $\mathbf{L}_t$ defined below satisfies the same property than U_t except that $\mathbf{L}_t$ is not necessarily continuous.

Definition 5.2.4. *Consider a random set G satisfying Condition $\mathbf{G}^0$. The liquidation value process associated to the process G is*

$$\mathbf{L}_t^G(z) := \sup\{\alpha \in \mathbf{R} : \ z - \alpha e_1 \in G_t\}. \tag{5.2.3}$$

Recall that G_t is assumed to be closed. Therefore, under Condition $\mathbf{G}^0$, $\mathbf{L}_t := \mathbf{L}_t^G$ can be rewritten as follows :

$$\mathbf{L}_t(z) := \max\{\alpha \in \mathbf{R} : \ z - \alpha e_1 \in G_t\}.$$

It is clear that $\mathbf{L}_t(z) \geq 0$, $\forall z \in G_t$. Reciprocally, if $\mathbf{L}_t(z) \geq 0$, this means that $z = \alpha e_1 + g_t \in G_t$ where $\alpha \geq 0$ and $g_t \in G_t$. This implies that G_t is represented by $\mathbf{L}_t$, i.e. $G_t(\omega) = \{x \in \mathbf{R}^d : \ \mathbf{L}_t(x) \geq 0\}$.

Definition 5.2.5. *We say that the random preorders $\succeq^t$ are represented by the liquidation value process $\mathbf{L}$ if the set of solvent positions at time t is given by $G_t(\omega) = \{x \in \mathbf{R}^d : \ \mathbf{L}_t(x) \geq 0\}$.*

We sumarize in the following proposition some basic properties of the liquidation value process.

Proposition 5.2.6. *Suppose that Condition $\mathbf{G}^0$ holds and the liquidation value process $\mathbf{L}$ is defined by (6.1.3). Then,*

(i) $\mathbf{L}_t$ satisfies the followings properties

$$\begin{aligned}
&\mathbf{L}_t(0) = 0,\\
&\mathbf{L}_t(x+y) \geq \mathbf{L}_t(x) + \mathbf{L}_t(y), \quad x, y \in \mathbf{R}^d,\\
&\mathbf{L}_t(x + \lambda e_1) = \mathbf{L}_t(x) + \lambda, \quad \lambda \in \mathbf{R},\, x \in \mathbf{R}^d,\\
&\mathbf{L}_t(\lambda x) \geq \lambda \mathbf{L}_t(x),\, \forall \lambda \geq 1.
\end{aligned}$$

(ii) $\mathbf{L}_t(x) < \infty$, $\forall x \in \mathbf{R}^d$. If G_t dominates $\mathbf{R}^d_+$, i.e. $\mathbf{R}^d_+ \setminus \{0\} \subseteq \operatorname{int} G_t$, then $\mathbf{L}_t(x) > -\infty$, $\forall x \in \mathbf{R}^d$.

(iii) $\{\mathbf{L}_t = 0\} \subseteq \partial G_t$. Consequently, $x - \mathbf{L}_t(x)e_1 \in \partial G_t, \quad x \in \mathbf{R}^d$.

(iv) $\mathbf{L}_t$ is upper-semicontinuous for all t.

(v) In the case where G is a convex cone dominating $\mathbf{R}^d_+$, the liquidation value function $\mathbf{L}_t$ is homogeneous, concave and continuous on $\mathbf{R}^d$. We

also have

$$\partial G_t = \{x \in \mathbf{R}^d : \mathbf{L}_t(x) = 0\}.$$

(vi) Suppose that the liquidation function $\mathbf{L}_t^K$ of K is finite on $\mathbf{R}^d$, then the concave hull function of $\mathbf{L}_t^G$ coincides $\mathbf{L}_t^K$ for all $t = 0, \cdots, T$.

See the proof in Appendix.

Corollary 5.2.7. *Suppose that* **Condition $\mathbf{G}^0$** *holds and the liquidation value process $\mathbf{L}$ is defined by (6.1.3). Let $\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Then, the mapping $\omega \mapsto \mathbf{L}_t(\omega, \gamma(\omega))$ is $\mathcal{F}_t$ measurable.*

See the proof in Appendix. The binary relation $x \succeq^{\omega,t} y$ iff $\mathbf{L}_t(x - y) \geq 0$ defines a preorder as soon as $\mathbf{L}_t$ satisfies Condition (d). This is the case when $\mathbf{L}_t$ is super additive, i.e. if $\mathbf{L}_t(x+y) \geq \mathbf{L}_t(x) + \mathbf{L}_t(y)$ for every $x, y \in \mathbf{R}^d$ a.s. Note that the liquidation value process given by (6.1.3) is super additive and upper semi-continuous. Reciprocally, observe that, if $\mathbf{L}_t$ is an upper semi-continuous and super additive function, then the set $G_t := \{\mathbf{L}_t \geq 0\}$ is closed and stable under addition.

Example 5.2.8. *Consider the following example in $\mathbf{R}^2$. Suppose that the financial market model consists of two underlying assets. The first asset is a cash account such that the bond is given by $S^1 = 1$ on $[0, T]$. The second asset is risky and modeled by a bid (sell) price S^b and an ask (buy) price S^a such that the bid-ask spread is given by $[S^b; S^a]$. Moreover, we suppose that there is a fixed cost for each transaction at time t we denote by $c_t > 0$. Besides, when the portfolio position is (x, y) such that $y \geq 0$, we suppose that the agent is rational enough not to deliberately sell the stock when the bid-price is too low to compensate for the fixed cost, i.e. when, at time t, $yS_t^b - c_t \leq 0$.*

Let us characterize $(x, y) \in G_t$. If $y \geq 0$, you sell the risky assets of your portfolio so that the position (x, y) is solvent meaning that $\max(x + S_t^b y - c_t, x) \geq 0$. If $y < 0$, you buy $|y|$ units of risky assets hence $(x, y) \in G_t$ when $x + S_t^a y - c_t$ is positive. We deduce that G_t is the intersection of two half-planes plus the first orthant $\mathbf{R}_+^2$ hence G_t is closed. We easily check that $G_t + G_t \subseteq G_t$ and $G_t \cap (-G_t) = \{0\}$. At last, observe that $\lambda G_t \subseteq G_t$ for all $\lambda \geq 1$. Note that $\overline{K}_t := \overline{\mathbf{R}_+ G_t}$ coincides with the solvency cone in the Kabanov model [72], i.e. such that $c = 0$. It is proper as soon as $S^b < S^a$.

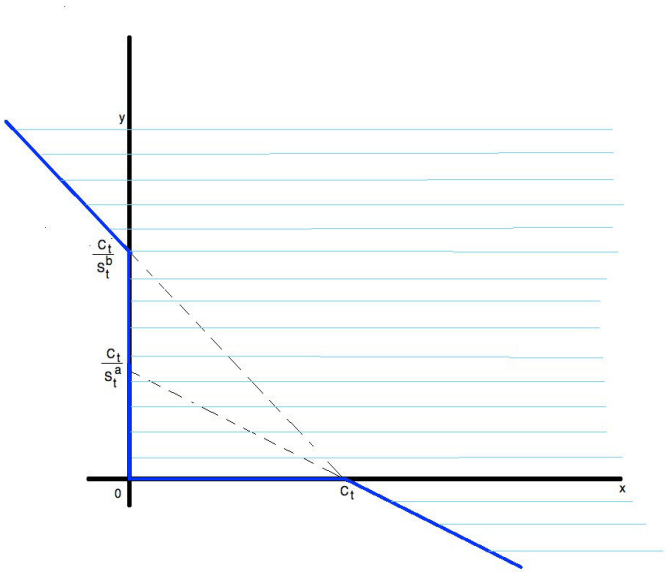

FIGURE 5.1 – The stripy domain corresponds to the set G_t of solvent points.

For such a model, the preorder $\geq^t$ is represented by the liquidation function

$$\mathbf{L}_t((x,y)) := \max(x, x + S_t^b y - c_t)1_{y\geq 0} + (x + S_t^a y - c_t)1_{y<0}.$$

We may easily check that $\mathbf{L}_t$ is a.s. continuous except the points of the form $(x,0)$ where $\mathbf{L}$ is only upper semi-continuous. Moreover, $\mathbf{L}_t$ is upper additive.

5.3 Absence of arbitrage opportunity

Arbitrage theory for financial markets with transaction costs is mainly considered under the multidimensional approach which consists in expressing the portfolio processes as well as the contingent claims in physical units [72]. In practice, agents acting on the market are essentially interested in the liquidation values of such portfolio processes. This is the approach we adopt here when considering arbitrage opportunities, which is also coherent with the arbitrage theory for frictionless market models.

In the following, we consider a financial market model defined by a random solvency set G satisfying Condition $\mathbf{G}^0$ or equivalently defined by the associated liquidation value process given by (6.1.3) which satisfies Proposition 5.2.6. In this case, a portfolio process is just an adapted process $(V_t)_{t=0,\cdots,T}$ satisfying $\Delta V_t := V_t - V_{t-1} \in -G_t$.

5.3.1 NA condition

Definition 5.3.1. *An adapted portfolio process $(V_u)_{u\geq t}$, $t \in \{0,\cdots,T\}$, realizes an arbitrage opportunity of the first kind if $V_T \geq^T 0$ and $\mathrm{P}(\mathbf{L}_T(V_T) >$*

$0) > 0$. *We say that* **NA** *Condition holds when there is no arbitrage opportunity of the first kind.*

We denote by R_T^0 the set of all terminal values V_T of portfolio processes V such that $V_{-1} = 0$. i.e.

$$\mathrm{R}_T^0 := \mathrm{R}_T^0(G) := \sum_{0 \le t \le T} L^0(-G_t, \mathcal{F}_t).$$

We also define

$$\mathrm{LV}_T^0 := \mathrm{LV}_T^0(G) := \{\mathbf{L}_T(V_T) : \ V_T \in \mathrm{R}_T^0\},$$

and more generally, we also define $\mathrm{R}_T^u := \sum_{u \le t \le T} L^0(-G_t, \mathcal{F}_t)$ and, similarly, LV_T^u. Observe that $L^0(-\mathbf{R}_+, \mathcal{F}_T) \subseteq \mathrm{LV}_T^0$. Let us introduce the convex cone

$$\mathrm{A}_T^0 := \{\sum_{i=1}^{n} \lambda_i X_i : \ \lambda_i \ge 0, \ X_i \in \mathrm{LV}_T^0\}.$$

Lemma 5.3.2.

$$\mathrm{A}_T^0 := \{\lambda X : \ \lambda \in [0,1], \ X \in \mathrm{LV}_T^0\}.$$

See the proof in Appendix. The following result shows that the NA condition may be reformulated as an absence of arbitrage opportunity for the convex model defined by the terminal values A_T^0. This could be an idea to obtain a characterization of the NA condition we leave for future research in a second paper on non convex market models.

Corollary 5.3.3.

$$\mathrm{NA} \Leftrightarrow \mathrm{A}_T^0 \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\} \Leftrightarrow \mathrm{A}_T^\infty \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$$

where $\mathrm{A}_T^\infty := \mathrm{A}_T^0 \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$.

The following lemma exhibits the relationship between Condition **NA** under the liquidation value approach and the condition $\mathbf{NA^w}$ [72] under its associated multidimensional approach.

Lemma 5.3.4. *If G_T dominates $\mathbf{R}_+^d$, i.e $\mathbf{R}_{++}^d \subseteq \operatorname{int} G_T$, then we have*

$$\mathrm{R}_T^0 \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\} \Leftrightarrow \mathrm{LV}_T^0 \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}.$$

Proof. ($\Rightarrow$) Let $\mathbf{L}_T(V_T) \in \mathrm{LV}_T^0 \cap L^0(\mathbf{R}_+, \mathcal{F}_T)$. We then have

$$\mathbf{L}_T(V_T)e_1 \in \mathrm{R}_T^0 \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\},$$

which implies that $\mathbf{L}_T(V_T) = 0$.

($\Leftarrow$) Let $V_T \in \mathrm{R}_T^0 \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T)$. Recall that G_T dominates $\mathbf{R}_+^d$. Hence, $\mathbf{L}_T \ge 0$ on $\mathbf{R}_+^d$ and, on $\mathbf{R}_+^d$, $\mathbf{L}_T(x) = 0$ if and only if $x = 0$. Therefore, if $V_T \in \mathbf{R}_{++}^d$ then $\mathbf{L}_T(V_T) \in \mathbf{R}_{++}$, which yields a contradiction. ■

Remark 5.3.5. The previous lemma states that, in the case where G_T dominates $\mathbf{R}^d_+$, Condition **NA** defined under the liquidation value viewpoint is equivalent to the classical notion $\mathbf{NA^w}$. Unfortunately, the requirement that G_T dominates $\mathbf{R}^d_+$ does not hold in general. Indeed, consider models with both fixed and proportional transaction costs (see Example 5.2.8). In this case, $\mathbf{L}_T(V_T) = 0$ even if $V_T \in \mathbf{R}^d_{++}$. In general, we only have the implication

$$\mathrm{R}^0_T \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\} \Rightarrow \mathrm{LV}^0_T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}.$$

In case of Example 5.2.8, observe that **NA** is equivalent to

$$\mathrm{R}^0_T \cap L^0(\mathbf{R}^2_+, \mathcal{F}_T) \subseteq L^0(\{0\} \times [0, \frac{c_T}{S^b_T}]). \tag{5.3.4}$$

The proof is obvious. Indeed, we first note that **NA** is equivalent to

$$\mathrm{R}^0_T \cap L^0(G_T, \mathcal{F}_T) \subseteq \{\mathbf{L}_T = 0\}.$$

Therefore, if **NA** holds and $X \in \mathrm{R}^0_T \cap L^0(\mathbf{R}^2_+, \mathcal{F}_T)$ we have

$$X \in \{\mathbf{L}_T = 0\} \cap L^0(\mathbf{R}^2_+, \mathcal{F}_T) = L^0(\{0\} \times [0, \frac{c_T}{S^b_T}]),$$

i.e. 5.3.4 holds. Conversely, if **NA** does not hold, there exists $X \in \mathrm{R}^0_T$ such that $\mathbf{L}_T(X) \geq 0$ and $\mathbf{L}_T(X) \neq 0$. Hence $\mathbf{L}_T(X)e_1 \in \mathrm{R}^0_T \cap L^0(\mathbf{R}^2_+, \mathcal{F}_T)$, a contradiction to 5.3.4.

Proposition 5.3.6. NA *holds if and only if for all* $V_T \in \mathrm{R}^0_T$*, there exists* $Z = Z^V \in L^\infty((0,\infty), \mathcal{F}_T)$ *such that* $\mathbb{E}Z\mathbf{L}_T(V_T) \leq 0$.

Proof. Suppose that $\mathbf{L}_T(V_T) \geq 0$. As $\mathbb{E}Z\mathbf{L}_T(V_T) \leq 0$, we get that $Z\mathbf{L}_T(V_T) = 0$ and $\mathbf{L}_T(V_T) = 0$, i.e. NA holds. Reciprocally, suppose that NA holds and consider w.l.o.g. that $\mathbf{L}_T(V_T) \neq 0$. Then, $\mathrm{P}(\{\mathbf{L}_T(V_T) < 0\}) > 0$. Indeed, otherwise, NA implies $\mathbf{L}_T(V_T) = 0$. To conclude, it is then sufficient to choose

$$Z := e^{-|\mathbf{L}_T(V_T)|}(\alpha 1_{\mathbf{L}_T(V_T)<0} + 1_{\mathbf{L}_T(V_T)\geq 0}),$$

where

$$\alpha > -\frac{\mathbb{E}\mathbf{L}_T(V_T)e^{-|\mathbf{L}_T(V_T)|}1_{\mathbf{L}_T(V_T)\geq 0}}{\mathbb{E}\mathbf{L}_T(V_T)e^{-|\mathbf{L}_T(V_T)|}1_{\mathbf{L}_T(V_T)<0}}.$$

■

5.3.2 NA2 condition

Definition 5.3.7. *The market defined by* G *satisfies the condition* **NA2** *if for all* $t \in [0,T]$*, for all initial endowment* $V_{t-1} \in \mathcal{F}_{t-1}$ *and all* $W_T \in \mathrm{R}^t_T$ *such that* $\mathbf{L}_T(V_{t-1} + W_T) \in L^\infty(\mathbf{R}_+, \mathcal{F}_T)$*, we have* $V_{t-1} \in G_{t-1}$ *a.s.*

Definition 5.3.8. *Given a family of sets of adapted processes $\Lambda = (\Lambda_t)_{0\leq t\leq T}$ such that $\Lambda_t \subseteq L^0(\mathbf{R}_+, \mathcal{F}_t)$, we say that a real-valued adapted process $(M_t)_{0\leq t\leq T}$ is a Λ-super martingale if, for all t, there exists $Z_{t+1} \in \Lambda_{t+1}$ such that*

$$\mathbb{E}(Z_{t+1}M_{t+1}|\mathcal{F}_t) \leq M_t.$$

Let $G_t^* := \{y \in \mathbf{R}^d : \ yx \geq 0, \forall x \in G_t\}$ be the positive dual of G_t. The following result is shown in Appendix.

Theorem 5.3.9. *In the conic Kabanov model,* **NA**2 *holds if and only if for all adapted portfolio process $(V_u)_{u\geq t}$, $t \in \{0, \cdots, T\}$, $(\mathbf{L}_u(V_u))_{u\geq t}$ is a Λ-super martingale where*

$$\Lambda_t = \{Z_t e_1 : \ Z_t \in L^0(G_t^*, \mathcal{F}_t)\}.$$

This theorem can be generalized to the case of non-conic models under mild conditions.

Theorem 5.3.10. *Suppose that the random set of solvable positions is given by $G_t := \{\mathbf{L}_t \geq 0\}$ where $\mathbf{L}$ is the liquidation value process which is super additive. Then,* **NA**2 *holds if and only if for all adapted portfolio process $(V_u)_{u\geq t}$, $t \in \{0, \cdots, T\}$, $(\mathbf{L}_u(V_u))_{u\geq t}$ is a Λ-super martingale for some Λ.*

Proof. The reverse direction of the equivalence is shown as in the previous theorem. Let us now suppose that **NA**2 holds and consider a portfolio process $(V_u)_{u\geq t}$. Let us define

$$Z_{u+1} := \frac{\mathbf{L}_u(V_u)}{\mathbf{L}_{u+1}(V_u)} 1_{\mathbf{L}_{u+1}(V_u)/\mathbf{L}_u(V_u)>0}.$$

Then, as $\mathbf{L}_{u+1}(V_u) \geq \mathbf{L}_{u+1}(V_{u+1}) + \mathbf{L}_{u+1}(V_u - V_{u+1}) \geq \mathbf{L}_{u+1}(V_{u+1})$, we deduce that

$$\begin{aligned} \mathbb{E}(Z_{u+1}\mathbf{L}_{u+1}(V_{u+1})|\mathcal{F}_u) &\leq \mathbb{E}(Z_{u+1}\mathbf{L}_{u+1}(V_u)|\mathcal{F}_u), \\ &\leq \mathbf{L}_u(V_u)\mathbb{E}(1_{\mathbf{L}_{u+1}(V_u)/\mathbf{L}_u(V_u)>0}|\mathcal{F}_u). \end{aligned}$$

It remains to show that

$$0 \leq \mathbf{L}_u(V_u)\mathbb{E}(1_{\mathbf{L}_{u+1}(V_u)/\mathbf{L}_u(V_u)\leq 0}|\mathcal{F}_u).$$

To do so, it is sufficient to prove that $\mathbb{E}(1_{\mathbf{L}_{u+1}(V_u)\geq 0}1_{\mathbf{L}_u(V_u)<0}|\mathcal{F}_u) = 0$. In the contrary case, there exists $F_u \in \mathcal{F}_u$ such that $1_{\mathbf{L}_{u+1}(V_u)\geq 0}1_{\mathbf{L}_u(V_u)<0} > 0$ on F_u. It follows that the portfolio process $(\tilde{V}_r)_{r=u,u+1}$ such that $\tilde{V}_{u-} = \tilde{V}_u$ and $\tilde{V}_{u+1} = \tilde{V}_u = V_u 1_{F_u}$ satisfies $\mathbf{L}_{u+1}(\tilde{V}_{u+1}) \geq 0$ while $\mathbf{L}_u(\tilde{V}_u) < 0$ on F_u, i.e. there is an arbitrage opportunity of the second kind hence a contradiction. ■

Corollary 5.3.11. *Suppose that the random set of solvent positions is given by $G_t := \{\mathbf{L}_t \geq 0\}$ where $\mathbf{L}$ is the liquidation value process which is super additive. Then, the following statements are equivalent :*

(i) **NA**2 *holds,*

(ii) For all t, $L^0(G_{t+1}, \mathcal{F}_t) \subseteq L^0(G_t, \mathcal{F}_t)$,
(iii) For every portfolio process and for every $t = 0, \cdots, T-1$,

$$\mathbf{L}_t(V_t) E\left(\frac{\mathbf{L}_{t+1}(V_{t+1})}{\mathbf{L}_{t+1}(V_t)} 1_{\frac{\mathbf{L}_{t+1}(V_t)}{\mathbf{L}_t(V_t)} > 0} - 1 | \mathcal{F}_t \right) \leq 0.$$

Proof. The implication (i)⇒ (ii) is trivial. Condition (iii) means that $V_t \geq \mathbb{E}(Z_{t+1} V_{t+1} | \mathcal{F}_t)$ where $Z_{t+1} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, which implies (i) by the previous theorem. The implication (ii)⇒ (iii) can be shown by following the proof of the last theorem since $\mathbf{L}_{u+1}(\tilde{V}_{u+1}) \geq 0$ means that $\tilde{V}_{u+1} \geq^{u+1} 0$ hence, by (ii), $\tilde{V}_u = V_u 1_{F_u} \geq^u 0$, i.e. a contradiction. ■

In Example 5.2.8, suppose that there is $Q \sim \mathrm{P}$ such that S^b and S^a are respectively Q-supermartingale and Q-submartingale. Then, if c is constant, the $NA2$ condition $L^0(G_{t+1}, \mathcal{F}_t) \subseteq L^0(G_t, \mathcal{F}_t)$ holds. To show it, let us first observe that for all t, $G_t = \mathbb{T}_t \cup (\overline{K}_t + ce_1)$ where $\mathbb{T}_t \subseteq \mathbf{R}^d_+$ and $\overline{K}_t$ is the solvency cone of the Kabanov model i.e. the same model but with $c = 0$. Consider $\gamma_t := (x_t, y_t) \in L^0(G_{t+1}, \mathcal{F}_t)$. Since $\mathbf{R}^d_+ \subseteq G_t$, we have $\gamma_t 1_{\gamma_t \in \mathbf{R}^d_+} \in G_t$ a.s. Otherwise, $(\gamma_t - ce_1) 1_{\gamma_t \notin \mathbf{R}^d_+} \in \overline{K}_{t+1}$, i.e.

$$\left(x_t - c + y_t S^b_{t+1} 1_{y_t \geq 0} + y_t S^a_{t+1} 1_{y_t < 0} \right) 1_{\gamma_t \notin \mathbf{R}^d_+} \geq 0.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ under Q and using the assumptions, we get that

$$\left(x_t - c + y_t S^b_t 1_{y_t \geq 0} + y_t S^a_t 1_{y_t < 0} \right) 1_{\gamma_t \notin \mathbf{R}^d_+} \geq 0,$$

i.e. $(\gamma_t - ce_1) 1_{\gamma_t \notin \mathbf{R}^d_+} \in \overline{K}_t 1_{\gamma_t \notin \mathbf{R}^d_+}$ hence $\gamma_t 1_{\gamma_t \notin \mathbf{R}^d_+} \in (\overline{K}_t + ce_1) 1_{\gamma_t \notin \mathbf{R}^d_+} \subseteq G_t$. The conclusion follows.

5.4 Super hedging prices

In the model we consider, recall that a stochastic basis $(\Omega, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathrm{P})$ is given. We consider a liquidation value process $\mathbf{L}$ such that the associated random set $G_t := \{z : \mathbf{L}_t(z) \geq 0\}$ is closed. We suppose that, for each t, $\mathbf{L}_t$ is upper additive and upper semi-continuous.

When we suppose that U_t is continuous (e.g. $U_t(x) = -d(x, G_t)$), we deduce that the associated preorder defined by $x \succeq^t y$ if $U_t(x - y) \geq 0$ is continuous hence admits a continuous multi-utility family [35]. Let us endow the set of $\mathbf{R}^d$-valued random variables with the following preference relation : for $\gamma_1, \gamma_2 \in L^0(\mathbf{R}^d, \mathcal{F})$, we write $\gamma_1 \succeq^t \gamma_2$ if $U_t(\gamma_1 - \gamma_2) \geq 0$ a.s. Observe that the relation $\succeq^t$ defines, for each t, a preorder on $L^0(\mathbf{R}^d, \mathcal{F})$ which is continuous. This means that the graph

$$\mathrm{GR}(t) := \{(\gamma_1, \gamma_2) \in L^0(\mathbf{R}^d, \mathcal{F}) \times L^0(\mathbf{R}^d, \mathcal{F}) : \ \gamma_2 \succeq^t \gamma_1\},$$

is closed in $L^0(\mathbf{R}^d, \mathcal{F}) \times L^0(\mathbf{R}^d, \mathcal{F})$. Indeed, as the function U_t satisfies Condition (a), it follows that the preorder $\succeq^t$ admits both a lower and upper semi-continuous multi-utility representation, [35]. But it is not clear whether there exists such a family which is countable. Similarly, for each ω in the probability space Ω, we may also consider the associated (random) preorder on $\mathbf{R}^d$ at time t, i.e. $x \succeq^{\omega,t} y$ if and only if $U_t(\omega, x - y) \geq 0$. Since $\mathbf{R}^d$ is both σ-compact and locally compact, by [59], we deduce that for each (ω, t), there exists a countable family of utility functions characterizing the preorder $\succeq^{\omega,t}$. Nevertheless, it is unclear whether there exists such a countable family whose utility functions are measurable in the sense given in [58]. When this is the case, we may define two concepts of *essential supremum* and *maximum* [58], [59]. The first notion, which is not a random variable but a family of vector-valued random variables, is a generalization of the real-valued essential supremum of a family of real-valued random variables to the set of vector-valued random variables equipped with a random preorder. This allows to characterize the *minimal* super-hedging prices of European and American contingent claims. In the present setting, the needed conditions do not seem to be satisfied so that we can not apply the existence theorems formulated in [58] and [59] which state that the essential supremum and maximum of a family of vector-valued random variables Γ are nonempty provided that Γ is bounded from above with respect to the random preorder considered. Actually, the mentioned results still hold in our case under mild conditions, as proved in Appendix by using a new approach. In the sequel, we use the notation $[a, b]^t$ when we denote an interval with respect to the preorder $\succeq^t$.

In the following, we assume that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]^t$, $\gamma^1, \gamma^2 \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, $\gamma_2 \succeq^t \gamma_1$, are compact a.s. This is the case when the cone $\overline{K}$ is proper as shown in the following lemma (see Appendix).

Lemma 5.4.1. *Suppose that the cone $\overline{K}_t$ is proper. Let us consider two random variables $\gamma^1, \gamma^2 \in L^0(\mathbf{R}^d, \mathcal{F})$ such that $\gamma_1 \preceq^t \gamma_2$. Then, the order interval $[\gamma^1, \gamma^2]^t$ is a.s. compact.*

In Example 5.2.8, $\overline{K}_t$ is proper. It follows that the (random) order intervals $[\gamma^1(\omega), \gamma^2(\omega)]^t$, $\gamma^1 \preceq^t \gamma^2$, with respect to G_t are compact a.s.

We consider the set $\mathcal{V}$ of $\mathbf{R}^d$-valued $\mathcal{F}_t$-adapted portfolio processes V such that the increments $\Delta V_t := V_t - V_{t-1} \in -G_t$ for all t. In this model, European contingent claims are d-dimensional random vectors and American contingent claims are adapted d-dimensional random processes. This means that the claims are expressed in physical units.

A portfolio process $V \in \mathcal{V}$ super-replicates a European claim Y_T if $V_T \succeq^T Y_T$. Moreover, it is called *minimal* if any portfolio process $W \in \mathcal{V}$ such that $W_T = Y_T$ and $V \succeq W$ coincides with V. The notation $\succeq$ means that we use the partial order $\succeq^{t+1}$ when comparing values of the processes at time $t < T$. Observe that under **NA2** Condition, the inequality $\hat{W}_t \preceq^{t+1} \hat{V}_t$ is equivalent to $\widehat{W_t} \preceq^{G_{t+1} \cap G_t} \widehat{V}_t$. We denote $\mathcal{V}^E_{min}(Y_T)$ the set of all minimal processes super replicating the European claim Y_T. Observe that the super

hedging problem $\mathbf{L}_T(V_T) \geq \xi$ where $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ reads as $V_T \succeq^T \xi e_1$. This is a restriction of the more general problem we consider.

Similarly, the value process $V \in \mathcal{V}$ super replicates an American claim $(Y_t)_{t=0,\cdots,T}$ if $V \succeq Y$, i.e. $V_t \succeq^t Y_t$ at any time t, and we define analogously $\mathcal{V}^A_{min}(Y)$. Before characterizing the set of all minimal super hedging portfolio processes and prices, we introduce the following concepts.

Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F} := \mathcal{F}_T$. We consider in the space $L^0(\mathbf{R}^d, \mathcal{F})$ of d-dimensional random variables a partial order defined by a Caratheodory function U satisfying the conditions $(a) \cdots (d)$ of Section 6.1. For instance, when $U = U_t$ as in Section 6.1, for elements $\gamma_1, \gamma_2 \in L^0(\mathbf{R}^d, \mathcal{F})$, the relation $\gamma_2 \succeq^t \gamma_1$ means that $U_t(\gamma_2 - \gamma_1) \geq 0$ a.s. In the following, for the sake of simplicity, we write $U = U_t$ whatever t is and we use the notation $\gamma_2 \succeq \gamma_1$. Recall that the solvency set is $G^U := \{x \in \mathbf{R}^d : \ U(x) \geq 0\}$ which defines a liquidation value function $\mathbf{L}$ given by (6.1.3).

Definition 5.4.2. *Let Γ be a subset of $L^0(\mathbf{R}^d, \mathcal{F})$. We denote by $\mathcal{H}$-Esssup Γ a subset $\hat{\Gamma}$ of $L^0(\mathbf{R}^d, \mathcal{H})$ such that the following conditions hold :*

(a) $\hat{\Gamma} \succeq \Gamma$,

(b) if $\gamma \in L^0(\mathbf{R}^d, \mathcal{H})$ and $\gamma \succeq \Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\gamma \succeq \hat{\gamma}$,

(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 = \hat{\gamma}_2$.

The following condition holds for the Kabanov model as well as the model in Example 5.2.8. Indeed, it suffices to consider $u_t(x, y) = \arctan(x + S^b_t y)$.

Condition II. There exists a measurable bounded continuous real-valued function u such that the following properties hold with $\tilde{G} = \{\mathbf{L} = 0\}$:

(i) : $x - y \in \tilde{G}$ implies that $u(x) \geq u(y)$,

(ii) $x - y \in \tilde{G}$ and $x \neq y$ implies that $u(x) > u(y)$.

Theorem 5.4.3. *Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a measurable liquidation value function L which is upper semi-continuous and upper additive and such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compact a.s. Suppose that Condition II holds. If a non-empty subset Γ is such that $\gamma^0 \succeq \Gamma$ for some $\gamma^0 \in L^0(\mathbf{R}^d, \mathcal{H})$, then $\mathcal{H}$-Esssup $\Gamma \neq \emptyset$.*

The proof is given in Appendix.

In the sequel, we denote by $\text{env}_{\mathcal{H}}\Gamma$ the smallest $\mathcal{H}$-decomposable [1] subset of $L^0(X, \mathcal{F})$ containing Γ and by $\text{cl}\,\text{env}_{\mathcal{H}}\Gamma$ its closure in $L^0(X, \mathcal{F})$.

Definition 5.4.4. *Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We put*

$$\mathcal{H}\text{-Essmax}\,\Gamma := \{\gamma \in \text{cl}\,\text{env}_{\mathcal{H}}\Gamma : \ \text{cl}\,\text{env}_{\mathcal{H}}\Gamma \cap [\gamma, \infty[= \{\gamma\}\}.$$

Definition 5.4.5. *Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}$-Essmax$_1\Gamma$ the unique subset $\hat{\Gamma} \subseteq \text{cl}\,\text{env}_{\mathcal{H}}\Gamma$ such that the following conditions hold :*

1. A set E is $\mathcal{H}$-decomposable if $A \in \mathcal{H}$ and $\gamma_1, \gamma_2 \in E$ implies that $1_A\gamma_1 + 1_{\Omega \setminus A}\gamma_2 \in E$.

(*i*) *if* $\gamma \in \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma$, *then there is* $\hat{\gamma} \in \hat{\Gamma}$ *such that* $\hat{\gamma} \succeq \gamma$;

(*ii*) *if* $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, *then* $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ *implies* $\hat{\gamma}_1 = \hat{\gamma}_2$.

Though our definitions are given for $\Gamma \in L^0(X, \mathcal{F})$, the most important is the case where $\Gamma \in L^0(X, \mathcal{H})$. Observe that $\mathcal{H}\text{-Essmax}_1\Gamma \subseteq \mathcal{H}\text{-Essmax}\,\Gamma$ and the equality holds provided that $\mathcal{H}\text{-Essmax}_1\,\Gamma \neq \emptyset$ due to (*i*).

Theorem 5.4.6. *Let* $\succeq$ *be a partial order in* $L^0(\mathbf{R}^d, \mathcal{F})$ *represented by a random liquidation value function* $\mathbf{L}$ *which is upper semi-continuous and upper additive and such that all order intervals* $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, *are compact a.s. Suppose that Condition II holds. Let* Γ *be a non-empty subset of* $L^0(\mathbf{R}^d, \mathcal{H})$. *Suppose that there exists* $\gamma^0 \in L^0(\mathbf{R}^d, \mathcal{H})$ *such that* $\gamma^0 \succeq \Gamma$. *Then* $\mathcal{H}\text{-Essmax}\,\Gamma = \mathcal{H}\text{-Essmax}_1\Gamma \neq \emptyset$.

The proof is given in Appendix. Following the proofs given in [58] and [59], Theorems 5.4.3 and 5.4.6 allow to deduce the following characterizations. We recall that $[x, \infty)^t := \{z : \ z \succeq^t x\}$ and, in the following, the random preorder we use to define an *essential supremum* or *minimum* is precised in parenthesis.

Proposition 5.4.7. *Suppose that* **NA**2 *holds and suppose there exits at least one* $V \in \mathcal{V}$ *such that* $V_T \succeq^T Y_T$. *Then* $\mathcal{V}^E_{min}(Y_T) \neq \emptyset$ *and it coincides with the set of solutions of backward inclusions*

$$V_t \in (\mathcal{F}_t, \succeq^{t+1})\text{-Esssup}\,\{V_{t+1}\}, \quad t \leq T-1, \quad V_T = Y_T. \tag{5.4.5}$$

Moreover, any $W \in \mathcal{V}$ *with* $W_T \succeq^T Y_T$ *is such that* $W \succeq V$ *for some* $V \in \mathcal{V}^E_{min}(Y_T)$.

Proof. The reasoning is very similar to Proposition 5.1 in [58]. We only need to check that if $x, y \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ are such that $x \succeq^{t+1} y$, i.e. $\mathbf{L}_{t+1}(x-y) \geq 0$, then $\mathbf{L}_t(x-y) \geq 0$, i.e. $x \succeq^t y$. This actually holds under the **NA**2 condition by Corollary 5.3.11. ■

Remark 5.4.8. Observe that, if a characterization of the minimal super hedging portfolio processes of any attainable European contingent claims is given by the set of solutions of backward inclusions above, then **NA**2 holds. Indeed, suppose that $\mathbf{L}_T(V_T) \geq 0$, $V \in \mathcal{V}$, i.e. $V_T \succeq^T 0$. With $Y_T = 0$, we easily deduce that there exists a unique minimal super hedging portfolio process of $Y_T = 0$, precisely the zero portfolio process. Therefore, $V \succeq 0$.

Definition 5.4.9. *Assume that the set* $\Gamma^E_{Y_T}$ *of superhedging price of* Y_T *is closed. We say that* V_0 *is a minimal super-hedging price of* Y_T *if the property* $V_0 \succeq^0 W_0$ *where* W_0 *is a super hedging price of* Y_T, *implies* $V_0 = W_0$. *This means that the minimal superhedging prices for* Y_T *coincide with* $G_0 - \mathrm{Min}\,\Gamma^E_{Y_T}$ *(with respect to the partial order generated by* G_0*).*

Proposition 5.4.10. *Assume that the set* $\Gamma^E_{Y_T}$ *of superhedging price of* Y_T *is closed. Any minimal superhedging price of* Y_T *is the initial value of a minimal superhedging portfolio process of* Y_T.

Proof. Consider a minimal superhedging price of Y_T. This is the initial value W_0 of a portfolio process W superhedging the payoff Y_T. By definition,

there exists a minimal superhedging portfolio process V for Y_T such that $W \succeq^G V$ hence $W_0 \succeq^{G_0} V_0$. This implies that $W_0 = V_0$ since W_0 is minimal. Therefore, $W_0 = V_0$ is the initial endowment of the minimal superhedging portfolio process V. ■

Corollary 5.4.11. *Assume that the set $\Gamma^E_{Y_T}$ of all superhedging prices for Y_T is closed. Let $\mathcal{V}^0_{\min}(Y_T)$ be the set of all initial values of minimal superhedging portfolio processes for Y_T. Then, $G_0 - \operatorname{Min}\Gamma_{Y_T} = G_0 - \operatorname{Min}\mathcal{V}^0_{\min}(Y_T)$.*

Observe that $\Gamma_{Y_T} = (G_0 - \operatorname{Min}\Gamma_{Y_T}) + G_0$. Therefore, the corollary above and Proposition 5.4.7 give a constructive approach to characterize Γ_{Y_T}.

Proposition 5.4.12. *Let Y be an American claim. Suppose there exists a process $V^0 \in \mathcal{V}$ such that $V^0 \succeq Y$. Then, the set $\mathcal{V}^A_{min}(Y)$ is non-empty and coincides with the set of solutions of backward inclusions*

$$V_t \in (\mathcal{F}_t, \succeq^{t+1})\text{-Essmin}L^0([Y_t,\infty)^t \cap [V_{t+1},\infty)^{t+1}, \mathcal{F}_t), \quad t \leq T-1, \quad V_T = Y_T. \tag{5.4.6}$$

Notice that by Proposition (5.4.7) or Proposition (5.4.12), the set of all super hedging prices is simply deduced by adding an element of G_0 to any initial value of a minimal super hedging portfolio process. Contrary to the traditional approach using convex analysis, in particular the Hahn-Banach separation theorem, it is not necessary to suppose that the financial market model is convex since we do not use fundamental elements like risk neutral probability measures or more generally consistent price systems in a dual approach. Here, it is sufficient to compute the minimal portfolio processes. As shown in Appendix, they are deduced by solving expected liquidation value minimization problems.

5.5 Appendix

Let us recall some elementary facts from convex analysis.

Lemma 5.5.1.

(i) Let K be a convex cone in $\mathbf{R}^d$ such that $\operatorname{int} K \neq \emptyset$. If $x \in \partial K$ and $y \in \operatorname{int} K$, then $x + y \in \operatorname{int} K$.

(ii) Let K be a convex set in $\mathbf{R}^d$ such that $\operatorname{int} K \neq \emptyset$. Then $\operatorname{int} \overline{K} \subseteq K$.

Proof of Proposition 5.2.6

(i) These properties are directly deduced from the condition $\mathbf{G}^0$.

(ii) If there is no $\alpha \geq 0$ such that $x - \alpha e_1 \in G_t$, we deduce that $\mathbf{L}_t(x) = -\infty$. Otherwise we assume that $\mathbf{L}_t(x) = \lim_n \uparrow \alpha^n$ where the sequence (α^n) satisfies $x - \alpha^n e_1 \in G_t$. Suppose that $\sup_n \alpha^n = +\infty$. Then, using the *normalization* procedure $\tilde{x}^n = x/\alpha^n$, we get that $\tilde{x}^n - e_1 \in \bar{K}_t$. Taking the limit, we get that $-e_1 \in \bar{K}_t \cap (-\bar{K}_t) = \{0\}$, i.e. a contradiction. Therefore, we may assume without loss of generality that $\alpha^n \mapsto \alpha^*$ and $\mathbf{L}_t(x) = \alpha^* < \infty$.

We now suppose that G_t dominates $\mathbf{R}^d_+$. Since $e_1 \in \operatorname{int} G_t$ we can chose $r > 0$ small enough such that $B(e_1, r) \subseteq G_t$. So, for all z such that $\|z\| \leq r$,

we have $\mathbf{L}_t(e_1+z)\geq 0$ which implies that $\mathbf{L}_t(z)>-\infty$, $\forall\, z, \|z\|\leq r$. Since $\lambda G_t\subseteq G_t\,, \forall \lambda\geq 1$, we easily deduce that $\mathbf{L}_t(x)>-\infty, \forall\, x\in\mathbf{R}^d$.

(iii) Suppose that there exists $x\in\{\mathbf{L}_t=0\}$ and $r>0$ such that $B_r(x)\subseteq G_t$. Since $x-re_1\in G_t$, we have $-r=\mathbf{L}_t(x-re_1)\geq 0$, i.e. a contradiction.

(iv) Consider a sequence $x^n\in\mathbf{R}^d$ which converges to $x_0\in\mathbf{R}^d$. Let us denote $\alpha^n:=\mathbf{L}_t(x^n)$. We have $x^n-\alpha^n e_1\in G_t$. Let us suppose that $\sup_n\alpha^n=\infty$. Then, using the *normalization* procedure, we get that $-e_1\in\bar{K}_t\cap(-\bar{K}_t)=\{0\}$, i.e. a contradiction. So, we may assume that $\alpha^\infty=\limsup\alpha^n\in\mathbf{R}$ and $x_0-\alpha^\infty e_1\in G_t$. It follows that $\mathbf{L}_t(x_0)\geq\limsup\mathbf{L}_t(x^n)$ which implies that $\mathbf{L}_t$ is upper-semicontinuous.

(v) If G is a cone, it is straightforward that $\mathbf{L}_t$ is homogeneous. Since $\mathbf{L}_t$ is super-additive, we deduce that $\mathbf{L}_t$ is concave, hence continuous (because $\mathbf{L}_t$ is finite on $\mathbf{R}^d$). We now prove the last statement. It is sufficient to verify that $\mathbf{L}_t(x)=0$, $\forall x\in\partial G_t$. We rewrite $x=g_t+\mathbf{L}_t(x)e_1$ such that $\mathbf{L}_t(g_t)=0$. If $\mathbf{L}_t(x)>0$, then $\mathbf{L}_t(x)e_1\in\operatorname{int}G_t$. By Lemma 5.5.1, $x\in\operatorname{int}G_t$, i.e. a contradiction. Hence, $\mathbf{L}_t(x)=0$. ■

(vi) Recall that the concave hull of $\mathbf{L}_t^G$ is given by

$$\operatorname{conc}\mathbf{L}_t^G(x):=\sup\{\sum\lambda_i l_t(x_i):\ \lambda_i\geq 0,\sum\lambda_i x_i=x\}.$$

Since $\mathbf{L}_t^G\leq\mathbf{L}_t^K$ and $\mathbf{L}_t^K$ is concave, we deduce that $\operatorname{conc}\mathbf{L}_t\leq\mathbf{L}_t^K$. Moreover,

$$\operatorname{conc}\mathbf{L}_t^G(x)\geq\sup_{\lambda>0}\frac{\mathbf{L}_t^G(\lambda x)}{\lambda}.$$

In order to prove that $\operatorname{conc}\mathbf{L}_t^G(x)\geq\mathbf{L}_t^K(x)$, it is sufficient to show that for all $x\in\mathbf{R}^d$ and $\varepsilon>0$, there exists $\delta>0$ such that

$$\frac{\mathbf{L}_t^G(\delta x)}{\delta}\geq\mathbf{L}_t^K(x)-\varepsilon.$$

To do so, write x as

$$x=k_t+\mathbf{L}_t^K(x)e_1,\quad k_t\in K_t.$$

Using the property of $\mathbf{L}_t^K$ on $\mathbf{R}e_1$, we have $\mathbf{L}_t^K(k_t)=0$. Define $g_\varepsilon:=k_t+\varepsilon e_1$. By Lemma 5.5.1, we deduce that $g_\varepsilon\in\operatorname{int}K_t\subset\mathbf{R}_+G_t$. We now rewrite k_t as

$$k_t=\lambda g-\varepsilon e_1,$$

where $g:=\frac{g_\varepsilon}{\lambda}\in G_t$ and $\lambda>0$. Since $\alpha G_t\subset G_t$, $\forall\alpha\geq 1$, we may assume, w.l.o.g that $\lambda\in(0,1]$.
Choosing $\delta=\lambda^{-1}$, we get that

$$\frac{\mathbf{L}_t^G(\delta x)}{\delta}\geq\frac{\mathbf{L}_t^G(g)}{\delta}+\mathbf{L}_t^K(x)-\varepsilon\geq\mathbf{L}_t^K(x)-\varepsilon.$$

The conclusion follows. ■

Proof of Corollary 5.2.7

Using the last lemma above, for all $c \in \mathbf{R}$, we get that

$$\{\omega :\ \mathbf{L}_t(\omega, \gamma(\omega)) > c\} = \bigcup_{q \in (c,\infty)\cap\mathbb{Q}} \{\omega :\ (\omega, \gamma(\omega) - qe_1) \in \operatorname{Gr} G_t\}$$

and the conclusion follows. ■

Proof of Lemma 5.3.2

Let $Y = \sum_{i=1}^n \lambda_i \mathbf{L}_T(V_T^i) \in \mathrm{A}_T^0$. We assume w.l.o.g. that $\lambda_i \neq 0$ for all i and we put $\alpha := \inf_i \lambda_i$. Therefore,

$$Y = \alpha \sum_{i=1}^n \frac{\lambda_i}{\alpha} \mathbf{L}_T(V_T^i) = \alpha \sum_{i=1}^n \tilde{\lambda}_i \mathbf{L}_T(\tilde{V}_T^i)$$

where $\tilde{\lambda}_i \geq 1$ and $\tilde{V}_T^i = \mathbf{L}_T(V_T^i)e_1 \in \mathrm{R}_T^0$. Observe that $\tilde{V}_T^i \in \mathrm{R}_T^0$ since $\mathbf{L}_T(V_T^i - \tilde{V}_T^i) = 0$. Moreover $\tilde{\lambda}_i \mathbf{L}_T(\tilde{V}_T^i) = \mathbf{L}_T(\tilde{\lambda}_i \tilde{V}_T^i)$ and $\tilde{\lambda}_i \tilde{V}_T^i \in \mathrm{R}_T^0$ as $\tilde{\lambda}_i \geq 1$ and $\tilde{\lambda}_i G_t \subseteq G_t$. By linearity of $\mathbf{L}_T$ on the line $\mathbf{R}e_1$, we deduce that

$$Y = \alpha \mathbf{L}_T(\sum_{i=1}^n \tilde{\lambda}_i \tilde{V}_T^i) =: \alpha \mathbf{L}_T(\tilde{V}_T)$$

where $\tilde{V}_T \in \mathrm{R}_T^0$. Indeed, as $G_t + G_t \subseteq G_t$, a finite sum of portfolio process is still a portfolio process. Therefore, it remains to show that an element of the form $Y = \alpha \mathbf{L}_T(V_T)$, $\alpha \geq 0$, can be rewritten in the same form where $\alpha \in [0, 1]$. To do so, we only consider the case $\alpha > 1$ and we may assume w.l.o.g. that $V_T \in \mathbf{R}e_1$. It follows that $Y = \mathbf{L}_T(\alpha V_T)$ where $\alpha V_T \in \mathrm{R}_T^0$ as $\alpha \geq 1$. ■

Proof of Theorem 5.3.9

Recall that in the Kabanov model [72],

$$\mathbf{L}_t(z) := \max\{\alpha :\ z - \alpha e_1 \in G_t\} = \min_{z_t \in G_t^*, z_t e_1 = 1} z_t z = z_t^* z$$

where $z_t^* \in L^0(G_t^*, \mathcal{F}_t)$ satisfies $z_t^* e_1 = 1$. Recall also that $G_t^* \subseteq \mathbf{R}_+^d$. Moreover, $z - \mathbf{L}_t(z)e_1 \in G_t$. Suppose that the supermartingale property holds and there is a portfolio process V such that $V_T \succeq^T 0$. As $\mathbf{L}_T(V_T) \geq 0$, we deduce that $\mathbf{L}_{T-1}(V_{T-1}) \geq \mathbb{E}(Z_T^1 \mathbf{L}_T(V_T)|\mathcal{F}_{T-1}) \geq 0$ hence $V_{T-1} \succeq^{T-1} 0$. We repeat the reasoning by induction and we get that **NA2** holds since $V_{t-1} \succeq^t V_t$. Reciprocally, suppose that **NA2** holds. By a measurable selection argument, we may write $\mathbf{L}_u(V_u) = Z_u V_u$ where $Z_u \in L^0(G_u^*, \mathcal{F}_u)$, $Z_u e_1 = 1$. By virtue of [91], Condition **NA2** [2] implies that Z_u is the initial value of a consistent price system $(Z_r)_{r \geq u}$, i.e. a martingale such that $Z_r \in G_r^*$, $r \geq u$. It follows that

$$Z_{u+1} e_1 \mathbf{L}_{u+1}(V_{u+1}) \leq Z_{u+1} V_{u+1} \leq Z_{u+1} V_u.$$

2. Note that the **NA2** condition as defined for the Kabanov model and the one we introduce for more general models coincide.

Taking the conditional expectation, we deduce that

$$\mathbb{E}(Z_{u+1}e_1\mathbf{L}_{u+1}(V_{u+1})|\mathcal{F}_u) \leq \mathbf{L}_u(V_u).$$

■

Proof of Lemma 5.4.1

Let us consider a sequence $x^n \in [\gamma^1, \gamma^2]^t$, i.e. such that $x^n - \gamma^1, \gamma^2 - x^n \in G_t$. Let us suppose that $\sup_n |x^n| = +\infty$. Using the normalisation procedure, we get a contradiction. ■

Proof of Theorem 5.4.3

Let us consider $\gamma \in L^0([\Gamma, \infty), \mathcal{H})$ and fix $\tilde{\gamma}^0 \in \Gamma$. For all $\zeta \in [\Gamma, \gamma]$, $\mathbf{L}(\tilde{\gamma}^0) \leq \mathbf{L}(\zeta) \leq \mathbf{L}(\gamma)$. From the inequality $\mathbf{L}(\gamma) \geq \mathbf{L}(\gamma - \zeta) + \mathbf{L}(\zeta)$, we deduce that

$$0 \leq \mathbf{L}(\gamma - \zeta) \leq \mathbf{L}(\gamma) - \mathbf{L}(\zeta) \leq \mathbf{L}(\gamma) - \mathbf{L}(\tilde{\gamma}^0).$$

With $dQ^\gamma = ce^{-|\mathbf{L}(\gamma)|-|\mathbf{L}(\tilde{\gamma}^0)|}$, c is a constant, we deduce that $\mathbf{L}(\gamma - \zeta)$ is uniformly Q^γ-integrable whatever $\zeta \in L^0([\Gamma, \gamma], \mathcal{H})$. We denote by $\mathbb{E}^\gamma$ the expectation under Q^γ. For any $\zeta \in L^0([\Gamma, \gamma], \mathcal{H})$, the mapping $\omega \mapsto \mathbf{L}(\omega, \gamma(\omega) - \zeta(\omega))$ is an $\mathcal{F}$-measurable random variable by Corollary 5.2.7. Put

$$a(\gamma) := \sup_{\zeta \in L^0([\Gamma,\gamma],\mathcal{H})} \mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta)$$

and consider a sequence $\zeta_n \in L^0([\Gamma, \gamma], \mathcal{H})$ such that $a(\gamma) = \lim_n \mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta_n)$. Without loss of generality, we may assume that the sequence of random variables $\mathbf{L}(\gamma - \zeta_n)$ is such that the conditional expectations $\mathbb{E}^\gamma(\mathbf{L}(\gamma - \zeta_n)|\mathcal{H})$ are increasing. Indeed, we can replace the sequence ζ_n by the sequence ζ'_n by putting $\zeta'_1 = \gamma$, and defining recursively the random variables

$$\begin{aligned} \zeta'_n &:= \zeta'_{n-1} 1_{\Omega^n} + \zeta_n I_{\Omega \setminus \Omega^n}, \\ \Omega^n &:= \{E^\gamma(\mathbf{L}(\gamma - \zeta'_{n-1})|\mathcal{H}) \geq E^\gamma(\mathbf{L}(\gamma - \zeta_n)|\mathcal{H})\}, \quad n \geq 2. \end{aligned}$$

Due to the assumption of the theorem, the order intervals $[\tilde{\gamma}_0(\omega), \gamma(\omega)]$ are compact (a.s.). It follows that $\sup_n |\zeta_n| < \infty$ a.s. By virtue of the lemma on converging subsequences (Lemma 2.1.2 [72]), there exists a strictly increasing sequence of $\mathcal{H}$-measurable integer-valued random variables τ_k such that the sequence ζ_{τ_k} converges a.s. to some ζ such that $\Gamma \preceq \zeta \preceq \gamma$. The monotonicity implies that

$$\mathbb{E}^\gamma(\mathbf{L}(\gamma - \zeta_{\tau_k})|\mathcal{H}) = \sum_{m \geq k} \mathbb{E}^\gamma(\mathbf{L}(\gamma - \zeta_m)|\mathcal{H}) I_{\{\tau_k = m\}} \geq \mathbb{E}^\gamma(\mathbf{L}(\gamma - \zeta_k)|\mathcal{H}).$$

It follows that

$$\mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta_{\tau_k}) \geq \mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta_k).$$

Using the upper semi-continuity of $\mathbf{L}(\omega, .)$ and the uniform integrability of the sequence $(\mathbf{L}(\gamma - \zeta_{\tau_k}))$, we have :

$$\mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta) \geq \limsup_k \mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta_{\tau_k}) \geq \lim_k \mathbb{E}^\gamma \mathbf{L}(\gamma - \zeta_k) = a(\gamma).$$

Thus,

$$a(\gamma) = \mathbb{E}^{\gamma}\mathbf{L}(\gamma - \zeta). \tag{5.5.7}$$

We denote by $\Lambda(\gamma)$ the set of all random variables $\zeta \in L^0([\Gamma, \gamma], \mathcal{H})$ verifying (5.5.7). For each $\bar{\gamma} \in \Lambda(\gamma)$, observe that $\tilde{\gamma} \leq \bar{\gamma}$ and $\tilde{\gamma} \in L^0([\Gamma, \infty), \mathcal{H})$ implies that $\tilde{\gamma} \in \Lambda(\gamma)$ and $\mathbf{L}(\bar{\gamma} - \tilde{\gamma}) = 0$ due to the definition of $a(\gamma)$ since the inequality $\mathbf{L}(\gamma - \tilde{\gamma}) \geq \mathbf{L}(\gamma - \bar{\gamma}) + \mathbf{L}(\bar{\gamma} - \tilde{\gamma})$ holds. So, $\bar{\gamma} - \tilde{\gamma} \in \tilde{G} = \{\mathbf{L} = 0\}$ a.s. Under Condition II, consider the corresponding function u and set

$$b(\bar{\gamma}) := \inf_{\tilde{\gamma} \in L^0([\Gamma, \bar{\gamma}), \mathcal{H})} \mathbb{E}u(\tilde{\gamma}).$$

As previously shown, the set $\hat{\Lambda}(\bar{\gamma})$ of all random variables $\hat{\gamma} \in L^0([\Gamma, \bar{\gamma}], \mathcal{H})$ verifying $b(\bar{\gamma}) = \mathbb{E}u(\bar{\gamma})$ is non empty. Define the set

$$\widehat{\Gamma} := \bigcup_{\gamma \in L^0([\Gamma, \infty[, \mathcal{H})} \bigcup_{\bar{\gamma} \in \Lambda(\gamma)} \hat{\Lambda}(\bar{\gamma}). \tag{5.5.8}$$

It remains to show that this set satisfies (a)–(c). Obviously, $\widehat{\Gamma} \succeq \Gamma$, i.e. (a) holds. If $\zeta \in L^0([\Gamma, \infty[, \mathcal{H})$, then $\zeta \succeq \Lambda(\zeta)$ by construction and for every $\bar{\gamma} \in \Lambda(\zeta)$, $\bar{\gamma} \succeq \hat{\Lambda}(\bar{\gamma})$ by construction as well i.e. (b) holds. At last, consider $\widehat{\zeta}_1, \widehat{\zeta}_2 \in \widehat{\Gamma}$ with $\widehat{\zeta}_1 \in \hat{\Lambda}(\bar{\gamma}_1)$ and $\widehat{\zeta}_2 \in \hat{\Lambda}(\bar{\gamma}_2)$, $\gamma_1, \gamma_2 \in [\Gamma, \infty[$, such that $\widehat{\zeta}_1 \succeq \widehat{\zeta}_2$. Then, $\widehat{\zeta}_2 \in L^0([\Gamma, \bar{\gamma}_1], \mathcal{H})$ and $\widehat{\zeta}_2 \preceq \widehat{\zeta}_1 \preceq \bar{\gamma}_1$. As already mentionned, we get that $\bar{\gamma}_1 - \widehat{\zeta}_1 \in \tilde{G}$ and $\bar{\gamma}_1 - \widehat{\zeta}_2 \in \tilde{G}$. We deduce that $\widehat{\zeta}_1 - \widehat{\zeta}_2 \in \tilde{G}$. It follows that $u(\widehat{\zeta}_1) - u(\widehat{\zeta}_2) \geq 0$ and the inequality is strict on a non-null set as soon as $\widehat{\zeta}_1 \neq \widehat{\zeta}_2$. It follows that there exists a non-null set $B \in \mathcal{H}$ on which

$$\mathbb{E}(u(\widehat{\zeta}_1) - u(\widehat{\zeta}_2)|\mathcal{H}) > 0.$$

Observe that, if $B \in \mathcal{H}$,

$$u(\widehat{\zeta}_2 I_B + \widehat{\zeta}_1 I_{B^c}) = u(\widehat{\zeta}_2) I_B + u(\widehat{\zeta}_1) I_{B^c},$$

and

$$\mathbb{E}(u(\widehat{\zeta}_2 I_B + \widehat{\zeta}_1 I_{B^c})|\mathcal{H}) = \mathbb{E}(u(\widehat{\zeta}_2)|\mathcal{H}) I_B + \mathbb{E}(u(\widehat{\zeta}_1)|\mathcal{H}) I_{B^c}.$$

It follows that

$$b(\bar{\gamma}_1) = \mathbb{E}u(\widehat{\zeta}_1) > \mathbb{E}u(\widehat{\zeta}_2 I_B + \widehat{\zeta}_1 I_{B^c}) \tag{5.5.9}$$

where $\widehat{\zeta}_2 I_B + \widehat{\zeta}_1 I_{B^c} \in [\Gamma, \bar{\gamma}_1]$. This is a contradiction. Hence, (c) also holds. ■

Proof of Theorem 5.4.6

The arguments are similar to those of Theorem 5.4.3. It remains only to verify (i). For $\gamma \in \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma$, we put

$$c(\gamma) := \sup_{\tilde{\gamma} \in \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma \cap L^0([\gamma, \infty), \mathcal{H})} \mathbb{E}\mathbf{L}(\tilde{\gamma} - \gamma).$$

Let $(\tilde{\gamma}_n)$ be a sequence on which the supremum in the above definition is attained. As the set $\mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{H})$ is decomposable, we may assume

without loss of generality (by applying Lemma 2.1.2 [72] on convergent subsequences) that the sequence of conditional expectations $\mathbb{E}(\mathbf{L}(\gamma_n - \gamma)|\mathcal{H})$ is increasing and $\tilde{\gamma}_n$ converges a.s. to some $\tilde{\gamma}_\infty \in \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma \cap L^0([\gamma, \infty), \mathcal{H})$ such that $c(\gamma) := \mathbb{E}\mathbf{L}(\tilde{\gamma}_\infty - \gamma)$. Indeed, observe that $\operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma \subseteq L^0((-\infty, \gamma^0], \mathcal{H})$ hence we use a compactness argument. For such an element $\tilde{\gamma}_\infty$, we solve

$$d(\gamma, \tilde{\gamma}_\infty) := \sup_{\tilde{\gamma} \in L^0([\tilde{\gamma}_\infty, \infty)) \cap \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma} \mathbb{E}u(\tilde{\gamma})$$

where u is given by Condition II. As previously, we obtain elements $\hat{\gamma} \in L^0([\tilde{\gamma}_\infty, \infty)) \cap \operatorname{cl}\operatorname{env}_{\mathcal{H}}\Gamma$ such that $d(\gamma, \tilde{\gamma}_\infty) = \mathbb{E}u(\hat{\gamma})$. By definition of d, using Condition II, we deduce that $\hat{\gamma} \in \mathcal{H}\text{-Essmax}_1\Gamma$ and the conclusion follows. ■

Chapitre 6

Arbitrage theory in non convex financial market models

Abstract

This chapter continues to study the general financial market model defined by a liquidation value process mentioned in the previous chapter. First, we provide a no arbitrage condition that charaterizes the existence of an equivalent risk neutral probability measure through which we establish the superhedging price of a european option for markets with proportional transaction costs. In the rest of the chapter, we review various notions of no arbitrage opportunities in this new model as well as study the relationship between this model and its extended model, i.e. market with only proportional transaction costs.

Keywords : Financial markets, Liquidation value, Transaction costs, European and American options, Hedging, Partial order.

Note. This chapter is based on an pre-article *Risk neutral probabilities and absence of arbitrage opportunity in non convex financial market models,* T. Tran, E. Lépinette, *Preprint.*

6.1 Introduction

The present chapter accompanies the previous one. For the motivation and history of the problematic as well as for references we refer in the sequel to Chapter 3 without further notice. Let us briefly recall some notations and definitions here. We are given a filtered probabilistic space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\cdots,T}, \mathrm{P})$ satisfying the usual conditions. Suppose that the portfolio processes we consider are expressed in physical units, i.e. the number of assets an agent holds. Moreover, we assumed that the first component of such portfolios corresponds to a cash account. Precisely, we assume without loss of generality that the underlying bond is $S^1 = 1$ so that the first component of a portfolio

position is an amount of cash. The model is defined by a adapted process of random sets $(G_t)_{t=0,\cdots,T}$ satisfying the following conditions

Condition $\mathbf{G}^0$:

(i) G_t is a closed set, $\forall 0 \leq t \leq T$,

(ii) $G_t + G_t \subseteq G_t$, $0 \leq t \leq T$,

(iii) $\lambda G_t \subseteq G_t$, $\forall \lambda \geq 1$,

(iv) $G_t + \mathbf{R}^d_+ = G_t$,

(v) The cone $\overline{\mathbf{R}_+ G_t}$ is proper , i.e. $\overline{\mathbf{R}_+ G_t} \cap (-\overline{\mathbf{R}_+ G_t}) = \{0\}$.

In financial terms, (G_t) are called to be solvency sets. A self-financing portfolio process $(V_t)_{t=0,\cdots,T}$ is an $(\mathcal{F}_t)_{t=0,\cdots,T}$-adapted process such that

$$V_{t-1} - V_t \in G_t, \quad \forall t = 0, \cdots, T \quad a.s. \tag{6.1.1}$$

The enlarged market is defined by $K = (K_t)_{t=0,\cdots,T}$, where

$$K_t := \operatorname{cone} G_t = \mathbf{R}_+ G_t \tag{6.1.2}$$

Here, $\operatorname{cone} G_t$ denotes the conic hull of G_t.The enlarged market $(K_t)_{t=0,\cdots,T}$ is therefore the minimal conic market that dominates $(G_t)_{t=0,\cdots,T}$. For instance, if $(G_t)_{t=0,\cdots,T}$ models a financial market with both fixed and proportional transaction costs, then $(K_t)_{t=0,\cdots,T}$ will be the market model with only proportional costs.

Consider a market $G = (G_t)_{t=0,\cdots,T}$ satisfying Condition G^0. The liquidation value process associated to G is defined by

$$\mathbf{L}_t(z) := \sup\{\alpha \in \mathbf{R} : \ z - \alpha e_1 \in G_t\}. \tag{6.1.3}$$

In other words, if x is a portfolio position, then $\mathbf{L}_t(x)$ is the largest amount of cash we get at time t, when liquidating the risky assets contained in the portfolio x. The new portfolio position after liquidation is therefore $\mathbf{L}_t(x)e_1$. We can prove that $(\mathbf{L}_t)$ defines (G_t) in the sense that

$$G_t = \{x \in \mathbf{R}^d : \mathbf{L}_t(x) \geq 0\}.$$

When G satisfies Condition G^0, the liquidation value process given by (6.1.3) is super additive and upper semi-continuous. As a consequence, we can define a pre-order on $\mathbf{R}^d$ by

$$x \succeq^{\omega,t} y \Leftrightarrow \mathbf{L}_t(x - y) \geq 0.$$

We denote by R^t_T the set of all terminal values V_T of portfolio processes V such that $V_{t-1} = 0$. i.e.

$$\mathrm{R}^t_T := \mathrm{R}^t_T(G) := \sum_{t \leq u \leq T} L^0(-G_u, \mathcal{F}_u).$$

We also define

$$\mathrm{LV}_T^t := \mathrm{LV}_T^t(G) := \{\mathbf{L}_T(V_T) : \ V_T \in \mathrm{R}_T^t\}.$$

Observe that $L^0(-\mathbf{R}_+, \mathcal{F}_T) \subseteq \mathrm{LV}_T^t$. Let us introduce the convex cone

$$\mathrm{A}_T^t := \{\sum_{i=1}^n \lambda_i X_i : \ \lambda_i \geq 0, \ X_i \in \mathrm{LV}_T^t\}.$$

6.2 Risk neutral probabilities in a general model

In the "standard model" of Kabanov for markets with transaction costs, the arbitrage theory is studied through a multidimensional conic cone R_T^0 consisting of final positions of portfolios starting from zero endownment. In this chapter, we provide another view based on the idea of liquidation function, which seems to be more intuitive. Moreover, it is also coherent to the classical arbitrage theory for frictionless markets that deals with real-valued portfolio positions.

For classical models without friction, but also for the Schachermayer model, absence of arbitrage opportunities is related to the existence of a *risk neutral probability measure*, precisely a probability measure $Q \sim P$ such that $E_Q[V_T] \leq 0$ for all terminal values V_T. Since the mentioned models are convex, this allows to formulate a dual characterization of the super-hedging prices of European type contingent claims. So, it is natural to characterize the existence of such a *risk neutral probability measure* in our model. As shown in this section, it appears that the latter is equivalent to the absence of asymptotic arbitrage opportunities but for the convex extended model. When this convex model satisfies absence of asymptotic arbitrage opportunities, we say that the *Extended* No Asymptotic Arbitrage opportunity condition **(ENAA)** holds for our initial model. We show that Condition **(ENAA)** is equivalent to the existence of $Q \sim P$ such that $E_Q[\mathbf{L}_T(V_T)] \leq 0$ for all terminal liquidation values of the portfolio processes V_T of our model. Not suprisingly, we also show that these conditions are equivalent to **ENFL**, the No Free Lunch condition for the Extended model.

Definition 6.2.1. *We say that the* Extended No Asymptotic Arbitrage *opportunity condition* **ENAA** *holds if there is no sequence* $(X^n) \in \mathrm{A}_T^0$ *such that* X^n *converges a.s. to* $\xi \in L^0(\mathbf{R}_+, \mathcal{F}_T)$ *such that* $P(\xi > 0) > 0$.

We need the following lemma

Lemma 6.2.2. *Let* $\mathcal{X}$ *be a convex cone in* $L^0(\mathbf{R}, \mathcal{F}_T)$ *containing* $-L_+^0$. *We denote by* $\overline{\mathcal{X}}^p$ *the closure of* $\mathcal{X}^p := \mathcal{X} \cap L^p(\mathbf{R}, \mathcal{F}_T)$ *for the* L^p *topology when* $1 \leq p < \infty$. *Let us introduce for any* $P' \sim P$ *and* $q \in (1, \infty]$

$$\mathcal{D}^q(P') := \{Q \sim P' : \ \frac{dQ}{dP'} \in L_{P'}^q, E_Q X \leq 0, \ \forall X \in \mathcal{X}^p\}.$$

The following conditions are equivalent

(i) $\overline{\mathcal{X}} \cap L^0_P(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$, *where* $\overline{\mathcal{X}}$ *is the closure of* $\mathcal{X}$ *in probability.*
(ii) $\forall P' \sim P$, $\overline{\mathcal{X}}^p(P') \cap L^p_{P'}(\mathbf{R}_+, \mathcal{F}_T) = \{0\}, 1 \leq p < \infty$.
(iii) $\forall P' \sim P$, $1 \leq p < \infty$, $\mathcal{D}^q(P') \neq \emptyset$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The implication $(i) \Rightarrow (ii)$ is straightforward. The implication $(ii) \Rightarrow (iii)$ is deduced from the Hahn–Banach separation theorem and the Kreps-Yan theorem. We now prove the implication $(iii) \Rightarrow (i)$. Suppose that (i) does not hold and let $(\psi_n)_{n\geq 0}$ be a sequence in $\mathcal{X}$ converging a.s to $\psi \in L^0_P(\mathbf{R}_+, \mathcal{F}_T) \setminus \{0\}$. Therefore, $\psi_n^- \to 0$ hence $\sup_n [\psi_n]^- < \infty$, P- a.s. We then choose $P' \sim P$ such that $\sup_n [\psi_n]^- \in Lp(P')$. We may also suppose without loss of generality that $\psi_n \in L_p(P')$ for all n. By assumption, there is $Q \sim P'$ such that $dQ/dP' \in L_q(P')$ and $E_Q \psi_n \leq 0$ for all n. Observe that $\sup_n [\psi_n]^- \in L_1(Q)$. Therefore, we may apply the Fatou lemma and deduce that

$$E_Q \psi \leq \liminf_n \psi_n \leq 0,$$

i.e. a contradiction. It follows that **ENAA** holds.

■

Let us denote by $\overline{\mathrm{A}^\infty_T}^w$ the closure of $\mathrm{A}^\infty_T := \mathrm{A}^0_T \cap L^\infty(\mathbf{R}, \mathcal{F}_T)$ with respect to the $\sigma(L^\infty, L^1_P)$-topology. By applying the above lemma with $\mathcal{X} = \mathrm{A}^0_T$ we obtain the following theorem.

Theorem 6.2.3. *The following conditions are equivalent*

(i) **ENAA** *holds, i.e.* $\overline{\mathrm{A}^0_b} \cap L^0_P(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$,

(ii) **ENFL** *holds, i.e.* $\overline{\mathrm{A}^\infty_T}^w \cap L^\infty(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$,

(iii) There exists a Risk Neutral Probability Measure

$$\mathcal{D}^1(P) := \{Q \sim P : \frac{dQ}{dP} \in L^1_P, E_Q X \leq 0 \, \forall X \in \mathrm{LV}^0_b\} \neq \emptyset, \tag{6.2.4}$$

where $\mathrm{LV}^0_b := \{X \in \mathrm{LV}^0_T : \|X^-\| < \infty\}$.

Proof. $(i) \Rightarrow (ii)$: By virtue of Lemma 6.2.2, if **ENAA** holds, the Kreps–Yann theorem asserts that there exists $Z \in L^\infty_P(\mathbf{R}_{++}, \mathcal{F}_T)$ such that $E[ZY] \leq 0$, $\forall Y \in \mathrm{A}^1_T$. By definition of the weak topology, we deduce that $E[ZY] \leq 0$, $\forall Y \in \overline{\mathrm{A}^\infty_T}^w$ hence, $\overline{\mathrm{A}^\infty_T}^w \cap L^\infty(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$.

$(ii) \Leftrightarrow (iii)$It is deduced from the Kreps-Yan theorem that **ENFL** is equivalent to

$$\widetilde{\mathcal{D}}^1(P) := \{Q \sim P : \frac{dQ}{dP} \in L^1_P, E_Q X \leq 0 \, \forall X \in \overline{\mathrm{A}^\infty_T}^w\} \neq \emptyset. \tag{6.2.5}$$

We will show that $\widetilde{\mathcal{D}}^1(P) = \mathcal{D}^1(P)$. Indeed, if $Q \in \widetilde{\mathcal{D}}^1(P)$ we have

$$E_Q X \leq 0 \, \forall X \in \mathrm{A}^\infty_T.$$

We aproximate each $Z \in \mathrm{LV}^{\mathrm{b}}_{\mathrm{T}}$ by $Z \wedge n \in \mathrm{A}^\infty_T$. By using the Fatou's lemma we get that $E_Q Z \leq 0$ which deduces that $Q \in \mathcal{D}^1(P)$.

We now consider $Q \in \mathcal{D}^1(P)$. By deffinition, we have

$$E_Q X \leq 0 \, \forall X \in \mathrm{LV}_T^\infty.$$

Since $\mathrm{A}_T^\infty := \{\lambda X, X \in \mathrm{LV}_T^\infty, \lambda \in [0,1]\}$, we then deduce that

$$E_Q X \leq 0 \, \forall X \in \mathrm{A}_T^\infty.$$

Note that the mapping $X \mapsto E_Q X$ is a continuous linear form with respect to the $\sigma(L^\infty, L_P^1)$- topology. We have $\mathrm{A}_T^\infty \subseteq \mathcal{X}(Q) := \{X : E_Q X \leq 0\}$ where the latter is weak-star closed, which implies that $\overline{\mathrm{A}_T^\infty}^w \subseteq \mathcal{X}(Q)$ or equivalently,

$$E_Q X \leq 0 \, \forall X \in \overline{\mathrm{A}_T^\infty}.$$

i.e. $\widetilde{\mathcal{D}}^1(P)$, hence $\widetilde{\mathcal{D}}^1(P) = \mathcal{D}^1(P)$.
$(iii) \Rightarrow (i)$: Recall a fact in Functional Analysis that , if $P' \sim P$ and $\mathcal{X}$ is a convex set in L^∞, then $\overline{\mathcal{X}}^w(P) = \overline{\mathcal{X}}^w(P')$. By applying this remark to $\mathcal{X} = \mathrm{A}_T^\infty$, we then conclude as in the proof of $(iii) \Rightarrow (i)$ in Lemma 6.2.2.
■

6.3 Pricing and non arbitrage theory : Kabanov's model

6.3.1 Super-hedge pricing

In this subsection we present the super hedging prices of European contingent claims for the Kabanov model. In the presence of transaction costs, contingent claims are traditionally supposed to be multidimensional. By using liquidation functions, we get back to the classical case where contingent claims are unidimensional, and super hedging prices are expressed in terms of cash rather than vector of physical delivery.
Let $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ be an abitrary contingent claim. For ease of arguments, we wil assume that ξ is bounded below. Let us define

$$\Gamma_\xi = \{x \in \mathbf{R} | \, \exists V_T \in R_T^0 : \, x + \mathbf{L}_T(V_T) \geq \xi\}.$$

The super hedging price of the contingent claim ξ is given by

$$V_0^\xi := \inf \Gamma_\xi.$$

Let us define

$$\mathcal{Q}^1 := \{Q \sim P : \, \frac{dQ}{dP} = Z_T^1, Z \in \mathcal{M}_T^0(G^* \backslash \{0\}), Z_0^1 = 1\}.$$

Here, $\mathcal{M}_T^0(G^* \backslash \{0\})$ denotes the set of martingale evolving in $G^* \backslash \{0\}$. When $\mathcal{Q}^1 \neq \emptyset$, it is clear that $\mathcal{Q}^1 \subseteq \mathcal{D}^1$ where $\mathcal{D}^1$ is defined in Theorem 6.2.3. We then get the following lower bounds of V_0^ξ :

$$V_0^\xi \geq \sup\{E_Q \xi : \, Q \in \mathcal{D}^1\} \geq \sup\{E_Q \xi : \, Q \in \mathcal{Q}^1\}. \tag{6.3.6}$$

In fact, we have a stronger result as follows

Theorem 6.3.1. *Suppose that* $\mathcal{M}_T^0(\operatorname{int} G^*) \neq \emptyset$. *We have* $\mathcal{D}^1 \neq \emptyset$ *and*

$$V_0^\xi = \sup_{Q \in \mathcal{D}^1} E_Q \xi = \sup_{Q \in \mathcal{Q}^1} E_Q \xi. \tag{6.3.7}$$

Proof. It suffices to verify that $V_0^\xi = \sup_{Q \in \mathcal{Q}^1} E_Q \xi$. Let us rewrite Γ_ξ in a multidimensional form :

$$\Gamma_\xi = \{x \in \mathbf{R} : \exists V_T \in R_T^0 : xe_1 + V_T \geq \xi e_1\}.$$

Define

$$D_\xi := \{x \in \mathbf{R} : Z_0^1 x \geq E Z_T^1 \xi \, \forall Z \in \mathcal{M}_T^0(G^* \backslash \{0\})\}.$$

Since $\mathcal{M}_T^0(\operatorname{int} G^*) \neq \emptyset$, the set R_T^0 is closed in probability. So, we can mimic the proof of Theorem 3.3.3 [72] to deduce that $\Gamma_\xi = D_\xi$. Consequently, we have $V_0^\xi = \sup_{Q \in \mathcal{Q}^1} E_Q \xi$. ■

6.3.2 NAVR and NABR

In this subsection, we suppose that the fixed cost is negligible, i.e. the Schachermayer-Kabanov model. By means of liquidation functions, we introduce two notions of non-arbitrage **NAVR** and **NABR** which are the analogs of **NFLVR** and **NFLBR** in the classical frictionless cases. A natural question is, how to link these non-arbitrage criteria to the coresponding ones for the multidimensional models ? Not suprisingly, it will be shown that there are indeed a strong connection between two frameworks.

Let us now denote $\overline{\mathrm{A}_T^\infty}$ and $\overline{\mathrm{A}_T^\infty}^w$ the closures of A_T^∞ in strong topology and weak-star topology of L^∞, respectively. We also denote $\overline{\mathrm{A}_T^\infty}^{sw}$ the sequential closure of A_T^∞ in weak-star topology of L^∞. We say that the market satisfies **NAVR**, **NFL** or **NABR** , if $\overline{\mathrm{A}_T^\infty}$, $\overline{\mathrm{A}_T^\infty}^w$ or $\overline{\mathrm{A}_T^\infty}^{sw}$ does not contain any non null random variable $x \in L_+^\infty$, respectivly. Since A_T^∞ is solide convex cone, by similar arguments as in [72], we can rewrite the conditions **NAVR** and **NABR** as follows

Lemma 6.3.2. *i) The market satisfies* **NAVR** *if and only if for a sequence* $(\xi_n) \subseteq \mathrm{A}_T^\infty$, *the condition* $\|\xi_n^-\|_\infty \to 0$ *implies that* $\xi_n \to 0$ *in probability.*
ii)The market satisfies **NABR** *if and only if for a sequence uniformly bounded below* $(\xi_n) \subseteq \mathrm{A}_T^\infty$, *the condition* $\xi_n^- \to 0$ *in probability implies that* $\xi_n \to 0$ *in probability.*

Note that the liquidation function expresses a final wealth in terms of cash. Therefore, it is natural to consider solvency sets in terms of *value*, i.e $\hat{G}_t := \phi_t G_t$ and $\hat{K}_t := \phi_t K_t$ where ϕ_t is the diagonal operator

$$\phi_t : (x_1, \ldots, x_d) \mapsto (x_1 S_t^1, \ldots, x_d S_t^d).$$

Let us now define $\widehat{R}_T^\infty := \phi_T R_T^\infty$. The following results establish a link between **NAVR** and **NABR** with the coresponding non-arbitrage criteria for the multidimensional market model

Theorem 6.3.3. *Assume that (G_t) is a conic model and G_T dominates $\mathbf{R}^d_+$, we have*

i) The market satisfies **NAVR** *if and only if*

$$\overline{\widehat{\mathrm{R}}^\infty_T} \cap L^\infty_+(\mathbf{R}^d, \mathcal{F}_T) = \{0\}.$$

ii) The market satisfies **NABR** *if and only if*

$$\overline{\widehat{\mathrm{R}}^\infty_T}^{sw} \cap L^\infty_+(\mathbf{R}^d, \mathcal{F}_T) = \{0\}.$$

Proof. i) As a basic property of the liquidation function, we have $\mathrm{A}^\infty_T e_1 \subseteq \widehat{\mathrm{R}}^\infty_T$ which deduces that $\overline{\mathrm{A}^\infty_T} e_1 \subseteq \overline{\widehat{\mathrm{R}}^\infty_T}$. The part "if" of the statement then follows. We now suppose that the market satisfies **NAVR** and that there exists a random variable $X \in \overline{\widehat{\mathrm{R}}^\infty_T} \cap L^\infty_+(\mathbf{R}^d, \mathcal{F}_T)$. Suppose that X is aproximated by a sequence $X_n \in \widehat{\mathrm{R}}^\infty_T$, i.e. $\|X_n - X\|_\infty \to 0$. This sequence is therefore bounded. Since $\mathbf{L}_T$ is uniformly continuous in any compact set, it is straightforward that $\|\mathbf{L}_T(X_n)^-\|_\infty \to 0$ and $\mathbf{L}_T(X_n) \to \mathbf{L}_T(X) \geq 0, \neq 0$. By applying the lemma 6.3.2 i) we deduce that $\mathbf{L}_T(X) = 0$, therefore $X = 0$.
ii) The part "if" is proved similarly as in *i*). Suppose **NABR** and let $X \in \overline{\widehat{\mathrm{R}}^\infty_T}^{sw} \cap L^\infty_+(\mathbf{R}^d, \mathcal{F}_T)$. There exists $X_n \in \widehat{R}^\infty_T$ such that $X_n \to X$ in weak-star topology. Hence (X_n) is uniformly bounded. By passing through convex combinations, we can suppose that $X_n \to X$ a.s. Since $(\mathbf{L}_T(X_n))$ is uniformly bounded below and $\mathbf{L}_T(X_n) \to \mathbf{L}_T(X) \geq 0, \neq 0$ it is deduced, by using Lemma 6.3.2 ii) that $\mathbf{L}_T(X) = 0$ or $X = 0$. ■

Remark 6.3.4. 1. *The theorem rises a natural question : is it true that* **NFL** *is equivalent to*

$$\overline{\widehat{\mathrm{R}}^\infty_T}^{w} \cap L^\infty_+(\mathbf{R}^d, \mathcal{F}_T) = \{0\}. \tag{6.3.8}$$

It is trivial that **NFL** *is deduced from 6.3.8, but the reverse direction is not necessarily true in general. This still remains as an open question. However, we have a affirmative answer for the two assets case, see Theorem 6.4.6 below.*

2. *It is well known that in some frictionless markets, the conditions* **NFLVR**, **NFLBR** *and* **NFL** *are equivalent. Furthermore, these conditions are also equivalent to the existence of a local martingale probability measure. In the presence of transaction costs, e.g in the Kabanov model, a similar result has not been known yet to date.*

3. *Suppose that the probability space is finite. Is it true that the conditions* **NWA** *and* **NA** *are equivalent for the conic model K ? (In this case* **NA** *is also equivalent to the existence of a separate probability measure $Q \sim P$). The answer is no. For example : Let us take a financial market model of one period. The probability is finite : $\Omega = \{\omega_1, \omega_2\}$ such that $P(\omega_i) = \frac{1}{2}$. Suppose that the interest rate is zero : $B_0 = B_1 = 1$ and the risky asset is $S_0 = 1, S_1 = 1 + \xi$ where $\xi(\omega_1) = 0, \xi(\omega_1) = 1$. The market is assumed to be*

frictioness. It is straightforward that this model admits an infinite number of Arbitrage Opportunity : If the number of risky asset hold at maturity $T = 1$ is $x > 0$, then the final wealth is $x\xi \geq 0$ and $x\xi \neq 0$. However, this model does not admit any Weark Arbitrage Opportunity. It is easily proved that the separation martingale measure in this case is the Dirac Measure concentrating at ω_1 which is not equivalent to P.

5. *Question : can we generalize the above result to the non convex case ? The answer would be no ! The reason is that the condition* $\overline{\mathrm{LV}_{t,T}}^{w} \cap L^{\infty}_{+}(\mathbf{R}, \mathcal{F}_T) = \{0\}$ *does not imply* $\overline{\widehat{\mathrm{R}}_{t,T}}^{w} \cap L^{\infty}_{+}(\mathbf{R}^d, \mathcal{F}_T) = \{0\}$ *as* $\mathbf{L}_T$ *is not continuous. For instance, let us take an example when the model is two dimensional : the sequence* $V^n_T := (\frac{c_T}{2}, -\frac{1}{n})$ *converges to* $(\frac{c_T}{2}, 0) \in L^{\infty}_{+}$ *but* $\mathbf{L}_T(V^n_T) \to -\frac{c_T}{2} \leq 0$.

6.4 Market with one risky asset

Let us consider a market model with two underlying assets : one risk-free asset whose value is constant over time : $B_t = 1$ for all t, and one risky asset of which the fixed cost process c_t is assumed to be bounded, and the bid-ask spreads are given by $[S^b_t, S^a_t]$ where S^a_t and S^b_t are two adapted processes. Denote the solvency set process of the initial market and the extended one by (G_t) and (K_t), respectively. For more details about the two dimensional case, see [52]. Observe that $K_t \supseteq \mathbf{R}^2_+$ is the closed proper proper cone generated by the bid-ask interval $[S^b_t, S^b_t]$, usually called the solvency cone in the Schachermayer model. We denote $C_t = c_t e_1$ and observe that $K_t + C_t \subseteq G_t$ consists of transactions that the investor has to pay fixed costs. The following lemma will be used frequently in this section. The proof is straightforward so we will leave it to readers.

Lemma 6.4.1. *i) Suppose that* $V_T = \sum_{t=0}^{T} \xi_t \in R^0_T(G)$ *where* $\xi_t \in -G_t$ *be a portfolio in the market G. Let us define*

$$\widetilde{V}_T := \sum_{t=0}^{T} [\xi_t + c_t 1_{-\xi_t \in K_t + C_t}].$$

Hence, $\widetilde{V}_T$ *is a terminal wealth in the market K satisfying* $\widetilde{V}_T \geq V_T$.

ii) Suppose that $\widetilde{V}_T = \sum_{t=0}^{T} \widetilde{\xi}_t \in R^0_T(K)$ *where* $\widetilde{\xi}_t \in -K_t$ *be a porfolio in the market K. Let us define*

$$V^k_T := \sum_{t=0}^{T} [k\widetilde{\xi}_t - c_t 1_{\xi_t \neq 0}], k \in \mathbf{R}.$$

Hence, V^k_T *is a terminal wealth in the market G.*

6.4.1 NWA condition

In this subsection, we study the relationship between the market G and it enlarged one K. We introduce a new notion of arbitrage called No Weak

Arbitrage Opportunity. The following definition can be applied for a general market model.

Definition 6.4.2. *In a financial market model defined by a liquidation value process L, we say that a portfolio process V starting from the initial endowment $V_{0-} = 0$ realizes a weak arbitrage opportunity if there exists $t \in \{0, 1, \cdots, T\}$ such that $V_u = 0$ for all $u \leq t-1$, there exists $B_t \in \mathcal{F}_t$ with $P(B_t) > 0$ such that $V_u 1_{\Omega \setminus B_t} = 0$ for all $u = t, \cdots, T$ and $\mathbf{L}_T(V_T) > m_t$ on B_t where $m_t \in L^0((0, \infty), \mathcal{F}_t)$. When there is no weak arbitrage opportunity, we say that the property No Weak Arbitrage (* **NWA** *) holds.*

Remark 6.4.3. The above definition is motivated from and designed for the model with fixed transaction costs as follows : if there is an arbitrage opportunity in the model with fixed transaction cost, there is a first moment $t = u$ when the agent holding the portfolio pays the fixed cost c_u with a strictly positive probability. Indeed, otherwise, the terminal liquidation value of the portfolio process is negative. Therefore, the terminal value of the portfolio process generated by the same strategy but for the model without fixed transaction cost is larger than c_u with a strictly positive probability.

Theorem 6.4.4. *Suppose that the fixed cost process is uniformly bounded above, i.e. $c_t \leq c_{\max} < \infty \, \forall t = 1, \ldots, T$. Then, G satisfies* **NWA** *if and only if K does.*

Proof. By using Lemma 6.4.1 i) we can easily deduce that G satisfies **NWA** if K does. Conversely, suppose that K does not satisfy **NWA**. By definition, there exists $B_t \in \mathcal{F}_t$ with $P(B_t) > 0$ and a fortfolio $\widetilde{V}_T = \sum_{t=0}^{T} \widetilde{\xi}_t \in R_T^0(K)$ such that $\widetilde{V}_u 1_{\Omega \setminus B_t} = 0$ for all $u = t, \cdots, T$ and $\mathbf{L}_T^K(V_T) > m_t$ on B_t where $m_t \in L^0((0, \infty), \mathcal{F}_t)$. Without lost of generality we can suppose that $m_t \geq \varepsilon > 0$ and $\widetilde{V}_T = \mathbf{L}_T^K(\widetilde{V}_T)e_1$. Lemma 6.4.1 ii) shows that we can construct a new portfolio V_T^k in the market G such that $V_T^k := \sum_{t=0}^{T} [k\widetilde{\xi}_t - c_t 1_{\xi_t \neq 0}], k > 0$. We then have $V_T^k = k\mathbf{L}_T^K(\widetilde{V}_T)e_1 - C_T^{\widetilde{V}_T}$ where $C_T^{\widetilde{V}_T}$ is the cumulated fixed cost which is bounded by $c_{\max} T$. Therefore

$$\mathbf{L}_T^G(V_T^k) \geq \mathbf{L}_T^G(k\widetilde{V}_T) - c_{\max} T \geq k\varepsilon - c_{\max} T > 0$$

for k large enough which deduces that V_T^k is a Weak Arbitrage Opportunity in the market G, i.e. G does not satisfy **NWA**. ■

Remark 6.4.5. The crucial point in the proof of the above proposition is that the fixed costs in the market G are bounded and do not depend on the size of transactions. This leads to the fact that if we scale the trading strategy up to a very large scalar, then the fixed costs become rather small in compare with the bid-ask effect. This observation is important because it show us the way to construct asymptotic arbitrage strategies for the market G from those of the market K. In a general setting we can also obtain the same results provided that the fixed costs in G are bounded.

6.4.2 NFL condition

The following theorem is one of the main result of this section. It establishes the equivalence between **ENAA** of the liquidative model and **NFL** of the

multidimensional conic model. Unfortunately, the technics of the proof are not able to be generalized to the multidimensional case.

Theorem 6.4.6. *Suppose that $S_t^a > S_t^b$ for all $t \in [0,T]$ a.s. Then, the following statements are equivalent :*

(i) (**ENAA***) holds for the model under both fixed and proportional transaction costs,*

(ii) (**ENAA***) holds for the same model under proportional transaction costs and $c = 0$,*

(iii) (**NFL***) holds for $\hat{K}$, i.e the multidimensional market model with only proportional transaction costs.*

(iv) For all $P' \sim P$, there exits a process $\tilde{S} \in [S^b, S^a]$ and $Q \sim P'$ such that dQ/dP' is bounded and $\tilde{S}$ is a Q-martingale.

Proof.

First, observe that (iii) and (iv) are equivalent. This is a classical result, see Proposition 2.1 [57]. Indeed, we may choose $dQ/dP = Z_T^1$ and $\tilde{S}_t = Z_T^2/E_P(Z_T^1|\mathcal{F}_t)$ where $Z = (Z^1, Z^2)$ is a consistent price system according to the associated Kabanov model. Moreover, it is clear that these conditions imply (ii) and (ii) also implies (i). Let us show that (i) $\Rightarrow$ (iv). By virtue of Theorem 6.2.2, there exists $Q \sim P'$ such that $dQ/dP \in L^\infty$ and $E_Q[\mathbf{L}_T(V_T)] \leq 0$ for all $\mathbf{L}_T(V_T) \in \mathrm{L}_T^0 \cap L^1(P')$. We may assume without loss of generality that $\sup_t(|S_t^a| + |S_t^b| + |c_t|)$ is P' integrable. Consider two stopping times σ and τ such that $\sigma \leq \tau \leq T$ a.s. and take $F_\sigma \in \mathcal{F}_\sigma$ and $N_\sigma \in L^\infty(\mathbf{R}_+, \mathcal{F}_\sigma)$. We define the portfolio process V such that $V_{0-} = 0$, $\Delta V = \Delta V 1_{\{\sigma\}} + \Delta V 1_{\{\tau\}}$ and

$$\begin{aligned} V_\sigma &= (-N_\sigma S_\sigma^a - c_\sigma 1_{N_\sigma \neq 0}, N_\sigma) 1_{F_\sigma}, \\ V_\tau &= \mathbf{L}_\tau(V_\sigma) e_1. \end{aligned}$$

Since $\mathbf{L}_\sigma(-V_\sigma) = 0$, we deduce that $\Delta V_\sigma \in -G_\sigma$. Moreover, $\Delta V_\tau \in -G_\tau$ since V_τ is the liquidation value of the frozen position V_σ. It follows that $\mathbf{L}_T(V_T) = \mathbf{L}_\tau(V_\sigma)$ where

$$\mathbf{L}_\tau(V_\sigma) = \left(-N_\sigma S_\sigma^a - c_\sigma 1_{N_\sigma \neq 0} + \left(N_\sigma S_\tau^b - c_\tau \right)^+ \right) 1_{F_\sigma}.$$

We deduce that

$$E_Q \left[\left(N_\sigma S_\tau^b - c_\tau \right)^+ | \mathcal{F}_\sigma \right] \leq N_\sigma S_\sigma^a + c_\sigma 1_{N_\sigma \neq 0}. \tag{6.4.9}$$

Similarly, we define the portfolio process V such that $V_{0-} = 0$, $\Delta V = \Delta V 1_{\{\sigma\}} + \Delta V 1_{\{\tau\}}$ and

$$\begin{aligned} V_\sigma &= ((N_\sigma S_\sigma^b - c_\sigma)^+, -N_\sigma) 1_{F_\sigma}, \\ V_\tau &= \mathbf{L}_\tau(V_\sigma) e_1. \end{aligned}$$

Let us explain how we obtain V_σ. First, we write

$$\mathbf{R}^2 \ni V_{\sigma-1} = 0 = (N_\sigma S^b_\sigma - c_\sigma, -N_\sigma)1_{F_\sigma} + (-N_\sigma S^b_\sigma + c_\sigma, N_\sigma)1_{F_\sigma}$$

and we liquidate the second position in the equality above at time σ. We get

$$\begin{aligned}\mathbf{L}_\sigma\left((-N_\sigma S^b_\sigma + c_\sigma, N_\sigma)1_{F_\sigma}\right)) &= \left(-N_\sigma S^b_\sigma + c_\sigma + (N_\sigma S^b_\sigma - c_\sigma)^+\right)1_{F_\sigma} \\ &= \max(-N_\sigma S^b_\sigma + c_\sigma; 0)1_{F_\sigma}.\end{aligned}$$

At last, we add this liquidation value to the first component of the portfolio position $(N_\sigma S^b_\sigma - c_\sigma, -N_\sigma)1_{F_\sigma}$. We then deduce that that $\mathbf{L}_T(V_T) = \mathbf{L}_\tau(V_\sigma)$ where

$$\mathbf{L}_\tau(V_\sigma) = \left((N_\sigma S^b_\sigma - c_\sigma)^+ - N_\sigma S^a_\tau - c_\tau\right)1_{F_\sigma}.$$

We deduce that

$$(N_\sigma S^b_\sigma - c_\sigma)^+ \leq E_Q\left[N_\sigma S^a_\tau + c_\tau | \mathcal{F}_\sigma\right]. \tag{6.4.10}$$

We then choose $N_\sigma = n \in \mathbb{N}$ and we make n converged to ∞. By virtue of Inequalities (6.4.9) and (6.4.10), we deduce that for all stopping times σ and τ, we have

$$E_Q\left[S^b_\tau | \mathcal{F}_\sigma\right] \leq S^a_\sigma, \tag{6.4.11}$$

$$S^b_\sigma \leq E_Q\left[S^a_\tau | \mathcal{F}_\sigma\right]. \tag{6.4.12}$$

Therefore, we deduce the existence of a Q-martingale $\tilde{S} \in [S^b, S^a]$ a.s. ■

6.4.3 Case $0 < c_{\min} < c_{\max} < \infty$

In this subsection, we study some types of arbitrage opportunities for a non convex market model. Inspired from Lemma 6.3.2, in the definition below we introduce two notions **NAVR** and **NWAVR** which naturally are asymptotic analogs of **NA** and **NWA**. We make use of LV^∞ instead of A^∞_T since the former has a naturally economical interpretation. More suprisingly is the theorem 6.4.8 below which states that **NAVR** and **NWAVR** are in fact not really a strict generalization of **NA** and **NWA** in the case $0 < c_{\min} < c_{\max} < \infty$.

Definition 6.4.7. *We say that the* **NAVR** *condition holds if, for all sequence $\xi^n \in \mathrm{R}^0_T$ such that $\mathbf{L}_T(\xi^n) \geq -\beta_n$ for all n where $\beta_n \to 0$ and $\xi^n \to \xi \in G_T$ a.s., we have $\mathbf{L}_T(\xi) = 0$.*

The market is said to be satisfied the **NWAVR** *condition if, for all $t = 1, \ldots, T$ and all sequence $\xi^n \in \mathrm{R}^0_{t,T}$ such that $\mathbf{L}_T(\xi^n) \geq -\beta_n$ for all n where $\beta_n \to 0$ and $\xi^n \to \xi$ a.s., $\mathbf{L}_T(\xi) \geq \psi_t \in L^0_+(\mathcal{F}_t)$ we have $\psi_t = 0$.*

Theorem 6.4.8. *If $0 < c_{\min} < c_{\max} < \infty$ then four conditions* **NA**, **NAVR**, **NWA** *and* **NWAVR** *are equivalent.*

Proof. By elementary arguments we can show that **NAVR** $\Rightarrow$ **NA** $\Rightarrow$ **NWA** and **NAVR** $\Rightarrow$ **NWAVR** $\Rightarrow$ **NWA**. It then suffices to verify that **NWA** $\Rightarrow$ **NAVR**. Suppose on the contrary that, if **NAVR** does not satisfy, then we can find a sequence $V_T^n \in \mathrm{R}_T^0$ such that $\mathbf{L}_T(V_T^n) \geq -\beta_n$ for all n where $\beta_n \to 0$ and $V_T^n \to X_T \in G_T$ a.s. such that $\mathbf{L}_T(X_T) \geq 0, \mathbf{L}_T(X_T) \neq 0$.
We now aim at constructing a new sequence of porfolio $\hat{V}_T^n := \sum_{t=1}^T \hat{\xi}_t^n$ such that $\hat{V}_T^n \succeq V_T^n$ and $\hat{\xi}_t^n = \hat{\xi}_t^n 1_{-\hat{\xi}_t^n \in K_t + C_t.}$ Let us define, for each $1 \leq t \leq T$:

$$\hat{V}_t^n := V_t^n + \sum_{u=1}^{t} [\mathbf{L}_u(V_{u-1}^n - V_u^n)e_1 + r_u^+(V_{u-1}^n - V_u^n)],$$

where r_t^+ is defined by $r_t^+(x) := r_t(x)1_{r_t(x) \in \mathbf{R}_+^d}, r_t(x) := x - \mathbf{L}_t(x)e_1$ denoting the remaining part of x after liquidating. The above equality shows that $\hat{V}_t^n \succeq V_t^n$. We also get that

$$\hat{V}_{t-1}^n - \hat{V}_t^n = V_{t-1}^n - V_t^n - \mathbf{L}_t(V_{t-1}^n - V_t^n)e_1 - r_t^+(V_{t-1}^n - V_t^n)].$$

We deduce from this equality that $-\hat{\xi}_t^n \in K_t + C_t \subseteq G_t$ where $\hat{\xi}_t^n = \hat{V}_t^n - \hat{V}_{t-1}^n$. Therefore, $\hat{V}_t^n$ is a portfolio satisfying all required properties.
We now choose n_0 large enough such that $\beta_n < -c_{\min} \forall\, n \geq n_0$. Since $\hat{V}_T^n \succeq V_T^n$ we have $\mathbf{L}_T(\hat{V}_T^n) \geq \mathbf{L}_T(V_T^n)$. We then get that

$$\liminf \mathbf{L}_T(\hat{V}_T^n) \geq \mathbf{L}_T(X_T) \geq 0, \mathbf{L}_T(X_T) \neq 0.$$

Therefore, we can deduce that there exists $n_1 \geq n_0$ such that $\hat{V}_T^{n_1} \neq 0$. We rewrite $\hat{V}_T^{n_1} = \sum_{t=0}^T \hat{\xi}_t^{n_1}$ where $\hat{\xi}_t^{n_1} \in -G_t$. For ease of notations we will denote $V_T^* = \hat{V}_T^{n_1}$. We also suppose that $V_T^* = \mathbf{L}_T(V_T^*)e_1$. Let us define

$$\pi^* := \min\{t \geq 0 :\ \xi_t^* \neq 0\},$$

and

$$t^* := \min\{t \geq 0 :\ P(\pi^* = t) > 0\}.$$

This means that we start making transactions from the instant t^* and do nothing before t^*. Denote $B_{t^*} := \{\xi_{t^*}^* \neq 0\}$, we have $P(B_{t^*}) > 0$. From the construction of V_T^* we have $-\xi_{t^*}^* \in K_t + C_t$ on B_{t^*} By using Lemma 6.4.1 i), we can construct a new porfolio $\tilde{V}_T$ in the anlarged market K associated to V_T^* such that $\tilde{V}_T = V_T^* + C_T^* e_1$ where C_T^* is the cumulated fixed costs incuring in the portfolio V_T^*. It is now clear that $C_T^* \geq c_{\min}$ on the event B_{t^*}. As $V_T^* = \mathbf{L}_T(V_T^*)e_1$ we get that

$$L_T^K(\tilde{V}_T) \geq c_{\min} - \beta_{n^1} =: \varepsilon > 0 \,\text{on}\, B_{t^*} : .$$

By Lemma 6.4.1 ii) we can construct a new porfolio V_T^k in the market G such that

$$V_T^k = k\tilde{V}_T - C_T e_1, k > 0,$$

where $C_T \leq c_{\max} T$. By choosing k large enough we have

$$\mathbf{L}_T(V_T^k) \geq k\varepsilon - c_{\max} T > 0 \,\text{on}\, B_{t^*} :,$$

i.e. V_T^k is a Weak Arbitrage Opportunity in the market G.

■

Corollary 6.4.9. *Suppose that* $0 < c_{\min} < c_{\max} < \infty$. *Then* **NAVR** *is equivalent to* $\overline{\mathrm{LV}_{0,T}^{\infty}} \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ *and* **NWAVR** *is equivalent to* $\overline{\mathrm{LV}_{t,T}^{\infty}} \cap L^0(\mathbf{R}_+, \mathcal{F}_t) = \{0\}$.

Chapitre 7

Utility Maximization under Target Risk Constraints

Abstract

This chapter studies the classical utility maximization problem in a general continuous-time financial markets model under target risk constraints, i.e. given in terms of some expected loss constraints imposed on the final wealth. The standard duality technics are used to solve the problem. In a complete market, we relate the utility maximization problem to some kind of hedging problem with multiple targets. The latter is solved by using both backward stochastic differential equation (BSDE) and convex duality technics.

Keywords : Utility maximization, Duality method, Shortfall risk, Convex risk measures, Optimal portfolio choice, Quantile hedging.

Note. This chapter is based on the article *Utility Maximization under Target Risk Constraints,* T. Tran. Preprint. It was done under the supervision of Bruno Bouchard and I want to thank him for his advice and helpful comments.

7.1 Introduction

In this chapter, we consider the utility maximization problem under expected loss constraints. First introduced by Merton (1971) in the non constraint case, the problem has attracted a lot of attentions from academicians and practitioners. Especially in the case of complete markets, it is solved by the martingale and duality methods, see for example Karatzas, Lehoczky and Shreve (1987) or Cox and Huang (1989). These technics appear to be very powerful even for incomplete markets. In the non-Markovian case without constraints, the existence of solutions and the characterizations of optimal strategies are given by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991). Similar results have been derived by Cvitanic and Karatzas (1992) or Cuoco (1997) in the case of portfolio constraints.

For financial institutions the measurement and management of downside risk is a key issue. Regulators, for example, might impose a risk constraint to certain companies, a manager of a firm might require his traders to stay within some risk limit, or an investor might wish to bound his own risk exposure. It is then natural to consider the utility maximization problem under risk constraints. Optimal investment policies under downside risk constraints in terms of value at risk and and a second risk functional have been studied in a Brownian setting by Basak & Shapiro (2001) and Gabih et al. (2005). A complete solution in a general semimartingale with utility-based shortfall risk constraints is given by Gundel and Weber (2005). In the latter, the authors reduce the problem to a static optimization one under constraints and then solve it by means of classical Lagrange multipliers. These mutipliers corespond to the expected loss and the budget constraints.

In this chapter, we investigate the utility maximization problem under expected loss constraints in a general setting, where the price process is described by a general semimartingale and the risk constraint is represented by a non decreasing concave function. This framework is more or less similar to that of Gundel and Weber (2005). The difference is that we chose the convex duality theory in the spirit of Kramkov and Schachermayer (1999) or Schachermayer (2001) to tackle the problem. Therefore, rather than providing a closed-form solution to the optimization problem, we only characterize the optimal solution through the duality relation between the primal and the dual problem by apealing to the set of equivalent local martingale measures. The duality technic is reviewed in different situations and with different ways. We consider both complete and incomplete markets ; the wealth might be negative or non-negative. We also consider both unconstrained and constrained Fenchel dual functions and provide the link between utility maximization problem and hedging-type problem in the complete market case.

The remainder of the chapter is structured as follows. In Section 6.2, we consider the problem with positive wealth constraint in an incomplete market framework. We show that the duality technic used in Kramkov and Schachermayer (1999) can be easily adapted to our new setting. Section 6.3 relaxes the positivity constraints by approximating the given utility function by a sequence of new utility functions bounded from below as being done in Schachermayer (2001). The case of complete markets is studied in Section 6.4 and Section 6.5 by two approaches which are still based on duality technics. By introducing new random shortfall thresholds which is inspired from the paper of Bouchard-Touzi-Elie (2009), in Section 6.4 we boil down the problem from one expected loss constraint to an infinite number of almost sure constraints. The latter is simpler to deal with by means of constrained Fenchel dual function, which leads to a new kind of duality in the case of complete markets. In Section 6.5, we relate the utility maximization problem to the hedging-type problem with multiple targets. This is a special kind of minimal solutions to a BSDE with weak terminal conditions as considered in Bouchard-Elie-Réveillac (2013).

7.2 Utility Maximization Problem on The Half Real Line

7.2.1 Formulation of the problem

We consider a financial market model over a finite time horizon $[0,T], T>0$ which consists of d risky assets and one risk-free asset defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{R})$ satisfying the usual conditions. Suppose that the price of the risk-free asset is constant over time and equal to 1, and the prices of the risky assets are described by a strictly positive $d-$ dimensional semi-martingale $(S_t)_{0\leq t\leq T}$ adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. Throughout this chapter we only consider the case where $(S_t)_{0\leq t\leq T}$ is a locally bounded process. The dynamics of the portfolio is given by

$$X_t^{x,\pi} = x + \int_0^t \pi_s dS_s, \quad 0\leq t\leq T, \tag{7.2.1}$$

where x is a positive initial endownment, and $(\pi_t)_{0\leq t\leq T}$ denotes the portfolio process that is assumed to be an element of $L(S)$, the set of all $\mathbf{R}^d$-valued predictable processes which are integrable with respect to S. We assume throughout this chapter that the family of equivalent local martingale measure $\mathcal{M}(S)$ is non empty.

In this section, we consider the case of non-negative wealth.The next section will be devoted to the case where wealth may be negative. Let us first denote

$$\mathcal{A}_0^+(x) := \{(\pi_t)_{0\leq t\leq T} \in L(S): \quad X_t^{x,\pi} \geq 0 \quad \forall 0\leq t\leq T\}, \tag{7.2.2}$$

At maturity, the wealth is subject to a loss constraint which is described by some *random utility-type function* ℓ

$$\mathcal{A}^+(x,m) := \{(\pi_t)_{0\leq t\leq T} \in \mathcal{A}_0^+(x): \quad E[\ell(X_T^{x,\pi})] \geq m\}, \tag{7.2.3}$$

where $\ell : \Omega \times \mathbf{R}_+ \to \mathbf{R}$ is a non decreasing random continuous concave function and m is some constant in $\operatorname{Im}\ell$. Throughout this chapter, we suppose that the inactive strategy belongs to $\mathcal{A}^+(x,m)$, i.e. $E[\ell(x)] \geq m$. A typical example for this kind of loss function is $\ell(X) := -l[(X-B)^-]$, where $l : \mathbf{R}_+ \to \mathbf{R}_+$ is a strictly increasing deterministic convex function and $B \in L_+^\infty$ may be some random liability or some market benchmark at maturity. This risk constraint measures the weighted expected shortfall loss from the level B. In particular, if l is the identity function then we turn back to the case of the usual expected shortfall risk measure ; and if $l = 1_{\mathbf{R}_+}$ we are led to the case of controlled loss probability constraints.

In the remainder of the chapter, we will assume that ℓ is bounded from above. This condition is natural from the practical point of view because, when it comes to risk management, we focus on the loss rather than the gain.

Let us now define the set of hedgeable positions :

$$\mathcal{C}^+(x,m) := \{X \in L_+^0(\mathcal{F}_T, P) - L_+^\infty(\mathcal{F}_T, P): \quad \exists \pi \in \mathcal{A}^+(x,m), X \leq X_T^{x,\pi}\} \tag{7.2.4}$$

We also denote $\mathcal{C}_0^+(x,m) := \mathcal{C}^+(x,m) - x$. Our goal is to study the utility maximization under target constraint :

$$u(x) = \sup_{X \in \mathcal{C}^+(x,m)} E[U(X)]. \tag{7.2.5}$$

Here, $U : \mathbf{R}^+ \to \mathbf{R}$ denotes the utility function which is assumed to be a strictly increasing concave C^1-function satisfying the Inada condition :

$$\lim_{r\to 0} U'(r) = +\infty \text{ and } \lim_{r\to+\infty} U'(r) = 0.$$

In order to avoid the trivial cases, we suppose in this chapter that $u(x) < U(\infty)$. Moreover, we need the following condition :

Assumption A : The asymptotic elasticity of the utility function at ∞ is strictly smaller than 1 :

$$\mathrm{AE}_\infty(U) := \mathrm{limsup}_{r\to\infty} \frac{rU'(r)}{U(r)} < 1. \tag{7.2.6}$$

We recall here the definition of Fenchel duality function

$$V(y) := \sup_{r>0}[U(r) - ry], y > 0.$$

The explicit form of V is given by

$$V(y) = U(I(y)) - yI(y),$$

where $I(y) := (U')^{-1} = -V'$. The duality function V is a decreasing and strictly convex function. Moreover $V(0) = U(\infty), V(\infty) = U(0)$ and, under the Inada conditions, we have $V'(0) = -\infty$ and $V'(\infty) = 0$.

7.2.2 Convex duality approach

We now follow the approach in [12] : Let $\mathcal{Y}^+ := \mathcal{Y}^+(x,m)$ be the duality domain defined by

$$\mathcal{Y}^+ := \{(y,Y) \in \mathbf{R}_+ \times L^1_+ : EY \le y, \delta(Y) := \sup_{X\in\mathcal{C}_0^+(x,m)} E[XY] < \infty\} \tag{7.2.7}$$

Here, δ is called to be support operator. From its definition, we can deduce that δ is non-negative and convex. As a consequence, $\mathcal{Y}^+$ is a convex set in $\mathbf{R}_+ \times L^1_+$. The following result characterizes the set of hedgeable claims :

Lemma 7.2.1. *Under the assumption $\mathcal{M}(S) \neq \emptyset$, we have*

$$\mathcal{C}^+(x,m) = \{X \in L^0_+ - L^\infty_+ : \sup_{(y,Y)\in\mathcal{Y}^+} \{E[XY] - \delta(Y) - xy\} \le 0\}. \tag{7.2.8}$$

Proof. Let $\mathcal{C}^+(x)$ be the set of hedgeable claims without loss constraints, i.e.

$$\mathcal{C}^+(x) := \{X \in L^0_+(\mathcal{F}_T, P) - L^\infty_+(\mathcal{F}_T, P) : \quad \exists \pi \in \mathcal{A}^+_0(x) : X \leq X^{x,\pi}_T\}.$$

It is well known that $\mathcal{C}^+(x)$ is Fatou closed when the set of equivalent local martingale measures is not empty. We first prove that $\mathcal{C}^+(x,m)$ is also Fatou closed. Indeed, let $(X_n) \in \mathcal{C}^+(x,m)$ be a sequence which is Fatou convergent to a random variable X. Obviously $X \in \mathcal{C}^+(x)$. By definition we can find a sequence of final wealth (X^{x,π_n}_T) such that $X_n \leq X^{x,\pi_n}_T$ and $E[\ell(X^{x,\pi_n}_T)] \geq m$ for all n. Since (X^{x,π_n}_T) is uniformly bounded from below we can construct a new sequence $\widetilde{X}^n$ such that $\widetilde{X}^n_T \in \text{conv}\,\{X^{x,\pi_n}_T, X^{x,\pi_{n+1}}_T, \ldots\}$ and $\widetilde{X}^n_T \to \widetilde{X}_T$ a.s. We have $\widetilde{X}_T \in \mathcal{C}^+(x)$ since $\mathcal{C}^+(x)$ is convex and Fatou closed. By virtue of the concavity of ℓ we first observe that $E[\ell(\widetilde{X}^n_T)] \geq m$ for all n. As $\ell(\widetilde{X}^n_T)$ is uniformly bounded from above we can apply the Fatou's Lemma to deduce that $E[\ell(\widetilde{X}_T)] \geq \limsup E[\ell(\widetilde{X}^n_T)] \geq m$. Therefore, $\widetilde{X}_T \in \mathcal{C}^+(x,m)$. Since $X_n \leq X^{x,\pi_n}_T$ for all n we also have $X \leq \widetilde{X}_T$ and then $X \in \mathcal{C}^+(x,m)$, which deduces that $\mathcal{C}^+(x,m)$ is Fatou closed.

Let's get back to the proof. Note that if $\overline{X} \in \mathcal{C}^+(x,m)$ then, by the definition of support operator δ we have

$$E[\overline{X}Y] - \delta(Y) - xy \leq 0 \,\forall (y,Y) \in \mathcal{Y}^+. \tag{7.2.9}$$

The first inclusion in (7.2.8) is therefore trivial. Let's take $\overline{X} \in L^0_+ - L^\infty_+$ satisfying (7.2.9). We will show that $\overline{X} \in \mathcal{C}^+(x,m)$. By considering $\overline{X} \wedge n$ instead of X and taking the limit when n tends to infinity, we can assume that $\overline{X} \in L^\infty$ (because $\mathcal{C}^+(x,m)$ is Fatou closed). Now, suppose on the contrary that $\overline{X} - x \notin \mathcal{C}^\infty_0(x,m) := \mathcal{C}^+_0(x,m) \cap L^\infty$. Since $\mathcal{C}^\infty_0(x,m) \supseteq -L^\infty_+(\mathcal{F}_T)$, by the Kreps-Yan theorem, we can find a separating probability measure $Q \sim P$ such that $Y := \frac{dQ}{dP} \in L^1$ and that

$$\sup_{X \in \mathcal{C}^\infty_0(x,m)} E[XY] < E[(\overline{X} - x)Y] < \infty.$$

Consequently, $\delta(Y) < \infty$ and therefore $(1, Y) \in \mathcal{Y}^+$. But the previous inequality also deduces that $E[\overline{X}Y] - \delta(Y) - x > 0$, a contradiction with (7.2.9). Hence, we can conclude that $\overline{X} \in \mathcal{C}^+(x,m)$. ■

Remark 7.2.2. *Additionally, if the portfolio is subject to another convex constraint, for instance, we require that $\pi_t \in K$ for all t, where K is some closed convex set in $\mathbf{R}^d$, then we also obtain the same result as in Lemma 7.2.8 above. We refer readers to [38], Proposition* 5.1. *and [88], Theorem* 4.1. *for more details.*

By using the definition of Fenchel dual function, we can easily prove the following lemma.

Lemma 7.2.3.

$$u(x) \leq \inf_{(y,Y) \in \mathcal{Y}^+} [v(Y) + xy], \tag{7.2.10}$$

where

$$v(Y) := E[V(Y)] + \delta(Y). \tag{7.2.11}$$

In the sequel we suppose that

$$\widetilde{\mathcal{Y}}^+ := \{(y, Y) \in \mathcal{Y}^+ : E[V(Y)] < \infty\} \neq \emptyset. \tag{7.2.12}$$

It is straightforward that $\widetilde{\mathcal{Y}}^+$ is also convex in $\mathbf{R}_+ \times L^1_+$. The following preliminary result will be frequently used. The interested reader can find its proof in, for example, Lemma 3.2 of [72].

Lemma 7.2.4. *The set* $\{V(Y)^- : Y \in \widetilde{\mathcal{Y}}^+\}$ *is uniformly integrable.*

The existence of solutions to dual problem is given in the following proposition.

Proposition 7.2.5. *Let* $w(x) := \inf_{(y,Y)\in\mathcal{Y}^+}[v(Y)+xy]$. *There exists unique* $(y^*, Y^*) \in \widetilde{\mathcal{Y}}^+$ *such that* $E[Y^*] = y^*$ *and*

$$w(x) = v(Y^*) + xy^*. \tag{7.2.13}$$

Proof. First, we note that if $(y^*, Y^*) \in \widetilde{\mathcal{Y}}^+$ is a minimizer of w, then necessarily $y^* = E[Y^*]$. Indeed, if this is not the case, i.e. $y^* > E[Y^*]$, we can replace (y^*, Y^*) by $(E[Y^*], Y^*) \in \widetilde{\mathcal{Y}}^+$ at which the function $v(Y) + xy$ takes a smaller value, contradicting with the optimality of (y^*, Y^*). Moreover, since V is strictly convex, there are no more than one minimizer to the dual problem.

Since V and δ are convex functions, $(y, Y) \to v(Y) + xy$ is also convex. Let (y_n, Y_n) be a minimizing sequence for the function $w(x)$, i.e.

$$w(x) = \lim_{n\to\infty}[v(Y_n) + xy_n] = \lim_{n\to\infty}[E[V(Y_n)] + \delta(Y_n) + xy_n].$$

We now prove that (y_n) and $(\delta(Y_n))$ are bounded. Indeed, for n sufficiently large we have

$$v(Y_n) + xy_n \leq w(x) + 1.$$

By using the Jensen's inequality and the decreasing property of V we get that

$$w(x) + 1 \geq V(E[Y_n]) + \delta(Y_n) + xy_n \geq V(y_n) + \delta(Y_n) + xy_n.$$

Since $x > 0$, by l'Hospital rule we have $\lim_{y\to\infty} \frac{V(y)}{xy} = \lim_{y\to\infty} \frac{V'(y)}{x} = 0$. Therefore, if (y_n) or $(\delta(Y_n))$ is not bounded, the expression on the right-hand side tends to ∞ as n tends to ∞, which is a contradiction. This shows that (y_n) and $(\delta(Y_n))$ are two bounded sequences. Therefore, we can apply the Komlos theorem for the sequence (y_n, Y_n) to find out another minimizing sequence that is, for ease of notation, still denoted by (y_n, Y_n) (by convexity of $\widetilde{\mathcal{Y}}^+$) such that $\lim_{n\to\infty}(y_n, Y_n) = (y^*, Y^*) \in \mathbf{R}_+ \times L^0_+$ a.s. By applying Fatou's Lemma, we have $E[Y^*] \leq y^* < \infty$. We now show that $(E[Y^*], Y^*)$ is a minimizer of w. Indeed, by using Lemma 7.2.4 and Fatou's Lemma, we have

$$E[V(Y^*)] \leq \liminf_{n\to\infty} E[V(Y_n)]. \tag{7.2.14}$$

Let $X_T^{x,\pi}$ be an arbitrary non negative terminal wealth in $\mathcal{C}^+(x,m)$. By Fatou's Lemma we get that

$$E[Y^*X_T^{x,\pi}] \leq \liminf_{n\to\infty} E[Y_nX_T^{x,\pi}] \leq \liminf_{n\to\infty}[\delta(Y_n) + xy_n].$$

By taking supremum over all non negative $X_T^{x,\pi} \in \mathcal{C}(x,m)$, we have

$$\delta(Y^*) + xE[Y^*] \leq \lim_{n\to\infty}[\delta(Y_n) + xy_n]. \tag{7.2.15}$$

By combining (7.2.14) and (7.2.15) we have

$$v(Y^*) + xE[Y^*] \leq \lim_{n\to\infty}[v(Y_n) + xy_n] = w(x). \tag{7.2.16}$$

Hence, $(E[Y^*], Y^*) \in \widetilde{\mathcal{Y}}^+$ and $v(Y^*) + xE[Y^*] = w(x)$. ■

The duality relation is reported in the following result.

Proposition 7.2.6. *Let (y^*, Y^*) be the minimizer of the dual problem. We have $y^* \neq 0$ and the duality holds*

$$u(x) = \inf_{(y,Y)\in\widetilde{\mathcal{Y}}^+}[v(Y) + xy] = v(Y^*) + xy^*. \tag{7.2.17}$$

Moreover, the optimal wealth is given by

$$X^* = I(Y^*) = -V'(Y^*). \tag{7.2.18}$$

Proof. We devide the proof into several steps.

Step 1 : We first show that

$$E[Y^*X^*] - \delta(Y^*) - xy^* = 0. \tag{7.2.19}$$

Indeed, we rewrite $Y^* = y^*Y_1$ such that $E[Y_1] = 1$ and let us define

$$L(y) := E[V(yY_1)] + y\delta(Y_1) + xy.$$

By virtue of Lemma 7.2.7 below, L is well-defined. It is straightforward that L is a convex function on $[0,\infty)$ and attains its global minimum at $y^* : L(y) \geq L(y^*) = w(x)$. Let us check that L is C^1 and

$$L'(y) = E[Y_1V'(yY_1)] + \delta(Y_1) + x. \tag{7.2.20}$$

Indeed, if the equality above is true, then $L'(y^*) = 0$ by optimality. Moreover, we can also show that $y^* > 0$ since L' is a decreasing function and $\lim_{x\to 0^+} L'(x) = \infty$. We then obtain the desired result. Let us now define

$$G(y) := V(yY_1) + y\delta(Y_1) + xy$$

which is a random convex function. We have $L(y) = E[G(y)]$ and

$$G'(y) = Y_1V'(yY_1)] + \delta(Y_1) + x,$$

where G' denotes the derivative of G in y. For each convex function $f : \mathbf{R} \to \mathbf{R}$ and $h, x \in \mathbf{R}$, denote

$$\Delta_h f(x) := \frac{f(x+h) - f(x)}{h} \text{ and } \Delta_\pm f(x) := \lim_{h \to 0\pm} \Delta_h f(x).$$

We recall from the theory of real convex functions that

$$\lim_{h \downarrow 0} \downarrow \Delta_h f(x) = \Delta_+ f(x) \geq \Delta_- f(x) = \lim_{h \uparrow 0} \uparrow \Delta_h f(x).$$

Therefore, by applying the monotone convergence theorem, in order to prove the (continuously) differentiability of L it suffices to show that $G'(y) \in L^1$, or equivalently, $Y_1 I(yY_1) \in L^1$. But this is done easily by using Lemma 7.2.7 below.

Step 2 : We now prove that $X^* \in \mathcal{C}^+(x, m)$. Indeed, let us consider, for each $0 < \varepsilon < 1$ and $(y, Y) \in \widetilde{\mathcal{Y}}^+$:

$$(y_\varepsilon, Y_\varepsilon) := (1 - \varepsilon)(y^*, Y^*) + \varepsilon(y, Y)$$

It is straightforward that $(y_\varepsilon, Y_\varepsilon) \in \widetilde{\mathcal{Y}}^+$. Note that (y^*, Y^*) is a global minimum point of the function

$$(y, Y) \to E[V(Y)] + \delta(Y) + xy.$$

We have, by the convexity

$$\begin{aligned} 0 &\geq E[V(Y^*) - V(Y^\varepsilon)] + \varepsilon(\delta(Y^*) - \delta(Y)) + x(y^* - y^\varepsilon) \\ &\geq E[(Y^* - Y^\varepsilon)V'(Y^\varepsilon)] + \varepsilon(\delta(Y^*) - \delta(Y)) + \varepsilon x(y^* - y) \\ &\geq \varepsilon E[(Y^* - Y)V'(Y^\varepsilon)] + \varepsilon(\delta(Y^*) - \delta(Y)) + \varepsilon x(y^* - y) \\ &= \varepsilon E[(Y - Y^*)I(Y^\varepsilon)] + \varepsilon(\delta(Y^*) - \delta(Y)) + \varepsilon x(y^* - y). \end{aligned}$$

Let us check that $[(Y - Y^*)I(Y^\varepsilon)]^-$ is uniformly integrable for $0 < \varepsilon \leq \varepsilon_0$, where ε_0 being small enough. Indeed, we have :

$$[(Y - Y^*)I(Y^\varepsilon)]^- \leq Y^* I(Y^\varepsilon) \leq Y^* I((1 - \varepsilon)Y^*).$$

The final term is uniformly integrable for ε positive small enough, by virtue of Lemma 7.2.7 below. We can then apply the Fatou's lemma to get that

$$E[(Y - Y^*)I(Y^*)] + (\delta(Y^*) - \delta(Y)) + x(y^* - y) \leq 0. \tag{7.2.21}$$

Or, equivalently,

$$E[YX^*] - \delta(Y) - xy \leq E[Y^*X^*] - \delta(Y^*) - xy^* = 0 \,\forall\, (y, Y) \in \widetilde{\mathcal{Y}}^+ \tag{7.2.22}$$

By using Lemma 7.2.1, this inequality deduces that $X^* \in \mathcal{C}^+(x, m)$.

Step 3 : We now prove that X^* is the optimal wealth of the primal problem for which the duality relation holds. Indeed, by using the equality (7.2.19) we have $E[X^*Y^*] = \delta(Y^*) + xy^*$. Therefore

$$v(Y^*) + xy^* = E[V(Y^*)] + E[Y^*X^*].$$

Moreover, by the definition of X^* (7.2.18), we have the duality

$$E[U(X^*)] = E[V(Y^*)] + E[Y^*X^*].$$

In combining the two equalities above we get that

$$E[U(X^*)] = v(Y^*) + xy^*.$$

This equality also implies that there actually exists a final wealth X_T^{x,π^*} which is the optimal solution to the primal problem. ■

Lemma 7.2.7. *Suppose that* $|E[V(Y)]| < \infty, Y \in \widetilde{\mathcal{Y}}^+$. *Let* $0 < t_0 < t_1$ *be two real constants, we have*

(i) $|E[V(tY)]| < \infty$ *for all* $t > 0$ *and* $\{V(tY) : t_0 \leq t \leq t_1\}$ *is uniformly integrable.*

(ii) $|E[YI(tY)]| < \infty$ *for all* $t > 0$ *and* $\{YI(tY) : t_0 \leq t \leq t_1\}$ *is uniformly integrable.*

Proof. These results are standard. We provide the proof for the completeness. Let us first recall some well-known facts about utility functions : If $\mathrm{AE}_\infty(U) < 1$ then there exists $y_0 > 0$ such that

$$\forall \mu \in (0,1)\, \exists C_\mu > 0 : V(\mu y) \leq C_\mu V(y)\, \forall y \in (0, y_0),$$

and

$$\exists C > 0 : yI(y) \leq CV(y)\, \forall y \in (0, y_0).$$

For the proofs, we refer to [72], Lemma 6.3 and Corrolary 6.1.

(i) By the Jensen's inequality we have $E[V(tY)] \geq V(ty) > -\infty$, where $y = EY$. We now prove that $E[V(tY)] < \infty$. It is sufficient to prove the statement for $0 < t < 1$.

$$\begin{aligned} V(tY) &= V(tY)1_{tY \geq y_0} + V(tY)1_{tY < y_0 \leq Y} + V(tY)1_{Y < y_0} \\ &\leq V(y_0)1_{tY \geq y_0} + V(ty_0)1_{tY < y_0 \leq Y} + C_t V(Y)1_{Y < y_0}. \end{aligned}$$

Note that all terms are integrable, hence we can conclude about the first statement of (i). The second statement is now an imediate consequence, since $t \to V(tY)$ is increasing and the two bounds $V(t_0Y), V(t_1Y)$ are in L^1.

(ii) Note that $YI(tY) > 0$. We then only care about the upper bounds for $YI(tY)$. We have

$$\begin{aligned} YI(tY) &= YI(tY)1_{tY \geq y_0} + YI(tY)1_{tY < y_0} \\ &\leq I(y_0)Y1_{tY \geq y_0} + \frac{C_t}{t} V(tY)1_{tY < y_0}. \end{aligned}$$

Therefore, (ii) is deduced directly from (i).

■

7.3 Utility Maximization Problem on The Whole Real Line

In this section we consider the utility maximization problem under target constraint, where the utility function is defined on the whole real line. In order to avoid arbitrage opportunities, the portfolio process is assumed to be uniformly bounded from below as usual. We then define the set of admissible strategies as follows

$$\mathcal{A}_0(x) := \{(\pi_t)_{0\leq t\leq T} \in L(S) : \quad X_t^{x,\pi} \geq -c^\pi \text{ a.s.} \quad \forall 0 \leq t \leq T\}, \tag{7.3.23}$$

where x denotes the initial endownment which might be negative, and c^π denotes some real constant that depends on the strategy π. By making an appropriate translation of variable for the utility function, we can suppose that $x > 0$.The set of constrained portfolios is defined by

$$\mathcal{A}(x,m) := \{(\pi_t)_{0\leq t\leq T} \in \mathcal{A}_0(x) : \quad E[\ell(X_T^{x,\pi})] \geq m\}. \tag{7.3.24}$$

Here, the expected loss constraint is similar than that in the previous section, except that the loss function now is defined on the whole real line. The set of hedgeable positions is given by

$$\mathcal{C}(x,m) := \{X \in L_+^0(\mathcal{F}_T) - L_+^\infty(\mathcal{F}_T) : \quad \exists \pi \in \mathcal{A}(x,m) : X \leq X_T^{x,\pi}\} \tag{7.3.25}$$

We also denotes $\mathcal{C}_0(x,m) := \mathcal{C}(x,m) - x$. The utility function U is assumed to satisfy the Inada conditions

$$\lim_{r\to-\infty} U'(r) = +\infty \text{ and } \lim_{r\to+\infty} U'(r) = 0.$$

As for the condition on the asymptotic elasticity, we require both **Assumption A** in the previous section, and :

Assumption B : The asymptotic elasticity of the utility function at $-\infty$ is strictly greater than 1 :

$$\text{AE}_{-\infty}(U) := \text{limsup}_{r\to-\infty} \frac{rU'(x)}{U(r)} < 1. \tag{7.3.26}$$

We still denote by V the Fenchel dual function

$$V(y) := \sup_{r\in\mathbf{R}}[U(r) - ry], y > 0.$$

It worths noting that in this setting, the utility maximization problem might not admit a solution in $\mathcal{C}(x,m)$, i.e. the maximizer is not necessarily uniformly bounded from below. Therefore, we need to expand the set of hedgeable claims $\mathcal{C}(x,m)$ to $\mathcal{C}_U(x,m)$ defined by

$$\mathcal{C}_U(x,m) := \{X \in L^0 : \exists (X_n) \subseteq \mathcal{C}(x,m), U(X_n) \to U(X) \text{ in } L^1\}.$$

We now define the utility maximization problem on the larger set $\mathcal{C}_U(x,m)$ instead of $\mathcal{C}(x,m)$ (But the value function u keeps unchanged by definition of $\mathcal{C}_U(x,m)$).

$$u(x) = \sup_{X \in \mathcal{C}_U(x,m)} E[U(X)]. \tag{7.3.27}$$

As we willl see, this reformulation ensure the existence of solution to the primal optimization problem. The dual domain is defined as follow

$$\mathcal{Y} := \{(y,Y) \in \mathbf{R}_+ \times L^1_+ : \, EY \leq y, \delta(Y) := \sup_{X \in \mathcal{C}_0(x,m)} E[XY] < \infty\} \tag{7.3.28}$$

We also suppose that

$$\widetilde{\mathcal{Y}} := \{(y,Y) \in \mathcal{Y} : \, E[V(Y)] < \infty\} \neq \emptyset. \tag{7.3.29}$$

The dual problem is

$$w(x) = \inf_{(y,Y) \in \mathcal{Y}} [v(Y) + xy], \tag{7.3.30}$$

where $v(Y) := E[V(Y)] + \delta(Y)$. It is straightforward that $u \leq w$.
We now follow the methodology of [93] and [12], consisting in approximating U by U_n whose domain is bounded from below :

$$U_n(x) := \begin{cases} U(x) & \text{if } x \geq -n \\ -\infty & \text{if } x < -n \end{cases} \tag{7.3.31}$$

Let us denote V_n the Fenchel duality of U_n. It is easy to show that

$$V_n(y) = \sup_{x \geq -n} [U(x) - xy] = U(I(y) \vee -n) - y[I(y) \vee -n]. \tag{7.3.32}$$

We also define the coresponding approximating optimization problem

$$u_n(x) = \sup_{X \in \mathcal{C}_U(x,m)} E[U_n(X)], \tag{7.3.33}$$

This problem is equivalent to

$$u_n(x) = \sup_{X \in \mathcal{C}_n(x,m)} E[U(X)], \tag{7.3.34}$$

where

$$\mathcal{C}_n(x,m) := \{X \in \mathcal{C}_U(x,m) : \, X \geq -n \text{ a.s}\}.$$

The dual domain for this problem is

$$\mathcal{Y}_n := \{(y,Y) \in \mathbf{R}_+ \times L^1_+ : \, EY \leq y, \delta_n(Y) := \sup_{X \in \mathcal{C}_n(x,m)} E[XY] < \infty\} \tag{7.3.35}$$

It is easy to show that $\mathcal{Y} \subseteq \mathcal{Y}_{n+1} \subseteq \mathcal{Y}_n \subseteq \mathcal{Y}^+$ for all $n \geq 0$. The dual approximating optimization problem is

$$w_n(x) = \inf_{(y,Y)\in\mathcal{Y}_n} [v_n(Y) + xy], \tag{7.3.36}$$

where $v_n(Y) := E[V_n(Y)] + \delta_n(Y)$.
The duality relation for the approximating problems is given in the following theorem

Theorem 7.3.1. *For each $n > 0$ the dual optimization problem (7.3.36) has a solution $(y_n, Y_n) \in \mathcal{Y}_n, y_n > 0$ such that $EY_n = y_n$. The primal optimization (7.3.34) also admits a solution X_n given by $X_n = I(Y_n) \vee -n \in \mathcal{C}_n(x, m)$. Moreover, the duality relation holds*

$$u_n(x) = \inf_{(y,Y)\in\mathcal{Y}_n} [v_n(Y) + xy].$$

Proof. The verification procedure is proceeded in a similar manner than in the case of non negative wealth, so it will be omitted here. We now briefly prove the existence of solutions to the dual problem (7.3.34). As before, we denote by $(y_n^k, Y_n^k)_{k\geq 1}$ be a minimizing sequence of the dual problem such that $E[Y_n^k] = y_n^k$. Assume that this sequence converges to $(y_n^*, Y_n^*) \in \mathbf{R}_+ \times L_+(\mathcal{F}_T)$ almost surely. We have to show that $(y_n, Y_n) := (E[Y_n^*], Y_n^*)$ is solution to the dual problem. Indeed, by the definition of V_n we have $V_n(y) - ny \geq U(-n)$. Consequently, $([V_n(Y) - nY]^-)_{Y\in\mathcal{Y}_n}$ is uniformly integrable. By using Fatou's Lemma, we have

$$E[V(Y_n) - nY_n] \leq \liminf_{k\to\infty} \{E[V(Y_n^k)] - ny_n^k\}. \tag{7.3.37}$$

Let $X_T^{x,\pi}$ be an arbitrary terminal wealth in $\mathcal{C}_n(x, m)$, we have $X_T^{x,\pi} + n \geq 0$. By Fatou's Lemma we get that

$$\begin{aligned} E[Y_n(X_T^{x,\pi} + n)] &\leq \liminf_{k\to\infty} E[Y_n^k(X_T^{x,\pi} + n)] \\ &\leq \liminf_{k\to\infty} [\delta(Y_n^k) + (n + x)y_n^k]. \end{aligned}$$

By taking supremum over all $X_T^{x,\pi} \in \mathcal{C}_n(x, m)$, we have

$$\delta(Y_n) + (n + x)E[Y_n] \leq \lim_{k\to\infty} [\delta(Y_n^k) + (n + x)y_n^k]. \tag{7.3.38}$$

By combining (7.3.37) and (7.3.38) we have

$$v(Y_n) + xE[Y_n] \leq \lim_{k\to\infty} [v(Y_n^k) + xy_n^k] = w_n(x). \tag{7.3.39}$$

Hence, $(y_n, Y_n) \in \widetilde{\mathcal{Y}}_n$ and $v(Y_n) + xy_n = w_n(x)$. ■

We now aim at linking the solutions of the approximating maximization problems to the initial ones. Observe that the two sequences of function $(u_n), (w_n)$ are identical and pointwise increasing (and bounded above by u, v). It is natural to expect that the limiting functions of (u_n) and (w_n) are u and v, respectively. In fact, this is the case. We first have the following result for the duality problem

Theorem 7.3.2. *There exists a subsequence (n_k) along which $(y_{n_k}, Y_{n_k}, \delta_{n_k}(Y_{n_k}))$ converges to $(y^*, Y^*, \delta(Y^*)) \in \mathcal{Y} \times \mathbf{R}_+$ in L^1 such that $EY^* = y^* > 0$. Moreover, (y^*, Y^*) is the unique minimizer for the duality problem*

$$w(x) = E[V(Y^*)] + \delta(Y^*) + xy^*.$$

Furthermore, $(V_{n_k}(Y_{n_k}))$ converges to $V(Y^*)$ in L^1 and therefore, w_n converges to w.

Proof. We first observe that $V(y) \geq U(0) \forall y \geq 0$. Since $V_n \leq V$ we have

$$w(x) \geq w_n(x) \geq U(0) + \delta_n(Y_n) + xy_n.$$

Hence, $(\delta_n(Y_n))$ and (y_n) are bounded sequences (since $x > 0$). We then deduce a subsequence (n_k) along which $y_n \to y^*$ and $\delta_n(Y_n) \to \delta^*$. For ease of notations we still denote n_k by n. By using the same arguments as in Step 2, Theorem 2.2 [93] we deduce the existence of Y^* such that Y_n converges to Y^* in L^1 and that $EY^* = y^*$.

We now verify $(y^*, Y^*) \in \mathcal{Y}$. Indeed, let us consider an arbitrary random variable $X \in \mathcal{C}_0(x, m)$ such that $X \geq -k$ a.s., where k is some positive integer. Since (Y_n) is uniformly integrable and X is bounded below, we can apply the Fatou's lemma to get that

$$E[Y^*X] \leq \liminf E[Y_nX] \leq \liminf \delta_n(Y_n) = \delta^*.$$

Therefore, by taking supremum over all $X \in \mathcal{C}_0(x, m)$ we have $\delta(Y^*) \leq \delta^* < \infty$. Hence $(y^*, Y^*) \in \mathcal{Y}$. By applying the Fatou's lemma one more time we obtain

$$w(x) \leq E[V(Y^*)] + \delta(Y^*) + xy^* \leq \liminf w_n(x) \leq w(x).$$

The above inequalities show that $w(x) = E[V(Y^*)] + \delta(Y^*) + xy^*$ and then (y^*, Y^*) is the unique minimizer of the duality problem. We also deduce that $V_n(Y_n) \to V(Y^*)$ in L^1 as $(V_n(Y_n))$ is uniformly bounded below and converges a.s to $V(Y^*)$ as well as $E[V_n(Y_n)] \to E[V(Y^*)]$. It is also straightforward that $\delta(Y^*) = \lim \delta_n(Y_n)$.

Finally, by the same arguments as in Step 1, Proposition 7.2.6, we can easily deduce that $y^* > 0$.

■

From the previous theorem and the fact that $u \leq w$ we deduce that $u_n \to u$ and $u = w$. Not supprisingly, we obtain the following

Theorem 7.3.3. *The sequence $(U(X_n))$ converges to $U(X^*)$ in L^1, where $X^* := I(Y^*) \in \mathcal{C}_U(x, m)$ is the maximizer of the primal optimization problem. We have $E[\ell(X^*)] \geq m$ and the duality relation holds*

$$u(x) = E[U(X^*)] = E[V(Y^*)] + \delta(Y^*) + xy^* = w(x). \tag{7.3.40}$$

The proof of theorem is basically the same as in the Step 7, Theorem 2.2, [93].

7.4 Complete Market : A Constrained Duality Approach

We now turn to the case where the market is complete, i.e. the set of equivalent martingale measures reduces to a singleton $\{Q\}$. In order to keep things as simple as possible, we consider in this section the case where the utility function is defined on the half positive real line. In general, the dual space $\mathcal{Y}^+(x,m)$ is quite large, even in the complete market case. It then turns out to be difficult to characterize the "optimal pricing measure" Y_1^*. Therefore, we need another technic which is still based on duality to cope with the problem. The idea is to replace the expected loss constraint by an infinite number of almost-sure shortfall constraints. We recall that in the case of shortfall constraint, i.e. constraint of the form $X_T^{x,\pi} \geq k$, where k denotes some constant threshold and $X_T^{x,\pi}$ denotes a terminal wealth, the optimal wealth to the utility maximization problem is given by $X_T^{x,\pi^*} := k \vee (U')^{-1}(y^* \frac{dQ}{dP})$. Here, y^* is determined by the equation $E_Q[X_T^{x,\pi^*}] = x$. We refer readers to [32] for more details.
Let us define the constrained Fenchel duality of the utility function U :

$$V_p(y) := \max_{r \geq p}[U(r) - ry], y \geq 0, p \in \mathbf{R}. \tag{7.4.41}$$

The explicit form of $V_p(y)$ is given by

$$V_p(y) = U(I(y) \vee p) - y[I(y) \vee p], \tag{7.4.42}$$

where $I = U'^{-1}$. The following lemma is easy to prove but it is useful for further analysis.

Lemma 7.4.1. *$V_p(y)$ is a non increasing convex function in y and a non increasing concave function in p. Moreover, its partial derivatives are continuous and given by*

$$\frac{dV_p}{dy}(y) = -[I(y) \vee p], \tag{7.4.43}$$

$$\frac{dV_p}{dp}(y) = -[U'(p) - y]^-. \tag{7.4.44}$$

We keep the setup and all necessary notations as in Section 6.2, except that the dual space now is $\mathcal{Y}^+ := \{yH : y > 0\}$ which does not depend on x, m. Here $H := \frac{dQ}{dP}$ denotes the risk-neutral density of the unique equivalent martingale measure Q. Let $\mathcal{C}^+(x)$ denote the set of hedgeable claims (which are bounded below) without target constraint. Hence $\mathcal{C}^+(x)$ is characterized by

$$\mathcal{C}^+(x) = \{X \in L^0_+(\mathcal{F}_T, P) - L^\infty_+(\mathcal{F}_T, P) : E_Q X \leq 0\}.$$

Let us define the set of shortfall thresholds as follows

$$\mathcal{M} = \mathcal{M}(x,m) := \{M \in L^0_+(\mathcal{F}_T) : E_Q M \leq x, E[\ell(M)] \geq m\}.$$

Let $X_T^{x,\pi}$ denote a terminal wealth satisfying the expected loss constraint. We have :

$$E[\ell(X_T^{x,\pi})] \geq m \Leftrightarrow \exists M \in \mathcal{M} : \ X_T^{x,\pi} \geq M. \tag{7.4.45}$$

Therefore, solving the primal problem :

$$u(x) := \sup_{X \in \mathcal{C}^+(x)} \{E[U(X)] : \ E[\ell(X_T^{x,\pi})] \geq m\} \tag{7.4.46}$$

boils down to solve the following

$$u_M(x) := \sup_{X \in \mathcal{C}^+(x)} \{E[U(X)] : \ X \geq M\}, M \in \mathcal{M}, \tag{7.4.47}$$

i.e. we have $u(x) = \sup_{M \in \mathcal{M}} u_M(x)$. By using (7.4.45) and the definition of the constrained Fenchel dual function (7.4.41), we have the following inequalities

$$u(x) \leq \sup_{M \in \mathcal{M}} \inf_{y>0} \{E[V_M(yH)] + xy\} \leq \inf_{y>0} \sup_{M \in \mathcal{M}} \{E[V_M(yH)] + xy\} \tag{7.4.48}$$

In order that the expressions in (7.4.48) are well-defined, we assume that $E[V(yH)] < \infty$ for all $y > 0$. Let us denote

$$w_M(x) := \inf_{y>0} \{E[V_M(yH)] + xy\}, \tag{7.4.49}$$

and

$$v(y) = v_{x,m}(y) := \sup_{M \in \mathcal{M}} E[V_M(yH)]. \tag{7.4.50}$$

We rewrite (7.4.48) as follows

$$u(x) \leq \sup_{M \in \mathcal{M}} w_M(x) \leq \inf_{y>0} [v(y) + xy]. \tag{7.4.51}$$

Our goal is to prove that under appropriate conditions, the inequalities in (7.4.51) become equalities. The first equality is established in the following theorem.

Theorem 7.4.2. *(i) For each $M \in \mathcal{M}$, there exists $0 < y_M^* \leq \infty$ such that*

$$w_M(x) = E[V_M(y_M^* H)] + xy_M^*. \tag{7.4.52}$$

Moreover, the optimal wealth of the utility maximization problem with shortfall constraint (7.4.47) is given by

$$X_M^* = M \vee I(y_M^* H), \tag{7.4.53}$$

and we also have $u_M = w_M$.

(ii) There exists M^* such that

$$w^1(x) := \sup_{M \in \mathcal{M}} w_M(x) = w_{M^*}(x).$$

Moreover, the duality holds, i.e. $u = w^1$; and the optimal wealth of the primal utility maximization problem (7.4.46) is given by

$$X^* = M^* \vee I(y^* H), \tag{7.4.54}$$

where $y^* := y^*_{M^*}$.

Proof.

(i) For each $M \in \mathcal{M}$, let us define

$$G_M(y) := V_M(yH) + xy,$$

and $L_M(y) := E[G_M(y)]$. By using the same arguments as in Proposition 7.2.6, we can show that L_M is C^1 and its derivative is given by

$$L'_M(y) = x - E_Q[M \vee I(yH)].$$

On the one hand, since $y \to M \vee I(yH)$ is an non-increasing positive function, $\{M \vee I(yH) : y \geq y_0 > 0\}$ is uniformly integrable for all strictly positive constant y_0 and we can apply the monotone convergence theorem to deduce that

$$\lim_{y \to \infty} L'_M(y) = x - E_Q M \geq 0.$$

On the other hand, since $\lim_{y \to 0} I(yH) = +\infty$, we can apply Fatou Lemma to get that

$$\lim_{y \to 0} L'_M(y) = -\infty.$$

Therefore, if $E_Q M < x$ then there exists $y^*_M > 0$ such that $L'_M(y^*_M) = 0$ or equivalently, $L_M(.)$ attains the global minimum at y^*_M by its convexity. Otherwise if $E_Q M = x$ then $L'_M(y) \geq 0$ for all $y > 0$, hence $L_M(.)$ attains its infimum at $y^*_M = \infty$. In both cases, $X^*_M := M \vee I(y^*_M H)$ satisfies $E_Q[X^*_M] = x$. By definition of the constrained dual function, we have $u_M(x) \leq w_M(x) \, \forall x > 0$ and it is straightforward to verify that X^*_M is the optimal solution to the utility maximization under shortfall constraint (7.4.47). As a result, we also have $u_M(x) = w_M(x)$.

(ii) Notice that since $M \to E[V_M(yH)] + xy$ is a concave function, $M \to w_M(x)$ is also a concave one. Let (M_n) be a maximizing sequence for $M \to w_M(x)$. That is

$$\lim_{n \to \infty} w_{M_n}(x) = w^1(x).$$

Since $M_n \geq 0$ for all n, by Lemma A.1.1 of Delbaen and Schachermayer (1994), there exists a sequence of $\mathcal{F}$-measurable random variables

$$\widetilde{M}_n \in \text{conv}(M_n, M_{n+1}, ...)$$

such that $\widetilde{M}_n$ converges almost surely to M^* which is $\mathcal{F}_T-$ measurable. We will prove that $M^* \in \mathcal{M}$ and M^* is solution to the optimization problem w^1. Indeed, by the concavity of $M \to E[V_M(yH)] + xy$ and the fact that

$$V_M(yH)^+ \leq V(yH)^+ \in L^1,$$

we can apply the Fatou's lemma to obtain that, for all $y > 0$:

$$E[V_{M^*}(yH)] + xy \geq \limsup_{n\to\infty}\{E[V_{\widetilde{M}_n}(yH)] + xy_n\} \geq \lim_{n\to\infty} w_{M_n}(x) = w^1(x).$$

Consequently, we have $w_{M^*}(x) \geq w^1(x)$.Therefore, we only need to verify that $M^* \in \mathcal{M}$. First, we have $M^* \geq 0$ and $E_Q[M^*] \leq x$ by the virtue of Fatou's Lemma. Second, since ℓ is continuous, concave and bounded from above, the Fatou's lemma gives

$$E[\ell(M^*)] \geq \limsup_{n\to\infty} E[\ell(\widetilde{M}_n)] \geq m.$$

We then deduce that $M^* \in \mathcal{M}$ and $w_{M^*}(x) = \max_{M\in\mathcal{M}} w_M(x)$. The remainder of the proof is straightforward.

■

The following theorem establishes the second dual equality.

Theorem 7.4.3. *(i) If there exists $\epsilon > 0$ such that $E[\ell(x-\epsilon)] \geq m$, then there exists $(\hat{y}, \widehat{M}) \in (0,\infty) \times \mathcal{M}$ such that*

$$w^2(x) := v(\hat{y}) + x\hat{y} = E[V_{\widehat{M}}(\hat{y}H)] + x\hat{y}. \tag{7.4.55}$$

(ii) Moreover, if $y^* < \infty$ then the minimax theorem holds, i.e.

$$\sup_{M\in\mathcal{M}} \inf_{y>0}\{E[V_M(yH)] + xy\} = \inf_{y>0} \sup_{M\in\mathcal{M}} \{E[V_M(yH)] + xy\}. \tag{7.4.56}$$

Consequently, the dual equalities hold, i.e. $u = w^1 = w^2$.

Proof. (i) Let $y > 0$ arbitrary. By similar argument as in the previous theorem, we can prove the existence of a $\widehat{M}_y \in \mathcal{M}$ such that $v(y) = E[V_{\widehat{M}_y}(yH)]$. Our goal now is to show that there exists a $\hat{y} \in (0,\infty)$ such that $w^2(x) = v(\hat{y}) + x\hat{y}$. Note that $y \to E[V_M(yH)]$ is convex for each $M \in \mathcal{M}$ fixed. We then deduce that $y \to v(y) + xy$ is also convex. Let y_n be a minimizing sequence for the function $w(x)$, i.e.

$$w(x) = \lim_{n\to\infty}[v(y_n) + xy_n].$$

Let us verify that (y_n) is bounded. Indeed, we deduce from the assumption that $x-\epsilon \in \mathcal{M}$. By using Jensen's Inequality for the convex function $V_{x-\epsilon}(.)$ we have

$$v(y_n) + xy_n \geq E[V_{x-\epsilon}(y_nH)] + xy_n \geq V_{x-\epsilon}(y_n) + xy_n \geq U(x-\epsilon) + \epsilon y_n.$$

Since $\{v(y_n) + xy_n\}_{n\geq 1}$ is bounded, it is obvious that $\{y_n\}_{n\geq 1}$ is also bounded. Therefore, we can suppose that $\lim_{n\to\infty} y_n = \hat{y} \geq 0$. We now verify the equality

$$w^2(x) = v(\hat{y}) + x\hat{y}.$$

To do so, let us rewrite

$$v(y) + xy = \sup_{M\in\mathcal{M}} \{E[V_M(yH) + yHM] - yE_Q M\}.$$

Since $V_p(y) + py \geq V(y)$, it is easy to check that $(V_M(y_nH) + y_nHM)^-$ is uniformly integrable for every $M \in \mathcal{M}$ (by virtue of the lemma 7.2.7). Therefore, we can apply Fatou's Lemma to get that

$$\begin{aligned} E[V_M(\hat{y}H)] &\leq \liminf_{n\to\infty} \{E[V_M(y_nH) + y_nHM] - y_nE_Q M\} \\ &\leq \liminf_{n\to\infty} \sup_{M\in\mathcal{M}} E[V_M(y_nH)] \\ &= \liminf_{n\to\infty} v(y_n). \end{aligned}$$

By taking the supremum over $M \in \mathcal{M}$ we have

$$v(\hat{y}) + x\hat{y} \leq \liminf_{n\to\infty} [v(y_n) + xy_n] = w(x).$$

Hence $w(x) = v(\hat{y}) + x\hat{y} = E[V_{\widehat{M}}(\hat{y}H)] + x\hat{y}$, where $\hat{M} = \hat{M}_{\hat{y}}$.
(ii) Observe that $y \to E[V_M(yH)] + xy$ is a continuous function. Moreover, since $V_M(yH) \leq V(yH) \in L^1(\mathcal{F}_T)$, we can easily check that $M \to E[V_M(yH)] + xy$ is an upper semi-continuous function in L^1-topology. Note also that $\mathcal{M}$ is a closed convex set of $L^1(\mathcal{F}_T)$. Hence, we can apply the minimax theorem (for reference, see for example [97], Theorem 45.8) to get that

$$\sup_{M\in\mathcal{M}} \inf_{y\in\mathcal{A}} \{E[V_M(yH)] + xy\} = \inf_{y\in\mathcal{A}} \sup_{M\in\mathcal{M}} \{E[V_M(yH)] + xy\},$$

where $A \subseteq (0,\infty)$ denotes an arbitrary closed convex compact set. By chossing $\mathcal{A}$ large enough such that $\mathcal{A}$ contains y^* (see the previous theorem), and such that $\{y_n\} \subseteq \mathcal{A}$, we then obtain the following minimax equality

$$\sup_{M\in\mathcal{M}} \inf_{y>0} \{E[V_M(yH)] + xy\} = \inf_{y>0} \sup_{M\in\mathcal{M}} \{E[V_M(yH)] + xy\}.$$

The theorem then follows.
■

7.5 Complete Market : Multiple Target Constraints and BSDEs

In this section we will consider the utility maximization problem under multiple constraints imposed on the final wealth :

$$u(x,m) := \sup\{E[U(X_T^{x,\pi})] : E[\ell(X_T^{x,\pi})] \succeq m\}, \tag{7.5.57}$$

where $\ell = (\ell_1, \dots, \ell_K)$ is a multidimensional loss function, each component of which is a scalar loss function as considered in the previous parts of the chapter; $m = (m_1, \dots, m_K) \in \operatorname{Im} \ell$ is a constant vector; the sign $\succeq$ is understood componentwisely. The initial endownment x is assumed to be large enough such that the set of admissible strategies is non-empty. For instance, we can suppose that $E[\ell(x)] \succeq m$. Let us consider the following problem :

$$v(y, m) := \inf\{x : \exists \pi : E[U(X_T^{x,\pi})] \geq y,\, E[\ell(X_T^{x,\pi})] \succeq m\} \tag{7.5.58}$$

Some elementary arguments show that $v(u(x, m), m) = x$, and symetrically, $u(v(y, m), m) = y$. In other words $u(., m) = v^{-1}(., m)$, where $v^{-1}(., m)$ denotes the right-inverse of v with respect to the $y-$ variable. This observation leads us to study the relation between the utility maximization problem and a hedging-type problem. Moreover, both of them are under multiple constraints. In order to calculate $u(., m)$, we only need to (theoretically) calculate $v(., m)$ and then invert it with respect to the $y-$variable.

For ease of notations, we can rewrite the conjugate problem in a multidimensional constraint form :

$$v(m) := \inf\{x : \exists \pi : E[\widetilde{\ell}(X_T^{x,\pi})] \succeq \widetilde{m}\}. \tag{7.5.59}$$

Here $\widetilde{\ell} = (\ell_0, \ell), \widetilde{m} = (m_0, m)$, where $\ell_0 := U$ and $m_0 := y$. In the sequel, we use the notations ℓ, m instead of $\widetilde{\ell}, \widetilde{m}$. To avoid non necessarily technical difficulties, we will assume that ℓ is a smooth deterministic function and m is some constant vector in the interior of $\operatorname{Im} \ell$, i.e. $l^k_{\min} < m_k < l^k_{\max} \, \forall k = 0, ..K$, where $l^k_{\min} := l_k(0)$ and $l^k_{\max} := \lim_{r \to \infty} l_k(r)$.

Assumption C : $\ell_k : \mathbf{R}_+ \to \mathbf{R}$ is C^1, strictly concave and $\lim_{x \to +\infty} \ell_k'(x) = 0 \, \forall \, k = 0, \dots, K$. Moreover, $\lim_{x \to 0+} \ell_0'(x) = +\infty$.

We consider a complete market model where the interest rate is r_t, the risk premium is θ_t and the volatility of the risky asset is σ_t which is an invertible matrix. We also suppose that $(r_t), (\theta_t), (\sigma_t)$ are bounded processes. The dynamics of the portfolio is given by

$$\begin{aligned} dX_t^{x,\pi} &= (r_t X_t^{x,\pi} + \theta_t \sigma_t \pi_t) dt + \sigma_t \pi_t dW_t. && (7.5.60) \\ X_0^{x,\pi} &= 0. && (7.5.61) \end{aligned}$$

The set of admissible strategies consist of adapted processes (π_t) such that

$$\int_0^T |\sigma_t . \pi_t|^2 dt < \infty \, P - \text{a.s.}$$

By a similar argument than in the unique constraint case, let us define the set of extra controls as follows.

$$\mathcal{M} := \{M = (M_0, \dots, M_K) \in L^1(\operatorname{Im} \ell) : \ell_k^{-1}(M_k) \in L^2_+, E[M_k] = m_k \, \forall \, k\}.$$

We have two remarks here. First, note that the notation of M in this section has a different meaning than that in the previous sections. Second, the requirements $\ell_k^{-1}(M_k) \in L_+^2 \forall k$ ensure that the BSDEs in this section are well-defined. We have

$$E[\ell(X_T^{x,\pi})] \succeq m \Leftrightarrow \exists M \in \mathcal{M} : X_T^{x,\pi} \geq F(M),$$

where $F(M) = \max\{\ell_0^{-1}(M_0), \ldots, \ell_K^{-1}(M_K)\}$ being a convex function of M. Let us define the Fenchel duality of F as follows

$$\widetilde{F}(p) := \sup_{r \in \mathrm{Im}\,\ell} [p.r - F(r)],\ p \in \mathbf{R}_+^{K+1}.$$

We can easily show that $\widetilde{F}$ is also convex and that

$$\widetilde{F}(p) = \sup_{t \geq 0} \sup_{F(r)=t} [p.r - F(r)] = \sup_{t \geq 0} [p.\ell(t) - t].$$

Here, $x.y$ denotes the scalar product of two vectors x and y. For each $p \in \mathbf{R}_+^{K+1}$, the function $f(t) := p.\ell(t) - t$ is strictly concave. We have $f'(t) := p.\ell'(t) - 1$ which is a strictly decreasing function and that

$$\lim_{t \to 0+} f'(t) = \infty,\ \lim_{t \to +\infty} f'(t) = -1,$$

which deduces the existence of a unique global maximal point of f , denoted by $t(p)$ which is an increasing function of p with respect to the natural order on $\mathbf{R}^{K+1}$ and solves the equation $p.\ell'(t(p)) = 1$. We then have

$$\widetilde{F}(p) = p.\ell(t(p)) - t(p).$$

Back to the main problem, let us introduce the BSDE version of the dynamics (7.5.60) as

$$X_t^{\xi} = \xi - \int_t^T (r_s X_s + \theta_s Z_s) ds - \int_t^T Z_s dW_s.$$

By the comparison principle, we have the following estimation

$$v(m) \geq \inf_{M \in \widetilde{\mathcal{M}}} \{X_0^{F(M)}\}. \tag{7.5.62}$$

Consider the state-price process

$$L_t = 1 + \int_0^t L_s r_s ds + \int_0^t L_s \theta_s dW_s.$$

Let Q the unique equivalent martingale measure defined by $\frac{dQ}{dP} = L_T$. For $M \in \mathcal{M}$, $(X_t^{F(M)} L_t)$ is a martingale, therefore $E_Q[F(M)] := E[F(M) L_T] = X_0^M$. By definition of the Fenchel dual function $\widetilde{F}$ we have, for all $p \in \mathbf{R}_+^{K+1}$:

$$X_0^{F(M)} = E[F(M) L_T] \geq E[p.M - \widetilde{F}(L_T^{-1} p) L_T].$$

We then get that

$$\inf_{M\in\widetilde{\mathcal{M}}}\{X_0^{F(M)}\} \geq \sup_{p\in\mathbf{R}_+^{K+1}}[p.m - E_Q[\widetilde{F}(L_T^{-1}p)]]. \tag{7.5.63}$$

By combining (7.5.62) and (7.5.63) we are led to

$$v(m) \geq \inf_{M\in\widetilde{\mathcal{M}}}\{X_0^{F(M)}\} \geq \sup_{p\in\mathbf{R}_+^{K+1}}[p.m - E_Q[\widetilde{F}(L_T^{-1}p)]]. \tag{7.5.64}$$

In fact, we can prove that the inequalities become equalities in (7.5.64). The following theorem is the main result of this section. Note that in the theorem, the notation $x \prec y$ means that $x_i < y_i$ for all i.

Theorem 7.5.1. *If the constant vector m satisfies $\ell_{\min} \prec m \prec \ell_{\max}$ then there exists a maximizer p^* for the optimization problem on the right-hand side of (7.5.63). Let us denote $X^* = t(L_T^{-1}p^*)$. Suppose that $p^* \succ 0$ and that X^* is square integrable. Then the duality holds, i.e.*

$$v(m) = \inf_{M\in\widetilde{\mathcal{M}}}\{X_0^{F(M)}\} = \sup_{p\in\mathbf{R}_+^{K+1}}[p.m - E_Q[\widetilde{F}(L_T^{-1}p)]]. \tag{7.5.65}$$

Moreover, the optimal wealth of the dual problem (7.5.59) is given by X^* and the coresponding optimal control is $M^* = \ell(X^*)$.

Proof. We first prove the existence of a maximizer $p^* = p^*(m)$ to the following optimization problem

$$\sup_{p\in\mathbf{R}_+^{K+1}}[p.m - E_Q[\widetilde{F}(L_T^{-1}p)]].$$

Notice that the expression $K(p) = K_m(p) := p.m - E_Q[\widetilde{F}(L_T^{-1}p)]]$ is a convex function of p whose derivative is given by

$$K'(p) = m - E[\nabla\widetilde{F}(L_T^{-1}p)].$$

By envelope theorem we have

$$\nabla\widetilde{F}(p) = \ell(t(p)).$$

Hence,

$$K'(p) = m - E[\ell(t(L_T^{-1}p))].$$

Therefore, $K(.)$ admits a maximizer $p^* \succ 0$ if and only if $K'(p^*) = 0$. Observe that $\widetilde{F}$ is a convex function, so we can apply Jensen's inequality to get that

$$K(p) \leq p.m - \widetilde{F}(E_Q[L_T^{-1}p]) = p.m - \widetilde{F}(p) =: J(p).$$

The application $p \to J(p)$ is also a concave function whose derivative is given by $J'(p) = m - \ell(t(p))$. Notice that $\lim_{\|p\|\to\infty} t(p) = \infty$ and that $m \prec \ell_{\max}$ by assumption, we obtain that $\lim_{\|p\|\to\infty} J'(p) \prec 0$. Therefore,

$J(.)$ converges to $-\infty$ when $\|p\|$ tends to ∞, and so does $K(.)$ Consequently, $K(.)$ attains its global maximum over $\mathbf{R}_+^{K+1}$ at p^*. If $p^* \succ 0$ we can deduce that $K'(p^*) = 0$, or $E[\ell(t(L_T^{-1}p^*))] = m$. Let us denote $X^* = t(L_T^{-1}p^*)$ and $M^* = \ell(X^*)$ we have $E[\ell(X^*)] = m$. By the duality we get that

$$\begin{aligned} E[L_T X^*] &= E[L_T F(M^*)] \\ &= E\left[L_T\left(L_T^{-1}p^*M^* - \widetilde{F}(L_T^{-1}p^*)\right)\right] \\ &= p^*m - E[\widetilde{F}(L_T^{-1}p^*)]. \end{aligned}$$

The theorem then follows. ■

Remark 7.5.2. *Consider the primal problem (7.5.57). It is deduced from the theorem above that if $p(x,m) := p^*(u(x,m),m)$ exists and lies in the interior of $\mathbf{R}_+^{K+1}$, then the optimal wealth of (7.5.57) is given by*

$$X^* = t(L_T^{-1}p(x,m)).$$

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