Subdivisions of digraphs

Ana Karolinna Maia de Oliviera

UNIVERSITÉ DE NICE - SOPHIA ANTIPOLIS
ÉCOLE DOCTORALE DES SCIENCES ET TECHNOLOGIES DE L'INFORMATION ET DE LA COMMUNICATION

P H D T H E S I S

to obtain the title of

Docteur en Sciences

de l'Université de Nice - Sophia Antipolis

Mention : Informatique

Defended by

Ana Karolinna Maia de Oliveira

Subdivisions of Digraphs

COATI Project
(INRIA, I3S (CNRS/UNS))
Advisor:
Frédéric Havet

Jury:

Reviewers:	Stéphane Bessy	-	Univeristé Montpellier II (LIRMM)
	Nicolas Trotignon	-	ENS Lyon
Examinators:	Frédéric Maffray	-	CNRS (Grenoble, France)
	Adrien Richard	-	CNRS (Sophia Antipolis, France)
	Yann Vaxés	-	Aix-Marseille Univeristé (LIF)

Subdivisions of Digraphs

Abstract: In this work, we consider the following problem: Given a directed graph D, does it contain a subdivision of a prescribed digraph F? We believe that there is a dichotomy between NP-complete and polynomial-time solvable instances of this problem. We present many examples of both cases. In particular, except for five instances, we are able to classify all the digraphs F of order 4.

While all NP-hardness proofs are made by reduction from some version of the 2-linkage problem in digraphs, we use different algorithmic tools for proving polynomial-time solvability of certain instances, some of them involving relatively complicated algorithms. The techniques vary from easy brute force algorithms, algorithms based on maximum-flow calculations, handle decompositions of strongly connected digraphs, among others.

Finally, we treat the very special case of F being the disjoint union of directed cycles. In particular, we show that the directed cycles of length at least 3 have the Erdős-Pósa Property: for every n, there exists an integer t_n such that for every digraph D, either D contains n disjoint directed cycles of length at least 3, or there is a set T of t_n vertices that meets every directed cycle of length at least 3. From this result, we deduce that if F is the disjoint union of directed cycles of length at most 3, then one can decide in polynomial time if a digraph contains a subdivision of F.

Subdivisions de Digraphes

Résumé : Dans ce travail, nous considérons le problème suivant : étant donné un graphe orienté D, contient-il une subdivision d'un digraphe fixé F ? Nous pensons qu'il existe une dichotomie entre les instances polynomiales et NP-complètes. Nous donnons plusieurs exemples pour les deux cas. En particulier, sauf pour cinq instances, nous sommes capable de classer tous les digraphes d'ordre 4.

Alors que toutes les preuves NP-complétude sont faites par réduction de une version du problème 2-linkage en digraphes, nous utilisons différents outils algorithmiques pour prouver la solvabilité en temps polynomial de certains cas, certains d'entre eux impliquant des algorithmes relativement complexes. Les techniques varient des simples algorithmes de force brute, aux algorithmes basés sur des calculs maximale de flot, et aux décompositions en anses des digraphes fortement connexes, entre autres.

Pour terminer, nous traitons le cas particulier où F étant une union disjointe de cycles dirigés. En particulier, nous montrons que les cycles dirigés de longueur au moins 3 possède la Propriété d'Erdős-Pósa : pour tout n, il existe un entier t_n tel que pour tout digraphe D, soit D a n cycles dirigés disjoints de longueur au moins 3, soit il y a un ensemble T d'au plus t_n sommets qui intersecte tous les cycles dirigés de longueur au moins 3. De ce résultat, nous déduisons que si F est l'union disjointe de cycles dirigés de longueur au plus 3, alors on peut décider en temps polynomial si un digraphe contient une subdivision de F.

Contents

Chapter 1

Introduction and preliminaries

1.1 Introduction

A *subdivision* of a graph $H = (V, E)$ is a graph obtained from H by replacing each edge $ab \in E(H)$ by a path between a and b of length at least 1. The subdivision (or subgraph homeomorphism) problem consists in deciding whether a given graph G has a subdivision of a prescribed graph H as a subgraph. The determination of its complexity was one of the open questions left by Garey and Johnson [28] and it was widely studied in the following years.

Observe that the subdivision problem is NP-complete if H is part of the input, since it includes the Hamiltonian cycle problem [28]. It remains NP-complete even for graphs G with bounded treewidth [44].

We can consider two variants when H is fixed: the correspondence between the vertices of H and G can be previously specified or not. Several polynomial-time algorithms for particular classes were proposed in both situations [18,38,42,50], and the problem was finally proved to be polynomial-time solvable for every fixed H by Robertson and Seymour linkage algorithm [48].

The linkage or disjoint paths problem is the following: given a graph $G = (V, E)$ and k pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices in $V(G)$, for a fixed k, answer whether there are k-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i to t_i. So, for finding a subdivision of a graph $H = (V, E)$ in a given graph $G = (V, E)$ with a fixed mapping of $V(H)$ in $V(G)$, we could simply look for $|E(H)|$ disjoint paths in G. Since there are $|V(G)|^{|V(H)|}$ possible ways of mapping the vertices of H in the vertices of G, we could still find the subdivision without predetermined mapping in polynomial time.

The linkage problem is NP-complete if k is allowed to vary on the input [36] and it is still NP-complete for planar graphs [43]. But contrary to the subdivision problem, it is polynomial-time solvable for graphs with bounded treewidth [44]. A lot of work was done before the definitive solution by Robertson and Seymour, specially for the case $k = 2$ where some algorithms were discovered independently [50, 52]. Indeed, the 2-linkage is a simpler case. It is proved that there are two disjoint (s_1, t_1)- and (s_2, t_2)-paths in G if and only if a small modification of G has a K_5-minor such that the K_5-model is "sufficiently" connected to the vertices s_1, s_2, t_1 and t_2. So, after making some connectivity reductions and be left with a 4-connected graph in which every K_5-model would have the desired property, the problem becomes equivalent to test if the graph is planar, by the result of Wagner [53] saying that a 4-connected graph has no K_5-minor if and only if it is planar. However, the algorithm for the general case of linkage is complicated and not practical, since the constants involved are huge. A recent work by Kawarabayashi, Kobayashi and Reed improves the complexity of the linkage algorithm from $O(n^3)$ to $O(n^2)$ [37]. And new results were also obtained concerning approximative and FPT algorithms for variations of the subdivision problem itself [1, 9, 41].

An important application of graph subdivisions arises in the fact that many interesting classes are defined by forbidding subgraphs, and this is also valid if we look at digraphs and induced graphs, two relevant variations of the subdivision problem. Planar graphs are a well-known example for the first case: they were characterized by Kuratowski as the graphs which do not have any subdivision of K_5 or $K_{3,3}$. Some examples regarding digraphs can be find in [29]. Another notorious example of the undirected case is the one of Perfect graphs, concerning induced subgraphs. According to the Perfect Graph Theorem, a graph is perfect if and only if it does not contain an induced subgraph which is either an odd cycle or the complement of an odd cycle [13]. A survey with some classes determined by forbidden induced subgraphs can be seen in [15]. Flow graph reducibility and programming schema are other applications of the subdivision problem [32, 33].

The aim of the present work is to investigate subdivisions in digraphs. We can define the problem similarly to the undirected case.

The *subdivision of an arc xy* of F is the replacement of xy by two arcs xz, zy, where z is a new vertex. If S can be obtained from F by repeatedly subdividing arcs (including the arcs previously subdivided), then S is a subdivision of F, also called an *F-subdivision*. Alternatively, a subdivision of a digraph F is a digraph obtained from F by replacing each arc xy of F by a directed (x, y)-path of length at least one.

We consider the following problem for a fixed digraph F.

F-SUBDIVISION
Input: A digraph D.
Question: Does D contain a subdivision of F as a subgraph?

Let $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ be distinct vertices of a digraph D. A *k-linkage* from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$ in D is a system of disjoint directed paths $P_1, P_2, \ldots, P_k$ such that P_i is an (x_i, y_i)-path in D. The k-LINKAGE problem is defined as follows.

k-LINKAGE
Input: A digraph D and $2k$ distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$.
Question: Is there a k-linkage from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$ in D?

Similarly to the situation for undirected graphs, the F-SUBDIVISION problem is related to k-LINKAGE. However, contrary to graphs, unless P=NP, k-LINKAGE cannot be solved in polynomial time in general digraphs. Fortune, Hopcroft and Wyllie [24] showed that already 2-LINKAGE is NP-complete. Using their result, we show that for lots of digraphs F, the F-SUBDIVISION problem is NP-complete.

In the same work [24], F-Subdivision for pre-setted mapping of the vertices of F in the vertices of D is completely classified: It is polynomial-time solvable for every tree of height at most 1 and NP-complete otherwise. Furthermore, they proved that if D is acyclic, then for every fixed integer k, there is a polynomial-time algorithm to solve k-Linkage. Mixed linkage problems (concerning digraphs and its respective underlying graphs) are considered in [5]. For a survey of linkages in digraphs, see [2, Chapter 10].

The digraph subdivision problem was solved for D being in some particular classes of graphs. As a consequence of Fortune et al. result for k-Linkage, for any fixed F, F-Subdivision is polynomial-time solvable for acyclic digraphs. The same can be said about digraphs of bounded directed tree-width [35] or bounded DAG-width [6]. In addition Chudnovsky, Scott and Seymour [14] showed that F-Subdivision is polynomial-time solvable when restricted to the class of tournaments. But to the best of our knowledge, there are no previous significant results on the general form of the problem.

We believe that there is a dichotomy between NP-complete and polynomial-time solvable instances of F-Subdivision. In this work, we present many examples of both cases. We start by giving, still in this chapter, a tool based on a reduction from the NP-complete 2-linkage problem in digraphs, which can be applied to conclude the NP-completeness of F-Subdivision for the majority of all digraphs F. In Chapter 2, we concentrate on the problem for F being in particular classes of graphs. For some of them, we are able to completely determine in which cases the F-subdivision problem is polynomial-time solvable and for which it is NP-complete. We turn to the digraphs of order 4 in Chapter 3, and except for 5 instances, we are able to classify all of them.

While all NP-hardness proofs are made by reduction from some version of the 2-linkage problem in digraphs, we describe different algorithmic tools for proving polynomial-time solvability of certain instances, some of them involving relatively complicated algorithms. The techniques vary from easy brute force algorithms, algorithms based on maximum-flow calculations, algorithms based on handle decompositions of strongly connected digraphs, to be detailed later in the chapter, between others. Finally, in Chapter 4 we treat the very special case of F being the disjoint union of cycles. In particular, we show that the directed cycles of length at least 3 have the Erdős-Pósa Property: for every n, there exists an integer t_n such that for every digraph D, either D contains n disjoint directed cycles of length at least 3, or there is a set T of at most t_n vertices that meets every directed cycle of length at least 3. From these result, we deduce that if F is the disjoint union of directed cycles of length at most 3, then one can decide in polynomial time if a digraph contains a subdivision of F.

As briefly mentioned before, the alternative problem in which we look for *induced* subdivisions of a prescribed graph H in an input graph G is also a question of interest. The undirected case is computationally harder than its equivalent for non-induced subdivisions: Lévêque at al. showed that it is NP-complete for $H = K_5$ [39]. Besides, this problem seems quite difficult to be addressed, since there are not many solved cases and so far the there is no indication of what would be the line between polynomial and NP-complete instances.

Despite the existence of trivial polynomial algorithms to find induced subdivisions in undirected graphs, like for cycles of length at least three, some other cases involve sophisticated techniques. They diversify from algorithms based on breadth-first search, as the one presented by Rose, Tarjan and Lueker to find an induced subdivision of a C_k ($k \geq 4$) efficiently ($O(n+m)$) [49], to the use of the *three-in-a-tree* algorithm of Chudnovsky and Seymour [16]. The three-in-a-tree problem consists in answer for a given graph and three pre-determined vertices of it if there is a tree containing such vertices. The algorithm to solve it, which has execution time of $O(n^4)$ [16], provide a general tool that can be used in many solutions of the subdivision problem, as it was done in the same paper for $K_{2,3}$. We use a similar approach to deal with some examples of the directed non-induced case in this thesis. But even being one of the main methods for the induced case, it does not seems to fit in the solutions of all the instances, like it was showed to subdivisions of the net graph (a cycle in that each of the vertices on it has a neighbour with degree 1 outside the cycle) [17]. Cleaning [10, 11] and decompositions [12, 19] are still other techniques used in the solution of the problem. In [40], Lévêque, Maffray and Trotignon present a decomposition theorem for graphs with no induced subdivision of K_4. But in this case, the theorem does not give directly a polynomial-time recognition algorithm. The complexity of finding induced subdivision for $H = K_4$ is still open.

The problem of finding an induced subdivision of a prescribed digraph F in a given digraph D, referred as INDUCED-F-SUBDIVISION, was also investigated. A lot more is known here than for undirected induced subdivisions. It turns out that there is a big difference in the complexity of the problem depending on whether the digraph is allowed to have 2-cycles or not, in which case it is called an *oriented graph*, as showed in [4]. In the latter case, the authors proved that INDUCED-F-SUBDIVISION is NP-complete for every oriented graph which is not the disjoint union of spiders (trees obtained from disjoint directed paths by identifying one end of each path into a vertex). Still in [4] it was conjectured that INDUCED-F-SUBDIVISION is NP-complete unless F is the disjoint union of spiders and at most one 2-cycle. The authors also consider the problem when D is an oriented graph, and they proved it to be polynomial-time solvable for some cases of transitive tournaments and oriented paths, among others.

1.2 Finding an F-subdivision

We are primarily interested in determining in which cases the problem is polynomial-time solvable or NP-complete. Lemma 1.1 implies that deciding if there is an F-subdivision in a digraph is polynomial-time solvable if and only if finding an F-subdivision in a digraph is polynomial-time solvable.

Lemma 1.1 (Havet, M. and Mohar)**.** *If F-*Subdivision *can be solved in $f(n,m)$ time, where f is non-decreasing in m, then there is an algorithm that finds an F-subdivision (if one exists) in a digraph in $((m+1)\cdot f(n,m)+m)$ time.*

Proof. Suppose that there exists an algorithm F-`decide`(D) that decides in $f(n,m)$ whether D contains an F-subdivision. We now construct an algorithm F-`find`(D) that finds an F-subdivision in D if there is one, and returns 'no' otherwise. It proceeds as follows.

Let $a_1,\dots,a_m$ be the arcs of D. If F-`decide`(D) returns 'no', then we also return 'no'. If not, then D contains an F-subdivision, we find it as follows: We initialize $D_0 := D$. For $i=1$ to m, $D_i := D_{i-1}-a_i$ if F-`decide`$(D_{i-1}-a_i)$ returns 'yes', and $D_i := D_{i-1}$ otherwise.

F-`find` is valid because at step i, we delete the arc a_i if and only if there is an F-subdivision not containing a_i. Hence at each step i, we are sure that D_i contains an F-subdivision, and that any F-subdivision must contain all the arcs of $D_i \cap \{a_1,\dots,a_i\}$.

F-`find` runs $(m+1)$ times the algorithm F-`decide` and removes at most m times an arc. Therefore, it runs in time $O(m)\cdot f(n,m)+m$. □

For sake of clarity, we only present algorithms for solving F-Subdivision as a decision problem. However, the proofs of validity of all given algorithms always rely on constructive claims. Hence each algorithm can be easily transformed into a polynomial-time algorithm for finding an F-subdivision in a given digraph, and then our algorithms for finding F-subdivisions have the same complexity as their decision versions.

1.3 Notation, known results and tools

In this section, we present the basic definitions needed for the global understanding of the work and the used notation. However, we assume that reader is familiar with fundamental concepts in graph theory, highlighting here those for digraphs in which the orientations are important. We also present the general digraphs subdivision problem and a few known techniques to be applied in solutions of some of its restricted cases along the text. We rely on [2,8] for additional standard information.

1.3.1 Elementary definitions

A *digraph* D consists of a set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of distinct vertices called *arcs*. Unless otherwise stated, in this text, the letters n and m will always denote the number of vertices and arcs (edges), respectively, of the input digraph (graph) of the problem in question. By *linear time*, we mean $O(n+m)$ time.

If (x,y) is an arc, then we say that x *dominates* y. In this case, x is the *tail*, y is the *head* and x,y are both *end-vertices* of (x,y). Moreover, x and y are said *adjacent*. To simplify, we frequently write xy instead of (x,y) to refer to an arc from x to y, and for a vertex x (resp. a subdigraph S of D), we abbreviate $\{x\}$ to x (resp. $V(S)$ to S) in the notation.

The set of vertices that dominate a vertex x in a digraph D is called its *in-neighbourhood* and denoted by $N_D^-(x)$ Similarly, $N_D^+(x)$ is the *out-neighbourhood* of x, that is, the set of vertices dominated by x. Let $N_D(x) = N_D^-(x) \cup N_D^+(x)$. The *in-degree* $d_D^-(x)$, *out-degree* $d_D^+(x)$ and *degree* $d_D(x)$ of a vertex x in D are the cardinality of $N_D^-(x)$, $N_D^+(x)$ and $N_D(x)$, respectively.

A *source* in D is a vertex of in-degree zero and a *sink* is a vertex of out-degree zero. A vertex x is said to be *small* if $d^-(x) \leq 2$, $d^+(x) \leq 2$ and $d(x) \leq 3$. A non-small vertex is called *big*.

Let D be a digraph. We call D a *multidigraph* if it has multiple arcs (pairs of arcs with the same tail and the same head). The *converse* of D is the digraph $\overline{D}$ obtained from D by reversing the orientation of all arcs. We denote by $UG(D)$ the *underlying (multi)graph* of D, that is, the (multi)graph we obtain by replacing each arc by an edge. To every graph G, we can associate a *symmetric digraph* by replacing every edge uv by the two arcs uv and vu.

An *oriented graph* is an orientation of an undirected graph. In other words, it is a digraph with no directed cycles of length 2. An *oriented path* is an orientation of an undirected path. Hence an oriented path P is a sequence $(x_1, a_1, x_2, a_2, \ldots, a_{n-1}, x_n)$, where the x_i are distinct vertices and for all $1 \leq j \leq n-1$, a_j is either the arc $x_j x_{j+1}$ or the arc $x_{j+1}x_j$. We often refer to such an oriented path P by the underlying undirected path $x_1 x_2 \ldots x_n$. This is a slight abuse, because the oriented path P is not completely determined by this sequence as there are two possible orientations for each edge. However, when we use this notation, either the orientation does not matter or it is clear from the context.

Let $P = x_1 x_2 \cdots x_n$ be an oriented path. We say that P is an *(x_1, x_n)-path.* The vertex x_1 is the *initial vertex* of P and x_n its *terminal vertex*. We denote the initial vertex of P by $s(P)$ and the terminal vertex of P by $t(P)$. The subpath $x_2 \cdots x_{n-1}$ is denoted by P°. If x_1x_2 is an arc, then P is an *outpath*, otherwise P is an *inpath*. The path P is *directed* if no vertex is the tail of two arcs in P nor the head of two arcs. In other words, all arcs are oriented in the same direction. We denote by P_k the directed path of length k. There are two kinds of directed paths, namely directed outpaths and directed inpaths. For convenience, a directed outpath is called a *dipath*. An *antidirected path* is an oriented path in which every vertex has either in-degree 0 or out-degree 0. The *blocks* of an oriented path P are the maximal directed subpaths of P. We often enumerate them from the initial vertex to the terminal vertex of the path. The number of blocks of P is denoted by $b(P)$. The *opposite path* of P, denoted $\overleftarrow{P}$, is the path $x_n x_{n-1} \cdots x_1$. For $1 \leq i \leq j \leq n$, we denote by $P[x_i, x_j]$ (resp. $P]x_i, x_j[$, $P]x_i, x_j]$, $P[x_i, x_j[$), the oriented subpath $x_i x_{i+1} \ldots x_j$ (resp. $x_{i+1}x_{i+2} \ldots x_{j-1}$, $x_{i+1}x_{i+2} \ldots x_j$, $x_i x_{i+1} \ldots x_{j-1}$).

The above definitions and notation can also be used for *oriented cycles*. If $C = x_1 x_2 \ldots x_n x_1$ is an oriented cycle, we shall assume that either C is a *directed cycle*, that is, $x_i x_{i+1}$ is an arc for all $1 \leq i \leq n$, where $x_{n+1} = x_1$, or both edges of C incident with x_1 are directed outwards, i.e. x_1x_2 and x_1x_n are arcs of C. A digraph D is said to be *acyclic* if it has no directed cycles. The directed cycle of length k is denoted by C_k.

Let X and Y be two sets of vertices in a digraph D. An (X,Y)-*dipath* is a dipath with initial vertex in X, terminal vertex in Y and all internal vertices in $V(D) \setminus (X \cup Y)$.

For a set X of vertices, the *out-section* of X in D, denoted by $S^+_D(X)$, is the set of vertices that are reachable from X by a dipath. The out-section of a set in a digraph can be found in linear time using Breadth-First Search. The directional dual notion, the *in-section* of X, in D is denoted by $S^-_D(X)$.

The digraph D is *connected* (resp. k-*connected*) if $UG(D)$ is a connected (resp. k-connected) graph, and the *connected components* of a D are the connected components of $UG(D)$. It is *strongly connected*, or *strong*, if for any two vertices x, y, there is a (x,y)-dipath in D, and D is *robust* if it is strong and $UG(D)$ is 2-connected. We use the notation $D[x,y]$ to denote an arbitrary (x,y)-dipath in D. The strong components of a digraph can also be found in linear time, using Depth-First Search.

The disjoint union of two digraphs D_1 and D_2 is denoted D_1+D_2. By *contracting* a set of vertices $X \subseteq V(D)$, we refer to the operation of first taking the digraph $D-X$, then adding new vertex v_X and adding the arc v_Xw for each $w \in V(D-X)$ with an in-neighbour in X and the arc uv_X for each $u \in V(D-X)$ with an out-neighbour in X. The *contraction* of a non-strong digraph D is the digraph obtained by contracting all strong components of D.

Let F be a digraph and u a vertex in F. In an F-subdivision S, the vertex corresponding to u is called the *u-vertex of S*. A vertex corresponding to some vertex $u \in F$ is called an *original* vertex.

For all notation given above, when it is clear from the context, we may omit the indices or parameters indicating the digraph or vertex to which it refers to.

1.3.2 Menger's Theorem

Let D be a digraph, and let x and y be distinct vertices of D. Two (x,y)-paths P and Q are *internally disjoint* if they have no internal vertices in common, that is if $V(P) \cap V(Q) = \{x,y\}$. A *k-separation* of (x,y) in D is a partition (W,S,Z) of its vertex set such that $x \in W$, $y \in Z$, $|S| \leq k$, each vertex in W can be reached from x by a dipath in $D[W]$, and there is no arc from W to Z.

One version of the celebrated Menger's Theorem is the following.

Theorem 1.2 (Menger)**.** *Let k be a positive integer, let D be a digraph, and let x and y be distinct vertices in D such that $xy \notin A(D)$. Then, in D, either there are $k+1$ pairwise internally disjoint (x,y)-dipaths, or there is a k-separation of (x,y).*

For any fixed k, there exist algorithms running in linear time that, given a digraph D and two distinct vertices x and y such that $xy \notin A(D)$, returns either $k+1$ internally disjoint (x,y)-dipaths in D or a k-separation (W,S,Z) of (x,y). Indeed, in such a particular case, any flow algorithm, like Ford–Fulkerson algorithm for example, performs at most $k+1$ incrementing-path searches, because it increments the flow by 1 each time, and we stop when the flow has value $k+1$, or if we find a cut of size less than $k+1$, which corresponds to a k-separation. Moreover, each incrementing-path search consists in a search (usually Breadth-First Search) in an auxiliary digraph of the same size, and so is done in linear time. For more details, we refer the reader to the book of Ford and Fulkerson [23] or Chapter 7 of [8]. We call such an algorithm a *Menger algorithm*.

Observe that using Menger algorithms, one can decide if there are k internally disjoint (x,y)-dipaths in a digraph D. If $xy \notin A(D)$, then we apply a Menger algorithm directly; if $xy \in A(D)$, then we check whether there are $k-1$ internally disjoint (x,y)-dipaths in $D \setminus xy$.

Let D be a digraph. Let X and Y be non-empty sets of vertices in D. Two (X,Y)-paths P and Q are *disjoint* if they have no vertices in common, that is if $V(P) \cap V(Q) = \emptyset$. A *$k$-separation* of (X,Y) in D is a partition (W,S,Z) of its vertex set such that $X \subseteq W \cup S$, $Y \subseteq Z \cup S$, $|S| \leq k$, all vertices of W can be reached from $X \setminus S$ by dipaths in $D[W]$, and there is no arc from W to Z.

Let x be a vertex of D and Y be a non-empty subset of $V(D) - \{x\}$. Two (x,Y)-paths P and Q are *independent* if $V(P) \cap V(Q) = \{x\}$. A *k-separation* of (x,Y) in D is a partition (W,S,Z) of its vertex set such that $x \in W$, $Y \subseteq Z \cup S$, $|S| \leq k$, all vertices of W can be reached from x by dipaths in $D[W]$, and there is no arc from W to Z.

Let y be a vertex of D and X be a non-empty subset of $V(D) - \{y\}$. Two (X,y)-paths are *independent* if $V(P) \cap V(Q) = \{y\}$. A *k-separation* of (X,y) in D is a partition (W,S,Z) of its vertex set such that W and Z are non-empty, $X \subseteq W \cup S$, $y \in Z$, $|S| \leq k$, all vertices of W can be reached from $X \setminus S$ by dipaths in $D[W]$, and there are no arcs from W to Z.

Let $W \subset V(D)$. The digraph D_W is the one obtained from D by adding a vertex s_W and the arcs $s_W w$ for all $w \in W$ and the digraph D^W is the one obtained from D by adding a vertex t_W and the arcs wt_W for all $w \in W$.

Applying Theorem 1.2 to D_X^Y and (s_X, t_Y) (resp. D^Y and (x, t_Y), D_X and (s_X, y)), we obtain the following version of Menger's Theorem.

Theorem 1.3 (Menger)**.** *Let k be a positive integer, and let D be a digraph. Then the following hold.*

(i) If X and Y are two non-empty subsets of $V(D)$, then, in D, either there are $k+1$ pairwise disjoint (X,Y)-dipaths, or there is a k-separation of (X,Y).

(ii) If x is a vertex of D and Y is a non-empty subset of $V(D)$, then, in D, either there are $k+1$ pairwise independent (x,Y)-dipaths in D, or there is a k-separation of (x,Y).

(iii) If X is a non-empty subset of $V(D)$ and y is a vertex of D and, then, in D, either there are $k+1$ pairwise independent (X,y)-dipaths in D, or there is a k-separation of (X,y).

Moreover, a Menger Algorithm applied to D_X^Y and (s_X, t_Y) (resp. D^Y and (x, t_Y), D_X and (s_X, Y)) finds in linear time the $k+1$ dipaths or the separation as described in Theorem 1.3 (i) (resp. (ii), (iii)).

1.3.3 Handle decomposition

Let D be a strongly connected digraph. A *handle* h of D is a directed path $(s, v_1, \dots, v_\ell, t)$ from s to t (where s and t may be identical) such that the digraph $D - h$ obtained from D by *suppressing* h, that is, removing the arcs and the internal vertices of h, is strongly connected. The vertex s is the *origin* of h and t its *terminus*.

Given a strongly connected digraph D, a *handle decomposition* (also known as *ear decomposition*) of D starting at $v \in V(D)$ is a triple $(v, (h_i)_{1\le i\le p}, (D_i)_{0\le i\le p})$, where $(D_i)_{0\le i\le p}$ is a sequence of strongly connected digraphs and $(h_i)_{1\le i\le p}$ is a sequence of handles such that:

- $V(D_0) = \{v\}$,
- for $1 \le i \le p$, h_i is a handle of D_i and D_i is the (arc-disjoint) union of D_{i-1} and h_i, and
- $D = D_p$.

A handle decomposition is uniquely determined by v and either $(h_i)_{1\le i\le p}$, or $(D_i)_{0\le i\le p}$. The number of handles p in any handle decomposition of D is exactly $|A(D)|-|V(D)|+1$. The value p is also called the *cyclomatic number* of D. Observe that $p = 0$ when D is a singleton and $p = 1$ when D is a directed cycle.

A handle decomposition $(v, (h_i)_{1\le i\le p}, (D_i)_{0\le i\le p})$ is *nice* if all handles except the first one h_1 have distinct end-vertices. The following proposition is well-known (see [8] Theorem 5.13). Recall that a digraph is robust if it is strong and $UG(D)$ is 2-connected.

Proposition 1.4. *Every robust digraph admits a nice handle decomposition.*

1.3.4 Linkage in digraphs

Recall that a 2-linkage from (x_1, x_2) to (y_1, y_2) in a digraph D is a pair P_1, P_2 of disjoint paths such that P_1 is a directed path from x_1 to y_1 and P_2 is a directed path from x_2 to y_2 in D.

2-LINKAGE
Input: A digraph D and 4 distinct vertices x_1, x_2, y_1, y_2.
Question: Is there a 2-linkage from (x_1, x_2) to (y_1, y_2) in D?

For sake of completeness, we reproduce the proof that 2-LINKAGE is NP-complete below.

Theorem 1.5 (Fortune, Hopcroft and Wyllie [24]). *The* 2-LINKAGE *problem is NP-complete.*

Proof. The proof is a reduction from 3-SAT. The next lemma is very important in the construction of the gadget.

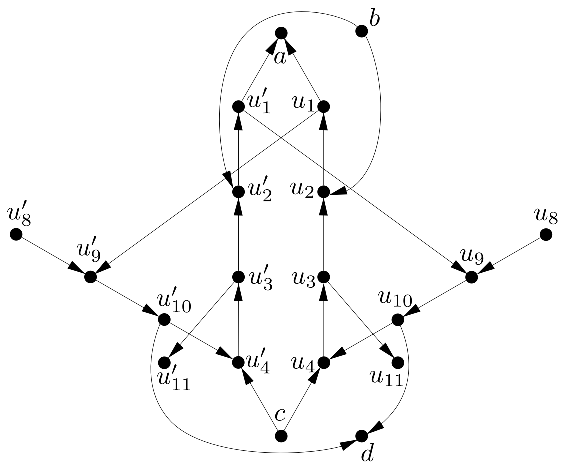


Figure 1.1: A switch.

Lemma 1.6. *Consider the digraph (called switch) of Figure 1.1. Suppose there are two disjoint dipaths P_1, P_2 passing through it, one leaving at vertex a (P_1) and another entering at vertex b (P_2). Then P_1 must have entered at c and P_2 must leave at d. Furthermore, there is exactly one more dipath disjoint from P_1, P_2 passing through the switch, either $L = u_8u_9u_{10}u_4u_3u_{11}$ or $R = u'_8u'_9u'_{10}u'_4u'_3u'_{11}$.*

Subproof. Suppose the first arc of P_2 is bu_2. Then necessarily $u_2u_1u'_9u'_{10}$ is part of P_2. If P_2 does not ends in d, its next arc should be $u'_{10}u'_4$. Since P_1 and P_2 need to be disjoint, P_1 can not have the arc u_1a, and in this case P_1 would go through the inverse of $u'1u'2u'3u'4$, and P_1, P_2 would intersect at u'_4. So, P_2 must end at d and P_1 starts at c. By symmetry of the digraph, a similar result is obtained if P_2 starts at a different arc. It is straightforward that the only other dipath passing through the switch is L or R, depending on the routing of $P_2(P_1)$. ♢

Consider an instance $\mathcal{F}$ of 3-SAT. Let $D_{\mathcal{F}}$ be the following digraph. For each clause $C_1, \ldots, C_r$ and variable $v_1, \ldots, v_k$ of $\mathcal{F}$, let $c_1, \ldots, c_r, v_1, \ldots, v_k$ be vertices of $D_{\mathcal{F}}$. Also add a vertices c_{r+1}, v_{k+1} to $V(D_{\mathcal{F}})$ and the arc $v_{k+1}c_1$. There should be three dipaths from c_i to c_{i+1}, $1 \leq i \leq r$, each corresponding to one variable of C_i, and two dipaths $Q_j, \bar{Q}_j$ from v_j to v_{j+1}, $1 \leq j \leq k$, representing the positive and negative occurrences of v_j, respectively.

For every clause C_i and variable $v_j(\bar{v}_j)$ in C_i, put a switch in $D_{\mathcal{F}}$ in a way that L is the dipath from c_i to c_{i+1} correspondent to $v_j(\bar{v}_j)$, and R is in $Q_j(\bar{Q}_j)$, now referred as $R^l_j(\bar{R}^l_j)$ if it represent the l-th occurrence of $v_j(\bar{v}_j)$ in F. Finally, let $Q_j =$

$v_j s(R_j^1) \cup t(R_j^1) s(R_j^2) \cup \ldots \cup t(R_j^p) v_{j+1}$ and $\bar{Q}_j = v_j s(\bar{R}_j^1) \cup t(\bar{R}_j^1) s(\bar{R}_j^2) \cup \ldots \cup t(\bar{R}_j^p) v_{j+1}$, if v_j and $\bar{v}_j$ appears p times in $\mathcal{F}$, respectively. Furthermore, connect the switches two by two by identifying the vertex $c(d)$ of one with $a(b)$ of the next and put an arc from the vertex d of the last switch to v_1. An example of $D_{\mathcal{F}}$ is showed in Figure 1.2.

Let x_2 be the vertex b of the first switch, $y_2 = c_{r+1}$, x_1 the the vertex c of the last switch and y_1 the vertex a of the first switch. We claim that $D_{\mathcal{F}}$ contains a 2-linkage from (x_1, x_2) to (y_1, y_2) if and only if $\mathcal{F}$ is satisfiable.

Suppose $\mathcal{F}$ is satisfiable. Let S_3 be the dipath from v_1 to v_k composed by the dipaths $\bar{Q}_j$ if v_j is true and Q_j otherwise. Since at least one literal in C_i is satisfied, there is a dipath S_4, disjoint from S_3, composed by the dipaths from c_i to c_{i+1} correspondent to a literal in C_i with value true. And by Lemma 1.6, there are two disjoint dipaths S_1, S_2 passing through the chain of switches from x_1 to y_1 and from x_2 to v_1, respectively, that are disjoint from S_3 and S_4, since each of those contains one (and only one) of the dipaths L or R of a switches. So, S_1 and $S_2 \cup S_3 \cup S_4$ is the desired 2-linkage.

Suppose now there are two disjoint dipaths S_1, S_2 from x_1 to y_1 and from x_2 to y_2, respectively. By Lemma 1.6, the dipaths S_1, S_2, arriving at y_1 and leaving x_2, can not start or end at $s(R), s(L)$ or $t(R), t(L)$ in the first switch, respec. They have to start and end at the vertices c and d in the first switch, that are the vertices a and b of the next. The same reasoning is valid the the following switches, and so S_1 and S_2 have to cross the chain of switches and finally S_1 starts at the vertex $c = x_1$ and S_2 goes through d of the last one. Let us call d' the vertex d of the last switch. Since $d'v_1$ is the only edge leaving this vertex, S_2 pass on it. Then necessarily S_2 contains a dipath from v_1 to v_{k+1} and the arc $v_{k+1}c_1$, since there is no way of reach one the vertices c_j, crossing one or more switches, by a dipath disjoint from S_1 and $S_2[x_2, d']$. Furthermore, from vertex v_j to v_{j+1}, the dipath is either Q_j or $\bar{Q}_j$, that is, the representation of a positive or negative occurrence of v_j, and it is composed by the union of dipaths R of switches. We claim that the assignment in which v_j is true if the chosen subpath for S_2 is $\bar{Q}_j$ and v_j is false otherwise satisfies the formula. Observe that, again, S_2 necessarily contains a dipath from c_1 to $c_r = y_2$. The dipath from c_i to c_{i+1}, corresponding to one of the literals $v_j(\bar{v}_j)$ in C_i, is a dipath L of a switch. Since in a switch exactly one of R and L is allowed to be used out of the dipaths S_1 and $S_2[x_2, d']$ (Lemma 1.6), the use of such dipath is just allowed because $\bar{Q}_j(Q_j)$ was the chosen dipath from v_j to v_{j+1}, and consequently the literal $v_j(\bar{v}_j)$ has value true, and C_i is satisfiable for every $1 \leq i \leq r$.

□

The problem is also NP-complete when restricted to some classes of digraphs. We use an easy modification of the 2-linkage problem as the basis for ours proofs.

Let us give some useful definitions before proceed. An *out-arborescence* is a tree in which all vertices have in-degree 1, except one special vertex, called *root*. A *switching out-arborescence* is an out-arborescence in which the root has out-degree 1, the leaves have out-degree 0 and all other vertices have out-degree 2. A *(switching)*

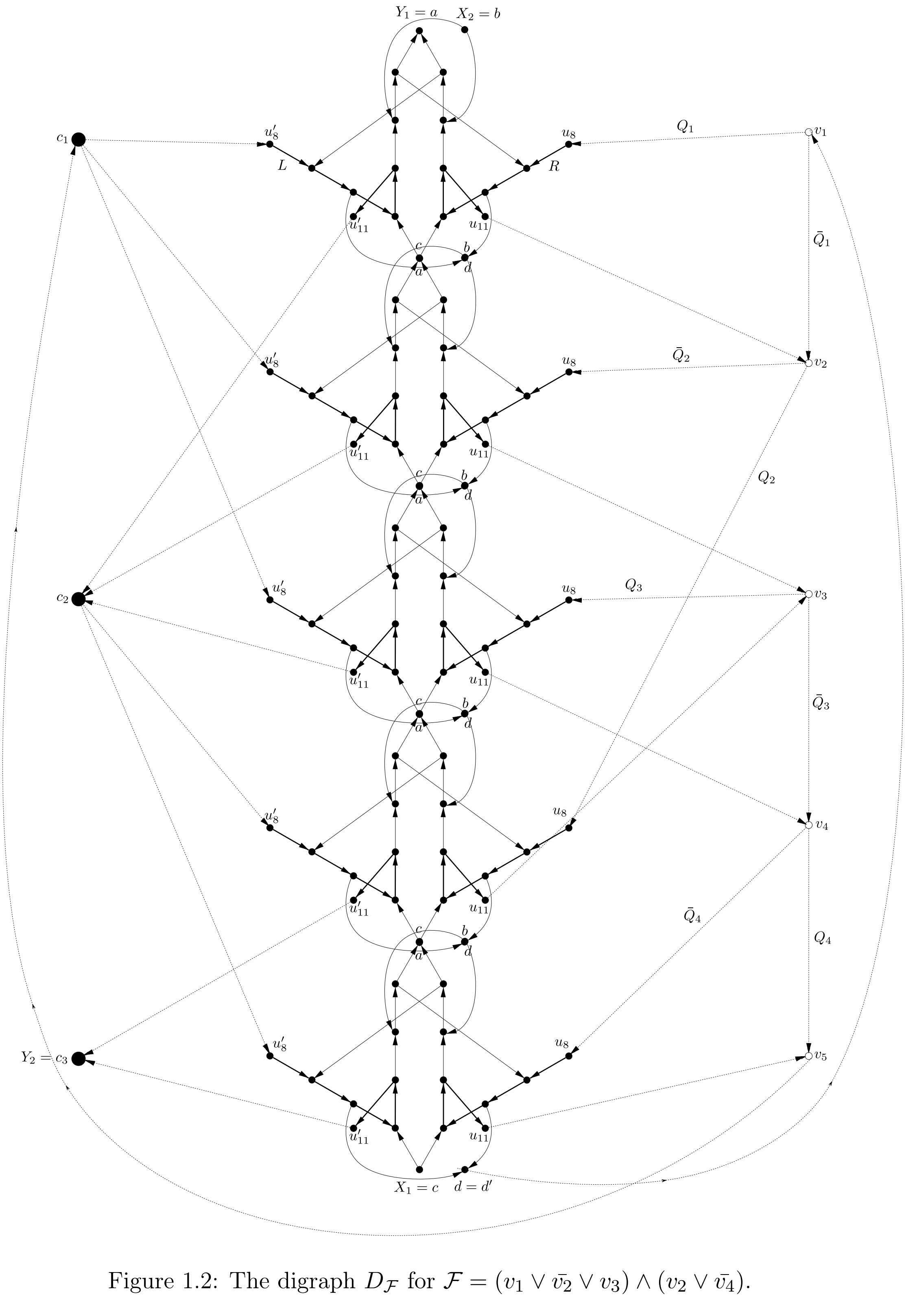


Figure 1.2: The digraph $D_{\mathcal{F}}$ for $\mathcal{F} = (v_1 \vee \bar{v_2} \vee v_3) \wedge (v_2 \vee \bar{v_4})$.

in-arborescence is the dual notion of (switching) out-arborescence.

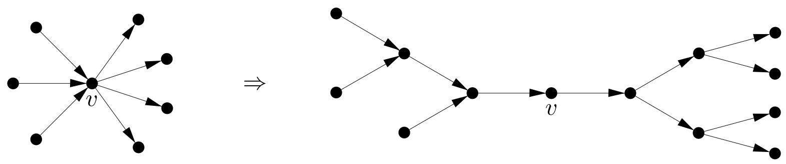


Figure 1.3: Replacement of $N^+(v)$ and $N^-(v)$ by switching out and in-arborescence with root v and leaves $N^+(v)$ and $N^-(v)$.

Consider the following problem.

RESTRICTED 2-LINKAGE
Input: A digraph D without big vertices in which x_1 and x_2 are sources and y_1 and y_2 are sinks.
Question: Is there a 2-linkage from (x_1, x_2) to (y_1, y_2) in D?

Theorem 1.7 (Bang-Jensen, Havet and M. [3]). *The* RESTRICTED 2-LINKAGE *problem is NP-complete.*

Proof. We will make a reduction from 2-LINKAGE in general digraphs. Let D and x_1, x_2, y_1, y_2 be an instance of 2-LINKAGE. Let D^* be the digraph obtained from D as follows. For every vertex v, replace all the arcs leaving v by a switching out-arborescence with root v and whose leaves corresponds to the out-neighbours of v in D, and replace all the arcs entering v by a switching in-arborescence with root v and whose leaves corresponds to the in-neighbours of v in D. Furthermore, delete all the arcs entering x_1 and x_2 and all the arcs leaving y_1 and y_2 in D. Because all vertices in a switching out(in)-arborescence are small, D^* has no big vertices and, moreover, it is clear that x_1 and x_2 are sources and y_1 and y_2 are sinks. Since every edge vu in D can be replaced by a path in the switching out-arborescence (for instance) with root v and leaf u in D^* and vice versa, it is straightforward that there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D if and only if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D^*. □

So far, all the subdivision NP-completeness proofs in this work are made by reduction from RESTRICTED 2-LINKAGE.

On the other hand, some digraphs can be proved tractable using linkage, because k-LINKAGE is polymomial-time solvable when restricted to some classes of digraphs. This is for example the case for acyclic digraphs, as shown Fortune, Hopcroft and Wyllie [24].

Theorem 1.8 (Fortune, Hopcroft and Wyllie [24]). *For every fixed k, the k-*LINKAGE *problem for acyclic digraphs can be solved in polynomial-time.*

Proof. Let $D = (V, A)$ be an acyclic digraph. Let $D' = (V', A')$ be the following digraph constructed from D: the vertices of D' are k-tuples of vertices of D, for every set of k vertices of $V(D)$ and every order of it. The set of arcs A' of D' is the following: There is an arc between $(v_1, \dots, v_{r-1}, v_r, v_{r+1}, \dots, v_k)$ to $(v_1, \dots, v_{r-1}, w, v_{r+1}, \dots, v_k)$ if there is no dipath from $\{v_1, \dots, v_{r-1}, v_{r+1}, \dots, v_k\}$ to v_r in D and, furthermore, w is an out-neighbour of v_r in D. We say that this arc is in the position r of the k-tuple.

Let D and $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$ be an instance of k-linkage. We claim there is a dipath P' from the vertex $(x_1, x_2, \dots, x_k)$ to $(y_1, y_2, \dots, y_k)$ in D' if and only if there is a k linkage from $(x_1, x_2, \dots, x_k)$ to $(y_1, y_2, \dots, y_k)$ in D.

Suppose there is such dipath P' in D'. Observe that, for each arc of P', there is a corresponding arc in D. Then let P_i be the dipath formed by the arcs in the position i of the k-tuples of P'. Then $P_1, \dots P_k$ is a k-linkage from $(x_1, x_2, \dots, x_k)$ to $(y_1, y_2, \dots, y_k)$, since each of them is a dipath from x_i to y_i and, moreover, they are disjoint. Suppose two of them have a common vertex w. Since there are no tuples with repeated vertices, w appears in different positions in different tuples. Consider the first time for which there is an arc between $(z_1, \dots, w, \dots, z_k)$ to $(z_1, \dots, u, \dots, z_k)$, meaning that u is an out-neighbour of w and there is no dipath from $\{z_1, \dots, z_k\}$ to w in D. But if w appear again in another tuple, it means one of z_j reach w by a dipath in D, a contradiction.

Suppose now there are k disjoint dipaths from $(x_1, x_2, \dots, x_k)$ to $(y_1, y_2, \dots, y_k)$ in D. Then the dipath P' of D' can be constructed like this: in the k-tuple $(z_1, \dots, z_k)$ (starting by $(x_1, x_2, \dots, x_k)$), take the vertex z_i such that there is no dipath from $\{z_1, \dots, z_{i-1}, z_{i-1}, \dots, z_k\}$ to z_i in D. Such vertex always exists because D is acyclic. Then take as next vertex of P' the k-tuple $(z_1, \dots, z_{i-1}, w_i, z_{i+1}, \dots, z_k)$, in which w_i is the vertex of P_i after z_i.

So, the problem of find a k-linkage in an acyclic digraph D can be reduced to the problem of finding a dipath in D'.

□

1.4 General NP-completeness results

We deduced a sufficient condition for F-Subdivision to be NP-complete. The next observations allow us to conclude that F-subdivision is "almost always" NP-complete.

For a digraph D, we denote by $B(D)$ the set of its big vertices. A *big path* in a digraph is a directed path whose end-vertices are big and whose internal vertices all have in- and out-degree one (in particular, an arc between two big vertices is a big path). Note also that two distinct big paths with the same end-vertices are necessarily internally disjoint. The *big paths digraph* of D, denoted $BP(D)$, is the multidigraph with vertex set $V(D)$ in which there are as many arcs between two vertices x and y as there are big (x, y)-paths in D. $BP(D)$ is well-defined and easy to construct in polynomial time given D.

Theorem 1.9 (Bang-Jensen, Havet and M. [3]). *Let F be a digraph. If F contains two arcs ab and cd whose end-vertices are big vertices and such that $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\}$ is not isomorphic to $BP(F)$, then F-*SUBDIVISION *is NP-complete.*

Proof. The proof is a reduction from 2-LINKAGE in digraphs with no big vertices in which x_1 and x_2 are sources and y_1 and y_2 are sinks.

Let D, x_1, x_2, y_1, y_2 be an instance of this problem. Let H be the digraph obtained from the disjoint union of $F \setminus \{ab, cd\}$ and D by adding the arcs ax_1, cx_2, y_1b, and y_2d. We claim that H has an F-subdivision if and only if D has a 2-linkage from (x_1, x_2) to (y_1, y_2).

If there is a 2-linkage P_1, P_2 in D, then the union of $F \setminus \{ab, cd\}$ and the paths $ax_1 \cup P_1 \cup y_1b$ and $cx_2 \cup P_2 \cup y_2d$ is a F-subdivision in H.

Conversely, suppose that H contains an F-subdivision S. Observe that in H, no vertex of D is big. Hence, since S has as many big vertices as F, F and S have the same set of big vertices.

Clearly, S contains as many big paths as F and thus there must be in D two disjoint directed paths between (x_1, x_2) and (y_1, y_2). These two paths cannot be an (x_1, y_2)- and an (x_2, y_1)-path, for otherwise $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\} = BP(S)$ would be isomorphic to $BP(F)$ since S is an F-subdivision. Hence, there is a 2-linkage from (x_1, x_2) to (y_1, y_2). $\square$

Remark 1.10. *Observe that if $BP(F)$ has two arcs ab and cd which are consecutive (i.e. $b = c$) or contains an antidirected path (a, b, c, d) of length 3, then $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\}$ is not isomorphic to $BP(F)$. Hence, by Theorem 1.10, F-*SUBDIVISION *is NP-complete.*

Corollary 1.11. *If F is a digraph with no small vertices, then F-*SUBDIVISION *is NP-complete.*

Proof. If F has no small vertices, then $BP(F) = F$. Moreover if F does not contain two consecutive arcs, then $V(F)$ can be partitioned into two sets A and B such that all arcs in F have tail in A and head in B. In this case, F contains an antidirected path of length 3 since its vertices are big, and then a vertex $b \in B$ contains at least two in-neighbours $a, c \in A$, each with at least one more out-neighbour in B. So by Remark 1.11, the F-SUBDIVISION problem is NP-complete. $\square$

1.5 The dichotomy conjecture and relatives

For many digraphs F, the condition of Theorem 1.10 is verified and so F-SUBDIVISION is NP-complete. However, there are graphs F that do not verify this condition but for which F-SUBDIVISION is NP-complete. We show it in Chapters 2 and 3. There are also many cases in which the F-SUBDIVISION problem is polynomial-time solvable. For example, a subdivision of the directed 2-cycle is a directed cycle. Hence a digraph has a C_2-subdivision if and only if it is not acyclic.

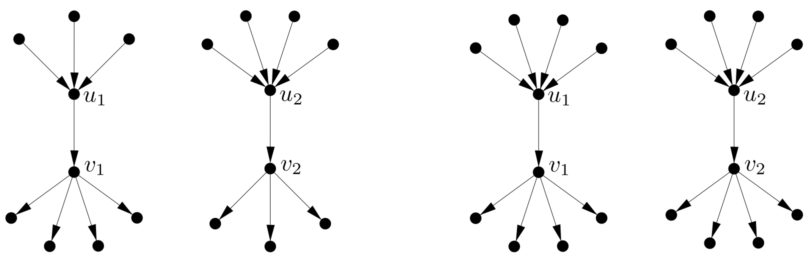
As one can check in linear time if a digraph is acyclic or not [2, Section 2.1], C_2-Subdivision is linear-time solvable.

We believe that there is a dichotomy between NP-complete and polynomial-time solvable instances.

Conjecture 1.12. *For every digraph F, the F-*Subdivision *problem is polynomial-time solvable or NP-complete.*

According to this conjecture, there are only two kinds of digraphs F: *hard* digraphs, for which F-Subdivision is NP-complete, and *tractable* digraphs, for which F-Subdivision is solvable in polynomial-time.

A first idea to prove this conjecture would be to try to establish for any digraph G and subdigraph F, that if F-Subdivision is NP-complete, then G-Subdivision is also NP-complete, and conversely, if G-Subdivision is polynomial-time solvable, then F-Subdivision is polynomial-time solvable. However, these two statements are false as shown by the two digraphs depicted Figure 1.4. The NP-completeness of A-Subdivision follows from Theorem 2.28. The fact that B-Subdivision is polynomial-time solvable is proved in Theorem 2.29.



(a) A (b) B

Figure 1.4: Digraphs A and B such that A is a subdigraph of B, A-Subdivision is NP-complete, and B-Subdivision is polynomial-time solvable.

Despite of the many examples of polynomial instances and NP-completeness proofs we present in this work, there is still no clear evidence of exactly which graph should be tractable and which one should be hard. Although, there are conjectures that give some outline. Motivated by directed tree-width and a conjecture of Johnson et al. [35], Seymour (private communication to J. Bang-Jensen, 2011) raised the following conjecture.

Conjecture 1.13 (Seymour). *F-*Subdivision *is polynomial-time solvable when F is a planar digraph with no big vertices.*

This conjecture would indeed be implied by the following conjecture. An arc uv in a digraph is *contractible* if $\min\{d^+(u), d^-(v)\} = 1$. A *minor* of a digraph D is any digraph $\tilde{D}$ which can be obtained from a subdigraph H of D by contracting zero or more contractible arcs of H. For $k = 1, 2, \ldots, k$ the digraph J_k is obtained from the union of k directed cycles (each of length $2k$) $C_1, C_2, \ldots, C_k$, where $C_i = u_{i,1}v_{i,1}u_{i,2}v_{i,2}\ldots u_{i,k}v_{i,k}u_{i,1}$, for $i = 1, 2\ldots, k$ and paths P_i, Q_i, $i = 1, 2\ldots, k$, where $P_i = u_{1,i}u_{2,i}\ldots u_{k,i}$ and $Q_i = v_{k,i}v_{k,i-1}\ldots v_{k,1}$ for $i = 1, 2\ldots, k$.

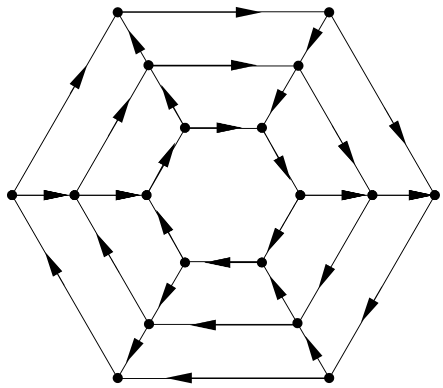


Figure 1.5: J_3.

Conjecture 1.14 (Johnson et al. [35]). *For every positive integer k there exists $N(k)$ such that the following holds: If a digraph D has directed treewidth more than $N(k)$, then D contains a minor isomorphic to J_k.*

If the directed tree-width of D is bounded, then F-Subdivision can be solved in polynomial time [35]. If, on the other hand, the directed tree-width of D is unbounded, then (if the algorithmic version of the conjecture also holds) we can find a minor isomorphic to J_k for a sufficiently large k and presumably use this to realize the desired subdivision using the fact the F is planar and has no big vertices [34].

We proved Seymour's Conjecture for graphs of order 3 and 4 in Chapter 3. We propose the following sort of counterpart to it.

Conjecture 1.15. *F-Subdivision is NP-complete for every non-planar digraph F.*

1.6 Disjoint directed cycles

A particular case of Conjecture 1.14 is the following.

Conjecture 1.16. *If F is a disjoint union of directed cycles, then F-Subdivision is polynomial-time solvable.*

Observe that if F is the disjoint union of n directed cycles of lengths $\ell_1, \ldots, \ell_n$, then a subdivision of F is the disjoint union of n directed cycles $C_1, \ldots, C_n$, each C_i being of length at least ℓ_i. We denote the directed cycle of length ℓ, or *directed ℓ-cycle*, by $\vec{C}_\ell$. A directed cycle of length at least ℓ is called *directed ℓ^+-cycle*.

A special case of Conjecture 1.17 is when all the directed cycles of F have the same length.

Conjecture 1.17. *For any two positive integers n and ℓ with $\ell \geq 2$, $n\vec{C}_\ell$-*SUBDIVISION *is polynomial-time solvable.*

In fact, we can show that Conjectures 1.17 and 1.18 are equivalent. Let us give some definitions before proceed.

A *feedback vertex set* or *cycle transversal* in a digraph D is a set of vertices S such that $D-S$ is acyclic. The minimum number of vertices in a cycle transversal of D is the *cycle-transversal number* and is denoted by $\tau(D)$. The maximum number of disjoint directed cycles in a digraph D is called the *cycle-packing number* and is denoted by $\nu(D)$.

For a digraph D and an integer $\ell \geq 2$, we denote by $\tau_\ell(D)$ the minimum t such that there exists $T \subseteq V(D)$ with $|T| = t$ meeting all directed cycles of length at least ℓ in D, and by $\nu_\ell(D)$ the maximum n such that D has n disjoint directed cycles of length at least ℓ.

Conjecture 1.18 is a particular case of Conjecture 1.17. We now show how Conjecture 1.17 can be deduced from Conjecture 1.18.

Lemma 1.18 (Havet and M. [31])**.** *Let F be a disjoint union of n directed cycles, all of length at most ℓ. If $m\vec{C}_\ell$-*SUBDIVISION *is polynomial-time solvable for all $1 \leq m \leq n$, then F-*SUBDIVISION *is also polynomial-time solvable.*

Proof. Let n be a positive integer. Assume that $m\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable for any $m \leq n$.

Let $F = \vec{C}_{\ell_1} + \cdots + \vec{C}_{\ell_n}$ with $\ell_1 \leq \cdots \leq \ell_n \leq \ell$. Any F-subdivision is a disjoint union of n directed cycles $\vec{C}_{p_1} + \cdots + \vec{C}_{p_n}$ with $p_1 \leq \cdots \leq p_n$ such that $\ell_i \leq p_i$ for all $1 \leq i \leq n$. The *threshold* of such a subdivision is the largest integer t such that $p_t < \ell$.

For $t = 0$ to n, we check whether there is an F-subdivision with threshold t with the following 'brute force' procedure. We enumerate all possible disjoint unions of directed cycles $U = \vec{C}_{p_1} + \cdots + \vec{C}_{p_t}$ with $p_1 \leq \cdots \leq p_t \leq \ell - 1$ and $\ell_i \leq p_i$ for all $1 \leq i \leq t$. There are at most $O\left(|V(D)|^{(t-1)(\ell-1)}\right)$ such U. For each such U, we check if $D - U$ contains an $(n-t)\vec{C}_\ell$-subdivision (whose union with U would be an F-subdivision with threshold t). This can be done in polynomial time by the hypothesis.

The algorithm is a succession of (at most) $n+1$ polynomial-time procedures, so it runs in polynomial time. □

Clearly, $\nu_\ell(D) \leq \tau_\ell(D)$. Proving the so-called Gallai-Younger Conjecture, Reed et al. [47] proved that there exists a minimum function f such that $\tau(D) \leq f(\nu(D))$. It is obvious that $f(1) = 1$ and McCuaig [45] proved that $f(2) = 3$. Reed et al. [47] proved the following result for the general case.

Theorem 1.19 (Reed et al. [47])**.** *For every integer $n \geq 0$, there exists an integer t_n such that for every digraph D, either $\nu(D) \geq n$ or $\tau(D) \leq t_n$.*

Determining $\nu(D)$ is NP-hard. Indeed, given a digraph D and an integer k, deciding whether D has at least k disjoint cycles is NP-complete (see Theorem 13.3.2 of [2]). As observed in [30], the problem parameterized with k is hard for the complexity class W[1] (this follows easily from the results of [51]). This means that, unless $FPT = W[1]$, there is no algorithm solving the problem with a $f(k) \cdot n^{O(1)}$ running time.

Theorem 1.20 is a directed analogue of the following theorem due to Erdös and Pósa.

Theorem 1.20 (Erdős and Pósa [21]). *Let n be a positive integer. There exists t_n^* such that for every graph G, either G has n pairwise-disjoint cycles, or there exists a set T of at most t_n^* vertices such that $G - T$ is acyclic.*

More precisely, Erdős and Pósa proved that there exist two absolute constants c_1 and c_2 such that $c_1 \cdot n \log n \leq t_n^* \leq c_2 \cdot n \log n$. Since we are interested only in the above version of the theorem, we reproduce here only the proof of the upper bound.

Some auxiliary results are needed, though. For the first one, consider a graph G and a set of vertices $x_1, \ldots, x_u$ of G to be called *principal vertices* (the others will be called *subsidiary vertices*). A *principal path* in G is a path whose end-vertices are principal and whose internal ones are subsidiary. Let $V_{max}(G)$ denote the maximum number of disjoint principal paths in G and $\pi_{min}(G)$ the minimum number of vertices that intersects all principal paths in G.

Theorem 1.21 (Gallai [27]). $\pi_{min}(G) \leq 2V_{max}(G)$.

The next result is also due to Erdős and Pósa.

Theorem 1.22 (Erdős and Pósa [20]). *There exists an absolute constant c_3 such that every graph with m vertices and $m + l$ edges contains at least $c_3 \cdot l/\log l$ edge-disjoint cycles.*

Observe that if two cycles are edge-disjoint but not disjoint, then every common vertex between two cycles has degree at least 4. So, if every vertex of the graph has degree at most 3, by Theorem 1.23, the graph has at least $c_3 \cdot l/\log l$ disjoint cycles.

Proof of Theorem 1.21. Assume that the maximum number of disjoint cycles in a graph G is n, and let C_i, $1 \leq i \leq n$, be these cycles. Consider the graph G_1 obtained from G by the deletion of the edges in all C_i, and let the vertices of all C_i be the principal vertices of G_1. Consider a set of principal paths S that maximizes the number of disjoint principal paths in G_1. Let G^* be the graph formed by the union of C_i, $1 \leq i \leq n$, and S, and let m be the number of vertices of G^*. As each principal path has one more edge than subsidiary vertex in G^*, the total amount of edges of G^* is $m + V_{max}(G_1)$. Since every vertex has degree at most 3 in G^*, by Theorem 1.23, G^* has at least $c_3 \cdot V_{max}(G_1)/\log V_{max}(G_1)$ disjoint cycles. This is also valid for G because G^* is a subgraph of G. So

$$\frac{c_3 \cdot V_{max}(G_1)}{\log V_{max}(G_1)} \leq n \text{ and then } V_{max}(G_1) \leq c_4 \cdot n \log n.$$

Let $y_1, \ldots, y_t$ be a set of vertices intersecting every principal path in G_1 such that t is minimum. Then, by Theorem 1.22 and the inequality above

$$t \leq 2c_4 \cdot n \log n.$$

Suppose there is a cycle D different from any C_j that does not contain any y_h. D must intersect some C_j. Suppose D intersects C_i but do not intersect any other C_j. Then D has only one vertex x_i in common with C_i, otherwise there would be a principal path not passing through any y_h. Suppose there were two cycles D_{i_1}, D_{i_2} of this kind intersecting C_i in different vertices x_{i_1}, x_{i_2}. If they were disjoint, replacing C_i by them would imply that the graph has more than n disjoint cycles. If they were not disjoint, then again there would be a principal path not passing through any y_h. The same would happen if D had vertices in common with more than on C_j. So each cycle different from any C_j that does not contain any y_h intersects a C_i in at most one vertex, the same vertex x_i for each C_i. The set composed by the vertices $y_1, \ldots, y_t, x_1, \ldots, x_k$ intersects all the cycles in G, and it contains at most

$$2c_4 \cdot n \log n + n \leq c_2 \cdot n \log n$$

vertices, and it completes the proof. □

An $n\vec{C}_2$-subdivision is the disjoint union of n directed cycles. Therefore Conjecture 1.18 for $\ell = 2$ can be deduced from Theorems 1.20 and 1.8. Using the result of Theorem 1.20, Reed et al. [47] gave a polynomial-time algorithm to decide for every fixed k whether a digraph D contains k disjoint directed cycles. Basically, it tests all possible sets T of $f(k)$ vertices. If none of them is a cycle transversal, then it returns 'yes'. If one of them is a cycle transversal, it reduces the problem to a finite number (but depending on k) of $f(k)$-linkage problem in $D - T$.

Theorem 1.23 (Reed et al. [47])**.** *Let k be a fixed integer. There is an algorithm running in time $O(n^{f(k)}(n+m))$ that decides whether there are k disjoint directed cycles in a digraph.*

In the undirected case, the complexity of finding even *two* disjoint *induced* cycles remains open [17]. Theorem 1.24 and Lemma 1.1 directly imply the following.

Corollary 1.24. *Let k be a fixed integer. There is an algorithm running in time $O(n^{f(k)}(n+m)m)$ that finds k disjoint directed cycles in a digraph if they exist, and returns 'no' otherwise.*

In fact, Reed et al. proved the following stronger statement than Theorem 1.24.

Theorem 1.25 (Reed et al. [47])**.** *For any digraph F, F-*Subdivision *is polynomial-time solvable when restricted to the class of digraphs with bounded cycle-transversal number.*

Note that this results is implied by the one of Berwanger et al. [6] stating that for every fixed k, k-LINKAGE is polynomial-time solvable on digraphs of bounded DAG-width.

We believe that a similar approach may be used to prove Conjecture 1.18 for all ℓ. The correspondent result for undirected graphs was showed by Birmelé, Bondy and Reed [7]. We show that Conjecture 1.18 for some ℓ is implied by the two following conjectures for the same ℓ.

The *circumference* of a non-acyclic digraph D, denoted $\mathrm{circ}(D)$, is the length of a longest directed cycle in D. If D is acyclic, then its *circumference* is defined by $\mathrm{circ}(D) = 1$.

Conjecture 1.26. *Let $\ell \geq 2$ be an integer. For any positive integer k, k-*LINKAGE *is polynomial-time solvable for digraphs with circumference at most $\ell - 1$.*

Evidently $\nu_\ell(D) \leq \tau_\ell(D)$ and Conjecture 1.28 states that for every fixed ℓ there exists a function f such that $\tau_\ell(D) \leq f(\nu_\ell(D))$.

Conjecture 1.27. *Let $\ell \geq 2$ be an integer. For every integer $n \geq 0$, there exists an integer $t_n = t_n(\ell)$ such that for every digraph D, either D has a n pairwise-disjoint directed ℓ^+-cycles, or there exists a set T of at most t_n vertices such that $D - T$ has no directed ℓ^+-cycles.*

Theorem 1.28 (Havet and M. [31]). *Let $\ell \geq 1$ be an integer. If Conjectures 1.27 and 1.28 hold for ℓ, then for every positive integer n, $n\vec{C}_\ell$-*SUBDIVISION *is polynomial-time solvable.*

Proof. Let D be a digraph. Let $t = t_n(\ell)$ with $t_n(\ell)$ as in Conjecture 1.28. We first check if $\tau_\ell(D) \leq t$. This can be done by brute force, testing for each subset T of $V(D)$ of size t whether it meets all directed ℓ^+-cycles. Such a test can be done by checking whether $D - T$ has circumference $\ell - 1$, that is, has no $\vec{C}_\ell$-subdivision. Since there are $O(|V(D)|^t)$ sets of size t, and $\vec{C}_\ell$-SUBDIVISION is polynomial-time solvable, this can be done in polynomial time.

If no t-subset T meets all directed ℓ^+-cycles, then $\tau_\ell(D) > t$. Therefore, because Conjecture 1.28 holds for ℓ, D contains an $n\vec{C}_\ell$-subdivision. So we return 'yes'.

If we find a set T of size t that meets all directed ℓ^+-cycles, then $\mathrm{circ}(D - T) \leq \ell - 1$. We use another brute force algorithm which is based on traces.

A *trace* is either a directed ℓ^+-cycle or a linkage. Observe that for any directed ℓ^+-cycle C and any subset Z of $V(D)$, the intersection $C \cap D[Z]$ is a trace. A trace contained in $D[Z]$ is called a *Z-trace*.

Now every ℓ^+-cycle intersects T in a non-empty trace because $\mathrm{circ}(D-T) \leq \ell-1$. We describe a polynomial-time procedure that, given a set of n pairwise disjoint traces $T_1 \ldots, T_n$, checks whether there is an $n\vec{C}_\ell$-subdivision $C_1 + \cdots + C_n$ such that $T_i = C_i \cap D[T]$ for all $1 \leq i \leq n$. Now since T has size t, there is a bounded number of possible sets of n pairwise disjoint traces T-traces (at most $\binom{t}{n+1}(B_t+1)$, where B_t is the is the number of partitions of a set of size t). Hence running the above procedure for all possible such set of T-traces, we obtain a polynomial-time algorithm that decides whether D contains an $n\vec{C}_\ell$-subdivision.

Let $\mathcal{T} = \{T_1, \dots, T_n\}$ be a set of n pairwise disjoint T-traces. Set $\overline{T} = V(D) \setminus T$. A *trace* is *suitable* if it has at least ℓ vertices, at most t components, and the initial and terminal vertices of all components are in $\overline{T}$.

For each T_i, we shall describe a set $\mathcal{T}_i$ of suitable traces such that a directed ℓ^+-cycle C such that $C \cap T = T_i$ contains at least one trace in $\mathcal{T}_i$. The set $\mathcal{T}_i$ is constructed as follows. Let $\mathcal{U}_i$ be the set of traces that can be obtained from T_i by extending each components P of T_i at both ends by an inneighbour of $s(P)$ and an outneighbour of $t(P)$ in $\overline{T}$. Clearly, $\mathcal{U}_i$ has size at most $|V(D)|^2k$, where k is the number of components of T_i. By construction, each trace of $\mathcal{U}_i$ has its initial and terminal vertices in $\overline{T}$ and has no more components than T_i. Moreover, a directed ℓ^+-cycle C such that $C \cap T = T_i$ contains one trace in $\mathcal{U}_i$. However, the set $\mathcal{U}_i$ might not be our set $\mathcal{T}_i$ because certain traces in it might be to small.

For any trace U, let $g(U)$ be set set of all possibles traces obtained from U by adding one vertex of $\overline{T}$ has outneighbour of a terminal vertex of one component of U. Clearly, $g(U)$ has size at most $k|V(D)|$, where k is the number of components of U, and a directed ℓ^+-cycle C containing U must contains a trace in $g(U)$. Moreover, every trace of $g(U)$ has size $|V(U)| + 1$, and no more components than U. Set $g^i(U) = \{U\}$ if i is a non-positive integer and for all positive integer i, define $g^i(U) = \bigcup_{U' \in g^{i-1}(U)} g(U')$. Now the set $\bigcup_{U \in \mathcal{U}_i} g^{\ell - |V(U)|}(U)$ is our desired $\mathcal{T}_i$. Moreover, $\mathcal{T}_i$ is of size at most $t^t \cdot |V(D)|^t$.

To have a polynomial-time procedure to decide whether there is an $n\vec{C}_\ell$-subdivision $C_1 + \cdots + C_n$ such that $T_i = C_i \cap T$ for all $1 \le i \le n$, it suffices to have a procedure that, given an n-tuple $(T'_1, \dots, T'_n)$ of disjoint traces such that $T'_i \in \mathcal{T}_i$, decides whether there is an $n\vec{C}_\ell$-subdivision $C_1 + \cdots + C_n$ such that T'_i is a subdigraph of C_i for all $1 \le i \le n$, and to run it on each possible such n-tuple. Such a procedure can be done as follows. Let $P^i_1, \dots, P^i_{k_i}$ be the components of T'_i. For each n-tuple of circular permutations $(\sigma_1, \dots, \sigma_n)$ of $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_n}$, one checks whether in the digraph D' induced by the vertices of $\overline{T}$ which are not internal vertices of any of the components of the union of the T'_i, if there is a linkage from

$$\left(s(P^1_1), \dots, s(P^1_{k_1}), s(P^2_1), \dots, s(P^2_{k_2}),\ \dots\ , s(P^n_1), \dots, s(P^n_{k_n})\right)$$

to

$$\left(t(P^1_{\sigma_1(1)}), \dots, t(P^1_{\sigma_1(k_1)}), t(P^2_{\sigma_2(1)}), \dots, t(P^2_{\sigma_2(k_2)}),\ \dots\ , t(P^n_{\sigma_n(1)}), \dots, t(P^n_{\sigma_n(k_n)})\right).$$

Now the digraph D' is a subdigraph of $D - T$ and so has circumference at most $\ell - 1$, and the linkage we are looking for has at most t components. Thus each of these instances of $(k_1 + \cdots + k_n)$-Linkage can be solved in time $O(|V(D)|^m)$ for some absolute constant m because Conjecture 1.27 holds for ℓ. □

In Chapter 4, we prove both Conjecture 1.27 and Conjecture 1.28 for $\ell = 3$ (Theorems 4.4 and 4.6), implying that $n\vec{C}_3$-Subdivision is polynomial-time solvable.

1.7 Operations preserving hardness or tractability

The next lemmas allow to extend NP-completeness results of F-Subdivision for some digraphs F to much larger classes.

Lemma 1.29 (Bang-Jensen, Havet and M. [3]). *Let F_1 and F_2 be two digraphs.*

*(i) If F_1-*Subdivision *is NP-complete, then $(F_1 + F_2)$-*Subdivision *is NP-complete.*

*(ii) If $(F_1 + F_2)$-*Subdivision *is polynomial-time solvable, then F_1-*Subdivision *is polynomial-time solvable.*

Proof. Let D be a digraph. We will prove that D contains an F_1-subdivision if and only if $D + F_2$ contains an $(F_1 + F_2)$-subdivision.

Clearly if D contains an F_1-subdivision S, then $S+F_2$ is an (F_1+F_2)-subdivision in $D + F_2$.

Conversely, assume that $D + F_2$ contains an $(F_1 + F_2)$-subdivision $S = S_1 + S_2$ with S_1 an F_1-subdivision and S_2 an F_2-subdivision. Let us consider such an $(F_1 + F_2)$-subdivision that maximizes the number of connected components of F_2 that are mapped (in S) into F_2 again (notice that since there are no arcs between D and F_2 in $D + F_2$, in the subdivision S every component of S_2 will either be entirely inside F_2 or entirely inside D). We claim that $S_2 = F_2$. Indeed suppose that some component T of S_2 is in D. Let C be the component of F_2 of which T is the subdivision. Let $U = S \cap C$. Then T contains a subdivision U' of U (because it is a subdivision of all of C). Hence replacing U by U' and T by C in S, we obtain a subdivision with one more component mapped on itself, a contradiction.

Hence $S_2 = F_2$, and so D contains S_1 which is an F_1-subdivision. □

Lemma 1.30 (Bang-Jensen, Havet and M. [3]). *Let F_1 and F_2 be two digraphs such that F_1 is strongly connected and F_2 contains no F_1-subdivision. Let F be obtained from F_1 and F_2 by adding some arcs with tail in $V(F_1)$ and head in $V(F_2)$.*

*(i) If F_1-*Subdivision *is NP-complete, then F-*Subdivision *is NP-complete.*

*(ii) If F-*Subdivision *is polynomial-time solvable, then F_1-*Subdivision *is polynomial-time solvable.*

Proof. We will prove that a digraph D contains an F_1-subdivision if and only if $D \mapsto F_2$ contains an F-subdivision, where $D \mapsto F_2$ is obtained from $D + F_2$ by adding all possible arcs from $V(D)$ to $V(F_2)$.

It is easy to see that if D contains an F_1-subdivision S, then $S+F_2$ together with some subset of the arcs from D to F_2 is an F-subdivision in $D \mapsto F_2$. Conversely, if $D \mapsto F_2$ contains an F subdivision S^*, then, since F_1 is strongly connected, the part of S^* forming a subdivision of F_1 has to lie entirely inside D or F_2. Since F_2 contains no F_1-subdivision, the subdivision of F_1 has to be inside D and hence we get that D has an F_1-subdivision. □

It is useful to look at Figure 1.4 again and notice that the digraphs A, B show that we need the assumption that F_1 is strongly connected in Lemma 1.31 (and the analogous version where the roles of F_1 and F_2 are interchanged).

Lemma 1.31 (Bang-Jensen, Havet and M. [3]). *Let F_1 and F_2 be two digraphs such that F_1 is robust and F_2 contains no F_1-subdivision. Let F be obtained from F_1 and F_2 by identifying one vertex of F_1 with one vertex of F_2.*

*(i) If F_1-*Subdivision *is NP-complete, then F-*Subdivision *is NP-complete.*

*(ii) If F-*Subdivision *is polynomial-time solvable, then F_1-*Subdivision *is polynomial-time solvable.*

Proof. Given a digraph D we form the digraph D^{F_2} by fixing one vertex x in F_2 and adding $|V(D)|$ disjoint copies of F_2 such that the ith copy has its copy of x identified with the ith vertex of D. Since F_2 contains no F_1-subdivision and $UG(F_1)$ is 2-connected, any subdivision of F_1 in D^{F_2} should be completely contained in D. It follows that D^{F_2} contains an F-subdivision if and only if D contains an F_1-subdivision. □

Lemma 1.32 (Bang-Jensen, Havet and M. [3]). *Let F be a digraph in which every vertex v satisfies $\max\{d^+(v), d^-(v)\} \geq 2$, and let S be a subdivision of F.*

*(i) If F-*Subdivision *is NP-complete, then S-*Subdivision *is NP-complete.*

*(ii) If S-*Subdivision *is polynomial-time solvable, then F-*Subdivision *is polynomial-time solvable.*

Proof. We will make a reduction from F-Subdivision to S-Subdivision.

Let D be an instance of F-Subdivision and p be the length of a longest path in S corresponding to an arc in F. Let D_p be the D-subdivision obtained by replacing every arc of D by a directed path of length p. Since every vertex v corresponding to one of F in S must be mapped onto a vertex corresponding to D in D_p because $\max\{d^+(v), d^-(v)\} \geq 2$, it follows that D has an F-subdivision if and only if D_p has an S-subdivision. □

We believe that the condition $\max\{d^+(v), d^-(v)\} \geq 2$ for all $v \in V(F)$ is may not necessary, although it is in our proof.

Conjecture 1.33 (Bang-Jensen, Havet and M. [3]). *Let F be a digraph, and let S be a subdivision of F.*

*(i) If F-*Subdivision *is NP-complete, then S-*Subdivision *is NP-complete.*

*(ii) If S-*Subdivision *is polynomial-time solvable, then F-*Subdivision *is polynomial-time solvable.*

The following property is also useful for the later chapters.

Lemma 1.34 (Havet, M. and Mohar)**.** *Let F be a digraph and let $u_1, \ldots, u_p$ be distinct vertices of F. Suppose that for every out-neighbour v of u_1, replacing the arc u_1v by a dipath u_1wv of length 2, where $w \notin V(F)$, always results in the same digraph F'. Suppose that for every given digraph D of order n and p vertices $x_1, \ldots, x_p$ in D, one can decide in $f(n)$ time whether there is an F-subdivision in D such that x_i is the u_i-vertex for every i. Then given a digraph D and p vertices $x_1, \ldots, x_p$, one can decide in $O\left(\binom{d^+(x_1)-1}{d^+(u_1)-1} \cdot \sum_{y \in N^+(x_1)} d^+(y) \cdot f(n-1)\right)$ time whether there is an F'-subdivision in D such that x_i is the u_i-vertex for every i.*

Proof. Set $q = d^+(u_1)$. For every set of q neighbours $y_1, \ldots, y_q$ of x_1 and every out-neighbour z of y_1, where $z \notin \{y_2, \ldots, y_q\}$, we shall give a procedure that verifies if D contains an F'-subdivision S' such that x_i is the u_i-vertex for all $1 \leq i \leq p$, and $\{x_1y_1, \ldots, x_1y_q, y_1z\} \subseteq A(S')$. Such an F'-subdivision is called *forced*.

Let D' be the digraph obtained from $D - y_1$ by deleting all arcs leaving x_1 except $x_1y_2, \ldots, x_1y_q$, and adding the arc x_1z.

Claim 1. *D has a forced F'-subdivision if and only if D' has an F-subdivision such that x_i is the u_i-vertex for every i.*

Subproof. Suppose that S is an F-subdivision in D' such that x_i is the u_i-vertex for all i. Since x_1 has outdegree q in D', we have $\{x_1y_2, \ldots, x_1y_q, x_1z\} \subseteq A(S)$. Let S' be the digraph obtained from S by replacing the arc x_1z by the dipath x_1y_1z. Because replacing the arc u_1v by a dipath of length 2 results in F' for any out-neighbour v of u_1, the digraph S' is an F'-subdivision in D. Thus S' is a forced F'-subdivision in D.

Conversely, assume that S' is a forced F'-subdivision in D. Then the digraph S obtained from S' by replacing the dipath x_1y_1z by the arc x_1z is an F-subdivision in D' such that x_i is the u_i-vertex for every i. $\Diamond$

This claim implies that deciding whether D contains a forced F'-subdivision can be done by checking whether D' has an F-subdivision such that x_i is the u_i-vertex for all i. This can be done in $f(n-1)$ time by assumption. By repeating this for every possible set $\{y_1, \ldots, y_q, z\}$ where the y_i are distinct out-neighbours of x_1 and $z \notin \{y_2, \ldots, y_q\}$ is an out-neighbour of y_1, we obtain an algorithm to decide whether there is an F'-subdivision in D such that x_i is the u_i-vertex for all i. Since there are at most $\binom{d^+(x_1)-1}{d^+(u_1)-1} \cdot \sum_{y \in N^+(x_1)} d^+(y)$ such sets, the running time of this algorithm is as claimed. $\square$

Chapter 2

F-Subdivision for some graph classes

In this chapter, we discuss F-Subdivision for F being in many different classes of graphs. For some of them, we are able to completely classify in which cases the problem is polynomial-time solvable and in which it is NP-complete. The polynomial cases illustrate the different techniques that can be used in their solutions.

2.1 Spiders

A *spider* is a tree obtained from disjoint directed paths by identifying one end of each path into a single vertex. This vertex is called the *body* of the spider.

If T is a spider, then every T-subdivision contains T as a subdigraph. Hence a digraph contains a T-subdivision if and only if it contains T as a subdigraph. This implies that T-Subdivision can be solved in $O(n^{|T|})$ time, and the same is valid if T is the disjoint union of spiders. We have then the following as a consequence.

Lemma 2.1 (Bang-Jensen, Havet and M. [3])**.** *Let F be a digraph and T a spider or the disjoint union of spiders. If F-*Subdivision *is polynomial-time solvable, then $(F+T)$-*Subdivision *is also polynomial-time solvable.*

Proof. For each set A of $|T|$ vertices, we check if the digraph $D\langle A\rangle$ induced by A contains T. Then, if yes, we check if $D-A$ has an F-subdivision. □

Gluing a spider T with body b to F at a vertex $u \in V(F)$ consists in taking the disjoint union of F and T and identifying u and b.

Lemma 2.2 (Havet, M. and Mohar)**.** *Let F be a digraph and u a vertex of F. If given a digraph D and a vertex v of D one can decide in polynomial time if there is an F-subdivision in D such that v is the u-vertex, then any digraph obtained from F by gluing a spider at u is tractable.*

Proof. Let T be a spider with body b and let F' be the digraph obtained by gluing T to F at u. Clearly, every F'-subdivision contains an F'-subdivision in which the arcs of T are not subdivided. Such an F'-subdivision is said to be *canonical*.

Consider the following algorithm. For every vertex v of D we repeat the following. For every set W of $|V(T)|-1$ vertices, we check whether $D[W\cup\{v\}]$ contains a copy of T with body v. This can be done in constant time. Then we check if

$D - W$ contains an F-subdivision with u-vertex v. This can be done in polynomial time by our assumption.

This algorithm clearly decides in polynomial time whether a given digraph D contains a canonical F'-subdivision. □

If the spider is specified in the input, the subdivision problem for spiders is NP-complete because it includes the Hamiltonian directed path problem. One could ask if in this case the problem can be solved in FPT time when parameterized by the spider T, that is, in $f(|V(T)|) \cdot n^c$ time, where f is a computable function and c an absolute constant. This question remains open until now.

2.2 Directed cycles

We already commented on the previous chapter about the C_2-SUBDIVISION, where it is enough to check if the graph is acyclic since the subdivision of a C_2 is simply a directed cycle. It is also easy to check the result for the general case.

Proposition 2.3 (Bang-Jensen, Havet and M. [3])**.** *For every $k \geq 2$, C_k-SUBDIVISION can be solved in time $O(n^k \cdot m)$.*

Proof. For every k-tuple $(x_1, x_2, \dots, x_k)$, we check if $(x_1, x_2, \dots, x_k)$ is a directed path and if yes, we check if there is a directed (x_k, x_1)-path in $D - \{x_2, \dots, x_{k-1}\}$. There are $O(n^k)$ k-tuples, so this can be done in $O(n^k \cdot m)$ time. □

The running time above is certainly not the best possible. We can also find a linear-time algorithm when $k = 3$.

Proposition 2.4 (Bang-Jensen, Havet and M. [3])**.** *C_3-SUBDIVISION can be solved in linear time.*

Proof. Let D be a digraph. If D has no directed 2-cycles, then D contains a C_3-subdivision if and only if it is not acyclic, which can be tested in linear time.

Assume now that D has some directed 2-cycles. Let H be the graph with vertex set $V(D)$ and edge-set $\{xy \mid (x, y, x) \text{ is a 2-cycle of } D\}$, that is, each edge of H induces a 2-cycle in D. The graph H can be constructed in linear time. We first check, in linear time, if H contains a cycle. If H contains a cycle, then it has length at least 3 and any of its two directed orientations is a directed cycle in D, so we return such a cycle, certifying that D is a 'yes'-instance.

If not, then H is a forest. If there is any single arc uv (an arc which is not part of a 2-cycle) in D such that both u and v belong to the same connected component of H, then we can produce a directed cycle of length at least 3 in D (following a path from u to v in H) so we may assume that all single arcs go between different components in H. Now it is easy to see that D contains a cycle of length at least 3 if and only if the digraph D' obtained by contracting (into a vertex) each connected component of H in D has a directed cycle: If D' has no cycles, then the only possible cycles of D are inside the contracted connected components of H in G, and therefore they are 2-cycles. In case we find a cycle in D', we can easily reproduce a directed cycle in D by replacing the contracted vertices by paths, and it has length at least 3, because the 2-cycles of D are in the contracted components of H in D and they do not appear in D'. $\square$

If k is not fixed but specified in the input, it is NP-complete to decide if a digraph has a directed cycle of length k because the Hamiltonian directed cycle is a particular case of it. Gabow and Nie proved that it is FPT to decide if a graph has a cycle of length at least k.

Theorem 2.5 (Gabow and Nie [25, 26]). *One can decide in $O(k^{3k} \cdot n \cdot m)$ time whether a digraph contains a directed cycle of length at least k.*

We let as open the problem of reproducing the result of Gabow and Nie but in linear time instead.

Problem 2.6. *For any fixed k, can we solve C_k-*Subdivision *in linear time? In other words, does there exists a computable function f such that one can decide in $O(f(k) \cdot (n+m))$ time whether a digraph contains a directed cycle of length at least k?*

2.3 Other oriented paths and cycles

We propose the following conjecture for paths and cycles in digraphs, which is a particular case of Seymour's Conjecture 1.14.

Conjecture 2.7 (Bang-Jensen, Havet and M. [3]). *If F is an oriented path or cycle, then F-*Subdivision *is polynomial-time solvable.*

Observe that if P is a directed path, as it happens for spiders (it can also be seen like a particular case of this class), every P-subdivision contains P as a subdigraph. Hence to solve the problem for a digraph D it is sufficient to check if D has P as a subdigraph. The case of directed cycles was discussed in the previous section.

Recall that an antidirected path is an oriented path in which every vertex has either in-degree 0 or out-degree 0.

Theorem 2.8 (Bang-Jensen, Havet and M. [3]). *If P is an antidirected path, then P-*Subdivision *is polynomial-time solvable.*

Proof. Let $P = (a_1, \dots, a_p)$ be an antidirected path. By directional symmetry, we may assume that a_i has in-degree 0 in P if and only if i is odd.

Let D be a digraph. For a p-tuple of vertices $(v_1, \dots, v_p)$ of D, we shall describe a procedure that either returns a P-subdivision, or returns that there exists no P-subdivision in which each v_i is the image of a_i. Then applying this procedure for all p-tuples of vertices, we obtain the desired algorithm to finding a P-subdivision.

The procedure is as follows: For all odd (resp. even) i, we remove all the arcs entering v_i (resp. leaving v_i) in D. Let D' be the resulting digraph. Clearly, D

contains a P-subdivision in which each v_i is the image of a_i if and only if D' does. In $UG(D')$, we check if there is a path $\tilde{Q}$ going through $v_1, \dots, v_p$ in this order. This can be done by checking for a linkage from $(v_1, v_2, \dots, v_{p-1})$ to $(v_2, v_3, \dots, v_p)$ and thus in polynomial time by Robertson and Seymour algorithm [48].

If no such $\tilde{Q}$ is found, then D' (and thus D) contains certainly no P-subdivision in which each v_i is the image of a_i.

If such a $\tilde{Q}$ is found, let Q be the oriented path corresponding to Q in D'. Since v_i is a source in D' when i is odd, and a sink in D' when i is even, the path Q has at least $p-1$ blocks (it can have more if some path $Q[v_i, v_{i+1}]$ is not directed), and so contains a subdivision of P. □

When we turn to antidirected cycles (defined similarly), the problem seems to become more complicated. We can not use the same technique of Theorem 2.8, because there we find a path Q with a least $p-1$ blocks, and this path contains necessarily a path with $p-1$ blocks, which is a subdivision of the antidirected path with p vertices: for instance, the first $p-1$ blocks of Q. But the an antidirected cycle with more than $p-1$ blocks does not contains the antidirected cycle on $p-1$ blocks. On the following, we show that $\hat{C}_4$, the antidirected cycle of length 4 (Figure 2.1), is tractable.

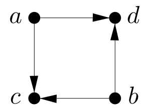


Figure 2.1: The antidirected cycle $\hat{C}_4$ of length 4.

Theorem 2.9 (Havet, M. and Mohar). *$\hat{C}_4$-Subdivision can be solved in $O(n^3 \cdot (n+m))$ time.*

Proof. We shall describe a polynomial-time procedure $\hat{C}_4$-`Subdivision`(a, b, D) that, given two vertices a, b, either finds a $\hat{C}_4$-subdivision (not necessarily with sources a and b) and in this case returns 'yes', or verifies that there is no $\hat{C}_4$-subdivision in D with a and b as sources and returns 'no'. Since a $\hat{C}_4$-subdivision has two sources, running this procedure for every pair $\{a, b\}$ of vertices yields an algorithm to decide whether D contains a $\hat{C}_4$-subdivision; in addition the algorithm runs in $O(n^3 \cdot (n+m))$ time, because the procedure $\hat{C}_4$-`Subdivision`(a, b, D) only needs $O(n \cdot (n+m))$ time.

First, we determine the out-sections $S_a = S^+_{D-b}(a)$ and $S_b = S^+_{D-a}(b)$. If there is a $\hat{C}_4$-subdivision with sources a and b in D, then its two sinks must be in $X = S_a \cap S_b$. Thus if $|X| \le 1$, we return 'no'. Henceforth, we assume that $|X| \ge 2$.

Let A (resp. B) be the set of vertices $x \in X$ such that there is an (a, x)-dipath in $D - b$ (resp. (b, x)-dipath in $D - a$) whose internal vertices are not in X. If there is a $\hat{C}_4$-subdivision with sources a and b in D, then A and B must both be of size at least 2. Thus if $|A| \le 1$ or $|B| \le 1$, we return 'no'. Henceforth, we assume that $|A| \ge 2$ and $|B| \ge 2$.

Claim 2. *Let D' be the digraph obtained from $D[X]$ by adding a, b and all arcs from a to A and from b to B. Then*

(i) *if D has a $\hat{C}_4$-subdivision with sources a and b, then so does D';*

(ii) *if D' has a $\hat{C}_4$-subdivision, then so does D.*

Subproof. (i) Assume that D contains a $\hat{C}_4$-subdivision S with sources a and b, and let c and d be the sinks of S. Let P_1 (resp. P_2, Q_1, and Q_2) be the (a, c)-dipath, (resp. (a, d)-dipath, (b, c)-dipath, and (b, d)-dipath) in S and let a_1 (resp. a_2, b_1, b_2) be the last vertex of A (resp. A, B, and B) on this path. Observe that $V(P_1[a_1, c]) \subseteq X$ and that a similar property holds for each of the paths $P_2[a_2, d]$, $Q_1[b_1, c]$, and $Q_2[b_2, d]$. This shows that the digraph which is the union of the four dipaths $aa_1P_1[a_1, c]$, $aa_2P_2[a_2, d]$, $bb_1Q_1[b_1, c]$ and $bb_2Q_2[b_2, d]$ is a $\hat{C}_4$-subdivision with sources a and b in D'.

(ii) Suppose that D' has a $\hat{C}_4$-subdivision S'. If $a \in V(S')$, let a_1 and a_2 be the two out-neighbours of a in S'. Clearly, $a_1, a_2 \in A$. Therefore in $D - b$, there exist an (a, a_1)-dipath P_1 and an (a, a_2)-dipath P_2 whose internal vertices are not in X. Let a' be the last vertex in $P_1 \cap P_2$ on P_1. We set $P' = \overleftarrow{P_1}[a_1, a']P_2[a', a_2]$.

Similarly, if $b \in V(S')$, denoting b_1 and b_2 the two out-neighbours of b in S', one can find a (b_1, b_2)-inpath with two blocks whose internal vertices are not in X. Call this path Q'.

Now replacing in S' the oriented path a_1aa_2 by P' if $a \in V(S')$ and the oriented path b_1bb_2 by Q' if $b \in V(S')$ results in a $\hat{C}_4$-subdivision in D. $\Diamond$

By Claim 2, we can replace D by D', i.e. we may assume henceforth that $D = D'$, $X = V(D) - \{a, b\}$, $A = N^+(a)$ and $B = N^+(b)$. Moreover, we will assume that $N^-(a) = N^-(b) = \emptyset$.

If $|A \cap B| \geq 2$, then we return 'yes'. Indeed, for any two distinct vertices c and d in $A \cap B$, the cycle $acbda$ is isomorphic to $\hat{C}_4$. Therefore, we may assume that $|A \cap B| \leq 1$.

If $|A \cap B| = 1$, say $A \cap B = \{d\}$, then we check with a Menger algorithm for each vertex $c \in V(D) - \{a, b, d\}$, whether there are independent $(\{a, b\}, c)$-dipaths. If there is a vertex c with two such dipaths P and Q, then we return 'yes'. Otherwise, then we return 'no'. This is valid by the following claim.

Claim 3. *If $A \cap B = \{d\}$, then D contains a $\hat{C}_4$-subdivision with sources a and b if and only if there is a vertex $c \in V(D) - \{a, b, d\}$ such that $D - d$ contains two independent $(\{a, b\}, c)$-dipaths.*

Subproof. If D contains a $\hat{C}_4$-subdivision S with sources a and b, then one of two oriented (a, b)-paths, say R, forming S does not contain d. Thus the sink in R is the desired vertex c.

If there is a vertex c as described above, then let P and Q be two independent $(\{a, b\}, c)$-dipaths with respective initial vertex a and b. Then $P\overleftarrow{Q}bda$ is a $\hat{C}_4$-subdivision. $\Diamond$

Assume now that $A \cap B = \emptyset$. We take a shortest (a, B)-dipath P_a (this can be in done in linear time by Breadth-First Search). Such a path exists because X is the out-section of a in $D - b$. Let c be the terminal vertex of P_a. We then search for a shortest $(a, B - \{c\})$-dipath in $D - c$. If we find such a path Q_a with terminal vertex d, then we return 'yes'. Indeed denoting by a' the last vertex in $P_a \cap Q_a$ on Q_a, the oriented cycle $P_a[a', c]cbd\overleftarrow{Q}_a[d, a']$ is a $\hat{C}_4$-subdivision.

Hence we may assume that every (a, B)-dipath goes through c. Let b' be a vertex in $B - \{c\}$, and let D^* be the digraph obtained by contracting $\{b, b'\}$ into a vertex b^* and removing all arcs entering b^*. We return $\hat{C}_4$-`Subdivision`(a, b^*, D^*). This is valid by Claim 4.

Claim 4. (i) *If there is a $\hat{C}_4$-subdivision with sources a and b in D, then there is a $\hat{C}_4$-subdivision with sources a and b^* in D^*.*

(ii) *If there is a $\hat{C}_4$-subdivision in D^*, then there is a $\hat{C}_4$-subdivision in D.*

Subproof. (i) Assume there is a $\hat{C}_4$-subdivision with sources a and b in D. Let S be such a subdivision with minimum number of vertices. Let b_1 and b_2 be the two out-neighbours of b in S.

If $b' \notin V(S)$, then the digraph obtained from S by replacing the vertex b and the arcs bb_1 and bb_2 by the vertex b^* and the arcs b^*b_1, b^*b_2 is a $\hat{C}_4$-subdivision in D^*.

Suppose now that $b' \in V(S)$. Then bb' is an arc of S. Indeed if it were not, then replacing the (b, b')-path in S not containing a by the arc bb', we would obtain a smaller $\hat{C}_4$-subdivision with sources a and b. Thus, we may assume that $b' = b_1$.

Now b' is not a sink in S. Indeed suppose it were. Let Q be the (a, b')-dipath in S. Necessarily, Q goes through c. Thus, the digraph obtained from S by replacing Q by $Q[a, c]$ and bb' with bc is a smaller $\hat{C}_4$-subdivision with sources a and b, a contradiction.

Hence, b' has an out-neighbour b'' is S. Then the digraph obtained from S by replacing the vertices b and b' and the arcs bb', $b'b''$ and bb_2 by the vertex b^* and the arcs b^*b'', b^*b_2 is a $\hat{C}_4$-subdivision in D^* with sources a and b^*.

(ii) Assume that S^* is a $\hat{C}_4$-subdivision in D^*. If b^* is not a vertex of S^*, then S^* is contained in D and we have the result. If b^* is a vertex in S^*, then it is a source since its in-degree in D^* is zero. Let s and t be its two out-neighbours in S^*. By definition of D^*, s and t are both in $N^+_D(b) \cup N^+_D(b')$. If s and t are both in $N^+_D(b)$ (resp. $N^+_D(b')$), then the digraph obtained from S^* by replacing the vertex b^* and the arcs b^*s and b^*t by the vertex b (resp. b') and the arcs bs and bt (resp. $b's$ and $b't$) respectively, is a $\hat{C}_4$-subdivision in D. If $s \in N^+_D(b)$ and $t \in N^+_D(b')$, then the digraph obtained from S^* by replacing the vertex b^* and the arcs b^*s and b^*t by the vertices b, b' and the arcs bs, bb' and $b't$ is a $\hat{C}_4$-subdivision in D. ◊

Let us now estimate the time complexity of $\hat{C}_4$-`Subdivision`. It first computes two out-sections, which can be done in linear time. Then either it leads a recursive call or it does not because it stops. In the preparation of a recursive call, it possibly computes a dipath (in the case $A \cap B = \emptyset$). Moreover, the order of digraph decreases by one in the call. In the second case, either it stops for some easy reason in $O(1)$ steps, or it stops after using a Menger algorithm which runs in linear time. Let r be the number of recursive calls made by the $\hat{C}_4$-`Subdivision`. Clearly $r \leq n$ and the procedure runs in $O(r \cdot (n+m) + (n+m))$ time, that is in $O(n(n+m))$ time. $\square$

The next step would be to investigate the subdivision problem for oriented paths and cycles with varied amount of blocks, that is, for the cases in which the paths in contrary directions have length bigger than one. We can show it is tractable for oriented paths with few blocks.

Proposition 2.10 (Bang-Jensen, Havet and M. [3])**.** *If P is an oriented path with at most four blocks, then P-Subdivision is polynomial-time solvable.*

Proof. Let $P = (a_1, \ldots, a_s)$ be an outpath (w.l.o.g) with four blocks and let a_p, a_r be the vertices of P with out-degree 0 and a_q be the vertex of P with in-degree 0, for $p < q < r$ (all the other vertices have in-degree and out-degree 1). Let $t_1 = q-p+1$ and $t_2 = r - q + 1$, that is, t_1 and t_2 are the number of vertices of $P[a_p, a_q]$ and $P[a_q, a_r]$, respectively. Suppose without loss of generality that $t_1 \geq t_2$ and say that $t = t_1 - t_2$.

Let D be a digraph. For every $(s + i)$-tuple of vertices $(v_1, \ldots, v_{s+i})$ of D, $0 \leq i \leq t$, we check if $(v_1, \ldots, v_p)$, $(v_q, \ldots, v_{p+1})$, $(v_q, \ldots, v_{q+t_2+i})$, $(v_{q+t_2+i}, \ldots, v_s)$ are directed paths. If yes, we look for a (v_{p+1}, v_p)-path in $D - (v_1, \ldots, v_{s+i})$. At this point, we have checked if there is any subdivision S of P in D in which the subdivision of $P[a_1, a_p]$ and $P[a_r, a_s]$ have exactly the same cardinality of them, the subdivision of $P[a_q, a_r]$ in S is path of order between t_2 and t_1 and the subdivision of $P[a_q, a_p]$ in S is any.

We then check for every $(s+t)$-tuple of vertices $(v_1, \ldots, v_{s+t})$ of D if $(v_1, \ldots, v_p)$, $(v_q, \ldots, v_{p+1})$, $(v_q, \ldots, v_{q+t_1-1})$, $(v_{q+t_1}, \ldots, v_s)$ are directed paths. If yes, we look for 2 internally disjoint directed paths starting in $\{v_{p+1}, v_{q+t_1}\}$ and ending in $\{v_p, v_{q+t_1}\}$ in $D - (v_1, \ldots, v_{s+t})$. This can be done using a Menger algorithm. So, at this point, we have checked if there is any subdivision S of P in D in which the subdivision of $P[a_1, a_p]$ and $P[a_r, a_s]$ have exactly the same cardinality of them, and the subdivision of $P[a_q, a_r]$ and $P[a_q, a_p]$ in S are paths of order bigger than t_1. This is true because we checked the fixed directed paths above and then since $(v_q, \ldots, v_{p+1})$, $(v_q, \ldots, v_{q+t_1-1})$ have both order $t_1 - 1$, does not matter from which vertex to which vertex the paths found by Menger's algorithm goes, in any case we will have the desired subdivision.

Observe that for any subdivision S of P in D, the part of S corresponding to $P[a_1, a_p]$ contains $P[a_1, a_p]$ as a subdigraph. The same happens to the part of S corresponding to $P[a_r, a_s]$. So, since we apply this procedure for all j-tuples of vertices, $s \leq j \leq s+t$, and there are $O(n^j)$ of them, we obtain the desired algorithm to finding a P-subdivision in $O(n^j \cdot (m+n))$ time. $\square$

The subdivision of oriented paths with two blocks, which includes the antidirected cycle of length 2, is a special case of the graphs discussed on the next section.

The case of oriented cycles with more blocks, as the case of oriented paths with more than four blocks, remains open.

2.4 Spindles

A $(k_1, \dots, k_p)$-*spindle* is the union of p pairwise internally disjoint directed (a, b)-paths $P_1, \dots, P_p$ of respective length $k_1, \dots, k_p$. Vertex a is said to be the *tail* of the spindle and b its *head*.

Proposition 2.11 (Bang-Jensen, Havet and M. [3]). *If F is a spindle, then F-Subdivision can be solved in $O\left(n^{|V(F)|} \cdot (n+m)\right)$ time.*

Proof. Let F be a spindle with tail a and head b. Let $a_1, \dots, a_p$ be the out-neighbours of a in F. An F-subdivision may be seen as an F-subdivision in which only the arcs aa_i, $1 \le i \le p$ are subdivided. The following algorithm takes advantage of this property.

Let D be a digraph. For each pair (S, a') where S is a set of $|V(F)| - 1$ vertices and a' a vertex of $D - S$, we first enumerate all the possible subdigraphs of $D\langle S\rangle$ isomorphic to $F - a$ with $a'_1, \dots, a'_p$ corresponding to $a_1, \dots, a_p$. We then check if, in $D - (S - \{a'_1, \dots, a'_p\})$, there exist p internally disjoint directed paths P_i, $1 \le i \le p$, each P_i starting in a' and ending in a'_i. This can be done using a Menger algorithm. Clearly, this algorithm decides if there is an F-subdivision in D. There are $O(n^{|V(F)|})$ possible pairs (S, a'), and for each of them we run at most $(|V(F)|-1)!$ times a Menger algorithm. Since such an algorithm runs in linear time, the time complexity of the above algorithm is $O\left(n^{|V(F)|} \cdot (n+m)\right)$. □

By the proof of Proposition 2.11, we can state the following:

Corollary 2.12. *Let F be a spindle with tail a and head b. Given a digraph D and two vertices a' and b', we can decide in polynomial time if T contains an F-subdivision with a-vertex a' and b-vertex b'.*

The complexity given in Proposition 2.11 is certainly not optimal. For example, it can be improved for spindles with paths of small lengths.

Proposition 2.13 (Bang-Jensen, Havet and M. [3]). *If F is a $(k_1, \dots, k_p)$-spindle and $k_i \le 2$ for all $1 \le i \le p$, then F-Subdivision can be solved in $O(n^2 \cdot (n+m))$ time.*

Proof. If some of the k_i, say k_1, equals 1, then finding an F-subdivision is equivalent to find p internally disjoint directed paths from some vertex a to some other vertex b, which by Menger's theorem is equivalent to check that the connectivity from a and b is at least p. For any pair (a, b), this can be done in linear time by a Menger algorithm.

If $k_i = 2$ for all $1 \le i \le 2$, then finding an F-subdivision is equivalent to find p internally disjoint directed paths of length at least two from some vertex a to some other vertex b. Such paths exist if and only if in $D \setminus ab$ there are p internally disjoint (a, b)-paths. For any pair (a, b), this can be checked in linear time by a Menger algorithm. □

A natural question is to ask about the complexity of deciding if a digraph contains a subdivision of a spindle, when the spindle is no more fixed but specified in the input.

Proposition 2.14 (Bang-Jensen, Havet and M. [3]). *The following problem is NP-complete*
SPINDLE-SUBDIVISION
Input: A spindle F and a digraph D.
Question: Does D contain a subdivision of F?

Proof. The proof is a reduction from the (undirected) Hamiltonian cycle problem.

Let G be an undirected graph. Let D be the symmetric digraph associated to G, that is, D is the digraph obtained from G by replacing every edge uv by the two arcs uv and vu. Let F be any (k_1, k_2)-spindle of the same order as G (and D). For order reason, the digraph contains an F-subdivision if and only if it contains F as a subgraph, and thus if and only if G has a Hamiltonian cycle, since the spindle F with two paths is going to be an oriented cycle containing all the vertices in D. □

In view of Proposition 2.14, one could ask whether it is possible to solve SPINDLE-SUBDIVISION in $f(|V(F)|) \cdot n^c$ time, where f is a computable function and c an absolute constant. This may be formulated in FPT setting as follows.

Problem 2.15. *Is the following problem fixed-paramater tractable?*
PARAMETERIZED SPINDLE-SUBDIVISION
Input: A spindle F and a digraph D.
Parameter: $|V(F)|$.
Question: Does D contain a subdivision of F?

2.5 Bispindles

The $(k_1, \dots, k_p; l_1, \dots, l_q)$-*bispindle*, denoted $B(k_1, \dots, k_p; l_1, \dots, l_q)$, is the digraph obtained from the disjoint union of a $(k_1, \dots, k_p)$-spindle with tail a_1 and head b_1 and a $(l_1, \dots, l_q)$-spindle with tail a_2 and head b_2 by identifying a_1 with b_2 into a vertex a, and a_2 with b_1 into a vertex b. The vertices a and b are called, respectively, the *left node* and the *right node* of the bispindle. The directed (a, b)-paths are called *forward paths*, while the directed (b, a)-paths are called *backward paths*.

We say that $(P_1, \dots, P_p; Q_1, \dots, Q_q)$ is a $(k_1, \dots, k_p; l_1, \dots, l_q)$-bispindle if, for each $1 \leq i \leq p$, P_i is a directed (c, d)-path of length k_i, for each $1 \leq j \leq q$, Q_j is a directed (d, c)-path of length l_j and the union of the P_i and Q_j is $B(k_1, \dots, k_p; l_1, \dots, l_q)$.

Let F be a bispindle with p forward paths and q backward paths. Consider the big paths multidigraph $BP(F)$. By Remark 1.11, we get the following.

Proposition 2.16. *Let F be a bispindle with p forward paths and q backward paths. If $p \geq 1$, $q \geq 1$, and $p + q \geq 4$, then F-*Subdivision *is NP-complete.*

On the other hand, if F has exactly one backward path and one forward path or no backward paths, then it is a directed cycle or a spindle, respectively. In both cases, F-Subdivision can be solved in polynomial time as shown in Sections 2.2 and 2.4, respectively.

In the next, we show that in the remaining cases, that is, when F is a bispindle with two forward paths and one backward path, F-Subdivision is polynomial-time solvable. This is done through the Fork problem, to be defined. We then present faster algorithms to solve $B(1,2;1)$- and $B(1,3;1)$-Subdivision.

2.5.1 The Fork Problem

In this section, to show that $B(k_1, k_2; l_1)$-Subdivision is polynomial-time solvable, we use an approach very similar to the one used by Chudnovsky and Seymour to find an induced subdivision of a $K_{2,3}$ in an undirected graph [16]. In their case, the solution is based on the algorithm they presented for the following problem, called three-in-a-tree: given a graph G and three vertices a, b and c of G, is there a tree passing through a, b and c that is an induced subgraph of G? Our algorithm is based on the following notion. A *fork* with *bottom vertex* a, top vertices b and c and centre t is a digraph in which

- a, b and c are distinct, and t is distinct from b and c (but possibly equal to a),
- every vertex except a has in-degree 1 and a has in-degree 0, and
- all vertices except b, c and t have out-degree 1 and b and c have out-degree 0 and t has out-degree 2.

Consider the following problem.

Fork
Input: A digraph D and three distinct vertices a, b and c.
Question: Does D contain a fork with bottom vertex a and top vertices b and c?

Lemma 2.17 (Bang-Jensen, Havet and M. [3])**.** Fork *can be solved in linear time.*

Proof. Assume that a digraph D contains a fork with bottom vertex a and top vertices b and c. Then, clearly, there are a directed (a, b)-path in $D - c$ and a directed (a, c)-path in $D - b$.

We claim that this necessary condition is also sufficient. Indeed, assume that there is a directed (a, b)-path P in $D - c$ and a directed (a, c)-path Q in $D - b$. Let t be the last vertex on P which also belongs to Q. Such a vertex exists because a is in P and Q. Then the union of P and $Q[t, c]$ is the desired fork.

Since one can decide in linear time if there is a directed (u, v)-path in a digraph, Fork can be solved in linear time. □

Theorem 2.18 (Bang-Jensen, Havet and M. [3]). *If F is a bispindle with two forward paths and one backward path, then F-*Subdivision *can be solved in $O(n^{|F|+1} \cdot (n+m))$ time.*

Proof. Let a be the left node of F and let b and c be its two out-neighbours in F.

For every subset S of $|F|$ vertices, we check if $D\langle S \rangle$ contains a copy of $F - \{ab, ac\}$ with a', b', c' corresponding to a, b, c, respectively. Then we check in $D - (S - \{a', b', c'\})$ if there is a fork with bottom vertex a' and top vertices b' and c'.

Since there are $O(n^{|F|})$ possible set S and Fork can be solved in linear time by Lemma 2.17, our algorithm runs in $O(n^{|F|+1} \cdot (n+m))$ time. □

Similarly to Proposition 2.14, one shows that given a digraph D and a bispindle F (with one forward paths and one backward path), deciding if D contains an F-subdivision is NP-complete. It is again natural to ask if it is FPT when parameterized by $|F|$.

Problem 2.19. *Is the following problem fixed-paramater tractable?*
Parameterized Bispindle-Subdivision
Input: A bispindle F and a digraph D.
Parameter: $|V(F)|$.
Question: Does D contain a subdivision of F?

2.5.2 Faster algorithms for subdivision of bispindles

The complexity given in Theorem 2.18 is certainly not best possible. In this subsection, using handle decomposition, we show algorithms to solve $B(1,2;1)$- and $B(1,3;1)$-Subdivision, whose running time is smaller than the complexity of Theorem 2.18.

Recall that a digraph D is *robust* if it is strongly connected and $UG(D)$ is 2-connected. The *robust components* of a digraph are its robust subdigraphs which are maximal by inclusion.

Because bispindles are robust, a subdivision S of a bispindle is also robust, and if a digraph D contains S, then S must be in a robust component of D. Finding the robust components of a digraph can be done in linear time, by finding the strong components and the 2-connected components of the underlying graphs of these. Therefore one can restrict our attention to subdivision of bispindles in robust digraphs.

2.5.2.1 Subdivision of the $(1,2;1)$-bispindle

A subdivision of the $(1,2;1)$-bispindle has cyclomatic number two. Conversely, observe that every robust digraph of cyclomatic number 2 is a subdivision of the $(1,2;1)$-bispindle. Hence, we have the following.

Proposition 2.20 (Bang-Jensen, Havet and M. [3]). *A digraph contains a subdivision of the* $(1,2;1)$*-bispindle if and only if one of its robust components has cyclomatic number at least two.*

Corollary 2.21 (Bang-Jensen, Havet and M. [3]). $B(1,2;1)$-Subdivision *can be solved in linear time.*

Proof. Finding the robust components can be done in linear time and computing the cyclomatic number of all of them in linear time as well. □

2.5.2.2 Subdivision of the $(1,3;1)$-bispindle

Observe that there is a C_4 in a $(1,3;1)$-bispindle. So, a digraph D that has no directed cycle of length greater than 3 contains no $B(1,3;1)$-subdivision.

Let D be a robust digraph and $C = (v_1,\dots,v_\ell,v_1)$ a directed cycle in D. A handle decomposition $(v,(h_i)_{1\le i\le p},(D_i)_{0\le i\le p})$ is said to be C-*bad* if

(i) $D_1 = C$;

(ii) for all $i \ge 2$, h_i has length 1 or 2, its end-vertices are on C and the distance between the origin and the terminus of h_i around C is 2.

(iii) If h_i is a (v_k, v_k+2)-path and h_j is a (v_{k-1}, v_k+1)-path (indices are taken modulo ℓ), then these two handles have length 1.

(iv) If $\ell \ge 5$, there no k such that (v_{k-2},v_k), (v_{k-1},v_{k+1}) and (v_k,v_{k+2}) are handles.

The notion of C-bad handle decomposition plays a crucial role for finding $B(1,3;1)$-subdivision as shown by the next two lemmas.

Lemma 2.22 (Bang-Jensen, Havet and M. [3]). *Let D be a digraph and C a directed cycle in D of length at least 4. Then one of the following holds:*

- *D contains a $B(1,3;1)$-subdivision,*
- *C is not a longest directed cycle in D, or*
- *D has a C-bad handle decomposition.*

Proof. Set $C = (v_1,\dots,v_\ell,v_1)$. Let $\mathcal{H} = (v,(h_i)_{1\le i\le p},(D_i)_{0\le i\le p})$ be a nice handle decomposition of D such that $D_1 = C$.

If $\mathcal{H}$ is not C-bad, then let k be the largest integer such that $\mathcal{H}_k = (v,(h_i)_{1\le i\le k},(D_i)_{0\le i\le k})$ is a C-bad handle decomposition. One of the following occurs:

(i) the origin s_{k+1} of h_{k+1} is the internal vertex of some h_i, $i \ge 2$. Since $\mathcal{H}_k$ is C-bad, then necessarily $h_i = (s_i, s_{k+1}, t_i)$, and there is a directed path (s_i, v_i, t_i) of length 2 in C. Let t_{k+1} be the terminus of h_{k+1}. If t_{k+1} is on C, we set $h^* = h_{k+1}$ and $t^* = t_{k+1}$. If not, then t_{k+1} has an out-neighbour t^* on C

and we let h^* be the concatenation of h_{k+1} and (t_{k+1}, t^*). In both cases, h^* is a directed (s_{k+1}, t^*)-path with no internal vertices in C. If $t^* = v_i$, then $h^* \cup (C \setminus \{s_i v_i\}) \cup (s_i, s_{k+1})$ is a directed cycle longer than C. If $t^* = s_i$, then $(C \cup h^* \cup (s_i, s_{k+1})) - v_i$ is a $B(1,3;1)$-subdivision with right node s_i and left node s_{k+1}. If $t^* = t_i$, then $C[t_i, s_i] \cup h^*$ is a directed cycle longer than C because in that case h^* has length at least 2. If $t^* \notin \{s_i, t_i, v_i\}$, then $C \cup h^* \cup (s_i, s_{k+1})$ is a $B(1,3;1)$-subdivision with left node s_i and right node t^*.

(ii) the terminus of h_{k+1} is the internal vertex of some h_i, $i \geq 2$. We get the result in a similar way to the preceding case.

(iii) h_{k+1} has length greater than 2 and its two end-vertices are on C. Then the union of C and h_{k+1} is a $B(1,3;1)$-subdivision.

(iv) $h_{k+1} = (s,t)$ with s,t and $C[s,t]$ has length at least 3. Then $C \cup (s,t)$ is a $B(1,3;1)$-subdivision with right node s and left node t.

(v) h_{k+1} is one of the two handles h and h', where h is a (v_{k-1}, v_{k+1})-handle and h' is a (v_k, v_{k+2}) for some k, and one of h and h' has length two. If h has length two, say (v_{k-1}, x_1, v_{k+1}), then the union of $(v_{k-1}, v_k) \cup h'$, $(v_{k-1}, x_1, v_{k+1}, v_{k+2})$ and $C[v_{k+2}, v_{k-1}]$ form a $B(1,3;1)$-subdivision. If h' has length two, say $h' = (v_k, x_2, v_{k+2})$, then the union of $h \cup (v_{k+1}, v_{k+2})$, $(v_{k-1}, v_k, x_2, v_{k+2})$ and $C[v_{k+2}, v_{k-1}]$ form a $B(1,3;1)$-subdivision.

(vi) h_{k+1} is one of the three handles (v_{k-2}, v_k), (v_{k-1}, v_{k+1}), (v_k, v_{k+2}) for some k and $p \geq 5$. In this case, the union of $(v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2})$, (v_{k-2}, v_k, v_{k+2}) and $C[v_{k+2}, v_{k-2}]$ form a $B(1,3;1)$-subdivision.

□

Lemma 2.23 (Bang-Jensen, Havet and M. [3]). *Let D be a robust digraph and C a directed cycle in D of length at least 4. If D has a C-bad handle decomposition, then it does not contain any $B(1,3;1)$-subdivision.*

Proof. By induction on the number p of handles of the handle decomposition, the result holding trivially if $p = 1$.

Set $C = (v_1, \ldots, v_\ell, v_1)$ and let $\mathcal{H} = (v, (h_i)_{1\leq i\leq p}, (D_i)_{0\leq i\leq p})$ be a C-bad handle decomposition of D.

By the induction hypothesis D_{p-1} does not have any $B(1,3;1)$-subdivision.

Suppose, by way of contradiction, that D_p contains a $B(1,3;1)$-subdivision S. Necessarily, h_p is a subdigraph of S. Free to rename, we may assume that v_1 and v_3 are the origin and the terminus, respectively, of h_p. If v_2 is not in S, then replacing h_p with (v_1, v_2, v_3) in S, we obtain a $B(1,3;1)$-subdivision contained in D_{p-1}, a contradiction. Hence $v_2 \in V(S)$. By the conditions *(iii)* and *(iv)* of a C-bad handle decomposition, there cannot be both a handle ending at v_2 and a handle starting at v_2. By directional symmetry, we may assume that v_2 has in-degree one, and so

$v_1v_2 \in A(S)$, and v_1 is the left node of S. Now, v_2v_3 is not an arc of S, for otherwise v_3 will be the right node of S, and the two directed (v_1, v_3)-paths in S have length at most 2, a contradiction. But, in S, there is an arc leaving v_2, it must be in a handle, and so by (iv) and (ii) of the definition of C-bad, this arc must be v_2v_4. Again by (iii) of the definition of C-bad, there is no arc leaving v_3 except v_3v_4. Hence $v_3v_4 \in A(S)$. Then v_4 is the right node of S, and the two directed (v_1, v_4)-paths in S have length 2, a contradiction. □

Theorem 2.24 (Bang-Jensen, Havet and M. [3]). *$B(1,3;1)$-Subdivision can be solved in $O(n \cdot m)$ time.*

Proof. Given a digraph D, we compute the robust components of D and solve the problem separately on each of them.

For each robust component, we first search for a directed cycle C_0 of length at least 4. This can be done in $O(n \cdot m)$ time by Theorem 2.5. If there is no such cycle, then we return 'no'. If not, then we build a handle decomposition starting from $C := C_0$. Each time, we add a new handle, one can mimick the proof of Lemma 2.22, we either find a $B(1,3;1)$-subdivision which we return, or a C-bad handle decomposition, or a directed cycle C' longer than the current C. Observe that in this case, it is easy to derive a C'-bad handle decomposition containing the vertices added so far from the C-bad one. This can be done in $O(n \cdot m)$ time because an arc has to be considered only when it is added in a handle, and we just need to keep a set of at most m handles.

At the end of this process, if no $B(1,3;1)$-subdivision has been returned, we end up with a C-bad decomposition of D. So, by Lemma 2.23, D has no $B(1,3;1)$-subdivision, and we can proceed to the next robust component, or return 'no' if there is none. □

2.6 Windmills

A *cycle windmill* is a digraph obtained from disjoint directed cycles by taking one vertex per cycle and identifying all of these. This vertex will be called the *axis* of the windmill.

Theorem 2.25 (Bang-Jensen, Havet and M. [3]). *If W is a cycle windmill, then W-Subdivision can be solved in $O(n^{|V(W)|} \cdot (n+m))$ time.*

Proof. Suppose W is a windmill with axis o and cycle lengths $a_1, a_2, \dots, a_p$. To check whether a given digraph $D = (V, A)$ contains a subdivision of W with axis at the vertex x we do the following (until success or all subsets have been tried): for all choices of disjoint ordered subsets $X_1, X_2, \dots, X_p$ of V such that $X_i = \{v_{i,1}, \dots, v_{i,a_i-1}\}$, $i = 1, 2, \dots, p$, check whether $Q_i = xv_{i,1}v_{i,2}\dots v_{i,a_i-1}$ is a directed (x, v_{i,a_i-1})-path. If this holds for all i, then delete all the vertices of $X_i - v_{i,a_i-1}$, $i = 1, 2, \dots, p$, and check whether the resulting digraph contains internally disjoint paths $P_1, P_2, \dots, P_p$ where P_i is a path from v_{i,a_i-1} to x using a

Menger algorithm. If these paths exist, then return the desired subdivision of W formed by the union of $Q_1, Q_2, \ldots, Q_p, P_1, P_2, \ldots, P_p$. Otherwise continue to the next choice for $X_1, X_2, \ldots, X_p$. Since the size of $X_1 \cup X_2 \cup \ldots \cup X_p$ is $|V(W)| - 1$, there are $O(n^{|V(W)|-1})$ choices for it, and there are n choices for x, hence the algorithm runs $O(n^{|V(W)|})$ times a Menger algorithm. Since a Menger algorithm runs in linear time, the overall complexity is $O(n^{|V(W)|} \cdot (n+m))$. □

Clearly, given as input a windmill W and a digraph D, deciding if D contains a W-subdivision is NP-complete because the Hamiltonian directed cycle problem is a particular case of it. Theorem 2.25 tells us that this problem parameterized by $|W|$ is in XP. But is it fixed-parameter tractable?

Problem 2.26. *Is the following problem fixed-paramater tractable?*
CYCLE-WINDMILL SUBDIVISION
Input: A cycle windmill W and a digraph D.
Parameter: $|V(W)|$.
Question: Does D contain a subdivision of W?

2.7 Dumbbells

A *dumbbell* is a digraph D with exactly two big vertices u and v which are connected by an induced oriented (u, v)-path P such that removing the internal vertices of P leaves a digraph with two connected components, one L containing u and one R containing the terminus v. The subdigraph L (resp. R) is the *left* (resp. *right*) *plate* of the dumbbell, vertex u is its *left clip*, vertex v its *right clip* and P its *bar*.

A *dumbbell set* is a disjoint union of dumbbells. In this section, we give some necessary conditions for F-SUBDIVISION to be NP-complete, F being a dumbbell set. We also show some particular cases in which F-SUBDIVISION is polynomial-time solvable.

Recall that we denote by $b(P)$ the number of blocks of a path P. A pair of oriented paths (P, Q) is a *bad pair* if one of the following holds:

- P and Q are both directed paths,
- $\{b(P), b(Q)\} = \{1, 2\}$,
- P and Q are both out-paths and $\{b(P), b(Q)\} \in \{\{2, 2\}; \{2, 4\}\}$, or
- P and Q are both in-paths and $\{b(P), b(Q)\} \in \{\{2, 2\}; \{2, 4\}\}$.

Lemma 2.27 (Bang-Jensen, Havet and M. [3]). *Let P and Q be two oriented paths. If (P, Q) is not a bad pair, then there exists $ab \in A(P)$ and $cd \in A(Q)$ such that the two oriented paths P' and Q' obtained from P and Q by replacing ab and cd by ad and cb verify $\{b(P), b(Q)\} \neq \{b(P'), b(Q')\}$.*

Proof. Let (P, Q) be a non-bad pair of paths. Without loss of generality, we may assume that $b(Q) \geq b(P)$. In particular this implies $b(Q) \geq 3$.

Assume that P is an out-path (resp. in-path) and Q is an in-path (resp. out-path). If $b(P) \geq 2$, then take ab as an arc of the first block of P and cd an arc of the first block of Q. Replacing ab and cd by ad and cb results necessarily in $b(P') = 1$ and $b(Q') = b(P) + b(Q) - 1$. If $b(P) = 1$, take ab as an arc of the first block of P and cd an arc of the second block of Q. Then $\{b(P'), b(Q')\} = \{2, b(Q) - 1\} \neq \{b(P), b(Q)\}$.

So we may assume that P and Q are both out-paths or both in-paths. Observe that this in particular implies that P and Q have an even number of blocks, because the opposite path (same digraph but starting form the terminus and ending at the origin) of an out-path with an odd number of blocks is an in-path with an odd number of blocks.

Take an arc ab of the first block of P and an arc cd of the second block of Q. Then one of P', Q' has two blocks and the other $b(P) + b(Q) - 2$ blocks. So if $\{b(P), b(Q)\} \neq \{2, b(P) + b(Q) - 2\}$, we have the result. Hence we may assume that $\{b(P), b(Q)\} = \{2, b(P) + b(Q) - 2\}$, so $b(P) = 2$ because $b(Q) \geq 3$.

Hence $b(Q) \geq 6$, because (P, Q) is not bad. Take ab be an arc of the first block of P and cd an arc of the third block of Q. Then one of P', Q' has four blocks and the other has $b(P) + b(Q) - 4$ blocks, so we have the result. $\square$

If two digraphs D and D' are isomorphic, we write $D \cong D'$, and if they are not, then we write $D \ncong D'$.

Theorem 2.28 (Bang-Jensen, Havet and M. [3]). *Let F be a dumbbell set. Let D_1 and D_2 be two dumbbells of F, and for $i = 1, 2$, let L_i, R_i, u_i, v_i and P_i be the left plate, right plate, left clip, right clip and bar of D_i. If one of the following holds*

(i) (P_1, P_2) *is not a bad pair,*

(ii) $L_1 \ncong L_2$, $L_1 \ncong R_2$, $R_1 \ncong L_2$ *and* $R_1 \ncong R_2$,

(iii) P_1 *and* P_2 *are both directed paths,* $L_1 \ncong L_2$ *and* $R_1 \ncong R_2$, *or*

(iv) P_1 *is a directed path and* P_2 *is an out-path (resp. in-path) with two blocks and* $L_1 \ncong L_2$ *or* $L_1 \ncong R_2$ *(resp.* $R_1 \ncong L_2$ *or* $R_1 \ncong R_2$*).*

then F-Subdivision is NP-complete.

Proof. By Lemma 1.30, it is sufficient to prove it when $F = D_1 + D_2$. We give a reduction from 2-linkage in digraphs with no big vertices in which x_1 and x_2 are sources and y_1 and y_2 are sinks.

Let D, x_1, x_2, y_1, y_2 be an instance of this problem. Let ab be an arc of the bar of D_1 and cd be an arc of the bar of D_2. Moreover, if (P_1, P_2) is not a bad pair, we choose ab and cd as decribed in Lemma 2.27. Let H be the digraph obtained from the disjoint union of $F \setminus \{ab, cd\}$ and D by adding the arcs ax_1, cx_2, y_1b, and y_2d. We can then show that H has an F-subdivision if and only if D has a 2-linkage from (x_1, x_2) to (y_1, y_2).

Clearly, if there is a 2-linkage Q_1, Q_2 in D, then the union of $F \setminus \{ab, cd\}$ and the paths $ax_1Q_1y_1b$ and $cx_2Q_2y_2d$ is an F-subdivision in H.

Conversely, suppose that H contains an F-subdivision S. For each vertex x of F, we denote by x^* the vertex corresponding to x in S and for any subdigraph G of F, we denote by G^* the subdigraph of S corresponding to the subdivision of G.

In H, no vertex of D is big, so the sole big vertices of D are the clips of D_1 and D_2. Hence $\{u_1^*, v_1^*, u_2^*, v_2^*\} = \{u_1, v_1, u_2, v_2\}$. Now in S, the paths P_1^* and P_2^* connect big vertices. For connectivity reasons these two paths must use $P_1 \setminus ab$ and $P_2 \setminus cd$. In particular, $(L_1 + L_2 + R_1 + R_2)^*$ is a subdigraph of $L_1 + L_2 + R_1 + R_2$. So $(L_1 + L_2 + R_1 + R_2)^* = L_1 + L_2 + R_1 + R_2$. So for any $G \in \{L_1, L_2, R_1, R_2\}$, the digraph G^* is isomorphic to G and is one of the subdigraphs L_1, L_2, R_1 and R_2.

Moreover the number of blocks $b(P_i^*) = b(P_i)$ for $i = 1, 2$. Hence, the subpaths of $P_1^* \cap D$ and $P_2^* \cap D$ must be two disjoint directed paths in D, with origins in $\{x_1, x_2\}$ and terminus in $\{y_1, y_2\}$, for otherwise $b(P_1^*) + b(P_2^*) > b(P_1) + b(P_2)$.

Let P_1' and P_2' be the oriented paths obtained from P_1 and P_2 by replacing ab and cd by ad and cb. By construction, if there is no 2-linkage from (x_1, x_2) to (y_1, y_2) in D, then P_1^* and P_2^* consist in a P_1'-subdivision and a P_2'-subdivision, and so $\{b(P_1'), b(P_2')\} = \{b(P_1^*), b(P_2^*)\}$. We consider then the previous cited cases for P_1, P_2, R_1, R_2, L_1, L_2.

(i) If (P_1, P_2) is not a bad pair, then by our choice of ab and cd, $\{b(P_1'), b(P_2')\} \neq \{b(P_1), b(P_2)\}$. Since $b(P_1^*) = b(P_1)$ and $b(P_2^*) = b(P_2)$, there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

(ii) If $L_1 \ncong L_2$ and $L_1 \ncong R_2$, then $L_1^* \in \{L_1, R_1\}$. Similarly, if $R_1 \ncong L_2$ and $R_1 \ncong R_2$, then $R_1^* \in \{L_1, R_1\}$. Hence P_1^* must go from u_1 to v_1, and so $P_1^* \cap D$ is a directed (x_1, y_1)-path. Hence there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

(iii) If P_1 and P_2 are both directed paths, then $\{u_1^*, u_2^*\} = \{u_1, u_2\}$ as there are the origin of P_1^* and P_2^*. Now, since $L_1 \ncong L_2$, we have $L_1^* = L_1$ and $L_2^* = L_2$. Similarly, $R_1^* = R_1$ and $R_2^* = R_2$. Hence, $P_1^* \cap D$ and $P_2^* \cap D$ form a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

(iv) Assume that P_1 is a directed path and that P_2 is an out-path with two blocks. (The proof is analoguous when P_2 is an in-path with two blocks.)

Assume that $L_1 \ncong L_2$. Then we can choose cd to be an arc of the first block of P_2. Necessarily, $v_1^* = v_1$ and $R_1^* = R_1$ since v_1^* is the only clip with out-degree 0 in $P_1^* \cup P_2^*$. It follows that $L_1^* \in \{L_1, L_2\}$, and so $L_1^* = L_1$ because $L_1 \ncong L_2$. Thus $P_1^* \cap D$ is a directed (x_1, y_1)-path and there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

If $L_1 \ncong R_2$, we get the result similarly by choosing cd to be an arc of the second block of P_2.

$\square$

A *palm tree* is a dumbbell whose left and right plates are spiders, and whose bar is a directed path of length one. Observe that in a palm tree, the two clips must be the bodies of the spiders. A *palm grove* is a disjoint union of palm trees. For example, the two graphs A and B depicted Figure 1.4 are palm groves.

By Theorem 2.28(c), if F is a palm grove having two palm trees whose left spiders are not isomorphic and whose right spiders are not isomorphic, then F-Subdivision is NP-complete. We shall now prove that it is indeed the only hard case. Observe that if a digraph contains a subdivision of a palm tree, then it contains a subdivision of this palm tree such that the only subdivided arc is the bar.

Theorem 2.29 (Bang-Jensen, Havet and M. [3]). *Let F be a palm grove. Then F-Subdivision is polynomial-time solvable if and only if all its left spiders are isomorphic or all its right spiders are isomorphic.*

Proof. If there are two left spiders that are not isomorphic and there are two right spiders that are not isomorphic, then there exist two palm trees such that their left spiders are not isomorphic and their right spiders are not isomorphic. Then, by Theorem 2.28-(c), F-Subdivision is NP-complete.

Assume now that all the right spiders are isomorphic to a spider R. Let $L_1, \dots, L_p$ be the left spiders (possibly some of them are isomorphic). We shall decribe an algorithm to solve F-Subdivision.

Let D be a digraph. By the above remark, if D contains an F-subdivision, then it contains an F-subdivision such that only the bars of the palm trees are subdivided. Hence we look for such a subdivision. Observe that such a subdivision is the disjoint union of copies of each of the L_i, $1 \le i \le p$, and p copies of R together with p disjoint directed paths from the bodies of the copies of the L_i to the bodies of the p copies of R. Hence to decide if D contains an F-subdivision, we try all possibilities for the disjoint union of spiders L_i, $1 \le i \le p$, and p spiders R and for each possibility we check via a Menger algorithm if there are disjoints directed paths from the bodies of the L_i to the bodies of the copies of R.

Formally, the algorithm is the following. For each set of distinct vertices $\{u_1, \dots u_p, v_1, \dots, v_p\}$ of D and family of disjoints subsets $\{U_1, \dots, U_p, V_1, \dots, V_p\}$ of D such that for $1 \le i \le p$, $u_i \in U_i$ and $v_i \in V_i$, we check if for all i, $D\langle U_i \rangle$ (resp. V_i) contains a spider isomorphic to L_i (resp. R) with body u_i (resp. v_i). If not we proceed to the next case. If yes, we check if there are p disjoint directed paths from $\{u_1, \dots, u_p\}$ to $\{v_1, \dots, v_p\}$ in the digraph $D - (\bigcup_{i=1}^{p}(U_i \cup V_i) - \{u_i, v_i\})$ via a Menger algorithm. If there are such paths, the union of them with the spiders is an F-subdivision and we return it. If such paths do not exist, we proceed to the next case.

The number of possible cases is $O(n^{|V(F)|})$ and each run of the Menger algorithm can be done in linear time. Hence the complexity of the algorithm is $O(n^{|V(F)|} \cdot (n + m))$. □

2.8 Wheels

The *wheel* W_k, also called k-wheel, is the graph obtained from the directed cycle C_k by adding a vertex, called the *centre*, dominating every vertex of C_k (the W_3 is represented in Figure 2.2) The cycle C_k is called the *rim* of W_k and the arcs incident to the centre are called the *spokes*. Similarly, if D' is a subdivision of a wheel D, the *centre* of D' is the vertex corresponding to the centre of D, the *rim* of D' is the directed path or cycle corresponding to the rim of D, and the *spokes* of D' are the directed paths corresponding to the spokes of D.

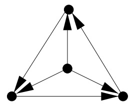


Figure 2.2: The 3-wheel W_3

We completely classify in which cases W_k-Subdivision is NP-complete and in which cases the problem is polynomial-time solvable.

Theorem 2.30 (Bang-Jensen, Havet and M. [3]). *For all $k \geq 4$, W_k-Subdivision is NP-complete.*

Proof. We show a reduction from 2-linkage in digraphs with no big vertices in which x_1 and x_2 are sources and y_1 and y_2 are sinks.

Let D, x_1, x_2, y_1, y_2 be an instance of this problem. Let D' be the graph obtained from D by adding $k+1$ new vertices z, $v_1, v_2, \dots, v_k$, the arcs zv_i, for $1 \geq i \geq k$, the arcs v_iv_{i+1} and $v_{k-1}v_k$, for $1 \geq i \geq k-3$, and finally y_1v_{k-1}, v_kx_2, y_2v_1 and $v_{k-2}x_1$.

Let us prove that D' has a W_4-subdivision if and only if D has a 2-linkage from (x_1, x_2) to (y_1, y_2).

If P_1, P_2 form the desired 2-linkage in D, then we take $P_1 \cup y_1v_{k-1} \cup v_{k-1}v_k \cup v_kx_2 \cup P_2 \cup y_2v_1 \cup v_1v_2 \dots v_{k-2} \cup v_{k-2}x_1$ as the rim and the arcs zv_i, for $1 \geq i \geq k$, as the spokes.

Conversely, suppose W is a subdivision of W_4 in D' and let C be its rim. The centre of W must be z as this is the only vertex of out-degree k in D'. Thus the k paths starting in z will start in the arcs zv_i, $1 \geq i \geq k$, respectively. Now observe that v_{k-1} must belong to C, otherwise the path containing zv_{k-1} could not be disjoint from the path containing zv_k (they would meet in v_k, since it is the only out-neighbour of v_{k-1}). Then v_{k-1} and v_k are in C. Similarly, $v_1, v_2, \dots, v_{k-3}$ must belong to C since otherwise the path containing zv_i could not be disjoint from the path containing zv_{i+1}, $1 \geq i \geq k-3$ (they would meet in v_{i+1}, for instance). Thus $v_1, v_2, \dots, v_{k-3}$ are on C and then v_{k-2} is on C since it is the only out-neighbour of v_{k-3}. Hence C contains the arc $v_{k-1}v_k$ and the path $v_1, v_2, \dots, v_{k-2}$, what implies that C contains the edges $v_kx_2, y_2v_1, v_{k-2}x_1, y_1, v_{k-1}$ and lastly the disjoint paths from x_1 to y_1 and x_2 to y_2, respectively. □

We now prove that the two remaining cases for wheels, that is, W_2 and W_3, are tractable. We start with the following proposition for W_2.

Proposition 2.31 (Bang-Jensen, Havet and M. [3]). *A digraph D contains a W_2-subdivision if and only if it contains some vertex z such that $D - z$ has a strong component S and two directed (z, S)-paths having only z in common.*

Proof. Suppose D contains a subdivision of W_2 with centre z and cycle C. Then the strong component of $D - z$ which contains C satisfies the required property.

Conversely, assume z is a vertex and S is a strong component of $D-z$ such that there are two directed (z, S)-paths P and Q having only z in common. Let x and y be the ends of P and Q, respectively.

Let R be a directed (x, y)-path in S and R' a directed (y, x)-path in S. (Such paths exists since S is a strong component.) If R and R' form a cycle we are done, with this cycle as rim and P, Q as spokes. Otherwise let q be the last vertex in $R' - \{x, y\}$ which is also on R. Then we have a W_2-subdivision with rim $R[x, q] \cup R'[q, x]$ and spokes P and $Q \cup R[y, q]$. □

Corollary 2.32 (Bang-Jensen, Havet and M. [3]). *W_2-Subdivision is solvable in $O(n \cdot (n + m))$ time.*

Proof. According to Proposition 2.31, to find a W_2-Subdivision in a digraph D it is sufficient to first calculate, for each vertex $z \in D$, the connected component of $D - z$. This can be done in $O(n \cdot m)$ using depth-first search. Then it checks by a Menger algorithm if there are two independent directed (z, S)-paths, what gives total time complexity of $O(n \cdot (n + m))$. □

Proposition 2.31 and Corolary 2.32 implies that, given a digraph D and a vertex v of D, one can decide in polynomial time if D contains a W_2-subdivision with centre z. We prove that we can also decide in polynomial time if there is a W_2-subdivision with two prescribed original vertices.

Lemma 2.33 (Havet, M. and Mohar). *Let W_2 be the 2-wheel with centre c and rim aba. Given a digraph D and two vertices b' and c', one can decide in $O(n^2 \cdot (n+m))$ time if there is a W_2-subdivision in D with b-vertex b' and c-vertex c'.*

Proof. Let us call a W_2-subdivision with b-vertex b' and c-vertex c' a (b', c')-*forced* W_2-subdivision. Let S be the strong component of b' in $D - c'$. The key element is the following claim.

Claim 5. *D contains a (b', c')-forced W_2 subdivision if and only if there exist distinct vertices x_1 and x_2 in $V(S)$ such that there are two independent $(c', \{x_1, x_2\})$-dipaths P_1 and P_2 in $D-(S-\{x_1, x_2\})$ and there are two independent $(\{x_1, x_2\}, b')$-dipaths Q_1 and Q_2 in S.*

Subproof. The existence of two vertices x_1, x_2 and four dipaths P_1, P_2, Q_1, Q_2 as in the statement is a necessary condition for the existence of a (b', c')-forced W_2-subdivision, because in such subdivision there would be two independent paths R_1 and R_2 (the subdivision of the spokes) to the a-vertex a' and to b', respectively. Since $C = a'b'a'$ is a directed cycle, a', b' are in the same strong component S, and x_1, x_2 would be the first vertices of $R_1 \cap S, R_2 \cap S$, respectively. So, $P_1 = R_1[c', x_1], P_2 = R_2[c', x_2], Q_1 = R_1[x_1, a'] \cup C[a', b'], Q_2 = R_2[x_2, b']$.

Assume now that such vertices x_1, x_2 and dipaths P_1, P_2, Q_1, Q_2 exist. Since S is strong, it contains a dipath R from b' to $(V(Q_1) \cup V(Q_2)) - \{b'\}$. (This set is not empty since it contains $\{x_1, x_2\} - \{b'\}$.) Then $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R$ is a (b', c')-forced W_2-subdivision. $\Diamond$

Our algorithm is the following. We first compute S, which can be done in linear time. Then for every pair $\{x_1, x_2\}$ of vertices of S, we check by running twice a Menger algorithm if the dipaths P_1 and P_2, and Q_1 and Q_2 as described in Claim 5 exist. If yes, we return 'yes', otherwise we return 'no'. The validity of this algorithm is given by Claim 5. Since there are $O(n^2)$ pairs of vertices $\{x_1, x_2\}$, the algorithm runs in $O(n^2 \cdot (n+m))$ time. $\square$

Theorem 2.34 (Havet, M. and Mohar)**.** *W_3-*SUBDIVISION *can be solved in $O(n^6 \cdot (n+m))$ time.*

The proof relies on the following notion. Let X be a set of three vertices. An *X-tripod* is a digraph which is the union of a directed cycle C and three disjoint dipaths P_1, P_2, P_3 with initial vertices in X and terminal vertices in C. If the P_i are (X, C)-dipaths, we say that the tripod is *unfolded*. Note that the dipaths P_i may be of length 0. We shall denote the tripod described above as the 4-tuple (C, P_1, P_2, P_3).

Proposition 2.35. *Let $X = \{x_1, x_2, x_3\}$ be a set of three distinct vertices. Any X-tripod contains an unfolded X-tripod.*

Proof. Let (C, P_1, P_2, P_3) be an X-tripod, where P_i has initial vertex x_i, for $i = 1, 2, 3$. Let y_i be the first vertex on C along P_i. Then $(C, P_1[x_1, y_1], P_2[x_2, y_2], P_3[x_3, y_3])$ is an unfolded X-tripod. $\square$

We shall consider the following decision problem.

TRIPOD
Input: A strong digraph D and a set X of three distinct vertices of D.
Question: Does D contain an X-tripod?

Observe that TRIPOD is also (as FORK of Lemma 2.17) resemblant to the "three-in-a-tree" problem for undirected graphs [16]. The problem of searching an induced X-tripod in an undirected graph G would allowed to solve the problem of finding an induced subdivision of a K_4 in G, and it remains an open question.

We show that TRIPOD is polynomial-time solvable in the next lemma. The scheme of the proof reminds an idea used in decompositions: we first find some cuts-like on the digraph, then we partitioned it in some (slightly) modified subdigraphs and finally we focus on the reduced problem.

Lemma 2.36. TRIPOD *can be solved in $O(n^2 \cdot (n+m))$ time.*

Proof. Let us describe a procedure `tripod`(D, X), solving Tripod.

We first look for a directed cycle of length at least 3 in D. This can be done in linear time. If there is no such cycle, then we return 'no'.

Otherwise we have a directed cycle C of length at least 3. We choose a set Y of three vertices in C and run a Menger algorithm between X and Y. If such an algorithm finds three disjoint (X, Y)-dipaths P_1, P_2, P_3, then we return the tripod (C, P_1, P_2, P_3). Otherwise, the Menger algorithm finds a 2-separation (W, S, Z) of (X, Y). Note that $|S| \geq 1$ because D is strong.

Assume first that $|S| = 1$, say $S = \{s\}$. Let D_1 be the digraph obtained from $D[W \cup S]$ by adding the arc sw for every vertex w in W having an in-neighbour $z \in Z$. We then make a recursive call to `tripod`(D_1, X). This is valid by virtue of the following claim.

Claim 6. *There is an X-tripod in D if and only if there is an X-tripod in D_1.*

Subproof. Suppose first that there is an X-tripod in D_1. Then D_1 contains an unfolded X-tripod T_1 by Proposition 2.35. If T_1 is contained in D, then we are done. So we may assume that it is not. Then T_1 contains an arc $sw \in A(D_1) \setminus A(D)$. It can contain only one such arc since every vertex has out-degree at most one in T_1 and all such arcs leave s. Furthermore, the head w of this arc is in W and w has an in-neighbour z in Z. Now, since D is strong, there is an (s, z)-dipath Q in D. Because there is no arc from W to Z, all internal vertices of Q are in Z. Hence the digraph T obtained from T_1 by replacing the arc sw by the dipath $Q \cup zw$ is an X-tripod in D.

Suppose now that D contains an X-tripod. Then it contains an unfolded X-tripod $T = (C, P_1, P_2, P_3)$ by Proposition 2.35. Since all (X, Z)-dipaths in D go through s, the terminal vertices of the P_i are in $W \cup S$, and $D[Z] \cap T$ is a dipath Q which is a subpath of one of the P_i or C. If Q is a (t, z)-dipath, then T contains arcs st and zw for some $w \in W$. Then the digraph T_1 obtained from T by replacing $st \cup Q \cup zw$ by the arc sw is an X-tripod in D_1. $\lozenge$

Assume now that $|S| = 2$, say $S = \{s_1, s_2\}$. If there is no arc from Z to W, let D_2 be the digraph obtained from $D[W \cup S]$ by adding the arc s_1s_2 (resp. s_2s_1) (if the arc is not already present in D) if there is an (s_1, s_2)-dipath (resp. (s_2, s_1)-dipath) in $D[Z \cup S]$. We then make a recursive call to `tripod`(D_2, X). This is valid by virtue of the following claim.

Claim 7. *There is an X-tripod in D if and only if there is an X-tripod in D_2.*

Subproof. Suppose first that there exists an X-tripod in D_2. Then there is an unfolded X-tripod T_2 in D_2, by Proposition 2.35. Then either it is an X-tripod in D, or T_2 contains exactly one of the arcs s_1s_2, s_2s_1 and this arc is not in $A(D)$. Without loss of generality, we may assume that this arc is s_1s_2. Since $s_1s_2 \in A(D_2) \setminus A(D)$, there is an (s_1, s_2)-dipath Q in $D[Z \cup S]$. Hence the digraph T obtained from T_2 by replacing the arc s_1s_2 by the dipath Q is an X-tripod in D.

Suppose now that D contains an X-tripod. Then it contains an unfolded X-tripod $T = (C, P_1, P_2, P_3)$ by Proposition 2.35. For $i = 1, 2, 3$, let y_i be the terminal vertex of P_i. Without loss of generality, we may assume that y_1, y_2, y_3 appear in this order along C. Since all $(X, Z \cup S)$-dipaths intersect S, one of the y_i, say y_3, must be in W. The three oriented paths P_2, $P_1C[y_1, y_2]$, and $\overline{C}[y_3y_2]$ are independent (W, y_2)-paths. But the underlying graph of D has no edges between W and Z, by the assumption made in the current subcase. So y_2 is in $W \cup S$. Similarly, y_1 is in $W \cup S$. It follows that $T \cap D[Z]$ is a dipath Q which is a subpath of one of the P_i or C. In addition, the in-neighbour in T of the initial vertex of Q is some vertex $s \in S$ (because there is no arc from W to Z) and the out-neighbour in T of the terminal vertex of Q is some vertex $s' \in S$ because there is no arc from Z to W). Furthermore $s \neq s'$ for otherwise $sQs' = C$ which is impossible as since $y_3 \in W \cap C$. Moreover, because sQs' is an (s, s')-dipath in $D[Z \cup S]$, ss' is an arc in D_2. Thus the digraph T_2 obtained from T by replacing sQs' by the arc ss' is an X-tripod in D_2. $\Diamond$

Now we may assume that there is an arc z_1w_1 with $z_1 \in Z$ and $w_1 \in W$. Since D is strong, there is a cycle C' containing the arc z_1w_1. Necessarily, the cycle C' must go through S and it contains at least three vertices.

Case 1: $S \subset V(C')$. Set $Y' = \{w_1, s_1, s_2\}$. We run a Menger algorithm between X and Y'. If such an algorithm finds three disjoint (X, Y')-dipaths P'_1, P'_2, P'_3, then we return the X-tripod (C', P'_1, P'_2, P'_3).

If not, we obtain a 2-separation (W', S', Z') of (X, Y'). We claim that $|W'| < |W|$. Indeed, no vertex $z \in Z$ is in W' because every (X, z)-dipath must go through S and thus through S'. Hence $W' \subseteq W - \{w_1\}$. Now, we replace C by C', Y by Y' and (W, S, Z) by (W', S', Z'), and then redo the procedure.

Case 2: $|S \cap V(C')| = 1$. Without loss of generality, we may assume $S \cap V(C') = \{s_1\}$. Set $Y' = \{w_1, s_1, z_1\}$. As in Case 1, we run a Menger algorithm between X and Y'. If such an algorithm finds three disjoint (X, Y')-dipaths P'_1, P'_2, P'_3, then we return the X-tripod (C', P'_1, P'_2, P'_3).

If not, the Menger algorithm returns a 2-separation (W', S', Z') for (X, Y'). Observe that there is a vertex $s'_1 \in S' \cap W$ because w_1 is reachable from X in $D[W]$. If S' contains a vertex s'_2 in Z, then one can see that there are no (X, Y')-dipaths in $D - \{s'_1, s_2\}$. Thus, there is a 2-separation (W'', S'', Z'') of (X, Y') where $S'' \subseteq \{s'_1, s_2\}$ and $s_1 \in Z''$. Hence, after possibly replacing the 2-separation (W', S', Z') by (W'', S'', Z''), we may assume that $S' \subset W \cup S$.

If $|W'| < |W|$, then we replace C by C', Y by Y' and (W, S, Z) by (W', S', Z') and redo the procedure.

If not, then the set $R = Z \cap W'$ is not empty. Set $L = Z - R = Z \cap Z'$. There is no arc from R to L, because (W', S', Z') is a 2-separation. Moreover, all (X, R)-dipaths must go through s_2. In particular, $s_2 \in W'$. Let D_3 be the digraph obtained from $D - L$ by adding an arc s_1w for every $w \in W$ having an in-neighbour in L. We then make a recursive call to `tripod`(D_3, X). This is valid by virtue of the following claim.

Claim 8. *There is an X-tripod in D if and only if there is an X-tripod in D_3.*

Subproof. Suppose first that D_3 contains an X-tripod. Then it contains an unfolded X-tripod T_3 by Proposition 2.35. If T_3 is contained in D, then we are done. So we may assume that T_3 is not contained in D. Then T_3 contains an arc in $s_1w \in A(D_3) \setminus A(D)$. It contains only one such arc since every vertex has out-degree at most one in T_3 and all arcs of $A(D_3) \setminus A(D)$ leave s_1. Furthermore the head w of this arc is in W and has an in-neighbour $z \in L$. Since D is strong, there is an (s_1, z)-dipath Q in D. Moreover since $s_2 \in W'$ all the (s_2, z)-dipaths must go through S'. But $S' \subseteq W \cup \{s_1\}$, so all (s_2, z)-dipaths must go through s_1. Thus Q does not go through s_2. It follows that all internal vertices of Q are in Z, because (W, S, Z) is a 2-separation, and so in L because there is no arc from R to L. Consequently, the digraph T obtained from T_3 by replacing the arc s_1w by the dipath Qzw is an X-tripod in D.

Suppose now that D contains an X-tripod. Then it contains an unfolded X-tripod $T = (C, P_1, P_2, P_3)$ by Proposition 2.35. For $i = 1, 2, 3$, let y_i be the terminal vertex of P_i. Without loss of generality, we may assume that y_1, y_2, y_3 appear in this order along C. If T is contained in $D - L$, then it is an X-tripod in D_3. Hence we may assume that T contains some vertices of L. Observe that the arcs entering L all leave s_1. Hence, y_i cannot be in L, since there are two (X, y_i)-dipaths in T, which are disjoint except for the common vertex y_i. Consequently, the intersection of T with $D[L]$ is a dipath Q which is a subpath of one of the P_i or C. Moreover, the in-neighbour in T of the initial vertex of Q is s_1 and the out-neighbour in T of the terminal vertex of Q is some vertex $w \in W \cup \{s_1\}$, because there is no arc from L to $R \cup \{s_2\}$. But $w \neq s_1$ for otherwise s_1Qs_1 would be C and would contain at most one of the y_i, a contradiction. Thus the digraph T_3 obtained from T by replacing s_1Qw by the arc s_1w is an X-tripod in D_3. ◊

Claims 6, 7 and 8 ensure that our algorithm is correct. Each time we do a recursive call, the number of vertices decreases. So we do at most n of them. Between two recursive calls, we first find a cycle of length at least 3 in linear time, and next run a sequence of Menger algorithms to produce a new 2-separation. At each step the size of the set W decreases. Therefore, we run at most n times the Menger algorithm between two recursive calls. Since a Menger algorithm runs in linear time, the time between two calls is at most $O(n \cdot (n+m))$ and so `tripod` runs in $O(n^2 \cdot (n+m))$ time. □

With Lemma 2.36 in hands, we now deduce Theorem 2.34.

Proof of Theorem 2.34. For every vertex v, we examine whether there is a W_3-subdivision with centre v in D. Observe that such a subdivision S is the union of a directed cycle C, and three internally disjoint (v, C)-dipaths P_1, P_2, P_3 with distinct terminal vertices y_1, y_2, y_3. The cycle C is contained in some strong component Γ of $D - v$. For $i = 1, 2, 3$, let x_i be the first vertex of P_i that belongs to Γ. Set

$X = \{x_1, x_2, x_3\}$. Then the paths $P_i[x_i, y_i]$, $i = 1, 2, 3$, and C form an X-tripod in Γ, and the $P_i[v, x_i]$, $i = 1, 2, 3$, are internally disjoint (v, X)-dipaths in $D-(\Gamma-X)$.

Hence for finding a W_3-subdivision with centre v, we use the following procedure to check whether there is a set X as above. First, we compute the strong components of $D - v$. Next, for every subset X of three vertices in the same strong component Γ, we run a Menger algorithm to check whether there are three independent (v, X)-dipaths in $D - (\Gamma - X)$. If yes, we check whether there is an X-tripod in Γ. If yes again, then we clearly have a W_3-subdivision with centre v, and we return 'yes'. If not, there is no such subdivision, and we proceed to the next triple.

For each vertex v, there are at most n^3 possible triples. And for each triple we run a Menger algorithm in time $O(n+m)$ and possibly `tripod` in time $O(n^2 \cdot (n+m))$. Hence the time spent on each vertex v is $O(n^5 \cdot (n + m))$. As we examine at most n vertices, the algorithm runs in $O(n^6 \cdot (n + m))$ time. □

2.9 Fans

The *fan* F_k is the graph obtained from the directed path P_k by adding a vertex, called the *centre*, dominated every vertex of P_k (it can also be defined as the wheels, with the center dominating the path, case in which F_k is W_k where one arc of the rim is deleted).

Theorem 2.37 (Bang-Jensen, Havet and M. [3]). *For all $k \geq 5$, F_k-*Subdivision *is NP-complete.*

Proof. Reduction from 2-Linkage in digraphs with no big vertices in which x_1 and x_2 are sources and y_1 and y_2 are sinks.

Let D, x_1, x_2, y_1 and y_2 be an instance of this problem. Let us denote by z the centre of F_k and by $(v_1, v_2, \dots, v_k)$ the directed path $F_k - z$. Let D_k be the digraph obtained from the disjoint union of D and F_k by removing the arcs v_1v_2 and v_3v_4 and adding the arcs v_1x_1, y_1v_2, v_3x_2 and y_2v_4.

We claim that D_k has an F_k-subdivision if and only if D has a linkage from (x_1, x_2) to (y_1, y_2).

Clearly, if there is a linkage (P_1, P_2) from (x_1, x_2) to (y_1, y_2) in D, then D_k contains an F_k-subdivision, obtained from F_k by replacing the arc v_1v_2 and v_3v_4 by the directed paths $(v_1, x_1) \cup P_1 \cup (y_1, v_2)$ and $(v_3, x_2) \cup P_2 \cup (y_2, v_4)$, respectively.

Suppose now that D_k contains an F_k-subdivision S in D_k. Since z is the unique vertex with in-degree k, the centre of S' is necessarily z. For $1 \leq i \leq k$, let v'_i be the vertex corresponding to v_i in S, and P_i be the directed (v'_i, z)-path in S.

Since z has in-degree exactly k in D_k, the v_i's are the penultimate vertices of the P_j's, each v_i on a different P_j. Since v_1 is a source in D_k, then $v_1 = v'_1$. Moreover, for $i = 3$ and $i \geq 5$, the path P'_j containing v_i must start at v_i because the unique in-neighbour of v_i is v_{i-1}. Hence $v_i = v'_j$. Furthermore, necessarily $v_{i-1} = v'_{j-1}$. Now, because v_k is a sink in $D_k - z$, then necessarily $v'_k = v_k$ and so for all $1 \leq i \leq k$, we have $v'_i = v_i$.

Let Q_1 and Q_2 be the directed (v_1, v_2)- and (v_3, v_4)-paths, respectively. Necessarily, the second vertex of Q_1 (resp. Q_2) is x_1, (resp. x_2) and its penultimate vertex is y_1 (resp. y_2). Hence $(Q_1[x_1, y_1], Q_2[x_2, y_2])$ is a linkage from (x_1, x_2) to (y_1, y_2) in D. □

Observe that F_2 is the $(1, 2)$-spindle. Thus F_2-Subdivision can be solved in $O(n^2 \cdot (n + m))$ time by Proposition 2.13. The next result shows that F_3-Subdivision is polynomial.

Let z be a vertex in a digraph D. A triple (x_1, x_2, x_3) is *F_3-nice* with respect to z in D if the following holds:

- x_1, x_2, x_3 are distinct vertices of $D - z$,
- x_3z is an arc,
- in $D - x_3$, there exist a directed (x_1, z)-path P_1 and a directed (x_2, z)-path P_2 which intersect only in z;
- in $D - \{x_3, z\}$, there is a directed (x_1, x_2)-path Q_1, and in $D - \{x_1, z\}$, there is a directed (x_2, x_3)-path Q_2.

Theorem 2.38 (Bang-Jensen, Havet and M. [3]). *A digraph contains an F_3-subdivision with centre z if and only if there is an F_3-nice triple with respect to z. In particular F_3-Subdivision is polynomial-time solvable.*

Proof. Trivially, if D contains an F_3-subdivision with centre z, then it contains an F_3-nice triple (x_1, x_2, x_3) with respect to z.

Conversely, assume that D contains an F_3-nice triple (x_1, x_2, x_3) with respect to z. Let P_1, P_2, Q_1 and Q_2 be the directed paths as defined in the definition of F_3-nice triple. We may assume that (x_1, x_2, x_3) is an F_3-nice triple (x_1, x_2, x_3) with respect to z that minimizes $\ell = \ell(P_1) + \ell(P_2) + \ell(Q_1) + \ell(Q_2)$, that is the sum of the lengths of these paths.

We shall prove that P_1, P_2, Q_1 and Q_2 are internally disjoint, implying that these paths and the arc x_3z form an F_3-subdivision with centre z.

a) Let us prove that Q_2 and P_1 are internally disjoint. Suppose not. Then let x_2' be the last vertex on Q_2 which also belongs to P_1. Then (x_2, x_2', x_3) is F_3-nice by the choice of paths $P_1' = P_2$, $P_2' = P_1[x_2', z]$, $Q_1' = Q_2[x_2, x_2']$ and $Q_2' = Q_2[x_2', x_3]$. Indeed, P_1' and P_2' are internally disjoint because P_1 and P_2 were, Q_1' does not go through x_3 nor z, because Q_2 is a directed (x_2, x_3)-path in $D - z$, and Q_2' does not go through x_2 nor z,for the same reason. This contradicts the minimality of ℓ.

b) Let us prove that Q_2 and P_2 are internally disjoint. Suppose not. Then let x_2' be the last vertex on Q_2 which also belongs to P_2. One easily verifies that (x_1, x_2', x_3) is F_3-nice by the choice of paths $P_1' = P_1$, $P_2' = P_2[x_2', z]$, Q_1' a directed (x_1, x'_2)-path included in $Q_1[x_1, x_2]Q_2[x_2, x'_2]$ (which can be a walk), and $Q'_2 = Q_2[x'_2, x_3]$. This contradicts the minimality of ℓ.

c) Let us prove that Q_1 and P_1 are internally disjoint. Suppose not. Then let x'_1 be the last vertex on Q_1 which also belongs to P_1. The path Q_2 does not go through x'_1 because Q_2 and P_1 are internally disjoint. Thus (x'_1, x_2, x_3) is F_3-nice with associated paths $P'_1 = P_1[x'_1, z]$, $P'_2 = P_2$, $Q'_1 = Q_1[x'_1, x_2]$, and $Q'_2 = Q_2$. This contradicts the minimality of ℓ.

d) Let us prove that Q_1 and P_2 are internally disjoint. Suppose not. Then let x'_2 be the last internal vertex on Q_1 which also belongs to P_2. Then (x_1, x'_2, x_3) is F_3-nice with associated paths $P'_1 = P_1$, $P'_2 = P_2[x'_2, z]$, $Q'_1 = Q_1[x_1, x'_2]$, and Q'_2 a directed (x_1, x'_2)-path included in $Q_1[x'_2, x_2]Q_2$ (which can be a walk). This contradicts the minimality of ℓ.

e) Let us prove that Q_1 and Q_2 are internally disjoint. Suppose not. Then let x'_2 be the last internal vertex on Q_2 which also belongs to Q_1. Then (x_1, x'_2, x_3) is a good triple with associated paths $P'_1 = P_1$, $P'_2 = Q_1[x'_2, x_2]P_2$, $Q'_1 = Q_1[x_1, x'_2]$, and $Q'_2 = Q_2[x'_2, x_3]$. Indeed, since P_2 and Q_1 are internally disjoint, P'_2 is a path, and since P_1 and Q_1 are internally disjoint, the paths P'_1 and P'_2 are also internally disjoint.

$\square$

Proposition 2.13 and Theorems 2.38 and 2.37 determine the complexity of F_k-Subdivision for all k except 4. So we are left with the following problem.

Problem 2.39. *What is the complexity of* F_4-Subdivision *?*

2.10 Tournaments

A *tournament* is an orientation of a complete graph. We denote by TT_k the acyclic tournament on k vertices, frequently called *transitive tournament*. The strongly connected tournament on k vertices, also referred to as *strong tournament*, is denoted by ST_k. Figure 2.3 shows a representation of the strong tournament on four vertices.

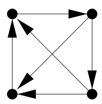


Figure 2.3: The strong tournament ST_4

Let T be a tournament. If $|T| \geq 5$, T-Subdivision is NP-complete by Corollary 1.12. On the other hand, if $|T| \leq 3$, T-Subdivision is polynomial-time solvable because the tournaments with 1 and 2 vertices are spiders, and those with 3 vertices are the TT_3, which is the $(1, 2)$-spindle, and the ST_3, which is the directed cycle C_3.

The tournaments of order 4 are ST_4, TT_4, W_3 and its converse. The last two were proved to be tractable on the previous section (Theorem 2.34). We shall now prove that ST_4 and TT_4-Subdivision are polynomial-time solvable.

Let us start by ST_4. We need the following auxiliary case to prove that ST_4-Subdivision is tractable. Consider the digraph W_2'-Subdivision, depicted in Figure 2.4. Using Lemmas 1.35 and 2.33, one can easily give a polynomial-time algorithm to solve it.

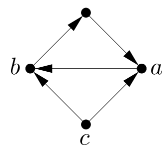


Figure 2.4: The digraph W_2'

Theorem 2.40 (Havet, M. and Mohar)**.** W_2'-Subdivision *can be solved in* $O(n^5(n+m))$ *time.*

Proof. By Lemma 2.33, applied to every vertex c' in D, for every vertex a' of D, one can decide in $O(n^3(n+m))$ if there is a W_2-subdivision with b-vertex b'. Observe that W_2' is obtained from W_2 by subdividing once the arc ba, which is the only arc leaving b. Therefore, Lemma 1.35 applies. Thus for every vertex b' one can decide in $O\left(\sum_{y\in N^+(b')} d^+(y)n^3(n+m)\right)$ if there is a W_2'-subdivision with b-vertex b'. Hence, one can decide in $O(n^5(n+m))$ time whether there is a W_2'-subdivision in D. □

We now give a more complicated but faster algorithm based on an algorithm deciding if there is a W_2'-subdivision with prescribed c-vertex. This proof uses in a simpler way the technique that we use in for proving that W_3 is tractable.

Theorem 2.41. *Given a digraph D and a vertex v of D, one can decide in $O(n^3(n+m))$ time if D contains a W_2'-subdivision with centre v. So W_2'-*Subdivision *can be solved in $O(n^4(n+m))$ time.*

The proof of this theorem relies on the following notion. Let X be a set of two vertices. An X*-bipod* is a digraph which is the union of a directed cycle C of length at least 3 and two disjoint dipaths P_1 and P_2 with initial vertices in X and terminal vertices in C. If the P_i are (X,C)-dipaths, we say that the bipod is *unfolded.* Note that the dipaths P_i may be of length 0. We often denote a bipod by the triple (C, P_1, P_2) described above.

Proposition 2.42. *Let $X = \{x_1, x_2\}$ be a set of two distinct vertices. Any X-bipod contains an unfolded X-bipod.*

Proof. Let (C, P_1, P_2) be an X-bipod, where P_i has initial vertex x_i, for $i = 1, 2$. Let y_i be the first vertex on C along P_i. Then $(C, P_1[x_1, y_1], P_2[x_2, y_2])$ is an unfolded X-bipod. □

We shall consider the following decision problem.

BIPOD
Input: A strong digraph D and a set X of two distinct vertices of D.
Question: Does D contain an X-bipod?

Lemma 2.43. BIPOD *can be solved in $O(n(n+m))$ time.*

Proof. Let us describe a procedure `bipod`(D, X), solving BIPOD.

We first look for a directed cycle of length at least 3 in D. This can be done in linear time. If there is no such cycle, then we return 'no'.

Otherwise we have a directed cycle C of length at least 3. We choose a set Y of two vertices in C and run a Menger algorithm between X and Y. If this algorithm finds two disjoint (X, Y)-dipaths P_1, P_2, then we return the bipod (C, P_1, P_2). Otherwise, the Menger algorithm finds a 1-separation (W, S, Z) of (X, Y). Note that $|S| = 1$ because D is strong. Set $S = \{s\}$.

Let D' be the digraph obtained from D by contracting Z into a vertex t. Note that D' is strong. We now make a recursive call to `bipod`(D', X). This is valid by virtue of the following claim.

Claim 9. *There is an X-bipod in D if and only if there is an X-bipod in D'.*

Subproof. Suppose first that there is an X-bipod in D'. Then D' contains an unfolded X-bipod B' by Proposition 2.42. If B' is contained in D, then we are done. So we may assume that it is not. Then B' contains a dipath stw for some $w \in W$. It contains only one such dipath since every vertex has out-degree at most one in B'. Moreover, t has in-degree 1 in D', so it has in-degree 1 also in B'. Since t was obtained by contraction of Z, w has an in-neighbour $z \in Z$. Now, since D is strong, there is an (s, z)-dipath Q in D. Because there is no arc from W to Z, all the internal vertices of Q are in Z. Hence the digraph B obtained from B' by replacing the dipath stw by the dipath Qzw is an X-bipod in D.

Suppose now that D contains an X-bipod. Then it contains an unfolded X-bipod $B = (C, P_1, P_2)$ by Proposition 2.42. Since all (X, Z)-dipaths in D go through s, the terminal vertices of the P_i are in $W \cup S$, and $D[Z] \cap B$ is a dipath Q which is a subpath of one of the P_i or C. If Q is a (u, z)-dipath, then B contains arcs su and zw for some $w \in W$. Then the digraph B' obtained from B by replacing $suQzw$ by the dipath stw is an X-bipod in D'. Indeed, if Q was a subpath of C, then the directed cycle in B' has length at least 3, as it contains the three vertices s, t and w. ◊

Each time we do a recursive call, the number of vertices decreases. So we do at most n of them. Between two recursive calls, we search for a directed cycle of length at least 3 and run a Menger algorithm. Both can be done in linear time. So `bipod` runs in $O(n(n+m))$ time. □

With Lemma 2.43 in hands, we now deduce Theorem 2.41.

Proof of Theorem 2.41. Let v be a vertex of D. Let us describe an algorithm that decides whether there is a W_2'-subdivision with centre v in D. Observe that such a subdivision S is the union of a directed cycle C of length at least 3, and two internally disjoint (v,C)-dipaths P_1, P_2 with distinct terminal vertices y_1, y_2. Since it is strong, the cycle C is contained in some strong component Γ of $D-v$. For $i=1,2$ let x_i be the first vertex of P_i in Γ. Set $X=\{x_1,x_2\}$. Then the paths $P_i[x_i,y_i]$, $i=1,2$, and C form an X-bipod in Γ, and the $P_i[v,x_i]$, $i=1,2$, are independent (v,X)-dipaths in $D-(\Gamma\setminus X)$. Hence for finding a W_2'-subdivision with centre v, the following procedure checks whether there is a set X as above. First, we compute the strong components of $D-v$. Next, for every subset X of two vertices in the same strong component Γ, we run a Menger algorithm to check whether there are two independent (v,X)-dipaths in $D-(\Gamma\setminus X)$. If yes, we check using `bipod` whether there is an X-bipod in Γ. If yes again, then we clearly have a W_2'-subdivision with centre v, and we return 'yes'. Otherwise, there is no such subdivision, and we proceed to the next pair.

There are at most n^2 possible pairs X. And for each pair we run a Menger algorithm in $O(n+m)$ time and possibly `bipod` in $O(n(n+m))$ time. Hence our algorithm decides whether there is W_2'-subdivision with centre v in D in $O(n^3(n+m))$ time.

To solve W_2'-subdivision, we check for every vertex v in turn if there is a W_2'-subdivision with centre v. As we examine at most n vertices, this algorithm runs in $O(n^4(n+m))$ time. □

We also shall need the following two lemmas.

Lemma 2.44 (Havet, M. and Mohar)**.** *Let D be a digraph, C a directed cycle in D, and x a vertex in $V(D)-V(C)$. If there are two (x,C)-dipaths P_1 and P_2 and a (C,x)-dipath Q such that $s(Q)$, $t(P_1)$ and $t(P_2)$ are distinct, then D contains an ST_4-subdivision.*

Proof. Assume first that P_1 and P_2 are independent. If $Q\cap(P_1^\circ\cup P_2^\circ)=\emptyset$, then $C\cup P_1\cup P_2\cup Q$ is an ST_4-subdivision, for instance with $t(P_1)$, $t(P_2)$, $s(Q)$ and x as a,b,c,d-vertex, respectively. If Q intersects $P_1^\circ\cup P_2^\circ$, then without loss of generality, we may assume that the first vertex y along Q in $P_1^\circ\cup P_2^\circ$ is on P_1°. Let z_2 be the first vertex in $V(Q)\cap V(P_2)$ along Q. Such a vertex exists because $x\in V(Q)\cap V(P_2)$. Let z_1 be the last vertex on $Q[y,z_2]$ which is on $P_1[y,t(P_1)]$. Now $Q[z_1,z_2]\cup P_1[y,t(P_1)]\cup Q[s(Q),y]\cup P_2[z_2,t(P_2)]\cup C$ is an ST_4-subdivision, for instance with $t(P_2)$, $t(P_1)$, $s(Q)$ and z_1 as a,b,c,d-vertex, respectively.

Assume now that P_1 and P_2 are not independent. Let x' be the last vertex in $P_1\cap P_2$ along P_1. Set $P_1'=P_1[x',t(P_1)]$, $P_2'=P_2[x',t(P_2)]$, and let Q' be the $(s(Q),x')$-dipath contained in the walk $QP_2[x,x']$. Then P_1' and P_2' are independent (x',C)-dipaths. Hence by the previous case, D contains an ST_4-subdivision. □

Lemma 2.45 (Havet, M. and Mohar)**.** *Let D be a strong digraph, C a directed cycle in D, x a vertex in $V(D-C)$. If there are three (x,C)-dipaths with distinct terminal vertices, then D contains an ST_4-subdivision.*

Subproof. Suppose that there are three (x, C)-dipaths P_1, P_2, P_3 such that $t(P_1)$, $t(P_2)$, and $t(P_3)$ are distinct. Since D is strong, there is a (C, x)-dipath Q. Without loss of generality, we may assume that $s(Q) \notin \{t(P_1), t(P_2)\}$. Thus by Lemma 2.44, D contains an ST_4-subdivision. $\diamondsuit$

Theorem 2.46 (Havet, M. and Mohar)**.** *ST_4-*Subdivision *can be solved in $O(n^5 \cdot (n+m))$ time.*

Proof. Since ST_4 is strong, its subdivisions are also strong. So we only need to prove the result for a strong input digraph D; if the digraph is non-strong, it suffices to check whether one of its strong components contains an ST_4-subdivision.

We shall describe a procedure ST_4-`Subdivision`(D, d'), that, given a strong digraph D and a vertex d', returns 'no' only if there is no ST_4-subdivision in D with d-vertex d' and returns 'yes' when it finds an ST_4-subdivision (not necessarily with d-vertex d'). Running this procedure for every vertex d' yields an algorithm to decide whether D contains an ST_4-subdivision; in addition, the algorithm runs in $O(n^5 \cdot (n+m))$ time, because the procedure ST_4-`Subdivision`(D, d') only needs $O(n^4 \cdot (n+m))$ time.

First, we check whether d' is the centre of a W_2'-subdivision. This can be done in $O(n^3 \cdot (n+m))$ time, according to Theorem 2.41. If not, then we return 'no' since every ST_4-subdivision with d-vertex d' contains a W_2'-subdivision with centre d'.

If there is a W_2'-subdivision with centre d', let us denote by C its directed cycle, and by P_1 and P_2 the two (d', C)-dipaths in it. For $i = 1, 2$, let x_i be the terminal vertex of P_i.

Let S^- and S^+ be the in-section and out-section, respectively, of d' in $D - \{x_1, x_2\}$. We compute S^- and S^+. If S^- contains a vertex in $V(C) - \{x_1, x_2\}$, then there is a (C, d')-dipath Q with initial vertex $x_3 \notin \{x_1, x_2\}$. So, by Lemma 2.44, there is an ST_4-subdivision in D, and we return 'yes'. Similarly, because of Lemma 2.45, we return 'yes' if S^+ contains a vertex in $V(C) - \{x_1, x_2\}$.

Assume now that $(S^- \cup S^+) \cap (V(C) - \{x_1, x_2\}) = \emptyset$. By the definition of out-section, no arc is leaving S^+ in $D - \{x_1, x_2\}$, so in D every arc leaving S^+ has its head in $\{x_1, x_2\}$. Similarly, all arcs entering S^- have tail in $\{x_1, x_2\}$. Moreover, because D is strong, for every vertex $s \in S^+$, there is an $(s, \{x_1, x_2\})$-dipath in $D[S^+ \cup \{x_1, x_2\}]$.

Since D is strong, there is a directed (C, d')-dipath in D. Its first arc goes from $\{x_1, x_2\}$ to S^-. Hence at least one vertex of $\{x_1, x_2\}$ has an out-neighbour in S^-.

Claim 10. *Suppose both x_1 and x_2 have an out-neighbour in S^-. If there is a (C, S^+)-dipath R with $s(R) \notin \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

Subproof. There is a $(t(R), \{x_1, x_2\})$-dipath P with internal vertices in S^+. Without loss of generality, we may assume that $t(P) = x_1$. Since x_2 has an out-neighbour in S^-, there is an $(x_2, t(R))$-dipath Q whose internal vertices are in $D - C$. Hence by the directional dual of Lemma 2.44 (ST_4 is isomorphic to its converse), D contains an ST_4-subdivision. $\diamondsuit$

Each x_i has an in-neighbour in P_i, and so an in-neighbour in S^+. Hence a similar reasoning as the proof of Claim 10 gives the following.

Claim 11. *If there is an (S^-, C)-dipath with terminal vertex $x_3 \notin \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

For $i = 1, 2$, let S_i^+ be the set of vertices s of S^+ for which there is an (s, x_i)-dipath with internal vertices in $V(D - C)$. In the very same way as Claim 10, one can prove the following claim.

Claim 12. *Suppose x_i has no out-neighbour in S^-. If there is a (C, S_i^+)-dipath with initial vertex $x_3 \notin \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

Case 1: Assume first that both x_1 and x_2 have an out-neighbour in S^-.

Let T^+ be the out-section of S^- in $D - \{x_1, x_2\}$, T^- the in-section of S^+ in $D - \{x_1, x_2\}$. The definition of T implies the following property:

(T_1) If $u \in V(D) - (T \cup \{x_1, x_2\})$ and Q is a (u, d')-path in D with at most two blocks, then Q contains a vertex in $\{x_1, x_2\}$.

Now, we compute $T = T^- \cup T^+$. If T contains a vertex of $V(C) - \{x_1, x_2\}$, then we return 'yes', since D contains an ST_4-subdivision by Claim 10 or 11. If not, then $T \cap (V(C) - \{x_1, x_2\}) = \emptyset$. Let D' be the digraph obtained from $D[T \cup \{x_1, x_2\}]$ by adding the arcs x_1x_2 and x_2x_1 if they were not in $A(D)$. Observe that D' has fewer vertices than D, because the vertices of $V(C) - \{x_1, x_2\}$ are not in $V(D')$ and this set is not empty because C has length at least 3. We then return ST_4-`Subdivision`(D', d'). The validity of this recursive call is established by the following claim.

Claim 13. *D contains an ST_4-subdivision with d-vertex d' if and only if D' does.*

Subproof. From every ST_4-subdivision in D' with d-vertex d', one can obtain an ST_4-subdivision in D with d-vertex d' by replacing the arc x_1x_2 (resp. x_2x_1) by $C[x_1, x_2]$ (resp. $C[x_2, x_1]$).

Assume now that D contains an ST_4-subdivision S with d-vertex d'. Let a', b', and c' be the vertices in S corresponding to a, b, and c, respectively.

Our first goal is to prove that $a', b', c' \in V(D')$. Let $u \in \{a', b', c'\}$ be one of these three vertices. Note that there are three internally disjoint paths in S joining u with d', and each of these paths has at most two blocks. If $u \notin V(D')$, then Property (T_1) stated above implies that each of these paths contains x_1 or x_2 as one of its internal vertices. Since the three paths are internally disjoint, this is not possible, and we conclude that $u \in V(D')$.

Hence, a', b', c', d' all belong to $V(D')$. Therefore, the intersection of S with $V(D) - T$ is a dipath P whose initial vertex is dominated by $x \in \{x_1, x_2\}$ and whose terminal vertex dominates the vertex x' of $\{x_1, x_2\} - \{x\}$. Hence, D' contains the ST_4-subdivision obtained from S by replacing xPx' by xx'. $\Diamond$

Case 2: Assume that one vertex in $\{x_1, x_2\}$, say x_1, has no out-neighbour in S^-.

Let T^+ be the out-section of S^- in $D - \{x_1, x_2\}$, T_1^- the in-section of S_1^+ in $D - \{x_1, x_2\}$ and $T = T^+ \cup T_1^-$. Observe that $S^+ \subseteq T$ because $d' \in S^-$. The definition of T implies the following property:

(T$_2$) If $u \in V(D) - (T \cup \{x_1, x_2\})$ and Q is a (u, d')-path with at most two blocks, then either Q contains a vertex in $\{x_1, x_2\}$, or Q has two blocks and there is a vertex $v \in S_2^+ - (S_1^+ \cup S^-)$ such that Q is composed of a (u, v)-dipath R_1 and a (d', v)-dipath R_2.

An ST_4-subdivision S is *special* if its d-vertex is d', its c-vertex is x_2, its a-vertex is not in $T \cup \{x_1, x_2\}$, and $x_1 \in V(S)$.

We check if D contains a special ST_4-subdivision. To do so, we check for every vertex a' in $V(D)-(T\cup\{x_1, x_2\})$, if there are two independent $(\{x_1, x_2\}, a')$-dipaths Q_1 and Q_2 in D and an (a', S^+)-dipath R in $D - \{x_1, x_2\}$. If we find a vertex $a' \in V(D) - (T \cup \{x_1, x_2\})$ such that three such dipaths exist, we return 'yes'. This is valid by the following claim.

Claim 14. *Let $a' \in V(D) - (T \cup \{x_1, x_2\})$. If there are two independent $(\{x_1, x_2\}, a')$-dipaths Q_1 and Q_2 in D and an (a', S^+)-dipath R in $D - \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

Subproof. The vertex $t(R)$ is in $S_2^+ - S_1^+$ because $a' \notin T_1^-$. Thus, there is a $(t(R), x_2)$-dipath R_1 with internal vertices in S^+. Let y_2 be an out-neighbour of x_2 in S^-. Since $a' \notin S^-$, the vertex y_2 is not on R_1. By definition of S^+ and S^-, there is a (y_2, R_1)-dipath R_2 in $D[S^+ \cup S^-]$.

Let C' be the directed cycle $x_2y_2 \cup R_2 \cup R_1[t(R_2), x_2]$. Since $y_2 \in S^-$, there is a directed (y_2, x_1)-dipath R_3 in $D[S^- \cup S^+ \cup \{x_1\}]$. This dipath does not intersect R_1 because $V(R_1)-\{x_2\} \subseteq S_2^+ - S_1^+$. Let z_2 be the last vertex along R_3 that lies in C'. The three vertices $x_2, z_2, t(R_2)$ are distinct. Moreover, the two dipaths Q_1 and Q_2 do not intersect C' for otherwise there would be a (y_2, a')-dipath in $D-\{x_1, x_2\}$ and a' would be in T^+. Thus $R_3 \cup Q_1$ contains a (y_3, a')-dipath R_3^* which is a (C', a')-dipath. Hence, we have two (C', a')-dipaths R_3^* and Q_2 and the (a', C')-dipath R' contained in $R \cup R_1[s(R_1), t(R_2)]$ whose vertices $s(R_3^*)$, $s(Q_2) = x_2$ and $t(R')$ on C' are pairwise distinct. Hence, by Lemma 2.44, D contains an ST_4-subdivision. $\Diamond$

If for every a' in $V(D) - (T \cup \{x_1, x_2\})$, three dipaths Q_1, Q_2, R as used above do not exist, then D has no special ST_4-subdivision, which we will assume henceforth. Let D' be the digraph obtained from $D[T \cup \{x_1, x_2\}]$ by adding the arcs x_1x_2 and x_2x_1 if they were not in $A(D)$. Observe that D' has fewer vertices than D, because the vertices of $V(C) - \{x_1, x_2\}$ are not in $V(D')$. We then return ST_4-`Subdivision`(D', d'). The validity of this recursive call is established by the following claim.

Claim 15. *D contains an ST_4-subdivision with d-vertex d' if and only if D' does.*

Subproof. From every ST_4-subdivision S' with d-vertex d' in D', one can obtain an ST_4-subdivision with d-vertex d' in D by replacing the arc x_1x_2 (resp. x_2x_1) by $C[x_1, x_2]$ (resp. $C[x_2, x_1]$).

Assume now that D contains an ST_4-subdivision S with d-vertex d'. Let a', b', and c' be the vertices in S corresponding to a, b, and c, respectively. Each arc in ST_4 corresponds to a dipath in S. We will denote these dipaths by $S[a', b']$, $S[b', c']$, etc.

Observe that in S, there are three internally disjoint directed paths (in both directions) between b' and d'. So $b' \in V(D')$, because directed paths between $V(D) - V(D')$ and d' must go through $\{x_1, x_2\}$ by Property (T$_2$).

Next, we claim that $a' \in V(D')$. Suppose for a contradiction that $a' \notin V(D')$. Then both paths $S[d', a']$ and $S[c', a'] \cup S[c', d']$ must go through $\{x_1, x_2\}$ by Property (T$_2$). The path $S[a', b'] \cup S[d', b']$ is thus disjoint from $\{x_1, x_2\}$, and by (T$_2$) we have that $b' \in S_2^+ - (S_1^+ \cup S^-)$. The path $S[b', c'] \cup S[c', d']$ must go through x_2 since $b' \notin S^- \cup S_1^+$. Thus, x_2 lies on $S[c', d']$. Since there is no special ST_4-subdivision in D, $c' \neq x_2$. Hence, $S[c', a']$ does not meet $\{x_1, x_2\}$, and the path $S[d', b'] \cup S[b', c'] \cup S[c', a']$ shows that $a' \in S^+$, a contradiction.

Let us prove that $c' \in V(D')$. Suppose for a contradiction that $c' \notin V(D')$. Then $c' \notin \{x_1, x_2\}$ and both, $S[d', b'] \cup S[b', c']$ and $S[c', d']$ must go through $\{x_1, x_2\}$. Moreover, x_2 is in $S[c', d']$ because x_1 has no out-neighbour in S^-. Since x_2 is also on $S[d', a'] \cup S[a', b'] \cup S[b', c']$, we conclude that $x_2 \in S[b', c']$. Now, the path $S[d', a'] \cup S[c', a']$ gives a contradiction to the property (T$_2$).

We have shown that $\{a', b', c', d'\} \subseteq V(D')$. Therefore, the part of S outside D' is a directed path P whose initial vertex is dominated by $x \in \{x_1, x_2\}$ and whose terminal vertex dominates the vertex x' of $\{x_1, x_2\} - \{x\}$. Hence, D' contains the ST_4-subdivision obtained from S by replacing xPx' by xx'. $\Diamond$

Each time we do a recursive call, the number of vertices decreases. So we do at most n of them. Between two recursive calls, we search for a W_2'-subdivision, which can be done in $O(n^3 \cdot (n+m))$ by Theorem 2.41, and we compute some out-sections and in-section, which can be done in linear time. The only part that may need more time is in Case 2, when we check for every a' in $V(D) - (T \cup \{x_1, x_2\})$ if D contains a special ST_4-subdivision. Each such test needs linear time by Claim 14. During this procedure, we either discover an ST_4-subdivision or not. If yes, we have spent at most $O(n \cdot (n + m))$ time for completing this task. Otherwise we spend linear time per vertex a', which is henceforth omitted when we proceed with the recursive call. This shows that ST_4-`Subdivision` runs in $O(n^4 \cdot (n + m))$ time. $\square$

For any non-negative integer p, let $TT_4(p)$ be the digraph obtained from TT_4 with source u and sink v by adding p new vertices dominated by u and dominating v. In particular, $TT_4(0) = TT_4$. We denote by $TT_4^*(p)$, the digraph obtained from $TT_4(p)$ by deleting the arc from its source u to its sink v. For simplicity, we abbreviate $TT_4^*(0)$ in TT_4^*.

We need the following definitions. Let X be a set of vertices in a digraph D.

The *out-section* generated by X in D is the set of vertices y to which there exists a directed path (possibly restricted to a single vertex) from $x \in X$; we denote this set by $S_D^+(X)$. For simplicity, we write $S_D^+(x)$ instead of $S_D^+(\{x\})$. The dual notion, the *in-section*, is denoted by $S_D^-(X)$. Note that the out-section and the in-section of a set may be found in linear time by any tree-search algorithm.

Theorem 2.47 (Bang-Jensen, Havet and M. [3])**.** *For every non-negative integer p, one can solve $TT_4(p)$-*Subdivision *in $O(n^3 \cdot (n+m))$-time.*

Proof. Let D be a digraph and let u and v be two distinct vertices of D. We shall describe a $O(n \cdot (n+m))$-time algorithm for finding a $TT_4(p)$-subdivision in D with source u and sink v, if one exists.

Observe that all vertices in such a subdivision are in $S_D^+(u) \cap S_D^-(v)$, hence we can restrict our search to the digraph D' induced by this set.

Then, using a maximum flow algorithm, we can find in D' a set of internally disjoint directed (u,v)-paths of maximum size in $O(n \cdot (n+m))$-time. Let $(P_1, \dots, P_k)$ denote this set. If $k < p+3$, then return 'no', because in any $TT_4(p)$-subdivision with source u and sink v, there are $p+3$ internally disjoint directed (u,v)-paths. Hence, we now assume that $k \geq 3$.

For $1 \leq i \leq k$, set $Q_i = P_i - \{u,v\}$, and set $H = D' - \{u,v\}$. For every vertex x in $V(H)$, we compute $S(x) = S_H^-(x) \cup S_H^+(x)$, and deduce $I(x) = \{i \mid V(Q_i) \cap S(x) \neq \emptyset\}$. If there exists x, such that $|I(x)| \geq 2$, then return 'yes'. Otherwise return 'no'.

The validity of this algorithm is proved by Claim 17.

Claim 16. *For all $x \in V(H)$, $I(x) \neq \emptyset$.*

Subproof. In D', there are directed (u,x)- and (x,v)-paths, whose concatenation contains a directed (u,v)-path R. Since $(P_1, \dots, P_k)$ is a set of internally disjoint directed (u,v)-paths of maximum size, $R - \{u,v\}$ must intersect one of the Q_i's, say Q_{i_0}. By definition, $V(R) - \{u,v\} \subseteq S(x)$, so $i_0 \in I(x)$. $\Diamond$

Claim 17. *D' contains a $TT_4(p)$-subdivision with source u and sink v if and only if there exists $x \in V(H)$ such that $|I(x)| \geq 2$.*

Subproof. Assume that $|I(x)| \geq 2$. Without loss of generality, $\{1,2\} \subset I(x)$. We shall prove that D' contains a $TT_4(p)$-subdivision with source u and sink v.

- Suppose first that $S_H^-(x) \cap Q_1 \neq \emptyset$ and $S_H^+(x) \cap Q_2 \neq \emptyset$. Then there is a directed (Q_1,x)-path and a directed (x,Q_2)-path whose concatenation contains a directed (Q_1,Q_2)-path R. Let y be the first vertex on R in $\bigcup_{i=2}^k Q_i$. Free to swap the names of Q_2 and the path Q_l containing y and taking the subpath of R from its origin to y instead of R, we may assume that y is the last vertex of R. Now the union of $P_1, \dots, P_{p+3}$, and R form a $TT_4(p)$-subdivision.

- If $S_H^-(x) \cap Q_2 \neq \emptyset$ and $S_H^+(x) \cap Q_1 \neq \emptyset$, the proof is similar to the previous case.

- Suppose now that $S^+_H(x) \cap Q_1 \neq \emptyset$ and $S^+_H(x) \cap Q_2 \neq \emptyset$. We may assume that $S^-_H(x) \cap \bigcup_{i=1}^k Q_i = \emptyset$, otherwise we are in one of the previous cases, and we get the result. Let R be a shortest (u, x)-path in D'. Then every vertex in $R - u$ is a vertex of $H - \bigcup_{i=1}^k Q_i$.

  Let S_1 be a shortest directed (x, Q_1)-path and S_2 be a shortest directed (x, Q_2)-path. For $i = 1, 2$, let z_i be the terminus of S_i. We may assume that all the internal vertices of S_1 and S_2 are in $H - \bigcup_{i=1}^k Q_i$ for otherwise one vertex z among z_1 and z_2 satisfies the condition of one of the previous cases (up to a permutation of the labels). Then the union of paths $P_2, \dots, P_{p+3}$, R, S_1, S_2 and $P_1[z_1, v]$ form a $TT_4(p)$-subdivision.

- If $S^-_H(x) \cap Q_1 \neq \emptyset$ and $S^-_H(x) \cap Q_2 \neq \emptyset$, the proof is similar to the previous case by directional symmetry.

Assume now that $|I(x)| < 2$ for all $x \in V(H)$. Then, by Claim 16, $|I(x)| = 1$ for all $x \in V(H)$. For $1 \leq i \leq k$, let $V_i = \{x \mid I(x) = \{i\}\}$. Then $(V_1, \dots V_k)$ is a partition of $V(H)$. Moreover, by definition, there is no arc between two distinct parts of this partition. In addition, in $D'\langle X_i \cup \{u, v\}\rangle$, there cannot be two internally disjoint directed (u, v)-paths, for otherwise it would contradict the maximality of $(P_1, \dots, P_k)$. Hence, D' contains no TT^*_4-subdivision, and so no $TT_4(p)$-subdivision. $\Diamond$

This finishes the proof of Theorem 2.47. □

Corollary 2.48 (Bang-Jensen, Havet and M. [3]). *For any non-negative integer p, the $TT^*_4(p)$-Subdivision problem can be solved in $O(n^3 \cdot (n + m))$.*

Proof. Observe that a graph D contains a $TT^*_4(p)$-subdivision with source u and sink v, if and only if the graph $D \cup \{uv\}$ contains a $TT_4(p)$-subdivision. Hence by just adding the arc uv to D if it does not exists in the above algorithm, we obtain a polynomial-time algorithm for $TT^*_4(p)$-Subdivision. □

2.11 $K_{3,3}$

A natural way to start trying to prove Conjecture 1.16 is look at what happens for subdivisions of orientations of K_5 and $K_{3,3}$. The first one is NP-complete in any case by Corollary 1.12. We prove next that the subdivision of any orientation of $K_{3,3}$ with at least one big vertex (Figure 2.5) is hard.

Proposition 2.49 (Bang-Jensen, Havet and M). *If F is an orientation of a $K_{3,3}$ with at least one big vertex, then F-Subdivision is NP-complete.*

Proof. An orientation of $K_{3,3}$ having at least on big vertex is one of the F_i, $0 \leq i \leq 9$, or the converse of one of them. F_0 and F_1 are hard by Theorem 1.10. In each other case, the problem is proved to be NP-complete by reduction from Restricted 2-Linkage. Let D, x_1, x_2, y_1 and y_2 be an instance of this problem. We construct a

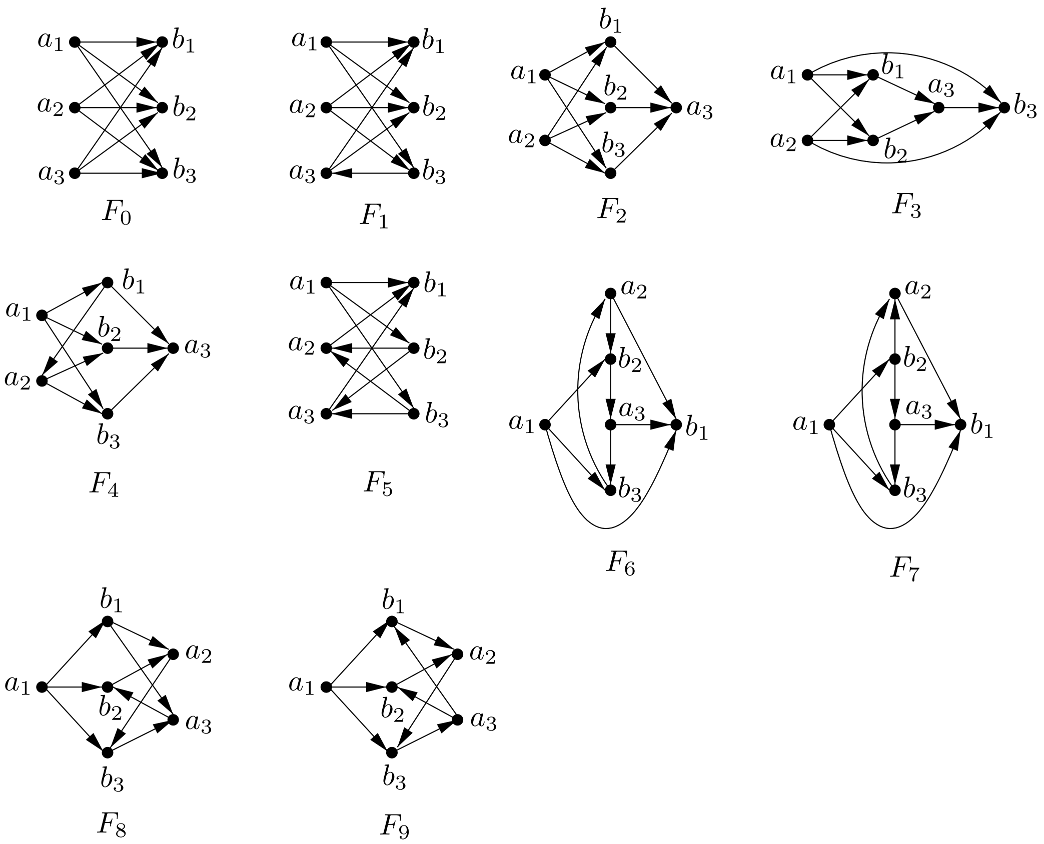


Figure 2.5: Orientations of $K_{3,3}$.

digraph D_i by placing D on two arcs $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of F_i (to be specified later), that is, by taking the disjoint union of D and F_i, removing the arcs e_1, e_2 of F_i and adding the arcs u_1x_1, y_1v_1, u_2x_2 and y_2v_2. We then show that D_i contains an F_i-subdivision if and only if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D. This implies that F_i-Subdivision is NP-complete.

Clearly, by construction of D_i, if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D, then D_i contains an F_i-subdivision. We now prove the converse for each $2 \leq i \leq 9$. In each case, we shall assume that D_i contains an F_i-subdivision S, and for $j = 1, 2, 3$, we shall denote by a'_j, b'_j the a_j, b_j-vertices of S, respectively.

- $i = 2$: We choose $e_1 = a_1b_1$ and $e_2 = a_2b_2$. Since D contains no big vertices, we have $a'_3 = a_3$ and $\{a'_1, a'_2\} = \{a_1, a_2\}$. By symmetry of F_2, we may assume that $a'_1 = a_1$ and $a'_2 = a_2$. Since $d^+_S(a'_1) = d^+_S(a'_1) = d^-_S(a'_3) = 3$, all arcs leaving a_1, a_2 and all arcs entering a_3 are in $A(S)$. Thus $\{b_1, b_2, b_3\} \in \{b'_1, b'_2, b'_3\}$. So, in S there are disjoint (a_1, b_1)-,(a_2, b_2)-dipaths, and these two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 3$: We choose $e_1 = a_1b_1$ and $e_2 = a_1b_3$. Since D contains no big vertices, we have $b'_3 = b_3$ and $\{a'_1, a'_2\} = \{a_1, a_2\}$. By symmetry of F_3, we may assume that

$a'_1 = a_1$ and $a'_2 = a_2$. Since $d^+_S(a'_1) = d^+_S(a'_1) = d^-_S(b'_3) = 3$, all arcs leaving a_1, a_2 and all arcs entering b_3 are in $A(S)$. Thus $\{b_2\} \in \{b'_2, b'_3\}$. Without loss of generality, we may assume $b_2 = b'_2$. Since $d^+_S(b'_2) = 1$, the arc b_2a_3 is an arc of S. Hence a_3 is the unique internal vertex in the (b'_2, b'_3)-dipath in S, so $a_3 = a'_3$. Now since $d^-_S(a'_3) = 2$, the arc b_1a_3 is an arc of S. Hence b_1 is the unique internal vertex in a (a'_2, a'_3)-dipath in S, and $b_1 \neq b'_2$, so $b_1 = b'_1$. Consequently, in S, there are disjoint (a_1, b_1)-,(a_1, b_3)-dipaths, and these two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 4$: We choose $e_1 = a_1b_1$ and $e_2 = b_3a_3$. Since D contains no big vertices, we have $a'_1 = a_1$ and $a'_3 = a_3$. Since $d^+_S(a'_1) = d^-_S(a'_3) = 3$, all arcs leaving a_1 and all arcs entering a_3 are in $A(S)$. Hence the dipath (a'_1, b_2, a'_3) is in S, so $b_2 \in \{b'_1, b'_2, b'_3\}$. For degree reason, necessarily $b_2 \in \{b'_2, b'_3\}$. By symmetry of F_4, we may assume that $b_2 = b'_2$. Hence the arc $a_2b_2 \in A(S)$ and since every vertex of $V(S) - \{a'_1\}$ has in-degree at least 1 in S, we have $b_1a_2 \in A(S)$. Therefore $d^+_S(b_1) \geq 2$, and then $b_1 \in a'_2, b'1$. But $b_1a_3 \in A(S)$ and the edge $a'_2a'_3$ should not exit in S. So, $b_1 = b'_1$. Moreover, a_2 is the unique internal vertex of the (b'_1, b'_2)-dipath in S, so $a_2 = a'_2$. Consequently, in S, there are disjoint (a_1, b_1)-,(b_3, a_3)-dipaths, and these two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 5$: We choose $e_1 = b_2a_2$ and $e_2 = b_3a_3$. Since D contains no big vertices, $a'_1 = a_1$ and $b'_1 = b_1$. Since $d^+_S(a'_1) = d^-_S(b'_1) = 3$, all arcs leaving a_1 and all arcs entering b_1 are in $A(S)$. In $S - \{a'_1b'_1\}$, there are two internally disjoint (a'_1, b'_1)-dipaths (Q_1, Q_2) with $\{b'_2, a'_2\} \in V(Q_1)$ and $\{b'_3, a'_3\} \in V(Q_2)$ and two internally disjoint (a'_1, b'_1)-dipaths (R_1, R_2) with $\{b'_2, a'_3\} \in V(R_1)$ and $\{b'_3, a'_2\} \in V(R_2)$. For one of these two pairs, b_2 and a_2 are in the same dipath and a_3 an b_3 on the others. Hence, in S, there are disjoint (b_2, a_2)-,(b_3, a_3)-dipaths, and these two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 6$: We choose $e_1 = a_2b_2$ and $e_2 = a_3b_3$. Since D contains no big vertices, $a'_1 = a_1$ and $b'_1 = b_1$. Since $d^+_S(a'_1) = d^-_S(b'_1) = 3$, all arcs leaving a_1, and all arcs entering b_1 are in $A(S)$. Hence a_2 and a_3 are in S and so b_3a_2 and b_2a_3 are arcs in S, because all vertices of S except a'_1 have in-degree ate least 1. Therefore, (a_1, b_1), (a_1, b_2, a_3, b_1) and (a_1, b_3, a_1, b_1) are the three internally disjoint (a_1, b_1)-dipaths in S. Hence $\{b_3a_2, b_2a_3\} = \{b'_3a'_2, b'_2a'_3\}$. Consequently, in S there are disjoint (a_2, b_2)-,(a_3, b_3)-dipaths, and these two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 7$: We choose $e_1 = a_1b_1$ and $e_2 = a_3b_3$. Since D contains no big vertices, we have $a'_1 = a_1$ and $b'_1 = b1$. In S, there are three internally disjoint (a'_1, b'_1)-dipath. Therefore the arcs b_3a_2 and b_2a_3 are in S. Both (a_1, b_2, a_3, b_1) and $(a_1, b_3, a_2, b1)$ are (a'_1, b'_1)-dipaths in S. Hence one of them does not correspond to the arc a_1b_1. Suppose that (a_1, b_2, a_3, b_1) is this dipath. Then necessarily

$b_2 = b'_2$ and $a_3 = a'_3$, so $b_2a_2 \in A(S)$. Hence $d^-_S(a_2) = 2$, so $a_2 = a'_2$ and consequently $b_3 = b'_3$. Similarly, if (a_1, b_3, a_2, b_1) does not correspond to the arc a_1b_1, then $a_2 = a'_2, b_3 = b'_3, b_2 = b'_2$ and $a_3 = a'_3$. In both cases, in S, there are disjoint (a_1, b_1)- and (a_3, b_3)-dipaths, which induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 8$: We choose $e_1 = b_1a_2$ and $e_2 = a_3b_2$. Since D contains no big vertices, we have $a'_1 = a_1$. Since $d^+_S(a'_1) = 3$, all the arcs leaving a_1 are in $A(S)$, and $\{b_1, b_2, b_3\} \in V(S)$. Every vertex in $V(S)$ has out-degree at least 1, so $b_2a_2, a_2b_3 \in A(S)$. Hence $b_3 \in \{b'_2, b'_3\}$. By symmetry of F_8, we may assume $b_3 = b'_3$. Now (a'_1, b_2, a_2, b'_3) is a dipath in S which correspond to the subdivision of (a_1, b_2, a_2, b_3) or (a_1, b_1, a_2, b_3). Therefore $a_2 = a'_2$ and $b_2 = b'_2$ because $d^+_{D_8}(b_2) = 1 < 2 = d^+_S(b'_1)$. Now in S (and D_8), the two paths corresponding to the subdivision of (a_1, b_1, a_2) and (b_3, a_3, b_2) are disjoint and induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

- $i = 9$: We choose $e_1 = b_1a_2$ and $e_2 = a_3b_2$. Since D contains no big vertices, we have $a'_1 = a_1$. Since $d^+_S(a'_1) = 3$, all the arcs leaving a_1 are in $A(S)$, and $\{b_1, b_2, b_3\} \in V(S)$. Every vertex in $V(S)$ has out-degree at least 1, so $b_2a_2, a_2b_3, b_3a_3 \in A(S)$. Hence $b_3 \in \{b'_1, b'_2, b'_3\}$. Therefore a_1b_3 is one of the arcs $\{a'_1b'_1, a'_1b'_2, a'_1b'_3\}$ and the dipath (a_1, b_2, a_2, b_3) correspond to the subdivision one of the dipath of length 3 from a_1 to $\{b_1, b_2, b_3\}$. Thus $a_2 \in \{a'_2, a'_3\}$ and so for degree reasons $a_2 = a'_2$. It follows that $b_2 \in \{b'_1, b'_2\}$. By symmetry of F_9, we may assume $b_2 = b'_2$. Now in S (and D_9), the two paths corresponding to the subdivision of $(a_1, b_1, a2)$ and (b_3, a_3, b_2) are disjoint and induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$\square$

Chapter 3

F-Subdivision for digraphs of order at most 4

In this chapter, we first conclude the classification of the digraphs of order at most 3. We turn then to digraphs of order 4, which we were able to classify all except five of them (up to directional duality). These are the digraphs O_i for $1 \le i \le 5$ depicted Figure 3.1.

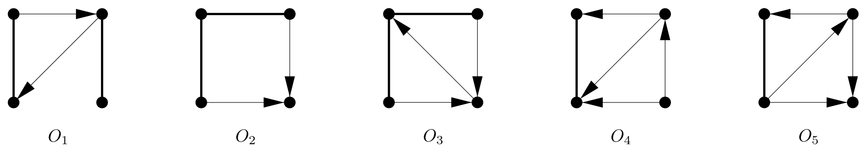


Figure 3.1: Digraphs on 4-vertices that are not known to be tractable or hard. Bold undirected edges represent directed 2-cycles.

In particular, we prove Seymour's Conjecture for digraphs of order at most 4.

In [17], the authors also checked the problem of detect an induced subdivision of an undirected graph with at most 4 vertices in a given graph G. For graphs with at most 3 vertices, the problem is polynomial-time solvable in that case. For the 12 graphs with 4 vertices, only the complexity of five of them is known, and the complexity of the others remains open.

Except for Section 1, the work in this chapter was done in cooperation with Frédéric Havet and Bojan Mohar.

3.1 Subdivisions of digraphs with three vertices

Let us denote by $\vec{K}_n$ the complete digraph on n vertices, in which there is an arc uv for any two distinct vertices u and v. Let D_3 be the digraph obtained from $\vec{K}_3$ by removing an arc, and the *lollipop* the digraph L with vertex set $\{x, y, z\}$ and arc set $\{xy, yz, zy\}$.

Theorem 3.1 (Bang-Jensen, Havet and M. [3]). *Let F be a digraph on three vertices. Then F-*Subdivision *is polynomial-time solvable unless $F = \vec{K}_3$ in which case it is NP-complete.*

Proof. If F is neither D_3 nor $\vec{K}_3$ nor the lollipop (or its converse), then it is either a disjoint union of spiders, or a spindle, or a bispindle, or a windmill, and so F-

Subdivision can be solved in polynomial time by virtue of the results of the previous sections. If $F = \vec{K}_3$, then F-Subdivision is NP-complete by Corollary 1.12.

It remains to prove that the subdivision problem for D_3 and the lollipop is polynomial-time solvable.

The *bulky vertex* of a D_3-subdivision S is the unique vertex of S with degree 4. We now give a procedure that given a vertex v, two of its out-neihbours s_1, s_2 and two of its in-neighbours t_1, t_2 check if there is a D_3-subdivision S in which v is the bulky vertex and $\{vs_1, vs_2, t_1v, t_2v\} \in A(S)$. Such a subdivision will be called *suitable*.

Applying a Menger algorithm, check if in $D - v$ there are two disjoint directed paths P_1 and P_2 from $\{s_1, s_2\}$ to $\{t_1, t_2\}$. If not, then D certainly does not contain any suitable D_3-subdivision. If yes, then check if there is a directed path Q from P_1 to P_2 or from P_2 to P_1. If such a Q exists, then P_1, P_2, Q together with v and the arcs vs_1, vs_2, t_1v, t_2v form a suitable D_3-subdivision. If not, then no suitable D_3-subdivision using the chosen arcs exists, because there is no vertex $s \in \{s_1, s_2\}$ such that there exists in $D - v$ both a directed (s, t_1)-path and a directed (s, t_2)-path.

A D_3-subdivision is clearly suitable with respect to its bulky vertex and its neighbours in this subdivision. Hence checking if there is a suitable D_3-subdivision for every 5-tuple (v, s_1, s_2, t_1, t_2) such that s_1, s_2 are out-neighbours of v and t_1, t_2 are out-neighbours yields a polynomial-time algorithm to decide if there is a D_3-subdivision in a digraph.

Consider now the lollipop. If D contains a strong component of cyclomatic number greater than 1, then it contains a lollipop. Indeed, the smallest directed cycle C in the component is induced and is not the whole strong component. Hence there must be a vertex v dominating a vertex of C thus forming a lollipop-subdivision. If not, then all the strong components are cycles. Thus D contains a lollipop if and only if one of its component is a directed cycle and is not an initial strong component (i.e some arc is entering it). All this can be checked in linear time. □

3.2 Oriented graphs of order 4

The aim of this section is to prove that every oriented graph of order 4 is tractable.

Theorem 3.2 (Havet, M. and Mohar). *If D is an oriented graph of order 4, then D-Subdivision is polynomial-time solvable.*

Proof. If D is a tournament, then it is either the transitive tournament TT_4, or the wheel W_3, or the converse of W_3, or the strong tournament ST_4, which in any case we prove to be tractable in Chapter 2.

If D is an orientation of $K_4 \setminus e$, the graph obtained from K_4 by removing one edge, then D must be one of the oriented graphs depicted Figure 3.2, or the converse of one of those. $S(1, 2, 2)$ is a spindle, $B(1, 2; 2)$ and $B(2, 2; 1)$ are bispindles and F_3 is the 3-fan. All these digraphs have been shown tractable in the previous chapter, as well as the digraphs TT_4^* and W_2'.

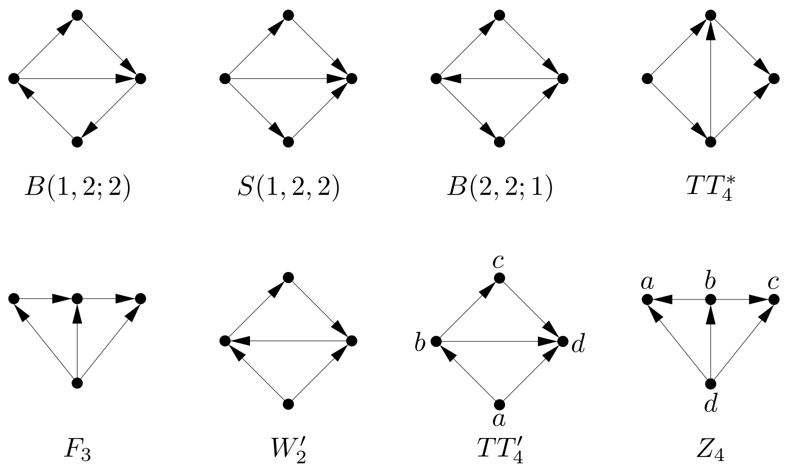


Figure 3.2: Some orientations of $K_4 \setminus e$

We prove in Subsection 3.2.1 that TT_4' is tractable, and in Subsection 3.2.3 that Z_4 is tractable.

If D is an oriented cycle, then it is either directed, or it has two blocks, or it is $\hat{C}_4$. In the first two cases, D-Subdivision has been shown polynomial-time solvable in [3] (Propositions 15 and 20), and latter case was presented in Chapter 2.

If D has at most four arcs and is not an oriented cycle, then it has been proved tractable in [3] except if D is the oriented graph Y_4 depicted Figure 3.3 or its converse. We show in Subsection 3.2.2 that Y_4 is tractable. □

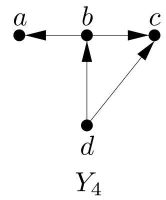


Figure 3.3: The oriented graph Y_4

3.2.1 TT_4'-subdivision

In this subsection, we prove that TT_4' is tractable. Our proof relies on the notion of *good triple*. A triple of distinct vertices (a', b', d') is *good* if there are an (a', b')-dipath Q in $D - d'$ and three internally disjoint dipaths P_1, P_2, P_3 with $s(P_1) = s(P_2) = b'$, $s(P_3) = a'$ and $t(P_1) = t(P_2) = t(P_3) = d'$.

Proposition 3.3. *A digraph D contains a TT_4'-subdivision if and only if it has a good triple.*

Proof. If D contains a TT_4'-subdivision, then the triple formed by its a-vertex, its b-vertex and its d-vertex is good.

Conversely, suppose that D contains a good triple. Let (a', b', d') be a good triple that minimizes the sum of the lengths of the paths Q, P_1, P_2, P_3 as named in the definition.

Assume for a contradiction that Q° intersects P_3°. Let a'' be a vertex of $Q^\circ \cap P_3^\circ$. Then the triple (a'', b', d') is good because of the paths $Q[a'', b']$, P_1, P_2, $P_3[a'', d']$, and contradicts the minimality of (a', b', d'). Hence Q° does not intersect P_3°.

Assume for a contradiction that Q° intersects $P_1^\circ \cup P_2^\circ$. Let b'' be the last vertex along Q° in $P_1^\circ \cup P_2^\circ$. Without loss of generality, we may assume that b'' is on P_1°. Then the triple (a', b'', d') is good because of the paths $Q[a', b'']$, $P_1[b'', d']$, $Q[b'', b']P_2$, P_3, and contradicts the minimality of (a', b', d'). Hence Q° does not intersect $P_1^\circ \cup P_2^\circ$.

Therefore the paths Q, P_1, P_2, P_3 are internally disjoint and the union of those dipaths is a TT_4'-subdivision. □

Corollary 3.4. TT_4'-SUBDIVISION *can be solved in $O(n^5(n+m))$ time.*

Proof. According to Proposition 3.3, TT_4'-SUBDIVISION is equivalent to deciding if D has a good triple.

Now one can decide if a triple (a', b', d')-triple in $O(n^2(n+m))$ time as follows. We check if there is an (a', b')-dipath Q in $D - d'$, and for every pair s_1, s_2 of distinct out-neighbours of b' in $D - a'$, we check if there are three independent $(\{s_1, s_2, a\}, d')$-dipath in $D - b'$ by a Menger algorithm.

Doing this procedure for the $O(n^3)$ triple of distinct vertices of D, one decides in $O(n^5(n+m)$ time whether D has a good triple. □

3.2.2 Y_4-subdivision

In this subsection, we prove that Y_4 is tractable.

Theorem 3.5. Y_4-SUBDIVISION *can be solved in $O(n^5(n+m))$ time.*

Proof. Let us describe a procedure that given three distinct vertices a', c', d', and two distinct arcs $d'u_1$ and $d'u_2$ in $D - a'$ decides whether a digraph D contains a Y_4-subdivision S with a-vertex a', c-vertex c', d-vertex d' such that $\{d'u_1, d'u_2\} \subseteq A(S)$. Such a subdivision is said to be $(a', c', d'u_1, d'u_2)$*-forced.*

We check whether there are two independent $(\{u_1, u_2\}, c')$-dipaths P_1, P_2 in $D - \{a', d'\}$ and a $(\{u_1, u_2\}, a')$-dipath Q in $D - \{c', d'\}$. This can be done in linear time using a Menger algorithm for each of the tasks. The existence of P_1, P_2, Q is clearly a necessary condition to contain an $(a', c', d'u_1, d'u_2)$-forced Y_4-subdivision. So if we do not find such dipaths, we return 'no'. If we have such dipaths, then we return 'yes'. Indeed the union of the dipaths $d'u_1, d'u_2, P_1, P_2$, and R, where R is the $(P_1 \cup P_2, a')$-subdipath of Q, is an $(a', c', d'u_1, d'u_2)$-forced Y_4-subdivision.

Doing this for every 5-tuple (a', c', d', u_1, u_2) of vertices, we obtain an algorithm solving Y_4-SUBDIVISION in $O(n^5(n+m))$ time. □

3.2.3 Z_4-subdivision

In this subsection, we show that Z_4 is tractable. The proof relies on the following lemma.

Lemma 3.6. *Let D be a digraph. There is a Z_4-subdivision in D if and only if there exists four distinct vertices a', b', c' and d' in D such that the following hold.*

(i) There are three independent $(d', \{a', b', c'\})$-dipaths.

(ii) There are two independent $(b', \{a', c'\})$-dipaths.

Proof. If D contains a Z_4-subdivision S, then the vertices a', b', c', d' corresponding to a, b, c, d (as indicated on Figure 3.2) clearly satisfy conditions (i) and (ii).

Conversely, suppose that D contains four vertices a', b', c', d' satisfying conditions (i) and (ii). Let P_1, P_2, P_3 be three independent $(d', \{a', b', c'\})$-dipaths with $t(P_1) = a'$, $t(P_2) = b'$ and $t(P_3) = c'$; let Q_1, Q_2 be two independent $(b', \{a', c'\})$-dipaths with $t(Q_1) = a'$ and $t(Q_2) = c'$.

We consider such vertices a', b', c', d' and dipaths such that the sum of the lengths of P_1, P_2, P_3, Q_1 and Q_2 is minimized.

Claim 18. $V(Q_1) \cap V(P_1) = \{a'\}$ *and* $V(Q_2) \cap V(P_3) = \{c'\}$.

Subproof. Suppose $V(Q_1) \cap V(P_1) \neq \{a'\}$. Then there is a vertex a'' distinct from a' in $V(Q_1) \cap V(P_1)$. The vertices a'', b', c', d' satisfy condition (i) with $P_1[d', a'']$, P_2, P_3 and condition (ii) with $Q_1[b', a'']$, Q_2. This contradicts our choice of a', b', c', d' and the corresponding paths, and so $V(Q_1) \cap V(P_1) = \{a'\}$.

The conclusion that $V(Q_2) \cap V(P_3) = \{c'\}$ is proved in the same way; the details are omitted. $\diamondsuit$

Claim 19. $(V(Q_1) \cup V(Q_2)) \cap V(P_2) = \{b'\}$.

Subproof. Suppose not. Then let b'' be the last vertex distinct from b' along P_2 which is in $V(Q_1) \cup V(Q_2)$. By symmetry, we may assume that $b'' \in V(Q_1)$. But the four vertices a', b'', c', d' satisfy condition (i) with P_1, $P_2[d', b'']$, P_3 and condition (ii) with $Q_1[b'', a']$, $P_2[b'', b']Q_2$. This contradicts our choice of a', b', c', d' and proves our claim. $\diamondsuit$

Claim 20. $V(Q_1) \cap V(P_3) = \emptyset$ *and* $V(Q_2) \cap V(P_1) = \emptyset$.

Subproof. Suppose not. Then $V(Q_1) \cap V(P_3)$ or $V(Q_2) \cap V(P_1)$ is not empty.

Assume first that these two sets are both non-empty. Let a'' be a vertex in $V(Q_2) \cap V(P_1)$ and c'' be a vertex in $V(Q_1) \cap V(P_3)$. Then the four vertices a'', b', c'', d' satisfy condition (i) with $P_1[d', a'']$, $P_3[d', c'']$, P_2 and condition (ii) with $Q_2[b', a'']$, $Q_1[b', c'']$. This contradicts our choice of a', b', c', d'.

Hence, exactly one of the two sets is empty. By symmetry, we may assume that $V(Q_1) \cap V(P_3) \neq \emptyset$, Let b'' be a vertex in $V(Q_1) \cap V(P_3)$. Now the four vertices a', b'', c', d' satisfy condition (i) with P_1, $P_3[d', b'']$, P_2Q_2 and condition (ii) with

$Q_1[b'', a']$, $P_3[b'', c']$. This contradicts our choice of a', b', c', d' and proves our claim. $\Diamond$

Claims 18, 19 and 20 imply that $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$ is a Z_4-subdivision. $\square$

Theorem 3.7. *Z_4-Subdivision can be solved in $O(n^4(n+m))$ time.*

Proof. By Lemma 3.6, Z_4-Subdivision is equivalent to deciding whether there are four vertices satisfying the condition (i) and (ii) of the lemma. But given four vertices a', b', c', d', one can check in linear time if conditions (i) and (ii) hold by running two Menger algorithms. Since there are $O(n^4)$ sets of four vertices in D, Z_4-Subdivision can be solved in $O(n^4(n+m))$ time. $\square$

3.3 Some hard digraphs

Theorem 1.10 implies that many digraphs on 4 vertices are hard. We now prove that some additional digraphs that are not covered by Theorem 1.10 are also hard. These graphs are depicted in Figure 3.4, where each of the bold edges without indicated direction represents a pair of oppositely directed arcs.

Proposition 3.8. *For each digraph N_i, $1 \leq i \leq 9$, depicted Figure 3.4, N_i-Subdivision is NP-complete.*

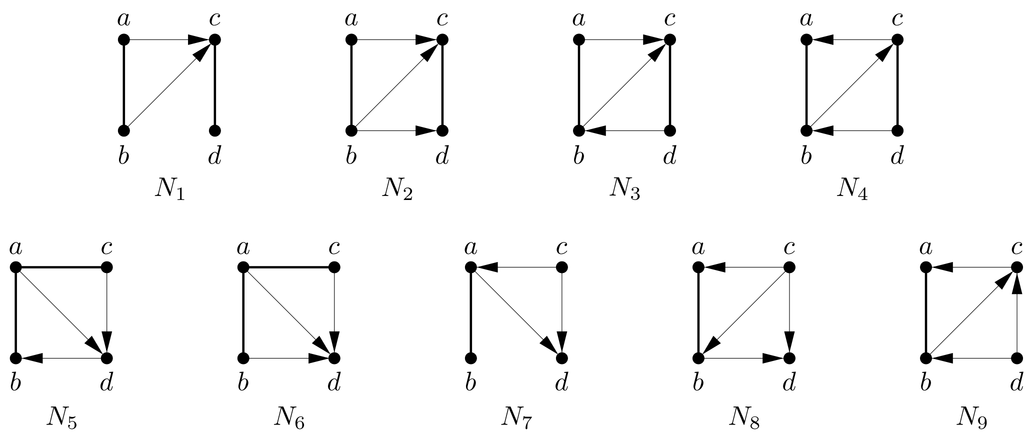


Figure 3.4: Some hard digraphs on 4-vertices. Bold undirected edges represent directed 2-cycles.

Proof. In each case, the problem is proved to be NP-complete by reduction from Restricted 2-Linkage. Let D, x_1, x_2, y_1 and y_2 be an instance of this problem. We construct a digraph D_i by *putting* D on two arcs $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of N_i (that will be specified later), that is by taking the disjoint union of D and N_i, by removing the arcs e_1 and e_2 and adding the arcs u_1x_1, y_1v_1, u_2x_2 and y_2v_2. We then show that D_i contains an N_i-subdivision if and only if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D. This implies that N_i-Subdivision is NP-complete.

Clearly, by construction of D_i, if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D, then D_i contains an N_i-subdivision. We now prove the converse for each i. In each case we shall assume that D_i contains an N_i-subdivision S, and we shall denote by a', b', c', d' the vertices in S corresponding to a, b, c, d, respectively.

$i = 1$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $c' = c$. Because $d^-_{D_1}(c) = 3$, the arcs ac, bc and dc are in S. Moreover, the arc ba is in S, because every vertex has in-degree at least 1 in S. Thus $d^+_S(b) \geq 2$, and so either $b = b'$ or $b = a'$. By symmetry between a and b in N_1, we may assume that $b = b'$. Then, necessarily, $a = a'$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i \in \{2, 3, 4\}$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $\{b, c\} = \{b', c'\}$. Therefore, the arc bc is contained in S, and this shows that $b' = b$ and $c' = c$. Now for degree reasons, all arcs incident to b and c must be in S. It follows that $a' = a$ and $d' = d$. (This is clear for N_3 and N_4. For N_2, we first conclude that $\{a', d'\} = \{a, d\}$ and then consider degrees of a and d to obtain the same conclusion.) Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induced a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 5$: We choose $e_1 = ba$ and $e_2 = cd$. Since D contains no big vertices, we have $a' = a$. Hence all the arcs incident to a are in $A(S)$. Therefore c is ether b' or c'. But $d^-(c) = 1$, so c cannot be b', and thus $c = c'$. All vertices have out-degree at least 1 in S, so $db \in A(S)$. Now there are two internally disjoint (a', b)-dipaths in $S - c'$, so necessarily, $b = b'$. Moreover, d' must be in one of those dipaths, so $d = d'$. Therefore, in S, there are internally disjoint (b, a)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 6$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $a' = a$ and $d' = d$. Hence all arcs incident to those two vertices are in S. Therefore $\{b', c'\} = \{b, c\}$. By symmetry of N_6, we may assume that $b' = b$ and $c' = c$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 7$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $a' = a$. Hence all arcs incident to a are in S. So c and d are in $V(S)$. Since $d^+_{D_7}(d) = 0$, we have $d = d'$; since $d^-_{D_7}(c) = 0$, we have $c = c'$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 8$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $b' = b$ and $c' = c$. Hence all arcs incident to those two vertices are in S. So $d \in V(S)$. Since $d^+_{D_8}(d) = 0$, it follows that $d = d'$. The arcs ba and ca show that $d^-_S(a) \geq 2$. Thus $a = a'$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 9$: We choose $e_1 = ab$ and $e_2 = dc$. Since D contains no big vertices, we have $b' = b$. Hence all arcs incident to b are in S. In particular $c, d \in V(S)$. Since $d^-_{D_9}(d) = 0$, we have $d' = d$. Since $d^+_S(c) \geq 1$, the arc ca is in $A(S)$, so $d^-_S(a) = 2$, and thus $a \in \{a', c'\}$. Since a' and c' are both in the out-section of d in $N_9 - b$, S contains a (d, a)-dipath disjoint from b. This dipath must pass through c and therefore the arc y_2c lies in S. This implies that $d_S^-(c) \geq 2$, so $c = c'$ and then we have $a = a'$. Consequently, in S, there are disjoint (a, b)- and (d, c)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D. □

3.4 Some tractable digraphs – easier cases

A *symmetric star* is a symmetric digraph associated to a star. The *centre* of a symmetric star is the centre of the star to which it is associated. A *superstar* is a digraph obtained from a symmetric star by adding an arc joining two non-central vertices. The *centre* of a superstar is the centre of the star from which it is derived. The symmetric star of order $k+1$ is denoted by SS_k and the superstar of order $k+1$ is denoted by SS_k^*. An SS_k-subdivision with centre a is the union of k internally disjoint (a, a)-handle. Therefore, on can decide if there is an SS_k-subdivision with centre a in linear time using a Menger algorithm. We showed that SS_3^*-Subdivision is polynomial-time solvable. This result can be extended to all superstars.

Theorem 3.9. *Let k be a positive integer. Given digraph D and a vertex v of D, on can decide in $O(n^{2k}(n + m))$-time whether D contains an SS_k^*-subdivision with centre v.*

Proof. We describe a procedure that given v, a set $X = \{x_1, \dots, x_k\}$ of k distinct out-neighbours of v and a set $Y = \{y_1, \dots, y_k\}$ of k distinct in-neighbours of v checks if there is an SS_k^*-subdivision S with centre v such that $\{vx_1, \dots, vx_k\} \cup \{y_1v, \dots, y_kv\} \in A(S)$. (Observe that it is allowed that $X \cap Y \neq \emptyset$.) Such a subdivision will be called (v, X, Y)*-forced*.

Applying a Menger algorithm, check whether in $D-v$ there are k disjoint dipaths $P_1, \dots, P_k$ from X to Y. If not, then D certainly does not contain any (v, X, Y)-forced SS_k-subdivision. If yes, then check whether there is a dipath Q from some P_i to a different P_j whose internal vertices are not in $\{v\} \cup \bigcup_{i=1}^k P_i$. This can be done in linear time by running a search on the digraph obtained from $D - v$ by contracting each path P_i into a single vertex. If such a Q exists, then $P_1, \dots, P_k$ and Q together with v and the arcs from v to X and from Y to v form a (v, X, Y)-forced SS_k^*-subdivision. If not, then no (v, X, Y)-forced SS_k^*-subdivision using the chosen arcs exists, because there is no vertex $x \in X$ with two vertices of Y in its out-section in $D - v$.

Applying this linear-time procedure, for every possible pair (X, Y), we can decide in $O(n^{2k}(n + m))$-time whether D contains an SS_k^*-subdivision with centre v. □

Corollary 3.10. *For every positive integer k, SS_k^*-Subdivision can be solved in $O(n^{2k+1}(n + m))$-time.*

Proposition 3.11. *For every $i \in \{3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15\}$, the digraph E_i depicted in Figure 3.5 is tractable.*

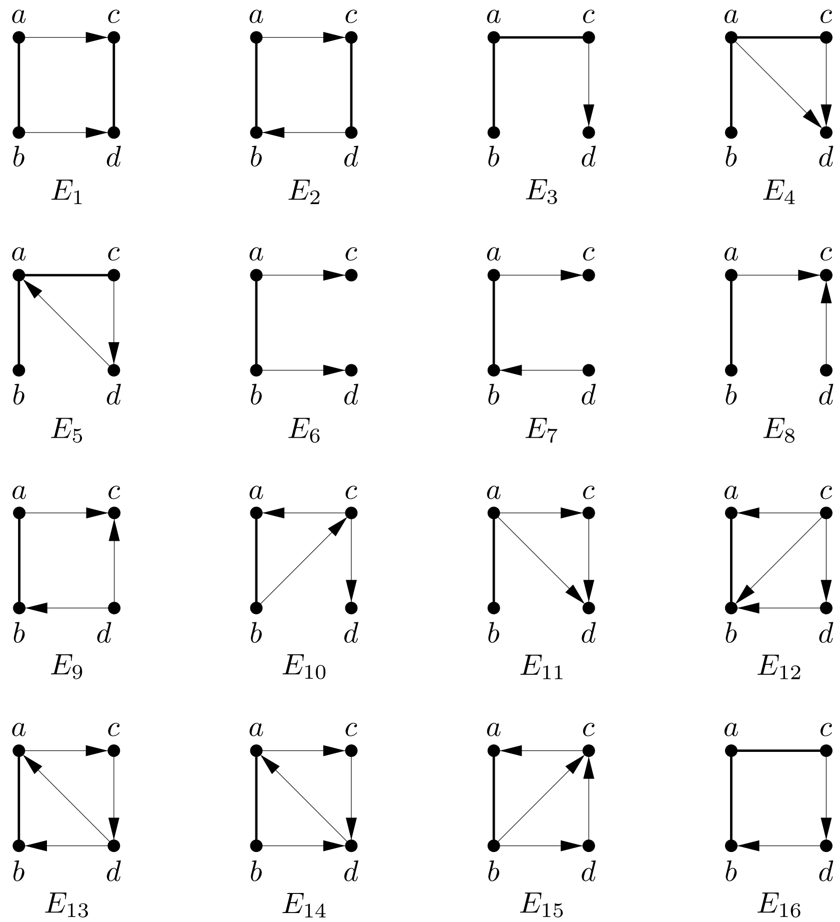


Figure 3.5: Some digraphs on 4-vertices, that are tractable. Bold undirected edges represent directed 2-cycles.

Proof. $i = 3$: Let us describe a procedure that, given two distinct vertices a' and d' in D and two out-neighbours s_1, s_2 of a' distinct from d', decides whether there is an E_3-subdivision with a-vertex a' and d-vertex d' such that $a's_1$ and $a's_2$ are arcs of S. Such a subdivision is said to be $(a's_1, a's_2, d)$-*forced*.

We check whether there is a dipath Q from $\{s_1, s_2\}$ to d' in $D - a'$, and with a Menger algorithm we check whether there are two independent $(\{s_1, s_2\}, a')$-dipaths P_1 and P_2 in $D - d'$. If these three dipaths do not exist, then D contains no $(a's_1, a's_2, d)$-forced E_3-subdivision, and we return 'no'. If the three paths Q, P_1, P_2 exist, then we return 'yes'. Indeed, denoting by c' the last vertex along Q in $P_1 \cup P_2$, the digraph $a's_1 \cup P_1 \cup a's_2 \cup P_2 \cup Q[c', d']$ is an $(a's_1, a's_2, d)$-forced E_3-subdivision.

Applying the above procedure for all possible triples $(a's_1, a's_2, d')$, one solves E_3-Subdivision in $O(n^4(n + m))$ time.

$i = 4$: Let us describe a procedure that given two distinct vertices a' and d' in D, a set $U = \{u_1, u_2, u_3\}$ of three out-neighbours of a', returns 'yes' if it finds an E_4-subdivision and returns 'no' only if there is no E_4-subdivision with a-vertex a'

and d-vertex d' such that $\{a'u_1, a'u_2, a'u_3\} \subseteq A(S)$. Such a subdivision is said to be (a', d', U)*-forced*.

We check with a Menger algorithm whether $|S^-_{D-a'}(d') \cap U| \geq 2$ and whether there are three internally disjoint dipaths P_1, P_2, P_3 with distinct initial vertices in U and with $t(P_1) = t(P_2) = a'$ and $t(P3) = d'$. If these two conditions are not both fulfilled, then D contains no (a', d', U)-forced E_4-subdivision, and we return 'no'. If these conditions are fulfilled, then we return 'yes'. Indeed consider three dipaths P_1, P_2, P_3 as above. Without loss of generality, $s(P_i) = u_i$ for $1 \leq i \leq 3$. Since $|S^-_{D-a'}(d') \cap U| \geq 2$, there exists a $(P_1 \cup P_2, P_3)$-dipath in $D - a'$. Let us denote its terminal vertex by d''. Then the union of the directed cycles $a'u_1P_1$, $a'u_2P_2$, and the dipaths $a'u_3P_3[u_3, d'']$, and Q is an E_4-subdivision.

Applying the above procedure for all possible triples (a', d', U), one solves E_4-Subdivision in $O(n^5(n + m))$ time.

$i = 5$: Let us describe a procedure that given two distinct vertices a' and d' in D and two out-neighbours s_1, s_2 of a' distinct from d', returns 'yes' when it finds an E_5-subdivision and returns 'no' only if there is no E_5-subdivision with a-vertex a' and d-vertex d' such that $\{a's_1, a's_2\} \subseteq S$. Such a subdivision is said to be $(a's_1, a's_2, d')$*-forced*.

We check whether there is an $(\{s_1, s_2\}, d')$ dipath Q in $D - a'$ and whether there are three independent $(\{s_1, s_2, d'\}, a')$-dipaths P_1, P_2, P_3 in D. If these two conditions are not both fulfilled, then D contains no $(a's_1, a's_2, d')$-forced E_5-subdivision, and we return 'no'. If these conditions are fulfilled then we return 'yes'.

Indeed, suppose there are four such dipaths Q, P_1, P_2, P_3. We may assume without loss of generality that $s(P_3) = d'$. Denote by c' the last vertex along Q in $P_1 \cup P_2$, and by d'' the first vertex in $Q[c', d']$ which is on P_3. Then the union of the two directed cycles $a's_1P_1a', a's_2P_2a'$ and the dipaths $Q[c', d'']$ and $P_3[d'', a']$ is an E_5-subdivision.

Applying the above procedure for all possible triples $(a's_1, a's_2, d')$, one solves E_5-Subdivision in $O(n^4(n + m))$ time.

$i = 6$: Observe first that if a digraph contains an E_6-subdivision, then it contains such an E_6-subdivision in which ac and bd are not subdivided. Henceforth, by E_6-subdivision, we mean an E_6-subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs $a'c'$ and $b'd'$, returns 'yes' if it finds an E_6-subdivision and returns 'no' only if there is no E_6-subdivision with a-vertex a', b-vertex b', c-vertex c' and d-vertex d'. Such a subdivision is said to be $(a'c', b'd')$*-forced*.

We check whether, in $D - \{c', d'\}$, there exists an (a', b')-dipath P and a (b', a')-dipath Q. If two such dipaths do not exist, then there is clearly no $(a'c', b'd')$-forced E_6-subdivision, and we return 'no'. If two such paths P and Q exist, then we return 'yes'. Indeed let b'' be the last vertex on $Q - a'$ that is in $P \cap Q$. If $b'' = b'$, then set $d'' = d'$, otherwise let d'' be the successor of b'' on P. Then the union of the directed cycle $P[a', b'']Q[b'', a']$ and the two arcs $a'c'$ and $b''d''$ form an E_6-subdivision.

Applying the above procedure for all possible pairs of distinct arcs $(a'c', b'd')$, one solves E_6-SUBDIVISION in $O(m^2(n+m))$ time.

$i = 7$: The proof is similar to the case $i = 6$, and we leave details to the reader.

$i = 8$: Observe first that if a digraph contains an E_8-subdivision, then it contains an E_8-subdivision in which dc is not subdivided. Henceforth, by E_8-subdivision, we mean an E_8-subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs sa' and $d'c'$, checks whether there is an E_8-subdivision S with a-vertex a', c-vertex c' and d-vertex d', and such that $sa' \in A(S)$. Such a subdivision is said to be *$(sa', d'c')$-forced*.

With a Menger algorithm, we check whether $D - d'$ contains two independent $(a', \{s, c'\})$-dipaths. If two such dipaths do not exist, then there is clearly no $(a'c', b'd')$-forced E_8-subdivision, and we return 'no'. If two such dipaths P and Q exist, then without loss generality $t(P) = s$ and $t(Q) = c$. The union of the directed cycle Psa', the dipath Q and the arc $d'c'$ is an $(sa', d'c')$-forced E_8-subdivision.

Applying the above procedure for all possible pairs of distinct arcs $(sa', d'c')$, one solves E_8-SUBDIVISION in $O(m^2(n+m))$ time.

$i = 11$: Let us describe a procedure that, given an arc sa' and a vertex $d' \notin \{s, a'\}$, checks whether there is an E_{11}-subdivision S with a-vertex a', d-vertex d', and such that $sa' \in A(S)$. Such a subdivision is said to be *(sa', d')-forced*.

We check with a Menger algorithm whether there are three independent $(a', \{s, d'\})$-dipaths, where two of the paths end up at d' and one at s. If three such dipaths do not exist, then there is clearly no (sa', d')-forced E_{11}-subdivision, and we return 'no'. If three such dipaths exist, then their union together with the arcs sa' form an (sa', d')-forced E_{11}-subdivision.

Applying the above procedure for all possible pairs (sa', d'), one solves E_{11}-SUBDIVISION in $O(mn(n+m))$ time.

$i = 12$: Let us describe a procedure that, given two distinct vertices b', c' and a set $S = \{s_1, s_2, s_3\}$ of three distinct in-neighbours of b' checks whether there is an E_{12}-subdivision S' with b-vertex b', c-vertex c', and such that $\{s_1b', s_2b', s_3b'\} \subset A(S')$. Such a subdivision is said to be *(b', c', S)-forced*.

We check with a Menger algorithm, if there are three independent (c', S)-dipaths P_1, P_2, P_3, and we check whether there is a $(b', S \setminus \{c'\})$-dipath Q in $D - c'$. If four such dipaths do not exist, then we return 'no' because there is no (b', c', S)-forced E_{12}-subdivision. If such dipaths P_1, P_2, P_3 and Q exist, then let x be the first vertex of Q in $P_1 \cup P_2 \cup P_3$. Then the union of $P_1, P_2, P_3, Q[b', x]$ and the three arcs s_1b', s_2b', s_3b' form a (b', c', S)-forced E_{12}-subdivision.

Applying the above procedure for all possible triples (a', b', S), one solves E_{12}-SUBDIVISION in $O(n^5(n+m))$ time.

$i = 13$: Observe that every E_{13}-subdivision may be seen as an E_{13}-subdivision in which the arc cd is not subdivided. Henceforth, by an E_{13}-subdivision, we mean such a subdivision.

Let us describe a procedure that, given two disjoint arcs, sb' $d'c'$, returns 'yes' if it finds an E_{13}-subdivision and returns 'no' only if there is no E_{13}-subdivision S with b-vertex b', c-vertex c', d-vertex d' and such that $\{sb', d'c'\} \subseteq A(S)$. Such a subdivision is called $(sb', d'c')$*-forced.*

Applying a Menger algorithm, we check whether in D there are three independent $(b', \{s, c', d'\})$-dipaths P_1, P_2, P_3 with $t(P_1) = s$ and applying a search we check whether there is a (c', s)-dipath Q in $D - \{b', d'\}$. Clearly, if four such dipaths do not exist, then D contains no $(sb', d'c')$-forced E_{13}-subdivision, so we return 'no'. Conversely, if these dipaths exist, then Q contains a (c', P_1)-subdipath R. Let c'' be the last vertex along R in $V(P_2 \cup P_3)$. Now in $P_2 \cup P_3 \cup R[c', c''] \cup d'c'$, there are two internally disjoint (b', c'')-dipaths P_2', P_3'. Thus $P_1 \cup sb' \cup P_2' \cup P_3' \cup R[c'', t(R)]$ is an E_{13}-subdivision, and we return 'yes'.

Doing this for every possible pair $(sb', d'c')$, one decides in $O(m^2(n + m))$ time whether D contains an E_{13}-subdivision.

$i = 14$: We proceed in two stages. We first check whether there is an E_{14}-subdivision in which the arc ab is not subdivided. Next we check whether there is an E_{14}-subdivision in which the arc ab is subdivided.

In the first stage we decide whether there is an E_{14}-subdivision with a-vertex a' and b-vertex b' for some arc $a'b'$. To do so, for every dipath $a'uv$ in $D - b'$, we check whether there is an E_{14}-subdivision with a-vertex a' and b-vertex b', and which contains the arcs of $\{a'u, uv, a'b'\}$. Such a subdivision is said to be $(a'uv, a'b')$*-forced.*

We proceed as follows. Applying a Menger algorithm, we check whether in $D - u$ there are independent $(\{v, b'\}, a')$-dipaths P_1 and P_2 with $s(P_1) = v$, and applying a search we check whether there is a (v, b')-dipath Q in $D - a' - u$. Clearly, if three such dipaths do not exist, then D contains no $(a'uv, a'b')$-forced E_{14}-subdivision, so we return 'no'. Conversely, if these dipaths exists, then Q contains a (P_1, P_2)-subdipath R. Then the union of P_1, P_2, R, $a'uv$, and $a'b'$ is an E_{14}-subdivision, and we return 'yes'. Doing this for every possible pair $(a'uv, a'b')$, one decides in $O(m^2(n + m))$ time that either D contains an E_{14}-subdivision, or that D contains no E_{14}-subdivision in which the arc ab is not subdivided.

Let F_{14} be the digraph obtained from E_{14} by subdividing the arc ab into a dipath awb of length 2. The second stage consists in deciding whether D contains an F_{14}-subdivision. We use a procedure similar to the one for detecting superstar subdivision. Given a pair $\{a'w_1x_1, a'w_2x_2\}$ of dipaths that are disjoint except for their initial vertex a', and two distinct in-neighbours y_1, y_2 of a' that are not in $\{w_1, w_2\}$ (allowing the possibility that $\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$), the procedure returns 'yes' if it finds an F_{14}-subdivision and returns 'no' only if there is no F_{14}-subdivision with a-vertex a' containing all arcs in $A' = \{a'w_1, w_1x_1, a'w_2, w_2x_2, y_1a', y_2a'\}$. Such a subdivision is called A'*-forced.*

The procedure proceeds as follows. With a Menger algorithm, we first check whether in $D - \{a', w_1, w_2\}$ there are two disjoint dipaths P_1, P_2 from $\{x_1, x_2\}$ to $\{y_1, y_2\}$. If not, then D certainly does not contain any A'-forced F_{14}-subdivision. If yes, then check whether there is a (P_1, P_2)-dipath Q in $D - \{a', w_1, w_2\}$. If such a dipath exists, then the union of the paths $P_1, P_2, Q, a'w_1x_1, a'w_2x_2$ and the arcs y_1a' and y_2a' is an F_{14}-subdivision and we return 'yes'. Next, we check if there is a (P_2, P_1)-dipath Q in $D - \{a', w_1, w_2\}$. If Q exists, we return 'yes'. If not, then no A'-forced F_{14}-subdivision exists, because there is no vertex $x \in \{x_1, x_2\}$ with two vertices of $\{y_1, y_2\}$ in its out-section in $D - \{a', w_1, w_2\}$. So we return 'no'.

This procedure runs in linear time. Thus, running it for every possible set A', one decides in $O(m^2n^3(n+m))$ time whether D contains an F_{14}-subdivision, which is nothing but an E_{14}-subdivision in which the arc ab is subdivided.

Doing the two stages one after another, we obtain an $O(m^2n^3(n+m))$-time algorithm for solving E_{14}-Subdivision.

$i = 15$: Similarly to the case $i = 14$, we proceed in two stages. We first check whether there is an E_{15}-subdivision in which the arc ab is not subdivided. Next we check whether there is an E_{15}-subdivision in which the arc ab is subdivided.

The first stage is the following. For every vertex a', every two distinct out-neighbours b', u, and every in-neighbour t' of a' distinct from b' and u, we run a procedure that returns 'yes' if it finds an E_{15}-subdivision, and return 'no' if there is no E_{15}-subdivision with a-vertex a' and b-vertex b' and whose arc set includes $\{t'a', a'b', a'u\}$. Such a subdivision is called $(t'a', a'b', a'u)$*-forced.* The procedure is the following. With a Menger algorithm, we check whether in $D - u$ there are two independent $(b', \{a', t'\})$-dipaths P_1, P_2 and whether there is a (u, t')-dipath Q in $D - \{a', b'\}$. If three such paths do not exist, then D certainly contains no $(t'a', a'b', a'u)$-forced E_{15}-subdivision and we return 'no'. If these three paths exist, we then we return 'yes'. Indeed let d' be the first vertex along Q in $P_1 \cup P_2$. Now the union of P_1, P_2, $Q[u, d']$, $a'b'$, $t'a'$ and $a'u$ is an E_{15}-subdivision with a-vertex a' and b-vertex b'.

Doing this for every possible triple $(t'a', a'b', a'u)$, one can decide in time $O(n^2m(n+m))$ whether there is an E_{15}-subdivision with in which the arc ab is not subdivided.

Observe that an E_{15}-subdivision in which ab is subdivided is an F_{14}-subdivision. Hence the second phase is exactly the same as the one for E_{14}.

Doing the two stages one after another, we obtain an $O(m^2n^3(n+m))$-time algorithm for solving E_{15}-Subdivision. □

3.5 More complicated tractable cases

3.5.1 E_1 is tractable

In this subsection we prove that the digraph E_1 depicted in Figure 3.5 is tractable.

Theorem 3.12. E_1-Subdivision *can be solved in* $O(n^4m(n+m))$*-time.*

Proof. Let D be a digraph and let x be a vertex of D. An E_1-subdivision is x-*suitable*, if x is on the subdivision of the directed cycle aba.

We shall present a procedure E_1-`Subdivision`(D, x), that given a digraph D and a vertex x returns 'no' only if there is no x-suitable E_1-subdivision, and returns 'yes' when it finds an E_1-subdivision (not necessarily x-suitable). Moreover, this procedure runs in $O(n^3m(n+m))$ time. Hence running E_1-`Subdivision`(D,x) for every vertex $x \in V(D)$, one solves E_1-Subdivision in $O(n^4m(n+m))$ time.

E_1-`Subdivision`(D,x) uses a subprocedure `Reduction`$(D,x,\mathcal{S})$ that, given a 1-separation $\mathcal{S} = (W_1,T,W_2)$ in D such that $x \in W_1$ and $W_2 \neq \emptyset$, reduces the problem to two smaller instances of E_1-`Subdivision`. `Reduction`$(D,x,\mathcal{S})$ proceeds as follows.

Let y be a vertex in W_2. We run a Menger algorithm that finds a 1-separation (W_1',T',W_2') of (x,y). The set W_1' is the set of vertices reachable from x in $D-T'$. We then replace $\mathcal{S}$ by $\mathcal{S}'$, that is, we set $W_1 := W_1'$, $T := T'$, and $W_2 := W_2'$. So now every vertex in W_1 can be reached from x.

If $T = \emptyset$, then we return E_1-`Subdivision`$(D[W_1],x)$. This is clearly valid since all the vertices of an x-suitable E_1-subdivision are in the out-section of x and thus cannot be in W_2 because there are no arcs from W_1 to W_2.

Suppose now that $|T| = 1$, say $T = \{t\}$. A vertex w_1 of W_1 is *W_2-reachable* if in D there exists a (t,w_1)-dipath whose internal vertices are all in W_2, and a vertex w_2 of W_2 is *W_1-reaching* if in D there exists a (w_2,t)-dipath whose internal vertices are all in W_1. Let D_1 be the digraph obtained from $D[W_1 \cup \{t\}]$ by adding the arc tw_1 (if it is not already in $A(D)$) for every W_2-reachable vertex $w_1 \in W_1$; let D_2 be the digraph obtained from $D[W_2 \cup \{t\}]$ by adding the arc w_2t (if it is not already in $A(D)$) for every W_1-reaching vertex w_2 of W_2.

`Reduction`$(D,x,\mathcal{S})$ returns (E_1-`Subdivision`(D_1,x) or E_1-`Subdivision`(D_2,t)).

The validity of the subprocedure `Reduction` is justified by the following claim.

Claim 21. (i) *If D contains an x-suitable E_1-subdivision, then either D_1 contains an x-suitable E_1-subdivision or D_2 contains a t-suitable E_1-subdivision.*

(ii) *For any $i = 1,2$, if D_i contains an E_1-subdivision, then D contains an E_1-subdivision.*

Subproof. (i) Assume that D contains an x-suitable E_1-subdivision S. Let C_1 and C_2 be the directed cycles in S corresponding to aba and cdc, respectively, and let P_1 and P_2 be the two disjoint $(V(C_1),V(C_2))$-dipaths in S. By definition, $x \in V(C_1)$.

We distinguish several cases according to the position of C_1 and C_2.

Assume first that C_1 is contained in $D[W_1 \cup \{t\}]$. Since all dipaths from W_1 to W_2 go through t, one of the P_i, say P_1, is in $D[W_1]$, and $V(C_2) \cap W_1 \neq \emptyset$.

- Suppose that C_2 is in $D[W_1]$. If $V(P_2) \cap W_2 = \emptyset$, then S is an x-suitable E_1-subdivision in D_1. If $V(P_2) \cap W_2 \neq \emptyset$, then there is a vertex w_1 of $W_1 \cap V(P_2)$ such that $P_2 \cap D[W_2] = P_2]t,w_1[$. Therefore the digraph S_1 obtained from S by replacing $P_2[t,w_1]$ by the arc tw_1 is an x-suitable E_1-subdivision in D_1.

- Suppose now that $V(C_2) \cap W_2 \neq \emptyset$. Then necessarily $t \in V(C_2)$, and P_1 and P_2 are in $D[W_1 \cup \{t\}]$. Moreover, there is a vertex w_1 of $W_1 \cap V(C_2)$ such that $C_2 \cap D[W_2] = C_2]t,w_1[$. Therefore the digraph S_1 obtained from S by replacing $C_2[t,w_1]$ by the arc tw_1 is an x-suitable E_1-subdivision in D_1.

Suppose now that $V(C_1) \cap W_2 \neq \emptyset$. Since $x \in V(C_1)$, C_1 necessarily contains t, because there is no arcs from W_1 to W_2. Moreover, there exist two vertices $w_1 \in W_1 \cap V(C_1)$ and $w_2 \in W_2 \cap V(C_1)$ such that $C_1 = C_1[t, w_2]w_2w_1C_1[w_1, t]$, $D[W_1] \cap C_1 = C_1[w_1, t[$ and $D[W_2] \cap C_1 = C_1]t, w_2]$. Now, C_2 is contained in $D[W_1 \cup W_2]$, and so C_2 is contained either in $D[W_1]$ or in $D[W_2]$.

- Assume that C_2 is in $D[W_1]$. Let w_1^i be the first vertex along P_i in W_1. Since there is no arc form W_1 to W_2, all vertices in $P_i[s(P_i), w_1^i[$ are in W_2 and all vertices in $P_i[w_1^i, t(P_i)]$ are in W_1.

  a) If $s(P_1) = w_1^1$ and $s(P_2) = w_1^2$, then the digraph S_1 obtained from S by replacing $C_1[t, w_1]$ by the arc tw_1 is an x-suitable E_1-subdivision in D_1.

  b) If $s(P_1) \neq w_1^1$ and $s(P_2) = w_1^2$, then the digraph S_1 obtained from S by replacing $C_1[t, w_1]$ and $P_1[s(P_1), w_1^1]$ by the arcs tw_1 and tw_1^1 is an x-suitable E_1-subdivision in D_1.

  c) Assume finally that $s(P_1) \neq w_1^1$ and $s(P_2) \neq w_1^2$. Then both $s(P_1)$ and $s(P_2)$ are in $W_2 \cup \{t\}$. Since every vertex of W_1 is reachable from x in $D[W_1]$, there is a $(V(C_1) \cap W_1, V(C_2))$-dipath Q in $D[W_1]$. Observe that $s(Q)$ is distinct from $s(P_1)$ and $s(P_2)$ because it is in W_1. If Q does not intersect $P_1 \cup P_2$, set $Q' := Q$. Otherwise, without loss of generality, the first vertex z along Q in $P_1 \cup P_2$ is in P_2. In this case, set $Q' = Q[s(Q), z]P_2[z, t(P_2)]$. In both cases, the subdigraph S' obtained from S by replacing P_2 by Q' is an E_1-subdivision. Now Subcase (b) applies to S', so D_1 contains an x-suitable E_1-subdivision in D_1.

- Assume that C_2 is in $D[W_2]$. Then P_1 and P_2 are in $D[W_2 \cup \{t\}]$, and so $C_1 \cap D[W_1] = C_1[w_1, t[$. Hence the digraph S_2 obtained from S by replacing $C_1[w_2, t]$ by the arc w_2t is a t-suitable E_1-subdivision in D_2.

(ii) Suppose that S_1 is an E_1-subdivision in D_1. By construction of D_1, all arcs of $A(S_1) \setminus A(D)$ are joining t to some W_2-reachable vertex. Since each vertex in E_1 has out-degree at most 2, there are at most two arcs in $A(S_1) \setminus A(D)$.

If there is no in arc in $A(S_1) \setminus A(D)$, then S_1 is an E_1-subdivision in D. If there is a unique arc tw_1 in $A(S_1) \setminus A(D)$, then the digraph S obtained from S_1 by replacing the arc tw_1 by a (t, w_1)-dipath with internal vertices in W_2 is an E_1-subdivision contained in D. Assume finally that $A(S_1) \setminus A(D)$ contains two arcs, tw_1 and tw_1'. Note that t has in-degree 1 and out-degree 2 in S_1. Let P (resp. P') be a (t, w_1)-dipath (resp. (t, w_1')-dipath) with all internal vertices in W_2. Let t' be the last vertex along P' which is in $V(P) \cap V(P')$. Now the digraph S obtained from S_1 by replacing tw_1 and tw_1' by the union of P and $P'[t', w_1']$ is an E_1-subdivision contained in D.

A similar argument shows that if D_2 contains an E_1-subdivision, then D contains an E_1-subdivision. $\Diamond$

Using `Reduction`, we construct another procedure `cleaning`(C_1, C_2, x) that given two disjoint directed cycles C_1 and C_2 and the vertex x, either reduces the problem or finds a pair of disjoint directed cycles (C'_1, C'_2) such that $x \in V(C'_1)$. This procedure proceeds as follows.

If C_1 contains x, then we set $(C'_1, C'_2) := (C_1, C_2)$. If C_2 contains x, then we set $(C'_1, C'_2) := (C_2, C_1)$.

Assume now that x is not in $V(C_1 \cup C_2)$. We first check whether there is a cycle C containing x. If not, then we return 'no' because D does certainly not contain any x-suitable E_1-subdivision. If C does not intersect C_1, then we set $(C'_1, C'_2) := (C, C_1)$. If C does not intersect C_2, then we set $(C'_1, C'_2) := (C, C_2)$. Henceforth, C intersects both C_1 and C_2. Let y be the first vertex after x in C that is in $V(C_1 \cup C_2)$, and let z be the last vertex before x in C that is in $V(C_1) \cup V(C_2)$. Free to permute the indices of C_1 and C_2, we may assume that $y \in V(C_1)$. Moreover, if $z \in V(C_1)$, then we set $(C'_1, C'_2) := (C[x, y]C_1[y, z]C[z, x], C_2)$. So we may assume that $z \in V(C_2)$. Using a Menger algorithm, we check whether there are two disjoint (x, C_2)-dipaths. If not, then we obtain a 1-separation $\mathcal{S} = (W_1, T, W_2)$ in D such that $x \in W_1$ and $V(C_2) \subseteq T \cup W_2$. In that case, we return $\texttt{Reduction}(D, x, \mathcal{S})$. Suppose now that there are two independent (x, C_2)-dipaths Q_1 and Q_2. If Q_i does not intersect C_1, then the closed walk $Q_iC_2[t(Q_i), y]C[y, x]$ contains a cycle through x. We return this cycle and C_1 as (C'_1, C'_2). If Q_1 and Q_2 both intersect C_1, then there are two disjoint (C_1, C_2)-dipaths, whose union with C_1 and C_2 is an E_1-subdivision. So we return 'yes'. This finishes the subprocedure $\texttt{cleaning}(C_1, C_2, x)$.

Finally, let us describe E_1-$\texttt{Subdivision}(D, x)$.

We first check whether there are two disjoint directed cycles in D. If not, then we return 'no' because D cannot contain an E_1-subdivision in this case. Henceforth, we may assume that there are two disjoint directed cycles Γ_1 and Γ_2.

We then run $\texttt{cleaning}(\Gamma_1, \Gamma_2, x)$. If the instance was not reduced by this procedure, we get two disjoint directed cycles (Γ'_1, Γ'_2) such that $x \in V(\Gamma'_1)$.

We run a Menger algorithm to check whether there are two disjoint $(V(\Gamma'_1), V(\Gamma'_2))$-dipaths. If two such dipaths P_1 and P_2 exist, then $\Gamma'_1 \cup \Gamma'_2 \cup P_1 \cup P_2$ is an x-suitable E_1-subdivision, and we return 'yes'. If not, then the Menger algorithm returns a 1-separation $\mathcal{S} = (W_1, T, W_2)$ of $(V(\Gamma'_1), V(\Gamma'_2))$. If $x \in W_1$, then we return $\texttt{Reduction}(D, x, \mathcal{S})$. If $x \notin W_1$, then $T = \{x\}$. In this case we proceed as described below.

From now on, we may assume that D contains no x-suitable E_1-subdivision in which the cycle through x lies in $D[W_2 \cup \{x\}]$ and the other cycle lies in $D[W_1]$. This is guaranteed either by nonexistence of C_1 or C_2, or by the outcome of the previous step.

Let D'_1 be the digraph obtained from D by contracting W_2 into a vertex u and let D_2 be the digraph obtained from $D[W_2 \cup \{x\}]$ by adding the arc w_2x (if it is not already in $A(D)$) for every W_1-reaching vertex w_2 of W_2. We return (E_1-$\texttt{Subdivision}(D'_1, x)$ or E_1-$\texttt{Subdivision}(D_2, x)$).

This is valid by the following claim whose proof is very similar to the one of Claim 21.

Claim 22. (i) *If D contains an x-suitable E_1-subdivision, then either D'_1 contains an x-suitable E_1-subdivision or D_2 contains a x-suitable E_1-subdivision.*

(ii) *If D'_1 or D_2 contains an E_1-subdivision, then D contains an E_1-subdivision.*

Subproof. (i) Assume that D contains an x-suitable E_1-subdivision S. Let C_1 and C_2 be the directed cycles in S corresponding to aba and cdc, respectively, and let P_1 and P_2 be the two disjoint $(V(C_1), V(C_2))$-dipaths in S. By definition, $x \in V(C_1)$.

We distinguish several cases according to the positions of C_1 and C_2. Suppose first that C_1 is contained in $D[W_1 \cup \{x\}]$. Since all dipaths from W_1 to W_2 go through x, one of the P_i, say P_1, is in $D[W_1]$, and $V(C_2) \cap W_1 \neq \emptyset$. Hence C_2 is in $D[W_1]$. If $V(P_2) \cap W_2 = \emptyset$, then S is an x-suitable E_1-subdivision in D'_1. If $V(P_2) \cap W_2 \neq \emptyset$, then P_2 contains x and there is a vertex w_1 of $W_1 \cap V(P_2)$ such that $P_2 \cap D[W_2] = P_2]x, w_1[$. Therefore the digraph S_1 obtained from S by replacing $P_2[x, w_1]$ by the dipath xuw_1 is an x-suitable E_1-subdivision in D'_1.

The second possibility is that $C_1 \subseteq D[W_2 \cup \{x\}]$. Since C_1 contains x, C_2 is contained in $D[W_1 \cup W_2]$, and so C_2 is contained either in $D[W_1]$ or in $D[W_2]$. If C_2 lies in $D[W_1]$, then the digraph S' obtained from S by contracting all the vertices of W_2 into U is an x-suitable E_1-subdivision in D'_1. If $C_2 \subseteq D[W_2]$, then S is an x-suitable E_1-subdivision in D_2.

Finally, suppose that $V(C_1) \cap W_2 \neq \emptyset$ and $V(C_1) \cap W_1 \neq \emptyset$. Then there exist two vertices $w_1 \in W_1 \cap V(C_1)$ and $w_2 \in W_2 \cap V(C_1)$ such that $C_1 = C_1[x, w_2]w_2w_1C_1[w_1, x]$, $D[W_1] \cap C_1 = C_1[w_1, x[$, and $D[W_2] \cap C_1 = C_1]x, w_2]$. Now C_2 is contained in $D[W_1 \cup W_2]$, and so C_2 is contained either in $D[W_1]$ or in $D[W_2]$.

- Suppose first that $C_2 \subseteq D[W_1]$. Let w_1^i be the first vertex along P_i in W_1. Since there is no arc form W_1 to W_2, all vertices of $P_i[s(P_i), w_1^i[$ are in W_2 and all vertices of $P_i[w_1^i, t(P_i)]$ are in W_1.

  a) If both P_1 and P_2 are contained in $D[W_1 \cup \{x\}]$, then the digraph S_1 obtained from S by replacing $C_1[x, w_1]$ by the dipath xuw_1 is an x-suitable E_1-subdivision in D'_1.

  b) If one of the two P_i's, say P_1 is contained in $D[W_1 \cup \{x\}]$, then the digraph S_1 obtained from S by replacing $C_1[x, w_1]$ and $P_1[s(P_1), w_1^1]$ by the dipaths xuw_1 and uw_1^1) is an x-suitable E_1-subdivision in D'_1.

  c) Assume now that P_1 and P_2 both intersect W_2. Both $s(P_1)$ and $s(P_2)$ are in $W_2 \cup \{x\}$. Observe that every vertex of W_1 is reachable from $\Gamma'_1 \setminus \{x\}$ in D_1, so there is an (x, C_2)-dipath Q in $D[W_1 \cup \{x\}]$. If Q does not intersect $P_1 \cup P_2$, then set $Q' := Q$. Otherwise, we may assume that the first vertex z in $V(P_1 \cup P_2)$ along $Q - x$ in P_2. Set $Q' := Q[x, z]P_2[z, t(P_2)]$. Observe that Q' and P_1 are disjoint except possibly in x. Now the digraph S_1 obtained from $C_1 \cup C_2 \cup P_1 \cup Q'$ by replacing $C_1[x, w_1]$ and $P_1[s(P_1), w_1^1]$ by the dipaths xuw_1 and uw_1^1 is an x-suitable E_1-subdivision in D'_1.

- The second possibility is that $C_2 \subseteq D[W_2]$. Then P_1 and P_2 are in $D[W_2 \cup \{x\}]$, and so $C_1 \cap D[W_1] = C_1[w_1, x[$. Hence the digraph S_2 obtained from S by replacing $C_1[w_2, x]$ by the arc w_2x is an x-suitable E_1-subdivision in D_2.

(ii) We already showed in Claim 21 that if D_2 contains an E_1-subdivision, then D contains an E_1-subdivision. Let us now prove that if D'_1 contains an E_1-subdivision, then D contains an E_1-subdivision.

Suppose that S_1 is an E_1-subdivision in D'_1. If $u \notin V(S_1)$, then S_1 is an E_1-subdivision in D. If $u \in V(S_1)$, then S_1 contains a dipath (x, u, w_1) for some $w_1 \in W_1$, and possibly one other arc uw'_1. By definition of D'_1, there are a (t, w_1)-dipath P and a (t, w'_1)-dipath P' with internal vertices in W_2. Let t' be the last vertex along P' which is in $V(P) \cap V(P')$. Then the digraph S obtained from S_1 by replacing tw_1 by P and tw'_1 (if it exists) by $P'[t', w'_1]$ is an E_1-subdivision contained in D. ◊

Let us now estimate the complexity of E_1-`Subdivision`(D, x). This procedure first finds two disjoint directed cycles and then runs a few Menger algorithms and either returns an answer or make a recursive call on two smaller instances, which are either D_1 and D_2, or D'_1 and D_2. Two disjoint directed cycles can be found in $O(n^3m(n + m)$ by Corollary 1.25.

The smaller instances D_1 and D'_1 can be constructed in linear time: indeed a vertex $w_1 \in W_1$ is W_2-reachable if and only if it is has an in-neighbour in the out-section of x in $D[W_2 \cup \{x\}]$, and so all W_2-reachable vertices can be found in linear time. Similarly, the set of W_1-reaching vertices in W_2 can be determined in linear time, and thus D_2 can be constructed in linear time. Hence E_1-`Subdivision`(D, x) makes at most cn^2 operations before calling recursively, for some absolute constant c.

Let us denote by $T(n)$ the maximum time for E_1-`Subdivision`(D, x) on a digraph with n vertices. Since $|V(D_1)|+|V(D_2)| = |V(D)|+1$ and $|V(D'_1)|+|V(D_2)| = |V(D)| + 2$, we have

$$T(n) \leq \max\{T(n_1) + T(n_2) + cn^2 \mid n_1, n_2 < n \text{ and } n_1 + n_2 \in \{n + 1, n + 2\}\}.$$

This implies that $T(n) \leq O(n^3)$.

Therefore E_1-`Subdivision` runs in time $O(n^3m(n + m))$. □

3.5.2 E_2 is tractable

The aim of this subsection is to prove that the digraph E_2 is tractable.

Theorem 3.13. E_2-Subdivision *can be solved in* $O(n^4m(n + m))$ *time.*

In order to prove Theorem 3.13, we need some preliminary results.

Let F be a subdigraph of a digraph D. An *ear* of F in D is an oriented path in D containing at least one edge, whose end-vertices lie in F but whose edges and internal vertices do not belong to F. A *directed ear* of F is an ear of F that is a directed path. A digraph is said to be *robust* if it is strong and 2-connected. The following lemma is well-known; it is very similar to Proposition 5.11 of [8].

Lemma 3.14. *Let F be a non-trivial strong subdigraph of a robust digraph D. Then F has a directed ear in D. Moreover such a directed ear can be found in time $O(n(n+m))$.*

Proof. Because D is 2-connected, F has an oriented ear in D. Among all such ears, we choose one in which the number of reverse arcs (those directed towards its initial vertex) is as small as possible. We show that this path P is in fact a directed ear.

Assume the contrary, and let uv be a backward arc of P. Because D is strong, there exist in D an (F, u)-dipath Q and a (v, F)-dipath R (one of which might be of length zero). The initial vertex of Q and the terminal vertex of R must be one and the same vertex, for otherwise the directed walk $QuvR$ would contain a directed ear of F, contradicting the choice of P and our assumption that P is not a directed ear. Let this common vertex be z. We may assume that $z \neq s(P)$ (the case $z \neq t(P)$ being analogous). Then the $(s(P), z)$-walk $P[s(P), v]Rz$ contains an oriented $(s(P), z)$-path that contradicts the choice of P. Thus P is indeed a directed ear of F.

A directed ear of F may be found by running a search from each vertex of F in $D \setminus A(F)$. Hence it can be found in $O(n(n+m))$ steps. $\square$

Let D_1 and D_2 be two subdigraphs in D. Two dipaths are (D_1, D_2)-*opposite* if they are disjoint and one of them is a (D_1, D_2)-dipath and the other is a (D_2, D_1)-dipath. Opposite dipaths play an important role in detecting E_2-subdivisions because of the following easy lemma.

Lemma 3.15. *Let D be a digraph and D_1 and D_2 two disjoint non-trivial strong subdigraphs of D. If there are (D_1, D_2)-opposite dipaths in D, then D contains an E_2-subdivision.*

Proof. Let P_1 and P_2 be two (D_1, D_2)-opposite dipaths, with P_1 a (D_1, D_2)-dipath and P_2 a (D_2, D_1)-dipath. Since D_i is strong, there is an $(t(P_{3-i}), s(P_i))$-dipath Q_i in D_i. For the same reason, there is a $(Q_i - s(Q_i), s(Q_i))$-dipath R_i in D_i. Now $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2$ is an E_2-subdivision in D. $\square$

Proof of Theorem 3.13. We shall present a recursive procedure E_2-`Subdivision`(D), that given a digraph D decides whether it contains an E_2-subdivision or not.

This procedure proceeds as follows. We first check whether D is robust. If not, then we solve the problem for each robust component separately. Henceforth, we may assume that D is robust.

We next check whether there are two disjoint directed cycles. If not, then we return 'no' since E_2 contains two disjoint directed cycles. If two such cycles C_1 and C_2 exist, then we compute the strong component D_1 of C_1 in $D - C_2$, and next the strong component D_2 of C_2 in $D - D_1$. Hence D_1 and D_2 are two disjoint non-trivial strong subdigraphs in D. Moreover, they satisfy the following property.

Claim 23. *If P is a (D_1, D_2)-dipath and Q is a (D_2, D_1)-dipath, then P and Q are internally disjoint.*

Subproof. Suppose, by way of contradiction, that P and Q have a common internal vertex x. The vertex x is in the strong component of D_1 in $D - D_2$. Hence it is in the strong component of C_1 in $D - C_2$. So $x \in D_1$, a contradiction. $\Diamond$

We check whether there are (D_1, D_2)-opposite dipaths in D. By using Claim 23 this task reduces to finding a (D_1, D_2)-dipath and a (D_2, D_1)-dipath whose end-vertices are disjoint. If there are such paths, then, by Lemma 3.15, D contains an E_2-subdivision and we return 'yes'. Henceforth we may assume that there are no (D_1, D_2)-opposite dipaths in D.

By Lemma 3.14, there is a directed ear P_1 of D_1. Since $D_1 \cup P_1$ is strong, P_1 must intersect D_2. Furthermore, the intersection of P_1 and D_2 is reduced to a single vertex, because there are no (D_1, D_2)-opposite dipaths. Let u_1 be the initial vertex of P_1, v_1 the terminal vertex of P_1, and let u_2 be the vertex of $P_1 \cap D_2$. By Lemma 3.14, there is a directed ear P_2 of D_2. If the terminal vertex of P_2 is u_2, then we consider the converse of D, D_1, D_2, P_1 and P_2. (This is valid since E_2 is its own converse.) Hence, we may assume that the terminal vertex v_2 of P_2 is different from u_2. Similarly to P_1, the directed ear P_2 intersects D_1 in a single vertex w_1. Necessarily, $w_1 = v_1$ for otherwise $P_1[u_2, v_1]$ and $P_2[w_1, v_2]$ are (D_1, D_2)-opposite, by Claim 23. Also, the initial vertex of P_2 is u_2, and thus we may assume that both ears have common segment $P_1[u_2, v_1] = P_2[u_2, v_1]$. Furthermore, $P_1[u_1, u_2]$ and $P_2[v_1, v_2]$ are disjoint for otherwise, there are two (D_1, D_2)-opposite dipaths.

Set $P = P_1P_2[v_1, v_2]$. We check whether $D - (D_1 \cup D_2)$ contains a non-trivial strong component D_3. If D_3 exists and intersects both $P[u_1, v_1]$ and $P[u_2, v_2]$, then we return 'yes'. This is valid by the following claim.

Claim 24. *If D_3 intersects both $P[u_1, v_1]$ and $P[u_2, v_2]$, then D contains an E_2-subdivision.*

Subproof. Suppose first that D_3 intersects $P[u_2, v_1]$. If D_3 also intersects $P[u_1, u_2]$, then there are two (D_1, D_3)-opposite dipaths, and so by Lemma 3.15, D contains an E_2-subdivision. Similarly, if D_3 also intersects $P[v_1, v_2]$, then there are two (D_2, D_3)-opposite dipaths, and so D contains an E_2-subdivision. Hence we may assume that D_3 does not intersect $P[u_1, u_2] \cup P[v_1, v_2]$. If D_3 intersects $P[u_2, v_1]$ in more than one vertex, then let u_3 (resp. v_3) be the first (resp. last) vertex of D_3 along $P[u_2, v_1]$. The two dipaths $P[u_1, u_3]$ and $P[v_3, v_1]$ are (D_1, D_3)-opposite. Hence, by Lemma 3.15, D contains an E_2-subdivision.

So D_3 intersects $P[u_2, v_1]$ in a unique vertex, say w_3. By Lemma 3.14, there is a directed ear P_3 of D_3. By definition of D_1 and D_2, P_3 intersects both D_1 and D_2. Now one of the two end-vertices of P_3, say u_3, is distinct from w_3.

If u_3 is the initial vertex of P_3, then consider the first vertex v_3 along P_3 in $D_1 \cup D_2$. By definition of D_2, $v_3 \in V(D_1)$. Now $P_3[u_3, v_3]$ is disjoint from $P[u_2, w_3]$ since D_3 is a strong component of $D - D_1$. Furthermore $P_3[u_3, v_3]$ is internally disjoint from $P[u_1, u_2]$ because the dipaths $P[u_1, u_2]$ and $P[u_2, w_3]D_3[w_3, u_3]P_3[u_3, v_3]$ are internally disjoint by Claim 23. If $v_3 \neq u_1$, then $P[u_1, w_3]$ and $P_3[u_3, v_3]$ are (D_1, D_3)-opposite, and so by Lemma 3.15, D contains an E_2-subdivision. Finally, if $v_3 = u_1$, then $P[v_1, v_2]$ is disjoint from $P_3[u_3, v_3]$. In this case, $P_3[u_3, v_3]$ and $P[v_1, v_2]D_2[v_2, u_2]P[u_2, w_3]$ are (D_1, D_3)-opposite paths giving an E_2-subdivision.

If u_3 is the terminal vertex of P_3, then we get the result analogously.

Suppose now that D_3 does not intersect $P[u_2, v_1]$. Then it must intersect both $P[u_1, u_2]$ and $P[v_1, v_2]$. Let u_3 be the first vertex of D_3 along $P[u_1, u_2]$ and v_3 be the last vertex of D_3 along $P[v_1, v_2]$. Now $P[u_1, u_3]$ and $P[v_3, v_2]D_2[v_2, u_2]P[u_2, v_1]$ are two (D_1, D_3)-opposite dipaths. Thus, by Lemma 3.15, D contains an E_2-subdivision. $\Diamond$

If D_3 exists, we are either done by Claim 24, or D_3 is disjoint from one of the paths, $P[u_1, v_1]$ or $P[u_2, v_2]$. Now, we replace D_1 by the strong component of $D - D_3$ containing $D_1 \cup D_2$. Observe that this makes the order of D_1 increase. Further, we replace D_2 by D_3 and replace C_2 by a cycle in this new strong digraph. By doing this, Claim 23 remains valid. By repeating the process as long as possible, we reach the situation where all strong components of $D - (D_1 \cup D_2)$ are trivial, that is $D - (D_1 \cup D_2)$ is acyclic. We check whether $D_2 - u_2$ contains a directed cycle. If it contains such a cycle C'_2, then let D'_1 be the strong component of C_1 in $D_2 - C'_2$ and D'_2 the strong component of C'_2 in $D - D'_1$. Clearly, D'_1 is a superdigraph of $D_1 \cup P_1$, so $|D'_1| > |D_1|$. Hence, we replace D_1, D_2 by D'_1 and D'_2, respectively, and repeat the procedure for the new pair D_1, D_2. So we may assume that there is no cycle in $D_2 - u_2$.

Moreover, if there is a (v_2, u_1)-dipath Q whose internal vertices are not in $V(D_1 \cup D_2)$, we also check whether there is a cycle in $D_2 - v_2$. If yes, then as above we find new pair of non-trivial strong digraphs (D'_1, D'_2) with $|D'_1| > |D_1|$. Hence in that case, we may also assume that there is no cycle in $D_2 - v_2$.

Let D^* be the digraph obtained from D by contracting D_2 into a single vertex w^*. We return E_2-`Subdivision`(D^*). The following claim shows that this recursive call is valid.

Claim 25. *D contains an E_2-subdivision if and only if D^* contains an E_2-subdivision.*

Subproof. Suppose that D contains an E_2-subdivision S. Let C_1 and C_2 be the two disjoint directed cycles in S corresponding to the subdivision of *aba* and *cdc*. Observe that each C_i intersects $D_1 \cup D_2$, because there is no strong component in $D - (D_1 \cup D_2)$.

C_1 and C_2 cannot be in both in D_2 for otherwise one of the two avoids u_2, which is impossible. Moreover, one of the cycles cannot be in D_1 while the other one is in D_2 for otherwise the (C_1, C_2)- and (C_2, C_1)-dipaths in S would contains two (D_1, D_2)-opposite dipaths in D, which is impossible. If C_1 and C_2 are both contained in D_1, then either S is contained in D_1, in which case it is also in D^*, or the arcs of S which are not in $A(D_1)$ induce a directed ear R which intersects with D_2 in a single vertex w_2, because there are no (D_1, D_2)-opposite dipaths. Hence the digraph S^* obtained from D by replacing the vertex w_2 by w^* is an E_2-subdivision in D^*. So we may assume that one of the cycles intersects D_1 and D_2.

Case 1: There is no (v_2, u_1)-dipath whose internal vertices are not in $V(D_1 \cup D_2)$. In that case, all (D_2, D_1)-dipaths are (u_2, v_1)-dipaths. Therefore, the two cycles C_1 and C_2 cannot both intersect D_1 and D_2. Thus one of them, say C_1, does not intersect both, and thus must be contained in D_1. Consequently, C_2 intersects both D_1 and D_2. Thus C_2 contains a (u_2, v_1)-dipath. Therefore, the (C_2, C_1)-dipath in S must be in D_1, and the (C_1, C_2)-dipath intersects D_2 in u_2. Therefore the digraph S^* obtained from S by contracting the vertices of $V(S) \cap V(D_2)$ into w^* is an E_2-subdivision in D^*.

Case 2: There is a (v_2, u_1)-dipath Q whose internal vertices are not in $V(D_1 \cup D_2)$. In this case, all (D_1, D_2)-dipaths are (u_1, u_2)- or (v_1, v_2)-dipaths because there are no (D_1, D_2)-opposite dipaths in D. For the same reason, all (D_2, D_1)-dipaths are (u_2, v_1)- or (v_2, u_1)-dipaths. Therefore, the two cycles C_1 and C_2 cannot both intersect D_1 and D_2. Thus one of them, say C_1, does not intersect both, and thus must be contained in D_1. Consequently, C_2 intersects both D_1 and D_2.

We are in one of the three following cases: $C_2 = P[u_1, v_1]D_1[v_1, u_1]$, $C_2 = P[u_2, v_2]D_2[v_2, u_2]$, or $C = PQ$. In each of these cases, one can see that the digraph S^* obtained from S by contracting the vertices of $V(S) \cap V(D_2)$ in w^* contains an E_2-subdivision in D^*.

Conversely, suppose that D^* contains an E_2-subdivision S^*. If S^* does not contains w^*, then it is contained in D. So we may assume that S^* contains w^*.

Suppose w^* has in-degree and out-degree 1 in S^*. Let u (resp. v) be the in-neighbour (resp. out-neighbour) of w^* in S^*. By definition of D^*, the vertex u has an out-neighbour u_2' in D_2 and the vertex v has an in-neighbour v_2' in D_2. Hence the digraph S obtained from S^* by replacing the dipath uw^*v by the dipath $uD[u_2', v_2']v$ is an E_2-subdivision in D.

Suppose w^* has in-degree 1 and out-degree 2 in S^*. Let u be the in-neighbour of w^* in S^* and let v and v' be the out-neighbours of w^* in S. By definition of D^*, the vertex u has an out-neighbour u_2' in D_2 and the vertex v (resp. v') has an in-neighbour v_2' (resp. v_2'') in D_2. Let P be a (u_2', v_2')-dipath in D_2 and Q be a (P, v_2'')-dipath in D. The digraph S obtained from S^* by replacing the vertex w^* by $P \cup Q$ is an E_2-subdivision in D.

If w^* has in-degree 2 and out-degree 1, we find an E_2-subdivision in D in a similar way. $\Diamond$

Let us now estimate the time complexity of E_2-`Subdivision`. The procedure first constructs the digraphs D_1 and D_2. It requires to find two disjoint directed cycles and then to compute two strong components. By Corollary 1.25 this can be done in time $O(n^3 m(n+m))$. Next, the algorithm checks a few times for opposite paths, and for directed cycles, before either increasing the order of D_1 or making a recursive call. Checking if there are (D_1, D_2)-opposite paths can be done in $O(n(n+m))$ time by running searches in $D \setminus A(D_1 \cup D_2)$ from each vertex, and finding if there is a directed cycle in a digraph can be done in $O(n(n+m))$ time by checking for each vertex v if there is a (v, v)-handle. Thus, since the order of D_1 increases at most

$O(n)$ times, there are at most $O(nm(n+m))$ such operations between two recursive calls. Hence the time between two recursive calls is at most $O(n^3m(n+m))$. At each call, the order of the instance digraph decreases. Hence the time complexity of E_2-`Subdivision` is $O(n^4m(n+m))$. □

3.5.3 E_9 is tractable

Theorem 3.16. *E_9 is tractable.*

Given two vertices c' and d' in D, a (c', d')*-forced* E_9-subdivision is an E_9-subdivision in D with c-vertex c' and d-vertex d'.

To prove Theorem 3.16, we shall describe a polynomial-time algorithm to solve E_9-Subdivision. The key ingredient of our algorithm is a polynomial-time procedure E_9-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$ whose input is a strong digraph D and two sets of two vertices $\{c_1, c_2\}, \{d_1, d_2\}$ ($c_1 \neq c_2$ and $d_1 \neq d_2$). Let $\hat{D}(\{c_1, c_2\}, \{d_1, d_2\})$ be the digraph obtained from D by adding two new vertices c'', d'' and the four arcs $c_1c'', c_2c'', d''d_1, d''d_2$. The procedure E_9-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$ returns 'yes' if it finds an E_9-subdivision in $\hat{D}$, and returns 'no' if $\hat{D}$ has no (c'', d'')-forced E_9-subdivision.

Before describing the procedure E_9-`Strong+`, let us describe the algorithm for E_9-Subdivision, assuming we have such a procedure.

3.5.3.1 The algorithm

For every vertex c' and d', we run a procedure E_9-`Forced`(c', d') that returns 'yes' if it finds an E_9-subdivision, and return 'no' if it finds evidence that there is no (c', d')*-forced* E_9-subdivision in D. Since there are $O(n^2)$ possible choices of c' and d', if E_9-`Forced` runs in polynomial time, the overall algorithm will also run in polynomial time.

E_9-`Forced`(c', d') proceeds as follows. We first compute the strong components $G_1, \dots, G_p$ of $D - \{c', d'\}$. Observe that the directed cycle (corresponding to aba) in an E_9-subdivision must be contained in one of the strong components. For each strong component G_i, we run a procedure E_9-`Suitable`(c', d', G_i) that returns 'yes' if it finds an E_9-subdivision and returns 'no' only if there is no (c', d')-forced E_9-subdivision whose directed cycle is in G_i. Such a subdivision is called (c', d', G_i)-suitable.

We first test if there is a (d', c')-dipath in $D - G_i$. We then run two separate procedures depending on whether or not such a path exists.

<u>Case 1:</u> Assume there is a (d', c')-dipath P in $D - G_i$. Let X be the set of vertices $x \in V(G_i)$ that are terminal vertices of a (d', G_i)-dipath in D. The set X can be computed in linear time by running a search from d' in the digraph obtained from D by deleting all the arcs having their tail in G_i. Let Y be the set of vertices $y \in V(G_i)$ that are initial vertices of a (G_i, c')-dipath in D. Similarly to X, the set Y can be determined in linear time.

If there are no two distinct vertices $x \in X$ and $y \in Y$, then we return 'no'. Otherwise we return 'yes'. This is valid according to the following claim.

Claim 26. (i) *If there are no two distinct vertices $x \in X$ and $y \in Y$, then D contains no (c', d', G_i)-suitable E_9-subdivision.*

(ii) *If there are two distinct vertices $x \in X$ and $y \in Y$, then D contains an E_9-subdivision.*

Subproof. (i) If D contains a (c', d', G_i)-suitable E_9-subdivision S, then consider the directed cycle C in S. This cycle is in G_i. Moreover, in S, there are two disjoint (d', C)- and (C, c')-dipaths Q and Q', respectively. Then the first vertex x in G_i along Q is in X and the last vertex y in G_i along Q' is in Y. Since Q and Q' are disjoint, x and y are distinct.

(ii) Suppose that there are vertices $x \in X$ and $y \in Y$, where $x \neq y$. Let Q be a (d', G_i)-dipath with terminal vertex x and let Q' be a (G_i, c')-dipath with initial vertex y. Observe that Q and Q' do not intersect because G_i is a strong component of $D - \{c', d'\}$. Since G_i is strong, there are an (x, y)-dipath R and a $(y, R-y)$-dipath R' in G_i. Now P contains a (Q, Q')-dipath P'. Thus $P' \cup Q[s(P'), x] \cup Q'[y, t(P')] \cup R \cup R'$ is an E_9-subdivision. $\lozenge$

Case 2: Assume now there is no (d', c')-dipath in $D - G_i$.

If D contains a (c', d', G_i)-suitable E_9-subdivision S, then the two independent (d', c')-dipaths in S intersect G_i. Since G_i is a strong component of $D - \{c', d'\}$, each of these two paths consists of three segments: a (d', G_i)-dipath, followed by a dipath in G_i, and ending with a (G_i, c')-dipath. The idea is to guess which are the first vertices d_1, d_2 and last vertices c_1, c_2 in G_i along these dipaths.

Hence, consider every pair of sets of two vertices, $\{c_1, c_2\}, \{d_1, d_2\}$, where $c_1 \neq c_2$ and $d_1 \neq d_2$. Observe that there must be two independent $(d', \{d_1, d_2\})$-dipaths whose internal vertices are not in G_i and two independent $(\{c_1, c_2\}, c')$-dipaths whose internal vertices are not in G_i. We can check for the existence of two independent $(d', \{d_1, d_2\})$-dipaths with internal vertices not in G_i by running a Menger algorithm in the digraph obtained from D by deleting all the arcs with tail in G_i. Similarly, we also check the existence of two independent $(\{c_1, c_2\}, c')$-dipaths with internal vertices not in G_i. If one of these pairs of dipaths do not exist, then we proceed to the next pair. If the two pairs of dipaths exist, they are internally disjoint from each other since G_i is a strong component. In that case, one can easily see that D contains a (c', d', G_i)-suitable E_9-subdivision S such that d_1, d_2 (resp. c_1, c_2) are the first (resp. last) vertices in G_i along the two independent (d', c')-dipaths in S if and only if the digraph $\hat{G}_i(\{c_1, c_2\}, \{d_1, d_2\})$ has no (c'', d'')-*forced* E_9-subdivision. Henceforth, we run E_9-`Strong+`$(G_i, \{c_1, c_2\}, \{d_1, d_2\})$. If this procedure returns 'yes', we also return 'yes'. If it returns 'no', we proceed to the next pair of sets $\{c_1, c_2\}, \{d_1, d_2\}$.

If all the pairs have been considered without returning 'yes', we return 'no'. This procedure is clearly valid provided that we have E_9-`Strong+` subroutine.

Hence our algorithm is valid and runs in polynomial time provided that the procedure E_9-`Strong+` is valid and runs in polynomial time. We now describe this subprocedure.

3.5.3.2 Detecting E_9 in strong digraphs

We now present procedure E_9-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$. Recall that procedure E_9-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$ returns 'yes' if it finds an E_9-subdivision in $\hat{D}$, and should returns 'no' if $\hat{D}$ has no (c'', d'')-forced E_9-subdivision. The assumption is that the input digraph D is strongly connected.

In the first phase, we treat the case when D is not 2-connected and reduce to the case when it is. Suppose that D has a cutvertex x. Let $X_1, \dots, X_p$ be the connected components of $D - x$, and for $1 \leq i \leq p$, let $D_i = D[X_i \cup \{x\}]$. Observe that each D_i is strong because D is strong.

Suppose first that c_1 and c_2 lie in different connected components of $D - x$, say X_1 and X_2 (respectively). Let P_1 be an (x, c_1)-dipath in D_1, P'_1 a $(c_1, P_1 - c_1)$-dipath, and P_2 a (c_2, x)-dipath in D_2. The digraph $P_1 \cup P'_1 \cup P_2 \cup c_1c'' \cup c_2c''$ is an E_9-subdivision in $\hat{D}$, and we return 'yes'. Similarly, if d_1 and d_2 are in different connected components of $D - x$, $\hat{D}$ contains an E_9-subdivision, and we return 'yes'.

Henceforth, we may assume that there is $i, j \in \{1, \dots, p\}$ such that $\{c_1, c_2\} \subseteq V(D_i)$ and $\{d_1, d_2\} \subseteq V(D_j)$. If $i \neq j$, then in D, there cannot be two internally disjoint (d'', c'')-dipaths, and thus there is no (c'', d'')-forced E_9-subdivision in $\hat{D}$. Therefore, we return 'no'. If $i = j$, then since E_9 is 2-connected, there is a (c'', d'')-forced E_9-subdivision in $\hat{D}$ if and only if there is a (c'', d'')-forced E_9-subdivision in $\hat{D}_i(\{c_1, c_2\}, \{d_1, d_2\})$. Hence we return E_9-`Strong+`$(D_i, \{c_1, c_2\}, \{d_1, d_2\})$.

Assume now that D is 2-connected, and so D is robust. The procedure uses a similar approach as the procedure E_2-`Subdivision`(D) to decide whether a digraph D contains an E_2-subdivision, and a key notion is the one of opposite dipaths. Recall that two dipaths are (D_1, D_2)-*opposite* if they are disjoint and one of them is a (D_1, D_2)-dipath and the other is a (D_2, D_1)-dipath. Since an E_2-subdivision contains an E_9-subdivision, Lemma 3.15 implies directly the following one.

Lemma 3.17. *Let D be a digraph and D_1 and D_2 disjoint non-trivial strong subdigraphs of D. If there are (D_1, D_2)-opposite paths in D, then D contains an E_9-subdivision.*

Lemma 3.18. *Suppose that D' is a strong subdigraph of D and R is a path in D with its end-vertices in D' and with its internal vertices in $D - D'$. If the path R has three blocks, then D contains an E_9-subdivision.*

Proof. Let $s = s(R)$, $t = t(R)$. Let Q be a (t, s)-dipath in D' and let Q' be an $(s, Q - s)$-dipath in D'. Then $Q \cup Q' \cup R$ is an E_9-subdivision in D. $\square$

Returning to the algorithm description, we first check if there are two disjoint directed cycles in D. If not, then one can solve the problem in polynomial time according to Theorem 1.26.

If two such cycles C_1 and C_2 exist, then we first compute the strong component D_1 of C_1 in $D-C_2$, and next we compute the strong component D_2 of C_2 in $D-D_1$. Hence D_1 and D_2 are two disjoint non-trivial strong subdigraphs in D. Moreover they satisfy the following property (Claim 23).

Claim 27. *If P is a (D_1, D_2)-dipath and Q is a (D_2, D_1)-dipath, then P and Q are internally disjoint.*

We check if there are (D_1, D_2)-opposite paths in D. If there are, then by Lemma 3.17, D contains an E_9-subdivision and we return 'yes'. Henceforth we may assume that there are no (D_1, D_2)-opposite paths in D.

By Lemma 3.14, there is a directed ear P_1 of D_1. Since $D_1 \cup P_1$ is strong, P_1 must intersect D_2. Furthermore, the intersection of P_1 and D_2 is reduced to a single vertex, because there are no (D_1, D_2)-opposite paths. Let u_1 be the initial vertex of P_1, v_1 the terminal vertex of P_1, and let u_2 be the vertex of $P_1 \cap D_2$. By Lemma 3.14, there is a directed ear P_2 of D_2. If the terminal vertex of P_2 is u_2, then we consider the converse of $\hat{D}$, P_1 and P_2 and exchange the roles of c'' and d'' (i.e. $(c'', d'') := (d'', c'')$) and their neighbours $((\{c_1, c_2\}, \{d_1, d_2\}) := (\{d_1, d_2\}, \{c_1, c_2\})$. (This is valid since E_9 is self-converse.) Hence, we may assume that the terminal vertex v_2 of P_2 is different from u_2. Similarly to P_1, the dipath P_2 intersects D_1 in a single vertex w_1. Clearly, $w_1 = v_1$ for otherwise $P_1[u_2, v_1]$ and $P_2[w_1, v_2]$ are (D_1, D_2)-opposite paths. Furthermore, $P_1[u_1, u_2]$ and $P_2[v_1, v_2]$ are disjoint for otherwise, there are two (D_1, D_2)-opposite dipaths.

Set $P = P_1P_2[v_1, v_2]$. We check whether $D - (D_1 \cup D_2)$ contains a non-trivial strong component D_3. If D_3 does not intersect $P[u_1, v_1]$, then we replace D_1 and D_2 by two disjoint non-trivial strong digraphs, $D_1 \cup D_2 \cup P[u_1, v_1]$ and D_3, respectively. Similarly, if D_3 does not intersect $P[u_2, v_2]$, then we replace D_1, D_2 by disjoint non-trivial strong digraphs $D_1 \cup D_2 \cup P[u_2, v_2]$ and D_3. In either case, we extend the first digraph to a strong component of $D - D_3$, while D_3 is already a strong component in the complement of the first digraph. Thus, Claim 27 remains valid. Observe that this change makes the order of D_1 increase. We also redefine u_1, v_1, u_2, v_2 and the path P if the change occurred.

If D_3 intersects both $P[u_1, v_1]$ and $P[u_2, v_2]$, then we return 'yes'. This is valid because in this case, by Claim 24, D contains an E_2-subdivision and so an E_9-subdivision.

Henceforth, we may assume that all strong components of $D - (D_1 \cup D_2)$ are trivial, that is $D - (D_1 \cup D_2)$ is acyclic.

Let F_{12} be the set of pairs $(x, y) \in V(D_1) \times V(D_2)$ such that there exists a (D_1, D_2)-dipath R with $s(R) = x$ and $t(R) = y$. Similarly, let F_{21} be the set of pairs $(y, x) \in V(D_2) \times V(D_1)$ such that there exists a (D_2, D_1)-dipath R with $s(R) = y$ and $t(R) = x$. By Claim 27 and because there are no (D_1, D_2)-opposite dipaths, we have one of the following two possible outcomes:

Case (A): $F_{21} = \{(u_2, v_1)\}$. In this case, $F_{12} = (U_1 \times \{u_2\}) \cup (\{v_1\} \times V_2)$, where $\{u_1\} \subseteq U_1 \subseteq V(D_1)$ and $\{v_2\} \subseteq V_2 \subseteq V(D_2)$.

Case (B): $F_{21} = \{(u_2, v_1), (v_2, u_1)\}$. In this case, $F_{12} = \{(u_1, u_2), (v_1, v_2)\}$. By setting $U_1 = \{u_1\}$ and $V_2 = \{v_2\}$, the set F_{12} can be written in the same way as in Case (A).

For each vertex $x \in V(D) \setminus V(D_1 \cup D_2)$, there is an $(x, D_1 \cup D_2)$-dipath and a $(D_1 \cup D_2, x)$-dipath. Since $D - (D_1 \cup D_2)$ has only trivial strong components, these two paths are internally disjoint and form a (D_1, D_2)-path R_x. We define $Z(U_1, u_2)$ as the set of all vertices $x \in V(D) \setminus V(D_1 \cup D_2)$, whose path R_x is a (U_1, u_2)-dipath. In the same way we define vertex-sets $Z(u_2, v_1)$, $Z(v_1, V_2)$, and $Z(v_2, u_1)$. Note that the latter set may be non-empty only when we have Case (B) and that these four sets partition $V(D) \setminus V(D_1 \cup D_2)$.

Next, we derive a sufficient condition for existence of E_9-subdivisions in $\hat{D}$.

Claim 28. *If there is a (D_1, D_2)-path with two or three blocks in $\hat{D}$, then $\hat{D}$ contains an E_9-subdivision.*

Subproof. Let R be a (D_1, D_2)-inpath with two blocks and let y be the vertex of out-degree 2 in R. Let w_1 be a vertex in $\{u_1, v_1\} \setminus \{s(R)\}$ and let $w_2 = u_2$ if $w_1 = u_1$ and $w_2 = v_2$ if $w_1 = v_1$. If $P[w_1, w_2]$ is disjoint from R, then the path $R \cup P[w_1, w_2] \cup D_2[w_2, t(R)]$ has three blocks and by Lemma 3.18, $\hat{D}$ contains an E_9-subdivision. On the other hand, if $P[w_1, w_2]$ intersects R, let z be the first vertex on $P[w_1, w_2]$ that lies on R. Since D_1 is a strong component of $D - D_2$, $z \in R[y, t(R)]$ and $z \neq y$. Therefore, $R[s(R), z]$ and $P[w_1, z]$ form a path with three blocks and we are done by Lemma 3.18.

Similarly, by directional duality, if there is a (D_1, D_2)-outpath with two blocks, then $\hat{D}$ contains an E_9-subdivision.

Suppose now that there is a (D_1, D_2)-outpath R in $\hat{D}$ with three blocks. Let $s = s(R)$, $t = t(R)$, and let x and y be vertices on R whose in-degree and out-degree (respectively) is equal to 2. $R[s, x] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$ because D_1 is a strong component of $D - D_2$ If $R[x, t] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$, then there is a (D_1, D_2)-path with two blocks and we have the result by the above case.

Hence, we may assume that R does not intersect $(Z(u_2, v_1) \cup Z(v_2, u_1)$. In particular, R and $P[U_2, v_1]$ are internally disjoint. If $s \neq v_1$, then $RD[t, u_2]P[u_2, v_1]$ is an ear of D_1 with three blocks and by Lemma 3.18 D contains an E_9-subdivision. Similarly, if $t \neq u_2$, then D contains an E_9-subdivision. Henceforth we may assume that $s = v_1$ and $t = u_2$.

If R is internally disjoint from $P[u_1, u_2]$, then $P[u_1, v_1] \cup D_1[v_1, u_1] \cup R$ is an E_9-subdivision in $\hat{D}$, and if R is internally disjoint from $P[v_1, v_2]$, then $P[u_2, v_2] \cup D_2[v_2, u_2] \cup R$ is an E_9-subdivision in $\hat{D}$. Thus, we may assume that R intersects both $P]u_1, u_2[$ and $P]v_1, v_2[$.

Let z be the first vertex on P that belongs to R. If $z \in R[s, y[$, then $P[u_1, z]R[z, t]$ is a (D_1, D_2)-dipath with three blocks, and we get the result as above because its initial vertex is not v_1. If $z \in R]y, t]$, then $P[u_1, z] \cup R \cup P[u_2, v_1] \cup D[v_1, u_1]$ is an E_9-subdivision. Therefore, we may assume that $z = y$.

Analogously, we may assume that the last vertex on $P[v_1, v_2]$ that belongs to R is x. Now $P[u_1, y]R[y, x]P[X, v_2]$ contains a (D_1, D_2)-dipath with initial vertex u_1 and terminal vertex v_2. By Claim 27, this dipath and $P[u_2, v_1]$ are (D_1, D_2)-opposite dipaths, a contradiction.

Similarly, by directional duality, if there is a (D_1, D_2)-inpath with three blocks, then $\hat{D}$ contains an E_9-subdivision. This completes the proof. $\lozenge$

Claim 29. (i) *For $i = 1, 2$, if $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$ and $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$, then $\hat{D}$ contains an E_9-subdivision.*

(ii) *For $i = 1, 2$, if $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$ and $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$, then $\hat{D}$ contains an E_9-subdivision.*

(iii) *If $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ or $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$, then $\hat{D}$ contains an E_9-subdivision.*

Subproof. (i) Suppose $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$ and $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$. Without loss of generality, we may assume that $i = 1$ and that $c_1 \in V(D_1)$ and $c_2 \notin V(D_1)$. Since D is strong, there is a $(D_1 \cup D_2, c_2)$-dipath Q and a $(c_2, D_1 \cup D_2)$-dipath R in D. If $s(Q) \in D_2$, then $c_1 c'' c_2 \overleftarrow{Q}$ is a (D_1, D_2)-path in $\hat{D}$ with two blocks, so by Claim 28, $\hat{D}$ contains an E_9-subdivision. If $s(Q) \in D_1$, then $t(R) \in D_1$, because D_1 is a strong component in $D - D_2$. Hence $c_1 c'' c_2 R$ is a (D_1, D_2)-path with two blocks, so by Claim 28, $\hat{D}$ contains an E_9-subdivision.

(ii) This claim is proved analogously to (i).

(iii) Assume that $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$. (The case when $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ is proved in the same way.) Since D is strong, there exist a $(D_1 \cup D_2, c_1)$-dipath Q_1, and a $(D_1 \cup D_2, c_2)$-dipath Q_2. If $s(Q_1)$ and $s(Q_2)$ do not lie in the same D_i, then there is a (D_1, D_2)-path (which is either contained in $Q_1 \cup Q_2$ if Q_1 and Q_2 intersect, or passes through c'' if they are disjoint) in $\hat{D}$ having two blocks, so by Claim 28, $\hat{D}$ contains an E_9-subdivision. Henceforth, we may assume that $s(Q_1)$ and $s(Q_2)$ are in the same D_i, say D_1.

Since D is strong, for $i = 1, 2$, there exists a $(c_i, D_1 \cup D_2,)$-dipath R_i. Its end-vertex $t(R_i)$ cannot be in D_1, because D_1 is a strong component of $D - D_2$. Thus $t(R_i) \in V(D_2)$. If R_1 intersects Q_2 and R_2 intersects Q_2, then c_1 and c_2 are in the same strong component of $D - (D_1 \cup D_2)$, which contradicts one of our previous assumptions. Therefore, without loss of generality, we may assume that R_2 does not intersect Q_1. Now $Q_1 \cup c_1 c'' \cup c'' c_2 \cup R_2$ is a (D_1, D_2)-path with three blocks. Thus by Claim 28, $\hat{D}$ contains an E_9-subdivision. $\lozenge$

In view of Claim 29, if $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$ and $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$ or $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$ and $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$ for some $i \in \{1, 2\}$, then we return 'yes'. The same holds if $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ or $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$. Thus, we may assume henceforth that there are indices $i_c, i_d \in \{1, 2\}$ such that $\{c_1, c_2\} \subseteq V(D_{i_c})$ and $\{d_1, d_2\} \subseteq V(D_{i_d})$.

We now run a procedure `2or3blocks`(D, D_1, D_2) for finding a (D_1, D_2)-path with two or three blocks in D. If such a path is found, we stop the main procedure

E_9-`strong+` by returning 'yes' since in this case we have an E_9-subdivision by Claim 28.

The procedure `2or3blocks`(D, D_1, D_2) proceeds as follows. Let S_1^- be the in-section of D_1 in $D - D_2$ and S_2^- the in-section of D_2 in $D - D_1$. It is easy to see that there is a (D_1, D_2)-inpath with two blocks, if and only if $S_1^- \cap S_2^-$ contains a vertex of $D - (D_1 \cup D_2)$. Therefore we compute $S_1^- \cap S_2^-$ and return 'yes' if this set contains a vertex in $D - (D_1 \cup D_2)$.

Similarly, to detect if there is a (D_1, D_2)-outpath with two blocks, we compute the out-section S_1^+ of D_1 in $D - D_2$ and the out-section S_2^+ of D_2 in $D - D_1$, and if $S_1^+ \cap S_2^+$ contains a vertex of $D - (D_1 \cup D_2)$, we return 'yes'.

Let us now describe how to discover paths with three blocks. Let tz be an arc and y be a vertex in D such that $t \in V(D - D_2)$, $y, z \in V(D - (D_1 \cup D_2))$, $y \notin \{t, z\}$. Arc tz and vertex y are said to be in 3-*block-position* if there are a (y, z)-dipath and a (y, D_2)-dipath in $D - (V(D_1) \cup \{t\})$ which are independent, and a (D_1, t)-dipath in $D - (D_2 \cup \{y, z\})$.

Claim 30. *There is a (D_1, D_2)-outpath in D with three blocks if and only if there are an arc tz and a vertex y in* 3-*block-position.*

Subproof. Trivially, if there is a (D_1, D_2)-outpath with three blocks, then there are an arc tz and a vertex y in 3-block-position.

Let us now prove the converse. Assume that tz and y are in 3-block-position. Let Q_1 and Q_2 be the two independent paths from y to z and D_2, respectively, and let R be the (D_1, t)-dipath in $D - (D_2 \cup \{y, z\})$.

If R does not intersect $Q_1 \cup Q_2$, then $R \cup tz \cup \overleftarrow{Q_1} \cup Q_2$ is a (D_1, D_2)-path with three blocks.

Assume now that R intersects $Q_1 \cup Q_2$. Let x be the first vertex along R in $V(Q_1 \cup Q_2)$. Note that $x \neq y$ by definition of R. If $x \in V(Q_1)$, then $R[s(R), x] \cup \overleftarrow{Q_1}[x, y] \cup Q_2$ is a (D_1, D_2)-path with three blocks. If $x \in V(Q_2)$, then consider a $(z, D_1 \cup D_2)$-dipath R' in D. Because D_1 is a strong component of $D - D_2$ and $RtzR'$ is a dipath, $t(R') \in D_2$. Furthermore R' does not meet $R \cup Q_1 \cup Q_2[y, x]$, because $D - (D_1 \cup D_2)$ is acyclic. Hence $R[s(R), x] \cup \overleftarrow{Q_2}[x, y] \cup Q_1 \cup R'$ is (D_1, D_2)-path with three blocks. ◊

Therefore, for every possible arc tz and vertex y such that $t \in V(D - D_2)$, $y, z \in V(D - (D_1 \cup D_2))$, $y \notin \{t, z\}$, we check if they are in 3-block-position. This can be done by running Menger algorithm. If we find an arc and a vertex in 3-block-position, then we return 'yes' because there is an E_9-subdivision by Claims 30 and 28.

We deal similarly with the (D_1, D_2)-inpaths with three blocks. This ends the procedure `2or3blocks`(D, D_1, D_2). After it, there is no (D_1, D_2)-path in $\hat{D}$ with two blocks and no (D_1, D_2)-path in D with three blocks.

We now show that we can reduce D to a digraph with vertex set $V(D_1) \cup V(D_2)$.

Let D^* be the digraph obtained from $D_1 \cup D_2$ by adding all arcs in $F_{12} \cup F_{21}$. In other words, we add all arcs x_1x_2 with $x_1 \in V(D_1)$ and $x_2 \in V(D_2)$ such that there is a (D_1, D_2)-dipath with initial vertex x_1 and terminal vertex x_2, and adding all arcs x_2x_1 with $x_1 \in V(D_1)$ and $x_2 \in V(D_2)$ for which there is a (D_2, D_1)-dipath with initial vertex x_2 and terminal vertex x_1. Set $\hat{D}^* = \hat{D}^*(\{c_1, c_2\}, \{d_1, d_2\})$.

Claim 31. *$\hat{D}$ contains a (c'', d'')-forced E_9-subdivision if and only if $\hat{D}^*$ contains a (c'', d'')-forced E_9-subdivision.*

Subproof. As mentioned above, D^* either contains the four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ (Case (B)), or contains the arcs uu_2 ($u \in U_1$), u_2v_1, and v_1v ($v \in V_2$), which is Case (A). For each of these arcs uv, there is a corresponding directed path R_{uv} in D. One can transform a (c'', d'')-forced E_9-subdivision S^* in $\hat{D}^*$ into an E_9-subdivision S of $\hat{D}$ by replacing each arc uv in S^* between D_1 and D_2 by the path R_{uv}. If all added paths R_{uv} are pairwise internally disjoint, this clearly gives rise to an E_9-subdivision in $\hat{D}$. The only possibility that two of such paths may not be internally disjoint (cf. Lemma 3.17) is that we have two paths R_{uu_2} and $R_{u'u_2}$ (where $u, u' \in U_1$ and $u \neq u'$) or two paths R_{v_1v} and $R_{v_1v'}$ (where $v, v' \in V_2$ and $v \neq v'$). However, since every vertex in E_9 has in- and out-degrees at most 2, there are at most two such paths entering u_2 and at most two leaving v_1. For two of them, we can always achieve that their intersection is a common subpath, and in that case, the resulting digraph is again an E_9-subdivision. Clearly, the resulting E_9-subdivision in $\hat{D}$ is (c'', d'')-forced.

Suppose now that $\hat{D}$ contains a (c'', d'')-forced E_9-subdivision S. Let a', b', be the vertices of S corresponding to a, b, respectively, and let C be the directed cycle in S. If $a', b' \in V(D_1) \cup V(D_2)$, then the arcs in S that are not in $D_1 \cup D_2$ form a collection of internally disjoint (D_1, D_2)- and (D_2, D_1)-dipaths. By replacing each of these dipaths by the corresponding arc in D^*, we obtain an E_9-subdivision in $\hat{D}^*$.

Assume now that $a' \notin V(D_1) \cup V(D_2)$. The cycle C must intersect both D_1 and D_2, and thus C contains a (D_2, D_1)-dipath. Without loss of generality, we may assume that the initial vertex of this dipath is u_2 and its terminal vertex is v_1.

Now let z_2 be the first vertex in $V(D_1 \cup D_2)$ along the (a', c'')-dipath in S. This vertex exists because $\{c_1, c_2\} \subseteq V(D_1 \cup D_2)$. Now since there are no (D_1, D_2)-opposite paths, and by definition of the D_i, $z_2 \in V(D_2)$, so a' does not lie in $C[u_2, v_1]$. Let y_2 be the first vertex after a' along C in $V(D_1 \cup D_2)$. For the same reason, $y_2 \in V(D_2)$ and so v_1 is the unique vertex in $C \cap D_1$, for otherwise, there would be (D_1, D_2)-opposite paths. Note that $y_2, z_2 \in V_2$.

If $\{c_1, c_2\} \subseteq V(D_1)$, then the (z_2, c'')-dipath in S contains a (D_2, D_1)-dipath that together with $C[v_1, y_2]$ gives (D_1, D_2)-opposite paths.

This shows that $\{c_1, c_2\} \subseteq V(D_2)$. The (d'', c'')-dipath in $S - E(C)$ must have all its internal vertices in D_2, because every (D_1, D_2)-dipath meets $\{u_2, v_1\}$. Therefore the digraph obtained from $S \cap (D_1 \cup D_2)$ by adding the arcs u_2v_1, v_1y_2, and v_1z_2 is an E_9-subdivision in $\hat{D}^*$.

If $b' \notin V(D_1) \cup V(D_2)$, we get the result analogously. $\Diamond$

In view of Claim 31, we replace D by D^*. Henceforth, now $V(D) = V(D_1) \cup V(D_2)$. Moreover, there are at most two arcs with tail in D_2 and head in D_1, namely

u_2v_1 and possibly v_2u_1.

Let D'_1 be the digraph obtained from D by contracting D_2 into a vertex z_2. If all four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ are present in D (Case (B)), then we also add into D'_1 the arcs u_1v_1 and v_1u_1 if they are not already contained in D_1. Similarly, we let D'_2 be the digraph obtained from D by contracting D_1 into a vertex z_1 and adding the arcs u_2v_2 and v_2u_2 if D contains all four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$. Observe that D'_1 and D'_2 are both strong, and contain fewer vertices than D.

If $i_c = i_d$, then we return E_9-`Strong+`$(D'_{i_c}, \{c_1, c_2\}, \{d_1, d_2\})$. This is valid by the following claim.

Claim 32. *If $i_c = i_d$, then $\hat{D}$ contains a (c'', d'')-forced E_9-subdivision if and only if $\hat{D}_{i_c}$ contains a (c'', d'')-forced E_9-subdivision.*

Subproof. We shall assume that $i_c = 1$. (The case when $i_c = 2$ is proved in the same way.) Suppose first that $\hat{D}$ contains a (c'', d'')-forced E_9-subdivision S. If S does not intersect D_2, then S is a (c'', d'')-forced E_9-subdivision in $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$. Henceforth we assume that S intersects D_2. Observe that vertices a' and b' corresponding to a and b in S belong to a (d'', c'')-dipath in S. Therefore, a' and b' belong to D_1, since there are no (D_1, D_2)-opposite dipaths in D. Consequently, every vertex in S lies on a (d'', c'')-dipath or on an (a', b')-dipath in S, and each such path intersects D_2 in at most one vertex. If S contains only one vertex v in D_2, the digraph S' obtained from S by replacing v by z_2, we obtain a (c'', d'')-forced E_9-subdivision in $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$. If S contains two vertices in D_2, then S contains all four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ and hence D'_1 contains the arcs u_1v_1 and v_1u_1. Moreover, $\{a', b'\} = \{u_1, v_1\}$ and the cycle in S is the 4-cycle $u_1u_2v_1v_2u_1$. Therefore, the arcs u_1v_1 and v_1u_1 are not both in S. Then we replace the cycle in S by the cycle $u_1v_1z_2u_1$ in D'_1 and obtain a (c'', d'')-forced E_9-subdivision in $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$.

Suppose now that $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$, contains a (c'', d'')-forced E_9-subdivision S'. Let us first assume that S' does not contain z_2. If S' contains an arc that is not in $A(D)$, then this is either the arc $e = u_1v_1$ or $e' = v_1u_1$. This arc was added to D'_1 only in Case (B). Thus, we can replace e in S' by the path $u_1u_2v_1$ and e' by the path $v_1v_2u_1$. By making these changes (if needed), we obtain a (c'', d'')-forced E_9-subdivision in $\hat{D}$. Henceforth we assume that S contains z_2.

If S' contains an arc $e \in \{v_1u_1, u_1, v_1\}$ and $e \notin A(D)$, then we have Case (B). In S', the vertex z_2 has in- and out-degree equal to 1, while each of u_1 and v_1 has either in- or out-degree equal to 2. Therefore, the arc in $\{v_1u_1, u_1, v_1\} \setminus \{e\}$ cannot be in S'. By replacing the path in S' joining u_1 and v_1 through z_2 by that arc, we obtain an E_9-subdivision that does not contain z_2, and we are done in the same way as above. Thus, we may assume that S' contains no edge in $\{v_1u_1, u_1, v_1\} \setminus A(D)$.

The vertex z_2 has an in-neighbour x_1 and an out-neighbour y_1 in S', and possibly has a third neighbour z_1. If z_1 exists, then we assume that the arcs x_1z_2 and z_2y_1 lie on the cycle in S'. By definition of contraction, x_1 has an out-neighbour x_2 in $V(D_2)$ and y_1 has an in-neighbour y_2 in $V(D_2)$. Moreover if z_1 exists, let w_2 be one of its out-neighbours (resp. in-neighbours) in $V(D_2)$ corresponding to the arc joining z_1 and z_2 in D'_1. Let Q be an (x_2, y_2)-dipath in D_2 and Q' be a (w_2, Q)-dipath (resp. (Q, w_2)-dipath) in D_2 if $z_2z_1 \in A(S')$ (resp. $z_1z_2 \in A(S')$). Now the digraph obtained from S' by replacing z_2 and the arcs incident to it by the paths Q and Q', and the arcs x_1x_2, y_1y_2 and z_1w_2 or w_2z_1, is a (c'', d'')-forced E_9-subdivision in $\hat{D}$. $\Diamond$

Henceforth, we have $i_c \neq i_d$. If $i_d = 2$ and $i_c = 1$, then a (c'', d'')-forced E_9-subdivision contains two disjoint arcs from D_2 to D_1. Thus, necessarily v_2u_1 is an arc, because there are no (D_1, D_2)-opposite paths. In this case, we consider exchanging the roles of D_1 and D_2. Thus, we may assume henceforth that we are in the case when $i_d = 1$ and $i_c = 2$.

Let $D_1^* = D[V(D_1) \cup \{u_2\}]$ and $D_2^* := D[V(D_2) \cup \{v_1\}]$. Observe that D_1^* and D_2^* are both strong.

A (c'', d'')-forced E_9-subdivision contains two internally disjoint (d'', c'')-dipaths. Therefore, using a Menger algorithm, we check if two such dipaths exist in $\hat{D}$. If two such dipaths do not exist, then we return 'no'. Otherwise there are two internally disjoint (d'', c'')-dipaths. Because there are no (D_1, D_2)-opposite paths, one of them say P_1^* must go through v_1 and the other, say P_2^*, through u_2. We return (E_9-`Strong+`$(D_1^*, \{v_1, u_2\}, \{d_1, d_2\})$ or E_9-`Strong+`$(D_2^*, \{c_1, c_2\}, \{v_1, u_2\})$. This is valid by the following claim.

Claim 33. *$\hat{D}$ contains a (c'', d'')-forced E_9-subdivision if and only if either $\hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$ or $\hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$ contains a (c'', d'')-forced E_9-subdivision.*

Subproof. Set $\hat{D}_1^* := \hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$ and $\hat{D}_2^* := \hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$.

A (c'', d'')-forced E_9-subdivision in $\hat{D}_1^*$ (or one in $\hat{D}_2^*$) can easily be transformed into a (c'', d'')-forced E_9-subdivision in $\hat{D}$ by replacing the arcs v_1c'' and u_2c'' by $P_1^*[v_1, c'']$ and $P_2^*[u_2, c'']$.

Suppose now that $\hat{D}$ contains a (c'', d'')-forced E_9-subdivision S. Let C be the directed cycle in S and let Q_1 and Q_2 be the two internally disjoint (d'', c'')-dipaths in S. Because there are no (D_1, D_2)-opposite paths, one of these dipaths, say Q_1 goes through v_1 and the other goes through u_2. Moreover, C intersects either D_1 or D_2 in at most one vertex. If D does not contain the arc v_2u_1 (Case (A)), then if C intersects D_1 (resp. D_2) in one vertex, this vertex must be v_1 (resp. u_2). We may assume that the same holds in Case (B) after possibly exchanging the roles of u_1 and v_1 and of u_2 and v_2. Hence the digraph obtained from S by replacing $Q_1[d'', v_1]$ and $Q_2[d'', u_2]$ (resp. $Q_1[v_1, c'']$ and $Q_2[u_2, c''])$ by the arcs $d''v_1$ and $d''u_2$ (resp. v_1c'' and u_2c'') is a (c'', d'')-forced E_9-subdivision in $\hat{D}_2^*$ (resp. $\hat{D}_1^*$). $\Diamond$

This completes the procedure E_9-`Strong+`. Let us now examine its time complexity. Let $T(n)$ be the maximum running time an a digraph with at most n vertices. Clearly, the running time between two recursive calls is bounded by a polynomial $P(n)$. When treating a graph D on n vertices, it then makes a recursive call either to a smaller digraph, or to two smaller digraphs D_1^* and D_2^* such that

$|D_1^*| + |D_2^*| \leq n + 2$. Hence $T(n)$ satisfies the inequality

$$T(n) \leq P(n) + \max\{T(n-1)\ ;\ \max\{T(n_1) + T(n_2) \mid n_1 + n_2 \leq n+2, n_1 < n, n_2 < n\}\}.$$

This implies that $T(n)$ is bounded above by a polynomial value in n.

3.5.3.3 Detecting E_{10}

Proposition 3.19. *The digraph E_{10} depicted in Figure 3.5 is tractable. More precisely, E_{10}-*Subdivision *can be solved in time $O(n^2m(n+m))$.*

Proof. Let D be a digraph. Observe that every E_{10}-subdivision contains an E_{10}-subdivision in which the arc cd is not subdivided. Henceforth by E_{10}-subdivision, we mean such a subdivision.

Given four distinct vertices a', b', c', d' such that $c'd'$ is an arc, we say that an E_{10}-subdivision is *$(a', b', c'd')$-forced* if a' is its a-vertex, b' its b-vertex, c' its c-vertex, and d' its d-vertex.

We shall present a procedure E_{10}-`Subdivision`$(D, a', b', c'd')$, that returns 'no' only if there is no $(a', b', c'd')$-forced E_{10}-subdivision in D, and returns 'yes' if it finds an E_{10}-subdivision in D (not necessarily one that is $(a', b', c'd')$-forced). We proceed as follows.

Suppose first that $a'b'$ is an arc. Using a Menger algorithm, we check whether there are two independent $(b', \{a', c'\})$-dipaths in $D - d'$, and using a search, we check whether there exists a (c', a')-dipath in $D' - \{b', d'\}$. If three such dipaths do not exist, then there is no $(a', b', c'd')$-forced E_{10}-subdivision in D, and we return 'no'. If three such dipaths exist, then we return 'yes'. This is valid by virtue of the following claim.

Claim 34. *If there are a (c', a')-dipath R in $D' - \{b', d'\}$ and two independent $(b', \{a', c'\})$-dipaths P_1, P_2 in $D - d'$, then D contains an E_{10}-subdivision.*

Subproof. Without loss of generality, we may assume that $t(P_1) = a'$ and $t(P_2) = c'$. The dipath R contains a subdipath R' with initial vertex s in P_2 and terminal vertex in $P_1]b', a']$. Let s^+ be the out-neighbour of s in $P_2 \cup c'd'$. Then $a'b' \cup P_1 \cup P_2[b', s] \cup R' \cup ss^+$ is an E_{10}-subdivision. ♢

Henceforth, we assume that $a'b'$ is not an arc in D.

If $d^+_{D-\{c',d'\}}(a') = 0$, then there is no (a', b')-dipath in $D - c' - d'$, and thus no $(a', b', c'd')$-forced E_{10}-subdivision. Hence we return 'no'.

If $d^+_{D-\{c',d'\}}(a') = 1$, then denote by a'' the unique out-neighbour of a' in $D - \{c', d'\}$. By our assumption, $a'' \neq b'$. Let D^* be the digraph obtained from D by first removing all arcs entering a'' and then identifying a' and a'' into a single vertex a^*. Note that a^* is dominated by the in-neighbours of a' in D and dominates the out-neighbours of a'' in D. We the return E_{10}-`Subdivision`$(D^*, a^*, b', c'd')$. The validity of this recursive call is shown by the following claim.

Claim 35. *If $d^+_{D-\{c',d'\}}(a') = 1$, then D contains an $(a', b', c'd')$-forced E_{10}-subdivision if and only if D^* contains an $(a^*, b', c'd')$-forced E_{10}-subdivision.*

Subproof. Assume that S is an $(a', b', c'd')$-forced E_{10}-subdivision in D. Since $d^+_{D-\{c',d'\}}(a') = 1$, S contains the arc $a'a''$ since a'' is the unique out-neighbour of a' in $D' - c' - d'$. Now the digraph S^* obtained from S by replacing a' and a'' and the four arcs $ua', va', a'a'', a''w$ by the vertex a^* and the three arcs ua^*, va^*, a^*w is an $(a^*, b', c'd')$-forced E_{10}-subdivision in D^*, because $a'' \neq b'$.

Conversely, if S^* is an $(a^*, b', c'd')$-forced E_{10}-subdivision in D^*, then the digraph S obtained from S^* by replacing a^* and its three incident arcs ua^*, va^*, a^*w by vertices a' and a'' and the four arcs $ua', va', a'a'', a''w$ is clearly an $(a', b', c'd')$-forced E_{10}-subdivision in D. $\diamondsuit$

Henceforth, we may assume that $d^+_{D-\{c',d'\}}(a') \geq 2$. Using a Menger algorithm, we check whether there are two independent $(b', \{a', c'\})$-dipaths in $D - d'$, and using a search we check whether there exists an (a', b')-dipath in $D - \{c', d'\}$, and whether there exists a (c', a')-dipath in $D - \{b', d'\}$. If four such dipaths do not exist, then there is no $(a', b', c'd')$-forced E_{10}-subdivision in D, and we return 'no'. If four such dipaths exist, then we return 'yes' by virtue of the following claim.

Claim 36. *If there are two independent $(b', \{a', c'\})$-dipaths P_1 and P_2 in $D - d'$, an (a', b')-dipath Q in $D - \{c', d'\}$, and a (c', a')-dipath R in $D - \{b', d'\}$, then D contains an E_{10}-subdivision.*

Subproof. Without loss of generality, we may assume that $t(P_1) = a'$ and $t(P_2) = c'$. Let v be the last vertex along $Q - b'$ that is in $P_1 \cup P_2$. We distinguish two cases according to whether v is on P_1 or P_2.

<u>Case 1:</u> $v \in V(P_1)$. Note that this is in particular the case when Q is internally disjoint from P_1 and P_2. Let C be the directed cycle formed by the union of $P_1[b'v]$ and $Q[v, b']$, let Q' be the (a', C)-subdipath in Q, and let $R' = RQ'$. The directed walk R' contains a subdipath R'' with initial vertex s in $P_2]b', c']$ and terminal vertex t in C. Let s^+ be the out-neighbour of s in $P_2 \cup c'd'$. Then $C \cup P_2[b', s] \cup R'' \cup ss^+$ is an E_{10}-subdivision.

<u>Case 2:</u> $v \in V(P_2)$. Let v^+ be the out-neighbour of v in $P_2 \cup c'd'$. The dipath $Q[a', v]$ contains a subdipath Q' with initial vertex u in P_1 and terminal vertex w in $P_2[b', v]$ whose internal vertices are not in $P_1 \cup P_2[b', v]$. Let C' be the directed cycle formed by the union of $P_2[b', v]$ and $Q[v, b']$. If $u \neq a'$, let u^+ be the out-neighbour of u in P_1. Then $C' \cup P_1[b', u^+] \cup Q'$ is an E_{10}-subdivision. Henceforth, we may assume that $u = a'$.

Let u' be the out-neighbour of a' in Q'. Now $d^+_{D-\{c',d'\}}(a') \geq 2$ and $a'b'$ is not an arc. Hence, a' has an out-neighbour z distinct from b', c', d', and u'.

- If $z \notin V(C' \cup P_1 \cup Q')$, then $C' \cup P_1 \cup Q' \cup a'z$ is an E_{10}-subdivision.
- If $z \in V(Q')$, then $C' \cup P_1 \cup a'z \cup Q'[z, w] \cup a'u'$ is an E_{10}-subdivision.

- Assume $z \in V(P_1)$. If $v^+ \notin V(Q')$, then $P_1 \cup a'z \cup Q' \cup P_2[w,v] \cup Q[v,b'] \cup vv^+$ is an E_{10}-subdivision. If $v^+ \in V(Q')$, then $v^+ \neq d'$ and so $P_2[v^+,c']$ is not an empty dipath. Denote by C'' the directed cycle $P_1[z,a'] \cup a'z$. The dipath $Q[v^+,b'] \cup P_1[b',z]$ contains a $(P_2[v^+,c'], C''-a')$-dipath Q''. Let s^+ be the out-neighbour of $s(Q'')$ in $P_2 \cup c'd'$. Now $C'' \cup Q[a',v^+] \cup P_2[v^+, s(Q'')] \cup Q'' \cup s(Q'')s^+$ is an E_{10}-subdivision.

- Assume $z \in V(Q]v,b'])$. Then one can replace the (a',b')-dipath Q by $a'zQ[z,b']$. This dipath is internally disjoint from P_1 and P_2, and we get the result by Case 1.

- Assume finally that $z \in V(P_2[b',v])$. If $u' \neq w$, then $C' \cup P_1 \cup a'z \cup a'u'$ is an E_{10}-subdivision. Henceforth we assume that $u' = w$, so $a'w$ is an arc. Without loss of generality, we may assume that z precedes w along P_2. For $i = 1,2$, let b_i^+ be the out-neighbour of b' in P_i. By the previous assumption, $b_2^+ \neq w$. Let t be the last vertex along $R - a'$ in $V(P_1 \cup P_2 \cup Q[v,b'])$.

  If $t \in V(P_1)$, then one of the two dipaths $P_1[t,a']$ and $R[t,a']$ has length at least 2. Let t^+ be the out-neighbour of t in this dipath, and let T be the other dipath. Now $C' \cup P_1[b',t] \cup T \cup a'w \cup tt^+$ is an E_{10}-subdivision.

  If $t \in V(Q[v,b'] \cup P_2[w,v])$, then $R[t,a'] \cup a'w \cup P_2[w,v] \cup Q[v,b'] \cup P_1 \cup b'b_2^+$ is an E_{10}-subdivision.

  If $t \in V(P_2[v,c'])$, then $R[t,a'] \cup a'w \cup P_2[w,t] \cup Q[v,b'] \cup P_1 \cup b'b_2^+$ is an E_{10}-subdivision.

  If $t \in V(P_2[z,w[)$, then $R[t,a'] \cup a'z \cup P_2[z,v] \cup Q[v,b'] \cup vv^+ \cup P_1$ is an E_{10}-subdivision.

  It remains to consider the case when $t \in V(P_2]b',z[)$. Let t^+ be the out-neighbour of t on P_2. Then $P_1 \cup P_2[b',t^+] \cup R[t,a'] \cup a'w \cup P_2[w,v] \cup Q[v,b']$ is an E_{10}-subdivision in D.

$\Diamond$

One can easily see that the procedure E_{10}-`Subdivision`$(D,a',b',c'd')$ runs in linear time as it either reduces the problem in constant time (when $d^+(a') = 1$) or runs a Menger algorithm and at most two searches, which can be done in linear time. Running this procedure for the $O(n^2m)$ possible choices of $(a',b',c'd')$, we obtain an algorithm with running time $O(n^2m(n+m))$ that solves E_{10}-Subdivision. $\square$

3.5.4 E_{16} is tractable

Theorem 3.20. E_{16}-Subdivision*can be solved in $O(n^7(n+m))$ time.*

The proof relies on the following notion. A *shunt* is a digraph composed by three dipaths P, Q and R such that $|R| \geq 2$, $s(R) \in P$, $t(R) \in Q$ and P, Q, R^0 are disjoint. We frequently refer to a shunt by the triple (P,Q,R). An (S,T)-*shunt* is a shunt (P,Q,R) such that $\{s(P), s(Q)\} = S$ and $\{t(P), t(Q)\} = T$.

We consider the following decision problem.

SHUNT
Input: A digraph D and four distinct vertices s_1, s_2, t_1, t_2.
Question: Does D contain an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt?

Assume that there are two disjoint dipaths P, Q from $\{s_1, s_2\}$ to $\{t_1, t_2\}$ in D. We now give some sufficient conditions considering P and Q for D to have a $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt .

Proposition 3.21. *If there is a dipath R of length at least 2 between P and Q, then D has an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.*

Proof. If such a dipath R exists, then (P, Q, R) or (Q, P, R) is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. □

For any vertex x in $V(P)$, an *x-bypass* is a dipath S internally disjoint from P and Q with initial vertex in $P[s(P), x[$ and terminal vertex in $P]x, t(P)]$. Similarly, for any vertex x in $V(Q)$, an *x-bypass* is a dipath S internally disjoint from P and Q with initial vertex in $Q[s(Q), x[$ and terminal vertex in $Q]x, t(Q)]$. An *arc bypass* is an x-bypass such that x is the end-vertex of an arc between P and Q.

Proposition 3.22. *If there is an arc bypass for some arc uv between P and Q, then D has an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.*

Proof. If S is a u-bypass, then $(P[s(P), s(S)] \cup S \cup P[t(S), t(P)], Q, P[s(S), u] \cup (u, v))$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt if $u \in V(P)$, and there is a shunt constructed analogously if $u \in V(Q)$.

If S is a v-bypass, $(Q, P[s(P), s(S)] \cup S \cup P[t(S), t(P)], (u, v) \cup P[v, t[S])$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt if $v \in V(P)$ (similarly if $v \in V(Q)$). □

A *crossing* (with respect to P and Q) is a pair of arcs $\{uv, u'v'\}$ such that u is before v' along P and u' is before v along Q. Moreover, if uv' is an arc of P and $u'v$ is an arc of Q, then the crossing is *tight*. Otherwise it is *loose*.

Proposition 3.23. *If there is a loose crossing, then there is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt in D.*

Proof. Let $\{uv, u'v'\}$ be a loose crossing. By symmetry, we may assume that uv' is not an arc. Then $(P[s(P), u] \cup (u, v) \cup Q[v, t(Q)], Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)], P[u, v'])$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. □

Let $C = \{uv, u'v'\}$ be a tight crossing. A *C-forward path* is a dipath internally disjoint from P and Q either with initial vertex in u and terminal vertex in v', or with initial vertex in u' and terminal vertex in v.

Proposition 3.24. *If there is a C-forward path, then there is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt in D.*

Proof. Observe that since a forward path S has length at least 2, because uv' and $u'v$ are arcs, then $(P[s(P),u]\cup(u,v)\cup Q[v,t(Q)], Q[s(Q),u']\cup(u',v')\cup P[v',t(P)], S)$ is an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt if S is a dipath from u to v', and $(Q[s(Q),u']\cup(u',v')\cup P[v',t(P)], P[s(P),u]\cup(u,v)\cup Q[v,t(Q)], S)$ is an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt if S is a dipath from u' to v. □

Still considering a tight crossing $C=\{uv,u'v'\}$, a *C-backward path* is a dipath internally disjoint from P and Q either with initial vertex in $P[v',t(P)]$ and terminal vertex in $P[s(P),u]$, or with initial vertex in $Q[v,t(Q)]$ and terminal vertex in $Q[s(Q),u']$. A *C-backward arc* is an arc that induces a C-backward path of length 1. A *C-bypass* is an x-bypass B such that x is an end-vertex of a C-backward arc and if $x\in P[s(P),u](Q[s(Q),u'])$, $t(B)$ is also in $P[s(P),u](Q[s(Q),u'])$, or if $x\in P[v',t(P)](Q[v,t(Q)])$, $s(B)$ is also in $P[v',t(P)](Q[v,t(Q)])$. A *crossing bypass* is a C-bypass for some tight crossing C.

Proposition 3.25. *If there is a backward path of length at least* 2 *or a crossing bypass in P or Q, then D has an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt.*

Proof. Suppose $C=\{uv,u'v'\}$ is a tight crossing. Then we first check if there is a C-backward path of length at least 2. If there is such a backward path R, the union of $P[s(P),u]\cup(u,v)\cup Q[v,t(Q)]$, $Q[s(Q),u']\cup(u',v')\cup P[v',t(P)])$, and R is an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt. So assume that all backward paths have length 1, and thus are arcs. For each tight crossing $C=\{uv,u'v'\}$, we check if there is a C-bypass. If there is such a C-bypass B, by symmetry and directional duality, we may assume that B is an x-bypass with $t(B)\in P[s(P),u]$ and that there is a C-backward arc a one end of which is x. If $a=wx$ for some $w\in P[v',t(P)]$, then $(Q[s(Q),u']\cup(u',v')\cup P[v',t(P)]), P[s(P),s(B)]\cup B\cup P[t(B),u]\cup(u,v)\cup Q[v,t(Q)], (w,x)\cup P[x,u])$ is an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt. If $a=xw$ for some $w\in P[v',t(P)]$, then $(Q[s(Q),u']\cup(u',v')\cup P[v',t(P)]), P[s(P),s(B)]\cup B\cup P[t(B),u]\cup(u,v)\cup Q[v,t(Q)], P[s(B),x]\cup(x,w))$ is an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt. □

We now prove that the conditions of Propositions 3.21, 3.22, 3.23, 3.24 and 3.25 are also necessary.

Lemma 3.26. *Let D be a digraph, and P and Q be two disjoint dipaths from $\{s_1,s_2\}$ to $\{t_1,t_2\}$. If there is no arc bypasses, no loose crossings, no forward paths, no backward paths of length at least* 2*, no crossing bypasses, and no dipaths of length at least* 2 *between P and Q, then D contains no $(\{s_1,s_2\},\{t_1,t_2\})$-shunt.*

Proof. Suppose for a contradiction that D is as in the statement, but it contains an $(\{s_1,s_2\},\{t_1,t_2\})$-shunt (P',Q',R'). Without loss of generality, we may assume that this shunt maximizes $|(A(P)\cup A(Q))\cap(A(P')\cup A(Q'))|$. Free to swap the names of P and Q, we may assume that $s(P)=s(P')$.

Consider the paths P and Q. Let u be the farthest vertex along P' such that $P'[s(P'),u]$ does not intersect Q. Necessarily $u\in V(P)$ for otherwise there would be a dipath of length at least 2 from P to Q. In addition, for the same reason, if

$u \neq t(P)$, then the out-neighbour v of u in P' must be in Q. Hence all vertices of $P'[s(P'), u] \cap P$ are in $P[s(P), u]$, for otherwise there would be a u-bypass in P, which would be an arc bypass for uv. Note also that for every vertex x in $P[s(P), u] - P'$ there is a subdipath of P' which is an x-bypass. So there is no $x \in Q'$ in $P[s(P), u]$, for otherwise there would be an arc bypass in P for the arc starting in $y \in Q$ and ending in x (there would be one since Q' and Q intersects at least in the first vertex) or a dipath of length at least 2 from Q to P. Let R'' be the shortest subdipath of P' with initial vertex in $V(P)$ and terminal vertex $s(R')$ if $s(R') \in P'[s(P'), u]$, and let R'' be the path of length 0 ($s(R')$) otherwise. Now, $(P'', Q'', R) = (P[s(P), u] \cup P'[u, t(P')], Q', R'' \cup R')$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. Moreover if $P'[s(P'), u] \neq P[s(P), u]$, then P'' and Q'' have more arcs in common with P and Q than P' and Q', which contradicts our choice of (P', Q', R'). Therefore $P'[s(P'), u] = P[s(P), u]$.

Let u' be the farthest vertex along Q' such that $Q'[s(Q'), u']$ does not intersect P. As above, one shows that $Q'[s(Q'), u'] = Q[s(Q), u']$.

If $u = t(P)$, then $P' = P$ and necessarily $Q = Q'$. Thus R' is a dipath of length at least 2 from P to Q as (P', Q', R') is a shunt, which is a contradiction. Therefore, we may assume that $u \neq t(P)$ and similarly $u' \neq t(Q)$. Furthermore the out-neighbour v of u in P' is in $V(Q)$ and the out-neighbour v' of u' is in $V(P)$. Since P' and Q' are disjoint, $P'[s(P'), u] = P[s(P), u]$ and $Q'[s(Q'), u'] = Q[s(Q), u']$, it follows that $C = \{uv, u'v'\}$ is a crossing with respect to P and Q, and thus a tight crossing.

Consider the dipath R'.

- Assume first that $s(R') \in P'[s(P'), u]$. Let S be the shortest subdipath of $R' \cup Q'[t(R'), t(Q')]$ such that $s(S) = s(R')$ and $t(S) \in V(P) \cup V(Q)$. Vertex $t(S)$ cannot be in $Q[s(Q), u']$ for otherwise $S = R'$ and it would be a dipath of length at least 2 between P and Q. Furthermore, $\{s(R')t(S), u'v'\}$ is a loose crossing, since the distance between u' and $t(S)$ in Q is at least 2 (u is between $s(R')$ and v and v is between u' and $t(S)$). Therefore $t(S) \in V(P)$ and so $t(S)$ is on $P[v', t(P)]$. But then S is a forward path or an arc bypass in P, a contradiction.

- Assume now that $s(R') \in P'[v, t(P')]$.

  Set $P^* = Q[s(Q), u'] \cup (u', v) \cup P'[v, t(P')]$ and $Q^* = P[s(P), u] \cup (u, v') \cup Q'[v', t(Q')]$. If $t(R') \in Q'[v', t(Q')]$, then (P^*, Q^*, R') is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. But P^* and Q^* have more arcs in common with P and Q than P' and Q', which contradicts our choice of (P', Q', R'). Therefore $t(R') \in Q'[s(Q'), u']$.

  Let S be the shortest subdipath of $P'[v, s(R')] \cup R'$ such that $t(S) = t(R')$ and $s(S) \in V(P) \cup V(Q)$.

  Assume first that $s(S) \in V(Q)$. Then S is a C-backward path. Hence it must have length 1. Therefore $s(S) \notin V(P') \cup V(Q')$ because R' has length at least 2. Let u_1 be the farthest vertex on $P'[v, t(P')]$ that is in $V(Q)$ and such that $P'[v, u_1]$ does not intersect P. Observe that u_1 appears before $s(S)$ in Q, for otherwise there would be a crossing bypass in P', as $s(S) \notin P'$. In particular, u_1 is not the terminal vertex of P'. Let v_1 be the first vertex after u_1 along P' which is on $P \cup Q$. It must be in $V(P)$ by the choice of u_1. Therefore u_1v_1 is an arc because there is no dipath of length at least 2 between Q and P. Let u_2 be the farthest vertex on $Q'[v', t(Q')] \cap P$ such that $Q'[v', u_2]$ does not intersect Q. Then v_1 is after u_2 along P, for otherwise there would be an arc bypass in P for u_1v_1. Thus u_2 is not the terminal vertex of Q'. Let v_2 be the first vertex after u_2 along Q' which is on $P \cup Q$. It must be in $V(Q)$ by the choice of u_2. Hence u_2v_2 is an arc because there is no dipath of length at least 2 between P and Q. Moreover, observe that for every vertex x in $Q[v, u_1] - P'$ there is a subdipath of P' which is an x-bypass. Therefore v_2 must be in $Q]u_1, t(Q)]$ for otherwise it would be an arc bypass. Hence $\{u_2v_2, u_1v_1\}$ is a crossing for $P \cup Q$, and so it must be tight. This implies in particular that $s(S) \in Q[v_2, t(Q)]$.

Set $P^+ = P'[s(P'), u] \cup (u, v') \cup Q[v', u_2] \cup (u_2, v_1) \cup P'[v_1, t(P')])$ and $Q^+ = Q'[s(Q), u'] \cup (u', v) \cup P'[v, u_1] \cup (u_1, v_2) \cup Q'[v_2, t(Q')])$. If $s(R') \in P'[v_1, t(P')])$, then (P^+, Q^+, R') is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. But P^+ and Q^+ have more arcs in common with P and Q than P' and Q', which contradicts our choice of (P', Q', R'). Therefore $s(R') \in P'[(v, u_1)]$. Now $P'[v, s(R')] \cup R'$ contains a subdipath T that is internally disjoint from P and Q and has initial vertex in $Q[v, u_1]$ and terminal vertex in $P \cup Q[v_2, t(Q)]$. Necessarily, $t(T) \in V(P)$ for otherwise T is an arc bypass. Hence T is an arc. Furthermore, $t(T)$ could not be in $P[v', u_2]$ for otherwise Q' would contain a $t(T)$-bypass, which would be an arc bypass. Hence $t(T) \in P]v_1, t(Q)]$ and $\{u_2v_2, T\}$ is a loose crossing, a contradiction.

Assume now that $s(S) \in V(P)$.Then it must be in $P[v', t(P)]$. Since there is no dipath of length at least 2 from P to Q, S has length 1. Moreover, since R' has length at least 2, $s(S)$ is an internal vertex of R', so it is not in $V(P' \cup Q')$. Let u_2 be the farthest vertex on $Q'[v', t(Q')]$ that is in $V(P)$ and such that $Q'[v', u_2]$ does not intersect Q. Then u_2 appears before $s(S)$ on P, for otherwise there would be an arc bypass for $s(S)t(S)$ in P and so u_2 is not the terminal vertex of Q'. Let v_2 be the first vertex after u_2 along Q' which is on $P \cup Q$. It must be in $V(Q)$ by the choice of u_2, and so on $Q[v, t(Q)]$. u_2v_2 is an arc for otherwise for otherwise there would be a dipath of length 2 from P to Q. Let u_1 be the farthest vertex on $P'[v, t(P')]$ that is also in $V(Q)$ such that $P'[v, u_1]$ does not intersect P. Vertex u_1 appears before v_2 in Q, for otherwise there would be an arc bypass for u_2v_2 in Q, and so u_1 is not the terminal vertex of P'. Let v_1 be the first vertex after u_1 along P' which is on $P \cup Q$. It must be in $V(Q)$ by the choice of u_1. Hence u_1v_1 is an arc because there is no dipath of length at least 2 between Q and P. Moreover, observe that for every vertex x in $P[v', u_2] - Q'$ there is a subdipath of P' which is an x-bypass. Therefore v_1 must be in $P]u_2, t(P)]$ for otherwise it would be an arc bypass. Hence $\{u_2v_2, u_1v_1\}$ is crossing for $P \cup Q$, and so it must be tight.

This implies in particular that $s(S) \in P[v_1, t(P)]$.

We then find a contradiction as in the previous case by considering P^+ and Q^+.

This finishes the proof of the claim. □

Corollary 3.27. SHUNT *can be solved in polynomial time.*

Proof. Now we describe the procedure $\texttt{shunt}(D, s_1, s_2, t_1, t_2)$, solving SHUNT and estimate its time complexity. The procedure then check, by a Menger algorithm, if there are two disjoint dipaths P, Q from $\{s_1, s_2\}$ to $\{t_1, t_2\}$, which runs in $O(n+m)$ time. Observe that the arcs s_1s_2 and s_2s_1 are useless, so we remove them from D if they exist. Then we should check if there are paths of length at least 2, arc bypasses, loose crossings, C-forward paths, backward paths of length at least 2 or crossing bypasses with respect to P and Q, according to propositions 3.21, 3.22, 3.23, 3.24, 3.25 . For every vertex $u \in P$ (and any vertex in Q, similarly), we do the following: if u has a neighbour in Q, we test if there is a path from $P[s(P), u[$ to $P]u, t(P)]$, which would be an arc bypass. Let v' be the last vertex of Q such that uv' is an arc (and such that $v'u$ is an arc, similarly). Then, for a vertex v in $P]u, t(P)]$, we check if there is a vertex u' in $Q[s(Q), v'[$ such that $u'v$ (vu') is an arc. Then if u, v and $u'v'$ have distance at least 2 in P and Q respectively, it would be a loose crossing. Otherwise, if such edges exists there is a tight crossing $C = \{uv', u'v\}$ containing u. We then run a Menger algorithm one more time, to test if there is a dipath from u to v in $D-P-Q$, which would be a forward path. So far, the running time of the algorithm is bounded by $O(n^2(n+m))$: the complexity of calculating the P and Q initially plus the complexity of, for each vertex in in $P \cup Q$, look for an arc bypass, plus the running time of analysing if each pair of vertices in P or Q are part of a loose crossing and finally plus the time of looking for a forward path. Then, still considering the same tight crossing C, for every vertex x in $P[v, t(P)]$, we check if there is a dipath to some y in $P[v, t(P)]$. If it is the case and xy is an arc, we then look for dipaths from $P[s(P), y[$ to $P]y, u]$ and from $P[v, x[$ to $P]x, t(P)]$. This can be done in $O(n^2(n+m))$: for every pair of vertices u and x, we uses Menger algorithm possibly three times to compute the dipaths above. So, $\texttt{shunt}(D, s_1, s_2, t_1, t_2)$ runs in $O(n^2(n+m))$ time in total. □

With Lemma 3.27 in hands, we now deduce Theorem 3.20.

Proof of Theorem 3.20. For every vertex v of D and for every set of two out-neighbours s_1, s_2 and two in-neighbours t_1, t_2 of v, we check if there is a $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt in D. Observe that there is an E_{16}-SUBDIVISIONin D in which v is the a-vertex if and only if there is a shunt for a pair of out-neighbours and a pair of in-neighbours of v. So, since there are n^5 possible choices for vertex v and its neighbours, and for each of them we apply the procedure $\texttt{shunt}$ that runs in $O(n^2(n+m))$ time, our algorithm decides whether there is an E_{16}-SUBDIVISION in D in $O(n^7(n+m))$ time.

□

Chapter 4

F-Subdivision for disjoint directed cycles

Since C_k-Subdivision can be solved in polynomial time for any fixed k, a natural question is to ask for the complexity of F-Subdivision when F is the disjoint union of directed cycles. We gave the foundations to this discussion in Section 1.6. This is not a simple problem as can be seen from the observation that a digraph D contains k disjoint directed cycles if and only if it contains an F-subdivision where F is the disjoint union of k directed 2-cycles. Hence, if F is the disjoint union of k directed 2-cycles, F-Subdivision is equivalent to deciding if $\nu(D) \geq k$ for a given digraph D. Reed et al. [47] proved that this can be done in polynomial time.

In this chapter, we first analyse the simpler case of $C_2 + C_3$. Then we prove that if F is the disjoint union of cycles of length at least 3, F-Subdivision is also polynomial-time solvable, based on the proof of Reed et al. for 2-cycles.

4.1 Disjoint union of C_2 and C_3

Theorem 4.1 (Bang-Jensen, Havet and M. [3]). *$(C_2 + C_3)$-Subdivision is polynomial-time solvable.*

Proof. Let D be a digraph. If D has no 2-cycles, then D has a $C_2 + C_3$-subdivision if and only if it contains two disjoint cycles. This can be checked in polynomial time by Theorem 1.24.

Assume now that D contains 2-cycles. For each 2-cycle (x, y, x), we check if $D - \{x, y\}$ has a directed cycle of length at least 3. This can be done in linear time according to Theorem 2.4. If the answer is 'yes' for one of them, then we return 'yes'.

Suppose now that the answer is 'no' for all 2-cycles. Let D' be the digraph obtained from D by deleting the arcs of all the 2-cycles.

Claim 37. *D contains a (C_2+C_3)-subdivision if and only if D' contains two disjoint directed cycles.*

Subproof. Suppose that D contains a $(C_2 + C_3)$-subdivision S. No cycle of S can contain two vertices x and y in a 2-cycle because $D - \{x, y\}$ contains no directed cycle of length at least 3. In particular, all the arcs of S are in D'.

Conversely, if D' contains two disjoint directed cycles, they form a $(C_2 + C_3)$-subdivision since D' has no 2-cycles. ◊

Hence we check if D' has two disjoint directed cycles, which can be done in polynomial time according to Theorem 1.24. $\square$

4.2 Disjoint union of 3-cycles

In this section, we prove the following theorem:

Theorem 4.2 (Havet and M. [31]). *For any positive integer n, $n\vec{C}_3$-Subdivision is polynomial-time solvable.*

We do this by proving both Conjecture 1.27 and Conjecture 1.28 for $\ell = 3$. Then the result is implied by virtue of Theorem 1.29.
Combined with Lemma 1.19, this result in turn implies the following.

Corollary 4.3 (Havet and M. [31]). *If F is the disjoint union of cycles of length at most 3, then F-Subdivision is polynomial-time solvable.*

4.2.1 Linkage in digraphs with circumference at most 2

The aim of this section is to prove the following theorem, that consists in Conjecture 1.27 for $\ell = 3$.

Theorem 4.4 (Havet and M. [31]). *For each fixed k, the k-Linkage problem is polynomial-time solvable for digraphs with circumference at most 2.*

We first prove the following lemma.

Lemma 4.5 (Havet and M. [31]). *Let $\mathcal{D}$ be a class of digraphs and $\mathcal{S}$ be the class of strong digraphs. If k'-Linkage is polynomial-time solvable on $\mathcal{D} \cap \mathcal{S}$ for any $k' \leq k$, then k-Linkage is polynomial-time solvable on $\mathcal{D}$.*

Proof. Let D be a digraph in $\mathcal{D}$. Let $\sim$ be the relation defined on $V(D)$ by $u \sim v$ if and only if u and v are in the same strong component. It is clearly an equivalence relation on $V(D)$ with equivalence classes the strong components of D. Let $D/\sim$ be the quotient of D by $\sim$, that is the digraph whose vertices are the strong components of D, and in which there is an arc from a strong component S to another S' if and only if there is an arc of D with tail in S and head in S'. One can also see $D/\sim$ as the digraph obtained by contracting each strong component into a vertex. It is well-known that $D/\sim$ is an acyclic digraph, therefore there is an ordering $S_1, \dots, S_p$ of the strong components such that there is no arc $S_jS_{j'}$ in $D/\sim$ with $j > j'$. This implies that for every $j > j'$, there is no directed (x, y)-path in D with $x \in S_j$ and y in $S_{j'}$. Let $\tilde{D}$ be the digraph $D \setminus \bigcup_{j=1}^{p} A(S_j)$, the digraph whose arcs are those between non-equivalent vertices with respect to $\sim$.

Form a new digraph $\mathbf{D}$ whose vertices are the k-tuples $\mathbf{v} = (v_1, \dots, v_k)$ of distinct vertices of D. For any such k-tuple $\mathbf{v}$, there is a minimum index m such that S_m intersects $\{v_1, \dots, v_k\}$. Let $I = \{i \mid v_i \in S_m\}$. Set $I = \{i_1, \dots, i_{k'}\}$ with $i_1 < i_2 < \cdots < i_{k'}$.

For each k'-tuple $(w_1, w_2, \dots, w_{k'})$ of distinct vertices of $V(D) \setminus \{v_1, v_2, \dots, v_k\}$ such that there exists a k'-tuple $(u_1, u_2, \dots, u_{k'})$ of vertices in $V(S_m)$ such that there is a linkage from $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}})$ to $(u_1, u_2, \dots, u_{k'})$ in S_m and $u_j w_j$ is an arc in $\tilde{D}$ for all $1 \leq j \leq k'$, we put an arc from $\mathbf{v}$ to the k-tuple obtained from it by replacing v_i by w_i for all $i \in I$. We say that such an arc in $\mathbf{D}$ is labelled by S_m.

Observe that there are $O(n^{k'})$ k'-tuples $(u_1, u_2, \dots, u_{k'})$ of $V(S_m)$, and for each of them one can decide in polynomial time whether there is a linkage from $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}})$ to $(u_1, u_2, \dots, u_{k'})$ because k'-LINKAGE is polynomial-time solvable on $\mathcal{D}$ by hypothesis. Hence in polynomial time, we can construct the digraph $\mathbf{D}$ which has polynomial size.

We now prove that for any two sets of k distinct vertices $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$, there is a k-linkage from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ if and only if there is a directed path from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ in $\mathbf{D}$.

Suppose first that there is a k-linkage $(P_1, \dots, P_k)$ from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$. Since, when $j > j'$, there are no directed (x, y)-paths in D with $x \in S_j$ and y in $S_{j'}$, each P_i goes through the strong components $S_1, \dots, S_p$ in that order, possibly avoiding some. For each $1 \leq m \leq p$ and each $1 \leq i \leq k$, let $v_i(m)$ the first vertex in $\bigcup_{j=m}^{p} S_j$ along P_i if $\bigcup_{j=m}^{p} S_j$ and P_i intersect, and $v_i(m) = y_i$ otherwise.

Let $M = \{m_1, \dots, m_r\}$ with $m_1 \leq m_2 \leq \cdots \leq m_r$, be the set of indices m such that $S_m \cap \bigcup_{i=1}^{k} P_i \neq \emptyset$. By definition of $\mathbf{D}$, $\mathbf{v}(m_q)\mathbf{v}(m_{q+1})$ is an arc in $\mathbf{D}$. Thus $\mathbf{v}(m_1)\mathbf{v}(m_2)\dots\mathbf{v}(m_r)$ is a directed path from $(x_1, x_2, \dots, x_k)$ to $(y_1, y_2, \dots, y_k)$ in $\mathbf{D}$.

Suppose now that $\mathbf{D}$ has a directed path $\mathbf{Q}$ from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ in $\mathbf{D}$. We construct directed walks P_i, $1 \leq i \leq k$, by the following procedure. At the beginning $P_i = (x_i)$ for all $1 \leq i \leq k$. For each arc $\mathbf{a} = \mathbf{vw}$ of $\mathbf{Q}$ one after another from the initial vertex to the terminal vertex of $\mathbf{Q}$, we do the following.

Let $I = \{i_1, \dots, i_{k'}\}$ be set of indices i such that $v_i \neq w_i$. By definition of $\mathbf{D}$, there is a strong component S of D, a k'-tuple $(u_1, \dots, u_{k'})$ of disjoint vertices of S, and such that there is a linkage $(R_1, \dots, R_{k'})$ from $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}})$ to $(u_1, u_2, \dots, u_{k'})$ in S and $u_j w_{i_j} \in A(\tilde{D})$ for all $1 \leq j \leq k'$. In that case, we extend each P_{i_j}, $1 \leq j \leq r$, by appending $R_j u_j w_{i_j}$ at the end of it. Observe that R_j might be a path reduced to the single vertex $v_{i_j} = u_j$.

Observe that in $\mathbf{D}$ an arc labelled by a strong component S_m enters a k-tuple of vertices that all belong to components S_j with $j > m$. In particular, $\mathbf{Q}$ contains at most one arc labelled with any strong component S_m. This implies that each P_i is a directed (x_i, y_i)-path. Combined with the fact that each $(R_1, \dots, R_{k'})$ as defined above is a linkage, it implies that the P_i are disjoint. □

We can easily derive Theorem 4.4 from Lemma 4.5.

Proof of Theorem 4.4. Let $\mathcal{C}_2$ be the class of digraphs with circumference at most 2. A strong digraph D in $\mathcal{C}_2$ is obtained from a tree T by replacing every edge by a directed 2-cycle. Hence there is a k-linkage from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ in D if and only if there is a k-linkage from $(x_1, \dots, x_k)$ to $(y_1, \dots, y_k)$ in T. Since

Undirected k-linkage is polynomial-time solvable, it follows that k-Linkage is polynomial-time solvable on $\mathcal{C}_2 \cap S$. Thus, by Lemma 4.5, it is polynomial-time solvable on $\mathcal{C}_2$. □

4.2.2 Packing directed 3^+-cycles

The aim of this section is to prove the following theorem, showing that the directed cycles of length at least 3 have the Erdős-Pósa Property. It consists in Conjecture 1.28 for $\ell = 3$.

Theorem 4.6 (Havet and M. [31]). *For every integer $n \geq 0$, there exists an integer $\hat{t}_n$ such that for every digraph D, either $\nu_3(D) \geq n$ or $\tau_3(D) \leq \hat{t}_n$.*

Our proof follows the same approach as the one used by Reed et al. [47] to demonstrate Theorem 1.20, an analogue of Theorem 4.6 in the case for which 2-cycles are also considered. Their theorem says that, for every integer $n \geq 0$, there exists an integer t_n such that, for every digraph D, either D has a n pairwise-disjoint directed cycles, or there exists a set T of at most t_n vertices such that $D - T$ is acyclic. So, it is easy to see that $t_n = 0$ for $n = 0$ or $n = 1$. The value of t_n for the case in which $n = 2$, much more complicated to determine, was established to be 3 by McCuaig [45].

The proof of Theorem 1.20 is done by induction on n. So it may be assumed then that $n \geq 1$ and the t_{n-1} exists. To show that t_n exists, the following two main lemmas of Reed et al. [47] are needed.

Lemma 4.7 (Reed et al. [47]). *Let $n \geq 1$ be an integer such that t_{n-1} exists, and let k be an integer. Then there exists an integer t such that the following holds. Let D be a digraph with $\nu(D) < n$ and $\tau(D) \geq t$. Then there are distinct vertices $a_1, \dots, a_k, b_1, \dots, b_k$ and two k-linkages L_1, L_2 in D so that*

- *(i) L_1 links $(a_1, \dots, a_k)$ to $(b_1, \dots, b_k)$,*
- *(ii) L_2 links $(b_1, \dots, b_k)$ to one of $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$,*
- *(iii) every directed cycle of $L_1 \cup L_2$ meets $\{a_1, \dots, a_k, b_1, \dots, b_k\}$.*

Lemma 4.8 (Reed et al. [47]). *For every non-negative integer n, there exists a positive integer k so that the following holds. Let D be a digraph, and let $a_1 \dots, a_k, b_1, \dots, b_k$ be distinct vertices of D. Let L_1, L_2 be linkages in D linking $(a_1, \dots, a_k)$ to $(b_1, \dots, b_k)$, and $(b_1, \dots, b_k)$ to one of $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$, respectively. Let every directed cycle of $L_1 \cup L_2$ meet $\{a_1, \dots, a_k, b_1, \dots, b_k\}$. Then $\nu(D) \geq n$.*

Take k as in Lemma 4.8 and t as in Lemma 4.7. Suppose there is a digraph for which $\nu(D) < n$ and $\tau(D) \geq t$. Then the items *(i)*, *(ii)*, *(iii)* of Lemma 4.7 are satisfied. But in this case, by Lemma 4.8, $\nu(D) \geq n$, a contradiction. So, for every digraph, either $\nu(D) \geq n$ or, if $\nu(D) < n$, $\tau(D) < t$, and then there is an upper bound t_n for $\tau(D)$ whose value is smaller than t.

We should establish equivalents of Lemmas 4.7 and 4.8 for 3^+-cycles. However, a key ingredient in their proof is that if P is a directed (a, b)-path and Q is a directed (b, a)-path, then $P \cup Q$ contains a directed cycle. But in such a case, $P \cup Q$ does not necessarily contain a 3^+-cycle. We claim that $P \cup Q$ does not contain a 3^+-cycle if and only if Q is the converse of P (Recall that the converse of a directed path $P = (x_1, \dots, x_m)$ is the directed path $(x_m, \dots, x_1)$).

Lemma 4.9 (Havet and M. [31]). *Let a and b two distinct vertices, and let P be a directed (a, b)-path and Q be a directed (b, a)-path. Then $P \cup Q$ contains a directed 3^+-cycle if and only if Q is not the converse of P.*

Proof. Clearly, if Q is the converse of P, then $P \cup Q$ contains no directed 3^+-cycle.

Conversely, we prove by induction on the length m of P that if Q is not the converse of P, then $P \cup Q$ contains a directed 3^+-cycle. It holds trivially if $m = 1$. So we may assume that $m \geq 1$. Let $P = (x_0, x_1, \dots, x_m)$. Let y be the penultimate vertex of Q. If $y = x_1$, then $Q - x_0$ is not the converse of $P - x_0$. Hence, by the induction hypothesis, there is a directed 3^+-cycle in $(P - x_0) \cup (Q - x_0)$, and so in $P \cup Q$. Assume now that $y \neq x_1$, then let z be the penultimate vertex in $V(P \cap Q)$ along Q. If $z = x_1$, then $Q[x_1, x_0]$ has length at least 2, and so $(x_0, x_1) \cup Q[x_1, x_0]$ is a directed 3^+-cycle on $P \cup Q$. If $z \neq x_1$, then $P[x_0, z] \cup Q[z, x_0]$ is a directed 3^+-cycle on $P \cup Q$. □

In the proof we will have to make sure that some directed paths are not converse of some others, emphasizing the extra work required to deal with directed 3^+-cycle.

4.2.2.1 Main proof

First replacing 'directed cycle' by 'directed 3^+-cycle' in the proof of Lemma 4.7 of [47] (Lemma (2.2)), we obtain the following analogue of (2.2) of [47].

Lemma 4.10. *Let $n \geq 1$ be an integer such that t_{n-1} exists, and let k be an integer. Then there exists an integer $t'(k)$ such that the following holds. Let D be a digraph with $\nu_3(D) < n$, and $\tau_3(D) \geq 2t'(k)$, and let T be a set of size $\tau_3(D)$ such that $D - T$ has no directed 3^+-cycles. For any disjoint subsets $A, B \subseteq T$ with $|A| = |B| = k$, there are distinct vertices $a_1, \dots, a_k$ in A and distinct vertices $b_1, \dots, b_k$ in B, and two k-linkages L_1, L_2 of D, so that*

(i) L_1 *links* $(a_1, \dots, a_k)$ *to* $(b_1, \dots, b_k)$,

(ii) L_2 *links* $(b_1, \dots, b_k)$ *to one of* $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$,

(iii) every directed 3^+*-cycle of* $L_1 \cup L_2$ *meets* $\{a_1, \dots, a_k, b_1, \dots, b_k\}$.

However, for our purpose we need an extra condition on the two linkages L_1 and L_2. This is in fact why we needed a stronger statement than Lemma 4.10.

Lemma 4.11 (Havet and M. [31]). *Let $n \geq 1$ be an integer such that t_{n-1} exists, and let k be an even integer. Then there exists an integer $t(k)$ such that the following holds. Let D be a digraph with $\nu_3(D) < n$, and $\tau_3(D) \geq t(k)$, and let T be a set of size $\tau_3(D)$ such that $D - T$ has no directed 3^+-cycles. Then there are distinct vertices $a_1, \dots, a_k, b_1, \dots, b_k$ in T, and two k-linkages L_1, L_2 of D so that*

(i) L_1 links $(a_1, \dots, a_k)$ to $(b_1, \dots, b_k)$,

(ii) L_2 links $(b_1, \dots, b_k)$ to one of $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$,

(iii) every directed 3^+-cycle of $L_1 \cup L_2$ meets $\{a_1, \dots, a_k, b_1, \dots, b_k\}$,

(iv) no component of L_1 is the converse of a component of L_2.

To prove this lemma, we will need Erdő-Pósa Theorem (Theorem 1.21) and the following lemma.

Lemma 4.12. *Let r be a positive integer. Let T be a tree and S a set of at least $3r - 2$ vertices of T. Then there exists a vertex x of T and two subsets A and B of S, both of size r such that every (A, B)-path in T goes through x.*

Proof. Let E_r be the set of edges e such that both components of $T \setminus e$ have at least r vertices of S. We divide the proof in two cases depending on whether or not E_r is empty.

Assume first that $E_r \neq \emptyset$. Let $e = xy$ be an edge of E_r, and let T_x be the component of $T \setminus e$ containing x and T_y containing y. Both T_x and T_y contain at least r vertices of S. Let A (resp. B) be a set of r vertices of $S \cap V(T_x)$ (resp. $S \cap V(T_y)$). Then every (A, B)-path in T goes through e and so through x.

Assume now that $E_r = \emptyset$.

Claim 38. *There exists a vertex x such that all components of $T - x$ have less than r vertices of S.*

Subproof. Let us orient the edges of T as follows. Let $e = uv$ be an edge of T. Since $e \notin E_r$ is empty, exactly one component of $T \setminus e$ contains less than r vertices of S. Without loss of generality, this component is the one containing v. Orient the edge e from u to v. Now every orientation of a tree contains a vertex x with out-degree 0. Consider a component C of $T - x$. It contains exactly one neighbour y of x, and it is precisely the component of $T \setminus xy$ containing y. Thus $|C \cap S| < r$ because the edge is oriented from x to y. Hence all components of $T - x$ have less than r vertices. $\Diamond$

Take a vertex x as in the above claim. Let $C_1, \dots, C_m$ be the components of $T - x$. Then $|C_j| \leq r - 1$ for all $1 \leq j \leq m$. Let i be the smallest integer such that $T_i = \bigcup_{j=1}^{i} C_j$ contains at least r vertices of S. Clearly, T_i contains at most $2r - 2$ vertices in S, and thus there are least r vertices in $T - T_i$. Let A (resp. B) be a set of r vertices in T_i (resp. $T - T_i$). Then x is in every (A, B)-path. $\square$

Remark 4.13. *The bound $3r-2$ in the above lemma is tight. Indeed consider a tree T with a set S of $3r-3$ leaves and four other vertices x, y_1, y_2 and y_3 such that for every $i \in \{1,2,3\}$, y_i is adjacent to x and $r-1$ leaves. One can check that for every vertex x and two sets A and B of r leaves there is an (A,B)-path avoiding x.*

Proof of Lemma 4.11. Let t_n^* be as in Erdős-Pósa Theorem (Theorem 1.21); let $r = k + 2t_n^* + 2$; let $t(k) = \max\{3r-2+t_n^*, 2t'(r)\}$, let where t' is as in Lemma 4.10. We claim that $t(k)$ satisfies the lemma.

Let G_2 be the undirected graph with vertex set $V(D)$ in which two vertices x and y are adjacent if and only if $D[\{x,y\}]$ is a directed 2-cycle. To each cycle in G_2 correspond two directed cycles in D, one in each direction. Thus G_2 has less than n disjoint cycles. Hence by Erdős-Pósa Theorem, there is a set $U \subset V(D)$ of size t_n^* such that $G_2 - U$ is acyclic.

Choose $T \subseteq V(D)$ with $|T| = \tau_3(D)$, meeting all directed 3^+-cycles of D. Since $t(k) \geq 3r-2+t_n^*$, there is a set S of size $3r-2$ in $T \setminus U$. Since $G_2 - U$ is acyclic, we can extend it into a tree T_2. Hence, by Lemma 4.12, there exists a vertex x in $V(T_2)$ and two sets A and B in S of size r such that every (A,B)-path in T_2 goes through x. Since $G_2 - U$ is a subgraph of T_2, every (A,B)-path in $G_2 - U$ goes through x.

Since $|T| \geq 2t'(r)$, by Lemma 4.10, there are distinct vertices $a_1, \dots, a_{k'}$ in A and distinct vertices $b_1, \dots, b_{k'}$ in B, and two k'-linkages L'_1, L'_2 of D so that

(i) L'_1 links $(a'_1, \dots, a'_{k'})$ to $(b'_1, \dots, b'_{k'})$,

(ii) L'_2 links $(b'_1, \dots, b'_{k'})$ to one of $(a'_1, \dots, a'_{k'})$, $(a'_{k'}, \dots, a'_1)$,

(iii) every directed 3^+-cycle of $L'_1 \cup L'_2$ meets $\{a'_1, \dots, a'_{k'}, b'_1, \dots, b'_{k'}\}$.

For $1 \leq i \leq k'$, let P_i be the component of L_1 with initial vertex a'_i and Q_i the component of L_2 with initial vertex b'_i.

Clearly, if L'_2 links $(b'_1, \dots, b'_{k'})$ to $(a'_{k'}, \dots, a'_1)$, then condition (iv) is also verified by L'_1 and L'_2, because k' is even as k is even. For $1 \leq i \leq k$, set $a_i = a'_{i+(k'-k)/2}$ and $b_i = b'_{i+(k'-k)/2}$, and let $L_1 = \{P_j \mid 1 + k'/2 - k/2 \leq j \leq k'/2 + k/2\}$ and $L_2 = \{Q_j \mid 1 + k'/2 - k/2 \leq j \leq k'/2 + k/2\}$. Then $a_1, \dots, a_k, b_1, \dots, b_k$, L_1 and L_2 satisfy the lemma.

Assume now that L'_2 links $(b'_1, \dots, b'_{k'})$ to one of $(a'_1, \dots, a'_{k'})$. At most t_n^* of the P_i intersect U and at most t_n^* of the Q_i intersect U. Thus, since $k' \geq k + 2t_n^* + 1$, there are at least $k+1$ indices i such that both P_i and Q_i do not intersect U. Without loss of generality, we may assume that these indices are $\{1, \dots, k+1\}$. Now for $1 \leq i \leq k+1$, if P_i is the converse of Q_i, then P_i is also a path in $G_2 - U$ and thus it must go through x. Hence there is at most one index i, say $k+1$, such that P_i is the converse of Q_i. Hence $a_1, \dots, a_k, b_1, \dots, b_k$, L_1 and L_2 satisfy the lemma. □

We say a digraph is *divalent* if every vertex has in-degree 2 and out-degree 2, or in-degree 1 and out-degree 1. In Subsection 4.2.2.2 we shall prove the following lemma which is the analogue of Lemma (2.3) of [47].

A pair $\{L_1, L_2\}$ of linkages is *fully intersecting* if each component of L_1 meets each component of L_2, and it is *acyclic* if $L_1 \cup L_2$ has no directed cycles.

Lemma 4.14. *For every positive integer n, there exists a positive integer k_1 such that for every divalent digraph D, if there is a fully intersecting and acyclic pair of k_1-linkages in D then $\nu_3(D) \geq n$.*

Lemma 4.14 is proved in Subsection 4.2.2.2. We assume it for the moment. We will show how to combine it with Lemma 4.11 to prove Theorem 4.6. First we prove the following lemma, which is the analogue of Lemma 4.8 of [47] (Lemma (2.4)).

Lemma 4.15 (Havet and M. [31]). *For every non-negative integer n, there exists a positive integer k so that the following holds. Let D be a digraph, and let $a_1 \dots, a_k, b_1, \dots, b_k$ be distinct vertices of D. Let L_1, L_2 be linkages in D linking $(a_1, \dots, a_k)$ to $(b_1, \dots, b_k)$, and $(b_1, \dots, b_k)$ to one of $(a_1, \dots, a_k)$, $(a_k, \dots, a_1)$, respectively, such that no component of L_2 is the converse of L_1. Let every directed 3^+-cycle of $L_1 \cup L_2$ meet $\{a_1, \dots, a_k, b_1, \dots, b_k\}$. Then $\nu_3(D) \geq n$.*

The proof of this lemma is similar to the one of Lemma 4.8 of [47] (Lemma (2.4)). However, some extra technical details are required. When reducing to a divalent digraph, we also have to get rid of directed 2-cycles, and Claim 39 is now required.

As in [47], we shall also need Ramsey's theorem [46], which can be stated as follows.

Theorem 4.16 (Ramsey [46]). *For all positive integers q, l, r, there exists a (minimum) integer $R_l(r; q)$ so that the following holds. Let Z be a set with $|Z| \geq R_l(r; q)$, let Q be a set with $|Q| = q$, and for each $X \subset Z$ with $|X| = l$ let $f(X) \in Q$. Then there exists $S \subseteq Z$ with $|S| = r$ and there exists $x \in Q$ so that $f(X) = x$ for all $X \subseteq S$ with $|X| = l$.*

Proof of Lemma 4.15, assuming Lemma 4.14. Let $n \geq 1$. Let $k' = \max\{k_1, \lceil n/4 \rceil\}$ with k_1 as in Lemma 4.14. Let $k = 2R_2(4k'; 9)$ defined as in Theorem 4.16. We claim that n and k satisfy Lemma 4.15.

For let $a_1 \dots, a_k, b_1, \dots, b_k, L_1, L_2$ be as in the statement of Lemma 4.15.

Let $G_i = P_i \cup P_{k+1-i} \cup Q_i \cup Q_{k+1-i}$.

We show by induction on $|E(D)| + |V(D)|$ that $\nu_3(D) \geq n$. If $L_1 \cup L_2 \neq D$, then the result follows immediately by induction, so we may assume that $L_1 \cup L_2 = D$.

Assume that either the arc $e = uv$ belongs to $P_i \cap Q_j$, or the arc uv is in P_i and the arc vu is in Q_j, then consider the graph D' and the two linkages L'_1 and L'_2 obtained by contracting uv. These two linkages clearly satisfy the hypothesis of Lemma 4.15 since directed cycles can only be shorten while contracting.

We therefore may assume that every arc of D belongs to exactly one of L_1, L_2, and that D has no directed 2-cycles. In particular, D is divalent and every directed cycle of D meets $\{a_1, \dots, a_k, b_1, \dots, b_k\}$.

For $1 \leq i \leq k$, let P_i be the component of L_1 with initial vertex a_i, and let Q_i be the component of L_2 with initial vertex b_i.

For $1 \le h < i \le k/2$, define $f(\{h,i\})$ as follows. If G_i and G_h are disjoint, let $f(\{h,i\}) = 0$. Otherwise, at least one of the eight digraphs $P_i \cap Q_h$, $P_{k+1-i} \cap Q_h$, $P_i \cap Q_{k+1-h}$, $P_{k+1-i} \cap Q_{k+1-h}$, $Q_i \cap P_h$, $Q_{k+1-i} \cap P_h$, $Q_i \cap P_{k+1-h}$, $Q_{k+1-i} \cap P_{k+1-h}$ is non-null. Number them $1, \dots, 8$ in order; we define $f(\{h,i\}) \in \{0, 1, \dots, 8\}$.

Since $k = 2R_2(4k'; 9)$, by Theorem 4.16, there exists $S \subseteq \{1, \dots, \frac{1}{2}k\}$ with $|S| = 4k'$ and x with $0 \le x \le 8$ such that $f(\{h,i\}) = x$ for all $h, i \in S$ with $h < i$.

If $x = 0$, then the subdigraphs G_i are pairwise disjoint for all $i \in S$. But as we shall prove in Claim 39, each G_i contains a directed 3^+-cycle, and so $\nu_3(D) \ge |S| = 4k' \ge n$.

Claim 39. *Each G_i contains a directed 3^+-cycle.*

Subproof. If Q_i is a directed (b_i, a_i)-path, then by assumption Q_i is not the converse of P_i. Thus by Lemma 4.9 $P_i \cup Q_i$ contains a directed 3^+-cycle, and so G_i also does.

Assume now that Q_i is a directed (b_i, a_{k+1-i})-path, and so Q_{k+1-i} is a directed (b_{k+1-i}, a_i)-path. Q_i contains a directed path R_1 with initial vertex u_1 in P_i and terminal vertex v_1 in P_{k+1-i} whose internal vertices are not in $P_i \cup P_{k+1-i}$. Now Q_{k+1-i} contains a directed path R_2 with initial vertex u_2 in $P_{k+1-i}[v_1, b_{k+1-i}]$ and terminal vertex v_2 in $P_i[a_1, u_1]$ whose internal vertices are not in $P_i \cup P_{k+1-i}$. Observe that u_1, u_2, v_1 and v_2 are all distinct because P_i and P_{k+1-i} are disjoint and Q_i and Q_{k+1-i} are disjoint. Hence the $R_1 \cup P_{k+1-i}[v_1, u_2] \cup R_2 \cup P_i[v_2, u_1]$ is a directed 4^+-cycle in G_i. ◊

Assume now that $x = 1$. Let $S = I \cup J$, where $|I| = k'$, $|J| = 3k'$ and $i < j$ for all $i \in I$ and $j \in J$. Then for all $i \in I$ and all $j \in J$, P_i meets Q_j. There are $2k'$ vertices that are end-vertices of paths $P_i, i \in I$, an each of them is an end-vertex of at most one $Q_j, j \in J$. Since $|J| \ge 3k'$, there exists $J' \subset J$ with $|J'| = k'$ so that P_i and Q_j have no common end-vertex for $i \in I$ and $j \in J'$. Let L'_1 be the union of the components $P_i, i \in I$ and L'_2 be the union of the components $Q_j, j \in J'$. Now every directed cycle in $L'_1 \cup L'_2$ meets $\{a_1 \dots, a_k, b_1, \dots, b_k\}$, and each of $a_1 \dots, a_k, b_1, \dots, b_k$ is incident with at most one arc of $L'_1 \cup L'_2$ since P_i and Q_j have no common end-vertex for $i \in I$ and $j \in J'$. Hence $L'_1 \cup L'_2$ has no directed cycles. We thus have the result by Lemma 4.14.

The cases $2 \le x \le 8$ are similar to the case $x = 1$. □

Proof of Theorem 4.6, assuming Lemma 4.14. We prove Theorem 4.6 by induction on n; we therefore assume that $n \ge 1$ and t_{n-1} exists, and we show that t_n exists. Let k be as in Lemma 4.15, and let t be as in Lemma 4.11. We claim that there is no digraph D with $\nu_3(D) < n$ and $\tau_3(D) \ge t$. For suppose that D is such a digraph. By Lemma 4.11, there exists $a_1 \dots, a_k, b_1, \dots, b_k$ and L_1, L_2 as in Lemma 4.10, and so $\nu_3(D) \ge n$ by Lemma 4.15, a contradiction. Thus there is no such D, and consequently t_n exists and $t_n < t$. □

4.2.2.2 Proving Lemma 4.14

In this section, following Section 3 of [47], we show that if a digraph D contains a kind of grid, with some additional paths, then $\nu_3(D)$ is large. We then use this lemma to prove Lemma 4.14.

Let p, q be positive integers. A (p, q)-*web* in a digraph D is a fully intersecting and acyclic pair (L_1, L_2) of linkages such that L_1 has p components and L_2 has q components.

Let p, q be positive integers. A (p, q)-*fence* in a digraph D is a sequence $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ with the following properties:

(i) $P_1, \dots, P_{2p}$ are pairwise disjoint directed paths of D, and so are $Q_1, \dots, Q_q$;

(ii) for $1 \leq i \leq 2p$ and $1 \leq j \leq q$, $P_i \cap Q_j$ is a directed path (and therefore non-null);

(iii) for $1 \leq j \leq q$, the directed paths $P_1 \cap Q_j, \dots, P_{2p} \cap Q_j$ are in order in Q_j, and the initial vertex of Q_j is in $V(P_1)$ and its terminal vertex is in $V(P_{2p})$;

(iv) for $1 \leq i \leq 2p$, if i is odd then $P_i \cap Q_1, \dots, P_i \cap Q_q$ are in order in P_i, and if i is even then $P_i \cap Q_q, \dots, P_i \cap Q_1$ are in order in P_i.

Let Q_j be a directed (a_j, b_j)-path $(1 \leq l \leq q)$; we call $\{a_1, \dots, a_q\}$ the *top* of the fence, and $\{b_1, \dots, b_q\}$ its *bottom*.

The following lemma is the analogue to Lemma (3.1) of [47]. It only differs in the conclusion $\nu_3(D) \geq n$, instead of $\nu(D) \geq n$.

Lemma 4.17. *For every positive integer n, there are positive integers p, r with the following property. For any $q \geq 2$, let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p, q)-fence in a digraph D, and let there be r disjoint paths in D from the bottom of the fence to the top. Then $\nu_3(D) \geq n$.*

Combining Lemmas (4.4), (4.5) and (4.7) of [47] we directly obtain the following lemma.

Lemma 4.18. *For all positive integers p, q, there are positive integers p' and q' so that for every digraph G, if D contains a (p', q')-web then it contains a (p, q)-fence.*

In exactly the same way that Reed et al. deduced Lemma (2.3) from Lemmas (3.1), (4.4), (4.5) and (4.7) in [47], one can deduce Lemma 4.14 from Lemmas 4.17 and 4.18.

Hence it only remains to prove Lemma 4.17.

4.2.2.3 Proof Lemma 4.17

Consider the following lemma (Lemma (3.2)) from [47].

Lemma 4.19 (Reed et al. [47]). *Let $(P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q)$ be a (p, q)-fence in a digraph D, with top A and bottom B. Let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| = |B'| = r$, for $r \leq p$. Then there are directed paths $Q'_1, \ldots, Q'_r$ in $P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q$ so that $(P_1, \ldots, P_{2p}, Q'_1, \ldots, Q'_q)$ is a (p, r)-fence with top A' and bottom B'.*

Remark 4.20. *In the proof of this lemma, the proven (p, r)-fence is a subgraph of the (p, q)-fence, with $A' \subseteq A$ and $B' \subseteq B$. Moreover, if $p \geq 2$, then Q_j has order at least 4 for $1 \leq j \leq q$, because Q_j intersects every P_i $1 \leq i \leq 2p$. So, if $p \geq 2$, Q'_l has size at least 4, for $1 \leq l \leq r$.*

We need also an analogue of Lemma (3.3) of [47]:

Lemma 4.21. *Let $n \geq 1$ be an integer, and let $p \geq 2n$ and $N \geq 2n^2 - 3n + 2$ be integers. For some integer $q \geq 1$ let $(P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q)$ be a (p, q)-fence in a digraph D. Let $R_1, \ldots, R_N$ be disjoint directed paths of D from the bottom of the fence to the top, so that each R_k has no vertex or arc in $P_1 \cup \cdots \cup P_{2p} \cup Q_1 \cup \cdots \cup Q_q$ except its end-vertices. Then $\nu_3(D) \geq n$.*

The proof of this lemma is exactly the same as the one of Lemma (3.3) of [47]. The disjoint directed cycles showed in the proof are of the form $Q'_j R_m$, for Q'_j in a (p, r)-fence $(P_1, \ldots, P_{2p}, Q'_1, \ldots, Q'_q)$, subgraph of the (p, q)-fence $(P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q)$. Since $p \geq 2n \geq 2$, by Remark 4.20 each Q'_j has length at least 2, and so $Q'_j R_m$ has length at least 3. Hence $\nu_3(D) \geq n$.

We prove Lemma 4.17 by induction on n. The proof is almost identical to the one of Lemma (3.1) in [47]. The only differences are the easy case $n = 1$, for which we need here to take $p = 2$ (instead of $p = 1$) to be sure that the directed cycle is of length at least 3, and the use in place of Lemma (3.3) of its analogue, namely Lemma 4.21.

Index of Definitions

Bibliography

[1] ALON, N., LINGAS, A., AND WAHLEN, M. Approximating the maximum clique minor and some subgraph homeomorphism problems. *Theoret. Comput. Sci. 374*, 1-3 (2007), 149–158. (Cited in page 2.)

[2] BANG-JENSEN, J., AND GUTIN, G. *Digraphs*, second ed. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2009. Theory, algorithms and applications. (Cited in pages 3, 5, 16 and 19.)

[3] BANG-JENSEN, J., HAVET, F., AND MAIA, A. K. Finding a subdivision of a digraph. Tech. rep., INRIA, 2012. (Cited in pages 13, 15, 23, 24, 25, 27, 28, 29, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44, 45, 46, 51, 52, 61, 62, 67, 69 and 107.)

[4] BANG-JENSEN, J., HAVET, F., AND TROTIGNON, N. Finding an induced subdivision of a digraph. *Theoret. Comput. Sci. 443* (2012), 10–24. (Cited in page 4.)

[5] BANG-JENSEN, J., AND KRIESELL, M. Disjoint directed and undirected paths and cycles in digraphs. *Theoret. Comput. Sci. 410*, 47-49 (2009), 5138–5144. (Cited in page 3.)

[6] BERWANGER, D., DAWAR, A., HUNTER, P., AND KREUTZER, S. DAG-width and parity games. In *STACS 2006*, vol. 3884 of *Lecture Notes in Comput. Sci.* Springer, Berlin, 2006, pp. 524–536. (Cited in pages 3 and 21.)

[7] BIRMELÉ, E., BONDY, J. A., AND REED, B. A. The Erdős-Pósa property for long circuits. *Combinatorica 27*, 2 (2007), 135–145. (Cited in page 21.)

[8] BONDY, J. A., AND MURTY, U. S. R. *Graph theory*, vol. 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008. (Cited in pages 5, 8, 9 and 84.)

[9] BOYER, J. M. Subgraph homeomorphism via the edge addition planarity algorithm. *J. Graph Algorithms Appl. 16*, 2 (2012), 381–410. (Cited in page 2.)

[10] CHUDNOVSKY, M., CORNUÉJOLS, G., LIU, X., SEYMOUR, P., AND VUŠKOVIĆ, K. Recognizing Berge graphs. *Combinatorica 25*, 2 (2005), 143–186. (Cited in page 4.)

[11] CHUDNOVSKY, M., KAWARABAYASHI, K.-I., AND SEYMOUR, P. Detecting even holes. *J. Graph Theory 48*, 2 (2005), 85–111. (Cited in page 4.)

[12] CHUDNOVSKY, M., PENEV, I., SCOTT, A., AND TROTIGNON, N. Excluding induced subdivisions of the bull and related graphs. *J. Graph Theory 71*, 1 (2012), 49–68. (Cited in page 4.)

[13] Chudnovsky, M., Robertson, N., Seymour, P., and Thomas, R. The strong perfect graph theorem. *Ann. of Math. (2) 164*, 1 (2006), 51–229. (Cited in page 2.)

[14] Chudnovsky, M., Scott, A., and Seymour, P. Disjoint paths in tournaments. Manuscript. (Cited in page 3.)

[15] Chudnovsky, M., and Seymour, P. Excluding induced subgraphs. In *Surveys in combinatorics 2007*, vol. 346 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2007, pp. 99–119. (Cited in page 2.)

[16] Chudnovsky, M., and Seymour, P. The three-in-a-tree problem. *Combinatorica 30*, 4 (2010), 387–417. (Cited in pages 4, 36 and 47.)

[17] Chudnovsky, M., Seymour, P., and Trotignon, N. Detecting an induced net subdivision. *J. Combin. Theory Ser. B 103*, 5 (2013), 630–641. (Cited in pages 4, 21 and 67.)

[18] Chung, M. J. $O(n^{2.5})$ time algorithms for the subgraph homeomorphism problem on trees. *J. Algorithms 8*, 1 (1987), 106–112. (Cited in page 1.)

[19] Conforti, M., Cornuéjols, G., Kapoor, A., and Vušković, K. Even-hole-free graphs. II. Recognition algorithm. *J. Graph Theory 40*, 4 (2002), 238–266. (Cited in page 4.)

[20] Erdős, P., and Pósa, L. On the maximal number of disjoint circuits of a graph. *Publ. Math. Debrecen 9* (1962), 3–12. (Cited in page 19.)

[21] Erdős, P., and Pósa, L. On independent circuits contained in a graph. *Canad. J. Math. 17* (1965), 347–352. (Cited in page 19.)

[22] Ford, Jr., L. R., and Fulkerson, D. R. *Flows in networks.* Princeton University Press, Princeton, N.J., 1962. (Cited in page 122.)

[23] Ford, Jr., L. R., and Fulkerson, D. R. *Flows in networks.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2010. Paperback edition [of [22]], With a new foreword by Robert G. Bland and James B. Orlin. (Cited in page 8.)

[24] Fortune, S., Hopcroft, J., and Wyllie, J. The directed subgraph homeomorphism problem. *Theoret. Comput. Sci. 10*, 2 (1980), 111–121. (Cited in pages 2, 3, 10 and 13.)

[25] Gabow, H. N., and Nie, S. Finding a long directed cycle. In *Proceedings of the Fifteenth Annual ACM-SIAM, Symposium on Discrete Algorithms* (New York, 2004), ACM, pp. 49–58 (electronic). (Cited in page 29.)

[26] Gabow, H. N., and Nie, S. Finding a long directed cycle. *ACM Trans. Algorithms 4*, 1 (2008), Art. 7, 21. (Cited in page 29.)

[27] Gallai, T. Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen. *Acta Math. Acad. Sci. Hungar. 12* (1961), 131–173. (Cited in page 19.)

[28] Garey, M. R., and Johnson, D. S. *Computers and intractability.* W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences. (Cited in page 1.)

[29] Granot, D., Granot, F., and Zhu, W. R. Naturally submodular digraphs and forbidden digraph configurations. *Discrete Appl. Math. 100*, 1-2 (2000), 67–84. (Cited in page 2.)

[30] Grohe, M., and Grüber, M. Parameterized approximability of the disjoint cycle problem. In *Automata, languages and programming*, vol. 4596 of *Lecture Notes in Comput. Sci.* Springer, Berlin, 2007, pp. 363–374. (Cited in page 19.)

[31] Havet, F., and Maia, A. K. On disjoint directed cycles with prescribed minimum lengths. Tech. rep., INRIA, 2013. (Cited in pages 18, 21, 108, 110, 111, 112 and 114.)

[32] Hecht, M. S., and Ullman, J. D. Flow graph reducibility. *SIAM J. Comput. 1*, 2 (1972), 188–202. (Cited in page 2.)

[33] Hunt, III, H. B., and Szymanski, T. G. Dichotomization, reachability, and the forbidden subgraph problem (extended abstract). In *Eighth Annual ACM Symposium on Theory of Computing (Hershey, Pa., 1976).* Assoc. Comput. Mach., New York, 1976, pp. 126–134. (Cited in page 2.)

[34] Itai, A., Papadimitriou, C. H., and Szwarcfiter, J. L. Hamilton paths in grid graphs. *SIAM J. Comput. 11*, 4 (1982), 676–686. (Cited in page 17.)

[35] Johnson, T., Robertson, N., Seymour, P. D., and Thomas, R. Directed tree-width. *J. Combin. Theory Ser. B 82*, 1 (2001), 138–154. (Cited in pages 3 and 17.)

[36] Karp, R. M. On the complexity of combinatorial problems. In *Networks* (Jan 1975), vol. 5, pp. 45–68. (Cited in page 1.)

[37] Kawarabayashi, K.-i., Kobayashi, Y., and Reed, B. The disjoint paths problem in quadratic time. *J. Combin. Theory Ser. B 102*, 2 (2012), 424–435. (Cited in page 2.)

[38] LaPaugh, A. S., and Rivest, R. L. The subgraph homeomorphism problem. *J. Comput. System Sci. 20*, 2 (1980), 133–149. ACM-SIGACT Symposium on the Theory of Computing (San Diego, Calif., 1978). (Cited in page 1.)

[39] Lévêque, B., Lin, D. Y., Maffray, F., and Trotignon, N. Detecting induced subgraphs. *Discrete Appl. Math. 157*, 17 (2009), 3540–3551. (Cited in page 3.)

[40] Lévêque, B., Maffray, F., and Trotignon, N. On graphs with no induced subdivision of K_4. *J. Combin. Theory Ser. B 102*, 4 (2012), 924–947. (Cited in page 4.)

[41] Lingas, A., and Wahlen, M. An exact algorithm for subgraph homeomorphism. *J. Discrete Algorithms 7*, 4 (2009), 464–468. (Cited in page 2.)

[42] Liu, P. C., and Geldmacher, R. C. An $O(\max(m, n))$ algorithm for finding a subgraph homeomorphic to K_4. In *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. II* (1980), vol. 29, pp. 597–609. (Cited in page 1.)

[43] Lynch, J. F. The equivalence of theorem proving and the interconnection problem. *SIGDA Newsl. 5*, 3 (Sept. 1975), 31–36. (Cited in page 1.)

[44] Matoušek, J., and Thomas, R. On the complexity of finding iso- and other morphisms for partial k-trees. *Discrete Math. 108*, 1-3 (1992), 343–364. Topological, algebraical and combinatorial structures. Frolík's memorial volume. (Cited in page 1.)

[45] McCuaig, W. Intercyclic digraphs. In *Graph structure theory (Seattle, WA, 1991)*, vol. 147 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1993, pp. 203–245. (Cited in pages 19 and 110.)

[46] Ramsey, F. P. On a Problem of Formal Logic. *Proc. London Math. Soc. S2-30*, 1, 264. (Cited in page 114.)

[47] Reed, B., Robertson, N., Seymour, P., and Thomas, R. Packing directed circuits. *Combinatorica 16*, 4 (1996), 535–554. (Cited in pages 19, 20, 21, 107, 110, 111, 113, 114, 116 and 117.)

[48] Robertson, N., and Seymour, P. D. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B 63*, 1 (1995), 65–110. (Cited in pages 1 and 30.)

[49] Rose, D. J., Tarjan, R. E., and Lueker, G. S. Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput. 5*, 2 (1976), 266–283. (Cited in page 4.)

[50] Shiloach, Y. A polynomial solution to the undirected two paths problem. *J. Assoc. Comput. Mach. 27*, 3 (1980), 445–456. (Cited in page 1.)

[51] Slivkins, A. Parameterized tractability of edge-disjoint paths on directed acyclic graphs. *SIAM J. Discrete Math. 24*, 1 (2010), 146–157. (Cited in page 19.)

[52] Thomassen, C. 2-linked graphs. *European J. Combin. 1*, 4 (1980), 371–378. (Cited in page 1.)

[53] Wagner, K. Bemerkungen zum vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung 46* (1936), 26–32. (Cited in page 1.)