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## Subdivisions of digraphs

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## Subdivisions of Digraphs

**Abstract:** In this work, we consider the following problem: Given a directed graph  $D$ , does it contain a subdivision of a prescribed digraph  $F$ ? We believe that there is a dichotomy between NP-complete and polynomial-time solvable instances of this problem. We present many examples of both cases. In particular, except for five instances, we are able to classify all the digraphs  $F$  of order 4.

While all NP-hardness proofs are made by reduction from some version of the 2-linkage problem in digraphs, we use different algorithmic tools for proving polynomial-time solvability of certain instances, some of them involving relatively complicated algorithms. The techniques vary from easy brute force algorithms, algorithms based on maximum-flow calculations, handle decompositions of strongly connected digraphs, among others.

Finally, we treat the very special case of  $F$  being the disjoint union of directed cycles. In particular, we show that the directed cycles of length at least 3 have the Erdős-Pósa Property: for every  $n$ , there exists an integer  $t_n$  such that for every digraph  $D$ , either  $D$  contains  $n$  disjoint directed cycles of length at least 3, or there is a set  $T$  of  $t_n$  vertices that meets every directed cycle of length at least 3. From this result, we deduce that if  $F$  is the disjoint union of directed cycles of length at most 3, then one can decide in polynomial time if a digraph contains a subdivision of  $F$ .

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## Subdivisions de Digraphes

**Résumé :** Dans ce travail, nous considérons le problème suivant : étant donné un graphe orienté  $D$ , contient-il une subdivision d'un digraphe fixé  $F$  ? Nous pensons qu'il existe une dichotomie entre les instances polynomiales et NP-complètes. Nous donnons plusieurs exemples pour les deux cas. En particulier, sauf pour cinq instances, nous sommes capable de classer tous les digraphes d'ordre 4.

Alors que toutes les preuves NP-complétude sont faites par réduction de une version du problème 2-linkage en digraphes, nous utilisons différents outils algorithmiques pour prouver la solvabilité en temps polynomial de certains cas, certains d'entre eux impliquant des algorithmes relativement complexes. Les techniques varient des simples algorithmes de force brute, aux algorithmes basés sur des calculs maximale de flot, et aux décompositions en anses des digraphes fortement connexes, entre autres.

Pour terminer, nous traitons le cas particulier où  $F$  étant une union disjointe de cycles dirigés. En particulier, nous montrons que les cycles dirigés de longueur au moins 3 possède la Propriété d'Erdős-Pósa : pour tout  $n$ , il existe un entier  $t_n$  tel que pour tout digraphe  $D$ , soit  $D$  a  $n$  cycles dirigés disjoints de longueur au moins 3, soit il y a un ensemble  $T$  d'au plus  $t_n$  sommets qui intersecte tous les cycles dirigés de longueur au moins 3. De ce résultat, nous déduisons que si  $F$  est l'union disjointe de cycles dirigés de longueur au plus 3, alors on peut décider en temps polynomial si un digraphe contient une subdivision de  $F$ .

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# Introduction and preliminaries

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## 1.1 Introduction

A *subdivision* of a graph  $H = (V, E)$  is a graph obtained from  $H$  by replacing each edge  $ab \in E(H)$  by a path between  $a$  and  $b$  of length at least 1. The subdivision (or subgraph homeomorphism) problem consists in deciding whether a given graph  $G$  has a subdivision of a prescribed graph  $H$  as a subgraph. The determination of its complexity was one of the open questions left by Garey and Johnson [28] and it was widely studied in the following years.

Observe that the subdivision problem is NP-complete if  $H$  is part of the input, since it includes the Hamiltonian cycle problem [28]. It remains NP-complete even for graphs  $G$  with bounded treewidth [44].

We can consider two variants when  $H$  is fixed: the correspondence between the vertices of  $H$  and  $G$  can be previously specified or not. Several polynomial-time algorithms for particular classes were proposed in both situations [18, 38, 42, 50], and the problem was finally proved to be polynomial-time solvable for every fixed  $H$  by Robertson and Seymour linkage algorithm [48].

The linkage or disjoint paths problem is the following: given a graph  $G = (V, E)$  and  $k$  pairs  $(s_1, t_1), \dots, (s_k, t_k)$  of vertices in  $V(G)$ , for a fixed  $k$ , answer whether there are  $k$ -disjoint paths  $P_1, \dots, P_k$  such that  $P_i$  connects  $s_i$  to  $t_i$ . So, for finding a subdivision of a graph  $H = (V, E)$  in a given graph  $G = (V, E)$  with a fixed mapping of  $V(H)$  in  $V(G)$ , we could simply look for  $|E(H)|$  disjoint paths in  $G$ . Since there are  $|V(G)|^{|V(H)|}$  possible ways of mapping the vertices of  $H$  in the vertices of  $G$ , we could still find the subdivision without predetermined mapping in polynomial time.

The linkage problem is NP-complete if  $k$  is allowed to vary on the input [36] and it is still NP-complete for planar graphs [43]. But contrary to the subdivision problem, it is polynomial-time solvable for graphs with bounded treewidth [44]. A lot of work was done before the definitive solution by Robertson and Seymour, specially for the case  $k = 2$  where some algorithms were discovered independently [50, 52]. Indeed, the 2-linkage is a simpler case. It is proved that there are two disjoint  $(s_1, t_1)$ - and  $(s_2, t_2)$ -paths in  $G$  if and only if a small modification of  $G$  has a  $K_5$ -minor such that the  $K_5$ -model is “sufficiently” connected to the vertices  $s_1, s_2, t_1$  and  $t_2$ . So, after making some connectivity reductions and be left with a 4-connected graph in which every  $K_5$ -model would have the desired property, the problem becomes equivalent to test if the graph is planar, by the result of Wagner [53] saying that a 4-connected graph has no  $K_5$ -minor if and only if it is planar. However, the algorithm for the general case of linkage is complicated and not practical, since the constants involved



are huge. A recent work by Kawarabayashi, Kobayashi and Reed improves the complexity of the linkage algorithm from  $O(n^3)$  to  $O(n^2)$  [37]. And new results were also obtained concerning approximative and FPT algorithms for variations of the subdivision problem itself [1, 9, 41].

An important application of graph subdivisions arises in the fact that many interesting classes are defined by forbidding subgraphs, and this is also valid if we look at digraphs and induced graphs, two relevant variations of the subdivision problem. Planar graphs are a well-known example for the first case: they were characterized by Kuratowski as the graphs which do not have any subdivision of  $K_5$  or  $K_{3,3}$ . Some examples regarding digraphs can be found in [29]. Another notorious example of the undirected case is the one of Perfect graphs, concerning induced subgraphs. According to the Perfect Graph Theorem, a graph is perfect if and only if it does not contain an induced subgraph which is either an odd cycle or the complement of an odd cycle [13]. A survey with some classes determined by forbidden induced subgraphs can be seen in [15]. Flow graph reducibility and programming schema are other applications of the subdivision problem [32, 33].

The aim of the present work is to investigate subdivisions in digraphs. We can define the problem similarly to the undirected case.

The *subdivision of an arc  $xy$*  of  $F$  is the replacement of  $xy$  by two arcs  $xz, zy$ , where  $z$  is a new vertex. If  $S$  can be obtained from  $F$  by repeatedly subdividing arcs (including the arcs previously subdivided), then  $S$  is a subdivision of  $F$ , also called an  *$F$ -subdivision*. Alternatively, a subdivision of a digraph  $F$  is a digraph obtained from  $F$  by replacing each arc  $xy$  of  $F$  by a directed  $(x, y)$ -path of length at least one.

We consider the following problem for a fixed digraph  $F$ .

#### $F$ -SUBDIVISION

Input: A digraph  $D$ .

Question: Does  $D$  contain a subdivision of  $F$  as a subgraph?

Let  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$  be distinct vertices of a digraph  $D$ . A  *$k$ -linkage* from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $D$  is a system of disjoint directed paths  $P_1, P_2, \dots, P_k$  such that  $P_i$  is an  $(x_i, y_i)$ -path in  $D$ . The  $k$ -LINKAGE problem is defined as follows.

#### $k$ -LINKAGE

Input: A digraph  $D$  and  $2k$  distinct vertices  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$ .

Question: Is there a  $k$ -linkage from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $D$ ?

Similarly to the situation for undirected graphs, the  $F$ -SUBDIVISION problem is related to  $k$ -LINKAGE. However, contrary to graphs, unless  $P=NP$ ,  $k$ -LINKAGE cannot be solved in polynomial time in general digraphs. Fortune, Hopcroft and Wyllie [24] showed that already 2-LINKAGE is NP-complete. Using their result, we show that for lots of digraphs  $F$ , the  $F$ -SUBDIVISION problem is NP-complete.

In the same work [24],  $F$ -SUBDIVISION for pre-setted mapping of the vertices of  $F$  in the vertices of  $D$  is completely classified: It is polynomial-time solvable for every tree of height at most 1 and NP-complete otherwise. Furthermore, they proved that if  $D$  is acyclic, then for every fixed integer  $k$ , there is a polynomial-time algorithm to solve  $k$ -LINKAGE. Mixed linkage problems (concerning digraphs and its respective underlying graphs) are considered in [5]. For a survey of linkages in digraphs, see [2, Chapter 10].

The digraph subdivision problem was solved for  $D$  being in some particular classes of graphs. As a consequence of Fortune et al. result for  $k$ -LINKAGE, for any fixed  $F$ ,  $F$ -SUBDIVISION is polynomial-time solvable for acyclic digraphs. The same can be said about digraphs of bounded directed tree-width [35] or bounded DAG-width [6]. In addition Chudnovsky, Scott and Seymour [14] showed that  $F$ -SUBDIVISION is polynomial-time solvable when restricted to the class of tournaments. But to the best of our knowledge, there are no previous significant results on the general form of the problem.

We believe that there is a dichotomy between NP-complete and polynomial-time solvable instances of  $F$ -SUBDIVISION. In this work, we present many examples of both cases. We start by giving, still in this chapter, a tool based on a reduction from the NP-complete 2-linkage problem in digraphs, which can be applied to conclude the NP-completeness of  $F$ -SUBDIVISION for the majority of all digraphs  $F$ . In Chapter 2, we concentrate on the problem for  $F$  being in particular classes of graphs. For some of them, we are able to completely determine in which cases the  $F$ -subdivision problem is polynomial-time solvable and for which it is NP-complete. We turn to the digraphs of order 4 in Chapter 3, and except for 5 instances, we are able to classify all of them.

While all NP-hardness proofs are made by reduction from some version of the 2-linkage problem in digraphs, we describe different algorithmic tools for proving polynomial-time solvability of certain instances, some of them involving relatively complicated algorithms. The techniques vary from easy brute force algorithms, algorithms based on maximum-flow calculations, algorithms based on handle decompositions of strongly connected digraphs, to be detailed later in the chapter, between others. Finally, in Chapter 4 we treat the very special case of  $F$  being the disjoint union of cycles. In particular, we show that the directed cycles of length at least 3 have the Erdős-Pósa Property: for every  $n$ , there exists an integer  $t_n$  such that for every digraph  $D$ , either  $D$  contains  $n$  disjoint directed cycles of length at least 3, or there is a set  $T$  of at most  $t_n$  vertices that meets every directed cycle of length at least 3. From these result, we deduce that if  $F$  is the disjoint union of directed cycles of length at most 3, then one can decide in polynomial time if a digraph contains a subdivision of  $F$ .

As briefly mentioned before, the alternative problem in which we look for *induced* subdivisions of a prescribed graph  $H$  in an input graph  $G$  is also a question of interest. The undirected case is computationally harder than its equivalent for non-induced subdivisions: Lévêque et al. showed that it is NP-complete for  $H = K_5$  [39]. Besides, this problem seems quite difficult to be addressed, since there are not many

solved cases and so far there is no indication of what would be the line between polynomial and NP-complete instances.

Despite the existence of trivial polynomial algorithms to find induced subdivisions in undirected graphs, like for cycles of length at least three, some other cases involve sophisticated techniques. They diversify from algorithms based on breadth-first search, as the one presented by Rose, Tarjan and Lueker to find an induced subdivision of a  $C_k$  ( $k \geq 4$ ) efficiently ( $O(n + m)$ ) [49], to the use of the *three-in-a-tree* algorithm of Chudnovsky and Seymour [16]. The three-in-a-tree problem consists in answer for a given graph and three pre-determined vertices of it if there is a tree containing such vertices. The algorithm to solve it, which has execution time of  $O(n^4)$  [16], provide a general tool that can be used in many solutions of the subdivision problem, as it was done in the same paper for  $K_{2,3}$ . We use a similar approach to deal with some examples of the directed non-induced case in this thesis. But even being one of the main methods for the induced case, it does not seem to fit in the solutions of all the instances, like it was showed to subdivisions of the net graph (a cycle in that each of the vertices on it has a neighbour with degree 1 outside the cycle) [17]. Cleaning [10, 11] and decompositions [12, 19] are still other techniques used in the solution of the problem. In [40], Lévêque, Maffray and Trotignon present a decomposition theorem for graphs with no induced subdivision of  $K_4$ . But in this case, the theorem does not give directly a polynomial-time recognition algorithm. The complexity of finding induced subdivision for  $H = K_4$  is still open.

The problem of finding an induced subdivision of a prescribed digraph  $F$  in a given digraph  $D$ , referred as INDUCED- $F$ -SUBDIVISION, was also investigated. A lot more is known here than for undirected induced subdivisions. It turns out that there is a big difference in the complexity of the problem depending on whether the digraph is allowed to have 2-cycles or not, in which case it is called an *oriented graph*, as showed in [4]. In the latter case, the authors proved that INDUCED- $F$ -SUBDIVISION is NP-complete for every oriented graph which is not the disjoint union of spiders (trees obtained from disjoint directed paths by identifying one end of each path into a vertex). Still in [4] it was conjectured that INDUCED- $F$ -SUBDIVISION is NP-complete unless  $F$  is the disjoint union of spiders and at most one 2-cycle. The authors also consider the problem when  $D$  is an oriented graph, and they proved it to be polynomial-time solvable for some cases of transitive tournaments and oriented paths, among others.

## 1.2 Finding an $F$ -subdivision

We are primarily interested in determining in which cases the problem is polynomial-time solvable or NP-complete. Lemma 1.1 implies that deciding if there is an  $F$ -subdivision in a digraph is polynomial-time solvable if and only if finding an  $F$ -subdivision in a digraph is polynomial-time solvable.

**Lemma 1.1** (Havet, M. and Mohar). *If  $F$ -SUBDIVISION can be solved in  $f(n, m)$  time, where  $f$  is non-decreasing in  $m$ , then there is an algorithm that finds an  $F$ -subdivision (if one exists) in a digraph in  $((m + 1) \cdot f(n, m) + m)$  time.*

*Proof.* Suppose that there exists an algorithm  $F$ -decide( $D$ ) that decides in  $f(n, m)$  whether  $D$  contains an  $F$ -subdivision. We now construct an algorithm  $F$ -find( $D$ ) that finds an  $F$ -subdivision in  $D$  if there is one, and returns ‘no’ otherwise. It proceeds as follows.

Let  $a_1, \dots, a_m$  be the arcs of  $D$ . If  $F$ -decide( $D$ ) returns ‘no’, then we also return ‘no’. If not, then  $D$  contains an  $F$ -subdivision, we find it as follows: We initialize  $D_0 := D$ . For  $i = 1$  to  $m$ ,  $D_i := D_{i-1} - a_i$  if  $F$ -decide( $D_{i-1} - a_i$ ) returns ‘yes’, and  $D_i := D_{i-1}$  otherwise.

$F$ -find is valid because at step  $i$ , we delete the arc  $a_i$  if and only if there is an  $F$ -subdivision not containing  $a_i$ . Hence at each step  $i$ , we are sure that  $D_i$  contains an  $F$ -subdivision, and that any  $F$ -subdivision must contain all the arcs of  $D_i \cap \{a_1, \dots, a_i\}$ .

$F$ -find runs  $(m + 1)$  times the algorithm  $F$ -decide and removes at most  $m$  times an arc. Therefore, it runs in time  $O(m) \cdot f(n, m) + m$ .  $\square$

For sake of clarity, we only present algorithms for solving  $F$ -SUBDIVISION as a decision problem. However, the proofs of validity of all given algorithms always rely on constructive claims. Hence each algorithm can be easily transformed into a polynomial-time algorithm for finding an  $F$ -subdivision in a given digraph, and then our algorithms for finding  $F$ -subdivisions have the same complexity as their decision versions.

## 1.3 Notation, known results and tools

In this section, we present the basic definitions needed for the global understanding of the work and the used notation. However, we assume that reader is familiar with fundamental concepts in graph theory, highlighting here those for digraphs in which the orientations are important. We also present the general digraphs subdivision problem and a few known techniques to be applied in solutions of some of its restricted cases along the text. We rely on [2, 8] for additional standard information.

### 1.3.1 Elementary definitions

A *digraph*  $D$  consists of a set  $V(D)$  of vertices and a set  $A(D)$  of ordered pairs of distinct vertices called *arcs*. Unless otherwise stated, in this text, the letters  $n$  and  $m$  will always denote the number of vertices and arcs (edges), respectively, of the input digraph (graph) of the problem in question. By *linear time*, we mean  $O(n + m)$  time.

If  $(x, y)$  is an arc, then we say that  $x$  *dominates*  $y$ . In this case,  $x$  is the *tail*,  $y$  is the *head* and  $x, y$  are both *end-vertices* of  $(x, y)$ . Moreover,  $x$  and  $y$  are said *adjacent*. To simplify, we frequently write  $xy$  instead of  $(x, y)$  to refer to an arc

from  $x$  to  $y$ , and for a vertex  $x$  (resp. a subdigraph  $S$  of  $D$ ), we abbreviate  $\{x\}$  to  $x$  (resp.  $V(S)$  to  $S$ ) in the notation.

The set of vertices that dominate a vertex  $x$  in a digraph  $D$  is called its *in-neighbourhood* and denoted by  $N_D^-(x)$ . Similarly,  $N_D^+(x)$  is the *out-neighbourhood* of  $x$ , that is, the set of vertices dominated by  $x$ . Let  $N_D(x) = N_D^-(x) \cup N_D^+(x)$ . The *in-degree*  $d_D^-(x)$ , *out-degree*  $d_D^+(x)$  and *degree*  $d_D(x)$  of a vertex  $x$  in  $D$  are the cardinality of  $N_D^-(x)$ ,  $N_D^+(x)$  and  $N_D(x)$ , respectively.

A *source* in  $D$  is a vertex of in-degree zero and a *sink* is a vertex of out-degree zero. A vertex  $x$  is said to be *small* if  $d^-(x) \leq 2$ ,  $d^+(x) \leq 2$  and  $d(x) \leq 3$ . A non-small vertex is called *big*.

Let  $D$  be a digraph. We call  $D$  a *multidigraph* if it has multiple arcs (pairs of arcs with the same tail and the same head). The *converse* of  $D$  is the digraph  $\bar{D}$  obtained from  $D$  by reversing the orientation of all arcs. We denote by  $UG(D)$  the *underlying (multi)graph* of  $D$ , that is, the (multi)graph we obtain by replacing each arc by an edge. To every graph  $G$ , we can associate a *symmetric digraph* by replacing every edge  $uv$  by the two arcs  $uv$  and  $vu$ .

An *oriented graph* is an orientation of an undirected graph. In other words, it is a digraph with no directed cycles of length 2. An *oriented path* is an orientation of an undirected path. Hence an oriented path  $P$  is a sequence  $(x_1, a_1, x_2, a_2, \dots, a_{n-1}, x_n)$ , where the  $x_i$  are distinct vertices and for all  $1 \leq j \leq n-1$ ,  $a_j$  is either the arc  $x_j x_{j+1}$  or the arc  $x_{j+1} x_j$ . We often refer to such an oriented path  $P$  by the underlying undirected path  $x_1 x_2 \dots x_n$ . This is a slight abuse, because the oriented path  $P$  is not completely determined by this sequence as there are two possible orientations for each edge. However, when we use this notation, either the orientation does not matter or it is clear from the context.

Let  $P = x_1 x_2 \dots x_n$  be an oriented path. We say that  $P$  is an  $(x_1, x_n)$ -*path*. The vertex  $x_1$  is the *initial vertex* of  $P$  and  $x_n$  its *terminal vertex*. We denote the initial vertex of  $P$  by  $s(P)$  and the terminal vertex of  $P$  by  $t(P)$ . The subpath  $x_2 \dots x_{n-1}$  is denoted by  $P^\circ$ . If  $x_1 x_2$  is an arc, then  $P$  is an *outpath*, otherwise  $P$  is an *inpath*. The path  $P$  is *directed* if no vertex is the tail of two arcs in  $P$  nor the head of two arcs. In other words, all arcs are oriented in the same direction. We denote by  $P_k$  the directed path of length  $k$ . There are two kinds of directed paths, namely directed outpaths and directed inpaths. For convenience, a directed outpath is called a *dipath*. An *antidirected path* is an oriented path in which every vertex has either in-degree 0 or out-degree 0. The *blocks* of an oriented path  $P$  are the maximal directed subpaths of  $P$ . We often enumerate them from the initial vertex to the terminal vertex of the path. The number of blocks of  $P$  is denoted by  $b(P)$ . The *opposite path* of  $P$ , denoted  $\overleftarrow{P}$ , is the path  $x_n x_{n-1} \dots x_1$ . For  $1 \leq i \leq j \leq n$ , we denote by  $P[x_i, x_j]$  (resp.  $P]x_i, x_j[$ ,  $P[x_i, x_j[$ ,  $P]x_i, x_j]$ ), the oriented subpath  $x_i x_{i+1} \dots x_j$  (resp.  $x_{i+1} x_{i+2} \dots x_{j-1}$ ,  $x_{i+1} x_{i+2} \dots x_j$ ,  $x_i x_{i+1} \dots x_{j-1}$ ).

The above definitions and notation can also be used for *oriented cycles*. If  $C = x_1 x_2 \dots x_n x_1$  is an oriented cycle, we shall assume that either  $C$  is a *directed cycle*, that is,  $x_i x_{i+1}$  is an arc for all  $1 \leq i \leq n$ , where  $x_{n+1} = x_1$ , or both edges of  $C$  incident with  $x_1$  are directed outwards, i.e.  $x_1 x_2$  and  $x_1 x_n$  are arcs of  $C$ . A

digraph  $D$  is said to be *acyclic* if it has no directed cycles. The directed cycle of length  $k$  is denoted by  $C_k$ .

Let  $X$  and  $Y$  be two sets of vertices in a digraph  $D$ . An  $(X, Y)$ -*dipath* is a dipath with initial vertex in  $X$ , terminal vertex in  $Y$  and all internal vertices in  $V(D) \setminus (X \cup Y)$ .

For a set  $X$  of vertices, the *out-section* of  $X$  in  $D$ , denoted by  $S_D^+(X)$ , is the set of vertices that are reachable from  $X$  by a dipath. The out-section of a set in a digraph can be found in linear time using Breadth-First Search. The directional dual notion, the *in-section* of  $X$ , in  $D$  is denoted by  $S_D^-(X)$ .

The digraph  $D$  is *connected* (resp. *k-connected*) if  $UG(D)$  is a connected (resp. *k-connected*) graph, and the *connected components* of a  $D$  are the connected components of  $UG(D)$ . It is *strongly connected*, or *strong*, if for any two vertices  $x, y$ , there is a  $(x, y)$ -dipath in  $D$ , and  $D$  is *robust* if it is strong and  $UG(D)$  is 2-connected. We use the notation  $D[x, y]$  to denote an arbitrary  $(x, y)$ -dipath in  $D$ . The strong components of a digraph can also be found in linear time, using Depth-First Search.

The disjoint union of two digraphs  $D_1$  and  $D_2$  is denoted  $D_1 + D_2$ . By *contracting* a set of vertices  $X \subseteq V(D)$ , we refer to the operation of first taking the digraph  $D - X$ , then adding new vertex  $v_X$  and adding the arc  $v_X w$  for each  $w \in V(D - X)$  with an in-neighbour in  $X$  and the arc  $uv_X$  for each  $u \in V(D - X)$  with an out-neighbour in  $X$ . The *contraction* of a non-strong digraph  $D$  is the digraph obtained by contracting all strong components of  $D$ .

Let  $F$  be a digraph and  $u$  a vertex in  $F$ . In an  $F$ -subdivision  $S$ , the vertex corresponding to  $u$  is called the *u-vertex* of  $S$ . A vertex corresponding to some vertex  $u \in F$  is called an *original* vertex.

For all notation given above, when it is clear from the context, we may omit the indices or parameters indicating the digraph or vertex to which it refers to.

### 1.3.2 Menger's Theorem

Let  $D$  be a digraph, and let  $x$  and  $y$  be distinct vertices of  $D$ . Two  $(x, y)$ -paths  $P$  and  $Q$  are *internally disjoint* if they have no internal vertices in common, that is if  $V(P) \cap V(Q) = \{x, y\}$ . A *k-separation* of  $(x, y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $x \in W$ ,  $y \in Z$ ,  $|S| \leq k$ , each vertex in  $W$  can be reached from  $x$  by a dipath in  $D[W]$ , and there is no arc from  $W$  to  $Z$ .

One version of the celebrated Menger's Theorem is the following.

**Theorem 1.2** (Menger). *Let  $k$  be a positive integer, let  $D$  be a digraph, and let  $x$  and  $y$  be distinct vertices in  $D$  such that  $xy \notin A(D)$ . Then, in  $D$ , either there are  $k + 1$  pairwise internally disjoint  $(x, y)$ -dipaths, or there is a  $k$ -separation of  $(x, y)$ .*

For any fixed  $k$ , there exist algorithms running in linear time that, given a digraph  $D$  and two distinct vertices  $x$  and  $y$  such that  $xy \notin A(D)$ , returns either  $k + 1$  internally disjoint  $(x, y)$ -dipaths in  $D$  or a  $k$ -separation  $(W, S, Z)$  of  $(x, y)$ . Indeed, in such a particular case, any flow algorithm, like Ford–Fulkerson algorithm for example, performs at most  $k + 1$  incrementing-path searches, because it increments

the flow by 1 each time, and we stop when the flow has value  $k + 1$ , or if we find a cut of size less than  $k + 1$ , which corresponds to a  $k$ -separation. Moreover, each incrementing-path search consists in a search (usually Breadth-First Search) in an auxiliary digraph of the same size, and so is done in linear time. For more details, we refer the reader to the book of Ford and Fulkerson [23] or Chapter 7 of [8]. We call such an algorithm a *Menger algorithm*.

Observe that using Menger algorithms, one can decide if there are  $k$  internally disjoint  $(x, y)$ -dipaths in a digraph  $D$ . If  $xy \notin A(D)$ , then we apply a Menger algorithm directly; if  $xy \in A(D)$ , then we check whether there are  $k - 1$  internally disjoint  $(x, y)$ -dipaths in  $D \setminus xy$ .

Let  $D$  be a digraph. Let  $X$  and  $Y$  be non-empty sets of vertices in  $D$ . Two  $(X, Y)$ -paths  $P$  and  $Q$  are *disjoint* if they have no vertices in common, that is if  $V(P) \cap V(Q) = \emptyset$ . A  $k$ -separation of  $(X, Y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $X \subseteq W \cup S$ ,  $Y \subseteq Z \cup S$ ,  $|S| \leq k$ , all vertices of  $W$  can be reached from  $X \setminus S$  by dipaths in  $D[W]$ , and there is no arc from  $W$  to  $Z$ .

Let  $x$  be a vertex of  $D$  and  $Y$  be a non-empty subset of  $V(D) - \{x\}$ . Two  $(x, Y)$ -paths  $P$  and  $Q$  are *independent* if  $V(P) \cap V(Q) = \{x\}$ . A  $k$ -separation of  $(x, Y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $x \in W$ ,  $Y \subseteq Z \cup S$ ,  $|S| \leq k$ , all vertices of  $W$  can be reached from  $x$  by dipaths in  $D[W]$ , and there is no arc from  $W$  to  $Z$ .

Let  $y$  be a vertex of  $D$  and  $X$  be a non-empty subset of  $V(D) - \{y\}$ . Two  $(X, y)$ -paths are *independent* if  $V(P) \cap V(Q) = \{y\}$ . A  $k$ -separation of  $(X, y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $W$  and  $Z$  are non-empty,  $X \subseteq W \cup S$ ,  $y \in Z$ ,  $|S| \leq k$ , all vertices of  $W$  can be reached from  $X \setminus S$  by dipaths in  $D[W]$ , and there are no arcs from  $W$  to  $Z$ .

Let  $W \subset V(D)$ . The digraph  $D_W$  is the one obtained from  $D$  by adding a vertex  $s_W$  and the arcs  $s_W w$  for all  $w \in W$  and the digraph  $D^W$  is the one obtained from  $D$  by adding a vertex  $t_W$  and the arcs  $wt_W$  for all  $w \in W$ .

Applying Theorem 1.2 to  $D_X^Y$  and  $(s_X, t_Y)$  (resp.  $D^Y$  and  $(x, t_Y)$ ,  $D_X$  and  $(s_X, y)$ ), we obtain the following version of Menger's Theorem.

**Theorem 1.3** (Menger). *Let  $k$  be a positive integer, and let  $D$  be a digraph. Then the following hold.*

- (i) *If  $X$  and  $Y$  are two non-empty subsets of  $V(D)$ , then, in  $D$ , either there are  $k + 1$  pairwise disjoint  $(X, Y)$ -dipaths, or there is a  $k$ -separation of  $(X, Y)$ .*
- (ii) *If  $x$  is a vertex of  $D$  and  $Y$  is a non-empty subset of  $V(D)$ , then, in  $D$ , either there are  $k + 1$  pairwise independent  $(x, Y)$ -dipaths in  $D$ , or there is a  $k$ -separation of  $(x, Y)$ .*
- (iii) *If  $X$  is a non-empty subset of  $V(D)$  and  $y$  is a vertex of  $D$  and, then, in  $D$ , either there are  $k + 1$  pairwise independent  $(X, y)$ -dipaths in  $D$ , or there is a  $k$ -separation of  $(X, y)$ .*

Moreover, a Menger Algorithm applied to  $D_X^Y$  and  $(s_X, t_Y)$  (resp.  $D^Y$  and  $(x, t_Y)$ ,  $D_X$  and  $(s_X, Y)$ ) finds in linear time the  $k + 1$  dipaths or the separation as described in Theorem 1.3 (i) (resp. (ii), (iii)).

### 1.3.3 Handle decomposition

Let  $D$  be a strongly connected digraph. A *handle*  $h$  of  $D$  is a directed path  $(s, v_1, \dots, v_\ell, t)$  from  $s$  to  $t$  (where  $s$  and  $t$  may be identical) such that the digraph  $D - h$  obtained from  $D$  by *suppressing*  $h$ , that is, removing the arcs and the internal vertices of  $h$ , is strongly connected. The vertex  $s$  is the *origin* of  $h$  and  $t$  its *terminus*.

Given a strongly connected digraph  $D$ , a *handle decomposition* (also known as *ear decomposition*) of  $D$  starting at  $v \in V(D)$  is a triple  $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$ , where  $(D_i)_{0 \leq i \leq p}$  is a sequence of strongly connected digraphs and  $(h_i)_{1 \leq i \leq p}$  is a sequence of handles such that:

- $V(D_0) = \{v\}$ ,
- for  $1 \leq i \leq p$ ,  $h_i$  is a handle of  $D_i$  and  $D_i$  is the (arc-disjoint) union of  $D_{i-1}$  and  $h_i$ , and
- $D = D_p$ .

A handle decomposition is uniquely determined by  $v$  and either  $(h_i)_{1 \leq i \leq p}$ , or  $(D_i)_{0 \leq i \leq p}$ . The number of handles  $p$  in any handle decomposition of  $D$  is exactly  $|A(D)| - |V(D)| + 1$ . The value  $p$  is also called the *cyclomatic number* of  $D$ . Observe that  $p = 0$  when  $D$  is a singleton and  $p = 1$  when  $D$  is a directed cycle.

A handle decomposition  $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$  is *nice* if all handles except the first one  $h_1$  have distinct end-vertices. The following proposition is well-known (see [8] Theorem 5.13). Recall that a digraph is robust if it is strong and  $UG(D)$  is 2-connected.

**Proposition 1.4.** *Every robust digraph admits a nice handle decomposition.*

### 1.3.4 Linkage in digraphs

Recall that a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in a digraph  $D$  is a pair  $P_1, P_2$  of disjoint paths such that  $P_1$  is a directed path from  $x_1$  to  $y_1$  and  $P_2$  is a directed path from  $x_2$  to  $y_2$  in  $D$ .

#### 2-LINKAGE

Input: A digraph  $D$  and 4 distinct vertices  $x_1, x_2, y_1, y_2$ .

Question: Is there a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ ?

For sake of completeness, we reproduce the proof that 2-LINKAGE is NP-complete below.



**Theorem 1.5** (Fortune, Hopcroft and Wyllie [24]). *The 2-LINKAGE problem is NP-complete.*

*Proof.* The proof is a reduction from 3-SAT. The next lemma is very important in the construction of the gadget.

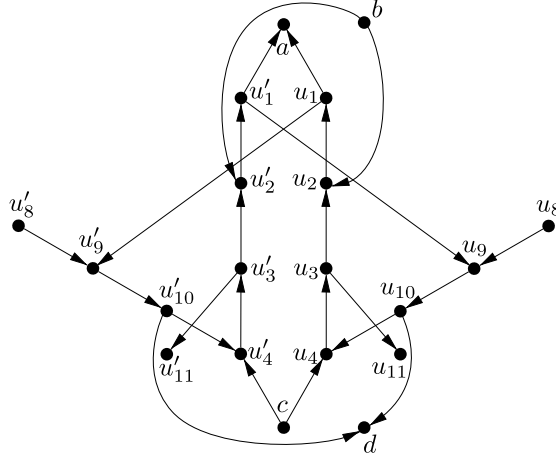


Figure 1.1: A switch.

**Lemma 1.6.** *Consider the digraph (called switch) of Figure 1.1. Suppose there are two disjoint dipaths  $P_1, P_2$  passing through it, one leaving at vertex  $a$  ( $P_1$ ) and another entering at vertex  $b$  ( $P_2$ ). Then  $P_1$  must have entered at  $c$  and  $P_2$  must leave at  $d$ . Furthermore, there is exactly one more dipath disjoint from  $P_1, P_2$  passing through the switch, either  $L = u_8 u_9 u_{10} u_4 u_3 u_{11}$  or  $R = u'_8 u'_9 u'_{10} u'_4 u'_3 u'_{11}$ .*

*Subproof.* Suppose the first arc of  $P_2$  is  $bu_2$ . Then necessarily  $u_2 u_1 u'_9 u'_{10}$  is part of  $P_2$ . If  $P_2$  does not end in  $d$ , its next arc should be  $u'_{10} u'_4$ . Since  $P_1$  and  $P_2$  need to be disjoint,  $P_1$  can not have the arc  $u_1 a$ , and in this case  $P_1$  would go through the inverse of  $u'_1 u'_2 u'_3 u'_4$ , and  $P_1, P_2$  would intersect at  $u'_4$ . So,  $P_2$  must end at  $d$  and  $P_1$  starts at  $c$ . By symmetry of the digraph, a similar result is obtained if  $P_2$  starts at a different arc. It is straightforward that the only other dipath passing through the switch is  $L$  or  $R$ , depending on the routing of  $P_2(P_1)$ .  $\diamond$

Consider an instance  $\mathcal{F}$  of 3-SAT. Let  $D_{\mathcal{F}}$  be the following digraph. For each clause  $C_1, \dots, C_r$  and variable  $v_1, \dots, v_k$  of  $\mathcal{F}$ , let  $c_1, \dots, c_r, v_1, \dots, v_k$  be vertices of  $D_{\mathcal{F}}$ . Also add a vertices  $c_{r+1}, v_{k+1}$  to  $V(D_{\mathcal{F}})$  and the arc  $v_{k+1} c_1$ . There should be three dipaths from  $c_i$  to  $c_{i+1}$ ,  $1 \leq i \leq r$ , each corresponding to one variable of  $C_i$ , and two dipaths  $Q_j, \bar{Q}_j$  from  $v_j$  to  $v_{j+1}$ ,  $1 \leq j \leq k$ , representing the positive and negative occurrences of  $v_j$ , respectively.

For every clause  $C_i$  and variable  $v_j(\bar{v}_j)$  in  $C_i$ , put a switch in  $D_{\mathcal{F}}$  in a way that  $L$  is the dipath from  $c_i$  to  $c_{i+1}$  correspondent to  $v_j(\bar{v}_j)$ , and  $R$  is in  $Q_j(\bar{Q}_j)$ , now referred as  $R_j^l(\bar{R}_j^l)$  if it represent the  $l$ -th occurrence of  $v_j(\bar{v}_j)$  in  $F$ . Finally, let  $Q_j =$

$v_j s(R_j^1) \cup t(R_j^1) s(R_j^2) \cup \dots \cup t(R_j^p) v_{j+1}$  and  $\bar{Q}_j = v_j s(\bar{R}_j^1) \cup t(\bar{R}_j^1) s(\bar{R}_j^2) \cup \dots \cup t(\bar{R}_j^p) v_{j+1}$ , if  $v_j$  and  $\bar{v}_j$  appears  $p$  times in  $\mathcal{F}$ , respectively. Furthermore, connect the switches two by two by identifying the vertex  $c(d)$  of one with  $a(b)$  of the next and put an arc from the vertex  $d$  of the last switch to  $v_1$ . An example of  $D_{\mathcal{F}}$  is showed in Figure 1.2.

Let  $x_2$  be the vertex  $b$  of the first switch,  $y_2 = c_{r+1}$ ,  $x_1$  the the vertex  $c$  of the last switch and  $y_1$  the vertex  $a$  of the first switch. We claim that  $D_{\mathcal{F}}$  contains a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  if and only if  $\mathcal{F}$  is satisfiable.

Suppose  $\mathcal{F}$  is satisfiable. Let  $S_3$  be the dipath from  $v_1$  to  $v_k$  composed by the dipaths  $\bar{Q}_j$  if  $v_j$  is true and  $Q_j$  otherwise. Since at least one literal in  $C_i$  is satisfied, there is a dipath  $S_4$ , disjoint from  $S_3$ , composed by the dipaths from  $c_i$  to  $c_{i+1}$  correspondent to a literal in  $C_i$  with value true. And by Lemma 1.6, there are two disjoint dipaths  $S_1, S_2$  passing through the chain of switches from  $x_1$  to  $y_1$  and from  $x_2$  to  $v_1$ , respectively, that are disjoint from  $S_3$  and  $S_4$ , since each of those contains one (and only one) of the dipaths  $L$  or  $R$  of a switches. So,  $S_1$  and  $S_2 \cup S_3 \cup S_4$  is the desired 2-linkage.

Suppose now there are two disjoint dipaths  $S_1, S_2$  from  $x_1$  to  $y_1$  and from  $x_2$  to  $y_2$ , respectively. By Lemma 1.6, the dipaths  $S_1, S_2$ , arriving at  $y_1$  and leaving  $x_2$ , can not start or end at  $s(R), s(L)$  or  $t(R), t(L)$  in the first switch, respec. They have to start and end at the vertices  $c$  and  $d$  in the first switch, that are the vertices  $a$  and  $b$  of the next. The same reasoning is valid the the following switches, and so  $S_1$  and  $S_2$  have to cross the chain of switches and finally  $S_1$  starts at the vertex  $c = x_1$  and  $S_2$  goes through  $d$  of the last one. Let us call  $d'$  the vertex  $d$  of the last switch. Since  $d'v_1$  is the only edge leaving this vertex,  $S_2$  pass on it. Then necessarily  $S_2$  contains a dipath from  $v_1$  to  $v_{k+1}$  and the arc  $v_{k+1}c_1$ , since there is no way of reach one the vertices  $c_j$ , crossing one or more switches, by a dipath disjoint from  $S_1$  and  $S_2[x_2, d']$ . Furthermore, from vertex  $v_j$  to  $v_{j+1}$ , the dipath is either  $Q_j$  or  $\bar{Q}_j$ , that is, the representation of a positive or negative occurrence of  $v_j$ , and it is composed by the union of dipaths  $R$  of switches. We claim that the assignment in which  $v_j$  is true if the chosen subpath for  $S_2$  is  $\bar{Q}_j$  and  $v_j$  is false otherwise satisfies the formula. Observe that, again,  $S_2$  necessarily contains a dipath from  $c_1$  to  $c_r = y_2$ . The dipath from  $c_i$  to  $c_{i+1}$ , corresponding to one of the literals  $v_j(\bar{v}_j)$  in  $C_i$ , is a dipath  $L$  of a switch. Since in a switch exactly one of  $R$  and  $L$  is allowed to be used out of the dipaths  $S_1$  and  $S_2[x_2, d']$  (Lemma 1.6), the use of such dipath is just allowed because  $\bar{Q}_j(Q_j)$  was the chosen dipath from  $v_j$  to  $v_{j+1}$ , and consequently the literal  $v_j(\bar{v}_j)$  has value true, and  $C_i$  is satisfiable for every  $1 \leq i \leq r$ . □

The problem is also NP-complete when restricted to some classes of digraphs. We use an easy modification of the 2-linkage problem as the basis for ours proofs.

Let us give some useful definitions before proceed. An *out-arborescence* is a tree in which all vertices have in-degree 1, except one special vertex, called *root*. A *switching out-arborescence* is an out-arborescence in which the root has out-degree 1, the leaves have out-degree 0 and all other vertices have out-degree 2. A (*switching*)

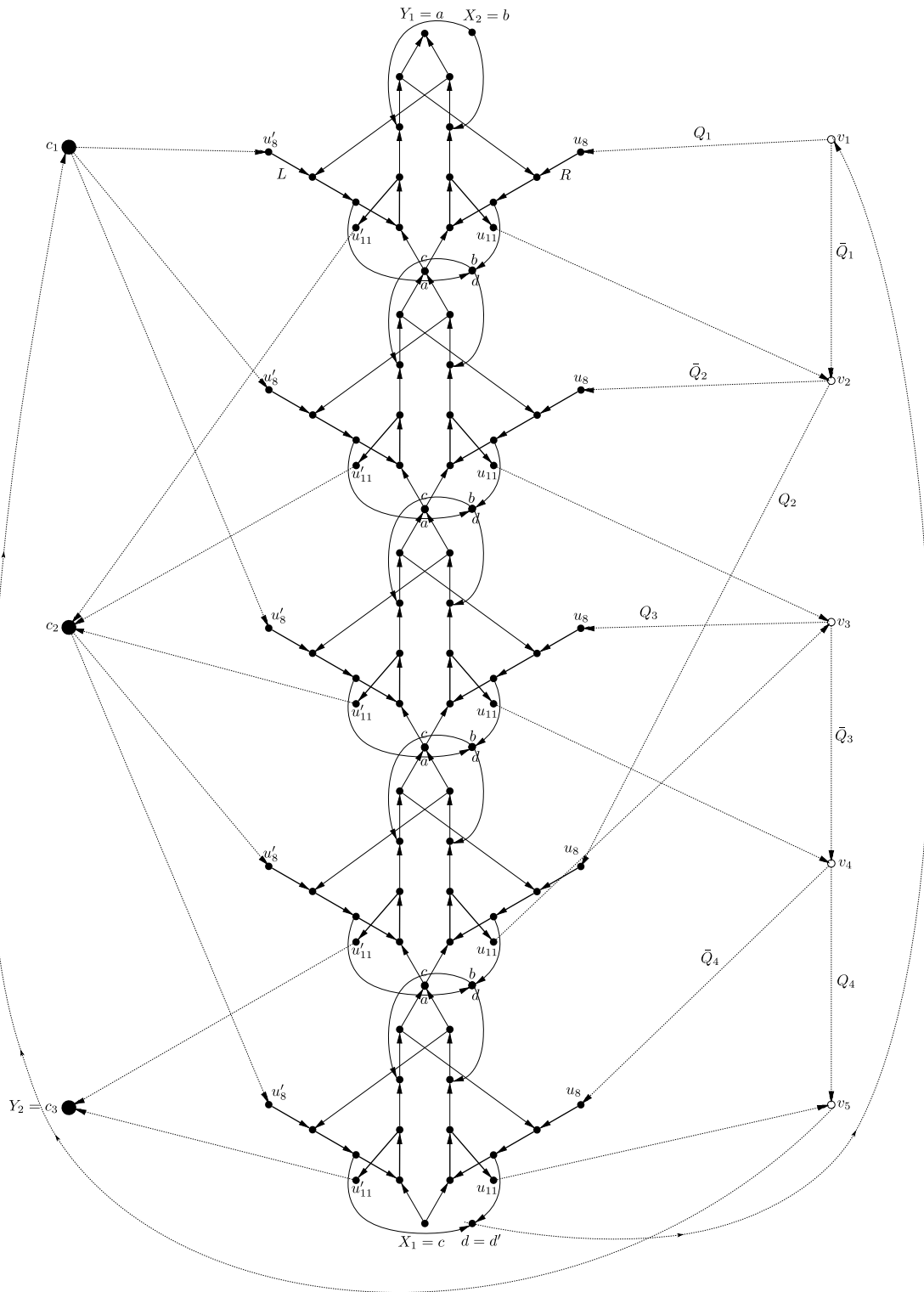


Figure 1.2: The digraph  $D_{\mathcal{F}}$  for  $\mathcal{F} = (v_1 \vee \bar{v}_2 \vee v_3) \wedge (v_2 \vee \bar{v}_4)$ .

*in-arborescence* is the dual notion of (switching) out-arborescence.

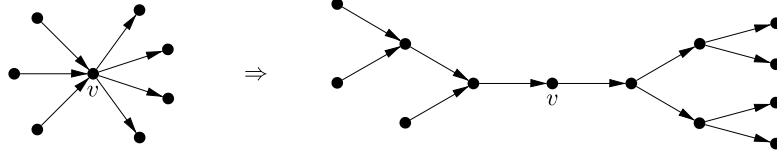


Figure 1.3: Replacement of  $N^+(v)$  and  $N^-(v)$  by switching out and in-arborescence with root  $v$  and leaves  $N^+(v)$  and  $N^-(v)$ .

Consider the following problem.

#### RESTRICTED 2-LINKAGE

Input: A digraph  $D$  without big vertices in which  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks.

Question: Is there a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ ?

**Theorem 1.7** (Bang-Jensen, Havet and M. [3]). *The RESTRICTED 2-LINKAGE problem is NP-complete.*

*Proof.* We will make a reduction from 2-LINKAGE in general digraphs. Let  $D$  and  $x_1, x_2, y_1, y_2$  be an instance of 2-LINKAGE. Let  $D^*$  be the digraph obtained from  $D$  as follows. For every vertex  $v$ , replace all the arcs leaving  $v$  by a switching out-arborescence with root  $v$  and whose leaves corresponds to the out-neighbours of  $v$  in  $D$ , and replace all the arcs entering  $v$  by a switching in-arborescence with root  $v$  and whose leaves corresponds to the in-neighbours of  $v$  in  $D$ . Furthermore, delete all the arcs entering  $x_1$  and  $x_2$  and all the arcs leaving  $y_1$  and  $y_2$  in  $D$ . Because all vertices in a switching out(in)-arborescence are small,  $D^*$  has no big vertices and, moreover, it is clear that  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks. Since every edge  $vu$  in  $D$  can be replaced by a path in the switching out-arborescence (for instance) with root  $v$  and leaf  $u$  in  $D^*$  and vice versa, it is straightforward that there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$  if and only if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D^*$ .  $\square$

So far, all the subdivision NP-completeness proofs in this work are made by reduction from RESTRICTED 2-LINKAGE.

On the other hand, some digraphs can be proved tractable using linkage, because  $k$ -LINKAGE is polynomial-time solvable when restricted to some classes of digraphs. This is for example the case for acyclic digraphs, as shown Fortune, Hopcroft and Wyllie [24].

**Theorem 1.8** (Fortune, Hopcroft and Wyllie [24]). *For every fixed  $k$ , the  $k$ -LINKAGE problem for acyclic digraphs can be solved in polynomial-time.*

*Proof.* Let  $D = (V, A)$  be an acyclic digraph. Let  $D' = (V', A')$  be the following digraph constructed from  $D$ : the vertices of  $D'$  are  $k$ -tuples of vertices of  $D$ , for every set of  $k$  vertices of  $V(D)$  and every order of it. The set of arcs  $A'$  of  $D'$  is the following: There is an arc between  $(v_1, \dots, v_{r-1}, v_r, v_{r+1}, \dots, v_k)$  to  $(v_1, \dots, v_{r-1}, w, v_{r+1}, \dots, v_k)$  if there is no dipath from  $\{v_1, \dots, v_{r-1}, v_{r+1}, \dots, v_k\}$  to  $v_r$  in  $D$  and, furthermore,  $w$  is an out-neighbour of  $v_r$  in  $D$ . We say that this arc is in the position  $r$  of the  $k$ -tuple.

Let  $D$  and  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$  be an instance of  $k$ -linkage. We claim there is a dipath  $P'$  from the vertex  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $D'$  if and only if there is a  $k$  linkage from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $D$ .

Suppose there is such dipath  $P'$  in  $D'$ . Observe that, for each arc of  $P'$ , there is a corresponding arc in  $D$ . Then let  $P_i$  be the dipath formed by the arcs in the position  $i$  of the  $k$ -tuples of  $P'$ . Then  $P_1, \dots, P_k$  is a  $k$ -linkage from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$ , since each of them is a dipath from  $x_i$  to  $y_i$  and, moreover, they are disjoint. Suppose two of them have a common vertex  $w$ . Since there are no tuples with repeated vertices,  $w$  appears in different positions in different tuples. Consider the first time for which there is an arc between  $(z_1, \dots, w, \dots, z_k)$  to  $(z_1, \dots, u, \dots, z_k)$ , meaning that  $u$  is an out-neighbour of  $w$  and there is no dipath from  $\{z_1, \dots, z_k\}$  to  $w$  in  $D$ . But if  $w$  appear again in another tuple, it means one of  $z_j$  reach  $w$  by a dipath in  $D$ , a contradiction.

Suppose now there are  $k$  disjoint dipaths from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $D$ . Then the dipath  $P'$  of  $D'$  can be constructed like this: in the  $k$ -tuple  $(z_1, \dots, z_k)$  (starting by  $(x_1, x_2, \dots, x_k)$ ), take the vertex  $z_i$  such that there is no dipath from  $\{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_k\}$  to  $z_i$  in  $D$ . Such vertex always exists because  $D$  is acyclic. Then take as next vertex of  $P'$  the  $k$ -tuple  $(z_1, \dots, z_{i-1}, w_i, z_{i+1}, \dots, z_k)$ , in which  $w_i$  is the vertex of  $P_i$  after  $z_i$ .

So, the problem of find a  $k$ -linkage in an acyclic digraph  $D$  can be reduced to the problem of finding a dipath in  $D'$ . □

## 1.4 General NP-completeness results

We deduced a sufficient condition for  $F$ -SUBDIVISION to be NP-complete. The next observations allow us to conclude that  $F$ -subdivision is “almost always” NP-complete.

For a digraph  $D$ , we denote by  $B(D)$  the set of its big vertices. A *big path* in a digraph is a directed path whose end-vertices are big and whose internal vertices all have in- and out-degree one (in particular, an arc between two big vertices is a big path). Note also that two distinct big paths with the same end-vertices are necessarily internally disjoint. The *big paths digraph* of  $D$ , denoted  $BP(D)$ , is the multidigraph with vertex set  $V(D)$  in which there are as many arcs between two vertices  $x$  and  $y$  as there are big  $(x, y)$ -paths in  $D$ .  $BP(D)$  is well-defined and easy to construct in polynomial time given  $D$ .

**Theorem 1.9** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a digraph. If  $F$  contains two arcs  $ab$  and  $cd$  whose end-vertices are big vertices and such that  $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\}$  is not isomorphic to  $BP(F)$ , then  $F$ -SUBDIVISION is NP-complete.*

*Proof.* The proof is a reduction from 2-LINKAGE in digraphs with no big vertices in which  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks.

Let  $D, x_1, x_2, y_1, y_2$  be an instance of this problem. Let  $H$  be the digraph obtained from the disjoint union of  $F \setminus \{ab, cd\}$  and  $D$  by adding the arcs  $ax_1$ ,  $cx_2$ ,  $y_1b$ , and  $y_2d$ . We claim that  $H$  has an  $F$ -subdivision if and only if  $D$  has a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$ .

If there is a 2-linkage  $P_1, P_2$  in  $D$ , then the union of  $F \setminus \{ab, cd\}$  and the paths  $ax_1 \cup P_1 \cup y_1b$  and  $cx_2 \cup P_2 \cup y_2d$  is a  $F$ -subdivision in  $H$ .

Conversely, suppose that  $H$  contains an  $F$ -subdivision  $S$ . Observe that in  $H$ , no vertex of  $D$  is big. Hence, since  $S$  has as many big vertices as  $F$ ,  $F$  and  $S$  have the same set of big vertices.

Clearly,  $S$  contains as many big paths as  $F$  and thus there must be in  $D$  two disjoint directed paths between  $(x_1, x_2)$  and  $(y_1, y_2)$ . These two paths cannot be an  $(x_1, y_2)$ - and an  $(x_2, y_1)$ -path, for otherwise  $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\} = BP(S)$  would be isomorphic to  $BP(F)$  since  $S$  is an  $F$ -subdivision. Hence, there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$ .  $\square$

**Remark 1.10.** *Observe that if  $BP(F)$  has two arcs  $ab$  and  $cd$  which are consecutive (i.e.  $b = c$ ) or contains an antidirected path  $(a, b, c, d)$  of length 3, then  $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\}$  is not isomorphic to  $BP(F)$ . Hence, by Theorem 1.10,  $F$ -SUBDIVISION is NP-complete.*

**Corollary 1.11.** *If  $F$  is a digraph with no small vertices, then  $F$ -SUBDIVISION is NP-complete.*

*Proof.* If  $F$  has no small vertices, then  $BP(F) = F$ . Moreover if  $F$  does not contain two consecutive arcs, then  $V(F)$  can be partitioned into two sets  $A$  and  $B$  such that all arcs in  $F$  have tail in  $A$  and head in  $B$ . In this case,  $F$  contains an antidirected path of length 3 since its vertices are big, and then a vertex  $b \in B$  contains at least two in-neighbours  $a, c \in A$ , each with at least one more out-neighbour in  $B$ . So by Remark 1.11, the  $F$ -SUBDIVISION problem is NP-complete.  $\square$

## 1.5 The dichotomy conjecture and relatives

For many digraphs  $F$ , the condition of Theorem 1.10 is verified and so  $F$ -SUBDIVISION is NP-complete. However, there are graphs  $F$  that do not verify this condition but for which  $F$ -SUBDIVISION is NP-complete. We show it in Chapters 2 and 3. There are also many cases in which the  $F$ -SUBDIVISION problem is polynomial-time solvable. For example, a subdivision of the directed 2-cycle is a directed cycle. Hence a digraph has a  $C_2$ -subdivision if and only if it is not acyclic.

As one can check in linear time if a digraph is acyclic or not [2, Section 2.1],  $C_2$ -SUBDIVISION is linear-time solvable.

We believe that there is a dichotomy between NP-complete and polynomial-time solvable instances.

**Conjecture 1.12.** *For every digraph  $F$ , the  $F$ -SUBDIVISION problem is polynomial-time solvable or NP-complete.*

According to this conjecture, there are only two kinds of digraphs  $F$ : *hard* digraphs, for which  $F$ -SUBDIVISION is NP-complete, and *tractable* digraphs, for which  $F$ -SUBDIVISION is solvable in polynomial-time.

A first idea to prove this conjecture would be to try to establish for any digraph  $G$  and subdigraph  $F$ , that if  $F$ -SUBDIVISION is NP-complete, then  $G$ -SUBDIVISION is also NP-complete, and conversely, if  $G$ -SUBDIVISION is polynomial-time solvable, then  $F$ -SUBDIVISION is polynomial-time solvable. However, these two statements are false as shown by the two digraphs depicted Figure 1.4. The NP-completeness of  $A$ -SUBDIVISION follows from Theorem 2.28. The fact that  $B$ -SUBDIVISION is polynomial-time solvable is proved in Theorem 2.29.

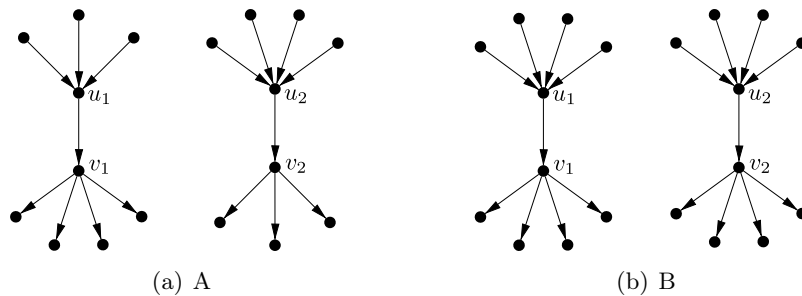


Figure 1.4: Digraphs  $A$  and  $B$  such that  $A$  is a subdigraph of  $B$ ,  $A$ -SUBDIVISION is NP-complete, and  $B$ -SUBDIVISION is polynomial-time solvable.

Despite of the many examples of polynomial instances and NP-completeness proofs we present in this work, there is still no clear evidence of exactly which graph should be tractable and which one should be hard. Although, there are conjectures that give some outline. Motivated by directed tree-width and a conjecture of Johnson et al. [35], Seymour (private communication to J. Bang-Jensen, 2011) raised the following conjecture.

**Conjecture 1.13** (Seymour).  *$F$ -SUBDIVISION is polynomial-time solvable when  $F$  is a planar digraph with no big vertices.*

This conjecture would indeed be implied by the following conjecture. An arc  $uv$  in a digraph is *contractible* if  $\min\{d^+(u), d^-(v)\} = 1$ . A *minor* of a digraph  $D$  is any digraph  $\tilde{D}$  which can be obtained from a subdigraph  $H$  of  $D$  by contracting zero or more contractible arcs of  $H$ . For  $k = 1, 2, \dots, k$  the digraph  $J_k$  is obtained

from the union of  $k$  directed cycles (each of length  $2k$ )  $C_1, C_2, \dots, C_k$ , where  $C_i = u_{i,1}v_{i,1}u_{i,2}v_{i,2} \dots u_{i,k}v_{i,k}u_{i,1}$ , for  $i = 1, 2, \dots, k$  and paths  $P_i, Q_i$ ,  $i = 1, 2, \dots, k$ , where  $P_i = u_{1,i}u_{2,i} \dots u_{k,i}$  and  $Q_i = v_{k,i}v_{k,i-1} \dots v_{k,1}$  for  $i = 1, 2, \dots, k$ .

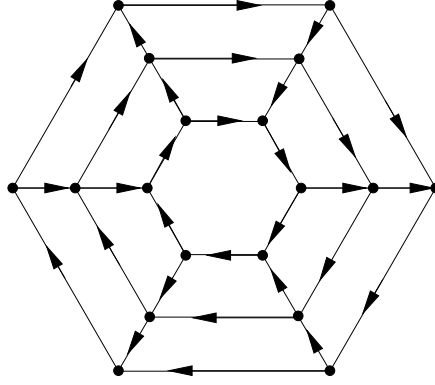


Figure 1.5:  $J_3$ .

**Conjecture 1.14** (Johnson et al. [35]). *For every positive integer  $k$  there exists  $N(k)$  such that the following holds: If a digraph  $D$  has directed treewidth more than  $N(k)$ , then  $D$  contains a minor isomorphic to  $J_k$ .*

If the directed tree-width of  $D$  is bounded, then  $F$ -SUBDIVISION can be solved in polynomial time [35]. If, on the other hand, the directed tree-width of  $D$  is unbounded, then (if the algorithmic version of the conjecture also holds) we can find a minor isomorphic to  $J_k$  for a sufficiently large  $k$  and presumably use this to realize the desired subdivision using the fact the  $F$  is planar and has no big vertices [34].

We proved Seymour's Conjecture for graphs of order 3 and 4 in Chapter 3. We propose the following sort of counterpart to it.

**Conjecture 1.15.**  *$F$ -SUBDIVISION is NP-complete for every non-planar digraph  $F$ .*

## 1.6 Disjoint directed cycles

A particular case of Conjecture 1.14 is the following.

**Conjecture 1.16.** *If  $F$  is a disjoint union of directed cycles, then  $F$ -SUBDIVISION is polynomial-time solvable.*

Observe that if  $F$  is the disjoint union of  $n$  directed cycles of lengths  $\ell_1, \dots, \ell_n$ , then a subdivision of  $F$  is the disjoint union of  $n$  directed cycles  $C_1, \dots, C_n$ , each  $C_i$  being of length at least  $\ell_i$ . We denote the directed cycle of length  $\ell$ , or *directed  $\ell$ -cycle*, by  $\vec{C}_\ell$ . A directed cycle of length at least  $\ell$  is called *directed  $\ell^+$ -cycle*.

A special case of Conjecture 1.17 is when all the directed cycles of  $F$  have the same length.



**Conjecture 1.17.** *For any two positive integers  $n$  and  $\ell$  with  $\ell \geq 2$ ,  $n\vec{C}_\ell$ -SUBDIVISION is polynomial-time solvable.*

In fact, we can show that Conjectures 1.17 and 1.18 are equivalent. Let us give some definitions before proceed.

A *feedback vertex set* or *cycle transversal* in a digraph  $D$  is a set of vertices  $S$  such that  $D - S$  is acyclic. The minimum number of vertices in a cycle transversal of  $D$  is the *cycle-transversal number* and is denoted by  $\tau(D)$ . The maximum number of disjoint directed cycles in a digraph  $D$  is called the *cycle-packing number* and is denoted by  $\nu(D)$ .

For a digraph  $D$  and an integer  $\ell \geq 2$ , we denote by  $\tau_\ell(D)$  the minimum  $t$  such that there exists  $T \subseteq V(D)$  with  $|T| = t$  meeting all directed cycles of length at least  $\ell$  in  $D$ , and by  $\nu_\ell(D)$  the maximum  $n$  such that  $D$  has  $n$  disjoint directed cycles of length at least  $\ell$ .

Conjecture 1.18 is a particular case of Conjecture 1.17. We now show how Conjecture 1.17 can be deduced from Conjecture 1.18.

**Lemma 1.18** (Havet and M. [31]). *Let  $F$  be a disjoint union of  $n$  directed cycles, all of length at most  $\ell$ . If  $m\vec{C}_\ell$ -SUBDIVISION is polynomial-time solvable for all  $1 \leq m \leq n$ , then  $F$ -SUBDIVISION is also polynomial-time solvable.*

*Proof.* Let  $n$  be a positive integer. Assume that  $m\vec{C}_\ell$ -SUBDIVISION is polynomial-time solvable for any  $m \leq n$ .

Let  $F = \vec{C}_{\ell_1} + \cdots + \vec{C}_{\ell_n}$  with  $\ell_1 \leq \cdots \leq \ell_n \leq \ell$ . Any  $F$ -subdivision is a disjoint union of  $n$  directed cycles  $\vec{C}_{p_1} + \cdots + \vec{C}_{p_n}$  with  $p_1 \leq \cdots \leq p_n$  such that  $\ell_i \leq p_i$  for all  $1 \leq i \leq n$ . The *threshold* of such a subdivision is the largest integer  $t$  such that  $p_t < \ell$ .

For  $t = 0$  to  $n$ , we check whether there is an  $F$ -subdivision with threshold  $t$  with the following ‘brute force’ procedure. We enumerate all possible disjoint unions of directed cycles  $U = \vec{C}_{p_1} + \cdots + \vec{C}_{p_t}$  with  $p_1 \leq \cdots \leq p_t \leq \ell - 1$  and  $\ell_i \leq p_i$  for all  $1 \leq i \leq t$ . There are at most  $O(|V(D)|^{(t-1)(\ell-1)})$  such  $U$ . For each such  $U$ , we check if  $D - U$  contains an  $(n - t)\vec{C}_\ell$ -subdivision (whose union with  $U$  would be an  $F$ -subdivision with threshold  $t$ ). This can be done in polynomial time by the hypothesis.

The algorithm is a succession of (at most)  $n + 1$  polynomial-time procedures, so it runs in polynomial time.  $\square$

Clearly,  $\nu_\ell(D) \leq \tau_\ell(D)$ . Proving the so-called Gallai-Younger Conjecture, Reed et al. [47] proved that there exists a minimum function  $f$  such that  $\tau(D) \leq f(\nu(D))$ . It is obvious that  $f(1) = 1$  and McCuaig [45] proved that  $f(2) = 3$ . Reed et al. [47] proved the following result for the general case.

**Theorem 1.19** (Reed et al. [47]). *For every integer  $n \geq 0$ , there exists an integer  $t_n$  such that for every digraph  $D$ , either  $\nu(D) \geq n$  or  $\tau(D) \leq t_n$ .*

Determining  $\nu(D)$  is NP-hard. Indeed, given a digraph  $D$  and an integer  $k$ , deciding whether  $D$  has at least  $k$  disjoint cycles is NP-complete (see Theorem 13.3.2 of [2]). As observed in [30], the problem parameterized with  $k$  is hard for the complexity class  $W[1]$  (this follows easily from the results of [51]). This means that, unless  $FPT = W[1]$ , there is no algorithm solving the problem with a  $f(k) \cdot n^{O(1)}$  running time.

Theorem 1.20 is a directed analogue of the following theorem due to Erdős and Pósa.

**Theorem 1.20** (Erdős and Pósa [21]). *Let  $n$  be a positive integer. There exists  $t_n^*$  such that for every graph  $G$ , either  $G$  has  $n$  pairwise-disjoint cycles, or there exists a set  $T$  of at most  $t_n^*$  vertices such that  $G - T$  is acyclic.*

More precisely, Erdős and Pósa proved that there exist two absolute constants  $c_1$  and  $c_2$  such that  $c_1 \cdot n \log n \leq t_n^* \leq c_2 \cdot n \log n$ . Since we are interested only in the above version of the theorem, we reproduce here only the proof of the upper bound.

Some auxiliary results are needed, though. For the first one, consider a graph  $G$  and a set of vertices  $x_1, \dots, x_u$  of  $G$  to be called *principal vertices* (the others will be called *subsidiary vertices*). A *principal path* in  $G$  is a path whose end-vertices are principal and whose internal ones are subsidiary. Let  $V_{max}(G)$  denote the maximum number of disjoint principal paths in  $G$  and  $\pi_{min}(G)$  the minimum number of vertices that intersects all principal paths in  $G$ .

**Theorem 1.21** (Gallai [27]).  $\pi_{min}(G) \leq 2V_{max}(G)$ .

The next result is also due to Erdős and Pósa.

**Theorem 1.22** (Erdős and Pósa [20]). *There exists an absolute constant  $c_3$  such that every graph with  $m$  vertices and  $m + l$  edges contains at least  $c_3 \cdot l / \log l$  edge-disjoint cycles.*

Observe that if two cycles are edge-disjoint but not disjoint, then every common vertex between two cycles has degree at least 4. So, if every vertex of the graph has degree at most 3, by Theorem 1.23, the graph has at least  $c_3 \cdot l / \log l$  disjoint cycles.

*Proof of Theorem 1.21.* Assume that the maximum number of disjoint cycles in a graph  $G$  is  $n$ , and let  $C_i$ ,  $1 \leq i \leq n$ , be these cycles. Consider the graph  $G_1$  obtained from  $G$  by the deletion of the edges in all  $C_i$ , and let the vertices of all  $C_i$  be the principal vertices of  $G_1$ . Consider a set of principal paths  $S$  that maximizes the number of disjoint principal paths in  $G_1$ . Let  $G^*$  be the graph formed by the union of  $C_i$ ,  $1 \leq i \leq n$ , and  $S$ , and let  $m$  be the number of vertices of  $G^*$ . As each principal path has one more edge than subsidiary vertex in  $G^*$ , the total amount of edges of  $G^*$  is  $m + V_{max}(G_1)$ . Since every vertex has degree at most 3 in  $G^*$ , by Theorem 1.23,  $G^*$  has at least  $c_3 \cdot V_{max}(G_1) / \log V_{max}(G_1)$  disjoint cycles. This is also valid for  $G$  because  $G^*$  is a subgraph of  $G$ . So

$$\frac{c_3 \cdot V_{max}(G_1)}{\log V_{max}(G_1)} \leq n \text{ and then } V_{max}(G_1) \leq c_4 \cdot n \log n.$$

Let  $y_1, \dots, y_t$  be a set of vertices intersecting every principal path in  $G_1$  such that  $t$  is minimum. Then, by Theorem 1.22 and the inequality above

$$t \leq 2c_4 \cdot n \log n.$$

Suppose there is a cycle  $D$  different from any  $C_j$  that does not contain any  $y_h$ .  $D$  must intersect some  $C_j$ . Suppose  $D$  intersects  $C_i$  but do not intersect any other  $C_j$ . Then  $D$  has only one vertex  $x_i$  in common with  $C_i$ , otherwise there would be a principal path not passing through any  $y_h$ . Suppose there were two cycles  $D_{i_1}, D_{i_2}$  of this kind intersecting  $C_i$  in different vertices  $x_{i_1}, x_{i_2}$ . If they were disjoint, replacing  $C_i$  by them would imply that the graph has more than  $n$  disjoint cycles. If they were not disjoint, then again there would be a principal path not passing through any  $y_h$ . The same would happen if  $D$  had vertices in common with more than one  $C_j$ . So each cycle different from any  $C_j$  that does not contain any  $y_h$  intersects a  $C_i$  in at most one vertex, the same vertex  $x_i$  for each  $C_i$ . The set composed by the vertices  $y_1, \dots, y_t, x_1, \dots, x_k$  intersects all the cycles in  $G$ , and it contains at most

$$2c_4 \cdot n \log n + n \leq c_2 \cdot n \log n$$

vertices, and it completes the proof.  $\square$

An  $n\vec{C}_2$ -subdivision is the disjoint union of  $n$  directed cycles. Therefore Conjecture 1.18 for  $\ell = 2$  can be deduced from Theorems 1.20 and 1.8. Using the result of Theorem 1.20, Reed et al. [47] gave a polynomial-time algorithm to decide for every fixed  $k$  whether a digraph  $D$  contains  $k$  disjoint directed cycles. Basically, it tests all possible sets  $T$  of  $f(k)$  vertices. If none of them is a cycle transversal, then it returns ‘yes’. If one of them is a cycle transversal, it reduces the problem to a finite number (but depending on  $k$ ) of  $f(k)$ -linkage problem in  $D - T$ .

**Theorem 1.23** (Reed et al. [47]). *Let  $k$  be a fixed integer. There is an algorithm running in time  $O(n^{f(k)}(n+m))$  that decides whether there are  $k$  disjoint directed cycles in a digraph.*

In the undirected case, the complexity of finding even *two* disjoint *induced* cycles remains open [17]. Theorem 1.24 and Lemma 1.1 directly imply the following.

**Corollary 1.24.** *Let  $k$  be a fixed integer. There is an algorithm running in time  $O(n^{f(k)}(n+m)m)$  that finds  $k$  disjoint directed cycles in a digraph if they exist, and returns ‘no’ otherwise.*

In fact, Reed et al. proved the following stronger statement than Theorem 1.24.

**Theorem 1.25** (Reed et al. [47]). *For any digraph  $F$ ,  $F$ -SUBDIVISION is polynomial-time solvable when restricted to the class of digraphs with bounded cycle-transversal number.*

Note that this result is implied by the one of Berwanger et al. [6] stating that for every fixed  $k$ ,  $k$ -LINKAGE is polynomial-time solvable on digraphs of bounded DAG-width.

We believe that a similar approach may be used to prove Conjecture 1.18 for all  $\ell$ . The correspondent result for undirected graphs was showed by Birmelé, Bondy and Reed [7]. We show that Conjecture 1.18 for some  $\ell$  is implied by the two following conjectures for the same  $\ell$ .

The *circumference* of a non-acyclic digraph  $D$ , denoted  $\text{circ}(D)$ , is the length of a longest directed cycle in  $D$ . If  $D$  is acyclic, then its *circumference* is defined by  $\text{circ}(D) = 1$ .

**Conjecture 1.26.** *Let  $\ell \geq 2$  be an integer. For any positive integer  $k$ ,  $k$ -LINKAGE is polynomial-time solvable for digraphs with circumference at most  $\ell - 1$ .*

Evidently  $\nu_\ell(D) \leq \tau_\ell(D)$  and Conjecture 1.28 states that for every fixed  $\ell$  there exists a function  $f$  such that  $\tau_\ell(D) \leq f(\nu_\ell(D))$ .

**Conjecture 1.27.** *Let  $\ell \geq 2$  be an integer. For every integer  $n \geq 0$ , there exists an integer  $t_n = t_n(\ell)$  such that for every digraph  $D$ , either  $D$  has a  $n$  pairwise-disjoint directed  $\ell^+$ -cycles, or there exists a set  $T$  of at most  $t_n$  vertices such that  $D - T$  has no directed  $\ell^+$ -cycles.*

**Theorem 1.28** (Havet and M. [31]). *Let  $\ell \geq 1$  be an integer. If Conjectures 1.27 and 1.28 hold for  $\ell$ , then for every positive integer  $n$ ,  $n\vec{C}_\ell$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Let  $D$  be a digraph. Let  $t = t_n(\ell)$  with  $t_n(\ell)$  as in Conjecture 1.28. We first check if  $\tau_\ell(D) \leq t$ . This can be done by brute force, testing for each subset  $T$  of  $V(D)$  of size  $t$  whether it meets all directed  $\ell^+$ -cycles. Such a test can be done by checking whether  $D - T$  has circumference  $\ell - 1$ , that is, has no  $\vec{C}_\ell$ -subdivision. Since there are  $O(|V(D)|^t)$  sets of size  $t$ , and  $\vec{C}_\ell$ -SUBDIVISION is polynomial-time solvable, this can be done in polynomial time.

If no  $t$ -subset  $T$  meets all directed  $\ell^+$ -cycles, then  $\tau_\ell(D) > t$ . Therefore, because Conjecture 1.28 holds for  $\ell$ ,  $D$  contains an  $n\vec{C}_\ell$ -subdivision. So we return ‘yes’.

If we find a set  $T$  of size  $t$  that meets all directed  $\ell^+$ -cycles, then  $\text{circ}(D - T) \leq \ell - 1$ . We use another brute force algorithm which is based on traces.

A *trace* is either a directed  $\ell^+$ -cycle or a linkage. Observe that for any directed  $\ell^+$ -cycle  $C$  and any subset  $Z$  of  $V(D)$ , the intersection  $C \cap D[Z]$  is a trace. A trace contained in  $D[Z]$  is called a  $Z$ -*trace*.

Now every  $\ell^+$ -cycle intersects  $T$  in a non-empty trace because  $\text{circ}(D - T) \leq \ell - 1$ . We describe a polynomial-time procedure that, given a set of  $n$  pairwise disjoint traces  $T_1 \dots, T_n$ , checks whether there is an  $n\vec{C}_\ell$ -subdivision  $C_1 + \dots + C_n$  such that  $T_i = C_i \cap D[T]$  for all  $1 \leq i \leq n$ . Now since  $T$  has size  $t$ , there is a bounded number of possible sets of  $n$  pairwise disjoint traces  $T$ -traces (at most  $\binom{t}{n+1}(B_t + 1)$ , where  $B_t$  is the number of partitions of a set of size  $t$ ). Hence running the

above procedure for all possible such set of  $T$ -traces, we obtain a polynomial-time algorithm that decides whether  $D$  contains an  $n\vec{C}_\ell$ -subdivision.

Let  $\mathcal{T} = \{T_1, \dots, T_n\}$  be a set of  $n$  pairwise disjoint  $T$ -traces. Set  $\bar{T} = V(D) \setminus T$ . A *trace* is *suitable* if it has at least  $\ell$  vertices, at most  $t$  components, and the initial and terminal vertices of all components are in  $\bar{T}$ .

For each  $T_i$ , we shall describe a set  $\mathcal{T}_i$  of suitable traces such that a directed  $\ell^+$ -cycle  $C$  such that  $C \cap T = T_i$  contains at least one trace in  $\mathcal{T}_i$ . The set  $\mathcal{T}_i$  is constructed as follows. Let  $\mathcal{U}_i$  be the set of traces that can be obtained from  $T_i$  by extending each components  $P$  of  $T_i$  at both ends by an inneighbour of  $s(P)$  and an outneighbour of  $t(P)$  in  $\bar{T}$ . Clearly,  $\mathcal{U}_i$  has size at most  $|V(D)|^2 k$ , where  $k$  is the number of components of  $T_i$ . By construction, each trace of  $\mathcal{U}_i$  has its initial and terminal vertices in  $\bar{T}$  and has no more components than  $T_i$ . Moreover, a directed  $\ell^+$ -cycle  $C$  such that  $C \cap T = T_i$  contains one trace in  $\mathcal{U}_i$ . However, the set  $\mathcal{U}_i$  might not be our set  $\mathcal{T}_i$  because certain traces in it might be too small.

For any trace  $U$ , let  $g(U)$  be set set of all possible traces obtained from  $U$  by adding one vertex of  $\bar{T}$  has outneighbour of a terminal vertex of one component of  $U$ . Clearly,  $g(U)$  has size at most  $k|V(D)|$ , where  $k$  is the number of components of  $U$ , and a directed  $\ell^+$ -cycle  $C$  containing  $U$  must contains a trace in  $g(U)$ . Moreover, every trace of  $g(U)$  has size  $|V(U)| + 1$ , and no more components than  $U$ . Set  $g^i(U) = \{U\}$  if  $i$  is a non-positive integer and for all positive integer  $i$ , define  $g^i(U) = \bigcup_{U' \in g^{i-1}(U)} g(U')$ . Now the set  $\bigcup_{U \in \mathcal{U}_i} g^{\ell-|V(U)|}(U)$  is our desired  $\mathcal{T}_i$ . Moreover,  $\mathcal{T}_i$  is of size at most  $t^\ell \cdot |V(D)|^\ell$ .

To have a polynomial-time procedure to decide whether there is an  $n\vec{C}_\ell$ -subdivision  $C_1 + \dots + C_n$  such that  $T_i = C_i \cap T$  for all  $1 \leq i \leq n$ , it suffices to have a procedure that, given an  $n$ -tuple  $(T'_1, \dots, T'_n)$  of disjoint traces such that  $T'_i \in \mathcal{T}_i$ , decides whether there is an  $n\vec{C}_\ell$ -subdivision  $C_1 + \dots + C_n$  such that  $T'_i$  is a subdigraph of  $C_i$  for all  $1 \leq i \leq n$ , and to run it on each possible such  $n$ -tuple. Such a procedure can be done as follows. Let  $P_1^i, \dots, P_{k_i}^i$  be the components of  $T'_i$ . For each  $n$ -tuple of circular permutations  $(\sigma_1, \dots, \sigma_n)$  of  $\mathfrak{S}_{k_1} \times \dots \times \mathfrak{S}_{k_n}$ , one checks whether in the digraph  $D'$  induced by the vertices of  $\bar{T}$  which are not internal vertices of any of the components of the union of the  $T'_i$ , if there is a linkage from

$$(s(P_1^1), \dots, s(P_{k_1}^1), s(P_1^2), \dots, s(P_{k_2}^2), \dots, s(P_1^n), \dots, s(P_{k_n}^n))$$

to

$$(t(P_{\sigma_1(1)}^1), \dots, t(P_{\sigma_1(k_1)}^1), t(P_{\sigma_2(1)}^2), \dots, t(P_{\sigma_2(k_2)}^2), \dots, t(P_{\sigma_n(1)}^n), \dots, t(P_{\sigma_n(k_n)}^n)).$$

Now the digraph  $D'$  is a subdigraph of  $D - T$  and so has circumference at most  $\ell - 1$ , and the linkage we are looking for has at most  $t$  components. Thus each of these instances of  $(k_1 + \dots + k_n)$ -LINKAGE can be solved in time  $O(|V(D)|^m)$  for some absolute constant  $m$  because Conjecture 1.27 holds for  $\ell$ .  $\square$

In Chapter 4, we prove both Conjecture 1.27 and Conjecture 1.28 for  $\ell = 3$  (Theorems 4.4 and 4.6), implying that  $n\vec{C}_3$ -SUBDIVISION is polynomial-time solvable.

## 1.7 Operations preserving hardness or tractability

The next lemmas allow to extend NP-completeness results of  $F$ -SUBDIVISION for some digraphs  $F$  to much larger classes.

**Lemma 1.29** (Bang-Jensen, Havet and M. [3]). *Let  $F_1$  and  $F_2$  be two digraphs.*

- (i) *If  $F_1$ -SUBDIVISION is NP-complete, then  $(F_1 + F_2)$ -SUBDIVISION is NP-complete.*
- (ii) *If  $(F_1 + F_2)$ -SUBDIVISION is polynomial-time solvable, then  $F_1$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Let  $D$  be a digraph. We will prove that  $D$  contains an  $F_1$ -subdivision if and only if  $D + F_2$  contains an  $(F_1 + F_2)$ -subdivision.

Clearly if  $D$  contains an  $F_1$ -subdivision  $S$ , then  $S + F_2$  is an  $(F_1 + F_2)$ -subdivision in  $D + F_2$ .

Conversely, assume that  $D + F_2$  contains an  $(F_1 + F_2)$ -subdivision  $S = S_1 + S_2$  with  $S_1$  an  $F_1$ -subdivision and  $S_2$  an  $F_2$ -subdivision. Let us consider such an  $(F_1 + F_2)$ -subdivision that maximizes the number of connected components of  $F_2$  that are mapped (in  $S$ ) into  $F_2$  again (notice that since there are no arcs between  $D$  and  $F_2$  in  $D + F_2$ , in the subdivision  $S$  every component of  $S_2$  will either be entirely inside  $F_2$  or entirely inside  $D$ ). We claim that  $S_2 = F_2$ . Indeed suppose that some component  $T$  of  $S_2$  is in  $D$ . Let  $C$  be the component of  $F_2$  of which  $T$  is the subdivision. Let  $U = S \cap C$ . Then  $T$  contains a subdivision  $U'$  of  $U$  (because it is a subdivision of all of  $C$ ). Hence replacing  $U$  by  $U'$  and  $T$  by  $C$  in  $S$ , we obtain a subdivision with one more component mapped on itself, a contradiction.

Hence  $S_2 = F_2$ , and so  $D$  contains  $S_1$  which is an  $F_1$ -subdivision.  $\square$

**Lemma 1.30** (Bang-Jensen, Havet and M. [3]). *Let  $F_1$  and  $F_2$  be two digraphs such that  $F_1$  is strongly connected and  $F_2$  contains no  $F_1$ -subdivision. Let  $F$  be obtained from  $F_1$  and  $F_2$  by adding some arcs with tail in  $V(F_1)$  and head in  $V(F_2)$ .*

- (i) *If  $F_1$ -SUBDIVISION is NP-complete, then  $F$ -SUBDIVISION is NP-complete.*
- (ii) *If  $F$ -SUBDIVISION is polynomial-time solvable, then  $F_1$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* We will prove that a digraph  $D$  contains an  $F_1$ -subdivision if and only if  $D \mapsto F_2$  contains an  $F$ -subdivision, where  $D \mapsto F_2$  is obtained from  $D + F_2$  by adding all possible arcs from  $V(D)$  to  $V(F_2)$ .

It is easy to see that if  $D$  contains an  $F_1$ -subdivision  $S$ , then  $S + F_2$  together with some subset of the arcs from  $D$  to  $F_2$  is an  $F$ -subdivision in  $D \mapsto F_2$ . Conversely, if  $D \mapsto F_2$  contains an  $F$  subdivision  $S^*$ , then, since  $F_1$  is strongly connected, the part of  $S^*$  forming a subdivision of  $F_1$  has to lie entirely inside  $D$  or  $F_2$ . Since  $F_2$  contains no  $F_1$ -subdivision, the subdivision of  $F_1$  has to be inside  $D$  and hence we get that  $D$  has an  $F_1$ -subdivision.  $\square$

It is useful to look at Figure 1.4 again and notice that the digraphs  $A, B$  show that we need the assumption that  $F_1$  is strongly connected in Lemma 1.31 (and the analogous version where the roles of  $F_1$  and  $F_2$  are interchanged).

**Lemma 1.31** (Bang-Jensen, Havet and M. [3]). *Let  $F_1$  and  $F_2$  be two digraphs such that  $F_1$  is robust and  $F_2$  contains no  $F_1$ -subdivision. Let  $F$  be obtained from  $F_1$  and  $F_2$  by identifying one vertex of  $F_1$  with one vertex of  $F_2$ .*

- (i) *If  $F_1$ -SUBDIVISION is NP-complete, then  $F$ -SUBDIVISION is NP-complete.*
- (ii) *If  $F$ -SUBDIVISION is polynomial-time solvable, then  $F_1$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Given a digraph  $D$  we form the digraph  $D^{F_2}$  by fixing one vertex  $x$  in  $F_2$  and adding  $|V(D)|$  disjoint copies of  $F_2$  such that the  $i$ th copy has its copy of  $x$  identified with the  $i$ th vertex of  $D$ . Since  $F_2$  contains no  $F_1$ -subdivision and  $UG(F_1)$  is 2-connected, any subdivision of  $F_1$  in  $D^{F_2}$  should be completely contained in  $D$ . It follows that  $D^{F_2}$  contains an  $F$ -subdivision if and only if  $D$  contains an  $F_1$ -subdivision.  $\square$

**Lemma 1.32** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a digraph in which every vertex  $v$  satisfies  $\max\{d^+(v), d^-(v)\} \geq 2$ , and let  $S$  be a subdivision of  $F$ .*

- (i) *If  $F$ -SUBDIVISION is NP-complete, then  $S$ -SUBDIVISION is NP-complete.*
- (ii) *If  $S$ -SUBDIVISION is polynomial-time solvable, then  $F$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* We will make a reduction from  $F$ -SUBDIVISION to  $S$ -SUBDIVISION.

Let  $D$  be an instance of  $F$ -SUBDIVISION and  $p$  be the length of a longest path in  $S$  corresponding to an arc in  $F$ . Let  $D_p$  be the  $D$ -subdivision obtained by replacing every arc of  $D$  by a directed path of length  $p$ . Since every vertex  $v$  corresponding to one of  $F$  in  $S$  must be mapped onto a vertex corresponding to  $D$  in  $D_p$  because  $\max\{d^+(v), d^-(v)\} \geq 2$ , it follows that  $D$  has an  $F$ -subdivision if and only if  $D_p$  has an  $S$ -subdivision.  $\square$

We believe that the condition  $\max\{d^+(v), d^-(v)\} \geq 2$  for all  $v \in V(F)$  is may not necessary, although it is in our proof.

**Conjecture 1.33** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a digraph, and let  $S$  be a subdivision of  $F$ .*

- (i) *If  $F$ -SUBDIVISION is NP-complete, then  $S$ -SUBDIVISION is NP-complete.*
- (ii) *If  $S$ -SUBDIVISION is polynomial-time solvable, then  $F$ -SUBDIVISION is polynomial-time solvable.*

The following property is also useful for the later chapters.

**Lemma 1.34** (Havet, M. and Mohar). *Let  $F$  be a digraph and let  $u_1, \dots, u_p$  be distinct vertices of  $F$ . Suppose that for every out-neighbour  $v$  of  $u_1$ , replacing the arc  $u_1v$  by a dipath  $u_1wv$  of length 2, where  $w \notin V(F)$ , always results in the same digraph  $F'$ . Suppose that for every given digraph  $D$  of order  $n$  and  $p$  vertices  $x_1, \dots, x_p$  in  $D$ , one can decide in  $f(n)$  time whether there is an  $F$ -subdivision in  $D$  such that  $x_i$  is the  $u_i$ -vertex for every  $i$ . Then given a digraph  $D$  and  $p$  vertices  $x_1, \dots, x_p$ , one can decide in  $O\left(\binom{d^+(x_1)-1}{d^+(u_1)-1} \cdot \sum_{y \in N^+(x_1)} d^+(y) \cdot f(n-1)\right)$  time whether there is an  $F'$ -subdivision in  $D$  such that  $x_i$  is the  $u_i$ -vertex for every  $i$ .*

*Proof.* Set  $q = d^+(u_1)$ . For every set of  $q$  neighbours  $y_1, \dots, y_q$  of  $x_1$  and every out-neighbour  $z$  of  $y_1$ , where  $z \notin \{y_2, \dots, y_q\}$ , we shall give a procedure that verifies if  $D$  contains an  $F'$ -subdivision  $S'$  such that  $x_i$  is the  $u_i$ -vertex for all  $1 \leq i \leq p$ , and  $\{x_1y_1, \dots, x_1y_q, y_1z\} \subseteq A(S')$ . Such an  $F'$ -subdivision is called *forced*.

Let  $D'$  be the digraph obtained from  $D - y_1$  by deleting all arcs leaving  $x_1$  except  $x_1y_2, \dots, x_1y_q$ , and adding the arc  $x_1z$ .

**Claim 1.**  *$D$  has a forced  $F'$ -subdivision if and only if  $D'$  has an  $F$ -subdivision such that  $x_i$  is the  $u_i$ -vertex for every  $i$ .*

*Subproof.* Suppose that  $S$  is an  $F$ -subdivision in  $D'$  such that  $x_i$  is the  $u_i$ -vertex for all  $i$ . Since  $x_1$  has outdegree  $q$  in  $D'$ , we have  $\{x_1y_2, \dots, x_1y_q, x_1z\} \subseteq A(S)$ . Let  $S'$  be the digraph obtained from  $S$  by replacing the arc  $x_1z$  by the dipath  $x_1y_1z$ . Because replacing the arc  $u_1v$  by a dipath of length 2 results in  $F'$  for any out-neighbour  $v$  of  $u_1$ , the digraph  $S'$  is an  $F'$ -subdivision in  $D$ . Thus  $S'$  is a forced  $F'$ -subdivision in  $D$ .

Conversely, assume that  $S'$  is a forced  $F'$ -subdivision in  $D$ . Then the digraph  $S$  obtained from  $S'$  by replacing the dipath  $x_1y_1z$  by the arc  $x_1z$  is an  $F$ -subdivision in  $D'$  such that  $x_i$  is the  $u_i$ -vertex for every  $i$ .  $\diamond$

This claim implies that deciding whether  $D$  contains a forced  $F'$ -subdivision can be done by checking whether  $D'$  has an  $F$ -subdivision such that  $x_i$  is the  $u_i$ -vertex for all  $i$ . This can be done in  $f(n-1)$  time by assumption. By repeating this for every possible set  $\{y_1, \dots, y_q, z\}$  where the  $y_i$  are distinct out-neighbours of  $x_1$  and  $z \notin \{y_2, \dots, y_q\}$  is an out-neighbour of  $y_1$ , we obtain an algorithm to decide whether there is an  $F'$ -subdivision in  $D$  such that  $x_i$  is the  $u_i$ -vertex for all  $i$ . Since there are at most  $\binom{d^+(x_1)-1}{d^+(u_1)-1} \cdot \sum_{y \in N^+(x_1)} d^+(y)$  such sets, the running time of this algorithm is as claimed.  $\square$





# $F$ -SUBDIVISION for some graph classes

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In this chapter, we discuss  $F$ -SUBDIVISION for  $F$  being in many different classes of graphs. For some of them, we are able to completely classify in which cases the problem is polynomial-time solvable and in which it is NP-complete. The polynomial cases illustrate the different techniques that can be used in their solutions.

## 2.1 Spiders

A *spider* is a tree obtained from disjoint directed paths by identifying one end of each path into a single vertex. This vertex is called the *body* of the spider.

If  $T$  is a spider, then every  $T$ -subdivision contains  $T$  as a subdigraph. Hence a digraph contains a  $T$ -subdivision if and only if it contains  $T$  as a subdigraph. This implies that  $T$ -SUBDIVISION can be solved in  $O(n^{|T|})$  time, and the same is valid if  $T$  is the disjoint union of spiders. We have then the following as a consequence.

**Lemma 2.1** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a digraph and  $T$  a spider or the disjoint union of spiders. If  $F$ -SUBDIVISION is polynomial-time solvable, then  $(F + T)$ -SUBDIVISION is also polynomial-time solvable.*

*Proof.* For each set  $A$  of  $|T|$  vertices, we check if the digraph  $D\langle A \rangle$  induced by  $A$  contains  $T$ . Then, if yes, we check if  $D - A$  has an  $F$ -subdivision.  $\square$

*Gluing* a spider  $T$  with body  $b$  to  $F$  at a vertex  $u \in V(F)$  consists in taking the disjoint union of  $F$  and  $T$  and identifying  $u$  and  $b$ .

**Lemma 2.2** (Havet, M. and Mohar). *Let  $F$  be a digraph and  $u$  a vertex of  $F$ . If given a digraph  $D$  and a vertex  $v$  of  $D$  one can decide in polynomial time if there is an  $F$ -subdivision in  $D$  such that  $v$  is the  $u$ -vertex, then any digraph obtained from  $F$  by gluing a spider at  $u$  is tractable.*

*Proof.* Let  $T$  be a spider with body  $b$  and let  $F'$  be the digraph obtained by gluing  $T$  to  $F$  at  $u$ . Clearly, every  $F'$ -subdivision contains an  $F$ -subdivision in which the arcs of  $T$  are not subdivided. Such an  $F$ -subdivision is said to be *canonical*.

Consider the following algorithm. For every vertex  $v$  of  $D$  we repeat the following. For every set  $W$  of  $|V(T)| - 1$  vertices, we check whether  $D[W \cup \{v\}]$  contains a copy of  $T$  with body  $v$ . This can be done in constant time. Then we check if

$D - W$  contains an  $F$ -subdivision with  $u$ -vertex  $v$ . This can be done in polynomial time by our assumption.

This algorithm clearly decides in polynomial time whether a given digraph  $D$  contains a canonical  $F'$ -subdivision.  $\square$

If the spider is specified in the input, the subdivision problem for spiders is NP-complete because it includes the Hamiltonian directed path problem. One could ask if in this case the problem can be solved in FPT time when parameterized by the spider  $T$ , that is, in  $f(|V(T)|) \cdot n^c$  time, where  $f$  is a computable function and  $c$  an absolute constant. This question remains open until now.

## 2.2 Directed cycles

We already commented on the previous chapter about the  $C_2$ -SUBDIVISION, where it is enough to check if the graph is acyclic since the subdivision of a  $C_2$  is simply a directed cycle. It is also easy to check the result for the general case.

**Proposition 2.3** (Bang-Jensen, Havet and M. [3]). *For every  $k \geq 2$ ,  $C_k$ -SUBDIVISION can be solved in time  $O(n^k \cdot m)$ .*

*Proof.* For every  $k$ -tuple  $(x_1, x_2, \dots, x_k)$ , we check if  $(x_1, x_2, \dots, x_k)$  is a directed path and if yes, we check if there is a directed  $(x_k, x_1)$ -path in  $D - \{x_2, \dots, x_{k-1}\}$ . There are  $O(n^k)$   $k$ -tuples, so this can be done in  $O(n^k \cdot m)$  time.  $\square$

The running time above is certainly not the best possible. We can also find a linear-time algorithm when  $k = 3$ .

**Proposition 2.4** (Bang-Jensen, Havet and M. [3]).  *$C_3$ -SUBDIVISION can be solved in linear time.*

*Proof.* Let  $D$  be a digraph. If  $D$  has no directed 2-cycles, then  $D$  contains a  $C_3$ -subdivision if and only if it is not acyclic, which can be tested in linear time.

Assume now that  $D$  has some directed 2-cycles. Let  $H$  be the graph with vertex set  $V(D)$  and edge-set  $\{xy \mid (x, y, x) \text{ is a 2-cycle of } D\}$ , that is, each edge of  $H$  induces a 2-cycle in  $D$ . The graph  $H$  can be constructed in linear time. We first check, in linear time, if  $H$  contains a cycle. If  $H$  contains a cycle, then it has length at least 3 and any of its two directed orientations is a directed cycle in  $D$ , so we return such a cycle, certifying that  $D$  is a 'yes'-instance.

If not, then  $H$  is a forest. If there is any single arc  $uv$  (an arc which is not part of a 2-cycle) in  $D$  such that both  $u$  and  $v$  belong to the same connected component of  $H$ , then we can produce a directed cycle of length at least 3 in  $D$  (following a path from  $u$  to  $v$  in  $H$ ) so we may assume that all single arcs go between different components in  $H$ . Now it is easy to see that  $D$  contains a cycle of length at least 3 if and only if the digraph  $D'$  obtained by contracting (into a vertex) each connected component of  $H$  in  $D$  has a directed cycle: If  $D'$  has no cycles, then the only possible cycles of  $D$  are inside the contracted connected components of  $H$  in  $G$ , and therefore

they are 2-cycles. In case we find a cycle in  $D'$ , we can easily reproduce a directed cycle in  $D$  by replacing the contracted vertices by paths, and it has length at least 3, because the 2-cycles of  $D$  are in the contracted components of  $H$  in  $D$  and they do not appear in  $D'$ .  $\square$

If  $k$  is not fixed but specified in the input, it is NP-complete to decide if a digraph has a directed cycle of length  $k$  because the Hamiltonian directed cycle is a particular case of it. Gabow and Nie proved that it is FPT to decide if a graph has a cycle of length at least  $k$ .

**Theorem 2.5** (Gabow and Nie [25, 26]). *One can decide in  $O(k^{3k} \cdot n \cdot m)$  time whether a digraph contains a directed cycle of length at least  $k$ .*

We let as open the problem of reproducing the result of Gabow and Nie but in linear time instead.

**Problem 2.6.** *For any fixed  $k$ , can we solve  $C_k$ -SUBDIVISION in linear time? In other words, does there exists a computable function  $f$  such that one can decide in  $O(f(k) \cdot (n + m))$  time whether a digraph contains a directed cycle of length at least  $k$ ?*

## 2.3 Other oriented paths and cycles

We propose the following conjecture for paths and cycles in digraphs, which is a particular case of Seymour's Conjecture 1.14.

**Conjecture 2.7** (Bang-Jensen, Havet and M. [3]). *If  $F$  is an oriented path or cycle, then  $F$ -SUBDIVISION is polynomial-time solvable.*

Observe that if  $P$  is a directed path, as it happens for spiders (it can also be seen like a particular case of this class), every  $P$ -subdivision contains  $P$  as a subdigraph. Hence to solve the problem for a digraph  $D$  it is sufficient to check if  $D$  has  $P$  as a subdigraph. The case of directed cycles was discussed in the previous section.

Recall that an antidirected path is an oriented path in which every vertex has either in-degree 0 or out-degree 0.

**Theorem 2.8** (Bang-Jensen, Havet and M. [3]). *If  $P$  is an antidirected path, then  $P$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Let  $P = (a_1, \dots, a_p)$  be an antidirected path. By directional symmetry, we may assume that  $a_i$  has in-degree 0 in  $P$  if and only if  $i$  is odd.

Let  $D$  be a digraph. For a  $p$ -tuple of vertices  $(v_1, \dots, v_p)$  of  $D$ , we shall describe a procedure that either returns a  $P$ -subdivision, or returns that there exists no  $P$ -subdivision in which each  $v_i$  is the image of  $a_i$ . Then applying this procedure for all  $p$ -tuples of vertices, we obtain the desired algorithm to finding a  $P$ -subdivision.

The procedure is as follows: For all odd (resp. even)  $i$ , we remove all the arcs entering  $v_i$  (resp. leaving  $v_i$ ) in  $D$ . Let  $D'$  be the resulting digraph. Clearly,  $D$

contains a  $P$ -subdivision in which each  $v_i$  is the image of  $a_i$  if and only if  $D'$  does. In  $UG(D')$ , we check if there is a path  $\tilde{Q}$  going through  $v_1, \dots, v_p$  in this order. This can be done by checking for a linkage from  $(v_1, v_2, \dots, v_{p-1})$  to  $(v_2, v_3, \dots, v_p)$  and thus in polynomial time by Robertson and Seymour algorithm [48].

If no such  $\tilde{Q}$  is found, then  $D'$  (and thus  $D$ ) contains certainly no  $P$ -subdivision in which each  $v_i$  is the image of  $a_i$ .

If such a  $\tilde{Q}$  is found, let  $Q$  be the oriented path corresponding to  $\tilde{Q}$  in  $D'$ . Since  $v_i$  is a source in  $D'$  when  $i$  is odd, and a sink in  $D'$  when  $i$  is even, the path  $Q$  has at least  $p - 1$  blocks (it can have more if some path  $Q[v_i, v_{i+1}]$  is not directed), and so contains a subdivision of  $P$ .  $\square$

When we turn to antirected cycles (defined similarly), the problem seems to become more complicated. We can not use the same technique of Theorem 2.8, because there we find a path  $Q$  with at least  $p - 1$  blocks, and this path contains necessarily a path with  $p - 1$  blocks, which is a subdivision of the antirected path with  $p$  vertices: for instance, the first  $p - 1$  blocks of  $Q$ . But an antirected cycle with more than  $p - 1$  blocks does not contain the antirected cycle on  $p - 1$  blocks. On the following, we show that  $\hat{C}_4$ , the antirected cycle of length 4 (Figure 2.1), is tractable.

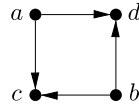


Figure 2.1: The antirected cycle  $\hat{C}_4$  of length 4.

**Theorem 2.9** (Havet, M. and Mohar).  $\hat{C}_4$ -SUBDIVISION can be solved in  $O(n^3 \cdot (n + m))$  time.

*Proof.* We shall describe a polynomial-time procedure  $\hat{C}_4$ -Subdivision( $a, b, D$ ) that, given two vertices  $a, b$ , either finds a  $\hat{C}_4$ -subdivision (not necessarily with sources  $a$  and  $b$ ) and in this case returns ‘yes’, or verifies that there is no  $\hat{C}_4$ -subdivision in  $D$  with  $a$  and  $b$  as sources and returns ‘no’. Since a  $\hat{C}_4$ -subdivision has two sources, running this procedure for every pair  $\{a, b\}$  of vertices yields an algorithm to decide whether  $D$  contains a  $\hat{C}_4$ -subdivision; in addition the algorithm runs in  $O(n^3 \cdot (n + m))$  time, because the procedure  $\hat{C}_4$ -Subdivision( $a, b, D$ ) only needs  $O(n \cdot (n + m))$  time.

First, we determine the out-sections  $S_a = S_{D-b}^+(a)$  and  $S_b = S_{D-a}^+(b)$ . If there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ , then its two sinks must be in  $X = S_a \cap S_b$ . Thus if  $|X| \leq 1$ , we return ‘no’. Henceforth, we assume that  $|X| \geq 2$ .

Let  $A$  (resp.  $B$ ) be the set of vertices  $x \in X$  such that there is an  $(a, x)$ -dipath in  $D - b$  (resp.  $(b, x)$ -dipath in  $D - a$ ) whose internal vertices are not in  $X$ . If there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ , then  $A$  and  $B$  must both be of size at least 2. Thus if  $|A| \leq 1$  or  $|B| \leq 1$ , we return ‘no’. Henceforth, we assume that  $|A| \geq 2$  and  $|B| \geq 2$ .

**Claim 2.** *Let  $D'$  be the digraph obtained from  $D[X]$  by adding  $a$ ,  $b$  and all arcs from  $a$  to  $A$  and from  $b$  to  $B$ . Then*

- (i) *if  $D$  has a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$ , then so does  $D'$ ;*
- (ii) *if  $D'$  has a  $\hat{C}_4$ -subdivision, then so does  $D$ .*

*Subproof.* (i) Assume that  $D$  contains a  $\hat{C}_4$ -subdivision  $S$  with sources  $a$  and  $b$ , and let  $c$  and  $d$  be the sinks of  $S$ . Let  $P_1$  (resp.  $P_2$ ,  $Q_1$ , and  $Q_2$ ) be the  $(a, c)$ -dipath, (resp.  $(a, d)$ -dipath,  $(b, c)$ -dipath, and  $(b, d)$ -dipath) in  $S$  and let  $a_1$  (resp.  $a_2$ ,  $b_1$ ,  $b_2$ ) be the last vertex of  $A$  (resp.  $A$ ,  $B$ , and  $B$ ) on this path. Observe that  $V(P_1[a_1, c]) \subseteq X$  and that a similar property holds for each of the paths  $P_2[a_2, d]$ ,  $Q_1[b_1, c]$ , and  $Q_2[b_2, d]$ . This shows that the digraph which is the union of the four dipaths  $aa_1P_1[a_1, c]$ ,  $aa_2P_2[a_2, d]$ ,  $bb_1Q_1[b_1, c]$  and  $bb_2Q_2[b_2, d]$  is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D'$ .

(ii) Suppose that  $D'$  has a  $\hat{C}_4$ -subdivision  $S'$ . If  $a \in V(S')$ , let  $a_1$  and  $a_2$  be the two out-neighbours of  $a$  in  $S'$ . Clearly,  $a_1, a_2 \in A$ . Therefore in  $D - b$ , there exist an  $(a, a_1)$ -dipath  $P_1$  and an  $(a, a_2)$ -dipath  $P_2$  whose internal vertices are not in  $X$ . Let  $a'$  be the last vertex in  $P_1 \cap P_2$  on  $P_1$ . We set  $P' = \overleftarrow{P_1}[a_1, a']P_2[a', a_2]$ .

Similarly, if  $b \in V(S')$ , denoting  $b_1$  and  $b_2$  the two out-neighbours of  $b$  in  $S'$ , one can find a  $(b_1, b_2)$ -inpath with two blocks whose internal vertices are not in  $X$ . Call this path  $Q'$ .

Now replacing in  $S'$  the oriented path  $a_1aa_2$  by  $P'$  if  $a \in V(S')$  and the oriented path  $b_1bb_2$  by  $Q'$  if  $b \in V(S')$  results in a  $\hat{C}_4$ -subdivision in  $D$ .  $\diamond$

By Claim 2, we can replace  $D$  by  $D'$ , i.e. we may assume henceforth that  $D = D'$ ,  $X = V(D) - \{a, b\}$ ,  $A = N^+(a)$  and  $B = N^+(b)$ . Moreover, we will assume that  $N^-(a) = N^-(b) = \emptyset$ .

If  $|A \cap B| \geq 2$ , then we return 'yes'. Indeed, for any two distinct vertices  $c$  and  $d$  in  $A \cap B$ , the cycle  $acbdba$  is isomorphic to  $\hat{C}_4$ . Therefore, we may assume that  $|A \cap B| \leq 1$ .

If  $|A \cap B| = 1$ , say  $A \cap B = \{d\}$ , then we check with a Menger algorithm for each vertex  $c \in V(D) - \{a, b, d\}$ , whether there are independent  $(\{a, b\}, c)$ -dipaths. If there is a vertex  $c$  with two such dipaths  $P$  and  $Q$ , then we return 'yes'. Otherwise, then we return 'no'. This is valid by the following claim.

**Claim 3.** *If  $A \cap B = \{d\}$ , then  $D$  contains a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  if and only if there is a vertex  $c \in V(D) - \{a, b, d\}$  such that  $D - d$  contains two independent  $(\{a, b\}, c)$ -dipaths.*

*Subproof.* If  $D$  contains a  $\hat{C}_4$ -subdivision  $S$  with sources  $a$  and  $b$ , then one of two oriented  $(a, b)$ -paths, say  $R$ , forming  $S$  does not contain  $d$ . Thus the sink in  $R$  is the desired vertex  $c$ .

If there is a vertex  $c$  as described above, then let  $P$  and  $Q$  be two independent  $(\{a, b\}, c)$ -dipaths with respective initial vertex  $a$  and  $b$ . Then  $P\overleftarrow{Q}bda$  is a  $\hat{C}_4$ -subdivision.  $\diamond$

Assume now that  $A \cap B = \emptyset$ . We take a shortest  $(a, B)$ -dipath  $P_a$  (this can be done in linear time by Breadth-First Search). Such a path exists because  $X$  is the out-section of  $a$  in  $D - b$ . Let  $c$  be the terminal vertex of  $P_a$ . We then search for a shortest  $(a, B - \{c\})$ -dipath in  $D - c$ . If we find such a path  $Q_a$  with terminal vertex  $d$ , then we return ‘yes’. Indeed denoting by  $a'$  the last vertex in  $P_a \cap Q_a$  on  $Q_a$ , the oriented cycle  $P_a[a', c]cb\overleftarrow{d}Q_a[d, a']$  is a  $\hat{C}_4$ -subdivision.

Hence we may assume that every  $(a, B)$ -dipath goes through  $c$ . Let  $b'$  be a vertex in  $B - \{c\}$ , and let  $D^*$  be the digraph obtained by contracting  $\{b, b'\}$  into a vertex  $b^*$  and removing all arcs entering  $b^*$ . We return  $\hat{C}_4\text{-Subdivision}(a, b^*, D^*)$ . This is valid by Claim 4.

**Claim 4.** (i) *If there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ , then there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b^*$  in  $D^*$ .*

(ii) *If there is a  $\hat{C}_4$ -subdivision in  $D^*$ , then there is a  $\hat{C}_4$ -subdivision in  $D$ .*

*Subproof.* (i) Assume there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ . Let  $S$  be such a subdivision with minimum number of vertices. Let  $b_1$  and  $b_2$  be the two out-neighbours of  $b$  in  $S$ .

If  $b' \notin V(S)$ , then the digraph obtained from  $S$  by replacing the vertex  $b$  and the arcs  $bb_1$  and  $bb_2$  by the vertex  $b^*$  and the arcs  $b^*b_1, b^*b_2$  is a  $\hat{C}_4$ -subdivision in  $D^*$ .

Suppose now that  $b' \in V(S)$ . Then  $bb'$  is an arc of  $S$ . Indeed if it were not, then replacing the  $(b, b')$ -path in  $S$  not containing  $a$  by the arc  $bb'$ , we would obtain a smaller  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$ . Thus, we may assume that  $b' = b_1$ .

Now  $b'$  is not a sink in  $S$ . Indeed suppose it were. Let  $Q$  be the  $(a, b')$ -dipath in  $S$ . Necessarily,  $Q$  goes through  $c$ . Thus, the digraph obtained from  $S$  by replacing  $Q$  by  $Q[a, c]$  and  $bb'$  with  $bc$  is a smaller  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$ , a contradiction.

Hence,  $b'$  has an out-neighbour  $b''$  in  $S$ . Then the digraph obtained from  $S$  by replacing the vertices  $b$  and  $b'$  and the arcs  $bb', b'b''$  and  $bb_2$  by the vertex  $b^*$  and the arcs  $b^*b'', b^*b_2$  is a  $\hat{C}_4$ -subdivision in  $D^*$  with sources  $a$  and  $b^*$ .

(ii) Assume that  $S^*$  is a  $\hat{C}_4$ -subdivision in  $D^*$ . If  $b^*$  is not a vertex of  $S^*$ , then  $S^*$  is contained in  $D$  and we have the result. If  $b^*$  is a vertex in  $S^*$ , then it is a source since its in-degree in  $D^*$  is zero. Let  $s$  and  $t$  be its two out-neighbours in  $S^*$ . By definition of  $D^*$ ,  $s$  and  $t$  are both in  $N_D^+(b) \cup N_D^+(b')$ . If  $s$  and  $t$  are both in  $N_D^+(b)$  (resp.  $N_D^+(b')$ ), then the digraph obtained from  $S^*$  by replacing the vertex  $b^*$  and the arcs  $b^*s$  and  $b^*t$  by the vertex  $b$  (resp.  $b'$ ) and the arcs  $bs$  and  $bt$  (resp.  $b's$  and  $b't$ ) respectively, is a  $\hat{C}_4$ -subdivision in  $D$ . If  $s \in N_D^+(b)$  and  $t \in N_D^+(b')$ , then the digraph obtained from  $S^*$  by replacing the vertex  $b^*$  and the arcs  $b^*s$  and  $b^*t$  by the vertices  $b, b'$  and the arcs  $bs, bb'$  and  $b't$  is a  $\hat{C}_4$ -subdivision in  $D$ .  $\diamond$

Let us now estimate the time complexity of  $\hat{C}_4\text{-Subdivision}$ . It first computes two out-sections, which can be done in linear time. Then either it leads a recursive call or it does not because it stops. In the preparation of a recursive call, it possibly

computes a dipath (in the case  $A \cap B = \emptyset$ ). Moreover, the order of digraph decreases by one in the call. In the second case, either it stops for some easy reason in  $O(1)$  steps, or it stops after using a Menger algorithm which runs in linear time. Let  $r$  be the number of recursive calls made by the  $\hat{C}_4$ -Subdivision. Clearly  $r \leq n$  and the procedure runs in  $O(r \cdot (n + m) + (n + m))$  time, that is in  $O(n(n + m))$  time.  $\square$

The next step would be to investigate the subdivision problem for oriented paths and cycles with varied amount of blocks, that is, for the cases in which the paths in contrary directions have length bigger than one. We can show it is tractable for oriented paths with few blocks.

**Proposition 2.10** (Bang-Jensen, Havet and M. [3]). *If  $P$  is an oriented path with at most four blocks, then  $P$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Let  $P = (a_1, \dots, a_s)$  be an outpath (w.l.o.g) with four blocks and let  $a_p, a_r$  be the vertices of  $P$  with out-degree 0 and  $a_q$  be the vertex of  $P$  with in-degree 0, for  $p < q < r$  (all the other vertices have in-degree and out-degree 1). Let  $t_1 = q - p + 1$  and  $t_2 = r - q + 1$ , that is,  $t_1$  and  $t_2$  are the number of vertices of  $P[a_p, a_q]$  and  $P[a_q, a_r]$ , respectively. Suppose without loss of generality that  $t_1 \geq t_2$  and say that  $t = t_1 - t_2$ .

Let  $D$  be a digraph. For every  $(s + i)$ -tuple of vertices  $(v_1, \dots, v_{s+i})$  of  $D$ ,  $0 \leq i \leq t$ , we check if  $(v_1, \dots, v_p)$ ,  $(v_q, \dots, v_{p+1})$ ,  $(v_q, \dots, v_{q+t_2+i})$ ,  $(v_{q+t_2+i}, \dots, v_s)$  are directed paths. If yes, we look for a  $(v_{p+1}, v_p)$ -path in  $D - (v_1, \dots, v_{s+i})$ . At this point, we have checked if there is any subdivision  $S$  of  $P$  in  $D$  in which the subdivision of  $P[a_1, a_p]$  and  $P[a_r, a_s]$  have exactly the same cardinality of them, the subdivision of  $P[a_q, a_r]$  in  $S$  is path of order between  $t_2$  and  $t_1$  and the subdivision of  $P[a_q, a_p]$  in  $S$  is any.

We then check for every  $(s + t)$ -tuple of vertices  $(v_1, \dots, v_{s+t})$  of  $D$  if  $(v_1, \dots, v_p)$ ,  $(v_q, \dots, v_{p+1})$ ,  $(v_q, \dots, v_{q+t_1-1})$ ,  $(v_{q+t_1}, \dots, v_s)$  are directed paths. If yes, we look for 2 internally disjoint directed paths starting in  $\{v_{p+1}, v_{q+t_1}\}$  and ending in  $\{v_p, v_{q+t_1}\}$  in  $D - (v_1, \dots, v_{s+t})$ . This can be done using a Menger algorithm. So, at this point, we have checked if there is any subdivision  $S$  of  $P$  in  $D$  in which the subdivision of  $P[a_1, a_p]$  and  $P[a_r, a_s]$  have exactly the same cardinality of them, and the subdivision of  $P[a_q, a_r]$  and  $P[a_q, a_p]$  in  $S$  are paths of order bigger than  $t_1$ . This is true because we checked the fixed directed paths above and then since  $(v_q, \dots, v_{p+1})$ ,  $(v_q, \dots, v_{q+t_1-1})$  have both order  $t_1 - 1$ , does not matter from which vertex to which vertex the paths found by Menger's algorithm goes, in any case we will have the desired subdivision.

Observe that for any subdivision  $S$  of  $P$  in  $D$ , the part of  $S$  corresponding to  $P[a_1, a_p]$  contains  $P[a_1, a_p]$  as a subdigraph. The same happens to the part of  $S$  corresponding to  $P[a_r, a_s]$ . So, since we apply this procedure for all  $j$ -tuples of vertices,  $s \leq j \leq s + t$ , and there are  $O(n^j)$  of them, we obtain the desired algorithm to finding a  $P$ -subdivision in  $O(n^j \cdot (m + n))$  time.  $\square$

The subdivision of oriented paths with two blocks, which includes the antidiirected cycle of length 2, is a special case of the graphs discussed on the next section.



The case of oriented cycles with more blocks, as the case of oriented paths with more than four blocks, remains open.

## 2.4 Spindles

A  $(k_1, \dots, k_p)$ -*spindle* is the union of  $p$  pairwise internally disjoint directed  $(a, b)$ -paths  $P_1, \dots, P_p$  of respective length  $k_1, \dots, k_p$ . Vertex  $a$  is said to be the *tail* of the spindle and  $b$  its *head*.

**Proposition 2.11** (Bang-Jensen, Havet and M. [3]). *If  $F$  is a spindle, then  $F$ -SUBDIVISION can be solved in  $O(n^{|V(F)|} \cdot (n + m))$  time.*

*Proof.* Let  $F$  be a spindle with tail  $a$  and head  $b$ . Let  $a_1, \dots, a_p$  be the out-neighbours of  $a$  in  $F$ . An  $F$ -subdivision may be seen as an  $F$ -subdivision in which only the arcs  $aa_i$ ,  $1 \leq i \leq p$  are subdivided. The following algorithm takes advantage of this property.

Let  $D$  be a digraph. For each pair  $(S, a')$  where  $S$  is a set of  $|V(F)| - 1$  vertices and  $a'$  a vertex of  $D - S$ , we first enumerate all the possible subdigraphs of  $D \langle S \rangle$  isomorphic to  $F - a$  with  $a'_1, \dots, a'_p$  corresponding to  $a_1, \dots, a_p$ . We then check if, in  $D - (S - \{a'_1, \dots, a'_p\})$ , there exist  $p$  internally disjoint directed paths  $P_i$ ,  $1 \leq i \leq p$ , each  $P_i$  starting in  $a'$  and ending in  $a'_i$ . This can be done using a Menger algorithm. Clearly, this algorithm decides if there is an  $F$ -subdivision in  $D$ . There are  $O(n^{|V(F)|})$  possible pairs  $(S, a')$ , and for each of them we run at most  $(|V(F)| - 1)!$  times a Menger algorithm. Since such an algorithm runs in linear time, the time complexity of the above algorithm is  $O(n^{|V(F)|} \cdot (n + m))$ .  $\square$

By the proof of Proposition 2.11, we can state the following:

**Corollary 2.12.** *Let  $F$  be a spindle with tail  $a$  and head  $b$ . Given a digraph  $D$  and two vertices  $a'$  and  $b'$ , we can decide in polynomial time if  $T$  contains an  $F$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$ .*

The complexity given in Proposition 2.11 is certainly not optimal. For example, it can be improved for spindles with paths of small lengths.

**Proposition 2.13** (Bang-Jensen, Havet and M. [3]). *If  $F$  is a  $(k_1, \dots, k_p)$ -spindle and  $k_i \leq 2$  for all  $1 \leq i \leq p$ , then  $F$ -SUBDIVISION can be solved in  $O(n^2 \cdot (n + m))$  time.*

*Proof.* If some of the  $k_i$ , say  $k_1$ , equals 1, then finding an  $F$ -subdivision is equivalent to find  $p$  internally disjoint directed paths from some vertex  $a$  to some other vertex  $b$ , which by Menger's theorem is equivalent to check that the connectivity from  $a$  and  $b$  is at least  $p$ . For any pair  $(a, b)$ , this can be done in linear time by a Menger algorithm.

If  $k_i = 2$  for all  $1 \leq i \leq p$ , then finding an  $F$ -subdivision is equivalent to find  $p$  internally disjoint directed paths of length at least two from some vertex  $a$  to some

other vertex  $b$ . Such paths exist if and only if in  $D \setminus ab$  there are  $p$  internally disjoint  $(a, b)$ -paths. For any pair  $(a, b)$ , this can be checked in linear time by a Menger algorithm.  $\square$

A natural question is to ask about the complexity of deciding if a digraph contains a subdivision of a spindle, when the spindle is no more fixed but specified in the input.

**Proposition 2.14** (Bang-Jensen, Havet and M. [3]). *The following problem is NP-complete*

SPINDLE-SUBDIVISION

Input: A spindle  $F$  and a digraph  $D$ .

Question: Does  $D$  contain a subdivision of  $F$ ?

*Proof.* The proof is a reduction from the (undirected) Hamiltonian cycle problem.

Let  $G$  be an undirected graph. Let  $D$  be the symmetric digraph associated to  $G$ , that is,  $D$  is the digraph obtained from  $G$  by replacing every edge  $uv$  by the two arcs  $uv$  and  $vu$ . Let  $F$  be any  $(k_1, k_2)$ -spindle of the same order as  $G$  (and  $D$ ). For order reason, the digraph contains an  $F$ -subdivision if and only if it contains  $F$  as a subgraph, and thus if and only if  $G$  has a Hamiltonian cycle, since the spindle  $F$  with two paths is going to be an oriented cycle containing all the vertices in  $D$ .  $\square$

In view of Proposition 2.14, one could ask whether it is possible to solve SPINDLE-SUBDIVISION in  $f(|V(F)|) \cdot n^c$  time, where  $f$  is a computable function and  $c$  an absolute constant. This may be formulated in FPT setting as follows.

**Problem 2.15.** *Is the following problem fixed-parameter tractable?*

PARAMETERIZED SPINDLE-SUBDIVISION

Input: A spindle  $F$  and a digraph  $D$ .

Parameter:  $|V(F)|$ .

Question: Does  $D$  contain a subdivision of  $F$ ?

## 2.5 Bispindles

The  $(k_1, \dots, k_p; l_1, \dots, l_q)$ -bispindle, denoted  $B(k_1, \dots, k_p; l_1, \dots, l_q)$ , is the digraph obtained from the disjoint union of a  $(k_1, \dots, k_p)$ -spindle with tail  $a_1$  and head  $b_1$  and a  $(l_1, \dots, l_q)$ -spindle with tail  $a_2$  and head  $b_2$  by identifying  $a_1$  with  $b_2$  into a vertex  $a$ , and  $a_2$  with  $b_1$  into a vertex  $b$ . The vertices  $a$  and  $b$  are called, respectively, the *left node* and the *right node* of the bispindle. The directed  $(a, b)$ -paths are called *forward paths*, while the directed  $(b, a)$ -paths are called *backward paths*.

We say that  $(P_1, \dots, P_p; Q_1, \dots, Q_q)$  is a  $(k_1, \dots, k_p; l_1, \dots, l_q)$ -bispindle if, for each  $1 \leq i \leq p$ ,  $P_i$  is a directed  $(c, d)$ -path of length  $k_i$ , for each  $1 \leq j \leq q$ ,  $Q_j$  is a directed  $(d, c)$ -path of length  $l_j$  and the union of the  $P_i$  and  $Q_j$  is  $B(k_1, \dots, k_p; l_1, \dots, l_q)$ .

Let  $F$  be a bispindle with  $p$  forward paths and  $q$  backward paths. Consider the big paths multidigraph  $BP(F)$ . By Remark 1.11, we get the following.

**Proposition 2.16.** *Let  $F$  be a bispindle with  $p$  forward paths and  $q$  backward paths. If  $p \geq 1$ ,  $q \geq 1$ , and  $p + q \geq 4$ , then  $F$ -SUBDIVISION is NP-complete.*

On the other hand, if  $F$  has exactly one backward path and one forward path or no backward paths, then it is a directed cycle or a spindle, respectively. In both cases,  $F$ -SUBDIVISION can be solved in polynomial time as shown in Sections 2.2 and 2.4, respectively.

In the next, we show that in the remaining cases, that is, when  $F$  is a bispindle with two forward paths and one backward path,  $F$ -SUBDIVISION is polynomial-time solvable. This is done through the FORK problem, to be defined. We then present faster algorithms to solve  $B(1, 2; 1)$ - and  $B(1, 3; 1)$ -SUBDIVISION.

### 2.5.1 The Fork Problem

In this section, to show that  $B(k_1, k_2; l_1)$ -SUBDIVISION is polynomial-time solvable, we use an approach very similar to the one used by Chudnovsky and Seymour to find an induced subdivision of a  $K_{2,3}$  in an undirected graph [16]. In their case, the solution is based on the algorithm they presented for the following problem, called three-in-a-tree: given a graph  $G$  and three vertices  $a, b$  and  $c$  of  $G$ , is there a tree passing through  $a, b$  and  $c$  that is an induced subgraph of  $G$ ? Our algorithm is based on the following notion. A *fork* with *bottom vertex*  $a$ , top vertices  $b$  and  $c$  and centre  $t$  is a digraph in which

- $a, b$  and  $c$  are distinct, and  $t$  is distinct from  $b$  and  $c$  (but possibly equal to  $a$ ),
- every vertex except  $a$  has in-degree 1 and  $a$  has in-degree 0, and
- all vertices except  $b, c$  and  $t$  have out-degree 1 and  $b$  and  $c$  have out-degree 0 and  $t$  has out-degree 2.

Consider the following problem.

FORK

Input: A digraph  $D$  and three distinct vertices  $a, b$  and  $c$ .

Question: Does  $D$  contain a fork with bottom vertex  $a$  and top vertices  $b$  and  $c$ ?

**Lemma 2.17** (Bang-Jensen, Havet and M. [3]). *FORK can be solved in linear time.*

*Proof.* Assume that a digraph  $D$  contains a fork with bottom vertex  $a$  and top vertices  $b$  and  $c$ . Then, clearly, there are a directed  $(a, b)$ -path in  $D - c$  and a directed  $(a, c)$ -path in  $D - b$ .

We claim that this necessary condition is also sufficient. Indeed, assume that there is a directed  $(a, b)$ -path  $P$  in  $D - c$  and a directed  $(a, c)$ -path  $Q$  in  $D - b$ . Let  $t$  be the last vertex on  $P$  which also belongs to  $Q$ . Such a vertex exists because  $a$  is in  $P$  and  $Q$ . Then the union of  $P$  and  $Q[t, c]$  is the desired fork.

Since one can decide in linear time if there is a directed  $(u, v)$ -path in a digraph, FORK can be solved in linear time.  $\square$

**Theorem 2.18** (Bang-Jensen, Havet and M. [3]). *If  $F$  is a bispindle with two forward paths and one backward path, then  $F$ -SUBDIVISION can be solved in  $O(n^{|F|+1} \cdot (n + m))$  time.*

*Proof.* Let  $a$  be the left node of  $F$  and let  $b$  and  $c$  be its two out-neighbours in  $F$ .

For every subset  $S$  of  $|F|$  vertices, we check if  $D\langle S \rangle$  contains a copy of  $F - \{ab, ac\}$  with  $a', b', c'$  corresponding to  $a, b, c$ , respectively. Then we check in  $D - (S - \{a', b', c'\})$  if there is a fork with bottom vertex  $a'$  and top vertices  $b'$  and  $c'$ .

Since there are  $O(n^{|F|})$  possible set  $S$  and FORK can be solved in linear time by Lemma 2.17, our algorithm runs in  $O(n^{|F|+1} \cdot (n + m))$  time.  $\square$

Similarly to Proposition 2.14, one shows that given a digraph  $D$  and a bispindle  $F$  (with one forward paths and one backward path), deciding if  $D$  contains an  $F$ -subdivision is NP-complete. It is again natural to ask if it is FPT when parameterized by  $|F|$ .

**Problem 2.19.** *Is the following problem fixed-parameter tractable?*

PARAMETERIZED BISPINDLE-SUBDIVISION

*Input:* A bispindle  $F$  and a digraph  $D$ .

*Parameter:*  $|V(F)|$ .

*Question:* Does  $D$  contain a subdivision of  $F$ ?

## 2.5.2 Faster algorithms for subdivision of bispindles

The complexity given in Theorem 2.18 is certainly not best possible. In this subsection, using handle decomposition, we show algorithms to solve  $B(1, 2; 1)$ - and  $B(1, 3; 1)$ -SUBDIVISION, whose running time is smaller than the complexity of Theorem 2.18.

Recall that a digraph  $D$  is *robust* if it is strongly connected and  $UG(D)$  is 2-connected. The *robust components* of a digraph are its robust subdigraphs which are maximal by inclusion.

Because bispindles are robust, a subdivision  $S$  of a bispindle is also robust, and if a digraph  $D$  contains  $S$ , then  $S$  must be in a robust component of  $D$ . Finding the robust components of a digraph can be done in linear time, by finding the strong components and the 2-connected components of the underlying graphs of these. Therefore one can restrict our attention to subdivision of bispindles in robust digraphs.

### 2.5.2.1 Subdivision of the $(1, 2; 1)$ -bispindle

A subdivision of the  $(1, 2; 1)$ -bispindle has cyclomatic number two. Conversely, observe that every robust digraph of cyclomatic number 2 is a subdivision of the  $(1, 2; 1)$ -bispindle. Hence, we have the following.

**Proposition 2.20** (Bang-Jensen, Havet and M. [3]). *A digraph contains a subdivision of the  $(1, 2; 1)$ -bispindle if and only if one of its robust components has cyclomatic number at least two.*

**Corollary 2.21** (Bang-Jensen, Havet and M. [3]).  *$B(1, 2; 1)$ -SUBDIVISION can be solved in linear time.*

*Proof.* Finding the robust components can be done in linear time and computing the cyclomatic number of all of them in linear time as well.  $\square$

### 2.5.2.2 Subdivision of the $(1, 3; 1)$ -bispindle

Observe that there is a  $C_4$  in a  $(1, 3; 1)$ -bispindle. So, a digraph  $D$  that has no directed cycle of length greater than 3 contains no  $B(1, 3; 1)$ -subdivision.

Let  $D$  be a robust digraph and  $C = (v_1, \dots, v_\ell, v_1)$  a directed cycle in  $D$ . A handle decomposition  $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$  is said to be  $C$ -bad if

- (i)  $D_1 = C$ ;
- (ii) for all  $i \geq 2$ ,  $h_i$  has length 1 or 2, its end-vertices are on  $C$  and the distance between the origin and the terminus of  $h_i$  around  $C$  is 2.
- (iii) If  $h_i$  is a  $(v_k, v_k + 2)$ -path and  $h_j$  is a  $(v_{k-1}, v_k + 1)$ -path (indices are taken modulo  $\ell$ ), then these two handles have length 1.
- (iv) If  $\ell \geq 5$ , there no  $k$  such that  $(v_{k-2}, v_k)$ ,  $(v_{k-1}, v_{k+1})$  and  $(v_k, v_{k+2})$  are handles.

The notion of  $C$ -bad handle decomposition plays a crucial role for finding  $B(1, 3; 1)$ -subdivision as shown by the next two lemmas.

**Lemma 2.22** (Bang-Jensen, Havet and M. [3]). *Let  $D$  be a digraph and  $C$  a directed cycle in  $D$  of length at least 4. Then one of the following holds:*

- $D$  contains a  $B(1, 3; 1)$ -subdivision,
- $C$  is not a longest directed cycle in  $D$ , or
- $D$  has a  $C$ -bad handle decomposition.

*Proof.* Set  $C = (v_1, \dots, v_\ell, v_1)$ . Let  $\mathcal{H} = (v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$  be a nice handle decomposition of  $D$  such that  $D_1 = C$ .

If  $\mathcal{H}$  is not  $C$ -bad, then let  $k$  be the largest integer such that  $\mathcal{H}_k = (v, (h_i)_{1 \leq i \leq k}, (D_i)_{0 \leq i \leq k})$  is a  $C$ -bad handle decomposition. One of the following occurs:

- (i) the origin  $s_{k+1}$  of  $h_{k+1}$  is the internal vertex of some  $h_i$ ,  $i \geq 2$ . Since  $\mathcal{H}_k$  is  $C$ -bad, then necessarily  $h_i = (s_i, s_{k+1}, t_i)$ , and there is a directed path  $(s_i, v_i, t_i)$  of length 2 in  $C$ . Let  $t_{k+1}$  be the terminus of  $h_{k+1}$ . If  $t_{k+1}$  is on  $C$ , we set  $h^* = h_{k+1}$  and  $t^* = t_{k+1}$ . If not, then  $t_{k+1}$  has an out-neighbour  $t^*$  on  $C$

and we let  $h^*$  be the concatenation of  $h_{k+1}$  and  $(t_{k+1}, t^*)$ . In both cases,  $h^*$  is a directed  $(s_{k+1}, t^*)$ -path with no internal vertices in  $C$ . If  $t^* = v_i$ , then  $h^* \cup (C \setminus \{s_i v_i\}) \cup (s_i, s_{k+1})$  is a directed cycle longer than  $C$ . If  $t^* = s_i$ , then  $(C \cup h^* \cup (s_i, s_{k+1})) - v_i$  is a  $B(1, 3; 1)$ -subdivision with right node  $s_i$  and left node  $s_{k+1}$ . If  $t^* = t_i$ , then  $C[t_i, s_i] \cup h^*$  is a directed cycle longer than  $C$  because in that case  $h^*$  has length at least 2. If  $t^* \notin \{s_i, t_i, v_i\}$ , then  $C \cup h^* \cup (s_i, s_{k+1})$  is a  $B(1, 3; 1)$ -subdivision with left node  $s_i$  and right node  $t^*$ .

- (ii) the terminus of  $h_{k+1}$  is the internal vertex of some  $h_i$ ,  $i \geq 2$ . We get the result in a similar way to the preceding case.
- (iii)  $h_{k+1}$  has length greater than 2 and its two end-vertices are on  $C$ . Then the union of  $C$  and  $h_{k+1}$  is a  $B(1, 3; 1)$ -subdivision.
- (iv)  $h_{k+1} = (s, t)$  with  $s, t$  and  $C[s, t]$  has length at least 3. Then  $C \cup (s, t)$  is a  $B(1, 3; 1)$ -subdivision with right node  $s$  and left node  $t$ .
- (v)  $h_{k+1}$  is one of the two handles  $h$  and  $h'$ , where  $h$  is a  $(v_{k-1}, v_{k+1})$ -handle and  $h'$  is a  $(v_k, v_{k+2})$  for some  $k$ , and one of  $h$  and  $h'$  has length two. If  $h$  has length two, say  $(v_{k-1}, x_1, v_{k+1})$ , then the union of  $(v_{k-1}, v_k) \cup h'$ ,  $(v_{k-1}, x_1, v_{k+1}, v_{k+2})$  and  $C[v_{k+2}, v_{k-1}]$  form a  $B(1, 3; 1)$ -subdivision. If  $h'$  has length two, say  $h' = (v_k, x_2, v_{k+2})$ , then the union of  $h \cup (v_{k+1}, v_{k+2})$ ,  $(v_{k-1}, v_k, x_2, v_{k+2})$  and  $C[v_{k+2}, v_{k-1}]$  form a  $B(1, 3; 1)$ -subdivision.
- (vi)  $h_{k+1}$  is one of the three handles  $(v_{k-2}, v_k)$ ,  $(v_{k-1}, v_{k+1})$ ,  $(v_k, v_{k+2})$  for some  $k$  and  $p \geq 5$ . In this case, the union of  $(v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2})$ ,  $(v_{k-2}, v_k, v_{k+2})$  and  $C[v_{k+2}, v_{k-2}]$  form a  $B(1, 3; 1)$ -subdivision.

□

**Lemma 2.23** (Bang-Jensen, Havet and M. [3]). *Let  $D$  be a robust digraph and  $C$  a directed cycle in  $D$  of length at least 4. If  $D$  has a  $C$ -bad handle decomposition, then it does not contain any  $B(1, 3; 1)$ -subdivision.*

*Proof.* By induction on the number  $p$  of handles of the handle decomposition, the result holding trivially if  $p = 1$ .

Set  $C = (v_1, \dots, v_\ell, v_1)$  and let  $\mathcal{H} = (v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$  be a  $C$ -bad handle decomposition of  $D$ .

By the induction hypothesis  $D_{p-1}$  does not have any  $B(1, 3; 1)$ -subdivision.

Suppose, by way of contradiction, that  $D_p$  contains a  $B(1, 3; 1)$ -subdivision  $S$ . Necessarily,  $h_p$  is a subdigraph of  $S$ . Free to rename, we may assume that  $v_1$  and  $v_3$  are the origin and the terminus, respectively, of  $h_p$ . If  $v_2$  is not in  $S$ , then replacing  $h_p$  with  $(v_1, v_2, v_3)$  in  $S$ , we obtain a  $B(1, 3; 1)$ -subdivision contained in  $D_{p-1}$ , a contradiction. Hence  $v_2 \in V(S)$ . By the conditions (iii) and (iv) of a  $C$ -bad handle decomposition, there cannot be both a handle ending at  $v_2$  and a handle starting at  $v_2$ . By directional symmetry, we may assume that  $v_2$  has in-degree one, and so

$v_1v_2 \in A(S)$ , and  $v_1$  is the left node of  $S$ . Now,  $v_2v_3$  is not an arc of  $S$ , for otherwise  $v_3$  will be the right node of  $S$ , and the two directed  $(v_1, v_3)$ -paths in  $S$  have length at most 2, a contradiction. But, in  $S$ , there is an arc leaving  $v_2$ , it must be in a handle, and so by (iv) and (ii) of the definition of  $C$ -bad, this arc must be  $v_2v_4$ . Again by (iii) of the definition of  $C$ -bad, there is no arc leaving  $v_3$  except  $v_3v_4$ . Hence  $v_3v_4 \in A(S)$ . Then  $v_4$  is the right node of  $S$ , and the two directed  $(v_1, v_4)$ -paths in  $S$  have length 2, a contradiction.  $\square$

**Theorem 2.24** (Bang-Jensen, Havet and M. [3]).  $B(1, 3; 1)$ -SUBDIVISION can be solved in  $O(n \cdot m)$  time.

*Proof.* Given a digraph  $D$ , we compute the robust components of  $D$  and solve the problem separately on each of them.

For each robust component, we first search for a directed cycle  $C_0$  of length at least 4. This can be done in  $O(n \cdot m)$  time by Theorem 2.5. If there is no such cycle, then we return ‘no’. If not, then we build a handle decomposition starting from  $C := C_0$ . Each time, we add a new handle, one can mimick the proof of Lemma 2.22, we either find a  $B(1, 3; 1)$ -subdivision which we return, or a  $C$ -bad handle decomposition, or a directed cycle  $C'$  longer than the current  $C$ . Observe that in this case, it is easy to derive a  $C'$ -bad handle decomposition containing the vertices added so far from the  $C$ -bad one. This can be done in  $O(n \cdot m)$  time because an arc has to be considered only when it is added in a handle, and we just need to keep a set of at most  $m$  handles.

At the end of this process, if no  $B(1, 3; 1)$ -subdivision has been returned, we end up with a  $C$ -bad decomposition of  $D$ . So, by Lemma 2.23,  $D$  has no  $B(1, 3; 1)$ -subdivision, and we can proceed to the next robust component, or return ‘no’ if there is none.  $\square$

## 2.6 Windmills

A *cycle windmill* is a digraph obtained from disjoint directed cycles by taking one vertex per cycle and identifying all of these. This vertex will be called the *axis* of the windmill.

**Theorem 2.25** (Bang-Jensen, Havet and M. [3]). If  $W$  is a cycle windmill, then  $W$ -SUBDIVISION can be solved in  $O(n^{|V(W)|} \cdot (n + m))$  time.

*Proof.* Suppose  $W$  is a windmill with axis  $o$  and cycle lengths  $a_1, a_2, \dots, a_p$ . To check whether a given digraph  $D = (V, A)$  contains a subdivision of  $W$  with axis at the vertex  $x$  we do the following (until success or all subsets have been tried): for all choices of disjoint ordered subsets  $X_1, X_2, \dots, X_p$  of  $V$  such that  $X_i = \{v_{i,1}, \dots, v_{i,a_i-1}\}$ ,  $i = 1, 2, \dots, p$ , check whether  $Q_i = xv_{i,1}v_{i,2} \dots v_{i,a_i-1}$  is a directed  $(x, v_{i,a_i-1})$ -path. If this holds for all  $i$ , then delete all the vertices of  $X_i - v_{i,a_i-1}$ ,  $i = 1, 2, \dots, p$ , and check whether the resulting digraph contains internally disjoint paths  $P_1, P_2, \dots, P_p$  where  $P_i$  is a path from  $v_{i,a_i-1}$  to  $x$  using a

Menger algorithm. If these paths exist, then return the desired subdivision of  $W$  formed by the union of  $Q_1, Q_2, \dots, Q_p, P_1, P_2, \dots, P_p$ . Otherwise continue to the next choice for  $X_1, X_2, \dots, X_p$ . Since the size of  $X_1 \cup X_2 \cup \dots \cup X_p$  is  $|V(W)| - 1$ , there are  $O(n^{|V(W)|-1})$  choices for it, and there are  $n$  choices for  $x$ , hence the algorithm runs  $O(n^{|V(W)|})$  times a Menger algorithm. Since a Menger algorithm runs in linear time, the overall complexity is  $O(n^{|V(W)|} \cdot (n + m))$ .  $\square$

Clearly, given as input a windmill  $W$  and a digraph  $D$ , deciding if  $D$  contains a  $W$ -subdivision is NP-complete because the Hamiltonian directed cycle problem is a particular case of it. Theorem 2.25 tells us that this problem parameterized by  $|W|$  is in XP. But is it fixed-parameter tractable?

**Problem 2.26.** *Is the following problem fixed-parameter tractable?*

CYCLE-WINDMILL SUBDIVISION

*Input:* A cycle windmill  $W$  and a digraph  $D$ .

*Parameter:*  $|V(W)|$ .

*Question:* Does  $D$  contain a subdivision of  $W$ ?

## 2.7 Dumbbells

A *dumbbell* is a digraph  $D$  with exactly two big vertices  $u$  and  $v$  which are connected by an induced oriented  $(u, v)$ -path  $P$  such that removing the internal vertices of  $P$  leaves a digraph with two connected components, one  $L$  containing  $u$  and one  $R$  containing the terminus  $v$ . The subdigraph  $L$  (resp.  $R$ ) is the *left* (resp. *right*) *plate* of the dumbbell, vertex  $u$  is its *left clip*, vertex  $v$  its *right clip* and  $P$  its *bar*.

A *dumbbell set* is a disjoint union of dumbbells. In this section, we give some necessary conditions for  $F$ -SUBDIVISION to be NP-complete,  $F$  being a dumbbell set. We also show some particular cases in which  $F$ -SUBDIVISION is polynomial-time solvable.

Recall that we denote by  $b(P)$  the number of blocks of a path  $P$ . A pair of oriented paths  $(P, Q)$  is a *bad pair* if one of the following holds:

- $P$  and  $Q$  are both directed paths,
- $\{b(P), b(Q)\} = \{1, 2\}$ ,
- $P$  and  $Q$  are both out-paths and  $\{b(P), b(Q)\} \in \{\{2, 2\}; \{2, 4\}\}$ , or
- $P$  and  $Q$  are both in-paths and  $\{b(P), b(Q)\} \in \{\{2, 2\}; \{2, 4\}\}$ .

**Lemma 2.27** (Bang-Jensen, Havet and M. [3]). *Let  $P$  and  $Q$  be two oriented paths. If  $(P, Q)$  is not a bad pair, then there exists  $ab \in A(P)$  and  $cd \in A(Q)$  such that the two oriented paths  $P'$  and  $Q'$  obtained from  $P$  and  $Q$  by replacing  $ab$  and  $cd$  by  $ad$  and  $cb$  verify  $\{b(P), b(Q)\} \neq \{b(P'), b(Q')\}$ .*



*Proof.* Let  $(P, Q)$  be a non-bad pair of paths. Without loss of generality, we may assume that  $b(Q) \geq b(P)$ . In particular this implies  $b(Q) \geq 3$ .

Assume that  $P$  is an out-path (resp. in-path) and  $Q$  is an in-path (resp. out-path). If  $b(P) \geq 2$ , then take  $ab$  as an arc of the first block of  $P$  and  $cd$  an arc of the first block of  $Q$ . Replacing  $ab$  and  $cd$  by  $ad$  and  $cb$  results necessarily in  $b(P') = 1$  and  $b(Q') = b(P) + b(Q) - 1$ . If  $b(P) = 1$ , take  $ab$  as an arc of the first block of  $P$  and  $cd$  an arc of the second block of  $Q$ . Then  $\{b(P'), b(Q')\} = \{2, b(Q) - 1\} \neq \{b(P), b(Q)\}$ .

So we may assume that  $P$  and  $Q$  are both out-paths or both in-paths. Observe that this in particular implies that  $P$  and  $Q$  have an even number of blocks, because the opposite path (same digraph but starting from the terminus and ending at the origin) of an out-path with an odd number of blocks is an in-path with an odd number of blocks.

Take an arc  $ab$  of the first block of  $P$  and an arc  $cd$  of the second block of  $Q$ . Then one of  $P', Q'$  has two blocks and the other  $b(P) + b(Q) - 2$  blocks. So if  $\{b(P), b(Q)\} \neq \{2, b(P) + b(Q) - 2\}$ , we have the result. Hence we may assume that  $\{b(P), b(Q)\} = \{2, b(P) + b(Q) - 2\}$ , so  $b(P) = 2$  because  $b(Q) \geq 3$ .

Hence  $b(Q) \geq 6$ , because  $(P, Q)$  is not bad. Take  $ab$  be an arc of the first block of  $P$  and  $cd$  an arc of the third block of  $Q$ . Then one of  $P', Q'$  has four blocks and the other has  $b(P) + b(Q) - 4$  blocks, so we have the result.  $\square$

If two digraphs  $D$  and  $D'$  are isomorphic, we write  $D \cong D'$ , and if they are not, then we write  $D \not\cong D'$ .

**Theorem 2.28** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a dumbbell set. Let  $D_1$  and  $D_2$  be two dumbbells of  $F$ , and for  $i = 1, 2$ , let  $L_i, R_i, u_i, v_i$  and  $P_i$  be the left plate, right plate, left clip, right clip and bar of  $D_i$ . If one of the following holds*

- (i)  $(P_1, P_2)$  is not a bad pair,
- (ii)  $L_1 \not\cong L_2, L_1 \not\cong R_2, R_1 \not\cong L_2$  and  $R_1 \not\cong R_2$ ,
- (iii)  $P_1$  and  $P_2$  are both directed paths,  $L_1 \not\cong L_2$  and  $R_1 \not\cong R_2$ , or
- (iv)  $P_1$  is a directed path and  $P_2$  is an out-path (resp. in-path) with two blocks and  $L_1 \not\cong L_2$  or  $L_1 \not\cong R_2$  (resp.  $R_1 \not\cong L_2$  or  $R_1 \not\cong R_2$ ).

then  $F$ -SUBDIVISION is NP-complete.

*Proof.* By Lemma 1.30, it is sufficient to prove it when  $F = D_1 + D_2$ . We give a reduction from 2-LINKAGE in digraphs with no big vertices in which  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks.

Let  $D, x_1, x_2, y_1, y_2$  be an instance of this problem. Let  $ab$  be an arc of the bar of  $D_1$  and  $cd$  be an arc of the bar of  $D_2$ . Moreover, if  $(P_1, P_2)$  is not a bad pair, we choose  $ab$  and  $cd$  as described in Lemma 2.27. Let  $H$  be the digraph obtained from the disjoint union of  $F \setminus \{ab, cd\}$  and  $D$  by adding the arcs  $ax_1, cx_2, y_1b$ , and  $y_2d$ . We can then show that  $H$  has an  $F$ -subdivision if and only if  $D$  has a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$ .

Clearly, if there is a 2-linkage  $Q_1, Q_2$  in  $D$ , then the union of  $F \setminus \{ab, cd\}$  and the paths  $ax_1Q_1y_1b$  and  $cx_2Q_2y_2d$  is an  $F$ -subdivision in  $H$ .

Conversely, suppose that  $H$  contains an  $F$ -subdivision  $S$ . For each vertex  $x$  of  $F$ , we denote by  $x^*$  the vertex corresponding to  $x$  in  $S$  and for any subdigraph  $G$  of  $F$ , we denote by  $G^*$  the subdigraph of  $S$  corresponding to the subdivision of  $G$ .

In  $H$ , no vertex of  $D$  is big, so the sole big vertices of  $D$  are the clips of  $D_1$  and  $D_2$ . Hence  $\{u_1^*, v_1^*, u_2^*, v_2^*\} = \{u_1, v_1, u_2, v_2\}$ . Now in  $S$ , the paths  $P_1^*$  and  $P_2^*$  connect big vertices. For connectivity reasons these two paths must use  $P_1 \setminus ab$  and  $P_2 \setminus cd$ . In particular,  $(L_1 + L_2 + R_1 + R_2)^*$  is a subdigraph of  $L_1 + L_2 + R_1 + R_2$ . So  $(L_1 + L_2 + R_1 + R_2)^* = L_1 + L_2 + R_1 + R_2$ . So for any  $G \in \{L_1, L_2, R_1, R_2\}$ , the digraph  $G^*$  is isomorphic to  $G$  and is one of the subdigraphs  $L_1, L_2, R_1$  and  $R_2$ .

Moreover the number of blocks  $b(P_i^*) = b(P_i)$  for  $i = 1, 2$ . Hence, the subpaths of  $P_1^* \cap D$  and  $P_2^* \cap D$  must be two disjoint directed paths in  $D$ , with origins in  $\{x_1, x_2\}$  and terminus in  $\{y_1, y_2\}$ , for otherwise  $b(P_1^*) + b(P_2^*) > b(P_1) + b(P_2)$ .

Let  $P_1'$  and  $P_2'$  be the oriented paths obtained from  $P_1$  and  $P_2$  by replacing  $ab$  and  $cd$  by  $ad$  and  $cb$ . By construction, if there is no 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ , then  $P_1^*$  and  $P_2^*$  consist in a  $P_1'$ -subdivision and a  $P_2'$ -subdivision, and so  $\{b(P_1'), b(P_2')\} = \{b(P_1^*), b(P_2^*)\}$ . We consider then the previous cited cases for  $P_1, P_2, R_1, R_2, L_1, L_2$ .

- (i) If  $(P_1, P_2)$  is not a bad pair, then by our choice of  $ab$  and  $cd$ ,  $\{b(P_1'), b(P_2')\} \neq \{b(P_1), b(P_2)\}$ . Since  $b(P_1^*) = b(P_1)$  and  $b(P_2^*) = b(P_2)$ , there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
- (ii) If  $L_1 \not\cong L_2$  and  $L_1 \not\cong R_2$ , then  $L_1^* \in \{L_1, R_1\}$ . Similarly, if  $R_1 \not\cong L_2$  and  $R_1 \not\cong R_2$ , then  $R_1^* \in \{L_1, R_1\}$ . Hence  $P_1^*$  must go from  $u_1$  to  $v_1$ , and so  $P_1^* \cap D$  is a directed  $(x_1, y_1)$ -path. Hence there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
- (iii) If  $P_1$  and  $P_2$  are both directed paths, then  $\{u_1^*, u_2^*\} = \{u_1, u_2\}$  as there are the origin of  $P_1^*$  and  $P_2^*$ . Now, since  $L_1 \not\cong L_2$ , we have  $L_1^* = L_1$  and  $L_2^* = L_2$ . Similarly,  $R_1^* = R_1$  and  $R_2^* = R_2$ . Hence,  $P_1^* \cap D$  and  $P_2^* \cap D$  form a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
- (iv) Assume that  $P_1$  is a directed path and that  $P_2$  is an out-path with two blocks. (The proof is analogous when  $P_2$  is an in-path with two blocks.)

Assume that  $L_1 \not\cong L_2$ . Then we can choose  $cd$  to be an arc of the first block of  $P_2$ . Necessarily,  $v_1^* = v_1$  and  $R_1^* = R_1$  since  $v_1^*$  is the only clip with out-degree 0 in  $P_1^* \cup P_2^*$ . It follows that  $L_1^* \in \{L_1, L_2\}$ , and so  $L_1^* = L_1$  because  $L_1 \not\cong L_2$ . Thus  $P_1^* \cap D$  is a directed  $(x_1, y_1)$ -path and there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

If  $L_1 \not\cong R_2$ , we get the result similarly by choosing  $cd$  to be an arc of the second block of  $P_2$ .

□

A *palm tree* is a dumbbell whose left and right plates are spiders, and whose bar is a directed path of length one. Observe that in a palm tree, the two clips must be the bodies of the spiders. A *palm grove* is a disjoint union of palm trees. For example, the two graphs  $A$  and  $B$  depicted Figure 1.4 are palm groves.

By Theorem 2.28(c), if  $F$  is a palm grove having two palm trees whose left spiders are not isomorphic and whose right spiders are not isomorphic, then  $F$ -SUBDIVISION is NP-complete. We shall now prove that it is indeed the only hard case. Observe that if a digraph contains a subdivision of a palm tree, then it contains a subdivision of this palm tree such that the only subdivided arc is the bar.

**Theorem 2.29** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a palm grove. Then  $F$ -SUBDIVISION is polynomial-time solvable if and only if all its left spiders are isomorphic or all its right spiders are isomorphic.*

*Proof.* If there are two left spiders that are not isomorphic and there are two right spiders that are not isomorphic, then there exist two palm trees such that their left spiders are not isomorphic and their right spiders are not isomorphic. Then, by Theorem 2.28-(c),  $F$ -SUBDIVISION is NP-complete.

Assume now that all the right spiders are isomorphic to a spider  $R$ . Let  $L_1, \dots, L_p$  be the left spiders (possibly some of them are isomorphic). We shall describe an algorithm to solve  $F$ -SUBDIVISION.

Let  $D$  be a digraph. By the above remark, if  $D$  contains an  $F$ -subdivision, then it contains an  $F$ -subdivision such that only the bars of the palm trees are subdivided. Hence we look for such a subdivision. Observe that such a subdivision is the disjoint union of copies of each of the  $L_i$ ,  $1 \leq i \leq p$ , and  $p$  copies of  $R$  together with  $p$  disjoint directed paths from the bodies of the copies of the  $L_i$  to the bodies of the  $p$  copies of  $R$ . Hence to decide if  $D$  contains an  $F$ -subdivision, we try all possibilities for the disjoint union of spiders  $L_i$ ,  $1 \leq i \leq p$ , and  $p$  spiders  $R$  and for each possibility we check via a Menger algorithm if there are disjoint directed paths from the bodies of the  $L_i$  to the bodies of the copies of  $R$ .

Formally, the algorithm is the following. For each set of distinct vertices  $\{u_1, \dots, u_p, v_1, \dots, v_p\}$  of  $D$  and family of disjoint subsets  $\{U_1, \dots, U_p, V_1, \dots, V_p\}$  of  $D$  such that for  $1 \leq i \leq p$ ,  $u_i \in U_i$  and  $v_i \in V_i$ , we check if for all  $i$ ,  $D \setminus \langle U_i \rangle$  (resp.  $V_i$ ) contains a spider isomorphic to  $L_i$  (resp.  $R$ ) with body  $u_i$  (resp.  $v_i$ ). If not we proceed to the next case. If yes, we check if there are  $p$  disjoint directed paths from  $\{u_1, \dots, u_p\}$  to  $\{v_1, \dots, v_p\}$  in the digraph  $D - (\bigcup_{i=1}^p (U_i \cup V_i) - \{u_i, v_i\})$  via a Menger algorithm. If there are such paths, the union of them with the spiders is an  $F$ -subdivision and we return it. If such paths do not exist, we proceed to the next case.

The number of possible cases is  $O(n^{|V(F)|})$  and each run of the Menger algorithm can be done in linear time. Hence the complexity of the algorithm is  $O(n^{|V(F)|} \cdot (n + m))$ .  $\square$

## 2.8 Wheels

The *wheel*  $W_k$ , also called  $k$ -wheel, is the graph obtained from the directed cycle  $C_k$  by adding a vertex, called the *centre*, dominating every vertex of  $C_k$  (the  $W_3$  is represented in Figure 2.2) The cycle  $C_k$  is called the *rim* of  $W_k$  and the arcs incident to the centre are called the *spokes*. Similarly, if  $D'$  is a subdivision of a wheel  $D$ , the *centre* of  $D'$  is the vertex corresponding to the centre of  $D$ , the *rim* of  $D'$  is the directed path or cycle corresponding to the rim of  $D$ , and the *spokes* of  $D'$  are the directed paths corresponding to the spokes of  $D$ .

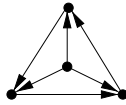


Figure 2.2: The 3-wheel  $W_3$

We completely classify in which cases  $W_k$ -SUBDIVISION is NP-complete and in which cases the problem is polynomial-time solvable.

**Theorem 2.30** (Bang-Jensen, Havet and M. [3]). *For all  $k \geq 4$ ,  $W_k$ -SUBDIVISION is NP-complete.*

*Proof.* We show a reduction from 2-LINKAGE in digraphs with no big vertices in which  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks.

Let  $D, x_1, x_2, y_1, y_2$  be an instance of this problem. Let  $D'$  be the graph obtained from  $D$  by adding  $k+1$  new vertices  $z, v_1, v_2, \dots, v_k$ , the arcs  $zv_i$ , for  $1 \leq i \leq k$ , the arcs  $v_i v_{i+1}$  and  $v_{k-1} v_k$ , for  $1 \leq i \leq k-3$ , and finally  $y_1 v_{k-1}, v_k x_2, y_2 v_1$  and  $v_{k-2} x_1$ .

Let us prove that  $D'$  has a  $W_4$ -subdivision if and only if  $D$  has a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$ .

If  $P_1, P_2$  form the desired 2-linkage in  $D$ , then we take  $P_1 \cup y_1 v_{k-1} \cup v_{k-1} v_k \cup v_k x_2 \cup P_2 \cup y_2 v_1 \cup v_1 v_2 \dots v_{k-2} \cup v_{k-2} x_1$  as the rim and the arcs  $zv_i$ , for  $1 \leq i \leq k$ , as the spokes.

Conversely, suppose  $W$  is a subdivision of  $W_4$  in  $D'$  and let  $C$  be its rim. The centre of  $W$  must be  $z$  as this is the only vertex of out-degree  $k$  in  $D'$ . Thus the  $k$  paths starting in  $z$  will start in the arcs  $zv_i$ ,  $1 \leq i \leq k$ , respectively. Now observe that  $v_{k-1}$  must belong to  $C$ , otherwise the path containing  $zv_{k-1}$  could not be disjoint from the path containing  $zv_k$  (they would meet in  $v_k$ , since it is the only out-neighbour of  $v_{k-1}$ ). Then  $v_{k-1}$  and  $v_k$  are in  $C$ . Similarly,  $v_1, v_2, \dots, v_{k-3}$  must belong to  $C$  since otherwise the path containing  $zv_i$  could not be disjoint from the path containing  $zv_{i+1}$ ,  $1 \leq i \leq k-3$  (they would meet in  $v_{i+1}$ , for instance). Thus  $v_1, v_2, \dots, v_{k-3}$  are on  $C$  and then  $v_{k-2}$  is on  $C$  since it is the only out-neighbour of  $v_{k-3}$ . Hence  $C$  contains the arc  $v_{k-1} v_k$  and the path  $v_1, v_2, \dots, v_{k-2}$ , what implies that  $C$  contains the edges  $v_k x_2, y_2 v_1, v_{k-2} x_1, y_1, v_{k-1}$  and lastly the disjoint paths from  $x_1$  to  $y_1$  and  $x_2$  to  $y_2$ , respectively.  $\square$

We now prove that the two remaining cases for wheels, that is,  $W_2$  and  $W_3$ , are tractable. We start with the following proposition for  $W_2$ .

**Proposition 2.31** (Bang-Jensen, Havet and M. [3]). *A digraph  $D$  contains a  $W_2$ -subdivision if and only if it contains some vertex  $z$  such that  $D - z$  has a strong component  $S$  and two directed  $(z, S)$ -paths having only  $z$  in common.*

*Proof.* Suppose  $D$  contains a subdivision of  $W_2$  with centre  $z$  and cycle  $C$ . Then the strong component of  $D - z$  which contains  $C$  satisfies the required property.

Conversely, assume  $z$  is a vertex and  $S$  is a strong component of  $D - z$  such that there are two directed  $(z, S)$ -paths  $P$  and  $Q$  having only  $z$  in common. Let  $x$  and  $y$  be the ends of  $P$  and  $Q$ , respectively.

Let  $R$  be a directed  $(x, y)$ -path in  $S$  and  $R'$  a directed  $(y, x)$ -path in  $S$ . (Such paths exist since  $S$  is a strong component.) If  $R$  and  $R'$  form a cycle we are done, with this cycle as rim and  $P, Q$  as spokes. Otherwise let  $q$  be the last vertex in  $R' - \{x, y\}$  which is also on  $R$ . Then we have a  $W_2$ -subdivision with rim  $R[x, q] \cup R'[q, x]$  and spokes  $P$  and  $Q \cup R[y, q]$ .  $\square$

**Corollary 2.32** (Bang-Jensen, Havet and M. [3]).  *$W_2$ -SUBDIVISION is solvable in  $O(n \cdot (n + m))$  time.*

*Proof.* According to Proposition 2.31, to find a  $W_2$ -SUBDIVISION in a digraph  $D$  it is sufficient to first calculate, for each vertex  $z \in D$ , the connected component of  $D - z$ . This can be done in  $O(n \cdot m)$  using depth-first search. Then it checks by a Menger algorithm if there are two independent directed  $(z, S)$ -paths, what gives total time complexity of  $O(n \cdot (n + m))$ .  $\square$

Proposition 2.31 and Corollary 2.32 implies that, given a digraph  $D$  and a vertex  $v$  of  $D$ , one can decide in polynomial time if  $D$  contains a  $W_2$ -subdivision with centre  $z$ . We prove that we can also decide in polynomial time if there is a  $W_2$ -subdivision with two prescribed original vertices.

**Lemma 2.33** (Havet, M. and Mohar). *Let  $W_2$  be the 2-wheel with centre  $c$  and rim  $aba$ . Given a digraph  $D$  and two vertices  $b'$  and  $c'$ , one can decide in  $O(n^2 \cdot (n + m))$  time if there is a  $W_2$ -subdivision in  $D$  with  $b$ -vertex  $b'$  and  $c$ -vertex  $c'$ .*

*Proof.* Let us call a  $W_2$ -subdivision with  $b$ -vertex  $b'$  and  $c$ -vertex  $c'$  a  $(b', c')$ -forced  $W_2$ -subdivision. Let  $S$  be the strong component of  $b'$  in  $D - c'$ . The key element is the following claim.

**Claim 5.**  *$D$  contains a  $(b', c')$ -forced  $W_2$  subdivision if and only if there exist distinct vertices  $x_1$  and  $x_2$  in  $V(S)$  such that there are two independent  $(c', \{x_1, x_2\})$ -dipaths  $P_1$  and  $P_2$  in  $D - (S - \{x_1, x_2\})$  and there are two independent  $(\{x_1, x_2\}, b')$ -dipaths  $Q_1$  and  $Q_2$  in  $S$ .*

*Subproof.* The existence of two vertices  $x_1, x_2$  and four dipaths  $P_1, P_2, Q_1, Q_2$  as in the statement is a necessary condition for the existence of a  $(b', c')$ -forced  $W_2$ -subdivision, because in such subdivision there would be two independent paths  $R_1$  and  $R_2$  (the subdivision of the spokes) to the  $a$ -vertex  $a'$  and to  $b'$ , respectively. Since  $C = a'b'a'$  is a directed cycle,  $a', b'$  are in the same strong component  $S$ ,

and  $x_1, x_2$  would be the first vertices of  $R_1 \cap S, R_2 \cap S$ , respectively. So,  $P_1 = R_1[c', x_1], P_2 = R_2[c', x_2], Q_1 = R_1[x_1, a'] \cup C[a', b'], Q_2 = R_2[x_2, b']$ .

Assume now that such vertices  $x_1, x_2$  and dipaths  $P_1, P_2, Q_1, Q_2$  exist. Since  $S$  is strong, it contains a dipath  $R$  from  $b'$  to  $(V(Q_1) \cup V(Q_2)) - \{b'\}$ . (This set is not empty since it contains  $\{x_1, x_2\} - \{b'\}$ .) Then  $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R$  is a  $(b', c')$ -forced  $W_2$ -subdivision.  $\diamond$

Our algorithm is the following. We first compute  $S$ , which can be done in linear time. Then for every pair  $\{x_1, x_2\}$  of vertices of  $S$ , we check by running twice a Menger algorithm if the dipaths  $P_1$  and  $P_2$ , and  $Q_1$  and  $Q_2$  as described in Claim 5 exist. If yes, we return ‘yes’, otherwise we return ‘no’. The validity of this algorithm is given by Claim 5. Since there are  $O(n^2)$  pairs of vertices  $\{x_1, x_2\}$ , the algorithm runs in  $O(n^2 \cdot (n + m))$  time.  $\square$

**Theorem 2.34** (Havet, M. and Mohar).  $W_3$ -SUBDIVISION can be solved in  $O(n^6 \cdot (n + m))$  time.

The proof relies on the following notion. Let  $X$  be a set of three vertices. An  $X$ -tripod is a digraph which is the union of a directed cycle  $C$  and three disjoint dipaths  $P_1, P_2, P_3$  with initial vertices in  $X$  and terminal vertices in  $C$ . If the  $P_i$  are  $(X, C)$ -dipaths, we say that the tripod is *unfolded*. Note that the dipaths  $P_i$  may be of length 0. We shall denote the tripod described above as the 4-tuple  $(C, P_1, P_2, P_3)$ .

**Proposition 2.35.** Let  $X = \{x_1, x_2, x_3\}$  be a set of three distinct vertices. Any  $X$ -tripod contains an unfolded  $X$ -tripod.

*Proof.* Let  $(C, P_1, P_2, P_3)$  be an  $X$ -tripod, where  $P_i$  has initial vertex  $x_i$ , for  $i = 1, 2, 3$ . Let  $y_i$  be the first vertex on  $C$  along  $P_i$ . Then  $(C, P_1[x_1, y_1], P_2[x_2, y_2], P_3[x_3, y_3])$  is an unfolded  $X$ -tripod.  $\square$

We shall consider the following decision problem.

TRIPOD

Input: A strong digraph  $D$  and a set  $X$  of three distinct vertices of  $D$ .

Question: Does  $D$  contain an  $X$ -tripod?

Observe that TRIPOD is also (as FORK of Lemma 2.17) resemblant to the “three-in-a-tree” problem for undirected graphs [16]. The problem of searching an induced  $X$ -tripod in an undirected graph  $G$  would allowed to solve the problem of finding an induced subdivision of a  $K_4$  in  $G$ , and it remains an open question.

We show that TRIPOD is polynomial-time solvable in the next lemma. The scheme of the proof reminds an idea used in decompositions: we first find some cuts-like on the digraph, then we partitioned it in some (slightly) modified subdigraphs and finally we focus on the reduced problem.

**Lemma 2.36.** TRIPOD can be solved in  $O(n^2 \cdot (n + m))$  time.

*Proof.* Let us describe a procedure  $\text{tripod}(D, X)$ , solving TRIPOD.

We first look for a directed cycle of length at least 3 in  $D$ . This can be done in linear time. If there is no such cycle, then we return ‘no’.

Otherwise we have a directed cycle  $C$  of length at least 3. We choose a set  $Y$  of three vertices in  $C$  and run a Menger algorithm between  $X$  and  $Y$ . If such an algorithm finds three disjoint  $(X, Y)$ -dipaths  $P_1, P_2, P_3$ , then we return the tripod  $(C, P_1, P_2, P_3)$ . Otherwise, the Menger algorithm finds a 2-separation  $(W, S, Z)$  of  $(X, Y)$ . Note that  $|S| \geq 1$  because  $D$  is strong.

Assume first that  $|S| = 1$ , say  $S = \{s\}$ . Let  $D_1$  be the digraph obtained from  $D[W \cup S]$  by adding the arc  $sw$  for every vertex  $w$  in  $W$  having an in-neighbour  $z \in Z$ . We then make a recursive call to  $\text{tripod}(D_1, X)$ . This is valid by virtue of the following claim.

**Claim 6.** *There is an  $X$ -tripod in  $D$  if and only if there is an  $X$ -tripod in  $D_1$ .*

*Subproof.* Suppose first that there is an  $X$ -tripod in  $D_1$ . Then  $D_1$  contains an unfolded  $X$ -tripod  $T_1$  by Proposition 2.35. If  $T_1$  is contained in  $D$ , then we are done. So we may assume that it is not. Then  $T_1$  contains an arc  $sw \in A(D_1) \setminus A(D)$ . It can contain only one such arc since every vertex has out-degree at most one in  $T_1$  and all such arcs leave  $s$ . Furthermore, the head  $w$  of this arc is in  $W$  and  $w$  has an in-neighbour  $z$  in  $Z$ . Now, since  $D$  is strong, there is an  $(s, z)$ -dipath  $Q$  in  $D$ . Because there is no arc from  $W$  to  $Z$ , all internal vertices of  $Q$  are in  $Z$ . Hence the digraph  $T$  obtained from  $T_1$  by replacing the arc  $sw$  by the dipath  $Q \cup zw$  is an  $X$ -tripod in  $D$ .

Suppose now that  $D$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T = (C, P_1, P_2, P_3)$  by Proposition 2.35. Since all  $(X, Z)$ -dipaths in  $D$  go through  $s$ , the terminal vertices of the  $P_i$  are in  $W \cup S$ , and  $D[Z] \cap T$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . If  $Q$  is a  $(t, z)$ -dipath, then  $T$  contains arcs  $st$  and  $zw$  for some  $w \in W$ . Then the digraph  $T_1$  obtained from  $T$  by replacing  $st \cup Q \cup zw$  by the arc  $sw$  is an  $X$ -tripod in  $D_1$ .  $\diamond$

Assume now that  $|S| = 2$ , say  $S = \{s_1, s_2\}$ . If there is no arc from  $Z$  to  $W$ , let  $D_2$  be the digraph obtained from  $D[W \cup S]$  by adding the arc  $s_1s_2$  (resp.  $s_2s_1$ ) (if the arc is not already present in  $D$ ) if there is an  $(s_1, s_2)$ -dipath (resp.  $(s_2, s_1)$ -dipath) in  $D[Z \cup S]$ . We then make a recursive call to  $\text{tripod}(D_2, X)$ . This is valid by virtue of the following claim.

**Claim 7.** *There is an  $X$ -tripod in  $D$  if and only if there is an  $X$ -tripod in  $D_2$ .*

*Subproof.* Suppose first that there exists an  $X$ -tripod in  $D_2$ . Then there is an unfolded  $X$ -tripod  $T_2$  in  $D_2$ , by Proposition 2.35. Then either it is an  $X$ -tripod in  $D$ , or  $T_2$  contains exactly one of the arcs  $s_1s_2, s_2s_1$  and this arc is not in  $A(D)$ . Without loss of generality, we may assume that this arc is  $s_1s_2$ . Since  $s_1s_2 \in A(D_2) \setminus A(D)$ , there is an  $(s_1, s_2)$ -dipath  $Q$  in  $D[Z \cup S]$ . Hence the digraph  $T$  obtained from  $T_2$  by replacing the arc  $s_1s_2$  by the dipath  $Q$  is an  $X$ -tripod in  $D$ .

Suppose now that  $D$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T = (C, P_1, P_2, P_3)$  by Proposition 2.35. For  $i = 1, 2, 3$ , let  $y_i$  be the terminal vertex of  $P_i$ . Without loss of generality, we may assume that  $y_1, y_2, y_3$  appear in this order along  $C$ . Since all  $(X, Z \cup S)$ -dipaths intersect  $S$ , one of the  $y_i$ , say  $y_3$ , must be in  $W$ . The three oriented paths  $P_2$ ,  $P_1C[y_1, y_2]$ , and  $\overline{C}[y_3y_2]$  are independent  $(W, y_2)$ -paths. But the underlying graph of  $D$  has no edges between  $W$  and  $Z$ , by the assumption made in the current subcase. So  $y_2$  is in  $W \cup S$ . Similarly,  $y_1$  is in  $W \cup S$ . It follows that  $T \cap D[Z]$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . In addition, the in-neighbour in  $T$  of the initial vertex of  $Q$  is some vertex  $s \in S$  (because there is no arc from  $W$  to  $Z$ ) and the out-neighbour in  $T$  of the terminal vertex of  $Q$  is some vertex  $s' \in S$  because there is no arc from  $Z$  to  $W$ ). Furthermore  $s \neq s'$  for otherwise  $sQs' = C$  which is impossible as since  $y_3 \in W \cap C$ . Moreover, because  $sQs'$  is an  $(s, s')$ -dipath in  $D[Z \cup S]$ ,  $ss'$  is an arc in  $D_2$ . Thus the digraph  $T_2$  obtained from  $T$  by replacing  $sQs'$  by the arc  $ss'$  is an  $X$ -tripod in  $D_2$ .  $\diamond$

Now we may assume that there is an arc  $z_1w_1$  with  $z_1 \in Z$  and  $w_1 \in W$ . Since  $D$  is strong, there is a cycle  $C'$  containing the arc  $z_1w_1$ . Necessarily, the cycle  $C'$  must go through  $S$  and it contains at least three vertices.

Case 1:  $S \subset V(C')$ . Set  $Y' = \{w_1, s_1, s_2\}$ . We run a Menger algorithm between  $X$  and  $Y'$ . If such an algorithm finds three disjoint  $(X, Y')$ -dipaths  $P'_1, P'_2, P'_3$ , then we return the  $X$ -tripod  $(C', P'_1, P'_2, P'_3)$ .

If not, we obtain a 2-separation  $(W', S', Z')$  of  $(X, Y')$ . We claim that  $|W'| < |W|$ . Indeed, no vertex  $z \in Z$  is in  $W'$  because every  $(X, z)$ -dipath must go through  $S$  and thus through  $S'$ . Hence  $W' \subseteq W - \{w_1\}$ . Now, we replace  $C$  by  $C'$ ,  $Y$  by  $Y'$  and  $(W, S, Z)$  by  $(W', S', Z')$ , and then redo the procedure.

Case 2:  $|S \cap V(C')| = 1$ . Without loss of generality, we may assume  $S \cap V(C') = \{s_1\}$ . Set  $Y' = \{w_1, s_1, z_1\}$ . As in Case 1, we run a Menger algorithm between  $X$  and  $Y'$ . If such an algorithm finds three disjoint  $(X, Y')$ -dipaths  $P'_1, P'_2, P'_3$ , then we return the  $X$ -tripod  $(C', P'_1, P'_2, P'_3)$ .

If not, the Menger algorithm returns a 2-separation  $(W', S', Z')$  for  $(X, Y')$ . Observe that there is a vertex  $s'_1 \in S' \cap W$  because  $w_1$  is reachable from  $X$  in  $D[W]$ . If  $S'$  contains a vertex  $s'_2$  in  $Z$ , then one can see that there are no  $(X, Y')$ -dipaths in  $D - \{s'_1, s_2\}$ . Thus, there is a 2-separation  $(W'', S'', Z'')$  of  $(X, Y')$  where  $S'' \subseteq \{s'_1, s_2\}$  and  $s_1 \in Z''$ . Hence, after possibly replacing the 2-separation  $(W', S', Z')$  by  $(W'', S'', Z'')$ , we may assume that  $S' \subset W \cup S$ .

If  $|W'| < |W|$ , then we replace  $C$  by  $C'$ ,  $Y$  by  $Y'$  and  $(W, S, Z)$  by  $(W', S', Z')$  and redo the procedure.

If not, then the set  $R = Z \cap W'$  is not empty. Set  $L = Z - R = Z \cap Z'$ . There is no arc from  $R$  to  $L$ , because  $(W', S', Z')$  is a 2-separation. Moreover, all  $(X, R)$ -dipaths must go through  $s_2$ . In particular,  $s_2 \in W'$ . Let  $D_3$  be the digraph obtained from  $D - L$  by adding an arc  $s_1w$  for every  $w \in W$  having an in-neighbour in  $L$ . We then make a recursive call to  $\text{tripod}(D_3, X)$ . This is valid by virtue of



the following claim.

**Claim 8.** *There is an  $X$ -tripod in  $D$  if and only if there is an  $X$ -tripod in  $D_3$ .*

*Subproof.* Suppose first that  $D_3$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T_3$  by Proposition 2.35. If  $T_3$  is contained in  $D$ , then we are done. So we may assume that  $T_3$  is not contained in  $D$ . Then  $T_3$  contains an arc in  $s_1w \in A(D_3) \setminus A(D)$ . It contains only one such arc since every vertex has out-degree at most one in  $T_3$  and all arcs of  $A(D_3) \setminus A(D)$  leave  $s_1$ . Furthermore the head  $w$  of this arc is in  $W$  and has an in-neighbour  $z \in L$ . Since  $D$  is strong, there is an  $(s_1, z)$ -dipath  $Q$  in  $D$ . Moreover since  $s_2 \in W'$  all the  $(s_2, z)$ -dipaths must go through  $S'$ . But  $S' \subseteq W \cup \{s_1\}$ , so all  $(s_2, z)$ -dipaths must go through  $s_1$ . Thus  $Q$  does not go through  $s_2$ . It follows that all internal vertices of  $Q$  are in  $Z$ , because  $(W, S, Z)$  is a 2-separation, and so in  $L$  because there is no arc from  $R$  to  $L$ . Consequently, the digraph  $T$  obtained from  $T_3$  by replacing the arc  $s_1w$  by the dipath  $Qzw$  is an  $X$ -tripod in  $D$ .

Suppose now that  $D$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T = (C, P_1, P_2, P_3)$  by Proposition 2.35. For  $i = 1, 2, 3$ , let  $y_i$  be the terminal vertex of  $P_i$ . Without loss of generality, we may assume that  $y_1, y_2, y_3$  appear in this order along  $C$ . If  $T$  is contained in  $D - L$ , then it is an  $X$ -tripod in  $D_3$ . Hence we may assume that  $T$  contains some vertices of  $L$ . Observe that the arcs entering  $L$  all leave  $s_1$ . Hence,  $y_i$  cannot be in  $L$ , since there are two  $(X, y_i)$ -dipaths in  $T$ , which are disjoint except for the common vertex  $y_i$ . Consequently, the intersection of  $T$  with  $D[L]$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . Moreover, the in-neighbour in  $T$  of the initial vertex of  $Q$  is  $s_1$  and the out-neighbour in  $T$  of the terminal vertex of  $Q$  is some vertex  $w \in W \cup \{s_1\}$ , because there is no arc from  $L$  to  $R \cup \{s_2\}$ . But  $w \neq s_1$  for otherwise  $s_1Qs_1$  would be  $C$  and would contain at most one of the  $y_i$ , a contradiction. Thus the digraph  $T_3$  obtained from  $T$  by replacing  $s_1Qw$  by the arc  $s_1w$  is an  $X$ -tripod in  $D_3$ .  $\diamond$

Claims 6, 7 and 8 ensure that our algorithm is correct. Each time we do a recursive call, the number of vertices decreases. So we do at most  $n$  of them. Between two recursive calls, we first find a cycle of length at least 3 in linear time, and next run a sequence of Menger algorithms to produce a new 2-separation. At each step the size of the set  $W$  decreases. Therefore, we run at most  $n$  times the Menger algorithm between two recursive calls. Since a Menger algorithm runs in linear time, the time between two calls is at most  $O(n \cdot (n + m))$  and so **tripod** runs in  $O(n^2 \cdot (n + m))$  time.  $\square$

With Lemma 2.36 in hands, we now deduce Theorem 2.34.

*Proof of Theorem 2.34.* For every vertex  $v$ , we examine whether there is a  $W_3$ -subdivision with centre  $v$  in  $D$ . Observe that such a subdivision  $S$  is the union of a directed cycle  $C$ , and three internally disjoint  $(v, C)$ -dipaths  $P_1, P_2, P_3$  with distinct terminal vertices  $y_1, y_2, y_3$ . The cycle  $C$  is contained in some strong component  $\Gamma$  of  $D - v$ . For  $i = 1, 2, 3$ , let  $x_i$  be the first vertex of  $P_i$  that belongs to  $\Gamma$ . Set

$X = \{x_1, x_2, x_3\}$ . Then the paths  $P_i[x_i, y_i]$ ,  $i = 1, 2, 3$ , and  $C$  form an  $X$ -tripod in  $\Gamma$ , and the  $P_i[v, x_i]$ ,  $i = 1, 2, 3$ , are internally disjoint  $(v, X)$ -dipaths in  $D - (\Gamma - X)$ .

Hence for finding a  $W_3$ -subdivision with centre  $v$ , we use the following procedure to check whether there is a set  $X$  as above. First, we compute the strong components of  $D - v$ . Next, for every subset  $X$  of three vertices in the same strong component  $\Gamma$ , we run a Menger algorithm to check whether there are three independent  $(v, X)$ -dipaths in  $D - (\Gamma - X)$ . If yes, we check whether there is an  $X$ -tripod in  $\Gamma$ . If yes again, then we clearly have a  $W_3$ -subdivision with centre  $v$ , and we return ‘yes’. If not, there is no such subdivision, and we proceed to the next triple.

For each vertex  $v$ , there are at most  $n^3$  possible triples. And for each triple we run a Menger algorithm in time  $O(n+m)$  and possibly `tripod` in time  $O(n^2 \cdot (n+m))$ . Hence the time spent on each vertex  $v$  is  $O(n^5 \cdot (n+m))$ . As we examine at most  $n$  vertices, the algorithm runs in  $O(n^6 \cdot (n+m))$  time.  $\square$

## 2.9 Fans

The *fan*  $F_k$  is the graph obtained from the directed path  $P_k$  by adding a vertex, called the *centre*, dominated every vertex of  $P_k$  (it can also be defined as the wheels, with the center dominating the path, case in which  $F_k$  is  $W_k$  where one arc of the rim is deleted).

**Theorem 2.37** (Bang-Jensen, Havet and M. [3]). *For all  $k \geq 5$ ,  $F_k$ -SUBDIVISION is NP-complete.*

*Proof.* Reduction from 2-LINKAGE in digraphs with no big vertices in which  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks.

Let  $D$ ,  $x_1$ ,  $x_2$ ,  $y_1$  and  $y_2$  be an instance of this problem. Let us denote by  $z$  the centre of  $F_k$  and by  $(v_1, v_2, \dots, v_k)$  the directed path  $F_k - z$ . Let  $D_k$  be the digraph obtained from the disjoint union of  $D$  and  $F_k$  by removing the arcs  $v_1v_2$  and  $v_3v_4$  and adding the arcs  $v_1x_1$ ,  $y_1v_2$ ,  $v_3x_2$  and  $y_2v_4$ .

We claim that  $D_k$  has an  $F_k$ -subdivision if and only if  $D$  has a linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$ .

Clearly, if there is a linkage  $(P_1, P_2)$  from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ , then  $D_k$  contains an  $F_k$ -subdivision, obtained from  $F_k$  by replacing the arc  $v_1v_2$  and  $v_3v_4$  by the directed paths  $(v_1, x_1) \cup P_1 \cup (y_1, v_2)$  and  $(v_3, x_2) \cup P_2 \cup (y_2, v_4)$ , respectively.

Suppose now that  $D_k$  contains an  $F_k$ -subdivision  $S$  in  $D_k$ . Since  $z$  is the unique vertex with in-degree  $k$ , the centre of  $S'$  is necessarily  $z$ . For  $1 \leq i \leq k$ , let  $v'_i$  be the vertex corresponding to  $v_i$  in  $S$ , and  $P_i$  be the directed  $(v'_i, z)$ -path in  $S$ .

Since  $z$  has in-degree exactly  $k$  in  $D_k$ , the  $v'_i$ 's are the penultimate vertices of the  $P_j$ 's, each  $v_i$  on a different  $P_j$ . Since  $v_1$  is a source in  $D_k$ , then  $v_1 = v'_1$ . Moreover, for  $i = 3$  and  $i \geq 5$ , the path  $P'_j$  containing  $v_i$  must start at  $v_i$  because the unique in-neighbour of  $v_i$  is  $v_{i-1}$ . Hence  $v_i = v'_i$ . Furthermore, necessarily  $v_{i-1} = v'_{j-1}$ . Now, because  $v_k$  is a sink in  $D_k - z$ , then necessarily  $v'_k = v_k$  and so for all  $1 \leq i \leq k$ , we have  $v'_i = v_i$ .

Let  $Q_1$  and  $Q_2$  be the directed  $(v_1, v_2)$ - and  $(v_3, v_4)$ -paths, respectively. Necessarily, the second vertex of  $Q_1$  (resp.  $Q_2$ ) is  $x_1$ , (resp.  $x_2$ ) and its penultimate vertex is  $y_1$  (resp.  $y_2$ ). Hence  $(Q_1[x_1, y_1], Q_2[x_2, y_2])$  is a linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .  $\square$

Observe that  $F_2$  is the  $(1, 2)$ -spindle. Thus  $F_2$ -SUBDIVISION can be solved in  $O(n^2 \cdot (n + m))$  time by Proposition 2.13. The next result shows that  $F_3$ -SUBDIVISION is polynomial.

Let  $z$  be a vertex in a digraph  $D$ . A triple  $(x_1, x_2, x_3)$  is  $F_3$ -nice with respect to  $z$  in  $D$  if the following holds:

- $x_1, x_2, x_3$  are distinct vertices of  $D - z$ ,
- $x_3z$  is an arc,
- in  $D - x_3$ , there exist a directed  $(x_1, z)$ -path  $P_1$  and a directed  $(x_2, z)$ -path  $P_2$  which intersect only in  $z$ ;
- in  $D - \{x_3, z\}$ , there is a directed  $(x_1, x_2)$ -path  $Q_1$ , and in  $D - \{x_1, z\}$ , there is a directed  $(x_2, x_3)$ -path  $Q_2$ .

**Theorem 2.38** (Bang-Jensen, Havet and M. [3]). *A digraph contains an  $F_3$ -subdivision with centre  $z$  if and only if there is an  $F_3$ -nice triple with respect to  $z$ . In particular  $F_3$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Trivially, if  $D$  contains an  $F_3$ -subdivision with centre  $z$ , then it contains an  $F_3$ -nice triple  $(x_1, x_2, x_3)$  with respect to  $z$ .

Conversely, assume that  $D$  contains an  $F_3$ -nice triple  $(x_1, x_2, x_3)$  with respect to  $z$ . Let  $P_1, P_2, Q_1$  and  $Q_2$  be the directed paths as defined in the definition of  $F_3$ -nice triple. We may assume that  $(x_1, x_2, x_3)$  is an  $F_3$ -nice triple  $(x_1, x_2, x_3)$  with respect to  $z$  that minimizes  $\ell = \ell(P_1) + \ell(P_2) + \ell(Q_1) + \ell(Q_2)$ , that is the sum of the lengths of these paths.

We shall prove that  $P_1, P_2, Q_1$  and  $Q_2$  are internally disjoint, implying that these paths and the arc  $x_3z$  form an  $F_3$ -subdivision with centre  $z$ .

- a) Let us prove that  $Q_2$  and  $P_1$  are internally disjoint. Suppose not. Then let  $x'_2$  be the last vertex on  $Q_2$  which also belongs to  $P_1$ . Then  $(x_2, x'_2, x_3)$  is  $F_3$ -nice by the choice of paths  $P'_1 = P_2, P'_2 = P_1[x'_2, z], Q'_1 = Q_2[x_2, x'_2]$  and  $Q'_2 = Q_2[x'_2, x_3]$ . Indeed,  $P'_1$  and  $P'_2$  are internally disjoint because  $P_1$  and  $P_2$  were,  $Q'_1$  does not go through  $x_3$  nor  $z$ , because  $Q_2$  is a directed  $(x_2, x_3)$ -path in  $D - z$ , and  $Q'_2$  does not go through  $x_2$  nor  $z$ , for the same reason. This contradicts the minimality of  $\ell$ .
- b) Let us prove that  $Q_2$  and  $P_2$  are internally disjoint. Suppose not. Then let  $x'_2$  be the last vertex on  $Q_2$  which also belongs to  $P_2$ . One easily verifies that  $(x_1, x'_2, x_3)$  is  $F_3$ -nice by the choice of paths  $P'_1 = P_1, P'_2 = P_2[x'_2, z], Q'_1$  a

directed  $(x_1, x'_2)$ -path included in  $Q_1[x_1, x_2]Q_2[x_2, x'_2]$  (which can be a walk), and  $Q'_2 = Q_2[x'_2, x_3]$ . This contradicts the minimality of  $\ell$ .

- c) Let us prove that  $Q_1$  and  $P_1$  are internally disjoint. Suppose not. Then let  $x'_1$  be the last vertex on  $Q_1$  which also belongs to  $P_1$ . The path  $Q_2$  does not go through  $x'_1$  because  $Q_2$  and  $P_1$  are internally disjoint. Thus  $(x'_1, x_2, x_3)$  is  $F_3$ -nice with associated paths  $P'_1 = P_1[x'_1, z]$ ,  $P'_2 = P_2$ ,  $Q'_1 = Q_1[x'_1, x_2]$ , and  $Q'_2 = Q_2$ . This contradicts the minimality of  $\ell$ .
- d) Let us prove that  $Q_1$  and  $P_2$  are internally disjoint. Suppose not. Then let  $x'_2$  be the last internal vertex on  $Q_1$  which also belongs to  $P_2$ . Then  $(x_1, x'_2, x_3)$  is  $F_3$ -nice with associated paths  $P'_1 = P_1$ ,  $P'_2 = P_2[x'_2, z]$ ,  $Q'_1 = Q_1[x_1, x'_2]$ , and  $Q'_2$  a directed  $(x_1, x'_2)$ -path included in  $Q_1[x'_2, x_2]Q_2$  (which can be a walk). This contradicts the minimality of  $\ell$ .
- e) Let us prove that  $Q_1$  and  $Q_2$  are internally disjoint. Suppose not. Then let  $x'_2$  be the last internal vertex on  $Q_2$  which also belongs to  $Q_1$ . Then  $(x_1, x'_2, x_3)$  is a good triple with associated paths  $P'_1 = P_1$ ,  $P'_2 = Q_1[x'_2, x_2]P_2$ ,  $Q'_1 = Q_1[x_1, x'_2]$ , and  $Q'_2 = Q_2[x'_2, x_3]$ . Indeed, since  $P_2$  and  $Q_1$  are internally disjoint,  $P'_2$  is a path, and since  $P_1$  and  $Q_1$  are internally disjoint, the paths  $P'_1$  and  $P'_2$  are also internally disjoint.

□

Proposition 2.13 and Theorems 2.38 and 2.37 determine the complexity of  $F_k$ -SUBDIVISION for all  $k$  except 4. So we are left with the following problem.

**Problem 2.39.** *What is the complexity of  $F_4$ -SUBDIVISION ?*

## 2.10 Tournaments

A *tournament* is an orientation of a complete graph. We denote by  $TT_k$  the acyclic tournament on  $k$  vertices, frequently called *transitive tournament*. The strongly connected tournament on  $k$  vertices, also referred to as *strong tournament*, is denoted by  $ST_k$ . Figure 2.3 shows a representation of the strong tournament on four vertices.



Figure 2.3: The strong tournament  $ST_4$

Let  $T$  be a tournament. If  $|T| \geq 5$ ,  $T$ -SUBDIVISION is NP-complete by Corollary 1.12. On the other hand, if  $|T| \leq 3$ ,  $T$ -SUBDIVISION is polynomial-time solvable because the tournaments with 1 and 2 vertices are spiders, and those with 3 vertices are the  $TT_3$ , which is the  $(1, 2)$ -spindle, and the  $ST_3$ , which is the directed cycle  $C_3$ .

The tournaments of order 4 are  $ST_4$ ,  $TT_4$ ,  $W_3$  and its converse. The last two were proved to be tractable on the previous section (Theorem 2.34). We shall now prove that  $ST_4$  and  $TT_4$ -SUBDIVISION are polynomial-time solvable.

Let us start by  $ST_4$ . We need the following auxiliary case to prove that  $ST_4$ -SUBDIVISION is tractable. Consider the digraph  $W'_2$ -SUBDIVISION, depicted in Figure 2.4. Using Lemmas 1.35 and 2.33, one can easily give a polynomial-time algorithm to solve it.

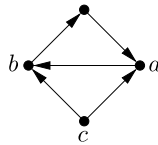


Figure 2.4: The digraph  $W'_2$

**Theorem 2.40** (Havet, M. and Mohar).  $W'_2$ -SUBDIVISION can be solved in  $O(n^5(n+m))$  time.

*Proof.* By Lemma 2.33, applied to every vertex  $c'$  in  $D$ , for every vertex  $a'$  of  $D$ , one can decide in  $O(n^3(n+m))$  if there is a  $W_2$ -subdivision with  $b$ -vertex  $b'$ . Observe that  $W'_2$  is obtained from  $W_2$  by subdividing once the arc  $ba$ , which is the only arc leaving  $b$ . Therefore, Lemma 1.35 applies. Thus for every vertex  $b'$  one can decide in  $O\left(\sum_{y \in N^+(b')} d^+(y)n^3(n+m)\right)$  if there is a  $W'_2$ -subdivision with  $b$ -vertex  $b'$ . Hence, one can decide in  $O(n^5(n+m))$  time whether there is a  $W'_2$ -subdivision in  $D$ .  $\square$

We now give a more complicated but faster algorithm based on an algorithm deciding if there is a  $W'_2$ -subdivision with prescribed  $c$ -vertex. This proof uses in a simpler way the technique that we use in for proving that  $W_3$  is tractable.

**Theorem 2.41.** Given a digraph  $D$  and a vertex  $v$  of  $D$ , one can decide in  $O(n^3(n+m))$  time if  $D$  contains a  $W'_2$ -subdivision with centre  $v$ . So  $W'_2$ -SUBDIVISION can be solved in  $O(n^4(n+m))$  time.

The proof of this theorem relies on the following notion. Let  $X$  be a set of two vertices. An  $X$ -bipod is a digraph which is the union of a directed cycle  $C$  of length at least 3 and two disjoint dipaths  $P_1$  and  $P_2$  with initial vertices in  $X$  and terminal vertices in  $C$ . If the  $P_i$  are  $(X, C)$ -dipaths, we say that the bipod is *unfolded*. Note that the dipaths  $P_i$  may be of length 0. We often denote a bipod by the triple  $(C, P_1, P_2)$  described above.

**Proposition 2.42.** Let  $X = \{x_1, x_2\}$  be a set of two distinct vertices. Any  $X$ -bipod contains an unfolded  $X$ -bipod.

*Proof.* Let  $(C, P_1, P_2)$  be an  $X$ -bipod, where  $P_i$  has initial vertex  $x_i$ , for  $i = 1, 2$ . Let  $y_i$  be the first vertex on  $C$  along  $P_i$ . Then  $(C, P_1[x_1, y_1], P_2[x_2, y_2])$  is an unfolded  $X$ -bipod.  $\square$

We shall consider the following decision problem.

BIPOD

Input: A strong digraph  $D$  and a set  $X$  of two distinct vertices of  $D$ .

Question: Does  $D$  contain an  $X$ -bipod?

**Lemma 2.43.** BIPOD can be solved in  $O(n(n+m))$  time.

*Proof.* Let us describe a procedure  $\text{bipod}(D, X)$ , solving BIPOD.

We first look for a directed cycle of length at least 3 in  $D$ . This can be done in linear time. If there is no such cycle, then we return ‘no’.

Otherwise we have a directed cycle  $C$  of length at least 3. We choose a set  $Y$  of two vertices in  $C$  and run a Menger algorithm between  $X$  and  $Y$ . If this algorithm finds two disjoint  $(X, Y)$ -dipaths  $P_1, P_2$ , then we return the bipod  $(C, P_1, P_2)$ . Otherwise, the Menger algorithm finds a 1-separation  $(W, S, Z)$  of  $(X, Y)$ . Note that  $|S| = 1$  because  $D$  is strong. Set  $S = \{s\}$ .

Let  $D'$  be the digraph obtained from  $D$  by contracting  $Z$  into a vertex  $t$ . Note that  $D'$  is strong. We now make a recursive call to  $\text{bipod}(D', X)$ . This is valid by virtue of the following claim.

**Claim 9.** *There is an  $X$ -bipod in  $D$  if and only if there is an  $X$ -bipod in  $D'$ .*

*Subproof.* Suppose first that there is an  $X$ -bipod in  $D'$ . Then  $D'$  contains an unfolded  $X$ -bipod  $B'$  by Proposition 2.42. If  $B'$  is contained in  $D$ , then we are done. So we may assume that it is not. Then  $B'$  contains a dipath  $stw$  for some  $w \in W$ . It contains only one such dipath since every vertex has out-degree at most one in  $B'$ . Moreover,  $t$  has in-degree 1 in  $D'$ , so it has in-degree 1 also in  $B'$ . Since  $t$  was obtained by contraction of  $Z$ ,  $w$  has an in-neighbour  $z \in Z$ . Now, since  $D$  is strong, there is an  $(s, z)$ -dipath  $Q$  in  $D$ . Because there is no arc from  $W$  to  $Z$ , all the internal vertices of  $Q$  are in  $Z$ . Hence the digraph  $B$  obtained from  $B'$  by replacing the dipath  $stw$  by the dipath  $Qzw$  is an  $X$ -bipod in  $D$ .

Suppose now that  $D$  contains an  $X$ -bipod. Then it contains an unfolded  $X$ -bipod  $B = (C, P_1, P_2)$  by Proposition 2.42. Since all  $(X, Z)$ -dipaths in  $D$  go through  $s$ , the terminal vertices of the  $P_i$  are in  $W \cup S$ , and  $D[Z] \cap B$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . If  $Q$  is a  $(u, z)$ -dipath, then  $B$  contains arcs  $su$  and  $zw$  for some  $w \in W$ . Then the digraph  $B'$  obtained from  $B$  by replacing  $suQzw$  by the dipath  $stw$  is an  $X$ -bipod in  $D'$ . Indeed, if  $Q$  was a subpath of  $C$ , then the directed cycle in  $B'$  has length at least 3, as it contains the three vertices  $s, t$  and  $w$ .  $\diamond$

Each time we do a recursive call, the number of vertices decreases. So we do at most  $n$  of them. Between two recursive calls, we search for a directed cycle of length at least 3 and run a Menger algorithm. Both can be done in linear time. So  $\text{bipod}$  runs in  $O(n(n+m))$  time.  $\square$

With Lemma 2.43 in hands, we now deduce Theorem 2.41.

*Proof of Theorem 2.41.* Let  $v$  be a vertex of  $D$ . Let us describe an algorithm that decides whether there is a  $W'_2$ -subdivision with centre  $v$  in  $D$ . Observe that such a subdivision  $S$  is the union of a directed cycle  $C$  of length at least 3, and two internally disjoint  $(v, C)$ -dipaths  $P_1, P_2$  with distinct terminal vertices  $y_1, y_2$ . Since it is strong, the cycle  $C$  is contained in some strong component  $\Gamma$  of  $D - v$ . For  $i = 1, 2$  let  $x_i$  be the first vertex of  $P_i$  in  $\Gamma$ . Set  $X = \{x_1, x_2\}$ . Then the paths  $P_i[x_i, y_i]$ ,  $i = 1, 2$ , and  $C$  form an  $X$ -bipod in  $\Gamma$ , and the  $P_i[v, x_i]$ ,  $i = 1, 2$ , are independent  $(v, X)$ -dipaths in  $D - (\Gamma \setminus X)$ . Hence for finding a  $W'_2$ -subdivision with centre  $v$ , the following procedure checks whether there is a set  $X$  as above. First, we compute the strong components of  $D - v$ . Next, for every subset  $X$  of two vertices in the same strong component  $\Gamma$ , we run a Menger algorithm to check whether there are two independent  $(v, X)$ -dipaths in  $D - (\Gamma \setminus X)$ . If yes, we check using `bipod` whether there is an  $X$ -bipod in  $\Gamma$ . If yes again, then we clearly have a  $W'_2$ -subdivision with centre  $v$ , and we return ‘yes’. Otherwise, there is no such subdivision, and we proceed to the next pair.

There are at most  $n^2$  possible pairs  $X$ . And for each pair we run a Menger algorithm in  $O(n + m)$  time and possibly `bipod` in  $O(n(n + m))$  time. Hence our algorithm decides whether there is  $W'_2$ -subdivision with centre  $v$  in  $D$  in  $O(n^3(n + m))$  time.

To solve  $W'_2$ -subdivision, we check for every vertex  $v$  in turn if there is a  $W'_2$ -subdivision with centre  $v$ . As we examine at most  $n$  vertices, this algorithm runs in  $O(n^4(n + m))$  time.  $\square$

We also shall need the following two lemmas.

**Lemma 2.44** (Havet, M. and Mohar). *Let  $D$  be a digraph,  $C$  a directed cycle in  $D$ , and  $x$  a vertex in  $V(D) - V(C)$ . If there are two  $(x, C)$ -dipaths  $P_1$  and  $P_2$  and a  $(C, x)$ -dipath  $Q$  such that  $s(Q)$ ,  $t(P_1)$  and  $t(P_2)$  are distinct, then  $D$  contains an  $ST_4$ -subdivision.*

*Proof.* Assume first that  $P_1$  and  $P_2$  are independent. If  $Q \cap (P_1^\circ \cup P_2^\circ) = \emptyset$ , then  $C \cup P_1 \cup P_2 \cup Q$  is an  $ST_4$ -subdivision, for instance with  $t(P_1)$ ,  $t(P_2)$ ,  $s(Q)$  and  $x$  as  $a, b, c, d$ -vertex, respectively. If  $Q$  intersects  $P_1^\circ \cup P_2^\circ$ , then without loss of generality, we may assume that the first vertex  $y$  along  $Q$  in  $P_1^\circ \cup P_2^\circ$  is on  $P_1^\circ$ . Let  $z_2$  be the first vertex in  $V(Q) \cap V(P_2)$  along  $Q$ . Such a vertex exists because  $x \in V(Q) \cap V(P_2)$ . Let  $z_1$  be the last vertex on  $Q[y, z_2]$  which is on  $P_1[y, t(P_1)]$ . Now  $Q[z_1, z_2] \cup P_1[y, t(P_1)] \cup Q[s(Q), y] \cup P_2[z_2, t(P_2)] \cup C$  is an  $ST_4$ -subdivision, for instance with  $t(P_2)$ ,  $t(P_1)$ ,  $s(Q)$  and  $z_1$  as  $a, b, c, d$ -vertex, respectively.

Assume now that  $P_1$  and  $P_2$  are not independent. Let  $x'$  be the last vertex in  $P_1 \cap P_2$  along  $P_1$ . Set  $P'_1 = P_1[x', t(P_1)]$ ,  $P'_2 = P_2[x', t(P_2)]$ , and let  $Q'$  be the  $(s(Q), x')$ -dipath contained in the walk  $QP_2[x, x']$ . Then  $P'_1$  and  $P'_2$  are independent  $(x', C)$ -dipaths. Hence by the previous case,  $D$  contains an  $ST_4$ -subdivision.  $\square$

**Lemma 2.45** (Havet, M. and Mohar). *Let  $D$  be a strong digraph,  $C$  a directed cycle in  $D$ ,  $x$  a vertex in  $V(D - C)$ . If there are three  $(x, C)$ -dipaths with distinct terminal vertices, then  $D$  contains an  $ST_4$ -subdivision.*

*Subproof.* Suppose that there are three  $(x, C)$ -dipaths  $P_1, P_2, P_3$  such that  $t(P_1)$ ,  $t(P_2)$ , and  $t(P_3)$  are distinct. Since  $D$  is strong, there is a  $(C, x)$ -dipath  $Q$ . Without loss of generality, we may assume that  $s(Q) \notin \{t(P_1), t(P_2)\}$ . Thus by Lemma 2.44,  $D$  contains an  $ST_4$ -subdivision.  $\diamond$

**Theorem 2.46** (Havet, M. and Mohar).  *$ST_4$ -SUBDIVISION can be solved in  $O(n^5 \cdot (n + m))$  time.*

*Proof.* Since  $ST_4$  is strong, its subdivisions are also strong. So we only need to prove the result for a strong input digraph  $D$ ; if the digraph is non-strong, it suffices to check whether one of its strong components contains an  $ST_4$ -subdivision.

We shall describe a procedure  $ST_4$ -SUBDIVISION( $D, d'$ ), that, given a strong digraph  $D$  and a vertex  $d'$ , returns ‘no’ only if there is no  $ST_4$ -subdivision in  $D$  with  $d$ -vertex  $d'$  and returns ‘yes’ when it finds an  $ST_4$ -subdivision (not necessarily with  $d$ -vertex  $d'$ ). Running this procedure for every vertex  $d'$  yields an algorithm to decide whether  $D$  contains an  $ST_4$ -subdivision; in addition, the algorithm runs in  $O(n^5 \cdot (n + m))$  time, because the procedure  $ST_4$ -SUBDIVISION( $D, d'$ ) only needs  $O(n^4 \cdot (n + m))$  time.

First, we check whether  $d'$  is the centre of a  $W'_2$ -subdivision. This can be done in  $O(n^3 \cdot (n + m))$  time, according to Theorem 2.41. If not, then we return ‘no’ since every  $ST_4$ -subdivision with  $d$ -vertex  $d'$  contains a  $W'_2$ -subdivision with centre  $d'$ .

If there is a  $W'_2$ -subdivision with centre  $d'$ , let us denote by  $C$  its directed cycle, and by  $P_1$  and  $P_2$  the two  $(d', C)$ -dipaths in it. For  $i = 1, 2$ , let  $x_i$  be the terminal vertex of  $P_i$ .

Let  $S^-$  and  $S^+$  be the in-section and out-section, respectively, of  $d'$  in  $D - \{x_1, x_2\}$ . We compute  $S^-$  and  $S^+$ . If  $S^-$  contains a vertex in  $V(C) - \{x_1, x_2\}$ , then there is a  $(C, d')$ -dipath  $Q$  with initial vertex  $x_3 \notin \{x_1, x_2\}$ . So, by Lemma 2.44, there is an  $ST_4$ -subdivision in  $D$ , and we return ‘yes’. Similarly, because of Lemma 2.45, we return ‘yes’ if  $S^+$  contains a vertex in  $V(C) - \{x_1, x_2\}$ .

Assume now that  $(S^- \cup S^+) \cap (V(C) - \{x_1, x_2\}) = \emptyset$ . By the definition of out-section, no arc is leaving  $S^+$  in  $D - \{x_1, x_2\}$ , so in  $D$  every arc leaving  $S^+$  has its head in  $\{x_1, x_2\}$ . Similarly, all arcs entering  $S^-$  have tail in  $\{x_1, x_2\}$ . Moreover, because  $D$  is strong, for every vertex  $s \in S^+$ , there is an  $(s, \{x_1, x_2\})$ -dipath in  $D[S^+ \cup \{x_1, x_2\}]$ .

Since  $D$  is strong, there is a directed  $(C, d')$ -dipath in  $D$ . Its first arc goes from  $\{x_1, x_2\}$  to  $S^-$ . Hence at least one vertex of  $\{x_1, x_2\}$  has an out-neighbour in  $S^-$ .

**Claim 10.** *Suppose both  $x_1$  and  $x_2$  have an out-neighbour in  $S^-$ . If there is a  $(C, S^+)$ -dipath  $R$  with  $s(R) \notin \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.*

*Subproof.* There is a  $(t(R), \{x_1, x_2\})$ -dipath  $P$  with internal vertices in  $S^+$ . Without loss of generality, we may assume that  $t(P) = x_1$ . Since  $x_2$  has an out-neighbour in  $S^-$ , there is an  $(x_2, t(R))$ -dipath  $Q$  whose internal vertices are in  $D - C$ . Hence by the directional dual of Lemma 2.44 ( $ST_4$  is isomorphic to its converse),  $D$  contains an  $ST_4$ -subdivision.  $\diamond$



Each  $x_i$  has an in-neighbour in  $P_i$ , and so an in-neighbour in  $S^+$ . Hence a similar reasoning as the proof of Claim 10 gives the following.

**Claim 11.** *If there is an  $(S^-, C)$ -dipath with terminal vertex  $x_3 \notin \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.*

For  $i = 1, 2$ , let  $S_i^+$  be the set of vertices  $s$  of  $S^+$  for which there is an  $(s, x_i)$ -dipath with internal vertices in  $V(D - C)$ . In the very same way as Claim 10, one can prove the following claim.

**Claim 12.** *Suppose  $x_i$  has no out-neighbour in  $S^-$ . If there is a  $(C, S_i^+)$ -dipath with initial vertex  $x_3 \notin \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.*

Case 1: Assume first that both  $x_1$  and  $x_2$  have an out-neighbour in  $S^-$ .

Let  $T^+$  be the out-section of  $S^-$  in  $D - \{x_1, x_2\}$ ,  $T^-$  the in-section of  $S^+$  in  $D - \{x_1, x_2\}$ . The definition of  $T$  implies the following property:

(T<sub>1</sub>) If  $u \in V(D) - (T \cup \{x_1, x_2\})$  and  $Q$  is a  $(u, d')$ -path in  $D$  with at most two blocks, then  $Q$  contains a vertex in  $\{x_1, x_2\}$ .

Now, we compute  $T = T^- \cup T^+$ . If  $T$  contains a vertex of  $V(C) - \{x_1, x_2\}$ , then we return ‘yes’, since  $D$  contains an  $ST_4$ -subdivision by Claim 10 or 11. If not, then  $T \cap (V(C) - \{x_1, x_2\}) = \emptyset$ . Let  $D'$  be the digraph obtained from  $D[T \cup \{x_1, x_2\}]$  by adding the arcs  $x_1x_2$  and  $x_2x_1$  if they were not in  $A(D)$ . Observe that  $D'$  has fewer vertices than  $D$ , because the vertices of  $V(C) - \{x_1, x_2\}$  are not in  $V(D')$  and this set is not empty because  $C$  has length at least 3. We then return  $ST_4$ -Subdivision( $D', d'$ ). The validity of this recursive call is established by the following claim.

**Claim 13.**  *$D$  contains an  $ST_4$ -subdivision with  $d$ -vertex  $d'$  if and only if  $D'$  does.*

*Subproof.* From every  $ST_4$ -subdivision in  $D'$  with  $d$ -vertex  $d'$ , one can obtain an  $ST_4$ -subdivision in  $D$  with  $d$ -vertex  $d'$  by replacing the arc  $x_1x_2$  (resp.  $x_2x_1$ ) by  $C[x_1, x_2]$  (resp.  $C[x_2, x_1]$ ).

Assume now that  $D$  contains an  $ST_4$ -subdivision  $S$  with  $d$ -vertex  $d'$ . Let  $a', b', c'$  be the vertices in  $S$  corresponding to  $a, b, c$ , respectively.

Our first goal is to prove that  $a', b', c' \in V(D')$ . Let  $u \in \{a', b', c'\}$  be one of these three vertices. Note that there are three internally disjoint paths in  $S$  joining  $u$  with  $d'$ , and each of these paths has at most two blocks. If  $u \notin V(D')$ , then Property (T<sub>1</sub>) stated above implies that each of these paths contains  $x_1$  or  $x_2$  as one of its internal vertices. Since the three paths are internally disjoint, this is not possible, and we conclude that  $u \in V(D')$ .

Hence,  $a', b', c', d'$  all belong to  $V(D')$ . Therefore, the intersection of  $S$  with  $V(D) - T$  is a dipath  $P$  whose initial vertex is dominated by  $x \in \{x_1, x_2\}$  and whose terminal vertex dominates the vertex  $x'$  of  $\{x_1, x_2\} - \{x\}$ . Hence,  $D'$  contains the  $ST_4$ -subdivision obtained from  $S$  by replacing  $xPx'$  by  $xx'$ .  $\diamond$

Case 2: Assume that one vertex in  $\{x_1, x_2\}$ , say  $x_1$ , has no out-neighbour in  $S^-$ .

Let  $T^+$  be the out-section of  $S^-$  in  $D - \{x_1, x_2\}$ ,  $T_1^-$  the in-section of  $S_1^+$  in  $D - \{x_1, x_2\}$  and  $T = T^+ \cup T_1^-$ . Observe that  $S^+ \subseteq T$  because  $d' \in S^-$ . The definition of  $T$  implies the following property:

(T<sub>2</sub>) If  $u \in V(D) - (T \cup \{x_1, x_2\})$  and  $Q$  is a  $(u, d')$ -path with at most two blocks, then either  $Q$  contains a vertex in  $\{x_1, x_2\}$ , or  $Q$  has two blocks and there is a vertex  $v \in S_2^+ - (S_1^+ \cup S^-)$  such that  $Q$  is composed of a  $(u, v)$ -dipath  $R_1$  and a  $(d', v)$ -dipath  $R_2$ .

An  $ST_4$ -subdivision  $S$  is *special* if its  $d$ -vertex is  $d'$ , its  $c$ -vertex is  $x_2$ , its  $a$ -vertex is not in  $T \cup \{x_1, x_2\}$ , and  $x_1 \in V(S)$ .

We check if  $D$  contains a special  $ST_4$ -subdivision. To do so, we check for every vertex  $a'$  in  $V(D) - (T \cup \{x_1, x_2\})$ , if there are two independent  $(\{x_1, x_2\}, a')$ -dipaths  $Q_1$  and  $Q_2$  in  $D$  and an  $(a', S^+)$ -dipath  $R$  in  $D - \{x_1, x_2\}$ . If we find a vertex  $a' \in V(D) - (T \cup \{x_1, x_2\})$  such that three such dipaths exist, we return ‘yes’. This is valid by the following claim.

**Claim 14.** *Let  $a' \in V(D) - (T \cup \{x_1, x_2\})$ . If there are two independent  $(\{x_1, x_2\}, a')$ -dipaths  $Q_1$  and  $Q_2$  in  $D$  and an  $(a', S^+)$ -dipath  $R$  in  $D - \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.*

*Subproof.* The vertex  $t(R)$  is in  $S_2^+ - S_1^+$  because  $a' \notin T_1^-$ . Thus, there is a  $(t(R), x_2)$ -dipath  $R_1$  with internal vertices in  $S^+$ . Let  $y_2$  be an out-neighbour of  $x_2$  in  $S^-$ . Since  $a' \notin S^-$ , the vertex  $y_2$  is not on  $R_1$ . By definition of  $S^+$  and  $S^-$ , there is a  $(y_2, R_1)$ -dipath  $R_2$  in  $D[S^+ \cup S^-]$ .

Let  $C'$  be the directed cycle  $x_2 y_2 \cup R_2 \cup R_1 [t(R_2), x_2]$ . Since  $y_2 \in S^-$ , there is a directed  $(y_2, x_1)$ -dipath  $R_3$  in  $D[S^- \cup S^+ \cup \{x_1\}]$ . This dipath does not intersect  $R_1$  because  $V(R_1) - \{x_2\} \subseteq S_2^+ - S_1^+$ . Let  $z_2$  be the last vertex along  $R_3$  that lies in  $C'$ . The three vertices  $x_2, z_2, t(R_2)$  are distinct. Moreover, the two dipaths  $Q_1$  and  $Q_2$  do not intersect  $C'$  for otherwise there would be a  $(y_2, a')$ -dipath in  $D - \{x_1, x_2\}$  and  $a'$  would be in  $T^+$ . Thus  $R_3 \cup Q_1$  contains a  $(y_2, a')$ -dipath  $R_3^*$  which is a  $(C', a')$ -dipath. Hence, we have two  $(C', a')$ -dipaths  $R_3^*$  and  $Q_2$  and the  $(a', C')$ -dipath  $R'$  contained in  $R \cup R_1 [s(R_1), t(R_2)]$  whose vertices  $s(R_3^*), s(Q_2) = x_2$  and  $t(R')$  on  $C'$  are pairwise distinct. Hence, by Lemma 2.44,  $D$  contains an  $ST_4$ -subdivision.  $\diamond$

If for every  $a'$  in  $V(D) - (T \cup \{x_1, x_2\})$ , three dipaths  $Q_1, Q_2, R$  as used above do not exist, then  $D$  has no special  $ST_4$ -subdivision, which we will assume henceforth. Let  $D'$  be the digraph obtained from  $D[T \cup \{x_1, x_2\}]$  by adding the arcs  $x_1 x_2$  and  $x_2 x_1$  if they were not in  $A(D)$ . Observe that  $D'$  has fewer vertices than  $D$ , because the vertices of  $V(C) - \{x_1, x_2\}$  are not in  $V(D')$ . We then return  $ST_4\text{-Subdivision}(D', d')$ . The validity of this recursive call is established by the following claim.

**Claim 15.**  *$D$  contains an  $ST_4$ -subdivision with  $d$ -vertex  $d'$  if and only if  $D'$  does.*

*Subproof.* From every  $ST_4$ -subdivision  $S'$  with  $d$ -vertex  $d'$  in  $D'$ , one can obtain an  $ST_4$ -subdivision with  $d$ -vertex  $d'$  in  $D$  by replacing the arc  $x_1x_2$  (resp.  $x_2x_1$ ) by  $C[x_1, x_2]$  (resp.  $C[x_2, x_1]$ ).

Assume now that  $D$  contains an  $ST_4$ -subdivision  $S$  with  $d$ -vertex  $d'$ . Let  $a'$ ,  $b'$ , and  $c'$  be the vertices in  $S$  corresponding to  $a$ ,  $b$ , and  $c$ , respectively. Each arc in  $ST_4$  corresponds to a dipath in  $S$ . We will denote these dipaths by  $S[a', b']$ ,  $S[b', c']$ , etc.

Observe that in  $S$ , there are three internally disjoint directed paths (in both directions) between  $b'$  and  $d'$ . So  $b' \in V(D')$ , because directed paths between  $V(D) - V(D')$  and  $d'$  must go through  $\{x_1, x_2\}$  by Property (T<sub>2</sub>).

Next, we claim that  $a' \in V(D')$ . Suppose for a contradiction that  $a' \notin V(D')$ . Then both paths  $S[d', a']$  and  $S[c', a'] \cup S[c', d']$  must go through  $\{x_1, x_2\}$  by Property (T<sub>2</sub>). The path  $S[a', b'] \cup S[d', b']$  is thus disjoint from  $\{x_1, x_2\}$ , and by (T<sub>2</sub>) we have that  $b' \in S_2^+ - (S_1^+ \cup S^-)$ . The path  $S[b', c'] \cup S[c', d']$  must go through  $x_2$  since  $b' \notin S^- \cup S_1^+$ . Thus,  $x_2$  lies on  $S[c', d']$ . Since there is no special  $ST_4$ -subdivision in  $D$ ,  $c' \neq x_2$ . Hence,  $S[c', a']$  does not meet  $\{x_1, x_2\}$ , and the path  $S[d', b'] \cup S[b', c'] \cup S[c', a']$  shows that  $a' \in S^+$ , a contradiction.

Let us prove that  $c' \in V(D')$ . Suppose for a contradiction that  $c' \notin V(D')$ . Then  $c' \notin \{x_1, x_2\}$  and both,  $S[d', b'] \cup S[b', c']$  and  $S[c', d']$  must go through  $\{x_1, x_2\}$ . Moreover,  $x_2$  is in  $S[c', d']$  because  $x_1$  has no out-neighbour in  $S^-$ . Since  $x_2$  is also on  $S[d', a'] \cup S[a', b'] \cup S[b', c']$ , we conclude that  $x_2 \in S[b', c']$ . Now, the path  $S[d', a'] \cup S[c', a']$  gives a contradiction to the property (T<sub>2</sub>).

We have shown that  $\{a', b', c', d'\} \subseteq V(D')$ . Therefore, the part of  $S$  outside  $D'$  is a directed path  $P$  whose initial vertex is dominated by  $x \in \{x_1, x_2\}$  and whose terminal vertex dominates the vertex  $x'$  of  $\{x_1, x_2\} - \{x\}$ . Hence,  $D'$  contains the  $ST_4$ -subdivision obtained from  $S$  by replacing  $xPx'$  by  $xx'$ .  $\diamond$

Each time we do a recursive call, the number of vertices decreases. So we do at most  $n$  of them. Between two recursive calls, we search for a  $W'_2$ -subdivision, which can be done in  $O(n^3 \cdot (n+m))$  by Theorem 2.41, and we compute some out-sections and in-section, which can be done in linear time. The only part that may need more time is in Case 2, when we check for every  $a'$  in  $V(D) - (T \cup \{x_1, x_2\})$  if  $D$  contains a special  $ST_4$ -subdivision. Each such test needs linear time by Claim 14. During this procedure, we either discover an  $ST_4$ -subdivision or not. If yes, we have spent at most  $O(n \cdot (n+m))$  time for completing this task. Otherwise we spend linear time per vertex  $a'$ , which is henceforth omitted when we proceed with the recursive call. This shows that  $ST_4$ -Subdivision runs in  $O(n^4 \cdot (n+m))$  time.  $\square$

For any non-negative integer  $p$ , let  $TT_4(p)$  be the digraph obtained from  $TT_4$  with source  $u$  and sink  $v$  by adding  $p$  new vertices dominated by  $u$  and dominating  $v$ . In particular,  $TT_4(0) = TT_4$ . We denote by  $TT_4^*(p)$ , the digraph obtained from  $TT_4(p)$  by deleting the arc from its source  $u$  to its sink  $v$ . For simplicity, we abbreviate  $TT_4^*(0)$  in  $TT_4^*$ .

We need the following definitions. Let  $X$  be a set of vertices in a digraph  $D$ .

The *out-section* generated by  $X$  in  $D$  is the set of vertices  $y$  to which there exists a directed path (possibly restricted to a single vertex) from  $x \in X$ ; we denote this set by  $S_D^+(X)$ . For simplicity, we write  $S_D^+(x)$  instead of  $S_D^+(\{x\})$ . The dual notion, the *in-section*, is denoted by  $S_D^-(X)$ . Note that the out-section and the in-section of a set may be found in linear time by any tree-search algorithm.

**Theorem 2.47** (Bang-Jensen, Havet and M. [3]). *For every non-negative integer  $p$ , one can solve  $TT_4(p)$ -SUBDIVISION in  $O(n^3 \cdot (n + m))$ -time.*

*Proof.* Let  $D$  be a digraph and let  $u$  and  $v$  be two distinct vertices of  $D$ . We shall describe a  $O(n \cdot (n + m))$ -time algorithm for finding a  $TT_4(p)$ -subdivision in  $D$  with source  $u$  and sink  $v$ , if one exists.

Observe that all vertices in such a subdivision are in  $S_D^+(u) \cap S_D^-(v)$ , hence we can restrict our search to the digraph  $D'$  induced by this set.

Then, using a maximum flow algorithm, we can find in  $D'$  a set of internally disjoint directed  $(u, v)$ -paths of maximum size in  $O(n \cdot (n + m))$ -time. Let  $(P_1, \dots, P_k)$  denote this set. If  $k < p + 3$ , then return ‘no’, because in any  $TT_4(p)$ -subdivision with source  $u$  and sink  $v$ , there are  $p + 3$  internally disjoint directed  $(u, v)$ -paths. Hence, we now assume that  $k \geq 3$ .

For  $1 \leq i \leq k$ , set  $Q_i = P_i - \{u, v\}$ , and set  $H = D' - \{u, v\}$ . For every vertex  $x$  in  $V(H)$ , we compute  $S(x) = S_H^-(x) \cup S_H^+(x)$ , and deduce  $I(x) = \{i \mid V(Q_i) \cap S(x) \neq \emptyset\}$ . If there exists  $x$ , such that  $|I(x)| \geq 2$ , then return ‘yes’. Otherwise return ‘no’.

The validity of this algorithm is proved by Claim 17.

**Claim 16.** *For all  $x \in V(H)$ ,  $I(x) \neq \emptyset$ .*

*Subproof.* In  $D'$ , there are directed  $(u, x)$ - and  $(x, v)$ -paths, whose concatenation contains a directed  $(u, v)$ -path  $R$ . Since  $(P_1, \dots, P_k)$  is a set of internally disjoint directed  $(u, v)$ -paths of maximum size,  $R - \{u, v\}$  must intersect one of the  $Q_i$ 's, say  $Q_{i_0}$ . By definition,  $V(R) - \{u, v\} \subseteq S(x)$ , so  $i_0 \in I(x)$ .  $\diamond$

**Claim 17.**  *$D'$  contains a  $TT_4(p)$ -subdivision with source  $u$  and sink  $v$  if and only if there exists  $x \in V(H)$  such that  $|I(x)| \geq 2$ .*

*Subproof.* Assume that  $|I(x)| \geq 2$ . Without loss of generality,  $\{1, 2\} \subset I(x)$ . We shall prove that  $D'$  contains a  $TT_4(p)$ -subdivision with source  $u$  and sink  $v$ .

- Suppose first that  $S_H^-(x) \cap Q_1 \neq \emptyset$  and  $S_H^+(x) \cap Q_2 \neq \emptyset$ . Then there is a directed  $(Q_1, x)$ -path and a directed  $(x, Q_2)$ -path whose concatenation contains a directed  $(Q_1, Q_2)$ -path  $R$ . Let  $y$  be the first vertex on  $R$  in  $\bigcup_{i=2}^k Q_i$ . Free to swap the names of  $Q_2$  and the path  $Q_i$  containing  $y$  and taking the subpath of  $R$  from its origin to  $y$  instead of  $R$ , we may assume that  $y$  is the last vertex of  $R$ . Now the union of  $P_1, \dots, P_{p+3}$ , and  $R$  form a  $TT_4(p)$ -subdivision.
- If  $S_H^-(x) \cap Q_2 \neq \emptyset$  and  $S_H^+(x) \cap Q_1 \neq \emptyset$ , the proof is similar to the previous case.

- Suppose now that  $S_H^+(x) \cap Q_1 \neq \emptyset$  and  $S_H^+(x) \cap Q_2 \neq \emptyset$ . We may assume that  $S_H^-(x) \cap \bigcup_{i=1}^k Q_i = \emptyset$ , otherwise we are in one of the previous cases, and we get the result. Let  $R$  be a shortest  $(u, x)$ -path in  $D'$ . Then every vertex in  $R - u$  is a vertex of  $H - \bigcup_{i=1}^k Q_i$ .

Let  $S_1$  be a shortest directed  $(x, Q_1)$ -path and  $S_2$  be a shortest directed  $(x, Q_2)$ -path. For  $i = 1, 2$ , let  $z_i$  be the terminus of  $S_i$ . We may assume that all the internal vertices of  $S_1$  and  $S_2$  are in  $H - \bigcup_{i=1}^k Q_i$  for otherwise one vertex  $z$  among  $z_1$  and  $z_2$  satisfies the condition of one of the previous cases (up to a permutation of the labels). Then the union of paths  $P_2, \dots, P_{p+3}$ ,  $R$ ,  $S_1$ ,  $S_2$  and  $P_1[z_1, v]$  form a  $TT_4(p)$ -subdivision.

- If  $S_H^-(x) \cap Q_1 \neq \emptyset$  and  $S_H^-(x) \cap Q_2 \neq \emptyset$ , the proof is similar to the previous case by directional symmetry.

Assume now that  $|I(x)| < 2$  for all  $x \in V(H)$ . Then, by Claim 16,  $|I(x)| = 1$  for all  $x \in V(H)$ . For  $1 \leq i \leq k$ , let  $V_i = \{x \mid I(x) = \{i\}\}$ . Then  $(V_1, \dots, V_k)$  is a partition of  $V(H)$ . Moreover, by definition, there is no arc between two distinct parts of this partition. In addition, in  $D' \langle X_i \cup \{u, v\} \rangle$ , there cannot be two internally disjoint directed  $(u, v)$ -paths, for otherwise it would contradict the maximality of  $(P_1, \dots, P_k)$ . Hence,  $D'$  contains no  $TT_4^*$ -subdivision, and so no  $TT_4(p)$ -subdivision.  $\diamond$

This finishes the proof of Theorem 2.47.  $\square$

**Corollary 2.48** (Bang-Jensen, Havet and M. [3]). *For any non-negative integer  $p$ , the  $TT_4^*(p)$ -SUBDIVISION problem can be solved in  $O(n^3 \cdot (n + m))$ .*

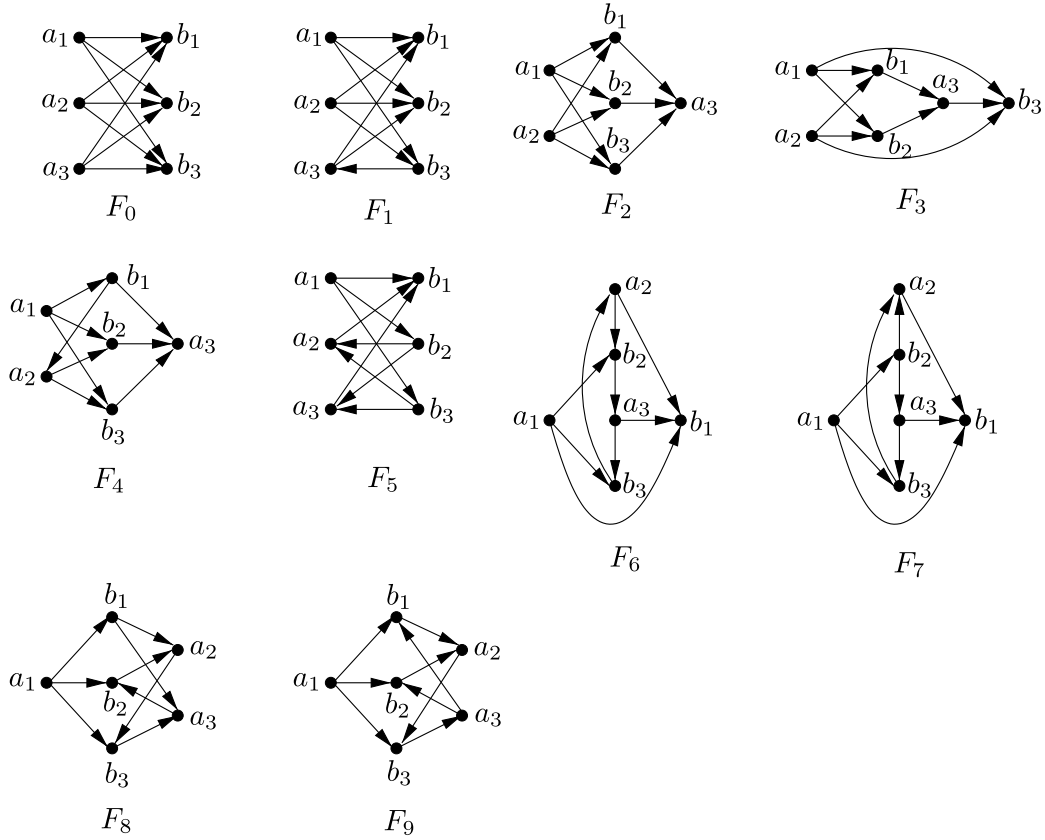
*Proof.* Observe that a graph  $D$  contains a  $TT_4^*(p)$ -subdivision with source  $u$  and sink  $v$ , if and only if the graph  $D \cup \{uv\}$  contains a  $TT_4(p)$ -subdivision. Hence by just adding the arc  $uv$  to  $D$  if it does not exist in the above algorithm, we obtain a polynomial-time algorithm for  $TT_4^*(p)$ -SUBDIVISION.  $\square$

## 2.11 $K_{3,3}$

A natural way to start trying to prove Conjecture 1.16 is look at what happens for subdivisions of orientations of  $K_5$  and  $K_{3,3}$ . The first one is NP-complete in any case by Corollary 1.12. We prove next that the subdivision of any orientation of  $K_{3,3}$  with at least one big vertex (Figure 2.5) is hard.

**Proposition 2.49** (Bang-Jensen, Havet and M). *If  $F$  is an orientation of a  $K_{3,3}$  with at least one big vertex, then  $F$ -SUBDIVISION is NP-complete.*

*Proof.* An orientation of  $K_{3,3}$  having at least one big vertex is one of the  $F_i$ ,  $0 \leq i \leq 9$ , or the converse of one of them.  $F_0$  and  $F_1$  are hard by Theorem 1.10. In each other case, the problem is proved to be NP-complete by reduction from RESTRICTED 2-LINKAGE. Let  $D, x_1, x_2, y_1$  and  $y_2$  be an instance of this problem. We construct a

Figure 2.5: Orientations of  $K_{3,3}$ .

digraph  $D_i$  by placing  $D$  on two arcs  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  of  $F_i$  (to be specified later), that is, by taking the disjoint union of  $D$  and  $F_i$ , removing the arcs  $e_1, e_2$  of  $F_i$  and adding the arcs  $u_1x_1, y_1v_1, u_2x_2$  and  $y_2v_2$ . We then show that  $D_i$  contains an  $F_i$ -subdivision if and only if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ . This implies that  $F_i$ -SUBDIVISION is NP-complete.

Clearly, by construction of  $D_i$ , if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ , then  $D_i$  contains an  $F_i$ -subdivision. We now prove the converse for each  $2 \leq i \leq 9$ . In each case, we shall assume that  $D_i$  contains an  $F_i$ -subdivision  $S$ , and for  $j = 1, 2, 3$ , we shall denote by  $a'_j, b'_j$  the  $a_j, b_j$ -vertices of  $S$ , respectively.

- $i = 2$ : We choose  $e_1 = a_1b_1$  and  $e_2 = a_2b_2$ . Since  $D$  contains no big vertices, we have  $a'_3 = a_3$  and  $\{a'_1, a'_2\} = \{a_1, a_2\}$ . By symmetry of  $F_2$ , we may assume that  $a'_1 = a_1$  and  $a'_2 = a_2$ . Since  $d_S^+(a'_1) = d_S^+(a'_2) = d_S^-(a'_3) = 3$ , all arcs leaving  $a_1, a_2$  and all arcs entering  $a_3$  are in  $A(S)$ . Thus  $\{b_1, b_2, b_3\} \in \{b'_1, b'_2, b'_3\}$ . So, in  $S$  there are disjoint  $(a_1, b_1)$ -,  $(a_2, b_2)$ -dipaths, and these two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
- $i = 3$ : We choose  $e_1 = a_1b_1$  and  $e_2 = a_1b_3$ . Since  $D$  contains no big vertices, we have  $b'_3 = b_3$  and  $\{a'_1, a'_2\} = \{a_1, a_2\}$ . By symmetry of  $F_3$ , we may assume that

- $a'_1 = a_1$  and  $a'_2 = a_2$ . Since  $d_S^+(a'_1) = d_S^-(a'_1) = d_S^-(b'_3) = 3$ , all arcs leaving  $a_1, a_2$  and all arcs entering  $b_3$  are in  $A(S)$ . Thus  $\{b_2\} \in \{b'_2, b'_3\}$ . Without loss of generality, we may assume  $b_2 = b'_2$ . Since  $d_S^+(b'_2) = 1$ , the arc  $b_2a_3$  is an arc of  $S$ . Hence  $a_3$  is the unique internal vertex in the  $(b'_2, b'_3)$ -dipath in  $S$ , so  $a_3 = a'_3$ . Now since  $d_S^-(a'_3) = 2$ , the arc  $b_1a_3$  is an arc of  $S$ . Hence  $b_1$  is the unique internal vertex in a  $(a'_2, a'_3)$ -dipath in  $S$ , and  $b_1 \neq b'_2$ , so  $b_1 = b'_1$ . Consequently, in  $S$ , there are disjoint  $(a_1, b_1)$ -,  $(a_1, b_3)$ -dipaths, and these two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
- $i = 4$ : We choose  $e_1 = a_1b_1$  and  $e_2 = b_3a_3$ . Since  $D$  contains no big vertices, we have  $a'_1 = a_1$  and  $a'_3 = a_3$ . Since  $d_S^+(a'_1) = d_S^-(a'_3) = 3$ , all arcs leaving  $a_1$  and all arcs entering  $a_3$  are in  $A(S)$ . Hence the dipath  $(a'_1, b_2, a'_3)$  is in  $S$ , so  $b_2 \in \{b'_1, b'_2, b'_3\}$ . For degree reason, necessarily  $b_2 \in \{b'_2, b'_3\}$ . By symmetry of  $F_4$ , we may assume that  $b_2 = b'_2$ . Hence the arc  $a_2b_2 \in A(S)$  and since every vertex of  $V(S) - \{a'_1\}$  has in-degree at least 1 in  $S$ , we have  $b_1a_2 \in A(S)$ . Therefore  $d_S^+(b_1) \geq 2$ , and then  $b_1 \in a'_2, b'_1$ . But  $b_1a_3 \in A(S)$  and the edge  $a'_2a'_3$  should not exit in  $S$ . So,  $b_1 = b'_1$ . Moreover,  $a_2$  is the unique internal vertex of the  $(b'_1, b'_2)$ -dipath in  $S$ , so  $a_2 = a'_2$ . Consequently, in  $S$ , there are disjoint  $(a_1, b_1)$ -,  $(b_3, a_3)$ -dipaths, and these two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
  - $i = 5$ : We choose  $e_1 = b_2a_2$  and  $e_2 = b_3a_3$ . Since  $D$  contains no big vertices,  $a'_1 = a_1$  and  $b'_1 = b_1$ . Since  $d_S^+(a'_1) = d_S^-(b'_1) = 3$ , all arcs leaving  $a_1$  and all arcs entering  $b_1$  are in  $A(S)$ . In  $S - \{a'_1, b'_1\}$ , there are two internally disjoint  $(a'_1, b'_1)$ -dipaths  $(Q_1, Q_2)$  with  $\{b'_2, a'_2\} \in V(Q_1)$  and  $\{b'_3, a'_3\} \in V(Q_2)$  and two internally disjoint  $(a'_1, b'_1)$ -dipaths  $(R_1, R_2)$  with  $\{b'_2, a'_3\} \in V(R_1)$  and  $\{b'_3, a'_2\} \in V(R_2)$ . For one of these two pairs,  $b_2$  and  $a_2$  are in the same dipath and  $a_3$  and  $b_3$  are on the others. Hence, in  $S$ , there are disjoint  $(b_2, a_2)$ -,  $(b_3, a_3)$ -dipaths, and these two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
  - $i = 6$ : We choose  $e_1 = a_2b_2$  and  $e_2 = a_3b_3$ . Since  $D$  contains no big vertices,  $a'_1 = a_1$  and  $b'_1 = b_1$ . Since  $d_S^+(a'_1) = d_S^-(b'_1) = 3$ , all arcs leaving  $a_1$ , and all arcs entering  $b_1$  are in  $A(S)$ . Hence  $a_2$  and  $a_3$  are in  $S$  and so  $b_3a_2$  and  $b_2a_3$  are arcs in  $S$ , because all vertices of  $S$  except  $a'_1$  have in-degree at least 1. Therefore,  $(a_1, b_1)$ ,  $(a_1, b_2, a_3, b_1)$  and  $(a_1, b_3, a_1, b_1)$  are the three internally disjoint  $(a_1, b_1)$ -dipaths in  $S$ . Hence  $\{b_3a_2, b_2a_3\} = \{b'_3a'_2, b'_2a'_3\}$ . Consequently, in  $S$  there are disjoint  $(a_2, b_2)$ -,  $(a_3, b_3)$ -dipaths, and these two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
  - $i = 7$ : We choose  $e_1 = a_1b_1$  and  $e_2 = a_3b_3$ . Since  $D$  contains no big vertices, we have  $a'_1 = a_1$  and  $b'_1 = b_1$ . In  $S$ , there are three internally disjoint  $(a'_1, b'_1)$ -dipath. Therefore the arcs  $b_3a_2$  and  $b_2a_3$  are in  $S$ . Both  $(a_1, b_2, a_3, b_1)$  and  $(a_1, b_3, a_2, b_1)$  are  $(a'_1, b'_1)$ -dipaths in  $S$ . Hence one of them does not correspond to the arc  $a_1b_1$ . Suppose that  $(a_1, b_2, a_3, b_1)$  is this dipath. Then necessarily

$b_2 = b'_2$  and  $a_3 = a'_3$ , so  $b_2a_2 \in A(S)$ . Hence  $d_S^-(a_2) = 2$ , so  $a_2 = a'_2$  and consequently  $b_3 = b'_3$ . Similarly, if  $(a_1, b_3, a_2, b_1)$  does not correspond to the arc  $a_1b_1$ , then  $a_2 = a'_2, b_3 = b'_3, b_2 = b'_2$  and  $a_3 = a'_3$ . In both cases, in  $S$ , there are disjoint  $(a_1, b_1)$ - and  $(a_3, b_3)$ -dipaths, which induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

- $i = 8$ : We choose  $e_1 = b_1a_2$  and  $e_2 = a_3b_2$ . Since  $D$  contains no big vertices, we have  $a'_1 = a_1$ . Since  $d_S^+(a'_1) = 3$ , all the arcs leaving  $a_1$  are in  $A(S)$ , and  $\{b_1, b_2, b_3\} \in V(S)$ . Every vertex in  $V(S)$  has out-degree at least 1, so  $b_2a_2, a_2b_3 \in A(S)$ . Hence  $b_3 \in \{b'_2, b'_3\}$ . By symmetry of  $F_8$ , we may assume  $b_3 = b'_3$ . Now  $(a'_1, b_2, a_2, b'_3)$  is a dipath in  $S$  which correspond to the subdivision of  $(a_1, b_2, a_2, b_3)$  or  $(a_1, b_1, a_2, b_3)$ . Therefore  $a_2 = a'_2$  and  $b_2 = b'_2$  because  $d_{D_8}^+(b_2) = 1 < 2 = d_S^+(b'_1)$ . Now in  $S$  (and  $D_8$ ), the two paths corresponding to the subdivision of  $(a_1, b_1, a_2)$  and  $(b_3, a_3, b_2)$  are disjoint and induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .
- $i = 9$ : We choose  $e_1 = b_1a_2$  and  $e_2 = a_3b_2$ . Since  $D$  contains no big vertices, we have  $a'_1 = a_1$ . Since  $d_S^+(a'_1) = 3$ , all the arcs leaving  $a_1$  are in  $A(S)$ , and  $\{b_1, b_2, b_3\} \in V(S)$ . Every vertex in  $V(S)$  has out-degree at least 1, so  $b_2a_2, a_2b_3, b_3a_3 \in A(S)$ . Hence  $b_3 \in \{b'_1, b'_2, b'_3\}$ . Therefore  $a_1b_3$  is one of the arcs  $\{a'_1b'_1, a'_1b'_2, a'_1b'_3\}$  and the dipath  $(a_1, b_2, a_2, b_3)$  correspond to the subdivision one of the dipath of length 3 from  $a_1$  to  $\{b_1, b_2, b_3\}$ . Thus  $a_2 \in \{a'_2, a'_3\}$  and so for degree reasons  $a_2 = a'_2$ . It follows that  $b_2 \in \{b'_1, b'_2\}$ . By symmetry of  $F_9$ , we may assume  $b_2 = b'_2$ . Now in  $S$  (and  $D_9$ ), the two paths corresponding to the subdivision of  $(a_1, b_1, a_2)$  and  $(b_3, a_3, b_2)$  are disjoint and induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

□





# $F$ -SUBDIVISION for digraphs of order at most 4

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In this chapter, we first conclude the classification of the digraphs of order at most 3. We turn then to digraphs of order 4, which we were able to classify all except five of them (up to directional duality). These are the digraphs  $O_i$  for  $1 \leq i \leq 5$  depicted Figure 3.1.

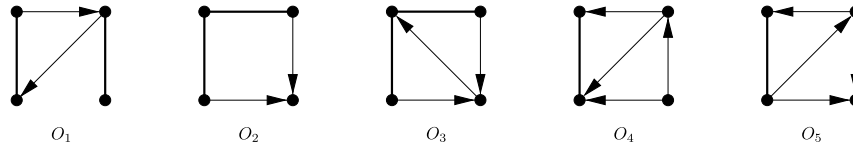


Figure 3.1: Digraphs on 4-vertices that are not known to be tractable or hard. Bold undirected edges represent directed 2-cycles.

In particular, we prove Seymour's Conjecture for digraphs of order at most 4.

In [17], the authors also checked the problem of detect an induced subdivision of an undirected graph with at most 4 vertices in a given graph  $G$ . For graphs with at most 3 vertices, the problem is polynomial-time solvable in that case. For the 12 graphs with 4 vertices, only the complexity of five of them is known, and the complexity of the others remains open.

Except for Section 1, the work in this chapter was done in cooperation with Frédéric Havet and Bojan Mohar.

## 3.1 Subdivisions of digraphs with three vertices

Let us denote by  $\vec{K}_n$  the complete digraph on  $n$  vertices, in which there is an arc  $uv$  for any two distinct vertices  $u$  and  $v$ . Let  $D_3$  be the digraph obtained from  $\vec{K}_3$  by removing an arc, and the *lollipop* the digraph  $L$  with vertex set  $\{x, y, z\}$  and arc set  $\{xy, yz, zy\}$ .

**Theorem 3.1** (Bang-Jensen, Havet and M. [3]). *Let  $F$  be a digraph on three vertices. Then  $F$ -SUBDIVISION is polynomial-time solvable unless  $F = \vec{K}_3$  in which case it is NP-complete.*

*Proof.* If  $F$  is neither  $D_3$  nor  $\vec{K}_3$  nor the lollipop (or its converse), then it is either a disjoint union of spiders, or a spindle, or a bispindle, or a windmill, and so  $F$ -

SUBDIVISION can be solved in polynomial time by virtue of the results of the previous sections. If  $F = \vec{K}_3$ , then  $F$ -SUBDIVISION is NP-complete by Corollary 1.12.

It remains to prove that the subdivision problem for  $D_3$  and the lollipop is polynomial-time solvable.

The *bulky vertex* of a  $D_3$ -subdivision  $S$  is the unique vertex of  $S$  with degree 4. We now give a procedure that given a vertex  $v$ , two of its out-neighbours  $s_1, s_2$  and two of its in-neighbours  $t_1, t_2$  check if there is a  $D_3$ -subdivision  $S$  in which  $v$  is the bulky vertex and  $\{vs_1, vs_2, t_1v, t_2v\} \in A(S)$ . Such a subdivision will be called *suitable*.

Applying a Menger algorithm, check if in  $D - v$  there are two disjoint directed paths  $P_1$  and  $P_2$  from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$ . If not, then  $D$  certainly does not contain any suitable  $D_3$ -subdivision. If yes, then check if there is a directed path  $Q$  from  $P_1$  to  $P_2$  or from  $P_2$  to  $P_1$ . If such a  $Q$  exists, then  $P_1, P_2, Q$  together with  $v$  and the arcs  $vs_1, vs_2, t_1v, t_2v$  form a suitable  $D_3$ -subdivision. If not, then no suitable  $D_3$ -subdivision using the chosen arcs exists, because there is no vertex  $s \in \{s_1, s_2\}$  such that there exists in  $D - v$  both a directed  $(s, t_1)$ -path and a directed  $(s, t_2)$ -path.

A  $D_3$ -subdivision is clearly suitable with respect to its bulky vertex and its neighbours in this subdivision. Hence checking if there is a suitable  $D_3$ -subdivision for every 5-tuple  $(v, s_1, s_2, t_1, t_2)$  such that  $s_1, s_2$  are out-neighbours of  $v$  and  $t_1, t_2$  are in-neighbours yields a polynomial-time algorithm to decide if there is a  $D_3$ -subdivision in a digraph.

Consider now the lollipop. If  $D$  contains a strong component of cyclomatic number greater than 1, then it contains a lollipop. Indeed, the smallest directed cycle  $C$  in the component is induced and is not the whole strong component. Hence there must be a vertex  $v$  dominating a vertex of  $C$  thus forming a lollipop-subdivision. If not, then all the strong components are cycles. Thus  $D$  contains a lollipop if and only if one of its component is a directed cycle and is not an initial strong component (i.e some arc is entering it). All this can be checked in linear time.  $\square$

## 3.2 Oriented graphs of order 4

The aim of this section is to prove that every oriented graph of order 4 is tractable.

**Theorem 3.2** (Havet, M. and Mohar). *If  $D$  is an oriented graph of order 4, then  $D$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* If  $D$  is a tournament, then it is either the transitive tournament  $TT_4$ , or the wheel  $W_3$ , or the converse of  $W_3$ , or the strong tournament  $ST_4$ , which in any case we prove to be tractable in Chapter 2.

If  $D$  is an orientation of  $K_4 \setminus e$ , the graph obtained from  $K_4$  by removing one edge, then  $D$  must be one of the oriented graphs depicted Figure 3.2, or the converse of one of those.  $S(1, 2, 2)$  is a spindle,  $B(1, 2; 2)$  and  $B(2, 2; 1)$  are bispindles and  $F_3$  is the 3-fan. All these digraphs have been shown tractable in the previous chapter, as well as the digraphs  $TT_4^*$  and  $W_2'$ .

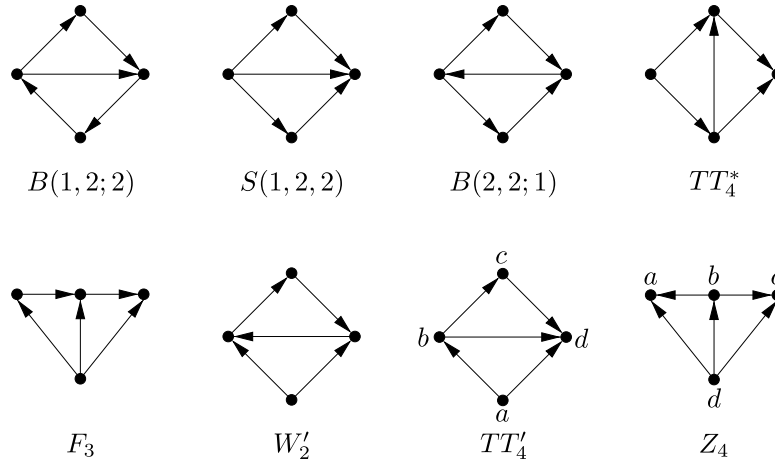


Figure 3.2: Some orientations of  $K_4 \setminus e$

We prove in Subsection 3.2.1 that  $TT'_4$  is tractable, and in Subsection 3.2.3 that  $Z_4$  is tractable.

If  $D$  is an oriented cycle, then it is either directed, or it has two blocks, or it is  $\hat{C}_4$ . In the first two cases,  $D$ -SUBDIVISION has been shown polynomial-time solvable in [3] (Propositions 15 and 20), and latter case was presented in Chapter 2.

If  $D$  has at most four arcs and is not an oriented cycle, then it has been proved tractable in [3] except if  $D$  is the oriented graph  $Y_4$  depicted Figure 3.3 or its converse. We show in Subsection 3.2.2 that  $Y_4$  is tractable.  $\square$

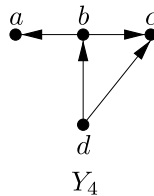


Figure 3.3: The oriented graph  $Y_4$

### 3.2.1 $TT'_4$ -subdivision

In this subsection, we prove that  $TT'_4$  is tractable. Our proof relies on the notion of *good triple*. A triple of distinct vertices  $(a', b', d')$  is *good* if there are an  $(a', b')$ -dipath  $Q$  in  $D - d'$  and three internally disjoint dipaths  $P_1, P_2, P_3$  with  $s(P_1) = s(P_2) = b'$ ,  $s(P_3) = a'$  and  $t(P_1) = t(P_2) = t(P_3) = d'$ .

**Proposition 3.3.** *A digraph  $D$  contains a  $TT'_4$ -subdivision if and only if it has a good triple.*

*Proof.* If  $D$  contains a  $TT'_4$ -subdivision, then the triple formed by its  $a$ -vertex, its  $b$ -vertex and its  $d$ -vertex is good.

Conversely, suppose that  $D$  contains a good triple. Let  $(a', b', d')$  be a good triple that minimizes the sum of the lengths of the paths  $Q, P_1, P_2, P_3$  as named in the definition.

Assume for a contradiction that  $Q^\circ$  intersects  $P_3^\circ$ . Let  $a''$  be a vertex of  $Q^\circ \cap P_3^\circ$ . Then the triple  $(a'', b', d')$  is good because of the paths  $Q[a'', b'], P_1, P_2, P_3[a'', d']$ , and contradicts the minimality of  $(a', b', d')$ . Hence  $Q^\circ$  does not intersect  $P_3^\circ$ .

Assume for a contradiction that  $Q^\circ$  intersects  $P_1^\circ \cup P_2^\circ$ . Let  $b''$  be the last vertex along  $Q^\circ$  in  $P_1^\circ \cup P_2^\circ$ . Without loss of generality, we may assume that  $b''$  is on  $P_1^\circ$ . Then the triple  $(a', b'', d')$  is good because of the paths  $Q[a', b''], P_1[b'', d'], Q[b'', b']P_2, P_3$ , and contradicts the minimality of  $(a', b', d')$ . Hence  $Q^\circ$  does not intersect  $P_1^\circ \cup P_2^\circ$ .

Therefore the paths  $Q, P_1, P_2, P_3$  are internally disjoint and the union of those dipaths is a  $TT'_4$ -subdivision.  $\square$

**Corollary 3.4.**  $TT'_4$ -SUBDIVISION can be solved in  $O(n^5(n+m))$  time.

*Proof.* According to Proposition 3.3,  $TT'_4$ -SUBDIVISION is equivalent to deciding if  $D$  has a good triple.

Now one can decide if a triple  $(a', b', d')$ -triple in  $O(n^2(n+m))$  time as follows. We check if there is an  $(a', b')$ -dipath  $Q$  in  $D - d'$ , and for every pair  $s_1, s_2$  of distinct out-neighbours of  $b'$  in  $D - a'$ , we check if there are three independent  $(\{s_1, s_2, a\}, d')$ -dipath in  $D - b'$  by a Menger algorithm.

Doing this procedure for the  $O(n^3)$  triple of distinct vertices of  $D$ , one decides in  $O(n^5(n+m))$  time whether  $D$  has a good triple.  $\square$

### 3.2.2 $Y_4$ -subdivision

In this subsection, we prove that  $Y_4$  is tractable.

**Theorem 3.5.**  $Y_4$ -SUBDIVISION can be solved in  $O(n^5(n+m))$  time.

*Proof.* Let us describe a procedure that given three distinct vertices  $a', c', d'$ , and two distinct arcs  $d'u_1$  and  $d'u_2$  in  $D - a'$  decides whether a digraph  $D$  contains a  $Y_4$ -subdivision  $S$  with  $a$ -vertex  $a', c$ -vertex  $c', d$ -vertex  $d'$  such that  $\{d'u_1, d'u_2\} \subseteq A(S)$ . Such a subdivision is said to be  $(a', c', d'u_1, d'u_2)$ -forced.

We check whether there are two independent  $(\{u_1, u_2\}, c')$ -dipaths  $P_1, P_2$  in  $D - \{a', d'\}$  and a  $(\{u_1, u_2\}, a')$ -dipath  $Q$  in  $D - \{c', d'\}$ . This can be done in linear time using a Menger algorithm for each of the tasks. The existence of  $P_1, P_2, Q$  is clearly a necessary condition to contain an  $(a', c', d'u_1, d'u_2)$ -forced  $Y_4$ -subdivision. So if we do not find such dipaths, we return ‘no’. If we have such dipaths, then we return ‘yes’. Indeed the union of the dipaths  $d'u_1, d'u_2, P_1, P_2$ , and  $R$ , where  $R$  is the  $(P_1 \cup P_2, a')$ -subdipath of  $Q$ , is an  $(a', c', d'u_1, d'u_2)$ -forced  $Y_4$ -subdivision.

Doing this for every 5-tuple  $(a', c', d', u_1, u_2)$  of vertices, we obtain an algorithm solving  $Y_4$ -SUBDIVISION in  $O(n^5(n+m))$  time.  $\square$

### 3.2.3 $Z_4$ -subdivision

In this subsection, we show that  $Z_4$  is tractable. The proof relies on the following lemma.

**Lemma 3.6.** *Let  $D$  be a digraph. There is a  $Z_4$ -subdivision in  $D$  if and only if there exists four distinct vertices  $a', b', c'$  and  $d'$  in  $D$  such that the following hold.*

- (i) *There are three independent  $(d', \{a', b', c'\})$ -dipaths.*
- (ii) *There are two independent  $(b', \{a', c'\})$ -dipaths.*

*Proof.* If  $D$  contains a  $Z_4$ -subdivision  $S$ , then the vertices  $a', b', c', d'$  corresponding to  $a, b, c, d$  (as indicated on Figure 3.2) clearly satisfy conditions (i) and (ii).

Conversely, suppose that  $D$  contains four vertices  $a', b', c', d'$  satisfying conditions (i) and (ii). Let  $P_1, P_2, P_3$  be three independent  $(d', \{a', b', c'\})$ -dipaths with  $t(P_1) = a', t(P_2) = b'$  and  $t(P_3) = c'$ ; let  $Q_1, Q_2$  be two independent  $(b', \{a', c'\})$ -dipaths with  $t(Q_1) = a'$  and  $t(Q_2) = c'$ .

We consider such vertices  $a', b', c', d'$  and dipaths such that the sum of the lengths of  $P_1, P_2, P_3, Q_1$  and  $Q_2$  is minimized.

**Claim 18.**  $V(Q_1) \cap V(P_1) = \{a'\}$  and  $V(Q_2) \cap V(P_3) = \{c'\}$ .

*Subproof.* Suppose  $V(Q_1) \cap V(P_1) \neq \{a'\}$ . Then there is a vertex  $a''$  distinct from  $a'$  in  $V(Q_1) \cap V(P_1)$ . The vertices  $a'', b', c', d'$  satisfy condition (i) with  $P_1[d', a''], P_2, P_3$  and condition (ii) with  $Q_1[b', a''], Q_2$ . This contradicts our choice of  $a', b', c', d'$  and the corresponding paths, and so  $V(Q_1) \cap V(P_1) = \{a'\}$ .

The conclusion that  $V(Q_2) \cap V(P_3) = \{c'\}$  is proved in the same way; the details are omitted.  $\diamond$

**Claim 19.**  $(V(Q_1) \cup V(Q_2)) \cap V(P_2) = \{b'\}$ .

*Subproof.* Suppose not. Then let  $b''$  be the last vertex distinct from  $b'$  along  $P_2$  which is in  $V(Q_1) \cup V(Q_2)$ . By symmetry, we may assume that  $b'' \in V(Q_1)$ . But the four vertices  $a', b'', c', d'$  satisfy condition (i) with  $P_1, P_2[d', b''], P_3$  and condition (ii) with  $Q_1[b'', a'], P_2[b'', b']Q_2$ . This contradicts our choice of  $a', b', c', d'$  and proves our claim.  $\diamond$

**Claim 20.**  $V(Q_1) \cap V(P_3) = \emptyset$  and  $V(Q_2) \cap V(P_1) = \emptyset$ .

*Subproof.* Suppose not. Then  $V(Q_1) \cap V(P_3)$  or  $V(Q_2) \cap V(P_1)$  is not empty.

Assume first that these two sets are both non-empty. Let  $a''$  be a vertex in  $V(Q_2) \cap V(P_1)$  and  $c''$  be a vertex in  $V(Q_1) \cap V(P_3)$ . Then the four vertices  $a'', b', c'', d'$  satisfy condition (i) with  $P_1[d', a''], P_3[d', c''], P_2$  and condition (ii) with  $Q_2[b', a''], Q_1[b', c'']$ . This contradicts our choice of  $a', b', c', d'$ .

Hence, exactly one of the two sets is empty. By symmetry, we may assume that  $V(Q_1) \cap V(P_3) \neq \emptyset$ . Let  $b''$  be a vertex in  $V(Q_1) \cap V(P_3)$ . Now the four vertices  $a', b'', c', d'$  satisfy condition (i) with  $P_1, P_3[d', b''], P_2Q_2$  and condition (ii) with

$Q_1[b'', a'], P_3[b'', c']$ . This contradicts our choice of  $a', b', c', d'$  and proves our claim.  $\diamond$

Claims 18, 19 and 20 imply that  $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$  is a  $Z_4$ -subdivision.  $\square$

**Theorem 3.7.**  $Z_4$ -SUBDIVISION can be solved in  $O(n^4(n+m))$  time.

*Proof.* By Lemma 3.6,  $Z_4$ -SUBDIVISION is equivalent to deciding whether there are four vertices satisfying the condition (i) and (ii) of the lemma. But given four vertices  $a', b', c', d'$ , one can check in linear time if conditions (i) and (ii) hold by running two Menger algorithms. Since there are  $O(n^4)$  sets of four vertices in  $D$ ,  $Z_4$ -SUBDIVISION can be solved in  $O(n^4(n+m))$  time.  $\square$

### 3.3 Some hard digraphs

Theorem 1.10 implies that many digraphs on 4 vertices are hard. We now prove that some additional digraphs that are not covered by Theorem 1.10 are also hard. These graphs are depicted in Figure 3.4, where each of the bold edges without indicated direction represents a pair of oppositely directed arcs.

**Proposition 3.8.** For each digraph  $N_i$ ,  $1 \leq i \leq 9$ , depicted Figure 3.4,  $N_i$ -SUBDIVISION is NP-complete.

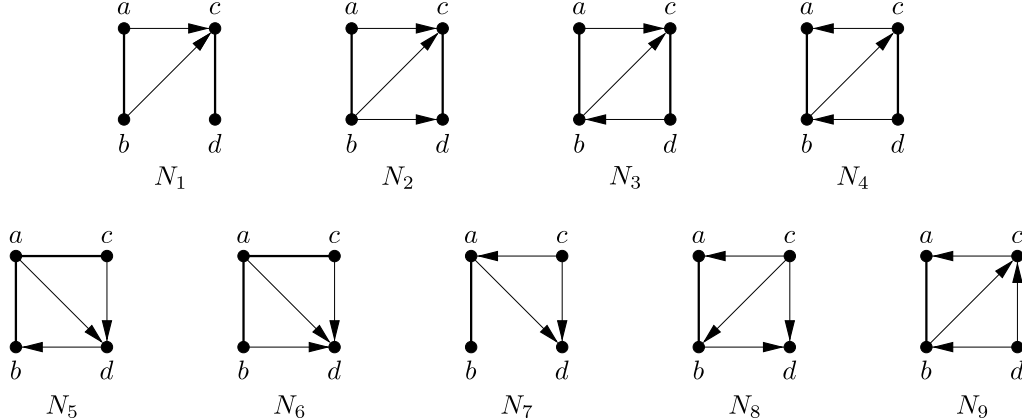


Figure 3.4: Some hard digraphs on 4-vertices. Bold undirected edges represent directed 2-cycles.

*Proof.* In each case, the problem is proved to be NP-complete by reduction from RESTRICTED 2-LINKAGE. Let  $D$ ,  $x_1$ ,  $x_2$ ,  $y_1$  and  $y_2$  be an instance of this problem. We construct a digraph  $D_i$  by putting  $D$  on two arcs  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  of  $N_i$  (that will be specified later), that is by taking the disjoint union of  $D$  and  $N_i$ , by removing the arcs  $e_1$  and  $e_2$  and adding the arcs  $u_1x_1$ ,  $y_1v_1$ ,  $u_2x_2$  and  $y_2v_2$ . We then show that  $D_i$  contains an  $N_i$ -subdivision if and only if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ . This implies that  $N_i$ -SUBDIVISION is NP-complete.

Clearly, by construction of  $D_i$ , if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ , then  $D_i$  contains an  $N_i$ -subdivision. We now prove the converse for each  $i$ . In each case we shall assume that  $D_i$  contains an  $N_i$ -subdivision  $S$ , and we shall denote by  $a', b', c', d'$  the vertices in  $S$  corresponding to  $a, b, c, d$ , respectively.

$i = 1$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $c' = c$ . Because  $d_{D_1}^-(c) = 3$ , the arcs  $ac, bc$  and  $dc$  are in  $S$ . Moreover, the arc  $ba$  is in  $S$ , because every vertex has in-degree at least 1 in  $S$ . Thus  $d_S^+(b) \geq 2$ , and so either  $b = b'$  or  $b = a'$ . By symmetry between  $a$  and  $b$  in  $N_1$ , we may assume that  $b = b'$ . Then, necessarily,  $a = a'$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i \in \{2, 3, 4\}$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $\{b, c\} = \{b', c'\}$ . Therefore, the arc  $bc$  is contained in  $S$ , and this shows that  $b' = b$  and  $c' = c$ . Now for degree reasons, all arcs incident to  $b$  and  $c$  must be in  $S$ . It follows that  $a' = a$  and  $d' = d$ . (This is clear for  $N_3$  and  $N_4$ . For  $N_2$ , we first conclude that  $\{a', d'\} = \{a, d\}$  and then consider degrees of  $a$  and  $d$  to obtain the same conclusion.) Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induced a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 5$ : We choose  $e_1 = ba$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $a' = a$ . Hence all the arcs incident to  $a$  are in  $A(S)$ . Therefore  $c$  is ether  $b'$  or  $c'$ . But  $d^-(c) = 1$ , so  $c$  cannot be  $b'$ , and thus  $c = c'$ . All vertices have out-degree at least 1 in  $S$ , so  $db \in A(S)$ . Now there are two internally disjoint  $(a', b)$ -dipaths in  $S - c'$ , so necessarily,  $b = b'$ . Moreover,  $d'$  must be in one of those dipaths, so  $d = d'$ . Therefore, in  $S$ , there are internally disjoint  $(b, a)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 6$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $a' = a$  and  $d' = d$ . Hence all arcs incident to those two vertices are in  $S$ . Therefore  $\{b', c'\} = \{b, c\}$ . By symmetry of  $N_6$ , we may assume that  $b' = b$  and  $c' = c$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 7$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $a' = a$ . Hence all arcs incident to  $a$  are in  $S$ . So  $c$  and  $d$  are in  $V(S)$ . Since  $d_{D_7}^+(d) = 0$ , we have  $d = d'$ ; since  $d_{D_7}^-(c) = 0$ , we have  $c = c'$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 8$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $b' = b$  and  $c' = c$ . Hence all arcs incident to those two vertices are in  $S$ . So  $d \in V(S)$ . Since  $d_{D_8}^+(d) = 0$ , it follows that  $d = d'$ . The arcs  $ba$  and  $ca$  show that  $d_S^-(a) \geq 2$ . Thus  $a = a'$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 9$ : We choose  $e_1 = ab$  and  $e_2 = dc$ . Since  $D$  contains no big vertices, we have  $b' = b$ . Hence all arcs incident to  $b$  are in  $S$ . In particular  $c, d \in V(S)$ . Since  $d_{D_9}^-(d) = 0$ , we have  $d' = d$ . Since  $d_S^+(c) \geq 1$ , the arc  $ca$  is in  $A(S)$ , so  $d_S^-(a) = 2$ ,



and thus  $a \in \{a', c'\}$ . Since  $a'$  and  $c'$  are both in the out-section of  $d$  in  $N_9 - b$ ,  $S$  contains a  $(d, a)$ -dipath disjoint from  $b$ . This dipath must pass through  $c$  and therefore the arc  $y_2c$  lies in  $S$ . This implies that  $d_S^-(c) \geq 2$ , so  $c = c'$  and then we have  $a = a'$ . Consequently, in  $S$ , there are disjoint  $(a, b)$ - and  $(d, c)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .  $\square$

### 3.4 Some tractable digraphs – easier cases

A *symmetric star* is a symmetric digraph associated to a star. The *centre* of a symmetric star is the centre of the star to which it is associated. A *superstar* is a digraph obtained from a symmetric star by adding an arc joining two non-central vertices. The *centre* of a superstar is the centre of the star from which it is derived. The symmetric star of order  $k+1$  is denoted by  $SS_k$  and the superstar of order  $k+1$  is denoted by  $SS_k^*$ . An  $SS_k$ -subdivision with centre  $a$  is the union of  $k$  internally disjoint  $(a, a)$ -handle. Therefore, one can decide if there is an  $SS_k$ -subdivision with centre  $a$  in linear time using a Menger algorithm. We showed that  $SS_3^*$ -SUBDIVISION is polynomial-time solvable. This result can be extended to all superstars.

**Theorem 3.9.** *Let  $k$  be a positive integer. Given digraph  $D$  and a vertex  $v$  of  $D$ , one can decide in  $O(n^{2k}(n+m))$ -time whether  $D$  contains an  $SS_k^*$ -subdivision with centre  $v$ .*

*Proof.* We describe a procedure that given  $v$ , a set  $X = \{x_1, \dots, x_k\}$  of  $k$  distinct out-neighbours of  $v$  and a set  $Y = \{y_1, \dots, y_k\}$  of  $k$  distinct in-neighbours of  $v$  checks if there is an  $SS_k^*$ -subdivision  $S$  with centre  $v$  such that  $\{vx_1, \dots, vx_k\} \cup \{y_1v, \dots, y_kv\} \in A(S)$ . (Observe that it is allowed that  $X \cap Y \neq \emptyset$ .) Such a subdivision will be called  $(v, X, Y)$ -forced.

Applying a Menger algorithm, check whether in  $D - v$  there are  $k$  disjoint dipaths  $P_1, \dots, P_k$  from  $X$  to  $Y$ . If not, then  $D$  certainly does not contain any  $(v, X, Y)$ -forced  $SS_k^*$ -subdivision. If yes, then check whether there is a dipath  $Q$  from some  $P_i$  to a different  $P_j$  whose internal vertices are not in  $\{v\} \cup \bigcup_{i=1}^k P_i$ . This can be done in linear time by running a search on the digraph obtained from  $D - v$  by contracting each path  $P_i$  into a single vertex. If such a  $Q$  exists, then  $P_1, \dots, P_k$  and  $Q$  together with  $v$  and the arcs from  $v$  to  $X$  and from  $Y$  to  $v$  form a  $(v, X, Y)$ -forced  $SS_k^*$ -subdivision. If not, then no  $(v, X, Y)$ -forced  $SS_k^*$ -subdivision using the chosen arcs exists, because there is no vertex  $x \in X$  with two vertices of  $Y$  in its out-section in  $D - v$ .

Applying this linear-time procedure, for every possible pair  $(X, Y)$ , we can decide in  $O(n^{2k}(n+m))$ -time whether  $D$  contains an  $SS_k^*$ -subdivision with centre  $v$ .  $\square$

**Corollary 3.10.** *For every positive integer  $k$ ,  $SS_k^*$ -SUBDIVISION can be solved in  $O(n^{2k+1}(n+m))$ -time.*

**Proposition 3.11.** *For every  $i \in \{3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15\}$ , the digraph  $E_i$  depicted in Figure 3.5 is tractable.*

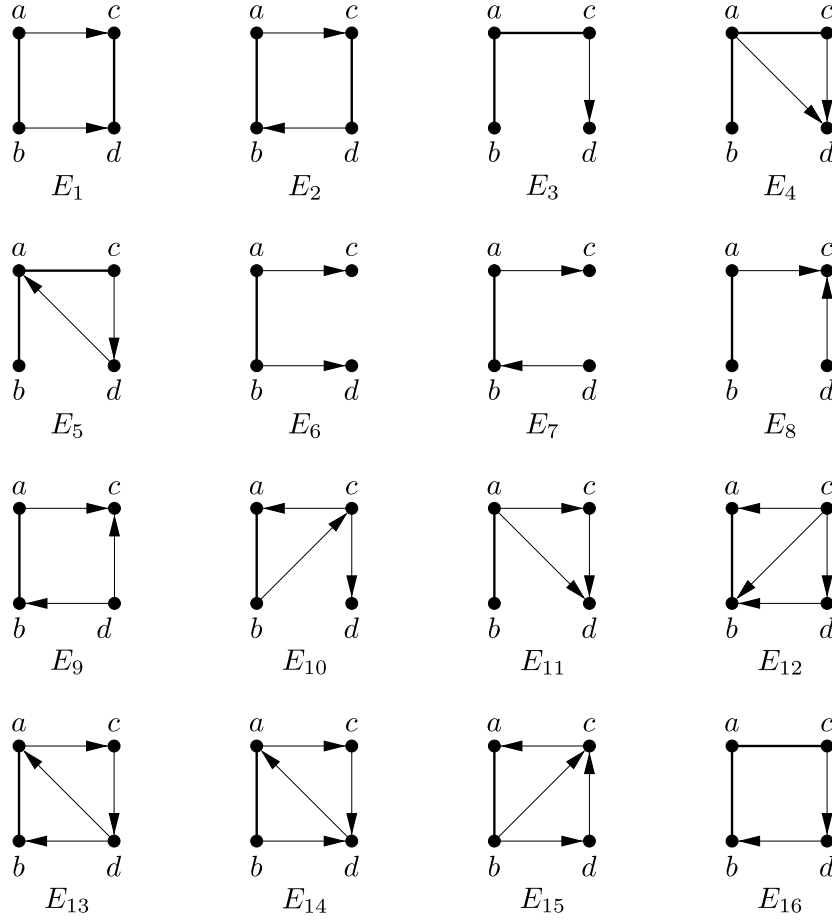


Figure 3.5: Some digraphs on 4-vertices, that are tractable. Bold undirected edges represent directed 2-cycles.

*Proof.*  $i = 3$ : Let us describe a procedure that, given two distinct vertices  $a'$  and  $d'$  in  $D$  and two out-neighbours  $s_1, s_2$  of  $a'$  distinct from  $d'$ , decides whether there is an  $E_3$ -subdivision with  $a$ -vertex  $a'$  and  $d$ -vertex  $d'$  such that  $a's_1$  and  $a's_2$  are arcs of  $S$ . Such a subdivision is said to be  $(a's_1, a's_2, d)$ -forced.

We check whether there is a dipath  $Q$  from  $\{s_1, s_2\}$  to  $d'$  in  $D - a'$ , and with a Menger algorithm we check whether there are two independent  $(\{s_1, s_2\}, a')$ -dipaths  $P_1$  and  $P_2$  in  $D - d'$ . If these three dipaths do not exist, then  $D$  contains no  $(a's_1, a's_2, d)$ -forced  $E_3$ -subdivision, and we return ‘no’. If the three paths  $Q, P_1, P_2$  exist, then we return ‘yes’. Indeed, denoting by  $c'$  the last vertex along  $Q$  in  $P_1 \cup P_2$ , the digraph  $a's_1 \cup P_1 \cup a's_2 \cup P_2 \cup Q[c', d']$  is an  $(a's_1, a's_2, d)$ -forced  $E_3$ -subdivision.

Applying the above procedure for all possible triples  $(a's_1, a's_2, d')$ , one solves  $E_3$ -SUBDIVISION in  $O(n^4(n + m))$  time.

$i = 4$ : Let us describe a procedure that given two distinct vertices  $a'$  and  $d'$  in  $D$ , a set  $U = \{u_1, u_2, u_3\}$  of three out-neighbours of  $a'$ , returns ‘yes’ if it finds an  $E_4$ -subdivision and returns ‘no’ only if there is no  $E_4$ -subdivision with  $a$ -vertex  $a'$

and  $d$ -vertex  $d'$  such that  $\{a'u_1, a'u_2, a'u_3\} \subseteq A(S)$ . Such a subdivision is said to be  $(a', d', U)$ -forced.

We check with a Menger algorithm whether  $|S_{D-a'}^-(d') \cap U| \geq 2$  and whether there are three internally disjoint dipaths  $P_1, P_2, P_3$  with distinct initial vertices in  $U$  and with  $t(P_1) = t(P_2) = a'$  and  $t(P_3) = d'$ . If these two conditions are not both fulfilled, then  $D$  contains no  $(a', d', U)$ -forced  $E_4$ -subdivision, and we return 'no'. If these conditions are fulfilled, then we return 'yes'. Indeed consider three dipaths  $P_1, P_2, P_3$  as above. Without loss of generality,  $s(P_i) = u_i$  for  $1 \leq i \leq 3$ . Since  $|S_{D-a'}^-(d') \cap U| \geq 2$ , there exists a  $(P_1 \cup P_2, P_3)$ -dipath in  $D - a'$ . Let us denote its terminal vertex by  $d''$ . Then the union of the directed cycles  $a'u_1P_1, a'u_2P_2$ , and the dipaths  $a'u_3P_3[u_3, d'']$ , and  $Q$  is an  $E_4$ -subdivision.

Applying the above procedure for all possible triples  $(a', d', U)$ , one solves  $E_4$ -SUBDIVISION in  $O(n^5(n+m))$  time.

$i = 5$ : Let us describe a procedure that given two distinct vertices  $a'$  and  $d'$  in  $D$  and two out-neighbours  $s_1, s_2$  of  $a'$  distinct from  $d'$ , returns 'yes' when it finds an  $E_5$ -subdivision and returns 'no' only if there is no  $E_5$ -subdivision with  $a$ -vertex  $a'$  and  $d$ -vertex  $d'$  such that  $\{a's_1, a's_2\} \subseteq S$ . Such a subdivision is said to be  $(a's_1, a's_2, d')$ -forced.

We check whether there is an  $(\{s_1, s_2\}, d')$  dipath  $Q$  in  $D - a'$  and whether there are three independent  $(\{s_1, s_2, d'\}, a')$ -dipaths  $P_1, P_2, P_3$  in  $D$ . If these two conditions are not both fulfilled, then  $D$  contains no  $(a's_1, a's_2, d')$ -forced  $E_5$ -subdivision, and we return 'no'. If these conditions are fulfilled then we return 'yes'.

Indeed, suppose there are four such dipaths  $Q, P_1, P_2, P_3$ . We may assume without loss of generality that  $s(P_3) = d'$ . Denote by  $c'$  the last vertex along  $Q$  in  $P_1 \cup P_2$ , and by  $d''$  the first vertex in  $Q[c', d']$  which is on  $P_3$ . Then the union of the two directed cycles  $a's_1P_1a', a's_2P_2a'$  and the dipaths  $Q[c', d'']$  and  $P_3[d'', a']$  is an  $E_5$ -subdivision.

Applying the above procedure for all possible triples  $(a's_1, a's_2, d')$ , one solves  $E_5$ -SUBDIVISION in  $O(n^4(n+m))$  time.

$i = 6$ : Observe first that if a digraph contains an  $E_6$ -subdivision, then it contains such an  $E_6$ -subdivision in which  $ac$  and  $bd$  are not subdivided. Henceforth, by  $E_6$ -subdivision, we mean an  $E_6$ -subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs  $a'c'$  and  $b'd'$ , returns 'yes' if it finds an  $E_6$ -subdivision and returns 'no' only if there is no  $E_6$ -subdivision with  $a$ -vertex  $a'$ ,  $b$ -vertex  $b'$ ,  $c$ -vertex  $c'$  and  $d$ -vertex  $d'$ . Such a subdivision is said to be  $(a'c', b'd')$ -forced.

We check whether, in  $D - \{c', d'\}$ , there exists an  $(a', b')$ -dipath  $P$  and a  $(b', a')$ -dipath  $Q$ . If two such dipaths do not exist, then there is clearly no  $(a'c', b'd')$ -forced  $E_6$ -subdivision, and we return 'no'. If two such paths  $P$  and  $Q$  exist, then we return 'yes'. Indeed let  $b''$  be the last vertex on  $Q - a'$  that is in  $P \cap Q$ . If  $b'' = b'$ , then set  $d'' = d'$ , otherwise let  $d''$  be the successor of  $b''$  on  $P$ . Then the union of the directed cycle  $P[a', b'']Q[b'', a']$  and the two arcs  $a'c'$  and  $b''d''$  form an  $E_6$ -subdivision.

Applying the above procedure for all possible pairs of distinct arcs  $(a'c', b'd')$ , one solves  $E_6$ -SUBDIVISION in  $O(m^2(n+m))$  time.

$i = 7$ : The proof is similar to the case  $i = 6$ , and we leave details to the reader.

$i = 8$ : Observe first that if a digraph contains an  $E_8$ -subdivision, then it contains an  $E_8$ -subdivision in which  $dc$  is not subdivided. Henceforth, by  $E_8$ -subdivision, we mean an  $E_8$ -subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs  $sa'$  and  $d'c'$ , checks whether there is an  $E_8$ -subdivision  $S$  with  $a$ -vertex  $a'$ ,  $c$ -vertex  $c'$  and  $d$ -vertex  $d'$ , and such that  $sa' \in A(S)$ . Such a subdivision is said to be  $(sa', d'c')$ -forced.

With a Menger algorithm, we check whether  $D - d'$  contains two independent  $(a', \{s, c'\})$ -dipaths. If two such dipaths do not exist, then there is clearly no  $(a'c', b'd')$ -forced  $E_8$ -subdivision, and we return ‘no’. If two such dipaths  $P$  and  $Q$  exist, then without loss generality  $t(P) = s$  and  $t(Q) = c$ . The union of the directed cycle  $Psa'$ , the dipath  $Q$  and the arc  $d'c'$  is an  $(sa', d'c')$ -forced  $E_8$ -subdivision.

Applying the above procedure for all possible pairs of distinct arcs  $(sa', d'c')$ , one solves  $E_8$ -SUBDIVISION in  $O(m^2(n+m))$  time.

$i = 11$ : Let us describe a procedure that, given an arc  $sa'$  and a vertex  $d' \notin \{s, a'\}$ , checks whether there is an  $E_{11}$ -subdivision  $S$  with  $a$ -vertex  $a'$ ,  $d$ -vertex  $d'$ , and such that  $sa' \in A(S)$ . Such a subdivision is said to be  $(sa', d')$ -forced.

We check with a Menger algorithm whether there are three independent  $(a', \{s, d'\})$ -dipaths, where two of the paths end up at  $d'$  and one at  $s$ . If three such dipaths do not exist, then there is clearly no  $(sa', d')$ -forced  $E_{11}$ -subdivision, and we return ‘no’. If three such dipaths exist, then their union together with the arcs  $sa'$  form an  $(sa', d')$ -forced  $E_{11}$ -subdivision.

Applying the above procedure for all possible pairs  $(sa', d')$ , one solves  $E_{11}$ -SUBDIVISION in  $O(mn(n+m))$  time.

$i = 12$ : Let us describe a procedure that, given two distinct vertices  $b', c'$  and a set  $S = \{s_1, s_2, s_3\}$  of three distinct in-neighbours of  $b'$  checks whether there is an  $E_{12}$ -subdivision  $S'$  with  $b$ -vertex  $b'$ ,  $c$ -vertex  $c'$ , and such that  $\{s_1b', s_2b', s_3b'\} \subset A(S')$ . Such a subdivision is said to be  $(b', c', S)$ -forced.

We check with a Menger algorithm, if there are three independent  $(c', S)$ -dipaths  $P_1, P_2, P_3$ , and we check whether there is a  $(b', S \setminus \{c'\})$ -dipath  $Q$  in  $D - c'$ . If four such dipaths do not exist, then we return ‘no’ because there is no  $(b', c', S)$ -forced  $E_{12}$ -subdivision. If such dipaths  $P_1, P_2, P_3$  and  $Q$  exist, then let  $x$  be the first vertex of  $Q$  in  $P_1 \cup P_2 \cup P_3$ . Then the union of  $P_1, P_2, P_3, Q[b', x]$  and the three arcs  $s_1b', s_2b', s_3b'$  form a  $(b', c', S)$ -forced  $E_{12}$ -subdivision.

Applying the above procedure for all possible triples  $(a', b', S)$ , one solves  $E_{12}$ -SUBDIVISION in  $O(n^5(n+m))$  time.

$i = 13$ : Observe that every  $E_{13}$ -subdivision may be seen as an  $E_{13}$ -subdivision in which the arc  $cd$  is not subdivided. Henceforth, by an  $E_{13}$ -subdivision, we mean such a subdivision.

Let us describe a procedure that, given two disjoint arcs,  $sb' d'c'$ , returns ‘yes’ if it finds an  $E_{13}$ -subdivision and returns ‘no’ only if there is no  $E_{13}$ -subdivision  $S$  with  $b$ -vertex  $b'$ ,  $c$ -vertex  $c'$ ,  $d$ -vertex  $d'$  and such that  $\{sb', d'c'\} \subseteq A(S)$ . Such a subdivision is called  $(sb', d'c')$ -forced.

Applying a Menger algorithm, we check whether in  $D$  there are three independent  $(b', \{s, c', d'\})$ -dipaths  $P_1, P_2, P_3$  with  $t(P_1) = s$  and applying a search we check whether there is a  $(c', s)$ -dipath  $Q$  in  $D - \{b', d'\}$ . Clearly, if four such dipaths do not exist, then  $D$  contains no  $(sb', d'c')$ -forced  $E_{13}$ -subdivision, so we return ‘no’. Conversely, if these dipaths exist, then  $Q$  contains a  $(c', P_1)$ -subdipath  $R$ . Let  $c''$  be the last vertex along  $R$  in  $V(P_2 \cup P_3)$ . Now in  $P_2 \cup P_3 \cup R[c', c''] \cup d'c'$ , there are two internally disjoint  $(b', c'')$ -dipaths  $P'_2, P'_3$ . Thus  $P_1 \cup sb' \cup P'_2 \cup P'_3 \cup R[c'', t(R)]$  is an  $E_{13}$ -subdivision, and we return ‘yes’.

Doing this for every possible pair  $(sb', d'c')$ , one decides in  $O(m^2(n + m))$  time whether  $D$  contains an  $E_{13}$ -subdivision.

$i = 14$ : We proceed in two stages. We first check whether there is an  $E_{14}$ -subdivision in which the arc  $ab$  is not subdivided. Next we check whether there is an  $E_{14}$ -subdivision in which the arc  $ab$  is subdivided.

In the first stage we decide whether there is an  $E_{14}$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$  for some arc  $a'b'$ . To do so, for every dipath  $a'uv$  in  $D - b'$ , we check whether there is an  $E_{14}$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$ , and which contains the arcs of  $\{a'u, uv, a'b'\}$ . Such a subdivision is said to be  $(a'uv, a'b')$ -forced.

We proceed as follows. Applying a Menger algorithm, we check whether in  $D - u$  there are independent  $(\{v, b'\}, a')$ -dipaths  $P_1$  and  $P_2$  with  $s(P_1) = v$ , and applying a search we check whether there is a  $(v, b')$ -dipath  $Q$  in  $D - a' - u$ . Clearly, if three such dipaths do not exist, then  $D$  contains no  $(a'uv, a'b')$ -forced  $E_{14}$ -subdivision, so we return ‘no’. Conversely, if these dipaths exist, then  $Q$  contains a  $(P_1, P_2)$ -subdipath  $R$ . Then the union of  $P_1, P_2, R, a'uv$ , and  $a'b'$  is an  $E_{14}$ -subdivision, and we return ‘yes’. Doing this for every possible pair  $(a'uv, a'b')$ , one decides in  $O(m^2(n + m))$  time that either  $D$  contains an  $E_{14}$ -subdivision, or that  $D$  contains no  $E_{14}$ -subdivision in which the arc  $ab$  is not subdivided.

Let  $F_{14}$  be the digraph obtained from  $E_{14}$  by subdividing the arc  $ab$  into a dipath  $awb$  of length 2. The second stage consists in deciding whether  $D$  contains an  $F_{14}$ -subdivision. We use a procedure similar to the one for detecting superstar subdivision. Given a pair  $\{a'w_1x_1, a'w_2x_2\}$  of dipaths that are disjoint except for their initial vertex  $a'$ , and two distinct in-neighbours  $y_1, y_2$  of  $a'$  that are not in  $\{w_1, w_2\}$  (allowing the possibility that  $\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$ ), the procedure returns ‘yes’ if it finds an  $F_{14}$ -subdivision and returns ‘no’ only if there is no  $F_{14}$ -subdivision with  $a$ -vertex  $a'$  containing all arcs in  $A' = \{a'w_1, w_1x_1, a'w_2, w_2x_2, y_1a', y_2a'\}$ . Such a subdivision is called  $A'$ -forced.

The procedure proceeds as follows. With a Menger algorithm, we first check whether in  $D - \{a', w_1, w_2\}$  there are two disjoint dipaths  $P_1, P_2$  from  $\{x_1, x_2\}$  to  $\{y_1, y_2\}$ . If not, then  $D$  certainly does not contain any  $A'$ -forced  $F_{14}$ -subdivision. If yes, then check whether there is a  $(P_1, P_2)$ -dipath  $Q$  in  $D - \{a', w_1, w_2\}$ . If such

a dipath exists, then the union of the paths  $P_1, P_2, Q, a'w_1x_1, a'w_2x_2$  and the arcs  $y_1a'$  and  $y_2a'$  is an  $F_{14}$ -subdivision and we return ‘yes’. Next, we check if there is a  $(P_2, P_1)$ -dipath  $Q$  in  $D - \{a', w_1, w_2\}$ . If  $Q$  exists, we return ‘yes’. If not, then no  $A'$ -forced  $F_{14}$ -subdivision exists, because there is no vertex  $x \in \{x_1, x_2\}$  with two vertices of  $\{y_1, y_2\}$  in its out-section in  $D - \{a', w_1, w_2\}$ . So we return ‘no’.

This procedure runs in linear time. Thus, running it for every possible set  $A'$ , one decides in  $O(m^2n^3(n+m))$  time whether  $D$  contains an  $F_{14}$ -subdivision, which is nothing but an  $E_{14}$ -subdivision in which the arc  $ab$  is subdivided.

Doing the two stages one after another, we obtain an  $O(m^2n^3(n+m))$ -time algorithm for solving  $E_{14}$ -SUBDIVISION.

$i = 15$ : Similarly to the case  $i = 14$ , we proceed in two stages. We first check whether there is an  $E_{15}$ -subdivision in which the arc  $ab$  is not subdivided. Next we check whether there is an  $E_{15}$ -subdivision in which the arc  $ab$  is subdivided.

The first stage is the following. For every vertex  $a'$ , every two distinct out-neighbours  $b', u$ , and every in-neighbour  $t'$  of  $a'$  distinct from  $b'$  and  $u$ , we run a procedure that returns ‘yes’ if it finds an  $E_{15}$ -subdivision, and return ‘no’ if there is no  $E_{15}$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$  and whose arc set includes  $\{t'a', a'b', a'u\}$ . Such a subdivision is called  $(t'a', a'b', a'u)$ -forced. The procedure is the following. With a Menger algorithm, we check whether in  $D - u$  there are two independent  $(b', \{a', t'\})$ -dipaths  $P_1, P_2$  and whether there is a  $(u, t')$ -dipath  $Q$  in  $D - \{a', b'\}$ . If three such paths do not exist, then  $D$  certainly contains no  $(t'a', a'b', a'u)$ -forced  $E_{15}$ -subdivision and we return ‘no’. If these three paths exist, we then we return ‘yes’. Indeed let  $d'$  be the first vertex along  $Q$  in  $P_1 \cup P_2$ . Now the union of  $P_1, P_2, Q[u, d'], a'b', t'a'$  and  $a'u$  is an  $E_{15}$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$ .

Doing this for every possible triple  $(t'a', a'b', a'u)$ , one can decide in time  $O(n^2m(n+m))$  whether there is an  $E_{15}$ -subdivision in which the arc  $ab$  is not subdivided.

Observe that an  $E_{15}$ -subdivision in which  $ab$  is subdivided is an  $F_{14}$ -subdivision. Hence the second phase is exactly the same as the one for  $E_{14}$ .

Doing the two stages one after another, we obtain an  $O(m^2n^3(n+m))$ -time algorithm for solving  $E_{15}$ -SUBDIVISION.  $\square$

## 3.5 More complicated tractable cases

### 3.5.1 $E_1$ is tractable

In this subsection we prove that the digraph  $E_1$  depicted in Figure 3.5 is tractable.

**Theorem 3.12.**  $E_1$ -SUBDIVISION can be solved in  $O(n^4m(n+m))$ -time.

*Proof.* Let  $D$  be a digraph and let  $x$  be a vertex of  $D$ . An  $E_1$ -subdivision is  $x$ -suitable, if  $x$  is on the subdivision of the directed cycle  $aba$ .

We shall present a procedure  $E_1$ -Subdivision( $D, x$ ), that given a digraph  $D$  and a vertex  $x$  returns ‘no’ only if there is no  $x$ -suitable  $E_1$ -subdivision, and returns

‘yes’ when it finds an  $E_1$ -subdivision (not necessarily  $x$ -suitable). Moreover, this procedure runs in  $O(n^3m(n+m))$  time. Hence running  $E_1$ -Subdivision( $D, x$ ) for every vertex  $x \in V(D)$ , one solves  $E_1$ -SUBDIVISION in  $O(n^4m(n+m))$  time.

$E_1$ -Subdivision( $D, x$ ) uses a subprocedure Reduction( $D, x, \mathcal{S}$ ) that, given a 1-separation  $\mathcal{S} = (W_1, T, W_2)$  in  $D$  such that  $x \in W_1$  and  $W_2 \neq \emptyset$ , reduces the problem to two smaller instances of  $E_1$ -Subdivision. Reduction( $D, x, \mathcal{S}$ ) proceeds as follows.

Let  $y$  be a vertex in  $W_2$ . We run a Menger algorithm that finds a 1-separation  $(W'_1, T', W'_2)$  of  $(x, y)$ . The set  $W'_1$  is the set of vertices reachable from  $x$  in  $D - T'$ . We then replace  $\mathcal{S}$  by  $\mathcal{S}'$ , that is, we set  $W_1 := W'_1$ ,  $T := T'$ , and  $W_2 := W'_2$ . So now every vertex in  $W_1$  can be reached from  $x$ .

If  $T = \emptyset$ , then we return  $E_1$ -Subdivision( $D[W_1], x$ ). This is clearly valid since all the vertices of an  $x$ -suitable  $E_1$ -subdivision are in the out-section of  $x$  and thus cannot be in  $W_2$  because there are no arcs from  $W_1$  to  $W_2$ .

Suppose now that  $|T| = 1$ , say  $T = \{t\}$ . A vertex  $w_1$  of  $W_1$  is  $W_2$ -reachable if in  $D$  there exists a  $(t, w_1)$ -dipath whose internal vertices are all in  $W_2$ , and a vertex  $w_2$  of  $W_2$  is  $W_1$ -reaching if in  $D$  there exists a  $(w_2, t)$ -dipath whose internal vertices are all in  $W_1$ . Let  $D_1$  be the digraph obtained from  $D[W_1 \cup \{t\}]$  by adding the arc  $tw_1$  (if it is not already in  $A(D)$ ) for every  $W_2$ -reachable vertex  $w_1 \in W_1$ ; let  $D_2$  be the digraph obtained from  $D[W_2 \cup \{t\}]$  by adding the arc  $w_2t$  (if it is not already in  $A(D)$ ) for every  $W_1$ -reaching vertex  $w_2$  of  $W_2$ .

Reduction( $D, x, \mathcal{S}$ ) returns  $(E_1$ -Subdivision( $D_1, x$ ) or  $E_1$ -Subdivision( $D_2, t$ )).

The validity of the subprocedure Reduction is justified by the following claim.

**Claim 21.** (i) *If  $D$  contains an  $x$ -suitable  $E_1$ -subdivision, then either  $D_1$  contains an  $x$ -suitable  $E_1$ -subdivision or  $D_2$  contains a  $t$ -suitable  $E_1$ -subdivision.*

(ii) *For any  $i = 1, 2$ , if  $D_i$  contains an  $E_1$ -subdivision, then  $D$  contains an  $E_1$ -subdivision.*

*Subproof.* (i) Assume that  $D$  contains an  $x$ -suitable  $E_1$ -subdivision  $S$ . Let  $C_1$  and  $C_2$  be the directed cycles in  $S$  corresponding to  $aba$  and  $cdc$ , respectively, and let  $P_1$  and  $P_2$  be the two disjoint  $(V(C_1), V(C_2))$ -dipaths in  $S$ . By definition,  $x \in V(C_1)$ .

We distinguish several cases according to the position of  $C_1$  and  $C_2$ .

Assume first that  $C_1$  is contained in  $D[W_1 \cup \{t\}]$ . Since all dipaths from  $W_1$  to  $W_2$  go through  $t$ , one of the  $P_i$ , say  $P_1$ , is in  $D[W_1]$ , and  $V(C_2) \cap W_1 \neq \emptyset$ .

- Suppose that  $C_2$  is in  $D[W_1]$ . If  $V(P_2) \cap W_2 = \emptyset$ , then  $S$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_1$ . If  $V(P_2) \cap W_2 \neq \emptyset$ , then there is a vertex  $w_1$  of  $W_1 \cap V(P_2)$  such that  $P_2 \cap D[W_2] = P_2]t, w_1[$ . Therefore the digraph  $S_1$  obtained from  $S$  by replacing  $P_2]t, w_1[$  by the arc  $tw_1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_1$ .
- Suppose now that  $V(C_2) \cap W_2 \neq \emptyset$ . Then necessarily  $t \in V(C_2)$ , and  $P_1$  and  $P_2$  are in  $D[W_1 \cup \{t\}]$ . Moreover, there is a vertex  $w_1$  of  $W_1 \cap V(C_2)$  such that  $C_2 \cap D[W_2] = C_2]t, w_1[$ . Therefore the digraph  $S_1$  obtained from  $S$  by replacing  $C_2]t, w_1[$  by the arc  $tw_1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_1$ .

Suppose now that  $V(C_1) \cap W_2 \neq \emptyset$ . Since  $x \in V(C_1)$ ,  $C_1$  necessarily contains  $t$ , because there is no arcs from  $W_1$  to  $W_2$ . Moreover, there exist two vertices  $w_1 \in W_1 \cap V(C_1)$  and  $w_2 \in W_2 \cap V(C_1)$  such that  $C_1 = C_1[t, w_2]w_2w_1C_1[w_1, t]$ ,  $D[W_1] \cap C_1 = C_1[w_1, t[$  and  $D[W_2] \cap C_1 = C_1]t, w_2]$ . Now,  $C_2$  is contained in  $D[W_1 \cup W_2]$ , and so  $C_2$  is contained either in  $D[W_1]$  or in  $D[W_2]$ .

- Assume that  $C_2$  is in  $D[W_1]$ . Let  $w_1^i$  be the first vertex along  $P_i$  in  $W_1$ . Since there is no arc from  $W_1$  to  $W_2$ , all vertices in  $P_i[s(P_i), w_1^i[$  are in  $W_2$  and all vertices in  $P_i[w_1^i, t(P_i)]$  are in  $W_1$ .
  - a) If  $s(P_1) = w_1^1$  and  $s(P_2) = w_1^2$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[t, w_1]$  by the arc  $tw_1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_1$ .
  - b) If  $s(P_1) \neq w_1^1$  and  $s(P_2) = w_1^2$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[t, w_1]$  and  $P_1[s(P_1), w_1^1]$  by the arcs  $tw_1$  and  $tw_1^1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_1$ .
  - c) Assume finally that  $s(P_1) \neq w_1^1$  and  $s(P_2) \neq w_1^2$ . Then both  $s(P_1)$  and  $s(P_2)$  are in  $W_2 \cup \{t\}$ . Since every vertex of  $W_1$  is reachable from  $x$  in  $D[W_1]$ , there is a  $(V(C_1) \cap W_1, V(C_2))$ -dipath  $Q$  in  $D[W_1]$ . Observe that  $s(Q)$  is distinct from  $s(P_1)$  and  $s(P_2)$  because it is in  $W_1$ . If  $Q$  does not intersect  $P_1 \cup P_2$ , set  $Q' := Q$ . Otherwise, without loss of generality, the first vertex  $z$  along  $Q$  in  $P_1 \cup P_2$  is in  $P_2$ . In this case, set  $Q' = Q[s(Q), z]P_2[z, t(P_2)]$ . In both cases, the subdigraph  $S'$  obtained from  $S$  by replacing  $P_2$  by  $Q'$  is an  $E_1$ -subdivision. Now Subcase (b) applies to  $S'$ , so  $D_1$  contains an  $x$ -suitable  $E_1$ -subdivision in  $D_1$ .
- Assume that  $C_2$  is in  $D[W_2]$ . Then  $P_1$  and  $P_2$  are in  $D[W_2 \cup \{t\}]$ , and so  $C_1 \cap D[W_1] = C_1[w_1, t[$ . Hence the digraph  $S_2$  obtained from  $S$  by replacing  $C_1[w_2, t]$  by the arc  $w_2t$  is a  $t$ -suitable  $E_1$ -subdivision in  $D_2$ .

(ii) Suppose that  $S_1$  is an  $E_1$ -subdivision in  $D_1$ . By construction of  $D_1$ , all arcs of  $A(S_1) \setminus A(D)$  are joining  $t$  to some  $W_2$ -reachable vertex. Since each vertex in  $E_1$  has out-degree at most 2, there are at most two arcs in  $A(S_1) \setminus A(D)$ .

If there is no in arc in  $A(S_1) \setminus A(D)$ , then  $S_1$  is an  $E_1$ -subdivision in  $D$ . If there is a unique arc  $tw_1$  in  $A(S_1) \setminus A(D)$ , then the digraph  $S$  obtained from  $S_1$  by replacing the arc  $tw_1$  by a  $(t, w_1)$ -dipath with internal vertices in  $W_2$  is an  $E_1$ -subdivision contained in  $D$ . Assume finally that  $A(S_1) \setminus A(D)$  contains two arcs,  $tw_1$  and  $tw_1'$ . Note that  $t$  has in-degree 1 and out-degree 2 in  $S_1$ . Let  $P$  (resp.  $P'$ ) be a  $(t, w_1)$ -dipath (resp.  $(t, w_1')$ -dipath) with all internal vertices in  $W_2$ . Let  $t'$  be the last vertex along  $P'$  which is in  $V(P) \cap V(P')$ . Now the digraph  $S$  obtained from  $S_1$  by replacing  $tw_1$  and  $tw_1'$  by the union of  $P$  and  $P'[t', w_1']$  is an  $E_1$ -subdivision contained in  $D$ .

A similar argument shows that if  $D_2$  contains an  $E_1$ -subdivision, then  $D$  contains an  $E_1$ -subdivision.  $\diamond$

Using **Reduction**, we construct another procedure  $\text{cleaning}(C_1, C_2, x)$  that given two disjoint directed cycles  $C_1$  and  $C_2$  and the vertex  $x$ , either reduces the



problem or finds a pair of disjoint directed cycles  $(C'_1, C'_2)$  such that  $x \in V(C'_1)$ . This procedure proceeds as follows.

If  $C_1$  contains  $x$ , then we set  $(C'_1, C'_2) := (C_1, C_2)$ . If  $C_2$  contains  $x$ , then we set  $(C'_1, C'_2) := (C_2, C_1)$ .

Assume now that  $x$  is not in  $V(C_1 \cup C_2)$ . We first check whether there is a cycle  $C$  containing  $x$ . If not, then we return ‘no’ because  $D$  does certainly not contain any  $x$ -suitable  $E_1$ -subdivision. If  $C$  does not intersect  $C_1$ , then we set  $(C'_1, C'_2) := (C, C_1)$ . If  $C$  does not intersect  $C_2$ , then we set  $(C'_1, C'_2) := (C, C_2)$ . Henceforth,  $C$  intersects both  $C_1$  and  $C_2$ . Let  $y$  be the first vertex after  $x$  in  $C$  that is in  $V(C_1 \cup C_2)$ , and let  $z$  be the last vertex before  $x$  in  $C$  that is in  $V(C_1) \cup V(C_2)$ . Free to permute the indices of  $C_1$  and  $C_2$ , we may assume that  $y \in V(C_1)$ . Moreover, if  $z \in V(C_1)$ , then we set  $(C'_1, C'_2) := (C[x, y]C_1[y, z]C[z, x], C_2)$ . So we may assume that  $z \in V(C_2)$ . Using a Menger algorithm, we check whether there are two disjoint  $(x, C_2)$ -dipaths. If not, then we obtain a 1-separation  $\mathcal{S} = (W_1, T, W_2)$  in  $D$  such that  $x \in W_1$  and  $V(C_2) \subseteq T \cup W_2$ . In that case, we return  $\text{Reduction}(D, x, \mathcal{S})$ . Suppose now that there are two independent  $(x, C_2)$ -dipaths  $Q_1$  and  $Q_2$ . If  $Q_i$  does not intersect  $C_1$ , then the closed walk  $Q_i C_2[t(Q_i), y]C[y, x]$  contains a cycle through  $x$ . We return this cycle and  $C_1$  as  $(C'_1, C'_2)$ . If  $Q_1$  and  $Q_2$  both intersect  $C_1$ , then there are two disjoint  $(C_1, C_2)$ -dipaths, whose union with  $C_1$  and  $C_2$  is an  $E_1$ -subdivision. So we return ‘yes’. This finishes the subprocedure  $\text{cleaning}(C_1, C_2, x)$ .

Finally, let us describe  $E_1$ -Subdivision( $D, x$ ).

We first check whether there are two disjoint directed cycles in  $D$ . If not, then we return ‘no’ because  $D$  cannot contain an  $E_1$ -subdivision in this case. Henceforth, we may assume that there are two disjoint directed cycles  $\Gamma_1$  and  $\Gamma_2$ .

We then run  $\text{cleaning}(\Gamma_1, \Gamma_2, x)$ . If the instance was not reduced by this procedure, we get two disjoint directed cycles  $(\Gamma'_1, \Gamma'_2)$  such that  $x \in V(\Gamma'_1)$ .

We run a Menger algorithm to check whether there are two disjoint  $(V(\Gamma'_1), V(\Gamma'_2))$ -dipaths. If two such dipaths  $P_1$  and  $P_2$  exist, then  $\Gamma'_1 \cup \Gamma'_2 \cup P_1 \cup P_2$  is an  $x$ -suitable  $E_1$ -subdivision, and we return ‘yes’. If not, then the Menger algorithm returns a 1-separation  $\mathcal{S} = (W_1, T, W_2)$  of  $(V(\Gamma'_1), V(\Gamma'_2))$ . If  $x \in W_1$ , then we return  $\text{Reduction}(D, x, \mathcal{S})$ . If  $x \notin W_1$ , then  $T = \{x\}$ . In this case we proceed as described below.

From now on, we may assume that  $D$  contains no  $x$ -suitable  $E_1$ -subdivision in which the cycle through  $x$  lies in  $D[W_2 \cup \{x\}]$  and the other cycle lies in  $D[W_1]$ . This is guaranteed either by nonexistence of  $C_1$  or  $C_2$ , or by the outcome of the previous step.

Let  $D'_1$  be the digraph obtained from  $D$  by contracting  $W_2$  into a vertex  $u$  and let  $D_2$  be the digraph obtained from  $D[W_2 \cup \{x\}]$  by adding the arc  $w_2x$  (if it is not already in  $A(D)$ ) for every  $W_1$ -reaching vertex  $w_2$  of  $W_2$ . We return  $(E_1\text{-Subdivision}(D'_1, x)$  or  $E_1\text{-Subdivision}(D_2, x))$ .

This is valid by the following claim whose proof is very similar to the one of Claim 21.

**Claim 22.** (i) If  $D$  contains an  $x$ -suitable  $E_1$ -subdivision, then either  $D'_1$  contains an  $x$ -suitable  $E_1$ -subdivision or  $D_2$  contains a  $x$ -suitable  $E_1$ -subdivision.

(ii) If  $D'_1$  or  $D_2$  contains an  $E_1$ -subdivision, then  $D$  contains an  $E_1$ -subdivision.

*Subproof.* (i) Assume that  $D$  contains an  $x$ -suitable  $E_1$ -subdivision  $S$ . Let  $C_1$  and  $C_2$  be the directed cycles in  $S$  corresponding to  $aba$  and  $cdc$ , respectively, and let  $P_1$  and  $P_2$  be the two disjoint  $(V(C_1), V(C_2))$ -dipaths in  $S$ . By definition,  $x \in V(C_1)$ .

We distinguish several cases according to the positions of  $C_1$  and  $C_2$ . Suppose first that  $C_1$  is contained in  $D[W_1 \cup \{x\}]$ . Since all dipaths from  $W_1$  to  $W_2$  go through  $x$ , one of the  $P_i$ , say  $P_1$ , is in  $D[W_1]$ , and  $V(C_2) \cap W_1 \neq \emptyset$ . Hence  $C_2$  is in  $D[W_1]$ . If  $V(P_2) \cap W_2 = \emptyset$ , then  $S$  is an  $x$ -suitable  $E_1$ -subdivision in  $D'_1$ . If  $V(P_2) \cap W_2 \neq \emptyset$ , then  $P_2$  contains  $x$  and there is a vertex  $w_1$  of  $W_1 \cap V(P_2)$  such that  $P_2 \cap D[W_2] = P_2[x, w_1[$ . Therefore the digraph  $S_1$  obtained from  $S$  by replacing  $P_2[x, w_1]$  by the dipath  $xuw_1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D'_1$ .

The second possibility is that  $C_1 \subseteq D[W_2 \cup \{x\}]$ . Since  $C_1$  contains  $x$ ,  $C_2$  is contained in  $D[W_1 \cup W_2]$ , and so  $C_2$  is contained either in  $D[W_1]$  or in  $D[W_2]$ . If  $C_2$  lies in  $D[W_1]$ , then the digraph  $S'$  obtained from  $S$  by contracting all the vertices of  $W_2$  into  $U$  is an  $x$ -suitable  $E_1$ -subdivision in  $D'_1$ . If  $C_2 \subseteq D[W_2]$ , then  $S$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_2$ .

Finally, suppose that  $V(C_1) \cap W_2 \neq \emptyset$  and  $V(C_1) \cap W_1 \neq \emptyset$ . Then there exist two vertices  $w_1 \in W_1 \cap V(C_1)$  and  $w_2 \in W_2 \cap V(C_1)$  such that  $C_1 = C_1[x, w_2]w_2w_1C_1[w_1, x]$ ,  $D[W_1] \cap C_1 = C_1[w_1, x[$ , and  $D[W_2] \cap C_1 = C_1[x, w_2]$ . Now  $C_2$  is contained in  $D[W_1 \cup W_2]$ , and so  $C_2$  is contained either in  $D[W_1]$  or in  $D[W_2]$ .

- Suppose first that  $C_2 \subseteq D[W_1]$ . Let  $w_1^i$  be the first vertex along  $P_i$  in  $W_1$ . Since there is no arc from  $W_1$  to  $W_2$ , all vertices of  $P_i[s(P_i), w_1^i[$  are in  $W_2$  and all vertices of  $P_i[w_1^i, t(P_i)]$  are in  $W_1$ .
  - a) If both  $P_1$  and  $P_2$  are contained in  $D[W_1 \cup \{x\}]$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[x, w_1]$  by the dipath  $xuw_1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D'_1$ .
  - b) If one of the two  $P_i$ 's, say  $P_1$  is contained in  $D[W_1 \cup \{x\}]$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[x, w_1]$  and  $P_1[s(P_1), w_1^1]$  by the dipaths  $xuw_1$  and  $uw_1^1$ ) is an  $x$ -suitable  $E_1$ -subdivision in  $D'_1$ .
  - c) Assume now that  $P_1$  and  $P_2$  both intersect  $W_2$ . Both  $s(P_1)$  and  $s(P_2)$  are in  $W_2 \cup \{x\}$ . Observe that every vertex of  $W_1$  is reachable from  $\Gamma'_1 \setminus \{x\}$  in  $D_1$ , so there is an  $(x, C_2)$ -dipath  $Q$  in  $D[W_1 \cup \{x\}]$ . If  $Q$  does not intersect  $P_1 \cup P_2$ , then set  $Q' := Q$ . Otherwise, we may assume that the first vertex  $z$  in  $V(P_1 \cup P_2)$  along  $Q - x$  in  $P_2$ . Set  $Q' := Q[x, z]P_2[z, t(P_2)]$ . Observe that  $Q'$  and  $P_1$  are disjoint except possibly in  $x$ . Now the digraph  $S_1$  obtained from  $C_1 \cup C_2 \cup P_1 \cup Q'$  by replacing  $C_1[x, w_1]$  and  $P_1[s(P_1), w_1^1]$  by the dipaths  $xuw_1$  and  $uw_1^1$  is an  $x$ -suitable  $E_1$ -subdivision in  $D'_1$ .
- The second possibility is that  $C_2 \subseteq D[W_2]$ . Then  $P_1$  and  $P_2$  are in  $D[W_2 \cup \{x\}]$ , and so  $C_1 \cap D[W_1] = C_1[w_1, x[$ . Hence the digraph  $S_2$  obtained from  $S$  by replacing  $C_1[w_2, x]$  by the arc  $w_2x$  is an  $x$ -suitable  $E_1$ -subdivision in  $D_2$ .

(ii) We already showed in Claim 21 that if  $D_2$  contains an  $E_1$ -subdivision, then  $D$  contains an  $E_1$ -subdivision. Let us now prove that if  $D'_1$  contains an  $E_1$ -subdivision, then  $D$  contains an  $E_1$ -subdivision.

Suppose that  $S_1$  is an  $E_1$ -subdivision in  $D'_1$ . If  $u \notin V(S_1)$ , then  $S_1$  is an  $E_1$ -subdivision in  $D$ . If  $u \in V(S_1)$ , then  $S_1$  contains a dipath  $(x, u, w_1)$  for some  $w_1 \in W_1$ , and possibly one other arc  $uw'_1$ . By definition of  $D'_1$ , there are a  $(t, w_1)$ -dipath  $P$  and a  $(t, w'_1)$ -dipath  $P'$  with internal vertices in  $W_2$ . Let  $t'$  be the last vertex along  $P'$  which is in  $V(P) \cap V(P')$ . Then the digraph  $S$  obtained from  $S_1$  by replacing  $tw_1$  by  $P$  and  $tw'_1$  (if it exists) by  $P'[t', w'_1]$  is an  $E_1$ -subdivision contained in  $D$ .  $\diamond$

Let us now estimate the complexity of  $E_1$ -Subdivision( $D, x$ ). This procedure first finds two disjoint directed cycles and then runs a few Menger algorithms and either returns an answer or make a recursive call on two smaller instances, which are either  $D_1$  and  $D_2$ , or  $D'_1$  and  $D_2$ . Two disjoint directed cycles can be found in  $O(n^3m(n+m))$  by Corollary 1.25.

The smaller instances  $D_1$  and  $D'_1$  can be constructed in linear time: indeed a vertex  $w_1 \in W_1$  is  $W_2$ -reachable if and only if it has an in-neighbour in the out-section of  $x$  in  $D[W_2 \cup \{x\}]$ , and so all  $W_2$ -reachable vertices can be found in linear time. Similarly, the set of  $W_1$ -reaching vertices in  $W_2$  can be determined in linear time, and thus  $D_2$  can be constructed in linear time. Hence  $E_1$ -Subdivision( $D, x$ ) makes at most  $cn^2$  operations before calling recursively, for some absolute constant  $c$ .

Let us denote by  $T(n)$  the maximum time for  $E_1$ -Subdivision( $D, x$ ) on a digraph with  $n$  vertices. Since  $|V(D_1)| + |V(D_2)| = |V(D)| + 1$  and  $|V(D'_1)| + |V(D_2)| = |V(D)| + 2$ , we have

$$T(n) \leq \max\{T(n_1) + T(n_2) + cn^2 \mid n_1, n_2 < n \text{ and } n_1 + n_2 \in \{n+1, n+2\}\}.$$

This implies that  $T(n) \leq O(n^3)$ .

Therefore  $E_1$ -Subdivision runs in time  $O(n^3m(n+m))$ .  $\square$

### 3.5.2 $E_2$ is tractable

The aim of this subsection is to prove that the digraph  $E_2$  is tractable.

**Theorem 3.13.**  $E_2$ -SUBDIVISION can be solved in  $O(n^4m(n+m))$  time.

In order to prove Theorem 3.13, we need some preliminary results.

Let  $F$  be a subdigraph of a digraph  $D$ . An *ear* of  $F$  in  $D$  is an oriented path in  $D$  containing at least one edge, whose end-vertices lie in  $F$  but whose edges and internal vertices do not belong to  $F$ . A *directed ear* of  $F$  is an ear of  $F$  that is a directed path. A digraph is said to be *robust* if it is strong and 2-connected. The following lemma is well-known; it is very similar to Proposition 5.11 of [8].

**Lemma 3.14.** *Let  $F$  be a non-trivial strong subdigraph of a robust digraph  $D$ . Then  $F$  has a directed ear in  $D$ . Moreover such a directed ear can be found in time  $O(n(n+m))$ .*

*Proof.* Because  $D$  is 2-connected,  $F$  has an oriented ear in  $D$ . Among all such ears, we choose one in which the number of reverse arcs (those directed towards its initial vertex) is as small as possible. We show that this path  $P$  is in fact a directed ear.

Assume the contrary, and let  $uv$  be a backward arc of  $P$ . Because  $D$  is strong, there exist in  $D$  an  $(F, u)$ -dipath  $Q$  and a  $(v, F)$ -dipath  $R$  (one of which might be of length zero). The initial vertex of  $Q$  and the terminal vertex of  $R$  must be one and the same vertex, for otherwise the directed walk  $QuvR$  would contain a directed ear of  $F$ , contradicting the choice of  $P$  and our assumption that  $P$  is not a directed ear. Let this common vertex be  $z$ . We may assume that  $z \neq s(P)$  (the case  $z \neq t(P)$  being analogous). Then the  $(s(P), z)$ -walk  $P[s(P), v]Rz$  contains an oriented  $(s(P), z)$ -path that contradicts the choice of  $P$ . Thus  $P$  is indeed a directed ear of  $F$ .

A directed ear of  $F$  may be found by running a search from each vertex of  $F$  in  $D \setminus A(F)$ . Hence it can be found in  $O(n(n+m))$  steps.  $\square$

Let  $D_1$  and  $D_2$  be two subdigraphs in  $D$ . Two dipaths are  $(D_1, D_2)$ -opposite if they are disjoint and one of them is a  $(D_1, D_2)$ -dipath and the other is a  $(D_2, D_1)$ -dipath. Opposite dipaths play an important role in detecting  $E_2$ -subdivisions because of the following easy lemma.

**Lemma 3.15.** *Let  $D$  be a digraph and  $D_1$  and  $D_2$  two disjoint non-trivial strong subdigraphs of  $D$ . If there are  $(D_1, D_2)$ -opposite dipaths in  $D$ , then  $D$  contains an  $E_2$ -subdivision.*

*Proof.* Let  $P_1$  and  $P_2$  be two  $(D_1, D_2)$ -opposite dipaths, with  $P_1$  a  $(D_1, D_2)$ -dipath and  $P_2$  a  $(D_2, D_1)$ -dipath. Since  $D_i$  is strong, there is an  $(t(P_{3-i}), s(P_i))$ -dipath  $Q_i$  in  $D_i$ . For the same reason, there is a  $(Q_i - s(Q_i), s(Q_i))$ -dipath  $R_i$  in  $D_i$ . Now  $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2$  is an  $E_2$ -subdivision in  $D$ .  $\square$

*Proof of Theorem 3.13.* We shall present a recursive procedure  $E_2$ -Subdivision( $D$ ), that given a digraph  $D$  decides whether it contains an  $E_2$ -subdivision or not.

This procedure proceeds as follows. We first check whether  $D$  is robust. If not, then we solve the problem for each robust component separately. Henceforth, we may assume that  $D$  is robust.

We next check whether there are two disjoint directed cycles. If not, then we return ‘no’ since  $E_2$  contains two disjoint directed cycles. If two such cycles  $C_1$  and  $C_2$  exist, then we compute the strong component  $D_1$  of  $C_1$  in  $D - C_2$ , and next the strong component  $D_2$  of  $C_2$  in  $D - D_1$ . Hence  $D_1$  and  $D_2$  are two disjoint non-trivial strong subdigraphs in  $D$ . Moreover, they satisfy the following property.

**Claim 23.** *If  $P$  is a  $(D_1, D_2)$ -dipath and  $Q$  is a  $(D_2, D_1)$ -dipath, then  $P$  and  $Q$  are internally disjoint.*

*Subproof.* Suppose, by way of contradiction, that  $P$  and  $Q$  have a common internal vertex  $x$ . The vertex  $x$  is in the strong component of  $D_1$  in  $D - D_2$ . Hence it is in the strong component of  $C_1$  in  $D - C_2$ . So  $x \in D_1$ , a contradiction.  $\diamond$

We check whether there are  $(D_1, D_2)$ -opposite dipaths in  $D$ . By using Claim 23 this task reduces to finding a  $(D_1, D_2)$ -dipath and a  $(D_2, D_1)$ -dipath whose end-vertices are disjoint. If there are such paths, then, by Lemma 3.15,  $D$  contains an  $E_2$ -subdivision and we return ‘yes’. Henceforth we may assume that there are no  $(D_1, D_2)$ -opposite dipaths in  $D$ .

By Lemma 3.14, there is a directed ear  $P_1$  of  $D_1$ . Since  $D_1 \cup P_1$  is strong,  $P_1$  must intersect  $D_2$ . Furthermore, the intersection of  $P_1$  and  $D_2$  is reduced to a single vertex, because there are no  $(D_1, D_2)$ -opposite dipaths. Let  $u_1$  be the initial vertex of  $P_1$ ,  $v_1$  the terminal vertex of  $P_1$ , and let  $u_2$  be the vertex of  $P_1 \cap D_2$ . By Lemma 3.14, there is a directed ear  $P_2$  of  $D_2$ . If the terminal vertex of  $P_2$  is  $u_2$ , then we consider the converse of  $D$ ,  $D_1$ ,  $D_2$ ,  $P_1$  and  $P_2$ . (This is valid since  $E_2$  is its own converse.) Hence, we may assume that the terminal vertex  $v_2$  of  $P_2$  is different from  $u_2$ . Similarly to  $P_1$ , the directed ear  $P_2$  intersects  $D_1$  in a single vertex  $w_1$ . Necessarily,  $w_1 = v_1$  for otherwise  $P_1[u_2, v_1]$  and  $P_2[w_1, v_2]$  are  $(D_1, D_2)$ -opposite, by Claim 23. Also, the initial vertex of  $P_2$  is  $u_2$ , and thus we may assume that both ears have common segment  $P_1[u_2, v_1] = P_2[u_2, v_1]$ . Furthermore,  $P_1[u_1, u_2]$  and  $P_2[v_1, v_2]$  are disjoint for otherwise, there are two  $(D_1, D_2)$ -opposite dipaths.

Set  $P = P_1 P_2[v_1, v_2]$ . We check whether  $D - (D_1 \cup D_2)$  contains a non-trivial strong component  $D_3$ . If  $D_3$  exists and intersects both  $P[u_1, v_1]$  and  $P[u_2, v_2]$ , then we return ‘yes’. This is valid by the following claim.

**Claim 24.** *If  $D_3$  intersects both  $P[u_1, v_1]$  and  $P[u_2, v_2]$ , then  $D$  contains an  $E_2$ -subdivision.*

*Subproof.* Suppose first that  $D_3$  intersects  $P[u_2, v_1]$ . If  $D_3$  also intersects  $P[u_1, u_2]$ , then there are two  $(D_1, D_3)$ -opposite dipaths, and so by Lemma 3.15,  $D$  contains an  $E_2$ -subdivision. Similarly, if  $D_3$  also intersects  $P[v_1, v_2]$ , then there are two  $(D_2, D_3)$ -opposite dipaths, and so  $D$  contains an  $E_2$ -subdivision. Hence we may assume that  $D_3$  does not intersect  $P[u_1, u_2] \cup P[v_1, v_2]$ . If  $D_3$  intersects  $P[u_2, v_1]$  in more than one vertex, then let  $u_3$  (resp.  $v_3$ ) be the first (resp. last) vertex of  $D_3$  along  $P[u_2, v_1]$ . The two dipaths  $P[u_1, u_3]$  and  $P[v_3, v_1]$  are  $(D_1, D_3)$ -opposite. Hence, by Lemma 3.15,  $D$  contains an  $E_2$ -subdivision.

So  $D_3$  intersects  $P[u_2, v_1]$  in a unique vertex, say  $w_3$ . By Lemma 3.14, there is a directed ear  $P_3$  of  $D_3$ . By definition of  $D_1$  and  $D_2$ ,  $P_3$  intersects both  $D_1$  and  $D_2$ . Now one of the two end-vertices of  $P_3$ , say  $u_3$ , is distinct from  $w_3$ .

If  $u_3$  is the initial vertex of  $P_3$ , then consider the first vertex  $v_3$  along  $P_3$  in  $D_1 \cup D_2$ . By definition of  $D_2$ ,  $v_3 \in V(D_1)$ . Now  $P_3[u_3, v_3]$  is disjoint from  $P[u_2, w_3]$  since  $D_3$  is a strong component of  $D - D_1$ . Furthermore  $P_3[u_3, v_3]$  is internally disjoint from  $P[u_1, u_2]$  because the dipaths  $P[u_1, u_2]$  and  $P[u_2, w_3] D_3[w_3, u_3] P_3[u_3, v_3]$  are internally disjoint by Claim 23. If  $v_3 \neq u_1$ , then  $P[u_1, w_3]$  and  $P_3[u_3, v_3]$  are  $(D_1, D_3)$ -opposite, and so by Lemma 3.15,  $D$  contains an  $E_2$ -subdivision. Finally,

if  $v_3 = u_1$ , then  $P[v_1, v_2]$  is disjoint from  $P_3[u_3, v_3]$ . In this case,  $P_3[u_3, v_3]$  and  $P[v_1, v_2]D_2[v_2, u_2]P[u_2, w_3]$  are  $(D_1, D_3)$ -opposite paths giving an  $E_2$ -subdivision.

If  $u_3$  is the terminal vertex of  $P_3$ , then we get the result analogously.

Suppose now that  $D_3$  does not intersect  $P[u_2, v_1]$ . Then it must intersect both  $P[u_1, u_2]$  and  $P[v_1, v_2]$ . Let  $u_3$  be the first vertex of  $D_3$  along  $P[u_1, u_2]$  and  $v_3$  be the last vertex of  $D_3$  along  $P[v_1, v_2]$ . Now  $P[u_1, u_3]$  and  $P[v_3, v_2]D_2[v_2, u_2]P[u_2, v_1]$  are two  $(D_1, D_3)$ -opposite dipaths. Thus, by Lemma 3.15,  $D$  contains an  $E_2$ -subdivision.

◇

If  $D_3$  exists, we are either done by Claim 24, or  $D_3$  is disjoint from one of the paths,  $P[u_1, v_1]$  or  $P[u_2, v_2]$ . Now, we replace  $D_1$  by the strong component of  $D - D_3$  containing  $D_1 \cup D_2$ . Observe that this makes the order of  $D_1$  increase. Further, we replace  $D_2$  by  $D_3$  and replace  $C_2$  by a cycle in this new strong digraph. By doing this, Claim 23 remains valid. By repeating the process as long as possible, we reach the situation where all strong components of  $D - (D_1 \cup D_2)$  are trivial, that is  $D - (D_1 \cup D_2)$  is acyclic. We check whether  $D_2 - u_2$  contains a directed cycle. If it contains such a cycle  $C'_2$ , then let  $D'_1$  be the strong component of  $C_1$  in  $D_2 - C'_2$  and  $D'_2$  the strong component of  $C'_2$  in  $D - D'_1$ . Clearly,  $D'_1$  is a superdigraph of  $D_1 \cup P_1$ , so  $|D'_1| > |D_1|$ . Hence, we replace  $D_1, D_2$  by  $D'_1$  and  $D'_2$ , respectively, and repeat the procedure for the new pair  $D_1, D_2$ . So we may assume that there is no cycle in  $D_2 - u_2$ .

Moreover, if there is a  $(v_2, u_1)$ -dipath  $Q$  whose internal vertices are not in  $V(D_1 \cup D_2)$ , we also check whether there is a cycle in  $D_2 - v_2$ . If yes, then as above we find new pair of non-trivial strong digraphs  $(D'_1, D'_2)$  with  $|D'_1| > |D_1|$ . Hence in that case, we may also assume that there is no cycle in  $D_2 - v_2$ .

Let  $D^*$  be the digraph obtained from  $D$  by contracting  $D_2$  into a single vertex  $w^*$ . We return  $E_2$ -Subdivision( $D^*$ ). The following claim shows that this recursive call is valid.

**Claim 25.**  $D$  contains an  $E_2$ -subdivision if and only if  $D^*$  contains an  $E_2$ -subdivision.

*Subproof.* Suppose that  $D$  contains an  $E_2$ -subdivision  $S$ . Let  $C_1$  and  $C_2$  be the two disjoint directed cycles in  $S$  corresponding to the subdivision of  $aba$  and  $cdc$ . Observe that each  $C_i$  intersects  $D_1 \cup D_2$ , because there is no strong component in  $D - (D_1 \cup D_2)$ .

$C_1$  and  $C_2$  cannot be in both in  $D_2$  for otherwise one of the two avoids  $u_2$ , which is impossible. Moreover, one of the cycles cannot be in  $D_1$  while the other one is in  $D_2$  for otherwise the  $(C_1, C_2)$ - and  $(C_2, C_1)$ -dipaths in  $S$  would contain two  $(D_1, D_2)$ -opposite dipaths in  $D$ , which is impossible. If  $C_1$  and  $C_2$  are both contained in  $D_1$ , then either  $S$  is contained in  $D_1$ , in which case it is also in  $D^*$ , or the arcs of  $S$  which are not in  $A(D_1)$  induce a directed ear  $R$  which intersects with  $D_2$  in a single vertex  $w_2$ , because there are no  $(D_1, D_2)$ -opposite dipaths. Hence the digraph  $S^*$  obtained from  $D$  by replacing the vertex  $w_2$  by  $w^*$  is an  $E_2$ -subdivision in  $D^*$ . So we may assume that one of the cycles intersects  $D_1$  and  $D_2$ .

Case 1: There is no  $(v_2, u_1)$ -dipath whose internal vertices are not in  $V(D_1 \cup D_2)$ . In that case, all  $(D_2, D_1)$ -dipaths are  $(u_2, v_1)$ -dipaths. Therefore, the two cycles  $C_1$  and  $C_2$  cannot both intersect  $D_1$  and  $D_2$ . Thus one of them, say  $C_1$ , does not intersect both, and thus must be contained in  $D_1$ . Consequently,  $C_2$  intersects both  $D_1$  and  $D_2$ . Thus  $C_2$  contains a  $(u_2, v_1)$ -dipath. Therefore, the  $(C_2, C_1)$ -dipath in  $S$  must be in  $D_1$ , and the  $(C_1, C_2)$ -dipath intersects  $D_2$  in  $u_2$ . Therefore the digraph  $S^*$  obtained from  $S$  by contracting the vertices of  $V(S) \cap V(D_2)$  into  $w^*$  is an  $E_2$ -subdivision in  $D^*$ .

Case 2: There is a  $(v_2, u_1)$ -dipath  $Q$  whose internal vertices are not in  $V(D_1 \cup D_2)$ . In this case, all  $(D_1, D_2)$ -dipaths are  $(u_1, u_2)$ - or  $(v_1, v_2)$ -dipaths because there are no  $(D_1, D_2)$ -opposite dipaths in  $D$ . For the same reason, all  $(D_2, D_1)$ -dipaths are  $(u_2, v_1)$ - or  $(v_2, u_1)$ -dipaths. Therefore, the two cycles  $C_1$  and  $C_2$  cannot both intersect  $D_1$  and  $D_2$ . Thus one of them, say  $C_1$ , does not intersect both, and thus must be contained in  $D_1$ . Consequently,  $C_2$  intersects both  $D_1$  and  $D_2$ .

We are in one of the three following cases:  $C_2 = P[u_1, v_1]D_1[v_1, u_1]$ ,  $C_2 = P[u_2, v_2]D_2[v_2, u_2]$ , or  $C = PQ$ . In each of these cases, one can see that the digraph  $S^*$  obtained from  $S$  by contracting the vertices of  $V(S) \cap V(D_2)$  in  $w^*$  contains an  $E_2$ -subdivision in  $D^*$ .

Conversely, suppose that  $D^*$  contains an  $E_2$ -subdivision  $S^*$ . If  $S^*$  does not contain  $w^*$ , then it is contained in  $D$ . So we may assume that  $S^*$  contains  $w^*$ .

Suppose  $w^*$  has in-degree and out-degree 1 in  $S^*$ . Let  $u$  (resp.  $v$ ) be the in-neighbour (resp. out-neighbour) of  $w^*$  in  $S^*$ . By definition of  $D^*$ , the vertex  $u$  has an out-neighbour  $u'_2$  in  $D_2$  and the vertex  $v$  has an in-neighbour  $v'_2$  in  $D_2$ . Hence the digraph  $S$  obtained from  $S^*$  by replacing the dipath  $uw^*v$  by the dipath  $uD[u'_2, v'_2]v$  is an  $E_2$ -subdivision in  $D$ .

Suppose  $w^*$  has in-degree 1 and out-degree 2 in  $S^*$ . Let  $u$  be the in-neighbour of  $w^*$  in  $S^*$  and let  $v$  and  $v'$  be the out-neighbours of  $w^*$  in  $S$ . By definition of  $D^*$ , the vertex  $u$  has an out-neighbour  $u'_2$  in  $D_2$  and the vertex  $v$  (resp.  $v'$ ) has an in-neighbour  $v'_2$  (resp.  $v''_2$ ) in  $D_2$ . Let  $P$  be a  $(u'_2, v'_2)$ -dipath in  $D_2$  and  $Q$  be a  $(P, v''_2)$ -dipath in  $D$ . The digraph  $S$  obtained from  $S^*$  by replacing the vertex  $w^*$  by  $P \cup Q$  is an  $E_2$ -subdivision in  $D$ .

If  $w^*$  has in-degree 2 and out-degree 1, we find an  $E_2$ -subdivision in  $D$  in a similar way.  $\diamond$

Let us now estimate the time complexity of  $E_2$ -Subdivision. The procedure first constructs the digraphs  $D_1$  and  $D_2$ . It requires to find two disjoint directed cycles and then to compute two strong components. By Corollary 1.25 this can be done in time  $O(n^3m(n+m))$ . Next, the algorithm checks a few times for opposite paths, and for directed cycles, before either increasing the order of  $D_1$  or making a recursive call. Checking if there are  $(D_1, D_2)$ -opposite paths can be done in  $O(n(n+m))$  time by running searches in  $D \setminus A(D_1 \cup D_2)$  from each vertex, and finding if there is a directed cycle in a digraph can be done in  $O(n(n+m))$  time by checking for each vertex  $v$  if there is a  $(v, v)$ -handle. Thus, since the order of  $D_1$  increases at most

$O(n)$  times, there are at most  $O(nm(n+m))$  such operations between two recursive calls. Hence the time between two recursive calls is at most  $O(n^3m(n+m))$ . At each call, the order of the instance digraph decreases. Hence the time complexity of  $E_2$ -Subdivision is  $O(n^4m(n+m))$ .  $\square$

### 3.5.3 $E_9$ is tractable

**Theorem 3.16.**  *$E_9$  is tractable.*

Given two vertices  $c'$  and  $d'$  in  $D$ , a  $(c', d')$ -forced  $E_9$ -subdivision is an  $E_9$ -subdivision in  $D$  with  $c$ -vertex  $c'$  and  $d$ -vertex  $d'$ .

To prove Theorem 3.16, we shall describe a polynomial-time algorithm to solve  $E_9$ -SUBDIVISION. The key ingredient of our algorithm is a polynomial-time procedure  $E_9$ -Strong $+(D, \{c_1, c_2\}, \{d_1, d_2\})$  whose input is a strong digraph  $D$  and two sets of two vertices  $\{c_1, c_2\}, \{d_1, d_2\}$  ( $c_1 \neq c_2$  and  $d_1 \neq d_2$ ). Let  $\hat{D}(\{c_1, c_2\}, \{d_1, d_2\})$  be the digraph obtained from  $D$  by adding two new vertices  $c'', d''$  and the four arcs  $c_1c'', c_2c'', d''d_1, d''d_2$ . The procedure  $E_9$ -Strong $+(D, \{c_1, c_2\}, \{d_1, d_2\})$  returns ‘yes’ if it finds an  $E_9$ -subdivision in  $\hat{D}$ , and returns ‘no’ if  $\hat{D}$  has no  $(c'', d'')$ -forced  $E_9$ -subdivision.

Before describing the procedure  $E_9$ -Strong $+$ , let us describe the algorithm for  $E_9$ -SUBDIVISION, assuming we have such a procedure.

#### 3.5.3.1 The algorithm

For every vertex  $c'$  and  $d'$ , we run a procedure  $E_9$ -Forced $(c', d')$  that returns ‘yes’ if it finds an  $E_9$ -subdivision, and return ‘no’ if it finds evidence that there is no  $(c', d')$ -forced  $E_9$ -subdivision in  $D$ . Since there are  $O(n^2)$  possible choices of  $c'$  and  $d'$ , if  $E_9$ -Forced runs in polynomial time, the overall algorithm will also run in polynomial time.

$E_9$ -Forced $(c', d')$  proceeds as follows. We first compute the strong components  $G_1, \dots, G_p$  of  $D - \{c', d'\}$ . Observe that the directed cycle (corresponding to  $aba$ ) in an  $E_9$ -subdivision must be contained in one of the strong components. For each strong component  $G_i$ , we run a procedure  $E_9$ -Suitable $(c', d', G_i)$  that returns ‘yes’ if it finds an  $E_9$ -subdivision and returns ‘no’ only if there is no  $(c', d')$ -forced  $E_9$ -subdivision whose directed cycle is in  $G_i$ . Such a subdivision is called  $(c', d', G_i)$ -suitable.

We first test if there is a  $(d', c')$ -dipath in  $D - G_i$ . We then run two separate procedures depending on whether or not such a path exists.

Case 1: Assume there is a  $(d', c')$ -dipath  $P$  in  $D - G_i$ . Let  $X$  be the set of vertices  $x \in V(G_i)$  that are terminal vertices of a  $(d', G_i)$ -dipath in  $D$ . The set  $X$  can be computed in linear time by running a search from  $d'$  in the digraph obtained from  $D$  by deleting all the arcs having their tail in  $G_i$ . Let  $Y$  be the set of vertices  $y \in V(G_i)$  that are initial vertices of a  $(G_i, c')$ -dipath in  $D$ . Similarly to  $X$ , the set  $Y$  can be determined in linear time.



If there are no two distinct vertices  $x \in X$  and  $y \in Y$ , then we return ‘no’. Otherwise we return ‘yes’. This is valid according to the following claim.

**Claim 26.** (i) *If there are no two distinct vertices  $x \in X$  and  $y \in Y$ , then  $D$  contains no  $(c', d', G_i)$ -suitable  $E_9$ -subdivision.*

(ii) *If there are two distinct vertices  $x \in X$  and  $y \in Y$ , then  $D$  contains an  $E_9$ -subdivision.*

*Subproof.* (i) If  $D$  contains a  $(c', d', G_i)$ -suitable  $E_9$ -subdivision  $S$ , then consider the directed cycle  $C$  in  $S$ . This cycle is in  $G_i$ . Moreover, in  $S$ , there are two disjoint  $(d', C)$ - and  $(C, c')$ -dipaths  $Q$  and  $Q'$ , respectively. Then the first vertex  $x$  in  $G_i$  along  $Q$  is in  $X$  and the last vertex  $y$  in  $G_i$  along  $Q'$  is in  $Y$ . Since  $Q$  and  $Q'$  are disjoint,  $x$  and  $y$  are distinct.

(ii) Suppose that there are vertices  $x \in X$  and  $y \in Y$ , where  $x \neq y$ . Let  $Q$  be a  $(d', G_i)$ -dipath with terminal vertex  $x$  and let  $Q'$  be a  $(G_i, c')$ -dipath with initial vertex  $y$ . Observe that  $Q$  and  $Q'$  do not intersect because  $G_i$  is a strong component of  $D - \{c', d'\}$ . Since  $G_i$  is strong, there are an  $(x, y)$ -dipath  $R$  and a  $(y, R - y)$ -dipath  $R'$  in  $G_i$ . Now  $P$  contains a  $(Q, Q')$ -dipath  $P'$ . Thus  $P' \cup Q[s(P'), x] \cup Q'[y, t(P')] \cup R \cup R'$  is an  $E_9$ -subdivision.  $\diamond$

Case 2: Assume now there is no  $(d', c')$ -dipath in  $D - G_i$ .

If  $D$  contains a  $(c', d', G_i)$ -suitable  $E_9$ -subdivision  $S$ , then the two independent  $(d', c')$ -dipaths in  $S$  intersect  $G_i$ . Since  $G_i$  is a strong component of  $D - \{c', d'\}$ , each of these two paths consists of three segments: a  $(d', G_i)$ -dipath, followed by a dipath in  $G_i$ , and ending with a  $(G_i, c')$ -dipath. The idea is to guess which are the first vertices  $d_1, d_2$  and last vertices  $c_1, c_2$  in  $G_i$  along these dipaths.

Hence, consider every pair of sets of two vertices,  $\{c_1, c_2\}, \{d_1, d_2\}$ , where  $c_1 \neq c_2$  and  $d_1 \neq d_2$ . Observe that there must be two independent  $(d', \{d_1, d_2\})$ -dipaths whose internal vertices are not in  $G_i$  and two independent  $(\{c_1, c_2\}, c')$ -dipaths whose internal vertices are not in  $G_i$ . We can check for the existence of two independent  $(d', \{d_1, d_2\})$ -dipaths with internal vertices not in  $G_i$  by running a Menger algorithm in the digraph obtained from  $D$  by deleting all the arcs with tail in  $G_i$ . Similarly, we also check the existence of two independent  $(\{c_1, c_2\}, c')$ -dipaths with internal vertices not in  $G_i$ . If one of these pairs of dipaths do not exist, then we proceed to the next pair. If the two pairs of dipaths exist, they are internally disjoint from each other since  $G_i$  is a strong component. In that case, one can easily see that  $D$  contains a  $(c', d', G_i)$ -suitable  $E_9$ -subdivision  $S$  such that  $d_1, d_2$  (resp.  $c_1, c_2$ ) are the first (resp. last) vertices in  $G_i$  along the two independent  $(d', c')$ -dipaths in  $S$  if and only if the digraph  $\hat{G}_i(\{c_1, c_2\}, \{d_1, d_2\})$  has no  $(c', d'')$ -forced  $E_9$ -subdivision. Henceforth, we run  $E_9$ -Strong $+$  $(G_i, \{c_1, c_2\}, \{d_1, d_2\})$ . If this procedure returns ‘yes’, we also return ‘yes’. If it returns ‘no’, we proceed to the next pair of sets  $\{c_1, c_2\}, \{d_1, d_2\}$ .

If all the pairs have been considered without returning ‘yes’, we return ‘no’. This procedure is clearly valid provided that we have  $E_9$ -Strong $+$  subroutine.

Hence our algorithm is valid and runs in polynomial time provided that the procedure  $E_9\text{-Strong+}$  is valid and runs in polynomial time. We now describe this subprocedure.

### 3.5.3.2 Detecting $E_9$ in strong digraphs

We now present procedure  $E_9\text{-Strong+}(D, \{c_1, c_2\}, \{d_1, d_2\})$ . Recall that procedure  $E_9\text{-Strong+}(D, \{c_1, c_2\}, \{d_1, d_2\})$  returns ‘yes’ if it finds an  $E_9$ -subdivision in  $\hat{D}$ , and should return ‘no’ if  $\hat{D}$  has no  $(c'', d'')$ -forced  $E_9$ -subdivision. The assumption is that the input digraph  $D$  is strongly connected.

In the first phase, we treat the case when  $D$  is not 2-connected and reduce to the case when it is. Suppose that  $D$  has a cutvertex  $x$ . Let  $X_1, \dots, X_p$  be the connected components of  $D - x$ , and for  $1 \leq i \leq p$ , let  $D_i = D[X_i \cup \{x\}]$ . Observe that each  $D_i$  is strong because  $D$  is strong.

Suppose first that  $c_1$  and  $c_2$  lie in different connected components of  $D - x$ , say  $X_1$  and  $X_2$  (respectively). Let  $P_1$  be an  $(x, c_1)$ -dipath in  $D_1$ ,  $P'_1$  a  $(c_1, P_1 - c_1)$ -dipath, and  $P_2$  a  $(c_2, x)$ -dipath in  $D_2$ . The digraph  $P_1 \cup P'_1 \cup P_2 \cup c_1 c'' \cup c_2 c''$  is an  $E_9$ -subdivision in  $\hat{D}$ , and we return ‘yes’. Similarly, if  $d_1$  and  $d_2$  are in different connected components of  $D - x$ ,  $\hat{D}$  contains an  $E_9$ -subdivision, and we return ‘yes’.

Henceforth, we may assume that there is  $i, j \in \{1, \dots, p\}$  such that  $\{c_1, c_2\} \subseteq V(D_i)$  and  $\{d_1, d_2\} \subseteq V(D_j)$ . If  $i \neq j$ , then in  $D$ , there cannot be two internally disjoint  $(d'', c'')$ -dipaths, and thus there is no  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}$ . Therefore, we return ‘no’. If  $i = j$ , then since  $E_9$  is 2-connected, there is a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}$  if and only if there is a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}_i(\{c_1, c_2\}, \{d_1, d_2\})$ . Hence we return  $E_9\text{-Strong+}(D_i, \{c_1, c_2\}, \{d_1, d_2\})$ .

Assume now that  $D$  is 2-connected, and so  $D$  is robust. The procedure uses a similar approach as the procedure  $E_2\text{-Subdivision}(D)$  to decide whether a digraph  $D$  contains an  $E_2$ -subdivision, and a key notion is the one of opposite dipaths. Recall that two dipaths are  $(D_1, D_2)$ -opposite if they are disjoint and one of them is a  $(D_1, D_2)$ -dipath and the other is a  $(D_2, D_1)$ -dipath. Since an  $E_2$ -subdivision contains an  $E_9$ -subdivision, Lemma 3.15 implies directly the following one.

**Lemma 3.17.** *Let  $D$  be a digraph and  $D_1$  and  $D_2$  disjoint non-trivial strong subdigraphs of  $D$ . If there are  $(D_1, D_2)$ -opposite paths in  $D$ , then  $D$  contains an  $E_9$ -subdivision.*

**Lemma 3.18.** *Suppose that  $D'$  is a strong subdigraph of  $D$  and  $R$  is a path in  $D$  with its end-vertices in  $D'$  and with its internal vertices in  $D - D'$ . If the path  $R$  has three blocks, then  $D$  contains an  $E_9$ -subdivision.*

*Proof.* Let  $s = s(R)$ ,  $t = t(R)$ . Let  $Q$  be a  $(t, s)$ -dipath in  $D'$  and let  $Q'$  be an  $(s, Q - s)$ -dipath in  $D'$ . Then  $Q \cup Q' \cup R$  is an  $E_9$ -subdivision in  $D$ .  $\square$

Returning to the algorithm description, we first check if there are two disjoint directed cycles in  $D$ . If not, then one can solve the problem in polynomial time according to Theorem 1.26.

If two such cycles  $C_1$  and  $C_2$  exist, then we first compute the strong component  $D_1$  of  $C_1$  in  $D - C_2$ , and next we compute the strong component  $D_2$  of  $C_2$  in  $D - D_1$ . Hence  $D_1$  and  $D_2$  are two disjoint non-trivial strong subdigraphs in  $D$ . Moreover they satisfy the following property (Claim 23).

**Claim 27.** *If  $P$  is a  $(D_1, D_2)$ -dipath and  $Q$  is a  $(D_2, D_1)$ -dipath, then  $P$  and  $Q$  are internally disjoint.*

We check if there are  $(D_1, D_2)$ -opposite paths in  $D$ . If there are, then by Lemma 3.17,  $D$  contains an  $E_9$ -subdivision and we return ‘yes’. Henceforth we may assume that there are no  $(D_1, D_2)$ -opposite paths in  $D$ .

By Lemma 3.14, there is a directed ear  $P_1$  of  $D_1$ . Since  $D_1 \cup P_1$  is strong,  $P_1$  must intersect  $D_2$ . Furthermore, the intersection of  $P_1$  and  $D_2$  is reduced to a single vertex, because there are no  $(D_1, D_2)$ -opposite paths. Let  $u_1$  be the initial vertex of  $P_1$ ,  $v_1$  the terminal vertex of  $P_1$ , and let  $u_2$  be the vertex of  $P_1 \cap D_2$ . By Lemma 3.14, there is a directed ear  $P_2$  of  $D_2$ . If the terminal vertex of  $P_2$  is  $u_2$ , then we consider the converse of  $\hat{D}$ ,  $P_1$  and  $P_2$  and exchange the roles of  $c''$  and  $d''$  (i.e.  $(c'', d'') := (d'', c'')$ ) and their neighbours  $(\{c_1, c_2\}, \{d_1, d_2\}) := (\{d_1, d_2\}, \{c_1, c_2\})$ . (This is valid since  $E_9$  is self-converse.) Hence, we may assume that the terminal vertex  $v_2$  of  $P_2$  is different from  $u_2$ . Similarly to  $P_1$ , the dipath  $P_2$  intersects  $D_1$  in a single vertex  $w_1$ . Clearly,  $w_1 = v_1$  for otherwise  $P_1[u_2, v_1]$  and  $P_2[w_1, v_2]$  are  $(D_1, D_2)$ -opposite paths. Furthermore,  $P_1[u_1, u_2]$  and  $P_2[v_1, v_2]$  are disjoint for otherwise, there are two  $(D_1, D_2)$ -opposite dipaths.

Set  $P = P_1 P_2[v_1, v_2]$ . We check whether  $D - (D_1 \cup D_2)$  contains a non-trivial strong component  $D_3$ . If  $D_3$  does not intersect  $P[u_1, v_1]$ , then we replace  $D_1$  and  $D_2$  by two disjoint non-trivial strong digraphs,  $D_1 \cup D_2 \cup P[u_1, v_1]$  and  $D_3$ , respectively. Similarly, if  $D_3$  does not intersect  $P[u_2, v_2]$ , then we replace  $D_1, D_2$  by disjoint non-trivial strong digraphs  $D_1 \cup D_2 \cup P[u_2, v_2]$  and  $D_3$ . In either case, we extend the first digraph to a strong component of  $D - D_3$ , while  $D_3$  is already a strong component in the complement of the first digraph. Thus, Claim 27 remains valid. Observe that this change makes the order of  $D_1$  increase. We also redefine  $u_1, v_1, u_2, v_2$  and the path  $P$  if the change occurred.

If  $D_3$  intersects both  $P[u_1, v_1]$  and  $P[u_2, v_2]$ , then we return ‘yes’. This is valid because in this case, by Claim 24,  $D$  contains an  $E_2$ -subdivision and so an  $E_9$ -subdivision.

Henceforth, we may assume that all strong components of  $D - (D_1 \cup D_2)$  are trivial, that is  $D - (D_1 \cup D_2)$  is acyclic.

Let  $F_{12}$  be the set of pairs  $(x, y) \in V(D_1) \times V(D_2)$  such that there exists a  $(D_1, D_2)$ -dipath  $R$  with  $s(R) = x$  and  $t(R) = y$ . Similarly, let  $F_{21}$  be the set of pairs  $(y, x) \in V(D_2) \times V(D_1)$  such that there exists a  $(D_2, D_1)$ -dipath  $R$  with  $s(R) = y$  and  $t(R) = x$ . By Claim 27 and because there are no  $(D_1, D_2)$ -opposite dipaths, we have one of the following two possible outcomes:

**Case (A):**  $F_{21} = \{(u_2, v_1)\}$ . In this case,  $F_{12} = (U_1 \times \{u_2\}) \cup (\{v_1\} \times V_2)$ , where  $\{u_1\} \subseteq U_1 \subseteq V(D_1)$  and  $\{v_2\} \subseteq V_2 \subseteq V(D_2)$ .

**Case (B):**  $F_{21} = \{(u_2, v_1), (v_2, u_1)\}$ . In this case,  $F_{12} = \{(u_1, u_2), (v_1, v_2)\}$ . By setting  $U_1 = \{u_1\}$  and  $V_2 = \{v_2\}$ , the set  $F_{12}$  can be written in the same way as in Case (A).

For each vertex  $x \in V(D) \setminus V(D_1 \cup D_2)$ , there is an  $(x, D_1 \cup D_2)$ -dipath and a  $(D_1 \cup D_2, x)$ -dipath. Since  $D - (D_1 \cup D_2)$  has only trivial strong components, these two paths are internally disjoint and form a  $(D_1, D_2)$ -path  $R_x$ . We define  $Z(U_1, u_2)$  as the set of all vertices  $x \in V(D) \setminus V(D_1 \cup D_2)$ , whose path  $R_x$  is a  $(U_1, u_2)$ -dipath. In the same way we define vertex-sets  $Z(u_2, v_1)$ ,  $Z(v_1, V_2)$ , and  $Z(v_2, u_1)$ . Note that the latter set may be non-empty only when we have Case (B) and that these four sets partition  $V(D) \setminus V(D_1 \cup D_2)$ .

Next, we derive a sufficient condition for existence of  $E_9$ -subdivisions in  $\hat{D}$ .

**Claim 28.** *If there is a  $(D_1, D_2)$ -path with two or three blocks in  $\hat{D}$ , then  $\hat{D}$  contains an  $E_9$ -subdivision.*

*Subproof.* Let  $R$  be a  $(D_1, D_2)$ -inpath with two blocks and let  $y$  be the vertex of out-degree 2 in  $R$ . Let  $w_1$  be a vertex in  $\{u_1, v_1\} \setminus \{s(R)\}$  and let  $w_2 = u_2$  if  $w_1 = u_1$  and  $w_2 = v_2$  if  $w_1 = v_1$ . If  $P[w_1, w_2]$  is disjoint from  $R$ , then the path  $R \cup P[w_1, w_2] \cup D_2[w_2, t(R)]$  has three blocks and by Lemma 3.18,  $\hat{D}$  contains an  $E_9$ -subdivision. On the other hand, if  $P[w_1, w_2]$  intersects  $R$ , let  $z$  be the first vertex on  $P[w_1, w_2]$  that lies on  $R$ . Since  $D_1$  is a strong component of  $D - D_2$ ,  $z \in R[y, t(R)]$  and  $z \neq y$ . Therefore,  $R[s(R), z]$  and  $P[w_1, z]$  form a path with three blocks and we are done by Lemma 3.18.

Similarly, by directional duality, if there is a  $(D_1, D_2)$ -outpath with two blocks, then  $\hat{D}$  contains an  $E_9$ -subdivision.

Suppose now that there is a  $(D_1, D_2)$ -outpath  $R$  in  $\hat{D}$  with three blocks. Let  $s = s(R)$ ,  $t = t(R)$ , and let  $x$  and  $y$  be vertices on  $R$  whose in-degree and out-degree (respectively) is equal to 2.  $R[s, x] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$  because  $D_1$  is a strong component of  $D - D_2$ . If  $R[x, t] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$ , then there is a  $(D_1, D_2)$ -path with two blocks and we have the result by the above case.

Hence, we may assume that  $R$  does not intersect  $(Z(u_2, v_1) \cup Z(v_2, u_1))$ . In particular,  $R$  and  $P[u_2, v_1]$  are internally disjoint. If  $s \neq v_1$ , then  $RD[t, u_2]P[u_2, v_1]$  is an ear of  $D_1$  with three blocks and by Lemma 3.18  $D$  contains an  $E_9$ -subdivision. Similarly, if  $t \neq u_2$ , then  $D$  contains an  $E_9$ -subdivision. Henceforth we may assume that  $s = v_1$  and  $t = u_2$ .

If  $R$  is internally disjoint from  $P[u_1, u_2]$ , then  $P[u_1, v_1] \cup D_1[v_1, u_1] \cup R$  is an  $E_9$ -subdivision in  $\hat{D}$ , and if  $R$  is internally disjoint from  $P[v_1, v_2]$ , then  $P[u_2, v_2] \cup D_2[v_2, u_2] \cup R$  is an  $E_9$ -subdivision in  $\hat{D}$ . Thus, we may assume that  $R$  intersects both  $P[u_1, u_2]$  and  $P[v_1, v_2]$ .

Let  $z$  be the first vertex on  $P$  that belongs to  $R$ . If  $z \in R[s, y[$ , then  $P[u_1, z]R[z, t]$  is a  $(D_1, D_2)$ -dipath with three blocks, and we get the result as above because its initial vertex is not  $v_1$ . If  $z \in R]y, t]$ , then  $P[u_1, z] \cup R \cup P[u_2, v_1] \cup D[v_1, u_1]$  is an  $E_9$ -subdivision. Therefore, we may assume that  $z = y$ .

Analogously, we may assume that the last vertex on  $P[v_1, v_2]$  that belongs to  $R$  is  $x$ . Now  $P[u_1, y]R[y, x]P[X, v_2]$  contains a  $(D_1, D_2)$ -dipath with initial vertex  $u_1$  and terminal vertex  $v_2$ . By Claim 27, this dipath and  $P[u_2, v_1]$  are  $(D_1, D_2)$ -opposite dipaths, a contradiction.

Similarly, by directional duality, if there is a  $(D_1, D_2)$ -inpath with three blocks, then  $\hat{D}$  contains an  $E_9$ -subdivision. This completes the proof.  $\diamond$

**Claim 29.** (i) For  $i = 1, 2$ , if  $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$  and  $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$ , then  $\hat{D}$  contains an  $E_9$ -subdivision.

(ii) For  $i = 1, 2$ , if  $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$  and  $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$ , then  $\hat{D}$  contains an  $E_9$ -subdivision.

(iii) If  $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$  or  $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ , then  $\hat{D}$  contains an  $E_9$ -subdivision.

*Subproof.* (i) Suppose  $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$  and  $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$ . Without loss of generality, we may assume that  $i = 1$  and that  $c_1 \in V(D_1)$  and  $c_2 \notin V(D_1)$ . Since  $D$  is strong, there is a  $(D_1 \cup D_2, c_2)$ -dipath  $Q$  and a  $(c_2, D_1 \cup D_2)$ -dipath  $R$  in  $D$ . If  $s(Q) \in D_2$ , then  $c_1 c'' c_2 \overleftarrow{Q}$  is a  $(D_1, D_2)$ -path in  $\hat{D}$  with two blocks, so by Claim 28,  $\hat{D}$  contains an  $E_9$ -subdivision. If  $s(Q) \in D_1$ , then  $t(R) \in D_1$ , because  $D_1$  is a strong component in  $D - D_2$ . Hence  $c_1 c'' c_2 R$  is a  $(D_1, D_2)$ -path with two blocks, so by Claim 28,  $\hat{D}$  contains an  $E_9$ -subdivision.

(ii) This claim is proved analogously to (i).

(iii) Assume that  $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ . (The case when  $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$  is proved in the same way.) Since  $D$  is strong, there exist a  $(D_1 \cup D_2, c_1)$ -dipath  $Q_1$ , and a  $(D_1 \cup D_2, c_2)$ -dipath  $Q_2$ . If  $s(Q_1)$  and  $s(Q_2)$  do not lie in the same  $D_i$ , then there is a  $(D_1, D_2)$ -path (which is either contained in  $Q_1 \cup Q_2$  if  $Q_1$  and  $Q_2$  intersect, or passes through  $c''$  if they are disjoint) in  $\hat{D}$  having two blocks, so by Claim 28,  $\hat{D}$  contains an  $E_9$ -subdivision. Henceforth, we may assume that  $s(Q_1)$  and  $s(Q_2)$  are in the same  $D_i$ , say  $D_1$ .

Since  $D$  is strong, for  $i = 1, 2$ , there exists a  $(c_i, D_1 \cup D_2, )$ -dipath  $R_i$ . Its end-vertex  $t(R_i)$  cannot be in  $D_1$ , because  $D_1$  is a strong component of  $D - D_2$ . Thus  $t(R_i) \in V(D_2)$ . If  $R_1$  intersects  $Q_2$  and  $R_2$  intersects  $Q_2$ , then  $c_1$  and  $c_2$  are in the same strong component of  $D - (D_1 \cup D_2)$ , which contradicts one of our previous assumptions. Therefore, without loss of generality, we may assume that  $R_2$  does not intersect  $Q_1$ . Now  $Q_1 \cup c_1 c'' \cup c'' c_2 \cup R_2$  is a  $(D_1, D_2)$ -path with three blocks. Thus by Claim 28,  $\hat{D}$  contains an  $E_9$ -subdivision.  $\diamond$

In view of Claim 29, if  $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$  and  $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$  or  $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$  and  $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$  for some  $i \in \{1, 2\}$ , then we return ‘yes’. The same holds if  $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$  or  $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ . Thus, we may assume henceforth that there are indices  $i_c, i_d \in \{1, 2\}$  such that  $\{c_1, c_2\} \subseteq V(D_{i_c})$  and  $\{d_1, d_2\} \subseteq V(D_{i_d})$ .

We now run a procedure `2or3blocks( $D, D_1, D_2$ )` for finding a  $(D_1, D_2)$ -path with two or three blocks in  $D$ . If such a path is found, we stop the main procedure

$E_9$ -strong+ by returning ‘yes’ since in this case we have an  $E_9$ -subdivision by Claim 28.

The procedure `2or3blocks`( $D, D_1, D_2$ ) proceeds as follows. Let  $S_1^-$  be the in-section of  $D_1$  in  $D - D_2$  and  $S_2^-$  the in-section of  $D_2$  in  $D - D_1$ . It is easy to see that there is a  $(D_1, D_2)$ -inpath with two blocks, if and only if  $S_1^- \cap S_2^-$  contains a vertex of  $D - (D_1 \cup D_2)$ . Therefore we compute  $S_1^- \cap S_2^-$  and return ‘yes’ if this set contains a vertex in  $D - (D_1 \cup D_2)$ .

Similarly, to detect if there is a  $(D_1, D_2)$ -outpath with two blocks, we compute the out-section  $S_1^+$  of  $D_1$  in  $D - D_2$  and the out-section  $S_2^+$  of  $D_2$  in  $D - D_1$ , and if  $S_1^+ \cap S_2^+$  contains a vertex of  $D - (D_1 \cup D_2)$ , we return ‘yes’.

Let us now describe how to discover paths with three blocks. Let  $tz$  be an arc and  $y$  be a vertex in  $D$  such that  $t \in V(D - D_2)$ ,  $y, z \in V(D - (D_1 \cup D_2))$ ,  $y \notin \{t, z\}$ . Arc  $tz$  and vertex  $y$  are said to be in *3-block-position* if there are a  $(y, z)$ -dipath and a  $(y, D_2)$ -dipath in  $D - (V(D_1) \cup \{t\})$  which are independent, and a  $(D_1, t)$ -dipath in  $D - (D_2 \cup \{y, z\})$ .

**Claim 30.** *There is a  $(D_1, D_2)$ -outpath in  $D$  with three blocks if and only if there are an arc  $tz$  and a vertex  $y$  in 3-block-position.*

*Subproof.* Trivially, if there is a  $(D_1, D_2)$ -outpath with three blocks, then there are an arc  $tz$  and a vertex  $y$  in 3-block-position.

Let us now prove the converse. Assume that  $tz$  and  $y$  are in 3-block-position. Let  $Q_1$  and  $Q_2$  be the two independent paths from  $y$  to  $z$  and  $D_2$ , respectively, and let  $R$  be the  $(D_1, t)$ -dipath in  $D - (D_2 \cup \{y, z\})$ .

If  $R$  does not intersect  $Q_1 \cup Q_2$ , then  $R \cup tz \cup \overleftarrow{Q_1} \cup Q_2$  is a  $(D_1, D_2)$ -path with three blocks.

Assume now that  $R$  intersects  $Q_1 \cup Q_2$ . Let  $x$  be the first vertex along  $R$  in  $V(Q_1 \cup Q_2)$ . Note that  $x \neq y$  by definition of  $R$ . If  $x \in V(Q_1)$ , then  $R[s(R), x] \cup \overleftarrow{Q_1}[x, y] \cup Q_2$  is a  $(D_1, D_2)$ -path with three blocks. If  $x \in V(Q_2)$ , then consider a  $(z, D_1 \cup D_2)$ -dipath  $R'$  in  $D$ . Because  $D_1$  is a strong component of  $D - D_2$  and  $RtzR'$  is a dipath,  $t(R') \in D_2$ . Furthermore  $R'$  does not meet  $R \cup Q_1 \cup Q_2[y, x]$ , because  $D - (D_1 \cup D_2)$  is acyclic. Hence  $R[s(R), x] \cup \overleftarrow{Q_2}[x, y] \cup Q_1 \cup R'$  is  $(D_1, D_2)$ -path with three blocks.  $\diamond$

Therefore, for every possible arc  $tz$  and vertex  $y$  such that  $t \in V(D - D_2)$ ,  $y, z \in V(D - (D_1 \cup D_2))$ ,  $y \notin \{t, z\}$ , we check if they are in 3-block-position. This can be done by running Menger algorithm. If we find an arc and a vertex in 3-block-position, then we return ‘yes’ because there is an  $E_9$ -subdivision by Claims 30 and 28.

We deal similarly with the  $(D_1, D_2)$ -inpaths with three blocks. This ends the procedure `2or3blocks`( $D, D_1, D_2$ ). After it, there is no  $(D_1, D_2)$ -path in  $\hat{D}$  with two blocks and no  $(D_1, D_2)$ -path in  $D$  with three blocks.

We now show that we can reduce  $D$  to a digraph with vertex set  $V(D_1) \cup V(D_2)$ .

Let  $D^*$  be the digraph obtained from  $D_1 \cup D_2$  by adding all arcs in  $F_{12} \cup F_{21}$ . In other words, we add all arcs  $x_1x_2$  with  $x_1 \in V(D_1)$  and  $x_2 \in V(D_2)$  such that

there is a  $(D_1, D_2)$ -dipath with initial vertex  $x_1$  and terminal vertex  $x_2$ , and adding all arcs  $x_2x_1$  with  $x_1 \in V(D_1)$  and  $x_2 \in V(D_2)$  for which there is a  $(D_2, D_1)$ -dipath with initial vertex  $x_2$  and terminal vertex  $x_1$ . Set  $\hat{D}^* = \hat{D}^*(\{c_1, c_2\}, \{d_1, d_2\})$ .

**Claim 31.**  $\hat{D}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision if and only if  $\hat{D}^*$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision.

*Subproof.* As mentioned above,  $D^*$  either contains the four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$  (Case (B)), or contains the arcs  $uu_2$  ( $u \in U_1$ ),  $u_2v_1$ , and  $v_1v$  ( $v \in V_2$ ), which is Case (A). For each of these arcs  $uv$ , there is a corresponding directed path  $R_{uv}$  in  $D$ . One can transform a  $(c'', d'')$ -forced  $E_9$ -subdivision  $S^*$  in  $\hat{D}^*$  into an  $E_9$ -subdivision  $S$  of  $\hat{D}$  by replacing each arc  $uv$  in  $S^*$  between  $D_1$  and  $D_2$  by the path  $R_{uv}$ . If all added paths  $R_{uv}$  are pairwise internally disjoint, this clearly gives rise to an  $E_9$ -subdivision in  $\hat{D}$ . The only possibility that two of such paths may not be internally disjoint (cf. Lemma 3.17) is that we have two paths  $R_{uu_2}$  and  $R_{u'v_2}$  (where  $u, u' \in U_1$  and  $u \neq u'$ ) or two paths  $R_{v_1v}$  and  $R_{v_1v'}$  (where  $v, v' \in V_2$  and  $v \neq v'$ ). However, since every vertex in  $E_9$  has in- and out-degrees at most 2, there are at most two such paths entering  $u_2$  and at most two leaving  $v_1$ . For two of them, we can always achieve that their intersection is a common subpath, and in that case, the resulting digraph is again an  $E_9$ -subdivision. Clearly, the resulting  $E_9$ -subdivision in  $\hat{D}$  is  $(c'', d'')$ -forced.

Suppose now that  $\hat{D}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision  $S$ . Let  $a', b'$  be the vertices of  $S$  corresponding to  $a, b$ , respectively, and let  $C$  be the directed cycle in  $S$ . If  $a', b' \in V(D_1) \cup V(D_2)$ , then the arcs in  $S$  that are not in  $D_1 \cup D_2$  form a collection of internally disjoint  $(D_1, D_2)$ - and  $(D_2, D_1)$ -dipaths. By replacing each of these dipaths by the corresponding arc in  $D^*$ , we obtain an  $E_9$ -subdivision in  $\hat{D}^*$ .

Assume now that  $a' \notin V(D_1) \cup V(D_2)$ . The cycle  $C$  must intersect both  $D_1$  and  $D_2$ , and thus  $C$  contains a  $(D_2, D_1)$ -dipath. Without loss of generality, we may assume that the initial vertex of this dipath is  $u_2$  and its terminal vertex is  $v_1$ .

Now let  $z_2$  be the first vertex in  $V(D_1 \cup D_2)$  along the  $(a', c'')$ -dipath in  $S$ . This vertex exists because  $\{c_1, c_2\} \subseteq V(D_1 \cup D_2)$ . Now since there are no  $(D_1, D_2)$ -opposite paths, and by definition of the  $D_i$ ,  $z_2 \in V(D_2)$ , so  $a'$  does not lie in  $C[u_2, v_1]$ . Let  $y_2$  be the first vertex after  $a'$  along  $C$  in  $V(D_1 \cup D_2)$ . For the same reason,  $y_2 \in V(D_2)$  and so  $v_1$  is the unique vertex in  $C \cap D_1$ , for otherwise, there would be  $(D_1, D_2)$ -opposite paths. Note that  $y_2, z_2 \in V_2$ .

If  $\{c_1, c_2\} \subseteq V(D_1)$ , then the  $(z_2, c'')$ -dipath in  $S$  contains a  $(D_2, D_1)$ -dipath that together with  $C[v_1, y_2]$  gives  $(D_1, D_2)$ -opposite paths.

This shows that  $\{c_1, c_2\} \subseteq V(D_2)$ . The  $(d'', c'')$ -dipath in  $S - E(C)$  must have all its internal vertices in  $D_2$ , because every  $(D_1, D_2)$ -dipath meets  $\{u_2, v_1\}$ . Therefore the digraph obtained from  $S \cap (D_1 \cup D_2)$  by adding the arcs  $u_2v_1, v_1y_2$ , and  $v_1z_2$  is an  $E_9$ -subdivision in  $\hat{D}^*$ .

If  $b' \notin V(D_1) \cup V(D_2)$ , we get the result analogously.  $\diamond$

In view of Claim 31, we replace  $D$  by  $D^*$ . Henceforth, now  $V(D) = V(D_1) \cup V(D_2)$ . Moreover, there are at most two arcs with tail in  $D_2$  and head in  $D_1$ , namely

$u_2v_1$  and possibly  $v_2u_1$ .

Let  $D'_1$  be the digraph obtained from  $D$  by contracting  $D_2$  into a vertex  $z_2$ . If all four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$  are present in  $D$  (Case (B)), then we also add into  $D'_1$  the arcs  $u_1v_1$  and  $v_1u_1$  if they are not already contained in  $D_1$ . Similarly, we let  $D'_2$  be the digraph obtained from  $D$  by contracting  $D_1$  into a vertex  $z_1$  and adding the arcs  $u_2v_2$  and  $v_2u_2$  if  $D$  contains all four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ . Observe that  $D'_1$  and  $D'_2$  are both strong, and contain fewer vertices than  $D$ .

If  $i_c = i_d$ , then we return  $E_9\text{-Strong}+(D'_{i_c}, \{c_1, c_2\}, \{d_1, d_2\})$ . This is valid by the following claim.

**Claim 32.** *If  $i_c = i_d$ , then  $\hat{D}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision if and only if  $\hat{D}_{i_c}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision.*

*Subproof.* We shall assume that  $i_c = 1$ . (The case when  $i_c = 2$  is proved in the same way.) Suppose first that  $\hat{D}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision  $S$ . If  $S$  does not intersect  $D_2$ , then  $S$  is a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$ . Henceforth we assume that  $S$  intersects  $D_2$ . Observe that vertices  $a'$  and  $b'$  corresponding to  $a$  and  $b$  in  $S$  belong to a  $(d'', c'')$ -dipath in  $S$ . Therefore,  $a'$  and  $b'$  belong to  $D_1$ , since there are no  $(D_1, D_2)$ -opposite dipaths in  $D$ . Consequently, every vertex in  $S$  lies on a  $(d'', c'')$ -dipath or on an  $(a', b')$ -dipath in  $S$ , and each such path intersects  $D_2$  in at most one vertex. If  $S$  contains only one vertex  $v$  in  $D_2$ , the digraph  $S'$  obtained from  $S$  by replacing  $v$  by  $z_2$ , we obtain a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$ . If  $S$  contains two vertices in  $D_2$ , then  $S$  contains all four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$  and hence  $D'_1$  contains the arcs  $u_1v_1$  and  $v_1u_1$ . Moreover,  $\{a', b'\} = \{u_1, v_1\}$  and the cycle in  $S$  is the 4-cycle  $u_1u_2v_1v_2u_1$ . Therefore, the arcs  $u_1v_1$  and  $v_1u_1$  are not both in  $S$ . Then we replace the cycle in  $S$  by the cycle  $u_1v_1z_2u_1$  in  $D'_1$  and obtain a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$ .

Suppose now that  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision  $S'$ . Let us first assume that  $S'$  does not contain  $z_2$ . If  $S'$  contains an arc that is not in  $A(D)$ , then this is either the arc  $e = u_1v_1$  or  $e' = v_1u_1$ . This arc was added to  $D'_1$  only in Case (B). Thus, we can replace  $e$  in  $S'$  by the path  $u_1u_2v_1$  and  $e'$  by the path  $v_1v_2u_1$ . By making these changes (if needed), we obtain a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}$ . Henceforth we assume that  $S$  contains  $z_2$ .

If  $S'$  contains an arc  $e \in \{v_1u_1, u_1, v_1\}$  and  $e \notin A(D)$ , then we have Case (B). In  $S'$ , the vertex  $z_2$  has in- and out-degree equal to 1, while each of  $u_1$  and  $v_1$  has either in- or out-degree equal to 2. Therefore, the arc in  $\{v_1u_1, u_1, v_1\} \setminus \{e\}$  cannot be in  $S'$ . By replacing the path in  $S'$  joining  $u_1$  and  $v_1$  through  $z_2$  by that arc, we obtain an  $E_9$ -subdivision that does not contain  $z_2$ , and we are done in the same way as above. Thus, we may assume that  $S'$  contains no edge in  $\{v_1u_1, u_1, v_1\} \setminus A(D)$ .

The vertex  $z_2$  has an in-neighbour  $x_1$  and an out-neighbour  $y_1$  in  $S'$ , and possibly has a third neighbour  $z_1$ . If  $z_1$  exists, then we assume that the arcs  $x_1z_2$  and  $z_2y_1$  lie on the cycle in  $S'$ . By definition of contraction,  $x_1$  has an out-neighbour  $x_2$  in  $V(D_2)$  and  $y_1$  has an in-neighbour  $y_2$  in  $V(D_2)$ . Moreover if  $z_1$  exists, let  $w_2$  be one of its out-neighbours (resp. in-neighbours) in  $V(D_2)$  corresponding to the arc



joining  $z_1$  and  $z_2$  in  $D'_1$ . Let  $Q$  be an  $(x_2, y_2)$ -dipath in  $D_2$  and  $Q'$  be a  $(w_2, Q)$ -dipath (resp.  $(Q, w_2)$ -dipath) in  $D_2$  if  $z_2 z_1 \in A(S')$  (resp.  $z_1 z_2 \in A(S')$ ). Now the digraph obtained from  $S'$  by replacing  $z_2$  and the arcs incident to it by the paths  $Q$  and  $Q'$ , and the arcs  $x_1 x_2$ ,  $y_1 y_2$  and  $z_1 w_2$  or  $w_2 z_1$ , is a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}$ .  $\diamond$

Henceforth, we have  $i_c \neq i_d$ . If  $i_d = 2$  and  $i_c = 1$ , then a  $(c'', d'')$ -forced  $E_9$ -subdivision contains two disjoint arcs from  $D_2$  to  $D_1$ . Thus, necessarily  $v_2 u_1$  is an arc, because there are no  $(D_1, D_2)$ -opposite paths. In this case, we consider exchanging the roles of  $D_1$  and  $D_2$ . Thus, we may assume henceforth that we are in the case when  $i_d = 1$  and  $i_c = 2$ .

Let  $D_1^* = D[V(D_1) \cup \{u_2\}]$  and  $D_2^* := D[V(D_2) \cup \{v_1\}]$ . Observe that  $D_1^*$  and  $D_2^*$  are both strong.

A  $(c'', d'')$ -forced  $E_9$ -subdivision contains two internally disjoint  $(d'', c'')$ -dipaths. Therefore, using a Menger algorithm, we check if two such dipaths exist in  $\hat{D}$ . If two such dipaths do not exist, then we return 'no'. Otherwise there are two internally disjoint  $(d'', c'')$ -dipaths. Because there are no  $(D_1, D_2)$ -opposite paths, one of them say  $P_1^*$  must go through  $v_1$  and the other, say  $P_2^*$ , through  $u_2$ . We return  $(E_9\text{-Strong}+(D_1^*, \{v_1, u_2\}, \{d_1, d_2\}))$  or  $(E_9\text{-Strong}+(D_2^*, \{c_1, c_2\}, \{v_1, u_2\}))$ . This is valid by the following claim.

**Claim 33.**  $\hat{D}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision if and only if either  $\hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$  or  $\hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision.

*Subproof.* Set  $\hat{D}_1^* := \hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$  and  $\hat{D}_2^* := \hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$ .

A  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}_1^*$  (or one in  $\hat{D}_2^*$ ) can easily be transformed into a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}$  by replacing the arcs  $v_1 c''$  and  $u_2 c''$  by  $P_1^*[v_1, c'']$  and  $P_2^*[u_2, c'']$ .

Suppose now that  $\hat{D}$  contains a  $(c'', d'')$ -forced  $E_9$ -subdivision  $S$ . Let  $C$  be the directed cycle in  $S$  and let  $Q_1$  and  $Q_2$  be the two internally disjoint  $(d'', c'')$ -dipaths in  $S$ . Because there are no  $(D_1, D_2)$ -opposite paths, one of these dipaths, say  $Q_1$  goes through  $v_1$  and the other goes through  $u_2$ . Moreover,  $C$  intersects either  $D_1$  or  $D_2$  in at most one vertex. If  $D$  does not contain the arc  $v_2 u_1$  (Case (A)), then if  $C$  intersects  $D_1$  (resp.  $D_2$ ) in one vertex, this vertex must be  $v_1$  (resp.  $u_2$ ). We may assume that the same holds in Case (B) after possibly exchanging the roles of  $u_1$  and  $v_1$  and of  $u_2$  and  $v_2$ . Hence the digraph obtained from  $S$  by replacing  $Q_1[d'', v_1]$  and  $Q_2[d'', u_2]$  (resp.  $Q_1[v_1, c'']$  and  $Q_2[u_2, c'']$ ) by the arcs  $d'' v_1$  and  $d'' u_2$  (resp.  $v_1 c''$  and  $u_2 c''$ ) is a  $(c'', d'')$ -forced  $E_9$ -subdivision in  $\hat{D}_2^*$  (resp.  $\hat{D}_1^*$ ).  $\diamond$

This completes the procedure  $E_9\text{-Strong}+$ . Let us now examine its time complexity. Let  $T(n)$  be the maximum running time on a digraph with at most  $n$  vertices. Clearly, the running time between two recursive calls is bounded by a polynomial  $P(n)$ . When treating a graph  $D$  on  $n$  vertices, it then makes a recursive call either to a smaller digraph, or to two smaller digraphs  $D_1^*$  and  $D_2^*$  such that

$|D_1^*| + |D_2^*| \leq n + 2$ . Hence  $T(n)$  satisfies the inequality

$$T(n) \leq P(n) + \max \{T(n-1) ; \max \{T(n_1) + T(n_2) \mid n_1 + n_2 \leq n + 2, n_1 < n, n_2 < n\}\}.$$

This implies that  $T(n)$  is bounded above by a polynomial value in  $n$ .

### 3.5.3.3 Detecting $E_{10}$

**Proposition 3.19.** *The digraph  $E_{10}$  depicted in Figure 3.5 is tractable. More precisely,  $E_{10}$ -SUBDIVISION can be solved in time  $O(n^2m(n+m))$ .*

*Proof.* Let  $D$  be a digraph. Observe that every  $E_{10}$ -subdivision contains an  $E_{10}$ -subdivision in which the arc  $cd$  is not subdivided. Henceforth by  $E_{10}$ -subdivision, we mean such a subdivision.

Given four distinct vertices  $a', b', c', d'$  such that  $c'd'$  is an arc, we say that an  $E_{10}$ -subdivision is  $(a', b', c'd')$ -forced if  $a'$  is its  $a$ -vertex,  $b'$  its  $b$ -vertex,  $c'$  its  $c$ -vertex, and  $d'$  its  $d$ -vertex.

We shall present a procedure  $E_{10}$ -SUBDIVISION( $D, a', b', c'd'$ ), that returns ‘no’ only if there is no  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision in  $D$ , and returns ‘yes’ if it finds an  $E_{10}$ -subdivision in  $D$  (not necessarily one that is  $(a', b', c'd')$ -forced). We proceed as follows.

Suppose first that  $a'b'$  is an arc. Using a Menger algorithm, we check whether there are two independent  $(b', \{a', c'\})$ -dipaths in  $D - d'$ , and using a search, we check whether there exists a  $(c', a')$ -dipath in  $D' - \{b', d'\}$ . If three such dipaths do not exist, then there is no  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision in  $D$ , and we return ‘no’. If three such dipaths exist, then we return ‘yes’. This is valid by virtue of the following claim.

**Claim 34.** *If there are a  $(c', a')$ -dipath  $R$  in  $D' - \{b', d'\}$  and two independent  $(b', \{a', c'\})$ -dipaths  $P_1, P_2$  in  $D - d'$ , then  $D$  contains an  $E_{10}$ -subdivision.*

*Subproof.* Without loss of generality, we may assume that  $t(P_1) = a'$  and  $t(P_2) = c'$ . The dipath  $R$  contains a subdipath  $R'$  with initial vertex  $s$  in  $P_2$  and terminal vertex in  $P_1]b', a']$ . Let  $s^+$  be the out-neighbour of  $s$  in  $P_2 \cup c'd'$ . Then  $a'b' \cup P_1 \cup P_2[b', s] \cup R' \cup ss^+$  is an  $E_{10}$ -subdivision.  $\diamond$

Henceforth, we assume that  $a'b'$  is not an arc in  $D$ .

If  $d_{D-\{c', d'\}}^+(a') = 0$ , then there is no  $(a', b')$ -dipath in  $D - c' - d'$ , and thus no  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision. Hence we return ‘no’.

If  $d_{D-\{c', d'\}}^+(a') = 1$ , then denote by  $a''$  the unique out-neighbour of  $a'$  in  $D - \{c', d'\}$ . By our assumption,  $a'' \neq b'$ . Let  $D^*$  be the digraph obtained from  $D$  by first removing all arcs entering  $a''$  and then identifying  $a'$  and  $a''$  into a single vertex  $a^*$ . Note that  $a^*$  is dominated by the in-neighbours of  $a'$  in  $D$  and dominates the out-neighbours of  $a''$  in  $D$ . We return  $E_{10}$ -SUBDIVISION( $D^*, a^*, b', c'd'$ ). The validity of this recursive call is shown by the following claim.

**Claim 35.** *If  $d_{D-\{c',d'\}}^+(a') = 1$ , then  $D$  contains an  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision if and only if  $D^*$  contains an  $(a^*, b', c'd')$ -forced  $E_{10}$ -subdivision.*

*Subproof.* Assume that  $S$  is an  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision in  $D$ . Since  $d_{D-\{c',d'\}}^+(a') = 1$ ,  $S$  contains the arc  $a'a''$  since  $a''$  is the unique out-neighbour of  $a'$  in  $D' - c' - d'$ . Now the digraph  $S^*$  obtained from  $S$  by replacing  $a'$  and  $a''$  and the four arcs  $ua', va', a'a'', a''w$  by the vertex  $a^*$  and the three arcs  $ua^*, va^*, a^*w$  is an  $(a^*, b', c'd')$ -forced  $E_{10}$ -subdivision in  $D^*$ , because  $a'' \neq b'$ .

Conversely, if  $S^*$  is an  $(a^*, b', c'd')$ -forced  $E_{10}$ -subdivision in  $D^*$ , then the digraph  $S$  obtained from  $S^*$  by replacing  $a^*$  and its three incident arcs  $ua^*, va^*, a^*w$  by vertices  $a'$  and  $a''$  and the four arcs  $ua', va', a'a'', a''w$  is clearly an  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision in  $D$ .  $\diamond$

Henceforth, we may assume that  $d_{D-\{c',d'\}}^+(a') \geq 2$ . Using a Menger algorithm, we check whether there are two independent  $(b', \{a', c'\})$ -dipaths in  $D - d'$ , and using a search we check whether there exists an  $(a', b')$ -dipath in  $D - \{c', d'\}$ , and whether there exists a  $(c', a')$ -dipath in  $D - \{b', d'\}$ . If four such dipaths do not exist, then there is no  $(a', b', c'd')$ -forced  $E_{10}$ -subdivision in  $D$ , and we return ‘no’. If four such dipaths exist, then we return ‘yes’ by virtue of the following claim.

**Claim 36.** *If there are two independent  $(b', \{a', c'\})$ -dipaths  $P_1$  and  $P_2$  in  $D - d'$ , an  $(a', b')$ -dipath  $Q$  in  $D - \{c', d'\}$ , and a  $(c', a')$ -dipath  $R$  in  $D - \{b', d'\}$ , then  $D$  contains an  $E_{10}$ -subdivision.*

*Subproof.* Without loss of generality, we may assume that  $t(P_1) = a'$  and  $t(P_2) = c'$ . Let  $v$  be the last vertex along  $Q - b'$  that is in  $P_1 \cup P_2$ . We distinguish two cases according to whether  $v$  is on  $P_1$  or  $P_2$ .

Case 1:  $v \in V(P_1)$ . Note that this is in particular the case when  $Q$  is internally disjoint from  $P_1$  and  $P_2$ . Let  $C$  be the directed cycle formed by the union of  $P_1[b'v]$  and  $Q[v, b']$ , let  $Q'$  be the  $(a', C)$ -subdipath in  $Q$ , and let  $R' = RQ'$ . The directed walk  $R'$  contains a subdipath  $R''$  with initial vertex  $s$  in  $P_2[b', c']$  and terminal vertex  $t$  in  $C$ . Let  $s^+$  be the out-neighbour of  $s$  in  $P_2 \cup c'd'$ . Then  $C \cup P_2[b', s] \cup R'' \cup ss^+$  is an  $E_{10}$ -subdivision.

Case 2:  $v \in V(P_2)$ . Let  $v^+$  be the out-neighbour of  $v$  in  $P_2 \cup c'd'$ . The dipath  $Q[a', v]$  contains a subdipath  $Q'$  with initial vertex  $u$  in  $P_1$  and terminal vertex  $w$  in  $P_2[b', v]$  whose internal vertices are not in  $P_1 \cup P_2[b', v]$ . Let  $C'$  be the directed cycle formed by the union of  $P_2[b', v]$  and  $Q[v, b']$ . If  $u \neq a'$ , let  $u^+$  be the out-neighbour of  $u$  in  $P_1$ . Then  $C' \cup P_1[b', u^+] \cup Q'$  is an  $E_{10}$ -subdivision. Henceforth, we may assume that  $u = a'$ .

Let  $u'$  be the out-neighbour of  $a'$  in  $Q'$ . Now  $d_{D-\{c',d'\}}^+(a') \geq 2$  and  $a'b'$  is not an arc. Hence,  $a'$  has an out-neighbour  $z$  distinct from  $b', c', d'$ , and  $u'$ .

- If  $z \notin V(C' \cup P_1 \cup Q')$ , then  $C' \cup P_1 \cup Q' \cup a'z$  is an  $E_{10}$ -subdivision.
- If  $z \in V(Q')$ , then  $C' \cup P_1 \cup a'z \cup Q'[z, w] \cup a'u'$  is an  $E_{10}$ -subdivision.

- Assume  $z \in V(P_1)$ . If  $v^+ \notin V(Q')$ , then  $P_1 \cup a'z \cup Q' \cup P_2[w, v] \cup Q[v, b'] \cup vv^+$  is an  $E_{10}$ -subdivision. If  $v^+ \in V(Q')$ , then  $v^+ \neq d'$  and so  $P_2[v^+, c']$  is not an empty dipath. Denote by  $C''$  the directed cycle  $P_1[z, a'] \cup a'z$ . The dipath  $Q[v^+, b'] \cup P_1[b', z]$  contains a  $(P_2[v^+, c'], C'' - a')$ -dipath  $Q''$ . Let  $s^+$  be the out-neighbour of  $s(Q'')$  in  $P_2 \cup c'd'$ . Now  $C'' \cup Q[a', v^+] \cup P_2[v^+, s(Q'')] \cup Q'' \cup s(Q'')s^+$  is an  $E_{10}$ -subdivision.
- Assume  $z \in V(Q[v, b'])$ . Then one can replace the  $(a', b')$ -dipath  $Q$  by  $a'zQ[z, b']$ . This dipath is internally disjoint from  $P_1$  and  $P_2$ , and we get the result by Case 1.
- Assume finally that  $z \in V(P_2[b', v])$ . If  $u' \neq w$ , then  $C' \cup P_1 \cup a'z \cup a'u'$  is an  $E_{10}$ -subdivision. Henceforth we assume that  $u' = w$ , so  $a'w$  is an arc. Without loss of generality, we may assume that  $z$  precedes  $w$  along  $P_2$ . For  $i = 1, 2$ , let  $b_i^+$  be the out-neighbour of  $b'$  in  $P_i$ . By the previous assumption,  $b_2^+ \neq w$ . Let  $t$  be the last vertex along  $R - a'$  in  $V(P_1 \cup P_2 \cup Q[v, b'])$ .

If  $t \in V(P_1)$ , then one of the two dipaths  $P_1[t, a']$  and  $R[t, a']$  has length at least 2. Let  $t^+$  be the out-neighbour of  $t$  in this dipath, and let  $T$  be the other dipath. Now  $C' \cup P_1[b', t] \cup T \cup a'w \cup tt^+$  is an  $E_{10}$ -subdivision.

If  $t \in V(Q[v, b'] \cup P_2[w, v])$ , then  $R[t, a'] \cup a'w \cup P_2[w, v] \cup Q[v, b'] \cup P_1 \cup b'b_2^+$  is an  $E_{10}$ -subdivision.

If  $t \in V(P_2[v, c'])$ , then  $R[t, a'] \cup a'w \cup P_2[w, t] \cup Q[v, b'] \cup P_1 \cup b'b_2^+$  is an  $E_{10}$ -subdivision.

If  $t \in V(P_2[z, w])$ , then  $R[t, a'] \cup a'z \cup P_2[z, v] \cup Q[v, b'] \cup vv^+ \cup P_1$  is an  $E_{10}$ -subdivision.

It remains to consider the case when  $t \in V(P_2[b', z])$ . Let  $t^+$  be the out-neighbour of  $t$  on  $P_2$ . Then  $P_1 \cup P_2[b', t^+] \cup R[t, a'] \cup a'w \cup P_2[w, v] \cup Q[v, b']$  is an  $E_{10}$ -subdivision in  $D$ .

◇

One can easily see that the procedure  $E_{10}\text{-Subdivision}(D, a', b', c'd')$  runs in linear time as it either reduces the problem in constant time (when  $d^+(a') = 1$ ) or runs a Menger algorithm and at most two searches, which can be done in linear time. Running this procedure for the  $O(n^2m)$  possible choices of  $(a', b', c'd')$ , we obtain an algorithm with running time  $O(n^2m(n+m))$  that solves  $E_{10}\text{-SUBDIVISION}$ .  $\square$

### 3.5.4 $E_{16}$ is tractable

**Theorem 3.20.**  $E_{16}\text{-SUBDIVISION}$  can be solved in  $O(n^7(n+m))$  time.

The proof relies on the following notion. A *shunt* is a digraph composed by three dipaths  $P$ ,  $Q$  and  $R$  such that  $|R| \geq 2$ ,  $s(R) \in P$ ,  $t(R) \in Q$  and  $P, Q, R^0$  are disjoint. We frequently refer to a shunt by the triple  $(P, Q, R)$ . An  $(S, T)$ -*shunt* is a shunt  $(P, Q, R)$  such that  $\{s(P), s(Q)\} = S$  and  $\{t(P), t(Q)\} = T$ .

We consider the following decision problem.

SHUNT

Input: A digraph  $D$  and four distinct vertices  $s_1, s_2, t_1, t_2$ .

Question: Does  $D$  contain an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt?

Assume that there are two disjoint dipaths  $P, Q$  from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$  in  $D$ . We now give some sufficient conditions considering  $P$  and  $Q$  for  $D$  to have a  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt .

**Proposition 3.21.** *If there is a dipath  $R$  of length at least 2 between  $P$  and  $Q$ , then  $D$  has an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.*

*Proof.* If such a dipath  $R$  exists, then  $(P, Q, R)$  or  $(Q, P, R)$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.  $\square$

For any vertex  $x$  in  $V(P)$ , an  $x$ -bypass is a dipath  $S$  internally disjoint from  $P$  and  $Q$  with initial vertex in  $P[s(P), x[$  and terminal vertex in  $P]x, t(P)]$ . Similarly, for any vertex  $x$  in  $V(Q)$ , an  $x$ -bypass is a dipath  $S$  internally disjoint from  $P$  and  $Q$  with initial vertex in  $Q[s(Q), x[$  and terminal vertex in  $Q]x, t(Q)]$ . An *arc bypass* is an  $x$ -bypass such that  $x$  is the end-vertex of an arc between  $P$  and  $Q$ .

**Proposition 3.22.** *If there is an arc bypass for some arc  $uv$  between  $P$  and  $Q$ , then  $D$  has an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.*

*Proof.* If  $S$  is a  $u$ -bypass, then  $(P[s(P), s(S)] \cup S \cup P[t(S), t(P)], Q, P[s(S), u] \cup (u, v))$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt if  $u \in V(P)$ , and there is a shunt constructed analogously if  $u \in V(Q)$ .

If  $S$  is a  $v$ -bypass,  $(Q, P[s(P), s(S)] \cup S \cup P[t(S), t(P)], (u, v) \cup P[v, t(S)])$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt if  $v \in V(P)$  (similarly if  $v \in V(Q)$ ).  $\square$

A *crossing* (with respect to  $P$  and  $Q$ ) is a pair of arcs  $\{uv, u'v'\}$  such that  $u$  is before  $v'$  along  $P$  and  $u'$  is before  $v$  along  $Q$ . Moreover, if  $uv'$  is an arc of  $P$  and  $u'v$  is an arc of  $Q$ , then the crossing is *tight*. Otherwise it is *loose*.

**Proposition 3.23.** *If there is a loose crossing, then there is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt in  $D$ .*

*Proof.* Let  $\{uv, u'v'\}$  be a loose crossing. By symmetry, we may assume that  $uv'$  is not an arc. Then  $(P[s(P), u] \cup (u, v) \cup Q[v, t(Q)], Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)], P[u, v'])$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.  $\square$

Let  $C = \{uv, u'v'\}$  be a tight crossing. A  $C$ -forward path is a dipath internally disjoint from  $P$  and  $Q$  either with initial vertex in  $u$  and terminal vertex in  $v'$ , or with initial vertex in  $u'$  and terminal vertex in  $v$ .

**Proposition 3.24.** *If there is a  $C$ -forward path, then there is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt in  $D$ .*

*Proof.* Observe that since a forward path  $S$  has length at least 2, because  $uv'$  and  $u'v$  are arcs, then  $(P[s(P), u] \cup (u, v) \cup Q[v, t(Q)], Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)], S)$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt if  $S$  is a dipath from  $u$  to  $v'$ , and  $(Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)], P[s(P), u] \cup (u, v) \cup Q[v, t(Q)], S)$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt if  $S$  is a dipath from  $u'$  to  $v$ .  $\square$

Still considering a tight crossing  $C = \{uv, u'v'\}$ , a  $C$ -backward path is a dipath internally disjoint from  $P$  and  $Q$  either with initial vertex in  $P[v', t(P)]$  and terminal vertex in  $P[s(P), u]$ , or with initial vertex in  $Q[v, t(Q)]$  and terminal vertex in  $Q[s(Q), u']$ . A  $C$ -backward arc is an arc that induces a  $C$ -backward path of length 1. A  $C$ -bypass is an  $x$ -bypass  $B$  such that  $x$  is an end-vertex of a  $C$ -backward arc and if  $x \in P[s(P), u](Q[s(Q), u'])$ ,  $t(B)$  is also in  $P[s(P), u](Q[s(Q), u'])$ , or if  $x \in P[v', t(P)](Q[v, t(Q)])$ ,  $s(B)$  is also in  $P[v', t(P)](Q[v, t(Q)])$ . A crossing bypass is a  $C$ -bypass for some tight crossing  $C$ .

**Proposition 3.25.** *If there is a backward path of length at least 2 or a crossing bypass in  $P$  or  $Q$ , then  $D$  has an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.*

*Proof.* Suppose  $C = \{uv, u'v'\}$  is a tight crossing. Then we first check if there is a  $C$ -backward path of length at least 2. If there is such a backward path  $R$ , the union of  $P[s(P), u] \cup (u, v) \cup Q[v, t(Q)]$ ,  $Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)]$ , and  $R$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. So assume that all backward paths have length 1, and thus are arcs. For each tight crossing  $C = \{uv, u'v'\}$ , we check if there is a  $C$ -bypass. If there is such a  $C$ -bypass  $B$ , by symmetry and directional duality, we may assume that  $B$  is an  $x$ -bypass with  $t(B) \in P[s(P), u]$  and that there is a  $C$ -backward arc  $a$  one end of which is  $x$ . If  $a = wx$  for some  $w \in P[v', t(P)]$ , then  $(Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)]), P[s(P), s(B)] \cup B \cup P[t(B), u] \cup (u, v) \cup Q[v, t(Q)], (w, x) \cup P[x, u])$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. If  $a = xw$  for some  $w \in P[v', t(P)]$ , then  $(Q[s(Q), u'] \cup (u', v') \cup P[v', t(P)]), P[s(P), s(B)] \cup B \cup P[t(B), u] \cup (u, v) \cup Q[v, t(Q)], P[s(B), x] \cup (x, w))$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.  $\square$

We now prove that the conditions of Propositions 3.21, 3.22, 3.23, 3.24 and 3.25 are also necessary.

**Lemma 3.26.** *Let  $D$  be a digraph, and  $P$  and  $Q$  be two disjoint dipaths from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$ . If there is no arc bypasses, no loose crossings, no forward paths, no backward paths of length at least 2, no crossing bypasses, and no dipaths of length at least 2 between  $P$  and  $Q$ , then  $D$  contains no  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.*

*Proof.* Suppose for a contradiction that  $D$  is as in the statement, but it contains an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt  $(P', Q', R')$ . Without loss of generality, we may assume that this shunt maximizes  $|(A(P) \cup A(Q)) \cap (A(P') \cup A(Q'))|$ . Free to swap the names of  $P$  and  $Q$ , we may assume that  $s(P) = s(P')$ .

Consider the paths  $P$  and  $Q$ . Let  $u$  be the farthest vertex along  $P'$  such that  $P'[s(P'), u]$  does not intersect  $Q$ . Necessarily  $u \in V(P)$  for otherwise there would be a dipath of length at least 2 from  $P$  to  $Q$ . In addition, for the same reason, if

$u \neq t(P)$ , then the out-neighbour  $v$  of  $u$  in  $P'$  must be in  $Q$ . Hence all vertices of  $P'[s(P'), u] \cap P$  are in  $P[s(P), u]$ , for otherwise there would be a  $u$ -bypass in  $P$ , which would be an arc bypass for  $uv$ . Note also that for every vertex  $x$  in  $P[s(P), u] - P'$  there is a subdipath of  $P'$  which is an  $x$ -bypass. So there is no  $x \in Q'$  in  $P[s(P), u]$ , for otherwise there would be an arc bypass in  $P$  for the arc starting in  $y \in Q$  and ending in  $x$  (there would be one since  $Q'$  and  $Q$  intersects at least in the first vertex) or a dipath of length at least 2 from  $Q$  to  $P$ . Let  $R''$  be the shortest subdipath of  $P'$  with initial vertex in  $V(P)$  and terminal vertex  $s(R')$  if  $s(R') \in P'[s(P'), u]$ , and let  $R''$  be the path of length 0 ( $s(R')$ ) otherwise. Now,  $(P'', Q'', R) = (P[s(P), u] \cup P'[u, t(P')], Q', R'' \cup R')$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. Moreover if  $P'[s(P'), u] \neq P[s(P), u]$ , then  $P''$  and  $Q''$  have more arcs in common with  $P$  and  $Q$  than  $P'$  and  $Q'$ , which contradicts our choice of  $(P', Q', R')$ . Therefore  $P'[s(P'), u] = P[s(P), u]$ .

Let  $u'$  be the farthest vertex along  $Q'$  such that  $Q'[s(Q'), u']$  does not intersect  $P$ . As above, one shows that  $Q'[s(Q'), u'] = Q[s(Q), u']$ .

If  $u = t(P)$ , then  $P' = P$  and necessarily  $Q = Q'$ . Thus  $R'$  is a dipath of length at least 2 from  $P$  to  $Q$  as  $(P', Q', R')$  is a shunt, which is a contradiction. Therefore, we may assume that  $u \neq t(P)$  and similarly  $u' \neq t(Q)$ . Furthermore the out-neighbour  $v$  of  $u$  in  $P'$  is in  $V(Q)$  and the out-neighbour  $v'$  of  $u'$  is in  $V(P)$ . Since  $P'$  and  $Q'$  are disjoint,  $P'[s(P'), u] = P[s(P), u]$  and  $Q'[s(Q'), u'] = Q[s(Q), u']$ , it follows that  $C = \{uv, u'v'\}$  is a crossing with respect to  $P$  and  $Q$ , and thus a tight crossing.

Consider the dipath  $R'$ .

- Assume first that  $s(R') \in P'[s(P'), u]$ . Let  $S$  be the shortest subdipath of  $R' \cup Q'[t(R'), t(Q')]$  such that  $s(S) = s(R')$  and  $t(S) \in V(P) \cup V(Q)$ . Vertex  $t(S)$  cannot be in  $Q[s(Q), u']$  for otherwise  $S = R'$  and it would be a dipath of length at least 2 between  $P$  and  $Q$ . Furthermore,  $\{s(R')t(S), u'v'\}$  is a loose crossing, since the distance between  $u'$  and  $t(S)$  in  $Q$  is at least 2 ( $u$  is between  $s(R')$  and  $v$  and  $v$  is between  $u'$  and  $t(S)$ ). Therefore  $t(S) \in V(P)$  and so  $t(S)$  is on  $P[v', t(P)]$ . But then  $S$  is a forward path or an arc bypass in  $P$ , a contradiction.
- Assume now that  $s(R') \in P'[v, t(P')]$ .

Set  $P^* = Q[s(Q), u'] \cup (u', v) \cup P'[v, t(P')]$  and  $Q^* = P[s(P), u] \cup (u, v') \cup Q'[v', t(Q')]$ . If  $t(R') \in Q'[v', t(Q')]$ , then  $(P^*, Q^*, R')$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. But  $P^*$  and  $Q^*$  have more arcs in common with  $P$  and  $Q$  than  $P'$  and  $Q'$ , which contradicts our choice of  $(P', Q', R')$ . Therefore  $t(R') \in Q'[s(Q'), u']$ .

Let  $S$  be the shortest subdipath of  $P'[v, s(R')] \cup R'$  such that  $t(S) = t(R')$  and  $s(S) \in V(P) \cup V(Q)$ .

Assume first that  $s(S) \in V(Q)$ . Then  $S$  is a  $C$ -backward path. Hence it must have length 1. Therefore  $s(S) \notin V(P') \cup V(Q')$  because  $R'$  has length at least 2. Let  $u_1$  be the farthest vertex on  $P'[v, t(P')]$  that is in  $V(Q)$  and such that  $P'[v, u_1]$  does not intersect  $P$ . Observe that  $u_1$  appears before  $s(S)$  in  $Q$ , for

otherwise there would be a crossing bypass in  $P'$ , as  $s(S) \notin P'$ . In particular,  $u_1$  is not the terminal vertex of  $P'$ . Let  $v_1$  be the first vertex after  $u_1$  along  $P'$  which is on  $P \cup Q$ . It must be in  $V(P)$  by the choice of  $u_1$ . Therefore  $u_1v_1$  is an arc because there is no dipath of length at least 2 between  $Q$  and  $P$ . Let  $u_2$  be the farthest vertex on  $Q'[v', t(Q')] \cap P$  such that  $Q'[v', u_2]$  does not intersect  $Q$ . Then  $v_1$  is after  $u_2$  along  $P$ , for otherwise there would be an arc bypass in  $P$  for  $u_1v_1$ . Thus  $u_2$  is not the terminal vertex of  $Q'$ . Let  $v_2$  be the first vertex after  $u_2$  along  $Q'$  which is on  $P \cup Q$ . It must be in  $V(Q)$  by the choice of  $u_2$ . Hence  $u_2v_2$  is an arc because there is no dipath of length at least 2 between  $P$  and  $Q$ . Moreover, observe that for every vertex  $x$  in  $Q[v, u_1] - P'$  there is a subdipath of  $P'$  which is an  $x$ -bypass. Therefore  $v_2$  must be in  $Q[u_1, t(Q)]$  for otherwise it would be an arc bypass. Hence  $\{u_2v_2, u_1v_1\}$  is a crossing for  $P \cup Q$ , and so it must be tight. This implies in particular that  $s(S) \in Q[v_2, t(Q)]$ .

Set  $P^+ = P'[s(P'), u] \cup (u, v') \cup Q[v', u_2] \cup (u_2, v_1) \cup P'[v_1, t(P')]$  and  $Q^+ = Q'[s(Q), u'] \cup (u', v) \cup P'[v, u_1] \cup (u_1, v_2) \cup Q'[v_2, t(Q')]$ . If  $s(R') \in P'[v_1, t(P')]$ , then  $(P^+, Q^+, R')$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. But  $P^+$  and  $Q^+$  have more arcs in common with  $P$  and  $Q$  than  $P'$  and  $Q'$ , which contradicts our choice of  $(P', Q', R')$ . Therefore  $s(R') \in P'[(v, u_1)]$ . Now  $P'[v, s(R')] \cup R'$  contains a subdipath  $T$  that is internally disjoint from  $P$  and  $Q$  and has initial vertex in  $Q[v, u_1]$  and terminal vertex in  $P \cup Q[v_2, t(Q)]$ . Necessarily,  $t(T) \in V(P)$  for otherwise  $T$  is an arc bypass. Hence  $T$  is an arc. Furthermore,  $t(T)$  could not be in  $P[v', u_2]$  for otherwise  $Q'$  would contain a  $t(T)$ -bypass, which would be an arc bypass. Hence  $t(T) \in P[v_1, t(Q)]$  and  $\{u_2v_2, T\}$  is a loose crossing, a contradiction.

Assume now that  $s(S) \in V(P)$ . Then it must be in  $P[v', t(P)]$ . Since there is no dipath of length at least 2 from  $P$  to  $Q$ ,  $S$  has length 1. Moreover, since  $R'$  has length at least 2,  $s(S)$  is an internal vertex of  $R'$ , so it is not in  $V(P' \cup Q')$ . Let  $u_2$  be the farthest vertex on  $Q'[v', t(Q')]$  that is in  $V(P)$  and such that  $Q'[v', u_2]$  does not intersect  $Q$ . Then  $u_2$  appears before  $s(S)$  on  $P$ , for otherwise there would be an arc bypass for  $s(S)t(S)$  in  $P$  and so  $u_2$  is not the terminal vertex of  $Q'$ . Let  $v_2$  be the first vertex after  $u_2$  along  $Q'$  which is on  $P \cup Q$ . It must be in  $V(Q)$  by the choice of  $u_2$ , and so on  $Q[v, t(Q)]$ .  $u_2v_2$  is an arc for otherwise there would be a dipath of length 2 from  $P$  to  $Q$ . Let  $u_1$  be the farthest vertex on  $P'[v, t(P')]$  that is also in  $V(Q)$  such that  $P'[v, u_1]$  does not intersect  $P$ . Vertex  $u_1$  appears before  $v_2$  in  $Q$ , for otherwise there would be an arc bypass for  $u_2v_2$  in  $Q$ , and so  $u_1$  is not the terminal vertex of  $P'$ . Let  $v_1$  be the first vertex after  $u_1$  along  $P'$  which is on  $P \cup Q$ . It must be in  $V(Q)$  by the choice of  $u_1$ . Hence  $u_1v_1$  is an arc because there is no dipath of length at least 2 between  $Q$  and  $P$ . Moreover, observe that for every vertex  $x$  in  $P[v', u_2] - Q'$  there is a subdipath of  $P'$  which is an  $x$ -bypass. Therefore  $v_1$  must be in  $P[u_2, t(P)]$  for otherwise it would be an arc bypass. Hence  $\{u_2v_2, u_1v_1\}$  is crossing for  $P \cup Q$ , and so it must be tight.



This implies in particular that  $s(S) \in P[v_1, t(P)]$ .

We then find a contradiction as in the previous case by considering  $P^+$  and  $Q^+$ .

This finishes the proof of the claim.  $\square$

**Corollary 3.27.** *SHUNT can be solved in polynomial time.*

*Proof.* Now we describe the procedure  $\text{shunt}(D, s_1, s_2, t_1, t_2)$ , solving SHUNT and estimate its time complexity. The procedure then check, by a Menger algorithm, if there are two disjoint dipaths  $P, Q$  from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$ , which runs in  $O(n+m)$  time. Observe that the arcs  $s_1s_2$  and  $s_2s_1$  are useless, so we remove them from  $D$  if they exist. Then we should check if there are paths of length at least 2, arc bypasses, loose crossings,  $C$ -forward paths, backward paths of length at least 2 or crossing bypasses with respect to  $P$  and  $Q$ , according to propositions 3.21, 3.22, 3.23, 3.24, 3.25. For every vertex  $u \in P$  (and any vertex in  $Q$ , similarly), we do the following: if  $u$  has a neighbour in  $Q$ , we test if there is a path from  $P[s(P), u[$  to  $P]u, t(P)]$ , which would be an arc bypass. Let  $v'$  be the last vertex of  $Q$  such that  $uv'$  is an arc (and such that  $v'u$  is an arc, similarly). Then, for a vertex  $v$  in  $P]u, t(P)]$ , we check if there is a vertex  $u'$  in  $Q[s(Q), v'[$  such that  $u'v$  ( $vu'$ ) is an arc. Then if  $u, v$  and  $u'v'$  have distance at least 2 in  $P$  and  $Q$  respectively, it would be a loose crossing. Otherwise, if such edges exists there is a tight crossing  $C = \{uv', u'v\}$  containing  $u$ . We then run a Menger algorithm one more time, to test if there is a dipath from  $u$  to  $v$  in  $D - P - Q$ , which would be a forward path. So far, the running time of the algorithm is bounded by  $O(n^2(n+m))$ : the complexity of calculating the  $P$  and  $Q$  initially plus the complexity of, for each vertex in in  $P \cup Q$ , look for an arc bypass, plus the running time of analysing if each pair of vertices in  $P$  or  $Q$  are part of a loose crossing and finally plus the time of looking for a forward path. Then, still considering the same tight crossing  $C$ , for every vertex  $x$  in  $P[v, t(P)]$ , we check if there is a dipath to some  $y$  in  $P[v, t(P)]$ . If it is the case and  $xy$  is an arc, we then look for dipaths from  $P[s(P), y[$  to  $P]y, u]$  and from  $P[v, x[$  to  $P]x, t(P)]$ . This can be done in  $O(n^2(n+m))$ : for every pair of vertices  $u$  and  $x$ , we uses Menger algorithm possibly three times to compute the dipaths above. So,  $\text{shunt}(D, s_1, s_2, t_1, t_2)$  runs in  $O(n^2(n+m))$  time in total.  $\square$

With Lemma 3.27 in hands, we now deduce Theorem 3.20.

*Proof of Theorem 3.20.* For every vertex  $v$  of  $D$  and for every set of two out-neighbours  $s_1, s_2$  and two in-neighbours  $t_1, t_2$  of  $v$ , we check if there is a  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt in  $D$ . Observe that there is an  $E_{16}$ -SUBDIVISION in  $D$  in which  $v$  is the  $a$ -vertex if and only if there is a shunt for a pair of out-neighbours and a pair of in-neighbours of  $v$ . So, since there are  $n^5$  possible choices for vertex  $v$  and its neighbours, and for each of them we apply the procedure  $\text{shunt}$  that runs in  $O(n^2(n+m))$  time, our algorithm decides whether there is an  $E_{16}$ -SUBDIVISION in  $D$  in  $O(n^7(n+m))$  time.  $\square$

# $F$ -SUBDIVISION for disjoint directed cycles

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Since  $C_k$ -SUBDIVISION can be solved in polynomial time for any fixed  $k$ , a natural question is to ask for the complexity of  $F$ -SUBDIVISION when  $F$  is the disjoint union of directed cycles. We gave the foundations to this discussion in Section 1.6. This is not a simple problem as can be seen from the observation that a digraph  $D$  contains  $k$  disjoint directed cycles if and only if it contains an  $F$ -subdivision where  $F$  is the disjoint union of  $k$  directed 2-cycles. Hence, if  $F$  is the disjoint union of  $k$  directed 2-cycles,  $F$ -SUBDIVISION is equivalent to deciding if  $\nu(D) \geq k$  for a given digraph  $D$ . Reed et al. [47] proved that this can be done in polynomial time.

In this chapter, we first analyse the simpler case of  $C_2 + C_3$ . Then we prove that if  $F$  is the disjoint union of cycles of length at least 3,  $F$ -SUBDIVISION is also polynomial-time solvable, based on the proof of Reed et al. for 2-cycles.

## 4.1 Disjoint union of $C_2$ and $C_3$

**Theorem 4.1** (Bang-Jensen, Havet and M. [3]).  *$(C_2 + C_3)$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* Let  $D$  be a digraph. If  $D$  has no 2-cycles, then  $D$  has a  $C_2 + C_3$ -subdivision if and only if it contains two disjoint cycles. This can be checked in polynomial time by Theorem 1.24.

Assume now that  $D$  contains 2-cycles. For each 2-cycle  $(x, y, x)$ , we check if  $D - \{x, y\}$  has a directed cycle of length at least 3. This can be done in linear time according to Theorem 2.4. If the answer is ‘yes’ for one of them, then we return ‘yes’.

Suppose now that the answer is ‘no’ for all 2-cycles. Let  $D'$  be the digraph obtained from  $D$  by deleting the arcs of all the 2-cycles.

**Claim 37.**  *$D$  contains a  $(C_2 + C_3)$ -subdivision if and only if  $D'$  contains two disjoint directed cycles.*

*Subproof.* Suppose that  $D$  contains a  $(C_2 + C_3)$ -subdivision  $S$ . No cycle of  $S$  can contain two vertices  $x$  and  $y$  in a 2-cycle because  $D - \{x, y\}$  contains no directed cycle of length at least 3. In particular, all the arcs of  $S$  are in  $D'$ .

Conversely, if  $D'$  contains two disjoint directed cycles, they form a  $(C_2 + C_3)$ -subdivision since  $D'$  has no 2-cycles.  $\diamond$

Hence we check if  $D'$  has two disjoint directed cycles, which can be done in polynomial time according to Theorem 1.24.  $\square$

## 4.2 Disjoint union of 3-cycles

In this section, we prove the following theorem:

**Theorem 4.2** (Havet and M. [31]). *For any positive integer  $n$ ,  $n\vec{C}_3$ -SUBDIVISION is polynomial-time solvable.*

We do this by proving both Conjecture 1.27 and Conjecture 1.28 for  $\ell = 3$ . Then the result is implied by virtue of Theorem 1.29.

Combined with Lemma 1.19, this result in turn implies the following.

**Corollary 4.3** (Havet and M. [31]). *If  $F$  is the disjoint union of cycles of length at most 3, then  $F$ -SUBDIVISION is polynomial-time solvable.*

### 4.2.1 Linkage in digraphs with circumference at most 2

The aim of this section is to prove the following theorem, that consists in Conjecture 1.27 for  $\ell = 3$ .

**Theorem 4.4** (Havet and M. [31]). *For each fixed  $k$ , the  $k$ -LINKAGE problem is polynomial-time solvable for digraphs with circumference at most 2.*

We first prove the following lemma.

**Lemma 4.5** (Havet and M. [31]). *Let  $\mathcal{D}$  be a class of digraphs and  $\mathcal{S}$  be the class of strong digraphs. If  $k'$ -LINKAGE is polynomial-time solvable on  $\mathcal{D} \cap \mathcal{S}$  for any  $k' \leq k$ , then  $k$ -LINKAGE is polynomial-time solvable on  $\mathcal{D}$ .*

*Proof.* Let  $D$  be a digraph in  $\mathcal{D}$ . Let  $\sim$  be the relation defined on  $V(D)$  by  $u \sim v$  if and only if  $u$  and  $v$  are in the same strong component. It is clearly an equivalence relation on  $V(D)$  with equivalence classes the strong components of  $D$ . Let  $D/\sim$  be the quotient of  $D$  by  $\sim$ , that is the digraph whose vertices are the strong components of  $D$ , and in which there is an arc from a strong component  $S$  to another  $S'$  if and only if there is an arc of  $D$  with tail in  $S$  and head in  $S'$ . One can also see  $D/\sim$  as the digraph obtained by contracting each strong component into a vertex. It is well-known that  $D/\sim$  is an acyclic digraph, therefore there is an ordering  $S_1, \dots, S_p$  of the strong components such that there is no arc  $S_j S_{j'}$  in  $D/\sim$  with  $j > j'$ . This implies that for every  $j > j'$ , there is no directed  $(x, y)$ -path in  $D$  with  $x \in S_j$  and  $y \in S_{j'}$ . Let  $\tilde{D}$  be the digraph  $D \setminus \bigcup_{j=1}^p A(S_j)$ , the digraph whose arcs are those between non-equivalent vertices with respect to  $\sim$ .

Form a new digraph  $\mathbf{D}$  whose vertices are the  $k$ -tuples  $\mathbf{v} = (v_1, \dots, v_k)$  of distinct vertices of  $D$ . For any such  $k$ -tuple  $\mathbf{v}$ , there is a minimum index  $m$  such that  $S_m$  intersects  $\{v_1, \dots, v_k\}$ . Let  $I = \{i \mid v_i \in S_m\}$ . Set  $I = \{i_1, \dots, i_{k'}\}$  with  $i_1 < i_2 < \dots < i_{k'}$ .

For each  $k'$ -tuple  $(w_1, w_2, \dots, w_{k'})$  of distinct vertices of  $V(D) \setminus \{v_1, v_2, \dots, v_k\}$  such that there exists a  $k'$ -tuple  $(u_1, u_2, \dots, u_{k'})$  of vertices in  $V(S_m)$  such that there is a linkage from  $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}})$  to  $(u_1, u_2, \dots, u_{k'})$  in  $S_m$  and  $u_j w_j$  is an arc in  $\tilde{D}$  for all  $1 \leq j \leq k'$ , we put an arc from  $\mathbf{v}$  to the  $k$ -tuple obtained from it by replacing  $v_i$  by  $w_i$  for all  $i \in I$ . We say that such an arc in  $\mathbf{D}$  is labelled by  $S_m$ .

Observe that there are  $O(n^{k'})$   $k'$ -tuples  $(u_1, u_2, \dots, u_{k'})$  of  $V(S_m)$ , and for each of them one can decide in polynomial time whether there is a linkage from  $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}})$  to  $(u_1, u_2, \dots, u_{k'})$  because  $k'$ -LINKAGE is polynomial-time solvable on  $\mathcal{D}$  by hypothesis. Hence in polynomial time, we can construct the digraph  $\mathbf{D}$  which has polynomial size.

We now prove that for any two sets of  $k$  distinct vertices  $\{x_1, \dots, x_k\}$  and  $\{y_1, \dots, y_k\}$ , there is a  $k$ -linkage from  $(x_1, \dots, x_k)$  to  $(y_1, \dots, y_k)$  if and only if there is a directed path from  $(x_1, \dots, x_k)$  to  $(y_1, \dots, y_k)$  in  $\mathbf{D}$ .

Suppose first that there is a  $k$ -linkage  $(P_1, \dots, P_k)$  from  $(x_1, \dots, x_k)$  to  $(y_1, \dots, y_k)$ . Since, when  $j > j'$ , there are no directed  $(x, y)$ -paths in  $D$  with  $x \in S_j$  and  $y$  in  $S_{j'}$ , each  $P_i$  goes through the strong components  $S_1, \dots, S_p$  in that order, possibly avoiding some. For each  $1 \leq m \leq p$  and each  $1 \leq i \leq k$ , let  $v_i(m)$  the first vertex in  $\bigcup_{j=m}^p S_j$  along  $P_i$  if  $\bigcup_{j=m}^p S_j$  and  $P_i$  intersect, and  $v_i(m) = y_i$  otherwise.

Let  $M = \{m_1, \dots, m_r\}$  with  $m_1 \leq m_2 \leq \dots \leq m_r$ , be the set of indices  $m$  such that  $S_m \cap \bigcup_{i=1}^k P_i \neq \emptyset$ . By definition of  $\mathbf{D}$ ,  $\mathbf{v}(m_q)\mathbf{v}(m_{q+1})$  is an arc in  $\mathbf{D}$ . Thus  $\mathbf{v}(m_1)\mathbf{v}(m_2)\dots\mathbf{v}(m_r)$  is a directed path from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $\mathbf{D}$ .

Suppose now that  $\mathbf{D}$  has a directed path  $\mathbf{Q}$  from  $(x_1, \dots, x_k)$  to  $(y_1, \dots, y_k)$  in  $\mathbf{D}$ . We construct directed walks  $P_i$ ,  $1 \leq i \leq k$ , by the following procedure. At the beginning  $P_i = (x_i)$  for all  $1 \leq i \leq k$ . For each arc  $\mathbf{a} = \mathbf{vw}$  of  $\mathbf{Q}$  one after another from the initial vertex to the terminal vertex of  $\mathbf{Q}$ , we do the following.

Let  $I = \{i_1, \dots, i_{k'}\}$  be set of indices  $i$  such that  $v_i \neq w_i$ . By definition of  $\mathbf{D}$ , there is a strong component  $S$  of  $D$ , a  $k'$ -tuple  $(u_1, \dots, u_{k'})$  of disjoint vertices of  $S$ , and such that there is a linkage  $(R_1, \dots, R_{k'})$  from  $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}})$  to  $(u_1, u_2, \dots, u_{k'})$  in  $S$  and  $u_j w_{i_j} \in A(\tilde{D})$  for all  $1 \leq j \leq k'$ . In that case, we extend each  $P_{i_j}$ ,  $1 \leq j \leq r$ , by appending  $R_j u_j w_{i_j}$  at the end of it. Observe that  $R_j$  might be a path reduced to the single vertex  $v_{i_j} = u_j$ .

Observe that in  $\mathbf{D}$  an arc labelled by a strong component  $S_m$  enters a  $k$ -tuple of vertices that all belong to components  $S_j$  with  $j > m$ . In particular,  $\mathbf{Q}$  contains at most one arc labelled with any strong component  $S_m$ . This implies that each  $P_i$  is a directed  $(x_i, y_i)$ -path. Combined with the fact that each  $(R_1, \dots, R_{k'})$  as defined above is a linkage, it implies that the  $P_i$  are disjoint.  $\square$

We can easily derive Theorem 4.4 from Lemma 4.5.

*Proof of Theorem 4.4.* Let  $\mathcal{C}_2$  be the class of digraphs with circumference at most 2. A strong digraph  $D$  in  $\mathcal{C}_2$  is obtained from a tree  $T$  by replacing every edge by a directed 2-cycle. Hence there is a  $k$ -linkage from  $(x_1, \dots, x_k)$  to  $(y_1, \dots, y_k)$  in  $D$  if and only if there is a  $k$ -linkage from  $(x_1, \dots, x_k)$  to  $(y_1, \dots, y_k)$  in  $T$ . Since

UNDIRECTED  $k$ -LINKAGE is polynomial-time solvable, it follows that  $k$ -LINKAGE is polynomial-time solvable on  $\mathcal{C}_2 \cap S$ . Thus, by Lemma 4.5, it is polynomial-time solvable on  $\mathcal{C}_2$ .  $\square$

## 4.2.2 Packing directed $3^+$ -cycles

The aim of this section is to prove the following theorem, showing that the directed cycles of length at least 3 have the Erdős-Pósa Property. It consists in Conjecture 1.28 for  $\ell = 3$ .

**Theorem 4.6** (Havet and M. [31]). *For every integer  $n \geq 0$ , there exists an integer  $\hat{t}_n$  such that for every digraph  $D$ , either  $\nu_3(D) \geq n$  or  $\tau_3(D) \leq \hat{t}_n$ .*

Our proof follows the same approach as the one used by Reed et al. [47] to demonstrate Theorem 1.20, an analogue of Theorem 4.6 in the case for which 2-cycles are also considered. Their theorem says that, for every integer  $n \geq 0$ , there exists an integer  $t_n$  such that, for every digraph  $D$ , either  $D$  has a  $n$  pairwise-disjoint directed cycles, or there exists a set  $T$  of at most  $t_n$  vertices such that  $D - T$  is acyclic. So, it is easy to see that  $t_n = 0$  for  $n = 0$  or  $n = 1$ . The value of  $t_n$  for the case in which  $n = 2$ , much more complicated to determine, was established to be 3 by McCuaig [45].

The proof of Theorem 1.20 is done by induction on  $n$ . So it may be assumed then that  $n \geq 1$  and the  $t_{n-1}$  exists. To show that  $t_n$  exists, the following two main lemmas of Reed et al. [47] are needed.

**Lemma 4.7** (Reed et al. [47]). *Let  $n \geq 1$  be an integer such that  $t_{n-1}$  exists, and let  $k$  be an integer. Then there exists an integer  $t$  such that the following holds. Let  $D$  be a digraph with  $\nu(D) < n$  and  $\tau(D) \geq t$ . Then there are distinct vertices  $a_1, \dots, a_k, b_1, \dots, b_k$  and two  $k$ -linkages  $L_1, L_2$  in  $D$  so that*

- (i)  $L_1$  links  $(a_1, \dots, a_k)$  to  $(b_1, \dots, b_k)$ ,
- (ii)  $L_2$  links  $(b_1, \dots, b_k)$  to one of  $(a_1, \dots, a_k), (a_k, \dots, a_1)$ ,
- (iii) every directed cycle of  $L_1 \cup L_2$  meets  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ .

**Lemma 4.8** (Reed et al. [47]). *For every non-negative integer  $n$ , there exists a positive integer  $k$  so that the following holds. Let  $D$  be a digraph, and let  $a_1, \dots, a_k, b_1, \dots, b_k$  be distinct vertices of  $D$ . Let  $L_1, L_2$  be linkages in  $D$  linking  $(a_1, \dots, a_k)$  to  $(b_1, \dots, b_k)$ , and  $(b_1, \dots, b_k)$  to one of  $(a_1, \dots, a_k), (a_k, \dots, a_1)$ , respectively. Let every directed cycle of  $L_1 \cup L_2$  meet  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ . Then  $\nu(D) \geq n$ .*

Take  $k$  as in Lemma 4.8 and  $t$  as in Lemma 4.7. Suppose there is a digraph for which  $\nu(D) < n$  and  $\tau(D) \geq t$ . Then the items (i), (ii), (iii) of Lemma 4.7 are satisfied. But in this case, by Lemma 4.8,  $\nu(D) \geq n$ , a contradiction. So, for every digraph, either  $\nu(D) \geq n$  or, if  $\nu(D) < n$ ,  $\tau(D) < t$ , and then there is an upper bound  $t_n$  for  $\tau(D)$  whose value is smaller than  $t$ .

We should establish equivalents of Lemmas 4.7 and 4.8 for  $3^+$ -cycles. However, a key ingredient in their proof is that if  $P$  is a directed  $(a, b)$ -path and  $Q$  is a directed  $(b, a)$ -path, then  $P \cup Q$  contains a directed cycle. But in such a case,  $P \cup Q$  does not necessarily contain a  $3^+$ -cycle. We claim that  $P \cup Q$  does not contain a  $3^+$ -cycle if and only if  $Q$  is the converse of  $P$  (Recall that the converse of a directed path  $P = (x_1, \dots, x_m)$  is the directed path  $(x_m, \dots, x_1)$ ).

**Lemma 4.9** (Havet and M. [31]). *Let  $a$  and  $b$  two distinct vertices, and let  $P$  be a directed  $(a, b)$ -path and  $Q$  be a directed  $(b, a)$ -path. Then  $P \cup Q$  contains a directed  $3^+$ -cycle if and only if  $Q$  is not the converse of  $P$ .*

*Proof.* Clearly, if  $Q$  is the converse of  $P$ , then  $P \cup Q$  contains no directed  $3^+$ -cycle.

Conversely, we prove by induction on the length  $m$  of  $P$  that if  $Q$  is not the converse of  $P$ , then  $P \cup Q$  contains a directed  $3^+$ -cycle. It holds trivially if  $m = 1$ . So we may assume that  $m \geq 1$ . Let  $P = (x_0, x_1, \dots, x_m)$ . Let  $y$  be the penultimate vertex of  $Q$ . If  $y = x_1$ , then  $Q - x_0$  is not the converse of  $P - x_0$ . Hence, by the induction hypothesis, there is a directed  $3^+$ -cycle in  $(P - x_0) \cup (Q - x_0)$ , and so in  $P \cup Q$ . Assume now that  $y \neq x_1$ , then let  $z$  be the penultimate vertex in  $V(P \cap Q)$  along  $Q$ . If  $z = x_1$ , then  $Q[x_1, x_0]$  has length at least 2, and so  $(x_0, x_1) \cup Q[x_1, x_0]$  is a directed  $3^+$ -cycle on  $P \cup Q$ . If  $z \neq x_1$ , then  $P[x_0, z] \cup Q[z, x_0]$  is a directed  $3^+$ -cycle on  $P \cup Q$ .  $\square$

In the proof we will have to make sure that some directed paths are not converse of some others, emphasizing the extra work required to deal with directed  $3^+$ -cycle.

#### 4.2.2.1 Main proof

First replacing ‘directed cycle’ by ‘directed  $3^+$ -cycle’ in the proof of Lemma 4.7 of [47] (Lemma (2.2)), we obtain the following analogue of (2.2) of [47].

**Lemma 4.10.** *Let  $n \geq 1$  be an integer such that  $t_{n-1}$  exists, and let  $k$  be an integer. Then there exists an integer  $t'(k)$  such that the following holds. Let  $D$  be a digraph with  $\nu_3(D) < n$ , and  $\tau_3(D) \geq 2t'(k)$ , and let  $T$  be a set of size  $\tau_3(D)$  such that  $D - T$  has no directed  $3^+$ -cycles. For any disjoint subsets  $A, B \subseteq T$  with  $|A| = |B| = k$ , there are distinct vertices  $a_1, \dots, a_k$  in  $A$  and distinct vertices  $b_1, \dots, b_k$  in  $B$ , and two  $k$ -linkages  $L_1, L_2$  of  $D$ , so that*

- (i)  $L_1$  links  $(a_1, \dots, a_k)$  to  $(b_1, \dots, b_k)$ ,
- (ii)  $L_2$  links  $(b_1, \dots, b_k)$  to one of  $(a_1, \dots, a_k), (a_k, \dots, a_1)$ ,
- (iii) every directed  $3^+$ -cycle of  $L_1 \cup L_2$  meets  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ .

However, for our purpose we need an extra condition on the two linkages  $L_1$  and  $L_2$ . This is in fact why we needed a stronger statement than Lemma 4.10.

**Lemma 4.11** (Havet and M. [31]). *Let  $n \geq 1$  be an integer such that  $t_{n-1}$  exists, and let  $k$  be an even integer. Then there exists an integer  $t(k)$  such that the following holds. Let  $D$  be a digraph with  $\nu_3(D) < n$ , and  $\tau_3(D) \geq t(k)$ , and let  $T$  be a set of size  $\tau_3(D)$  such that  $D - T$  has no directed  $3^+$ -cycles. Then there are distinct vertices  $a_1, \dots, a_k, b_1, \dots, b_k$  in  $T$ , and two  $k$ -linkages  $L_1, L_2$  of  $D$  so that*

- (i)  $L_1$  links  $(a_1, \dots, a_k)$  to  $(b_1, \dots, b_k)$ ,
- (ii)  $L_2$  links  $(b_1, \dots, b_k)$  to one of  $(a_1, \dots, a_k), (a_k, \dots, a_1)$ ,
- (iii) every directed  $3^+$ -cycle of  $L_1 \cup L_2$  meets  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ ,
- (iv) no component of  $L_1$  is the converse of a component of  $L_2$ .

To prove this lemma, we will need Erdős-Pósa Theorem (Theorem 1.21) and the following lemma.

**Lemma 4.12.** *Let  $r$  be a positive integer. Let  $T$  be a tree and  $S$  a set of at least  $3r - 2$  vertices of  $T$ . Then there exists a vertex  $x$  of  $T$  and two subsets  $A$  and  $B$  of  $S$ , both of size  $r$  such that every  $(A, B)$ -path in  $T$  goes through  $x$ .*

*Proof.* Let  $E_r$  be the set of edges  $e$  such that both components of  $T \setminus e$  have at least  $r$  vertices of  $S$ . We divide the proof in two cases depending on whether or not  $E_r$  is empty.

Assume first that  $E_r \neq \emptyset$ . Let  $e = xy$  be an edge of  $E_r$ , and let  $T_x$  be the component of  $T \setminus e$  containing  $x$  and  $T_y$  containing  $y$ . Both  $T_x$  and  $T_y$  contain at least  $r$  vertices of  $S$ . Let  $A$  (resp.  $B$ ) be a set of  $r$  vertices of  $S \cap V(T_x)$  (resp.  $S \cap V(T_y)$ ). Then every  $(A, B)$ -path in  $T$  goes through  $e$  and so through  $x$ .

Assume now that  $E_r = \emptyset$ .

**Claim 38.** *There exists a vertex  $x$  such that all components of  $T - x$  have less than  $r$  vertices of  $S$ .*

*Subproof.* Let us orient the edges of  $T$  as follows. Let  $e = uv$  be an edge of  $T$ . Since  $e \notin E_r$  is empty, exactly one component of  $T \setminus e$  contains less than  $r$  vertices of  $S$ . Without loss of generality, this component is the one containing  $v$ . Orient the edge  $e$  from  $u$  to  $v$ . Now every orientation of a tree contains a vertex  $x$  with out-degree 0. Consider a component  $C$  of  $T - x$ . It contains exactly one neighbour  $y$  of  $x$ , and it is precisely the component of  $T \setminus xy$  containing  $y$ . Thus  $|C \cap S| < r$  because the edge is oriented from  $x$  to  $y$ . Hence all components of  $T - x$  have less than  $r$  vertices.  $\diamond$

Take a vertex  $x$  as in the above claim. Let  $C_1, \dots, C_m$  be the components of  $T - x$ . Then  $|C_j| \leq r - 1$  for all  $1 \leq j \leq m$ . Let  $i$  be the smallest integer such that  $T_i = \bigcup_{j=1}^i C_j$  contains at least  $r$  vertices of  $S$ . Clearly,  $T_i$  contains at most  $2r - 2$  vertices in  $S$ , and thus there are least  $r$  vertices in  $T - T_i$ . Let  $A$  (resp.  $B$ ) be a set of  $r$  vertices in  $T_i$  (resp.  $T - T_i$ ). Then  $x$  is in every  $(A, B)$ -path.  $\square$

**Remark 4.13.** *The bound  $3r - 2$  in the above lemma is tight. Indeed consider a tree  $T$  with a set  $S$  of  $3r - 3$  leaves and four other vertices  $x, y_1, y_2$  and  $y_3$  such that for every  $i \in \{1, 2, 3\}$ ,  $y_i$  is adjacent to  $x$  and  $r - 1$  leaves. One can check that for every vertex  $x$  and two sets  $A$  and  $B$  of  $r$  leaves there is an  $(A, B)$ -path avoiding  $x$ .*

*Proof of Lemma 4.11.* Let  $t_n^*$  be as in Erdős-Pósa Theorem (Theorem 1.21); let  $r = k + 2t_n^* + 2$ ; let  $t(k) = \max\{3r - 2 + t_n^*, 2t'(r)\}$ , let where  $t'$  is as in Lemma 4.10. We claim that  $t(k)$  satisfies the lemma.

Let  $G_2$  be the undirected graph with vertex set  $V(D)$  in which two vertices  $x$  and  $y$  are adjacent if and only if  $D[\{x, y\}]$  is a directed 2-cycle. To each cycle in  $G_2$  correspond two directed cycles in  $D$ , one in each direction. Thus  $G_2$  has less than  $n$  disjoint cycles. Hence by Erdős-Pósa Theorem, there is a set  $U \subset V(D)$  of size  $t_n^*$  such that  $G_2 - U$  is acyclic.

Choose  $T \subseteq V(D)$  with  $|T| = \tau_3(D)$ , meeting all directed  $3^+$ -cycles of  $D$ . Since  $t(k) \geq 3r - 2 + t_n^*$ , there is a set  $S$  of size  $3r - 2$  in  $T \setminus U$ . Since  $G_2 - U$  is acyclic, we can extend it into a tree  $T_2$ . Hence, by Lemma 4.12, there exists a vertex  $x$  in  $V(T_2)$  and two sets  $A$  and  $B$  in  $S$  of size  $r$  such that every  $(A, B)$ -path in  $T_2$  goes through  $x$ . Since  $G_2 - U$  is a subgraph of  $T_2$ , every  $(A, B)$ -path in  $G_2 - U$  goes through  $x$ .

Since  $|T| \geq 2t'(r)$ , by Lemma 4.10, there are distinct vertices  $a_1, \dots, a_{k'}$  in  $A$  and distinct vertices  $b_1, \dots, b_{k'}$  in  $B$ , and two  $k'$ -linkages  $L'_1, L'_2$  of  $D$  so that

- (i)  $L'_1$  links  $(a'_1, \dots, a'_{k'})$  to  $(b'_1, \dots, b'_{k'})$ ,
- (ii)  $L'_2$  links  $(b'_1, \dots, b'_{k'})$  to one of  $(a'_1, \dots, a'_{k'}), (a'_{k'}, \dots, a'_1)$ ,
- (iii) every directed  $3^+$ -cycle of  $L'_1 \cup L'_2$  meets  $\{a'_1, \dots, a'_{k'}, b'_1, \dots, b'_{k'}\}$ .

For  $1 \leq i \leq k'$ , let  $P_i$  be the component of  $L_1$  with initial vertex  $a'_i$  and  $Q_i$  the component of  $L_2$  with initial vertex  $b'_i$ .

Clearly, if  $L'_2$  links  $(b'_1, \dots, b'_{k'})$  to  $(a'_{k'}, \dots, a'_1)$ , then condition (iv) is also verified by  $L'_1$  and  $L'_2$ , because  $k'$  is even as  $k$  is even. For  $1 \leq i \leq k$ , set  $a_i = a'_{i+(k'-k)/2}$  and  $b_i = b'_{i+(k'-k)/2}$ , and let  $L_1 = \{P_j \mid 1 + k'/2 - k/2 \leq j \leq k'/2 + k/2\}$  and  $L_2 = \{Q_j \mid 1 + k'/2 - k/2 \leq j \leq k'/2 + k/2\}$ . Then  $a_1, \dots, a_k, b_1, \dots, b_k, L_1$  and  $L_2$  satisfy the lemma.

Assume now that  $L'_2$  links  $(b'_1, \dots, b'_{k'})$  to one of  $(a'_1, \dots, a'_{k'})$ . At most  $t_n^*$  of the  $P_i$  intersect  $U$  and at most  $t_n^*$  of the  $Q_i$  intersect  $U$ . Thus, since  $k' \geq k + 2t_n^* + 1$ , there are at least  $k + 1$  indices  $i$  such that both  $P_i$  and  $Q_i$  do not intersect  $U$ . Without loss of generality, we may assume that these indices are  $\{1, \dots, k + 1\}$ . Now for  $1 \leq i \leq k + 1$ , if  $P_i$  is the converse of  $Q_i$ , then  $P_i$  is also a path in  $G_2 - U$  and thus it must go through  $x$ . Hence there is at most one index  $i$ , say  $k + 1$ , such that  $P_i$  is the converse of  $Q_i$ . Hence  $a_1, \dots, a_k, b_1, \dots, b_k, L_1$  and  $L_2$  satisfy the lemma.  $\square$

We say a digraph is *divalent* if every vertex has in-degree 2 and out-degree 2, or in-degree 1 and out-degree 1. In Subsection 4.2.2.2 we shall prove the following lemma which is the analogue of Lemma (2.3) of [47].



A pair  $\{L_1, L_2\}$  of linkages is *fully intersecting* if each component of  $L_1$  meets each component of  $L_2$ , and it is *acyclic* if  $L_1 \cup L_2$  has no directed cycles.

**Lemma 4.14.** *For every positive integer  $n$ , there exists a positive integer  $k_1$  such that for every divalent digraph  $D$ , if there is a fully intersecting and acyclic pair of  $k_1$ -linkages in  $D$  then  $\nu_3(D) \geq n$ .*

Lemma 4.14 is proved in Subsection 4.2.2.2. We assume it for the moment. We will show how to combine it with Lemma 4.11 to prove Theorem 4.6. First we prove the following lemma, which is the analogue of Lemma 4.8 of [47] (Lemma (2.4)).

**Lemma 4.15** (Havet and M. [31]). *For every non-negative integer  $n$ , there exists a positive integer  $k$  so that the following holds. Let  $D$  be a digraph, and let  $a_1, \dots, a_k, b_1, \dots, b_k$  be distinct vertices of  $D$ . Let  $L_1, L_2$  be linkages in  $D$  linking  $(a_1, \dots, a_k)$  to  $(b_1, \dots, b_k)$ , and  $(b_1, \dots, b_k)$  to one of  $(a_1, \dots, a_k)$ ,  $(a_k, \dots, a_1)$ , respectively, such that no component of  $L_2$  is the converse of  $L_1$ . Let every directed  $3^+$ -cycle of  $L_1 \cup L_2$  meet  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ . Then  $\nu_3(D) \geq n$ .*

The proof of this lemma is similar to the one of Lemma 4.8 of [47] (Lemma (2.4)). However, some extra technical details are required. When reducing to a divalent digraph, we also have to get rid of directed 2-cycles, and Claim 39 is now required.

As in [47], we shall also need Ramsey's theorem [46], which can be stated as follows.

**Theorem 4.16** (Ramsey [46]). *For all positive integers  $q, l, r$ , there exists a (minimum) integer  $R_l(r; q)$  so that the following holds. Let  $Z$  be a set with  $|Z| \geq R_l(r; q)$ , let  $Q$  be a set with  $|Q| = q$ , and for each  $X \subset Z$  with  $|X| = l$  let  $f(X) \in Q$ . Then there exists  $S \subseteq Z$  with  $|S| = r$  and there exists  $x \in Q$  so that  $f(X) = x$  for all  $X \subseteq S$  with  $|X| = l$ .*

*Proof of Lemma 4.15, assuming Lemma 4.14.* Let  $n \geq 1$ . Let  $k' = \max\{k_1, \lceil n/4 \rceil\}$  with  $k_1$  as in Lemma 4.14. Let  $k = 2R_2(4k'; 9)$  defined as in Theorem 4.16. We claim that  $n$  and  $k$  satisfy Lemma 4.15.

For let  $a_1, \dots, a_k, b_1, \dots, b_k, L_1, L_2$  be as in the statement of Lemma 4.15.

Let  $G_i = P_i \cup P_{k+1-i} \cup Q_i \cup Q_{k+1-i}$ .

We show by induction on  $|E(D)| + |V(D)|$  that  $\nu_3(D) \geq n$ . If  $L_1 \cup L_2 \neq D$ , then the result follows immediately by induction, so we may assume that  $L_1 \cup L_2 = D$ .

Assume that either the arc  $e = uv$  belongs to  $P_i \cap Q_j$ , or the arc  $uv$  is in  $P_i$  and the arc  $vu$  is in  $Q_j$ , then consider the graph  $D'$  and the two linkages  $L'_1$  and  $L'_2$  obtained by contracting  $uv$ . These two linkages clearly satisfy the hypothesis of Lemma 4.15 since directed cycles can only be shorten while contracting.

We therefore may assume that every arc of  $D$  belongs to exactly one of  $L_1, L_2$ , and that  $D$  has no directed 2-cycles. In particular,  $D$  is divalent and every directed cycle of  $D$  meets  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ .

For  $1 \leq i \leq k$ , let  $P_i$  be the component of  $L_1$  with initial vertex  $a_i$ , and let  $Q_i$  be the component of  $L_2$  with initial vertex  $b_i$ .

For  $1 \leq h < i \leq k/2$ , define  $f(\{h, i\})$  as follows. If  $G_i$  and  $G_h$  are disjoint, let  $f(\{h, i\}) = 0$ . Otherwise, at least one of the eight digraphs  $P_i \cap Q_h, P_{k+1-i} \cap Q_h, P_i \cap Q_{k+1-h}, P_{k+1-i} \cap Q_{k+1-h}, Q_i \cap P_h, Q_{k+1-i} \cap P_h, Q_i \cap P_{k+1-h}, Q_{k+1-i} \cap P_{k+1-h}$  is non-null. Number them  $1, \dots, 8$  in order; we define  $f(\{h, i\}) \in \{0, 1, \dots, 8\}$ .

Since  $k = 2R_2(4k'; 9)$ , by Theorem 4.16, there exists  $S \subseteq \{1, \dots, \frac{1}{2}k\}$  with  $|S| = 4k'$  and  $x$  with  $0 \leq x \leq 8$  such that  $f(\{h, i\}) = x$  for all  $h, i \in S$  with  $h < i$ .

If  $x = 0$ , then the subdigraphs  $G_i$  are pairwise disjoint for all  $i \in S$ . But as we shall prove in Claim 39, each  $G_i$  contains a directed  $3^+$ -cycle, and so  $\nu_3(D) \geq |S| = 4k' \geq n$ .

**Claim 39.** *Each  $G_i$  contains a directed  $3^+$ -cycle.*

*Subproof.* If  $Q_i$  is a directed  $(b_i, a_i)$ -path, then by assumption  $Q_i$  is not the converse of  $P_i$ . Thus by Lemma 4.9  $P_i \cup Q_i$  contains a directed  $3^+$ -cycle, and so  $G_i$  also does.

Assume now that  $Q_i$  is a directed  $(b_i, a_{k+1-i})$ -path, and so  $Q_{k+1-i}$  is a directed  $(b_{k+1-i}, a_i)$ -path.  $Q_i$  contains a directed path  $R_1$  with initial vertex  $u_1$  in  $P_i$  and terminal vertex  $v_1$  in  $P_{k+1-i}$  whose internal vertices are not in  $P_i \cup P_{k+1-i}$ . Now  $Q_{k+1-i}$  contains a directed path  $R_2$  with initial vertex  $u_2$  in  $P_{k+1-i}[v_1, b_{k+1-i}]$  and terminal vertex  $v_2$  in  $P_i[a_1, u_1]$  whose internal vertices are not in  $P_i \cup P_{k+1-i}$ . Observe that  $u_1, u_2, v_1$  and  $v_2$  are all distinct because  $P_i$  and  $P_{k+1-i}$  are disjoint and  $Q_i$  and  $Q_{k+1-i}$  are disjoint. Hence the  $R_1 \cup P_{k+1-i}[v_1, u_2] \cup R_2 \cup P_i[v_2, u_1]$  is a directed  $4^+$ -cycle in  $G_i$ .  $\diamond$

Assume now that  $x = 1$ . Let  $S = I \cup J$ , where  $|I| = k', |J| = 3k'$  and  $i < j$  for all  $i \in I$  and  $j \in J$ . Then for all  $i \in I$  and all  $j \in J$ ,  $P_i$  meets  $Q_j$ . There are  $2k'$  vertices that are end-vertices of paths  $P_i, i \in I$ , and each of them is an end-vertex of at most one  $Q_j, j \in J$ . Since  $|J| \geq 3k'$ , there exists  $J' \subset J$  with  $|J'| = k'$  so that  $P_i$  and  $Q_j$  have no common end-vertex for  $i \in I$  and  $j \in J'$ . Let  $L'_1$  be the union of the components  $P_i, i \in I$  and  $L'_2$  be the union of the components  $Q_j, j \in J'$ . Now every directed cycle in  $L'_1 \cup L'_2$  meets  $\{a_1 \dots, a_k, b_1, \dots, b_k\}$ , and each of  $a_1 \dots, a_k, b_1, \dots, b_k$  is incident with at most one arc of  $L'_1 \cup L'_2$  since  $P_i$  and  $Q_j$  have no common end-vertex for  $i \in I$  and  $j \in J'$ . Hence  $L'_1 \cup L'_2$  has no directed cycles. We thus have the result by Lemma 4.14.

The cases  $2 \leq x \leq 8$  are similar to the case  $x = 1$ .  $\square$

*Proof of Theorem 4.6, assuming Lemma 4.14.* We prove Theorem 4.6 by induction on  $n$ ; we therefore assume that  $n \geq 1$  and  $t_{n-1}$  exists, and we show that  $t_n$  exists. Let  $k$  be as in Lemma 4.15, and let  $t$  be as in Lemma 4.11. We claim that there is no digraph  $D$  with  $\nu_3(D) < n$  and  $\tau_3(D) \geq t$ . For suppose that  $D$  is such a digraph. By Lemma 4.11, there exists  $a_1 \dots, a_k, b_1, \dots, b_k$  and  $L_1, L_2$  as in Lemma 4.10, and so  $\nu_3(D) \geq n$  by Lemma 4.15, a contradiction. Thus there is no such  $D$ , and consequently  $t_n$  exists and  $t_n < t$ .  $\square$

### 4.2.2.2 Proving Lemma 4.14

In this section, following Section 3 of [47], we show that if a digraph  $D$  contains a kind of grid, with some additional paths, then  $\nu_3(D)$  is large. We then use this lemma to prove Lemma 4.14.

Let  $p, q$  be positive integers. A  $(p, q)$ -web in a digraph  $D$  is a fully intersecting and acyclic pair  $(L_1, L_2)$  of linkages such that  $L_1$  has  $p$  components and  $L_2$  has  $q$  components.

Let  $p, q$  be positive integers. A  $(p, q)$ -fence in a digraph  $D$  is a sequence  $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$  with the following properties:

- (i)  $P_1, \dots, P_{2p}$  are pairwise disjoint directed paths of  $D$ , and so are  $Q_1, \dots, Q_q$ ;
- (ii) for  $1 \leq i \leq 2p$  and  $1 \leq j \leq q$ ,  $P_i \cap Q_j$  is a directed path (and therefore non-null);
- (iii) for  $1 \leq j \leq q$ , the directed paths  $P_1 \cap Q_j, \dots, P_{2p} \cap Q_j$  are in order in  $Q_j$ , and the initial vertex of  $Q_j$  is in  $V(P_1)$  and its terminal vertex is in  $V(P_{2p})$ ;
- (iv) for  $1 \leq i \leq 2p$ , if  $i$  is odd then  $P_i \cap Q_1, \dots, P_i \cap Q_q$  are in order in  $P_i$ , and if  $i$  is even then  $P_i \cap Q_q, \dots, P_i \cap Q_1$  are in order in  $P_i$ .

Let  $Q_j$  be a directed  $(a_j, b_j)$ -path ( $1 \leq l \leq q$ ); we call  $\{a_1, \dots, a_q\}$  the *top* of the fence, and  $\{b_1, \dots, b_q\}$  its *bottom*.

The following lemma is the analogue to Lemma (3.1) of [47]. It only differs in the conclusion  $\nu_3(D) \geq n$ , instead of  $\nu(D) \geq n$ .

**Lemma 4.17.** *For every positive integer  $n$ , there are positive integers  $p, r$  with the following property. For any  $q \geq 2$ , let  $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$  be a  $(p, q)$ -fence in a digraph  $D$ , and let there be  $r$  disjoint paths in  $D$  from the bottom of the fence to the top. Then  $\nu_3(D) \geq n$ .*

Combining Lemmas (4.4), (4.5) and (4.7) of [47] we directly obtain the following lemma.

**Lemma 4.18.** *For all positive integers  $p, q$ , there are positive integers  $p'$  and  $q'$  so that for every digraph  $G$ , if  $D$  contains a  $(p', q')$ -web then it contains a  $(p, q)$ -fence.*

In exactly the same way that Reed et al. deduced Lemma (2.3) from Lemmas (3.1), (4.4), (4.5) and (4.7) in [47], one can deduce Lemma 4.14 from Lemmas 4.17 and 4.18.

Hence it only remains to prove Lemma 4.17.

### 4.2.2.3 Proof Lemma 4.17

Consider the following lemma (Lemma (3.2)) from [47].

**Lemma 4.19** (Reed et al. [47]). *Let  $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$  be a  $(p, q)$ -fence in a digraph  $D$ , with top  $A$  and bottom  $B$ . Let  $A' \subseteq A$  and  $B' \subseteq B$  with  $|A'| = |B'| = r$ , for  $r \leq p$ . Then there are directed paths  $Q'_1, \dots, Q'_r$  in  $P_1, \dots, P_{2p}, Q_1, \dots, Q_q$  so that  $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_r)$  is a  $(p, r)$ -fence with top  $A'$  and bottom  $B'$ .*

**Remark 4.20.** *In the proof of this lemma, the proven  $(p, r)$ -fence is a subgraph of the  $(p, q)$ -fence, with  $A' \subseteq A$  and  $B' \subseteq B$ . Moreover, if  $p \geq 2$ , then  $Q_j$  has order at least 4 for  $1 \leq j \leq q$ , because  $Q_j$  intersects every  $P_i$   $1 \leq i \leq 2p$ . So, if  $p \geq 2$ ,  $Q'_l$  has size at least 4, for  $1 \leq l \leq r$ .*

We need also an analogue of Lemma (3.3) of [47]:

**Lemma 4.21.** *Let  $n \geq 1$  be an integer, and let  $p \geq 2n$  and  $N \geq 2n^2 - 3n + 2$  be integers. For some integer  $q \geq 1$  let  $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$  be a  $(p, q)$ -fence in a digraph  $D$ . Let  $R_1, \dots, R_N$  be disjoint directed paths of  $D$  from the bottom of the fence to the top, so that each  $R_k$  has no vertex or arc in  $P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_q$  except its end-vertices. Then  $\nu_3(D) \geq n$ .*

The proof of this lemma is exactly the same as the one of Lemma (3.3) of [47]. The disjoint directed cycles showed in the proof are of the form  $Q'_j R_m$ , for  $Q'_j$  in a  $(p, r)$ -fence  $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_r)$ , subgraph of the  $(p, q)$ -fence  $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ . Since  $p \geq 2n \geq 2$ , by Remark 4.20 each  $Q'_j$  has length at least 2, and so  $Q'_j R_m$  has length at least 3. Hence  $\nu_3(D) \geq n$ .

We prove Lemma 4.17 by induction on  $n$ . The proof is almost identical to the one of Lemma (3.1) in [47]. The only differences are the easy case  $n = 1$ , for which we need here to take  $p = 2$  (instead of  $p = 1$ ) to be sure that the directed cycle is of length at least 3, and the use in place of Lemma (3.3) of its analogue, namely Lemma 4.21.

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