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# Goodness-of-fit tests in reliability: Weibull distribution and imperfect maintenance models

Meryam Krit

► **To cite this version:**

Meryam Krit. Goodness-of-fit tests in reliability: Weibull distribution and imperfect maintenance models. General Mathematics [math.GM]. Université de Grenoble, 2014. English. NNT: 2014GRENM038 . tel-01126901

**HAL Id: tel-01126901**

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## THÈSE

Pour obtenir le grade de

## DOCTEUR DE L'UNIVERSITÉ DE GRENOBLE

Spécialité : **Mathématiques Appliquées**

Arrêté ministériel : 7 août 2006

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# Goodness-of-fit tests in reliability: Weibull distribution and imperfect maintenance models.

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## Remerciements

Entre un mélange de sentiment de deuil et de culpabilité. Je souhaite dédier ma thèse à ma tante Rkia que j'ai perdue quelques jours avant ma soutenance sans que ma famille et moi ne le sachions. Elle voulait que je lui envoie mon manuscrit de thèse, elle se sentait capable de le lire et de le comprendre s'il était en français!

Elle n'arrêtait pas de nous faire rire avec ses blagues. Je me souviens encore de notre dernier fou rire car elle adorait coudre des poches à ses robes à partir des manches. Elle nous a beaucoup séduit par sa générosité voire son altruisme ... Donner sans attendre en retour. Je pense à ses dons pour les gens dans le besoin, ses petites balades et ses cadeaux aux jeunes adolescents défavorisés.

Je pense à toute ma famille malgré ma colère contre eux d'avoir attendu pour nous annoncer la triste nouvelle. Je pense à mon cousin Adnane, ses petits enfants Nour et Khalid, mes tantes Mellouki, Souad et Fatima, mes oncles Simo et Boubker, mes cousines Salma et Mouna et à ma mère. Je tiens à leur exprimer tout mon soutien et mon amour.

Je tiens à remercier mon directeur de thèse Olivier Gaudoin pour tout son soutien et son attention depuis mes premiers cours de statistiques à l'Ensimag, ses encouragements pendant la thèse et même après la thèse. Mais aussi pour la qualité de son encadrement et surtout sa gentillesse et son ouverture d'esprit. Je remercie aussi mes deux encadrants Laurent Doyen et Emmanuel Remy pour leurs relectures minutieuses et leur encadrement pendant ces trois ans de thèse.

Laurent Bordes et Bo Lindqvist m'ont fait l'honneur d'être rapporteurs de ma thèse et ont pris le temps de se déplacer. Pour tout cela je les remercie. Je remercie également Jean-Yves Dauxois pour avoir accepté de présider mon jury.

Mes remerciements vont aussi à tous les thésards du LJK : Ester, Gildas, Christine, Jonathon, Farida, Kévin, Nadia, Federico, Amine, Matthias, Pierre Olivier, Chloé, Meryem, Nelson, Roland, Margaux, Morgane, Pierre Jean, Nhu. Je remercie Anne et Laurence de leur gentillesse.

Je remercie spécialement, Maha Moussa pour sa générosité, sa gentillesse et son amour infini, merci pour le pot et la décoration.

Il m'est impossible d'oublier toute la communauté libanaise de Grenoble : Roland, Makieh, Rida, Ali, Hassan, Sandra, Hind, Mahmoud, Wael, Jihad, Hassan et Sara Bazzi. J'adresse mes sincères remerciements à l'adorable Wafa pour sa présence et son soutien continu et à Jeremy pour nos débats sur les questions existentielles qui ne finissent jamais. Je remercie également mes copines Leyla, Siham et Aliae qui sont comme des soeurs pour moi. Merci de m'avoir toujours soutenu avant et pendant la thèse, d'avoir fait le déplacement de Paris pour ma soutenance et pour votre attention sans fin.

Je souhaite aussi remercier certaines personnes qui ont beaucoup marqué ma vie. Malgré la distance je pense à eux : Chourouk, Keiko, Oum lfadl, Karn, Leyla-san, Jihene, Nada, la famille Sahil spécialement Wafa Sahil et sa maman Latifa pour leur accueil à Agadir et leur générosité, la famille Bami particulièrement Nabila et sa maman Leyla. Je remercie aussi la famille El Bahri pour leur accueil à Grenoble, la famille Françon pour leur gentillesse et leur accueil chez eux à Montélimar.

Je tiens à remercier particulièrement tout le département MRI d'EDF R&D pour leur accueil et pour les conditions de travail privilégiées qui m'ont été offertes et tous les thésards, en premier mes co-bureaux Jean-Baptiste et Guillaume et aussi Jeanne et Vincent.

Je voudrais également remercier ma deuxième famille de France, la famille des champions Calandreau pour leur soutien continu. Je remercie Steph, Emeric, Benix, Juliette et mes parents de France Alain et Véronique. Je sais que je dois m'entraîner dur pour faire les cross avec vous et pas seulement venir encourager. On aura encore pleins de compétitions, de tours du lac et de voironnaises à organiser ... sans courir!

Mes plus profonds remerciements vont à mes parents qui m'ont toujours soutenu et encouragé pendant mon cursus scolaire et je les remercie de nous avoir donné (mes soeurs, mon frère et moi) toutes les chances pour réussir. La thèse n'est qu'un aboutissement de leurs efforts, leur dévouement et leur amour infini. J'en profite pour leur exprimer ma plus grande gratitude.

Une pensée pour ma grand-mère qui j'espère sera fière de moi là où elle est, même si je ne suis pas devenue ministre d'industrie du Maroc et que je ne sais pas encore cuisiner comme elle l'espérait. Je remercie également mon frère Badr, ma belle-sœur Nawal et mes sœurs Bouchra, Sara et Kawtar de me faire rire tout les jours avec leurs messages pleins d'humour et d'amour.

À Julien

# Contents

<b>Table of contents</b>	<b>4</b>
<b>1 Introduction</b>	<b>11</b>
1.1 Industrial context . . . . .	11
1.2 Operation feedback data and reliability models . . . . .	12
1.3 Goodness-of-fit tests . . . . .	15
1.4 Structure of the dissertation . . . . .	17
<b>2 Exponential distribution: basic properties and usual GOF tests</b>	<b>19</b>
2.1 The Exponential distribution: definition and properties . . . . .	19
2.2 GOF tests for the Exponential distribution: complete samples . . . . .	21
2.2.1 Principles of GOF tests . . . . .	22
2.2.2 Test based on the probability plot . . . . .	23
2.2.3 Shapiro-Wilk test . . . . .	23
2.2.4 Tests based on the empirical distribution function . . . . .	24
2.2.5 Tests based on the normalized spacings . . . . .	24
2.2.6 Tests based on a transformation to exponentials or uniforms . . . . .	26
2.2.7 Likelihood based tests . . . . .	26
2.2.8 Tests based on the Laplace transform . . . . .	27
2.2.9 Tests based on the characteristic function . . . . .	28
2.2.10 Test based on the entropy . . . . .	29
2.2.11 Tests based on the mean residual life . . . . .	30
2.2.12 Tests based on the integrated distribution function . . . . .	30
2.3 GOF tests for the Exponential distribution: censored samples . . . . .	31
2.3.1 Tests based on the normalized spacings . . . . .	31
2.3.2 Tests based on the lack of trend . . . . .	32
2.3.3 Tests based on the empirical distribution function . . . . .	32
2.3.4 Test based on the Kullback-Leibler information . . . . .	33
2.4 Comparison of the GOF tests for the Exponential distribution . . . . .	34
2.4.1 Complete samples . . . . .	34
2.4.2 Censored samples . . . . .	38
<b>3 Weibull distribution: basic properties and usual GOF tests</b>	<b>43</b>
3.1 The Weibull distribution: definition and properties . . . . .	43
3.2 Usual GOF tests for the Weibull distribution . . . . .	47
3.2.1 Tests based on probability plots . . . . .	47
3.2.2 Shapiro-Wilk type tests . . . . .	48

3.2.3	Tests based on the empirical distribution function . . . . .	49
3.2.4	Tests based on the normalized spacings . . . . .	50
3.2.5	Generalized smooth tests . . . . .	51
3.2.6	Tests based on the Kullback-Leibler information . . . . .	52
3.2.7	Tests based on the Laplace transform . . . . .	53
<b>4</b>	<b>Simplified likelihood based GOF tests for the Weibull distribution</b>	<b>55</b>
4.1	Generalized Weibull distributions $\mathcal{GW}$ . . . . .	55
4.2	Likelihood based goodness-of-fit tests . . . . .	58
4.3	Test statistics for several $\mathcal{GW}$ distributions . . . . .	61
4.3.1	Tests based on the Exponentiated Weibull distribution . . . . .	61
4.3.2	Tests based on the Generalized Gamma distribution . . . . .	62
4.3.3	Tests based on the Additive Weibull distribution . . . . .	64
4.3.4	Tests based on the Burr Generalized Weibull distribution . . . . .	65
4.3.5	Tests based on the Marshall-Olkin extended Weibull distribution . . . . .	66
4.3.6	Test based on the Modified Weibull distribution . . . . .	67
4.3.7	Tests based on the Power Generalized Weibull distribution . . . . .	68
4.4	Simulation and comparison . . . . .	69
4.4.1	The simulation framework . . . . .	69
4.4.2	Results and discussion . . . . .	71
4.5	Asymptotic properties of some test statistics . . . . .	73
4.5.1	Asymptotic properties of $\widehat{GG}_s^1$ and $\widehat{GG}_w^1$ . . . . .	73
4.5.2	Asymptotic property of $\widehat{GG}_l^1$ . . . . .	80
<b>5</b>	<b>GOF tests for the Weibull distribution based on the Laplace transform</b>	<b>83</b>
5.1	Reminder of previous works . . . . .	83
5.2	A new test combining the approaches of Henze and Cabaña-Quiroz . . . . .	84
5.3	Cabaña and Quiroz statistics with Maximum Likelihood Estimators . . . . .	87
5.4	Simulation and comparison . . . . .	90
5.5	Results and discussion . . . . .	93
<b>6</b>	<b>Comprehensive comparison of the Weibull GOF tests</b>	<b>95</b>
6.1	Simulation framework . . . . .	95
6.2	Combination of GOF tests . . . . .	98
6.3	Results and discussion . . . . .	98
<b>7</b>	<b>Repairable systems: preliminary results</b>	<b>109</b>
7.1	Introduction . . . . .	109
7.2	Repairable systems with CM only . . . . .	110
7.2.1	Point processes . . . . .	110
7.2.2	Non Homogeneous Poisson processes (NHPP) . . . . .	112
7.2.3	AGAN model . . . . .	114
7.3	Repairable systems with CM and PM . . . . .	115
7.3.1	Notations . . . . .	115
7.3.2	Imperfect maintenance: virtual age models . . . . .	115
7.4	Exact conditional GOF tests for NHPP . . . . .	121

<b>8</b>	<b>Exact conditional GOF tests for the <math>ARA_1</math>–LLP imperfect maintenance model</b>	<b>125</b>
8.1	Principle of the test . . . . .	125
8.1.1	Sufficient statistic . . . . .	126
8.2	Parameters estimation . . . . .	127
8.3	Extension of the sufficient statistic . . . . .	129
8.4	Conditional sampling given the large sufficient statistic . . . . .	129
8.4.1	First step . . . . .	129
8.4.2	Second step . . . . .	131
8.5	Transformation to uniforms . . . . .	136
8.6	Test statistics . . . . .	137
8.7	Simulation results . . . . .	138
<b>9</b>	<b>Application to real data sets</b>	<b>141</b>
9.1	Non repairable systems . . . . .	141
9.1.1	Literature data . . . . .	141
9.1.2	EDF data . . . . .	146
9.2	Repairable systems . . . . .	151
<b>10</b>	<b>Prospects</b>	<b>155</b>
	<b>Bibliographie</b>	<b>156</b>
	<b>Appendices</b>	<b>167</b>
	<b>Appendix A</b>	<b>167</b>
A.1	Power results of the GOF tests for the Exponential distribution: complete samples . . . . .	167
A.2	Power results of the GOF tests for the Exponential distribution: censored samples . . . . .	184
A.3	Power results of the simplified likelihood GOF tests for the Weibull distribution . . . . .	192
	<b>Appendix B</b>	<b>199</b>
B.1	Documentation of EWGoF package . . . . .	199



# Acronyms

<b>ABAO</b>	As Bad As Old
<b>AGAN</b>	As Good As New
<b>ARA<sub>1</sub></b>	Arithmetic Reduction of Age model with memory one
<b>ARA<sub>∞</sub></b>	Arithmetic Reduction of Age model with infinite memory
<b>BFGS</b>	Broyden-Fletcher-Goldfarb-Shanno algorithm
<b>BP</b>	Brown-Proschan model
<b>BP<sub>l</sub></b>	Brown-Proschan model with log-linear failure intensity
<b>BP<sub>p</sub></b>	Brown-Proschan model with power-law failure intensity
<b>BT</b>	bathtub shaped hazard rate
<b>cdf</b>	cumulative distribution function
<b>CM</b>	corrective maintenance
<b>DHR</b>	decreasing hazard rate
<b>GOF</b>	goodness-of-fit
<b>GRA</b>	Geometric Reduction of Age
<b>HPP</b>	Homogeneous Poisson Process
<b>IHR</b>	increasing hazard rate
<b>iid</b>	independent and identically distributed
<b>LLP</b>	log-linear process
<b>LSE</b>	least squared estimator
<b>ME</b>	moment estimator
<b>MLE</b>	maximum likelihood estimator
<b>NHPP</b>	Non Homogeneous Poisson Process
<b>pdf</b>	probability density function
<b>PLP</b>	power-law process
<b>PM</b>	preventive maintenance
<b>RP</b>	renewal process
<b>UBT</b>	upside-down bathtub shaped hazard rate



# Chapter 1

## Introduction

---

### 1.1 Industrial context

Risk management of industrial facilities, such as EDF power plants, needs to accurately assess and predict systems reliability. Depending on the available knowledge, three main types of approaches are commonly used to assess systems reliability. If operation feedback data is available, the classical frequentist statistical approach can be used. When the operation feedback data is not informative enough, the Bayesian statistical approach is a convenient alternative since it allows adding knowledge from expert judgment [51]. When the systems failure has never been observed during the operation time period, a structural reliability analysis can be carried out to assess risk indicators from numerical models representing the physical behavior of the systems [86, 32].

In this dissertation, one considers the situation where operation feedback data is available and is the only source of knowledge about the systems reliability: thus the classical frequentist statistical approach is our scope of work. Sometimes one can obtain useful results using non parametric techniques that do not require any choice of a probabilistic model. It is the case for instance when estimating a Mean Time to Failure (MTTF) by the mean value of the observed operation lifetimes of the systems that failed. But, if one is able to choose an appropriate probabilistic parametric model, this presents several advantages:

- the hypothesis of the model may allow to better understand the nature of the random observed phenomenon
- the estimation of the reliability indicators is of a better quality
- an adapted model allows to make predictions outside the operation feedback data set which can not be accomplished by a non parametric method.

This parametric approach consists in three main steps illustrated in figure 1.1. The first phase requires the building of relevant probabilistic parametric models in order to reflect the randomness of the occurrence of systems failures. These models must be complex enough to be able to represent the way the systems are operated. In a second stage, statistical inference of the parameters of the developed models must be carried out, based on the available operation feedback data. When these two steps are carried out,

a final stage, as important as the previous ones, consists in firstly validating the fitted models using statistical criteria and secondly comparing the different competing models. The subject of the PhD thesis falls within this last stage of model validation and selection, which is a crucial issue.

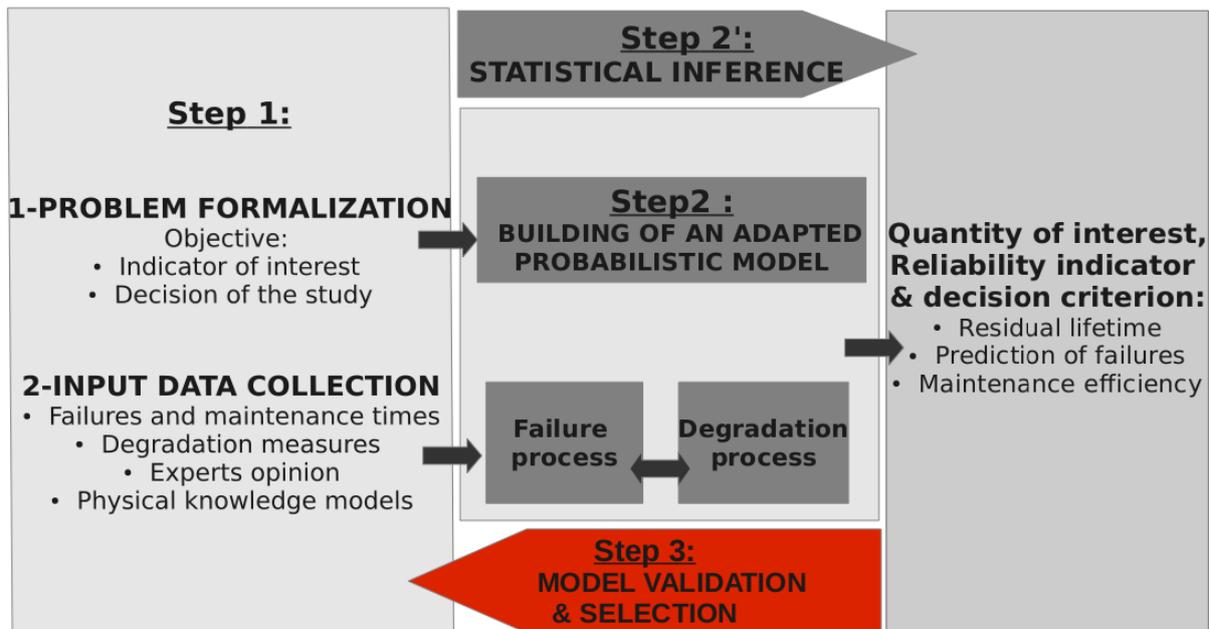


Figure 1.1: Three main steps of the approach

Indeed from a regulatory point of view, electric utilities have to present convincing quantitative arguments to regulatory authorities in order to justify systems reliability. It is thus essential to ensure the fitted models are relevant (even the best) given the operation feedback data. From a performance point of view, the misspecification of the systems reliability models can lead to establish inappropriate preventive maintenance plans resulting in poor availability and economic performance of the power plants.

That is why it is crucial for EDF to have efficient probabilistic and statistical techniques to determine the closest reliability models to the reality and prove their relevance and quality.

## 1.2 Operation feedback data and reliability models

Depending on the characteristics of the studied systems, specific operation feedback data are observed and appropriate probabilistic reliability models must be used to represent real-life condition. The simplest case is the one of non repairable systems, generally components. The quantity of interest is the operation time before the (unique) failure of the systems. The feedback data from the operation of a fleet of components is made of:

- complete data associated with the operation times of the components that failed;

- censored data relative, for instance, to the lifetimes of the components that did not break down during the operation time period.

When the components are identical (from design, manufacturing, operation, maintenance, environmental ... points of view) and independent (no common cause failures), the operation feedback data is thus compounded of observations which constitute a sample of independent random variables following the same distribution (identically distributed). For instance, table 1.1 presents a classical data set of the literature [2]. It gives the failure times of 50 devices.

Table 1.1: Failure data of 50 devices (Aarset data)

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

The most usual distribution used to represent the lifetime of components are the Exponential and the Weibull distributions. These distributions are widely used to model the lifetimes of non repairable systems. The Exponential distribution represents the disadvantage of having a constant failure rate. The Weibull distribution is a more flexible model since it allows decreasing, constant and increasing failure rates. It is then essential to be able to check the relevance of these two distributions for a given data set. In this work, we focus on the two-parameter Weibull distribution.

It is important to highlight that even if the Weibull distribution is popular in reliability survival and analysis, it is also frequently used in many other technical fields: one can mention environmental sciences (weather forecasting and hydrology), insurance, geology, chemistry, physics, medicine, economics and geography. Due to its close link to the extreme value distribution, the Weibull model also appears in the extreme value theory. Last but not least, the founding work of Waloddi Weibull [128] in the field of structural mechanics stresses the relevance of using the Weibull distribution to model physical parameters such as the mechanical toughness (or strength) of a material and the length of defects. EDF is also interested by data of that kind. For instance, table 1.2 presents measures of toughness of EDF material at a specific temperature  $\delta_2$ . These data have been modified for confidentiality reasons.

Table 1.2: Toughness data at  $\delta_2$

14.13	67.54	70.68	98.96	102.10	105.24
105.24	149.22	171.21	177.49	183.78	190.06
205.77	240.33	252.89	268.60	284.31	293.73
300.02	303.16	312.58	362.85	369.13	409.97

The case of repairable systems is more complex. Firstly let us suppose no preventive maintenance (PM) is carried out on the system (the "run to failure" strategy is adopted).

After a failure, a repair (or corrective maintenance - CM) is carried out so that the system can perform its function again. Throughout the thesis, we will consider that repair times are negligible or not taken into account, so failure times and CM times are identical. For a given piece of equipment, one is interested in the time sequence of the successive CM. It is a sequence of recurrent events which can be modeled by a univariate point process. Figure 1.2 illustrates the occurrence of CM for a repairable system. Table 1.3 represents CM times (in days) of some type of pipes within the boiler of an EDF coal-fired power station. The welds of the straps holding these pipes are subjected to corrosion leading to the initiation then propagation of flaws that may endanger the stability of the pipes. Since it has no major impact on the operation of the plant, a run to failure maintenance plan is carried out.



Figure 1.2: Occurrence of CM of a repairable system

Table 1.3: CM times of a repairable system

28	732	1152	1243	1327	1446	1502	1509	1739	1809
1852	2005	2198	2234	2521	2646	2804	2839	2873	2943
3090	3111								

Now let us make the assumption that PM is also carried out on the system. PM intends to slow down the wear process and reduce the frequency of occurrence of the system failures. PM can be carried out at specific times previously fixed (planned maintenance strategy) or depend on the monitoring of the system state (condition based maintenance strategy). For a given system, one is interested in the sequence of both types of maintenance times (CM and PM). In this case, a multivariate point process must be used to model the sequence of events illustrated in figure 1.3.

The second case study deals with a specific family of components within the boiler of an EDF coal-fired power station. The welds holding these components are subjected to thermal fatigue leading to the initiation then propagation of flaws that may generate leaks. These require the immediate shutdown of the boiler, and consequently of the plant, which may be critical from a performance point of view if the energy demand is huge. That is why a specific maintenance plan combining corrective and preventive actions is carried out:

1. when a leak occurs during plant operation, the incriminated weld is repaired without delay to restart the plant as soon as possible.

2. scheduled preventive inspections of the hazard zones of the system are carried out periodically and the detected cracks are scoured.

Table 1.4 gives the PM and CM times of these welds [35].



Figure 1.3: Occurrence of CM and PM of a repairable system

Table 1.4: CM and PM times of a repairable system

25	50	93	109	114	141	163	164	195	225	264
PM	CM	CM	CM	PM	CM	CM	CM	CM	PM	PM

For maintained systems, the maintenance effect naturally impacts the system reliability. A first classical approach to take into account this impact is to assume that the maintenance is minimal, which means it leaves the system in the same state as it was just before. It characterizes a maintenance effect that neither improves nor damages the system. It is called As Bad As Old (ABAO) maintenance and the corresponding random process family is the Non Homogeneous Poisson Processes (NHPP). A second basic hypothesis consists in assuming that the maintenance is perfect, which means that it perfectly repairs the system and leaves it as if it was new. The latter is "As Good As New" (AGAN) after maintenance and the system is comparable to a similar new system put into operation just after the previous maintenance. The corresponding random process family is the renewal processes. Obviously standard maintenance reduces failure intensity but does not systematically leave the system as good as new: reality is between the two extreme cases previously presented. In the literature, models enabling to take into account a maintenance effect between ABAO and AGAN are known as imperfect maintenance models. Many models have been suggested [63, 20] and among them the most popular are the virtual age models, for which the maintenance rejuvenates the system [63]. The Arithmetic Reduction of Age (ARA) models are one of those and are based on an arithmetic reduction of what is called the virtual age of the system [33, 34, 35].

In order to take into account the diversity of the types of systems which are installed within EDF power plants, it is necessary to have validation and selection statistical indicators adapted for the different probabilistic models which have just been presented.

### 1.3 Goodness-of-fit tests

As already mentioned, it is fundamental to be able to choose an adapted parametric model to a given data set and choose the best fitted model from a large range of candidate models.

It is a classical statistical issue known as model validation and selection. Goodness-of-fit (GOF) tests are a useful tool to achieve this goal.

There is a wide literature on GOF tests for the Exponential distribution, but very little attention was paid to GOF tests for parametric models suitable in the field of industrial reliability, such as the Weibull distribution and the imperfect maintenance models that have been presented in the previous section.

Moreover, in nuclear electricity generation industry, systems failures are rare events, leading to small and highly censored data sets which make the use of standard statistical techniques difficult (even impossible). That is why the subject of the thesis, “GOF tests in reliability: Weibull distribution and imperfect maintenance models”, is as challenging as the imposed industrial constraints which require the development of new methods.

The first aim of the dissertation is to develop GOF tests for basic models like samples of independent and identically distributed (iid) random variables, in order to answer the question whether an iid sample comes from a specific distribution (the Exponential or the Weibull distributions) or not. The second aim answers the same question for more sophisticated models: Non Homogeneous Poisson processes (NHPP), imperfect maintenance models, ...

For non-repairable systems, we consider  $n$  similar systems operating independently to each others. Their lifetimes are considered to be realizations of random variables  $X_1, \dots, X_n$  independent and identically distributed.

If all the lifetimes of the  $n$  systems are observed, they constitute a complete sample. When not all the lifetimes are observed, it is a censored sample. There exist several kinds of censoring: left or right, type I or type II, simple or multiple, etc ...

For non-repairable systems, we will be interested basically in complete samples and in some cases simple type II censored samples. Type II left-censoring occurs when the smallest  $s$  lifetimes are not observed and type II right-censoring occurs when the largest  $r$  lifetimes are not observed.

For repairable systems, we consider that we are studying one system that can be subject to CM or PM. The quantities of interest are the CM times of the given system. The PM are considered to be deterministic. The CM are considered to be the realizations of a random point process. The question is still to find the best fitted point process to model the occurrence of the failures. The system is assumed to be repaired after each failure so we consider in all the studied cases that we have type I right-censoring which means the observation stops after a given censoring time  $T$ .

The example of data in table 1.1 presents realizations of iid random variables. The problem of interest is to find a model which fits well this data set. The problem is expressed as a statistical test. We denote  $F$  the unknown distribution function of the sample. This distribution is assumed to be continuous. In the case of discrete distributions, the presented procedures need some arrangements that are not always simple. The GOF tests for discrete distribution are detailed in chapter 7 of [18].

We distinguish two cases, depending on whether we want to test the goodness-of-fit to an entirely specified distribution or to a family of distributions.

- GOF tests to an entirely specified distribution:

$$H_0 : “F = F_0” \text{ vs } H_1 : “F \neq F_0”. \quad (1.1)$$

- GOF tests to a family of distributions:

$$H_0 : "F \in \mathcal{F}" \text{ vs } H_1 : "F \notin \mathcal{F}." \quad (1.2)$$

Often, family  $\mathcal{F}$  is a parametric family:  $\mathcal{F} = \{F(\cdot; \theta); \theta \in \Theta\}$ . It is the case when we test whether the Aarset data comes from a Weibull distribution without precising specific values of the parameters. If a Weibull distribution is adapted, we can estimate lately its parameters.

The examples in tables 1.3 and 1.4 give CM and PM times of a repairable system. The observations in this case are realizations of a point process. We want to find a relevant model for this process. We denote  $\lambda$  the unknown intensity function of the point process. The GOF test in this case has the following hypotheses:

$$H_0: "\lambda \in \mathcal{I}" \text{ vs } H_1 : "\lambda \notin \mathcal{I}"$$

where the family  $\mathcal{I}$  is a parametric family:  $\mathcal{I} = \{\lambda(\cdot; \theta); \theta \in \Theta\}$ . For instance, one may want to test a NHPP with a specific intensity function, either a power law intensity or a NHPP with log-linear intensity function.

## 1.4 Structure of the dissertation

The thesis is structured in two parts of unequal size. Chapters 2 to 6 are devoted to non-repairable systems and Chapters 7 and 8 to repairable systems. Two appendices provide tables of results and a documentation of the R package EWGoF we have developed.

Chapter 2 presents a review of existing GOF tests for the Exponential distribution, for complete and censored samples. A comprehensive comparison study is done, using Monte-Carlo simulations. It leads to identify the best of these tests.

Chapter 3 is the first of 4 chapters dedicated to the two-parameter Weibull distribution. First it gives the definitions and main properties of this distribution, that will be used throughout the dissertation. Then, it presents a review of existing GOF tests for the Weibull distribution.

In Chapters 4 and 5, we propose two new families of GOF tests for the Weibull distribution. Chapter 4 is dedicated to likelihood-based tests. These tests consist in nesting the two-parameter Weibull distribution in three-parameter generalized Weibull families and testing the value of the third parameter by using the Wald, score and likelihood ratio procedures. We simplify the usual likelihood based tests by getting rid of the nuisance parameters, using three estimation methods, maximum likelihood, least squares and moments.

Chapter 5 presents a second new family of GOF tests for the Weibull distribution, based on the Laplace transform. These tests merge the ideas of Cabaña and Quiroz [22] and those introduced by Henze [53] for testing the Exponential distribution. We also introduce new versions of the statistics of Cabaña and Quiroz, using maximum likelihood estimators instead of moment estimators.

Chapter 6 presents a comprehensive comparison study of all GOF tests for the Weibull distribution. This comparison includes the usual GOF tests presented in Chapter 3 and the new ones developed in Chapters 4 and 5. The idea of combining GOF tests is also

introduced. Recommendations about the most powerful tests are given, according to the characteristics of the tested data. The best tests that we have identified are little known and rarely used.

In Chapter 7, we move to the repairable systems case. This chapter gives some preliminary results about nonhomogeneous Poisson processes and imperfect maintenance models, when both corrective and preventive maintenances are performed. The tests proposed by Lindqvist and Rannestad [79] for testing the fit of NHPPs are presented. They are based on conditional sampling given a sufficient statistic.

Chapter 8 is a first attempt to building GOF tests for imperfect maintenance models. The considered model assumes that the corrective maintenances are minimal (ABAO) with a log-linear initial intensity. It also assumes that the preventive maintenances are done at deterministic times and that their effect is of the Arithmetic Reduction of Age with memory one ( $ARA_1$ ) type. In this case, a sufficient statistic exists and the tests of Lindqvist and Rannestad [79] can be generalized.

Chapter 9 presents the application of this study to real data sets, some from the literature and some from EDF. These data sets are from both non repairable and repairable systems. The practical use of the tests in an industrial context is detailed.

Appendix A contains a huge number of simulation results, which aim to assess the power of GOF tests for the Exponential and Weibull distributions.

Appendix B gives a detailed documentation on the R package EWGoF that we have developed. This package implements all the GOF tests for non repairable systems presented in this dissertation: GOF tests for the Exponential distribution of Chapter 2 and GOF tests for the Weibull distribution of Chapters 3, 4 and 5. An important feature of these tests is that they are all exact: the critical values needed for performing the tests are obtained by Monte-Carlo simulation and no asymptotic results are used. Then, the GOF tests can be applied for any sample size. All the simulation results and applications to real data presented in the thesis have been done using the EWGoF package.

# Chapter 2

## Exponential distribution: basic properties and usual GOF tests

---

This chapter is dedicated to the Exponential distribution. First, some definitions and basic properties of this distribution are given. Then, we present a quick review of GOF tests for the Exponential distribution, based on different approaches: probability plots, empirical distribution function, normalized spacings, Laplace transform, characteristic function, entropy, integrated distribution function, likelihood based tests, ... Complete and censored samples are treated. Finally, an extensive comparison study is done which leads to identify the best GOF tests for the Exponential distribution.

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### 2.1 The Exponential distribution: definition and properties

A random variable  $X$  is from the Exponential distribution of parameter  $\lambda$ , denoted  $\exp(\lambda)$ , if and only if its cumulative distribution function (cdf) is:

$$F(x; \lambda) = 1 - \exp(-\lambda x), \quad x \geq 0, \lambda > 0. \quad (2.1)$$

- The probability density function (pdf) is:

$$f(x; \lambda) = \lambda \exp(-\lambda x), \quad x \geq 0, \lambda > 0. \quad (2.2)$$

- The reliability is  $R(x) = 1 - F(x, \lambda) = \exp(-\lambda x)$ .
- The expectation (or the Mean time to failure MTTF) is:  $\text{MTTF} = \mathbb{E}[X] = \frac{1}{\lambda}$ .
- The variance is  $\text{Var}[X] = \frac{1}{\lambda^2}$ .

- The hazard rate is  $h(x) = \frac{f(x)}{R(x)} = \frac{\lambda \exp(-\lambda x)}{\exp(-\lambda x)} = \lambda$ .
- The mean residual life is  $m(x) = \mathbb{E}[X - x | X > x] = \frac{1}{\lambda} = \mathbb{E}[X]$ .
- The Laplace transform is  $\psi(t) = \mathbb{E}[\exp(-tX)] = \frac{\lambda}{\lambda + t}$ .
- The characteristic function is  $\varphi(t) = \mathbb{E}[\exp(itX)] = \frac{\lambda}{\lambda - it}$ .
- If  $X$  is from  $\exp(\lambda)$ ,  $Y = \lambda X$  follows a standard Exponential distribution  $\exp(1)$ .

The Exponential distribution is without memory. It means if that the system did not fail yet at time  $t$ , then it behaves as if it was new at this time. Indeed, the random variable  $X$  obeys the following relation:

$$\forall x \geq 0, \quad P(X > t + x | X > t) = P(X > x). \quad (2.3)$$

In reliability, it means that the Exponential distribution is suitable for systems which are not deteriorating neither improving with time.

Let  $x_1, \dots, x_n$  be realizations of independent and identically distributed (iid) random variables  $X_1, \dots, X_n$  with the  $\exp(\lambda)$  distribution. The likelihood function is :

$$\mathcal{L}(\lambda; x_1, \dots, x_n) = \prod_{i=1}^n f(x_i) = \lambda^n \exp\left(-\lambda \sum_{i=1}^n x_i\right). \quad (2.4)$$

Maximizing this function, we obtain that the Maximum Likelihood Estimator (MLE) of  $\lambda$  is:

$$\hat{\lambda}_n = \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\bar{X}_n}. \quad (2.5)$$

After estimating  $\lambda$  by  $\hat{\lambda}_n = \frac{1}{\bar{X}_n}$ , we will be interested in the random variables  $\hat{Y}_i = \hat{\lambda}_n X_i = \frac{X_i}{\bar{X}_n}$  that have a distribution that should be “close” to  $\exp(1)$ .

The vector  $(\hat{Y}_1, \dots, \hat{Y}_n) / n$  has the Dirichlet distribution  $D(1, \dots, 1)$ . This allows to prove that asymptotically, distribution of this vector is independent of the parameter  $\lambda$ . Therefore, each statistic built as a function of  $(\hat{Y}_i)_{1 \leq i \leq n}$  can be a GOF test statistic.

Let  $X_1^* \leq \dots \leq X_n^*$  be the order statistics of the sample  $X_1, \dots, X_n$ , and  $X_0^* = 0$ . The distribution of the  $(X_i^*)_{1 \leq i \leq n}$  has location and scale parameters  $\mu$  and  $\sigma$ , if the distribution

of  $\frac{X_i - \mu}{\sigma}$  does not depend on  $\mu$  nor on  $\sigma$ . For such a distribution, the normalized spacings are defined as:

$$E_i = \frac{X_i^* - X_{i-1}^*}{\mathbb{E} \left[ \frac{X_i^* - \mu}{\sigma} \right] - \mathbb{E} \left[ \frac{X_{i-1}^* - \mu}{\sigma} \right]}, \forall i \in \{1, \dots, n\}. \quad (2.6)$$

The expectations at the denominator of  $E_i$  do not depend on  $\mu$  and  $\sigma$ , then the  $E_i$  are observed. The normalized spacings can be written as follows:

$$E_i = \sigma \frac{X_i^* - X_{i-1}^*}{\mathbb{E} [X_i^* - X_{i-1}^*]} = \sigma \frac{\frac{X_i^* - \mu}{\sigma} - \frac{X_{i-1}^* - \mu}{\sigma}}{\mathbb{E} \left[ \frac{X_i^* - \mu}{\sigma} \right] - \mathbb{E} \left[ \frac{X_{i-1}^* - \mu}{\sigma} \right]}. \quad (2.7)$$

Any statistic written as  $\sum_i a_i E_i / \sum_j b_j E_j$  is distributed independently of the parameters  $\mu$  and  $\sigma$ , so it can be used to build a GOF test.

When the sample  $X_1, \dots, X_n$  comes from  $\exp(\lambda)$  ( $\mu = 0$  and  $\sigma = \frac{1}{\lambda}$ ), the normalized spacings are defined in this case as:

$$E_i = (n - i + 1)(X_i^* - X_{i-1}^*), i \in \{1, \dots, n\}. \quad (2.8)$$

Under the Exponential assumption, the  $(E_i)_{1 \leq i \leq n}$  are iid with the same distribution  $\exp(\lambda)$ .

In the case of censored samples, when only the lowest  $n-r$  failure times  $x_1^* \leq \dots \leq x_{n-r}^*$  are observed, the likelihood function in this case is:

$$\begin{aligned} \mathcal{L}(\lambda; x_1^*, \dots, x_{n-r}^*) &= \prod_{i=1}^{n-r} f(x_i^*) [1 - F(x_{n-r}^*)]^r \\ &= \lambda^{n-r} \exp \left( -\lambda \sum_{i=1}^{n-r} x_i^* - \lambda r x_{n-r}^* \right). \end{aligned}$$

Thus, the maximum likelihood estimator of  $\lambda$  is:

$$\hat{\lambda}_n = \frac{n-r}{\sum_{i=1}^{n-r} X_i^* + r X_{n-r}^*}. \quad (2.9)$$

## 2.2 GOF tests for the Exponential distribution: complete samples

In this section, we present a review of GOF tests for the Exponential distribution for complete samples. There is a wide literature on GOF tests for the Exponential distribution from the 50's until now. Several review papers were published through time: Epstein [40,

41], Spurrier [117], Ascher [7], Henze-Meintanis [53], chapter 10 of D'Agostino-Stephens [31] and chapter 13 of Balakrishnan-Basu [10]. In all what follows, the studied GOF tests have the most general alternative hypothesis. There are some GOF tests that aim to test the Exponential distribution against a specific distribution such as the work of Muralidharan [91], Basu-Mitra [14] and Gatto-Jammalamadaka [46].

The GOF tests families presented are the families of tests based on the probability plot, the empirical distribution function, the normalized spacings, the likelihood, the Laplace transform, the characteristic function, the entropy, the mean residual life and the integrated distribution function.

### 2.2.1 Principles of GOF tests

Let  $X_1, \dots, X_n$  be iid random variables and  $F$  their cumulative distribution function. For the Exponential distribution, a GOF test is a statistical test of hypothesis  $H_0: "F \in \mathcal{F}"$  vs  $H_1: "F \notin \mathcal{F}"$ , where  $\mathcal{F}$  is the family of the cdfs of the Exponential distributions.

The type I error consists in wrongly rejecting the null hypothesis  $H_0$ . Here, it means concluding that the distribution is not Exponential while it is Exponential indeed. The significance level of the test,  $\alpha$ , is the probability of type I error. It is generally set to  $\alpha = 5\%$ . The type II error consists in not rejecting the Exponential hypothesis while the distribution is indeed not Exponential. The power of the test is the probability of not committing the type II error. It measures the test ability of concluding correctly that the distribution is not Exponential.

A GOF test is generally based on a test statistic  $Z$  which is a measure of the distance between two quantities: a theoretical one which characterizes the tested hypothesis  $H_0$  and an empirical one computed from the studied data set. The null hypothesis in this case is rejected when  $Z$  is too large. The critical region is the set of values of  $Z$  for which  $H_0$  is rejected. If the observed value of  $Z$ ,  $z_{obs}$ , belongs to the critical region, the conclusion of the test is the rejection of  $H_0$ .

The determination of the critical region is based on the distribution of the test statistic under  $H_0$ . When the rejection is done for large values of the statistic, it means that, for a fixed level  $\alpha$ ,  $H_0$  is rejected when  $z_{obs} > q_{1-\alpha}$ , where  $q_{1-\alpha}$  is the quantile of order  $1 - \alpha$  of the distribution of  $Z$  under  $H_0$ :  $P_{H_0}(Z > q_{1-\alpha}) = \alpha$ . The test in this case is a one-sided test. Some tests are two-sided:  $H_0$  is rejected when  $Z$  is either larger than the quantile of order  $1 - \alpha/2$  or lower than the quantile of order  $\alpha/2$ .

In most cases, the distribution of the test statistics under  $H_0$  is not known. Then, their quantiles are computed using simulations. We simulate a large number  $K$  of samples from the Exponential distribution. For each  $k \in \{1, \dots, K\}$ , the value of the test statistic  $Z_k$  is computed. The quantile of order  $1 - \alpha$  is approximated by the  $(1 - \alpha)^{\text{th}}$  empirical quantile of the sample  $Z_1, \dots, Z_K$ .

The p-value of the test is the probability under  $H_0$  that the test statistic is greater than its observed value:  $p_{obs} = P_{H_0}(Z > z_{obs})$ . If the distribution of  $Z$  is not known,  $p_{obs}$  is estimated by the frequency of simulated values of  $Z$  which are greater than  $z_{obs}$ :

$$\hat{p}_{obs} = \frac{1}{K} \sum_{i=1}^K \mathbb{1}_{\{Z_i > z_{obs}\}}.$$

The distribution of the test statistics under  $H_0$  has to be known or computable. Then, it cannot depend on the parameters of the tested distribution. This is a very important

point, on which we will focus in the following.

### 2.2.2 Test based on the probability plot

The probability plot is a graph that can be used to evaluate the fit of a distribution  $F(\cdot; \theta)$  to the observations. The principle is to look for a linear relationship such as  $h_1[F(x; \theta)] = \alpha_1(\theta)h_2(x) + \alpha_2(\theta)$  where  $h_1$  and  $h_2$  are functions that do not depend on  $\theta$ . Thus, if the real cdf is  $F(\cdot; \theta)$ , then  $h_1[\mathbb{F}_n(x)]$  should be close to  $\alpha_1(\theta)h_2(x) + \alpha_2(\theta)$  where

$\mathbb{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}}$  is the empirical distribution function.

Let  $x_1^* < \dots < x_n^*$  be order statistics of the observations  $x_1, \dots, x_n$ . For  $x = x_i^*$ ,  $h_1[\mathbb{F}_n(x_i^*)] = h_1\left(\frac{i}{n}\right)$ . When  $F$  is the real cdf, the points of the plot  $(h_2(x_i^*), h_1\left(\frac{i}{n}\right))$  should be approximately aligned. For the Exponential distribution,  $F(x; \lambda) = 1 - \exp(-\lambda x)$  then,  $\ln(1 - F(x; \lambda)) = -\lambda x$ . Thus, the probability plot of the Exponential distribution is the plot of points [10]:

$$\left(x_i^*, \ln\left(1 - \frac{i}{n}\right)\right), i \in \{1, \dots, n-1\}. \tag{2.10}$$

Patwardhan [99] worked on a variant of the probability plot based on the expectations of the order statistics of the standard Exponential distribution [99]:

$$\left(\sum_{j=1}^i \frac{1}{n-j+1}, x_i^*\right), i \in \{1, \dots, n\}. \tag{2.11}$$

For all  $i$ , let  $\delta_i = \sum_{j=1}^i \frac{1}{n-j+1}$  and  $\hat{Y}_i^* = \frac{X_i^*}{\bar{X}_n}$ . Under the Exponential assumption,

these points should be approximately on the line  $y = x$ . Patwardhan suggested a statistic  $Pa_n$  that measures the proximity between vectors  $(\delta_1, \dots, \delta_n)$  and  $(\hat{Y}_1^*, \dots, \hat{Y}_n^*)$ . This statistic can also be written as a function of the normalized spacings  $E_i$ :

$$Pa_n = n(n+1) \frac{\sum_{i=1}^n E_i^2}{\left[\sum_{i=1}^n E_i\right]^2}. \tag{2.12}$$

The null hypothesis  $H_0$  is rejected for large values of  $Pa_n$ .

### 2.2.3 Shapiro-Wilk test

The Shapiro-Wilk test [113] is based on the ratio of two estimators of  $1/\lambda$ . Their procedure is applied to Exponential distribution with a location parameter and can not be applied to standard Exponential distribution. Stephens in [119] adapted Shapiro-wilk statistic for the Exponential distribution with a null location parameter. The test statistic is:

$$SW_n = \frac{\bar{X}_n^2}{(n+1)S_n^2 + \bar{X}_n^2}, \text{ where } \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \text{ and } S_n^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}_n^2. \quad (2.13)$$

The rejection of the null hypothesis  $H_0$  is done for too large or too small values of the test statistic.

## 2.2.4 Tests based on the empirical distribution function

These tests are based on a measure of the departure between the empirical distribution function  $\mathbb{F}_n$  and the estimated theoretical distribution function  $\hat{F}_0(x) = F(x; \hat{\lambda}_n) = 1 - \exp(-\hat{\lambda}_n x)$ . The null hypothesis is rejected when this difference is too large.

The best known statistics are [31]:

- Kolmogorov-Smirnov statistic (KS):

$$\begin{aligned} KS_n &= \sqrt{n} \sup_{x \in \mathbb{R}} \left| \mathbb{F}_n(x) - \hat{F}_0(x) \right| \\ &= \sqrt{n} \max \left[ \max \left\{ \frac{i}{n} - U_i^* \right\}, \max \left\{ U_i^* - \frac{i-1}{n} \right\} \right] \end{aligned} \quad (2.14)$$

- Cramer-von Mises statistic (CM):

$$\begin{aligned} CM_n &= n \int_{-\infty}^{+\infty} \left[ \mathbb{F}_n(x) - \hat{F}_0(x) \right]^2 d\hat{F}_0(x) \\ &= \sum_{i=1}^n \left( \hat{U}_i^* - \frac{2i-1}{2n} \right)^2 + \frac{1}{12n} \end{aligned} \quad (2.15)$$

- Anderson-Darling statistic (AD):

$$\begin{aligned} AD_n &= n \int_{-\infty}^{+\infty} \frac{\left[ \mathbb{F}_n(x) - \hat{F}_0(x) \right]^2}{\hat{F}_0(x) (1 - \hat{F}_0(x))} d\hat{F}_0(x) \\ &= -n + \frac{1}{n} \sum_{i=1}^n \left[ (2i-1-2n) \ln(1 - \hat{U}_i^*) - (2i-1) \ln(\hat{U}_i^*) \right] \end{aligned} \quad (2.16)$$

where  $U_i = \hat{F}_0(X_i) = 1 - \exp(-X_i/\bar{X}_n)$ .

## 2.2.5 Tests based on the normalized spacings

Several statistics have been developed using the normalized spacings  $E_i = (n-i+1)(X_i^* - X_{i-1}^*)$ . Gnedenko in [49] suggested the following one:

$$Gn(l) = \frac{(n-l) \sum_{j=1}^l E_j}{l \sum_{j=l+1}^n E_j}. \quad (2.17)$$

The statistic  $Gn$  has, under  $H_0$ , the Fisher-Snedecor distribution  $F(2l, 2(n-l))$ . A second test statistic is proposed by Harris [52]:

$$Gn^*(l) = \frac{(n-2l) \left( \sum_{j=1}^l E_j + \sum_{j=n-l+1}^n E_j \right)}{2l \sum_{j=l+1}^{n-l} E_j}. \quad (2.18)$$

The test statistics  $Gn(l)$  and  $Gn^*(l)$  are functions of the parameter  $l$ . We will use the recommended values of the parameter  $l$  given in [52]:  $l = [n/2]$  for  $Gn$  and  $l = [n/4]$  for  $Gn^*$ . Gail and Gastwirth [45] proposed the Gini statistic:

$$GG_n = \frac{\sum_{i=1}^{n-1} iE_{i+1}}{(n-1) \sum_{i=1}^n E_i}. \quad (2.19)$$

For the previous three tests, the Exponential hypothesis is rejected for large and small values of the statistics.

Lin and Mudholkar in [77] used separately both terms of the Harris statistic  $Gn^*(l)$ :

$$LM_1(l) = \frac{(n-2l) \sum_{i=1}^l E_i}{l \sum_{j=l+1}^n E_j} \quad (2.20)$$

$$LM_2(l) = \frac{(n-2l) \sum_{j=l+1}^{n-l} E_j}{l \sum_{j=l+1}^n E_j}. \quad (2.21)$$

The Exponential hypothesis is rejected if at least one of the two statistics  $LM_1$  and  $LM_2$  is too large or too small. The test is denoted  $LM(l)$ . We choose  $l = \lfloor \frac{(n-1)}{10} \rfloor$  as in [77].

### 2.2.6 Tests based on a transformation to exponentials or uniforms

Some transformations can be applied to the original sample  $X_1, \dots, X_n$ . For example the normalized spacings  $E_i = (n - i + 1)(X_i^* - X_{i-1}^*)$ ,  $i \in \{1, \dots, n - 1\}$ , are random variables composing a new iid sample from  $\exp(\lambda)$ . Stephens in [31] called it the transformation N. All the previous GOF tests for the Exponential distribution applied to  $X_1, \dots, X_n$  can also be applied to  $E_1, \dots, E_n$ .

A second approach consists in transforming an iid sample from  $\exp(\lambda)$  to an iid sample from the uniform distribution over  $[0, 1]$ ,  $\mathcal{U}[0, 1]$ . Therefore, testing the exponentiality

of the sample  $X_1, \dots, X_n$  is equivalent to testing the uniformity of  $\frac{\sum_{j=1}^i E_j}{\sum_{j=1}^n E_j}$ . The last

transformation is called by Stephens in [31] the K transformation.

### 2.2.7 Likelihood based tests

The likelihood based tests consist in including the tested distribution in a larger parametric family and testing a specific value of the parameter of this family using some procedures such as the score and likelihood ratio tests. In our case, the Exponential distribution  $\exp(\lambda)$  is included in the family of Weibull distributions  $\mathcal{W}(1/\lambda, \beta)$ . The idea is to test exponentiality by testing  $H_0$ : “ $\beta = 1$ ” and  $H_1$ : “ $\beta \neq 1$ ”, where  $\beta$  is the shape parameter of the Weibull distribution and  $\lambda$  is a nuisance parameter. The test proposed by Cox and Oakes [29] is the score test using the observed Fisher information instead of the exact Fisher information. The rejection of the null hypothesis  $H_0$  is done for large values of the statistics. The likelihood based test statistics are as follows:

- Score test:

$$Sc_n = \frac{6}{n\pi^2} \left[ n + \sum_{i=1}^n \ln X_i - \frac{1}{\bar{X}_n} \sum_{i=1}^n (\ln X_i) X_i \right]^2 \quad (2.22)$$

- Cox-Oakes test:

$$CO_n = \frac{\left[ n + \sum_{i=1}^n \ln X_i - \frac{1}{\bar{X}_n} \sum_{i=1}^n (\ln X_i) X_i \right]^2}{n + \frac{1}{\bar{X}_n} \sum_{i=1}^n (\ln X_i / \bar{X}_n)^2 X_i - \frac{1}{n\bar{X}_n^2} \left[ \sum_{i=1}^n (\ln X_i / \bar{X}_n) X_i \right]^2} \quad (2.23)$$

- Likelihood ratio test:

$$LR_n = 2n \ln \frac{\hat{\beta}_n \sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^{\hat{\beta}_n}} + 2(\hat{\beta}_n - 1) \sum_{i=1}^n \ln X_i \quad (2.24)$$

where  $\hat{\beta}_n$  is the MLE of  $\beta$  defined in equation (3.5). The rejection of  $H_0$  is done for large values of the statistics.

### 2.2.8 Tests based on the Laplace transform

Henze [53] proposed GOF tests for the Exponential distribution based on the Laplace transform. The building of the test is based on the measure of the difference between the empirical Laplace transform and its theoretical version.

Henze used the fact that the sample  $Y_i = \lambda X_i, \forall i \in \{1, \dots, n\}$  is a sample from the unit Exponential distribution. Its Laplace transform is:

$$\psi(t) = \mathbb{E}[\exp(-tY_i)] = \frac{1}{1+t}. \quad (2.25)$$

Since  $\lambda$  is unknown, it can be estimated by the MLE  $\hat{\lambda}_n$ . The distribution of  $\hat{Y}_1, \dots, \hat{Y}_n$  is independent of  $\lambda$ .

Henze's idea [53] is to reject the hypothesis that  $X_1, \dots, X_n$  are exponentially distributed if the empirical Laplace transform  $\psi_n(t) = \frac{1}{n} \sum_{i=1}^n \exp(-t\hat{Y}_i)$  is too far from the theoretical Laplace transform of a standard Exponential  $\psi(t)$ . The closeness between both functions is measured by a test statistic of the form:

$$He_{n,a} = n \int_0^{+\infty} \left[ \psi_n(t) - \frac{1}{(1+t)} \right]^2 w(t; a) dt \quad (2.26)$$

where  $w(t; a) = \exp(-at)$  is a weight function. Using the integration by parts, the test statistic turns out to be:

$$He_{n,a} = \frac{1}{n} \sum_{i,j=1}^n \frac{1}{\hat{Y}_i + \hat{Y}_j + a} - 2 \sum_{j=1}^n \exp(\hat{Y}_j + a) E_1(\hat{Y}_j + a) + n(1 - ae^a E_1(a)) \quad (2.27)$$

where  $E_1(z) = \int_z^{+\infty} \frac{\exp(-t)}{t} dt$ .

The choice of the parameter  $a$  allows to build powerful GOF tests for a large range of alternatives.

Baringhaus and Henze [12] proposed to use the fact that  $\psi(t)$  is solution of the differential equation  $(\lambda + t) \psi'(t) + \psi(t) = 0$ . The corresponding test statistics is:

$$BH_{n,a} = n \int_0^{+\infty} [(1+t) \psi'_n(t) + \psi_n(t)]^2 w(t; a) dt. \quad (2.28)$$

The integral defining  $BH_{n,a}$  can be computed and expressed as an explicit function of the  $\hat{Y}_i$ :

$$BH_{n,a} = \frac{1}{n} \sum_{j,k=1}^n \left[ \frac{(1 - \hat{Y}_j)(1 - \hat{Y}_k)}{\hat{Y}_j + \hat{Y}_k + a} - \frac{\hat{Y}_j + \hat{Y}_k}{(\hat{Y}_j + \hat{Y}_k + a)^2} + \frac{2\hat{Y}_j \hat{Y}_k}{(\hat{Y}_j + \hat{Y}_k + a)^2} + \frac{2\hat{Y}_j \hat{Y}_k}{(\hat{Y}_j + \hat{Y}_k + a)^3} \right]. \quad (2.29)$$

Both tests reject the Weibull assumption for large values of the statistics.

### 2.2.9 Tests based on the characteristic function

The characteristic function of the Exponential distribution is

$$\varphi(t) = \mathbb{E}[\exp(itX)] = \frac{\lambda}{\lambda - it} = C(t) + iS(t) = \frac{\lambda^2}{\lambda^2 + t^2} + i\frac{\lambda t}{\lambda^2 + t^2}. \quad (2.30)$$

Epps and Pulley [39] proposed to compare the characteristic function of the standard Exponential distribution to the empirical characteristic function of the sample  $\hat{Y}_1, \dots, \hat{Y}_n$ ,  $\varphi_n(t) = \frac{1}{n} \sum_{j=1}^n \exp(-it\hat{Y}_j) = C_n(t) + iS_n(t)$ , where  $C_n(t) = \frac{1}{n} \sum_{j=1}^n \cos(t\hat{Y}_j)$  and  $S_n(t) = \frac{1}{n} \sum_{j=1}^n \sin(t\hat{Y}_j)$ . The expression of their statistic simplifies to:

$$EP_n = \sqrt{48n} \left[ \frac{1}{n} \sum_{i=1}^n \exp(-\hat{Y}_i) - \frac{1}{2} \right]. \quad (2.31)$$

Henze and Meintanis [54] suggested to build a test based on the equation verified by the real and the imaginary parts of the characteristic function:  $S(t) - tC(t)/\lambda = 0$ . This idea applied to the  $\hat{Y}_j$  leads to a statistic of the form:

$$HM_{n,a} = n \int_{-\infty}^{+\infty} [S_n(t) - tC_n(t)]^2 w(t; a) dt \quad (2.32)$$

Two weight functions are used:  $w_1(t; a) = \exp(-at)$  and  $w_2(t; a) = \exp(-at^2)$ . The corresponding statistics are denoted  $HM_{n,a}^{(1)}$  and  $HM_{n,a}^{(2)}$ . The integral in (2.32) can be computed and expressed as an explicit function of the  $(\hat{Y}_j)_{1 \leq j \leq n}$ :

$$HM_{n,a}^{(1)} = \frac{a}{2n} \sum_{j,k=1}^n \left[ \frac{1}{a^2 + (\hat{Y}_j - \hat{Y}_k)^2} - \frac{1}{a^2 + (\hat{Y}_j + \hat{Y}_k)^2} - \frac{4(\hat{Y}_j + \hat{Y}_k)}{(a^2 + (\hat{Y}_j - \hat{Y}_k)^2)^2} + \frac{2a^2 - 6(\hat{Y}_j - \hat{Y}_k)^2}{(a^2 + (\hat{Y}_j + \hat{Y}_k)^2)^3} + \frac{2a^2 - 6(\hat{Y}_j + \hat{Y}_k)^2}{(a^2 + (\hat{Y}_j + \hat{Y}_k)^2)^3} \right] \quad (2.33)$$

$$HM_{n,a}^{(2)} = \frac{\sqrt{\pi}}{4n\sqrt{a}} \sum_{j,k=1}^n \left[ \left( 1 + \frac{2a - (\hat{Y}_j - \hat{Y}_k)^2}{4a^2} \right) \exp\left( -\frac{(\hat{Y}_j - \hat{Y}_k)^2}{4a} \right) + \left( \frac{2a - (\hat{Y}_j + \hat{Y}_k)^2}{4a^2} - \frac{\hat{Y}_j + \hat{Y}_k}{a} - 1 \right) \exp\left( -\frac{(\hat{Y}_j + \hat{Y}_k)^2}{4a} \right) \right]. \quad (2.34)$$

Henze and Meintanis [55, 56] used a similar technique inspired by the fact, reported by Meintanis and Iliopoulos [84], that  $|\varphi(t)|^2 = C(t)$ . The statistic has the form:

$$MI_{n,a} = n \int_{-\infty}^{+\infty} [|\varphi_n(t)|^2 - C_n(t)]^2 w(t; a) dt. \quad (2.35)$$

As before, both weight functions  $w_1(t; a) = \exp(-at)$  and  $w_2(t; a) = \exp(-at^2)$  are used.

The corresponding statistics are denoted  $MI_{n,a}^{(1)}$  and  $MI_{n,a}^{(2)}$  and have the following explicit expressions:

$$\begin{aligned} MI_{n,a}^{(1)} = & \frac{a}{n} \sum_{j,k=1}^n \left[ \frac{1}{a^2 + \hat{Y}_{jk-}^2} + \frac{1}{a^2 + \hat{Y}_{jk+}^2} \right] \\ & - \frac{2a}{n^2} \sum_{j,k=1}^n \sum_{l=1}^n \left[ \frac{1}{a^2 + [\hat{Y}_{jk-} - \hat{Y}_l]^2} + \frac{1}{a^2 + [\hat{Y}_{jk-} + \hat{Y}_l]^2} \right] \\ & + \frac{a}{n^3} \sum_{j,k=1}^n \sum_{l,m=1}^n \left[ \frac{1}{a^2 + [\hat{Y}_{jk-} - \hat{Y}_{lm-}]^2} + \frac{1}{a^2 + [\hat{Y}_{jk-} + \hat{Y}_{lm-}]^2} \right] \end{aligned} \quad (2.36)$$

and

$$\begin{aligned} MI_{n,a}^{(2)} = & \frac{1}{2n} \sqrt{\frac{\pi}{a}} \sum_{j,k=1}^n \left[ \exp\left(-\frac{\hat{Y}_{jk-}^2}{4a}\right) + \exp\left(-\frac{\hat{Y}_{jk+}^2}{4a}\right) \right] \\ & - \frac{1}{n^2} \sqrt{\frac{\pi}{a}} \sum_{j,k=1}^n \sum_{l=1}^n \left[ \exp\left(-\frac{[\hat{Y}_{jk-} - \hat{Y}_l]^2}{4a}\right) + \exp\left(-\frac{[\hat{Y}_{jk-} + \hat{Y}_l]^2}{4a}\right) \right] \\ & + \frac{1}{2n^3} \sqrt{\frac{\pi}{a}} \sum_{j,k=1}^n \sum_{l,m=1}^n \left[ \exp\left(-\frac{[\hat{Y}_{jk-} - \hat{Y}_{lm-}]^2}{4a}\right) + \exp\left(-\frac{[\hat{Y}_{jk-} + \hat{Y}_{lm-}]^2}{4a}\right) \right] \end{aligned} \quad (2.37)$$

where  $\hat{Y}_{jk-} = \hat{Y}_j - \hat{Y}_k$  and  $\hat{Y}_{jk+} = \hat{Y}_j + \hat{Y}_k$ .

For all the previous tests,  $H_0$  is rejected for large values of the statistics.

### 2.2.10 Test based on the entropy

The entropy of a random variable  $X$  whose pdf is  $f$ , is defined by:

$$H(X) = \mathbb{E}[-\ln f(X)] = - \int_{-\infty}^{+\infty} f(x) \ln f(x) dx.$$

For all the positive random variables,  $H(X) \leq 1 + \ln \mathbb{E}[X]$ , which is equivalent to  $\exp(H(X))/\mathbb{E}[X] \leq e$ . The equality in the previous inequation is verified only for the Exponential distribution.

Grzegorzewski and Wieczorkowski [50] suggested a test that rejects the Exponential hypothesis when an estimation of  $\exp(H[X])/E[X]$  is too small. One of the known estimators of the entropy used in [50] is Vasicek estimator [124] defined as:

$$\hat{H}_{m,n} = \frac{1}{n} \sum_{i=1}^n \ln \frac{n}{2m} (X_{i+m}^* - X_{i-m}^*) \quad (2.38)$$

where  $m$  is an integer less than  $n/2$ ,  $X_i^* = X_1^*$  for  $i < 1$  and  $X_i^* = X_n^*$  for  $i > n$ .

The corresponding statistic is:

$$GW_{m,n} = \frac{n}{2m\bar{X}_n} \left[ \prod_{i=1}^n (X_{i+m}^* - X_{i-m}^*) \right]^{\frac{1}{n}}. \quad (2.39)$$

It can be rewritten as:

$$GW_{m,n} = \frac{n}{2m\hat{\bar{Y}}_n} \left[ \prod_{i=1}^n (\hat{Y}_{i+m}^* - \hat{Y}_{i-m}^*) \right]^{\frac{1}{n}}. \quad (2.40)$$

Approximated formulas to compute the quantiles are given in [50].

### 2.2.11 Tests based on the mean residual life

The mean residual life of the Exponential distribution is:

$$m(t) = \mathbb{E}[X - t | X > t] = \mathbb{E}[X] = 1/\lambda, \quad \forall t \geq 0. \quad (2.41)$$

This property is equivalent to  $\mathbb{E}[\min(X, t)] = F(t)\mathbb{E}[X], \forall t \geq 0$ . Then, Baringhaus and Henze [13] proposed to build a GOF test based on the comparison between an estimator of  $\mathbb{E}[\min(X, t)]$  and an estimator of  $F(t)\mathbb{E}[X]$ .

Two statistics are suggested, using Kolmogorov-Smirnov and Cramer-Von Mises type metrics:

$$BHK_n = \sqrt{n} \sup_{t \geq 0} \left| \frac{1}{n} \sum_{i=1}^n \min(\hat{Y}_i, t) - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\hat{Y}_i \leq t\}} \right| \quad (2.42)$$

$$BHC_n = n \int_0^\infty \left[ \frac{1}{n} \sum_{i=1}^n \min(\hat{Y}_i, t) - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\hat{Y}_i \leq t\}} \right]^2 \exp(-t) dt. \quad (2.43)$$

### 2.2.12 Tests based on the integrated distribution function

The integrated distribution function of the standard Exponential distribution is:

$$\Psi(t, \lambda) = \mathbb{E}[\max(X - t, 0)] = \int_t^{+\infty} R(x) dx = \frac{e^{-\lambda t}}{\lambda}. \quad (2.44)$$

Klar [65] proposed to build a GOF test based on the Cramer-Von Mises distance between the estimated  $\Psi(t; \hat{\lambda}_n)$  and the empirical integrated distribution function  $\Psi_n(t) = \frac{1}{n} \sum_{j=1}^n \max(X_j - t, 0)$ . The statistic has the expression:

$$Kl_n = n\hat{\lambda}_n^3 \int_0^{+\infty} \left( \Psi_n(t) - \Psi(t; \hat{\lambda}_n) \right)^2 dt \quad (2.45)$$

The statistic  $Kl_n$  can be written as a function of  $(\hat{Y}_i)_{1 \leq i \leq n}$  which proves the fact that the null distribution of  $Kl_n$  does not depend on the parameter  $\lambda$ :

$$Kl_n = n \int_0^{+\infty} \left( \frac{1}{n} \sum_{i=1}^n (\hat{Y}_i - u) \mathbb{1}_{\{\hat{Y}_i > u\}} - \exp(-u) \right)^2 du. \quad (2.46)$$

The use of a weight function usually allows to increase the power of the test. The statistic will have the form:

$$Kl_{a,n} = na^3 \hat{\lambda}_n^3 \int_0^{+\infty} [\Psi_n(t) - \Psi(t; \hat{\lambda}_n)]^2 \exp(-a \hat{\lambda}_n t) dt. \quad (2.47)$$

The statistic  $Kl_{n,a}$  can be written using the sample  $(\hat{Y}_i)_{1 \leq i \leq n}$  :

$$Kl_{n,a} = \frac{2(3a+2)n}{(2+a)(1+a)^2} - 2a^3 \sum_{i=1}^n \frac{\exp(-(1+a)\hat{Y}_i)}{(1+a)^2} - \frac{2}{n} \sum_{i=1}^n \exp(-a\hat{Y}_i) + \frac{2}{n} \sum_{i < j} [a(\hat{Y}_j^* - \hat{Y}_i^*) - 2] \exp(-a\hat{Y}_i^*). \quad (2.48)$$

The Exponential hypothesis is rejected for large values of the statistic  $Kl_{a,n}$ .

## 2.3 GOF tests for the Exponential distribution: censored samples

In this section we give a short bibliographical review of some GOF tests for the Exponential distribution in the case of simply type II censored samples.  $s$  and  $r$  denote respectively the number of the left and right censored observations. Let us remind that it means that only  $X_{s+1}^*, \dots, X_{n-r}^*$  are observed.

### 2.3.1 Tests based on the normalized spacings

In the case of censored observations  $X_{s+1}^*, \dots, X_{n-r}^*$ , the observed normalized spacings are  $E_{s+2}, \dots, E_{n-r}$ . They constitute a sample of size  $n-r-s-1$  of the  $\exp(\lambda)$  distribution. So all the previous GOF tests for the Exponential distribution can be applied to this sample. In the simulations presented in section 2.4.1, we apply the GOF tests  $Gn, Gn^*, LM$  and  $CO$  to the spacings  $E_{s+2}, \dots, E_{n-r}$ .

### 2.3.2 Tests based on the lack of trend

Two test statistics were suggested by Brain and Shapiro in the case of doubly censored samples [19].

Under the Exponential assumption, the  $E_i$  are iid, so they do not exhibit a trend. This lack of trend can be tested using the Laplace test statistic:

$$BS_1 = \frac{\sum_{i=1}^{m-1} (i - m/2)(E_{i+s+1} - \bar{E})}{\sum_{i=1}^{m-1} E_{s+i+1} \left\{ (i - m/2)^2 / m(m-1) \right\}^{1/2}} \quad (2.49)$$

where  $\bar{E} = \sum_{i=1}^{m-1} E_{s+i+1} / (m-1)$  and  $m = n - r + 1$ . The Exponentiality assumption is rejected for large and small values of the statistics. The distribution of  $BS_1$  under  $H_0$  converges to the standard normal distribution when  $m$  goes to infinity. The statistic can be rewritten as:

$$BS_1 = [12(m-2)]^{1/2} (\bar{U} - 1/2)$$

where  $T_i = \sum_{j=1}^i E_{s+j+1}$ ,  $i = 1, \dots, m-1$ ,  $U_i = T_i / T_{m-1}$ ,  $i = 1, \dots, m-2$ ,  $\bar{U} = \sum_{i=1}^{m-2} U_i / (m-2)$ .

The last expression of the statistic  $BS_1$  is the usual expression of the Laplace test statistic applied to the uniform order statistics  $U_i$ ,  $i = 1, \dots, m-2$ .

A second statistic  $BS^*$  is introduced. It is built as the sum of squares of two components, the first one associated to  $BS_1$  and the second one to  $BS_2$  obtained by replacing in the previous expression  $(i - m/2)$  by  $(i - m/2)^2 - m(m-2)/12$ . The aim is to build a test sensitive to non-monotonic hazard functions.

$$BS_2 = [5(m-2)(m+1)(m-3)]^{1/2} (m-3 + 6(m-1)\bar{U} - 12 \sum_{i=1}^{m-2} iU_i / (m-2)) \quad (2.50)$$

The combined statistic is  $BS^*$  as follows:

$$BS^* = BS_1^2 + BS_2^2. \quad (2.51)$$

The distribution of  $BS^*$  under  $H_0$  can be approximated by the  $\chi^2$  distribution. The null hypothesis  $H_0$  is rejected when the statistic is too large. This idea of combining two test statistics will be used later in section 6.2.

### 2.3.3 Tests based on the empirical distribution function

Pettitt and Stephens [100] introduced versions of the Cramer-von Mises, Watson and Anderson-Darling statistics in the case of simple right censoring. The statistics are ob-

tained by modifying the upper limit of integration in their definitions in subsection 2.2.4. After simplification, the statistics have the following expressions [31]:

- Cramer-von-Mises statistic (CM):

$$CM = \sum_{i=1}^{n-r} \left( U_i^* - \frac{2i-1}{2n} \right)^2 + \frac{n-r}{12n^2} + \frac{n}{3} \left( U_{n-r}^* - \frac{n-r}{n} \right)^3 \quad (2.52)$$

- Watson statistic (W):

$$W = CM - nU_{n-r}^* \left[ \frac{n-r}{n} - \frac{U_{n-r}^*}{2} - \frac{(n-r)\bar{U}}{nU_{n-r}^*} \right]^2 \quad (2.53)$$

- Anderson-Darling statistic (AD):

$$AD = -\frac{1}{n} \sum_{i=1}^{n-r} (2i-1) [\ln U_i^* - \ln(1-U_i^*)] - 2 \sum_{i=1}^{n-r} \ln(1-U_i^*) - \frac{1}{n} [r^2 \ln(1-U_{n-r}^*) - (n-r)^2 \ln(U_{n-r}^*) + n^2 U_{n-r}^*] \quad (2.54)$$

- Kolmogorov-Smirnov statistic (KS) can also be adapted for censored data:

$$KS = \sup_{1 \leq i \leq n-r} \left| \frac{i-0.5}{n} - U_i^* \right| + \frac{0.5}{n} \quad (2.55)$$

where  $U_i = 1 - e^{-\hat{\lambda}_n X_i}$ ,  $\hat{\lambda}_n = \frac{n-r}{\sum_{i=1}^{n-r} X_i^* + rX_{n-r}^*}$  and  $U_1^*, \dots, U_{n-r}^*$  are the order statistics of

the sample  $U_1, \dots, U_{n-r}$ .

The same statistics can be applied in the case of left-censored samples. We use the transformation  $V_i^* = 1 - U_{n+1-i}^*$ ,  $i = 1, \dots, n-s$ , where  $s = r$  is the number of censored observations. The exponentiality hypothesis is rejected for large values of the statistics.

### 2.3.4 Test based on the Kullback-Leibler information

This test is based on the Kullback-Leibler information. It was proposed in order to test the exponentiality in the case of progressively censored samples of type II [11]. It can be applied to the special case of simply right-censored samples:

$$KL = -H(w, m, n) + \frac{m-1}{n} \left[ \ln \left( \frac{1}{m-1} \sum_{i=1}^{m-1} X_i \right) + 1 \right]^2 \quad (2.56)$$

where  $m = n - r$  and

$$H(w, m, n) = \frac{1}{n} \sum_{i=1}^m \ln \frac{(n+1)(X_{\min(i+w, m-1)}^* - X_{\max(i-w, 1)}^*)}{\min(i+w, m-1) - \max(i-w, 1)} + \left( 1 - \frac{m-1}{n} \right) \ln \left( 1 - \frac{m-1}{n} \right).$$

The choice of  $w$  is given as a function of the sample size. We will use the value recommended in [38]. The rejection of the Exponential hypothesis is done for large values of the test statistic.

## 2.4 Comparison of the GOF tests for the Exponential distribution

In this section, we make an exhaustive comparison of all the previous GOF tests for the Exponential distribution. The comparisons are based on Monte-Carlo simulations. Some reviews were already done for complete samples, by Henze-Meintanis [55], Spurrier [117] and Ascher [7]. The review presented here is more complete with more compared GOF tests, more alternatives with various hazard rates shapes and more sample sizes. All the GOF tests studied in this section have been implemented in the R package EWGoF that we have developed.

### 2.4.1 Complete samples

For complete sample the comparison includes the following test statistics:

- *Pa*: Patwardhan test statistic defined in (2.12).
- *SW*: Shapiro-Wilk test statistic defined in (2.13).
- *KS*: Kolmogorov-Smirnov test statistic defined in (2.14).
- *CM*: Cramer-von-Mises test statistic defined in (2.15).
- *AD*: Anderson-Darling test statistic defined in (2.16).
- *Gn*: Gnedenko test statistic defined in (2.17).
- *Gn\**: Harris test statistic defined in (2.18).
- *GG*: Gini test statistic defined in (2.19).
- *LM<sub>1</sub>* and *LM<sub>2</sub>*: Lin-Mudholkar test statistics defined respectively in (2.20) and (2.21).
- *Sc*: Score test statistic defined in (2.22).
- *CO*: Cox and Oakes test statistic defined in (2.23).
- *LR*: Likelihood ratio test statistic defined in (2.24).
- *He*: Henze test statistic defined in (2.27).
- *BH*: Baringhaus-Henze test statistic defined in (2.29).
- *EP*: Epps-Pulley test statistic defined in (2.31).
- *HM<sup>(1)</sup>* and *HM<sup>(2)</sup>*: Henze and Meintanis test statistics defined respectively in equations (2.33) and (2.34).
- *MI<sup>(1)</sup>* and *MI<sup>(2)</sup>*: Meintanis and Iliopoulos test statistics defined respectively in equations (2.36) and (2.37).

- *GW*: Grzegorzewski and Wieczorkowski test statistic defined in (2.40).
- *BHK* and *BHC*: Baringhaus and Henze test statistics based on the mean residual life defined in (2.42).
- *Kl*: Klar test statistic defined in (2.48).

We first simulate iid exponentially distributed samples to verify that the rejection percentage of the Exponential distribution is close to the theoretical significance level. Then, we simulate samples with the following alternative distributions. For each distribution we give their pdfs  $f(x)$  and hazard rate  $h(x)$  when it has an explicit expression:

- The Gamma distribution  $\mathcal{G}(\alpha, \lambda)$ :

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \exp(-\lambda x) x^{\alpha-1}$$

- The Lognormal distribution  $\mathcal{LN}(m, \sigma^2)$ :

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2} (\ln x - m)^2\right)$$

- The Uniform distribution  $\mathcal{U}[0, a]$ :

$$f(x) = \frac{1}{a} \mathbb{1}_{[0,a]}(x)$$

$$h(x) = \frac{1}{a-x} \mathbb{1}_{[0,a]}(x)$$

- The Inverse-Gamma distribution  $\mathcal{IG}(\alpha, \beta)$ :

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right).$$

For the sake of simplicity, we adopt the following conventions: scale parameters of the Weibull, Gamma and Inverse-Gamma distribution (respectively  $\eta$ ,  $\lambda$  and  $\beta$ ) are arbitrarily set to 1 and the parameter  $m$  of the Lognormal distribution is set to 0. The corresponding distributions are denoted  $\mathcal{W}(1, \beta) \equiv \mathcal{W}(\beta)$ ,  $\mathcal{G}(\alpha, 1) \equiv \mathcal{G}(\alpha)$ ,  $\mathcal{IG}(\alpha, 1) \equiv \mathcal{IG}(\alpha)$  and  $\mathcal{LN}(0, \sigma^2) \equiv \mathcal{LN}(\sigma^2)$ . Parameters of the simulated distributions are selected to obtain different shapes of the hazard rate:

- IHR: increasing hazard rate
- DHR: decreasing hazard rate
- BT: bathtub-shaped hazard rate
- UBT: upside-down bathtub-shaped hazard rate.

Table 2.1: Simulated distributions

Exponential	exp(0.2)	exp(1)	exp(2)	exp(42)
IHR	$\mathcal{W}(1.5)$	$\mathcal{W}(3)$	$\mathcal{U}[0, 2]$	$\mathcal{G}(2)$
DHR	$\mathcal{W}(0.5)$	$\mathcal{W}(0.8)$	$\mathcal{W}(0.98)$	$\mathcal{G}(0.5)$
UBT	$\mathcal{LN}(0.6)$	$\mathcal{LN}(0.8)$	$\mathcal{LN}(1.4)$	

For the Exponential case, we use only UBT alternatives. BT alternatives will be also used for the Weibull case in the following chapter. Table 2.1 gives the values of the parameters and the notations used for all the simulated distributions:

For a given alternative with fixed parameters and a fixed sample size, we simulate 50000 samples of size  $n \in \{5, 10, 20, 50\}$ .

All the GOF tests are applied with a significance level set to 5%. The power of the tests is assessed by the percentage of rejection of the null hypothesis. The quantiles of the distribution of the test statistics under  $H_0$  are obtained mainly by simulations. For instance, let us consider the Cox-Oakes test, the test statistic is given by (2.23):

$$CO_n = \frac{\left[ n + \sum_{i=1}^n \ln X_i - \frac{1}{\bar{X}_n} \sum_{i=1}^n (\ln X_i) X_i \right]^2}{n + \frac{1}{\bar{X}_n} \sum_{i=1}^n (\ln X_i / \bar{X}_n)^2 X_i - \frac{1}{n \bar{X}_n^2} \left[ \sum_{i=1}^n (\ln X_i / \bar{X}_n) X_i \right]^2}.$$

For a given sample size  $n$ , we simulate  $X_1, \dots, X_n$  from  $\exp(1)$ , then we compute the corresponding value of  $CO_n$ . This process is done  $m = 100000$  times. The quantiles of the distribution of  $CO_n$  under  $H_0$  are given by the empirical quantiles of the  $m$  values of  $CO_n$ . Table 2.2 gives some quantiles for several values of  $n$ . We observe that, for small  $n$ , the distribution of  $CO_n$  under  $H_0$  may be quite far from the  $\chi_1^2$  distribution. So it is important to be able to apply these GOF tests without using the chi-square approximation especially for small samples.

Table 2.2: Quantiles of the distribution of  $CO_n$  under  $H_0$ 

$n$	75%	80%	85%	90%	95%	99%
5	1.548	1.847	2.214	2.697	3.422	5.079
10	1.460	1.762	2.164	2.728	3.658	5.853
20	1.379	1.702	2.128	2.729	3.777	6.318
50	1.335	1.656	2.079	2.690	3.810	6.537
$\chi_1^2$	1.323	1.642	2.072	2.705	3.841	6.635

For the power study, we simulate a sample  $X_1, \dots, X_n$  of size  $n$  of a given distribution. For  $n = 50$ , the Exponential assumption is rejected at the level 5% if  $CO_n > 3.810$ . This process is done  $K = 50000$  times. The percentage of rejection of  $H_0$  is an estimation of the power of the test for this alternative. For instance, we see in Table 2.4 that the power of the  $CO_n$  test for simulated  $\mathcal{LN}(0, 0.8)$  samples and  $n = 50$  is estimated at 63.8%. The

higher the rejection percentage is, the better the test is. We will observe that the results are tightly linked to the tested alternatives. In order to evaluate globally the power of the tests, we compute for each test the mean value of the rejection percentage for all the alternatives. The power tables of the studied GOF tests are given in Appendix A in order to avoid a complex and long dissertation in this chapter.

In a first step, the tests are compared inside each family. The choice of parameters such as  $a$  and  $m$  is discussed. In a second step, the best tests of each family are compared.

Tables A.1 and A.2 present the power results for the GOF tests based on the empirical distribution function (KS, CM and AD) with and without the application of the K transformation.  $AD$  is the best and  $KS$  is the worst of the three. The use of the K transformation gives better results for some special cases such as the Weibull,  $\mathcal{LN}(1.4)$  and uniform distributions.

Tables from A.3 to A.5 present the power results of the tests based on the normalized spacings.  $GG$  has the best performance followed by  $LM$ .

Tables A.6 and A.7 compare the power results for the three likelihood based tests ( $Sc$ ,  $CO$  and  $LR$ ). It seems clearly that the score test  $Sc$  is more appropriate for the DHR alternatives and the test  $LR$  based on the likelihood ratio is powerful for the IHR alternatives. The test  $CO$  has never been the best one for specific alternatives, but it represents an excellent compromise by giving generally good results.

Tables from A.8 to A.11 present the power results of Henze test based on the Laplace transform. Small values of the parameter  $a$  are appropriate for DHR alternatives ( $\mathcal{W}(0.5)$ ,  $\mathcal{W}(0.8)$ ,  $\mathcal{G}(0.5)$ ), while moderately large values of  $a$ , are appropriate to IHR alternatives ( $\mathcal{W}(1.5)$ ,  $\mathcal{W}(3)$ ,  $\mathcal{G}(2)$ , uniform). The best compromise is made for  $a = 1$ .

Tables from A.12 to A.15, present the power results of Baringhaus-Henze test based on the Laplace transform. The conclusions are similar to those of the previous test. We recommend also the value  $a = 1$ . Baringhaus-Henze test is slightly more powerful than the test of Henze.

Tables from A.16 to A.19 present the power results of Henze-Meintanis tests based on the characteristic function. The power difference between the two tests  $HM_{n,a}^{(1)}$  and  $HM_{n,a}^{(2)}$  can be very important for some alternatives, for instance: 82.3% and 28.9% for  $\mathcal{LN}(0.8)$  distribution with  $n = 50$ . Generally, the test  $HM_{n,a}^{(1)}$  is more powerful than  $HM_{n,a}^{(2)}$ . But for DHR alternatives, we recommend the use of  $HM_{n,a}^{(2)}$  with large value of the parameter  $a$ . If nothing is known about the tested alternatives, a good compromise is to choose the test  $HM_{n,a}^{(1)}$  with  $a = 1.5$  for  $n \leq 10$  and  $a = 1$  for  $n > 10$ .

Tables from A.20 to A.22 present the power results of Meintanis-Iliopoulos tests based on the characteristic function. The fact that the statistics  $MI_{n,a}^{(1)}$  and  $MI_{n,a}^{(2)}$  have more complex expression than the previous ones, slows down the simulations. These tests present the characteristic to have extremely weak powers for DHR alternatives and good ones for IHR alternatives. There is no significant difference between  $MI_{n,a}^{(1)}$  and  $MI_{n,a}^{(2)}$ . We choose  $a = 2.5$ , even if the choice of the parameter  $a$  has no significant effect on the results.

Table A.23 presents the power results of Grzegorzewski-Wieczorkowski test based on the entropy. The choice of the parameter  $m$  depends slightly on the tested alternatives. We recommend  $m = 4$  to have the best compromise. This test and  $Pa$  test are not very powerful that is why they will not be presented later in the comparison tables.

Tables A.24 to A.27 present the power results of the test of Klar. For small size samples, we should absolutely avoid to choose large values of the parameter  $a$  which give

null rejection percentages for some alternatives. For  $n \geq 20$ , the best suitable values of the parameter  $a$  depend on the used alternatives. The best compromise is obtained for  $a = 5$ .

After finding the best GOF tests within each family, tables A.28 to A.33 of the appendix are given to compare all the selected GOF tests for the sample sizes  $n \in \{5, 10, 20\}$ . The following tables 2.3 and 2.4 give the power results for  $n = 50$ . Our first conclusion is that none of these tests is always powerful. The performances of the tests depend strongly on the alternatives used in the simulations. Secondly, the family of the likelihood based tests gives globally the best results. The test  $Sc$  is recommended for the DHR alternatives and  $LR$  test is rather recommended for the IHR alternatives. The test  $CO$  gives a good compromise and can be recommended in all cases. Besides their good performances, the likelihood based GOF test statistics do not require any parameter to be chosen and have simple expressions.

altern.	$KS$	$CM$	$AD$	$GG$	$SW$	$BHK$	$BHC$	$K_{5,n}$	$BH_{n,0.1}$	$He_{n,1}$
exp(0.2)	5.1	5.7	4.9	4.9	4.8	4.5	5.2	5.1	5.1	5.1
exp(1)	4.7	4.9	5.6	5	4.6	5.4	5.4	5.1	4.9	4.8
exp(2)	5.3	5.3	5.1	4.9	4.8	4.8	5.1	5	5.2	5.1
exp(42)	4.9	4.8	5.2	5	4.7	4.8	4.9	5	5.1	5
$\mathcal{W}(0.5)$	99.9	99.9	100	100	98.6	99.9	99.9	100	100	100
$\mathcal{W}(0.8)$	36.4	41.8	50.8	48.2	31.5	34.5	46.3	49.1	53.6	50
$\mathcal{W}(0.98)$	5.3	5.5	5.9	5.6	5	4.8	4.3	5.5	6	5.4
$\mathcal{G}(0.5)$	83.3	89.7	90.5	89.2	63.3	86.4	84.9	94.5	97.6	93
$\mathcal{W}(1.5)$	79.4	89.5	91.4	92.6	89.1	88.1	87.5	91.2	87.7	93.9
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	92.1	98.5	98.2	99	99.8	99.4	98.6	86.2	80.4	96.1
$\mathcal{G}(2)$	81.7	90.4	91.4	89.2	79.9	81.2	82.1	94.5	93.6	93.6
$\mathcal{LN}(0.6)$	99.2	99.6	99.1	99.2	88.8	100	98.2	100	100	99.9
$\mathcal{LN}(0.8)$	71.1	75.9	85.5	46.5	28.2	64.5	61.7	82.1	93.6	61.9
$\mathcal{LN}(1.4)$	81.5	85.1	87.4	88.4	84.8	83.6	87.2	80.7	77.6	86.9
Mean	75.4	79.6	81.8	78	69.9	76.6	77.3	80.3	80.9	80.1

Table 2.3: Exponential distribution - Tests comparison,  $n = 50 - 1$

Without any information about the tested alternative, we recommend the test  $CO$  followed by the tests  $AD$  and  $Kl$ . For the IHR alternatives, the test  $LR$  is the best followed by  $MI$  and  $CO$ . For the DHR alternatives, the test  $Sc$  is the best followed by  $BH$ ,  $AD$  and  $CO$ . For the alternatives with upside-down bathtub shaped hazard rate, the two tests  $Kl$  and  $BH$  are powerful. Even though the comparison study presented here is larger than those of Ascher [7] and Henze-Meintanis [55], the conclusions are globally similar.

## 2.4.2 Censored samples

For censored samples, to our knowledge similar reviews have never been done. In this subsection, we compare the following tests:

altern.	$MI_{n,2.5}^{(1)}$	$MI_{n,2.5}^{(2)}$	$HM_{n,1.5}^{(1)}$	$HM_{n,0.5}^{(2)}$	$EP$	$Sc$	$CO$	$LR$
exp(0.2)	5.2	5	4.9	5.3	5	5	5	5
exp(1)	5.4	5.2	5.1	5.2	5.1	5	4.9	5
exp(2)	5.1	5.2	5	5.2	5.2	4.9	5.1	5
exp(42)	5.5	5.2	5.1	5	5.4	4.9	4.8	5.1
$\mathcal{W}(0.5)$	98.5	98.8	99.8	99.6	99.9	100	100	100
$\mathcal{W}(0.8)$	24.8	31.2	38.1	36	47.8	56.4	52.8	48.4
$\mathcal{W}(0.98)$	4	4.2	5.2	5.3	5.6	5.9	5.8	5.1
$\mathcal{G}(0.5)$	71.8	74	83.4	79.5	89.3	97.4	97.2	96.5
$\mathcal{W}(1.5)$	93.8	92.6	88.5	87.9	92.9	93.4	94.9	96.3
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	99.8	99.9	99.5	99.7	98.2	90.1	93.7	95
$\mathcal{G}(2)$	90	85.6	86.8	83.4	90.4	94.8	95.7	96.5
$\mathcal{LN}(0.6)$	99.3	95.7	99.8	98.4	98.9	99.7	99.6	99.7
$\mathcal{LN}(0.8)$	49.1	32.2	56.6	39.9	45.1	64.6	63.8	66
$\mathcal{LN}(1.4)$	60.7	72.8	82.7	82.5	88.7	82.3	78.1	76.1
Mean	72	71.6	76.4	73.9	77.9	80.4	80.1	79.9

Table 2.4: Exponential distribution - Tests comparison,  $n = 50 - 2$ 

- $Gn$ : Gnedenko test statistic defined in (2.17) applied to the normalized spacings.
- $Gn^*$ : Harris test statistic defined in (2.18) applied to the normalized spacings.
- $LM_1$  and  $LM_2$ : Lin-Mudholkar test statistics defined respectively in (2.20) and (2.21) applied to the normalized spacings.
- $CO$ : Cox and Oakes test statistic defined in (2.23) applied to the normalized spacings.
- $BS_1$  and  $BS^*$ : Brain and Shapiro test statistics defined respectively in (2.49) and (2.51).
- $CM$ : Cramer-von-Mises test statistic defined in (2.52).
- $W$ : Watson test statistic defined in (2.53).
- $AD$ : Anderson-Darling test statistic defined in (2.54).
- $KS$ : Kolmogorov-Smirnov test statistic defined in (2.55).
- $KL$ : Test based on the Kullback-Leibler information defined in (3.38).

As previously, we first simulate iid exponentially distributed samples to verify that the rejection percentage of the Exponential distribution is close to the theoretical significance level. Then, we simulate samples with the alternatives given in table 2.5.

For a given alternative with fixed parameters and a fixed sample size, we simulate 50000 samples of size  $n \in \{10, 20, 50\}$  and we consider only simple type II right-censoring where  $r \in \{\lfloor \frac{n}{8} \rfloor, \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor\}$ .

Table 2.5: Simulated distributions for the censored samples

Exponential	exp(0.2)	exp(1)	exp(2)	exp(42)
IHR	$\mathcal{W}(1.5)$	$\mathcal{W}(3)$	$\mathcal{U}[0, 2]$	$\mathcal{G}(2)$
DHR	$\mathcal{W}(0.5)$	$\mathcal{W}(0.8)$	$\mathcal{W}(0.98)$	$\mathcal{G}(0.5)$
UBT	$\mathcal{LN}(0.6)$ $\mathcal{IG}(0.5)$	$\mathcal{LN}(0.8)$ $\mathcal{IG}(1.5)$	$\mathcal{LN}(1.4)$ $\mathcal{IG}(2)$	$\mathcal{LN}(2.4)$ $\mathcal{IG}(3)$

As before, all the GOF tests are applied with a significance level set to 5%. The power of the tests is assessed by the percentage of rejection of the null hypothesis. Table 2.6 shows the power results for  $n = 50$  and  $r = 25$ .

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5.1	5.1	4.9	4.9	5	5	5	4.9	5.1	5
exp(1)	5.1	5	5	4.9	5	5.1	5	4.8	5.1	4.8
exp(2)	5	5	5	4.9	5	5.1	5.1	4.9	5.1	5.1
exp(42)	5	5	5.1	5	5	5	4.9	4.8	5.3	5.1
$\mathcal{W}(0.5)$	80.4	75.2	83.9	0	94.1	79.8	74.3	71	44.4	53.8
$\mathcal{W}(0.8)$	15.8	12.4	15.4	0	23.6	16.7	14.7	12.2	5.1	6.7
$\mathcal{W}(0.98)$	5	5.1	4.9	4.1	5.2	5.2	5.2	5.2	5	5.2
$\mathcal{G}(0.5)$	60	53.4	64.4	0	83.4	61.3	61.9	58.2	24.1	36.1
$\mathcal{W}(1.5)$	28.1	24.5	47.9	58.3	46.5	28.4	25.8	23.7	29.1	6.7
$\mathcal{W}(3)$	84.7	79	100	100	100	98.9	97.3	96.4	99.4	73.9
$\mathcal{U}[0, 2]$	14	10.7	17.2	23.8	13.2	12.6	7.8	7.3	8.9	5.1
$\mathcal{G}(1.5)$	15.5	14.4	27.1	35.7	26	15	15.7	15.7	19.2	5.3
$\mathcal{G}(2)$	31.8	29.1	63.9	72.5	65.5	36.4	37.8	39.7	44.2	8.8
$\mathcal{G}(3)$	53.7	49.6	96.6	97.8	97.5	74.5	76.6	80.8	84.3	27.6
$\mathcal{LN}(0.6)$	46.5	45	99.8	99.8	99.9	86.7	93.8	98.4	98.2	53.2
$\mathcal{LN}(0.8)$	24.8	26.2	84.3	86.5	89.8	44.4	61	76.1	72.7	16.1
$\mathcal{LN}(1.4)$	7.5	8.4	5.8	4.2	5	4.1	3.9	7.1	11.6	5
$\mathcal{LN}(1.8)$	29.5	23.7	23.3	0	24	23.4	6.2	9.5	11.5	6.3
$\mathcal{LN}(2.4)$	71.3	62.4	70.5	0	78.1	68.4	39.3	28.7	31.6	23.2
$\mathcal{IG}(0.5)$	26.5	25.6	15.4	3.5	14.5	7.8	6.4	19	24.4	6.1
$\mathcal{IG}(1.5)$	13.8	18.7	91.9	88.6	96.7	44.4	76.4	95.1	19.2	27.6
$\mathcal{IG}(2)$	22.6	26.5	99.1	98.1	99.8	68.8	91.5	99.4	43.9	47
$\mathcal{IG}(3)$	37.3	38.6	100	100	100	92.9	99.1	100	83.8	77.7
Mean	35.2	33.1	58.5	45.9	61.2	45.8	47.1	49.7	40	25.9

Table 2.6: Exponential distribution - Tests comparison,  $n = 50$  and  $r = 25$ 

Tables from A.34 to A.41 present other power results. Mostly the same results come out whatever the size and the rate of the censoring. For DHR alternatives,  $AD$  followed by  $CM$  are the best tests; for IHR alternatives, the test  $W$  is recommended and for UBT alternatives,  $CM$  is the best test. The  $CO$  test applied to the normalized spacings  $E_1, \dots, E_{n-r}$ , is the worst test. Generally, the two tests based on the empirical distribution function  $AD$  and  $CM$  have the best performances, unlike the test of Watson  $W$  that is biased in some cases.

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To sum up, for the censored samples, Anderson-Darling test has the best performances among all the studied ones. For the complete samples, the GOF tests of Anderson-Darling  $AD$ , Cox-Oakes  $CO$  and the tests based on the empirical characteristic function  $BH$  seem to have the best performances. The comparisons were done among 60 GOF tests for complete samples and 10 GOF tests for censored samples. All the previous GOF tests for censored samples are implemented in the R package we have developed EWGoF. A part of this work has been presented in ESREL 2012 conference [70].

The good performance of Cox-Oakes  $CO$  test has attracted our attention. That is why we have developed new GOF tests based on the likelihood for the Weibull distribution (chapter 4).

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# Chapter 3

## Weibull distribution: basic properties and usual GOF tests

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This chapter is dedicated to the two-parameter Weibull distribution. Some definitions and basic properties of this distribution are given. Then we present a quick review of the usual GOF test for the Weibull distribution. Several GOF tests families are presented such as tests based on the probability plots, Shapiro-wilk tests, tests based on the empirical distribution function, tests based on the normalized spacings, generalized smooth tests, tests based on the Kullback-Leibler information and tests based on the Laplace transform.

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### 3.1 The Weibull distribution: definition and properties

A random variable  $X$  is from the two-parameter Weibull distribution  $\mathcal{W}(\eta, \beta)$ , if and only if its cdf is:

$$F(x; \eta, \beta) = 1 - \exp(-(x/\eta)^\beta), \quad x \geq 0, \eta > 0, \beta > 0. \quad (3.1)$$

- The pdf of  $\mathcal{W}(\eta, \beta)$  is:

$$f(x; \eta, \beta) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1} \exp(-(x/\eta)^\beta), \quad x \geq 0, \eta > 0, \beta > 0. \quad (3.2)$$

- The reliability is  $R(x) = \exp(-(x/\eta)^\beta)$ .
- The expectation is:  $\text{MTTF} = \mathbb{E}[X] = \eta \Gamma\left(\frac{1}{\beta} + 1\right)$ .
- The variance is:  $\text{Var}(X) = \eta^2 \Gamma\left(\frac{2}{\beta} + 1\right) - \eta^2 \Gamma^2\left(\frac{1}{\beta} + 1\right)$ .

- The hazard rate is  $h(x) = \frac{f(x)}{R(x)} = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1}$ . Figure 3.1 illustrates the different shapes of the hazard rate depending on parameter  $\beta$ :
  - $\beta < 1$ :  $h$  is decreasing and the system is improving,
  - $\beta > 1$ :  $h$  is increasing and the system is wearing,
  - $\beta = 1$ :  $h$  is constant and we find the special case of the Exponential distribution.
- $\left(\frac{X}{\eta}\right)^\beta$  has the standard Exponential distribution  $\exp(1)$ .

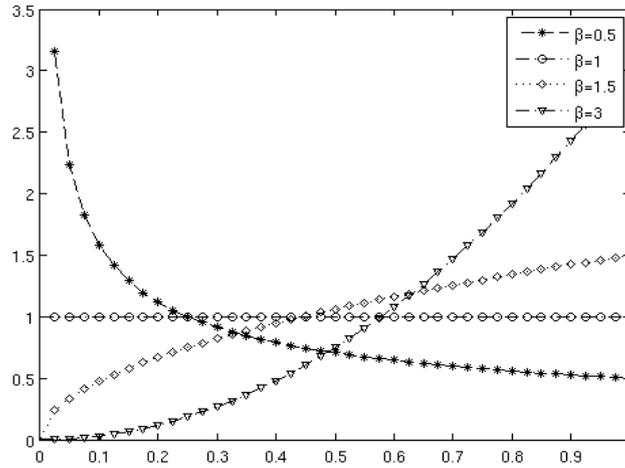


Figure 3.1: Hazard rate function for the  $\mathcal{W}(\eta, \beta)$  distribution

When  $X$  is a random variable from the  $\mathcal{W}(\eta, \beta)$  distribution,  $\ln X$  has the type I Extreme Value distribution  $\mathcal{EV}_1(\mu, \sigma)$  with cdf:

$$G(y; \mu, \sigma) = 1 - \exp(-\exp((y - \mu)/\sigma)), \quad y \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0 \quad (3.3)$$

where  $\mu = \ln \eta$  and  $\sigma = 1/\beta$ .

- The pdf of  $\mathcal{EV}_1(\mu, \sigma)$  is:

$$g(y; \mu, \sigma) = \frac{1}{\sigma} \exp((y - \mu)/\sigma - \exp((y - \mu)/\sigma)), \quad y \in \mathbb{R}. \quad (3.4)$$

- The reliability is  $S(y) = \exp(-\exp((y - \mu)/\sigma))$ .
- The expectation is:  $\mathbb{E}[\ln X] = \mu - \gamma_E \sigma$  where  $\gamma_E = 0.577\dots$  is the Euler constant.
- The variance is  $Var[\ln X] = \frac{\pi^2}{6} \sigma^2$ .

- The hazard rate is  $h(y) = \frac{1}{\sigma} \exp(-(y - \mu)/\sigma)$ .
- The Laplace transform is  $\psi(t) = \mathbb{E}[\exp(-t \ln X)] = \Gamma(1 - \sigma t) \exp(\mu t)$ ,  $\forall t > 0$ .
- $Y = \beta \ln(X/\eta) = (\ln X - \mu)/\sigma$  follows  $\mathcal{E}\mathcal{V}_1(0, 1)$ .

Let  $X_1, \dots, X_n$  be  $n$  (iid) random variables from the  $\mathcal{W}(\eta, \beta)$  distribution. We consider three methods for estimating the parameters  $\eta$  and  $\beta$ : the maximum likelihood, least squares and moment methods.

- The maximum likelihood estimators (MLEs) of  $\eta$  and  $\beta$ ,  $\hat{\eta}_n$  and  $\hat{\beta}_n$ , are solutions of the following equations:

$$\begin{cases} \hat{\eta}_n = \left( \frac{1}{n} \sum_{i=1}^n X_i^{\hat{\beta}_n} \right)^{1/\hat{\beta}_n} \\ \frac{n}{\hat{\beta}_n} + \sum_{i=1}^n \ln X_i - \frac{n}{\sum_{i=1}^n X_i^{\hat{\beta}_n}} \sum_{i=1}^n X_i^{\hat{\beta}_n} \ln X_i = 0. \end{cases} \quad (3.5)$$

- The Weibull probability plot (WPP) [92] is the plot of points:

$$(\ln X_i^*, c_i), i \in \{1, \dots, n\} \quad (3.6)$$

where  $c_i = \ln[-\ln(1 - p_i)]$  and  $p_i, i \in \{1, \dots, n\}$ , are approximations of the order statistics of a uniform sample. Usual choices are symmetrical ranks  $p_i = (i - 0.5)/n$  and mean ranks  $p_i = i/(n + 1)$ . Under the Weibull assumption, these points should be approximately on a straight line [31].

The least squares estimators (LSEs) based on the WPP,  $\tilde{\eta}_n$  and  $\tilde{\beta}_n$ , are defined as follows [76]:

$$\begin{cases} \tilde{\beta}_n = \frac{\sum_{i=1}^n (c_i - \bar{c})^2}{\sum_{i=1}^n (\ln X_i - \overline{\ln X})(c_i - \bar{c})} \\ \ln \tilde{\eta}_n = \overline{\ln X} - \frac{\bar{c}}{\tilde{\beta}_n} \end{cases} \quad (3.7)$$

where  $\overline{\ln X} = \frac{1}{n} \sum_{i=1}^n \ln X_i$  and  $\bar{c} = \frac{1}{n} \sum_{i=1}^n c_i$ .

- The moment estimators (MEs),  $\check{\eta}_n$  and  $\check{\beta}_n$ , are defined as follows [111]:

$$\begin{cases} \check{\beta}_n = \frac{\pi}{\sqrt{6}} \left[ \frac{1}{n-1} \sum_{i=1}^n (\ln X_i - \overline{\ln X})^2 \right]^{-1/2} \\ \ln \check{\eta}_n = \overline{\ln X} + \frac{\gamma_E}{\check{\beta}_n} \end{cases} \quad (3.8)$$

For all  $i \in \{1, \dots, n\}$ ,  $Y_i = \beta \ln(X_i/\eta)$  has the  $\mathcal{EV}_1(0, 1)$  distribution. The order statistics of this sample are denoted  $Y_1^* \leq \dots \leq Y_n^*$ .

Since  $\eta$  and  $\beta$  are unknown, it will be useful in the following to replace them by the above estimators. For all  $i$ , let  $\hat{Y}_i = \hat{\beta}_n \ln(X_i/\hat{\eta}_n)$ ,  $\tilde{Y}_i = \tilde{\beta}_n \ln(X_i/\tilde{\eta}_n)$  and  $\check{Y}_i = \check{\beta}_n \ln(X_i/\check{\eta}_n)$ . It is expected that the distributions of  $\hat{Y}_i$ ,  $\tilde{Y}_i$  and  $\check{Y}_i$  will not be far from the  $\mathcal{EV}_1(0, 1)$  distribution.

From [6], the distribution of  $(\hat{Y}_1, \dots, \hat{Y}_n)$  does not depend on  $\eta$  and  $\beta$ . From [76], it is also the case of the distribution of  $(\tilde{Y}_1, \dots, \tilde{Y}_n)$ . The following property proves the same result for  $(\check{Y}_1, \dots, \check{Y}_n)$ .

**Property 3.1** *The distribution of  $(\check{Y}_1, \dots, \check{Y}_n)$  does not depend on  $\eta$  and  $\beta$ .*

**Proof:** We know that  $\forall i \in \{1, \dots, n\}$ ,  $\ln X_i = \frac{Y_i}{\beta} + \ln \eta$ . So:

$$\check{Y}_i = \check{\beta}_n (\ln X_i - \ln \check{\eta}_n) = \check{\beta}_n \left( \frac{Y_i}{\beta} - \frac{\bar{Y}}{\beta} - \frac{\gamma_E}{\check{\beta}_n} \right) \quad (3.9)$$

where  $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i = \beta(\overline{\ln X} - \ln \eta)$ .

Moreover,  $S^2 = \frac{S_Y^2}{\beta^2}$ , so  $\check{\beta}_n = \beta \frac{\pi}{\sqrt{6} S_Y}$ , where  $S_Y = \left[ \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 \right]^{1/2}$ . Hence:

$$\check{Y}_i = \frac{\pi}{\sqrt{6} S_Y} (Y_i - \bar{Y}) - \gamma_E. \quad (3.10)$$

Since the distribution of  $(Y_1, \dots, Y_n)$  does not depend on  $\eta$  and  $\beta$ , it is also the case for the distribution of  $(\check{Y}_1, \dots, \check{Y}_n)$  and the property is proved. ■

The fact that the distributions of the samples  $\hat{Y}_i$ ,  $\tilde{Y}_i$  and  $\check{Y}_i$  are independent of the parameters of the underlying Weibull distribution is a very fundamental property since it allows to build GOF test statistics as functions of these samples. If a statistic  $S$  is a function of the  $Y_i$ , we will denote  $\hat{S}$ ,  $\tilde{S}$  and  $\check{S}$  the same statistic as a function of respectively the  $\hat{Y}_i$ ,  $\tilde{Y}_i$  and  $\check{Y}_i$ .

The normalized spacings of the Extreme Value distribution  $E_i$  are:

$$\forall i \in \{1, \dots, n\}, E_i = \frac{\ln X_i^* - \ln X_{i-1}^*}{\mathbb{E} \left[ \frac{\ln X_i^* - \mu}{\sigma} \right] - \mathbb{E} \left[ \frac{\ln X_{i-1}^* - \mu}{\sigma} \right]} = \sigma \frac{Y_i^* - Y_{i-1}^*}{\mathbb{E} [Y_i^* - Y_{i-1}^*]}. \quad (3.11)$$

Pyke [104] proved that for any distribution verifying given conditions, an appropriate subset of the normalized spacings  $E_i$  constitutes a vector whose components are asymptotically independent and exponentially distributed. These results are exact in the case of the Exponential distribution and asymptotic for the Extreme Value distribution. The values of the expectations  $\mathbb{E}[Y_i^*]$  are known and have explicit expressions given in [121].

## 3.2 Usual GOF tests for the Weibull distribution

In this section, we present families of GOF tests for the Weibull distribution. There exist several families such as tests based on the Weibull probability plot, tests based on the empirical distribution function, tests based on the normalized spacings, tests based on the Laplace transform, ... Thanks to a logarithmic transformation, these GOF tests can also be used as GOF tests for the Extreme Value distribution.

### 3.2.1 Tests based on probability plots

We remind that the Weibull probability plot is the plot of points:

$$(\ln X_i^*, c_i), i \in \{1, \dots, n\}$$

where  $c_i = \ln[-\ln(1 - p_i)]$  and  $p_i, i \in \{1, \dots, n\}$ , are approximations of the order statistics of a uniform sample. Under the Weibull assumption, these points should be approximately on a straight line.

Smith and Bain [115] used the determination coefficient  $R_{SB}^2$  of the ordinary least squares to build a GOF test. The test is based on the statistic  $Z^2 = n(1 - R_{SB}^2)$ , where

$$R_{SB}^2 = \frac{\left[ \sum_{i=1}^n (\ln X_i^* - \overline{\ln X}) (c_i - \bar{c}_n) \right]^2}{\sum_{i=1}^n (\ln X_i^* - \overline{\ln X})^2 \sum_{i=1}^n (c_i - \bar{c}_n)^2} \quad (3.12)$$

with  $c_i = \ln[-\ln(1 - p_i)], i = 1, \dots, n, \bar{c}_n = \frac{1}{n} \sum_{i=1}^n c_i$  and  $p_i = \frac{i}{n+1}$  (mean ranks).

The test rejects the null hypothesis when  $Z^2$  is too close to 1. Evans, Johnson and Green [42] proposed a GOF test based on a statistic similar to  $R_{SB}^2$ :

$$R_{EJG}^2 = \frac{\left[ \sum_{i=1}^n (\ln X_i^* - \overline{\ln X}) M_i \right]^2}{\sum_{i=1}^n (\ln X_i^* - \overline{\ln X})^2 \sum_{i=1}^n (M_i - \bar{M}_n)^2} \quad (3.13)$$

where  $M_i = \frac{1}{\hat{\beta}_n} \ln \left[ -\ln \left( 1 - \frac{i - 0.3175}{n + 0.365} \right) \right]$ .

The stabilized probability plot [87] is an alternative to the probability plot, which aims to stabilize the variance of the plotted points. It is defined as the sets of points  $(r_i, s_i)$ ,  $i \in \{1, \dots, n\}$ , where:

$$r_i = \frac{2}{\pi} \arcsin \sqrt{\frac{i-0.5}{n}} \quad \text{and} \quad s_i = \frac{2}{\pi} \arcsin \sqrt{1 - \exp\left(-\left(\frac{X_i^*}{\hat{\eta}_n}\right)^{\hat{\beta}_n}\right)}. \quad (3.14)$$

Under the Weibull assumption, these points should be approximately on the line  $y = x$ . Coles [28] proposed to measure the departure from this line by the statistic:

$$SPP_n = \max_{1 \leq i \leq n} |r_i - s_i|. \quad (3.15)$$

He used Blom estimators [17] instead of MLEs. Kimber [64] gave the quantiles tables of the statistic  $SPP_n$  using an approximation of the best unbiased linear estimators.

### 3.2.2 Shapiro-Wilk type tests

Shapiro and Brain [115] and Öztürk and Korukoğlu [97] adapted the idea of Shapiro-Wilk GOF test of exponentiality. The idea of the test is to compute the ratio of two estimators of  $\frac{1}{\beta}$ . The first estimator is a linear combination of the  $\ln X_i^*$ , suggested by d'Agostino [30]:

$$\frac{1}{n} \sum_{i=1}^n [0.6079 w_{n+i} - 0.257 w_i] \ln X_i^* \quad (3.16)$$

where the  $w_i$  are defined as follows:

$$\begin{aligned} w_i &= \ln \frac{n+1}{n-i+1} & \forall i \in \{1, \dots, n-1\} \\ w_n &= n - \sum_{i=1}^{n-1} w_i \\ w_{n+i} &= w_i(1 + \ln w_i) - 1 & \forall i \in \{1, \dots, n-1\} \\ w_{2n} &= 0.4228n - \sum_{i=1}^{n-1} w_{n+i}. \end{aligned}$$

The second estimator proposed by Shapiro and Brain [112] is the empirical standard deviation of the  $\ln X_i^*$ . The corresponding statistic has the following expression:

$$SB_n = \frac{\left(\frac{1}{n} \sum_{i=1}^n [0.6079 w_{n+i} - 0.257 w_i] \ln X_i^*\right)^2}{\frac{1}{n} \sum_{i=1}^n (\ln X_i^* - \overline{\ln X})^2}. \quad (3.17)$$

The second estimator used by Öztürk and Korukoğlu [97] is another linear combination of the  $\ln X_i^*$ :

$$\frac{\sum_{i=1}^n (2i - n - 1) \ln X_i^*}{\ln 2n(n-1)}. \quad (3.18)$$

The corresponding statistic is:

$$OK_n = \frac{\ln 2 (n-1) \sum_{i=1}^n [0.6079 w_{n+i} - 0.257 w_i] \ln X_i^*}{\sum_{i=1}^n (2i - n - 1) \ln X_i^*}. \quad (3.19)$$

Öztürk and Korukoğlu recommended using the following standardized statistic that improves the performance of the test:

$$OK_n^* = \frac{OK_n - 1 - 0.13/\sqrt{n} + 1.18/n}{0.49/\sqrt{n} - 0.36/n}. \quad (3.20)$$

For the two GOF tests  $SB_n$  and  $OK_n^*$ , the null hypothesis  $H_0$  is rejected when the value of the statistic (the ratio of two estimators of  $\frac{1}{\beta}$ ) is too far from 1. Thus the rejection is pronounced for too small or too large values of the statistic.

### 3.2.3 Tests based on the empirical distribution function

These tests are based on a measure of the departure between the empirical cdf of the  $\ln X_i$ :  $\mathbb{G}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\ln X_i \leq x\}}$  and the estimated theoretical cdf using the MLEs  $\hat{G}_0(y) = G(y; \ln \hat{\eta}_n, 1/\hat{\beta}_n)$ . The null hypothesis is rejected when this difference is too large. The usual tests based on the empirical cdf can be used only when the tested distribution has location-scale parameters. It is the case of the Extreme Value distribution that is why we use the empirical cdf of  $\ln X_i$  instead of the one of  $X_i$ . The best known statistics [25, 31] as those for the Exponential distribution previously defined in subsection 2.2.4 are:

- Kolmogorov-Smirnov statistic (KS):

$$KS_n = \sqrt{n} \sup_{y \in \mathbb{R}} \left| \mathbb{G}_n(y) - \hat{G}_0(y) \right| \quad (3.21)$$

$$= \sqrt{n} \max \left[ \max \left( \frac{i}{n} - \hat{U}_i^*, i = 1 : n \right), \max \left( \hat{U}_i^* - \frac{i-1}{n}, i = 1 : n \right) \right] \quad (3.22)$$

- Cramer-von Mises statistic (CM):

$$CM_n = n \int_{-\infty}^{+\infty} \left[ \mathbb{G}_n(y) - \hat{G}_0(y) \right]^2 d\hat{G}_0(y) = \sum_{i=1}^n \left( \hat{U}_i^* - \frac{2i-1}{2n} \right)^2 + \frac{1}{12n} \quad (3.23)$$

- Anderson-Darling statistic (AD):

$$AD_n = n \int_{-\infty}^{+\infty} \frac{\left[ \mathbb{G}_n(y) - \hat{G}_0(y) \right]^2}{\hat{G}_0(y)(1 - \hat{G}_0(y))} d\hat{G}_0(y) \quad (3.24)$$

$$= -n + \frac{1}{n} \sum_{i=1}^n \left[ (2i-1-2n) \ln(1 - \hat{U}_i^*) - (2i-1) \ln(\hat{U}_i^*) \right] \quad (3.25)$$

where  $\hat{U}_i = \hat{G}_0(\ln X_i) = 1 - \exp(-\exp(\hat{\beta}_n(\ln X_i - \ln \hat{\eta}_n))) = 1 - \exp(-\exp(\hat{Y}_i))$ .

Liao and Shimokawa [76] have proposed to replace the MLEs by the LSEs in (3.22), (3.23) and (3.25). For instance, the  $AD_n$  statistic will be denoted  $\widehat{AD}_n$  when used with the MLEs and  $\widetilde{AD}_n$  when used with the LSEs. Liao and Shimokawa [76] also combined the ideas of Kolmogorov-Smirnov and Anderson-Darling statistics with the LSEs instead of the MLEs. They proposed the statistic:

$$\widetilde{LS}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\max \left| \mathbb{G}_n(\ln X_i) - \widetilde{G}_0(\ln X_i) \right|}{\sqrt{\widetilde{G}_0(\ln X_i)(1 - \widetilde{G}_0(\ln X_i))}} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\max \left[ \frac{i}{n} - \widetilde{U}_i^*, \widetilde{U}_i^* - \frac{i-1}{n} \right]}{\sqrt{\widetilde{U}_i^*(1 - \widetilde{U}_i^*)}} \quad (3.26)$$

where  $\widetilde{G}_0(y) = G(y; \ln \widetilde{\eta}_n, 1/\widetilde{\beta}_n) = 1 - \exp(-\exp(\widetilde{\beta}_n(y - \ln \widetilde{\eta}_n)))$  and  $\widetilde{U}_i = \widetilde{G}_0(\ln X_i)$ .

They recommended the use of  $\widetilde{LS}_n$  or  $\widetilde{AD}_n$ .

### 3.2.4 Tests based on the normalized spacings

The normalized spacings are defined as:

$$\forall i \in \{1, \dots, n\}, E_i = \frac{\ln X_i^* - \ln X_{i-1}^*}{\mathbb{E} \left[ \frac{\ln X_i^* - \mu}{\sigma} \right] - \mathbb{E} \left[ \frac{\ln X_{i-1}^* - \mu}{\sigma} \right]} = \sigma \frac{Y_i^* - Y_{i-1}^*}{\mathbb{E} [Y_i^* - Y_{i-1}^*]}. \quad (3.27)$$

Every statistic of the form  $\frac{\sum_i a_i E_i}{\sum_j b_j E_j}$  can be used as a test statistic because it has a

distribution that is independent of the two parameters  $\eta$  and  $\beta$ .

Mann, Scheuer and Fertig [82] used the fact that for  $i \in \{1, \dots, n\}$ , the  $E_i/\sigma$  are asymptotically independent and distributed according to a standard Exponential distribution [104]. The authors proposed the statistic:

$$MSF_n = \frac{\sum_{j=\lfloor \frac{n}{2} \rfloor + 2}^n E_j}{\sum_{j=2}^n E_j} \quad (3.28)$$

where  $\lfloor x \rfloor$  is the floor of  $x$ . Under  $H_0$ ,  $MSF_n$  is asymptotically distributed according to the beta distribution  $\beta(\lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)$ . The Weibull assumption is rejected for both large and small values of the statistic. For usual alternatives, Mann, Sheuer and Fertig

recommended rejecting the null hypothesis only for large values of the statistic. This wrong use of the test improves its power in some cases, but leads to a biased test.

Tiku and Singh [121] suggested to reject the Weibull hypothesis for both large and small values of the statistic:

$$TS_n = \frac{2 \sum_{i=2}^{n-1} (n-i) E_i}{(n-2) \sum_{i=2}^n E_i}. \quad (3.29)$$

Under  $H_0$ ,  $\sqrt{3(n-2)}(TS_n - 1)$  has asymptotically the normal distribution  $N(0, 1)$ .

Lockhart, O'Reilly and Stephens [81] used the random variables:

$$Z_j = \sum_{i=2}^j E_i / \sum_{i=2}^n E_i, j = 2, \dots, n-1. \quad (3.30)$$

Under  $H_0$ , the  $Z_j$  are approximately distributed as the order statistics of the uniform distribution  $\mathcal{U}[0, 1]$ . Then, Lockhart et al proposed a GOF test based on the Anderson-Darling statistic computed for the  $Z_j$ :

$$LOS_n = 2 - n - \frac{1}{n-2} \sum_{i=2}^{n-1} [(2(i-n) + 1) \ln(1 - Z_i) - (2i - 3) \ln Z_i]. \quad (3.31)$$

The previous statistics  $MSF_n$  and  $TS_n$  are based respectively on the median and the mean value of the sample  $Z_i$  as it was discussed in [81]. Other test statistics can be applied to test the uniformity of the sample  $Z_i$  such as the Cramer-Von-Mises statistic.

These tests require the computation of the expectations of the order statistics of the standard Extreme Value distributions ( $\mathbb{E}[Y_i^*]$ ) which makes the use of these statistics not straightforward.

### 3.2.5 Generalized smooth tests

The principle of these tests consists in nesting the tested pdf  $g$  of the sample  $\ln X_1, \dots, \ln X_n$  within an order  $k$  alternative [107]:

$$g_k(y; \theta, \mu, \sigma) = C(\theta, \mu, \sigma) \exp \left( \sum_{i=1}^k \theta_i h_i(y; \mu, \sigma) \right) g(y; \mu, \sigma) \quad (3.32)$$

where  $\theta^T = (\theta_1, \dots, \theta_k)$  and  $\{h_i(y, \mu, \sigma)\}$ ,  $i \in \{1, \dots, k\}$ , are orthonormal functions on  $g(y; \mu, \sigma)$  and  $C(\theta, \mu, \sigma)$  is a normalizing constant. Testing the Weibull assumption is

equivalent to test  $H_0$ : " $\theta = 0$ " vs  $H_1$ : " $\theta \neq 0$ ". Generalized score test statistic of order  $k$  can be used to test the null hypothesis  $H_0$ . The test statistic is a quadratic form of the vector  $(ST_1, \dots, ST_k)$  where  $ST_i = \frac{1}{\sqrt{n}} h_i(Y_i)$ . Each single component  $ST_i$  of the test

statistic can be used as a test statistic.

The orthonormal polynomial functions used in the case of the Extreme Value distribution are given in Appendix C of [107]. Using the moment estimators of the parameters given in equation (3.8), the test statistics have simpler expressions and can be easily interpreted. It has been shown in [16] that the two first non-zero components are related to the skewness and the kurtosis coefficients of the sample  $Y_i$ . They have the following expressions:

$$\check{S}T_3 = -\left\{\sqrt{b_1} + 1.139547\right\}/\sqrt{20/n} \quad (3.33)$$

$$\check{S}T_4 = \left\{b_2 + 7.55\sqrt{b_1} + 3.21\right\}/\sqrt{219.72/n} \quad (3.34)$$

where  $\sqrt{b_1}$  and  $b_2$  are, respectively, the sample skewness and kurtosis defined by:

$$\sqrt{b_1} = \frac{1}{n} \sum_{j=1}^n (\ln X_j - \overline{\ln X})^3 / \left[ \frac{1}{n} \sum_{j=1}^n (\ln X_j - \overline{\ln X})^2 \right]^{3/2} \quad (3.35)$$

$$b_2 = \frac{1}{n} \sum_{j=1}^n (\ln X_j - \overline{\ln X})^4 / \left[ \frac{1}{n} \sum_{j=1}^n (\ln X_j - \overline{\ln X})^2 \right]^2. \quad (3.36)$$

The rejection of  $H_0$  is done for large values of the statistics.

### 3.2.6 Tests based on the Kullback-Leibler information

The following GOF test for the Extreme Value distribution is based on the Kullback-Leibler information [123]. The test can be applied to the sample  $\ln X_1, \dots, \ln X_n$  with the pdf  $g$ . The Kullback-Leibler discrimination information between  $g$  and  $\hat{g}$  is:

$$KL(g, \hat{g}) = \int_{-\infty}^{+\infty} g(y) \ln \frac{g(y)}{g(y; \hat{\mu}_n, \hat{\sigma}_n)} dy = \int_{-\infty}^{+\infty} g(y) \ln(g(y)) dy - \int_{-\infty}^{+\infty} g(y) \ln(\hat{g}(y)) dy. \quad (3.37)$$

To evaluate  $KL(g, \hat{g})$ , the first term of (3.37) is estimated by the Vasicek estimator

$$H_{m,n} = \frac{1}{n} \sum_{i=1}^n \ln \left[ \frac{n}{2m} (\ln X_{i+m}^* - \ln X_{i-m}^*) \right] \text{ where } m < \lfloor n/2 \rfloor, X_j^* = X_1^* \text{ if } j < 1, X_j^* = X_n^* \text{ if } j > n. \text{ Note that this estimation is computed for the } \ln X_i \text{ while the similar one in equation (2.38) was computed for the } Y_i. \text{ The second term is estimated by } \frac{1}{n} \sum_{i=1}^n \ln g(\ln X_i; \hat{\mu}_n, \hat{\sigma}_n). \text{ Thus } KL \text{ can be estimated by:}$$

$$\widehat{KL}_{m,n} = -\frac{1}{n} \sum_{i=1}^n \ln \left[ \frac{n}{2m} (\hat{Y}_{i+m}^* - \hat{Y}_{i-m}^*) \right] - \frac{1}{n} \sum_{i=1}^n \hat{Y}_i + \frac{1}{n} \sum_{i=1}^n e^{\hat{Y}_i}. \quad (3.38)$$

The null hypothesis  $H_0$  is rejected for large values of the statistic  $\widehat{KL}_{m,n}$ . The values of the parameter  $m$  are given in [38]. This test can have two additional versions  $\widetilde{KL}$  and  $\check{KL}$  using respectively the least squares and the moment estimators.

### 3.2.7 Tests based on the Laplace transform

Cabaña and Quiroz [22] used the Laplace transform to build GOF tests for the Weibull and type I Extreme Value distributions.

We remind that the Laplace transform of a sample  $Y_1, \dots, Y_n$  from the  $\mathcal{EV}_1(0, 1)$  distribution is:

$$\psi(t) = \mathbb{E} [e^{-tY}] = \Gamma(1 - t), \forall t < 1. \quad (3.39)$$

Cabaña and Quiroz proposed to estimate  $\psi(t)$  by the empirical Laplace transform  $\psi_n(t) = \frac{1}{n} \sum_{j=1}^n e^{-t\check{Y}_j}$  using the moment estimators.

Their statistic is based on the closeness between the empirical and theoretical Laplace transform which is measured by the empirical moment generating process:

$$\check{v}_n(s) = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^n e^{-\check{Y}_j s} - \Gamma(1 - s) \right). \quad (3.40)$$

They proved the convergence, under  $H_0$ , of  $\check{v}_n(s)$ ,  $s \in J$ , to a Gaussian process  $\check{G}_p(s)$ , for some interval  $J$ . They recommended to choose  $J \subseteq [-2.5, 0.49]$  and suggested two test statistics based on  $\check{v}_n$ . The first one has the following quadratic form:

$$\check{C}Q_n = \check{v}_{n,S} V^{-1}(S) {}^t \check{v}_{n,S} \quad (3.41)$$

where  $\check{v}_{n,S} = (\check{v}_n(s_1), \dots, \check{v}_n(s_k))$ ,  $S = \{s_1, \dots, s_k\} \subset J$  and  $V(S)$  is the limiting covariance matrix of  $\check{v}_{n,S}$  given in equation (2.7) in [22]. The statistic  $\check{C}Q_n$  has a limiting  $\chi^2$  distribution with  $k$  degrees of freedom under the null hypothesis  $H_0$ . The second test statistic is similar to the test of Henze: it is based on a weighted  $L^2$  norm. The only difference being the choice of the weight function that is different from the one used by Henze:

$$\check{S}_n = \int_J \check{v}_n^2(s) / V(s) ds \quad (3.42)$$

where  $V$  is the limiting variance of  $\check{v}_n$ . The asymptotic distribution of the test statistic  $\check{S}_n$  converges to the distribution of  $\int_J \check{G}_p^2(s) / V(s) ds$ . The rejection of the Weibull assumption is done for large values of the statistics.

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In this chapter we presented a complete review of existing GOF tests for the Weibull distribution. All the previous GOF tests will be compared later in chapter 6 with the new developed GOF tests presented in chapters 4 and 5. All these GOF tests are implemented in our R package EWGoF.

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# Chapter 4

## Simplified likelihood based GOF tests for the Weibull distribution

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The aim of this chapter is to present new likelihood based GOF tests for the Weibull distribution. These tests consist in nesting the Weibull distribution in three-parameter generalized Weibull families [92, 101] and testing the value of the third parameter by using the Wald, score and likelihood ratio procedures. We simplify the usual likelihood based tests by getting rid of the nuisance parameters, using the three estimation methods presented previously in chapter 3: maximum likelihood, least squares and moment methods. The proposed tests are exact.

The Weibull distribution is embedded in generalized Weibull distributions, which have been proposed in order to take into account bathtub-shaped or upside-down bathtub shaped hazard rates. A comprehensive comparison study is presented. Theoretical asymptotic results are derived when the Weibull distribution is nested in the generalized Gamma distribution.

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### 4.1 Generalized Weibull distributions $\mathcal{GW}$

Many generalizations of the Weibull distribution have been proposed in order to take into account more general shapes of the hazard rate such as bathtub-shaped or upside-down bathtub-shaped [92, 101], until very recently [96, 114, 4]. Most of them have three parameters  $\boldsymbol{\theta} = (\theta, \eta, \beta)'$ , and, for a value  $\theta_0$  of  $\theta$ , they reduce to the two-parameter Weibull distribution. These distributions will be called here generalized Weibull distributions and will be denoted  $\mathcal{GW}(\theta, \eta, \beta)$ . Then,  $\mathcal{GW}(\theta_0, \eta, \beta) = \mathcal{W}(\eta, \beta)$ .

For a given data set, it is interesting to determine if the simple Weibull distribution is an adapted model or if a more sophisticated model is needed. Then, within a particular  $\mathcal{GW}(\theta, \eta, \beta)$  model, it is worthwhile to build a statistical test of  $H_0: \theta = \theta_0$  vs  $H_1: \theta \neq \theta_0$ . Such a test can be considered as a GOF test for the Weibull distribution, because  $H_0$  corresponds to the assumption that the distribution of the data is Weibull, even if  $H_1$  is a more constrained hypothesis than the initial alternative. This approach is similar to that of the Cox-Oakes test defined in equation (2.23), which is a GOF test

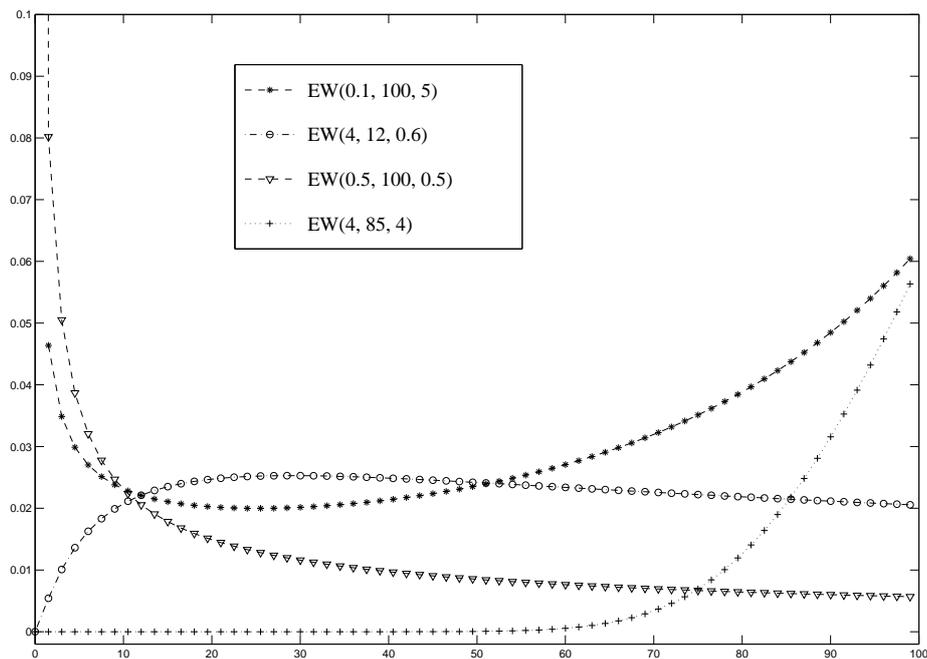
of exponentiality based on the inclusion of the Exponential distribution in the Weibull family. We showed in chapter 2 that this test is one of the most powerful GOF tests for the Exponential distribution. That is why we use the same approach to build new GOF tests for the Weibull distribution.

Table 4.1 presents the generalized Weibull distributions used in this work, mainly selected from [92]. All these distributions are defined on  $\mathbb{R}^+$ , as the standard Weibull, so in the table,  $x \geq 0$ . For each distribution, the table gives its name, its cdf  $F_X$ , the value  $\theta_0$  of parameter  $\theta$  for which the distribution is Weibull, and describes the possible shapes of the hazard rate.

For instance, Figures 4.1 and 4.2 present the 4 types of shape of the hazard rate of respectively the Exponentiated Weibull and the Generalized Gamma distributions [88, 118].

For the Generalized Weibull distribution introduced in [89], we consider only the case of unbounded lifetimes, corresponding to  $\lambda > 0$ . In this case, the distribution is linked to the Burr type XII distribution [21], so we will call it the Burr Generalized Weibull distribution.

For the Additive Weibull distribution, we consider the special case with three parameters instead of four, which happens to be also the  $\mathcal{B}$  distribution [15].



\*

Figure 4.1: Shapes of the hazard rate for the Exponentiated Weibull distribution

Table 4.1: Generalized Weibull distributions

Name	Cdf	Characteristics
Exponentiated Weibull $\mathcal{EW}(\theta, \eta, \beta)$ [88]	$F_X(x; \theta, \eta, \beta) = \left[1 - e^{-(x/\eta)^\beta}\right]^\theta$ $\theta, \eta, \beta > 0$	Weibull if $\theta = 1$ DHR if $\beta < 1$ and $\theta < 1$ IHR if $\beta > 1$ and $\theta > 1$ BT or IHR if $\beta > 1$ and $\theta < 1$ UBT or DHR if $\beta < 1$ and $\theta > 1$
Generalized Gamma $\mathcal{GG}(k, \eta, \beta)$ [118]	$F_X(x; k, \eta, \beta) = \frac{1}{\Gamma(k)} \gamma(k, (x/\eta)^\beta)$ $k, \eta, \beta > 0,$ $\gamma(s, x) = \int_0^x v^{s-1} e^{-v} dv$	Weibull if $k = 1$ if $\frac{1-k\beta}{\beta-1} > 0$ , $\begin{cases} \text{BT if } \beta > 1 \\ \text{UBT if } 0 < \beta < 1 \end{cases}$ otherwise $\begin{cases} \text{IHR if } \beta > 1 \\ \text{DHR if } 0 < \beta < 1 \end{cases}$
Additive Weibull $\mathcal{AW}(\xi, \eta, \beta)$ [129, 15]	$F_X(x; \xi, \eta, \beta) = 1 - e^{-\xi x - (\frac{x}{\eta})^\beta}$ $\xi, \eta, \beta > 0$	Weibull if $\xi \rightarrow 0$ IHR if $\beta > 1$ DHR if $\beta < 1$
Burr Generalized Weibull $\mathcal{BGW}(\lambda, \eta, \beta)$ [89]	$F_X(x; \lambda, \eta, \beta) = 1 - [1 + \lambda(x/\eta)^\beta]^{-\frac{1}{\lambda}}$ $\lambda, \eta, \beta > 0$	Weibull if $\lambda \rightarrow 0$ DHR if $\beta < 1$ UBT if $\beta > 1$
Marshall-Olkin Extended Weibull $\mathcal{MO}(\alpha, \eta, \beta)$ [83]	$F_X(x; \alpha, \eta, \beta) = 1 - \frac{\alpha e^{-(x/\eta)^\beta}}{1 - (1-\alpha)e^{-(x/\eta)^\beta}}$ $\alpha, \eta, \beta > 0$	Weibull if $\alpha = 1$ IHR if $\alpha \geq 1$ and $\beta \geq 1$ DHR if $\alpha \leq 1$ and $\beta \leq 1$ other shapes
Modified Weibull $\mathcal{MW}(\rho, \eta, \beta)$ [72]	$F_X(x; \rho, \eta, \beta) = 1 - e^{-(\frac{x}{\eta})^\beta e^{\rho x}}$ $\rho, \eta, \beta > 0$	Weibull if $\rho \rightarrow 0$ IHR if $\beta > 1$ BT if $0 < \beta < 1$
Power Generalized Weibull $\mathcal{PGW}(\nu, \eta, \beta)$ [94]	$F_X(x; \nu, \eta, \beta) = 1 - e^{1 - (1 + (x/\eta)^\beta)^{\frac{1}{\nu}}}$ $\nu, \eta, \beta > 0$	Weibull if $\nu = 1$ IHR if $\beta > 1$ and $\beta > \nu$ DHR if $0 < \beta < 1$ and $\beta \leq \nu$ BT if $0 < \nu < \beta < 1$ UBT if $\nu > \beta > 1$

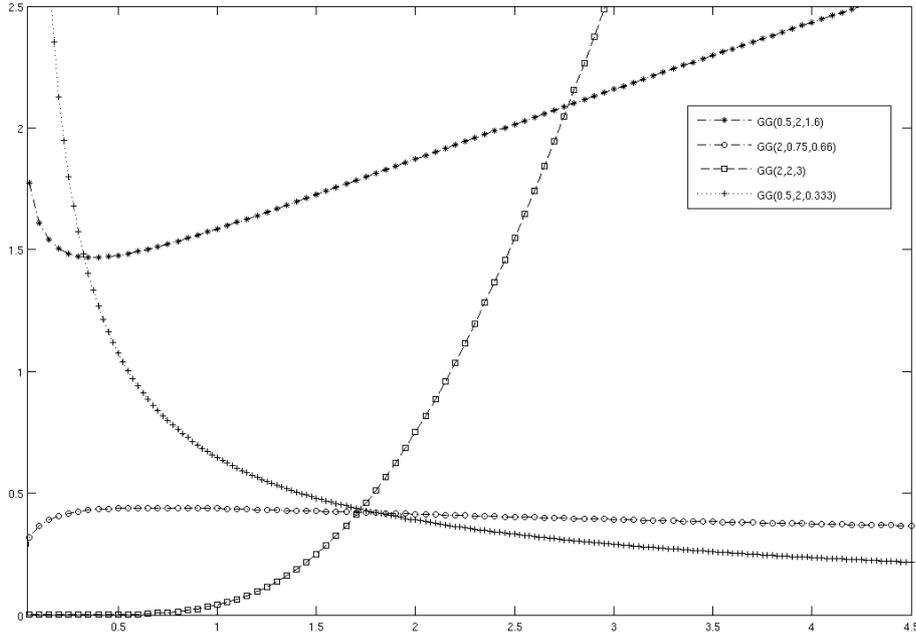


Figure 4.2: Shapes of the hazard rate for the Generalized Gamma distribution

## 4.2 Likelihood based goodness-of-fit tests

Let us assume that  $X_1, \dots, X_n$  are from a generalized Weibull distribution  $\mathcal{GW}(\boldsymbol{\theta})$ , with pdf  $f(x; \boldsymbol{\theta})$ , where  $\boldsymbol{\theta} = (\theta, \eta, \beta)'$ . When  $\theta = \theta_0$ , the  $X_i$  have the  $\mathcal{W}(\eta, \beta)$  distribution. So we want to test the null hypothesis  $H_0: \theta = \theta_0$  vs  $H_1: \theta \neq \theta_0$ , where  $(\eta, \beta)$  is a nuisance parameter.

The tests studied are the Wald, score and likelihood ratio tests, which are based on the asymptotic properties of the maximum likelihood estimators. The likelihood function for  $\boldsymbol{\theta}$  is:

$$L(\boldsymbol{\theta}) = \prod_{i=1}^n f(x_i; \boldsymbol{\theta}). \quad (4.1)$$

Let  $\hat{\boldsymbol{\theta}}_n = (\hat{\theta}_n, \hat{\eta}_n, \hat{\beta}_n)'$  be the MLE of  $\boldsymbol{\theta}$ , value at which  $L(\boldsymbol{\theta})$  is maximized. The score vector is  $U(\boldsymbol{\theta}) = \nabla \ln L(\boldsymbol{\theta})$ .

The Fisher matrix is the covariance matrix of the score. Since it is often difficult to compute, it is more convenient to use the observed Fisher information matrix  $I(\boldsymbol{\theta})$ . Since  $\boldsymbol{\theta}$  is partitioned in  $\theta$  and  $(\eta, \beta)$ , we partition  $U(\boldsymbol{\theta})$  and  $I(\boldsymbol{\theta})$  in the same way, so that the score and observed information matrix can be written:

$$U(\boldsymbol{\theta}) = \begin{pmatrix} U_1(\boldsymbol{\theta}) \\ U_2(\boldsymbol{\theta}) \end{pmatrix} \quad \text{and} \quad I(\boldsymbol{\theta}) = \begin{pmatrix} I_{11}(\boldsymbol{\theta}) & I_{12}(\boldsymbol{\theta}) \\ I_{21}(\boldsymbol{\theta}) & I_{22}(\boldsymbol{\theta}) \end{pmatrix}. \quad (4.2)$$

The inverse of  $I(\boldsymbol{\theta})$  is denoted:

$$I(\boldsymbol{\theta})^{-1} = \begin{pmatrix} I^{11}(\boldsymbol{\theta}) & I^{12}(\boldsymbol{\theta}) \\ I^{21}(\boldsymbol{\theta}) & I^{22}(\boldsymbol{\theta}) \end{pmatrix}.$$

For a given value  $\theta_0$  of  $\theta$ , let  $(\tilde{\eta}_n(\theta_0), \tilde{\beta}_n(\theta_0))$  denotes the profile MLE of  $(\eta, \beta)$ , obtained by maximizing the profile likelihood  $L(\theta_0, \eta, \beta)$  with respect to  $(\eta, \beta)$ . Let  $\tilde{\theta}_n = (\theta_0, \tilde{\eta}_n(\theta_0), \tilde{\beta}_n(\theta_0))$ .

The Wald, score and likelihood ratio statistics are respectively [74]:

$$W = \frac{(\hat{\theta}_n - \theta_0)^2}{I^{11}(\hat{\theta}_n)}, \quad (4.3)$$

$$Sc = U_1(\tilde{\theta}_n)^2 I^{11}(\tilde{\theta}_n), \quad (4.4)$$

$$LR = -2 \ln \frac{L(\tilde{\theta}_n)}{L(\hat{\theta}_n)}. \quad (4.5)$$

Under the null hypothesis  $H_0$ , these statistics converge to the  $\chi_1^2$  distribution when  $n$  tends to infinity [74]. The (asymptotic) tests consist in rejecting the Weibull assumption  $H_0$  at the significance level  $\alpha$ , if the statistics are greater than the quantile of order  $(1 - \alpha)$  of the  $\chi_1^2$  distribution.

This approach has been used by Mudholkar-Srivastava [88], Mudholkar et al [89], Bousquet et al [15] and Caroni [23], who nested the Weibull distribution in respectively the exponentiated Weibull [88], Burr generalized Weibull [89], a particular additive Weibull [15] and the Marshall-Olkin extended Weibull [83] distributions. In these papers, the aim of the tests is to confirm that the considered generalized distribution is needed instead of the basic Weibull. In each case, the asymptotic tests are used, with the  $\chi_1^2$  distribution. But this can lead to significant errors for small samples. For instance, Caroni showed that the chi-square approximation for the distribution of the score statistics is still not satisfactory for  $n = 200$ . Moreover, a simulation study showed that the distributions under  $H_0$  of the Wald, score and likelihood ratio statistics may depend on  $\eta$  and  $\beta$  in the case of small samples. Finally, the computation of  $\tilde{\theta}_n$  and  $\hat{\theta}_n$  may be difficult for some generalized Weibull distributions.

Since we want to use this kind of tests for small samples and for a large spectrum of generalized Weibull distributions, we propose another approach. The first idea is to move from the Weibull to the extreme value distribution with the logarithmic transformation. Let us recall that the  $Y_i = \beta \ln(X_i/\eta)$  are a sample of the  $\mathcal{EV}_1(0, 1)$  distribution. The nice fact is that this distribution has no unknown parameter. Using the idea of likelihood based tests, the  $\mathcal{EV}_1(0, 1)$  distribution can be included in a larger family with only one parameter  $\theta$ . We can call these families generalized extreme value distributions and denote them  $\mathcal{GEV}(\theta)$ . It is possible to derive likelihood based tests of  $H_0: " \theta = \theta_0 "$  vs  $H_1: " \theta \neq \theta_0 "$  in these families. This situation is simpler than the previous one because there are no nuisance parameters.

In fact, these tests can not be used directly since  $\eta$  and  $\beta$  are unknown, so the  $Y_i$  are not observed. Given that the  $Y_i$ , can be approximated by the  $\hat{Y}_i$ ,  $\tilde{Y}_i$  and  $\check{Y}_i$ , previously defined in section 3.1, our proposal is to test that the  $X_i$  come from a Weibull distribution by testing that the  $\hat{Y}_i$ ,  $\tilde{Y}_i$  or  $\check{Y}_i$  come from the  $\mathcal{EV}_1(0, 1)$  distribution using the Wald, score or likelihood ratio tests.

Now we have to choose a particular  $\mathcal{GEV}(\theta)$  family. Since only parameter  $\theta$  is concerned, we can use the link between Weibull and extreme value distributions when  $\eta =$

$\beta = 1$ . So we will define a  $\mathcal{G}\mathcal{E}\mathcal{V}(\theta)$  distribution as the distribution of  $\ln X$ , when  $X$  has a  $\mathcal{GW}(\theta, 1, 1)$  distribution.

This approach is summarized in the following steps:

1. Choose a generalized Weibull family  $\mathcal{GW}(\theta, \eta, \beta)$ . Let  $F_X(x; \theta, \eta, \beta)$  be its cdf and  $f_X(x; \theta, \eta, \beta)$  be its pdf.
2. Compute the pdf of  $Y = \ln X$  when  $\eta = \beta = 1$ :

$$f_Y(y; \theta) = e^y f_X(e^y; \theta, 1, 1).$$

3. Compute the likelihood  $L(\theta) = \prod_{i=1}^n f_Y(y_i; \theta)$  and the MLE of  $\theta$ ,  $\hat{\theta}_n$ . The computation of  $\hat{\theta}_n$  is of course much easier than that of  $\hat{\boldsymbol{\theta}}_n$  in the first approach.
4. Compute the score and observed information:

$$U(\theta) = \frac{\partial \ln L(\theta)}{\partial \theta}, \quad (4.6)$$

$$I(\theta) = -\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}. \quad (4.7)$$

Note that the score and information are here unidimensional, so their computation is much easier than in the first approach (4.2).

5. The likelihood based statistics are:

$$W = I(\theta_0)(\hat{\theta}_n - \theta_0)^2, \quad (4.8)$$

$$Sc = \frac{U^2(\theta_0)}{I(\theta_0)}, \quad (4.9)$$

$$LR = -2 \ln \frac{L(\theta_0)}{L(\hat{\theta}_n)}. \quad (4.10)$$

The expressions of these statistics are much simpler than that of the corresponding statistics in the first approach. The fact that  $Sc$  does not depend on  $\hat{\theta}_n$  is generally very convenient, but sometimes, this leads to undefined statistics. Then, it will be useful to use Slutsky's theorem and define  $W$  and  $Sc$  as:

$$W = I(\hat{\theta}_n)(\hat{\theta}_n - \theta_0)^2, \quad (4.11)$$

$$Sc = \frac{U^2(\theta_0)}{I(\hat{\theta}_n)}. \quad (4.12)$$

6. In the expressions of  $\hat{\theta}_n$ ,  $W$ ,  $Sc$  and  $LR$ , replace  $Y_i$  by  $\hat{Y}_i$ . If  $T$  denotes a particular  $\mathcal{GW}$  model chosen, the corresponding statistics are denoted  $\hat{T}_w$ ,  $\hat{T}_s$  and  $\hat{T}_l$ .

Do the same thing with  $\check{Y}_i$  and  $\breve{Y}_i$  and derive  $\check{T}_w$ ,  $\check{T}_s$ ,  $\check{T}_l$ ,  $\breve{T}_w$ ,  $\breve{T}_s$  and  $\breve{T}_l$ .

7. For each possible  $\mathcal{GW}$  model, we have 9 test statistics. In section 3.1, we have shown that, under  $H_0$ , the distributions of the vectors  $(\hat{Y}_1, \dots, \hat{Y}_n)$ ,  $(\tilde{Y}_1, \dots, \tilde{Y}_n)$  and  $(\check{Y}_1, \dots, \check{Y}_n)$  are independent of the parameters of the underlying Weibull distribution. So the distribution of the test statistics under  $H_0$  is also independent of  $\eta$  and  $\beta$ . Then, it is possible to compute any quantile of these distributions. Since it is very difficult to obtain a closed form expression of these quantiles for finite  $n$ , we will compute them by Monte Carlo simulations. Using the fact that the distributions do not depend on  $\eta$  and  $\beta$ , the quantiles will be obtained by simply simulating samples  $X_1, \dots, X_n$  from the Exponential distribution with parameter 1.

When  $n$  tends to infinity, the distributions of the test statistics (4.8) to (4.12) under  $H_0$  converge to the  $\chi_1^2$  distribution when they are computed with the  $(Y_i)_{1 \leq i \leq n}$ . However, nothing proves that it will be also the case when the statistics are computed with the  $(\hat{Y}_i)_{1 \leq i \leq n}$ ,  $(\tilde{Y}_i)_{1 \leq i \leq n}$  and  $(\check{Y}_i)_{1 \leq i \leq n}$ .

8. Finally, a GOF test consists in rejecting the Weibull assumption at the significance level  $\alpha$  if a statistic is greater than the corresponding quantile of order  $1 - \alpha$ .

Section 4.3 derives the test statistics for all the  $\mathcal{GW}$  distributions proposed in section 4.1. The detail of the computations is given for the first distribution studied, the Exponentiated Weibull distribution. The expressions of the statistics for the other families will be given directly.

## 4.3 Test statistics for several $\mathcal{GW}$ distributions

### 4.3.1 Tests based on the Exponentiated Weibull distribution

If  $X$  has the  $\mathcal{EW}(\theta, \eta, \beta)$  distribution, its cdf and pdf are respectively [88]:

$$F_X(x; \theta, \eta, \beta) = \left[1 - e^{-(x/\eta)^\beta}\right]^\theta, \quad (4.13)$$

$$f_X(x; \theta, \eta, \beta) = \theta \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^\beta} \left[1 - e^{-(x/\eta)^\beta}\right]^{\theta-1}. \quad (4.14)$$

So the pdf of  $Y = \ln X$  when  $\eta = \beta = 1$  is:

$$f_Y(y; \theta) = e^y f_X(e^y; \theta, 1, 1) = \theta (1 - e^{-e^y})^{\theta-1} e^{y-e^y}. \quad (4.15)$$

The Weibull distribution for  $X$  or  $\mathcal{EV}_1(0, 1)$  for  $Y$  is obtained for  $\theta = 1$ , so we have to test  $H_0$ : “ $\theta = 1$ ” vs  $H_1$ : “ $\theta \neq 1$ ”. The log-likelihood function of a sample  $Y_1, \dots, Y_n$  from the distribution with pdf  $f_Y$  is:

$$\ln L(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^n \ln(1 - e^{-e^{Y_i}}) + \sum_{i=1}^n Y_i - \sum_{i=1}^n e^{Y_i}. \quad (4.16)$$

The score and observed information have the following expressions:

$$U(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \ln(1 - e^{-e^{Y_i}}), \quad (4.17)$$

$$I(\theta) = \frac{n}{\theta^2}. \quad (4.18)$$

The MLE of  $\theta$  is:

$$\hat{\theta}_n = -\frac{n}{\sum_{i=1}^n \ln(1 - e^{-Y_i})}. \quad (4.19)$$

Then, the general expressions of the likelihood based statistics can be easily derived:

- Wald:

$$EW_w = I(1)(\hat{\theta}_n - 1)^2 = n(\hat{\theta}_n - 1)^2, \quad (4.20)$$

- score:

$$EW_s = U^2(1)/I(1) = n \left(1 - \frac{1}{\hat{\theta}_n}\right)^2, \quad (4.21)$$

- likelihood ratio:

$$EW_l = -2 \ln \frac{L(1)}{L(\hat{\theta}_n)} = 2n \left( \ln \hat{\theta}_n - 1 + \frac{1}{\hat{\theta}_n} \right). \quad (4.22)$$

Each of the statistics (4.20), (4.21), (4.22) can be computed by using  $\hat{Y}_i$ ,  $\check{Y}_i$  and  $\check{\check{Y}}_i$  instead of  $Y_i$ ,  $i \in \{1, \dots, n\}$ . Then we obtain nine test statistics  $\widehat{EW}_w$ ,  $\widehat{EW}_s$ ,  $\widehat{EW}_l$ ,  $\widetilde{EW}_w$ ,  $\widetilde{EW}_s$ ,  $\widetilde{EW}_l$ ,  $\check{EW}_w$ ,  $\check{EW}_s$  and  $\check{EW}_l$ . A remarkable fact is the extreme simplicity of the expressions of the test statistics. The quantiles of their distributions under  $H_0$  are easily obtained by simulating samples  $X_1, \dots, X_n$  from the Exponential distribution with parameter 1. The power of the corresponding nine GOF tests will be assessed in Section 4.4.

### 4.3.2 Tests based on the Generalized Gamma distribution

The cdf of the Generalized Gamma distribution  $\mathcal{GG}(k, \eta, \beta)$  is [118]:

$$F_X(x; k, \eta, \beta) = \frac{1}{\Gamma(k)} \gamma(k, (x/\eta)^\beta). \quad (4.23)$$

The pdf of  $Y$  is:

$$f_Y(y; k) = \frac{1}{\Gamma(k)} e^{ky - e^y}. \quad (4.24)$$

We have to test  $H_0$ : “ $k = 1$ ” vs  $H_1$ : “ $k \neq 1$ ”. The log-likelihood, score and information are respectively given by:

$$\ln L(k) = -n \ln \Gamma(k) + k \sum_{i=1}^n Y_i - \sum_{i=1}^n e^{Y_i} \quad (4.25)$$

$$U(k) = -n\varphi(k) + \sum_{i=1}^n Y_i \quad (4.26)$$

$$I(k) = n\varphi'(k) \quad (4.27)$$

where  $\varphi = \Gamma'/\Gamma$  and  $\varphi'$  are the digamma and trigamma functions [3].  $\varphi(1) = -\gamma_E$ , where  $\gamma_E$  is the Euler constant and  $\varphi'(1) = \pi^2/6$ .

The MLE  $\hat{k}_n$  of  $k$  verifies the following equation:

$$\varphi(\hat{k}_n) = \frac{1}{n} \sum_{i=1}^n Y_i. \quad (4.28)$$

The likelihood based statistics are:

- Wald:

$$GG_w^1 = \frac{n\pi^2}{6} (\hat{k}_n - 1)^2 \quad (4.29)$$

- score:

$$GG_s^1 = \frac{6}{n\pi^2} \left( \sum_{i=1}^n Y_i + n\gamma_E \right)^2 \quad (4.30)$$

- likelihood ratio:

$$GG_l^1 = -2n \ln \Gamma(\hat{k}_n) + 2(\hat{k}_n - 1) \sum_{i=1}^n Y_i. \quad (4.31)$$

The  $\widetilde{GG}$  and  $\check{GG}$  statistics are, respectively, functions of  $\sum_{i=1}^n \widetilde{Y}_i = n\bar{c} = n \sum_{i=1}^n \ln[-\ln[1 - (i - 0.5)/n]]$  and  $\sum_{i=1}^n \check{Y}_i = -n\gamma_E$ . Therefore, the MLE  $\hat{k}_n$  and the test statistics are constant, so the corresponding GOF tests are not defined.

That is why it is interesting to use a new parametrization of the Generalized Gamma distribution, proposed in [73]: we consider now the  $\mathcal{GG}(k, \eta^{1/\sqrt{k}}, \beta\sqrt{k})$  distribution.

Its cdf is:

$$F_X(x; k, \eta, \beta) = \frac{1}{\Gamma(k)} \gamma(k, (x/k)^{\beta\sqrt{k}} \eta^{-\beta}). \quad (4.32)$$

The pdf of  $Y$  is:

$$f_Y(y; k) = \frac{k^{k-0.5}}{\Gamma(k)} e^{\sqrt{ky} - ke^{y/\sqrt{k}}}. \quad (4.33)$$

We have to test  $H_0: "k = 1"$  vs  $H_1: "k \neq 1"$ . The log-likelihood, score and information are respectively given by:

$$\ln L(k) = \left(k - \frac{1}{2}\right) n \ln k - n \ln \Gamma(k) + \sqrt{k} \sum_{i=1}^n Y_i - k \sum_{i=1}^n e^{\frac{Y_i}{\sqrt{k}}} \quad (4.34)$$

$$U(k) = n \ln k + n - \frac{n}{2k} - n\varphi(k) + \frac{1}{2\sqrt{k}} \sum_{i=1}^n Y_i - \sum_{i=1}^n \left(1 - \frac{Y_i}{2\sqrt{k}}\right) e^{\frac{Y_i}{\sqrt{k}}} \quad (4.35)$$

$$I(k) = -\frac{n}{k} - \frac{n}{2k^2} + n\varphi'(k) - \frac{1}{4k^{\frac{3}{2}}} \sum_{i=1}^n Y_i e^{\frac{Y_i}{\sqrt{k}}} + \frac{1}{4k^{\frac{3}{2}}} \sum_{i=1}^n Y_i + \frac{1}{4k^2} \sum_{i=1}^n Y_i^2 e^{\frac{Y_i}{\sqrt{k}}} \quad (4.36)$$

where  $\varphi$  and  $\varphi'$  are the digamma and trigamma functions defined above.

The likelihood based statistics are:

- Wald:

$$GG_w^2 = (\hat{k}_n - 1)^2 \left( -\frac{3n}{2} + n\varphi'(1) - \frac{1}{4} \sum_{i=1}^n Y_i e^{Y_i} + \frac{1}{4} \sum_{i=1}^n Y_i + \frac{1}{4} \sum_{i=1}^n Y_i^2 e^{Y_i} \right) \quad (4.37)$$

- score:

$$GG_s^2 = \frac{\left( \frac{n}{2} - n\varphi(1) + \frac{1}{2} \sum_{i=1}^n Y_i - \sum_{i=1}^n e^{Y_i} + \frac{1}{2} \sum_{i=1}^n Y_i e^{Y_i} \right)^2}{-\frac{3n}{2} + n\varphi'(1) - \frac{1}{4} \sum_{i=1}^n Y_i e^{Y_i} + \frac{1}{4} \sum_{i=1}^n Y_i + \frac{1}{4} \sum_{i=1}^n Y_i^2 e^{Y_i}} \quad (4.38)$$

- likelihood ratio:

$$GG_l^2 = (2\hat{k}_n - 1)n \ln \hat{k}_n - 2n \ln \Gamma(\hat{k}_n) + 2(\sqrt{\hat{k}_n} - 1) \sum_{i=1}^n Y_i + 2 \sum_{i=1}^n e^{Y_i} - 2\hat{k}_n \sum_{i=1}^n e^{\frac{Y_i}{\sqrt{\hat{k}_n}}}. \quad (4.39)$$

### 4.3.3 Tests based on the Additive Weibull distribution

The cdf of the Additive Weibull distribution with 3 parameters considered  $\mathcal{AW}(\xi, \eta, \beta)$  is [129, 15]:

$$F_X(x; \xi, \eta, \beta) = 1 - e^{-\xi x - (x/\eta)^\beta}. \quad (4.40)$$

The pdf of  $Y$  is:

$$f_Y(y; \xi) = (\xi + 1) e^{y - (\xi + 1)e^y}. \quad (4.41)$$

$Y$  has the  $\mathcal{EV}_1(-\ln(\xi + 1), 1)$  distribution. We have to test  $H_0$ : “ $\xi = 0$ ” vs  $H_1$ : “ $\xi \neq 0$ ”. The log-likelihood, score and information are respectively given by:

$$\ln L(\xi) = n \ln(\xi + 1) + \sum_{i=1}^n Y_i - (\xi + 1) \sum_{i=1}^n e^{Y_i}, \quad (4.42)$$

$$U(\xi) = \frac{n}{\xi + 1} - \sum_{i=1}^n e^{Y_i}, \quad (4.43)$$

$$I(\xi) = \frac{n}{(\xi + 1)^2}. \quad (4.44)$$

The MLE  $\hat{\xi}_n$  of  $\xi$  is:

$$\hat{\xi}_n = \frac{n}{\sum_{i=1}^n e^{Y_i}} - 1. \quad (4.45)$$

The likelihood based statistics are:

- Wald:

$$AW_w = n\hat{\xi}_n^2 \quad (4.46)$$

- score:

$$AW_s = \frac{n\hat{\xi}_n^2}{(\hat{\xi}_n + 1)^2} \quad (4.47)$$

- likelihood ratio:

$$AW_l = 2n \left( \ln(\hat{\xi}_n + 1) - \frac{\hat{\xi}_n}{\hat{\xi}_n + 1} \right). \quad (4.48)$$

When the vector  $(\hat{Y}_1, \dots, \hat{Y}_n)$  is used,  $\sum_{i=1}^n e^{\hat{Y}_i} = \sum_{i=1}^n (X_i/\hat{\eta}_n)^{\hat{\beta}_n} = n$ , so  $\hat{\xi}_n = 0$ . Then, only the  $\widetilde{AW}$  and  $\check{AW}$  statistics are defined in this case.

#### 4.3.4 Tests based on the Burr Generalized Weibull distribution

The cdf of the Burr Generalized Weibull distribution  $\mathcal{BGW}(\lambda, \eta, \beta)$  is [89]:

$$F_X(x; \lambda, \eta, \beta) = 1 - [1 + \lambda(x/\eta)^\beta]^{-\frac{1}{\lambda}}. \quad (4.49)$$

The pdf of  $Y$  is:

$$f_Y(y; \lambda) = e^y(1 + \lambda e^y)^{-\frac{1}{\lambda}-1}. \quad (4.50)$$

We have to test  $H_0: “\lambda \sim 0”$  vs  $H_1: “\lambda \neq 0”$ . The log-likelihood, score and information are respectively given by:

$$\ln L(\lambda) = \sum_{i=1}^n Y_i - \left( \frac{1}{\lambda} + 1 \right) \sum_{i=1}^n \ln(1 + \lambda e^{Y_i}) \quad (4.51)$$

$$U(\lambda) = \frac{1}{\lambda^2} \sum_{i=1}^n \ln(1 + \lambda e^{Y_i}) - \left( \frac{1}{\lambda} + 1 \right) \sum_{i=1}^n \frac{1}{e^{-Y_i} + \lambda} \quad (4.52)$$

$$I(\lambda) = \frac{2}{\lambda^3} \sum_{i=1}^n \ln(1 + \lambda e^{Y_i}) - \frac{2}{\lambda^2} \sum_{i=1}^n \frac{1}{e^{-Y_i} + \lambda} - \left( \frac{1}{\lambda} + 1 \right) \sum_{i=1}^n \frac{1}{(e^{-Y_i} + \lambda)^2}. \quad (4.53)$$

The MLE  $\hat{\lambda}_n$  of  $\lambda$  verifies:

$$\sum_{i=1}^n \ln(1 + \hat{\lambda}_n e^{Y_i}) = \hat{\lambda}_n(1 + \hat{\lambda}_n) \sum_{i=1}^n \frac{1}{e^{-Y_i} + \hat{\lambda}_n}. \quad (4.54)$$

It happens that  $\lim_{\lambda \rightarrow 0} I(\lambda) = +\infty$ , but

$$I(\hat{\lambda}_n) = \frac{2}{\hat{\lambda}_n} \sum_{i=1}^n \frac{1}{e^{-Y_i} + \hat{\lambda}_n} - \left( \frac{1}{\hat{\lambda}_n} + 1 \right) \sum_{i=1}^n \frac{1}{(e^{-Y_i} + \hat{\lambda}_n)^2}. \quad (4.55)$$

So for the Wald and score tests, we will use expressions (4.11) and (4.12) instead of (4.8) and (4.9). Then, the likelihood based statistics are:

- Wald:

$$BGW_w = 2\hat{\lambda}_n \sum_{i=1}^n \frac{1}{e^{-Y_i} + \hat{\lambda}_n} - \hat{\lambda}_n(\hat{\lambda}_n + 1) \sum_{i=1}^n \frac{1}{(e^{-Y_i} + \hat{\lambda}_n)^2} \quad (4.56)$$

- score:

$$BGW_s = \frac{\frac{1}{2} \sum_{i=1}^n e^{2Y_i} - \sum_{i=1}^n e^{Y_i}}{\frac{2}{\hat{\lambda}_n} \sum_{i=1}^n \frac{1}{e^{-Y_i} + \hat{\lambda}_n} - \left( \frac{1}{\hat{\lambda}_n} + 1 \right) \sum_{i=1}^n \frac{1}{(e^{-Y_i} + \hat{\lambda}_n)^2}} \quad (4.57)$$

- likelihood ratio:

$$BGW_l = -2 \left( \frac{1}{\hat{\lambda}_n} + 1 \right) \sum_{i=1}^n \ln(1 + \hat{\lambda}_n e^{Y_i}) + 2 \sum_{i=1}^n e^{Y_i}. \quad (4.58)$$

### 4.3.5 Tests based on the Marshall-Olkin extended Weibull distribution

The cdf of the Marshall-Olkin extended Weibull distribution  $\mathcal{MO}(\alpha, \eta, \beta)$  is [83]:

$$F_X(x; \alpha, \eta, \beta) = 1 - \frac{\alpha e^{-(x/\eta)^\beta}}{1 - (1 - \alpha) e^{-(x/\eta)^\beta}}. \quad (4.59)$$

The pdf of  $Y$  is:

$$f_Y(y; k) = \frac{\alpha e^{y - e^y}}{[1 - (1 - \alpha) e^{-e^y}]^2}. \quad (4.60)$$

We have to test  $H_0: \alpha = 1$  vs  $H_1: \alpha \neq 1$ . The log-likelihood, score and information are respectively given by:

$$\ln L(\alpha) = n \ln \alpha + \sum_{i=1}^n Y_i - \sum_{i=1}^n e^{Y_i} - 2 \sum_{i=1}^n \ln [1 - (1 - \alpha) e^{-e^{Y_i}}], \quad (4.61)$$

$$U(\alpha) = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{e^{-e^{Y_i}}}{1 - (1 - \alpha) e^{-e^{Y_i}}}, \quad (4.62)$$

$$I(\alpha) = \frac{n}{\alpha^2} - 2 \sum_{i=1}^n \frac{e^{-2e^{Y_i}}}{[1 - (1 - \alpha) e^{-e^{Y_i}}]^2}. \quad (4.63)$$

The maximum likelihood estimator  $\hat{\alpha}_n$  of  $\alpha$  verifies the following equation:

$$U(\hat{\alpha}_n) = 0. \quad (4.64)$$

The likelihood based statistics are:

- Wald:

$$MO_w = (\hat{\alpha}_n - 1)^2 \left( n - 2 \sum_{i=1}^n e^{-2e^{Y_i}} \right) \quad (4.65)$$

- score:

$$MO_s = \frac{\left( n - 2 \sum_{i=1}^n e^{-e^{Y_i}} \right)^2}{n - 2 \sum_{i=1}^n e^{-2e^{Y_i}}} \quad (4.66)$$

- likelihood ratio:

$$MO_l = 2n \ln \hat{\alpha}_n - 4 \sum_{i=1}^n \ln \left[ 1 - (1 - \hat{\alpha}_n) e^{-e^{Y_i}} \right]. \quad (4.67)$$

### 4.3.6 Test based on the Modified Weibull distribution

The cdf of the Modified Weibull distribution  $\mathcal{MW}(\rho, \eta, \beta)$  is [72]:

$$F_X(x; \rho, \eta, \beta) = 1 - e^{-(x/\eta)^\beta e^{\rho x}}. \quad (4.68)$$

The pdf of  $Y$  is:

$$f_Y(y; \rho) = (1 + \rho e^y) e^{y + \rho e^y - e^{y + \rho e^y}}. \quad (4.69)$$

We have to test  $H_0: \rho = 0$  vs  $H_1: \rho \neq 0$ . The log-likelihood, score and information are respectively given by:

$$\ln L(\rho) = \sum_{i=1}^n Y_i + \rho \sum_{i=1}^n e^{Y_i} + \sum_{i=1}^n \ln(1 + \rho e^{Y_i}) - \sum_{i=1}^n e^{Y_i + \rho e^{Y_i}} \quad (4.70)$$

$$U(\rho) = \sum_{i=1}^n e^{Y_i} + \sum_{i=1}^n \frac{e^{Y_i}}{1 + \rho e^{Y_i}} - \sum_{i=1}^n e^{2Y_i + \rho e^{Y_i}} \quad (4.71)$$

$$I(\rho) = \sum_{i=1}^n \frac{e^{2Y_i}}{(1 + \rho e^{Y_i})^2} + \sum_{i=1}^n e^{3Y_i + \rho e^{Y_i}}. \quad (4.72)$$

The likelihood based statistics are:

- Wald:

$$MW_w = \hat{\rho}_n^2 \left[ \sum_{i=1}^n e^{2Y_i} + \sum_{i=1}^n e^{3Y_i} \right], \quad (4.73)$$

- score:

$$MW_s = \frac{\left[ 2 \sum_{i=1}^n e^{Y_i} - \sum_{i=1}^n e^{2Y_i} \right]^2}{\sum_{i=1}^n e^{2Y_i} + \sum_{i=1}^n e^{3Y_i}}, \quad (4.74)$$

- likelihood ratio:

$$MW_l = 2(\hat{\rho}_n + 1) \sum_{i=1}^n e^{Y_i} + 2 \sum_{i=1}^n \ln(1 + \hat{\rho}_n e^{Y_i}) - 2 \sum_{i=1}^n e^{Y_i + \hat{\rho}_n e^{Y_i}}. \quad (4.75)$$

### 4.3.7 Tests based on the Power Generalized Weibull distribution

The cdf of the Power Generalized Weibull distribution  $\mathcal{PGW}(\nu, \eta, \beta)$  is [94]:

$$F_X(x; \nu, \eta, \beta) = 1 - e^{1 - (1 + (x/\eta)^\beta)^{\frac{1}{\nu}}}. \quad (4.76)$$

The pdf of  $Y$  is:

$$f_Y(y; \nu) = \frac{1}{\nu} (1 + e^y)^{\frac{1}{\nu} - 1} e^{y + 1 - (1 + e^y)^{\frac{1}{\nu}}}. \quad (4.77)$$

We have to test  $H_0: \nu = 1$  vs  $H_1: \nu \neq 1$ . The log-likelihood, score and information are respectively given by:

$$\ln L(\nu) = -n \ln \nu + \sum_{i=1}^n Y_i + \left( \frac{1}{\nu} - 1 \right) \sum_{i=1}^n \ln(1 + e^{Y_i}) + n - \sum_{i=1}^n (1 + e^{Y_i})^{\frac{1}{\nu}}, \quad (4.78)$$

$$U(\nu) = -\frac{n}{\nu} + \frac{1}{\nu^2} \sum_{i=1}^n \ln(1 + e^{Y_i}) \left[ (1 + e^{Y_i})^{\frac{1}{\nu}} - 1 \right], \quad (4.79)$$

$$I(\nu) = -\frac{n}{\nu^2} - \frac{2}{\nu^3} \sum_{i=1}^n \ln(1 + e^{Y_i}) + \frac{2}{\nu^3} \sum_{i=1}^n \ln(1 + e^{Y_i}) (1 + e^{Y_i})^{\frac{1}{\nu}} \left[ 1 + \frac{1}{2\nu} \ln(1 + e^{Y_i}) \right]. \quad (4.80)$$

The MLE  $\hat{\nu}_n$  of  $\nu$  verifies the following equation:

$$\hat{\nu}_n = \frac{1}{n} \sum_{i=1}^n \ln(1 + e^{Y_i}) \left[ (1 + e^{Y_i})^{\frac{1}{\hat{\nu}_n}} - 1 \right]. \quad (4.81)$$

The likelihood based statistics are:

- Wald:

$$PGW_w = (\hat{\nu}_n - 1)^2 \left[ -n + 2 \sum_{i=1}^n \ln(1 + e^{Y_i}) e^{Y_i} + \sum_{i=1}^n [\ln(1 + e^{Y_i})]^2 (1 + e^{Y_i}) \right], \quad (4.82)$$

- score:

$$PGW_s = \frac{\left[ -n + \sum_{i=1}^n \ln(1 + e^{Y_i}) e^{Y_i} \right]^2}{-n + 2 \sum_{i=1}^n \ln(1 + e^{Y_i}) e^{Y_i} + \sum_{i=1}^n [\ln(1 + e^{Y_i})]^2 (1 + e^{Y_i})}, \quad (4.83)$$

- likelihood ratio:

$$PGW_l = -2n \ln \hat{\nu}_n + 2 \left( \frac{1}{\hat{\nu}_n} - 1 \right) \sum_{i=1}^n \ln(1 + e^{Y_i}) - 2 \sum_{i=1}^n (1 + e^{Y_i})^{\frac{1}{\hat{\nu}_n}} + 2 \sum_{i=1}^n e^{Y_i} + 2n. \quad (4.84)$$

## 4.4 Simulation and comparison

Section 4.3 has proposed a large amount of likelihood based GOF tests for the Weibull distribution. It is then important to select the best of them. One criterion is the simplicity of the computation of the test statistics. In this case, the  $EW$  and  $AW$  tests are the most interesting because they use an explicit estimator of the parameter. But the most important criterion is the power of the tests. This section presents the results of an intensive Monte-Carlo simulation study in order to assess the power of all the likelihood based tests. The best of them will be compared with the usual GOF tests for the Weibull distribution in chapter 6. All these GOF tests are implemented in our R package EWGoF.

### 4.4.1 The simulation framework

The study is carried out using a broad class of alternative distributions. For each distribution, we simulate 50000 samples of size  $n \in \{20, 50\}$ . All the GOF tests are applied with a significance level set to 5%. The power of the tests is assessed by the percentage of rejection of the null hypothesis.

We first simulate Weibull samples, in order to check that the percentage of rejection is close to the nominal significance level 5%. For the other simulations, we have chosen usual alternatives of the Weibull distribution (Gamma  $\mathcal{G}$ , Lognormal  $\mathcal{LN}$ , Inverse-Gamma  $\mathcal{IG}$ ) and generalized Weibull distributions ( $\mathcal{EW}$ ,  $\mathcal{GG}$ ,  $\mathcal{AW}$ ,  $\mathcal{PGW}$ ). For the sake of simplicity, the scale parameters of the Weibull, Gamma and Inverse-Gamma distributions are set to 1 and the mean of the Lognormal distribution is set to 0. Parameters of the simulated distributions are selected to obtain different shapes of the hazard rate. Table 4.2 gives the values of the parameters and the notations used for all the simulated distributions.

Table 4.2: Simulated distributions

Weibull	$\exp(1)$	$\mathcal{W}(1, 0.5)$	$\mathcal{W}(1, 3)$
IHR	$\mathcal{G}(3, 1)$	$\mathcal{AW}1 \equiv \mathcal{AW}(10, 0.02, 5.2)$	$\mathcal{EW}1 \equiv \mathcal{EW}(6.5, 20, 6)$
DHR	$\mathcal{G}(0.5, 1)$	$\mathcal{AW}2 \equiv \mathcal{AW}(2, 20, 0.1)$	$\mathcal{EW}2 \equiv \mathcal{EW}(0.1, 0.01, 0.95)$
BT	$\mathcal{EW}3 \equiv \mathcal{EW}(0.1, 100, 5)$ $\mathcal{PGW}1 \equiv \mathcal{PGW}(0.01, 200, 0.9)$	$\mathcal{GG}1 \equiv \mathcal{GG}(0.1, 1, 4)$	$\mathcal{GG}2 \equiv \mathcal{GG}(0.2, 1, 3)$
UBT	$\mathcal{LN}(0, 0.8)$ $\mathcal{GG}3 \equiv \mathcal{GG}(10, 0.01, 0.2)$	$\mathcal{IG}(3, 1)$ $\mathcal{PGW}2 \equiv \mathcal{PGW}(4, 1, 3)$	$\mathcal{EW}4 \equiv \mathcal{EW}(4, 12, 0.6)$

For instance, let us consider the Wald test based on the Exponentiated Weibull distribution, with maximum likelihood estimators. The test statistics is given by (4.20):

$$\widehat{EW}_w = n(\hat{\theta}_n - 1)^2 = n \left[ 1 + n / \sum_{i=1}^n \ln(1 - e^{-e^{\hat{Y}_i}}) \right]^2. \quad (4.85)$$

For a given sample size  $n$ , we simulate  $X_1, \dots, X_n$  from the Exponential distribution with parameter 1. For all  $i$ , we compute  $\hat{Y}_i = \hat{\beta}_n \ln(X_i / \hat{\eta}_n)$ , then we derive  $\widehat{EW}_w$ , as detailed in section 4.3. This process is done  $m = 100000$  times. The quantiles of the distribution of  $\widehat{EW}_w$  under  $H_0$  are given by the empirical quantiles of the  $m$  values of

$\widehat{EW}_w$ . Table 4.3 gives some quantiles for several values of  $n$ , and the same quantiles for the  $\chi_1^2$  distribution. We observe that, even for very large  $n$ , the distribution of  $\widehat{EW}_w$  under  $H_0$  is very far from the  $\chi_1^2$  distribution. So it is important to be able to apply these GOF tests without using the chi-square approximation.

Table 4.3: Quantiles of the distribution of  $\widehat{EW}_w$  under  $H_0$

$n$	90%	95%	97.5%	99%
20	0.107	0.154	0.204	0.276
50	0.123	0.176	0.232	0.315
100	0.127	0.182	0.240	0.323
1000	0.132	0.188	0.247	0.328
$\chi_1^2$	2.706	3.841	5.024	6.635

For the power study, we simulate a sample  $X_1, \dots, X_n$  of size  $n$  of a given distribution. For  $n = 50$ , the Weibull assumption is rejected at the level 5% if  $\widehat{EW}_w > 0.176$ . This process is done  $K = 50000$  times. The percentage of rejection of  $H_0$  is an estimation of the power of the test for this alternative. For instance, we see in Table 4.4 that the power of the  $\widehat{EW}_w$  test for simulated  $\mathcal{LN}(0, 0.8)$  samples and  $n = 20$  is estimated at 29.8%. These percentages are given in Tables A.42 to A.53 of the appendix, in order to assess the power of the likelihood based tests within each family of generalized Weibull distributions.

Table 4.4: Power results for the tests based on the Exponentiated Weibull distribution,  $n = 20$

altern.	$\widehat{EW}_w$	$\widehat{EW}_s$	$\widehat{EW}_l$	$\check{EW}_w$	$\check{EW}_s$	$\check{EW}_l$	$\check{EW}_w$	$\check{EW}_s$	$\check{EW}_l$	mean
exp(1)	5	5	4.9	5	4.9	5	5	5	5	5
$\mathcal{W}(1, 0.5)$	5.3	5.3	5.3	5.2	5.1	5.1	5	4.9	5	5.1
$\mathcal{W}(1, 3)$	5	5	5	5.1	5	5	5	5	5	5
$\mathcal{G}(3, 1)$	9.7	7.2	8	5.1	5.9	5.7	5	6	5.6	6.5
$\mathcal{AW}1$	49.9	53.7	52.5	46.4	44.5	45.3	48.4	46.5	47.2	48.3
$\mathcal{EW}1$	21.5	16.6	18.2	10.9	12.9	12.3	10.4	12.5	11.7	14.1
$\mathcal{G}(0.5, 1)$	8.6	10.8	10	9.2	8.5	8.8	8.7	8	8.2	9
$\mathcal{AW}2$	79.8	84.1	82.7	32.7	30.3	31.3	41.4	38.6	39.5	51.2
$\mathcal{EW}2$	13.7	18.2	16.6	4.6	4	4.2	4.3	3.7	3.9	8.2
$\mathcal{EW}3$	13.5	18	16.5	4.6	4	4.2	4.3	3.7	3.9	8.1
$\mathcal{GG}1$	29.2	34.8	33	18	16.5	17.1	17.5	16	16.5	22.1
$\mathcal{GG}2$	21.4	26	24.5	15.9	14.6	15.2	15	13.5	13.9	17.8
$\mathcal{PGW}1$	11.2	14.1	13.1	11	10.1	10.5	10.2	9.4	9.6	11
$\mathcal{LN}(0, 0.8)$	29.8	23.8	25.8	16.5	19.3	18.5	15	18	16.9	20.4
$\mathcal{IG}(3, 1)$	56.2	49.9	52.3	44.9	48.9	48.9	36.5	41.3	39.9	46.5
$\mathcal{EW}4$	15.7	11.9	13.2	7.6	9	8.6	7.2	8.9	8.3	10.1
$\mathcal{GG}3$	16.9	12.6	14.1	8.3	9.9	9.4	8	9.7	9.1	10.9
$\mathcal{PGW}2$	28.7	22.7	24.2	16.7	19.3	18.5	16.1	19	18	20.4
mean	27.1	27	27	16.8	17.2	17.2	16.5	17	16.8	20.3

Table 4.5: Power results for the tests based on the Exponentiated Weibull distribution,  $n = 50$ 

altern.	$\widehat{EW}_w$	$\widehat{EW}_s$	$\widehat{EW}_l$	$\widetilde{EW}_w$	$\widetilde{EW}_s$	$\widetilde{EW}_l$	$\check{EW}_w$	$\check{EW}_s$	$\check{EW}_l$	mean
exp(1)	5	5.1	5.1	5	5	5	5.1	5.1	5.1	5
$\mathcal{W}(1, 0.5)$	4.9	5	5	5.1	5.2	5.1	4.9	4.9	4.9	5
$\mathcal{W}(1, 3)$	5	5	5	5.1	5.1	5.1	5.1	5	5	5
$\mathcal{G}(3, 1)$	20	17	18.1	11.6	12.9	12.4	9.9	11.3	10.7	13.8
$\mathcal{AW}1$	81.8	83.4	83	80.2	79.2	79.4	81	80.1	80.4	80.9
$\mathcal{EW}1$	53	48.5	50.2	23.7	25.8	25	31.4	34.4	33.3	36.2
$\mathcal{G}(0.5, 1)$	14.6	17.4	16.6	11.7	10.9	11	11.9	11.1	11.3	12.9
$\mathcal{AW}2$	99.7	99.8	99.8	55.4	53.4	53.8	70.9	68.5	69.3	74.5
$\mathcal{EW}2$	41	46.9	45.2	1.9	1.6	1.6	2.3	1.9	2	16
$\mathcal{EW}3$	40.6	46.6	44.9	1.8	1.5	1.6	2.3	1.9	2	15.9
$\mathcal{GG}1$	69.5	73.6	72.4	29.9	28.3	28.7	31.1	29.3	29.8	43.6
$\mathcal{GG}2$	51.5	56.4	55	24.9	23.4	23.7	25.2	23.7	24.1	34.2
$\mathcal{PGW}1$	23.9	27.7	26.6	14.9	13.9	14.2	14.9	13.9	14.1	18.2
$\mathcal{LN}(0, 0.8)$	68.5	64.3	65.9	56.6	59.4	59.3	49.2	52.7	51.3	58.6
$\mathcal{IG}(3, 1)$	94.6	93.2	93.8	95	95.7	95.5	88	89.7	89.1	92.7
$\mathcal{EW}4$	38.3	33.8	35.7	23.2	25.4	24.6	20	22.4	21.5	27.2
$\mathcal{GG}3$	41.2	36.9	38.6	27.4	29.8	28.9	22.9	25.6	24.6	30.7
$\mathcal{PGW}2$	66.5	61.9	63.5	53.8	56.3	55.4	48	51.2	49.8	56.3
mean	53.6	53.8	53.9	34.1	34.5	34.4	33.9	34.5	34.2	40.8

For comparison purpose, the tables give also two additional results:

- The last row gives the mean of the rejection percentages of each test for all simulated alternative distributions, except the Weibull ones. This allows to identify the best tests for a broad range of alternatives.
- The last column gives the mean of the rejection percentages of all tests for each simulated alternative distribution. This allows to identify the simulated distributions for which the Weibull assumption is rejected easily or with difficulty.

#### 4.4.2 Results and discussion

The most striking result of the analysis of these tables is that the performance of the tests is strongly linked to the shape of the hazard rate of the simulated distribution. More precisely, we see that the same kind of behavior appears for, on one hand the IHR and UBT alternatives, and on the other hand the DHR and BT alternatives. This link is not surprising since a UBT hazard rate starts by increasing and a BT hazard rate starts by decreasing.

Another important remark is that many of these tests appear to be biased: for some alternatives, their power is smaller than the significance level 5%. This fact was yet noticed in [121] for the Mann-Scheuer-Fertig test. In fact, many tests which are very powerful for IHR-UBT alternatives are biased for DHR-BT alternatives and vice versa.

We can also notice that the powers of Weibull GOF tests are significantly lower than the ones of the Exponential GOF tests.

In the following, we compare the performance of the GOF tests within each Generalized Weibull family.

- *Tests based on the Exponentiated Weibull distribution (tables 4.4 and 4.5).* These tests have globally good performance and none of them is biased. The tests  $\widehat{EW}$  based on the MLE are the most powerful.  $\widehat{EW}_s$  is slightly the best for the DHR-BT alternatives and  $\widehat{EW}_w$  is slightly the best in the IHR-UBT case.
- *Tests based on the Generalized Gamma distribution (tables A.42 and A.43).* Many of these tests are biased. The three  $\widehat{GG}^1$  tests have similar good performance. For DHR-BT alternatives,  $\widehat{GG}_s^2$  is very good, except for  $\mathcal{AW}2$  samples. In the UBT case,  $\widehat{GG}_w^2$  is the best. In the IHR case,  $\widehat{GG}_s^2$  is a good choice.  $\widehat{GG}_l^2$  is globally the best test.
- *Tests based on the Additive Weibull distribution (tables A.44 and A.45).* The performances of all tests are very bad for the DHR-BT alternatives. However the three versions of the  $\widehat{AW}$  tests have good power in the UBT case.  $\widehat{AW}_w$  is better than the others for  $\mathcal{AW}1$  samples.
- *Tests based on the Burr Generalized Weibull distribution (tables A.46 and A.47).* The comments are similar to the previous ones. These tests are highly biased for the DHR-BT alternatives. They have globally the worst performance of all families of distributions, but their power is satisfactory for some IHR and UBT alternatives.
- *Tests based on the Marshall-Olkin distribution (tables A.48 and A.49).* Only a few tests are biased.  $\widehat{MO}_w$  is the most powerful for the DHR-BT alternatives and  $\widehat{MO}_s$  is the best in the UBT case. Globally, the best test is  $\widehat{MO}_w$ , it is close to the best tests and never gives very bad results.
- *Tests based on the Modified Weibull distribution (tables A.50 and A.51).* Many tests are biased, but only for IHR-UBT alternatives. In this case,  $\widehat{MW}_s$  is the best.  $\widehat{MW}_w$  has a particularly good power for DHR-BT alternatives. Globally,  $\widehat{MW}_s$  can be recommended.
- *Tests based on the Power Generalized Weibull distribution (tables A.52 and A.53).* Half of the tests are biased.  $\widehat{PGW}_s$  has the best performance for DHR-BT alternatives and is not biased.  $\widehat{PGW}_w$  is the best in the IHR-UBT case, except for  $\mathcal{AW}1$  samples, but it is biased. Globally,  $\widehat{PGW}_l$  and  $\widehat{PGW}_w$  are the most powerful tests.

From this analysis, we can derive the following conclusions.

- Among the 3 methods of estimation, the maximum likelihood provides generally more powerful tests than the least squares and moment methods.
- Among the 3 likelihood-based statistics, the Wald statistic gives generally better results than the score and likelihood-ratio.

- Among the 7 generalized Weibull distributions, the best average results are obtained for the Power Generalized Weibull, Marshall-Olkin and Generalized Gamma distributions.
- The tests with the best global performance are  $\widehat{GG}_l^2$ ,  $\widetilde{MO}_w$  and the families of  $\widehat{PGW}$ ,  $\widehat{GG}^1$  and  $\widehat{EW}$ .
- The best test for DHR-BT alternatives is  $\widehat{MW}_w$ , but it is biased. Among the unbiased tests,  $\widehat{PGW}_s$  is the best.
- The best tests for UBT alternatives are  $P\check{G}W_w$  and  $\widetilde{AW}_s$ . Both are biased. Among the unbiased tests,  $\widetilde{MW}_s$  is the best.
- For IHR alternatives, the results are not so clear.  $P\check{G}W_w$  is the best test for Gamma and  $\mathcal{EW}$  alternatives, but is biased for  $\mathcal{AW}1$  samples. Then,  $\widetilde{GG}_s^2$ ,  $\widetilde{AW}_s$ ,  $\widetilde{MW}_s$  and  $\widetilde{PGW}_s$  are good choices for general IHR alternatives. Note that these tests are all based on least squares estimators and the score statistics.  $\widehat{GG}_l^2$  has also good power in this case.
- A test based on a given  $\mathcal{GW}$  family is not more powerful than other tests for data simulated according to this particular distribution.
- The power of the tests is very poor for Gamma samples. It means that it is difficult to discriminate the Weibull and Gamma distributions, which is not surprising. The Weibull assumption is easily rejected for  $\mathcal{AW}$  and  $\mathcal{IG}$  samples. The power is intermediate for the other distributions.

## 4.5 Asymptotic properties of some test statistics

General results on asymptotic properties of the previous tests are not available. Practically, they are exact and can be used for small samples: there is no need to use the asymptotic distribution of these statistics since the exact quantiles can be found by Monte-Carlo simulations. Furthermore, Monte-Carlo simulations of the quantiles show that the convergence to the asymptotic distribution is quite long as shown in table 4.3.

However it is interesting from a theoretical point of view to study the asymptotic properties of GOF test statistics. In this section we study the particular case when the Weibull distribution is nested in the Generalized Gamma distribution. The Delta method is used to prove the convergence of the Wald, score and likelihood ratio test statistics  $\widehat{GG}_w^1$ ,  $\widehat{GG}_s^1$  and  $\widehat{GG}_l^1$  to a scaled chi-squared distribution.

### 4.5.1 Asymptotic properties of $\widehat{GG}_s^1$ and $\widehat{GG}_w^1$

**Property 4.1** *Under  $H_0$ , the statistics  $\widehat{GG}_s^1$  and  $\widehat{GG}_w^1$  converge asymptotically to weighted  $\chi_1^2$  distributions:*

$$\frac{\widehat{GG}_s^1}{1 - 6/\pi^2 - 36/\pi^4} \xrightarrow[n \rightarrow \infty]{d} \chi_1^2 \quad (4.86)$$

$$\frac{\widehat{GG}_w^1}{1 - 6/\pi^2 - 36/\pi^4} \xrightarrow[n \rightarrow \infty]{d} \chi_1^2. \quad (4.87)$$

**Proof:**

For the demonstration, we will use the following Delta method [122].

**Property 4.2** [Delta method]

Suppose  $n^b(\theta_n - \theta_0) \xrightarrow[n \rightarrow \infty]{d} Y$  where  $\theta_n$  and  $Y$  are  $k$ -random vectors,  $\theta_0$  is a non-random  $k$ -vector,  $b > 0$ . Suppose  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^m$  is a differentiable function in a neighborhood of  $\theta_0$  and  $\nabla\phi(\theta_0)$  exists and is different from  $0_k$ . Then,

$$n^b(\phi(\theta_n) - \phi(\theta_0)) \xrightarrow[n \rightarrow \infty]{d} \nabla\phi(\theta_0)Y. \quad (4.88)$$

A special case of the property 4.2 is given when  $k = 3$ ,  $b = 1/2$  and  $Y$  is from a Normal distribution  $N(0, V)$ :

$$\sqrt{n}(\phi(\theta_n) - \phi(\theta_0)) \xrightarrow[n \rightarrow \infty]{d} N(0, \nabla\phi(\theta_0)V {}^t\nabla\phi(\theta_0)). \quad (4.89)$$

The two statistics  $\widehat{GG}_s^1$  and  $\widehat{GG}_w^1$  can be expressed thanks to two differentiable functions  $\phi_s$  and  $\phi_w$  of the vector  $\theta_n$ , where:

$$\theta_n = \begin{pmatrix} \hat{\eta}_n \\ \hat{\beta}_n \\ \frac{1}{n} \sum_{i=1}^n \ln X_i \end{pmatrix}. \quad (4.90)$$

Indeed:

$$\widehat{GG}_s^1 = \frac{6}{\pi^2} (\sqrt{n}\phi_s(\theta_n))^2 \quad (4.91)$$

where

$$\phi_s : \mathbb{R}_+^* \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto x_2 x_3 - x_2 \ln x_1 + \gamma_E \quad (4.92)$$

and

$$\widehat{GG}_w^1 = \frac{\pi^2}{6} (\sqrt{n}\phi_w(\theta_n))^2 \quad (4.93)$$

where

$$\phi_w : \mathbb{R}_+^* \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \varphi^{-1}(x_2 x_3 - x_2 \ln x_1) - 1. \quad (4.94)$$

**Property 4.3** *The random vector  $\theta_n$  is asymptotically a Gaussian vector and we have the following asymptotic result:*

$$\sqrt{n}(\theta_n - \theta) \xrightarrow[n \rightarrow \infty]{d} N(0, V) \quad (4.95)$$

where

$$V = \left[ \begin{array}{c|c} I_{\eta, \beta}^{-1} & \begin{matrix} v_{1,3} \\ v_{2,3} \end{matrix} \\ \hline \begin{matrix} v_{1,3} & v_{2,3} \end{matrix} & v_3 \end{array} \right] \quad (4.96)$$

where  $I_{\eta, \beta}^{-1}$  is the Fisher information matrix of  $\mathcal{W}(\eta, \beta)$ :

$$I_{\eta, \beta}^{-1} = \left[ \begin{array}{cc} (\eta/\beta)^2(1 + \frac{6}{\pi^2}(1 - \gamma_E)^2) & \frac{6\eta}{\pi^2}(1 - \gamma_E) \\ \frac{6\eta}{\pi^2}(1 - \gamma_E) & \frac{6\beta^2}{\pi^2} \end{array} \right] \quad (4.97)$$

and

$$\begin{cases} v_{1,3} = (\eta/\beta^2)(1 + \frac{6}{\pi^2}(1 - \gamma_E)) \\ v_{2,3} = \frac{6}{\pi^2} \\ v_3 = \frac{\pi^2}{6\beta^2} \end{cases}$$

$$\theta = \begin{pmatrix} \eta \\ \beta \\ \ln \eta - \frac{\gamma_E}{\beta} \end{pmatrix}. \quad (4.98)$$

**Proof:**

The maximum likelihood estimators  $\hat{\eta}_n$  and  $\hat{\beta}_n$  verify asymptotically the condition (Theorem 5.39, page 65 [122]):

$$\sqrt{n} \begin{pmatrix} \hat{\eta}_n - \eta \\ \hat{\beta}_n - \beta \end{pmatrix} = \frac{1}{\sqrt{n}} I_{\eta, \beta}^{-1} \sum_{i=1}^n \begin{pmatrix} \frac{\partial \ln f}{\partial \eta}(X_i; \eta, \beta) \\ \frac{\partial \ln f}{\partial \beta}(X_i; \eta, \beta) \end{pmatrix} + o_p(1). \quad (4.99)$$

We know that

$$\frac{\partial \ln f}{\partial \eta}(X_i; \eta, \beta) = \frac{\beta}{\eta} \left( \left( \frac{X_i}{\eta} \right)^\beta - 1 \right) \quad (4.100)$$

and

$$\frac{\partial \ln f}{\partial \beta}(X_i; \eta, \beta) = \frac{1}{\beta} \left( 1 + \ln \left( \left( \frac{X_i}{\eta} \right)^\beta \right) \left( 1 - \left( \frac{X_i}{\eta} \right)^\beta \right) \right). \quad (4.101)$$

Consequently:

$$\sqrt{n}(\theta_n - \theta) = \sqrt{n} \left[ \begin{array}{c|c} I_{\eta, \beta}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array} \right] \frac{1}{n} \sum_{i=1}^n A_i + o_p(1) \quad (4.102)$$

where

$$A_i = \begin{pmatrix} \frac{\beta}{\eta} (\exp(Y_i) - 1) \\ \frac{1}{\beta} (1 + Y_i(1 - \exp(Y_i))) \\ \frac{1}{\beta} (Y_i + \gamma_E) \end{pmatrix} \quad (4.103)$$

and  $\exp(Y_i) = \left(\frac{X_i}{\eta}\right)^\beta$ ,  $1 \leq i \leq n$  follows a standard Exponential distribution and the  $Y_i$ ,  $1 \leq i \leq n$  follows  $\mathcal{EV}(0, 1)$  (see section 3.1). Then,

$$\begin{aligned}\mathbb{E}[\exp(Y_i)] &= 1 \\ \mathbb{E}[Y_i] &= \gamma_E.\end{aligned}\tag{4.104}$$

Moreover,

$$\begin{aligned}\mathbb{E}[Y_i(1 - \exp(Y_i))] &= \mathbb{E}[Y_i] - \mathbb{E}[Y_i \exp(Y_i)] \\ &= -\gamma_E - \int_0^{+\infty} x \ln x \exp(-x) dx \\ &= -\gamma_E - \Gamma'(2) \\ &= -\gamma_E - (1 - \gamma_E) \\ &= -1.\end{aligned}\tag{4.105}$$

Then from (4.104) and (4.105), we have  $\mathbb{E}[A_i] = 0_3$ .

Moreover using the Central limit theorem we have the asymptotic result:

**Lemma 1**

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (A_i - 0_3) \xrightarrow[n \rightarrow \infty]{d} N(0_3, \Sigma)\tag{4.106}$$

where

$$\Sigma = \begin{bmatrix} \left(\frac{\beta}{\eta}\right)^2 & \frac{\gamma_E - 1}{\eta} & \frac{1}{\eta} \\ \frac{\gamma_E - 1}{\eta} & \frac{\pi^2}{6\beta^2} \left(1 + \frac{6}{\pi^2} (1 - \gamma_E)^2\right) & \frac{\gamma_E}{\beta^2} \\ \frac{1}{\eta} & \frac{\gamma_E}{\beta^2} & \frac{\pi^2}{6\beta^2} \end{bmatrix}.\tag{4.107}$$

**Proof:**

The results are found from the following equations, using some properties of the Gamma function. For a fixed index  $i \in \{1, \dots, n\}$ :

$$\begin{cases} \text{Var}(Y_i) & = \pi^2/6 \\ \text{Cov}(\exp(Y_i), Y_i) & = 1 \\ \text{Cov}(Y_i(1 - \exp(Y_i)), Y_i) & = \gamma_E \\ \text{Var}(Y_i(1 - \exp(Y_i))) & = \pi^2/6 + (1 - \gamma_E)^2 \\ \text{Cov}(Y_i(1 - \exp(Y_i)), \exp(Y_i)) & = \gamma_E - 1 \end{cases}\tag{4.108}$$

Indeed,

$$\begin{aligned}\text{Cov}(\exp(Y_i), Y_i) &= \mathbb{E}[\exp(Y_i)Y_i] - \mathbb{E}[Y_i]\mathbb{E}[\exp(Y_i)] \\ &= \int_0^{+\infty} x \ln x \exp(-x) dx - (-\gamma_E) \times 1 \\ &= \Gamma'(2) + \gamma_E \\ &= 1\end{aligned}\tag{4.109}$$

and

$$\begin{aligned}
\mathbb{E}[Y_i^2(1 - \exp(Y_i))] &= \mathbb{E}[Y_i^2] - \mathbb{E}[Y_i^2 \exp(Y_i)] \\
&= \frac{\pi^2}{6} + \gamma_E^2 - \int_0^{+\infty} \ln^2(y)y \exp(-y) dy \\
&= \frac{\pi^2}{6} + \gamma_E^2 - \Gamma''(2) \\
&= 2\gamma_E.
\end{aligned} \tag{4.110}$$

Using equations (4.109) and (4.110), we have:

$$\begin{aligned}
\text{Cov}(Y_i(1 - \exp(Y_i)), Y_i) &= \mathbb{E}[Y_i^2(1 - \exp(Y_i))] - \mathbb{E}[Y_i(1 - \exp(Y_i))]\mathbb{E}[Y_i] \\
&= 2\gamma_E - (-1) \times (-\gamma_E) \\
&= \gamma_E
\end{aligned} \tag{4.111}$$

$$\begin{aligned}
\text{Cov}(Y_i(1 - \exp(Y_i)), \exp(Y_i)) &= \mathbb{E}[Y_i \exp(Y_i)(1 - \exp(Y_i))] - \mathbb{E}[Y_i(1 - \exp(Y_i))]\mathbb{E}[\exp(Y_i)] \\
&= \int_0^{+\infty} x \ln x(1 - x) \exp(-x) dx - \mathbb{E}[Y_i(1 - \exp(Y_i))]\mathbb{E}[\exp(Y_i)]
\end{aligned}$$

From equation (4.105) and the properties of the Gamma function, we obtain:

$$\begin{aligned}
\text{Cov}(Y_i(1 - \exp(Y_i)), \exp(Y_i)) &= \Gamma'(2) - \Gamma'(3) + 1 \\
&= 1 - \gamma_E - (3 - 2\gamma_E) + 1 \\
&= \gamma_E - 1.
\end{aligned}$$

By definition and using equation (4.109), we have:

$$\begin{aligned}
\text{Var}(Y_i(1 - \exp(Y_i))) &= \mathbb{E}[Y_i^2(1 - \exp(Y_i))^2] - \mathbb{E}[Y_i(1 - \exp(Y_i))]^2 \\
&= \int_0^{+\infty} \ln^2(x)(1 - x)^2 \exp(-x) dx - \mathbb{E}[Y_i(1 - \exp(Y_i))]^2 \\
&= \Gamma''(1) - 2\Gamma''(2) + \Gamma''(3) - 1 \\
&= (1 - \gamma_E)^2 + \frac{\pi^2}{6}.
\end{aligned} \tag{4.112}$$

Thus, we can deduce from results in (4.102) and (4.106) the asymptotic property of  $\theta_n$ : ■

$$\sqrt{n}(\theta_n - \theta) \xrightarrow[n \rightarrow \infty]{d} N(0_3, V) \tag{4.113}$$

$$V = \left[ \begin{array}{c|c} I_{\eta, \beta}^{-1} & 0 \\ \hline 0 & 1 \end{array} \right] \Sigma \left[ \begin{array}{c|c} I_{\eta, \beta}^{-1} & 0 \\ \hline 0 & 1 \end{array} \right]. \tag{4.114}$$

After computation, the value of  $V$  is:

$$V = \left[ \begin{array}{ccc} \left(\frac{\eta}{\beta}\right)^2 \left(1 + \frac{6}{\pi^2}(1 - \gamma_E)^2\right) & \left(\frac{6\eta}{\pi^2}\right)(1 - \gamma_E) & \left(\frac{\eta}{\beta^2}\right) \left(1 + \frac{6}{\pi^2}(1 - \gamma_E)\right) \\ \left(\frac{6\eta}{\pi^2}\right)(1 - \gamma_E) & \frac{6\beta^2}{\pi^2} & \frac{6}{\pi^2} \\ \left(\frac{\eta}{\beta^2}\right) \left(1 + \frac{6}{\pi^2}(1 - \gamma_E)\right) & \frac{6}{\pi^2} & \frac{\pi^2}{6\beta^2} \end{array} \right]. \tag{4.115}$$

As expected the first  $2 \times 2$  blocks of  $V$  is  $I_{\eta, \beta}^{-1}$ . ■

Now, we apply the Delta method to  $\theta_n$ ,  $\phi_s$  and  $\phi_w$ :

$$\sqrt{n}(\phi_s(\theta_n) - \overbrace{\phi_s(\theta)}^{=0}) \xrightarrow[n \rightarrow \infty]{d} N\left(0, \nabla \phi_s(\theta) V {}^t \nabla \phi_s(\theta)\right) \quad (4.116)$$

$$\sqrt{n}(\phi_w(\theta_n) - \overbrace{\phi_w(\theta)}^{=0}) \xrightarrow[n \rightarrow \infty]{d} N\left(0, \nabla \phi_w(\theta) V {}^t \nabla \phi_w(\theta)\right). \quad (4.117)$$

Indeed, we use  $\gamma_E = -\varphi(1)$  to obtain:

$$\phi_s(\theta) = \beta \left( \ln \eta - \frac{\gamma_E}{\beta} \right) - \beta \ln \eta + \gamma_E = 0$$

$$\begin{aligned} \phi_w(\theta) &= \varphi^{-1} \left( \beta \left( \ln \eta - \frac{\gamma_E}{\beta} \right) - \beta \ln \eta \right) \\ &= \varphi^{-1}(-\gamma_E) - 1 \\ &= \varphi^{-1}(\varphi(1)) - 1 = 0 \end{aligned}$$

We know that  $\gamma_E = -\varphi(1)$  and  $(\varphi^{-1})'(x) = \frac{1}{\varphi'(\varphi^{-1}(x))}$ . So we have:

$$\begin{aligned} \nabla \phi_s(\theta) &= -\left( \frac{\beta}{\eta}, \frac{\gamma_E}{\beta}, -\beta \right) \\ \nabla \phi_w(\theta) &= -\left( \frac{\beta}{\eta} (\varphi^{-1})'(-\gamma_E), \frac{\gamma_E}{\beta} (\varphi^{-1})'(-\gamma_E), -\beta (\varphi^{-1})'(-\gamma_E) \right) \\ &= \frac{1}{\varphi'(\varphi^{-1}(\varphi(1)))} \nabla \phi_s(\theta) \\ &= \frac{1}{\varphi'(1)} \nabla \phi_s(\theta) \\ &= \frac{6}{\pi^2} \nabla \phi_s(\theta). \end{aligned} \quad (4.118)$$

Using equations (4.91) and (4.93), we have:

$$\begin{aligned} \sqrt{\widehat{GG}_s^1} \xrightarrow[n \rightarrow \infty]{d} N\left(0, (6/\pi^2) \nabla \phi_s(\theta) V {}^t \nabla \phi_s(\theta)\right) \\ \sqrt{\widehat{GG}_w^1} \xrightarrow[n \rightarrow \infty]{d} N\left(0, (6/\pi^2) \nabla \phi_s(\theta) V {}^t \nabla \phi_s(\theta)\right). \end{aligned} \quad (4.119)$$

Furthermore:

$$\nabla \phi_s(\theta) V {}^t \nabla \phi_s(\theta) = \frac{\pi^2}{6} - \frac{6}{\pi^2} - 1. \quad (4.120)$$

Consequently, we have the convergence:

$$\begin{aligned} \sqrt{\widehat{GG}_s^1} \xrightarrow[n \rightarrow \infty]{d} N\left(0, 1 - 6/\pi^2 - 36/\pi^4\right) \\ \sqrt{\widehat{GG}_w^1} \xrightarrow[n \rightarrow \infty]{d} N\left(0, 1 - 6/\pi^2 - 36/\pi^4\right). \end{aligned} \quad (4.121)$$

Finally, we have the following convergence of the two statistics  $\widehat{GG}_s^1$  and  $\widehat{GG}_w^1$ :

$$\frac{\widehat{GG}_s^1}{1 - 6/\pi^2 - 36/\pi^4} \xrightarrow[n \rightarrow \infty]{d} \chi_1^2 \quad (4.122)$$

$$\frac{\widehat{GG}_w^1}{1 - 6/\pi^2 - 36/\pi^4} \xrightarrow[n \rightarrow \infty]{d} \chi_1^2. \quad (4.123)$$

■

We notice that the asymptotic distributions of  $\widehat{GG}_s^1$  and  $\widehat{GG}_w^1$  are very far from the  $\chi_1^2$  distribution as  $1 - 6/\pi^2 - 36/\pi^4 = 0.022$ . So it is wrong to use the  $\chi_1^2$  distribution to apply the tests. Tables 4.6 and 4.7 give the quantiles for several values of  $n$  of  $\frac{\widehat{GG}_s^1}{1 - 6/\pi^2 - 36/\pi^4}$  and  $\frac{\widehat{GG}_w^1}{1 - 6/\pi^2 - 36/\pi^4}$ , under  $H_0$ . Quantiles for finite  $n$  are obtained by Monte-Carlo simulations. We observe that, for small  $n$ , these distributions are far from the  $\chi_1^2$  distribution. So it is important to be able to use the quantiles found by simulation instead of the asymptotic quantiles especially for small  $n$ .

Table 4.6: Quantiles of  $\frac{\widehat{GG}_s^1}{1 - 6/\pi^2 - 36/\pi^4}$  under  $H_0$

$n$	90%	92.5%	95%	97.5%	99%
10	1.843	2.169	2.651	3.522	4.703
20	2.207	2.608	3.207	4.265	5.650
50	2.466	2.886	3.523	4.657	6.244
100	2.583	3.035	3.712	4.874	6.435
1000	2.668	3.131	3.805	5.021	6.627
$\chi_1^2$	2.705	3.170	3.841	5.023	6.634

Table 4.7: Quantiles of  $\frac{\widehat{GG}_w^1}{1 - 6/\pi^2 - 36/\pi^4}$  under  $H_0$

$n$	90%	92.5%	95%	97.5%	99%
10	1.870	2.212	2.728	3.681	5.136
20	2.221	2.633	3.257	4.356	5.865
50	2.474	2.896	3.534	4.693	6.332
100	2.589	3.044	3.720	4.901	6.4913
1000	2.666	3.134	3.810	5.025	6.614
$\chi_1^2$	2.705	3.170	3.841	5.023	6.634

### 4.5.2 Asymptotic property of $\widehat{GG}_l^1$

**Property 4.4** Under  $H_0$ , the test statistic  $\widehat{GG}_l^1$  converges asymptotically to a weighted  $\chi_1^2$  distribution:

$$\frac{\widehat{GG}_l^1}{1 - 6/\pi^2 - 36/\pi^4} \xrightarrow[n \rightarrow \infty]{d} \chi_1^2. \quad (4.124)$$

**Proof:**

The test statistic  $\widehat{GG}_l^1$  can be expressed as a value of a differentiable function  $\phi_l$  computed in the vector  $\theta_n$  defined in (4.90).

$$\widehat{GG}_l^1 = -2n \phi_l(\theta_n) \quad (4.125)$$

where

$$\phi_l : \mathbb{R}_+^* \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \ln \left( \Gamma(\varphi^{-1}(x_2(x_3 - \ln x_1))) \right) - (\varphi^{-1}(x_2(x_3 - \ln x_1)) - 1)(x_2(x_3 - \ln x_1)). \quad (4.126)$$

Since  $\varphi^{-1}(-\gamma_E) = 1$ ,  $\nabla \phi_l(\theta) = (\varphi^{-1}(-\gamma_E) - 1)(\frac{\beta}{\eta}, \frac{\gamma_E}{\beta}, -\beta) = 0_3$ , the first-order Delta method in property 4.2 can not be applied. That is why we use the second-order Delta method [122]:

**Property 4.5** [Second-order Delta method]

Suppose  $\phi$  is two-times differentiable in a neighborhood of  $\theta_0$  and  $\nabla \phi(\theta_0) = 0_k$ .

Then  $n^b(\theta_n - \theta_0) \xrightarrow[n \rightarrow \infty]{d} Y$  implies that

$$n^{2b}(\phi(\theta_n) - \phi(\theta_0)) \xrightarrow[n \rightarrow \infty]{d} \frac{1}{2} {}^t Y H_{\theta_0} Y \quad (4.127)$$

where  $H_{\theta_0}$  is Hessian matrix of  $\phi$  computed in  $\theta_0$ .

We compute the Hessian matrix  $H$  of  $\phi_l$  in  $\theta$ . The computation is simplified thanks to the equation  $(\varphi^{-1})'(-\gamma_E) = \frac{1}{\varphi'(1)} = \frac{6}{\pi^2}$ :

$$H = -\frac{6}{\pi^2} \begin{bmatrix} \left(\frac{\beta}{\eta}\right)^2 & \frac{\gamma_E}{\eta} & -\frac{\beta^2}{\eta} \\ \frac{\gamma_E}{\eta} & \left(\frac{\gamma_E}{\beta}\right)^2 & -\gamma_E \\ -\frac{\beta^2}{\eta} & -\gamma_E & \beta^2 \end{bmatrix}. \quad (4.128)$$

Since  $\nabla \phi_l(\theta) = 0_3$  and the convergence in (4.95):

$$\sqrt{n}(\theta_n - \theta) \xrightarrow[n \rightarrow \infty]{d} N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \sim N(0, V). \quad (4.129)$$

We can apply the second-order Delta method ( $b = \frac{1}{2}$ ):

$$n(\phi_l(\theta_n) - \overbrace{\phi_l(\theta)}^{=0}) \xrightarrow[n \rightarrow \infty]{d} \frac{1}{2} {}^tNHN. \quad (4.130)$$

Indeed,

$$\phi_l(\theta) = \ln \left( \Gamma(\varphi^{-1}(-\gamma_E)) \right) + \gamma_E(\varphi^{-1}(-\gamma_E) - 1) = 0. \quad (4.131)$$

Finally,

$$\widehat{GG}_l^1 \xrightarrow[n \rightarrow \infty]{d} - {}^tNHN. \quad (4.132)$$

The matrix  $H$  is symmetric, then it is orthogonally diagonalizable. We have to solve:

$$\det|H - \lambda I_3| = 0, \quad \lambda \in \mathbb{R},$$

We evaluate the determinant by expanding along the first row. After simplification, we have:

$$\begin{aligned} \det|H - \lambda I_3| &= \det \begin{vmatrix} -6\left(\frac{\beta}{\eta\pi}\right)^2 - \lambda & \frac{6\gamma_E}{\eta\pi^2} & \frac{6\beta^2}{\eta\pi^2} \\ -\frac{6\gamma_E}{\eta\pi^2} & -6\left(\frac{\gamma_E}{\beta\pi}\right)^2 - \lambda & \frac{6\gamma_E}{\pi^2} \\ \frac{6\beta^2}{\eta\pi^2} & \frac{6\gamma_E}{\pi^2} & -\frac{6\beta^2}{\pi^2} - \lambda \end{vmatrix} \\ &= \lambda^2 \left( \frac{\pi^2}{6} \left( \left(\frac{\beta}{\eta}\right)^2 + \left(\frac{\gamma_E}{\beta}\right)^2 + \beta^2 \right) + \lambda \right). \end{aligned}$$

The zeros of the previous equation are the two eigenvalues, the first one is:  $\lambda_0 = -\frac{6}{\pi^2} \left( \left(\frac{\beta}{\eta}\right)^2 + \left(\frac{\gamma_E}{\beta}\right)^2 + \beta^2 \right)$  with the corresponding eigenvector  ${}^tP = (p_1 \ p_2 \ p_3) = \left( \frac{\beta}{\eta} \ \frac{\gamma_E}{\beta} \ -\beta \right)$ . Indeed, we can easily verify that:

$$HP = \lambda_0 P.$$

The second eigenvalue is equal to 0 with order 2.

The quadratic form  ${}^tNHN$  is, then, simplified using the diagonalization to:

$${}^tNHN = \lambda_0 \left( \sum_{i=1}^3 N_i p_i \right)^2 / \sum_1^3 p_i^2 \quad (4.133)$$

$$\begin{aligned} {}^tNHN &= -\frac{6}{\pi^2} \left( \left(\frac{\beta}{\eta}\right) N_1 + \left(\frac{\gamma_E}{\beta}\right) N_2 - \beta N_3 \right)^2 \\ &= -\frac{6}{\pi^2} ({}^tPN)^2. \end{aligned} \quad (4.134)$$

From equation (4.129) and since  ${}^tPVP = \frac{\pi^2}{6} - \frac{6}{\pi^2} - 1$ ,

$${}^tPN \xrightarrow[n \rightarrow \infty]{d} N \left( 0, \frac{\pi^2}{6} - \frac{6}{\pi^2} - 1 \right). \quad (4.135)$$

Finally from equations (4.132), (4.134) and (4.135) and after normalization, we obtain that:

$$\frac{\widehat{GG}_l^1}{1 - 6/\pi^2 - 36/\pi^4} \xrightarrow[n \rightarrow \infty]{d} \chi_1^2. \quad (4.136)$$

■

We notice that the test statistic  $\widehat{GG}_l^1$  has asymptotically the same distribution as  $\widehat{GG}_s^1$  and  $\widehat{GG}_w^1$ . As we mentioned before, this distribution is far from the  $\chi_1^2$  distribution because  $1 - 6/\pi^2 - 36/\pi^4 = 0.022$ . Table 4.8 gives the quantiles for several values of  $n$  of  $\frac{\widehat{GG}_l^1}{1 - 6/\pi^2 - 36/\pi^4}$ , under  $H_0$ . We notice the convergence of these quantiles to those of the  $\chi_1^2$  distribution. But for small  $n$ , we have to use the quantiles found by simulation instead of the asymptotic quantiles.

Table 4.8: Simulated and asymptotic quantiles of  $\frac{\widehat{GG}_l^1}{1 - 6/\pi^2 - 36/\pi^4}$  under  $H_0$

$n$	90%	92.5%	95%	97.5%	99%
10	1.841	2.187	2.694	3.611	5.059
20	2.211	2.594	3.192	4.279	5.764
50	2.458	2.892	3.495	4.629	6.252
100	2.590	3.027	3.692	4.856	6.449
1000	2.685	3.138	3.805	5.014	6.689
$\chi_1^2$	2.705	3.170	3.841	5.023	6.634

---

This chapter introduced new likelihood based GOF tests for the Weibull distribution. Three estimation methods were used to get rid of the Weibull parameter: maximum likelihood, least squares and moment methods. The proposed tests are exact.

A comprehensive comparison study is presented. It compares 54 likelihood based GOF tests and recommends those with the best performances. Theoretical asymptotic results are derived when the Weibull distribution is nested in the generalized Gamma distribution.

The main part of this chapter has been presented in a paper to appear in *Communications in Statistics - Simulation and Computation* [71]. The asymptotic results of section 4.5 have been accepted for presentation in the MIMAR 2014 conference [68].

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# Chapter 5

## GOF tests for the Weibull distribution based on the Laplace transform

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The aim of this chapter is to present new GOF tests for the Weibull distribution based on the Laplace transform. These tests merge the ideas of Cabaña and Quiroz [22] and those introduced by Henze [53] for testing the Exponential distribution. We also introduce new versions of the two statistics of Cabaña and Quiroz using the maximum likelihood estimators instead of the moment estimators. The convergence of the distribution of one of these statistics to the chi-squared distribution is established. The proposed tests are not asymptotic and can be applied to small samples. Finally a comprehensive comparison study is carried out.

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### 5.1 Reminder of previous works

Henze proposed GOF tests for the Exponential distribution based on the Laplace transform previously presented in subsection 2.2.8. The building of the test is based on the measure of the difference between the empirical Laplace transform and its theoretical version.

Henze proposed to compare the Laplace transform of the standard ( $\lambda = 1$ ) Exponential distribution,  $\frac{1}{1+t}$ , to the empirical Laplace transform of the sample  $(\hat{Y}_i)_{1,\dots,n}$  (here  $\hat{Y}_i = \frac{X_i}{\bar{X}_n}$ ),  $\psi_n(t) = \frac{1}{n} \sum_{i=1}^n \exp(-t\hat{Y}_i)$ . He built the following statistic (previously given in (2.26)):

$$He_{n,a} = n \int_0^{+\infty} \left[ \psi_n(t) - \frac{1}{1+t} \right]^2 w(t; a) dt \quad (5.1)$$

where  $w(t; a) = \exp(-at)$  is a weight function and  $a$  is a parameter to be chosen. The

integrals defining  $He_{n,a}$  can be computed and expressed as explicit functions of the  $\hat{Y}_i$  (see (2.27)).

The work of Cabaña and Quiroz [22], previously presented in subsection 3.2.7, uses the Laplace transform to build GOF tests for the Weibull and type I extreme value distributions. We remind that the distribution of  $\check{Y}_i = \check{\beta}_n \ln \frac{X_i}{\check{\eta}}$  is close to  $\mathcal{EV}_1(0, 1)$  distribution and the Laplace transform of a sample  $Y_1, \dots, Y_n$  from the  $\mathcal{EV}_1(0, 1)$  distribution is:

$$\psi(t) = \Gamma(1 - t), \forall t < 1.$$

The tests are based on the closeness between the empirical Laplace transform  $\psi_n(t) = \frac{1}{n} \sum_{i=1}^n \exp(-t\check{Y}_i)$  and the theoretical Laplace transform  $\psi(t)$ . This closeness is measured by the empirical moment generating process  $\check{v}_n(s)$ :

$$\check{v}_n(s) = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^n e^{-\check{Y}_j s} - \Gamma(1 - s) \right). \tag{5.2}$$

Cabaña and Quiroz proved the convergence, under  $H_0$ , of  $\check{v}_n(s)$ , to a zero mean, continuous Gaussian process  $\check{G}_p(s)$  for  $s \in [-\delta', \eta']$ ,  $\delta' > 0, \eta' < 0.5$ . They suggested two test statistics that are functions of the stochastic process  $\check{v}_n$  given in previous equations (3.41) and (3.42):

$$\check{C}Q_n = \check{v}_{n,S} V^{-1}(S) {}^t \check{v}_{n,S} \tag{5.3}$$

$$\check{S}_n = \int_J \check{v}_n^2(s) / V(s) ds. \tag{5.4}$$

In the following we combine both approaches, the one of Henze based on the weighted  $L^2$  norm and the one of Cabaña and Quiroz based on the difference between the empirical Laplace transform of the transformed data  $Y_i, i \in \{1, \dots, n\}$ , and the Laplace transform of the  $\mathcal{EV}_1(0, 1)$  distribution.

## 5.2 A new test combining the approaches of Henze and Cabaña-Quiroz

Combining both approaches of Henze in (5.1) and of Cabaña and Quiroz in (5.4), we propose a test statistic of the following form:

$$n \int_I \left( \frac{1}{n} \sum_{j=1}^n e^{-Y_j t} - \Gamma(1 - t) \right)^2 w_a(t) dt = \int_I v_n^2(t) w_a(t) dt \tag{5.5}$$

where  $w_a$  is a weight function and  $I \subset ] - \infty, 1[$  is a bounded interval for which the above integral is convergent. The function  $w_a$  depends on a parameter  $a$  that can be chosen to obtain the best performance of the test as in Henze's work [53].

Henze chose  $w_a(t) = e^{-at}$ . This choice was justified by the fact of using a test of Cramer-Von-Mises type which gives an explicit expression of the statistics and a good

power for different alternatives by adjusting the value of  $a$ . It is common in Cramer-Von-Mises and Anderson-Darling tests (statistics defined in equations (3.23) and (3.24)) to use as a weight function the probability density function tested. Thus, we use as a weight function the probability density function of the  $\mathcal{EV}_1(0, 1)$  after dilatation with parameter  $a$ ,  $w_a(t) = e^{at-e^{at}}$ .

For the Exponential distribution, it was possible to find an explicit and simple expression of Henze's statistic as a function of the sample  $Y_j$  (see (2.27)). But, for the Weibull distribution, the integral (5.5) is not easy to compute since  $\Gamma(1 - t)$  is more complex than  $\frac{1}{1 + t}$ . We can compute the integral using Simpson or Monte Carlo integration or we can simply compare the theoretical Laplace transform and the empirical one by discretizing the integral on an appropriately chosen interval  $I$ . For instance, with a discretization on  $[0, 1[$ , we obtain the following test statistic:

$$LT_{a,m} = \sum_{k=1}^{m-1} v_n^2(k/m)w_a(k/m) = n \sum_{k=1}^{m-1} \left[ \frac{1}{n} \sum_{j=1}^n e^{-Y_j k/m} - \Gamma(1 - k/m) \right]^2 w_a(k/m). \tag{5.6}$$

The statistic  $LT_{a,m}$  can be written as a quadratic form, as the first statistic of Cabaña and Quiroz:

$$LT_{a,m} = v_{n,m} W_a {}^t v_{n,m} \tag{5.7}$$

where  $v_{n,m} = (v_n(\frac{1}{m}), \dots, v_n(\frac{m-1}{m}))$  and  $W_a = \begin{bmatrix} w_a(\frac{1}{m}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_a(\frac{m-1}{m}) \end{bmatrix}$  is a diagonal weight matrix.

Equation (5.5) is similar to (5.4) and (5.7) is similar to (5.3): thus these tests are linked to those of Cabaña and Quiroz. But they are much simpler since they do not require the computation of the covariance matrix  $V(s)$ .

We tried different range values of  $t$  by discretizing the intervals  $[-50, 1[$ ,  $[-10, 1[$ ,  $[-1, 1[$ ,  $[0, 1[$ ,  $[-1, 0]$ ,  $[-10, 0]$  and  $[-50, 0]$ . We used normalizing factors in some cases in order to have usual orders of magnitude of the statistics. The power results are similar for the statistics based on the discretizations of  $[-50, 1[$ ,  $[-10, 1[$ ,  $[-1, 1[$  and  $[0, 1[$ . Similarly the statistics based on discretizing  $[-1, 0]$ ,  $[-10, 0]$  and  $[-50, 0]$  have a comparable performance. That is why we use only the discretizations of  $[0, 1[$  and  $[-1, 0]$ . The two corresponding statistics are respectively denoted  $LT^1$  and  $LT^2$ :

$$LT_{a,m}^1 = n \sum_{k=1}^{m-1} \left[ \frac{1}{n} \sum_{j=1}^n e^{-Y_j k/m} - \Gamma(1 - k/m) \right]^2 w_a(k/m) \tag{5.8}$$

$$LT_{a,m}^2 = n \sum_{k=-m}^{-1} \left[ \frac{1}{n} \sum_{j=1}^n e^{-Y_j k/m} - \Gamma(1 - k/m) \right]^2 w_a(k/m). \tag{5.9}$$

For a comparison purpose, let  $LT_{a,m}^3$  be a third test statistic based on the discretization of the interval  $[-2.5, 0.49]$  recommended by Cabaña and Quiroz [22]:

$$LT_{a,m}^3 = n \sum_{k=-2.5m}^{0.49m} \left[ \frac{1}{n} \sum_{j=1}^n e^{-Y_j k/m} - \Gamma(1 - k/m) \right]^2 w_a(k/m). \tag{5.10}$$

Each of the statistics (5.8), (5.9), (5.10) can be computed using  $\hat{Y}_1, \dots, \hat{Y}_n$  or  $\check{Y}_1, \dots, \check{Y}_n$  or  $\tilde{Y}_1, \dots, \tilde{Y}_n$  instead of  $Y_1, \dots, Y_n$ . The corresponding statistics are denoted respectively  $\widehat{LT}^i$ ,  $\widetilde{LT}^i$  and  $\check{LT}^i$ ,  $i \in \{1, 2, 3\}$ .

Using the moment estimators, we can conclude from the convergence result of  $\check{\nu}_n(s)$ ,  $s \in J$  [22], and the continuous mapping theorem, that  $\check{LT}^i$ ,  $i \in \{1, 2, 3\}$ , converges under the null hypothesis  $H_0$ , to the distribution of:

$$\sum_{s \in I^i(m)} \check{G}_p^2(s) w_a(s)$$

where  $I^1(m) = \{\frac{1}{m}, \dots, \frac{m-1}{m}\}$ ,  $I^2(m) = \{-1, \frac{-m+1}{m}, \dots, \frac{-1}{m}\}$  and  $I^3(m) = \{-2.5, \frac{-2.5m+1}{m}, \dots, 0.49\}$ .

We have the same asymptotic convergence of the statistics  $\widehat{LT}^i$  to  $\sum_{s \in I^i(m)} \widehat{G}_p^2(s) w_a(s)$ , where  $\widehat{G}_p(s)$  is a zero mean continuous Gaussian process with a specific covariance matrix that will be derived later in section 5.3. Indeed, theorem 2.1 in [22] can be applied to the empirical process  $\hat{\nu}_n$  using MLEs instead of the MEs.

The behavior of the test statistics depends on the choice of the parameter value  $a$  of the weight function. It is impossible to find a value of parameter  $a$  that maximizes the power of the GOF tests whatever the tested alternative. Indeed the behavior of the tests depends in theory on the alternative tested and the sample size. After several simulations with different values of  $a$ , we recommend the use of  $a = -5$  for both  $\widehat{LT}_{a,m}^1$  and  $\widehat{LT}_{a,m}^2$ . We will use this value for the remaining test statistics.

Concerning the choice of parameter  $m$ , it was set in all the simulations to  $m = 100$ . However  $m = 100$  is not in all the cases the optimal value that gives the best performance. For instance, we studied the Monte Carlo estimation of the power of the test  $\widehat{LT}_{-5,m}^1$  for a sample simulated from the Gamma distribution with parameters (1, 2). Figure 5.1 shows that the optimal value is  $m = 70$  in this case. But this satisfactory value could have been different if we had simulated another distribution. Choosing a large value of  $m$  guarantees satisfying results in a large range of cases.

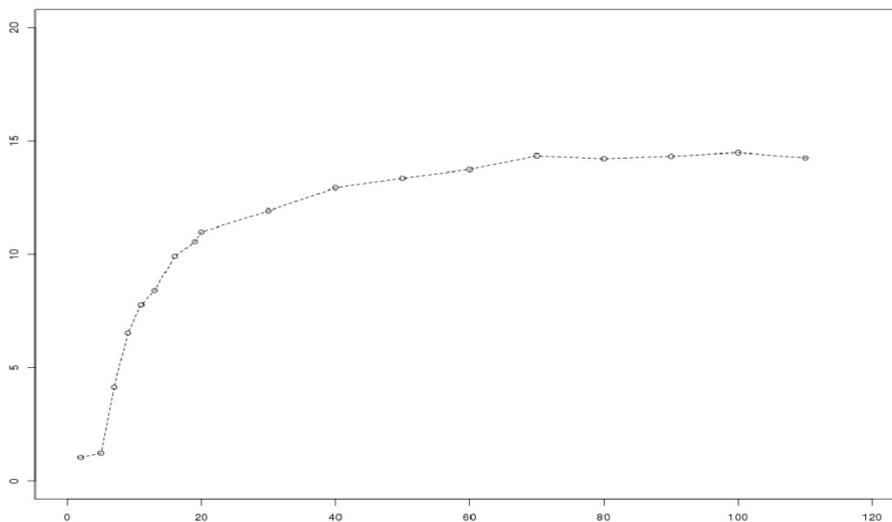


Figure 5.1: The power of the test  $\widehat{LT}_{-5,m}^1$  as a function of  $m$

Given the expression of the new GOF test statistics as the distance between the theoretical and the empirical Laplace transforms, the null hypothesis  $H_0$  is rejected when the statistics are too large. The Weibull assumption is rejected at the level  $\alpha$  if the statistics are greater than the quantile of order  $1 - \alpha$  of its distribution under  $H_0$ . These quantiles can be easily obtained by simulation.

### 5.3 Cabaña and Quiroz statistics with Maximum Likelihood Estimators

The results of Cabaña and Quiroz are valid for affine invariant estimators of  $\mu = \ln \eta$  and  $\sigma = \frac{1}{\beta}$  which are satisfying a condition denoted (2.6) in [22]. Cabaña and Quiroz showed that this condition is fulfilled by the moment estimators, and obtained the test statistics  $\check{C}Q_n$  and  $\check{S}_n$ .

In this section, we prove that the MLEs verify condition (2.6) in [22]. So we are able to build the corresponding test statistics  $\widehat{C}Q_n$  and  $\widehat{S}_n$ . This condition is given in equation (5.11) of the following theorem 5.1 (theorem 2.1 of [22]):

**Theorem 5.1** *Suppose that the parameters  $\mu$  and  $\sigma$  are estimated by an affine pair of estimators  $\hat{\mu}$  and  $\hat{\sigma}$ , such that linearly independent functions  $K_1$  and  $K_2$  exist in  $\mathcal{L}^2(P)$  satisfying:*

$$\|\sqrt{n} \ ^t(\hat{\mu}, \hat{\sigma} - 1) - A \frac{1}{\sqrt{n}} \sum_{i \leq n} \ ^t(K_1(Y_i), K_2(Y_i))\| = o_p(1) \tag{5.11}$$

where  $A$  is a non-singular,  $2 \times 2$  matrix. Then, under the null hypothesis,  $\hat{v}_n$ , as a stochastic process indexed in  $J$ , converges in distribution to a zero-mean, sample continuous Gaussian process  $G_p(s)$ , with covariance structure given by:

$$\begin{aligned} \mathbb{E}[G_p(u)G_p(v)] &= \Gamma(1 - u - v) - \Gamma(1 - u)\Gamma(1 - v) \\ &+ \nabla(v)A \ ^t\mathbb{E}[K_1(Y) \exp(uY), K_2(Y) \exp(uY)] \\ &+ \nabla(u)A \ ^t\mathbb{E}[K_1(Y) \exp(vY), K_2(Y) \exp(vY)] \\ &+ \nabla(u)ACov(K_1(Y), K_2(Y)) \ ^tA \ ^t\nabla(v) \end{aligned}$$

where  $Y$  is a variable with the  $\mathcal{E}V_1(0, 1)$  distribution and  $\nabla(u) = u(-\Gamma(1 - u), \Gamma'(1 - u))$ .

We know that the MLEs verify asymptotically the following property (theorem 5.39, page 65 [122]):

$$\sqrt{n} \ ^t(\hat{\mu}_n, \hat{\sigma}_n - 1) = \frac{1}{\sqrt{n}} I_{(\mu=0, \sigma=1)}^{-1} \sum_{i=1}^n \ ^t\left(\frac{\partial \ln g}{\partial \mu}(Y_i, \mu = 0, \sigma = 1), \frac{\partial \ln g}{\partial \sigma}(Y_i, \mu = 0, \sigma = 1)\right) + o_p(1) \tag{5.12}$$

where  $I^{-1}$  is the inverse of the Fisher information matrix of the  $\mathcal{E}V_1(0, 1)$  distribution which can be derived as:

$$I_{(\mu=0, \sigma=1)}^{-1} = \begin{vmatrix} 1 + \frac{6}{\pi^2}(1 - \gamma_E)^2 & \frac{6}{\pi^2}(\gamma_E - 1) \\ \frac{6}{\pi^2}(\gamma_E - 1) & \frac{6}{\pi^2} \end{vmatrix}.$$

Property (5.12) becomes:

$$\sqrt{n} \quad {}^t(\hat{\mu}_n, \hat{\sigma}_n - 1) = \frac{1}{\sqrt{n}} I_{(\mu=0, \sigma=1)}^{-1} \sum_{i=1}^n {}^t(-1 + e^{Y_i}, -1 - Y_i + Y_i e^{Y_i}) + o_p(1). \quad (5.13)$$

The two functions  $K_1(y) = -1 + e^y$  and  $K_2(y) = -1 - y + ye^y$  are linearly independent. Then, condition (5.11) is fulfilled for the MLEs and we can apply theorem 5.1. Under the null hypothesis  $H_0$ ,  $\hat{v}_n(s) = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^n e^{-s\hat{Y}_j} - \Gamma(1-s) \right)$ , as a stochastic process indexed on  $J$ , converges in distribution to a zero mean, sample continuous Gaussian process  $\widehat{G}_p(s)$  with covariance structure given by:

$$\begin{aligned} \mathbb{E}[\widehat{G}_p(v)\widehat{G}_p(u)] &= \Gamma(1-u-v) - \Gamma(1-u)\Gamma(1-v) \\ &\quad + \nabla(v)I^{-1} \quad {}^t\mathbb{E} [(-1 + e^Y)e^{-uY}, (-1 - Y + Ye^Y)e^{-uY}] \\ &\quad + \nabla(u)I^{-1} \quad {}^t\mathbb{E} [(-1 + e^Y)e^{-vY}, (-1 - Y + Ye^Y)e^{-vY}] \\ &\quad + \nabla(u)I^{-1}Cov(-1 + e^Y, -1 - Y + Ye^Y) \quad {}^tI^{-1} \quad {}^t\nabla(v) \end{aligned}$$

where  $\nabla(u) = u(-\Gamma(1-u), \Gamma'(1-u))$  and  $Y$  is a variable with the  $\mathcal{EV}_1(0, 1)$  distribution.

After computation, the limiting covariance structure is as follows:

$$\begin{aligned} \mathbb{E}[\widehat{G}_p(v)\widehat{G}_p(u)] &= \Gamma(1-u-v) - \Gamma(1-u)\Gamma(1-v) \\ &\quad + \nabla(v)I^{-1} \begin{pmatrix} \Gamma(2-u) - \Gamma(1-u) \\ -\Gamma(1-u) - \Gamma'(1-u) + \Gamma'(2-u) \end{pmatrix} \\ &\quad + \nabla(u)I^{-1} \begin{pmatrix} \Gamma(2-v) - \Gamma(1-v) \\ -\Gamma(1-v) - \Gamma'(1-v) + \Gamma'(2-v) \end{pmatrix} \\ &\quad + \nabla(u) \quad {}^tI^{-1} \quad {}^t\nabla(v). \end{aligned}$$

We use the following results similar to those presented in section 4.5:

$$\left\{ \begin{array}{l} \mathbb{E}[Ye^{-vY}] = \Gamma'(1-v) \\ \mathbb{E}[Y^2e^{-uY}] = \Gamma''(1-u) \\ \mathbb{E}[(-1 + e^Y)e^{-vY}] = \Gamma(2-v) - \Gamma(1-v) \\ \mathbb{E}[(-1 - Y + Ye^Y)e^{-uY}] = -\Gamma(1-u) - \Gamma'(1-u) + \Gamma'(2-u) \\ \text{Var}(-1 + e^Y) = 1 \\ \text{Var}(-1 - Y + Ye^Y) = \frac{\pi^2}{6} + (\gamma_E - 1)^2 \\ \text{Cov}(-1 + e^Y, -1 - Y + Ye^Y) = 1 - \gamma_E. \end{array} \right. \quad (5.14)$$

Hence, we can define new versions of the Cabaña and Quiroz statistics based on the MLEs instead of MEs:

$$\widehat{CQ}_n = \widehat{v}_{n,S} \widehat{V}^{-1}(S) \quad {}^t\widehat{v}_{n,S} \quad (5.15)$$

$$\hat{S}_n = \int_J \hat{v}_n^2(s) / \widehat{V}(s) ds \tag{5.16}$$

where  $\widehat{v}_{n,S} = (\hat{v}_n(s_1), \dots, \hat{v}_n(s_k))$ ,  $S = \{s_1, \dots, s_k\} \subset J$  and  $\widehat{V}(S)$  is the limiting covariance matrix of  $\widehat{v}_{n,S}$  given above. Statistic  $\widehat{CQ}_n$  has a limiting chi-squared distribution with  $k$  degrees of freedom. Figure 5.2 shows that the limiting variance of  $\hat{v}_n$  grows very fast when  $s$  goes to  $-\infty$  and the same when  $s$  approaches 0.5. In this case, we recommend that the interval  $J$  should be included in  $[-1.5, 0.49]$ . In the simulations presented in section 5.4, we will use the test  $\widehat{CQ}_n$  with the following values:  $k = 2$ ,  $s_1 = -0.1$  and  $s_2 = 0.02$ .

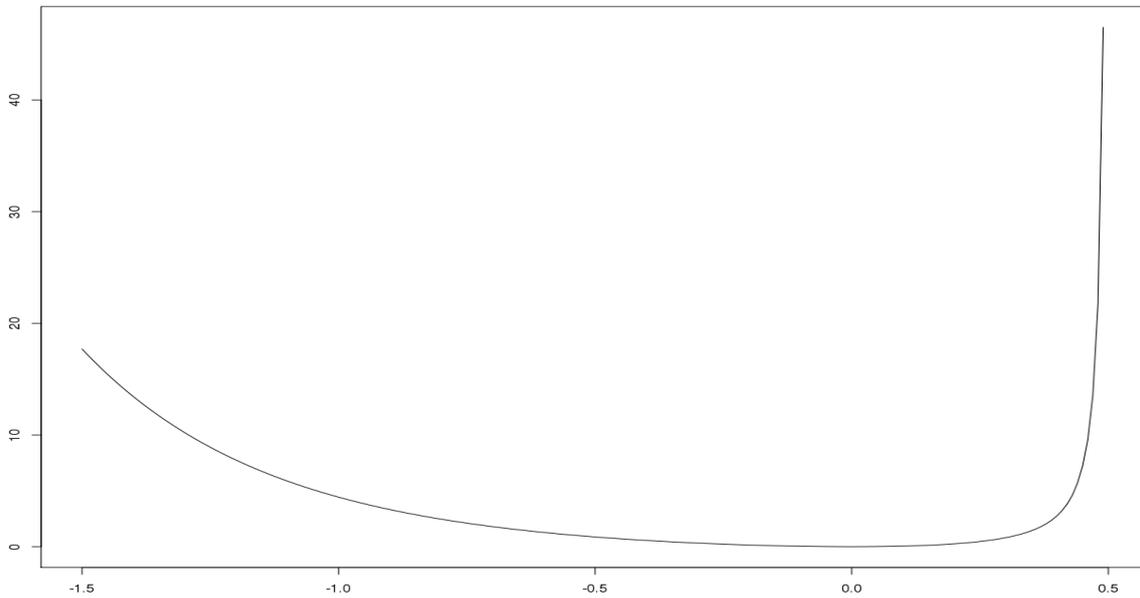


Figure 5.2: The asymptotic variance of  $\hat{v}_n(s)$  as a function of  $s$

Since the test statistics can be used for small values of the sample sizes, the asymptotic results are not often relevant in practice. That is why we had the idea of using a different version of the test statistic  $\widehat{CQ}_n$  that we denote  $\widehat{CQ}_n^*$ , whose expression is given by using any non singular matrix  $A$ :

$$\widehat{CQ}_n^* = \widehat{v}_{n,S} A^{-1} {}^t \widehat{v}_{n,S}. \tag{5.17}$$

In this case, we have no more convergence of the test statistic distribution to a chi-squared distribution, but this is not important since we use simulated quantiles for a given sample size. Nevertheless we still have the property that the distribution of  $\widehat{CQ}_n^*$  is independent of the parameters of the Weibull distribution under  $H_0$ . In the simulations in the next section, we will use the test  $\widehat{CQ}_n^*$  where  $k = 2$ ,  $S = \{-0.1, 0.02\}$  and we fix the following matrix, found after several simulations:  $A = \begin{bmatrix} 1.59 & 0.91 \\ 0.91 & 0.53 \end{bmatrix}$ .

## 5.4 Simulation and comparison

The previous section has proposed new GOF tests for the Weibull distribution. As in chapter 4, it is then important to select the best of them and compare them with the best GOF tests of the literature. This section presents the results of an intensive Monte Carlo simulation study in order to assess the power of the new GOF tests.

The study is done using a broad range of alternative distributions. We have four classes depending on the shape of the hazard rates (IHR, DHR, BT and UBT).

As previously, for each distribution, we simulate 50,000 samples of size  $n \in \{10, 20, 50, 100\}$ . All the GOF tests are applied with a significance level set to 5%. The tests reject the Weibull hypothesis when the statistic is greater than the quantile of order 95% of its distribution under  $H_0$ . These quantiles are obtained by simulation, thus the asymptotic results are not used in this case.

The power of the tests is assessed by the percentage of rejection of the null hypothesis. The algorithms have been written in R and are included in the package EWGoF that we have developed.

We first simulate Weibull samples, in order to check that the percentage of rejection is close to the nominal significance level 5%. For the other simulations, we have chosen the following distributions:

- Gamma  $\mathcal{G}$
- Lognormal  $\mathcal{LN}$
- Inverse-Gamma  $\mathcal{IG}$
- Generalized Weibull distributions (see table 4.1):
  - Exponentiated Weibull distribution  $\mathcal{EW}(\theta, \eta, \beta)$
  - Generalized Gamma distribution  $\mathcal{GG}(k, \eta, \beta)$
  - Additive Weibull distribution  $\mathcal{AW}(\xi, \eta, \beta)$ .

As before, for the sake of simplicity, the scale parameters of the Weibull, Gamma and Inverse-Gamma distributions are set to 1 and the mean of the Lognormal distribution is set to 0. The choice of the parameters of the simulated distributions is done in order to obtain different shapes of the hazard rate. Table 5.1 gives the values of the parameters and the notation used for all the simulated distributions.

Table 5.1: Simulated distributions

Weibull	$exp(1)$	$\mathcal{W}(1, 0.5) \equiv \mathcal{W}(0.5)$	$\mathcal{W}(1, 3) \equiv \mathcal{W}(3)$
IHR	$\mathcal{G}(2, 1) \equiv \mathcal{G}(2)$	$\mathcal{G}(3, 1) \equiv \mathcal{G}(3)$	$\mathcal{AW}1 \equiv \mathcal{AW}(10, 0.02, 5.2)$
DHR	$\mathcal{G}(0.2, 1) \equiv \mathcal{G}(0.2)$	$\mathcal{AW}2 \equiv \mathcal{AW}(2, 20, 0.1)$	$\mathcal{EW}1 \equiv \mathcal{EW}(0.1, 0.01, 0.95)$
BT	$\mathcal{EW}2 \equiv \mathcal{EW}(0.1, 100, 5)$	$\mathcal{GG}1 \equiv \mathcal{GG}(0.1, 1, 4)$	$\mathcal{GG}2 \equiv \mathcal{GG}(0.2, 1, 3)$
UBT	$\mathcal{LN}(0, 0.8) \equiv \mathcal{LN}(0.8)$	$\mathcal{LN}(0, 2.4) \equiv \mathcal{LN}(2.4)$	$\mathcal{LN}(0, 3) \equiv \mathcal{LN}(3)$
	$\mathcal{IG}(3, 1) \equiv \mathcal{IG}(3)$	$\mathcal{GG}3 \equiv \mathcal{GG}(10, 0.01, 0.2)$	

We remind the values of the parameters used for the new test statistics:

- For  $LT^i$ ,  $i \in \{1, 2, 3\}$ :  $m = 100$  and  $a = -5$
- For  $\check{C}Q$ :  $k = 2$ ,  $S = \{-1, 0.4\}$
- For  $\widehat{C}Q$ :  $k = 2$ ,  $S = \{-0.1, 0.02\}$
- For  $\widehat{C}Q^*$ :  $k = 2$ ,  $S = \{-0.1, 0.02\}$  and  $A = \begin{bmatrix} 1.59 & 0.91 \\ 0.91 & 0.53 \end{bmatrix}$ .

For the power study, the percentage of rejection of  $H_0$  is an estimation of the power of the test for this alternative. For instance, we see in table 5.4 that the power of the  $\widehat{LT}^1$  test for simulated  $\mathcal{LN}(0, 0.8)$  samples and  $n = 20$  is estimated at 37.1%.

In the following tables, we assess the powers of the new GOF statistics  $LT^i$ ,  $i \in \{1, 2, 3\}$ , with the three estimation methods and the new version of Cabaña and Quiroz test  $\widehat{C}Q$ .

We compare the performance of these new GOF tests to the one suggested by Cabaña and Quiroz  $\check{C}Q$  defined in (3.41). The last rows of tables 5.2, 5.3, 5.4 and 5.5 give the mean of rejection percentages of each test for all simulated alternative distributions, except the Weibull ones. This allows to identify the best tests for a broad range of alternatives.

Table 5.2: Power results for the tests based on Laplace transform,  $n = 100$

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	$\check{C}Q$	$\widehat{C}Q$	$\widehat{C}Q^*$
$exp(1)$	5.1	5.1	4.9	5	5	5.3	5.1	5	5.1	5.1	5.1	5
$\mathcal{W}(0.5)$	5.1	5.1	4.9	4.9	4.9	5.1	5	5	4.9	4.9	5	4.8
$\mathcal{W}(3)$	5.3	5.1	5.1	5.1	5	5.2	5.1	5.1	4.9	5	5.1	5.1
$\mathcal{G}(2)$	22.1	17.1	2.2	23.2	39.5	6.8	17.9	10.8	9.2	22.8	11.3	19.6
$\mathcal{G}(3)$	38.4	31.7	5.7	27.5	39.5	12.3	28.9	21.6	17.1	40.4	23.9	34.7
$\mathcal{AW}1$	84.9	94.1	97.9	35.9	10.8	91.1	19.6	96	83	96.4	98.3	94.3
$\mathcal{G}(0.2)$	16.3	84.7	61.5	2.5	0.2	22.9	1.4	36.4	17.2	45.5	76.9	87.7
$\mathcal{AW}2$	60.7	100	99.7	6.8	0.3	66.2	2.8	86.1	36.3	98.9	100	100
$\mathcal{EW}1$	0	95.2	14.2	0.2	0.2	0	0.3	0.7	0	7.5	50.4	88.4
$\mathcal{EW}2$	0	95.3	14.4	0.1	0.2	0	0.3	0.7	0	7.7	50.4	88.5
$\mathcal{GG}1$	21.1	96.6	73.9	1.3	0.2	27.5	1.5	44.4	19.4	59.6	89.3	97.5
$\mathcal{GG}2$	16.1	84.4	61.2	2.5	0.2	22.7	1.4	36.8	16.7	45.6	77.2	88.2
$\mathcal{LN}(0.8)$	97.5	89.7	70.5	87.2	97.6	78.8	89.9	92.8	86	97.7	92.3	93.2
$\mathcal{LN}(2.4)$	97.6	89.9	70.9	87.4	97.7	78.5	89.8	92.5	86.2	97.8	92.5	93.1
$\mathcal{LN}(3)$	97.4	89.7	70.5	87.3	97.6	78.7	90	92.7	86.4	97.6	92.3	93.3
$\mathcal{IG}(3)$	100	99.7	99.2	99.9	100	99.9	100	100	100	100	99.9	99.9
$\mathcal{GG}3$	76.3	63.7	27.2	56.5	76.8	37.9	59.1	58.2	46.5	78.5	61.8	69.5
mean	52	80.8	54.9	37.1	40.1	44.5	35.9	55	43.2	64	72.6	82

Table 5.3: Power results for the tests based on Laplace transform,  $n = 50$

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	4.9	5.3	5.1	5	4.9	4.8	4.8	5	4.9	5.1	4.8	5.1
$\mathcal{W}(0.5)$	5	5.2	5	5	5	5	4.9	5	4.9	5.1	5	5
$\mathcal{W}(3)$	5.1	5.2	5	5	4.9	5	4.8	5.2	5	5.1	5	5
$\mathcal{G}(2)$	14.5	11.7	1.4	13.4	15.6	7.6	13.6	6.7	10.2	15.4	4.7	11.6
$\mathcal{G}(3)$	23.2	18.2	1.4	19.8	24.2	11.9	20.2	10.7	15.8	25.1	8.4	18.6
$\mathcal{AW}1$	64.1	75.8	86.2	18.8	3.1	67.8	6	79.6	53.1	74.7	87.4	76.4
$\mathcal{G}(0.2)$	11.6	52.9	40.4	1.3	0.2	14.1	0.4	23.5	9.4	17.7	49.6	57.2
$\mathcal{AW}2$	44.8	99.9	93.1	3.6	0.3	41.7	0.5	61.1	20.3	74	98.3	99.9
$\mathcal{EW}1$	0.1	65.3	11.5	0.2	0.3	0.2	0.3	1.3	0.1	1.6	26.1	53.9
$\mathcal{EW}2$	0.1	65.3	11.2	0.2	0.3	0.2	0.3	1.3	0.1	1.6	26.5	54.4
$\mathcal{GG}1$	15	73.8	51.4	1.4	0.2	16.5	0.3	28.5	10.5	24.2	63.2	75.6
$\mathcal{GG}2$	11.8	53.1	40.9	1.2	0.2	13.9	0.3	23.4	9.3	17.9	49.9	57.1
$\mathcal{LN}(0.8)$	78.8	62.6	19.9	65.8	78.2	55.9	68.2	59.9	66	79.3	53.4	66.3
$\mathcal{LN}(2.4)$	78.7	62.4	20	65.8	78.4	56.3	67.8	60	66.2	79.4	52.7	65.7
$\mathcal{LN}(3)$	78.5	62.1	19.7	66.2	78.6	55.7	67.8	60.1	65.5	79.7	52.8	66
$\mathcal{IG}(3)$	98.6	91.3	66.7	96.7	98.8	93.1	97.5	95.5	97.3	98.3	91.2	93.5
$\mathcal{GG}3$	48.1	37.1	5.4	38.9	48.6	27.2	38.9	28.1	34.9	50.3	23.8	39.4
mean	40.6	59.4	33.5	28.1	30.5	33.1	27.3	38.6	32.8	45.6	49.2	59.7

Table 5.4: Power results for the tests based on Laplace transform,  $n = 20$

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	4.9	4.9	4.9	5	5.1	5	4.9	5	5.1	5	5	5.4
$\mathcal{W}(0.5)$	4.8	4.9	5	5	5.1	5.1	4.9	5	5	4.9	5.2	5.3
$\mathcal{W}(3)$	4.9	5	5	5	5.1	4.9	5	5	4.9	5	5	5.5
$\mathcal{G}(2)$	9	7	1.9	9.5	10.1	8.5	9.3	3.8	9.7	10	2.8	6.6
$\mathcal{G}(3)$	12.1	8.7	1.1	12.7	13.6	11	4.6	0.9	13.3	5	2.4	8.4
$\mathcal{AW}1$	33.4	45.1	56.2	3.1	3.1	27.9	0.7	47.4	12.6	27.5	57.8	49.6
$\mathcal{G}(0.2)$	6.9	22.1	23.5	0.5	0.6	5	0.7	14.8	1.6	3.8	26.5	26
$\mathcal{AW}2$	25.9	87.9	66.7	0.5	0.8	16.4	0	37.9	4.1	19.4	75.4	87.1
$\mathcal{EW}1$	1	22.7	11	0.4	0.7	0.7	0.6	4.2	0.5	0.5	15.1	20.8
$\mathcal{EW}2$	0.9	22.8	10.8	0.4	0.7	0.7	0.6	3.9	0.5	0.4	14.8	21.3
$\mathcal{GG}1$	8.6	32.2	28.9	0.3	0.4	5.7	0.4	17.6	1.6	5	33.3	35.4
$\mathcal{GG}2$	6.8	22.6	23.6	0.5	0.6	5	0.6	14.8	1.6	3.9	26.7	25.9
$\mathcal{LN}(0.8)$	37.1	26.5	1.8	35.3	39.3	27.5	35.8	17.7	38.1	40.7	10.1	27.2
$\mathcal{LN}(2.4)$	37.3	26.6	1.8	35.3	39.7	27.1	35.5	17.7	38.2	40.5	9.7	26.7
$\mathcal{LN}(3)$	37.4	26.7	1.8	35.4	39.3	27	35.3	17.3	37.9	40.4	9.8	26.9
$\mathcal{IG}(3)$	68.9	51.6	10.5	67.7	71.9	45.3	68.6	47	71.4	70.3	31.1	53.3
$\mathcal{GG}3$	21.8	15.5	0.6	21.2	23.3	17	20.9	8.2	22.7	24.1	4.2	15.4
mean	21.9	29.9	17.2	15.9	17.4	16.1	15.3	18.1	18.1	21.4	22.8	30.8

Table 5.5: Power results for the tests based on Laplace transform,  $n = 10$

altern.	$\widehat{LT}^1$	$\widehat{LT}^2$	$\widehat{LT}^3$	$\widetilde{LT}^1$	$\widetilde{LT}^2$	$\widetilde{LT}^3$	$\check{LT}^1$	$\check{LT}^2$	$\check{LT}^3$	$\check{CQ}$	$\widehat{CQ}$	$\widehat{CQ}^*$
$exp(1)$	5.1	5	4.9	4.9	5.1	5.4	5.2	5	4.8	4.9	5.1	5.2
$\mathcal{W}(0.5)$	5.4	5.1	4.8	5	5	5.2	5.1	4.8	4.9	5	5.1	5.1
$\mathcal{W}(3)$	5.1	5	5	5.1	5.2	5	5	4.9	5.1	5.1	5	5.2
$\mathcal{G}(2)$	7.6	5	2.6	7.8	7.9	7.4	7.8	3.1	7.7	7.8	2.7	4.5
$\mathcal{G}(3)$	9.3	5.5	1.8	9.3	9.4	8.2	9.1	2.6	9.3	9.6	1.9	4.7
$\mathcal{AW1}$	15.6	27.4	33.8	1.9	2.6	6.5	1.8	28.8	1.8	11.7	34.6	31.9
$\mathcal{G}(0.2)$	4	13.5	15.4	1.2	1.4	2.2	1.3	12.3	1.2	2.7	16.4	15.2
$\mathcal{AW2}$	14	56	43.3	0.1	0.9	3.6	0.2	29.6	0.1	9.2	47.5	53.9
$\mathcal{EW1}$	1.8	12.1	10.1	0.9	1.1	1.4	1.1	7.1	0.9	1.1	11.2	11.6
$\mathcal{EW2}$	1.9	12.1	10.1	1.1	1.4	1.4	1.2	7.3	1	1.2	11.3	11.8
$\mathcal{GG1}$	4.4	17.6	18.3	0.7	1	1.6	0.9	14.1	0.8	2.7	19.9	19.2
$\mathcal{GG2}$	4.1	13.4	15.5	1.2	1.3	2	1.4	12.3	1.2	2.7	16.3	15.2
$\mathcal{LN}(0.8)$	20.2	11	0.3	19.5	19.8	12.1	19.8	3.5	20.2	21	1	8.9
$\mathcal{LN}(2.4)$	20.4	11	0.3	19.7	20.1	12	19.7	3.4	20.2	20.3	0.9	9.2
$\mathcal{LN}(3)$	20.4	10.8	0.4	19.6	20	12.3	19.7	3.4	20.2	20.5	0.9	9
$\mathcal{IG}(3)$	37.3	21.9	0.2	35.7	37.2	13.8	37.4	9.4	37.8	37.1	0.9	19.2
$\mathcal{GG3}$	13.5	7.3	0.2	13.4	13.5	9.9	13.3	2.4	13.5	13.6	1.2	5.7
mean	12.5	16.1	10.9	9.4	9.8	6.7	9.6	9.9	9.7	11.5	12.1	15.7

### 5.5 Results and discussion

The first obvious result of the analysis of these tables is that, as for the likelihood based tests, the performance of the tests is strongly linked to the shape of the hazard rate of the simulated distribution. We see the same behaviour of the tests that appears for, on one hand the IHR and UBT alternatives and on the other hand the DHR and BT alternatives.

The second important remark is that the new GOF tests are biased for some alternatives except the test  $\widetilde{LT}^2$ ; their power is smaller than the significance level 5%.

The tests based on the LSEs  $\widetilde{LT}^i, i \in \{1, 2\}$ , and one based on the MEs  $\check{LT}^1$  are powerful for IHR-UBT alternatives and biased for DHR-BT alternatives. The tests  $\widehat{LT}^1, \widetilde{LT}^3, \check{LT}^i, i \in \{2, 3\}$  and  $\check{CQ}$  are biased for Exponentiated Weibull distributions ( $\mathcal{EW1}$  and  $\mathcal{EW2}$ ) for large  $n (\geq 20)$ . For small values of the sample size  $n \leq 10$ , the tests  $\widehat{LT}^1, \widetilde{LT}^i, i \in \{1, 2, 3\}, \check{LT}^i, i \in \{2, 3\}$ , and  $\check{CQ}$  are biased for the DHR-BT alternatives and the tests  $\widehat{LT}^3$  and  $\widehat{CQ}$  become biased for IHR-UBT alternatives (except for the alternative  $\mathcal{AW1}$  for  $n = 10$ ).

The two tests  $\widehat{CQ}$  and  $\check{CQ}$  depend on the choice of the values of  $S$ . The test  $\widehat{CQ}_n^*$  depends on both the value of  $S$  and the choice of matrix  $A$ . Thus we do not guarantee the representativeness of the results and we may have better performances for different values than those used for the comparison. The tables comparison shows that the test  $\widehat{CQ}$  is more powerful than  $\check{CQ}$ , but the results can be very different depending on the choice of  $S$ . The test  $\widehat{CQ}^*$  is the most powerful among both  $\check{CQ}$  and  $\widehat{CQ}$ .

The only non biased test for all the sample sizes is the test based on the MLEs  $\widehat{LT}^2$ . The performance of the test statistics is very dependent on the shape of the hazard rate.

The GOF tests have the following behaviour:

- For the IHR alternatives:  $\widetilde{LT}^2$  is powerful except for the alternative  $\mathcal{AW}1$  where the power is very low and biased for  $n \leq 50$ .
- For the DHR-BT alternatives: the new GOF tests based on the LSEs  $\widetilde{LT}^i, i \in \{1, 2\}$ , and the MEs  $\check{LT}^1$  are biased for all the DHR-BT alternatives. The two tests  $\widetilde{LT}^3$  and  $\check{LT}^3$  become biased for small values of  $n \leq 20$  not only for the alternatives  $\mathcal{EW}i, i = 1, 2$ , but for all the remaining DHR-BT alternatives.
- For the UBT alternatives: the three tests  $\widehat{LT}^1, \widetilde{LT}^2$  and  $\check{CQ}$  are very powerful. For  $n \leq 20$ , the test  $\check{CQ}$  becomes powerful for the UBT alternatives and loses the performance it has against DHR-BT alternatives and becomes biased in this case.

For the majority of the studied alternatives, there exists a new GOF test that is significantly powerful but no test is uniformly the best. Globally, the two best tests among all the new GOF tests are  $\widehat{LT}^2$  and  $\widehat{CQ}^*$ .

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This chapter presents new GOF test (11 tests) for the Weibull distribution based on the Laplace transform. The implementation of these tests was done in the package EWGoF.

It also introduces new versions of Cabaña and Quiroz test statistics using the maximum likelihood estimators and proves the convergence of the distribution of one of these statistics to the chi-squared distribution.

This work has been presented in JSFds conference [66] and has been published in the Journal de la Société Française de Statistique [67].

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# Chapter 6

## Comprehensive comparison of the Weibull GOF tests

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The aim of this chapter is to present a comprehensive comparison study of all the GOF tests for the Weibull distribution. We also propose to combine GOF tests with complementary behaviors to build better tests. Recommendations about the most powerful tests are given.

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### 6.1 Simulation framework

The purpose of this section is to present the framework of Monte Carlo simulations in order to assess the performance of all the presented tests and to compare them all. The tests have been implemented in our R package EWGoF.

The study is done using a broad class of alternative distributions. For each distribution, we simulate 50000 samples of size  $n \in \{10, 20, 50\}$ . All the GOF tests are applied with a significance level set to 5%. The power of the tests is assessed by the percentage of rejection of the null hypothesis.

As in previous chapters, first, Weibull samples are simulated in order to check that the percentage of rejection is close to the nominal significance level 5%. For the other simulations, we have chosen a broad range of alternative distributions (Table 6.1): with increasing hazard rate (IHR), decreasing hazard rate (DHR), bathtub hazard rate (BT) and upside-down hazard rate (UBT). We have chosen usual alternatives (Gamma  $\mathcal{G}$ , Lognormal  $\mathcal{LN}$ , Inverse-Gamma  $\mathcal{IG}$ ) and Generalized Weibull distributions  $\mathcal{EW}$ ,  $\mathcal{GG}$  and  $\mathcal{AW}$ . We added some new alternatives in order to have various ones with different hazard rates monotony and that are not Generalized Weibull distributions. These alternatives have never been tested before in the literature, which gives more originality to the comparison study.

- Distribution I of Dhillon [36]  $\mathcal{D1}(\beta, b)$  with the cdf:

$$F(x) = 1 - e^{-(e^{(\beta x)^b} - 1)}, b, \beta > 0, x \geq 0$$

- Distribution II of Dhillon [36]  $\mathcal{D2}(\lambda, b)$  with the cdf:

$$F(x) = 1 - e^{-(\ln(\lambda x + 1))^{b+1}}, \lambda > 0, b \geq 0, x \geq 0$$

- Inverse Gaussian distribution [44]  $\mathcal{IS}(\mu, \lambda)$  with the cdf:

$$F(x) = \Phi((\lambda/x)^{\frac{1}{2}}(1 + x/\mu)) + e^{2\lambda/\mu}\Phi(-(\lambda/x)^{\frac{1}{2}}(1 + x/\mu)), \mu > 0, x \geq 0$$

where  $\Phi$  is the cdf of the standard normal distribution

- Hjorth distribution [58]  $\mathcal{H}(\beta, \delta, \theta)$  with the cdf:

$$F(x) = 1 - \frac{e^{-\delta x^2/2}}{(1 + \beta x)^{\theta/\beta}}, x \geq 0$$

- Chen's distribution [26]  $\mathcal{C}(\lambda, \beta)$  with the cdf:

$$F(x) = 1 - e^{\lambda(1 - e^{x^\beta})}, \lambda, \beta > 0, x > 0.$$

Table 6.1: Simulated distributions

<u>WEIBULL</u>	$\exp(1)$	$\mathcal{W}(0.5) \equiv \mathcal{W}(1, 0.5)$	$\mathcal{W}(3) \equiv \mathcal{W}(1, 3)$
<u>IHR</u>	$\mathcal{G}(2) \equiv \mathcal{G}(2, 1)$ $\mathcal{EW1} \equiv \mathcal{EW}(6.5, 20, 6)$	$\mathcal{G}(3) \equiv \mathcal{G}(3, 1)$ $\mathcal{D2}(2) \equiv \mathcal{D2}(1, 2)$	$\mathcal{AW1} \equiv \mathcal{AW}(10, 0.02, 5.2)$
<u>UBT</u>	$\mathcal{LN}(0.8) \equiv \mathcal{LN}(0, 0.8)$ $\mathcal{IS}(0.25) \equiv \mathcal{IS}(1, 0.25)$	$\mathcal{IG}(3) \equiv \mathcal{IG}(3, 1)$ $\mathcal{IS}(4) \equiv \mathcal{IS}(1, 4)$	$\mathcal{EW4} \equiv \mathcal{EW}(4, 12, 0.6)$
<u>DHR</u>	$\mathcal{G}(0.2) \equiv \mathcal{G}(0.2, 1)$ $\mathcal{H}(0) \equiv \mathcal{H}(0, 1, 1)$	$\mathcal{AW2} \equiv \mathcal{AW}(2, 20, 0.1)$ $\mathcal{D2}(0) \equiv \mathcal{D2}(1, 0)$	$\mathcal{EW2} \equiv \mathcal{EW}(0.1, 0.01, 0.95)$
<u>BT</u>	$\mathcal{EW3} \equiv \mathcal{EW}(0.1, 100, 5)$ $\mathcal{C}(0.4) \equiv \mathcal{C}(2, 0.4)$	$\mathcal{GG1} \equiv \mathcal{GG}(0.1, 1, 4)$ $\mathcal{D1}(0.8) \equiv \mathcal{D1}(1, 0.8)$	$\mathcal{GG2} \equiv \mathcal{GG}(0.2, 1, 3)$

For the sake of simplicity, the scale parameters of the Weibull, Gamma and Inverse-Gamma distribution are set to 1 and the mean of the lognormal distribution is set to 0. The choice of the parameters of the simulated distributions is done to obtain different shapes of the hazard rate. Table 4.2 gives the values of the parameters and the notations used for all the simulated distributions.

We studied the following tests:

- $Z^2$ : test based on the correlation coefficient of Smith and Bain defined in (3.12)
- $R_{EJG}^2$ : test based on the correlation coefficient of Evans-Johnson-Green defined in (3.13)
- $SPP$ : test based on the stabilized probability plot defined in (3.15)
- $SB$ : test of Shapiro and Brain defined in (3.17)
- $OK^*$ : test of Öztürk and Korukoğlu defined in (3.20)

- $KS$ : test of Kolmogorov-Smirnov defined in (3.21)
- $CM$ : test of Cramer-von Mises defined in (3.23)
- $\widehat{AD}$  and  $\widetilde{AD}$ : test of Anderson-Darling defined in (3.24) computed using respectively the MLEs and LSEs as recommended in [76]
- $LS$ : test of Liao and Shimokawa defined in (3.26)
- $MSF$ : one-sided test of Mann-Scheuer-Fertig defined in (3.28). This test should be two-sided, but, for comparison purposes, we use the one-sided version, as recommended by Mann, Scheuer and Fertig.
- $TS$ : test of Tiku-Singh defined in (3.29)
- $LOS$ : test of Lockhart-O'Reilly-Stephens defined in (3.31)
- $\check{S}T_3$  and  $\check{S}T_4$ : generalized smooth tests defined in (3.33) and (3.34)
- $\hat{K}L$  and  $\check{K}L$ : tests based on the Kullback-Leibler information defined in (3.38) computed with respectively the MLEs and MEs
- $\check{C}Q$ ,  $\widehat{C}Q$  and  $\widehat{C}Q^*$ : test of Cabaña and Quiroz and our generalizations defined respectively in (3.41), (5.15) and (5.17)
- $\widehat{LT}^2$ : test based on the Laplace transform defined in (5.9)
- $\widehat{EW}_w$ ,  $\widehat{GG}_l^2$ ,  $\widetilde{MO}_w$  and  $\widehat{PGW}_w$ : likelihood based tests defined respectively in (4.20), (4.39), (4.65) and (4.82).

We remind the values of the parameters used for some test statistics:

- For  $\widehat{LT}^2$ :  $m = 100$  and  $a = -5$
- For  $\check{C}Q$ :  $k = 2$ ,  $S = \{-1, 0.4\}$
- For  $\widehat{C}Q$ :  $k = 2$ ,  $S = \{-0.1, 0.02\}$
- For  $\widehat{C}Q^*$ :  $k = 2$ ,  $S = \{-0.1, 0.02\}$  and  $A = \begin{bmatrix} 1.59 & 0.91 \\ 0.91 & 0.53 \end{bmatrix}$  as in section 5.3.

## 6.2 Combination of GOF tests

Complementary behaviors of some likelihood based GOF tests have been mentioned in section 4.4.2 (see tables A.50 and A.52). For instance, when the statistic  $P\check{G}W_w$  has a very low power against a fixed alternative, the statistic  $\widehat{MW}_w$  has, conversely, very high power against the same alternative and vice versa. Building a GOF test that combines both statistics might help to get rid of the bias and give a global good performance for a large range of the tested alternatives. This approach can be applied to any test statistics with complementary behaviors.

For instance, we combine here the two test statistics  $\widehat{MW}_w$  and  $P\check{G}W_w$ . In order to keep the same order of magnitude, we center each statistic  $\widehat{MW}_w$  and  $P\check{G}W_w$  by its mean value (respectively  $\overline{\widehat{MW}_w}$  and  $\overline{P\check{G}W_w}$ ) and normalize it by its standard deviation (respectively  $sd(\widehat{MW}_w)$  and  $sd(P\check{G}W_w)$ ). These two last quantities are computed using simulations. For instance, in order to compute  $sd(\widehat{MW}_w)$  and  $\overline{\widehat{MW}_w}$  for a fixed sample size  $n$ , we simulate a large number  $K$  of  $X_1^k, \dots, X_n^k, k \in \{1, \dots, K\}$ , iid samples from the  $\exp(1)$  distribution. Indeed, since the distributions of the test statistics under  $H_0$  do not depend on the value of the parameters  $\eta$  and  $\beta$  of the Weibull distribution, the simulation can be carried out using the simplest Weibull distribution, which is  $\exp(1)$ . For each sample, we compute the statistic  $\widehat{MW}_w$  defined previously in equation (4.73). Then, we obtain a sample of size  $K$   $\widehat{MW}_w^1, \dots, \widehat{MW}_w^K$ . Both quantities  $sd(\widehat{MW}_w)$  and  $\overline{\widehat{MW}_w}$  are approximated respectively by the empirical standard deviation and the mean value of the sample  $\widehat{MW}_w^1, \dots, \widehat{MW}_w^K$ .

Different combinations can be used to obtain new GOF tests. We noticed previously in equation (2.51) of subsection 2.3.2 that the sum was used by Brain and Shapiro in [19] as a way of combining two statistics. In our case, the maximum and the sum of the standardized statistics can be used as GOF test statistics:

$$T_1 = \max \left( \frac{1}{sd(\widehat{MW}_w)} |\widehat{MW}_w - \overline{\widehat{MW}_w}|, \frac{1}{sd(P\check{G}W_w)} |P\check{G}W_w - \overline{P\check{G}W_w}| \right) \quad (6.1)$$

$$T_2 = \frac{1}{sd(\widehat{MW}_w)} |\widehat{MW}_w - \overline{\widehat{MW}_w}| + \frac{1}{sd(P\check{G}W_w)} |P\check{G}W_w - \overline{P\check{G}W_w}|. \quad (6.2)$$

The Weibull assumption is rejected for large values of the statistics. The quantiles of the distributions of  $T_1$  and  $T_2$ , under  $H_0$ , are given by Monte-Carlo simulations so the tests can be applied to any sample size. The two test statistics  $T_1$  and  $T_2$  will be added to the compared GOF tests presented in section 6.1.

## 6.3 Results and discussion

Tables 6.2 to 6.7 present the power results of all the compared tests. These powers are significantly lower than the ones of the Exponential GOF tests studied in subsection 2.4.1.

Table 6.2: Power results for the Weibull GOF tests,  $n = 50$

altern.	$Z^2$	$R_{EJG}^2$	$SPP$	$SB$	$OK^*$	$KS$	$CM$	$\widehat{AD}$	$\widetilde{AD}$	$\widetilde{LS}$	$MSF$	$TS$	$LOS$
$exp(0.5)$	4.8	4.9	5	5.1	4.8	5	4.9	4.8	4.9	4.8	5.1	5.3	4.9
$\mathcal{W}(0.5)$	4.8	4.9	5	4.9	5.1	4.9	5	5	5	4.9	5.1	5.1	4.9
$\mathcal{W}(3)$	4.9	5.1	4.9	5	4.8	5.1	5	5	5	5	5.1	5.2	4.9
<u>INCREASING HAZARD RATE</u>													
$\mathcal{G}(2)$	2.3	3.8	10.3	8.2	11.7	7.3	8.4	8.6	11.2	16.2	14.5	11.7	10.7
$\mathcal{G}(3)$	2.3	5.2	15.6	12.3	19.1	10.1	12.3	13.2	17	25.4	21.7	20.1	17.5
$\mathcal{AW}1$	80.1	79.6	65	54.2	80.2	57.2	64.3	71.3	77.9	67.5	0.1	82.1	81.8
$\mathcal{EW}1$	10.6	20.4	42.6	36.8	52.9	25.2	34.7	39.2	46.2	61.2	50.1	55.4	51.9
$\mathcal{D}2(2)$	2.6	4.9	14.8	11.2	18.9	9.7	11.9	12.7	15.5	24	20.2	18.9	16.8
<u>UPSIDE-DOWN BATHTUB HAZARD RATE</u>													
$\mathcal{LN}(0.8)$	21.9	37.5	60.1	55.6	68.9	36.3	50.1	55.8	64.7	78.2	65.3	72.8	70.2
$\mathcal{IG}(3)$	75.7	87.5	94.9	94.7	95.3	76	88.1	91.5	96.1	98.3	93.3	97.2	97.2
$\mathcal{EW}4$	5.1	11.5	29.2	23.5	37.7	17.4	23.6	26.1	31.8	45.3	36.8	39.1	35.3
$\mathcal{IS}(0.25)$	73.5	88.7	95.9	97.2	91.9	67.8	84.1	89.5	96.9	98.6	91.4	96.3	97.1
$\mathcal{IS}(4)$	24.2	42	64.7	63.2	70.2	37	50.2	56.4	69.7	81.3	67	75.4	72.9
<u>DECREASING HAZARD RATE</u>													
$\mathcal{G}(0.2)$	23.2	23.3	32.1	15.8	59.7	30.3	38.4	45.8	30.8	13	0.1	55.4	55.2
$\mathcal{AW}2$	85.4	88.7	99.8	34.4	99.9	99.5	99.9	99.9	99.6	95.3	0	99.9	99.9
$\mathcal{EW}2$	2.7	4	40.5	0.4	58.6	34.4	47.1	56.5	23.9	7.8	0	48.3	57.5
$\mathcal{H}(0)$	33.2	43.5	64.5	46	77.9	52.5	65	68.7	65.1	80.1	70.4	77.2	74.5
$\mathcal{D}2(0)$	33.1	43.5	63.8	45.5	77.6	52.1	65.3	68.9	65	80	70.6	76.7	74.9
<u>BATHTUB HAZARD RATE</u>													
$\mathcal{EW}3$	2.6	4	40.7	0.4	58.5	34.4	47.2	56.2	23.7	7.7	0	48.1	57.3
$\mathcal{GG}1$	29.5	29.6	56.7	18.8	80.4	46	58.3	67.8	46.5	20.8	0	74.1	77.3
$\mathcal{GG}2$	23.2	23.2	31.4	15.5	58.8	30.5	38.1	45.6	31	13.2	0	55.2	54.9
$\mathcal{C}(0.4)$	9.8	9.2	7.1	7.1	15.4	8.8	10.4	12	7.7	2.9	1.1	14.7	14.3
$\mathcal{D}1(0.8)$	14.5	14	12.6	10.6	28.1	14	16.7	20.3	13.1	5	0.4	26.8	25.2
Mean	27.8	32.7	47.1	32.6	58.1	37.3	45.7	50.3	46.7	46.1	30.1	57.3	57.1

Table 6.3: Power results for the Weibull GOF tests,  $n = 50$

altern.	$\check{S}T_3$	$\check{S}T_4$	$\widehat{KL}_5$	$\check{K}L_5$	$\check{C}Q$	$\widehat{C}Q$	$\widehat{C}Q^*$	$\widehat{L}T^2$	$\widehat{E}W_w$	$\widehat{G}G_l^2$	$\widetilde{M}O_w$	$\widehat{P}G\widehat{W}_w$	$T_1$	$T_2$
$exp(0.5)$	4.9	5.1	5.1	5	5.3	5	5.1	5.1	5	5.1	5	5.1	5.1	5
$\mathcal{W}(0.5)$	5.1	5.1	5.2	5	5.3	5	4.8	5.1	4.8	5.1	5.1	4.9	5.2	5
$\mathcal{W}(3)$	5.2	5.2	5.1	5.1	5.3	5.1	5	5	4.9	5.1	5	5.1	5	5
<u>INCREASING HAZARD RATE</u>														
$\mathcal{G}(2)$	10	15	6.7	10.9	1.8	4.7	11.8	11.8	12.4	13	12.4	11.3	12.5	13
$\mathcal{G}(3)$	15.6	23.7	8.6	17	2.3	8.3	18.8	18.2	19.9	21.2	20.3	18.6	20.6	21.8
$\mathcal{A}W1$	67.5	69.8	56.6	73.3	86.9	87.3	76.4	75.8	81.9	82.4	68	80.6	74.5	76.2
$\mathcal{E}W1$	47.6	61.1	22.6	48.6	14.9	34.7	50.1	47.5	52.6	55.9	54.4	50.2	56.9	57.5
$\mathcal{D}2(2)$	14.5	22.3	9.9	16	2.5	8.3	18.5	18.6	19.6	20.7	19	18.4	19.9	20.5
<u>UPSIDE-DOWN BATHTUB HAZARD RATE</u>														
$\mathcal{L}\mathcal{N}(0.8)$	68.3	79.1	35.4	67.5	28.2	52.7	66	62.8	68.6	72.3	72.1	65.8	73.7	74.7
$\mathcal{I}\mathcal{G}(3)$	97.5	98.7	79.7	96.8	75.7	90.7	93.5	91.5	94.3	96.2	96.5	93.1	97.1	97.3
$\mathcal{E}W4$	31.2	43.9	15.2	33.2	7.3	20.9	36.2	34.9	38.1	40.5	38.9	36.1	40.6	41.2
$\mathcal{I}\mathcal{S}(0.25)$	98.1	98.9	91.4	97.3	59.7	85.3	87.9	83.3	89.6	93.7	95.2	87.1	96.1	96.3
$\mathcal{I}\mathcal{S}(4)$	73.6	83.1	52.1	71.8	27.5	54	65.9	62	69.2	73.1	73.7	65.3	76.2	77.1
<u>DECREASING HAZARD RATE</u>														
$\mathcal{G}(0.2)$	13.7	17.5	28.5	41	42.7	49.7	57	53.1	51.5	55.8	45.1	56.4	59.7	57.7
$\mathcal{A}W2$	36.1	69.9	99.7	99.9	94.2	98.3	99.9	99.9	99.7	99.8	99.8	99.9	100	100
$\mathcal{E}W2$	0.2	2.1	44.5	55.7	13.2	26.2	54.5	64.9	40.9	47.2	53.1	51.9	63.6	62.1
$\mathcal{H}(0)$	62.6	75.9	51.8	69.8	44.3	63.3	77.1	76.1	78.3	79.5	76.1	77.1	77.7	78.3
$\mathcal{D}2(0)$	62.9	75.8	52	69.8	43.5	62.9	77.1	76.6	78.3	79.2	76.2	76.8	77.2	77.9
<u>BATHTUB HAZARD RATE</u>														
$\mathcal{E}W3$	0.2	2	44.6	56	13.3	26.4	54.5	65.2	40.6	47.3	52.9	51.7	63.8	62.3
$\mathcal{G}\mathcal{G}1$	16.3	23.6	51	67.4	53.2	63	75.3	73.8	69.1	73.3	63.2	75.1	81.6	79.6
$\mathcal{G}\mathcal{G}2$	13.7	17.4	28.5	41.4	43	50.2	56.8	52.9	51.7	55.7	45.9	56.7	59.5	58.1
$\mathcal{C}(0.4)$	6.4	4.9	9.4	9.8	16	17.1	15.2	13.5	12.9	15.5	11.6	15	14.6	13.7
$\mathcal{D}1(0.8)$	8.9	8.3	14.6	16.8	25.3	27.8	27.3	23.7	24.5	27.1	20.4	27.4	27.2	25.7
Mean	37.2	44.6	40.1	53	34.8	46.6	56	55.3	54.7	57.5	54.7	55.7	59.6	59.6

Table 6.4: Power results for the Weibull GOF tests,  $n = 20$

altern.	$Z^2$	$R_{EJG}^2$	$SPP$	$SB$	$OK^*$	$KS$	$CM$	$\widehat{AD}$	$\widetilde{AD}$	$\widetilde{LS}$	$MSF$	$TS$	$LOS$
$exp(0.5)$	4.9	4.9	4.8	5.1	4.9	5.1	5.1	5	5.1	5	5.1	5.2	5.1
$\mathcal{W}(0.5)$	4.9	5	4.9	5.2	4.9	5	5.1	4.9	4.9	4.8	5.1	5	5
$\mathcal{W}(3)$	4.9	4.9	4.9	5.1	5	5	4.9	4.8	5	5	5.2	5	4.9
<u>INCREASING HAZARD RATE</u>													
$\mathcal{G}(2)$	3.3	4.4	6.5	6.1	7.1	5.7	6.4	5.9	8.2	11	9.6	6.5	6.7
$\mathcal{G}(3)$	3.3	5	7.8	7.1	9.5	6.7	7.6	7.1	10.9	15.2	12.5	8.6	8.5
$\mathcal{AW}1$	49.5	47.5	36.5	33.6	46.9	30.1	33.9	40.1	39.1	14.6	0.8	49.3	48.1
$\mathcal{EW}1$	6.6	11.2	15.7	15.4	22	12.3	14.9	15	22.7	31.6	24.3	20.5	20.4
$\mathcal{D}2(2)$	3.4	5.1	7.9	7.1	9.1	6.7	7.3	7	10.2	14.8	11.8	8.7	8.4
<u>UPSIDE-DOWN BATHTUB HAZARD RATE</u>													
$\mathcal{LN}(0.8)$	10.5	17	21.7	22.1	29.9	16	20.9	21.4	31.4	41.7	32.2	29.3	29
$\mathcal{IG}(3)$	35.6	46	48.7	53	58	35.3	46.3	48.4	62.2	71.4	58.9	60.3	59.7
$\mathcal{EW}4$	4.7	7.9	11.9	11.2	15.6	9.2	11.2	11	16.4	23.7	18.6	14.6	14.6
$\mathcal{IS}(0.25)$	28.8	41.9	44.2	53.9	49.5	29	39.5	41.5	61.2	68.3	54.7	53.6	55.7
$\mathcal{IS}(4)$	10.6	18.1	22.4	23.9	30	16	21	21.6	32.9	43	33.1	29.8	29.6
<u>DECREASING HAZARD RATE</u>													
$\mathcal{G}(0.2)$	15.2	14.2	12.4	11.1	23.8	14.5	16.7	20.1	10.4	1	0.6	23.9	22.9
$\mathcal{AW}2$	52.9	52.3	75.9	23.8	89.4	77.3	84.6	88.8	69.4	16.8	0	86.8	90.7
$\mathcal{EW}2$	4.9	4.9	11.1	2.7	19.4	14.8	18.4	21.7	6.7	0.4	0.6	17.8	19.4
$\mathcal{H}(0)$	17.8	23.5	28.9	23.3	39.4	23.3	30	30.7	35.6	47.7	36.4	36.3	35.9
$\mathcal{D}2(0)$	18.1	23.3	29.2	23.7	38.8	22.9	29.8	30.6	35.9	47.8	36.3	36.3	36.1
<u>BATHTUB HAZARD RATE</u>													
$\mathcal{EW}3$	5.1	4.7	11.1	2.6	19.6	14.5	18.3	21.6	6.8	0.4	0.6	17.9	20.1
$\mathcal{GG}1$	18.4	17.1	17.6	12.8	35.5	19.9	24	29.1	15	1.2	0.4	33.9	33.6
$\mathcal{GG}2$	15.3	13.9	12.3	10.9	24.1	14.4	16.4	19.8	10.5	1	0.6	24	22.3
$\mathcal{C}(0.4)$	8.1	7.4	5.4	6.3	7.9	6.6	6.9	7.6	4.9	1.6	2.2	8.5	7.9
$\mathcal{D}1(0.8)$	10.9	9.6	7.2	8.1	11.9	8.4	9	10.6	6	1.2	1.5	12.7	11.9
Mean	16.1	18.7	21.7	17.9	29.4	19.2	23.1	25	24.8	22.7	16.8	29	29.1

Table 6.5: Power results for the Weibull GOF tests,  $n = 20$

altern.	$\check{S}T_3$	$\check{S}T_4$	$\widehat{KL}_5$	$\check{K}L_5$	$\check{C}Q$	$\widehat{C}Q$	$\widehat{C}Q^*$	$\widehat{L}T^2$	$\widehat{E}W_w$	$\widehat{G}G_l^2$	$\widetilde{M}O_w$	$\widehat{P}G\widehat{W}_w$	$T_1$	$T_2$
$exp(0.5)$	5.1	5.1	4.9	5.1	4.6	5.3	5.1	5.1	5	4.9	5	4.7	5.1	5
$\mathcal{W}(0.5)$	5	5	4.9	4.9	4.6	5.1	5.4	5	5.1	5	5	4.9	5.1	4.9
$\mathcal{W}(3)$	5	5	5.1	5	4.9	5.1	5.4	4.9	5.2	5	5.1	4.8	5.1	4.9
<u>INCREASING HAZARD RATE</u>														
$\mathcal{G}(2)$	8.9	9.9	5.9	7.4	1.5	2.5	6.5	7.1	7.1	7.3	7.8	6.3	8.3	8.3
$\mathcal{G}(3)$	11.9	13.5	6.5	9.6	0.9	5.4	8.7	9.2	9.8	9.5	10.7	8.2	11.1	11.5
$\mathcal{AW}1$	25.9	26.1	27.3	36.6	54.7	58	49.4	45	49.9	49.7	39.6	49.8	42.9	40.7
$\mathcal{EW}1$	26.3	29.7	11.6	20.7	0.2	5.8	19.5	19.6	21.6	21.9	23.8	18.6	24.9	25.5
$\mathcal{D}2(2)$	11.6	13.3	6.8	9.2	1	2.8	8.9	8.9	9.8	9.5	10.5	8.1	11.1	11.3
<u>UPSIDE-DOWN BATHTUB HAZARD RATE</u>														
$\mathcal{LN}(0.8)$	36.4	40.1	16	28.9	0.5	10	26.9	26.6	29.8	30.5	33.4	26.2	34.5	35
$\mathcal{IG}(3)$	69.7	72	39.4	60.2	4.3	30.8	53.1	51.5	56.1	59.1	63.5	51.7	63.6	64.9
$\mathcal{EW}4$	19.5	22.3	8.7	14.8	0.4	3.7	14.3	14.5	15.7	16.2	17.4	13.6	18.4	18.8
$\mathcal{IS}(0.25)$	67.4	68.8	44.2	58.3	0.7	19.1	41.7	39.6	45.6	50.1	58.3	39.9	56.2	57.5
$\mathcal{IS}(4)$	38.7	42.3	20.2	30.3	0.3	9.2	26.3	25.8	29.1	30.6	33.8	25.1	34.7	35.5
<u>DECREASING HAZARD RATE</u>														
$\mathcal{G}(0.2)$	4.8	3.5	14.6	15.6	22.1	26.5	26.3	22.2	20.9	22.5	16.4	24.1	20.6	19.1
$\mathcal{AW}2$	12.9	15.5	82.6	86.5	63.9	75.2	87.3	87.7	79.9	82.3	79.3	85.6	88.3	87.4
$\mathcal{EW}2$	0.7	0.4	20.7	17.9	9.7	15.2	21.2	22.8	13.7	15	16.3	18.2	18	16.5
$\mathcal{H}(0)$	39.7	44.3	22.4	34.7	2.7	17.9	37.4	37.4	39.3	39.3	39.8	36.2	42.2	42.4
$\mathcal{D}2(0)$	39.5	44.1	22.1	34.4	2.5	18	36.9	37.5	39.6	39.2	39.7	36.4	41.7	42.8
<u>BATHTUB HAZARD RATE</u>														
$\mathcal{EW}3$	0.7	0.5	20.8	18.3	9.4	14.7	21.2	22.7	13.7	15.2	16.1	18.2	18.1	16.2
$\mathcal{GG}1$	5.6	4.5	24	26.2	27.3	32.8	36.3	32.4	29.3	30.6	23.5	33.5	31.8	28.8
$\mathcal{GG}2$	4.9	3.7	14.5	15.7	22.3	26.2	26.1	22.6	21.2	21.9	16.6	24.4	20.7	18.5
$\mathcal{C}(0.4)$	3.5	2.7	6.5	5.8	9.9	11.5	9.4	7.6	7.5	7.8	5.9	8.1	9.8	6
$\mathcal{D}1(0.8)$	3.8	2.8	8.3	8.1	14	16.3	14.2	11.5	11.5	11.7	8.6	12.7	10.3	9.1
Mean	21.6	23	21.1	27	12.4	20.1	28.6	27.6	27.5	28.5	28	27.2	30.4	29.8

Table 6.6: Power results for the Weibull GOF tests,  $n = 10$

altern.	$Z^2$	$R_{EJG}^2$	$SPP$	$SB$	$OK^*$	$KS$	$CM$	$\widehat{AD}$	$\widetilde{AD}$	$\widetilde{LS}$	$MSF$	$TS$	$LOS$
$exp(0.5)$	5.1	5	5.1	5.2	5.1	5	4.9	5	5.2	5.1	5.2	5.1	5
$\mathcal{W}(0.5)$	5	4.9	5.1	5.1	5	5.1	4.9	4.9	5.1	5.1	5.2	5.2	5.1
$\mathcal{W}(3)$	5	4.9	5.1	5.1	5.2	5.1	5	5.1	5.2	5.1	5	5.1	5.2
<u>INCREASING HAZARD RATE</u>													
$\mathcal{G}(2)$	4.5	5.2	5.3	5.2	5.5	5.3	5.4	4.9	7.2	8.2	7.6	5.6	5.6
$\mathcal{G}(3)$	4.5	5.5	5.8	5.7	6.4	5.5	5.9	5.4	8.5	9.9	8.8	6.3	6.1
$\mathcal{AW}1$	27.6	25	22.1	22.1	26.7	18.6	20.3	24.8	16.5	2	2	28.1	26.2
$\mathcal{EW}1$	6.5	8.8	8.5	8.7	10.7	7.8	8.6	7.6	14	16.9	14	10.5	9.8
$\mathcal{D}2(2)$	4.5	5.6	5.8	5.6	6.4	5.5	5.9	6.2	8.4	9.8	8.4	6.3	6.1
<u>UPSIDE-DOWN BATHTUB HAZARD RATE</u>													
$\mathcal{LN}(0.8)$	8.6	11.4	10.2	10.9	13.6	9.2	10.7	9.7	17.6	21.4	17.2	13.6	12.5
$\mathcal{IG}(3)$	19.2	24.3	19.8	23.6	26.9	17.3	21.8	20.9	33.1	38.3	31	28	26.5
$\mathcal{EW}4$	5.4	6.9	7.3	7	8.6	6.8	7.2	6.5	11.2	13.7	11.4	8.4	7.9
$\mathcal{IS}(0.25)$	16.1	20.6	16.3	21.6	21.7	14.8	18.1	16.8	30.8	34.5	28.4	23.5	23
$\mathcal{IS}(4)$	8.6	11.4	10.1	11.3	13.1	9.5	10.9	10.1	18.1	21.7	17.5	13.7	13.1
<u>DECREASING HAZARD RATE</u>													
$\mathcal{G}(0.2)$	10.6	8.7	9.1	9.3	12	9.5	9.7	11.9	5	0.9	1.5	11.9	11.3
$\mathcal{AW}2$	32.7	29.4	40.5	21.7	53.3	42.9	48.6	54.6	26.1	0.4	0.2	50.9	54.3
$\mathcal{EW}2$	6.1	5.3	7.4	5.5	9.4	8.7	9.6	11.1	3.5	0.8	1.5	8.9	9.5
$\mathcal{H}(0)$	12.4	14.7	14.1	13.3	18.3	12.3	14.3	13.5	20.8	25.7	19.5	18.3	16.5
$\mathcal{D}2(0)$	12.2	14.9	14.1	13.3	17.9	11.9	14.5	13.7	20.7	25.6	19.8	17.7	16.4
<u>BATHTUB HAZARD RATE</u>													
$\mathcal{EW}3$	6	4.9	7.2	5.3	9.2	8.7	9.4	11.6	3.4	0.7	1.5	8.9	9.5
$\mathcal{GG}1$	12.1	10.2	11.3	10.8	16.1	11.9	12.7	15.7	5.9	0.6	1.2	15.9	15.6
$\mathcal{GG}2$	10.6	8.6	9.1	9.2	12.1	9.3	9.5	11.8	4.9	1.1	1.5	12	11.4
$\mathcal{C}(0.4)$	6.4	5.4	5.4	5.9	6.2	5.8	5.6	6.4	4	2.5	3.2	6.2	5.9
$\mathcal{D}1(0.8)$	7.7	6.6	6.7	7.2	7.8	6.6	6.7	8.1	3.9	1.9	2.5	7.7	7.5
Mean	11.1	11.7	11.8	11.2	15.1	11.4	12.8	13.6	13.2	11.8	9.9	15.1	14.7

Table 6.7: Power results for the Weibull GOF tests,  $n = 10$

altern.	$\check{S}T_3$	$\check{S}T_4$	$\widehat{KL}_3$	$\check{K}L_3$	$\check{C}Q$	$\widehat{C}Q$	$\widehat{C}Q^*$	$\widehat{L}T^2$	$\widehat{E}W_w$	$\widehat{G}G_l^2$	$\widehat{M}O_w$	$\widehat{P}G\widehat{W}_w$	$T_1$	$T_2$
$exp(0.5)$	5	5	5.1	4.9	5	5.1	5.1	5.2	4.9	5.1	5	4.9	5	5
$\mathcal{W}(0.5)$	4.9	5	5.1	4.9	4.9	5.1	5.1	5	5.1	4.9	5.1	5	5.1	5.1
$\mathcal{W}(3)$	5	5	4.9	4.9	4.9	5	5.1	5	4.9	4.9	4.9	5.1	5.2	4.9
<u>INCREASING HAZARD RATE</u>														
$\mathcal{G}(2)$	7.6	7.7	5.4	6.5	2.1	2.6	4.5	4.9	5.3	5.4	6.3	4.6	6.2	6.4
$\mathcal{G}(3)$	9.1	9.3	5.8	7.6	1.5	1.9	4.7	5.5	5.8	6.1	7.4	4.7	6.9	7.1
$\mathcal{AW}1$	1.9	1.9	13.8	17.3	31.5	34.3	31.1	27.9	28.8	28.8	23.4	30.7	25.8	24.6
$\mathcal{EW}1$	16.1	16.5	7.8	12.4	0.5	1.1	7	8.5	9.4	9.8	12.5	7.4	11.9	12.2
$\mathcal{D}2(2)$	9.3	9.5	6.5	7.5	1.6	2.1	4.7	5.5	5.8	5.7	7.5	4.9	6.9	7.1
<u>UPSIDE-DOWN BATHTUB HAZARD RATE</u>														
$\mathcal{LN}(0.8)$	20.8	21.1	9.5	15.8	0.3	0.9	9.2	11.1	12.4	12.5	15.9	9.5	15.1	15.4
$\mathcal{IG}(3)$	38.6	38.7	18.6	31.3	0	2.5	19	21.5	24.3	25.3	30.9	20.2	29.7	30.1
$\mathcal{EW}4$	13	13.3	6.9	9.8	0.9	1.3	5.7	6.9	7.7	7.8	9.9	6	9.4	9.9
$\mathcal{IS}(0.25)$	35.2	34.8	23.2	28.7	0	1.3	13.5	16.4	18.4	20.3	26.7	14.6	24.3	24.8
$\mathcal{IS}(4)$	21	21.2	12.7	16.2	0.2	0.7	8.9	10.8	11.9	12.3	16.1	9.5	15.2	15.4
<u>DECREASING HAZARD RATE</u>														
$\mathcal{G}(0.2)$	1.1	1	8.7	6.9	13.8	16.5	14.8	13.3	11.7	11.9	8.8	14	11	10.1
$\mathcal{AW}2$	0.1	0	45.4	44.1	40.2	46.7	53.8	55.9	44.7	46.6	40.5	52	50	48.5
$\mathcal{EW}2$	0.8	0.7	10.5	6.6	8.6	11.2	11.5	11.8	8.1	8.3	6.9	10.5	8.6	7.7
$\mathcal{H}(0)$	24.6	25.4	13.9	19.1	0.8	2.6	13.7	15.9	17.3	17.5	20.6	14.2	20.2	20.5
$\mathcal{D}2(0)$	24.4	25.2	14	19.2	0.8	2.6	13.7	16.1	17.2	17.4	20.4	14.4	20.1	20.3
<u>BATHTUB HAZARD RATE</u>														
$\mathcal{EW}3$	0.8	0.7	10.4	6.5	8.7	11.1	11.7	12.2	7.9	8.4	6.7	10.4	8.4	7.8
$\mathcal{GG}1$	0.7	0.6	12	9.7	16.4	19.5	19.3	17.7	14.8	15.3	11	18.1	14.5	13.6
$\mathcal{GG}2$	1.2	1.1	8.5	6.8	13.5	16.3	15.5	13.7	12.1	12	8.7	14.3	10.8	10.1
$\mathcal{C}(0.4)$	2.5	2.4	5	4.5	7.1	8.6	7.6	6.7	6.3	6.3	5.2	7.1	5.8	5.3
$\mathcal{D}1(0.8)$	1.8	1.8	5.3	4.8	9.4	11.6	10.3	8.6	7.9	8	6.2	9	7.2	6.5
Mean	11.5	11.6	12.2	14.1	7.9	9.8	14	14.5	13.9	14.3	14.6	13.8	15.4	15.2

Indeed, it is more difficult to detect a departure from the Weibull distribution than from the Exponential distribution.

Similarly to previous chapters (2, 4 and 5), the first remark is that the powers of the Weibull GOF tests are close to the significance level 5% when the alternative used is the Weibull or the Exponential distribution. The second one is that some of the GOF tests, such as  $SB$ ,  $\check{S}T_3$  and  $\check{S}T_4$ , seem to be biased especially when the sample size  $n$  is lower than 20. Indeed, some powers are smaller than the significance level 5%. This remark has already been discussed in [121] for the Mann-Sheuer-Fertig test.

We also notice that the performance of the test statistics is tightly linked to the shape of the hazard rate. The behavior appears to be the same for the DHR and BT hazard rates on one hand and the IHR and UBT hazard rates on the other hand, with some special cases. In the following, we compare the GOF tests within each specific family.

- **Tests based on probability plots:** These tests have globally bad performance. The test  $SPP$  based on the stabilized probability plot is the best among these tests. It is the only unbiased test, unlike  $Z^2$  and  $R_{EJG}$ . This result proves that one can not rely only on the Weibull probability plot to assess the fitness of the Weibull distribution to a data set, as it is often done in industry.
- **Shapiro-Wilk type tests:** The modified test  $OK^*$  of Öztürk and Korukoğlu is significantly more powerful than the biased test  $SB$ , as it is also shown in [97]. The test  $OK^*$  is among the most powerful studied tests. It has globally a good performance and it can be recommended whatever the shape of the hazard rate.
- **Tests based on the empirical distribution function:**  $\widehat{AD}$  is globally the best test in this family. The test  $\widetilde{LS}$  seems to be very powerful for IHR and UBT alternatives but very bad for DHR and BT alternatives; it is a biased test. The test  $\widetilde{AD}$  of Anderson-Darling based on the LSEs is better than the test  $\widehat{AD}$  only for IHR and BT distributions but it is less powerful for the DHR and UBT alternatives; this test can be biased for small  $n$ . These results may seem contradictory with what Liao and Shimokawa have concluded in [76]. This is mainly due to the limited range of alternatives they chose to test these GOF tests: the comparisons we made seem to use a broader range of alternatives with different shapes of the hazard rate.
- **Generalized smooth tests:** Both tests  $\check{S}T_3$  and  $\check{S}T_4$  are biased and have generally bad performance compared to the other tests.  $\check{S}T_4$  is very powerful for UBT alternatives and some special IHR alternatives. Those two tests are generally not recommended; they are related to the skewness and the kurtosis of the tested sample. Similar results have been found in [16].
- **Tests based on normalized spacings:** The test of Tiku-Singh  $TS$  has the best performance. The test  $LOS$  is slightly better than  $CMZ$  because generally the Anderson-Darling test is more powerful than the Cramer-Von-Mises test. Mann-Sheuer-Fertig test  $MSF$  is the only biased test among those based on the normalized spacings. This is essentially due to the wrong use of the test as a one-sided test

like it was recommended in [82]. The test  $TS$  is the most powerful of all these tests and even among all those studied. But the test  $TS$ , as well as all the other tests based on the normalized spacings, are not recommended for samples with ties such as Aarset data presented previously in table 1.1. Some test statistics are not even defined in this case, for instance,  $LOS$ . When the statistic is defined (for example the case of  $TS$  and  $MSF$ ), the use of one of these statistics may lead to some wrong conclusions. This point will be detailed in chapter 9.

- **Tests based on the Kullback-Leibler information:** Generally the two tests  $\widehat{KL}$  and  $\check{KL}$  are less powerful than the best ones. Using the MEs improves the performance of the test, but they are still less powerful than  $TS$  and  $OK^*$ .
- **Tests based on the Laplace transform:** The test of Cabaña and Quiroz  $\check{CQ}$  is biased and has the lowest mean rejection percentage. The new test  $\widehat{CQ}$  is more powerful than  $\check{CQ}$  but is still biased for small  $n$  and less powerful than both  $\widehat{CQ}^*$  and  $\widehat{LT}^2$ . These two tests  $\widehat{CQ}^*$  and  $\widehat{LT}^2$  are the best ones to recommend within this family. Indeed, they have globally very good performance that is very close to the one of  $TS$  and  $OK^*$ . Moreover, their expressions are simple than the ones of Cabaña and Quiroz  $\check{CQ}$  and Tiku-Singh test statistic  $TS$ .
- **Tests based on the likelihood:** The tests  $\widehat{EW}_w$ ,  $\widehat{GG}_l^2$ ,  $\widehat{MO}_w$  and  $\widehat{PGW}_w$  have the advantage of being unbiased GOF tests. For  $n \leq 20$ , the test  $\widehat{MO}_w$  is the best one for IHR and UBT alternatives and  $\widehat{PGW}_w$  is the best for DHR and BT alternatives. The test  $\widehat{GG}_l^2$  has the best performance among them which is very close to the one of  $TS$  and sometimes better.
- **Combined likelihood based tests:** The two tests  $T_1$  and  $T_2$  have very good performance and they have even the highest mean of rejection percentages compared with all the previous tests. These tests are combinations of both tests  $\check{PGW}_w$  and  $\widehat{MW}_w$ . These latest two tests have complementary performances: they are both biased, but when the power of one test is high, the other test has a null power [71]. That is why, combining the two tests gives an unbiased GOF test with very good performances. The test  $T_1$  uses the maximum of the absolute value of the two tests after they have been centered and normalized. The test  $T_2$  uses the sum of the centered and normalized tests. The reason why these two tests are powerful may also be the fact that the estimation of the two Weibull parameters is done in two different ways (MLEs and LSEs). The use of these two tests could be recommended for all kinds of alternatives.

The principle of combining two tests with complementary behavior is then a promising way of improving the performance of the tests. Further work is needed in order to identify the best possible combinations.

From this analysis, we can have the following conclusions.

- Among all these tests, the tests  $TS$  and  $OK^*$  are very powerful. The test  $OK^*$  has the simplest expression which is more convenient for practical purposes.

- The test  $T_1$  is the most powerful test, because it combines two GOF test statistics with two different methods of estimating the parameters.
- According to the shape of the hazard rate, the recommended tests are:
  - For IHR alternatives:  $\widetilde{LS}$
  - For UBT alternatives:  $\check{ST}_4$
  - For DHR-BT alternatives:  $T_1$ .

---

This chapter presents a comprehensive comparison of the GOF tests for the Weibull distribution. Until now 84 GOF tests were compared. Preliminary comparisons were done in chapter 4 to compare all the likelihood based GOF tests (54 GOF tests) between them and a second comparison was done in chapter 5, comparing the tests based on the Laplace transform within the same family (11 GOF tests). The best of the previous GOF tests within each family were finally compared to the GOF tests of the literature (19 GOF tests). The R package EWGoF includes all these GOF tests.

Some new alternatives were tested using these GOF tests, conclusions and recommendations about the use of these tests are given depending on the shape of the hazard rate.

To our knowledge, the best tests that we have identified are little known and rarely used. So the main outcome of our study is to advise the users of the Weibull and Extreme Value distributions to apply these tests before any study using these distributions.

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# Chapter 7

## Repairable systems: preliminary results

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This chapter gives some preliminary results about Non-Homogenous Poisson processes (NHPP) and imperfect maintenance models. Properties of NHPP are detailed. Lindqvist and Rannestad suggested exact GOF tests to check the relevance of a NHPP using a sufficient statistic. When both CM and deterministic PM are performed, some imperfect maintenance models based on the idea of virtual age are presented.

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### 7.1 Introduction

The context of the study in this chapter is different from what we have presented until now. We are interested this time in systems that are repairable and subject to maintenance. There are several kinds of maintenance:

- Corrective maintenance (CM), also called repair, is carried out after a failure and intends to put the system into a state in which it can perform its function again.
- Preventive maintenance (PM) is carried out when the system is operating and intends to slow down the wear process and reduce the frequency of occurrence of system failures.

Mathematically, the failure times of a repairable system are random variables and so are the CM. The PM are, in our case, fixed before the system is put into service and they are consequently carried out at deterministic times.

These maintenances can have different effects on the system reliability. The basic assumptions on maintenance efficiency are known as minimal repair or As Bad As Old (ABAO) and perfect repair or As Good As New (AGAN). In the ABAO case, each maintenance leaves the system in the state it was before maintenance. In the AGAN case, each maintenance is perfect and leaves the system as if it were new. It is well known that reality is between these two extreme cases: standard maintenance reduces failure intensity but does not leave the system AGAN. This is known as imperfect maintenance.

The mathematical modeling of the occurrence and efficiency of maintenance is done using random point processes. In this framework, the model is completely characterized by its failure intensity. The likelihood function can be written as a function of this intensity.

The most known and used models for repairable systems are Non Homogeneous Poisson Processes (NHPP). They assume that the effect of the CM is ABAO. The two classical intensities are the power-law and the log-linear intensity functions. The objective of our study, as it was mentioned in chapter 1, is to be able to measure the fitness of a given data set to a given maintenance model. Lindqvist and Rannestad [79] suggested exact Monte-Carlo GOF tests for NHPP with both power-law and log-linear intensity functions.

## 7.2 Repairable systems with CM only

### 7.2.1 Point processes

The point processes allow to model the occurrence of events in time. In general, the times between occurrences are neither independent nor identically distributed. The maintenance durations are assumed to be negligible, or not taken into account, then the failures and CM times are the same. So in the following, failures and CM times are confounded.

Let  $\{T_i\}_{i \geq 1}$  be the CM times, that is to say the times of the events occurrences of a random point process and let  $T_0 = 0$ . After each CM, the system is imperfectly repaired and then is put again into service.

**Definition 7.1** Let  $N_t$  be the random variable that denotes the number of failures in the interval  $[0, t]$ .  $\{N_t\}_{t \geq 0}$  is called a counting process [5] if it verifies:

- $N_0 = 0$  a.s.
- $\{N_t\}_{t \geq 0}$  is an integer.
- The trajectories of  $\{N_t\}_{t \geq 0}$  are increasing, constant piecewise functions, and right continuous with left hand limits.

We consider that the process  $\{N_t\}_{t \geq 0}$  is simple, i.e we can not have more than one failure at once:

$$\forall t \geq 0, \forall \Delta t \geq 0, P(N_{t+\Delta t} - N_t \geq 2) = o(\Delta t). \quad (7.1)$$

A failure process is defined equivalently by one of the following three random processes [5, 27] (see figure 7.1):

- $\{T_i\}_{i \geq 1}$  the failure times of the system, with  $T_0 = 0$ .  $\mathbf{T}_n$  denotes the vector of first  $n$  failure times  $\mathbf{T}_n = (T_1, \dots, T_n)$ .
- $\{X_i\}_{i \geq 1}$  the times between CM where

$$\forall i \geq 1, X_i = T_i - T_{i-1} \quad (7.2)$$

- $\{N_t\}_{t \geq 1}$  the counting process of the failures.

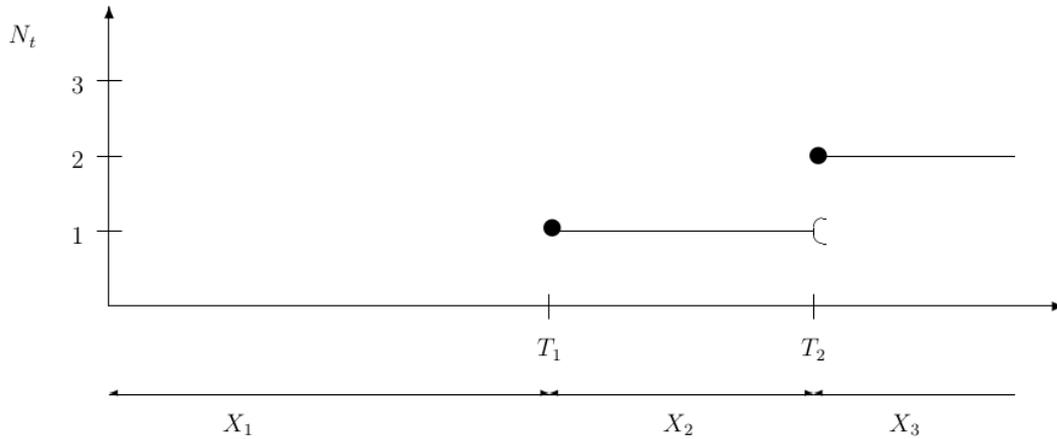


Figure 7.1: Observations of a counting process and corresponding notations

$N_{t-}$  denotes the left hand limit of  $N_t$ , it means the number of failures occurred in  $[0, t[$ . We assume that  $\forall t \in \mathbb{R}^+$ ,  $P(N_t < +\infty) = 1$  which means that the number of CM occurred, at each instant, is always finite.

In order to be able to predict the future of the process, we need its history. That is why we need to introduce, formally, the notion of filtration [27]. We consider first that all the random variables  $N_t$ ,  $t > 0$ , are defined in the same probability space  $(\Omega, \mathcal{A}, \mathcal{P})$ . A filtration  $\mathcal{H} = \{\mathcal{H}_t\}_{t \geq 0}$  is an increasing sequence of sub- $\sigma$ -algebras of  $\mathcal{A}$ :

$$s < t \Rightarrow \mathcal{H}_s \subset \mathcal{H}_t. \quad (7.3)$$

The process  $\{N_t\}_{t \geq 0}$  is  $\mathcal{H}$ -adapted if and only if for all  $t \geq 0$ ,  $N_t$  is  $\mathcal{H}$ -measurable. This means that the filtration  $\mathcal{H}_t$  includes all the information of the history at time  $t$  that is likely to influence the random variable  $N_t$ . Let  $\mathcal{H}_{t-} = \bigcap_{s < t} \mathcal{H}_s$ .

Since the process  $\{N_t\}_{t \geq 0}$  is a piecewise constant function that changes its values only at the times  $\{T_i\}_{i \geq 1}$ , its history at time  $t$  is entirely known by the number and the times of CM occurred between 0 and  $t$ . Thus  $\mathcal{H}_t$  is the  $\sigma$ -algebra generated by the history of the process at time  $t$ :

$$\mathcal{H}_t = \sigma(N_t, T_1, \dots, T_{N_t}). \quad (7.4)$$

In this case the future of the process depends only on its history  $\mathcal{H}_t$ . It is called self-exciting process [116].

**Definition 7.2** *The failure intensity function of the counting process  $\{N_t\}_{t \geq 0}$  [5] is:*

$$\begin{aligned} \forall t \geq 0, \quad \lambda_t &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(N_{(t+\Delta t)-} - N_{t-} = 1 | \mathcal{H}_{t-}) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(t \leq T_{N_{t-}+1} < t + \Delta t | \mathcal{H}_{t-}) \end{aligned} \quad (7.5)$$

The failure intensity function expresses the propension of the system to have a failure at  $[t, t + \Delta t[$ , given  $\mathcal{H}_{t-}$  which represents all the available information just before  $t$ . A self exciting process is completely characterized by its failure intensity [27].

In a parametric approach, we assume that the failure intensity is specified using a vector parameters  $\theta$ . The failure intensity is either denoted  $\lambda_t$  or  $\lambda_t(\theta)$ . The integral of

the failure intensity is called the cumulative intensity function, denoted  $\Lambda_t$  [5, 27]:

$$\Lambda_t = \int_0^t \lambda_s ds. \quad (7.6)$$

The estimation of  $\theta$  can be done thanks to maximum likelihood method.

**Property 7.1** *Let  $t > 0$  be a deterministic time (time censoring). The likelihood function associated to the observation of the failure process over  $[0, t]$  is [27]:*

$$L_t(\theta) = \left[ \prod_{i=1}^{N_t} \lambda_{T_i}(\theta) \right] \exp(-\Lambda_t(\theta)). \quad (7.7)$$

The log-likelihood is the logarithm of the likelihood:

$$\mathcal{L}_t(\theta) = \sum_{i=1}^{N_t} \ln(\lambda_{T_i}(\theta)) - \int_0^t \lambda_s(\theta) ds. \quad (7.8)$$

The maximum likelihood estimator  $\hat{\theta}_t$  is defined as the value of  $\mathcal{I}_0$  that maximizes the likelihood or equivalently the log-likelihood:

$$\hat{\theta}_t = \operatorname{argmax}_{\theta \in \mathcal{I}_0} \mathcal{L}_t(\theta). \quad (7.9)$$

## 7.2.2 Non Homogeneous Poisson processes (NHPP)

A special case of the previous counting processes are the Non-Homogenous Poisson processes (NHPP). They are characterized by their failure intensities that are deterministic functions of the time:  $\lambda(t)$  and  $\Lambda_t = \Lambda(t)$ . CM are carried out at random dates, while the intensity function is deterministic. Therefore, the CM can not have any effect on the intensity function: they are ABAO. Then, repair is minimal.

In what follows, we give some useful properties of the NHPP.

**Property 7.2** *The NHPP  $\{N_t\}_{t \geq 0}$  verifies the properties [27]:*

- $\{N_t\}_{t \geq 0}$  has independent increments.
- $\forall t \geq 0$ ,  $N_t$  has Poisson distribution  $\mathcal{P}(\Lambda(t))$  where:

$$\forall n \in \mathbb{N}, P(N_t = n) = \exp(-\Lambda(t)) \frac{(\Lambda(t))^n}{n!}. \quad (7.10)$$

- $\forall (s, t), 0 \leq s \leq t$ ,  $N_t - N_s$  follows a Poisson distribution:

$$\mathcal{P}(\Lambda(t) - \Lambda(s)) = \mathcal{P}\left(\int_s^t \lambda(u) du\right).$$

**Property 7.3** Given the times of the  $n$  first CM, the conditional distribution of the next time between CM  $X_{n+1}$  is given by [5]:

$$\forall x \geq 0, P(X_{n+1} > x | \mathbf{T}_n) = \exp\left(-\int_{T_n}^{T_n+x} \lambda(u) du\right) \tag{7.11}$$

The simplest model is the Homogeneous Poisson Process (HPP). It assumes that the system does not deteriorate neither improve.

**Definition 7.3** The HPP is a NHPP with a constant intensity:

$$\forall t \geq 0, \lambda_t = \lambda.$$

It will be denoted  $HPP(\lambda)$ .

**Property 7.4** The joint pdf of failure times  $\mathbf{T}_n$  from a  $HPP(\lambda)$  is [27]:

$$f_{\mathbf{T}_n}(t_1, \dots, t_n) = \lambda^n \exp(-\lambda t_n) \mathbb{1}_{\{0 < t_1 < \dots < t_n\}} \tag{7.12}$$

The following properties 7.5 and 7.6 allow transformations from NHPPs to HPPs [27].

**Property 7.5** If  $T_1, \dots, T_n$  are the successive event times of a NHPP with cumulative intensity  $\Lambda$ , then  $\Lambda(T_1), \dots, \Lambda(T_n)$  are event times of a  $HPP(1)$ .

**Property 7.6** If the NHPP is observed on the interval time  $[0, t]$ , then conditionally to  $N_t = n$ , the transformed times  $V_i = \frac{\Lambda(T_i)}{\Lambda(t)}, i = 1, \dots, n$ , are distributed as order statistics of  $n$  iid variables from  $\mathcal{U}[0, 1]$ .

The most popular parametrizations of NHPPs are the power law and log-linear processes [110]:

- Power law process (PLP):

$$\lambda_t(\alpha, \beta) = \alpha\beta t^{\beta-1}, \alpha, \beta > 0, t > 0. \tag{7.13}$$

For repairable systems, this model is equivalent to the Weibull distribution for non repairable systems. Parameter  $\alpha$  is a scale parameter. Parameter  $\beta$  is a shape parameter which characterizes the wear out of the system:

- $\beta > 1$ : wear out or aging;
- $\beta < 1$ : improvement or rejuvenating;
- $\beta = 1$ : stability ( $HPP(\alpha)$ ).

Figure 7.2: Power law failure intensity

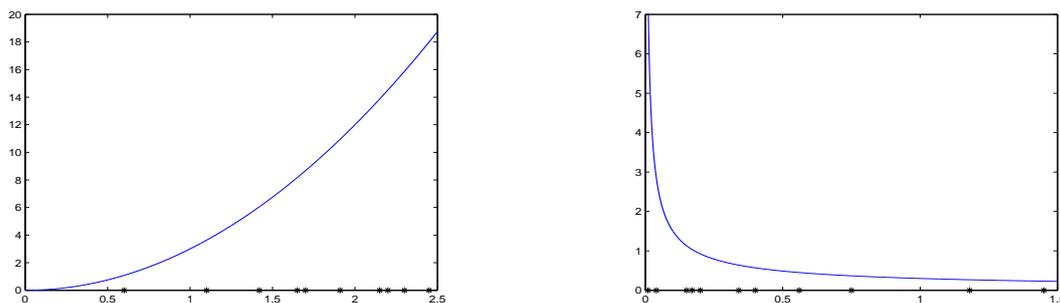


Figure (7.2) gives illustrations of the shapes of the power-law failure intensity function, the stars on the abscissa representing the CM times. The left figure shows the power law intensity for the parameters  $(\alpha = 1, \beta = 3.6)$  and the right is plotted for  $(\alpha = 1, \beta = 0.3)$ .

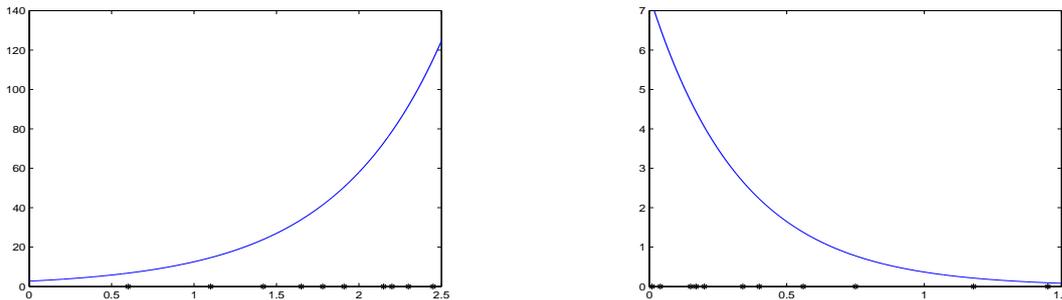
- Log-linear process:

$$\lambda_t(a, b) = \exp(a + bt), \quad a, b \in \mathbb{R}, \quad t > 0. \quad (7.14)$$

Illustrations of different shapes of its intensity are given in figure 7.3 (left figure  $(a = 1, b = 1.53)$ , right figure  $(a = 1, b = -3)$ ). Parameter  $a$  is a scale parameter and the parameter  $b$  is a shape parameter. The sign of  $b$  characterizes the wear of the system:

- $b > 0$ : wear or aging;
- $b < 0$ : improvement or rejuvenating;
- $b = 0$ : stability (HPP( $\exp(a)$ )).

Figure 7.3: Log-linear failure intensity



### 7.2.3 AGAN model

The perfect maintenance model considers that each maintenance is perfect and leaves the system as it were new (see figure 7.4). The random process is then a renewal process (RP). The times between two maintenance actions are independent and identically distributed. This implies that the failure intensity can be written as [78]:

$$\lambda_t = \lambda(t - T_{N_t^-}). \quad (7.15)$$

Figure 7.4 represents a trajectory of the failure intensity of a system with AGAN maintenance in the case of power-law intensity function with the parameters  $\alpha = 1, \beta = 3.6$ . The CM times are the times when the intensity function jumps. After maintenance, the intensity function starts from zero in parallel to the initial intensity function.

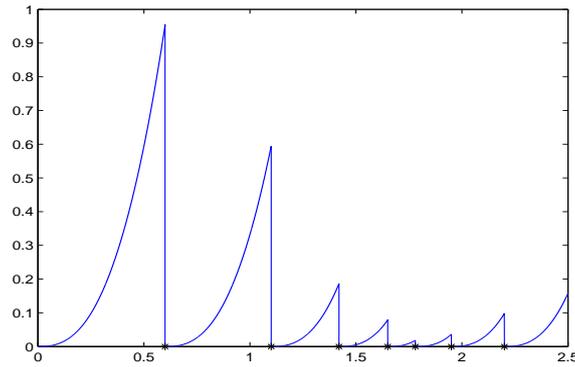


Figure 7.4: Power Law failure intensity and AGAN effect

## 7.3 Repairable systems with CM and PM

### 7.3.1 Notations

For a system preventively maintained at predetermined deterministic times  $\{\tau_i\}_{i \geq 1}$ , the number of PM at  $t$  is denoted by  $m_t$ . CM are done at unpredictable random times  $\{T_i\}_{i \geq 1}$ . The associated counting process is denoted by  $\{N_t\}_{t \geq 0}$ . The duration of maintenance (PM and CM) actions is not taken into account. The counting process of both CM and PM is denoted by  $\{K_t\}_{t \geq 0}$ . Finally,  $\{C_i\}_{i \geq 1}$  and  $\{W_i\}_{i \geq 1}$  denote respectively the maintenance times (PM and CM) and the times between maintenance.  $\{X_i\}_{i \geq 1}$  denotes the times between two successive CM ( $X_i = T_i - T_{i-1}$ ) and the  $\{\chi_i\}_{i \geq 1}$  denotes the times between two successive PM ( $\chi_i = \tau_i - \tau_{i-1}$ ). Figure 7.5 illustrates all the previous notations.

Before the first failure, the failure intensity is assumed to be a not always null function, non decreasing, deterministic, from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ , called initial intensity and denoted by  $\lambda(t)$ . The initial intensity represents the intrinsic wear out which means the wear out in the absence of maintenance actions. When the initial intensity is known, an imperfect maintenance model is only characterized by the effect of maintenance actions on the failure intensity. Deterministic PM is a particular case of planned PM, for which Doyen and Gaudoin [35] have proposed a general framework for simultaneous modeling and assessment of aging and maintenance efficiency.

In this context, the PM-CM process is completely defined by its failure intensity  $\lambda_t$  which has the same expression as in (7.5):

$$\forall t \geq 0, \quad \lambda_t = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(N_{(t+\Delta t)^-} - N_{t^-} = 1 | \mathcal{H}_{t^-})$$

### 7.3.2 Imperfect maintenance: virtual age models

In practice, the effect of maintenance is neither minimal (ABAO) nor maximal (AGAN), it is between these to extreme situations. Indeed, it is more reasonable to think that the maintenance has an effect more than minimal, which means that the system after repair is better than old. It is also less likely that the maintenance leaves the system as good as

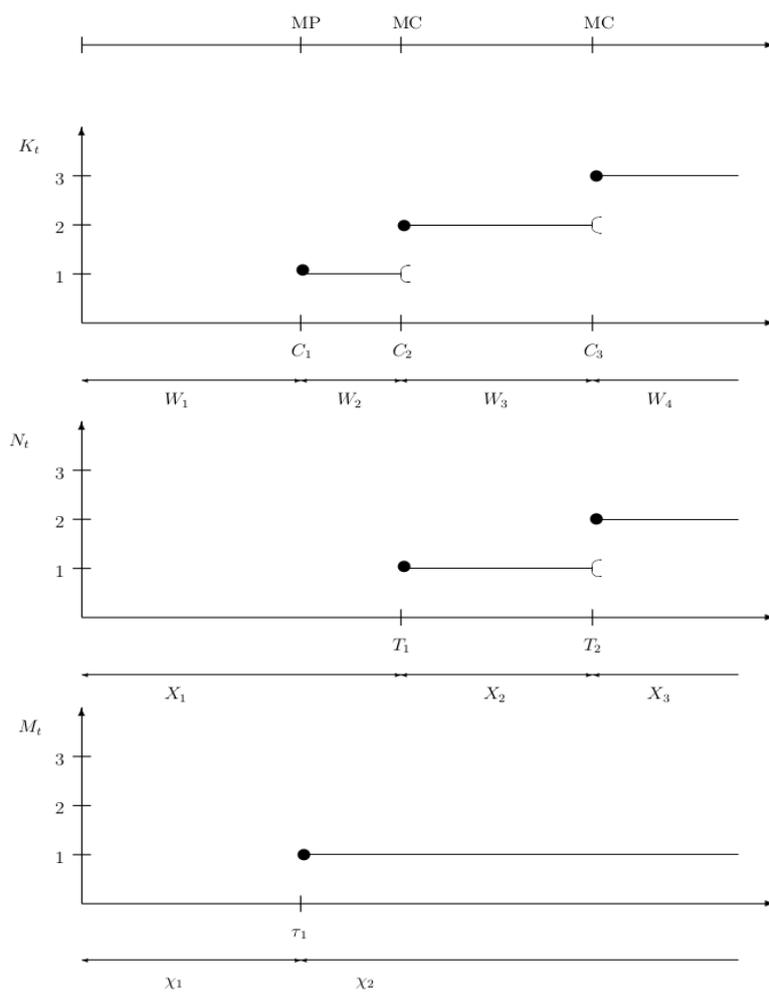


Figure 7.5: Observations of a counting process and the corresponding notations

new. The system in this case after repair is worse than new. This situation is known as Better than minimal repair or as imperfect maintenance. Many imperfect maintenance models have been proposed [102]. Virtual age [63] models are among these imperfect maintenance models. They assume that after the  $i^{\text{th}}$  maintenance the system behaves like a new one that has survived without failure until  $A_i$  [35]:

$$P(W_{i+1} > w | W_1, \dots, W_i, A_i) = P(Y > A_i + w | Y > A_i, A_i) \tag{7.16}$$

where  $Y$  is a random variable independent of  $A_i$  and with the same distribution as the time to failure of the new unmaintained system. The corresponding failure intensity is [35]:

$$\lambda_t = \lambda(A_{K_t^-} + t - C_{K_t^-}). \tag{7.17}$$

$A_{K_t}$  is called the effective age at time  $t$  and  $A_{K_t} + t - C_{K_t}$  is the virtual age at time  $t$ . The effective age is the virtual age of the system just after the last maintenance action. The idea that repair actions reduce the age of the system is the basis of Kijima's virtual age models [63]. Several models can be derived. Some of them will be presented in the following and illustrated by a trajectory of the corresponding intensity function (chosen to be power-law with parameters  $\alpha = 1, \beta = 3.6$ ), the stars on the abscissa representing the CM times and the circles the PM times .

- AGAN PM-AGAN CM: each maintenance is supposed to be AGAN. Effective ages are then equal to zero  $A_i = 0, \forall i \geq 1$ . The failure intensity is:

$$\lambda_t = \lambda(t - C_{K_t^-}). \tag{7.18}$$

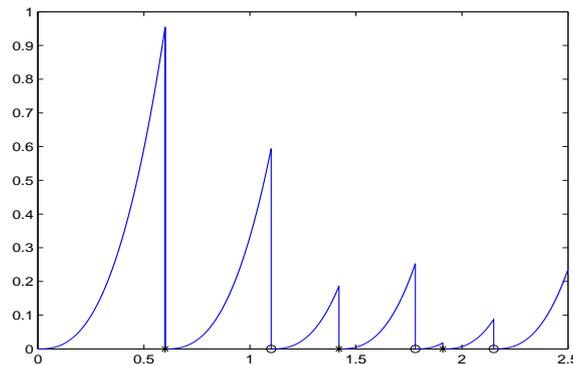


Figure 7.6: Failure intensity in the case of AGAN PM-AGAN CM

- ABAO PM-ABAO CM: each maintenance is supposed to be minimal. Then effective ages are equal to the last maintenance times  $A_i = C_i, \forall i \geq 1$ . The failure intensity is only a function of time, and the failure process is a NHPP:

$$\lambda_t = \lambda(t) \tag{7.19}$$

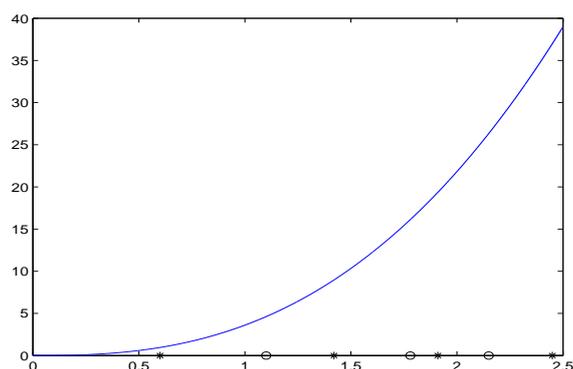


Figure 7.7: Failure intensity in the case of ABAO PM-ABAO CM

- ABAO PM-AGAN CM: Each preventive maintenance is minimal, while each corrective maintenance renews the system. The effective age is equal to the times elapsed between the last maintenance and the last perfect maintenance:  $A_i = C_i - T_{N_{C_i}}$ . The failure process is a renewal process with failure intensity:

$$\lambda_t = \lambda(t - T_{N_{t-}}). \quad (7.20)$$

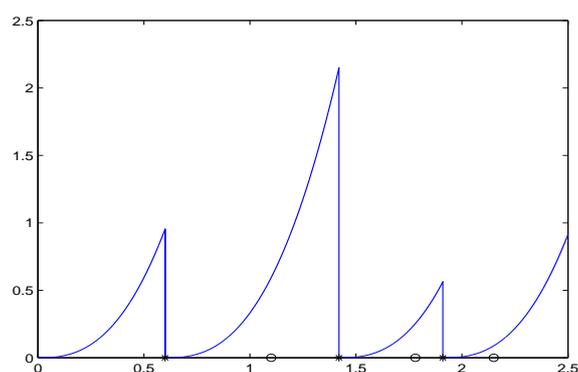


Figure 7.8: Failure intensity in the case of ABAO PM-AGAN CM

- AGAN PM-ABAO CM: the preventive maintenances are perfect and the corrective maintenances are ABAO, then the effective ages are  $A_i = C_i - \tau_{m_{C_i}}$  and the failure intensity is:

$$\lambda_t = \lambda(t - \tau_{m_{t-}}). \quad (7.21)$$

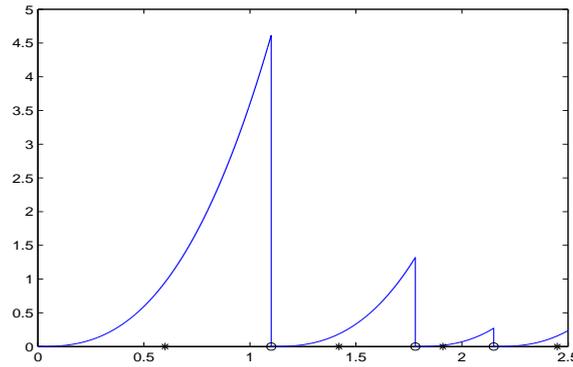


Figure 7.9: Failure intensity in the case of AGAN PM-ABAO CM

- Virtual age PM effect-ABAO CM: the effective age is equal to the effective age at the time of the last PM plus the time elapsed since the last PM. In this case, the effective ages are  $A_i = A_{K\tau_{m_{C_i}}} + C_i - \tau_{m_{C_i}}$ . Then the failure intensity is:

$$\lambda_t = \lambda(A_{K\tau_{m_{t-}}} + t - \tau_{m_{t-}}). \quad (7.22)$$

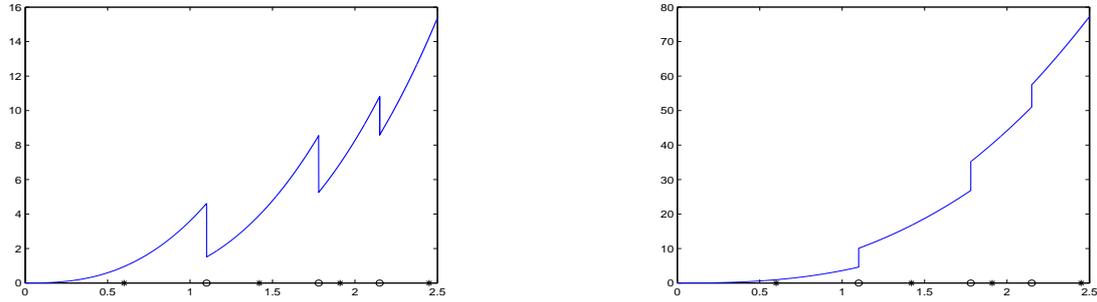
- $ARA_1$  PM-ABAO CM: when preventive maintenance are considered to have the Arithmetic Reduction of Age effect with memory one ( $ARA_1$ ) [33], the effective ages are  $A_i = A_{i-1} + (1 - \rho)(\tau_i - \tau_{i-1}) = (1 - \rho)\tau_i$ . The failure intensity is:

$$\lambda_t = \lambda(t - \rho\tau_{m_{t-}}). \quad (7.23)$$

We have the following special cases when the initial failure intensity is increasing (the system wears out with time and usage):

- $\rho = 0$ : minimal PM (ABAO),
- $\rho = 1$ : perfect PM (AGAN),
- $0 < \rho < 1$ : imperfect PM,
- $\rho < 0$ : harmful PM,
- According to the choice of the initial intensity, it may be possible to have  $\rho > 1$  corresponding to a “better than new” PM. This is possible for a log-linear intensity (because  $\exp(a + bt) > 0, \forall t < 0$ ) but not for the power law intensity (because  $\alpha\beta t^{\beta-1}$  is not defined for  $t < 0$ ).

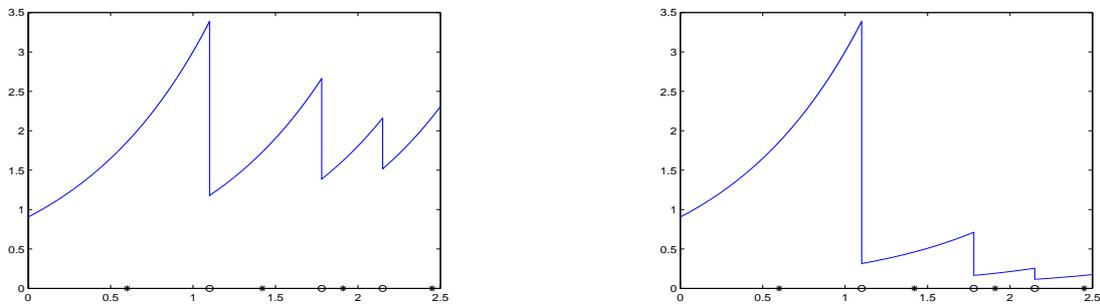
Figure 7.10 illustrates trajectories of the power law failure intensity in the case of  $ARA_1$  PM-ABAO CM. We can notice that at each PM times, the intensity function is reduced in the case ( $\rho = 0.35$ ) and increased in the case ( $\rho = -0.35$ ). This figure illustrates the fact that the maintenance efficiency depends on the sign of the parameter  $\rho$ .

Figure 7.10: Failure intensity in the case of  $ARA_1$  PM-ABAO CM

In the next chapter, we will use this model with a log-linear intensity function:

$$\lambda_t(a, b, \rho) = \exp(a + b(t - \rho\tau_{m_t-})). \quad (7.24)$$

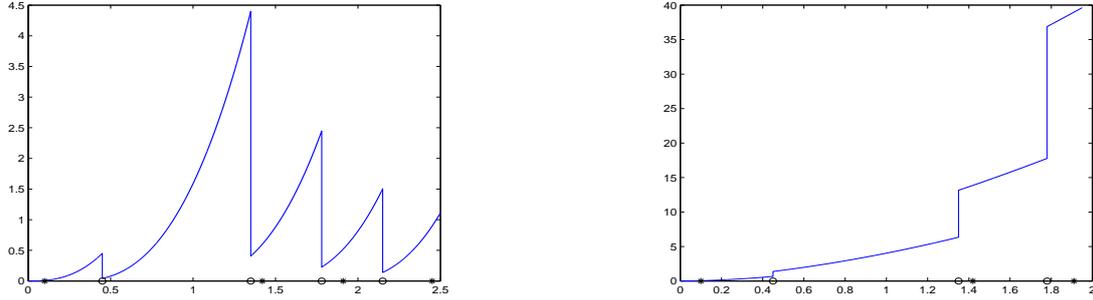
The model will be denoted  $ARA_1$ -LLP. Figure 7.11 illustrates a trajectory of  $ARA_1$ -LLP with the parameter values:  $a = -0.1, b = 1.2$  (at left  $\rho = 0.8$  and at right  $\rho = 1.2$ ). The fact that PM is better than new for  $\rho > 1$  can be seen on the right figure.

Figure 7.11: Intensity failure of the model  $ARA_1$ -LLP

- $ARA_\infty$  PM-ABAO CM: when the effects of the preventive maintenance are considered to have the Arithmetic Reduction of Age effect with infinite memory ( $ARA_\infty$ ) [33], the effective ages are  $A_i = (1 - \rho)(A_{i-1} + (\tau_i - \tau_{i-1}))$ . Recursively the failure intensity of  $ARA_\infty$  is [33]:

$$\lambda_t = \lambda \left( t - \rho \sum_{j=0}^{m_t-1} (1 - \rho)^j \tau_{m_t-j} \right). \quad (7.25)$$

Figure 7.12 shows the failure intensity of  $ARA_\infty$  PM-ABAO CM (left  $\rho = 0.7$  and right  $\rho = -0.5$ ).

Figure 7.12: Failure intensity in the case of  $ARA_\infty$  PM-ABAO CM


- ABAO CM-Brown Proschan PM [20]: this model is defined by external random variables  $B = \{B_i\}_{i \geq 1}$ , independent of Bernoulli distribution with parameter  $p$ .  $B_i$  represents the efficiency of the  $i^{\text{th}}$  repair:

$$B_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ PM is AGAN} \\ 0 & \text{if the } i^{\text{th}} \text{ PM is ABAO} \end{cases} \quad (7.26)$$

The effective ages are  $A_i = \sum_{j=1}^i \left[ \prod_{k=j}^i (1 - B_k) \chi_j \right]$ . The failure intensity is:

$$\lambda_t = \lambda(t - \tau_{m_{t-}} + A_{m_{t-}}). \quad (7.27)$$

## 7.4 Exact conditional GOF tests for NHPP

Our aim is to build GOF tests for the maintenance models presented before. Very few work exist for imperfect maintenance models [80, 131], but some work have been done for NHPP, i.e maintenance with ABAO effects [98, 9, 47, 48]. Here we will focus on a recent work by Lindqvist-Rannestad [79]. Their GOF tests are developed for any NHPP having sufficient statistic, particularly, for both the power-law and the log-linear NHPP models.

Suppose that the failure process is a NHPP with intensity function  $\lambda(t)$ .

**Definition 7.4** A statistic  $S = s(\mathbf{T}_n)$  is a sufficient statistic for the parameter  $\theta$  if it contains all the information about  $\theta$  that is available from the data. The conditional distribution of  $\mathbf{T}_n$  given  $S$  does not depend on  $\theta$  i.e:

$$P(\mathbf{T}_n = \mathbf{t}_n | S(\mathbf{T}_n) = s, \theta) = P(\mathbf{T}_n = \mathbf{t}_n | S(\mathbf{T}_n) = s) \quad (7.28)$$

where  $\mathbf{t}_n = (t_1, \dots, t_n)$ .

In practice the sufficient statistic can be found using the factorization theorem [24].

**Theorem 7.1** Factorization theorem

A statistic  $S = s$  is sufficient for  $\theta$  if and only if there exist functions  $g(s, \theta)$  and  $h(t)$  such that the likelihood function  $L_t$  is factorized as follows:

$$L_t(\theta) = g(s, \theta) h(t).$$

In accordance with the Factorization Theorem 7.1, we will compute the joint pdf resulting from the observed failure times  $T_1, \dots, T_n$  observed in  $[0, T]$  in order to find the sufficient statistics for both power law and log-linear process. We will take into account that the failure times are time truncated. The corresponding log-likelihood function is derived by replacing  $\lambda_t$  by  $\lambda(t)$  in (7.8). Then, the corresponding sufficient statistics of both cases are deduced.

- The power law case:

$$\begin{aligned} \mathcal{L}_T(\alpha, \beta) &= \ln \left( \prod_{j=1}^{N_T} \alpha \beta T_j^{\beta-1} \exp \left( - \int_0^T \alpha \beta s^{\beta-1} ds \right) \right) \\ &= N_T (\ln(\alpha) + \ln(\beta)) + (\beta - 1) \sum_{j=1}^{N_T} \ln(T_j) - \alpha T^\beta. \end{aligned} \quad (7.29)$$

Then, the sufficient statistic is:

$$S_p = \left( N_T, \sum_{j=1}^{N_T} \ln(T_j) \right). \quad (7.30)$$

- The log-linear case:

$$\begin{aligned} \mathcal{L}_T(a, b) &= \ln \left( \prod_{j=1}^{N_T} \exp(a + bT_j) \exp \left( - \int_0^T \exp(a + bs) ds \right) \right) \\ &= aN_T + b \sum_{j=1}^{N_T} T_j - (\exp(a)/b)(\exp(bT) - 1). \end{aligned} \quad (7.31)$$

Then, the sufficient statistic is:

$$S_l = \left( N_T, \sum_{j=1}^{N_T} T_j \right). \quad (7.32)$$

The tested null hypothesis is:

$$H_0: \lambda \in \mathcal{I} \text{ vs } H_1: \lambda \notin \mathcal{I}.$$

where the family  $\mathcal{I} = \{NHPP(\lambda(\cdot; \theta)); \theta \in \Theta\}$  is the family of NHPPs with intensity function  $\lambda(t; \theta)$  depending on some unknown parameters  $\theta$ . Let  $D = (N_T, T_1, \dots, T_{N_T})$  be the data observed at time  $T$ . Lindqvist-Rannestad test is based on the following sufficiency property: if  $S$  is a sufficient statistic, the distribution of  $D|S$  is independent of the parameters  $\theta$ .

Let  $Z$  be a GOF test statistic that has the ability to reveal departure from  $H_0$ . For example, we can consider that large values of  $Z$  correspond to the rejection of the null hypothesis.

Given  $S = s_{obs}$ , a test with a significance level  $\alpha$  rejects  $H_0$  when  $Z > k(s_{obs})$ , where  $k(s_{obs})$  is a critical value verifying  $P_{H_0}(Z \geq k(s_{obs})|S = s_{obs}) = \alpha$ . The critical value

$k(s_{obs})$  depends on the sufficient statistic and then on the tested data. Practically, the conditional  $p$ -value is considered:

$$p_{obs} = P_{H_0}(Z \geq z_{obs} | S = s_{obs}) \quad (7.33)$$

where  $s_{obs}$  and  $z_{obs}$  are the observed values of respectively the sufficient and the test statistics.

We suppose that we are able to simulate  $K$  realizations  $Z_1^*, \dots, Z_K^*$  of  $Z$  given  $S = s_{obs}$ . Using Monte Carlo method, the  $p$ -value in (7.33) can be estimated by the frequency of the event  $Z \geq z_{obs}$ :

$$\hat{p}_{obs} = \frac{1}{K} \sum_{i=1}^K \mathbb{1}_{\{Z_i^* \geq z_{obs}\}}. \quad (7.34)$$

The conditional sampling is straightforward when the intensity admits pivotal statistics such as in the power law case [79]. The conditional sampling can be done using the Gibbs algorithm.

To sum up, in order to build Lindqvist-Rannestad exact conditional GOF test for NHPPs, we need:

1. existence of a sufficient statistic;
2. conditional simulations of  $D|S = s_{obs}$  (that can be done either by using an explicit expression or by Gibbs algorithm);
3. computation of a GOF test statistic  $Z$ . Lindqvist and Rannestad chose for  $Z$  a statistic that tests the fitness of  $\frac{\Lambda(T_i)}{\Lambda(T)}$  to the uniform distribution. The transformation defined in proposition 7.6 is applied to the CM times  $T_1, \dots, T_n$  in order to reduce the problem to a GOF test for the uniform distribution. Lindqvist and Rannestad proposed to use classical GOF test statistics  $Z$  such as Laplace, Greenwood, Cramer-Von Mises, ...

Then the exact (non asymptotic)  $p$ -value of the test can be estimated by Monte Carlo simulations.

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We give in this chapter the principal properties of NHPP and we introduced the imperfect maintenance models. We presented Lindqvist-Rannestad GOF tests. In the next chapter, we aim to generalize these methods in order to derive GOF tests for imperfect maintenance models with deterministic PM.

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# Chapter 8

## Exact conditional GOF tests for the $ARA_1$ –LLP imperfect maintenance model

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In this chapter, we propose a generalization of Lindqvist-Rannestad GOF tests for a particular imperfect maintenance model with both CM and PM. The CM are assumed to be minimal (ABAO) with a log-linear initial intensity. It also assumes that the PM are carried out at deterministic times and that their effect is of the  $ARA_1$  type. In this case, a sufficient statistic exists and the tests of Lindqvist-Rannestad [79] can be generalized.

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### 8.1 Principle of the test

The construction of an exact conditional GOF test for the  $ARA_1$ –LLP imperfect maintenance model is possible. In this model, PM effect is of the  $ARA_1$  type and the failure intensity is considered to be log-linear. We remind the failure intensity of  $ARA_1$ –LLP:

$$\lambda_t(a, b, \rho) = \exp(a + b(t - \rho\tau_{m_t-})). \quad (8.1)$$

The CM effects are assumed to be ABAO. This assumption is meaningful because CM aims to quickly restore the system in working order. It is also common [80] and absolutely necessary in order to be able to apply Lindqvist-Rannestad [79] GOF test method since a NHPP is needed.

The GOF test in this case has the following hypotheses:

$$H_0 : \lambda_t(\theta) \in \mathcal{I} \text{ vs } H_1 : \lambda_t(\theta) \notin \mathcal{I}$$

where  $\mathcal{I}$  is the family of failure intensities defined in (8.1) for all  $(a, b, \rho) \in \mathbb{R}^3$ .

The considered model needs also to have a sufficient statistic. The  $ARA_1$ –LLP model has this property. There exists a second model that has a sufficient statistic (PM- $GRA_1$  (Geometric Reduction of age [33]) and CM ABAO with power-law intensity), but this model presents some drawbacks, so in all what follows we study only the model  $ARA_1$ –LLP.

In order to apply the same approach as Lindqvist and Rannestad [79] to  $ARA_1$ -LLP model we need:

- existence of a sufficient statistic;
- conditional simulation of  $D|S = s_{obs}$ ;
- computation of a GOF test statistic  $Z$ .

### 8.1.1 Sufficient statistic

**Property 8.1** *The cumulative intensity function of the  $ARA_1$ -LLP model is:*

$$\begin{aligned} \Lambda_t(a, b, \rho) &= \frac{\exp(a)}{b} \sum_{m=1}^{m_t} \exp(-b\rho\tau_{m-1}) \left[ \exp(b\tau_m) - \exp(b\tau_{m-1}) \right] \\ &\quad + \frac{\exp(a)}{b} \exp(-b\rho\tau_{m_t}) \left[ \exp(bt) - \exp(b\tau_{m_t}) \right] \quad \text{if } b \neq 0 \\ \Lambda_t(a, 0, \rho) &= \exp(a)t \end{aligned}$$

**Proof:**

By definition and equation (8.1), if  $b \neq 0$ , the cumulative intensity function is as follows:

$$\begin{aligned} \Lambda_t(a, b, \rho) &= \int_0^t \exp(a + bs - b\rho\tau_{m_s}) ds \\ &= \exp(a) \sum_{m=1}^{m_t} \int_{\tau_{m-1}}^{\tau_m} \exp(bs - b\rho\tau_{m-1}) ds + \exp(a) \int_{\tau_{m_t}}^t \exp(bs - b\rho\tau_{m_t}) ds \\ &= \frac{\exp(a)}{b} \sum_{m=1}^{m_t} \exp(-b\rho\tau_{m-1}) \left[ \exp(b\tau_m) - \exp(b\tau_{m-1}) \right] \\ &\quad + \frac{\exp(a)}{b} \exp(-b\rho\tau_{m_t}) \left[ \exp(bt) - \exp(b\tau_{m_t}) \right] \end{aligned}$$

and  $\Lambda_t(a, 0, \rho) = \exp(a)t$ . ■

We consider that the failure times are observed on the time interval  $[0, T]$ . For simplification reasons we denote  $\tau_{m_T+1} = T$  and we will use this notation in all the following.

**Property 8.2** *The log-likelihood function is:*

$$\mathcal{L}_T(a, b, \rho) = aN_T + b \sum_{T_i \leq T} T_i - b\rho \sum_{m=2}^{m_T+1} \tau_{m-1} (N_{\tau_m} - N_{\tau_{m-1}}) - \Lambda_T(a, b, \rho).$$

**Proof:**

Using previous property 7.1, the likelihood function evaluated in  $\theta = (a, b, \rho)$  at  $T$  denoted  $L_T(a, b, \rho)$  is:

$$\begin{aligned} L_T(a, b, \rho) &= \left[ \prod_{T_i \leq T} \lambda(T_i) \right] \exp(-\Lambda_T(a, b, \rho)) \\ &= \left[ \prod_{T_i \leq T} \exp(a + bT_i - b\rho\tau_{m_{T_i}}) \right] \exp(-\Lambda_T(a, b, \rho)) \end{aligned}$$

Therefore, property 8.2 is proved applying the logarithm to the previous expression. Indeed, we can easily prove that  $\sum_{i=1}^{N_T} \tau_{m_{T_i}} = \sum_{m=2}^{m_T+1} \tau_{m-1} (N_{\tau_m} - N_{\tau_{m-1}})$ . ■

Since  $\Lambda_T$  is a deterministic function, we apply the factorization theorem (7.1) and deduce the three components of the sufficient statistic  $S = (S_1, S_2, S_3)$ .

**Property 8.3** *The sufficient statistic of the  $ARA_1$ -LLP model exists and is:*

$$S = \left( N_T, \sum_{T_i \leq T} T_i, \sum_{m=2}^{m_T+1} \tau_{m-1} (N_{\tau_m} - N_{\tau_{m-1}}) \right). \tag{8.2}$$

## 8.2 Parameters estimation

In this section we derive the maximum likelihood estimation of the parameters.

**Property 8.4** *Let  $\hat{a}$ ,  $\hat{b}$ , and  $\hat{\rho}$  be respectively the MLEs of the parameters  $a$ ,  $b$  and  $\rho$  of the  $ARA_1$ -LLP model. They verify the following equations:*

$$\begin{cases} \hat{a} = \ln(S_1) - \ln(\tilde{\Lambda}_T(\hat{b}, \hat{\rho})) \\ -S_1 \left( \frac{\partial \tilde{\Lambda}_T}{\partial b}(\hat{b}, \hat{\rho}) \right) / \tilde{\Lambda}_T(\hat{b}, \hat{\rho}) + S_2 - \hat{\rho} S_3 = 0 \\ \hat{b} S_3 + S_1 \left( \frac{\partial \tilde{\Lambda}_T}{\partial \rho}(\hat{b}, \hat{\rho}) \right) / \tilde{\Lambda}_T(\hat{b}, \hat{\rho}) = 0 \end{cases} \tag{8.3}$$

where  $\tau_0 = 0$ ,  $\tau_{m_T+1} = T$ ,

$$\begin{aligned} \tilde{\Lambda}_T(b, \rho) &= \exp(-a)\Lambda_T(a, b, \rho) \\ &= \frac{1}{b} \sum_{m=1}^{m_T+1} \exp(-b\rho\tau_{m-1}) \left[ \exp(b\tau_m) - \exp(b\tau_{m-1}) \right] \end{aligned} \tag{8.4}$$

and

$$\left\{ \begin{aligned} \frac{\partial \tilde{\Lambda}_T}{\partial b}(b, \rho) &= -\frac{1}{b^2} \sum_{m=1}^{m_T+1} \exp(-b\rho\tau_{m-1}) (\exp(b\tau_m) - \exp(b\tau_{m-1})) \\ &+ \frac{1}{b} \sum_{m=1}^{m_T+1} (\tau_m - \rho\tau_{m-1}) \exp(b\tau_m - b\rho\tau_{m-1}) + \frac{1}{b}(\rho - 1) \sum_{m=1}^{m_T+1} \tau_{m-1} \exp(-b(\rho - 1)\tau_{m-1}) \\ \frac{\partial \tilde{\Lambda}_T}{\partial \rho}(b, \rho) &= - \sum_{m=1}^{m_T+1} \tau_{m-1} \exp(-b\rho\tau_{m-1}) (\exp(b\tau_m) - \exp(b\tau_{m-1})). \end{aligned} \right. \quad (8.5)$$

**Proof:**

We have  $\Lambda_T(a, b, \rho) = \exp(a)\tilde{\Lambda}_T(b, \rho)$ . The MLEs  $\hat{a}$ ,  $\hat{b}$  and  $\hat{\rho}$  are solutions of the maximization problem:

$$(\hat{a}, \hat{b}, \hat{\rho}) = \underset{(a,b,\rho) \in \mathbb{R}}{\operatorname{argmax}} \mathcal{L}_T(a, b, \rho)$$

which is equivalent to solving

$$\nabla \mathcal{L}_T(\hat{a}, \hat{b}, \hat{\rho}) = 0.$$

Thus,

$$\frac{\partial \mathcal{L}_T}{\partial a}(a, b, \rho) = S_1 - \frac{\partial \Lambda_T}{\partial a}(a, b, \rho)$$

$\Rightarrow$

$$S_1 - \exp(\hat{a})\tilde{\Lambda}_T(b, \rho) = 0$$

$\Rightarrow$

$$\hat{a}(b, \rho) = \ln \left( \frac{S_1}{\tilde{\Lambda}_T(b, \rho)} \right).$$

After substituting the value of  $\hat{a}$  in the log-likelihood function, the two remaining parameters estimators maximize the function:

$$\mathcal{L}_T(\hat{a}(b, \rho), b, \rho) = S_1 \ln \frac{S_1}{\tilde{\Lambda}_T(b, \rho)} + bS_2 - b\rho S_3 - S_1.$$

The two first partial derivatives of  $\mathcal{L}_T(\hat{a}(b, \rho), b, \rho)$  are given in (8.6). The MLEs set these partial derivatives to zero and verify the equations in (8.3).

$$\left\{ \begin{aligned} \frac{\partial \mathcal{L}_T}{\partial b}(\hat{a}(b, \rho), b, \rho) &= -S_1 \left( \frac{\partial \tilde{\Lambda}_T}{\partial b}(b, \rho) \right) / \tilde{\Lambda}_T(b, \rho) + S_2 - \rho S_3 \\ \frac{\partial \mathcal{L}_T}{\partial \rho}(\hat{a}(b, \rho), b, \rho) &= -bS_3 - S_1 \left( \frac{\partial \tilde{\Lambda}_T}{\partial \rho}(b, \rho) \right) / \tilde{\Lambda}_T(b, \rho). \end{aligned} \right. \quad (8.6)$$

■

The classical BFGS algorithm method [43] is used to solve the optimization problem and obtain the MLEs of the model parameters.

### 8.3 Extension of the sufficient statistic

The conditional sampling given the sufficient statistic  $S$  is too difficult especially given the third component. That is why we use a larger sufficient statistic in order to make the conditional sampling possible. The new sufficient statistic has the following expression:

$$\tilde{S} = \left( N_{\tau_1}, \dots, N_{\tau_{m_T}}, N_T, \sum_{T_i \leq T} T_i \right). \tag{8.7}$$

It is obvious that there is no loss of information when conditioning by the statistic  $\tilde{S}$  defined in equation (8.7) instead of  $S$  defined in (8.2). Apparently there is no need to know explicitly  $(N_{\tau_1}, \dots, N_{\tau_{m_T}}, N_T)$  in order to know  $S_3 = \sum_{m=2}^{m_T+1} \tau_{m-1}(N_{\tau_m} - N_{\tau_{m-1}})$ , but it is not always true.

The following example illustrates this fact. We consider 4 PM at times  $\tau_1 = 1.833, \tau_2 = 2.404, \tau_3 = 2.985, \tau_4 = 3.538$  and the CM are such that  $N_{\tau_1} = 6, N_{\tau_2} = 17, N_{\tau_3} = 19, N_{\tau_4} = 24$ . Let  $T = \tau_4$ . Then,  $N_{\tau_4} = 24$  and  $S_3 = \sum_{m=2}^4 \tau_{m-1}(N_{\tau_m} - N_{\tau_{m-1}}) = 39.896$ . It can be proved that  $(6, 17, 19, 24)$  is the only value of  $(N_{\tau_1}, N_{\tau_2}, N_{\tau_3}, N_{\tau_4})$  which verifies  $N_{\tau_4} = 24$  and  $S_3 = 39.896$ . So in this case  $S$  and  $\tilde{S}$  are equivalent. This fact will also happen in many cases for which the PM times are not periodic, because we have to find integer solutions to real coefficients equations.

### 8.4 Conditional sampling given the large sufficient statistic

#### 8.4.1 First step

The conditional sampling is done using the statistic  $\tilde{S}$ . We will use a classical trick for computational distributions given the sufficient statistic which consists in choosing the parameters values that give rise to particular simple models. This can be done since the conditional distribution, given the sufficient statistic, is the same whatever the parameter values are (definition 7.4). For simplification reasons, we shall use parameter values  $(a = b = 0)$  for which the model  $ARA_1$ -LLP is an HPP(1).

The objective is to be able to simulate HPP(1) conditionally to the sufficient statistic. Since the statistic in (8.7) includes the number of observed failures at each PM time, our first objective is to condition by  $\mathbf{N}_T = (N_{\tau_1}, \dots, N_{\tau_{m_T}}, N_T)$ . We will use the following property.

**Property 8.5** *Conditionally on  $\mathbf{N}_T$ , the event times of HPP(1) are distributed like  $(m_T + 1)$  independent samples, the  $i^{th}$  sample having the distribution of independent order statistics of  $(N_{\tau_i} - N_{\tau_{i-1}})$  variables uniformly distributed on  $[\tau_{i-1}, \tau_i]$ , for  $i \in \{1, \dots, m_T + 1\}$  where  $\tau_0 = 0, \tau_{m_T+1} = T$ .*

**Proof:**

We denote  $f_{\mathbf{T}_n|\mathbf{N}_T}$  the conditional pdf of  $\mathbf{T}_n|\mathbf{N}_T$  and  $\mathbf{n} = (n_1, \dots, n_{m_T}, n)$  where  $n_{m_T+1}$ :

$$f_{\mathbf{T}_n|\mathbf{N}_T=\mathbf{n}}(t_1, \dots, t_n) = P(N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n | T_1 = t_1, \dots, T_n = t_n) \times f_{\mathbf{T}_n}(t_1, \dots, t_n) / P(N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n).$$

$\mathbf{T}_n$  are failure times from  $HPP(1)$ . Property 7.4 gives the joint pdf of  $\mathbf{T}_n$ :

$$f_{\mathbf{T}_n}(t_1, \dots, t_n) = \exp(-t_n) \mathbb{1}_{\{0 < t_1 < \dots < t_n\}}.$$

Using property 7.2, we have:

$$P(N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n | T_1 = t_1, \dots, T_n = t_n) = \mathbb{1}_{\{0 < t_1 < \dots < t_{n_1} < \tau_1 < t_{n_1+1} < \dots < T\}} \times P(N_T - N_{t_n} = 0)$$

and

$$P(N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n) = P(N_{\tau_1} = n_1) P(N_{\tau_2} - N_{\tau_1} = n_2 - n_1) \times \dots \times P(N_T - N_{\tau_{m_T}} = n - n_{m_T}).$$

Since  $\{N_t\}_{t \geq 0}$  is a particular case of NHPP, we will use property 7.2 that justifies the independence of the increments and gives their probabilities. Then,

$$f_{\mathbf{T}_n|\mathbf{N}_T=\mathbf{n}}(t_1, \dots, t_n) = \exp(-T) \mathbb{1}_{\{0 < t_1 < \dots < t_{n_1} < \tau_1 < t_{n_1+1} < \dots < T\}} / \left[ \exp(-\tau_1) \frac{\tau_1^{n_1}}{n_1!} \exp(-(\tau_2 - \tau_1)) \frac{(\tau_2 - \tau_1)^{n_2 - n_1}}{(n_2 - n_1)!} \times \dots \times \exp(-(T - \tau_{m_T})) \frac{(T - \tau_{m_T})^{n - n_{m_T}}}{(n - n_{m_T})!} \right].$$

Finally,

$$f_{\mathbf{T}_n|\mathbf{N}_T=\mathbf{n}}(t_1, \dots, t_n) = n_1! \left[ \prod_{i=1}^{n_1} \frac{1}{\tau_1} \mathbb{1}_{\{0 \leq t_i \leq \tau_1\}} \right] \mathbb{1}_{\{0 < t_1 < \dots < t_{n_1} < \tau_1\}} \times (n_2 - n_1)! \left[ \prod_{i=n_1+1}^{n_2} \frac{1}{\tau_2 - \tau_1} \mathbb{1}_{\{\tau_1 \leq t_i \leq \tau_2\}} \right] \mathbb{1}_{\{\tau_1 < t_{n_1+1} < \dots < t_{n_2} < \tau_2\}} \times \dots \times (n - n_{m_T})! \left[ \prod_{i=n_{m_T}+1}^n \frac{1}{T - \tau_{m_T}} \mathbb{1}_{\{\tau_{m_T} \leq t_i \leq T\}} \right] \mathbb{1}_{\{\tau_{m_T} < t_{n_{m_T}+1} < \dots < t_n < T\}} \quad (8.8)$$

The distribution in (8.8) is the distribution of  $(m_T + 1)$  independent samples of order statistics of uniforms in  $[\tau_{i-1}, \tau_i]$  for  $i \in \{1, \dots, m_T + 1\}$  and  $\tau_{m_T+1} = T$ . We have consequently proved property 8.5. ■

Thanks to property 8.5, the simulation of  $\mathbf{T}_n$  conditionally to  $\mathbf{N}_T$  is reduced to simulating independent order statistics of uniforms  $(U_1, \dots, U_n)$ . Our next objective is to

simulate these uniforms  $(U_1, \dots, U_n)$  conditionally to the remaining components of the sufficient statistic  $\tilde{S}$  which is  $\sum_{i=1}^n U_i = \sum_{i=1}^n T_i = s_2$ .

The simulation problem is then transformed into a problem of conditional sampling of uniform variables. The purpose of the next subsection is to show how this conditional sampling can be carried out.

### 8.4.2 Second step

We consider the desired sample  $U_1, \dots, U_n$  composed of  $(m_T + 1)$  independent samples of iid random variables. Each sample  $i$  is, respectively, of size  $(n_i - n_{i-1})$  and follows  $\mathcal{U}[\tau_i, \tau_{i-1}]$ ,  $i \in \{1, \dots, m_T + 1\}$ , where  $n_0 = 0$ . There is apparently no simple direct way of sampling from the conditional distribution of the uniforms  $U_1, \dots, U_n$  given  $\sum_{i=1}^n U_i = s_2$ .

Inspired from [79], we use Gibbs sampler algorithm to simulate the desired samples. As explained in [79], there is no simple expression for the pdf of  $\sum_{i=1}^n U_i$ . The algorithm is based on the following property 8.6 which gives the conditional distribution of a pair  $(U_i, U_j)$  with  $i \neq j$  given  $\sum_{i=1}^n U_i = s_2$  and on the values of all  $U_k$  for  $k \neq i$  and  $k \neq j$ . Since

the conditional distribution of  $U_1, \dots, U_n$  given  $\sum_{i=1}^n U_i = s_2$  is singular, in order to have a proper conditional pdf we have to leave out one variable, for example  $U_j$ . We consider then the conditional distribution of  $U_1, \dots, U_{j-1}, U_j, \dots, U_n$  given  $\sum_{k=1}^n U_k = s_2$  and deduce

$$U_j = s_2 - \sum_{k \neq j} U_k.$$

We use a modified Gibbs algorithm where in each iteration two of the vector components  $(U_i, U_j)$ ,  $i \neq j$ , are updated. The algorithm consists in simulating at iteration  $m$  the conditional pdf of  $U_i^m | U_k^{m-1} = u_k^{m-1}, k \neq i, k \neq j, \sum_{k=1}^n U_k^{m-1} = s_2$ . This last simulation is equivalent to the simulation of  $U_i^m | U_i^{m-1} + U_j^{m-1} = s_2 - \sum_{k \neq i, j} u_k^{m-1}$ . That is why we will compute in the following the conditional cdf of  $U_i | U_i + U_j$ .

**Property 8.6** *Let  $0 \leq c_i^1 < c_i^2, 0 \leq c_j^1 < c_j^2, U_i$  and  $U_j$  two independent random variables from respectively  $\mathcal{U}[c_i^1, c_i^2]$  and  $\mathcal{U}[c_j^1, c_j^2]$ .*

*Then the conditional distribution of  $U_i$  given  $U_i + U_j = s$  is uniform on  $I$  where:*

- $I = [c_i^1, s - c_j^1]$  if  $c_i^1 + c_j^1 \leq s \leq \min(c_i^2 + c_j^1, c_i^1 + c_j^2)$
- $I = [c_i^1, c_i^2]$  if  $c_i^2 + c_j^1 \leq s \leq c_i^1 + c_j^2$
- $I = [s - c_j^2, s - c_j^1]$  if  $c_i^1 + c_j^2 \leq s \leq c_i^2 + c_j^1$
- $I = [s - c_j^2, c_i^1]$  if  $\max(c_j^1 + c_i^2, c_i^1 + c_j^2) \leq s \leq c_i^2 + c_j^2$ .

**Proof:**

We know that the conditional pdf of  $U_i|U_i + U_j = s$  is:

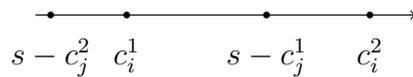
$$f_{U_i|U_i+U_j=s}(u) = \frac{f_{U_i}(u)}{f_{U_i+U_j}(s)} f_{U_i+U_j|U_i=u}(s). \quad (8.9)$$

Let us now compute each of the pdfs of  $U_i + U_j|U_i = u$  and  $U_i + U_j$ . By definition, we have:

$$\begin{aligned} f_{U_i+U_j|U_i=u}(s) &= f_{U_j|U_i=u}(s-u) \\ &= f_{U_j}(s-u) \\ &= \begin{cases} \frac{1}{c_j^2 - c_j^1} & \text{if } u \in [c_i^1, c_i^2] \cap [s - c_j^2, s - c_j^1] = I \\ 0 & \text{else} \end{cases} \end{aligned} \quad (8.10)$$

The values of the interval bounds  $I$  depend on the order of  $s$ ,  $c_i^1$ ,  $c_i^2$ ,  $c_j^1$  and  $c_j^2$ . We will check all the possible configurations.

If



then

$$I = [c_i^1, s - c_j^1] \text{ and } c_i^1 + c_j^1 \leq s \leq \min(c_i^1 + c_j^2, c_i^2 + c_j^1).$$

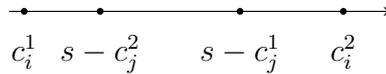
If



then

$$I = [c_i^1, c_i^2] \text{ and } c_i^2 + c_j^1 \leq s \leq c_i^1 + c_j^2.$$

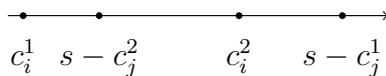
If



then

$$I = [s - c_j^2, s - c_j^1] \text{ and } c_i^1 + c_j^2 \leq s \leq c_i^2 + c_j^1.$$

If



then

$$I = [s - c_j^2, c_i^2] \text{ and } \max(c_i^1 + c_j^2, c_i^2 + c_j^1) \leq s \leq c_i^2 + c_j^2.$$

If



then

$$I = \emptyset \text{ and } s \geq c_i^2 + c_j^2 \text{ or } s \leq c_i^1 + c_j^1.$$

Only 4 cases from 5 are explored. Indeed, the two cases:  $c_i^2 + c_j^1 \leq s \leq c_i^1 + c_j^2$  and  $c_i^1 + c_j^2 \leq s \leq c_i^2 + c_j^1$  are exclusive. Now, we compute the pdf of  $U_i + U_j$ :

$$f_{U_i+U_j}(s) = \int_{c_i^1}^{c_i^2} f_{U_i+U_j|U_i=u}(s) f_{U_i}(u) du = \frac{1}{(c_i^2 - c_i^1)(c_j^2 - c_j^1)} \int_I du. \quad (8.11)$$

From equations (8.9), (8.10) and (8.11), we can deduce the conditional pdf of  $U_i|U_i + U_j = s$ :

$$\begin{aligned} f_{U_i|U_i+U_j=s}(u) &= \frac{f_{U_i}(u) f_{U_j}(s-u)}{f_{U_i+U_j}(s)} \\ &= \frac{1}{c_i^2 - c_i^1} \mathbb{1}_{[c_i^1, c_i^2]}(u) \frac{1}{c_j^2 - c_j^1} \mathbb{1}_{[s-c_j^2, s-c_j^1]}(s-u) / \left( \frac{1}{(c_j^2 - c_j^1)(c_i^2 - c_i^1)} \int_I du \right) \\ &= \frac{\mathbb{1}_I(u)}{\int_I du}. \end{aligned} \quad (8.12)$$

Hence the distribution of  $U_i|U_i + U_j = s$  is the uniform distribution on  $I$ . ■

Finally, the Gibbs sampler algorithm is given next in algorithm 2 for the model  $ARA_1$ -LLP. It makes conditional sampling of  $\mathbf{T}_n|N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n, \sum_{i=1}^n T_i = s_2$ .

For the initialization of the algorithm, Lindqvist and Rannestad used the same value  $s_2/n$  of all the components:  $(t_1^0, \dots, t_n^0) = (s_2/n, \dots, s_2/n)$ . We propose here in algorithm 1 a random initialization which guarantees  $\sum_{i=1}^n t_i^0 = s_2$  and  $N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n$ .

This initialization is independent of the first configuration of the tested data, which makes the convergence of the Gibbs algorithm faster. Furthermore, our procedure guarantees the independence of the successive simulated values of  $\mathbf{T}_n| \sum_{i=1}^n T_i = s_2$ . Then it is not necessary to use a burn in period as in [79].

Let  $n = n_{m_T+1}$  and  $n_0 = 0$ .

---

**Algorithm 1** Initialization of Gibbs sampler algorithm

---

```

for all  $j \in \{1, \dots, m_T + 1\}$  do
  for all  $i \in \{n_{j-1}, \dots, n_j\}$  do
    draw  $u_i^0 \sim U[\tau_{j-1}, \tau_j]$ 
     $d_i^1 \leftarrow u_i^0 - \tau_{j-1}$ 
     $d_i^2 \leftarrow \tau_j - u_i^0$ 
  end for
end for
if  $\sum_{i=1}^n u_i^0 > s_2$  then
  for all  $i \in \{1, \dots, n\}$  do
    
$$t_i^0 \leftarrow u_i^0 - d_i^1 \frac{\sum_{i=1}^n u_i^0 - s_2}{\sum_{i=1}^n d_i^1}$$

  end for
else
  for all  $i \in \{1, \dots, n\}$  do
    
$$t_i^0 \leftarrow u_i^0 + d_i^2 \frac{s_2 - \sum_{i=1}^n u_i^0}{\sum_{i=1}^n d_i^2}$$

  end for
end if
sort  $t_1^0, \dots, t_n^0$ 
return  $t_1^0, \dots, t_n^0$ 

```

---

**Proof:** We will prove that  $t_i^0 \in [\tau_{m(i)-1}, \tau_{m(i)}], i \in \{1, \dots, n\}$  where  $m(i) = \underset{j \in \{1, \dots, m_T+1\}}{\operatorname{argmin}} \{i \leq n_j\}$

and  $\sum_{i=1}^n t_i^0 = s_2$ . We have:

$$\sum_{i=1}^n t_i^0 = \begin{cases} \sum_{i=1}^n u_i^0 - \left( \sum_{i=1}^n d_i^1 \right) \frac{\sum_{i=1}^n u_i^0 - s_2}{\sum_{i=1}^n d_i^1} = s_2 & \text{if } \sum_{i=1}^n u_i^0 > s_2 \\ \sum_{i=1}^n u_i^0 + \left( \sum_{i=1}^n d_i^2 \right) \frac{s_2 - \sum_{i=1}^n u_i^0}{\sum_{i=1}^n d_i^2} = s_2 & \text{if } \sum_{i=1}^n u_i^0 < s_2 \end{cases}$$

Since  $s_2$  is observable then,

$$\sum_{i=1}^n \tau_{m(i)-1} \leq s_2 = \sum_{i=1}^n T_i \leq \sum_{i=1}^n \tau_{m(i)} \tag{8.13}$$

$\Rightarrow$

$$-\sum_{i=1}^n d_i^1 \leq s_2 - \sum_{i=1}^n u_i^0 \leq \sum_{i=1}^n d_i^2. \tag{8.14}$$

If  $\sum_{i=1}^n u_i^0 > s_2$  then,

$$0 < \frac{\sum_{i=1}^n u_i^0 - s_2}{\sum_{i=1}^n d_i^1} \leq 1$$

and if  $\sum_{i=1}^n u_i^0 < s_2$  then,

$$0 < \frac{s_2 - \sum_{i=1}^n u_i^0}{\sum_{i=1}^n d_i^2} \leq 1.$$

We can deduce easily that  $t_i^0 \in [\tau_{m(i)-1}, \tau_{m(i)}], i \in \{1, \dots, n\}$ .

■

Let  $nb$  be a large number to guarantee the convergence of the Gibbs algorithm.

**Algorithm 2** Gibbs sampler algorithm for conditional sampling of  $\mathbf{T}_n | N_{\tau_1} = n_1, \dots, N_{\tau_{m_T}} = n_{m_T}, N_T = n, \sum_{i=1}^n T_i = s_2$

---

```

Start with initializing  $t_i^0, i = 1, \dots, n$  (algorithm 1)
for all  $k \in \{1, \dots, nb\}$  do
     $t_i^{k-1} \leftarrow t_i^k, i = 1, \dots, n$ 
    draw integers  $1 \leq i < j \leq n$  randomly
    let  $n_i$  and  $n_j$  of  $\{n_1, \dots, n_{m_t}, n\}^2$  be such that  $n_{i-1} < i \leq n_i$  and  $n_{j-1} < j \leq n_j$ 
    let  $s \leftarrow t_i^{k-1} + t_j^{k-1}, c_i^1 \leftarrow \tau_{n_{i-1}}, c_i^2 \leftarrow \tau_{n_i}, c_j^1 \leftarrow \tau_{n_{j-1}}, c_j^2 \leftarrow \tau_{n_j}$ 
    if  $c_i^1 + c_j^1 \leq s \leq \min(c_i^2 + c_j^1, c_i^1 + c_j^2)$  then
        draw  $t_i^k \sim U[c_i^1, s - c_j^1]$ 
    else if  $c_i^2 + c_j^1 \leq s \leq c_i^1 + c_j^2$  then
        draw  $t_i^k \sim U[c_i^1, c_i^2]$ 
    else if  $c_i^1 + c_j^2 \leq s \leq c_i^2 + c_j^1$  then
        draw  $t_i^k \sim U[s - c_j^2, s - c_j^1]$ 
    else if  $\max(c_j^1 + c_i^2, c_i^1 + c_j^2) \leq s \leq c_i^2 + c_j^2$  then
        draw  $t_i^k \sim U[s - c_j^2, c_i^1]$ 
    end if
     $t_j^k \leftarrow s - t_i^k$ 
end for
return  $t_1^{nb}, \dots, t_n^{nb}$ 
    
```

---

It has been shown in [24] that the distribution of the sample  $(t_1^k, \dots, t_n^k)$  converges to the target distribution, whatever the starting vector is. The successive simulated samples are from a Markov chain, and the target distribution is the stationary distribution of this Markov chain. 'Burn in' samples are needed before the samples can be taken to be from the correct distribution.

## 8.5 Transformation to uniforms

When the conditional sampling is done, a GOF test is chosen to detect the departure from the tested model. We will present later the GOF tests used and we first use the previous transformation given in property 7.6.

Since the parameters  $(a, b, \rho)$  of the model  $ARA_1$ –LLP are unknown, we use the MLEs  $(\hat{a}, \hat{b}, \hat{\rho})$  defined in property (8.4). Let  $\hat{\Lambda}(\cdot)$  be an estimate of the cumulative intensity function  $\Lambda(\cdot)$  based on the observation  $(T_1, \dots, T_n, N_T)$ , defined as:

$$\hat{\Lambda}_t = \Lambda_t(\hat{a}, \hat{b}, \hat{\rho}). \tag{8.15}$$

We consider the estimated transformed times defined as follows:  $\hat{V}_i = \frac{\hat{\Lambda}(T_i)}{\hat{\Lambda}(T)}$ . The distribution of the last sample  $\hat{V}_1, \dots, \hat{V}_n$  is very close to order statistics of uniforms (property 7.6). We can use the classical GOF tests for the uniform distribution to suggest GOF tests for the tested model based on the  $\hat{V}_i$ . This was already the approach of Lindqvist-Rannestad [79] and before him of Baker in [9]. For  $ARA_1$ –LLP model, we

check the uniformity of the variables  $\hat{V}_i, i = 1, \dots, n$ :

$$\hat{V}_i = \frac{\sum_{m=1}^{m_{T_i}} \exp(-\hat{b}\hat{\rho}\tau_{m-1}) \left[ \exp(\hat{b}\tau_m) - \exp(\hat{b}\tau_{m-1}) \right] + \exp(-\hat{b}\hat{\rho}\tau_{m_{T_i}}) \left[ \exp(\hat{b}T_i) - \exp(\hat{b}\tau_{m_{T_i}}) \right]}{\sum_{m=1}^{m_T+1} \exp(-\hat{b}\hat{\rho}\tau_{m-1}) \left[ \exp(\hat{b}\tau_m) - \exp(\hat{b}\tau_{m-1}) \right]} \quad (8.16)$$

where  $(\hat{b}, \hat{\rho})$  are the maximum likelihood estimators of parameters  $(b, \rho)$  in property 8.4.

After obtaining  $(\hat{V}_1, \dots, \hat{V}_n)$ , instead of using these random variables, we can transform the simulated samples to another iid order statistics on  $\mathcal{U}[0, 1]$ . Since  $\hat{\Lambda}$  is an increasing function,  $\forall j \in \{1, \dots, m_T\}, \forall i \in \{n_j, \dots, n_{j+1}\}$ , let  $V_\tau = \frac{\hat{\Lambda}(\tau)}{\hat{\Lambda}(T)}$ , we have:

$$\hat{V}_{\tau_j} = \frac{\hat{\Lambda}(\tau_j)}{\hat{\Lambda}(T)} \leq \frac{\hat{\Lambda}(T_i)}{\hat{\Lambda}(T)} \leq \frac{\hat{\Lambda}(\tau_{j+1})}{\hat{\Lambda}(T)} = \hat{V}_{\tau_{j+1}}.$$

Then, all the order statistics  $(\hat{V}_i)_{n_j, \dots, n_{j+1}}$  follow  $\mathcal{U}[\hat{V}_{\tau_j}, \hat{V}_{\tau_{j+1}}]$ . We define for all  $i \in \{1, \dots, n\}$ :

$$\tilde{V}_i = \left( \hat{V}_i - \hat{V}_{\tau_{m(i)}} \right) / \left( \hat{V}_{\tau_{m(i)+1}} - \hat{V}_{\tau_{m(i)}} \right). \quad (8.17)$$

The following approaches can be applied on either the samples  $(\hat{V}_1, \dots, \hat{V}_n)$  or  $(\tilde{V}_1, \dots, \tilde{V}_n)$ .

## 8.6 Test statistics

In all the following simulations, we apply to the transformed samples  $(\hat{V}_j)_{j=1, \dots, n}$  the classical test statistics for the uniform distribution as it was done in [79].

- Laplace statistic:

$$L = \sqrt{\frac{12}{n}} \sum_{j=1}^n \left( \hat{V}_j - \frac{1}{2} \right) \quad (8.18)$$

- Greenwood statistic:

$$G = \sum_{j=1}^{n+1} (\hat{V}_j - \hat{V}_{j-1})^2 \quad (8.19)$$

- Modified Cramer Von Mises statistic:

$$CM = \sum_{j=1}^n \left[ \hat{V}_j - \frac{(2j-1)}{2n} \right]^2 + \frac{1}{12n} \quad (8.20)$$

- Modified Anderson Darling statistic:

$$AD = -\frac{1}{n} \left[ \sum_{j=1}^n (2j-1) [\ln(\hat{V}_j) + \ln(1 - \hat{V}_{n+1-j})] \right] - n \quad (8.21)$$

- Modified Kolmogorov-Smirnov statistic:

$$KS = \max \left[ \max_{1 \leq j \leq n} \left( \frac{j}{n} - \hat{V}_j \right), \max_{1 \leq j \leq n} \left( \hat{V}_j - \frac{j-1}{n} \right) \right]. \quad (8.22)$$

To sum up the whole approach, if we have failure times  $T_1, \dots, T_n$  and we wish to test if these instants are from the  $ARA_1$ -LLP model, we apply the following steps in algorithm 3. Let  $K$  be a large number to guarantee the computation of the p-value estimated by:

$$\hat{p}_{obs} = \frac{1}{K} \sum_{k=1}^K \mathbb{1}_{\{Z_k^* \geq z_{obs}\}}. \quad (8.23)$$

---

**Algorithm 3** Steps to follow for the computation of the p-value

---

Compute the observed sufficient statistic  $\tilde{S} = \tilde{s}_{obs}$ .

Compute the test statistic  $z_{obs}$  from the observation.

Compute the MLEs  $(\hat{a}, \hat{b}, \hat{\rho})$  using property 8.4.

**for all**  $k \in \{1, \dots, K\}$  **do**

apply algorithm 2 to simulate  $T_1, \dots, T_n | (\tilde{S} = \tilde{s}_{obs})$

compute the transformation to the uniforms given in (8.16)

compute the test statistic  $Z_k$  given in subsection 8.6

**end for**

Compute the p-value  $\sum_{k=1}^K \mathbb{1}_{\{Z_k^* \geq z_{obs}\}} / K$

---

## 8.7 Simulation results

In this section, we use all the previous tools to assess the performance of the developed GOF tests for the  $ARA_1$ -LLP model.

The power study is carried out using different alternative models chosen to be Brown-Proschan models. The significance level in all the simulations results is fixed at 5% and 4 PM are made at the times ( $\tau_1 = 1.833, \tau_2 = 2.404, \tau_3 = 2.985, \tau_4 = 3.538$ ). These values are the same as those used in section 8.3. They are chosen such that there is an equivalence between  $S$  and  $\tilde{S}$ .

The power of the test is assessed by the percentage of rejection of  $H_0$  over the total number of simulated samples. We set the number of the simulated samples from each tested alternative to 5000. We apply the approach presented in algorithm 3: we simulate  $K = 10000$  samples of  $\hat{V}_i, i = 1, \dots, n$ . The 'burn in' period is set to 200.

The GOF tests used here are Laplace  $L$ , Greenwood  $G$ , Cramer-Von-Mises  $CM$ , Anderson-Darling  $AD$  and Kolmogorov-Smirnov  $KS$  tests. We denote respectively the GOF tests  $\tilde{L}, \tilde{G}, \tilde{CM}, \tilde{AD}$  and  $\tilde{KS}$  the same tests based on the transformation in equation (8.17).

We first simulate samples from the  $ARA_1$ -LLP model, in order to check that the percentage of rejection is close to the nominal significance level 5%. All the simulations of

the alternative models are done using the inverse of the conditional cdf given in property 7.3.

The difficulty we faced for the simulations is how to find a compromise between the number of simulated CM between two successive PM and the choice of the parameters values. Indeed, the simulated sample sizes depend on the parameters values and the PM times. In all the simulations, we tried to adjust the parameters in a way we can both test large and small sample sizes and to have at least one CM in average between two successive PM.

Tables 8.1 to 8.3 show the computed rejection percentages (powers) of the GOF tests. The first column gives the parameter values of the simulated models and the last column gives the mean value of the number of the simulated CM.

Table 8.1 gives the powers when the simulated model is  $ARA_1$ -LLP. The first column presents the values of parameters  $(a, b, \rho)$ . We obviously obtain the significance level 5% as a power of the test when we simulate the model under the null hypothesis  $H_0$ .

alternatives	$L$	$G$	$CM$	$AD$	$KS$	$\tilde{L}$	$\tilde{G}$	$\widetilde{CM}$	$\widetilde{AD}$	$\widetilde{KS}$	$n$
(1, 0.1, 0.9)	4.8	5	4.7	5.3	5	5.1	5.7	4.3	4.7	4.5	115
(1, 0.8, 0.75)	5.4	4.1	4.8	4.9	4.1	4.5	5	4.1	4.3	4.3	28
(1, 2.8, 1.2)	5	5.8	4.8	5	5.5	5.4	5	5.3	5.2	5	160

Table 8.1: Power results:  $ARA_1$ -LLP under  $H_0$

Among several simulated alternatives, we choose to present the following ones.

1. The Brown-Proschan model defined previously in section 7.3.2 with a log-linear initial intensity function, denoted  $BP_l(a, b, p)$ . When  $p = 0$  or  $p = 1$ , this model is a special case of the model  $ARA_1$ -LLP with  $\rho = 0$  (if  $p = 0$  ABAO) and  $\rho = 1$  (if  $p = 1$  AGAN). If  $0 < p < 1$ , we expect to have a model that should be rejected by the developed test. Unfortunately, the power of the tests is close to 5% for all the values of probability  $p$ . Table 8.2 shows the power results. The GOF tests in this case did not detect the departure from  $H_0$ .

alternatives	$L$	$G$	$CM$	$AD$	$KS$	$\tilde{L}$	$\tilde{G}$	$\widetilde{CM}$	$\widetilde{AD}$	$\widetilde{KS}$	$n$
$BP_l(1, 0.9, 0)$	5.7	5.2	5.6	5.5	4.4	4.9	5.1	4.8	5.3	4.9	70
$BP_l(1, 0.9, 1)$	5.7	5.5	5.8	5.3	5.1	4.7	4.1	4.4	5.4	5.3	18
$BP_l(1, 0.9, 0.5)$	4.1	4	4.2	4.4	5.9	3.1	5.5	4.7	4.5	5.6	45
$BP_l(1, 0.9, 0.7)$	5	5.8	5	4.5	4.3	4.9	4.5	4.9	5.3	4.9	34
$BP_l(1, 0.9, 0.3)$	5	5	5.2	4.5	5.9	5.1	4	4.7	5	4.9	55
$BP_l(3, 0.5, 0.5)$	4.4	4.4	5	4.6	5	4.5	5.5	4.2	4.4	4.8	145

Table 8.2: Power results:  $BP_l(a, b, p)$  as alternative

2. The Brown-Proschan model with a power-law intensity function, denoted  $BP_p(\alpha, \beta, p)$ . Table 8.3 gives the power results. For  $\beta = 1$ , the  $BP_p$  is an HPP, which is a special case of the  $ARA_1$ -LLP model. So it is normal to find a power close to the significance level 5%. We also notice that this model is rejected by the developed GOF tests and their performances depend tightly on the chosen parameter values.

alternatives	$L$	$G$	$CM$	$AD$	$KS$	$\tilde{L}$	$\tilde{G}$	$\widetilde{CM}$	$\widetilde{AD}$	$\widetilde{KS}$	$n$
$BP_p(3.8, 1, 0)$	4.9	4.3	4.6	5	5	5.1	5.6	4.2	5.1	5.2	13
$BP_p(3.8, 1, 0.6)$	5.2	5	4.6	5	5	5.1	5.6	5.2	5.2	5.4	13
$BP_p(3, 1, 0)$	5.2	5.6	5.4	5.4	4.6	4.9	5.2	4.9	5.2	5.2	10
$BP_p(3.8, 0.1, 0.6)$	72.4	23.4	12.3	11.9	9.6	75.9	46.3	32.8	81	34.9	10
$BP_p(3.8, 0.1, 0.3)$	71.5	23.3	5.3	5.3	4.2	73.1	35.6	18.3	51.4	18.3	8
$BP_p(3.8, 0.1, 0.8)$	72.7	24.3	14	14.3	11	81.1	55.3	44.2	90.7	44.5	12
$BP_p(3.8, 0.1, 1)$	72.6	26.4	18.1	18.6	14.7	83.7	58.1	58.5	98.2	58.5	15
$BP_p(3.8, 0.2, 0.3)$	50.7	16.5	3.6	4.5	3.4	51.2	17.5	8.7	29.1	7	8
$BP_p(20, 0.2, 0.3)$	82.3	12.5	0.4	0.4	1.7	75.5	30.3	7.1	34.9	7.6	40
$BP_p(3.8, 1.5, 0.5)$	5.6	4.9	4.6	5	7.1	5.2	6.2	5.5	5.5	6.3	20
$BP_p(10, 1.5, 0.5)$	9.6	7.8	4.7	3.8	11.1	6.6	5.1	7	6.1	8.7	50
$BP_p(1, 2.4, 0.5)$	5.1	5.1	4.3	4.1	5.7	5.2	6.1	2.1	2.3	3.8	13
$BP_p(10, 2.4, 0.5)$	28.4	22.4	3.4	3.2	23.8	22.3	6.3	4.3	3.7	6.5	130

Table 8.3: Power results:  $BP_p$  as alternative

We notice that some of the GOF tests are biased with powers less than the significance level (for instance  $CM$ ,  $AD$  and  $KS$  when the tested alternative is  $BP_p(20, 0.2, 0.3)$ ). Moreover, the rejection percentages do not always increase with the sample size. The departure from  $H_0$  is detected for a decreasing intensity function i.e when  $\beta < 1$ . But when  $\beta > 1$ , the GOF tests are not able to reject the null hypothesis: the power is less or equal than the significance level. The GOF test  $\widetilde{AD}$  has the highest rejection percentage for even small samples (90.7% and 98.2% for respectively  $n = 12$  and  $n = 15$ ).

The two tests  $\tilde{L}$  and  $L$  are among the most powerful ones with high rejection percentages.

It seems that the 'adapted' GOF test  $\widetilde{AD}$  has a very good performance. This is due to the effect of the transformation in equation (8.17) on the repartition of the sample  $\tilde{V}_i$ , but this test has the disadvantage of being biased on some cases.

From the previous power results (tables 8.2 and 8.3), we deduce that the GOF tests are able to detect the shape of the tested initial failure intensity  $\lambda(\cdot)$  but unable to detect the different effects of the PM. This is may be due to the sufficient statistic that includes a lot of information and deteriorates the tests performances.

We have started to study the case of periodic PM. In this case, the problem of section 8.3 may have several solutions. Then, it is possible to perform the conditional sampling given the minimal sufficient statistic. This is a promising prospect, but further research is needed

---

In this chapter, we developed a generalization of Lindqvist-Rannestad GOF tests applied to a specific imperfect maintenance model with deterministic PM. The power results are mitigated, but some tests are reasonably powerful. Then, this is a promising area of future research. The work has been presented in the MMR 2013 conference [69].

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# Chapter 9

## Application to real data sets

---

In this chapter, we apply the presented GOF tests to real data sets. Firstly, we check the relevance of the Weibull distribution for some data sets from the literature representing lifetimes of non repairable systems and data sets from EDF representing physical quantities: toughness and length of defects. Secondly, GOF tests for repairable systems are also applied to real data sets from EDF consisting in sequences of CM and PM times.

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### Warning

For confidentiality reasons, EDF data used in this chapter have been modified. However the results presented here are consistent with the results of the original data.

## 9.1 Non repairable systems

### 9.1.1 Literature data

In this section, the GOF tests for the Exponential and Weibull distributions are applied to two data sets. The first data set (Xie data) deals with the time to failure of 18 electronic devices [130] (Table 9.1).

Table 9.1: Failure data of 18 electronic devices (Xie data)

5	11	21	31	46	75
98	122	145	165	195	224
245	293	321	330	350	420

The second data set (Aarset data) is a classical data set of the literature [2] previously introduced in chapter 1. It represents failure times of 50 devices. Table 9.2 represents these data.

Table 9.2: Failure data of 50 devices (Aarset data)

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Figure 9.1 presents the Weibull probability plots (WPP) of the two previous data sets. For Aarset data, we can notice easily that the points are not aligned, so we can deduce that the Weibull assumption is rejected for this data set. This decision will be confirmed later using the GOF tests. For Xie data, the graphical approach is not concluding, the judgment about the points alignment is not straightforward. As a consequence, any decision based on the WPP is not robust enough. In this case, the use of the GOF tests is needed.

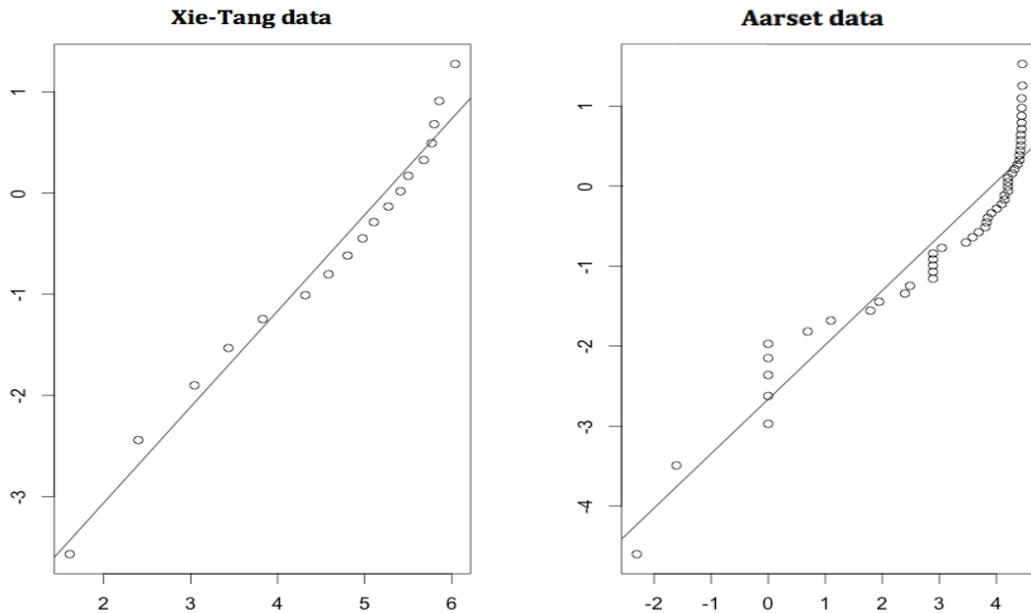


Figure 9.1: Weibull probability plots of the data sets

We remind that the previous chapters 4, 5 and 6 showed that the GOF tests performances depend on the monotony of the hazard rate. Non parametric estimation of the hazard rates of the tested data sets is then useful. In all that follows, we use a non parametric estimation of the hazard rate. A smoothed estimator of the hazard rate [90]

using the cumulative hazard Nelson-Aalen estimator [93, 1]  $H_n = \sum_{i=1}^n \frac{\mathbb{1}_{\{X_i \leq t\}}}{n - i + 1}$  has the expression:

$$\begin{aligned}\hat{h}_n(t) &= \int \frac{1}{b} K\left(\frac{t-x}{b}\right) dH_n(t) \\ &= \frac{1}{b} \sum_{i=1}^n K\left(\frac{t-X_i}{b}\right) \frac{1}{n-i+1}\end{aligned}$$

We chose to use a bi-weight (quartic) kernel  $K(x) = \frac{15}{16}(1-x^2)^2 \mathbb{1}_{|x| \leq 1}$  and an optimal bandwidth  $b$  that minimizes the mean integrated squared error  $\mathbb{E} \left[ \int (\hat{h}_n(t) - h(t))^2 dt \right]$  [90]. The same kernel is used for the density estimation:

$$\hat{f}_n(t) = \frac{1}{nb} \sum_{i=1}^n K\left(\frac{t-X_i}{b}\right).$$

Figures 9.2 and 9.3 show respectively the estimated hazard rates and pdfs of the previous data sets. For Xie data, the estimated hazard rate can be considered as increasing then according to chapter 6, the best GOF test to use is  $\widehat{LS}$ . For Aarset data, the estimated hazard rate is not BT neither IHR then the recommended GOF test in this case is  $T_1$ . The estimated pdf of Xie data in figure 9.3 looks like the pdf of an exponential distribution, while the estimated hazard rate is not constant. This contradiction is probably due the bad performance of the non parametric estimation of the hazard rate for small data set (18 observations). For Aarset data the estimated pdf is bimodal, thus far from the pdf of the Weibull distribution.

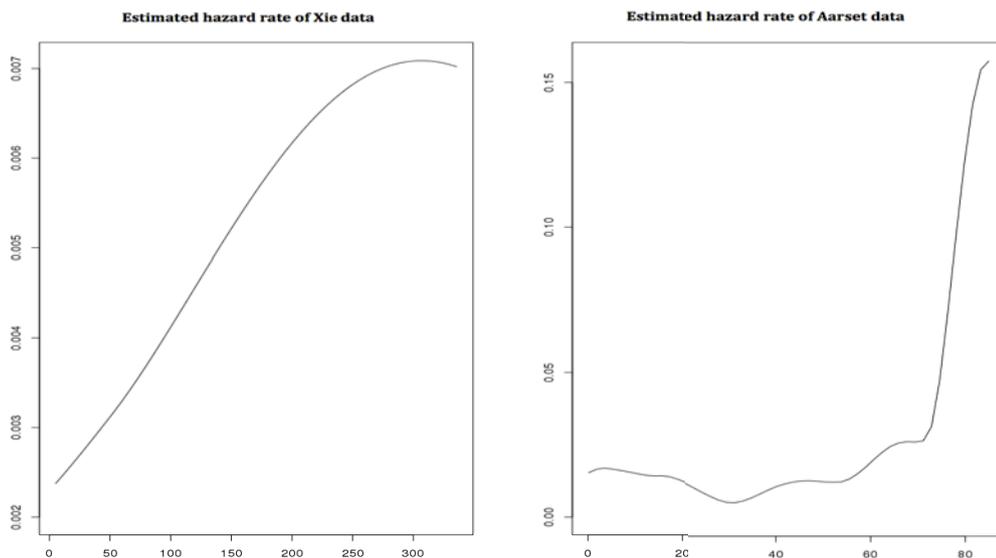


Figure 9.2: Estimated hazard rates of the data sets

In order to check the previous remark about the Xie data pdf that looks like an Exponential distribution, we apply the best GOF tests for the Exponential distribution.

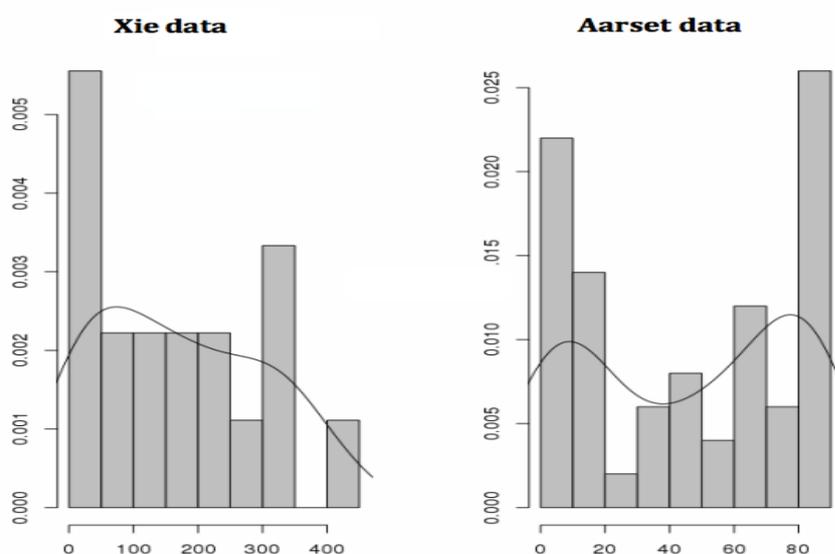


Figure 9.3: Estimated pdfs of the data sets

We use the R package EWGoF to apply the GOF tests. The example below illustrates the use of the functions of the EWGoF package:

```
> LK.test(Xie.data, type='CO')
```

Test of Cox and Oakes for the Exponential distribution

```
data: Xie.data
```

```
S =0.4062, p-value=0.556
```

```
sample estimates:
```

```
[1] 0.005812076
```

```
> CF.test(Xie.data, type='EP')
```

Test of Epps and Pulley for the Exponential distribution

```
data: Xie.data
```

```
S =-0.8985, p-value=0.737
```

```
sample estimates:
```

```
[1] 0.005812076
```

Each function from EWGoF package is dedicated to a special family of the GOF tests, for instance **LK.test** is the function that includes all the likelihood based tests for the Exponential distribution such as *CO*, *Sc* and *LR*. The function **CF.test** includes the GOF tests based on the characteristic function: *EP*,  $HM^{(1)}$ ,  $HM^{(2)}$ ,  $MI^{(1)}$ , ... The input 'type' in each function should be specified. In addition to the p-value of the test, the outputs of each function are: the value of the test statistic, the maximum likelihood estimation of the parameter  $\hat{\lambda}_n$ , the name of the applied test and the data's name. Detailed documentation of the EWGoF package is given in appendix B.

Table 9.3 gives the p-values of the best GOF tests. These p-values are high, thus the Exponential distribution is not rejected for Xie data set.

Table 9.3: P-values of the GOF tests for the Exponential distribution (Xie data)

GOF tests	<i>AD</i>	<i>CO</i>	<i>BH</i>	<i>He</i>
P-values	0.616	0.556	0.517	0.523

Since the Exponential distribution is not rejected for Xie data, the Weibull distribution should not be rejected either. We apply the best GOF tests for the Weibull distribution to Xie data set in order to check that the Weibull distribution is not rejected and to Aarset data in order to test whether or not the Weibull distribution is a relevant distribution. The example below illustrates the application of the Weibull GOF tests to Aarset data. As for the Exponential distribution, the functions of the R package are dedicated to the families of the GOF tests for the Weibull distribution. The functions names start with the letter W for Weibull. For instance, for the tests based on the empirical distribution function, the function name in the package is **WEDF.test**. The input 'funEstimate' precises the method used for parameters estimation (LSE, MLE or ME). By default the MLEs are used. The output of the function are the p-value of the test, the value of the test statistics and the estimates of parameters  $\eta$  and  $\beta$  (LSE, MLE, ME).

```
> WEDF.test(Aarset,type='AD')
```

Test of Anderson and Darling for the Weibull distribution using the MLEs

```
data: Aarset
S = 3.5877, p-value < 2.2e-16
sample estimates:
eta beta
44.9125175 0.9490436
```

```
> WEDF.test(Aarset, type='LS', funEstimate='LSE')
```

Test of Liao and Shimokawa for the Weibull distribution using the LSEs

```
data: Aarset
S = 1.5971, p-value = 0.014
sample estimates:
eta beta
46.4540001 0.7523908
```

```
> WNS.test(Aarset,type='TS')
```

Test of Tiku and Singh for the Weibull distribution

```
data: Aarset
```

S = 1.3715, p-value < 2.2e-16  
 sample estimates:  
 eta beta  
 44.9125175 0.9490436

The p-values of the tests are given in table 9.4 for Xie data and table 9.5 for Aarset data.

Table 9.4: P-values of the best GOF tests for Weibull distribution (Xie data)

GOF tests	$OK^*$	$SPP$	$\widehat{AD}$	$\widetilde{LS}$	$TS$	$LOS$	$\check{S}T_4$
P-values	0.118	0.643	0.297	0.587	0.148	0.238	0.693
GOF tests	$\widehat{CQ}^*$	$\widehat{GG}_l^2$	$\widehat{EW}_w$	$\widehat{PGW}_w$	$\widetilde{MO}_w$	$T_1$	$T_2$
P-values	0.111	0.15	0.161	0.11	0.172	0.896	0.896

Table 9.5: P-values of the best GOF tests for Weibull distribution (Aarset data)

GOF tests	$OK^*$	$SPP$	$\widehat{AD}$	$\widetilde{LS}$	$TS$	$LOS$	$\check{S}T_4$
P-values	$< 2.2 \cdot 10^{-16}$	$< 2.2 \cdot 10^{-16}$	$< 2.2 \cdot 10^{-16}$	0.014	$< 2.2 \cdot 10^{-16}$	not defined	0.09
GOF tests	$\widehat{CQ}^*$	$\widehat{GG}_l^2$	$\widehat{EW}_w$	$\widehat{PGW}_w$	$\widetilde{MO}_w$	$T_1$	$T_2$
P-values	$10^{-7}$	$< 2.2 \cdot 10^{-16}$	$10^{-5}$	$< 2.2 \cdot 10^{-16}$	$4 \cdot 10^{-3}$	$10^{-7}$	$10^{-5}$

- For Xie data, the p-values are quite high. The recommended GOF test in this case,  $\widetilde{LS}$ , has a large p-value of 58.7%. Some of the GOF tests such as  $OK^*$ ,  $\widehat{CQ}^*$  and  $\widehat{PGW}_w$  have lower p-values around 11% but they are high enough, so we can not reject the Weibull distribution. This is consistent with the previous results of the Exponential GOF tests. The given estimated scale and shape parameters of the Weibull distribution are  $\hat{\eta} = 179.59$  and  $\hat{\beta} = 1.14$ . Since  $\hat{\beta}_n$  is close to 1, the Exponential distribution is a distribution that fits well Xie data.
- For Aarset data, the p-values are all very low. All the GOF tests strongly reject the Weibull assumption at very low significance level, except the  $\check{S}T_4$  that has a larger p-value (9%). Since this test is biased, we consider that the Weibull assumption is rejected. These results confirm the previous work of Lai et al [72] which showed that the new modified Weibull distribution presented in section 4.3.6 can fit better this data set.

### 9.1.2 EDF data

In this section, the application is carried out to EDF data sets. The observations in this case are not the lifetimes of components but some physical quantities and mechanical characteristics of the studied component. As said in chapter 1, the work of Weibull himself [127] justifies the relevance of using Weibull distribution for this kind of data.

The case study deals with the mechanical performance of a passive component within EDF power plant. The reliability of the component depends on two main characteristics: the length of the defects and the toughness of the material. Under severe stress conditions, the preexisting flaws, which uneventfully remain non-progressive through the operation of the structure, might initiate if the toughness is not high enough. Examinations have been performed, resulting in 150 measures of the length of the defects, 143 and 24 measures of the toughness, under respectively, fixed temperatures  $\delta_1$  and  $\delta_2$ . Table 9.6 gives the data of the toughness at  $\delta_2$ .

Table 9.6: Toughness data at  $\delta_2$ 

14.13	67.54	70.68	98.96	102.10	105.24
105.24	149.22	171.21	177.49	183.78	190.06
205.77	240.33	252.89	268.60	284.31	293.73
300.02	303.16	312.58	362.85	369.13	409.97

The first question to be answered is whether or not the Exponential distribution is adapted for the studied variables. Figure 9.4 shows Exponential probability plots of the data sets. The alignment of the points is clearly questionable for all the data sets. This doubt about the Exponential distribution has to be confirmed by a GOF test.

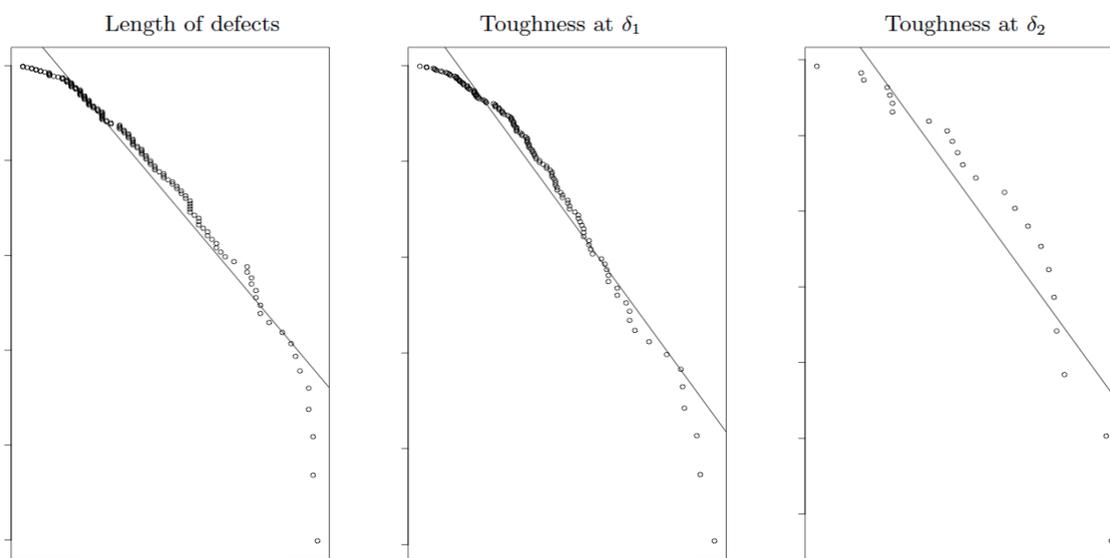


Figure 9.4: Exponential probability plots of the data sets

We apply the best recommended GOF tests for the Exponential distribution to the previous three data sets. Tables 9.7, 9.8 and 9.9 give the p-values of the GOF tests. The p-values are very small so the Exponential distribution is strongly rejected for the three data sets.

Table 9.7: P-values of the GOF tests for the Exponential distribution (length of defects)

GOF tests	$AD$	$CO$	$BH$	$He$
P-values	$< 2.2 \cdot 10^{-16}$			

Table 9.8: P-values of the GOF tests for the Exponential distribution (toughness at  $\delta_1$ )

GOF tests	$AD$	$CO$	$BH$	$He$
P-values	$< 2.2 \cdot 10^{-16}$	$< 2.2 \cdot 10^{-16}$	$10^{-2}$	$< 2.2 \cdot 10^{-16}$

Table 9.9: P-values of the GOF tests for the Exponential distribution (toughness at  $\delta_2$ )

GOF tests	$AD$	$CO$	$BH$	$He$
P-values	$3 \cdot 10^{-2}$	$3 \cdot 10^{-2}$	$4.5 \cdot 10^{-2}$	$6.5 \cdot 10^{-2}$

Since the Exponential distribution is rejected, we check now the relevance of the Weibull distribution for the three data sets. Figure 9.5 shows the Weibull probability plots. The alignment of the points is striking for length of defects and toughness at  $\delta_1$ , but less obvious for toughness at  $\delta_2$ .

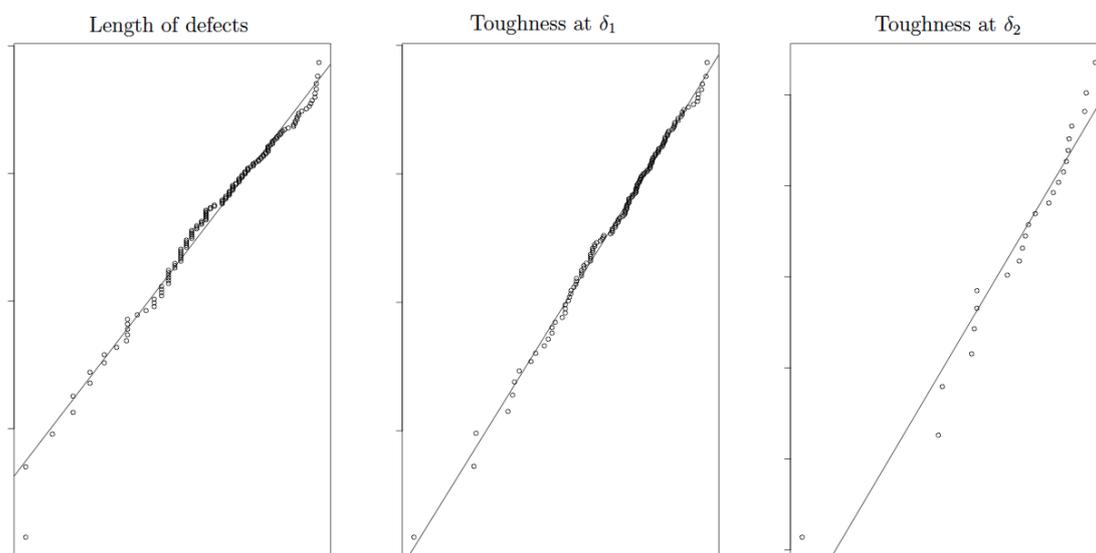


Figure 9.5: Weibull probability plots of the data sets

Graphically in figure 9.6, the estimated hazard rates (kernel non-parametric estimation) of the data sets show that they have increasing hazard rates and that the Weibull distribution can be a candidate distribution. The estimated pdfs in figure 9.7 show the same thing (the Weibull distribution may be a possible model).

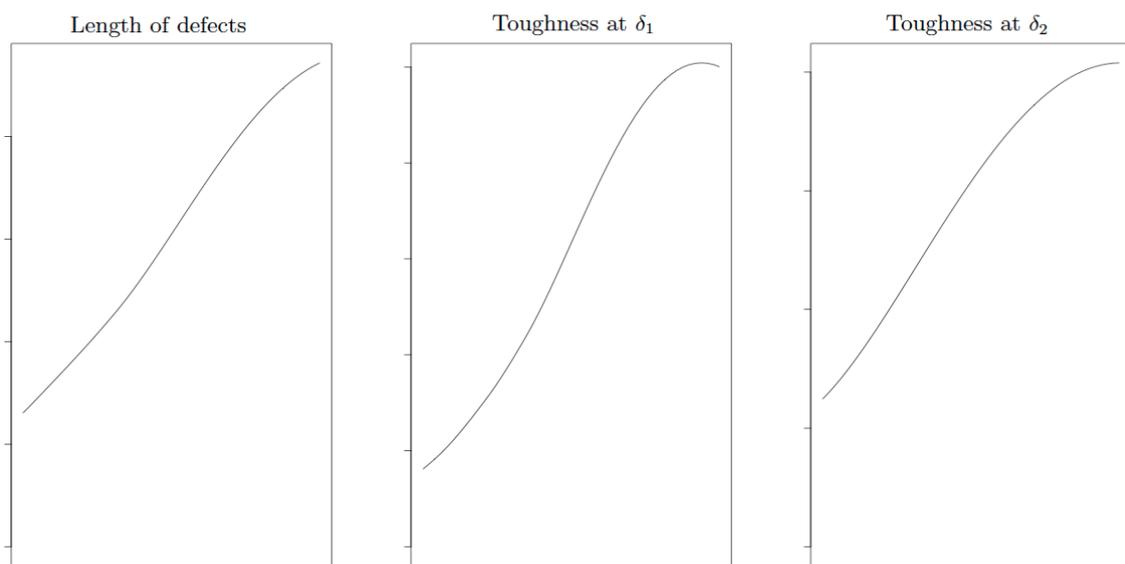


Figure 9.6: Estimated hazard rates of the data sets

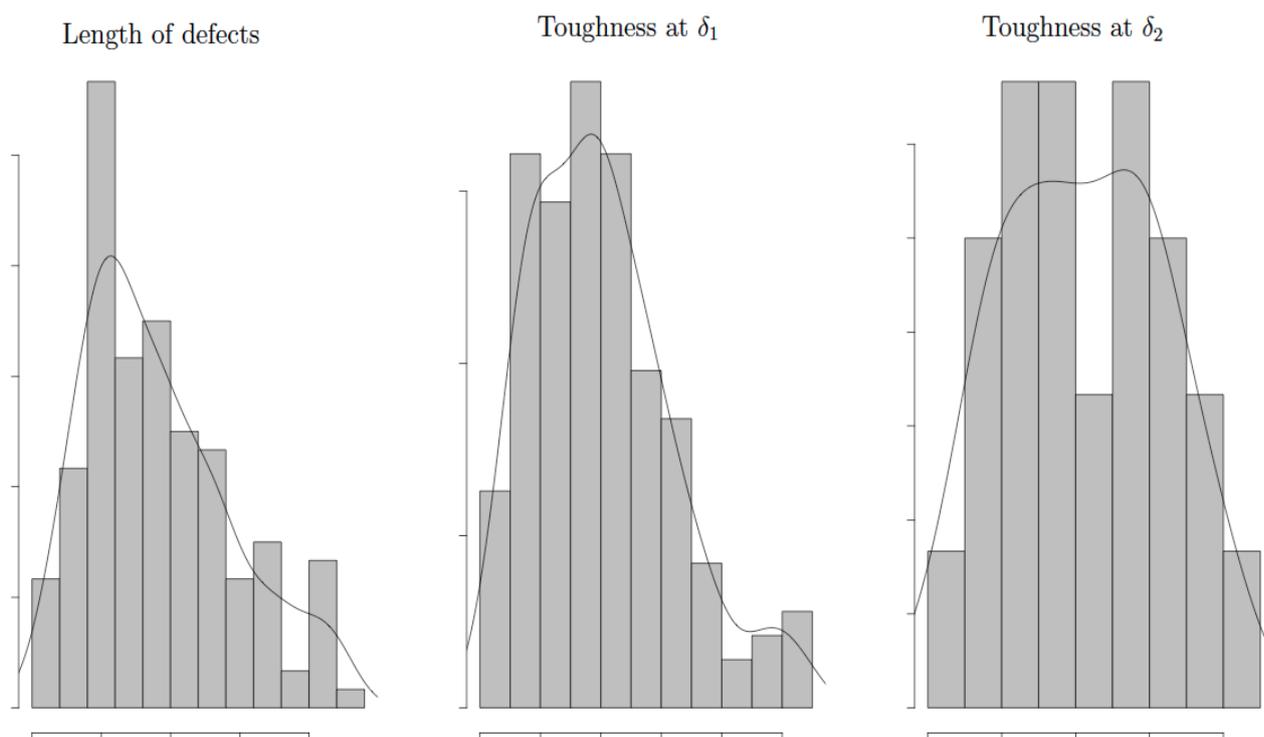


Figure 9.7: Estimated pdfs of the data sets

Table 9.10: P-values of the GOF tests for Weibull distribution (length of the defects)

GOF tests	$OK^*$	$SPP$	$\widehat{AD}$	$\widetilde{LS}$	$TS$	$LOS$	$\check{S}T_4$
P-values	0.626	0.218	0.184	0.125	0.31	not defined	0.193
GOF tests	$\widehat{CQ}^*$	$\widehat{GG}_l^2$	$\widehat{EW}_w$	$\widehat{PGW}_w$	$\widetilde{MO}_w$	$T_1$	$T_2$
P-values	0.569	0.446	0.498	0.576	0.227	0.41	0.126

Table 9.11: P-values of the GOF tests for Weibull distribution (toughness at  $\delta_1$ ,  $n = 143$ )

GOF tests	$OK^*$	$SPP$	$\widehat{AD}$	$\widetilde{LS}$	$TS$	$LOS$	$\check{S}T_4$
P-values	0.956	0.435	0.867	0.907	0.885	0.94	0.908
GOF tests	$\widehat{CQ}^*$	$\widehat{GG}_l^2$	$\widehat{EW}_w$	$\widehat{PGW}_w$	$\widehat{MO}_w$	$T_1$	$T_2$
P-values	0.915	0.885	0.891	0.908	0.861	0.488	0.321

Table 9.12: P-values of the GOF tests for Weibull distribution (toughness at  $\delta_2$ ,  $n = 24$ )

GOF tests	$OK^*$	$SPP$	$\widehat{AD}$	$\widetilde{LS}$	$TS$	$LOS$	$\check{S}T_4$
P-values	0.169	0.761	0.431	0.546	0.172	0.266	0.336
GOF tests	$\widehat{CQ}^*$	$\widehat{GG}_l^2$	$\widehat{EW}_w$	$\widehat{PGW}_w$	$\widehat{MO}_w$	$T_1$	$T_2$
P-values	0.161	0.164	0.168	0.15	0.301	0.946	0.974

We apply the best GOF tests for the Weibull distribution. The p-values are given in tables 9.10, 9.11 and 9.12.

Table 9.13 gives the estimation of the two parameters of the Weibull distribution for the length of defects and the toughness data.

Table 9.13: Parameters estimation

Data	$\hat{\eta}$	$\hat{\beta}$
Length of defects	10.31	1.8
Toughness at $\delta_1$	226.8	1.78
Toughness at $\delta_2$	235.7	2.03

For the toughness data, the p-values are very high especially for the toughness at  $\delta_1$ . Thus, the Weibull distribution is clearly not rejected in this case. It is consistent with the habit of using the Weibull distribution to model the materials toughness [127]. For the length of defects data, the p-values are not too high, especially for some GOF tests such as  $\widehat{AD}$ ,  $\widetilde{LS}$ ,  $\widehat{MO}_w$  and  $T_1$ , but still large enough to not reject the Weibull distribution.

The p-value of  $TS$  is not so high either. This can be explained by the fact that the tested data set has a lot of ties. Since the statistic  $TS$  is based on the spacings, a large number of these spacings are null. So the p-value is not too high, but it is still high enough to lead to the same conclusion. The GOF test  $LOS$  is not defined. This is also due to the presence of tied observations, since the expression of the test statistic (3.31) is based on logarithms. For instance for Aarset data, the two largest values of the sample are equal  $X_{n-1}^* = X_n^* = 86$ . Then the normalized spacing  $E_n$  is equal to 0 and the random variable defined in (3.30):  $Z_{n-1} = 1$ . Then  $\ln(1 - Z_{n-1})$  is infinite, so the statistic  $LOS$  is not defined. The presence of tied observations presents the weakness of the GOF tests based on the normalized spacings in spite their global good performances.

To sum up, for the three data sets (length of defects and the toughness under  $\delta_1$  and  $\delta_2$ ) the Weibull distribution can be used to model these quantities.

## 9.2 Repairable systems

The first case study deals with the welds of three systems within the boiler of an EDF coal-fired power station. The welds are subjected to thermal fatigue leading to the initiation then propagation of cracks that may generate leaks. These require the immediate shutdown of the boiler, and consequently of the plant, which may be critical for the installation and electric power transmission operators when the energy demand is huge and the power grid is over-stretched. In order to manage the technical and economic risks, a specific maintenance plan is carried out:

- When a leak occurs during plant operation, the incriminated weld is repaired without delay to restart the plant as soon as possible.
- Scheduled preventive inspections of the hazard zones of the system are carried out and the detected cracks are scoured.

The operation feedback data of three systems are available but not since the commission date of the plant: it only covers the operation period 1997 – 2006. Over this time interval, CM times and PM times are at our disposal. Table 9.14 shows data of the first system. The data is not expressed in calendar time, but in the (cumulative) number of cold starts (CS) over the observation period of the plant: indeed, from a physical point of view, this operation parameter is known to be one of the most influential on the wear-out of the system welds. Thus, considering year 1997 as the time of reference, for the first system:

- 7 CM occurred, the first one after 50 CS and the last one after 195 CS.
- 3 scheduled PM actions were carried out.
- the ending time of the OF data 1 (right censoring) is  $T = 264$  CS. This time is considered to be the last PM.

In [108], the data set 1 was studied and the PM AGAN-CM  $ARA_\infty$  model was adopted.

Table 9.14: Available OF data 1

<b>Number of CS</b>	25	50	93	109	114	141	163	164	195	225	264
<b>Maintenance type</b>	PM	CM	CM	CM	PM	CM	CM	CM	CM	PM	PM

We will assume that the system was new in 1997, as data has only been available since then. Consequently, the presented results will characterize the relative wear-out of the system since 1997 rather than its absolute one.

The data of the two other systems are given in the following tables 9.15 and 9.16:

We want to check whether the data are consistent with the model  $ARA_1$ -LLP using the statistics presented in chapter 8 ( $L$ ,  $G$ ,  $CM$ ,  $AD$  and  $KS$ ). For data 1, we first apply Laplace statistic  $L$  and compute its value for the observed data (OF data 1)  $L_{obs}^1 = 0.273$ . After, we simulate the distribution of  $L$ : this is done by simulating 100000 samples of

Table 9.15: Available OF data 2

<b>Number of CS</b>	43	156	180	189	190	214	243	257
<b>Maintenance type</b>	PM	CM	PM	CM	CM	CM	CM	PM

Table 9.16: Available OF data 3

<b>Number of CS</b>	2	130	149	161	178	181	225	227	235
<b>Maintenance type</b>	PM	PM	CM	CM	CM	CM	CM	PM	PM

$\hat{V}_i$  and for each simulated sample we compute the value of the statistic  $L$ . The resulting histograms of some statistics are given in figure 9.8. The resulting p-value is 81.1% (twice the tail to the left of the observed since the test is two-sided). Hence the Laplace statistic does not reject the  $ARA_1$ -LLP model.

The estimated p-values of the statistics are given in table 9.17. All the p-values are quite high and all conclude the acceptance of the null hypothesis. The lower p-values are 14.2% and 14.1% respectively, given by  $CM$  and  $AD$ . The estimates of the parameters are given by the MLEs:  $\hat{a} = -3.995$ ,  $\hat{b} = 0.0131$  and  $\hat{\rho} = 1.253$ . We notice that  $\hat{\rho} > 1$  which means that the PM are better than new, which is consistent with the AGAN assumption in [108].

Table 9.17: P-values of the GOF tests

GOF tests	$L$	$G$	$CM$	$AD$	$KS$
P-values	0.811	0.708	0.142	0.141	0.374
GOF tests	$\tilde{L}$	$\tilde{G}$	$\widetilde{CM}$	$\widetilde{AD}$	$\widetilde{KS}$
P-values	0.943	0.796	0.578	0.791	0.413

The previous GOF tests were applied with  $nb = 100000$ . The same application of the GOF tests were carried-out to the two data sets data 2 and data 3. Tables 9.18 and 9.19 give the p-values of the presented GOF tests applied to respectively data 2 and data 3. The p-values are high which implies the non-rejection of the  $ARA_1$ -LLP model of  $H_0$ .

Table 9.18: P-values of the GOF tests for OF data 2

GOF tests	$L$	$G$	$CM$	$AD$	$KS$
P-values	0.289	0.554	0.355	0.433	0.446
GOF tests	$\tilde{L}$	$\tilde{G}$	$\widetilde{CM}$	$\widetilde{AD}$	$\widetilde{KS}$
P-values	0.311	0.587	0.412	0.390	0.421

The estimates of the parameters are  $\hat{a} = -6.0534$ ,  $\hat{b} = 0.00380468$  and  $\hat{\rho} = -3.314$  for the OF data 2.

Figure 9.8: Simulated distribution of some test statistics

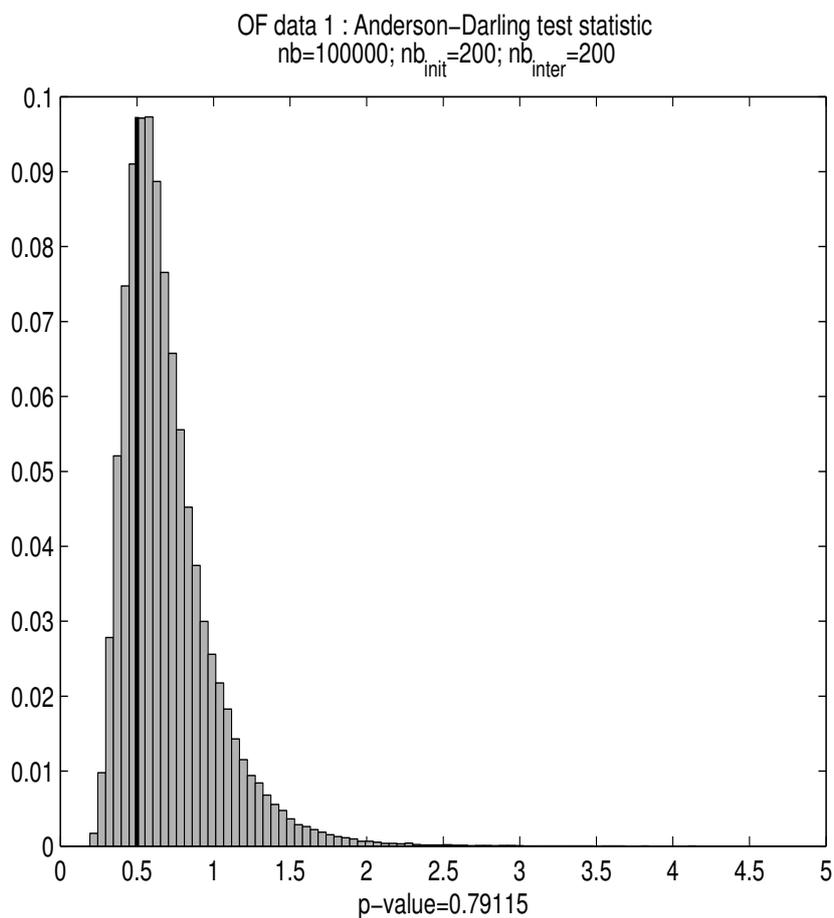
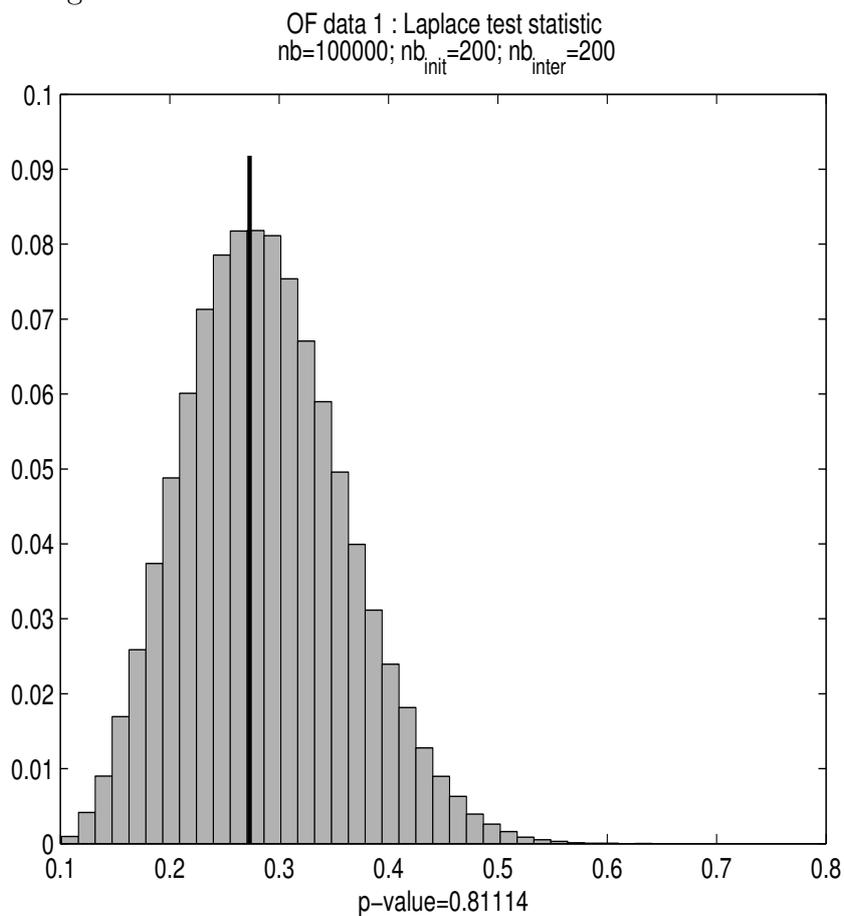


Table 9.19: P-values of the GOF tests for OF data 3

GOF tests	$L$	$G$	$CM$	$AD$	$KS$
P-values	0.979	0.712	0.381	0.429	0.263
GOF tests	$\tilde{L}$	$\tilde{G}$	$\widetilde{CM}$	$\widetilde{AD}$	$\widetilde{KS}$
P-values	0.979	0.187	0.246	0.378	0.135

The estimated  $\hat{a} = -5.894$ ,  $\hat{b} = 0.00872862$  and  $\hat{\rho} = -0.849999$  for the OF data 3.

# Chapter 10

## Prospects

This work constitutes a contribution to both theoretical and practical mathematics applied to reliability. It deals with goodness-of-fit tests for non repairable and repairable systems. These tests represents an efficient tool to check the relevance of a model to a given data set.

The developed works presented in this dissertation open doors to new prospects. First, concerning the new GOF tests for the two-parameter Weibull distribution, we showed that some of these new GOF tests are very competitive with the classical ones. Until now only complete samples were studied, future work can be done to adapt these GOF tests for type II simple censoring. For instance, for the likelihood based tests, the censoring can be considered when estimating the Weibull parameters. The maximum likelihood estimators for simple censoring exist and can be computed and plugged in the expression of the statistics. GOF tests are needed for data with other kinds of censoring such as multiple and interval censoring.

Secondly, we showed that the combination of the test statistics with complementary behavior is a promising way of improving the tests performances. A future work is needed to identify the best combinations to be used.

Third, in some industrial cases, the use of the three parameter Weibull distribution is usual. However, GOF tests for this distribution are less developed in the literature. Future work can be inspired from the GOF tests for the two-parameter distribution.

Finally, multiple GOF tests can be done. It might be interesting to nest the Weibull distribution in more than one Genrelized Weibull distributions and test all the parameters at once.

Concerning repairable systems, more GOF tests need to be developed for more imperfect maintenance models. The performance of the GOF tests based on the sufficient statistic can be improved in the case of periodic and semi periodic PM. Indeed, the conditional sampling given the minimal sufficient statistic seems to be possible.

The application to EDF data sets brings the need to explore more research tracks and raise questions that the GOF tests can not answer. Indeed, when the null hypothesis is rejected, we are not able to know which model to adapt. Sometimes, we have more than one candidate models to test and we need to choose the closest one from a range of possible models. These kinds of questions can be answered using the model selection

tools.

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# Appendix A

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This appendix contains all the simulation results which aim to assess the power of the GOF tests for the Exponential and Weibull distributions.

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## A.1 Power results of the GOF tests for the Exponential distribution: complete samples

$n$	$n = 5$			$n = 5$			$n = 10$			$n = 10$		
altern.	$KS$	$CM$	$AD$	$KS$	$CM$	$AD$	$KS$	$CM$	$AD$	$KS$	$CM$	$AD$
exp(0.2)	4.9	5.5	5.4	4.9	4.8	4.7	4.8	4.9	5.1	4.5	5.2	4.9
exp(1)	5.2	5.6	5.4	4.8	5.2	5.2	5	5	5.1	5.2	4.7	5.2
exp(2)	4.9	5.4	5.2	5.1	5.1	4.8	5.1	5.1	5.2	4.8	4.9	5.2
exp(42)	4.8	5.2	5.2	5.2	4.7	5	5.2	5.2	5.2	5.1	4.8	5.1
$\mathcal{W}(0.5)$	34.5	37.6	47.4	27.5	30.2	51	60	64.7	74.3	56.2	61	77
$\mathcal{W}(0.8)$	9.4	10.4	11.6	6.9	6.6	13.5	13.2	14.6	19.7	9.6	11.6	17.9
$\mathcal{W}(0.98)$	5.3	5.7	5.7	5	4.2	5.8	5.2	5.4	5.6	4.4	3.9	5.4
$\mathcal{W}(1.5)$	9	10.5	9.4	11.6	12.6	8.3	18.9	22	21.1	21.2	22.9	17.2
$\mathcal{W}(3)$	47.9	57.2	52.4	53.2	62.1	50.7	90.3	96.1	96	89.5	96.6	95.4
$\mathcal{LN}(0.6)$	18	20.6	17.1	23.7	23.9	16.9	42.5	47.9	42.6	48.7	51.9	47.5
$\mathcal{LN}(0.8)$	8.5	9.3	7.5	10.4	10.3	6.8	14.1	15.3	13.1	16.3	16.6	14.8
$\mathcal{LN}(1.4)$	16.7	17.8	16.1	13.7	12.2	16	31.4	32.6	30.7	26.7	26.6	27
$\mathcal{U}[0, 2]$	13	15.1	16.8	15.4	17.3	12.8	28.5	33.8	40.7	27.8	37.7	31.9
$\mathcal{G}(0.5)$	17.9	19.7	27.5	13.7	14.3	31.1	28	31.5	42.3	23.8	30.3	48.9
$\mathcal{G}(2)$	9.2	10.5	9.2	12.1	12.3	8	18.8	21.5	20	21.2	24.4	21.2
Mean	17.2	19.5	20.1	17.5	18.7	20.1	31.9	35.1	36.6	31.4	34.8	36.7

Table A.1: Exponential distribution - Power results of the GOF tests  $KS$ ,  $CM$  et  $AD$ , with the transformation K (left) and without the transformation K (right) for  $n = 5$  and  $n = 10$

$n$	$n = 20$			$n = 20$			$n = 50$			$n = 50$		
	$KS$	$CM$	$AD$									
exp(0.2)	5.1	5.2	5.3	5.2	4.9	5	4.9	4.9	5	5.1	5.7	4.9
exp(1)	4.9	5	5	4.7	4.7	5.1	4.9	4.9	5	4.7	4.9	5.6
exp(2)	5	4.9	5.1	4.8	5.1	5.3	5	5.1	5.2	5.3	5.3	5.1
exp(42)	5.2	5.1	5.1	5	5	5.4	4.9	4.9	4.9	4.9	4.8	5.2
$\mathcal{W}(0.5)$	88.3	91.2	95	86.4	89.8	95.7	99.8	100	100	99.9	99.9	100
$\mathcal{W}(0.8)$	20.3	22.9	25.7	17.4	20	26.9	40.6	46.2	50.5	36.4	41.8	50.8
$\mathcal{W}(0.98)$	5.4	5.5	5.6	5.4	5.6	5.8	5.5	5.6	5.7	5.3	5.5	5.9
$\mathcal{W}(1.5)$	40	47	40.3	40.1	47.9	44.7	83.7	90.6	91.2	79.4	89.5	91.4
$\mathcal{W}(3)$	100	100	100	99.8	100	99.9	100	100	100	100	100	100
$\mathcal{LN}(0.6)$	80.3	84.1	82.4	84.6	88.9	89.7	100	100	100	99.2	99.6	99.1
$\mathcal{LN}(0.8)$	25.1	27.3	25.6	28.9	33.7	34.1	60.4	61.5	67.8	71.1	75.9	85.5
$\mathcal{LN}(1.4)$	53.6	56	54	45.8	51.2	51.1	87.2	89	88.4	81.5	85.1	87.4
$\mathcal{U}[0, 2]$	60.9	69	80	51.8	67.6	63.8	98.3	99.3	100	92.1	98.5	98.2
$\mathcal{G}(0.5)$	48.4	53.6	65.9	39.8	48.2	45.8	85.4	89.3	94.9	83.3	89.7	90.5
$\mathcal{G}(2)$	38.6	45.5	44.3	39.8	48.2	45.8	82.9	89.4	90.2	81.7	90.4	91.4
Mean	51	50	56.8	49.5	54.6	54.8	76.7	79.2	80.8	75.4	79.6	81.8

Table A.2: Exponential distribution - Power results of the GOF tests  $KS$ ,  $CM$  et  $AD$ , with the transformation K (left) and without the transformation K (right) for  $n = 20$  and  $n = 50$

altern.	$Gn$	$Gn^*$	$LM_1$	$LM_2$	$LM$	$GG$
exp(0.2)	5.1	4.9	5.1	5.2	4.9	5.2
exp(1)	5.1	4.9	5	5	4.8	5.5
exp(2)	4.9	5.1	5	5	5	4.7
exp(42)	5	5.1	5.2	5.1	4.9	4.8
$\mathcal{W}(0.5)$	61.2	48.9	44.4	28.2	49.3	65.6
$\mathcal{W}(0.8)$	13	10.4	9	9.3	10.4	14
$\mathcal{W}(0.98)$	5.3	5.1	5.1	5.1	5.1	5.3
$\mathcal{G}(0.5)$	30.4	33.6	32.5	11.8	29.6	31.7
$\mathcal{W}(1.5)$	13.7	13	15.9	4.2	11.5	21.7
$\mathcal{W}(3)$	78.5	70.4	74.2	6.7	64.9	97.3
$\mathcal{U}[0, 2]$	32.3	12.2	12.7	12.8	14.4	36.5
$\mathcal{G}(2)$	12.2	14.4	18.7	3.9	12.6	23.4
$\mathcal{LN}(0.6)$	19.9	36.5	52	4.6	37.5	48.3
$\mathcal{LN}(0.8)$	6.7	13.9	20.1	7.4	15	14.5
$\mathcal{LN}(1.4)$	24.7	2.8	1.2	23.8	19.3	34.3
Mean	27.1	23.7	25.9	10.7	24.5	35.7

Table A.3: Exponential distribution - Power results of  $Gn$ ,  $Gn^*$ ,  $LM_1$ ,  $LM_2$ ,  $LM$  and  $GG$ ,  $n = 10$

altern.	$Gn$	$Gn^*$	$LM_1$	$LM_2$	$LM$	$GG$
exp(0.2)	4.9	5	5	5.2	4.8	5.8
exp(1)	5.1	5	5	5	5.1	5
exp(2)	5	5	5.1	5	5	4.9
exp(42)	5	5.2	5	5.1	4.8	5
$\mathcal{W}(0.5)$	85.6	71.5	74.3	45.1	78.9	91.3
$\mathcal{W}(0.8)$	19.5	14.5	12.8	11.2	14.4	23.8
$\mathcal{W}(0.98)$	5.1	5.1	5.2	5.4	5.1	5.4
$\mathcal{G}(0.5)$	46.5	48.1	52.4	15.5	48.4	54.7
$\mathcal{W}(1.5)$	30.8	22.8	29.6	5.9	23.4	49.9
$\mathcal{W}(3)$	99.5	95.5	96.7	15.6	95	100
$\mathcal{U}[0, 2]$	61.6	18.1	17.8	49.4	42.4	71.3
$\mathcal{G}(2)$	25.5	25.4	36.6	4.7	28	47.3
$\mathcal{LN}(0.6)$	42.6	64.4	89.1	5.3	80.1	80.1
$\mathcal{LN}(0.8)$	9.8	23.9	43.5	9.8	32.7	24.4
$\mathcal{LN}(1.4)$	46	5.6	1.2	39.5	33.3	55
Mean	42.9	35.9	41.7	18.8	43.8	54.8

Table A.4: Exponential distribution - Power results of the tests based on the normalized spacings,  $n = 20$

altern.	$Gn$	$Gn^*$	$LM_1$	$LM_2$	$LM$	$GG$
exp(0.2)	5	5	5	5.2	4.9	4.9
exp(1)	5.2	5	5	5	5	5
exp(2)	4.9	4.8	5	5	5.2	4.9
exp(42)	5.1	5	5	5	5	5
$\mathcal{W}(0.5)$	99.5	97.3	98.5	75.8	99.4	100
$\mathcal{W}(0.8)$	37.2	26.4	26.2	16.6	27.6	48.2
$\mathcal{W}(0.98)$	5.4	5.2	5.4	5.2	5.2	5.6
$\mathcal{G}(0.5)$	78.8	81.3	87.1	24.1	84.6	89.2
$\mathcal{W}(1.5)$	73.4	57.5	64.2	13.6	57.5	92.6
$\mathcal{W}(3)$	100	100	100	57.5	100	100
$\mathcal{U}[0, 2]$	99.1	64.9	32.3	98.7	97.9	99
$\mathcal{G}(2)$	64	65.3	35.1	7.8	69.9	89.2
$\mathcal{LN}(0.6)$	86.9	99.1	100	6	99.9	99.2
$\mathcal{LN}(0.8)$	18.7	63.9	89.2	15	81.8	46.5
$\mathcal{LN}(1.4)$	82.2	10.5	2.4	71	65	88.4
Mean	67.7	61	58.2	34.4	68.6	78

Table A.5: Exponential distribution - Power results of the tests based on the normalized spacings,  $n = 50$

$n$	$n = 5$			$n = 10$		
	$Sc$	$CO$	$LR$	$Sc$	$CO$	$LR$
altern.						
exp(0.2)	5.2	5	5	5.1	5	4.8
exp(1)	5	4.9	5.2	5	5	5.1
exp(2)	5	5	5	5	5	4.8
exp(42)	5	5.2	5	5	5	5
$\mathcal{W}(0.5)$	52.7	46.8	27	79.5	75.7	65.7
$\mathcal{W}(0.8)$	12.9	10.2	4.3	18.6	14.8	8.4
$\mathcal{W}(0.98)$	5.2	5.2	4.8	5.4	5.2	4.7
$\mathcal{G}(0.5)$	32.2	27.6	13	50.4	45.3	33.4
$\mathcal{W}(1.5)$	7.6	11.6	17.6	20.3	27.2	33.5
$\mathcal{W}(3)$	49.3	62.6	75.3	96	98.2	99.2
$\mathcal{U}[0, 2]$	11.9	16.7	23.6	24.9	33.5	40.3
$\mathcal{G}(2)$	7.5	11.9	18.1	20.9	27.3	34
$\mathcal{LN}(0.6)$	15.4	23	32.4	49.8	57	63.1
$\mathcal{LN}(0.8)$	5.9	9.7	14.5	14.9	19.5	24
$\mathcal{LN}(1.4)$	14.7	10.7	5	28.1	21.8	14.1
Mean	19.6	21.4	21.4	37.1	38.6	38.2

Table A.6: Exponential distribution - Power results of the likelihood based tests,  $n = 5$  and  $n = 10$

$n$	$n = 20$			$n = 50$		
	$Sc$	$CO$	$LR$	$Sc$	$CO$	$LR$
altern.						
exp(0.2)	4.9	5	5	5	5	5
exp(1)	4.9	4.8	5.1	5	4.9	5
exp(2)	4.9	5	5.3	4.9	5.1	5
exp(42)	5	5	5	4.9	4.8	5.1
$\mathcal{W}(0.5)$	96.7	96.2	94.3	100	100	100
$\mathcal{W}(0.8)$	29.2	25.4	19	56.4	52.8	48.4
$\mathcal{W}(0.98)$	5.5	5.6	4.8	5.9	5.8	5.1
$\mathcal{G}(0.5)$	74.6	71.9	64.8	97.4	97.2	96.5
$\mathcal{W}(1.5)$	48.7	56.2	62.7	93.4	94.9	96.3
$\mathcal{W}(3)$	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	49.5	59.1	66.1	90.1	93.7	95
$\mathcal{G}(2)$	51.1	57.5	63.2	94.8	95.7	96.5
$\mathcal{LN}(0.6)$	87.2	88.2	90	99.7	99.6	99.7
$\mathcal{LN}(0.8)$	31.8	34.3	39.2	64.6	63.8	66
$\mathcal{LN}(1.4)$	48.5	41.5	35.5	82.3	78.1	76.1
Mean	56.5	57.8	58.1	80.4	80.1	79.9

Table A.7: Exponential distribution - Power results of the likelihood based tests,  $n = 20$  and  $n = 50$

altern.	$a = 0.1$	$a = 1$	$a = 2.5$
exp(0.2)	4.8	4.9	5
exp(1)	4.8	5.1	4.9
exp(2)	4.7	4.9	5.1
exp(42)	4.7	4.9	5
$\mathcal{W}(0.5)$	55.1	38	34.5
$\mathcal{W}(0.8)$	15	9	8.7
$\mathcal{W}(0.98)$	5.5	5	5.1
$\mathcal{G}(0.5)$	34.3	19.5	17.2
$\mathcal{W}(1.5)$	18.1	11.2	11.5
$\mathcal{W}(3)$	18.2	60.2	62.6
$\mathcal{U}[0, 2]$	4.4	15.6	16.7
$\mathcal{G}(2)$	1.8	11.1	11.8
$\mathcal{LN}(0.6)$	3.8	22.1	23.2
$\mathcal{LN}(0.8)$	14	9.3	9.7
$\mathcal{LN}(1.4)$	15.6	15.1	15.7
Mean	16.9	19.6	19.7

Table A.8: Exponential distribution - Power results of the tests  $He$ ,  $n = 5$ 

altern.	$a = 0.1$	$a = 1$	$a = 2.5$
exp(0.2)	4.7	5.1	5.3
exp(1)	4.8	4.8	5.1
exp(2)	4.8	4.8	5.2
exp(42)	4.8	4.7	5.1
$\mathcal{W}(0.5)$	79.3	68.4	65.3
$\mathcal{W}(0.8)$	19.5	14.2	14.5
$\mathcal{W}(0.98)$	5.4	5	5.1
$\mathcal{G}(0.5)$	50.7	35.5	31.6
$\mathcal{W}(1.5)$	15	24.7	24.6
$\mathcal{W}(3)$	90	97.5	97.9
$\mathcal{U}[0, 2]$	16.6	29.8	33.7
$\mathcal{G}(2)$	15.8	25.4	24.3
$\mathcal{LN}(0.6)$	44.6	55.4	51
$\mathcal{LN}(0.8)$	11.9	18.2	17.1
$\mathcal{LN}(1.4)$	24.9	30	32.6
Mean	34	36.7	36.2

Table A.9: Exponential distribution - Power results of the tests  $He$ ,  $n = 10$

altern.	$a = 0.1$	$a = 1$	$a = 2.5$
exp(0.2)	4.8	5	5.1
exp(1)	5	5.1	5.1
exp(2)	4.8	5.2	5
exp(42)	4.9	5	4.9
$\mathcal{W}(0.5)$	96.3	93.2	91.2
$\mathcal{W}(0.8)$	28.4	24.8	23.6
$\mathcal{W}(0.98)$	5.4	5.2	5.5
$\mathcal{G}(0.5)$	74.1	60.4	54.7
$\mathcal{W}(1.5)$	41.7	52.7	51
$\mathcal{W}(3)$	100	100	100
$\mathcal{U}[0, 2]$	33.7	60	66.2
$\mathcal{G}(2)$	47.9	53.7	49.1
$\mathcal{LN}(0.6)$	94.3	88.5	80.1
$\mathcal{LN}(0.8)$	39.1	33.1	26.7
$\mathcal{LN}(1.4)$	40.9	53	56.1
Mean	54.7	56.8	55

Table A.10: Exponential distribution - Power results of the tests  $He$ ,  $n = 20$ 

altern.	$a = 0.1$	$a = 1$	$a = 2.5$
exp(0.2)	5.1	5.1	5.1
exp(1)	4.9	54.8	5.1
exp(2)	5.1	5.1	4.8
exp(42)	5	5	4.9
$\mathcal{W}(0.5)$	100	100	100
$\mathcal{W}(0.8)$	54.3	50	48.3
$\mathcal{W}(0.98)$	5.9	5.4	5.6
$\mathcal{G}(0.5)$	97.3	93	89.5
$\mathcal{W}(1.5)$	89.3	93.9	93.1
$\mathcal{W}(3)$	100	100	100
$\mathcal{U}[0, 2]$	74.7	96.1	98.5
$\mathcal{G}(2)$	94.9	93.6	90.7
$\mathcal{LN}(0.6)$	100	99.9	98.9
$\mathcal{LN}(0.8)$	89.8	61.9	46.2
$\mathcal{LN}(1.4)$	73.2	86.9	89.1
Mean	80	80.1	78.2

Table A.11: Exponential distribution - Power results of the tests  $He$ ,  $n = 50$

altern.	$a = 0.1$	$a = 1$	$a = 1.5$	$a = 2.5$	$a = 10$
exp(0.2)	5.2	4.9	4.9	4.8	5
exp(1)	5.1	4.9	5.2	5	4.7
exp(2)	4.9	5	4.9	4.8	5.1
exp(42)	4.9	4.9	4.9	4.8	4.9
$\mathcal{W}(0.5)$	57.5	39.1	37.5	35.2	31.4
$\mathcal{W}(0.8)$	15.8	9.5	9.3	9.1	8.4
$\mathcal{W}(0.98)$	5.4	5.1	5.2	5.1	5.1
$\mathcal{G}(0.5)$	37.6	20.1	19.1	17.6	15.9
$\mathcal{W}(1.5)$	0.2	11	12.1	11.4	11.1
$\mathcal{W}(3)$	0.2	60.1	62.8	61.6	60.8
$\mathcal{U}[0, 2]$	15.6	15.5	16.8	16.1	16.1
$\mathcal{G}(2)$	0.07	11.1	11.9	11.5	10.8
$\mathcal{LN}(0.6)$	0.03	22.4	23.7	22.4	21.1
$\mathcal{LN}(0.8)$	0.2	9.3	10.3	9.5	9.4
$\mathcal{LN}(1.4)$	14.7	15.5	15.7	15.5	16.5
Mean	13.4	19.9	20.4	19.5	18.8

Table A.12: Exponential distribution - Power results of the tests  $BH$ ,  $n = 5$ 

altern.	$a = 0.1$	$a = 1$	$a = 1.5$	$a = 2.5$	$a = 10$
exp(0.2)	5.2	4.9	4.9	5	4.8
exp(1)	5.1	4.8	4.8	5.0	4.9
exp(2)	5.1	5.1	4.9	5.1	4.9
exp(42)	4.9	5.1	4.9	4.9	5.1
$\mathcal{W}(0.5)$	80	69.8	67.7	64.9	60.9
$\mathcal{W}(0.8)$	19.9	14.8	14.3	14.5	14.6
$\mathcal{W}(0.98)$	5.7	5	5.2	5.3	5.4
$\mathcal{G}(0.5)$	52.7	36.2	34.4	31.9	28.3
$\mathcal{W}(1.5)$	12.3	25	24.7	24	19.3
$\mathcal{W}(3)$	86.9	97.6	97.8	97.7	96.4
$\mathcal{U}[0, 2]$	14.5	30.1	31.6	33.2	31.2
$\mathcal{G}(2)$	12.6	25.4	25	23.8	18.5
$\mathcal{LN}(0.6)$	38.5	55.8	53.6	50.2	40.7
$\mathcal{LN}(0.8)$	9.1	18.3	17.8	16.7	13.2
$\mathcal{LN}(1.4)$	25.5	30.1	30.9	32.4	34.6
Mean	32.5	37.1	36.6	35.9	33

Table A.13: Exponential distribution - Power results of the tests  $BH$ ,  $n = 10$

altern.	$a = 0.1$	$a = 1$	$a = 1.5$	$a = 2.5$	$a = 10$
exp(0.2)	4.9	4.8	4.8	5.2	4.7
exp(1)	5.1	4.9	4.8	5.0	4.8
exp(2)	5	4.9	4.7	5.1	4.7
exp(42)	5.1	4.7	4.8	5.1	4.8
$\mathcal{W}(0.5)$	96.4	93.7	92.5	91.3	87.6
$\mathcal{W}(0.8)$	29.2	24.5	23.9	24	23.2
$\mathcal{W}(0.98)$	5.6	5.2	5.2	5.4	5.4
$\mathcal{G}(0.5)$	74.9	61.5	58.4	54.7	48.1
$\mathcal{W}(1.5)$	38.1	52.2	51.9	50.8	40.6
$\mathcal{W}(3)$	100	100	100	100	100
$\mathcal{U}[0, 2]$	34.1	59.2	62.7	66.6	66
$\mathcal{G}(2)$	43.8	53.2	51.5	48.9	36.9
$\mathcal{LN}(0.6)$	93.2	89.1	86	81	65.7
$\mathcal{LN}(0.8)$	36.6	33.2	30.1	26.8	18.7
$\mathcal{LN}(1.4)$	42.5	52.8	54.5	56.1	58.5
Mean	54.1	56.8	56.1	55.1	50.1

Table A.14: Exponential distribution - Power results of the tests  $BH$ ,  $n = 20$ 

altern.	$a = 0.1$	$a = 1$	$a = 1.5$	$a = 2.5$	$a = 10$
exp(0.2)	5.1	4.9	5	5.2	5.1
exp(1)	4.9	4.9	4.8	5.0	5.1
exp(2)	5.2	5.1	4.9	5	4.8
exp(42)	5.1	4.9	4.8	4.9	5.1
$\mathcal{W}(0.5)$	100	100	100	100	99.8
$\mathcal{W}(0.8)$	53.6	50.2	49.3	48.4	45.5
$\mathcal{W}(0.98)$	6	5.6	5.6	5.7	6
$\mathcal{G}(0.5)$	97.6	93.6	92	89.6	82.2
$\mathcal{W}(1.5)$	87.7	93.5	93.3	93.1	88.6
$\mathcal{W}(3)$	100	100	100	100	100
$\mathcal{U}[0, 2]$	80.4	96.2	97.5	98.5	99.2
$\mathcal{G}(2)$	93.6	93.7	92.7	91	81.3
$\mathcal{LN}(0.6)$	100	100	99.8	99.3	93.3
$\mathcal{LN}(0.8)$	93.6	67	57.6	47.9	30.4
$\mathcal{LN}(1.4)$	77.6	86.9	87.9	89	90
Mean	80.9	80.6	79.6	78.4	74.2

Table A.15: Exponential distribution - Power results of the tests  $BH$ ,  $n = 50$

a	$a = 0.5$		$a = 0.75$		$a = 1$		$a = 1.5$		$a = 2.5$	
altern.	$HM_{n,a}^{(1)}$	$HM_{n,a}^{(2)}$								
exp(0.2)	5.2	5	5	5.2	5	4.9	5.1	5.4	5	5.1
exp(1)	4.8	5	5.1	5.3	5	4.9	5	5.1	5.2	5
exp(2)	5	5	5	5	5.1	4.8	5	5.2	5	5.1
exp(42)	5	5.1	5	5	5.2	5.1	4.9	5.1	5.3	5.2
$\mathcal{W}(0.5)$	28.4	25.5	28.4	29.5	27.8	32.3	26	34.8	28.4	34.8
$\mathcal{W}(0.8)$	7	6.8	6.6	8.8	6.7	10.1	6.7	11.6	8.8	11.8
$\mathcal{W}(0.98)$	5.2	5.2	5	5.2	5	5.3	5	5.5	5.3	5.6
$\mathcal{G}(0.5)$	14.5	11.8	13.9	14.8	13.3	17	12	18.8	14.4	19.4
$\mathcal{W}(1.5)$	9.2	13.7	11.9	9.9	13.8	7.1	14.2	4.8	10.8	2.9
$\mathcal{W}(3)$	37	66.6	55.7	57.3	64.7	46.7	67.5	35	59.6	22.9
$\mathcal{U}[0, 2]$	13.8	20.7	17.8	16.6	20.6	12.3	20.8	8.6	17.4	6.6
$\mathcal{G}(2)$	9.2	13.1	12.5	10.3	14	7.1	13.9	4.7	10.8	2.8
$\mathcal{LN}(0.6)$	17.1	24.3	23	18.8	25.5	13.9	25.3	9.1	20.5	5.4
$\mathcal{LN}(0.8)$	8.9	11.2	10.5	8.8	11.1	6.8	11.5	5.1	9.4	4.1
$\mathcal{LN}(1.4)$	9.3	13.3	10.3	16.9	11	18.7	12.4	20.6	16.7	20.9
Mean	14.5	19.3	17.8	17.9	19.4	16.1	19.6	14.4	18.4	12.5

Table A.16: Exponential distribution - Power results of the tests  $HM$ ,  $n = 5$ 

a	$a = 0.5$		$a = 0.75$		$a = 1$		$a = 1.5$		$a = 2.5$	
altern.	$HM_{n,a}^{(1)}$	$HM_{n,a}^{(2)}$								
exp(0.2)	5.2	4.9	5.3	4.8	5	5	5.1	5.1	4.8	4.8
exp(1)	5	5	5.2	4.9	5.2	5	5.1	5.1	4.9	4.9
exp(2)	4.9	5.1	5.4	5	5	4.9	5.1	5	5.1	4.9
exp(42)	5	5	5.2	4.8	5	4.9	5.2	5.1	5	5
$\mathcal{W}(0.5)$	54.8	51.2	54.6	52.1	54.8	55.2	54.6	58.4	56.1	59.3
$\mathcal{W}(0.8)$	9.6	9.8	9.8	10.8	10	12.6	10.1	15.3	12.2	16.9
$\mathcal{W}(0.98)$	5.2	5.1	5.2	4.9	5	5.2	5	5.5	5.1	5.6
$\mathcal{G}(0.5)$	26.7	21.3	25.9	22.6	24.9	24.8	24	28.2	25	29.5
$\mathcal{W}(1.5)$	14.9	25.4	21.7	22.5	24.4	19	25.8	11.9	21.6	1.5
$\mathcal{W}(3)$	76.4	97.5	93.1	96.7	96.1	95.1	97.5	88.3	96.5	41.4
$\mathcal{U}[0, 2]$	20.6	45.5	31.2	44.6	37.1	41.7	42.8	31.4	42.2	7.2
$\mathcal{G}(2)$	16.2	23.6	23	20.6	25.3	17.5	25.1	10.1	19.6	1.5
$\mathcal{LN}(0.6)$	40.1	42.4	54	37	54.9	32	49	21.3	37.1	3.6
$\mathcal{LN}(0.8)$	17.3	14.1	20.5	13	19.2	11.2	16.4	8.2	12.6	4.2
$\mathcal{LN}(1.4)$	15.5	24.6	18.8	27.9	21.2	31.7	25.5	36.1	30.7	38.1
Mean	27.1	32.8	32.5	32.1	33.9	31.4	34.2	28.6	32.6	18.9

Table A.17: Exponential distribution - Power results of the tests  $HM$ ,  $n = 10$

$a$	$a = 0.5$		$a = 0.75$		$a = 1$		$a = 1.5$		$a = 2.5$	
	$HM_{n,a}^{(1)}$	$HM_{n,a}^{(2)}$								
exp(0.2)	5.1	5	5	4.7	5	5	5	4.7	4.9	5
exp(1)	5.1	4.9	4.9	4.7	4.8	4.7	5	5	5	4.9
exp(2)	4.7	4.7	5.1	4.9	5.1	4.9	5	4.8	4.9	4.9
exp(42)	5	5	4.9	4.9	5	4.9	5.1	5.1	4.8	5.2
$\mathcal{W}(0.5)$	83.7	81.9	85.4	82	85.1	82.3	85.1	83.2	84.6	83.4
$\mathcal{W}(0.8)$	14.3	15.8	15.2	16.9	15.9	18.3	16.9	20.5	18.7	23.2
$\mathcal{W}(0.98)$	5	5	4.9	4.5	5	5.2	5	5.5	5.2	5.1
$\mathcal{G}(0.5)$	48.1	40.7	46.7	40.1	46	39.9	44.6	41.6	42.8	42.8
$\mathcal{W}(1.5)$	27.8	47.6	39.2	45.3	45	42	48.3	35.6	44.5	20.6
$\mathcal{W}(3)$	99.3	100	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	37.7	79.3	55.6	81.9	65.9	82.6	72.3	80.6	80.5	69.3
$\mathcal{G}(2)$	32.6	43.2	43	38.8	46.4	35.2	45.8	28.4	38.5	15.8
$\mathcal{LN}(0.6)$	83.8	71.2	89.7	61.2	88.5	54	82.1	45.7	63.5	28.9
$\mathcal{LN}(0.8)$	38.1	20.1	38.8	16.3	65.9	82.6	26.5	12.6	17.5	8.9
$\mathcal{LN}(1.4)$	26.5	45.6	34.3	49.5	46	39.9	45.7	55.8	53.3	60.1
Mean	45.2	50	50.2	48.8	55.4	51	52	46.3	49.9	41.6

Table A.18: Exponential distribution - Power results of the tests  $HM$ ,  $n = 20$ 

$a$	$a = 0.5$		$a = 0.75$		$a = 1$		$a = 1.5$		$a = 2.5$	
	$HM_{n,a}^{(1)}$	$HM_{n,a}^{(2)}$								
exp(0.2)	4.7	5.3	5.1	5	5.1	5	4.9	5	5.1	5.2
exp(1)	4.8	5.2	5.1	5	5	5	5.1	5.1	5	5.2
exp(2)	4.7	5.2	5.1	5.1	5.2	5	5	5	4.9	5.2
exp(42)	4.7	5	5	5	5.1	5.1	5.1	5.1	5	5.1
$\mathcal{W}(0.5)$	99.7	99.6	99.8	99.5	99.8	99.4	99.8	99.4	99.6	99.2
$\mathcal{W}(0.8)$	29.7	36	34.1	36.9	36.2	37.7	38.1	38.3	39.3	39.6
$\mathcal{W}(0.98)$	4.9	5.3	5.1	5.3	5.2	5.2	5.2	5.6	5.4	6
$\mathcal{G}(0.5)$	85.6	79.5	85.9	77.1	85.4	75.8	83.4	74.3	78.9	71.6
$\mathcal{W}(1.5)$	66.5	87.9	81.3	87.1	85.9	85.8	88.5	82.3	87.5	74.5
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	82.4	99.7	95.3	99.8	98.3	99.9	99.5	99.9	99.9	99.9
$\mathcal{G}(2)$	77.3	83.4	86.1	79.7	88	75.8	86.8	68.9	80.4	57.9
$\mathcal{LN}(0.6)$	100	98.4	100	95.2	100	90.9	99.8	81.9	96.4	67.8
$\mathcal{LN}(0.8)$	86.1	39.9	82.3	28.9	74.3	22.9	56.6	17.8	31.1	14
$\mathcal{LN}(1.4)$	56.1	82.5	69.5	85.2	75.8	86.2	82.7	87.9	87.6	89.4
Mean	71.6	73.9	76.3	72.3	77.2	70.9	76.4	68.8	73.3	65.4

Table A.19: Exponential distribution - Power results of the tests  $HM$ ,  $n = 50$

$a$	$a = 0.5$		$a = 0.75$		$a = 1$		$a = 1.5$	
altern.	$MI_{5,0.5}^{(1)}$	$MI_{5,0.5}^{(2)}$	$MI_{5,0.75}^{(1)}$	$MI_{5,0.75}^{(2)}$	$MI_{5,1}^{(1)}$	$MI_{5,1}^{(2)}$	$MI_{5,1.5}^{(1)}$	$MI_{5,1.5}^{(2)}$
exp(0.2)	5.2	5.3	4.8	4.8	4.8	4.9	4.9	5
exp(1)	5.3	5.4	5.2	5.2	5.3	5.3	5.2	5.2
exp(2)	4.9	4.8	5.1	5.1	4.7	4.8	4.8	5
exp(42)	4.7	4.9	5.4	5.5	5	5.1	5	4.9
$\mathcal{W}(0.5)$	0.5	0.5	0.5	0.4	0.5	0.5	0.5	0.4
$\mathcal{W}(0.8)$	2.3	2.2	2.5	2.4	2.6	2.7	2.3	2.3
$\mathcal{W}(0.98)$	4.2	4.2	4.6	4.5	4.7	4.6	4.7	4.8
$\mathcal{G}(0.5)$	1	0.9	1.1	0.9	1	0.9	1.1	1.1
$\mathcal{W}(1.5)$	16.5	17.6	17.6	17.7	17.4	17.4	17.7	18
$\mathcal{W}(3)$	70.3	74.8	74.4	75.5	75.7	76.3	74.4	75
$\mathcal{U}[0, 2]$	22	25.7	25	25.7	24.9	25.1	25.1	25.1
$\mathcal{G}(2)$	16.6	17.5	17	16.7	17.5	17.6	18.1	18.3
$\mathcal{LN}(0.6)$	31.8	31.7	32.7	32	31.3	31.4	30.8	31.5
$\mathcal{LN}(0.8)$	14.5	14.3	15	14.8	14.1	14.2	13.8	14.1
$\mathcal{LN}(1.4)$	2.4	2.3	2.2	2.2	2.2	2.2	2	2
Mean	16.6	17.4	17.5	17.5	17.5	17.5	17.3	17.5

Table A.20: Exponential distribution - Power results of the tests  $MI$ ,  $n = 5$ 

$a$	$a = 0.5$		$a = 0.75$		$a = 1$		$a = 1.5$	
altern.	$MI_{5,0.5}^{(1)}$	$MI_{5,0.5}^{(2)}$	$MI_{5,0.75}^{(1)}$	$MI_{5,0.75}^{(2)}$	$MI_{5,1}^{(1)}$	$MI_{5,1}^{(2)}$	$MI_{5,1.5}^{(1)}$	$MI_{5,1.5}^{(2)}$
exp(0.2)	5.1	5.1	4.9	4.7	5.4	5.2	5.1	5
exp(1)	4.7	5	4.8	4.8	5	4.8	4.4	4.5
exp(2)	5.3	5.4	5	4.9	5.2	5	4.8	4.8
exp(42)	5	4.7	4.7	4.7	5	4.9	5.2	5.2
$\mathcal{W}(0.5)$	0	0	0	0	0	0	0	4.8
$\mathcal{W}(0.8)$	1.5	1.4	1.7	1.7	1.3	1.2	1.3	1.4
$\mathcal{W}(0.98)$	5	4.6	4.2	4.5	4.5	4.5	4.4	4.4
$\mathcal{G}(0.5)$	0.4	0.4	0.4	0.4	0.3	0.4	0.4	1.4
$\mathcal{W}(1.5)$	29.6	33.3	31.2	32.9	33.3	33.8	33.5	33.6
$\mathcal{W}(3)$	97.7	98.8	98.4	99.1	98.7	99	98.9	99.1
$\mathcal{U}[0, 2]$	40.5	53.8	44.5	53.3	47.1	53.4	50.8	53.1
$\mathcal{G}(2)$	31	30.8	32.3	30.9	32.8	31.7	32.6	32.3
$\mathcal{LN}(0.6)$	64	53.8	62.8	51.9	60.4	51.6	57.8	53.1
$\mathcal{LN}(0.8)$	26.6	19.3	24.7	19.4	23.3	19.1	20.8	19.4
$\mathcal{LN}(1.4)$	1.5	1.1	1.3	0.9	0.9	0.8	0.8	1.5
Mean	27.1	27	27.4	26.8	27.5	26.9	27.4	27.6

Table A.21: Exponential distribution - Power results of the tests  $MI$ ,  $n = 10$

$n$	$n = 20$				$n = 50$			
altern.	$MI_{20,1.5}^{(1)}$	$MI_{20,1.5}^{(2)}$	$MI_{20,2.5}^{(1)}$	$MI_{20,2.5}^{(2)}$	$MI_{50,1.5}^{(1)}$	$MI_{50,1.5}^{(2)}$	$MI_{50,2.5}^{(1)}$	$MI_{50,2.5}^{(2)}$
exp(0.2)	4.4	4.5	4.5	5	5.2	5.1	5.2	5
exp(1)	5.1	4.7	5.1	5.2	5.2	5.5	5.4	5.2
exp(2)	5	5	5.2	4.9	5.2	5.4	5.1	5.2
exp(42)	4.6	4.9	4.8	4.8	5	5	5.5	5.2
$\mathcal{W}(0.5)$	16.1	52.1	50	69	96.5	98.1	98.5	98.8
$\mathcal{W}(0.8)$	1	5.4	3.6	9.8	17.3	26.8	24.8	31.2
$\mathcal{W}(0.98)$	4.1	4.2	4.2	4.6	4.1	4.8	4	4.2
$\mathcal{G}(0.5)$	3.4	19.8	16.3	30	65.7	71.9	71.8	74
$\mathcal{W}(1.5)$	58.9	58.9	59.6	56.2	93.5	93.6	93.8	92.6
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	82.7	87.3	85.9	85.8	99.6	99.9	99.8	99.9
$\mathcal{G}(2)$	56.7	52.5	53.8	49.2	92.4	88.7	90	85.6
$\mathcal{LN}(0.6)$	86.7	74.1	79.7	69.7	99.9	98.1	99.3	95.7
$\mathcal{LN}(0.8)$	34.2	24.5	28.3	23.2	65.5	40.4	49.1	32.2
$\mathcal{LN}(1.4)$	1.1	14.6	11.9	29.6	38.4	60.8	60.7	72.8
Mean	40.4	44.8	44.8	48	70.3	71.2	72	71.6

Table A.22: Exponential distribution - Power results of the tests  $MI$ ,  $n = 20$  et  $n = 50$ 

altern.	$m = 2$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$
exp(0.2)	4.8	5.3	5.1	4.4	5.4	4.8
exp(2)	5.5	5.5	4.5	4.1	4.4	4.9
exp(42)	4.7	5.6	4.7	3.8	4.7	4.9
$\mathcal{W}(0.8)$	7.2	5.7	1.2	1.4	0.5	0.4
$\mathcal{W}(0.98)$	5.1	5	4.1	3.5	3.7	3.5
$\mathcal{G}(0.5)$	27.7	22.8	6.5	5.3	0.2	0.1
$\mathcal{W}(1.5)$	35.7	46.5	52.7	50.4	58.3	57.8
$\mathcal{U}[0, 2]$	77.1	85.1	88.2	87.9	91.3	91.3
$\mathcal{G}(2)$	38	47.6	52.3	50.6	56.8	52.3
$\mathcal{LN}(0.6)$	84.6	89.9	88.6	88.3	86.9	83.8
$\mathcal{LN}(0.8)$	35.9	43.1	39.3	38.3	37.7	34.9
$\mathcal{LN}(1.4)$	29.2	27.2	11.4	10.4	1.7	0.5
Mean	37.8	41.4	38.2	37.3	37.4	36.1

Table A.23: Exponential distribution - Power results of the tests  $GW$ ,  $n = 20$

altern.	$KL_n$	$a = 1$	$a = 5$	$a = 10$	$a = 20$
exp(0.2)	5	5	5	4.9	4.9
exp(1)	5	5.2	5	5	5.1
exp(2)	5.3	4.8	5.1	4.9	5
exp(42)	5	5.1	5	4.8	5
$\mathcal{W}(0.5)$	32.1	32.9	41.2	53.7	56.4
$\mathcal{W}(0.8)$	9.4	8.4	10	15.3	15.3
$\mathcal{W}(0.98)$	5.1	5.2	5	5.6	5.7
$\mathcal{G}(0.5)$	16.1	15.6	22.3	33.2	37.2
$\mathcal{W}(1.5)$	9.4	12.2	10.7	0.1	0
$\mathcal{W}(3)$	55.9	63.5	57.4	0	0
$\mathcal{U}[0, 2]$	14.8	17.6	14.5	1	1.3
$\mathcal{G}(2)$	9.3	12.3	10.9	0	0
$\mathcal{LN}(0.6)$	18.4	23.2	21.7	0	0
$\mathcal{LN}(0.8)$	8.6	10.2	8.9	0	0
$\mathcal{LN}(1.4)$	17.6	15.6	14.9	16.8	13.1
Mean	17.8	19.7	19.8	11.4	11.7

Table A.24: Exponential distribution - Power results of the tests  $KL$ ,  $n = 5$ 

altern.	$KL_n$	$a = 1$	$a = 5$	$a = 10$	$a = 20$
exp(0.2)	5.1	4.9	4.9	5.5	5.1
exp(1)	4.9	4.8	4.9	5.4	4.9
exp(2)	5	5	4.8	5.4	5
exp(42)	5.1	5	5	5.4	5
$\mathcal{W}(0.5)$	60.1	61.5	70.5	76.4	79.6
$\mathcal{W}(0.8)$	15.3	13.6	14.6	18.3	21
$\mathcal{W}(0.98)$	5.5	5.3	5.2	5.9	5.9
$\mathcal{G}(0.5)$	28.8	28.7	38.2	48	54.5
$\mathcal{W}(1.5)$	14.6	23.4	23.6	16.8	0.07
$\mathcal{W}(3)$	93.7	97.5	95.3	84.8	0
$\mathcal{U}[0, 2]$	29.2	37	23.4	14.6	0.8
$\mathcal{G}(2)$	13.6	22.3	25.6	18.6	0.01
$\mathcal{LN}(0.6)$	31	45.5	61.1	53.2	0
$\mathcal{LN}(0.8)$	10.7	15.3	20.1	16	0.01
$\mathcal{LN}(1.4)$	35.6	32.2	26.7	24.9	19.7
Mean	30.8	34.7	36.7	34.4	16.5

Table A.25: Exponential distribution - Power results of the tests  $KL$ ,  $n = 10$

altern.	$KL_n$	$a = 1$	$a = 5$	$a = 10$	$a = 20$
exp(0.2)	4.8	4.9	4.9	5	4.9
exp(1)	4.8	5	5	5.1	5.1
exp(2)	4.8	4.9	5.1	4.9	5.1
exp(42)	5.2	4.9	5.1	4.9	5.1
$\mathcal{W}(0.5)$	86.2	89.3	93.8	95.1	95.7
$\mathcal{W}(0.8)$	23.5	21.8	23.1	26.1	29.5
$\mathcal{W}(0.98)$	5.6	5.3	5	5.7	6
$\mathcal{G}(0.5)$	45.8	50.4	63.5	70.2	75.2
$\mathcal{W}(1.5)$	32.2	48.1	50	37.9	17.9
$\mathcal{W}(3)$	100	100	100	99.8	91.6
$\mathcal{U}[0, 2]$	68	72.3	44.7	24.8	10
$\mathcal{G}(2)$	27.9	44.2	54.3	46.7	24.1
$\mathcal{LN}(0.6)$	53.9	75.6	94.7	96.2	83.2
$\mathcal{LN}(0.8)$	14.6	22.7	41.7	46.6	27.6
$\mathcal{LN}(1.4)$	59.1	56	46.3	38.6	29
Mean	46.9	53.2	56.1	53.5	44.7

Table A.26: Exponential distribution - Power results of the tests  $KL$ ,  $n = 20$ 

altern.	$KL_n$	$a = 1$	$a = 5$	$a = 10$	$a = 20$
exp(0.2)	5.1	5.2	5	4.9	4.9
exp(1)	4.9	4.7	5.1	5	5
exp(2)	4.9	5.1	4.9	5	4.9
exp(42)	4.9	5.1	4.9	5	4.9
$\mathcal{W}(0.5)$	99.6	99.8	100	100	60
$\mathcal{W}(0.8)$	42.8	45.4	49.1	49.7	48.6
$\mathcal{W}(0.98)$	6	5.4	5.5	5.6	5.9
$\mathcal{G}(0.5)$	78.6	86.5	94.5	96.1	96.8
$\mathcal{W}(1.5)$	81.3	91.4	91.2	84.2	67.3
$\mathcal{W}(3)$	100	100	100	100	100
$\mathcal{U}[0, 2]$	99.7	99.4	86.2	58.2	29.5
$\mathcal{G}(2)$	70.6	87.5	94.5	92.9	84.2
$\mathcal{LN}(0.6)$	90	99	100	100	100
$\mathcal{LN}(0.8)$	25	42.4	82.1	93.3	96.2
$\mathcal{LN}(1.4)$	90	89.2	80.7	69.1	50.2
Mean	71.2	76.9	80.3	77.2	67.1

Table A.27: Exponential distribution - Power results of the tests  $KL$ ,  $n = 50$

altern.	$KS$	$CM$	$AD$	$GG$	$SW$	$BH_{n,1.5}$	$HE_{n,1}$	$Kl_{n,5}$	$BHK$	$BHC$
exp(0.2)	4.9	4.8	4.7	5.1	4.9	4.9	4.9	5	5	5.1
exp(1)	4.8	5.2	5.2	4.8	5.2	5.2	5.1	5	5	5.1
exp(2)	5.1	5.1	4.8	5.3	5	4.9	4.9	5.1	5.1	5
exp(42)	5.2	4.7	5	5.1	5.1	4.9	4.9	5	5	5
$\mathcal{W}(0.5)$	27.5	30.2	51	37.6	29.8	37.5	38	41.2	11.4	32.5
$\mathcal{W}(0.8)$	6.9	6.6	13.5	10.3	8.4	9.3	9	10	3.4	8.2
$\mathcal{W}(0.98)$	5	4.2	5.8	5.1	5.2	5.2	5	5	4.7	5.2
$\mathcal{G}(0.5)$	13.7	14.3	31.1	18.9	14.5	19.1	19.5	22.3	4.7	15.7
$\mathcal{W}(1.5)$	11.6	12.6	8.3	10.1	11.8	12.1	11.2	10.7	15.4	13
$\mathcal{W}(3)$	53.2	62.1	50.7	57.6	62.9	62.8	60.2	57.4	63.4	65
$\mathcal{U}[0, 2]$	15.4	17.3	12.8	15.5	17.3	16.8	15.6	14.5	21.3	18.2
$\mathcal{G}(2)$	12.1	12.3	8	10.1	11.6	11.9	11.1	10.9	15	12.9
$\mathcal{LN}(0.6)$	23.7	23.9	16.9	20.1	22.8	23.7	22.1	21.7	27.1	24.5
$\mathcal{LN}(0.8)$	10.4	10.3	6.8	8.2	10.1	10.3	9.3	8.9	12.4	10.7
$\mathcal{LN}(1.4)$	13.7	12.2	16	16.8	15.5	15.7	15.1	14.9	6.5	15
Mean	17.5	18.7	20.1	19.1	19.1	20.4	19.6	19.8	16.8	20.1

Table A.28: Exponential distribution - Tests comparison,  $n = 5 - 1$ 

altern.	$MI_{n,1.5}^{(1)}$	$MI_{n,1.5}^{(2)}$	$HM_{n,1.5}^{(1)}$	$HM_{n,0.5}^{(2)}$	$EP$	$Sc$	$CO$	$LR$
exp(0.2)	4.9	5	5.1	5	5.2	5.2	5	5
exp(1)	5.2	5.2	5	5	5.2	5	4.9	5.2
exp(2)	4.8	5	5	5	5.1	5	5	5
exp(42)	5	4.9	4.9	5.1	5	5	5.2	5
$\mathcal{W}(0.5)$	0.5	0.4	26	25.5	37.1	52.7	46.8	27
$\mathcal{W}(0.8)$	2.3	2.3	6.7	6.8	10.4	12.9	10.2	4.3
$\mathcal{W}(0.98)$	4.7	4.8	5	5.2	5.2	5.2	5.2	4.8
$\mathcal{G}(0.5)$	1.1	1.1	12	11.8	19.2	32.2	27.6	13
$\mathcal{W}(1.5)$	17.7	18	14.2	13.7	10.6	7.6	11.6	17.6
$\mathcal{W}(3)$	74.4	75	67.5	66.6	58.6	49.3	62.6	75.3
$\mathcal{U}[0, 2]$	25.1	25.1	20.8	20.7	15.2	11.9	16.7	23.6
$\mathcal{G}(2)$	18.1	18.3	13.9	13.1	10.3	7.5	11.9	18.1
$\mathcal{LN}(0.6)$	30.8	31.5	25.3	24.3	20.4	15.4	23	32.4
$\mathcal{LN}(0.8)$	13.8	14.1	11.5	11.2	8.9	5.9	9.7	14.5
$\mathcal{LN}(1.4)$	2	2	12.4	13.3	16.8	14.7	10.7	5
Mean	17.3	17.5	19.6	19.3	19.3	19.6	21.4	21.4

Table A.29: Exponential distribution - Tests comparison,  $n = 5 - 2$

altern.	$KS$	$CM$	$AD$	$GG$	$SW$	$K_{n,5}$	$BH_{n,1}$	$He_{n,1}$	$BHK$	$BHC$
exp(0.2)	4.5	5.2	4.9	5.2	4.8	4.9	4.9	5.1	5	5
exp(1)	5.2	4.7	5.2	5.5	4.9	5	4.8	4.8	4.8	4.8
exp(2)	4.8	4.9	5.2	4.7	4.8	4.9	5.1	4.8	5.1	5
exp(42)	5.1	4.8	5.1	4.8	4.8	5.1	5.1	4.7	5.1	5
$\mathcal{W}(0.5)$	56.2	61	77	65.6	52.3	70.5	69.8	68.4	45.4	62.5
$\mathcal{W}(0.8)$	9.6	11.6	17.9	14	11.7	14.6	14.8	14.2	6.4	12.5
$\mathcal{W}(0.98)$	4.4	3.9	5.4	5.3	5	5.2	5	5	4.6	5.1
$\mathcal{G}(0.5)$	23.8	30.3	48.9	31.7	22.3	38.2	36.2	35.5	15.8	29.1
$\mathcal{W}(1.5)$	21.2	22.9	17.2	21.7	21.7	23.6	25	24.7	27.2	23.9
$\mathcal{W}(3)$	89.5	96.6	95.4	97.3	97.3	95.3	97.6	97.5	95.2	97.2
$\mathcal{U}[0, 2]$	27.8	37.7	31.9	36.5	36.3	23.4	30.1	29.8	42	36.9
$\mathcal{G}(2)$	21.2	24.4	21.2	23.4	20.8	25.6	25.4	25.4	25.6	23.8
$\mathcal{LN}(0.6)$	48.7	51.9	47.5	48.3	42.1	61.1	55.8	55.4	51.7	50.1
$\mathcal{LN}(0.8)$	16.3	16.6	14.8	14.5	14.6	20.1	18.3	18.2	17.8	16.9
$\mathcal{LN}(1.4)$	26.7	26.6	27	34.3	30.7	26.7	30.1	30	20	30.8
Mean	31.4	34.8	36.7	35.7	32.2	36.7	37.1	36.7	32	35.3

Table A.30: Exponential distribution - Tests comparison,  $n = 10 - 1$ 

altern.	$MI_{n,1.5}^{(1)}$	$MI_{n,1.5}^{(2)}$	$HM_{n,1.5}^{(1)}$	$HM_{n,0.5}^{(2)}$	$EP$	$Sc$	$CO$	$LR$
exp(0.2)	5.1	5	5.1	4.9	5	5.1	5	4.8
exp(1)	4.4	4.5	5.1	5	5.1	5	5	5.1
exp(2)	4.8	4.8	5.1	5.1	5	5	5	4.8
exp(42)	5.2	5.2	5.2	5	5.1	5	5	5
$\mathcal{W}(0.5)$	0	4.8	54.6	51.2	65.3	79.5	75.7	65.7
$\mathcal{W}(0.8)$	1.3	1.4	10.1	9.8	14.3	18.6	14.8	8.4
$\mathcal{W}(0.98)$	4.4	4.4	5	5.1	5.2	5.4	5.2	4.7
$\mathcal{G}(0.5)$	0.4	1.4	24	21.3	32.1	50.4	45.3	33.4
$\mathcal{W}(1.5)$	33.5	33.6	25.8	25.4	22.9	20.3	27.2	33.5
$\mathcal{W}(3)$	98.9	99.1	97.5	97.5	97.6	96	98.2	99.2
$\mathcal{U}[0, 2]$	50.8	53.1	42.8	45.5	32.6	24.9	33.5	40.3
$\mathcal{G}(2)$	32.6	32.3	25.1	23.6	23.2	20.9	27.3	34
$\mathcal{LN}(0.6)$	57.8	53.1	49	42.4	48.9	49.8	57	63.1
$\mathcal{LN}(0.8)$	20.8	19.4	16.4	14.1	16.1	14.9	19.5	24
$\mathcal{LN}(1.4)$	0.8	1.5	25.5	24.6	32.6	28.1	21.8	14.1
Mean	27.4	27.6	34.2	32.8	35.5	37.1	38.6	38.2

Table A.31: Exponential distribution - Tests comparison ,  $n = 10 - 2$

altern.	$KS$	$CM$	$AD$	$GG$	$SW$	$BHK$	$BHC$	$K_{n,5}$	$BH_{n,1}$	$He_{n,1}$
exp(0.2)	5.2	4.9	5	5.8	4.7	5.2	5.3	4.9	4.8	5
exp(1)	4.7	4.7	5.1	5	4.7	5.1	4.8	5	4.9	5.1
exp(2)	4.8	5.1	5.3	4.9	4.7	5.1	4.8	5.2	4.9	5.2
exp(42)	5	5	5.4	5	4.9	5.4	5.3	5.1	4.7	5
$\mathcal{W}(0.5)$	86.4	89.8	95.7	91.3	77	82.4	90.6	93.9	93.7	93.2
$\mathcal{W}(0.8)$	17.4	20	26.9	23.8	16.2	13.4	19.3	23.1	24.5	24.8
$\mathcal{W}(0.98)$	5.4	5.6	5.8	5.4	4.9	5.8	4.9	5.1	5.2	5.2
$\mathcal{G}(0.5)$	45.9	53.2	71.2	54.7	33.5	37.6	56.7	63.6	61.5	60.4
$\mathcal{W}(1.5)$	40.1	47.9	44.7	49.9	47	46.1	47.8	50	52.2	52.7
$\mathcal{W}(3)$	99.8	100	99.9	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	51.8	67.6	63.8	71.3	76.7	72.6	72.6	44.7	59.2	60
$\mathcal{G}(2)$	39.8	48.2	45.8	47.3	42.2	45.6	55.5	54.3	53.2	53.7
$\mathcal{LN}(0.6)$	84.6	88.9	89.7	80.1	66.4	84.4	84.7	94.7	89.1	88.5
$\mathcal{LN}(0.8)$	28.9	33.7	34.1	24.4	20.9	28.6	25.9	41.8	33.2	33.1
$\mathcal{LN}(1.4)$	45.8	51.2	51.1	55	51.5	45.5	53.4	46.4	52.8	53
Mean	49.6	55.1	57.1	54.8	48.7	51.1	56	56.1	56.8	56.8

Table A.32: Exponential distribution - Tests comparison,  $n = 20 - 1$ 

altern.	$MI_{n,2.5}^{(1)}$	$MI_{n,2.5}^{(2)}$	$HM_{n,1.5}^{(1)}$	$HM_{n,0.5}^{(2)}$	$EP$	$Sc$	$CO$	$LR$
exp(0.2)	4.5	5	5	5	5	4.9	5	5
exp(1)	5.1	4.7	5	4.7	4.9	4.9	4.8	5.1
exp(2)	5.2	5	5	4.9	5.2	4.9	5	5.3
exp(42)	4.8	4.9	5.1	4.9	4.9	5	5	5
$\mathcal{W}(0.5)$	50	69	85.1	81.9	91.1	96.7	96.2	94.3
$\mathcal{W}(0.8)$	3.6	9.8	16.9	15.8	24	29.2	25.4	19
$\mathcal{W}(0.98)$	4.2	4.6	5	5	5.4	5.5	5.6	4.8
$\mathcal{G}(0.5)$	16.3	30	44.6	40.7	54.7	74.6	71.9	64.8
$\mathcal{W}(1.5)$	59.6	56.2	48.3	47.6	50.5	48.7	56.2	62.7
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	85.9	85.8	72.3	79.3	66.7	49.5	59.1	66.1
$\mathcal{G}(2)$	53.8	49.2	45.8	43.2	48.4	51.1	57.5	63.2
$\mathcal{LN}(0.6)$	79.7	69.7	85.1	71.2	79.8	87.2	88.2	90
$\mathcal{LN}(0.8)$	28.3	23.2	26.5	20.1	25.5	31.8	34.3	39.2
$\mathcal{LN}(1.4)$	11.9	29.6	45.7	45.6	55.7	48.5	41.5	35.5
Mean	44.8	48	52	50	54.7	56.5	57.8	58.1

Table A.33: Exponential distribution - Tests comparison,  $n = 20 - 2$

## A.2 Power results of the GOF tests for the Exponential distribution: censored samples

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5	5	4.9	5	4.9	5	4.9	5	5	5
exp(1)	4.9	5	5	5.1	5.1	5	5.1	4.9	5	5
exp(2)	4.8	4.7	4.9	5.1	5	5	5	5.1	5.2	5
exp(42)	4.8	5	5.1	4.9	5	5.1	5	5	5.1	5
$\mathcal{W}(0.5)$	41.4	32.1	52.2	0	70.2	54.7	43.6	43.5	11.7	38.2
$\mathcal{W}(0.8)$	9.4	8.1	10.6	1.4	16.4	11.8	9.6	9.4	3.4	6.7
$\mathcal{W}(0.98)$	5	5	5	4.4	5.3	5.2	5	4.9	4.7	5
$\mathcal{G}(0.5)$	19	14.2	27	0.4	45.5	31	31.1	28.6	3.8	20.3
$\mathcal{W}(1.5)$	9.7	9	20.1	30	13.4	12.6	11	9.9	18.5	4.7
$\mathcal{W}(3)$	32.8	26.7	91.6	97.3	85.3	71.6	59.6	57.7	81.4	25.5
$\mathcal{U}[0, 2]$	18.8	13	22.5	34.4	14.9	20.3	9.6	10.1	17.3	6.5
$\mathcal{G}(1.5)$	5.9	6.1	10.7	16.2	6.7	6.9	6.9	6.4	11.6	4.8
$\mathcal{G}(2)$	8.2	7.8	21.1	30.8	14.5	12.1	12.4	11.5	20.5	5
$\mathcal{G}(3)$	12.7	11.6	46.9	60.5	36.4	25.5	25.9	25.5	40.5	6.8
$\mathcal{LN}(0.6)$	8.5	8.9	52	61.2	41.7	23.6	31.3	34.2	47.8	7.2
$\mathcal{LN}(0.8)$	6	7.3	18.9	25.4	13.2	8.6	12.8	13	22.1	4.6
$\mathcal{LN}(1.4)$	18.8	17	16.6	1.6	17	12.1	2.5	11.1	7.6	7.1
$\mathcal{LN}(1.8)$	35.9	30.3	38.5	0.4	44	35.1	9.4	19.4	12.2	16.6
$\mathcal{LN}(2.4)$	59.1	50	66.4	0	75.8	66.4	33.5	39.1	25.1	43.4
$\mathcal{IG}(0.5)$	61.3	55.5	59.8	0.5	61	53.5	6.7	38.3	3.8	37.9
$\mathcal{IG}(1.5)$	13.3	14	17.9	15.7	13.4	5.6	11.6	17.8	11.6	5.6
$\mathcal{IG}(2)$	8.7	10.3	27.2	30	20.7	9.4	19.9	24.2	20.9	5.2
$\mathcal{IG}(3)$	7	8.4	53.1	57.1	43.9	20.5	35.9	42.7	40.9	8.2
Mean	20.1	17.7	34.6	24.6	33.6	25.6	19.9	23.5	21.3	13.6

Table A.34: Exponential distribution - Tests comparison,  $n = 10$  and  $r = 1$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn(4)$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5	5	4.9	4.8	4.7	5	4.9	5	5	5
exp(1)	5	4.9	4.8	4.7	5	5	4.8	4.8	5	5.1
exp(2)	5	5	4.7	4.7	4.8	4.9	4.8	4.9	5	5
exp(42)	5	5	5.1	5	5	5	5	4.9	5.1	5.1
$\mathcal{W}(0.5)$	26.7	20.2	34	0.1	55.2	42	33.5	35.3	7.8	28.8
$\mathcal{W}(0.8)$	7.5	6.9	7.4	1.7	12.5	5.2	5.1	8.7	3.5	6.5
$\mathcal{W}(0.98)$	5	5.1	4.9	4.5	5.1	5.2	5.1	5	4.9	4.8
$\mathcal{G}(0.5)$	15.2	11.5	20.4	0.5	39.1	27.9	27.2	25.9	4.1	19.1
$\mathcal{W}(1.5)$	7.2	6.9	15.6	21.1	10.1	9.5	10.3	7.6	15	5
$\mathcal{W}(3)$	16.5	18.8	77.3	85.4	66.8	50.4	51.8	40.3	64.4	15.4
$\mathcal{U}[0,2]$	8.7	7.3	12.1	16.5	8.2	9.8	7.9	6.6	10.8	5.4
$\mathcal{G}(1.5)$	5.4	5.5	9.7	13.3	5.9	6.2	7	5.5	10.7	5.1
$\mathcal{G}(2)$	7.1	6.9	18	24	12	10	12.5	9.1	17.4	5
$\mathcal{G}(3)$	9.4	8.8	37.4	46.5	27.5	19.5	25.8	17.6	32.5	5.8
$\mathcal{LN}(0.6)$	7.4	7.4	47	55.4	36.6	21.8	36.5	24.5	41.5	6.5
$\mathcal{LN}(0.8)$	5.6	6	19.9	25.9	13.7	9.5	16.4	11.1	20.2	4.8
$\mathcal{LN}(1.4)$	8.7	8.5	6.4	2.9	5.9	4.7	2.2	6	5.4	5.1
$\mathcal{LN}(1.8)$	15.8	13.5	15.4	0.8	19	15.1	2	9.6	5.1	7.2
$\mathcal{LN}(2.4)$	31.3	24.6	35.6	0.2	47.5	39.7	11.3	21.8	10.1	20
$\mathcal{IG}(0.5)$	25.2	21.9	21.1	1.8	21.1	15.3	2.2	15.9	4	10.8
$\mathcal{IG}(1.5)$	6	6.7	19.3	23.8	13.6	7.8	19.4	13.2	10.6	4.5
$\mathcal{IG}(2)$	5.7	6.4	31.5	37.3	23.7	12.6	29.7	19.2	17.5	5
$\mathcal{IG}(3)$	6.2	6.4	54.4	60.2	44.5	23.5	48.5	32.8	32.4	7.6
Mean	11.6	10.5	25.6	22.2	24.6	11.6	10.5	16.6	16.7	9.1

Table A.35: Exponential distribution - Tests comparison,  $n = 10$  and  $r = 3$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	4.9	4.9	5.1	5.1	5.1	5	4.9	4.7	5	5.1
exp(1)	5.1	5	5.1	5.3	5	4.9	4.9	4.8	5	5.2
exp(2)	5	4.9	5.1	5.2	5	5	4.9	4.9	5	5.1
exp(42)	5	5	5.2	5.2	5.3	4.9	4.8	4.8	5	4.9
$\mathcal{W}(0.5)$	15.6	13.7	16.4	0.5	39.6	31.5	26.5	27.4	4.1	22.5
$\mathcal{W}(0.8)$	6	5.9	4.9	2.5	10.2	8.9	7.7	7.7	3.6	6.4
$\mathcal{W}(0.98)$	5.2	5.2	5.1	5	5.4	5	5	5	4.8	5.1
$\mathcal{G}(0.5)$	11.6	10.2	11.7	0.9	31.7	24.8	22.9	22.8	3.2	17.7
$\mathcal{W}(1.5)$	5.8	5.8	13.5	16.1	8.1	6.9	7.1	5.5	11.8	5.1
$\mathcal{W}(3)$	9.6	9.4	58.1	64.3	43.7	31.5	29	20	42.4	9.1
$\mathcal{U}[0,2]$	5.5	5.4	8.9	10.1	6.4	6.4	5.8	5.2	7.5	5.1
$\mathcal{G}(1.5)$	5.2	5.2	9.9	11.7	5.7	5.3	5.4	4.5	9.3	5.3
$\mathcal{G}(2)$	5.5	5.5	16.6	19.9	10.1	8	8.4	6.3	13.5	5.2
$\mathcal{G}(3)$	6.4	6.5	30.7	37.5	20.2	14	14.7	10.3	23.6	5.2
$\mathcal{LN}(0.6)$	5.9	5.9	41.4	47	28.8	18.5	21.9	14.4	31.9	5.6
$\mathcal{LN}(0.8)$	5.2	5.2	20.7	24.7	37.6	9.3	11.6	7.9	18.1	4.9
$\mathcal{LN}(1.4)$	5.5	5.5	5	4.9	13	3	2.9	4.2	5.7	5.1
$\mathcal{LN}(1.8)$	8	7.6	4.9	5.1	3.4	6	2.3	5.4	4.1	5.1
$\mathcal{LN}(2.4)$	13.6	12.4	5.2	2.3	7.5	18.8	7.1	11.6	4.1	9.8
$\mathcal{IG}(0.5)$	8.8	8.6	6.9	5.6	5.6	3.1	3.7	6.6	3.2	5.2
$\mathcal{IG}(1.5)$	4.9	5	24.6	28.7	15.9	9.8	14.7	9.5	9.1	4.5
$\mathcal{IG}(2)$	5	5.2	34.7	39.6	23.4	14.2	20.4	12.7	13.8	4.9
$\mathcal{IG}(3)$	5.6	5.7	52.2	57.2	38.3	22.5	30.1	18.7	23.6	6.8
Mean	7.3	7	19.5	20.2	17.7	13	13	10.8	12.5	7.3

Table A.36: Exponential distribution - Tests comparison,  $n = 10$  and  $r = 5$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5.1	5	4.8	4.9	5.1	4.9	5.1	5	5	5.1
exp(1)	5	5	4.9	5	4.9	4.9	5	5	5.1	5
exp(2)	4.9	4.9	5.1	5	5.1	4.9	5	5.1	5	5.1
exp(42)	4.9	4.8	4.9	5.1	5	4.9	5.1	4.9	5.1	5
$\mathcal{W}(0.5)$	76.9	70	82.8	0	92.2	77.9	68.4	55.1	38.9	57.8
$\mathcal{W}(0.8)$	15.4	12.1	16.7	0.6	23.8	16.7	13.4	10.2	4.3	7.1
$\mathcal{W}(0.98)$	5.2	5.1	5.2	4.3	5.7	5.2	5.4	5.3	4.7	5
$\mathcal{G}(0.5)$	39.9	32.4	49.1	0	68.5	44.3	48.1	37.2	11.3	27.6
$\mathcal{W}(1.5)$	23.6	19.5	39.5	54.6	34.4	24.5	21.7	18.4	29.2	6.3
$\mathcal{W}(3)$	77.8	68.2	99.9	100	99.9	97.7	93.7	85.3	98.9	65.4
$\mathcal{U}[0,2]$	43.4	31.1	42.6	62	33.3	40.1	14.2	12.4	29.9	10.1
$\mathcal{G}(1.5)$	10.4	9.7	17.3	26.5	14.4	10.3	11.4	10.4	16.2	5.2
$\mathcal{G}(2)$	19.1	17.1	42.6	55.9	38.8	22.5	25.7	24.4	33.9	6.3
$\mathcal{G}(3)$	34.2	30.4	83.7	91.2	82.1	50.8	56.8	57	69.6	15.4
$\mathcal{LN}(0.6)$	19.5	20.6	88.5	89.8	89	43.7	68.9	80.3	80.3	19.5
$\mathcal{LN}(0.8)$	8.1	11.3	36.3	41.8	35.9	12.3	27.7	36.5	38.6	5.9
$\mathcal{LN}(1.4)$	33.9	30.2	29.2	0.5	28.8	24.2	3.2	13.7	14.2	8.9
$\mathcal{LN}(1.8)$	66.8	59.2	66.3	0	70.1	61.8	19.6	23.8	29.9	26.6
$\mathcal{LN}(2.4)$	90.7	86.7	92.6	0	95.5	89.9	61.6	45.9	64.1	67.4
$\mathcal{IG}(0.5)$	89	85.8	86.1	0	86.1	82.8	13.8	47.8	10.7	60.3
$\mathcal{IG}(1.5)$	18.9	23.2	30.5	19.2	30.8	6.4	23.7	43.3	15.9	8.3
$\mathcal{IG}(2)$	10.7	16.1	51.6	43.5	53.7	11.6	43.9	67.5	34.1	10.4
$\mathcal{IG}(3)$	10.8	15.8	88	81.7	86.7	34.6	75.5	93	69.6	23.5
Mean	36.5	33.9	55.2	35.3	56.3	39.9	36.7	40.4	36.5	23

Table A.37: Exponential distribution - Tests comparison,  $n = 20$  and  $r = 2$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5	5	5	4.9	4.9	4.8	4.9	5.1	4.8	5.1
exp(1)	5	5.1	5.1	5.1	5	5	5	4.9	5.1	5
exp(2)	5	5.1	5.1	5.2	5	5	5	5	5	4.9
exp(42)	5	5	5.1	5	5	5	5	4.9	4.9	4.9
$\mathcal{W}(0.5)$	61.8	52.4	69.8	0	84.2	67.6	60.7	48.4	22.8	45.8
$\mathcal{W}(0.8)$	12.2	9.5	13	0	19.1	13.9	11.9	9.7	3.6	6.3
$\mathcal{W}(0.98)$	5.2	5.1	5	4.4	5.4	5.5	5	5	4.7	5
$\mathcal{G}(0.5)$	34.2	26.5	42.4	0.1	62.5	4.5	44.9	35.3	8.2	25.8
$\mathcal{W}(1.5)$	17	14.6	31.9	43.7	27.1	19.3	18.4	15.2	22.8	5.7
$\mathcal{W}(3)$	60.8	52.2	99.5	99.9	99.2	92.1	87.7	79.5	94.4	47.9
$\mathcal{U}[0,2]$	21	14.9	23.6	35	17.4	19.7	10.4	8.7	13.8	5.8
$\mathcal{G}(1.5)$	8.7	8.6	16	23.5	12.9	9.5	10.8	9.6	14.3	5.1
$\mathcal{G}(2)$	15.2	14.3	37.2	48.9	33.5	20.1	23.5	21.1	28.9	5.7
$\mathcal{G}(3)$	26.6	24.1	75.9	84.3	73.8	45.5	51.6	49.3	59.6	12.5
$\mathcal{LN}(0.6)$	18.4	18.6	86.5	89.3	87	47.6	68.3	74	75.7	17.5
$\mathcal{LN}(0.8)$	8.8	10.3	39.5	47.2	38.5	16.1	29.9	33.3	38.1	5.8
$\mathcal{LN}(1.4)$	16.2	14.7	12.6	1.6	11.1	9.5	2.3	8	8.8	5.5
$\mathcal{LN}(1.8)$	40.5	32.8	38.7	0.1	41.6	35.4	9.4	13.7	13.3	11.6
$\mathcal{LN}(2.4)$	71.7	62.6	75.1	0	81.9	72.6	41.8	29.1	32.9	39
$\mathcal{IG}(0.5)$	59.6	53.4	51.9	0.4	49.6	43.5	3.1	25.4	8.2	21.6
$\mathcal{IG}(1.5)$	7.9	11	34.2	34.4	35.1	9.9	32.3	44.7	14.2	6.4
$\mathcal{IG}(2)$	7.5	10.4	60.8	60	62.6	21	52.1	67.3	28.6	10.6
$\mathcal{IG}(3)$	11.6	13.7	90.8	89	92.1	46.3	79.8	91.2	59.6	23.9
Mean	26.6	23.7	47.6	34.8	49.2	31.6	33.9	35.2	29.1	16.2

Table A.38: Exponential distribution - Tests comparison,  $n = 20$  and  $r = 5$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	4.9	4.9	5.2	5.1	5.1	5.1	5	5.1	5.1	4.9
exp(1)	5	5	5	5	5	4.9	4.9	4.7	5.1	4.8
exp(2)	5	5	5.2	5.2	5.2	5	5	5	5.1	4.9
exp(42)	5.2	5	5	5	4.9	5.1	4.9	4.9	5.3	5.1
$\mathcal{W}(0.5)$	36.1	26.8	41.1	0	64.6	46.9	43.8	38.7	11	31.4
$\mathcal{W}(0.8)$	8.4	6.9	7.7	1.5	13.6	10.4	9.6	8.7	3.6	6.2
$\mathcal{W}(0.98)$	5.1	5	4.9	4.5	5.3	5.2	5.1	5	5.1	4.9
$\mathcal{G}(0.5)$	23.6	17.1	27.1	0.2	50.2	33.6	35.6	31.4	6.4	22.5
$\mathcal{W}(1.5)$	9.7	9.1	22.9	28.7	17.7	11.1	12.2	10.2	16.6	5.1
$\mathcal{W}(3)$	30.8	26.6	92.2	95.2	89.5	61.9	64.4	59.3	75.8	23.8
$\mathcal{U}[0, 2]$	7.5	6.6	10.9	13.7	8	7.7	6.5	5.6	7.9	5
$\mathcal{G}(1.5)$	6.8	6.9	13.9	18.4	10.4	7.4	8.4	7.4	12.2	5.1
$\mathcal{G}(2)$	9.9	9.5	29.2	36.4	24	13.1	16.3	14.4	22.2	5.1
$\mathcal{G}(3)$	15.5	14.4	60.5	68.1	55.3	27.4	35.5	32.5	44.6	8.3
$\mathcal{LN}(0.6)$	8.1	8.5	79.2	83.4	77.2	35.1	53.8	54.9	65.1	12.8
$\mathcal{LN}(0.8)$	6.5	6.6	41.4	48.1	37.6	15.5	26.3	25	34.6	5.5
$\mathcal{LN}(1.4)$	13.3	8.2	5	4.9	3.9	3.7	3.3	5.2	7.5	5.1
$\mathcal{LN}(1.8)$	24.4	11.1	9.7	1.1	11.9	11.7	3	6.8	6	5.3
$\mathcal{LN}(2.4)$	30.8	23.7	30.2	0.1	41.4	35.3	15.2	14.9	9.7	13.1
$\mathcal{IG}(0.5)$	14.3	13.3	8.9	4.9	7.6	5.7	4.6	10.1	6.4	5.3
$\mathcal{IG}(1.5)$	6	7.1	48.7	53.1	47.2	15.2	35.6	38.6	11.9	6.7
$\mathcal{IG}(2)$	7.8	8.6	69.1	71.6	68.1	24.6	50.5	55.1	22.4	10.5
$\mathcal{IG}(3)$	10.6	10.8	90.1	90.7	90	41.5	71.3	77.8	44.6	20
Mean	14.5	11.9	36.4	32.9	38.1	21.7	26.4	26.4	21.8	10.6

Table A.39: Exponential distribution - Tests comparison,  $n = 20$  and  $r = 10$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5.1	5.1	4.9	4.9	5.1	4.7	4.9	5	5	5
exp(1)	5	5	5	4.9	5	5.1	4.9	5	5.1	5
exp(2)	4.9	5	5	4.7	5	4.9	4.8	5.1	5	5
exp(42)	5.1	5.1	4.9	4.9	5	4.9	5	5	5.1	5
$\mathcal{W}(0.5)$	99.1	98.8	99.5	0	99.9	97.7	94	96.8	90.3	85.6
$\mathcal{W}(0.8)$	32.8	26	33.8	0	43.2	29.3	23.2	21.5	9.3	7.9
$\mathcal{W}(0.98)$	5.3	5.2	5.4	3.7	5.8	5.3	5.2	5.2	4.7	5.1
$\mathcal{G}(0.5)$	80.6	78.5	86	0	94.9	74.8	76.8	79.8	45.9	45.3
$\mathcal{W}(1.5)$	65.3	56.5	79.9	90.2	79.6	59.6	47.6	48	54.7	11.5
$\mathcal{W}(3)$	99.8	99.4	100	100	100	100	100	100	100	98.9
$\mathcal{U}[0, 2]$	84.4	77.9	81.1	93	73.6	74.8	24.6	44.1	57.9	18.5
$\mathcal{G}(1.5)$	26.1	23.9	38.8	51.8	39.5	22.3	23.2	24.4	26.4	5.7
$\mathcal{G}(2)$	57	53.1	84.9	91.4	87.1	56.7	57.3	63.7	65.3	13.6
$\mathcal{G}(3)$	84.9	80.6	99.9	100	99.9	93.8	94.1	97.1	97.9	50
$\mathcal{LN}(0.6)$	60.1	63.4	100	99.9	100	90.4	98.4	99.9	99.8	68.2
$\mathcal{LN}(0.8)$	19.6	30	80.9	75.1	89	30	63.3	83.6	77.5	16.7
$\mathcal{LN}(1.4)$	61	55.4	52.9	0	50.2	45.8	5.2	27.1	30.8	11.2
$\mathcal{LN}(1.8)$	95	92	94.1	0	94.7	91	44.7	58.7	69.6	44.4
$\mathcal{LN}(2.4)$	99.9	99.8	99.9	0	100	99.7	91.8	96.5	97.7	92.4
$\mathcal{IG}(0.5)$	99.6	99.4	99.2	0	99.1	98.6	28.6	86.7	45.5	86.4
$\mathcal{IG}(1.5)$	21.5	38.4	70.1	28.1	83.9	6.9	54.6	86.2	26.9	24
$\mathcal{IG}(2)$	11.3	32.6	95.1	72.8	98.6	26.2	85.4	98.9	64.4	41.8
$\mathcal{IG}(3)$	28.6	45.3	100	99.2	100	78.2	99.3	100	97.9	80.9
Mean	59.6	60.8	79	47.6	81	62.2	58.8	69.4	61.1	42.5

Table A.40: Exponential distribution - Tests comparison,  $n = 50$  and  $r = 6$

altern.	$BS_1$	$BS^*$	$CM$	$W$	$AD$	$Gn$	$Gn^*$	$LM$	$KL$	$CO$
exp(0.2)	5.1	5	5	4.9	5	5.1	5	4.9	4.8	5
exp(1)	5	5	5.1	5	5	5	5.2	5	5.1	4.7
exp(2)	5.1	5	4.9	5.1	5.1	5	5.1	5	5	5.1
exp(42)	5.1	5	5	5.1	5	5	5.1	5	4.9	5
$\mathcal{W}(0.5)$	96.8	96	97.9	0	99.5	94.6	91	91	78.1	76
$\mathcal{W}(0.8)$	26.3	20.8	27.3	0	36.1	24.3	20.5	16.6	6.8	7.4
$\mathcal{W}(0.98)$	5.5	5.2	5.2	4	5.7	5.2	5.3	5.2	4.7	5
$\mathcal{G}(0.5)$	75.6	72.2	80.8	0	92.5	70.8	74.2	73	37.6	42.6
$\mathcal{W}(1.5)$	52.5	45.2	70.2	82	70.5	48.4	41.7	39	45.2	9.3
$\mathcal{W}(3)$	98.7	97.2	100	100	100	100	99.9	99.8	100	96.1
$\mathcal{U}[0, 2]$	55.2	43.7	54.1	71	45.8	47.4	16.9	18.4	27.9	9
$\mathcal{G}(1.5)$	22.9	21	34.5	47.2	35.8	20	22	21.8	24.2	5.5
$\mathcal{G}(2)$	49.1	45.8	79.1	86.8	82.3	50.3	54.4	56.7	58.4	11.8
$\mathcal{G}(3)$	76.8	72.5	99.7	99.8	99.8	89.7	91.9	94.2	95.3	43.1
$\mathcal{LN}(0.6)$	58.8	59.5	100	99.9	100	90.4	98.4	99.8	99.6	65.2
$\mathcal{LN}(0.8)$	23.7	29.8	83.7	82.4	91	37	67.4	84.1	77.2	17.4
$\mathcal{LN}(1.4)$	33.5	30.2	26.8	0.5	23.1	21.9	3	13.7	18.3	6.4
$\mathcal{LN}(1.8)$	80.2	73.1	77.4	0	78.4	72.6	27.4	30.5	41.3	21.2
$\mathcal{LN}(2.4)$	98.7	98	98.9	0	99.4	97.5	81.1	78.2	85.9	72.8
$\mathcal{IG}(0.5)$	93	90.6	88.1	0	86.5	83.1	7.3	54.7	37.9	46.3
$\mathcal{IG}(1.5)$	9.1	22.4	77.2	56.3	89.5	15.7	68.2	92.9	24.4	23.3
$\mathcal{IG}(2)$	14.5	27.6	97.5	89.3	99.5	44.1	91.2	99.5	58.6	44.9
$\mathcal{IG}(3)$	35.9	44.5	100	99.8	100	87.2	99.6	100	95.6	82.5
Mean	53	52.4	73.6	48.4	75.5	57.9	55.9	61.5	53.6	36.1

Table A.41: Exponential distribution - Tests comparison,  $n = 50$  and  $r = 12$

### A.3 Power results of the simplified likelihood GOF tests for the Weibull distribution

altern.	$\widehat{GG}_w^1$	$\widehat{GG}_s^1$	$\widehat{GG}_l^1$	$\widehat{GG}_w^2$	$\widehat{GG}_s^2$	$\widehat{GG}_l^2$	$\widetilde{GG}_w^2$	$\widetilde{GG}_s^2$	$\widetilde{GG}_l^2$	$\check{GG}_w^2$	$\check{GG}_s^2$	$\check{GG}_l^2$	mean
exp(1)	5.1	5.1	5	5.1	5	5.7	5.2	5	5	5	5.1	5	5.1
$\mathcal{W}(1, 0.5)$	5.1	5.1	5.1	5.1	5	5.5	5.1	5	5	5.1	5.1	5.1	5.1
$\mathcal{W}(1, 3)$	5	5	5	5.1	4.9	5.6	5	4.8	4.8	5.1	5.1	5.1	5
$\mathcal{G}(3, 1)$	7.9	6.9	7.2	15.4	1.1	10.4	15.6	13.5	15.2	15.2	13.5	15.3	11.4
$\mathcal{AW}1$	53	54.1	53.7	0.7	27.9	49.6	0.7	35.2	0.7	0.7	16.5	0.8	24.5
$\mathcal{EW}1$	17.9	16.2	16.7	32.4	0.3	22.4	32.7	29.3	32.1	32.4	28.9	32.3	24.5
$\mathcal{G}(0.5, 1)$	10	10.7	10.4	1.6	12.7	9.4	1.6	4.1	1.5	1.4	4.3	1.4	5.8
$\mathcal{AW}2$	81.5	83.3	82.7	0	20.6	82.3	0	71.7	0	0	63.7	0	40.5
$\mathcal{EW}2$	15.9	18	17.3	0.1	24.3	15.3	0.1	8.5	0	0.1	12.4	0.1	9.3
$\mathcal{EW}3$	16.3	17.9	17.3	0	24.4	16.9	0.1	8.6	0.1	0.1	12.7	0.1	9.5
$\mathcal{GG}1$	32.2	34.5	33.8	0.1	32	30.7	0	16.2	0	0	16.9	0	16.4
$\mathcal{GG}2$	24.1	26.1	25.4	0.2	26.3	21.9	0.2	10.6	0.2	0.2	11	0.2	12.2
$\mathcal{PGW}1$	13.2	14.5	14.1	0.7	17.5	11.6	0.8	4.9	0.8	0.9	5.5	0.9	7.1
$\mathcal{LN}(0, 0.8)$	25.3	22.8	23.5	42.3	0	30.5	42.7	38.7	41.9	42	38.2	42	32.5
$\mathcal{IG}(3, 1)$	52.1	48.9	49.8	70.5	0	60.9	72	68.4	71.2	70.4	66.3	69.8	58.4
$\mathcal{EW}4$	13	11.7	12.2	24.3	0.4	15.7	24.6	21.9	24.1	24.5	21.7	24.4	18.2
$\mathcal{GG}3$	13.5	12.1	12.5	25.9	0.4	17.1	26.1	23	25.5	26.2	23.3	26.1	19.3
$\mathcal{PGW}2$	23.9	21.7	22.5	41.5	0.1	29.8	41.8	37.9	40.9	41.4	37.3	41.1	31.6
mean	26.6	26.6	26.6	17.1	12.5	28.3	17.2	26.1	16.9	17.1	24.8	16.9	21.2

Table A.42: Power results for the tests based on the Generalized Gamma distribution,  $n = 20$

altern.	$\widetilde{GG}_w^1$	$\widetilde{GG}_s^1$	$\widetilde{GG}_l^1$	$\widetilde{GG}_w^2$	$\widetilde{GG}_s^2$	$\widetilde{GG}_l^2$	$\check{GG}_w^2$	$\check{GG}_s^2$	$\check{GG}_l^2$	$\check{GG}_w^2$	$\check{GG}_s^2$	$\check{GG}_l^2$	mean
exp(1)	5.1	5.1	5.1	5	5	5.5	5	5	4.9	5	5	5	5
$\mathcal{W}(1, 0.5)$	5.1	5	5	5	5.2	5.6	4.9	4.9	4.9	4.9	5	4.9	5
$\mathcal{W}(1, 3)$	5.1	5	5	5	5	5.3	5	5	5.1	5	5	4.9	5
$\mathcal{G}(3, 1)$	18.2	16.8	17.2	29	1	21.1	28.9	25.2	28.8	29	25	29.2	22.5
$\mathcal{AW}1$	83.7	84.1	83.9	0	62.4	82.3	0	73.5	4.7	0	44	0	47.8
$\mathcal{EW}1$	50.7	49	49.6	66.1	7.3	56.3	66.1	61.7	65.8	65.5	60.3	65.4	55.3
$\mathcal{G}(0.5, 1)$	16.8	17.6	17.2	0.5	24.9	16.7	0.5	9.3	0.5	0.4	9.6	0.4	9.5
$\mathcal{AW}2$	99.8	99.8	99.8	0	13.1	99.8	0	99.8	27.8	0	96	2.7	53.2
$\mathcal{EW}2$	44.1	46.2	45.5	0	63.3	47.4	0	39.6	0	0	45.1	0	27.6
$\mathcal{EW}3$	43.7	45.3	44.6	0	63.3	47.5	0	39.9	0	0	45.9	0	27.5
$\mathcal{GG}1$	71.7	73.4	72.9	0	77.4	73.3	0	62.1	0.2	0	61.1	0	41
$\mathcal{GG}2$	54.9	56.7	56.2	0	66.4	55.8	0	41.6	0	0	41.9	0	31.1
$\mathcal{PGW}1$	26.9	28.4	27.9	0.1	38.2	27	0.2	16.3	0.1	0.2	16.8	0.2	15.2
$\mathcal{LN}(0, 0.8)$	66.9	65.3	65.8	80.5	15.7	72.5	80.8	77.4	80.5	80.3	76.3	80.4	70.2
$\mathcal{IG}(3, 1)$	94.2	93.6	93.7	97.9	57.7	96.2	98.2	97.6	98.2	98	97.1	98	93.4
$\mathcal{EW}4$	35.8	33.7	34.3	50.6	3.3	40.6	50.3	46	50.2	50.1	44.9	50.4	40.9
$\mathcal{GG}3$	38.8	37.1	37.7	54.4	3.8	44.1	54.5	49.9	54.3	54.4	49	54.3	44.4
$\mathcal{PGW}2$	64.6	62.5	63.1	78.4	12.3	69.9	78.9	75.3	78.7	78.6	74.4	78.5	67.9
mean	54.1	54	54	30.5	34	56.7	30.6	54.4	32.7	30.4	52.5	30.6	43.1

Table A.43: Power results for the tests based on the Generalized Gamma distribution,  $n = 50$

altern.	$\widetilde{AW}_w$	$\widetilde{AW}_s$	$\widetilde{AW}_l$	$\check{AW}_w$	$\check{AW}_s$	$\check{AW}_l$	mean
exp(1)	5.1	5.1	5	5	5.1	5	5
$\mathcal{W}(1, 0.5)$	5.1	5	5	5.1	5.2	5.1	5.1
$\mathcal{W}(1, 3)$	4.9	4.9	4.8	4.8	5	5	4.9
$\mathcal{G}(3, 1)$	14.8	14.8	14.7	10.7	14.2	13.4	13.8
$\mathcal{AW}1$	12.6	0.7	0.7	41.5	23.7	31	18.4
$\mathcal{EW}1$	31.5	31.6	31.5	24.2	30.7	28.9	29.7
$\mathcal{G}(0.5, 1)$	1.9	1.6	1.6	5.8	2.6	3.6	2.8
$\mathcal{AW}2$	5.3	0	0	35.7	13.8	21.5	12.7
$\mathcal{EW}2$	0.2	0.2	0.2	2.3	0.3	0.7	0.6
$\mathcal{EW}3$	0.2	0.2	0.2	2.3	0.3	0.7	0.6
$\mathcal{GG}1$	1.5	0.1	0.1	12.9	4.2	6.9	4.3
$\mathcal{GG}2$	1.3	0.3	0.3	10.7	3.3	5.4	3.6
$\mathcal{PGW}1$	1.4	0.9	0.9	6.8	2.5	3.7	2.7
$\mathcal{LN}(0, 0.8)$	42.4	42.4	42.4	33.5	40.7	38.8	40
$\mathcal{IG}(3, 1)$	72.8	72.9	72.8	63.9	70.7	69	70.4
$\mathcal{EW}4$	23.7	23.7	23.7	17.3	22.8	21.3	22.1
$\mathcal{GG}3$	25.8	25.9	25.8	19	24.8	23.2	24.1
$\mathcal{PGW}2$	41.1	41.2	41.1	32.9	40.5	38.5	39.2
mean	18.4	17.1	17.1	21.2	19.7	20.4	19

Table A.44: Power results for the tests based on the Additive Weibull distribution,  $n = 20$

altern.	$\widetilde{AW}_w$	$\widetilde{AW}_s$	$\widetilde{AW}_l$	$\check{AW}_w$	$\check{AW}_s$	$\check{AW}_l$	mean
exp(1)	4.9	4.8	4.8	5	5.1	5.1	4.9
$\mathcal{W}(1, 0.5)$	5.1	5	5	5	5	5	5
$\mathcal{W}(1, 3)$	5	4.9	5	4.9	5	5	5
$\mathcal{G}(3, 1)$	26.1	27.8	27.7	19.5	24.6	23.2	24.8
$\mathcal{AW}1$	70.1	42.7	55.7	80.1	72.3	75.6	66.1
$\mathcal{EW}1$	63.8	65.8	65.6	55.7	62.5	60.5	62.3
$\mathcal{G}(0.5, 1)$	5	1.3	2.2	10.1	5.5	7.1	5.2
$\mathcal{AW}2$	51.4	17.1	30.2	77.6	60.5	67.5	50.7
$\mathcal{EW}2$	0.4	0	0	2.9	0.6	1.1	0.8
$\mathcal{EW}3$	0.4	0	0	2.8	0.6	1.1	0.8
$\mathcal{GG}1$	19.5	3.8	8.6	32.2	19.6	24.2	18
$\mathcal{GG}2$	14.3	2.7	6.1	25	15.1	18.7	13.6
$\mathcal{PGW}1$	6.9	1.1	2.6	13.7	7.6	9.7	6.9
$\mathcal{LN}(0, 0.8)$	81	82.3	82.2	73.6	79.2	77.6	79.3
$\mathcal{IG}(3, 1)$	98.8	98.9	98.9	97.5	98.4	98.2	98.5
$\mathcal{EW}4$	47.1	49.2	49	39.2	46	44.1	45.8
$\mathcal{GG}3$	51.8	53.9	53.7	43.4	50.5	48.4	50.3
$\mathcal{PGW}2$	77.4	78.7	78.6	70.6	76.4	74.7	76.1
mean	40.9	35	37.4	42.9	41.2	42.1	39.9

Table A.45: Power results for the tests based on the Additive Weibull distribution,  $n = 50$ 

altern.	$\widehat{BGW}_w$	$\widehat{BGW}_s$	$\widehat{BGW}_l$	$\check{BGW}_w$	$\check{BGW}_s$	$\check{BGW}_l$	$\check{BGW}_w$	$\check{BGW}_s$	$\check{BGW}_l$	mean
exp(1)	4.9	5	4.9	5	4.9	5	5.1	5	5.1	5
$\mathcal{W}(1, 0.5)$	4.9	5	4.9	5	5	4.9	5.3	5	5.3	5
$\mathcal{W}(1, 3)$	4.9	4.9	4.9	5.1	5	5	5.1	4.9	5.2	5
$\mathcal{G}(3, 1)$	14.3	14.2	14.2	15.1	15	15	15.5	15.1	15.5	14.9
$\mathcal{AW}1$	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
$\mathcal{EW}1$	28.5	28.3	28.3	31.8	31.5	31.6	32.4	31.7	32.4	30.7
$\mathcal{G}(0.5, 1)$	1.5	1.4	1.5	1.6	1.5	1.6	1.4	1.5	1.5	1.5
$\mathcal{AW}2$	0	0	0	0	0	0	0	0	0	0
$\mathcal{EW}2$	0	0	0	0.1	0.1	0.1	0	0.1	0	0
$\mathcal{EW}3$	0	0	0	0.1	0	0.1	0.1	0	0.1	0
$\mathcal{GG}1$	0	0	0	0.1	0.1	0	0	0	0	0
$\mathcal{GG}2$	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$\mathcal{PGW}1$	0.7	0.7	0.7	1	0.9	0.9	0.8	0.7	0.8	0.8
$\mathcal{LN}(0, 0.8)$	35.8	35.6	35.8	42.2	41.8	42	42.2	41.4	42.1	39.9
$\mathcal{IG}(3, 1)$	60.2	59.8	60	71.6	70.9	71.2	70.4	69.4	70.2	67.1
$\mathcal{EW}4$	21.8	21.7	21.7	23.9	23.9	23.9	24	23.4	24	23.1
$\mathcal{GG}3$	23.1	22.9	22.9	25.9	25.8	25.8	26.1	25.5	26.1	24.9
$\mathcal{PGW}2$	34.9	34.6	34.7	41.6	41	41.3	41.3	40.3	41.3	39
mean	14.8	14.7	14.7	17	16.9	16.9	17.3	16.7	17	16.2

Table A.46: Power results for the tests based on the Burr Generalized Weibull distribution,  $n = 20$

altern.	$\widehat{BGW}_w$	$\widehat{BGW}_s$	$\widehat{BGW}_l$	$\widetilde{BGW}_w$	$\widetilde{BGW}_s$	$\widetilde{BGW}_l$	$B\check{G}W_w$	$B\check{G}W_s$	$B\check{G}W_l$	mean
exp(1)	4.9	5	4.9	4.9	4.9	5	5	4.9	5	5
$\mathcal{W}(1, 0.5)$	4.9	4.9	4.9	4.9	5.1	5	5	4.9	5	5
$\mathcal{W}(1, 3)$	5	4.9	5	4.9	4.9	5	5.1	4.9	5.2	5
$\mathcal{G}(3, 1)$	25	24.7	25	26.8	26.9	27	25.8	25.6	25.7	25.8
$\mathcal{AW}1$	0.1	0.1	0.1	0	0	0	0	0	0	0
$\mathcal{EW}1$	55.2	54.5	55.1	62.8	62.6	62.9	60	59.7	59.9	59.2
$\mathcal{G}(0.5, 1)$	0.5	0.5	0.5	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$\mathcal{AW}2$	0	0	0	0	0	0	0	0	0	0
$\mathcal{EW}2$	0	0	0	0	0	0	0	0	0	0
$\mathcal{EW}3$	0	0	0	0	0	0	0	0	0	0
$\mathcal{GG}1$	0	0	0	0	0	0	0	0	0	0
$\mathcal{GG}2$	0	0	0	0	0	0	0	0	0	0
$\mathcal{PGW}1$	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$\mathcal{LN}(0, 0.8)$	68.3	67.5	68.1	79.5	79.3	79.4	76.7	76.4	76.4	74.6
$\mathcal{IG}(3, 1)$	91.4	90.8	91.2	98.5	98.5	98.5	97.9	97.7	97.9	95.8
$\mathcal{EW}4$	43.4	42.8	43.3	46.9	46.8	47.1	44.8	44.5	44.7	44.9
$\mathcal{GG}3$	44.8	44.3	44.8	51.2	51.1	51.3	49.2	48.8	49.1	48.3
$\mathcal{PGW}2$	66.2	65.4	66.1	77.5	77.2	77.5	74.6	74.3	74.5	72.6
mean	26.3	26.1	26.3	29.6	29.5	29.7	28.6	28.5	28.6	28.1

Table A.47: Power results for the tests based on the Burr Generalized Weibull distribution,  $n = 50$

altern.	$\widehat{MO}_w$	$\widehat{MO}_s$	$\widehat{MO}_l$	$\widetilde{MO}_w$	$\widetilde{MO}_s$	$\widetilde{MO}_l$	$\check{M}O_w$	$\check{M}O_s$	$\check{M}O_l$	mean
exp(1)	5	5.1	5	5.1	5.1	5.1	5.1	5	5.1	5.1
$\mathcal{W}(1, 0.5)$	5.1	5	5.1	5	5	5	4.9	4.9	4.9	5
$\mathcal{W}(1, 3)$	4.8	5	4.9	4.9	4.9	4.9	5.1	5	5.1	5
$\mathcal{G}(3, 1)$	1	5.7	3.7	10.4	14.1	13.7	14.8	14.7	14.8	10.3
$\mathcal{AW}1$	57.6	53.3	55.6	39.5	15.3	25.5	2.2	0.8	0.8	27.7
$\mathcal{EW}1$	0.6	12.1	7.6	24.3	30.8	29.8	31.3	30.1	31.3	22
$\mathcal{G}(0.5, 1)$	13.7	10.6	12.2	6.6	2.1	3.1	1.8	1.6	1.6	5.9
$\mathcal{AW}2$	86.9	82.7	85.1	79.5	38.3	58.7	18	0	0	49.9
$\mathcal{EW}2$	23.8	17.7	20.6	16	2.5	6.3	1.7	0.2	0.2	9.9
$\mathcal{EW}3$	23.6	17.5	20.6	16.2	2.3	6.4	1.9	0.2	0.2	9.9
$\mathcal{GG}1$	39.9	33	36.6	23.6	3.9	9.7	1.8	0.1	0.1	56.5
$\mathcal{GG}2$	31.1	25.1	28	16.5	2.6	6.5	1.2	0.3	0.3	12.4
$\mathcal{PGW}1$	17.9	14	16	8.6	1.8	3.6	1.3	1.1	1.1	7.3
$\mathcal{LN}(0, 0.8)$	0.7	17.1	11.2	33.5	40.8	39.7	41.6	41.4	41.6	29.7
$\mathcal{IG}(3, 1)$	3.1	38.2	28.3	63.6	70.4	69.4	71.5	71.5	71.6	54.2
$\mathcal{EW}4$	0.6	8.7	5.5	17.7	22.7	21.8	23.4	23.2	23.4	16.3
$\mathcal{GG}3$	0.5	9.6	6.1	19.2	24.6	23.7	25.3	25.1	25.3	17.7
$\mathcal{PGW}2$	0.8	18.1	12	32.8	40.3	39.1	41.3	40.9	41.1	29.6
mean	20.1	24.2	23.3	27.2	20.8	23.8	18.6	16.8	16.9	21.3

Table A.48: Power results for the tests based on the Marshall-Olkin distribution,  $n = 20$

altern.	$\widehat{MO}_w$	$\widehat{MO}_s$	$\widehat{MO}_l$	$\widetilde{MO}_w$	$\widetilde{MO}_s$	$\widetilde{MO}_l$	$\check{MO}_w$	$\check{MO}_s$	$\check{MO}_l$	mean
exp(1)	4.9	5	5	5.1	5	5	5	4.9	4.9	5
$\mathcal{W}(1, 0.5)$	4.9	4.8	4.8	5	4.8	4.9	5	5	5	4.9
$\mathcal{W}(1, 3)$	4.8	4.8	4.9	5.2	5	4.9	5.1	5.1	4.9	5
$\mathcal{G}(3, 1)$	5.7	12.6	10.4	20.2	25	23.7	25.4	27.3	26.8	19.7
$\mathcal{AW}1$	86	83.6	84.5	68	54.7	60.1	13.7	3.3	6.2	51.1
$\mathcal{EW}1$	21.4	37.9	33.3	54.3	60.5	59	61.8	64	63.5	50.6
$\mathcal{G}(0.5, 1)$	21.2	16.9	18.6	12.3	6	8.2	4.8	1	1.8	10.1
$\mathcal{AW}2$	99.7	99.6	99.7	99.8	98.8	99.4	79.2	56.4	66.9	88.8
$\mathcal{EW}2$	47.4	39.7	42.7	52.7	35.6	42.5	37.4	10.7	19.6	36.5
$\mathcal{EW}3$	47.2	39.5	42.5	53.2	36.2	42.8	37.1	10.6	19.3	36.5
$\mathcal{GG}1$	73.1	67.2	69.6	63	44.3	51.9	35.7	9.6	18.4	48.2
$\mathcal{GG}2$	58.7	52	54.6	45	27.9	34.3	22.6	4.5	9.7	34.4
$\mathcal{PGW}1$	31.8	26.1	28.3	20.2	10.1	13.5	8.5	1.2	2.8	15.8
$\mathcal{LN}(0, 0.8)$	33.4	53.2	47.9	71.5	76.6	75.5	77.3	79.3	79	66
$\mathcal{IG}(3, 1)$	64.8	79.1	75.8	96.5	97.5	97.3	97.7	98.1	98	89.4
$\mathcal{EW}4$	13	25.6	21.3	38.9	45.2	43.4	45.5	47.8	47.3	36.5
$\mathcal{GG}3$	14.7	28.4	24.3	42.5	48.9	47.3	49.9	52.2	51.7	40
$\mathcal{PGW}2$	35.8	54.8	49.7	71	76.5	75.1	76.9	78.6	78.2	66.3
mean	43.6	47.7	46.8	53.9	49.6	51.6	44.9	36.3	39.3	45.9

Table A.49: Power results for the tests based on the Marshall-Olkin distribution,  $n = 50$

altern.	$\widehat{MW}_w$	$\widehat{MW}_s$	$\widehat{MW}_l$	$\widetilde{MW}_w$	$\widetilde{MW}_s$	$\widetilde{MW}_l$	$\check{MW}_w$	$\check{MW}_s$	$\check{MW}_l$	mean
exp(1)	5.1	5.1	5.1	5.1	5.3	5.1	5	5.1	5.1	5.1
$\mathcal{W}(1, 0.5)$	5.1	5.1	5.1	5.2	5.1	5.2	5	5.1	5.1	5.1
$\mathcal{W}(1, 3)$	5	4.9	5	5.1	5.1	5.1	5	5	5.1	5
$\mathcal{G}(3, 1)$	1.2	4.4	1.2	1	11.9	1	0.9	9.6	0.9	3.6
$\mathcal{AW}1$	53.4	51.7	53.5	58.3	44.6	58.4	58.2	50	58.5	54.1
$\mathcal{EW}1$	0.3	9.5	0.3	0.2	26.6	0.2	0.1	21.6	0.2	6.5
$\mathcal{G}(0.5, 1)$	14.3	12.6	14.3	13.6	6	13.6	13	7.9	13.1	12
$\mathcal{AW}2$	95	94	94.9	82.8	60.3	82.2	82	67.5	81.6	82.3
$\mathcal{EW}2$	35.2	31.5	35	20.3	5.2	19.9	18.3	7.9	18.3	21.3
$\mathcal{EW}3$	35	31.3	34.9	20.4	5.3	20	18.5	8	18.4	22.4
$\mathcal{GG}1$	49.5	45.8	49.3	38.9	18.6	38.5	36.9	23.5	36.9	37.5
$\mathcal{GG}2$	35.9	32.8	35.8	30.1	13.4	30	29.1	18.1	29.2	28.3
$\mathcal{PGW}1$	19.7	17.6	19.7	17.9	7.3	17.9	17.5	10.4	17.8	18.4
$\mathcal{LN}(0, 0.8)$	0.1	13.9	0.1	0	36.5	0	0	30	0	9
$\mathcal{IG}(3, 1)$	0	32.4	0	0.2	66.1	0	0	58.4	0	17.5
$\mathcal{EW}4$	0.6	7	0.6	0.3	19.9	0.3	0.4	16	0.4	5
$\mathcal{GG}3$	0.5	7.3	0.5	0.3	21.3	0.2	0.2	17.1	0.2	5.3
$\mathcal{PGW}2$	0.2	12.6	0.2	0.1	35.5	0.1	0.1	29.1	0.1	8.9
mean	12.7	26.9	22.7	19	25.2	18.8	18.4	25	18.4	20.8

Table A.50: Power results for the tests based on the Modified Weibull distribution,  $n = 20$

altern.	$\widehat{MW}_w$	$\widehat{MW}_s$	$\widehat{MW}_l$	$\widetilde{MW}_w$	$\widetilde{MW}_s$	$\widetilde{MW}_l$	$\check{MW}_w$	$\check{MW}_s$	$\check{MW}_l$	mean
exp(1)	5	5.1	4.9	5.1	4.9	5.1	4.9	4.9	4.8	5
$\mathcal{W}(1, 0.5)$	5	5.1	4.9	4.9	5.1	5.1	5	5.1	5	5
$\mathcal{W}(1, 3)$	5.3	5.3	5.1	4.9	4.9	5	5	5	4.9	5
$\mathcal{G}(3, 1)$	0.4	12.1	0.4	0.3	22.8	0.3	0.2	19.5	0.2	6.3
$\mathcal{AW}1$	81.1	78.1	81.1	88.1	81.6	88.1	88.2	83.6	88.1	84.3
$\mathcal{EW}1$	0	35.2	0	0	59.1	0	0	53	0	16.4
$\mathcal{G}(0.5, 1)$	24.3	19.6	24.4	23	11.3	23.2	22.5	14	22.3	20.5
$\mathcal{AW}2$	100	100	100	99.6	97.9	99.6	99.7	98.5	99.6	99.4
$\mathcal{EW}2$	78.8	71.6	78.5	44.9	18.6	44.1	41.6	21.5	40.4	48.9
$\mathcal{EW}3$	78.9	71.9	78.6	44.6	18.7	43.8	41.9	21.7	40.7	49
$\mathcal{GG}1$	89.9	86	89.8	75.7	55.5	75.3	73.4	58.1	72.7	75.2
$\mathcal{GG}2$	73.2	66.9	73	61.2	40.5	60.7	58.2	43.4	58.2	59.5
$\mathcal{PGW}1$	40.1	33.2	39.9	34.7	18.6	34.5	32.7	21.2	32.4	31.9
$\mathcal{LN}(0, 0.8)$	0	48.3	0	0	75.3	0	0	69.9	0	21.5
$\mathcal{IG}(3, 1)$	0	81.8	0	2.1	97.5	0	0.4	95.6	0	30.8
$\mathcal{EW}4$	0	24.1	0	0	42.6	0	0	37.5	0	11.6
$\mathcal{GG}3$	0	26.8	0	0	47	0	0	41.6	0	12.8
$\mathcal{PGW}2$	0	44.2	0	0	72.2	0	0	66.9	0	20.4
mean	37.8	53.3	37.7	31.6	50.5	31.3	30.6	49.7	30.3	39.2

Table A.51: Power results for the tests based on the Modified Weibull distribution,  $n = 50$ 

altern.	$\widehat{PGW}_w$	$\widehat{PGW}_s$	$\widehat{PGW}_l$	$\widetilde{PGW}_w$	$\widetilde{PGW}_s$	$\widetilde{PGW}_l$	$\check{PGW}_w$	$\check{PGW}_s$	$\check{PGW}_l$	mean
exp(1)	4.8	4.8	4.8	4.8	5.1	4.9	5	5	4.9	4.9
$\mathcal{W}(1, 0.5)$	4.9	4.9	4.8	5	5.2	5	5.1	5.2	5.1	5
$\mathcal{W}(1, 3)$	5	4.9	5	4.8	5.1	4.9	5	5	5	5
$\mathcal{G}(3, 1)$	8.1	5.7	6.6	14.9	13.4	14.8	15.4	9.3	13.5	11.3
$\mathcal{AW}1$	46.7	52.7	51.9	0.7	38.3	8.7	0.8	48.7	35.5	31.6
$\mathcal{EW}1$	18.4	13.6	15.3	31.2	28.9	31.3	32.2	21.4	28.9	24.6
$\mathcal{G}(0.5, 1)$	9.3	11.2	10.7	1.6	4.8	1.7	1.5	7.5	3.9	5.8
$\mathcal{AW}2$	85.8	88.7	88	0	36.5	3.7	0	52.9	31	43
$\mathcal{EW}2$	18.5	22.8	21.7	0.2	1.9	0.2	0	5.2	1.1	7.9
$\mathcal{EW}3$	17.8	22.4	21.2	0.2	1.9	0.2	0.1	5.1	1.1	7.7
$\mathcal{GG}1$	33.7	38.7	37.3	0.1	11.6	0.6	0	19.5	8.8	16.7
$\mathcal{GG}2$	24.1	28.2	27.1	0.2	9	0.6	0.2	15.2	6.8	12.4
$\mathcal{PGW}1$	12.8	15.5	14.7	0.9	5.5	1	0.7	9.3	4.3	7.2
$\mathcal{LN}(0, 0.8)$	25.7	19.7	21.9	42	39.5	42.1	42.7	30.3	39.2	33.7
$\mathcal{IG}(3, 1)$	51.6	43.6	46.8	72	69.9	72.1	71.1	58.8	67.9	61.5
$\mathcal{EW}4$	13.5	9.6	10.9	23.7	21.7	23.8	24.1	15.1	21.4	18.2
$\mathcal{GG}3$	14.5	10.3	11.8	25.8	23.6	25.8	25.9	16.4	23.1	19.7
$\mathcal{PGW}2$	24.9	18.8	21.1	41.4	38.7	41.5	41.1	28.8	37.6	28.2
mean	27.2	26.7	27.1	17	23	17.8	17.1	22.9	21.6	22.3

Table A.52: Power results for the tests based on the Power Generalized Weibull distribution,  $n = 20$

altern.	$\widehat{PGW}_w$	$\widehat{PGW}_s$	$\widehat{PGW}_l$	$\widetilde{PGW}_w$	$\widetilde{PGW}_s$	$\widetilde{PGW}_l$	$P\check{G}W_w$	$P\check{G}W_s$	$P\check{G}W_l$	mean
exp(1)	4.9	4.9	5	4.9	4.9	5	5	5.1	5.2	5
$\mathcal{W}(1, 0.5)$	5	5	5	5	5.1	5.1	5	5.1	5.2	5
$\mathcal{W}(1, 3)$	5	5	5.1	5.1	5	5.1	4.9	5	5	5
$\mathcal{G}(3, 1)$	18.6	15.6	16.7	28.5	23.7	27.2	28.9	18.5	23.9	22.4
$\mathcal{AW}1$	80.6	82.2	81.8	0	79.2	68.8	0	83.5	78.2	58.1
$\mathcal{EW}1$	49.6	44.8	46.7	66.5	60.9	65.4	66.8	53.2	60.6	57.2
$\mathcal{G}(0.5, 1)$	16.1	18.6	17.7	0.5	8.7	3.7	0.5	12.7	8	9.6
$\mathcal{AW}2$	99.9	99.9	99.9	0	84.8	66.9	0	91.9	83.9	69.6
$\mathcal{EW}2$	52.2	57	55.8	0	4.8	0.7	0	8.8	3.1	20.3
$\mathcal{EW}3$	51.8	56.6	55.4	0	4.9	0.7	0	8.9	3.1	20.2
$\mathcal{GG}1$	75.3	78.4	77.6	0	37.7	20.5	0	44.6	31.9	40.7
$\mathcal{GG}2$	56.3	60.6	59.4	0	28	13.8	0	34	23.3	30.6
$\mathcal{PGW}1$	26.9	30.2	29.3	0.2	13.1	5.5	0.2	18.3	11.5	15
$\mathcal{LN}(0, 0.8)$	65.5	60.9	62.7	82.2	77.8	81.4	82.5	71.6	78	73.6
$\mathcal{IG}(3, 1)$	93.3	91.6	92.4	98.8	98.2	98.7	98.6	96.5	97.8	96.2
$\mathcal{EW}4$	35.8	31.4	33.2	50.2	44	48.8	51.2	37.6	44.9	41.9
$\mathcal{GG}3$	39	34.5	36.3	54.5	48.5	53.4	55.6	41.4	49.2	45.8
$\mathcal{PGW}2$	63	58.1	59.9	79.7	75	78.6	79.7	68.4	75	70.8
mean	54.9	54.6	55	30.7	45.9	42.3	30.9	46	44.8	45

Table A.53: Power results for the tests based on the Power Generalized Weibull distribution,  $n = 50$

altern.	$\widehat{GG}_w^1$	$\widehat{GG}_s^1$	$\widehat{GG}_l^2$	$\widehat{MO}_w$	$\widehat{MW}_w$	$\widehat{PGW}_w$	$\widehat{PGW}_s$	$\widehat{PGW}_l$	$P\check{G}W_w$	$AD$	$MSF$	$TS$
exp(1)	5.1	5.1	5.7	5.1	5.1	4.8	4.8	4.8	5	5.6	4.9	5.1
$\mathcal{W}(1, 0.5)$	5.1	5.1	5.5	5	5.1	4.9	4.9	4.8	5.1	5.6	4.9	5.2
$\mathcal{W}(1, 3)$	5	5	5.6	4.9	5	5	4.9	5	5	5.6	4.8	5
$\mathcal{G}(3, 1)$	7.9	6.9	10.4	10.4	1.2	8.1	5.7	6.6	15.4	8.5	8.3	8.7
$\mathcal{AW}1$	53	54.1	49.6	39.5	53.4	46.7	52.7	51.9	0.8	42.1	32.6	49.5
$\mathcal{EW}1$	17.9	16.2	22.4	24.3	0.3	18.4	13.6	15.3	32.2	16.5	16.9	19.9
$\mathcal{G}(0.5, 1)$	10	10.7	9.4	6.6	14.3	9.3	11.2	10.7	1.5	8.8	7.7	9.2
$\mathcal{AW}2$	81.5	83.3	82.3	79.5	95	85.8	88.7	88	0	89.7	85.7	87
$\mathcal{EW}2$	15.9	18	15.3	16	35.2	18.5	22.8	21.7	0	23.3	19.9	18.1
$\mathcal{EW}3$	16.3	17.9	16.9	16.2	35	17.8	22.4	21.2	0.1	23.7	19.6	17.6
$\mathcal{GG}1$	32.2	34.5	30.7	23.6	49.5	33.7	38.7	37.3	0	31.7	28.2	34.1
$\mathcal{GG}2$	24.1	26.1	21.9	16.5	35.9	24.1	28.2	27.1	0.2	21.6	19	24
$\mathcal{PGW}1$	13.2	14.5	11.6	8.6	19.7	12.8	15.5	14.7	0.7	11.7	10.1	12.6
$\mathcal{LN}(0, 0.8)$	25.3	22.8	30.5	33.5	0.1	25.7	19.7	21.9	42.7	22.8	23	28.8
$\mathcal{IG}(3, 1)$	52.1	48.9	60.9	63.6	0	51.6	43.6	46.8	71.1	50.5	47.9	59.7
$\mathcal{EW}4$	13	11.7	15.7	17.7	0.6	13.5	9.6	10.9	24.1	12.1	12.4	14.4
$\mathcal{GG}3$	13.5	12.1	17.1	19.2	0.5	14.5	10.3	11.8	25.9	12.9	13.2	15.7
$\mathcal{PGW}2$	23.9	21.7	29.8	32.8	0.2	24.9	18.8	21.1	41.1	23.2	23.2	28.6
mean	26.6	26.6	28.3	27.2	12.7	27.2	26.7	27.1	17.1	26.6	24.5	28.5

Table A.54: Comparison with usual GOF tests,  $n = 20$

altern.	$\widehat{GG}_w^1$	$\widehat{GG}_s^1$	$\widehat{GG}_l^2$	$\widehat{MO}_w$	$\widehat{MW}_w$	$\widehat{PGW}_w$	$\widehat{PGW}_s$	$\widehat{PGW}_l$	$P\check{G}W_w$	$AD$	$MSF$	$TS$
exp(1)	5.1	5.1	5.5	5.1	5	4.9	4.9	5	5	5.6	5.3	4.9
$\mathcal{W}(1, 0.5)$	5.1	5	5.6	5	5	5	5	5	5	5.4	5.2	5
$\mathcal{W}(1, 3)$	5.1	5	5.3	5.2	5.3	5	5	5.1	4.9	5.3	5.2	5.1
$\mathcal{G}(3, 1)$	18.2	16.8	21.1	20.2	0.4	18.6	15.6	16.7	28.9	14.6	15.7	18.9
$\mathcal{AW}1$	83.7	84.1	82.3	68	81.1	80.6	82.2	81.8	0	72.2	63.4	82.2
$\mathcal{EW}1$	50.7	49	56.3	54.3	0	49.6	44.8	46.7	66.8	40.8	41.4	55.2
$\mathcal{G}(0.5, 1)$	16.8	17.6	16.7	12.3	24.3	16.1	18.6	17.7	0.5	13.5	10.6	15.5
$\mathcal{AW}2$	99.8	99.8	99.8	99.8	100	99.9	99.9	99.9	0	99.9	99.6	99.6
$\mathcal{EW}2$	44.1	46.2	47.4	52.7	78.8	52.2	57	55.8	0	57.9	39.9	49.4
$\mathcal{EW}3$	43.7	45.3	47.5	53.2	78.9	51.8	56.6	55.4	0	58.1	39.8	49.8
$\mathcal{GG}1$	71.7	73.4	73.3	63	89.9	75.3	78.4	77.6	0	69.4	57.5	74.9
$\mathcal{GG}2$	54.9	56.7	55.8	45	73.2	56.3	60.6	59.4	0	48.3	39.9	56.2
$\mathcal{PGW}1$	26.9	28.4	27	20.2	40.1	26.9	30.2	29.3	0.2	21.1	17.4	27.2
$\mathcal{LN}(0, 0.8)$	66.9	65.3	72.5	71.5	0	65.5	60.9	62.7	82.5	56.5	56.2	72
$\mathcal{IG}(3, 1)$	94.2	93.6	96.2	96.5	0	93.3	91.6	92.4	98.6	92.3	89.9	96.9
$\mathcal{EW}4$	35.8	33.7	40.6	38.9	0	35.8	31.4	33.2	51.2	27.9	28.3	38.9
$\mathcal{GG}3$	38.8	37.1	44.1	42.5	0	39	34.5	36.3	55.6	30.1	31.6	42.9
$\mathcal{PGW}2$	64.6	62.5	69.9	71	0	63	58.1	59.9	79.7	56.9	57.1	71.6
mean	54.1	54	56.7	53.9	37.8	54.9	54.6	55	30.9	50.6	45.9	56.7

Table A.55: Comparison with usual GOF tests,  $n = 50$

## Appendix B

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This appendix presents a documentation on the R package EWGoF that we have developed. it can be downloaded at the link (<http://cran.r-project.org/web/packages/EWGoF/index.html>).

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### B.1 Documentation of EWGoF package

# Package ‘EWGoF’

July 2, 2014

**Type** Package

**Title** Goodness-of-fit tests for the Exponential and two-parameter Weibull distributions

**Version** 2.0

**Date** 2014-01-13

**Author** Meryam KRIT

**Maintainer** Meryam KRIT <meryam.krit@imag.fr>

**Description** This package contains a large number of the goodness-of-fit tests for the Exponential and Weibull distributions classified into families: the tests based on the empirical distribution function, the tests based on the probability plot, the tests based on the normalized spacings, the tests based on the Laplace transform and the likelihood based tests.

**License** GPL (>= 2.0)

**Imports** Rcpp (>= 0.10.3), maxLik, miscTools

**LinkingTo** Rcpp

**LazyLoad** yes

**NeedsCompilation** yes

**Repository** CRAN

**Date/Publication** 2014-04-08 09:47:41

## R topics documented:

EWGoF-package	2
BLOMEst	3
CF.test	4
EDF_NS.test	5
LK.test	7
LRI.test	8
LSEst	9

MEst . . . . .	11
MLEst . . . . .	12
WEDF.test . . . . .	13
WLK.test . . . . .	14
WLP.test . . . . .	16
WNS.test . . . . .	17
WPP.test . . . . .	19
<b>Index</b>	<b>21</b>

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EWGoF-package	<i>Goodness-of-fit tests for the Exponential and two-parameter Weibull distributions</i>
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### Description

This package contains a large number of the goodness-of-fit tests for the Exponential and Weibull distributions classified into families: the tests based on the empirical distribution function, the tests based on the probability plot, the tests based on the normalized spacings, the tests based on the Laplace transform and the likelihood based tests ...

### Details

Package:	EWGoF
Type:	Package
Version:	1.0
Date:	2014-01-13
License:	GPL (>=2.0)
Imports:	Rcpp (>= 0.10.3), inline, maxLik
LazyLoad:	yes
LinkingTo:	Rcpp

Computes the p-value of the chosen test statistic, the value of the observed statistic and an estimation of the distribution parameters (either the parameter of the Exponential distribution or the shape and the scale Weibull parameters).

### Author(s)

Meryam KRIT Maintainer: Meryam KRIT <meryam.krit@imag.fr>

### References

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- Cabana A. and Quiroz A.J., Using the empirical moment generating function in testing the Weibull and type 1 Extreme Value distributions, *Test*, 14(2), 417-431, 2005.
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BLOMEst

*Blom's estimators of the two parameters of the Weibull distribution***Description**

Computes the Blom's estimators of the shape and scale parameters of the Weibull distribution from an i.i.d sample  $x$ . It also gives the sample  $\tilde{y}$  after using the logarithmic transformation ( $\tilde{y} = (shape) \ln(x/scale)$ , where *shape* and *scale* are the estimated shape and scale parameters).

**Usage**

BLOMEst(x)

**Arguments**

x                      a numeric vector of data values.

**Value**

A list containing the following elements:

eta                    the Blom's estimator of the scale parameter of the Weibull distribution ( $\check{scale}$ ).

beta                   the Blom's estimator of the shape parameter of the Weibull distribution ( $\check{shape}$ ).

y                        the pseudo-observations  $\tilde{y}$  after using the logarithmic transformation and the Blom's estimators.

**Author(s)**

Meryam KRIT

**References**

Blom G., Statistical Estimates and Transformed Beta-variables. New York: Wiley, 1958.

**Examples**

```
x <- rweibull(50,2,3)

#Value of the Blom's estimator of the scale parameter
BLOMEst(x)$eta

#Value of the Blom's estimator of the shape parameter
BLOMEst(x)$beta
```

---

CF.test	<i>GoF tests based on the empirical characteristic function for the Exponential distribution</i>
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---

**Description**

Computes the GoF tests based on the characteristic function of the Exponential distribution: Epps-Pulley (EP), Henze-Meintanis (W1, W2) and Meintanis-Iliopoulos test statistics (T1, T2).

**Usage**

```
CF.test(x, type = "EP", a = 1, nsim = 1000)
```

**Arguments**

x	a numeric vector of data values.
type	the type of the test statistic used. "EP" is the default used test of Epps-Pulley, "W1" and "W2" for Henze and Meintanis, "T1" and "T2" for Meintanis-Iliopoulos test statistics.
a	parameter value to be adjusted for the test statistics ("W1", "W2", "T1" and "T2").
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Details**

The computation time of this function is quite long for the test statistics "W1", "W2", "T1" and "T2" because of their complex expression. The Monte-Carlo simulations take more time compared to the other tests. These tests are not defined for a=0.

**Value**

An object of class htest.

**Author(s)**

Meryam KRIT

**References**

Epps T.W. and Pulley L.B., A test for exponentiality vs. monotone hazard alternatives derived from the empirical characteristic function, *Journal of the Royal Statistical Society, Series B*, 48, 206-213, 1986.

Henze N. and Meintanis S.G., Recent and classical tests for exponentiality: partial review with comparisons, *Metrika*, 61, 29-45, 2005.

Henze N. and Meintanis S.G., Goodness-of-fit tests based on a new characterization of the exponential distribution, *Communications in Statistics, Theory and Methods*, 31, 1479-1497, 2002.

Meintanis S.G. and Iliopoulos G., Characterizations of the exponential distribution based on certain properties of its characteristic function, *Kybernetika*, 39 (3), 295-298, 2003.

**Examples**

```
x <- rgamma(10,0.3)

#Apply the Epps-Pulley test
CF.test(x,type="EP")

# Apply the test of Meintanis-Iliopoulos
CF.test(x,type="T1",nsim=200)

# Apply the test of Henze-Meintanis
CF.test(x,type="W1",nsim=200)
```

EDF\_NS.test

*GoF tests based on the empirical distribution function, the normalized spacings and the probability plots for the Exponential distribution*

**Description**

Computes the Exponential GoF tests based on the empirical distribution function: the Kolmogorov-Smirnov (KS), Cramer-Von-Mises (CM) and Anderson-Darling (AD) tests, the tests based on the probability plot: Shapiro-Wilk (SW) and Patwardhan (PA) tests and the tests based on the normalized spacings: Gnedenko (Gn) and Gini (G) tests.

**Usage**

```
EDF_NS.test(x, type = "AD", nsim = 2000)
```

**Arguments**

x	a numeric vector of data values.
type	the type of the test statistic used. "AD" is the default used test of Anderson-Darling, "KS" for Kolmogorov-Smirnov, "CM" for Cramer-Von-Mises, "SW" for Shapiro-Wilk, "PA" for Patwardhan, "Gn" for Gnedenko and "G" for Gini test statistic.
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Details**

This function computes the GoF test statistics of three different families: the tests based on the empirical distribution function, the tests based on the probability plots and the tests based on the normalized spacings. The p-value of the tests is computed using Monte-Carlo simulations because only the asymptotic distribution of the previous statistics is known. Therefore the tests can be applied to small samples.

**Value**

An object of class htest.

**Author(s)**

Meryam KRIT

**References**

- D'Agostino R.B. and Stephens M.A., *Goodness-of-fit techniques*, Marcel Dekker, 1986.
- Gail M.H. and Gastwirth J.L., A scale-free goodness-of-fit test for the exponential distribution based on the Gini statistic, *Journal of the Royal Statistical Society, Series B*, 40, 350-357, 1978.
- Gnedenko B.V., Belyayev Y.K. and Solovyev A.D., *Mathematical Models of Reliability Theory*, Academic Press, 1969.
- Shapiro S.S. and Wilk M.B., An analysis of variance test for the exponential distribution (complete samples), *Technometrics*, 14, 355-370, 1972.
- Patwardhan G., Tests for exponentiality, *Communications in Statistics, Theory and Methods*, 17, 3705-3722, 1988.

**Examples**

```
x1 <- rexp(50,2)

#Apply the Kolmogorov-Smirnov test
EDF_NS.test(x1,type="KS")

x2 <- rlnorm(50,0.2)

#Apply the Patwardhan test
```

*LK.test*

7

```
EDF_NS.test(x2,type="PA")

#Apply the Cramer-von Mises test
EDF_NS.test(x2,type="CM")

#Apply the Gini test
EDF_NS.test(x2,type="G")
```

---

*LK.test**GoF tests based on the likelihood for the Exponential distribution*

---

**Description**

Computes the Exponential GoF tests based on the three following statistics: the score of Cox-Oakes (CO), Wald (W) and likelihood ratio (LR) GoF tests. These tests include the Exponential distribution in the Weibull distribution and apply a parametric test to check whether the shape parameter is equal to one.

**Usage**

```
LK.test(x, type = "CO", nsim = 1000)
```

**Arguments**

x	a numeric vector of data values.
type	the type of the test statistic used. "CO" is the default used test of Cox-Oakes, "W" for Wald, "LR" for likelihood ratio test statistic.
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Details**

The test statistic of Cox and Oakes is equivalent to the score test applied to the shape parameter of the Weibull distribution. The expression of the score is given using the observed information matrix and not the exact one.

**Value**

An object of class `htest`.

**Author(s)**

Meryam KRIT

**References**

Cox D.R. and Oakes D., *Analysis of survival data*, Chapman and Hall, 1984.

**Examples**

```
x <- rgamma(20,0.3)

#Apply the Cox and Oakes test (score) test
LK.test(x,type="CO",nsim=200)

# Apply the test based on Wald
LK.test(x,type="W",nsim=200)

# Apply the test based on the likelihood ratio
LK.test(x,type="LR",nsim=200)
```

---

LRI.test

*GoF tests based on the Laplace transform, the mean residual life and the integrated distribution function for the Exponential distribution*

---

**Description**

Computes the Weibull GoF tests based on the Laplace transform: Baringhaus-Henze (BH) and Henze (He). The test statistic of Klar (Kl) is based on the integrated distribution function. Two tests are based on the mean residual life (BHC, BHK).

**Usage**

```
LRI.test(x, type = "BH", a = 1, nsim = 2000)
```

**Arguments**

x	a numeric vector of data values.
type	the type of the test statistic used. "BH" is the default used test of Baringhaus-Henze,"He" for Henze, "Kl" for Klar, "BHC" and "BHK" for the tests based on the integrated distribution function.
a	parameter value to be adjusted for the test statistics.
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Value**

An object of class htest.

**Author(s)**

Meryam KRIT

## References

Baringhaus L. and Henze N., Tests of fit for exponentiality based on a characterization via the mean residual life function, *Statistical Papers*, 41, 225-236, 2000.

Baringhaus L. and Henze N., A class of consistent tests for exponentiality based on the empirical Laplace transform, *Annals of the Institute of Statistical Mathematics*, 43, 551-564, 1991.

Henze N., A new flexible class of omnibus tests for exponentiality, *Communications in Statistics, Theory and Methods*, 22, 115-133, 1993.

Klar B., Goodness-of-fit tests for the exponential and normal distribution based on the integrated distribution function, *Annals of the Institute of Statistical Mathematics*, 53, 338-353, 2001.

## Examples

```
x <- rlnorm(50,0.3)

#Apply the Baringhaus-Henze test
LRI.test(x,type="BH",nsim=200)

# Apply the test of Henze
LRI.test(x,type="He",nsim=200)

# Apply the test of Klar
LRI.test(x,type="K1")

# Apply the test of Barighaus based on the integrated distribution function
LRI.test(x,type="BHC")
```

---

LSEst

*Least Squares Estimators of the two parameters of the Weibull distribution*

---

## Description

This function computes the Least Squares Estimators (LSEs) of the shape and scale parameters of the Weibull distribution, based on the probability plot, from an i.i.d sample  $x$ . It also gives the sample  $\tilde{y}$  after using the logarithmic transformation ( $\tilde{y} = (\widehat{shape}) \ln(x/\widehat{scale})$ , where  $\widehat{shape}$  and  $\widehat{scale}$  are the estimated shape and scale parameters).

## Usage

```
LSEst(x)
```

## Arguments

$x$  a numeric vector of data values.

**Details**

The elements of the numeric vector should be positive. The support of the Weibull distribution is  $R^+$ . These estimators are used by Liao and Shimokawa; they are based on the probability plot and symmetrical ranks.

**Value**

A list containing the following elements:

eta	the least squares estimator of the scale parameter of the Weibull distribution ( <i>scale</i> ).
beta	the least squares estimator of the shape parameter of the Weibull distribution ( <i>shape</i> ).
y	the pseudo-observations $\tilde{y}$ after using the logarithmic transformation and the LSEs.

**Author(s)**

Meryam KRIT

**References**

Liao M. and Shimokawa T., A new goodness-of-fit test for type-I extreme-value and 2-parameter Weibull distributions with estimated parameters, *Journal of Statistical Computation and Simulation*, 64 (1), 23-48, 1999.

Krit M., Gaudoin O., Xie M. and Remy E., Simplified likelihood goodness-of-fit tests for the Weibull distribution, *Communications in Statistics - Simulation and Computation*.

**Examples**

```
x <- rweibull(50,2,3)
Est <- LSEst(x)
#Value of the least squares estimator of the scale parameter
Est$eta
#Value of the least squares estimator of the shape parameter
Est$beta
```

**Description**

This function computes the Moment Estimators (MEs) of the shape and scale parameters of the Weibull distribution from an i.i.d sample  $x$ . It also gives the sample  $\check{y}$  after using the logarithmic transformation ( $\check{y} = (shape) \ln(x/scale)$ , where *shape* and *scale* are the estimated shape and scale parameters).

**Usage**

```
MEst(x)
```

**Arguments**

$x$  a numeric vector of data values.

**Details**

The elements of the numeric vector should be positive. The support of the Weibull distribution is  $\mathbb{R}^+$ . These estimators are based on the moments of the extreme value distribution.

**Value**

A list containing the following elements:

$\eta$  the moment estimator of the scale parameter of the Weibull distribution ( $\check{scale}$ ).  
 $\beta$  the moment estimator of the shape parameter of the Weibull distribution ( $\check{shape}$ ).  
 $y$  the pseudo-observations  $\check{y}$  after using the logarithmic transformation and the MEs.

**Author(s)**

Meryam KRIT

**References**

Rinne H., *The Weibull distribution - A handbook*, CRC-Chapman & Hall, 2009.

**Examples**

```
x <- rweibull(50,2,3)

Est <- MEst(x)

#Value of the moment estimator of the scale parameter
Est$eta
```

#Value of the moment estimator of the shape parameter  
Est\$beta

---

MLEst	<i>Maximum Likelihood Estimators of the two parameters of the Weibull distribution</i>
-------	--

---

### Description

This function computes the Maximum Likelihood Estimators (MLEs) of the shape and scale parameters of the Weibull distribution from an i.i.d sample  $x$ . It also gives the sample  $\hat{y}$  after using the logarithmic transformation ( $\hat{y} = (\hat{shape}) \ln(x/\hat{scale})$ , where  $\hat{shape}$  and  $\hat{scale}$  are the estimated shape and scale parameters).

### Usage

MLEst(x)

### Arguments

$x$  a numeric vector of data values.

### Details

The elements of the numeric vector should be positive. The support of the Weibull distribution is  $\mathbb{R}^+$ .

### Value

A list containing the following elements:

eta	the maximum likelihood estimator of the scale parameter of the Weibull distribution ( $\hat{scale}$ ).
beta	the maximum likelihood estimator of the shape parameter of the Weibull distribution ( $\hat{shape}$ ).
y	the pseudo-observations $\hat{y}$ after using the logarithmic transformation and the MLEs.

### Author(s)

Meryam KRIT

### References

D'Agostino R.B. and Stephens M.A., *Goodness-of-fit techniques*, Marcel Dekker, 1986.  
Krit M., Gaudoin O., Xie M. and Remy E., Simplified likelihood goodness-of-fit tests for the Weibull distribution, *Communications in Statistics - Simulation and Computation*.

*WEDF.test***Examples**

```
x <- rweibull(50,2,3)

Est <- MLEst(x)

#Value of the maximum likelihood estimator of the scale parameter
Est$eta

#Value of the maximum likelihood estimator of the shape parameter
Est$beta
```

---

WEDF.test	<i>GoF tests based on the empirical distribution function for the Weibull distribution</i>
-----------	--

---

**Description**

Computes the Weibull GoF tests based on the empirical distribution function: Kolmogorov-Smirnov (KS), Cramer-Von-Mises (CM), Watson (W), Anderson-Darling (AD) and Liao-Shimokawa (LS) statistics using the three following estimation methods: Maximum Likelihood Estimators (MLE), Least Squares Estimators (LSE) and Moments Estimators (ME). A test statistic (KL) is added to this family but it is based on the Kullback-Leibler information; this test can also be computed with the three estimation methods.

**Usage**

```
WEDF.test(x, type = "AD", funEstimate = "MLE", paramKL = 2, nsim = 2000)
```

**Arguments**

x	a numeric vector of data values.
type	the type of the test statistic used, "AD" is the default used test of Anderson-Darling, "KS" for Kolmogorov-Smirnov, "CM" for Cramer-Von-Mises, "W" for Watson, "LS" for Liao-Shimokawa and "KL" for Kullback-Leibler.
funEstimate	the method used to estimate the two Weibull parameters. "MLE" is the default used method based on the maximum likelihood estimators, "LSE" for the least squares estimators and "ME" for the moment estimators.
paramKL	the value of the parameter m used in the expression of the statistic KL.
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Details**

The above test statistics are known in the literature by using the maximum likelihood estimators except the test "LS" that Liao and Shimokawa recommend to use with the least squares estimators. Each of the tests can have three versions, depending on the estimation method used.

All these tests statistics can be used for small samples. The asymptotic distributions of the tests are known in some cases but the use of their quantiles for small samples may lead to wrong conclusions. That is why we use Monte-Carlo simulations to apply the tests.

**Value**

An object of class htest.

**Author(s)**

Meryam KRIT

**References**

- D'Agostino R.B. and Stephens M.A., *Goodness-of-fit techniques*, Marcel Dekker, 1986.
- Liao M. and Shimokawa T., A new goodness-of-fit test for type-I extreme-value and 2-parameter Weibull distributions with estimated parameters, *Journal of Statistical Computation and Simulation*, 64 (1), 23-48, 1999.
- Chandra M., Singpurwalla N.D and Stephens M.A, Kolmogorov statistics for tests of fit for the Extreme Value and Weibull distributions, *Journal of American Statistical Association*, 76 (375), 729-731, 1981.
- Vaquera-Huerta P.P. and Villasenor-Alva J.A., A Goodness-of-fit test for the Gumbel distribution based on Kullback-Leibler information, *Communications in Statistics, Theory and Methods*, 38: 842-855, 2009.

**Examples**

```
x1 <- rweibull(50,2,3)

#Apply Kolmogorov-Smirnov test with maximum likelihood estimators
WEDF.test(x1,type="KS",funEstimate="MLE")

x2 <- rlnorm(50,0.2)

#Apply the Liao and Shimokawa test using the least squares estimators
WEDF.test(x2,type="LS",funEstimate="LSE")

#Apply the Cramer-von Mises test with the moment estimators
WEDF.test(x2,type="CM",funEstimate="ME")

#Apply the test based on the Kullback-Leibler information with the moment estimators
WEDF.test(x2,type="KL",funEstimate="ME")
```

---

WLK.test

*GoF tests based on the likelihood for the Weibull distribution*

---

**Description**

Computes the Weibull GoF tests based on the three following statistics: the score, Wald and likelihood ratio GoF tests. These tests include the Weibull distribution in larger statistics and apply a parametric test to the additional parameter.

**Usage**

```
WLK.test(x, type = "GG1", funEstimate = "MLE", procedure = "S", nsim = 500)
```

**Arguments**

x	a numeric vector of data values.
type	the type of the test statistic used: <ul style="list-style-type: none"> <li>• "GG1" is the default used test based on the Generalized Gamma distribution,</li> <li>• "GG2" is also based on the Generalized Gamma distribution after a transformation,</li> <li>• "EW" for the tests based on the Exponentiated Weibull,</li> <li>• "PGW" for the tests based on the Power Generalized Weibull,</li> <li>• "MO" for the tests based on the Marshall-Olkin distribution,</li> <li>• "MW" for the tests based on the Modified Weibull distribution,</li> <li>• "T" for a combination of two tests "PGW" and "MW".</li> </ul>
funEstimate	the method used to estimate the two Weibull parameters. "MLE" is the default used method based on the Maximum Likelihood Estimators, "LSE" for the Least Squares Estimators and "ME" for the Moment Estimators.
procedure	the procedure used as a default is the score "S". The procedure can be either "W" for the Wald test or "LR" for the test based on the likelihood ratio procedure.
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Details**

The tests are based on different generalized Weibull families: the tests (GG1) and (GG2) are based on the Generalized Gamma distribution, the tests (EW) are based on the Exponentiated Weibull, (PGW) on the Power Generalized Weibull, (MO) on the Marshall-Olkin distribution and (MW) are based on the Modified Weibull distribution. Each family can have nine versions depending on the procedure used (score, Wald or likelihood ratio statistic) and on the parameters estimation methods: maximum likelihood, moment or least squares method, except GG1 which has only three versions using the maximum likelihood estimators.

The test statistics  $T$  is a combination between two Wald test statistics: PGW with ME ( $P\check{G}W_w$ ) and MW with MLE ( $M\hat{W}_w$ ) after they are centered with their mean values ( $\overline{P\check{G}W_w}$  and  $\overline{M\hat{W}_w}$ ) and normalized by their standard deviations (respectively  $sd(P\check{G}W_w)$  and  $sd(M\hat{W}_w)$ ).

The expression of the statistic  $T$  is as follows:

$$T = 0.5 \left| \frac{P\check{G}W_w - \overline{P\check{G}W_w}}{sd(P\check{G}W_w)} \right| + 0.5 \left| \frac{M\hat{W}_w - \overline{M\hat{W}_w}}{sd(M\hat{W}_w)} \right|$$

**Value**

An object of class htest.

**Author(s)**

Meryam KRIT

## References

Krit M., Gaudoin O., Xie M. and Remy E., Simplified likelihood goodness-of-fit tests for the Weibull distribution, *Communications in Statistics - Simulation and Computation*.

## Examples

```
x <- rlnorm(50,.3)

#Apply some likelihood based tests
WLK.test(x,type="GG1",funEstimate="MLE",procedure="W")
WLK.test(x,type="PGW",funEstimate="ME",procedure="S")
WLK.test(x,type="MO",funEstimate="LSE",procedure="LR")
```

---

WLP.test	<i>GoF tests based on the Laplace transform for the Weibull distribution</i>
----------	--

---

## Description

Computes the Weibull GoF tests based on the Laplace transform: (CQ) Cabana-Quiroz with the two versions either with the Maximum Likelihood Estimators (MLE) or the Moment Estimators (ME) and (LT) based on the Laplace transform with discretization which has three versions, it can be computed either with the MLE, ME or the Least Squares Estimators (LSE).

## Usage

```
WLP.test(x,type = "CQ",funEstimate="MLE",s1=-1,s2=0.4,mr=100,a=-5,nsim=500)
```

## Arguments

x	a numeric vector of data values.
type	the type of the test statistic used. "CQ" is the default used test of Cabana and Quiroz and "LT" is the test statistic based on the Laplace transform discretization.
funEstimate	the method used to estimate the two Weibull parameters. "MLE" is the default used method based on the maximum likelihood estimators, "LSE" for the least squares estimators and "ME" for the moment estimators. The test statistic "CQ" can be computed for MLE or ME only.
s1	the first value where the quadratic form of the statistic "CQ" is computed.
s2	the second value where the quadratic form of the statistic "CQ" is computed.
mr	the number of the discretizations done to compute the test "LT"; mr = 100 is the default value recommended.
a	the value of the adjusting parameter used in the statistic "LT"; a = -5 is the default value used.
nsim	an integer specifying the number of replicates used in Monte Carlo.

WNS.test

### Value

An object of class htest.

### Author(s)

Meryam KRIT

### References

Cabana A. and Quiroz A.J., Using the empirical moment generating function in testing the Weibull and type 1 Extreme Value distributions, *Test*, 14(2), 417-431, 2005.

Krit M., Goodness-of-fit tests for the Weibull distribution based on the Laplace transform, *J-SFdS*, 2014.

### Examples

```
x<-rlnorm(50,3)

#Apply test based on the discretized Laplace transform
WLP.test(x,type="LT",funEstimate="MLE")

#Apply test of Cabana and Quiroz with maximum likelihood estimators
WLP.test(x, type="CQ",funEstimate="MLE",s1=-0.1,s2=0.02)

#Apply test of Cabana and Quiroz with the moment estimators
WLP.test(x, type="CQ",funEstimate="ME",s1=-1,s2=0.4)
```

---

WNS.test	<i>GoF tests based on the normalized spacings for the Weibull distribution</i>
----------	--

---

### Description

Computes the Weibull GoF tests based on the normalized spacings: (TS) Tiku-Singh, (MSF) Mann-Scheuer-Fertig only with right censoring and (LOS) Lockhart-O'Reilly-Stephens test statistics.

### Usage

```
WNS.test(x, type = "TS", s = 0, r = 0, nsim = 2000)
```

### Arguments

x	a numeric vector of data values.
type	the type of the test statistic used. "TS" is the default used test of Tiku-Singh, "MSF" for Mann-Scheuer-Fertig and "LOS" for Lockhart-O'Reilly-Stephens test statistic.
s	the index of the smallest observed value of x (s=0 if no left censoring).
r	the index of the largest observed value of x (r=0 if no right censoring).
nsim	an integer specifying the number of replicates used in Monte Carlo.

**Details**

For these tests statistics, there is no need to estimate the unknown values of the Weibull parameters and they are among the most powerful ones especially TS and LOS.

The p-value computed is not the exact p-value: the null hypothesis distribution of some statistics is known only asymptotically and sometimes is not even known. The asymptotic approximation is not correct especially for small samples. That is why Monte Carlo simulation is needed to compute the p-value.

**Value**

An object of class `htest`.

**Author(s)**

Meryam KRIT

**References**

Tiku M.L. and Singh M., Testing the two-parameter Weibull distribution, *Communications in Statistics*, 10, 907-918, 1981.

Mann N.R., Scheuer E.M. and Fertig K.W., A new goodness-of-fit test for the two-parameter Weibull or extreme-value distribution, *Communications in Statistics*, 2, 383-400, 1973.

Lockhart R.A., O'Reilly F. and Stephens M.A., Tests for the extreme-value and Weibull distributions based on normalized spacings, *Naval Research Logistics Quarterly*, 33, 413-421, 1986.

**Examples**

```
x1 <- rweibull(50,2,3)

#Apply the Tiku-Singh test with censored sample: 4 left censored and 5 right censored values
WNS.test(x1,type="TS",s=4,r=5)

#Apply the Lockahrt-O'reilly and Stephens test to the complete sample
WNS.test(x1,type="LOS")

x2 <- rlnorm(100,.3)

#Apply the Mann-Scheuer and Fertig test
WNS.test(x2,type="MSF")

#Apply Tiku-Singh test to the complete sample x2
WNS.test(x2)
```

---

WPP.test	<i>GoF tests based on the Weibull probability plot for the Weibull distribution</i>
----------	---

---

### Description

Computes the GoF tests based on the Weibull probability plot (WPP). The test statistics are similar to the coefficient of determination of the regression on the WPP: (RSB) test statistic of Smith and Bain, (REJG) test statistic of Evans, Johnson and Green, (SPP) test statistic of Coles based on the stabilized probability plot.

The second family includes the Shapiro-Wilk type test statistics: (SB) Shapiro and Brain and (OK) Ozturk and Korukoglu test statistic.

Two additional statistics can also be computed: the first one (ST1) is based on the kurtosis and the second one (ST2) is based on the skewness coefficient.

### Usage

```
WPP.test(x, type = "SB", nsim = 2000)
```

### Arguments

x	a numeric vector of data values.
type	the type of the test statistic used. "OK" is the test statistic of Ozturk and Korukoglu, "RSB" the test statistic of Smith and Bain, "REJG" the test statistic of Evans-Johnson and Green, "SPP" the test statistic of Coles based on the stabilized probability plot, "SB" the test statistic of Shapiro and Brain, "ST1" the test statistic based on the kurtosis and "ST2" the test statistic based on the skewness.
nsim	an integer specifying the number of replicates used in Monte Carlo.

### Value

An object of class htest.

### Author(s)

Meryam KRIT

### References

- Coles S.G., On goodness-of-fit tests for the two-parameter Weibull distribution derived from the stabilized probability plot, *Biometrika*, 76 (3), 593-598, 1989.
- Evans J.W., Johnson R.A. and Green D.W., *Two and three parameter Weibull goodness-of-fit tests*, Research paper FPL-RP-493, U.S. Forest Products Laboratory, Madison, WI, 1989.
- Ozturk A. and Korukoglu S., A new test for the extreme value distribution, *Communications in Statistics - Simulation and Computation*, 17, 1375-1393, 1988.

- Smith R.M. and Bain L.J., Correlation type goodness-of-fit statistics with censored sampling, *Communications in statistics*, 5, 119-132, 1976.
- Best D.J., Rayner J.C.W and Thas O., Comparison of five tests of fit for the extreme value distribution, *Journal of Statistical Theory and Practice*, 1 (1), 89-99, 2007.

**Examples**

```
x <- rlnorm(50,3)

#Apply the Ozturk-Korukoglu test
WPP.test(x,type="OK")

# Apply the test based on the stabilized probability plot
WPP.test(x,type="SPP")

# Apply the generalized smooth test based on the kurtosis
WPP.test(x,type="ST1")

#Apply the test of Shapiro and Brain
WPP.test(x)
```

# Index

- \*Topic **Anderson-Darling test**  
WEDF. test, 13
- \*Topic **Blom's estimators**  
BLOMEst, 3
- \*Topic **Cabana-Quiroz test**  
WLP. test, 16
- \*Topic **Cox-Oakes**  
LK. test, 7
- \*Topic **Cramer-von-Mises test**  
WEDF. test, 13
- \*Topic **EWGoF**  
EWGoF-package, 2
- \*Topic **Empirical characteristic function**  
CF. test, 4
- \*Topic **Empirical distribution function**  
EDF\_NS. test, 5  
WEDF. test, 13
- \*Topic **Epps-Pulley**  
CF. test, 4
- \*Topic **Gini**  
EDF\_NS. test, 5
- \*Topic **Gndenko**  
EDF\_NS. test, 5
- \*Topic **Henze-Meintanis**  
CF. test, 4
- \*Topic **Henze**  
LRI. test, 8
- \*Topic **Klar**  
LRI. test, 8
- \*Topic **Kolmogorov-Smirnov test**  
WEDF. test, 13
- \*Topic **Kullback-Lebler information**  
WEDF. test, 13
- \*Topic **Laplace transform**  
LRI. test, 8  
WLP. test, 16
- \*Topic **Least squares estimators**  
LSEst, 9
- \*Topic **Liao and Shimokawa test**  
WEDF. test, 13
- \*Topic **Liao-shimokawa**  
LSEst, 9
- \*Topic **Likelihood based tests**  
WLK. test, 14
- \*Topic **Likelihood ratio statistic**  
WLK. test, 14
- \*Topic **Likelihood ratio test**  
LK. test, 7
- \*Topic **Lockhart-O'Reilly-Stephens test statistic**  
WNS. test, 17
- \*Topic **Mann-Scheuer-Fertig statistic**  
WNS. test, 17
- \*Topic **Maximum likelihood estimators**  
MLEst, 12
- \*Topic **Meintanis-Iliopoulos**  
CF. test, 4
- \*Topic **Moment estimators**  
MEst, 11
- \*Topic **Ozturk-Korukoglu test**  
WPP. test, 19
- \*Topic **Patwardhan**  
EDF\_NS. test, 5
- \*Topic **Score statistic**  
WLK. test, 14
- \*Topic **Score test**  
LK. test, 7
- \*Topic **Shapiro-Wilk GoF tests**  
WPP. test, 19
- \*Topic **Shapiro-Wilk**  
EDF\_NS. test, 5
- \*Topic **Smith-Bain test**  
WPP. test, 19
- \*Topic **Stabilized Weibull probability plot**

- WPP.test, 19
  - \*Topic **Tiku-Singh statistic**
    - WNS.test, 17
  - \*Topic **WPP**
    - WPP.test, 19
  - \*Topic **Wald statistic**
    - WLK.test, 14
  - \*Topic **Wald test**
    - LK.test, 7
  - \*Topic **Weibull probability plot**
    - WPP.test, 19
  - \*Topic **integrated distribution function**
    - LRI.test, 8
  - \*Topic **kurtosis**
    - WPP.test, 19
  - \*Topic **linear unbiased estimators**
    - BLOMEst, 3
  - \*Topic **mean residual life**
    - LRI.test, 8
  - \*Topic **normalized spacings**
    - WNS.test, 17
  - \*Topic **skewness**
    - WPP.test, 19
  - \*Topic **smooth tests**
    - WPP.test, 19
- BLOMEst, 3
- CF.test, 4
- EDF\_NS.test, 5
- EWGoF (EWGoF-package), 2
- EWGoF-package, 2
- LK.test, 7
- LRI.test, 8
- LSEst, 9
- MEst, 11
- MLEst, 12
- WEDF.test, 13
- WLK.test, 14
- WLP.test, 16
- WNS.test, 17
- WPP.test, 19

**Abstract:** This work deals with goodness-of-fit (GOF) tests in reliability for both non repairable and repairable systems. GOF tests are efficient techniques to check the relevance of a model for a given data set.

For non repairable systems, the Exponential and Weibull distributions are the most used lifetimes distributions in reliability. A comprehensive comparison study of the GOF tests for the Exponential distribution is presented for complete and censored samples followed by recommendations about the use of the tests.

The two-parameter Weibull distribution allows decreasing and increasing failure rates unlike the Exponential distribution that makes the assumption of a constant hazard rate. Yet, there exist less GOF tests in the literature for the Weibull distribution. A comprehensive review of the existing GOF tests is done and two new families of exact GOF tests are introduced. The first family is the likelihood based GOF tests and the second is the family of tests based on the Laplace transform. Theoretical asymptotic properties of some new tests statistics are established.

A comprehensive comparison study of the GOF tests for the Weibull distribution is done. Recommendations about the most powerful tests are given depending on the characteristics of the tested data sets.

For repairable systems, new GOF tests are developed for imperfect maintenance models when both corrective maintenance and deterministic preventive maintenance are performed. These tests are exact and can be applied to small data sets.

Finally, illustrative applications to real data sets from industry are carried out for repairable and non repairable systems.

**Key words:** goodness-of-fit test, statistics, reliability, Weibull distribution, imperfect maintenance.

**Résumé :** Ce travail porte sur les tests d'adéquation en fiabilité, à la fois pour les systèmes non réparables et les systèmes réparables. Les tests d'adéquation sont des outils efficaces pour vérifier la pertinence d'un modèle pour un jeu de données.

Pour les systèmes non réparables, la loi exponentielle et la loi de Weibull sont les lois de durée de vie les plus utilisées en fiabilité. Une comparaison exhaustive des tests d'adéquation pour la loi exponentielle est présentée pour des données complètes et censurées, suivie par des recommandations d'utilisation de ces tests.

La loi de Weibull à deux paramètres permet de modéliser des taux de hasard décroissants et croissants contrairement à la loi exponentielle qui suppose un taux de hasard constant. Cependant, il existe moins de tests d'adéquation à la loi de Weibull dans la littérature. Une revue exhaustive des tests existant est effectuée et deux familles de tests exacts sont présentées. La première famille est la famille des tests basés sur la vraisemblance et la deuxième est la famille des tests basés sur la transformée de Laplace. Des propriétés asymptotiques des nouvelles statistiques de tests sont établies.

Une comparaison complète des tests d'adéquation pour la loi de Weibull est effectuée. Des recommandations sur les tests les plus puissants sont données en fonction des caractéristiques du jeu de données testé.

Pour les systèmes réparables, de nouveaux tests d'adéquation sont développés pour des modèles de maintenance imparfaite avec à la fois des maintenances correctives et des maintenances préventives déterministes. Ces tests sont exacts et peuvent être appliqués à des petits jeux de données.

Finalement, des applications à de vrais jeux de données issus de l'industrie sont effectuées pour des systèmes réparables et des systèmes non réparables.

**Mots clés :** test d'adéquation, statistique, fiabilité, loi de Weibull, maintenance imparfaite.