Observateur par intervalles et observateur positif

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"Il faut savoir douter où il faut, assurer où il faut et se soumettre où il faut. Qui ne fait ainsi n'entend pas la force de la raison."

Blaise Pascal. Mathématicien et Philosophe. 1623 - 1662.

UNIVERSITÉ PARIS-SUD

Résumé Etendu

Laboratoire des Signaux et Systèmes
ECOLE DOCTORALE STITS

Docteur en Physique

Observateur par intervalles et observateur positif

par Thach N. DINH (Đinh Ngọc Thạch)

Lorsque est abordée l'étude d'un phénomène physique ou biologique, une première étape peut consister à fournir une modélisation mathématique de ce phénomène sous la forme d'une équation différentielle ordinaire :

$$\dot{x}(t) \;=\; f(t, x(t)) \;.$$

Les équations aux différences sont également souvent employées. Le modèle est alors un système de la forme :

$$x_{k+1} \;=\; f(k, x_k) \;.$$

Le vecteur $x \in \mathbb{R}^n$ représente l'état du système, t et k sont le temps de l'évolution du système, appartenant respectivement à $[0, +\infty)$ et $\mathbb{N}$.

L'ensemble des informations disponibles sur l'état du système aux cours du temps est contenu dans un vecteur y évoluant avec le temps. Dans le cas où les mesures sont effectuées en temps continu et qu'aucun retard n'est présent, ce vecteur y peut être modélisé par une fonction de l'état du système de la forme :

$$y(t) = h(t, x(t)) \;,$$

et dans le cas où les mesures sont fournies de façon discrètes et à intervalle régulier, le vecteur y est alors :

$$y_k = h(k, x_k) \;.$$

A partir de cette modélisation et des informations y, un des objectifs de l'automatique est de répondre à la question fondamentale suivante :

Quel est l'état actuel du système ?

Dans ce contexte, une classe d'algorithmes d'estimation en temps réel a été particulièrement étudiée depuis de nombreuses années : les observateurs dynamiques. Ceux-ci sont des algorithmes fournissant une estimation $\hat{x}(t)$ pour la version à temps continu (respectivement $\hat{x}_k$ pour la version à temps discret) obtenue par intégration d'un système dynamique dépendant de y de la forme :

$$\dot{z}(t) = \theta(t, z(t), y(t)) \ , \ \hat{x}(t) = \psi(t, z(t), y(t)) \ ,$$

dans le cas continu et de la forme :

$$z_{k+1} = \theta(k, z_k, y_k) \ , \ \hat{x}_k = \psi(k, z_k, y_k) \ ,$$

dans le cas discret.

Le problème est alors de trouver les fonctions θ et ψ garantissant une convergence asymptotique de $\hat{x}(t)$ (resp. $\hat{x}_k$) vers $x(t)$ (resp. x_k) au cours du temps t (resp. k). Notons que ce problème n'admet pas toujours de solution.

La littérature sur cette thématique est abondante et celle-ci est toujours actuellement un sujet de recherche actif. A titre d'exemple nous pouvons citer les approches suivantes (qui ne recouvrent pas l'ensemble des techniques disponibles) : les approches "grand gain" [48] (voir aussi [135] pour une approche grand gain utilisant le formalisme par inégalités matricielles affines, appelé "LMI"), les approches par contractions et extension de dynamiques [6], les approches pour les non linéarités monotones [10], les observateurs par intervalles [51] et enfin les approches préservant les symétries du système [18].

Problématique

Cette thèse est construite autour de deux types d'estimation de l'état d'un système, traités séparément. Le premier problème abordé concerne la construction d'observateurs positifs basés sur la métrique de Hilbert. Le second traite de la synthèse d'observateurs par intervalles pour différentes familles de systèmes dynamiques et la construction de lois de commande robustes qui stabilisent ces systèmes.

Un système positif est un système dont les variables d'état sont toujours positives ou nulles lorsque celles-ci ont des conditions initiales qui le sont. Les systèmes positifs apparaissent souvent de façon naturelle dans des applications pratiques où les variables d'état représentent des quantités qui n'ont pas de signification si elles ont des valeurs négatives. Dans ce contexte, il parait naturel de rechercher des observateurs fournissant des estimées elles aussi positives ou nulles. Dans un premier temps, nous complétons les travaux de S. Bonnabel et al. [17]. Notre contribution réside dans la mise au point d'une nouvelle méthode de construction d'observateurs positifs sur l'orthant positif. L'analyse de convergence est basée sur la métrique de Hilbert. L'avantage concurrentiel de notre méthode par rapport à celle de l'article [17] est que la vitesse de convergence peut être contrôlée.

Notre étude concernant la synthèse d'observateurs par intervalles est basée sur la théorie des systèmes dynamiques positifs. Les observateurs par intervalles constituent un type d'observateurs très particuliers. Ce sont des outils développés depuis moins de 15 ans seulement : ils trouvent leur origine dans les travaux de Gouzé et al. en 2000 [51] et se développent très rapidement dans de nombreuses directions. Un observateur par intervalles consiste en un système dynamique auxiliaire fournissant un intervalle dans lequel se trouve l'état, en considérant que l'on connaît des bornes pour la condition initiale et pour les quantités incertaines.

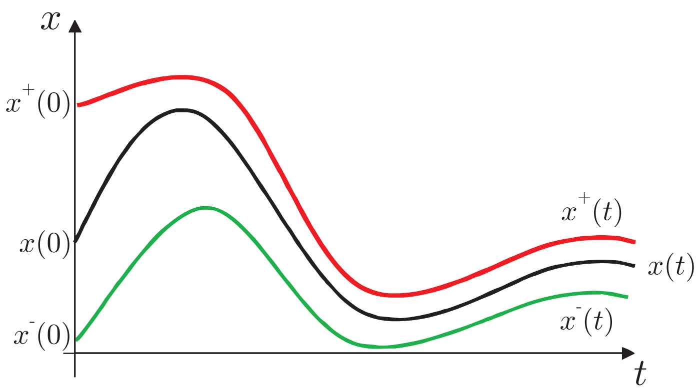

FIGURE 1 – Solution (ligne noire). Bornes données par l'observateur par intervalles : ligne rouge et ligne verte.

Les observateurs par intervalles donnent la possibilité de considérer le cas où des perturbations importantes sont présentes et fournissent certaines informations à tout instant.

Objectifs et résultats de la thèse

Notre objectif principal porte sur la construction d'observateurs par intervalles pour différentes familles de systèmes dynamiques en temps continu, en temps discret et aussi dans un contexte "continu-discret" (une classe de systèmes dynamiques en temps continu et dont la sortie est constante par morceaux). Un autre de nos objectifs a été la construction de lois de commande qui stabilisent pour les systèmes considérés en utilisant les observateurs par intervalles. Nos travaux sur le sujet se décomposent de la façon suivante :

1. Observateurs par intervalles pour des systèmes en temps continu, non linéaires et affines en les variables non mesurées [36].

2. Observateurs par intervalles pour des systèmes linéaires et non linéaires à temps discret [94–97].
3. Observateurs par intervalles continu-discrets pour des systèmes en temps continu avec sortie constante par morceaux [92, 93].

En parallèle, nous avons étudié pour des systèmes linéaires positifs des observateurs positifs de type Luenberger [35]. Le point de départ de l'approche est un théorème proposé par G. Birkhoff [21] qui date des années 50. La construction d'un observateur positif est basée sur l'utilisation des coordonnées polaires généralisées dans l'orthant positif. C'est-à-dire, on assure premièrement que la direction de l'erreur d'estimation est exponentiellement reconstruit et en suite sa norme l'est aussi. Les propriétés de contraction de l'erreur d'estimation sont étudiées grâce à la métrique projective de Hilbert.

Organisation de la thèse

Cette thèse est divisée en sept chapitres. Pour faciliter la lecture, nous les résumons ci-après.

Chapitre 1 : Introduction aux méthodes d'estimation d'état robustes

L'objectif de ce chapitre est d'introduire certains concepts de base qui seront utiles dans la suite de notre travail. Nous présentons l'état de l'art concernant les méthodes d'estimation robuste. Nous présentons tout d'abord les méthodes d'estimation classiques et ensuite nous décrivons certaines innovations récentes. Ce chapitre fixe également le cadre théorique des systèmes dynamiques monotones et positifs, et donne des notations d'analyse fonctionnelle (métrique, diamètre projectif...).

Chapitre 2 : Construction d'observateur positif basée sur la métrique de Hilbert

A partir de ce chapitre, nous présentons plus particulièrement notre contribution. Dans le Chapitre 2 une nouvelle méthode permettant de construire des observateurs positifs sur l'orthant positif est proposée. L'analyse de convergence est basée sur la métrique de Hilbert. Un avantage clé de cette approche est la simplicité de la construction d'observateurs positifs (nous construisons les observateurs positifs de type Luenberger).

Les résultats sont publiés dans l'article [35].

Chapitre 3 : Observateurs par intervalles pour des systèmes non linéaires à temps continu

Dans le Chapitre 3, nous fournissons une nouvelle technique de construction d'observateurs par intervalles pour une famille de systèmes affines en les variables non mesurées. Cette technique permet de considérer la présence de perturbations additives. La partie dynamique des observateurs par intervalles est simplement composée de deux "copies" d'un observateur classique. Aussi pouvons nous les utiliser pour stabiliser les systèmes au moyen d'une commande par retour de sortie dynamique.

Les résultats sont publiés dans l'article [36].

Chapitre 4 : Observateurs par intervalles pour des systèmes à temps discret

Dans ce chapitre, nous considérons des systèmes en temps discret. Des observateurs par intervalles sont proposés pour une famille de systèmes non linéaires autonomes et ensuite nous montrons que pour tout système linéaire discret autonome exponentiellement stable avec des perturbations additives, des observateurs par intervalles exponentiellement stables à temps discret et variant dans le temps peuvent être construits. Le dernier résultat repose sur la construction de changements de coordonnées dépendant du temps qui transforment un système linéaire en un système linéaire positif.

Les résultats sont publiés dans les articles [94, 95].

Chapitre 5 : Observateurs par intervalles robustes pour des systèmes non linéaires en temps discret avec une entrée et une sortie

Ce chapitre complète le chapitre précédent. Nous étendons les résultats développés à une famille de systèmes non linéaires en temps discret avec une entrée, une sortie et un terme d'incertitudes. La caractéristique principale du résultat présenté est que les observateurs par intervalles proposés sont composés de deux "copies" d'observateurs classiques auxquels sont associées deux sorties particulières. Par ailleurs, ces observateurs par intervalles appliqués en présence de termes non linéaires bornés inconnus et de perturbations peuvent être utilisés pour obtenir un résultat de stabilité asymptotique au moyen du choix approprié de commandes dynamiques de sortie.

Les résultats sont publiés dans les articles [96, 97].

Chapitre 6 : Observateurs par intervalles continu-discrets.

Après avoir construit des observateurs par intervalles pour des systèmes en temps continu et en temps discret, nous traitons d'un cas en lequel se rencontrent des spécificités temps continu et temps discret. Dans ce chapitre, nous considérons des systèmes linéaires à temps continu avec une entrée, une sortie et des perturbations additives lorsque la mesure est seulement disponible à des instants discrets. Pour ces systèmes, nous résolvons un problème d'estimation d'état en construisant une famille d'observateurs par intervalles continus-discrets. Ils sont composés de quatre "copies" du système étudié accompagnées de sorties appropriées, qui donnent des bornes supérieures et inférieures pour les solutions.

Les résultats sont publiés dans les articles [92, 93].

Chapitre 7 : Conclusion et Perspectives.

Ce chapitre termine le mémoire en donnant les conclusions et les perspectives de ce travail.

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Etat des publications

Articles dans des revues avec comité de lecture

1. **T.N. Dinh**, V. Andrieu, M. Nadri, U. Serres, "Continuous-discrete time observer design for Lipschitz systems with sampled measurements," A paraitre dans IEEE Transactions on Automatic Control, accepté 2014.[⋆]
2. F. Mazenc, **T.N. Dinh**, "Construction of Interval Observers for Continuous-time Systems with Discrete Measurements," Automatica, Vol. 50, pp. 2555-2560, 2014.
3. F. Mazenc, **T.N. Dinh**, S.-I. Niculescu, "Interval Observers For Discrete-time Systems," International Journal of Robust and Nonlinear Control, Vol. 24, pp. 2867-2890, 2014.
4. F. Mazenc, **T.N. Dinh**, S.-I. Niculescu, "Robust Interval Observers and Stabilization Design for Discrete-time Systems with Input and Output," Automatica, Vol. 49, pp. 3490-3497, 2013.
5. F. Mazenc, M. Malisoff, **T.N. Dinh**, "Robustness of Nonlinear Systems with Respect to Delay and Sampling of the Controls," Automatica, Vol. 49, pp. 1925-1931, 2013.[⋆]

Actes de conférence avec comité de lecture

6. **T.N. Dinh**, F. Mazenc, S.-I. Niculescu, "Interval Observer Composed of Observers for Nonlinear Systems," in Proceedings of the 13th European Control Conference, Strasbourg, France, pp. 660-665, 2014.
7. **T.N. Dinh**, S. Bonnabel, R. Sepulchre, "Positive observer design based on the Hilbert metric," in Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, Italy, pp. 6574-6579, 2013.
8. F. Mazenc, **T.N. Dinh**, "Continuous-Discrete Interval Observers for Systems with Discrete Measurements," in Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, Italy, pp. 787-792, 2013.
9. F. Mazenc, **T.N. Dinh**, S.-I. Niculescu, "Robust Interval Observers for Discrete-time Systems of Luenberger type," in Proceedings of the American Control Conference, Washington, USA, pp. 2484-2489, 2013.
10. F. Mazenc, M. Malisoff, **T.N. Dinh**, "Uniform Global Asymptotic Stability for Nonlinear Systems under Input Delays and Sampling of the Controls," in Proceedings of the American Control Conference, Washington, USA, pp. 4857-4861, 2013.[⋆]

11. F. Mazenc, **T.N. Dinh**, S.-I. Niculescu, "Interval Observers For Discrete-time Systems," in Proceedings of the 51st IEEE Conference on Decision and Control, Maui, USA, pp.6755-6760, 2012.
12. **T.N. Dinh**, V. Andrieu, M. Nadri, U. Serres, "Un observateur continu-discret pour les systèmes avec des non-linéarités de Lipschitz et des mesures discrètes," dans les actes de la Conférence Internationale Francophone d'Automatique, Grenoble, France, 2012.[⋆]

[⋆]. **Les articles ne sont pas inclus dans la partie principale de ce mémoire.**

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Notations

$\mathbb{R}$	Corps des nombres réels
$\mathbb{R}^n$	Espace des vecteurs à n entiers réelles
$\mathbb{R}^{m\times n}$	Espace des matrices réelles de dimension $m \times n$
$\mathcal{C}^k$	Ensemble des fonctions dont la k-ième dérivée est continue
$\mathbb{C}$	Corps des nombres complexes
I_n	Matrice identité d'ordre n
$A^\top$	Transposée de matrice
A^{-1}	Matrice inverse de A
$\langle\ \rangle$	Produit scalaire
$\lvert \cdot \rvert$	Norme Euclidienne

Dành tặng Bố Mẹ tôi...

Chapitre 1

Introduction aux méthodes d'estimation d'état robustes

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1.1 Introduction

L'objectif de ce chapitre est d'introduire certains concepts de base qui seront utiles dans la suite de notre travail. Nous présentons l'état de l'art concernant les méthodes d'estimation robuste. Nous parlerons en premier lieu des méthodes d'estimation classiques et puis décrirons certaines innovations récentes. Ce chapitre défini également le cadre théorique des notions de contrôlabilité et d'observabilité, des notions de stabilité et des méthodes de stabilisation, des systèmes dynamiques monotones et positifs, et donne de plus des notations d'analyse fonctionnelle (métrique, diamètre projectif...). Il fournit un fond suffisant avant d'entrer dans la partie principale de ce mémoire.

Le présent chapitre est organisé comme suit : nous présentons tout d'abord les notions de contrôlabilité et d'observabilité pour des systèmes linéaires à temps continu et à temps discret. Nous proposons en suite les notions de stabilité, particulièrement la stabilité asymptotique, la stabilité exponentielle et la stabilité entrée-état. Nous évoquons aussi de la commande par retour d'état, le théorème de Lyapunov, la technique d'ajout d'intégrateur. Nous rappelons les observateurs classiques comme l'observateur de Luenberger, le filtre de Kalman, le filtre de Kalman étendu, les approches "grand gain" et les observateurs continu-discrets. Les systèmes monotones et positifs sont ensuite présentés au centre de l'avant-dernière section. Pour finir, nous présentons un type d'observateurs particuliers, appelé les observateurs par intervalles pour des systèmes à temps continu et à temps discret. Ceux-ci jouent un rôle essentiel dans ce rapport de thèse.

1.2 Contrôlabilité et observabilité de systèmes linéaires

La contrôlabilité et l'observabilité représentent deux concepts majeurs de la théorie de contrôle. Ils ont été introduits par R. Kalman en 1960. Ils peuvent être définis en gros de la façon suivante.

Contrôlabilité : Si un système dynamique permet de se déplacer d'un point quelconque de son espace à un autre point quelconque au moyen d'une commande appropriée, on dit alors que ce système est commandable.

Observabilité : Si la connaissance d'une sortie d'un système dynamique permet de reconstituer tout l'état de celui-ci, alors on dit que le système avec cette sortie est observable.

Dans cette section nous montrons que les concepts de contrôlabilité et d'observabilité pour des systèmes linéaires sont liés à des propriétés des équations algébriques linéaires. Il est bien connu qu'un système d'équations algébriques linéaires admet une et une seule solution si, et seulement si, la matrice du système est de rang plein. Les tests de contrôlabilité et d'observabilité sont reliés à des tests du rang de certaines matrices, les matrices de contrôlabilité et d'observabilité.

1.2.1 Observabilité de systèmes discrets

Considérons un système autonome linéaire, en temps discret de forme

$$x_{k+1} = A_d x_k, \ k \in \mathbb{N} \tag{1.1}$$

avec les mesures

$$y_k = C_d x_k \tag{1.2}$$

où $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, A_d et C_d sont des matrices constantes de dimensions appropriées. Une question naturelle se pose : peut-on connaître entièrement le comportement des solutions du système (1.1) en utilisant seulement des informations des mesures (1.2) ? Si l'on connaît x_0, alors la formule récurrente (1.1) nous donne la connaissance complète des variables d'état à tout instant. Par conséquent, la seule chose que nous devons déterminer à partir des mesures d'état (1.2) est le vecteur d'état initial x_0.

Puisque le vecteur $x_0 \in \mathbb{R}^n$, on peut prévoir que n mesures sont suffisantes pour déterminer x_0. Prenons $k = 0, 1, ..., n-1$ dans (1.1), (1.2), et résolvons le système suivant

$$\begin{array}{lcl} y_0 & = & C_d x_0 \\ y_1 & = & C_d x_1 = C_d A_d x_0 \\ y_2 & = & C_d x_2 = C_d A_d x_1 = C_d A_d^2 x_0 \\ \vdots & & \\ y_{n-1} & = & C_d x_{n-1} = C_d A_d^{n-1} x_0 \end{array} \tag{1.3}$$

Ces égalités sont équivalentes à

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ \vdots \\ C_d A_d^{n-1} \end{bmatrix}^{(np)\times n} \times x_0 \tag{1.4}$$

La théorie algébrique des systèmes linéaires garanti que le système linéaire (1.4) admet une et une seule solution si et seulement si la matrice associée au système est de rang n :

$$\text{rang} \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ \vdots \\ C_d A_d^{n-1} \end{bmatrix} = n \tag{1.5}$$

En d'autres termes, la condition initiale x_0 peut être déterminée si la matrice d'observabilité

$$\mathcal{O}(A_d, C_d) = \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ \vdots \\ C_d A_d^{n-1} \end{bmatrix}^{(np)\times n} \tag{1.6}$$

est de rang n.

Théorème 1.1. *Le système linéaire en temps discret (1.1) avec les mesures (1.2) est observable si et seulement si le rang de la matrice d'observabilité (1.6) est égal à n.*

1.2.2 Observabilité de systèmes continus

Nous considérons cette fois des systèmes à temps continu. Nous allons étudier l'observabilité d'un système sans entrée

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{1.7}$$

avec $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ et les mesures

$$y(t) = Cx(t) \ . \tag{1.8}$$

Le problème auquel nous sommes confrontés est celui de trouver $x(0)$ à partir des mesures disponibles (1.8). La technique en temps continu peut être décrite comme suit. Commençons par observer que :

$$\begin{array}{lcl} y(0) & = & Cx(0) \\ \dot{y}(0) & = & C\dot{x}(0) = CAx(0) \\ \ddot{y}(0) & = & C\ddot{x}(0) = CA^2x(0) \\ \vdots & & \\ y^{(n-1)}(0) & = & Cx^{(n-1)}x(0) = CA^{n-1}x(0) \end{array} \tag{1.9}$$

Les équations (1.9) forment un système de np équations algébriques linéaires. Elles peuvent être mis sous forme matricielle comme suit

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}^{(np)\times n} \times x(0) = \mathcal{O}x(0) \tag{1.10}$$

où $\mathcal{O}$ est la matrice d'observabilité défini dans (1.6). Il apparait que la condition initiale $x(0)$ peut être déterminée au moyen de (1.10) si et seulement si la matrice d'observabilité est de rang plein, c'est-à-dire si $\operatorname{rang} \mathcal{O} = n$.

Théorème 1.2. *Le système linéaire en temps continu (1.7) avec les mesures (1.8) est observable si et seulement si la matrice d'observabilité est de rang plein.*

Il est important de noter que l'ajout de dérivées d'ordre supérieur dans (1.10) ne peut pas augmenter le rang de la matrice d'observabilité car d'après le théorème de Cayley-Hamilton, pour tout $k \geq n$, nous avons

$$A^k = \sum_{i=0}^{n-1} \alpha_i A^i \tag{1.11}$$

Par conséquent, les équations ajoutées seraient linéairement dépendantes des n équations définies précédemment (1.10).

1.2.3 Contrôlabilité de systèmes discrets

Nous considérons maintenant les systèmes linéaires, en temps discret autonome du type

$$x_{k+1} = A_d x_k + B_d u_k, \;\; k \in \mathbb{N} \tag{1.12}$$

où l'état est $x \in \mathbb{R}^n$, la commande est $u \in \mathbb{R}^m$ et les matrices A_d et B_d sont constantes et de tailles $n \times n$ et $n \times m$, respectivement. La contrôlabilité du système est la capacité d'aller d'un état initial quelconque x_0 à un état final désiré $x_{k_1} = x_f$ en temps fini, c'est-à-dire pour $k_1 < \infty$. La question qui se pose est alors la suivante :

Pouvons nous trouver un ensemble de lois de contrôle $u_0, u_1, ..., u_{k-1}$ tel que $x_k = x_f$?

Nous allons commencer par un problème simplifié, c'est-à-dire nous allons supposer que l'entrée u_k est un scalaire, et donc que la matrice B_d est un vecteur, que nous notons par

b_d. En conséquence, nous avons

$$x_{k+1} = A_d x_k + b_d u_k \ . \tag{1.13}$$

En prenant $k = 0, 1, 2, ..., n$ dans (1.13), nous obtenons l'ensemble d'équations suivant :

$$\begin{aligned} x_1 &= A_d x_0 + b_d u_0 \\ x_2 &= A_d x_1 + b_d u_1 = A_d^2 x_0 + A_d b_d u_0 + b_d u_1 \\ &\vdots \\ x_n &= A_d^n x_0 + A_d^{n-1} b_d u_0 + ... + b_d u_{n-1} \end{aligned} \tag{1.14}$$

Les équations (1.14) peuvent être exprimées sous la forme matricielle suivante

$$x_n - A_d^n x_0 = \left[\begin{array}{c|c|c|c} b_d & A_d b_d & \cdots & A_d^{n-1} b_d \end{array}\right] \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} \tag{1.15}$$

La matrice carrée $\mathcal{C}(A_d, b_d) = \left[\begin{array}{c|c|c|c} b_d & A_d b_d & \cdots & A_d^{n-1} b_d \end{array}\right]$ de taille $n \times n$ est dite matrice de commandabilité de Kalman. Si elle est inversible, alors nécessairement

$$\begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} = \mathcal{C}^{-1} \left(x_n - A_d^n x_0 \right) \tag{1.16}$$

ce qui détermine les valeurs de la commande u_k de façon unique de $k = 0$ à $k = n - 1$.

Nous concluons que pour $x_n = x_f$ quelconque, l'expression (1.16) détermine la séquence de contrôle qui transfère l'état initial x_0 à l'état désiré x_f dans n étapes. Il en résulte que la condition de commandabilité, dans ce cas, est équivalente à l'inversibilité de la matrice de commandabilité $\mathcal{C}$.

Dans le cas général, quand l'entrée u_k est un vecteur de dimension m, la répétition de la même procédure que dans (1.13) - (1.15) conduit au résultat suivant

$$x_n - A_d^n x_0 = \left[\begin{array}{c|c|c|c} B_d & A_d B_d & \cdots & A_d^{n-1} B_d \end{array}\right] \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} \tag{1.17}$$

La matrice de commandabilité, dans le cas général, est définie par

$$\mathcal{C}(A_d, B_d) = \left[\begin{array}{c|c|c|c} B_d & A_d B_d & \cdots & A_d^{n-1} B_d \end{array}\right] \tag{1.18}$$

La matrice est de dimension $n \times (mn)$. Le système d'équations algébriques linéaires correspondant est donné par

$$\mathcal{C}(A_d, B_d) \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}^{(nm)\times 1} = x_n - A_d^n x_0 = x_f - A_d^n x_0 \tag{1.19}$$

et possède une solution pour un x_f quelconque si et seulement si la matrice $\mathcal{C}$ est de rang plein.

En résumé, nous avons le théorème de commandabilité suivant.

Théorème 1.3. *Le système (1.12) est commandable si et seulement si la matrice de commandabilité est de rang n :*

$$\operatorname{rang} \mathcal{C}(A_d, B_d) = n, \tag{1.20}$$

où la matrice $\mathcal{C}(A_d, B_d)$ est définie par (1.18).

1.2.4 Contrôlabilité de systèmes continus

L'étude du concept de commandabilité en temps continu est plus difficile qu'en temps discret. Pour commencer, nous allons appliquer la même stratégie que dans la Section 1.2.3 afin de clarifier les difficultés que nous allons rencontrer lorsque nous étudierons la version en temps continu. En suite, nous allons montrer comment trouver une commande qui nous donne la possibilité de transférer notre système d'un état initial quelconque à un état final souhaité quelconque.

Considérons les systèmes linéaires en temps continu de la forme

$$\dot{x} = Ax + bu, \quad x(0) = x_0 \tag{1.21}$$

avec $x \in \mathbb{R}^n$, $u \in \mathbb{R}$.

En dérivant n fois l'état x, nous obtenons le système suivant

$$\begin{array}{lcl} \dot{x} & = & Ax + bu \\ \ddot{x} & = & A^2x + Abu + b\dot{u} \\ \vdots & & \\ x^{(n)} & = & A^n x + A^{n-1}bu + A^{n-2}b\dot{u} + \ldots + bu^{(n-1)} \end{array} \tag{1.22}$$

La dernière équation de (1.22) peut être écrite comme suit

$$x^{(n)}(t) - A^n x(t) = \mathcal{C} \begin{bmatrix} u^{(n-1)}(t) \\ u^{(n-2)}(t) \\ \vdots \\ \dot{u}(t) \\ u(t) \end{bmatrix} \tag{1.23}$$

où $\mathcal{C} = \left[\begin{array}{c|c|c|c} b & Ab & \cdots & A^{n-1}b \end{array} \right]$.

Notons que l'expression (1.23) est satisfaite pour tout $t \in [0, t_f]$ avec t_f quelconque fini. Si la matrice de commandabilité $\mathcal{C}$ est inversible, alors (1.23) détermine les valeurs de la commande $u(t)$ et toutes ses dérivées d'ordre 1 à $n-1$ de façons unique, pour tout $t < t_f$.

Dans le cas général avec $u(t) \in \mathbb{R}^m$, considérons le système linéaire suivant

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{1.24}$$

Par un raisonnement identique à celui fait au dessus, nous avons

$$\mathcal{C}^{n\times(nm)} \begin{bmatrix} u^{(n-1)}(t) \\ u^{(n-2)}(t) \\ \vdots \\ \dot{u}(t) \\ u(t) \end{bmatrix}^{(nm)\times 1} = x^{(n)}(t) - A^n x(t) = \gamma(t) \tag{1.25}$$

où $\mathcal{C} = \left[\begin{array}{c|c|c|c} B & AB & \cdots & A^{n-1}B \end{array} \right]$.

Une solution de (1.25) existe pour tout $\gamma(t)$ - un état désiré quelconque à l'instant t - si et seulement si

$$\operatorname{rang} \mathcal{C} = n \tag{1.26}$$

Les équations (1.23) et (1.25) établissent les relations entre l'état et les variables d'entrée. Néanmoins nous ne pouvons déduire de (1.23) et (1.25) une réponse explicite concernant la fonction d'entrée qui nous permet de transférer le système d'un état initial quelconque $x(0)$ à n'importe quel état final $x(t_1) = x_f$. Par conséquent, la même méthode appliquée pour le cas en temps discret dans la section précédente ne peut pas être complètement étendue pour la version en temps continu. Une autre approche qui est mathématiquement plus compliquée, est demandée dans ce cas. Elle sera présentée dans la partie restante de cette section.

Nous savons que la solution de (1.24) admet la représentation :

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

À l'instant t_1 nous avons

$$x(t_1) = x_f = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

d'où

$$e^{-At_1}x_f - x(0) = \int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

En utilisant le théorème de Cayley-Hamilton, nous obtenons alors

$$e^{-A\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)A^i \tag{1.27}$$

où $\alpha_i(\tau)$, $i = 0, 1, ..., n-1$, sont les fonctions scalaire. Nous en déduisons que

$$e^{-At_1}x_f - x(0) = \sum_{i=0}^{n-1} A^iB \int_0^{t_1} \alpha_i(\tau)u(\tau)d\tau$$

Donc

$$e^{-At_1}x_f - x(0) = \left[\begin{array}{c|c|c|c} B & AB & \cdots & A^{n-1}B \end{array}\right] \begin{bmatrix} \int_0^{t_1} \alpha_0(\tau)u(\tau)d\tau \\ \int_0^{t_1} \alpha_1(\tau)u(\tau)d\tau \\ \vdots \\ \int_0^{t_1} \alpha_{n-1}(\tau)u(\tau)d\tau \end{bmatrix}$$

Nous remarquons que les quantités à la gauche de cette équation sont connues. Par conséquent, nous pouvons réécrire celle-là sous forme

$$\mathrm{const}^{n\times 1} = \mathcal{C}(A,B)^{n\times(nm)} \begin{bmatrix} \int_0^{t_1} \alpha_0(\tau)u(\tau)d\tau \\ \int_0^{t_1} \alpha_1(\tau)u(\tau)d\tau \\ \vdots \\ \int_0^{t_1} \alpha_{n-1}(\tau)u(\tau)d\tau \end{bmatrix}^{(nm)\times 1} , \qquad \tau \in (0,t_1) \tag{1.28}$$

Une solution de (1.28) existe si et seulement si $\operatorname{rang}\mathcal{C}(A,B) = n$, qui est la condition établie dans (1.26). En général, il est très difficile de résoudre cette équation. L'une des nombreuses solutions possibles de (1.28) peut être donnée en termes du grammien de contrôlabilité. Le grammien de contrôlabilité est donné par l'intégral suivant

$$W(0,t_1) = \int_0^{t_1} e^{-A\tau}BB^\top e^{-A^\top\tau}d\tau \tag{1.29}$$

Les résultats présentés dans cette section peuvent être résumés par le théorème

Théorème 1.4. *Le système linéaire en temps continu est commandable si et seulement si la matrice de commandabilité $\mathcal{C}$ est de rang plein, i.e. $\operatorname{rang}\mathcal{C} = n$.*

Nous avons montré que l'observabilité et la commandabilité des systèmes linéaires en temps discret et en temps continu peuvent être exprimées en terme de la matrice d'observabilité (1.6) et de la matrice de commandabilité (1.18), respectivement. Pour abréger, on dit souvent que la paire (A_d, C_d) est observable pour dire que le rang de la matrice d'observabilité $\mathcal{O}(A_d, C_d)$ est maximum et la paire (A_d, B_d) est commandable pour dire que le rang de la matrice de commandabilité $\mathcal{C}(A_d, B_d)$ est maximum. Examiner le rang de la matrice d'observabilité et de la matrice de commandabilité est un critère algébrique qui donne la possibilité de tester l'observabilité ainsi la commandabilité d'un système.

1.3 Stabilisation, feedback

Dans cette section, nous présentons les notions de stabilité, la commande par retour d'état, le théorème de Lyapunov et les commandes non linéaires robustes via la technique d'ajout d'intégrateur.

1.3.1 Notions de stabilité

La notion de stabilité s'attache à formaliser l'intuition suivante : un point d'équilibre sera dit stable si un petit déséquilibre initial n'entraîne que de petits écarts pour tout temps postérieur, en bref de petites causes n'ont que de petites conséquences.

Soit un système dynamique non linéaire et non autonome de la forme :

$$\begin{cases} \dot{x} & = & f(t,x) \\ x(t,t_0) & = & x_0 \end{cases} \tag{1.30}$$

où $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ est une fonction continue en t et localement lipschitzienne [1] en x sur $[0,+\infty) \times D$ et $D \subset \mathbb{R}^n$ est un domaine qui contient l'origine $x = 0$.

Définition 1.5. L'origine est un point d'équilibre de (1.30) si

$$f(t,0) = 0, \ \forall t \geq 0.$$

Définition 1.6. Le point d'équilibre $x = 0$ de (1.30) est

- stable si pour chaque $\epsilon > 0$, il existe $\delta = \delta(\epsilon, t_0) > 0$ tel que

$$|x(t_0,t_0)| < \delta \Rightarrow |x(t,t_0)| < \epsilon, \forall t \geq t_0 \geq 0. \tag{1.31}$$

- uniformément stable si, pour chaque $\epsilon > 0$, il existe $\delta = \delta(\epsilon) > 0$, indépendant de t_0, tel que la relation (1.31) est satisfaite.
- instable s'il n'est pas stable.
- asymptotiquement stable s'il est stable et s'il existe $c = c(t_0) > 0$ tel que $\lim_{t\to\infty} x(t,t_0) = 0$ pour tout $|x(t_0,t_0)| < c$.
- uniformément asymptotiquement stable s'il est uniformément stable et s'il existe $c > 0$, indépendant de t_0, tel que pour chaque $\epsilon > 0$, il existe $T = T(\epsilon) > 0$ tel que

$$|x(t,t_0)| < \epsilon, \ \forall t \geq t_0 + T(\epsilon), \ \forall |x(t_0,t_0)| < c.$$

1. On dit que f est localement lipschitzienne si, pour tout x de D, il existe un voisinage V de x, et un réel k, tel que, pour tous u, v de V, $|f(u) - f(v)| \leq k|u - v|$.

- globalement uniformément asymptotiquement stable s'il est uniformément stable et si toutes paires de réels positifs ϵ, c, il existe $T = T(\epsilon, c) > 0$ tel que

$$|x(t,t_0)| < \epsilon, \ \forall t \geq t_0 + T(\epsilon, c), \ \forall |x(t_0, t_0)| < c.$$

Définition 1.7. Une fonction $\alpha : [0, a) \to [0, +\infty)$ est dite appartenir à la classe des fonctions $\mathcal{K}$ si celle-ci est continue, strictement croissante et $\alpha(0) = 0$. Elle est dite appartenir à la classe de fonction $\mathcal{K}_\infty$ si $a = +\infty$ et

$$\lim_{r \to +\infty} \alpha(r) = +\infty.$$

Définition 1.8. Une fonction continue $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ est dite appartenir à la classe des fonctions $\mathcal{KL}$ si, pour tout s fixé, la fonction $\beta(r, s)$ est de classe $\mathcal{K}$ par rapport à r et, pour tout $r > 0$ fixé, la fonction $\beta(r, s)$ est strictement décroissante par rapport à s et

$$\lim_{s \to +\infty} \beta(r, s) = 0.$$

Stabilité asymptotique et exponentielle

Définition 1.9. Le point d'équilibre $x = 0$ de (1.30) est uniformément asymptotiquement stable s'il existe une fonction β de classe $\mathcal{KL}$ et une constate $c > 0$ telles que

$$|x(t,t_0)| \leq \beta(|x(t_0,t_0)|, t - t_0), \ \forall t \geq t_0 \geq 0, \ \forall |x(t_0,t_0)| < c. \tag{1.32}$$

Il est globalement uniformément asymptotiquement stable si l'inégalité (1.32) est satisfaite pour toute condition initiale $x(t_0, t_0)$.

Définition 1.10. Le point d'équilibre $x = 0$ de (1.30) est exponentiellement stable si l'inégalité (1.32) est satisfaite avec

$$\beta(r, s) = kr\,\mathrm{e}^{-\gamma s}$$

où k, γ sont deux constantes strictement positives. Il est globalement exponentiellement stable si cette condition est satisfaite pour toute condition initiale.

Stabilité entrée-état et stabilité intégrale entrée-état

Un outil fondamental pour l'analyse des systèmes non linéaires interconnectés est la stabilité entrée-état, appelée en anglais input-to-state stability (ISS). Cette notion, introduite à la fin des années 80 par E.D. Sontag [127], impose que la norme de l'état à chaque instant soit bornée par la somme d'une fonction de l'amplitude du signal exogène appliqué et d'un terme non négatif fonction de la norme de l'état initial et tendant vers zéro avec le temps. Cette propriété concerne les systèmes non linéaires de la forme :

$$\dot{x} = f(x, u) \tag{1.33}$$

Définition 1.11. Le système (1.33) est dit stable entrée-état (ISS) s'il existe une fonction β de classe $\mathcal{KL}$ et une fonction γ de classe $\mathcal{K}_\infty$ (appelée gain ISS) telles que, pour toute condition initiale $x_0 \in \mathbb{R}^n$ et toute entrée admissible u, sa solution satisfait

$$|x(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma\left(|u|_{[0,t]}\right), \ \forall t \geq 0.$$

Malgré l'indéniable succès de cette notion, le fait que l'état soit borné pour tout signal d'entrée borné constitue une contrainte très forte en pratique. Une alternative à l'ISS est la stabilité entrée-état intégrale, appelée en anglais integral input-to-state stability (iISS), introduite dans [128], qui évalue l'impact de l'énergie du signal appliqué (plutôt que son amplitude) sur le système considéré.

Définition 1.12. Le système (1.33) est dit intégralement stable entrée-état s'il existe une fonction β de class $\mathcal{KL}$, et deux fonctions μ_1, μ_2 de classe $\mathcal{K}_\infty$ telles que, pour toute condition initiale $x_0 \in \mathbb{R}^n$ et toute entrée admissible u, sa solution satisfait

$$|x(t, x_0, u)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2\left(|u(s)|\right) ds\right), \ \forall t \geq 0.$$

μ_2 est alors appelée gain iISS de (1.33).

1.3.2 Le retour d'état

Soit le système linéaire stationnaire défini par

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{1.34}$$

où $x(t) \in \mathbb{R}^n$ représente l'état du système, $u(t) \in \mathbb{R}^m$ est la commande et $y(t) \in \mathbb{R}^p$ est la sortie mesurée du système.

Une loi de commande possible est celle définie par : $u(t) = Kx(t)$, où $K \in \mathbb{R}^{m\times n}$ est une matrice appelée gain du retour d'état. C'est une commande en boucle fermée car elle dépend des signaux internes du système, même si elle ne prend pas en compte directement la sortie du système $y(t)$.

Le système en boucle fermée s'écrit donc :

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & (A+BK)x(t) \\ y(t) & = & Cx(t) \end{array}\right. \tag{1.35}$$

Remarque 1.13. Nous pouvons modifier au moyen d'un choix approprié de K tous les pôles du système si et seulement si le système est commandable.

Lorsqu'on veut stabiliser exponentiellement le système (1.34), on peut déterminer le gain K tel que $A + BK$ est Hurwitz. Ces idées se généralisent à des systèmes non linéaires, mais la détermination de lois de commande stabilisant asymptotiquement des points d'équilibre est en général plus difficile. Il n'existe pas de méthode systématique procurant ces lois de commandes.

1.3.3 Méthode de Lyapunov et technique d'ajout d'intégrateur

La notion de stabilité au sens de Lyapunov apparait dans l'étude des systèmes dynamiques. Une des approches de la commande robuste, consiste à rechercher une fonction de Lyapunov quadratique optimale au moyen d'un algorithme d'optimisation convexe, sur la base d'inégalités matricielles linéaires (appelées en anglais Linear Matrix Inequalities - LMIs).

Les formulations LMI

Le terme "inégalité matricielle linéaire" est, de nos jours, au cœur d'un nombre important de problèmes d'analyse et de commande des systèmes dynamiques, voir [20], [121]. Nous rappelons quelques notions LMIs.

Définition 1.14. Une inégalité matricielle linéaire est une expression de la forme

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succ 0 \tag{1.36}$$

où $x = (x_i)_{i=\overline{1,m}}$ est un vecteur de nombres réels (variables de décisions) et $(F_i)_{i=\overline{1,m}}$ sont des matrices réelles symétriques, c'est à dire $F_i = F_i^\top \in \mathbb{R}^{n\times n}$.

Remarque 1.15. L'inégalité "$\succ$" dans (1.36) signifie "définie positive", c'est à dire $u^\top F(x)u > 0$ pour tout $u \in \mathbb{R}^n$, $u \neq 0$, ce qui, en raison de la symétrie de $F(x)$, est équivalent au fait que la plus petite valeur propre de $F(x)$ est strictement positive.

Définition 1.16. Une inégalité matricielle linéaire est une inégalité

$$F(x) \succ 0 \tag{1.37}$$

où F est une fonction affine d'un espace vectoriel de dimension fini $\mathbb{V}$ vers un ensemble

$$\mathbb{S}^n := \{M \in \mathbb{R}^{n\times n} \ : M = M^\top\} \tag{1.38}$$

de matrices réelles symétriques.

Remarque 1.17. 1. L'objectif est de trouver $x \in \mathbb{V}$ satisfaisant l'inégalité. Ce choix de x est appelé le **problème de faisabilité**.

2. Le terme **"inégalité matricielle linéaire"** est utilisé dans la littérature relative aux systèmes et au contrôle, mais la terminologie peut surprendre dans la mesure où la fonction F vérifiant $F(x) \succ 0$ n'est pas linéaire.
3. Une LMI non stricte est une LMI telle que l'inégalité dans (1.37) est non stricte et (1.37) est remplacée par $F(x) \succeq 0$. Les inégalités matricielles $F(x) \prec 0$ et $F(x) \prec G(x)$ où F, G étant des fonctions affines, sont des cas particuliers de (1.36) puisqu'elles peuvent être reformulées ainsi : $-F(x) \succ 0$ et $G(x) - F(x) \succ 0$.

Théorème 1.18 (Convexité). *L'existence d'une solution à la LMI (1.36) définit un problème dit convexe car l'ensemble des solutions*

$$\mathbb{S} = \{x \in \mathbb{R}^m \ : F(x) \succ 0\} \tag{1.39}$$

est convexe.

Démonstration. On veut montrer que, $x_1, x_2 \in \mathbb{S}, \lambda \in [0, 1]$ implique

$$\lambda x_1 + (1 - \lambda)x_2 \in \mathbb{S} \tag{1.40}$$

F étant affine, alors,

$$F(\lambda x_1 + (1 - \lambda)x_2) = \lambda F(x_1) + (1 - \lambda)F(x_2) \succ 0 \tag{1.41}$$

□

Définition 1.19 (Optimisation). Une fonction $f : \mathbb{S} \to \mathbb{R}$ est dite convexe si,

1. $\mathbb{S}$ est convexe,
2. Pour tout $x_1, x_2 \in \mathbb{S}$ et $\alpha \in [0, 1]$, on a :

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2) \tag{1.42}$$

Soit maintenant, $f : \mathbb{S} \to \mathbb{R}$ où on suppose $\mathbb{S} = \{x \in \mathbb{R}^m \ : F(x) \succ 0\}$. Le problème de la détermination du

$$V_{\text{opt}} = \inf_{x\in\mathbb{S}} f(x) \tag{1.43}$$

est appelé problème d'optimisation avec contraintes LMI.

Lemme 1.20 (Complément de Schur)**.** *La LMI*

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succ 0 \tag{1.44}$$

où $Q = Q^\top$ *et* $R = R^\top$ *est équivalente à* $R \succ 0$, $Q - SR^{-1}S^\top \succ 0$ *ou encore* $Q \succ 0$, $R - SQ^{-1}S^\top \succ 0$

Démonstration. La preuve se fait facilement en multipliant (1.44) à droite par,

$$\begin{bmatrix} I & 0 \\ -R^{-1}S^\top & I \end{bmatrix}$$

et à gauche par la transposée de cette dernière matrice. On obtient alors,

$$\begin{bmatrix} Q - SR^{-1}S^\top & 0 \\ 0 & R \end{bmatrix} \succ 0.$$

□

La LMI la plus célèbre en automatique, mentionnée ci-dessus, est celle qui résulte de l'application de la seconde méthode de Lyapunov à la stabilité d'un système dont la présentation est $\dot{x} = Ax$.

On montre alors que le système est asymptotiquement stable si et seulement s'il existe une matrice $P = P^\top \succ 0$ telle que la LMI (également appelée l'inégalité de Lyapunov) suivante est vérifiée :

$$A^\top P + PA \prec 0 \tag{1.45}$$

Jury et Ahn [64] montrent qu'il est possible de choisir sans restriction n'importe quelle matrice $Q = Q^\top \succ 0$ et de résoudre l'équation linéaire suivante, dite de Lyapunov

$$A^\top P + PA = -Q. \tag{1.46}$$

Ogata [108] montre un résultat similaire pour les systèmes discrets $x_{k+1} = Ax_k$. Ce dernier système est asymptotiquement stable si et seulement si pour toute matrice $Q = Q^\top \succ 0$, il existe une unique matrice $P = P^\top \succ 0$ telle que :

$$A^\top PA - P = -Q. \tag{1.47}$$

Pour terminer, on parle de la résolution de la LMI. Afin de rendre les solvers de la LMI facilement utilisables pour les problèmes de l'automatique, des interfaces ont été développées permettant d'écrire les problèmes sous des formes matricielles simples. On peut citer l'outil YALMIP [129] qui permet de définir un problème LMI et de le résoudre avec n'importe quel solver installé sur notre machine. Les trois problèmes classiques que ces outils résolvent sont :

1. La faisabilité (ou existence),
2. La minimisation d'une fonction linéaire,
3. Le problème de la valeur propre généralisée.

Fonction de Lyapunov

Définition 1.21. Considérons le système $\dot{x} = f(t,x)$ introduit en (1.30). Une fonction $V : [0,+\infty) \times \mathbb{R}^n \to [0,+\infty)$, pour laquelle il existe des fonctions $\alpha_1, \alpha_2, \alpha_4 \in \mathcal{K}_\infty$ et $\alpha_3 : \mathbb{R}^n \to [0,+\infty)$ telles que pour tout $x \in \mathbb{R}$ et tout $t \geq 0$ ce qui suit est vérifié :

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|), \tag{1.48}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha_3(x), \tag{1.49}$$

$$\left| \frac{\partial V}{\partial x}(t,x) \right| \leq \alpha_4(|x|), \tag{1.50}$$

est appelée fonction de Lyapunov pour le système (1.30) si α_3 est semi-définie positive, et fonction de Lyapunov stricte pour le système (1.30) si α_3 est définie positive.

Théorème 1.22 (Théorème classique de Lyapunov)**.** *Soit $x = 0$, un point d'équilibre de (1.30) et $D \subset \mathbb{R}^n$ un domaine contenant $x = 0$. Soit $V : [0,+\infty) \times D \to R$ une fonction continument différentiable telle que :*

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|), \ \forall t \in [0,+\infty), \ \forall x \in D \tag{1.51}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq -\alpha_3(x), \ \forall t \in [0,+\infty), \ \forall x \in D \tag{1.52}$$

où $\alpha_1, \alpha_2, \alpha_3$ sont des fonctions définies positives et continues sur D. Alors $x = 0$ est uniformément asymptotiquement stable. Si $D = \mathbb{R}^n$ et α_1 est de classe $\mathcal{K}_\infty$, alors $x = 0$ est globalement uniformément asymptotiquement stable.

L'une des grandes forces de l'ISS et de l'iISS réside dans leurs caractérisations en termes de fonctions de dissipation, inspirées par les fonctions de Lyapunov. Ces caractérisations ont été respectivement établies dans [126] et [9].

Définition 1.23. Supposons que V est une fonction de Lyapunov candidate dont la dérivée le long des trajectoires du système (1.33) satisfait, pour tout $x \in \mathbb{R}^n$ et tout $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x}(x)f(x,u) \leq -\alpha(|x|) + \gamma(|u|),$$

où γ désigne une fonction de classe $\mathcal{K}_\infty$. L'existence d'une telle fonction est équivalente à l'ISS si le taux de dissipation α est de classe $\mathcal{K}_\infty$, et à l'iISS si α est une fonction définie positive.

La technique d'ajout d'intégrateur

La technique d'ajout d'intégrateur, appelée en anglais "backstepping", introduite par Byrnes et Isidori ([22]), Kolesnikov ([72]), Tsinias ([132]), Coron et Praly ([28]), est devenue l'une des méthodes de base de construction de lois de commande stabilisantes. Les avantages inhérents à cette technique sont bien connus. Observons en particulier qu'elle procure une grande famille de lois de commande globalement asymptotiquement stabilisantes, permet de résoudre divers problèmes de robustesse et de commande adaptative ([77]). Cette approche répond essentiellement au problème de la construction de feedbacks qui stabilisent globalement asymptotiquement l'origine d'un système de forme particulière. Plus précisément cette approche répond essentiellement à la question de savoir quand est ce que la stabilisabilité asymptotique du système :

$$\dot{y} = f(y,u), \tag{1.53}$$

où u est l'entrée, implique celle du système :

$$\left\{ \begin{array}{lcl} \dot{y} & = & f(y,x) \\ \dot{x} & = & h(x,y,u) \end{array} \right. , \tag{1.54}$$

où u est l'entrée. De la réponse à ce problème, on peut déduire une preuve de la stabilisabilité asymptotique globale de systèmes sous forme dite feedback, c'est à dire ayant la structure récurrente particulière suivante :

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & f_1(x_1,x_2) \\ \dot{x}_2 & = & x_3 + f_2(x_1,x_2) \\ & \vdots & \\ \dot{x}_n & = & u + f_n(x_1,\ldots,x_n) \end{array} \right. , \tag{1.55}$$

avec pour seule hypothèse celle de la stabilisabilité asymptotique globale du système $\dot{x} = f_1(x,u)$ par une loi de commande suffisamment régulière et la régularité des fonctions f_i, $i = 1$ à $n-1$.

1.4 Méthodes d'estimation d'état classique

Le problème de la conception d'observateurs se pose tout naturellement pour l'étude de systèmes dynamiques, dès lors que l'on a besoin de certaines informations internes à partir des mesures externes qui sont directement disponibles. En effet, il est clair que l'on ne peut pas toujours utiliser autant de capteurs que de signaux d'intérêt (pour des raisons de coût, des contraintes technologiques, etc.). D'ailleurs ces signaux peuvent être nombreux et de différents types : les signaux à temps variant caractérisant le système (les variables d'état), les signaux constants (les paramètres), et ceux externes non mesurés (les perturbations).

Le besoin d'information interne peut être motivé par des objectifs différents : la modélisation (l'identification), la surveillance (la détection de défaut), ou la conduite (le contrôle) du système. Tous ces objectifs sont des étapes essentielles au maintien d'un système sous contrôle, ce qui est résumé par la Figure 1.1 au dessous.

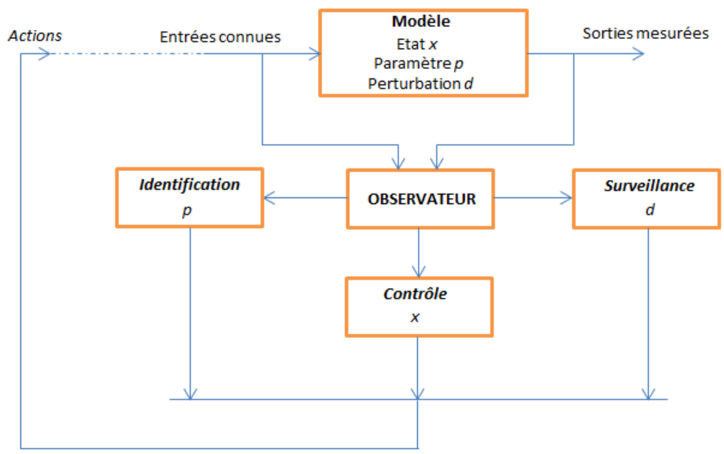

FIGURE 1.1 – Observateur comme le cœur des systèmes de contrôle.

Observons qu'un observateur est un système dynamique constitué assez souvent d'une copie du modèle et d'un terme de correction permettant de faire converger l'observateur vers l'état réel du système.

En ce qui concerne le modèle considéré, il peut être à temps continu ou à temps discret, déterministe ou stochastique, de dimension finie ou infinie, lisse ou "avec des singularités".

La théorie de l'observateur d'état déterministe a été introduite dans les années soixante par Luenberger pour les systèmes linéaires. Kalman a également formulé un observateur en considérant un système linéaire stochastique. Pour les systèmes non linéaires, l'observation reste un domaine où la recherche est très active, comme par exemple [74], [75], [69], [76], [6], [7], [8], mais l'utilisation d'observateurs de type filtre de Kalman étendu est la plus commune.

Dans l'article de Kerner et Isidori en 1983 [74], ils ont donné des conditions nécessaires et suffisantes pour une linéarisation de l'erreur de l'observation des modèles non linéaires afin de leur appliquer l'observateur de Luenberger. Cependant, leurs résultats ne s'appliquent qu'à une classe réduite de systèmes non linéaires.

A la même époque Fliess et Kupka [46] fournissent des conditions suffisantes et nécessaires pour une linéarisation exacte des systèmes non linéaires en mettant en évidence le concept de l'immersion.

Un autre résultat dû à Krener et Repondek [75] vient élargir la classe des systèmes dynamiques étudiés par Krener et Isidori [74] en permettant un difféomorphisme sur la sortie.

Dans le paragraphe qui suit on va introduire la notion d'observateurs connus dans la littérature : l'observateur de Luenberger, le filtre de Kalman, les approches "grand gain" et l'observateur continu-discret.

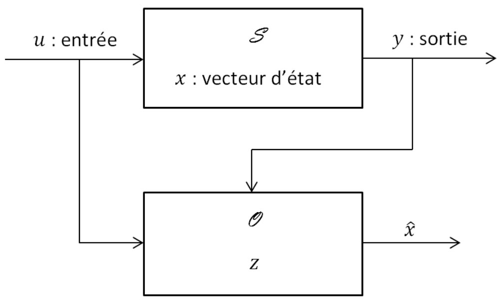

FIGURE 1.2 – Observateur dynamique.

1.4.1 Observateur de Luenberger

Luenberger [82] a étudié le problème d'observateur dans le cas d'un système linéaire. Si un système est linéaire et complètement observable, son vecteur d'état peut être reconstitué approximativement par la construction d'un observateur qui est lui-même un système linéaire conduit par les sorties et les entrées du système d'origine. Le vecteur d'état d'un système d'ordre n avec m sorties indépendantes peut être reconstruit avec un observateur d'ordre $n - m$.

Considérons le système linéaire suivant :

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases} \tag{1.56}$$

où $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$.

Luenberger a proposé un observateur pour le système (1.56) comme suit :

$$\dot{z} = Rz + Sy, \quad \hat{x} = Tz, \tag{1.57}$$

où $\hat{x}$ désigne la variable estimée correspondant à la variable réelle x, $T = D^{-1}$ et D est la solution de $DA = RD + SC$ où R est Hurwitz (c.-à-d. possède des valeurs propres à parties réelles strictement négatives).

A partir de (1.57), on obtient :

$$\begin{aligned}\dot{\hat{x}} &= T\dot{z} \\ &= TRz + TSy = TRD\hat{x} + TSy.\end{aligned}$$

Puisque $DA = RD+SC$, l'égalité $A = TRD+TSC$ est vérifiée et donc $TRD = A-TSC$.

On en déduit

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} - TSC\hat{x} + TSy \\ &= A\hat{x} + TS(y - C\hat{x}).\end{aligned}$$

Posons $K = TS$. Nous obtenons :

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K(y(t) - C\hat{x}(t))$$

Par ailleurs, nous avons $A = TRD + TSC$. Donc $R = D(A - KC)D^{-1}$.

Par conséquent, pour avoir convergence exponentielle de l'observateur, il faut déterminer le gain K tel que $(A - KC)$ soit Hurwitz .

1.4.2 Filtre de Kalman

Le filtre de Kalman ([65], [4]) est très connu dans le cadre des systèmes linéaires, il peut être considéré comme un observateur de Luenberger avec un gain variant dans le temps. Cela permet de minimiser la variance a priori de l'erreur d'estimation.

Filtre de Kalman de systèmes linéaires

On considère le modèle dynamique du système linéaire défini par :

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t) \end{cases} \tag{1.58}$$

où $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ et $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^p$ sont deux bruits blancs gaussiens d'espérance nulle, de covariances respectives $Q(t)$ et $R(t)$. Ces bruits sont supposés non corrélés. Les matrices du systèmes sont de dimensions appropriées, et les conditions initiales sont définies par $x(0) = x_0$.

Nous supposons que le système est observable et que la distribution initiale est gaussienne tels que :

$$E[x_0] = \hat{x}_0, \quad E\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^\top\right] = P_0 \tag{1.59}$$

où E représente la valeur attendue et P_0 est la matrice de covariance de l'erreur initiale. Le filtre est établi dans quatre étapes ([49]) :

1. L'initialisation :

$$E[x_0] = \hat{x}_0, \quad E\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^\top\right] = P_0 \tag{1.60}$$

2. Le vecteur d'estimation d'état :

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - Cx(t)) \tag{1.61}$$

3. La propagation de covariance de l'erreur (l'équation de Riccati) :

$$\dot{P}(t) = AP(t) + P(t)A^\top - P(t)C^\top R(t)^{-1}CP(t) + Q(t) \tag{1.62}$$

4. Le calcul du gain :

$$K(t) = P(t)C^\top R(t)^{-1} \tag{1.63}$$

Remarque 1.24. • Le filtre de Kalman peut être appliqué dans le cas où les matrices A et C dépendent du temps (l'observabilité du système doit être toujours assurée).

- L'estimation des matrices définies positives R, Q, P_0 est souvent très délicate, surtout lorsque les propriétés du bruit ne sont pas connues.
- Cet observateur consiste à minimiser l'intégrale de 0 à t du carré de l'erreur.

Filtre de Kalman Etendu

Le filtre de Kalman étendu est l'une des techniques d'estimation les plus populaires et largement étudiées dans le domaine d'estimation d'état des systèmes dynamiques non linéaires. Ce filtre étendu consiste à utiliser les équations du filtre de Kalman standard au modèle non linéaire linéarisé par la formule de Taylor au premier ordre.

Ce filtre étendu a été appliqué avec succès sur différents types de procédés non linéaires. Malheureusement les preuves de stabilité et de convergence établies dans le cas des systèmes linéaires, ne peuvent en général pas être étendues au cas des systèmes non linéaires. Dans un environnement déterministe, une preuve de la convergence du filtre de Kalman étendu a été établie dans [61] et [24] pour la classe des systèmes non linéaires à temps discret. Cependant, cette convergence n'est que locale.

1.4.3 Approches "grand gain"

Il existe une grande classe d'observateurs non linéaires qui présentent d'excellentes propriétés globales : ce sont les observateurs "grand gain" dont la référence historique est l'article [48]. Contrairement au filtre de Kalman étendu, ces observateurs sont des observateurs globaux à convergence exponentielle. En revanche, ils ne s'appliquent qu'à une classe restreinte de systèmes non linéaires. Dans cet article, les auteurs considèrent les systèmes à une observation sous la forme :

$$\begin{cases} \dot{x} &= f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \\ y &= h(x) \end{cases} \tag{1.64}$$

où $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$.

Ces systèmes sont affines en l'entrée. Nous supposons deux hypothèses suivantes :

Hypothèse 1.25. *Le système (1.64) est uniformément observable.*

Hypothèse 1.26. *Les fonctions $f_i(x)$, $i = 0$ à n dans (1.64), sont globalement Lipschitziennes.*

La construction d'un observateur exige certaines propriétés d'observabilité. Quand un système linéaire est observable, il est observable indépendamment de l'entrée $u(t)$. Pour les systèmes non linéaires, cela n'est pas le cas. En général, les systèmes non linéaires ont des entrées singulières qui les rendent non observables. Par conséquent, lorsque les entrées sont égales à "ces mauvaises entrées", l'observation est impossible, et lorsque les entrées sont à proximité de "ces mauvaises entrées", l'observation devient de plus en plus difficile.

L'article [48] dit que sous les deux Hypothèses (1.25) et (1.26), il existe, pour (1.64), un observateur de type d'observateur de Luenberger étendu et cet observateur est connu sous le nom d'observateur à grand gain.

Afin de rendre le résultat global, nous supposons de plus que ϕ est un difféomorphisme de Ω dans $\phi(\Omega)$, où Ω représente le domaine physique qui correspond à l'évolution de x. Nous considérons le changement de coordonnées :

$$\begin{aligned} \phi :\ & \mathbb{R}^n \to \mathbb{R}^n \\ & x \to \begin{bmatrix} h(x) & L_{f_0}h(x) & \dots & L_{f_0}^{n-1}h(x) \end{bmatrix}^\top , \end{aligned} \tag{1.65}$$

avec :

$$\begin{aligned} L_{f_0}h(x) &= \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i} f_{0i} , \\ L_{f_0}^{j}h(x) &= L_{f_0}\left(L_{f_0}^{j-1}h(x)\right) , \end{aligned}$$

et :

$$L_{f_0}^{0}h(x) = h(x).$$

En posant $z = \phi(x)$, nous pouvons donc écrire le système (1.64) dans ces nouvelles coordonnées (en admettant qu'il existe une forme analytique pour $\phi^{-1}(z)$) :

$$\begin{array}{rcccl} \dot{z}_1 & = & \frac{\partial h(x)}{\partial x}\dot{x} & = & L_{f_0}h(x) + \sum_{i=1}^{m} L_{f_i}h(x)u_i \\ & & & = & z_2 + \sum_{i=1}^{m} L_{f_i}L_{f_0}^{0}h\left(\phi^{-1}(z)\right)u_i \\ \dot{z}_2 & = & \frac{\partial L_{f_0}h(x)}{\partial x}\dot{x} & = & L_{f_0}^{2}h(x) + \sum_{i=1}^{m} L_{f_i}L_{f_0}h(x)u_i \\ & & & = & z_3 + \sum_{i=1}^{m} L_{f_i}L_{f_0}h\left(\phi^{-1}(z)\right)u_i \\ \vdots & & & & \\ \dot{z}_{n-1} & = & \frac{\partial L_{f_0}^{n-2}h(x)}{\partial x}\dot{x} & = & L_{f_0}^{n-1}h(x) + \sum_{i=1}^{m} L_{f_i}L_{f_0}^{n-2}h(x)u_i \\ & & & = & z_n + \sum_{i=1}^{m} L_{f_i}L_{f_0}^{n-2}h\left(\phi^{-1}(z)\right)u_i \\ \dot{z}_n & = & \frac{\partial L_{f_0}^{n-1}h(x)}{\partial x}\dot{x} & = & L_{f_0}^{n}h\left(\phi^{-1}(z)\right) + \sum_{i=1}^{m} L_{f_i}L_{f_0}^{n-1}h\left(\phi^{-1}(z)\right)u_i \\ y & = & h(x) & = & z_1 \end{array}$$

Nous obtenons ainsi la forme canonique du système initial :

$$\left\{\begin{array}{lcl} \dot{z} & = & Az + \psi(z) + \sum_{i=1}^{m} u_i \Phi_i(z) \\ y & = & Cz \end{array}\right. , \tag{1.66}$$

où $A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, $\psi(z) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Psi_n(z) \end{bmatrix}$, et

$$\left\{\begin{array}{lcl} \Phi_i(z) & = & \begin{bmatrix} \Phi_{i_1}(z_1) \\ \Phi_{i_2}(z_1, z_2) \\ \vdots \\ \Phi_{i_n}(z_1, \dots, z_n) \end{bmatrix}, \ \forall i = \{1, \dots, m\} \\ \Phi_{i_j}(z) & = & L_{f_i} L_{f_0}^{j-1} h\left(\phi^{-1}(z)\right), \ \forall j = \{1, \dots, n\} \end{array}\right. .$$

Ce changement de variables permet de construire un observateur de type "grand gain" (le résultat principal de [48]) et dont la forme est la suivante :

$$\dot{\hat{z}} = A\hat{z} + \psi(z) + \sum_{i=1}^{m} u_i \Phi_i(\hat{z}) - S_\theta^{-1} C^\top (C\hat{z} - y),$$

où S_θ est la matrice solution de l'équation de Lyapunov :

$$\theta S_\theta + A^\top S_\theta + S_\theta A = C^\top C. \tag{1.67}$$

L'article [31] est l'extension de l'article [48] où les auteurs construisent un observateur exponentiel à "grand gain" pour les systèmes non linéaires affines avec une seule sortie et observables pour toute entrée. Dans [31], les auteurs établissent les résultats de la stabilité pour les filtres de Kalman étendu dans le cas de systèmes continus aux mesures continues (systèmes continus-continus), ainsi que dans le cas de systèmes continus aux mesures discrètes (systèmes continus-discrets) à partir de l'observateur construit en [48]. Nous remarquons qu'il y a une modification du fait que le gain varie : S_θ est la solution d'une équation différentielle.

Nous nous intéressons surtout au résultat suivant : dans le cas continu-discret, si le paramètre θ (défini en [48]) est suffisamment grand et le temps de mesure échantillonnée δ est suffisamment petit, alors l'observateur à "grand gain" construit en [48] converge.

1.4.4 Observateur continu-discret

L'objectif des observateurs continu-discrets est de fournir une estimation de l'état d'un système dont la dynamique est en temps continu en connaissant sa sortie donnée par des mesures en temps discret. Des observateurs continu-discrets ont été étudiés par Jazwinski qui a introduit le filtre de Kalman continu-discret pour résoudre un problème de filtrage pour les systèmes stochastiques en temps continu-discret (voir [61]). Inspiré par cette approche, Deza et ses collègues ([31]) ont proposé un observateur pour l'estimation de l'état des systèmes non linéaires observables pour toute entrée de la forme :

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & f(x) + g(x)u \\ y & = & h(x) \end{array}\right. \tag{1.68}$$

Pour ce type de systèmes, on sait construire un observateur continu que l'on écrit sous la forme suivante :

$$\dot{\hat{x}}(t) = f(\hat{x}) + g(\hat{x})u - w_0(\hat{x})(h(\hat{x}) - y), \tag{1.69}$$

où

$$w_0(\hat{x}) = \left(\frac{\partial \phi(x)}{\partial x}|_{x=\hat{x}}\right)^{-1} S_\theta^{-1} C^\top. \tag{1.70}$$

avec ϕ, C, S_θ définis dans (1.65), (1.66) et (1.67).

On suppose maintenant que les mesures $y(t)$ du système (1.68) sont données sous forme discrète, c'est-à-dire $y(t_k) = h(x(t_k))$, où $t_k = k\delta$, δ étant la période d'échantillonnage de $y(t)$.

Dans ce cas l'observateur continu-discret est décrit par

i) Une étape de prédiction si aucune mesure n'est disponible, c'est-à-dire lorsque $t \in [t_k, t_{k+1})$, l'estimation est obtenue en intégrant le modèle :

$$\left\{\begin{array}{rcl} \dot{\hat{x}}(t) & = & f(\hat{x}) + g(\hat{x})u \\ \dot{S}(t) & = & -\theta S(t) - A^\top S(t) - S(t)A \end{array}\right. , \tag{1.71}$$

où $A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}$.

ii) Lorsque survient une mesure, c'est-à-dire à l'instant $t = t_{k+1}$, l'observateur fait une correction impulsive de l'estimation :

$$\left\{\begin{array}{rcl} S(t_{k+1}) & = & S(t_{k+1}^-) + \delta C^\top C \\ \hat{x}(t_{k+1}) & = & \hat{x}(t_{k+1}^-) - K w_0(\hat{x}(t_{k+1})) \left[h(\hat{x}(t_{k+1}^-)) - y(t_{k+1})\right] \end{array}\right. , \tag{1.72}$$

où

$$w_0(\hat{x}(t_{k+1})) = \left(\frac{\partial \phi(x)}{\partial x}|_{x=\hat{x}(t_{k+1}^-)}\right)^{-1} S^{-1}(t_{k+1})C^\top, \tag{1.73}$$

avec $S(t_{k+1}^-) = \lim\limits_{t \to t_{k+1}^-} S(t)$, $S(t_{k+1})$ est une matrice symétrique définie positive, variante dans le temps et K est un paramètre qu'on introduit pour le réglage de l'observateur.

Dans [31] les auteurs ont démontré que cet observateur converge exponentiellement pour δ suffisamment petit et pour $\theta \in [\theta_0, \theta_1]$, où θ_0 et θ_1 sont deux constantes positives.

1.5 Systèmes monotones et positifs

L'objectif de cette section est de rappeler certains concepts élémentaires concernant l'étude des systèmes positifs et une analyse de ces systèmes. Nous rappelons également dans la dernière partie de la section 1.5 le Théorème de Birkhoff concernant la relation entre la métrique projective de Hilbert et le diamètre projectif. Ce théorème est au cœur de notre construction des observateurs positifs.

1.5.1 Les systèmes positifs

Les systèmes positifs sont les systèmes pour lesquels la trajectoire d'état est toujours positive quand l'état initial est positif. De manière générale, les systèmes positifs jouent un rôle particulièrement important en ingénierie (ex. quantité d'informations à faire transiter par un réseau [123]), en économie (les variables descriptives peuvent être, par exemple, des prix de marchandises, des quantités de stocks [79, 131]), en sciences sociales (ex. nombres d'individus, des taux de satisfaction, de consommation [119]), en biologie (ex. dynamique de populations, épidémiologie [32], bioréacteurs [13, 106]), en physique (ex. problèmes de réservoirs d'eau, systèmes d'irrigation [56]), en chimie (ex. réacteurs chimiques, colonnes à distiller [29]). Dans ces domaines, les variables d'état sont représentées par des grandeurs qui n'ont de sens que lorsqu'elles sont positives.

Nous présentons maintenant les définitions de base de la positivité qui seront utilisées tout au long de notre travail.

Définition 1.27. L'inégalité au sens classique dans $\mathbb{R}^n$: $x \geq y \Leftrightarrow \forall i, x_i \geq y_i$.

Définition 1.28. La positivité au sens classique dans $\mathbb{R}^n$: $x \geq 0 \Leftrightarrow \forall i, x_i \geq 0$.

Définition 1.29. L'orthant positif des vecteurs réels : $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n, x \geq 0\}$.

Définition 1.30. La stricte positivité au sens classique dans $\mathbb{R}^n$: $x > 0 \Leftrightarrow \forall i, x_i > 0$.

Définition 1.31. L'orthant strictement positif des vecteurs réels : $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n, x > 0\}$.

Définition 1.32. $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ est une matrice positive si $\forall i, j : a_{ij} \geq 0$, autrement dit toutes ses entrées sont positives. Nous noterons une telle matrice par : $A \geq 0$ ou encore, $A \in \mathbb{R}^{n\times m}_{\geq 0}$.

Définition 1.33. $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ est une matrice strictement positive si $\forall i, j : a_{ij} > 0$, autrement dit toutes ses entrées sont strictement positives. Nous noterons une telle matrice par : $A > 0$ ou encore, $A \in \mathbb{R}^{n\times m}_{>0}$.

Définition 1.34. $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ est une matrice **coopérative** ou une matrice de **Metzler** si $\forall i \neq j : a_{ij} \geq 0$, autrement dit toutes ses entrées hors diagonales sont positives.

Proposition 1.35. *A est une matrice de Metzler si et seulement si, pour tout $t \geq 0$, $\mathrm{e}^{At} \in \mathbb{R}^{n\times n}_{\geq 0}$.*

Démonstration. • Nécessité : supposons que A est une matrice de Metzler, alors on peut trouver un réel $\lambda > 0$ tel que $A + \lambda I_n \geq 0$.

Nous avons :

$$\begin{aligned} \mathrm{e}^{At} &= \mathrm{e}^{(A+\lambda I_n - \lambda I_n)t} \\ &= \mathrm{e}^{(A+\lambda I_n)t}\, \mathrm{e}^{-\lambda I_n t} \end{aligned}$$

On peut en déduire que $e^{At} \in \mathbb{R}^{n\times n}_{\geq 0}$, $\forall t \geq 0$ du fait que $\mathrm{e}^{(A+\lambda I_n)t} \in \mathbb{R}^{n\times n}_{\geq 0}$ et $\mathrm{e}^{-\lambda I_n t} \in \mathbb{R}^{n\times n}_{\geq 0}$, $\forall t \geq 0$.

• Suffisance : supposons que pour tout $t \geq 0$, $\mathrm{e}^{At} \geq 0$. Ainsi, nous avons :

$$A = \frac{d}{dt}(\mathrm{e}^{At})_{t=0} = \lim_{t\to 0^+} \frac{\mathrm{e}^{At} - I}{t}$$

Nous prenons e_i, e_j deux vecteurs de la base canonique, et obtenons pour tout $i \neq j$:

$$\begin{aligned} a_{ij} &= \langle A e_j, e_i \rangle \\ &= \lim_{t\to 0^+} \left\langle \frac{\mathrm{e}^{At} - I}{t} e_j, e_i \right\rangle \\ &= \lim_{t\to 0^+} \left\{ \frac{\left\langle \mathrm{e}^{At}\, e_j, e_i \right\rangle}{t} - \frac{\langle e_j, e_i \rangle}{t} \right\} \end{aligned}$$

Puisque $\langle e_j, e_i \rangle = 0$, alors on en déduit que $a_{ij} = \lim_{t\to 0^+} \left\{ \frac{\left\langle \mathrm{e}^{At}\, e_j, e_i \right\rangle}{t} \right\} \geq 0$, $\forall i \neq j$. Par conséquent A est une matrice de Metzler.

□

Maintenant, nous considérons la positivité de systèmes dynamiques. Considérons le système à temps continu sous la forme d'une équation différentielle ordinaire :

$$\begin{cases} \dot{x}(t) &= f(x(t)) \\ x(0) &= x_0 \end{cases} \tag{1.74}$$

et les équations aux différence dans le cas où le système est à temps discret :

$$x_{k+1} = f(x_k) \tag{1.75}$$

où $x \in \mathbb{R}^n$ et $f \in \mathcal{C}^1$.

La positivité au sens classique : le système (1.74) (resp. (1.75)) est un système positif si et seulement si $x_0 \geq 0 \Rightarrow \forall t \geq 0$ (resp. $\forall k \geq 0$), $x(t, x_0) \geq 0$ (resp. $x(k, x_0) \geq 0$). C'est-à-dire l'orthant positif $\mathbb{R}^n_{\geq 0}$ est invariant.

Proposition 1.36 ([45, 84])**.** *Le système (1.74) est positif si et seulement si :*

$$\forall x \geq 0,\ x_i = 0 \Rightarrow \dot{x}_i \geq 0. \tag{1.76}$$

avec $i \in \{1, ..., n\}$.

Les systèmes linéaires positifs à temps continu

Cette section traite de définitions et de quelques résultats de positivité pour les systèmes linéaires à temps continu. Nous verrons dans la partie suivante, les résultats adaptés au temps discret.

Considérons le système linéaire autonome suivant :

$$\begin{cases} \dot{x} &= Ax + b \\ x(0) &= x_0 \end{cases} \tag{1.77}$$

où $x \in \mathbb{R}^n$.

Théorème 1.37. *Le système (1.77) est positif si et seulement si A est une matrice de Metzler et $b \geq 0$.*

Démonstration. La Proposition 1.36 implique que garantir la positivité de $\dot{x}_i$ en $x_i = 0$ (avec $x(t) \geq 0$) permet de garantir que le système (1.77) est positif.

Donc pour tout i :

$$\dot{x}_i = a_{ii}x_i + \sum_{j \neq i} a_{ij}x_j + b_i \ .$$

Quand $x_i = 0$, $\dot{x}_i \geq 0$ est équivalent à

$$\sum_{j \neq i} a_{ij} x_j + b_i \geq 0 \ .$$

On en déduit que le système (1.77) est positif si et seulement si $\forall i$, $\forall j \neq i$, $a_{ij} \geq 0$ et $b_i \geq 0$. Ce qui termine la démonstration. □

Le théorème suivant, proposé par Luenberger en 1979, montre le lien entre l'existence et la stabilité d'un équilibre positif $x^\star$.

Théorème 1.38 ([84])**.** *Si le système (1.77) est positif et la matrice A est Hurwitz, alors :*

$$\exists x^\star \geq 0 \ : Ax^\star + b = 0 \tag{1.78}$$

Les systèmes linéaires positifs à temps discret

Dans cette section, nous considérons le système linéaire en temps discret suivant :

$$x_{k+1} = Ax_k + b \tag{1.79}$$

où $x \in \mathbb{R}^n$.

Théorème 1.39. *Le système (1.79) est positif si et seulement si A est une matrice positive et $b \geq 0$.*

Démonstration. Notons que la solution x_k de (1.79) est donnée par :

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^i b$$

- Nécessité : supposons que A est une matrice positive, alors A^k est une matrice positive pour tout k positif, et si $b \geq 0$, $x_0 \in \mathbb{R}^n_{\geq 0}$, alors il s'ensuit que $x_k \in \mathbb{R}^n_{\geq 0}$ pour tout k positif, ce qui implique que le système (1.79) est positif.
- Suffisance : maintenant supposons que le système (1.79) est positif. Notons qu'avec $x_0 = 0$, cela implique que $b \geq 0$. Par ailleurs, nous avons :

  $$x_1 = Ax_0 + b$$

  Il s'ensuit que si x_1 est positif pour tout $x_0 \in \mathbb{R}^n_{\geq 0}$, alors A est une matrice positive. Ce qui termine la démonstration.

□

1.5.2 Principe des systèmes coopératifs

La classe des systèmes dynamiques "coopératifs" a initialement été mise en valeur par E. Kamke [66], reprise ensuite par J.K. Hale [53] puis par M.W. Hirsch [57–59] qui leur donne le nom de "systèmes coopératifs" et exploite leurs caractéristiques. Ces résultats ont ensuite été repris et étendus par H.L. Smith dans une monographie sur cette structure de systèmes [124]. Nous donnons ici certains résultats fondamentaux relatifs aux systèmes coopératifs.

Définissons tout d'abord les systèmes dynamiques coopératifs. Considérons le système dynamique dans un domaine ouvert convexe $U \subset \mathbb{R}^n$:

$$\dot{x}(t) = f(t,x),\ x(0) = x_0 \tag{1.80}$$

Par la suite nous noterons $x(t, x_0)$ la trajectoire du système (1.80) issue de la condition initiale x_0.

Définition 1.40. Le système dynamique (1.80) est coopératif si et seulement si la fonction $f(t,x)$ est telle que :

$$\forall i \neq j,\ \forall t \geq 0,\ \forall x \in U,\ \frac{\partial f_i}{\partial x_j}(t,x) \geq 0,$$

c'est-à-dire la matrice Jacobienne du système (1.80) est une matrice de Metzler.

Un résultat très important concernant les systèmes coopératifs est celui assurant que deux trajectoires d'un système coopératif initialisées en $x_0 \geq y_0$ conservent cette relation au cours du temps. On trouvera une preuve de ce résultat, repris dans le théorème suivant :

Théorème 1.41 (voir [124])**.** *Considérons le système coopératif (1.80). Soient x_0 et y_0 appartenant à $\mathbb{R}^n$ tels que : $x_0 \geq y_0$. Alors :*

$$\forall t \geq 0,\ x(t,x_0) \geq y(t,y_0)$$

Un autre résultat d'importance concerne cette propriété de conservation des relations d'ordre pour deux systèmes encadrant un système coopératif par la relation "$\geq$". Nous rappelons ce résultat dans le théorème suivant :

Théorème 1.42. *Nous considérons les systèmes différentiels du type :*

$$\dot{x}(t) = f(t, x), \ x(0) = x_0 \tag{1.81}$$

$$\dot{z}(t) = g(t, z), \ z(0) = z_0 \tag{1.82}$$

définis sur un domaine convexe $U \subset \mathbb{R}^n$.

Si les inégalités suivante sont satisfaites pour chaque composante :

- $\forall a \in U, \forall t \geq 0, \ f(t, a) \leq g(t, a)$
- g *est coopérative*
- $x_0 \leq z_0$

alors $x(t) \leq z(t)$ pour $t \geq 0$.

Démonstration. Voir [125] pour la démonstration. □

1.5.3 Théorème de Birkhoff

Cette section fournit un outil important pour le Chapitre 2. Nous présentons maintenant quelque notations d'analyse fonctionnelle et en suite le théorème de Birkhoff proposé en 1957.

Définition 1.43. Pour deux vecteurs $x, y \in \mathbb{R}^n_{\geq 0}$, la métrique projective de Hilbert dans $\mathbb{R}^n_{\geq 0}$ est définie par :

$$d(x, y) = \max_{i,j} \log \left(\frac{x_i y_j}{x_j y_i} \right)$$

Elle est comprise comme une métrique projective dans le sens que pour tous $\lambda, \mu > 0$ et $x, y \in \mathbb{R}^n_{\geq 0}$, nous avons $d(\lambda x, \mu y) = d(x, y)$.

Le théorème suivant est au cœur de notre prochain chapitre :

Théorème 1.44 (Théorème de Birkhoff [21])**.** *Si $A \in \mathbb{R}^{n \times n}_{\geq 0}$, nous avons pour tous $x, y \in \mathbb{R}^n_{\geq 0}$, l'inégalité :*

$$d(Ax, Ay) \leq \left(\tanh \left(\frac{\Delta(A)}{4} \right) \right) d(x, y) \ ,$$

où $\Delta(A) = \max \left\{ \log \left(\frac{a_{ij} a_{kr}}{a_{ir} a_{kj}} \right) : 1 \leq i, j, k, r \leq n \right\}$ est le diamètre projectif de la matrice A.

Nous allons rappeler et expliquer plus en détails dans la Section 2.2 du Chapitre 2 ce théorème ainsi que le lien entre la métrique projective de Hilbert et le diamètre projectif.

1.6 Observateurs par intervalles basés sur la théorie des systèmes coopératifs

Les observateurs par intervalles sont une approche de l'estimation d'état. Cette technique est basée sur des méthodes garanties. Elle consiste en un système dynamique auxiliaire fournissant une borne supérieure et une borne inférieure pour les solutions du système considéré, en considérant que l'on connaît des bornes pour la condition initiale et pour des quantité incertaines. Une telle méthode permet de faire face à des perturbations importantes et donne une information, composante par composante, sur les solutions possibles. Ainsi, cette approche est fondamentalement différente des techniques classiques d'analyse de stabilité robuste ou de construction des lois de commande pour des systèmes perturbés à temps continu et à temps discret, telles que celles qui sont présentées, par exemple dans [73].

A notre connaissance, les méthodes garanties pour l'estimation d'état débutent par les travaux de Schweppe en 1968 [122] et qui ont été suivis par de nombreuses contributions, par exemple [1, 25, 26, 71, 102]. Mais la notion d'*observateur par intervalles* est plus récente. Elle a été introduite pour la première fois par Gouzé et al. en 2000 ([51]). La technique est en suite développée dans plusieurs directions car l'estimation d'état est essentielle pour la surveillance, la détection des défauts et le contrôle (plus d'explications peuvent être trouvées notamment dans [2] et les références qui y sont citées). Par ailleurs, les observateurs par intervalles ont été appliqués avec succès à beaucoup de problèmes de la vie réelle (voir les articles [2, 15, 50, 105]). Ainsi, au cours de la dernière décennie, un certain nombre de contributions qui présentent des constructions d'observateurs par intervalles pour différents types de systèmes ont été proposées. Par exemple, les articles [27, 88, 99] sont consacrés à des classes de systèmes linéaires à temps continu et les articles [89, 105, 116, 117] sont consacrés à diverses familles de systèmes non linéaires à temps continu. Les observateurs par intervalles pour des systèmes à temps discret ont été récemment envisagés (voir par exemple les articles [38, 94]) et les articles [93, 98] sont consacrés à des systèmes en temps continu avec sortie constante par morceaux. Pour plus de détails sur la technique d'observateurs par intervalles, le lecteur peut se référer aux contributions [3, 88, 99, 116, 117], pour n'en citer que quelques-uns.

Dans cette section, nous introduisons d'abord les définitions générales d'observateurs par intervalles pour des systèmes à temps continu et à temps discret. Nous présentons également les résultats de base dans la construction de ce type d'observateurs. Nous présentons ces résultats préliminaires avant d'aborder les Chapitres 3, 4, 5, 6 où nous exposerons nos contributions pour ce domaine.

Nous considérons maintenant le système à temps continu suivant :

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t), w(t)), \; x(0) \;=\; x_0 \\ y(t) &= h(x(t), v(t)) \end{cases} \tag{1.83}$$

et dans le cas temps discret le système suivant :

$$\begin{cases} x_{k+1} &= f(x_k, u_k, w_k), \; k \in \mathbb{N} \\ y_k &= h(x_k, v_k) \end{cases} \tag{1.84}$$

avec $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, f, h sont deux fonctions et où $w \in \mathbb{R}^r$, $v \in \mathbb{R}^q$ sont les perturbations de l'état et de la sortie respectivement. L'état initial x_0 n'est pas supposé connu. Les perturbations et l'état initial sont bornés par des quantités connues telles que pour tout $t \geq 0$ (resp. $k \geq 0$) :

$$w^-(t) \leq w(t) \leq w^+(t) \qquad (\text{resp. } w_k^- \leq w_k \leq w_k^+), \tag{1.85}$$

$$v^-(t) \leq v(t) \leq v^+(t) \qquad (\text{resp. } v_k^- \leq v_k \leq v_k^+), \tag{1.86}$$

$$x_0^- \leq x_0 \leq x_0^+ \tag{1.87}$$

Alors si f vérifie des conditions supplémentaires, des observateurs par intervalles pour les systèmes (1.83) et (1.84) peuvent être établis.

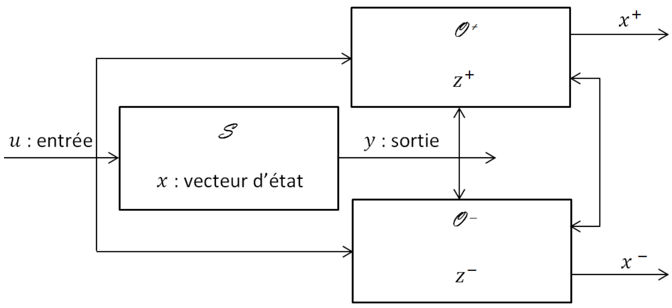

FIGURE 1.3 – Principe de l'observateur par intervalles.

1.6.1 Version à temps continu

Définition 1.45. Considérons le système autonome en temps continu (1.83) avec les conditions (1.85), (1.86) et (1.87). Le système suivant :

$$\begin{cases} \dot{z}^+ &= f^+(z^-, z^+, w^-, w^+, v^-, v^+, u, y), \; z_0^+ = g^+(x_0^-, x_0^+) \\ \dot{z}^- &= f^-(z^-, z^+, w^-, w^+, v^-, v^+, u, y), \; z_0^- = g^-(x_0^-, x_0^+) \\ x^+ &= h^+(z^-, z^+, w^-, w^+, v^-, v^+, u, y) \\ x^- &= h^-(z^-, z^+, w^-, w^+, v^-, v^+, u, y) \end{cases} \tag{1.88}$$

où $f^-, f^+, h^-, h^+, g^-, g^+$ sont les fonctions dans les domaines appropriés, est un encadreur du système (1.83) si pour toute condition initiale $x_0^- \leq x_0 \leq x_0^+$ nous avons $x^-(t) \leq x(t) \leq x^+(t)$.

Ce système est un observateur par intervalles de (1.83) si dans le cas où $w = v = 0$, il existe un réel positif M tel qu'il existe $T > 0$ tel que :

$$\|x^+(t) - x^-(t)\| \leq M, \qquad \forall t \geq T \tag{1.89}$$

La Figure 1.4 montre les simulations d'observateur par intervalles (à gauche) et d'encadreur (à droite).

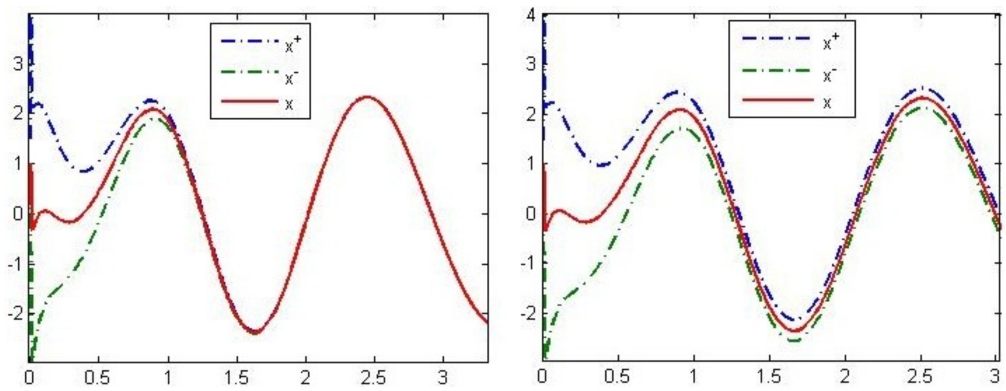

FIGURE 1.4 – Encadreur et Observateur par intervalles pour les systèmes à temps continu.

Application aux systèmes coopératifs

Nous considérons ici le système non linéaire stationnaire du type :

$$\dot{x}(t) = f(x, u) \tag{1.90}$$

Théorème 1.46. *Si f est coopérative, alors le système différentiel suivant encadre les solutions du système (1.90) :*

$$\begin{cases} \dot{z}^+ &= f(z^+, u),\ z_0^+ = x_0^+ \\ \dot{z}^- &= f(z^-, u),\ z_0^- = x_0^- \\ x^+ &= z^+ \\ x^- &= z^- \end{cases} \tag{1.91}$$

Si $x_0^- \leq x_0 \leq x_0^+$, alors $x^-(t) \leq x(t) \leq x^+(t)$ pour tout $t \geq 0$.

Si le système (1.90) est globalement asymptotiquement stable, c'est un observateur par intervalles.

Démonstration. Nous pouvons directement montrer que (1.91) est un encadreur de (1.90) grâce au Théorème 1.42.

Nous savons que f est coopérative et $z_0^- = x_0^- \leq x_0 \leq x_0^+ = z_0^+$. Donc d'après le Théorème 1.42, les relations d'ordre pour deux systèmes encadrant un système coopératif par la relation "$\geq$" sont conservées. Nous obtenons donc, $x^-(t) = z^-(t) \leq x(t) \leq z^+(t) = x^+(t)$.

Par ailleurs, le système (1.90) est globalement asymptotiquement stable alors il existe un réel positif M et un temps T tels que :

$$\|f(z^+, u) - f(z^-, u)\| \leq M, \qquad \forall t \geq T \tag{1.92}$$

Par conséquent (1.91) est un observateur par intervalles de (1.90). □

Le cas linéaire

Considérons maintenant le système linéaire :

$$\dot{x} = Ax + w \tag{1.93}$$

où w est un terme d'incertitudes inconnu tel que : $w^-(t) \leq w(t) \leq w^+(t)$, $\forall t \geq 0$ avec w^+, w^- sont les deux bornes connues.

Théorème 1.47. *Si la matrice $A \in \mathbb{R}^{n\times n}$ est coopérative, alors le système différentiel suivant encadre les solutions du système (1.93) :*

$$\begin{cases} \dot{z}^+ &= Az^+ + w^+,\ z_0^+ = x_0^+ \\ \dot{z}^- &= Az^- + w^-,\ z_0^- = x_0^- \\ x^+ &= z^+ \\ x^- &= z^- \end{cases} \tag{1.94}$$

Si $x_0^- \leq x_0 \leq x_0^+$, alors $x^-(t) \leq x(t) \leq x^+(t)$ pour tout $t \geq 0$.

Si la matrice $A \in \mathbb{R}^{n\times n}$ est Hurwitz, c'est un observateur par intervalles.

Démonstration. Les systèmes suivants sont positifs :

$$\dot{z}^+ - \dot{x} = A(z^+ - x) + w^+ - w \tag{1.95}$$

$$\dot{x} - \dot{z}^- = A(x - z^-) + w - w^- \tag{1.96}$$

car la matrice $A \in \mathbb{R}^{n\times n}$ est coopérative et $w^+ - w \geq 0$, $w - w^- \geq 0$ pour $t \geq 0$ (voir le Théorème 1.37).

Si $z_0^- = x_0^- \leq x_0 \leq x_0^+ = z_0^+$, alors $x^-(t) = z^-(t) \leq x(t) \leq z^+(t) = x^+(t)$ pour tout $t \geq 0$.

Par ailleurs, dans le cas où $w = 0$, le système $\dot{z}^+ - \dot{z}^- = A(z^+ - z^-)$ est globalement asymptotiquement stable car la matrice $A \in \mathbb{R}^{n\times n}$ est Hurwitz. Ce qui termine la démonstration.

□

1.6.2 Version à temps discret

Définition 1.48. Considérons le système autonome en temps continu (1.84) avec les conditions (1.85), (1.86) et (1.87). Le système suivant :

$$\begin{cases} z_{k+1}^+ &= f^+(z_k^-, z_k^+, w_k^-, w_k^+, v_k^-, v_k^+, u_k, y_k), \\ & \qquad\qquad\qquad\qquad\qquad\qquad z_0^+ = g^+(x_0^-, x_0^+) \\ z_{k+1}^- &= f^-(z_k^-, z_k^+, w_k^-, w_k^+, v_k^-, v_k^+, u_k, y_k), \\ & \qquad\qquad\qquad\qquad\qquad\qquad z_0^- = g^-(x_0^-, x_0^+) \\ x_k^+ &= h^+(z_k^-, z_k^+, w_k^-, w_k^+, v_k^-, v_k^+, u_k, y_k) \\ x_k^- &= h^-(z_k^-, z_k^+, w_k^-, w_k^+, v_k^-, v_k^+, u_k, y_k) \end{cases} \tag{1.97}$$

où $f^-, f^+, h^-, h^+, g^-, g^+$ sont les fonctions dans les domaines appropriés, est un encadreur du système (1.84) si pour toute condition initiale $x_0^- \leq x_0 \leq x_0^+$ nous avons $x_k^- \leq x_k \leq x_k^+$.

Ce système est un observateur par intervalles de (1.83) si dans le cas où $w = v = 0$, il existe un réel positif M et un temps K tels que :

$$\|x_k^+ - x_k^-\| \leq M, \qquad \forall k \geq K \tag{1.98}$$

La Figure 1.5 montre les simulations d'observateur par intervalles (à gauche) et d'encadreur (à droite).

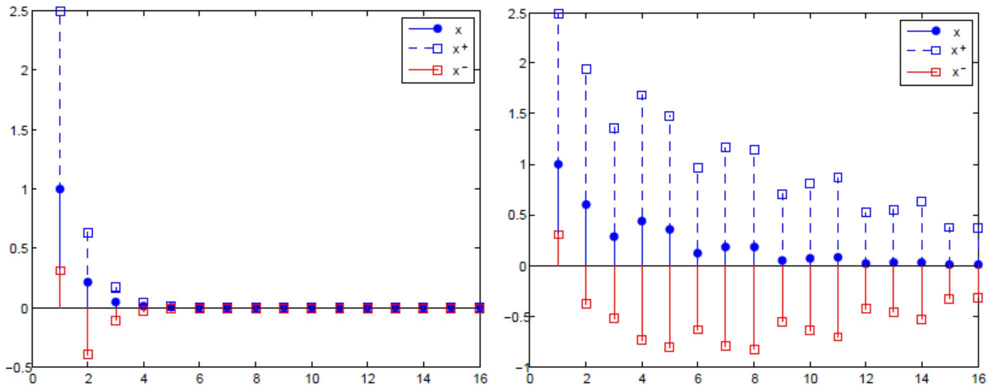

FIGURE 1.5 – Encadreur et Observateur par intervalles pour les systèmes à temps discret.

Le cas linéaire

Nous définissons d'abord le rayon spectral de la matrice et la condition de stabilité au sens de Schur.

Définition 1.49. On appelle rayon spectral de $A \in \mathbb{R}^{n\times n}$, la quantité

$$\rho(A) = \max_{i=1,\dots,n} |\lambda_i(A)|$$

où $\lambda_i(A), i = 1, ..., n$ sont les valeurs propres de A.

Définition 1.50. Une matrice carrée $A \in \mathbb{R}^{n\times n}$ est appelée matrice de stabilité au sens de Schur si $\rho(A) < 1$. Si A est une matrice de stabilité au sens de Schur, alors l'équation différentielle ordinaire $x_{k+1} = Ax_k$ est exponentiellement stable, c'est-à-dire $x_k \to 0$ quand $k \to +\infty$.

On considère un système linéaire sous forme d'état :

$$x_{k+1} = Ax_k + w_k, \;\; k \in \mathbb{N} \tag{1.99}$$

où w_k est un terme d'incertitudes inconnu tel que : $w_k^- \leq w_k \leq w_k^+$, pour tout $k \geq 0$ où w_k^+, w_k^- sont les deux bornes connues.

Théorème 1.51. *Si la matrice $A \in \mathbb{R}^{n\times n}$ est positive, alors les équations aux différences suivantes encadrent les solutions du système (1.99) :*

$$\begin{cases} z_{k+1}^+ &= Az_k^+ + w_k^+,\ z_0^+ = x_0^+ \\ z_{k+1}^- &= Az_k^- + w_k^-,\ z_0^- = x_0^- \\ x_k^+ &= z_k^+ \\ x_k^- &= z_k^- \end{cases} \tag{1.100}$$

Si $x_0^- \leq x_0 \leq x_0^+$, alors $x_k^- \leq x_k \leq x_k^+$ pour tout $k \geq 0$.

Si la matrice $A \in \mathbb{R}^{n\times n}$ est une matrice de stabilité au sens de Schur, c'est un observateur par intervalles.

Démonstration. Les systèmes suivants sont positifs :

$$z_{k+1}^+ - x_{k+1} = A(z_k^+ - x_k) + w_k^+ - w_k \tag{1.101}$$

$$x_{k+1} - z_{k+1}^- = A(x_k - z_k^-) + w_k - w_k^- \tag{1.102}$$

car la matrice $A \in \mathbb{R}^{n\times n}$ est positive et $w_k^+ - w_k \geq 0$, $w_k - w_k^- \geq 0$ pour $k \geq 0$ (voir le Théorème 1.39).

Si $z_0^- = x_0^- \leq x_0 \leq x_0^+ = z_0^+$, alors $x_k^- = z_k^- \leq x_k \leq z_k^+ = x_k^+$ pour tout $k \geq 0$.

Par ailleurs, dans le cas où $w_k = 0$, le système $z_{k+1}^+ - z_{k+1}^- = A(z_k^+ - z_k^-)$ est globalement asymptotiquement stable car la matrice $A \in \mathbb{R}^{n\times n}$ est une matrice de stabilité au sens de Schur. □

1.7 Conclusion

Ce chapitre a présenté différents concepts de base qui seront utiles dans la suite de ce mémoire. La littérature sur la thématique de l'estimation d'état est abondante. Ce sujet est encore un thème de recherche actif. Il est à noter que les systèmes positifs apparaissent souvent dans les problèmes de la vie réelle. C'est pourquoi l'étude des observateurs basés sur la théorie des systèmes dynamiques positifs attire toujours l'attention de beaucoup de chercheurs.

A partir du prochain chapitre, nous allons commencer à présenter nos contributions à ce domaine de recherche. Nous montrerons tout d'abord une nouvelle méthode de construction d'observateurs positifs sur l'orthant positif. En suite l'essentiel de notre travail est consacré à introduire la performance d'observateurs par intervalles dans la résolution d'estimation robuste d'état pour des systèmes qui sont perturbés par des termes d'incertitudes.

The thesis is now continued in English

Chapter 2

Contraction-based design of positive observers

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2.1 Introduction

Positive systems arise in several areas, where each coordinate of the state represents a positive quantity such as a population or a concentration, that does not have any physical meaning if it is not positive, see, for example, [45, 60, 124]. Positive observers aim at estimating the state variables from output measurements in such a way that the estimates are always positive, and thus admit a physical interpretation at all times, that is, even when the estimation error is not small. Positive observers for linear systems can be built under specific structural assumptions. In [109] and [55] structural properties, including observability, of positive systems have been explored, and observers for compartmental systems have been developed [133]. In [12] the positive observer design problem has been dealt with using coordinates transformations and the theory of positive realization [14, 109], thus generalizing the results in [133] and relaxing the conditions under which positive observers exist.

In a previous paper [17], the authors have advocated the use of generalized polar coordinates (that is, direction and norm) in solid cones on Banach spaces to build positive observers.

The starting point of the approach is a theorem by G. Birkhoff which dates back to the 1950s, and which proves that a large class of positive mappings on cones admit contraction properties in the projective space for the so-called Hilbert metric. Thus, the idea underlying the method of [17] is that for those systems, a mere copy of the dynamics may yield an observer in the projective space, as the contraction properties of the dynamics imply that two arbitrary solutions (namely the true solution and the observer which, being a copy of the system, is also a solution) converge exponentially towards each other [80]. Once the direction of the true state is correctly estimated (i.e. the state is correctly estimated in the projective space), very mild assumptions on the output map allow to reconstruct its norm, and thus to estimate the whole state.

Our work in this chapter builds upon previous paper [17] and focuses on the case of discrete-time positive time-varying linear systems in the orthant $\mathbb{R}^n_{\geq 0}$ [2] in the standard form $x_{k+1} = A_k x_k + B_k u_k$ with output $y_k = C_k x_k$. The proposed Luenberger type candidate observer of the form $w_{k+1} = A_k w_k + B_k u_k + L_k(C_k w_k - y_k)$ contains correction terms in addition to the copy of the system dynamics in order to accelerate its *contraction rate*. The design is based on the use of Birkhoff theorem again, but here the goal is to reconstruct the linear estimation error $e_k = w_k - x_k$ in the projective space. Once the direction of the error e_k is correctly estimated, its norm can be estimated in turn. This latter step yields a nonlinear estimated error $\hat{e}_k$ and the final positive observer is merely the sum $\hat{x}_k = w_k + \hat{e}_k$.

At each step, the contraction rate of the error system is directly linked to the so-called projective diameter of the matrix $A_k + L_k C_k$. The problem of finding the optimal gain matrix L_k that achieves the maximal contraction rate for the error system in the projective space is posed. It is non-convex and nonlinear, but it is shown that simple heuristics suffice to determine a matrix gain L_k such that the contraction rate of $A_k + L_k C_k$ is higher than the one of A_k, implying that the observer converges faster that a mere copy of the dynamics. The design is in sharp contrast with the classical design of a Luenberger observer that focuses on the eigenvalues assignment (or spectral radius) of the linear error dynamics.

The chapter is organized as follows : in Section 2.2, the definition of positive linear observers and the Birkhoff-Bushell theorem are recalled. In Section 2.3, a class of positive nonlinear observers on the positive orthant is proposed. Section 2.4 addresses the problem of gain tuning to enforce contraction. A numerical example is given in Section 2.5. Concluding remarks are drawn in Section 2.6.

2. See the Definition 1.29.

2.2 Positive linear systems and the Birkhoff theorem

Recall that a positive linear system is a linear system whose solutions live in the positive orthant at all times. In this section, the operators "$>$", "$\geq$" are understood as a set of inequalities applied componentwise [3]. The orthant $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ are defined in the Definitions 1.29, 1.31 respectively. A is said nonnegative mapping if $A : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$, and if $A : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ we say that A is positive.

Now we consider a linear time-varying system :

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned} \tag{2.1}$$

where

- A_k is a $n \times n$ positive matrix [4] for all $k \geq 0$.
- B_k is a $n \times m$ nonnegative matrix [5] for all $k \geq 0$.
- C_k is a $p \times n$ nonnegative matrix for all $k \geq 0$.

If the three conditions above are met, the linear system (2.1) is said to be positive in the following sense: if the initial state x_0 is nonnegative and the control input u_k is nonnegative for all $k \geq 0$, then the state x_k and the output y_k are nonnegative at all time $k \geq 0$ [6].

We recall the Hilbert projective metric for any two vectors $x, y \in \mathbb{R}^n_{>0}$ which is presented in the Definition 1.43 :

$$d(x, y) = \max_{i,j} \log\left(\frac{x_i y_j}{x_j y_i}\right). \tag{2.2}$$

It is understood as a projective metric in the sense that for all $\lambda, \mu > 0$ and x, y in the orthant we have $d(\lambda x, \mu y) = d(x, y)$. The projective diameter of a positive mapping A is defined as the diameter in the sense of the Hilbert metric of the image of the orthant by A, i.e.,

$$\Delta(A) = \sup\{d(Ax, Ay) | x, y \in \mathbb{R}^n_{>0}\} \tag{2.3}$$

and it may be infinite (take for instance the identity function). In the case of a positive linear mapping on $\mathbb{R}^n_{>0}$ the projective diameter can be expressed as [21]

$$\Delta(A) = \max\left\{\log\left(\frac{a_{ij} a_{kr}}{a_{ir} a_{kj}}\right) : 1 \leq i, j, k, r \leq n\right\} \tag{2.4}$$

where $A = (a_{ij})$ is the corresponding $n \times n$ matrix with positive entries. The latter formula can be understood in the following way. Note that, each column, say $a_{\bullet,j}$ for $1 \leq j \leq n$, of

3. See the Definitions 1.27, 1.28, 1.30.
4. See the Definition 1.33.
5. See the Definition 1.32.
6. This one is an extended case of the Theorem 1.39.

the matrix A represents the image of the j-th vector of the canonical basis of $\mathbb{R}^n$. But from (2.2), the distance between two vectors $x, y \in \mathbb{R}^n_{>0}$ can be written as $\max_{ij} \log\left(\frac{x_i y_j}{x_j y_i}\right)$. As the diameter of A is defined as the diameter of the image of the orthant, that is the diameter of the convex hull of the images $a_{\bullet,j}$'s of the base vectors, finding the projective diameter of A is equivalent to finding the maximal Hilbert metric between pairs of columns of A. The diameter is thus easily shown to be

$$\begin{aligned}\Delta(A) = \max_{j,r} d(a_{\bullet,j}, a_{\bullet,r}) &= \max_{j,r} \max_{i,k} \log\left(\frac{a_{ij}a_{kr}}{a_{kj}a_{ir}}\right) \\ &= \max_{i,j,k,r} \log\left(\frac{a_{ij}a_{kr}}{a_{kj}a_{ir}}\right)\end{aligned}$$

Note that, it is easily seen that the projective diameter will be finite if and only if all entries of A are positive.

The Birkhoff theorem 1.44 is at the core of our approach to the problem of positive observer design. This one says that if A is a positive linear mapping we have for all $x, y \in \mathbb{R}^n_{>0}$:

$$d(Ax, Ay) \leq \left(\tanh\left(\frac{\Delta(A)}{4}\right)\right) d(x, y).$$

As the hyperbolic tangent of any nonnegative number is always smaller than 1, we see that any positive linear mapping is a contraction. Moreover, if the projective diameter is finite, i.e. $\Delta(A) < \infty$, then A is a strict contraction. In particular, the Hilbert distance between two arbitrary solutions of the time-invariant system $x_{k+1} = Ax_k$ tends exponentially to zero with rate $\gamma = \tanh\left(\frac{\Delta(A)}{4}\right) < 1$. This means, that for two arbitrary solutions $x_k, \tilde{x}_k$ we have $d(x_k, \tilde{x}_k) \leq \gamma^k d(x_0, \tilde{x}_0)$. Note that, the celebrated Perron-Froebenius theorem can be viewed as a consequence of the Birkhoff theorem as in the projective space A is proved to be a contraction. This implies using the Banach fixed point theorem any solution x_k will converge in direction to a vector, this vector being in fact an eigenvector of A. This is the reason why Birkhoff theorem is often considered as a generalization of the Perron-Froebenius theorem to general solid cones, and to homogeneous (possibly time-varying) nonlinear positive maps [21]. As a final remark, note that, the contraction rate is monotonically linked to the diameter of the map A: a smaller diameter ensures a higher contraction rate.

2.3 A class of positive observers for positive systems

The reference [17] advocates a different observer design methodology for a class of positive linear systems on solid cones based on the Hilbert metric and Theorem 1.44. In the case where the considered solid cone is the positive orthant, those systems are of the form $x_{k+1} = Ax_k$

where A is characterized by either a finite diameter, or A admits a power, say N, such that A^N has a finite diameter. For those systems, a mere copy of the system provides a simple positive observer which converges exponentially in the projective space. This is a straightforward consequence of Birkhoff's theorem. When the output is linear, it is then rather easy to reconstruct the norm of the true state as soon as the direction is sufficiently well estimated. Building upon this method, and focusing on the case of linear systems in the orthant, we propose a novel design methodology for positive observers allowing to embrace a broader class of time-varying systems of the form (2.1), where the observers include correction terms and gain matrices such that the the gains can be tuned to accelerate the convergence of the observer. As a result, the maps A_k (or their N-th power) need not necessarily have a finite projective diameter.

2.3.1 Nonlinear Luenberger type positive observers for positive linear systems

For positive systems, a natural requirement is that the observers should provide state estimates that are also nonnegative so they can be given a physical meaning at all times. In the present section we are concerned with nonlinear Luenberger-type positive observers for such systems.

Consider the system (2.1) in $\mathbb{R}^n_{>0}$. We propose to build a class of positive nonlinear observers based on the following steps. First consider the Luenberger positive observer defined by

$$w_{k+1} = (A_k + L_k C_k) w_k + B_k u_k - L_k y_k \tag{2.5}$$

where w_k plays the role of a "surrogate" estimated state and where we assume $0 < w_0 < x_0$ (such an initial condition can always be found if we suppose that we have a lower bound on each coordinate of the state). The matrix gain $L_k \in \mathbb{R}^{n \times p}$ should satisfy the two following conditions:

(a) $A_k + L_k C_k$ is a $n \times n$ positive matrix having finite projective diameter for all $k \geq 0$.

(b) L_k satisfies the linear inequality $L_k y_k \leq B_k u_k$ for all $k \geq 0$.

Note that those two conditions are equivalent to the componentwise *linear* inequalities $L_k C_k > -A_k$ and $L_k y_k \leq B_k u_k$ at all times. Now consider the surrogate estimation error

$$e_k = x_k - w_k \tag{2.6}$$

For system (2.1) and observer (2.5), the error dynamics satisfy

$$e_{k+1} = (A_k + L_k C_k) e_k \text{ with } e_0 > 0 \tag{2.7}$$

The idea now is to use the Birkhoff theorem to reconstruct the direction of this surrogate error. Indeed consider the following positive system

$$\hat{e}_{k+1} = (A_k + L_k C_k)\hat{e}_k \text{ with } \hat{e}_0 > 0 \tag{2.8}$$

From the Birkhoff theorem we have $d(e_k, \hat{e}_k) \to 0$ exponentially as long as the diameter of $A_k + L_k C_k$ is bounded above by some fixed quantity $R > 0$. Thus, the normalized error $\frac{\hat{e}_k}{\|\hat{e}_k\|}$ will in this case provide a righteous estimation of the direction of the error. The final step consists of defining the state estimate as:

$$\hat{x}_k = w_k + \mu_k \frac{\hat{e}_k}{\|\hat{e}_k\|} \tag{2.9}$$

where for all $k \geq 0$, μ_k is defined by

$$\mu_k = \frac{\| y_k - C_k w_k \|}{\| C_k \hat{e}_k \|} \|\hat{e}_k\|$$

Note that, the quantity $\frac{\|y_k - C_k w_k\|}{\|C_k \hat{e}_k\|}$ can be replaced by the ratio of any component of the vector $y_k - C_k w_k$ with the same component of the vector $C_k \hat{e}_k$ (provided it is large enough to avoid problems in the division). This small modification can be advantageous when there is noise in the system and where it is not desirable to square the noise via a norm calculation.

The idea underlying the latter definition (2.9), is that, as soon as the gain matrix is well designed, $\hat{e}_k$ should allow to estimate the direction of the error e_k in the projective space, whereas μ_k allows to estimate its norm. The vector $\mu_k \frac{\hat{e}_k}{\|\hat{e}_k\|}$ is then a righteous estimation of the true error e_k.

The resulting observer is nonlinear and positive, as proved by the following result :

Proposition 2.1. *For all $k \geq 0$, the estimated state defined by (2.9) satisfies $\hat{x}_k \geq 0$.*

Proof. First, note that, because of assumptions (a) and (b), and the fact that $w_0 > 0$ initially, the estimate w_k defined by (2.5) is always positive. Then, because of the assumption that $w_0 < x_0$, and of assumption (a), we have that e_k is always positive. The estimate $\hat{x}_k$ defined by (2.9) is thus positive. □

2.3.2 Convergence issues

The asymptotic convergence of the error e_k is characterized by the following proposition.

Proposition 2.2. *Consider the system (2.1) and suppose there exists a non-positive $n \times p$ matrix L_k satisfying assumptions (a) and (b), and such that $\Delta(A_k + L_k C_k) \leq R$ for some*

$R > 0$, $\forall k \in \mathbb{N}$. *Then the direction of error e_k is asymptotically well estimated as*

$$d(\hat{e}_k, e_k) = d\left(\frac{\hat{e}_k}{\|\hat{e}_k\|}, e_k\right) \to 0 \quad \textit{exponentially.}$$

where d is the Hilbert projective metric. Moreover, if there exist $\alpha, \beta, \epsilon > 0$ such that for all $k > 0$, $S\left(\frac{\hat{e}_k}{\|\hat{e}_k\|}, \epsilon\right) \subset \mathbb{R}^n_{\geq 0}$, where $S(z, \epsilon)$ is the ball of centre z and radius ϵ, and $\alpha \leq \|C_k\| \leq \beta$ then the norm of the error is asymptotically well estimated as

$$\left|\frac{\|e_k\|}{\mu_k} - 1\right| \to 0 \quad \textit{exponentially.}$$

Before proving the proposition, we discuss its interpretation. First, the theorem indicates the convergence of the estimation error to zero in generalized polar coordinates. Indeed, the first part ensures that the direction of the error e_k is asymptotically reconstructed at an exponentially fast rate, and the second part that its norm is also asymptotically reconstructed exponentially fast, for a natural norm discrepancy based on the norms ratio. The norm ratio is a dimensionless quantity, and it is a meaningful way to measure discrepancy between norms, as unstable eigenvalues are often encountered in positive systems, and tends make norms of both the true and the estimated state go to infinity (see also [17] which advocates using this natural alternative error).

Secondly, if the ratio between the norms of the errors is sufficiently close to 1, the theorem implies that $d(\hat{x}_k, x_k)$ tends exponentially to zero. Indeed, we know from Theorem 3.5 in [21] that if $\|e_k\| = \mu_k$, then $d(\hat{x}_k, x_k) \leq d(\hat{e}_k, e_k)$. If the norm ratio is close to 1, the inequality may be violated, but in practice it is to be expected that $d(\hat{x}_k, x_k)$ be close to $d(\hat{e}_k, e_k)$, and thus exponentially convergent.

Finally, note that, the assumption of the proposition that $S\left(\frac{\hat{e}_k}{\|\hat{e}_k\|}, \epsilon\right) \subset \mathbb{R}^n_{\geq 0}$ essentially means that $\hat{e}_k$ does not converge to the boundary of the cone. It could have been replaced with the same condition on the true error e_k, but the choice is justified by the fact that the assumption as stated in the proposition can at all times be checked by the user, whereas the true error e_k is unknown. Note also that when the system is time-invariant, i.e., A, L, C are fixed, this latter assumption is automatically satisfied thanks to Birkhoff's theorem and the Banach fixed point theorem, which states then that there exits a positive vector v that is a fixed point of $A + LC$ in the projective space, and such that $d(e_k, v) \to 0$ exponentially.

Proof. The errors defined by (2.7) and (2.8) are solutions of the same positive system. Because of the assumption on the diameter of $A_k + L_k C_k$, and because of the Birkhoff-Bushell theorem, the considered system is contractive with a contraction ratio bounded away from 1, implying exponential convergence of the quantity $d(e_k, \hat{e}_k)$. As concerns the second result, note that

$$\mu_k = \frac{\|C_k e_k\|}{\|C_k \hat{e}_k\|}\|\hat{e}_k\|$$

letting $z_k = e_k/\|e_k\|$ and $\hat{z}_k = \hat{e}_k/\|\hat{e}_k\|$. Thus we have

$$\mu_k = \frac{\|C_k z_k\|}{\|C_k \hat{z}_k\|}\|e_k\|$$

Thus

$$\frac{\mu_k}{\|e_k\|} \leq \frac{1}{\alpha\cos\theta}|\|C_k z_k\| - \|C_k \hat{z}_k\|| \leq \frac{\beta}{\alpha\cos\theta}\|z_k - \hat{z}_k\|$$

where we have defined θ as the maximal angle between $\hat{z}_k$ and any line of the positive matrix C_k, whose cosine is bounded away from 0 as $\hat{z}_k$ has been assumed to be bounded away from the boundary of the cone. Using the fact that $z_k, \hat{z}_k$ have unit norm we can exploit a result of [21] which says that

$$\|z_k - \hat{z}_k\| \leq e^{d(z_k,\hat{z}_k)} - 1 = e^{d(e_k,\hat{e}_k)} - 1$$

Exponential convergence thus stems from the exponential convergence of the Hilbert distance $d(e_k, \hat{e}_k)$. □

2.4 Gain tuning issues

As presented in the previous section, the advantage of using correction terms L_k over the method proposed in [17] is that convergence speed can be managed. The lower the projective diameter of $A_k + L_k C_k$ is, the faster the convergence speed is. The goal is to find a $n \times p$ matrix L_k which must satisfy the two linear inequalities (a) - (b) and such that $\Delta(A_k + L_k C_k)$ is the lowest. At least this should be smaller than the value of $\Delta(A)$, the ideal case being for $\Delta(A_k + L_k C_k) = 0$.

2.4.1 Optimization problem involved

To simplify exposition, we omit the subscripts of A_k, B_k, L_k and we also assume a scalar output $y_k \in \mathbb{R}_{\geq 0}$, ie. $C \in \mathbb{R}^{1\times n}$. Given A, C at each time step k in order to optimize the contraction rate, one can consider the following problem :

$$\begin{aligned} \underset{L}{\text{minimize}} \quad & \Delta(A + LC) \\ \text{subject to} \quad & (\alpha), (\beta). \end{aligned}$$

where L satisfies

(α) $\ell_{i1} y \leq \sum_{k=1}^{m} b_{ik} u_{k1}$ for all $i \in \{1, ..., n\}$.

(β) $a_{ij} + \ell_{i1} c_{1j} > 0$ for all $i \in \{1, ..., n\}$.

and where

$$\Delta(A + LC) = \max_{i,j,k,r} \left\{ \log \left(\frac{(a_{ij} + \ell_{i1} c_{1j})(a_{kr} + l_{k1} c_{1r})}{(a_{ir} + \ell_{i1} c_{1r})(a_{kj} + \ell_{k1} c_{1j})} \right) \right\}. \tag{2.10}$$

Note that the optimization problem is always feasible with $L = 0$. It is a minmax optimization problem, and can be handled numerically via standard optimization methods, including brute force algorithms if the dimension of the state space is not too large.

2.4.2 A geometrical method to decrease $\Delta(A + LC)$

When the matrices A_k are known in advance, typically in the time-invariant case, the optimization problem which consists of minimizing $\Delta(A_k + L_k C_k)$ at each step k can be handled numerically, and the optimal gain matrices L_k can be computed offline. However, when the system is time-varying and the matrix sequence $(A_k)_{k \geq 0}$ is supposed not to be known in advance, for the observer to work in real-time the gain matrices L_k must be computed online.

Computational power limitations can forbid the use of costly optimization methods online. We propose a simple geometric heuristic that guarantees at each step (omitting the subscipt k) $\Delta(A + LC) \leq \Delta(A)$. This implies that with correction terms the error equation is proved to converge faster than without such terms. The method is simple and can be suboptimal, however it proves to be optimal in the two-dimensional case, as shown in the following subsection.

For the sake of illustration, suppose that the scalar output measures the first coordinate of the state vector, i.e., $C_k = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{1 \times n}$ for all $k \geq 0$. The idea is then as follows : consider the orthogonal projection of the first column $a_{\bullet,1}$ onto the linear space spanned by the remaining columns $a_{\bullet,j}$ for $2 \leq j \leq n$. In turn, this vector can be projected onto the convex space $\{a_{\bullet,1} + l_{\bullet,1}\}$, where L satisfies conditions $(\alpha), (\beta)$. The corresponding correction matrix L lies in the feasible set, and the angle between the vector $a_{\bullet,1} + l_{\bullet,1}$ and any remaining column $a_{\bullet,j}$ for $2 \leq j \leq n$ is less than the angle between $a_{\bullet,1}$ and $a_{\bullet,j}$. But the function $y \to d(x, y)$ for fixed x is an increasing function of the angle between x and y. As a result, the Hilbert distance between two arbitrary columns of $A + LC$ is less than the distance between the corresponding columns in A. Recalling the characterization of the projective diameter in previous sections, this implies that $\Delta(A + LC) \leq \Delta(A)$.

2.4.3 Two-dimensional linear case

In this section, we propose an explicit construction of positive observers for two-dimensional systems. The proposed observer maximizes the rate of contraction as measured by the Hilbert projective metric. It turns out to be a dead-beat observer. Once again we assume $y = x_1$ for simplicity.

Proposition 2.3. *Consider the time-varying positive system*

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C x_k \end{aligned} \tag{2.11}$$

where A_k is a matrix in $\mathbb{R}^{2\times 2}$ with positive entries, B_k is a nonnegative matrix in $\mathbb{R}^{2\times m}$ and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then there exists a nonpositive 2×1 matrix L_k such that the observer (2.9) is dead-beat. We have indeed $\hat{x}_k = x_k$ for all $k \geq 1$.

Proof. At each time step k, one can always choose a matrix $L = \begin{pmatrix} \ell_{11} \\ \ell_{21} \end{pmatrix} \in \mathbb{R}^{2\times 1}$ such that

$$-a_{11} < \ell_{11} < \min\left(\frac{-\det(A)}{a_{22}}, 0\right) \tag{2.12}$$

$$\ell_{21} = \frac{\det(A) + \ell_{11} a_{22}}{a_{12}} \tag{2.13}$$

where $A = (a_{i,j}) > 0$ for $i, j \in \{1, 2\}$. Now we verify the two conditions $(\alpha) - (\beta)$

(α) From (2.12), we have $\ell_{11} < 0$. From the fact that $\ell_{11} < -\frac{\det(A)}{a_{22}}$ and combining with (2.13), we can deduce that $\ell_{21} < 0$. Consequently $L_k y_k \leq B_k u_k$ at any instant k where u_k is a positive control.

(β) From (2.12) and (2.13), we can easily deduce that $a_{11} + \ell_{11},\ a_{21} + \ell_{21} > 0$. Then $A_k + L_k C_k > 0$ at any time $k \geq 0$.

Furthermore, we have $\det(A + LC) = (a_{11} + \ell_{11})a_{22} - (a_{21} + \ell_{21})a_{12} = 0$. Thus, $\operatorname{rank}(A + LC) = 1$. Then $\Delta(A_k + L_k C_k) = 0$ for all $k \in \mathbb{N}$. The optimization problem is solved. As a result, $\|e_k\| = \mu_k$. Then $d(x_k, \hat{x}_k) = 0$ and $\left|\frac{\|x_k\|}{\|\hat{x}_k\|} - 1\right| = 0$ for $k \geq 1$. The proposition follows. $\square$

2.5 Illustrative example

Consider for example the discrete-time system:

$$x_{k+1} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} x_k \ , \quad y_k = \begin{pmatrix} 1 & 0 \end{pmatrix} x_k \tag{2.14}$$

We have $\det(A) = 1$. From (2.12) and (2.13), $\ell_{11} = -2$, $\ell_{21} = -1$ can be taken. The following conditions are obviously satisfied :

(α) $A + LC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is a positive matrix.

(β) $L = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ is a negative matrix. Thus, $Ly_k < Bu_k$ for all $k \in \mathbb{N}$.

Moreover $\det(A+LC) = 0$ proving $\Delta(A+LC) = 0$. A dead bit observer is therefore readily obtained:

$$\hat{x}_k = w_k + \frac{x_{1,k} - w_{1,k}}{\hat{e}_{1,k}} \hat{e}_k \tag{2.15}$$

where

$$\begin{pmatrix} w_{1,k+1} \\ w_{2,k+1} \end{pmatrix} = \begin{pmatrix} w_{1,k} + w_{2,k} + 2x_{1,k} \\ w_{1,k} + w_{2,k} + x_{1,k} \end{pmatrix} \tag{2.16}$$

and

$$\begin{pmatrix} \hat{e}_{1,k+1} \\ \hat{e}_{2,k+1} \end{pmatrix} = \begin{pmatrix} \hat{e}_{1,k} + \hat{e}_{2,k} \\ \hat{e}_{1,k} + \hat{e}_{2,k} \end{pmatrix}. \tag{2.17}$$

Figures 2.1 and 2.2 below illustrate results of observer (2.15) for system (2.14). A trajectory with $x_0 = \left[\frac{11}{10}, 20\right]^\top$, $w_0 = \left[\frac{1}{15}, 1\right]^\top$, $e_0 = \left[\frac{31}{30}, 19\right]^\top$ and $\hat{e}_0 = [1, 10]^\top$ as initial conditions is simulated. The plot 2.1 shows that the ratio between the norms of x_k and its estimate $\hat{x}_k$ are equal to 1 after a single iteration, whereas the plot 2.1 shows that their the Hilbert distance between their directions $d(x_k, \hat{x}_k)$ is null after one iteration also, thanks to the convergence of $d(e_k, \hat{e}_k)$.

Remark 2.4. An interesting remark concerns the dynamical behavior of the error system: note that, the matrix $A + LC$ is not stable. Thus the corresponding Luenberger observer for (2.11) is not stable, whereas our technique allows a dead bit estimation of the state. This illustrates well the difference between our approach based on a two-step estimation where first the direction must be estimated, independently from the norm that can grow exponentially. We believe that this approach to estimation problems in the positive orthant indeed suits more the geometry of the state space.

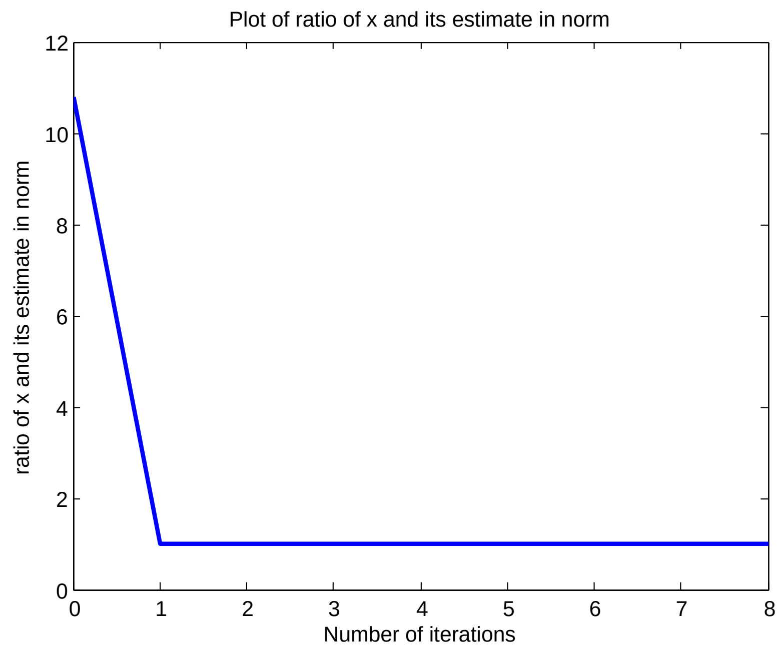

FIGURE 2.1: Plot of the quantity $\frac{\|\hat{x}_k\|}{\|x_k\|}$.

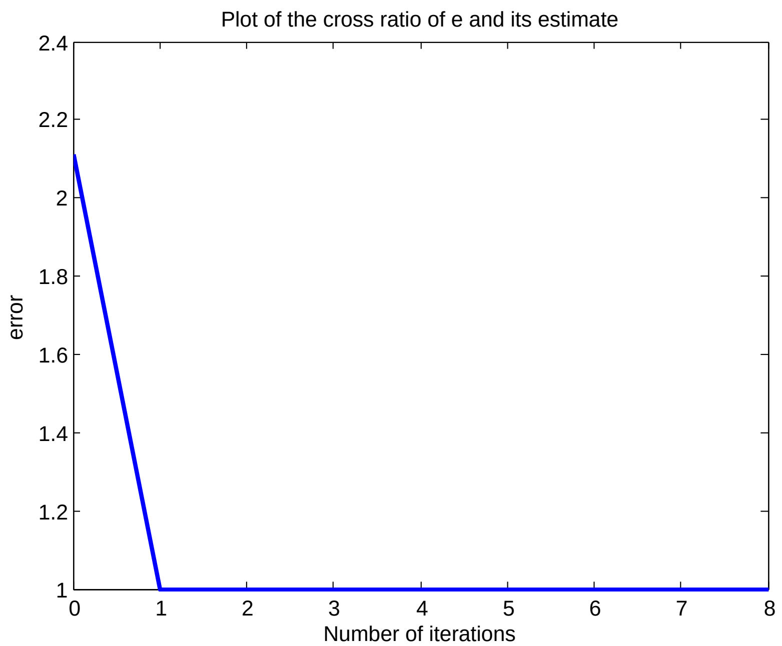

FIGURE 2.2: Plot of the cross ratio $\frac{\hat{e}_{2k}e_{1k}}{\hat{e}_{1k}e_{2k}}$.

2.6 Conclusion

In this chapter, we proposed a new design method for positive observers in the positive orthant. The observer design mimics the Luenberger observer design except that the gain is tuned to maximize the contraction rate of the error as measured by the Hilbert projective metric rather than the usual Euclidean metric that we believe does not suit ideally the geometry of the considered state space. This leads to a counter intuitive result that the observer can converge exponentially although the error system is unstable.

The key ingredient in our approach is the Hilbert metric, that distorts the distances in the orthant so that the boundary is at an infinite distance of any point, leading to a natural preservation of positiveness. We anticipate interesting extensions in the nonlinear case and in more general cones.

Chapter 3

Interval Observer Composed of Observers for Nonlinear Systems

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3.1 Introduction

From this chapter onwards, interval observers are our focus. We will construct them for various types of systems such that continuous-time systems (Chapter 3), discrete-time systems (Chapter 4 and Chapter 5), continuous-time systems with discrete measurement (Chapter 6). The state of the art and the motivation to study interval observers can be traced back to the section 1.6 of Chapter 1.

In spite of the results already available in the literature, much remains to be done to complete the theory of interval observers, especially in the context of systems with input and output. The purpose of the present chapter is to complement the design techniques of interval observers available for this type of systems, and in particular [116]. In [116], interval observers have been constructed for a general family of nonlinear continuous-time systems with input and output. The result relies on partial linearization and technical assumptions on the resulting nonlinear terms. In the present chapter, we consider another fundamental family of systems for which classical observers can be constructed : the family of nonlinear systems affine in the unmeasured part of the state variables, which comprises the state-affine systems and the linear systems with additive output nonlinearity for which observers are proposed in [16]. More precisely, our objective is to develop interval observers for the systems

$$\begin{cases} \dot{x} &= \alpha(y)x + \beta(y,u) + \delta(t), \\ y &= \mathcal{C}x, \end{cases} \tag{3.1}$$

with $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^p$, the input $u \in \mathbb{R}^q$, where α and β are Lipschitz continuous nonlinear functions, where $\mathcal{C} \in \mathbb{R}^{p\times n}$ is a constant matrix and δ is a Lipschitz continuous function bounded by two known Lipschitz continuous functions δ^-, δ^+: for all $t \geq 0$, $\delta^-(t) \leq \delta(t) \leq \delta^+(t)$. For this system, under standard assumptions which ensure existence of a classical asymptotic observer, we construct interval observers without resorting to linearization or partial linearization. As in [96, 97] for discrete-time systems of another type, we will show how two copies of classical observers associated with suitably selected initial conditions and outputs compose an interval observer. The key advantage of the new interval observer we propose is the simplicity of its dynamics. Each copy of observer, or its associated error equation, does not possess the property of being a cooperative or a nonnegative system (see, for instance, the Section 1.5.2 of Chapter 1 for the definition of cooperative system). This fact may sound surprising since most of the designs of interval observers available in the literature are carried out for cooperative systems (sometimes after a coordinate transformation). In fact, we will use the notion of nonnegative and cooperative system as well, but only indirectly to select for the interval observer appropriate initial conditions and upper and lower bounds for the solutions of the studied system. This feature of our design, which is shared with those of [96, 97] and [93], is crucial: it is the reason why the equations of interval observers we propose are very simple. In addition to their simplicity, the interval observers we present offer the possibility to construct a bundle of interval observers, as done for instance in [15], without having to introduce extra dynamics, simply by proposing several choices of initial conditions and bounding outputs. Another key advantage of our technique is that, under some assumptions and an appropriate choice of output feedback, global asymptotic stability of the origin of the system with its interval observer is obtained. We establish this stability result through the construction of a strict Lyapunov function.

Finally, let us observe that the motivation for studying systems affine in the unmeasured part of the state is due to the fact that many systems belong to this family: let us mention for instance the TORA system [62], the nonlinear pendulum (see [43]), the model of electromechanical system described in [30] when, in both cases, the position variables are the measured variables and the nonholonomic system with output studied in [63]. Important results on the problem of constructing classical observers for this type of systems exist: see for instance [78, 81, 107, 113]. But to the best of our knowledge, no interval observers have been proposed.

The rest of this chapter is organized as follows. Notation and definitions are given in Section 3.2. The Section 3.3 is devoted to a new interval observer design. An example borrowed from [30] illustrates the technique in Section 3.4. Concluding remarks are drawn in Section 3.5.

3.2 Notation, definitions, basic result

For the convenience of the reader, we resume in this section some notation, definitions and basic results which will be used throughout the Chapter 3.

- The notation will be simplified whenever no confusion can arise from the context.
- Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0.
- The Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions are denoted $|\cdot|$.
- All the inequalities must be understood *componentwise* (see Definition 1.27 of Chapter 1).
- For two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$, $\max\{A, B\}$ is the matrix where each entry is $m_{ij} = \max\{a_{ij}, b_{ij}\}$.
- For a matrix $A \in \mathbb{R}^{r\times s}$, $A^+ = \max\{A, 0\}$, $A^- = \max\{-A, 0\}$.
- A square matrix is said to be *cooperative* or Metzler if its off-diagonal entries are nonnegative[7].
- We recall a notion of nonnegative system which has been presented in Section 1.5.1 of Chapter 1. A system $\dot{x}(t) = f(t, x(t))$, for which the solutions are unique and defined over $[0, +\infty)$, is said to be nonnegative if for any initial condition $x_0 \geq 0$ at any initial time $t_0 \geq 0$, the solution $x(t)$ is nonnegative for all $t \geq t_0$.
- Let $\mathcal{K}$ denote the set of all continuous functions $\alpha : [0, +\infty^{8}) \to [0, +\infty)$ for which (i) $\alpha(0) = 0$ and (ii) α is increasing.

For the sake of generality, we introduce a general definition of framer and interval observer for time-varying nonlinear systems. This notion has been introduced, with slightly different features, in Definition 1.45.

Definition 3.1. Consider a system:

$$\dot{x}(t) = f_1(t, x(t), \delta(t)), \tag{3.2}$$

with $x \in \mathbb{R}^n$, with an output $y = m(x) \in \mathbb{R}^p$, and where f_1 and m are two nonlinear locally Lipschitz functions. The uncertainty $\delta(t) \in \mathbb{R}^\ell$ is a Lipschitz continuous function such that (3.2) is forward complete and such that there exist two known bounds $\delta^+(t) \in \mathbb{R}^\ell$,

7. Definition 1.34.
8. instead of a in Definition 1.7.

$\delta^-(t) \in \mathbb{R}^\ell$, Lipschitz continuous, and such that, for all $t \geq 0$, $\delta^-(t) \leq \delta(t) \leq \delta^+(t)$. The initial condition at the instant t_0, $x(t_0) = x_0$, is assumed to be bounded by two known bounds:

$$x_0^- \leq x_0 \leq x_0^+. \tag{3.3}$$

Then, the dynamical system

$$\dot{z}(t) = f_2\left(t, z(t), y(t), \overline{\delta}(t)\right), \tag{3.4}$$

with $\overline{\delta} = (\delta^+, \delta^-)$, associated with the initial condition $z_0 = g(t_0, x_0^+, x_0^-) \in \mathbb{R}^{n_z}$ and bounds for the solution $x(t)$: $x^+(t) = h^+(t, z(t))$, $x^-(t) = h^-(t, z(t))$, where f_2, g, h^+ and h^- are Lipschitz continuous nonlinear functions, is called

(i) a framer for (3.2) if for any vectors x_0, x_0^- and x_0^+ in $\mathbb{R}^n$ satisfying (3.3), the solutions of (3.2)-(3.4) with respectively x_0, $z_0 = g(t_0, x_0^+, x_0^-)$ as initial condition at $t = t_0$, denoted respectively x and z, satisfy, for all $t \geq t_0$, the inequalities

$$x^-(t) = h^-(t, z(t)) \leq x(t) \leq h^+(t, z(t)) = x^+(t), \tag{3.5}$$

(ii) an interval observer for (3.2) if in addition the system (3.2)-(3.4) admits the origin as a global asymptotically stable equilibrium point when $\overline{\delta}$ is identically equal to zero.

Lemma 3.2. *Consider the system*

$$\dot{z} = A(t)z + \delta(t), \tag{3.6}$$

where $z \in \mathbb{R}^n$ and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\delta : \mathbb{R} \to \mathbb{R}^n$ are continuous. Assume that, for all $t \geq 0$, the matrix $A(t)$ is Metzler and $\delta(t) \geq 0$, for all $t \geq 0$. Then the system (3.6) is nonnegative.

Proof. A proof for time-invariant systems is given in Theorem 1.37. Similarly, this case of time-varying systems can be handled in the same way. □

The following lemma will be used in the next section. It is interesting for its own sake because it gives conditions ensuring that a system is nonnegative without implying that it is cooperative.

Lemma 3.3. *Consider the system*

$$\begin{aligned} \dot{z}_1 &= f(t, z_1, z_2), \\ \dot{z}_2 &= g(z_1, z_2)z_2 + \delta(t), \end{aligned} \tag{3.7}$$

where $z_1 \in \mathbb{R}^{n_1}$, $z_2 \in \mathbb{R}^{n_2}$, the functions f, g are Lipschitz continuous and δ is continuous. Assume that this system is forward complete, $\delta(t) \geq 0$, for all $t \geq 0$ and, for all $z_1 \in \mathbb{R}^{n_1}$,

$z_2 \in \mathbb{R}^{n_2}$, the matrix $g(z_1, z_2)$ is Metzler. Then, if at the instant t_0, $z_2(t_0) \geq 0$, then $z_2(t) \geq 0$ for all $t \geq t_0$.

Proof. Consider a solution (z_1, z_2) of (3.7) with an initial condition such that $z_2(t_0) \geq 0$. Since the system (3.7) is forward complete, we can define for all $t \geq 0$ the function A by $A(t) = g(z_1(t), z_2(t))$. Moreover, this matrix is Metzler for all $t \geq 0$. It follows from Lemma 3.2 that all the solutions of

$$\dot{\xi} = A(t)\xi + \delta(t), \tag{3.8}$$

with $\xi(t_0) \geq 0$ are nonnegative for all $t \geq t_0$. Therefore, in particular, the solution ξ_p of (3.8) with $\xi_p(t_0) = z_2(t_0)$ is nonnegative since $z_2(t_0) \geq 0$. Now observe that the solutions ξ_p and z_2 satisfy, for all $t \geq t_0$,

$$\dot{\xi}_p(t) = A(t)\xi_p(t) + \delta(t), \quad \dot{z}_2(t) = A(t)z_2(t) + \delta(t),$$

and $\xi_p(t_0) = z_2(t_0)$. Uniqueness of the solutions implies that $\xi_p(t) = z_2(t)$ for all $t \geq t_0$. Consequently, $z_2(t) \geq 0$ for all $t \geq t_0$. $\square$

3.3 A new interval observer design

This section is devoted to the construction of interval observers for the system (3.1). We introduce some assumptions needed for our development.

Assumption 3.4. *There exist a Lipschitz continuous function $\lambda(y)$, a positive definite radially unbounded C^1 function V and a continuous positive definite function ω such that:*

$$\frac{\partial V}{\partial \xi}(\xi)[\alpha(y) + \lambda(y)\mathcal{C}]\xi \leq -\omega(\xi), \tag{3.9}$$

for all $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^p$.

Assumption 3.5. *There exists an invertible constant matrix $R \in \mathbb{R}^{n\times n}$ such that, for all $y \in \mathbb{R}^p$, the matrix*

$$\Gamma(y) = R[\alpha(y) + \lambda(y)\mathcal{C}]R^{-1} \tag{3.10}$$

is Metzler.

Assumption 3.6. *There exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}$, a Lipschitz continuous feedback $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$, a continuous positive definite function μ and a function γ of class $\mathcal{K}$ such that for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, the inequality*

$$\frac{\partial U}{\partial x}(x)\left[\alpha(\mathcal{C}x)x + \beta(\mathcal{C}x, u_s(\mathcal{C}x, x + d))\right] \leq -\mu(x) + \gamma\left(|d|\right) \tag{3.11}$$

holds. Moreover there exists a nonnegative and increasing continuous function $\kappa : [0, +\infty) \to [0, +\infty)$ *such that, for all* $d \in \mathbb{R}^n$*,*

$$\gamma(|d|) \leq \kappa(V(d))\,\omega(d). \tag{3.12}$$

Now, we state and prove the following result:

Theorem 3.7. *Let the system (3.1) satisfy Assumptions 3.4 to 3.6. Then this system with the dynamic extension*

$$\left\{\begin{array}{lcl} \dot{\hat{x}}^+ & = & \alpha(y)\hat{x}^+ + \beta(y,u) + \lambda(y)[\mathcal{C}\hat{x}^+ - y] + S\left[R^+\delta^+ - R^-\delta^-\right], \\ \dot{\hat{x}}^- & = & \alpha(y)\hat{x}^- + \beta(y,u) + \lambda(y)[\mathcal{C}\hat{x}^- - y] + S\left[R^+\delta^- - R^-\delta^+\right], \end{array}\right. \tag{3.13}$$

in closed-loop with the feedback $u_s(y, \hat{x}^+)$ *is globally asymptotically stable when* $\delta^+(t) = \delta^-(t) = 0$ *for all* $t \geq 0$*. Moreover, the system (3.13) associated with the initial conditions*

$$\hat{x}_0^+ = S\left[R^+x_0^+ - R^-x_0^-\right]\ ,\ \hat{x}_0^- = S\left[R^+x_0^- - R^-x_0^+\right], \tag{3.14}$$

the bounds for the solutions x

$$x^+ = S^+R\hat{x}^+ - S^-R\hat{x}^-\ ,\ x^- = S^+R\hat{x}^- - S^-R\hat{x}^+, \tag{3.15}$$

where $S = R^{-1}$*, is an interval observer for system (3.1) in closed loop with* $u_s(y, \hat{x}^+)$*.*

Discussion of Theorem 3.7.

• Assumptions 3.4 and 3.6 are technical assumptions which ensure the global asymptotic stabilizability by dynamic output feedback of the system (3.1). Assumption 3.4 implies that, for any constant vector $y \in \mathbb{R}^p$, the origin of $\dot{\chi} = [\alpha(y) + \lambda(y)\mathcal{C}]\chi$ is globally asymptotically stable.

• Assumption 3.5 ensures the existence of a time-invariant change of coordinates that transforms for all $y \in \mathbb{R}^p$ the matrix $\alpha(y) + \lambda(y)\mathcal{C}$ into a Metzler matrix. This assumption is not restrictive in the particular case where there is a scalar function $\varsigma(y)$ and a constant matrix α_o such that $\alpha(y) = \varsigma(y)\alpha_o$ and the pair $(\alpha_o, \mathcal{C})$ is observable: in such a case Assumption 3.5 is satisfied with $\lambda(y) = \varsigma(y)\lambda_o$, where λ_o is so that all the eigenvalues of $\alpha_o + \lambda_o\mathcal{C}$ are distinct. The case mentioned about is important since it encompasses the case of the systems with a constant function α, which studied in particular in [51].

• For the system (3.1) in closed loop with $u_s(\mathcal{C}x, x + d)$ and no additive uncertainties δ, Assumption 3.6 is a robustness condition which guarantees that the system is iISS with respect to d because the function U is a iISS Lyapunov function (see for instance Definition

1.23 for the definition of iISS Lyapunov function). Such a Lyapunov function is not unique and the condition (3.12) depends on the selected function U. We conjecture that if there is a function U such that (3.11) is satisfied, then there always exists another function U such that both (3.11) and (3.12) are satisfied. In many cases, the system under study is locally exponentially stabilizable, both V and ω are positive definite quadratic functions and one can select a function U lower and upper bounded by positive definite quadratic functions in a neighborhood of the origin. Then, γ can be taken equal to a quadratic function in a neighborhood of zero and then one can always determine a function κ so that (3.12) is satisfied by using the tools given in [86, Appendix A].

• One can check easily that if an input u is so that all the solutions of the system (3.1) are defined over $[0, +\infty)$ and belong to a compact set, then $\lim\limits_{t\to+\infty} |\hat{x}^+(t) - \hat{x}^-(t)| = 0$. But, even if in addition (3.13) is a framer for the system (3.1), it does not follow that it is an interval observer for this system because the asymptotic stability of the system (3.1) is not guaranteed. This motivates the selection of the special feedback u_s.

Proof. The proof splits up into two parts. The first is devoted to the stability analysis of the system (3.1)-(3.13) in closed-loop with the dynamic output feedback $u_s(y, \hat{x}^+)$. The second consists in showing that (3.13) associated to the initial conditions (3.14) and the bounds (3.15) is an interval observer for the system (3.1) in closed-loop with $u_s(y, \hat{x}^+)$.

1. *Stability of the systems (3.1) - (3.13) with the feedback $u_s(y, \hat{x}^+)$ in the absence of δ, δ^+ and δ^-.* Let us prove that the system

$$\left\{\begin{array}{lcl} \dot{x} & = & \alpha(y)x + \beta(y, u_s(y, \hat{x}^+)), \\ \dot{\hat{x}}^+ & = & \alpha(y)\hat{x}^+ + \beta(y, u_s(y, \hat{x}^+)) + \lambda(y)[\mathcal{C}\hat{x}^+ - y], \\ \dot{\hat{x}}^- & = & \alpha(y)\hat{x}^- + \beta(y, u_s(y, \hat{x}^+)) + \lambda(y)[\mathcal{C}\hat{x}^- - y], \end{array}\right. \tag{3.16}$$

admits the origin as a globally asymptotically stable equilibrium point.

This system is transformed by the change of coordinates $p = \hat{x}^+ - x$, $q = \hat{x}^+ - \hat{x}^-$ into

$$\left\{\begin{array}{lcl} \dot{x} & = & \alpha(y)x + \beta(y, u_s(y, x + p)), \\ \dot{p} & = & [\alpha(y) + \lambda(y)\mathcal{C}]\, p, \\ \dot{q} & = & [\alpha(y) + \lambda(y)\mathcal{C}]\, q. \end{array}\right. \tag{3.17}$$

We establish the global asymptotic stability of the origin of the system (3.17) through a Lyapunov approach. Let $L(p, q) = V(p) + V(q)$, where V is the function in Assumption 3.4. The derivative of L along the trajectories of (3.17) satisfies

$$\dot{L}(p, q) \leq -\omega(p) - \omega(q). \tag{3.18}$$

Moreover, by Assumption 3.6, we know that the derivative of the function U along the trajectories of (3.17) satisfies

$$\dot{U}(x) \leq -\mu(x) + \gamma\left(|p|\right). \tag{3.19}$$

Now, we consider the function

$$M(x,p,q) = \int_0^{L(p,q)} [\kappa(\ell)+1]d\ell + U(x). \tag{3.20}$$

where κ is the function in Assumption 3.6. The inequalities (3.18), (3.19) and Assumption 3 imply that

$$\begin{aligned}
\dot{M}(x,p,q) &\leq -\omega(p) - \omega(q) - \mu(x) - \kappa(L(p,q))\omega(q) + [\kappa(V(p)) - \kappa(L(p,q))]\,\omega(p) \\
&\leq -\omega(p) - \omega(q) - \mu(x) + [\kappa(V(p)) - \kappa(V(p)+V(q))]\,\omega(p) \\
&\leq -\varpi(x,p,q),
\end{aligned} \tag{3.21}$$

where $\varpi(x,p,q) = \omega(p) + \omega(q) + \mu(x)$. This function is positive definite and the function M is positive definite and radially unbounded. This allows us to conclude by invoking the Lyapunov theorem (see [120], [86]) .

2. *Property of framer.* First, consider vectors $x_0, x_0^+, x_0^-, \hat{x}_0^+, \hat{x}_0^-$ in $\mathbb{R}^n$ such that

$$x_0^- \leq x_0 \leq x_0^+ \tag{3.22}$$

and

$$\hat{x}_0^+ = S\left[R^+x_0^+ - R^-x_0^-\right] \;,\; \hat{x}_0^- = S\left[R^+x_0^- - R^-x_0^+\right]. \tag{3.23}$$

Then, by Lemma A.5 in Appendix A,

$$R\hat{x}_0^- \leq Rx_0 \leq R\hat{x}_0^+. \tag{3.24}$$

Second, for an initial instant $t_0 \geq 0$, consider the solution $(x, \hat{x}^+, \hat{x}^-)$ of the system (3.1)-(3.13) in closed-loop with $u_s(y, \hat{x}^+)$ satisfying

$$x(t_0) = x_0,\ \hat{x}^+(t_0) = \hat{x}_0^+,\ \hat{x}^-(t_0) = \hat{x}_0^-. \tag{3.25}$$

For all $t \geq t_0$, we have

$$\left\{\begin{aligned}
\dot{x} &= [\alpha(y) + \lambda(y)\mathcal{C}]\,x + \beta(y, u_s(y,\hat{x}^+)) - \lambda(y)y + \delta, \\
\dot{\hat{x}}^+ &= [\alpha(y) + \lambda(y)\mathcal{C}]\,\hat{x}^+ + \beta(y, u_s(y,\hat{x}^+)) - \lambda(y)y + S\left[R^+\delta^+ - R^-\delta^-\right], \\
\dot{\hat{x}}^- &= [\alpha(y) + \lambda(y)\mathcal{C}]\,\hat{x}^- + \beta(y, u_s(y,\hat{x}^+)) - \lambda(y)y + S\left[R^+\delta^- - R^-\delta^+\right].
\end{aligned}\right. \tag{3.26}$$

From (3.10) in Assumption 3.5, it follows that

$$\begin{array}{rcl} R\dot{x} & = & R\left([\alpha(y)+\lambda(y)\mathcal{C}]x+\beta(y,u_s(y,\hat{x}^+))\right)-R\lambda(y)y+R\delta \\ & = & \Gamma(y)Rx+R\beta(y,u_s(y,\hat{x}^+))-R\lambda(y)y+\left(R^+-R^-\right)\delta \end{array} \tag{3.27}$$

and, similarly,

$$\begin{array}{rcl} R\dot{\hat{x}}^+ & = & \Gamma(y)R\hat{x}^+ + R\beta(y,u_s(y,\hat{x}^+))-R\lambda(y)y+R^+\delta^+-R^-\delta^-, \\ R\dot{\hat{x}}^- & = & \Gamma(y)R\hat{x}^- + R\beta(y,u_s(y,\hat{x}^+))-R\lambda(y)y+R^+\delta^- - R^-\delta^+. \end{array} \tag{3.28}$$

Thus,

$$\begin{array}{rcl} R\dot{\hat{x}}^+ - R\dot{x} & = & \Gamma(y)(R\hat{x}^+ - Rx)+\nu_1, \\ R\dot{x}-R\dot{\hat{x}}^- & = & \Gamma(y)(Rx-R\hat{x}^-)+\nu_2, \end{array} \tag{3.29}$$

with $\nu_1(t)=R^+(\delta^+(t)-\delta(t))+R^-(\delta(t)-\delta^-(t))$ and $\nu_2(t)=R^+(\delta(t)-\delta^-(t))+R^-(\delta^+(t)-\delta(t))$.

From (3.24) and (3.25), the fact that, for all $y\in\mathbb{R}^p$, $\Gamma(y)$ is Metzler and $\nu_1(t)\geq 0$ and $\nu_2(t)\geq 0$, for all $t\geq 0$, we deduce from Lemma 3.3 that $R\hat{x}^-(t)\leq Rx(t)\leq R\hat{x}^+(t)$, $\forall t\geq t_0$. Since the matrices S^+ and S^- are nonnegative, it follows from Lemma A.5 in Appendix A that, for all $t\geq t_0$,

$$S^+R\hat{x}^-(t)-S^-R\hat{x}^+(t)\leq \quad SRx(t)\leq \quad S^+R\hat{x}^+(t)-S^-R\hat{x}^-(t). \tag{3.30}$$

From the definitions of x^+ and x^- in (3.15), it follows that, for all $t\geq t_0$, $x^-(t)\leq x(t)\leq x^+(t)$. This allows us to conclude. □

3.4 Illustrative example

In this section, we illustrate Theorem 3.7 with the model of electromechanical system described in [30]:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & b_1x_3-a_1\sin(x_1)-a_2x_2, \\ \dot{x}_3 & = & b_0u-a_3x_2-a_4x_3, \end{array}\right. \tag{3.31}$$

with $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, with the output $y=x_1$ and where the b_i's and the a_i's are positive real numbers. Assuming that the variable x_1 is measured is realistic from an applied point of view because x_1 represents the angular motor position of the device.

Let us check that Assumption 3.4 is satisfied. With the notation of previous sections and choosing $\lambda(y) = [-1\ 0\ 0]^\top$, we obtain

$$\alpha(y) + \lambda(y)\mathcal{C} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -a_2 & b_1 \\ 0 & -a_3 & -a_4 \end{bmatrix}. \tag{3.32}$$

Assumption 3.4 is satisfied with the positive definite quadratic function

$$V(\xi) = a_2 a_3 \xi_1^2 + a_3 \xi_2^2 + b_1 \xi_3^2.$$

Indeed, one can check easily that the inequality

$$\frac{\partial V}{\partial \xi}(\xi)[\alpha(y) + \lambda(y)\mathcal{C}]\xi \leq -\omega(\xi),$$

with $\omega(\xi) = a_2 a_3(\xi_1^2 + \xi_2^2) + 2b_1 a_4 \xi_3^2$ is satisfied and ω is a positive definite quadratic function.

Let us check that Assumption 3.5 is satisfied. The matrix $\alpha(y) + \lambda(y)\mathcal{C}$ is not Metzler because $a_3 > 0$. But we show that, with the numerical values given below, this matrix can be transformed into a Metzler matrix through a time-invariant transformation. With the notation of [30], we have $b_1 = \frac{1}{M}$, $a_2 = \frac{B}{M}$, $a_3 = \frac{K_B}{L}$, $a_4 = \frac{R}{L}$, with $B = \frac{B_0}{K_\tau}$, $M = \frac{J}{K_\tau} + \frac{mL_0^2}{3K_\tau} + \frac{M_0 L_0^2}{K_\tau} + \frac{2M_0 R_0^2}{5K_\tau}$. The numerical values given in [30] are: $M_0 = 0.434$, $J = \frac{1.625}{1000}$, $m = 0.506$, $R_0 = 0.023$, $B_0 = 0.01625$, $L_0 = 0.305$, $L = 0.025$, $R = 5$, $K_\tau = K_B = 0.9$. Therefore $0.018 \leq B \leq 0.0181$ and $0.06419 \leq M \leq 0.0642$.

We obtain $15.576 \leq b_1 \leq 15.579, 0.280 \leq a_2 \leq 0.282, a_3 = 36, a_4 = 200$.

To simplify, let us choose values close to those given in [30]: $b_1 = 15, a_2 = \frac{1}{4}, a_3 = 36, a_4 = 200$. We replace their own values in (3.32), then the equality

$$R[\alpha(y) + \lambda(y)\mathcal{C}]R^{-1} = \begin{bmatrix} -1 & \frac{13}{2} & 37 \\ 0 & -160 & \frac{225}{4} \\ 0 & 104 & -\frac{161}{4} \end{bmatrix}$$

with

$$R = \begin{bmatrix} 1 & 6 & 1 \\ 0 & 1 & 6 \\ 0 & -1 & -4 \end{bmatrix}$$

is satisfied. Therefore $R[\alpha(y) + \lambda(y)\mathcal{C}]R^{-1}$ is Metzler and Assumption 3.5 is satisfied.

Let us check that Assumption 3.6 is satisfied. The system (3.31) in closed-loop with the feedback

$$u_s(y,x) = \frac{1}{b_0 b_1}\left[-b_1 a_2 a_3 y + a_1 a_4 \sin(y) + a_1 \cos(y) x_2\right] \tag{3.33}$$

is

$$\begin{cases} \dot{x}_1 &=& x_2, \\ \dot{x}_2 &=& b_1 x_3 - a_1 \sin(x_1) - a_2 x_2, \\ \dot{x}_3 &=& -a_2 a_3 x_1 + \frac{a_1 a_4}{b_1}\sin(x_1) + \frac{a_1}{b_1}\cos(x_1) x_2 - a_3 x_2 - a_4 x_3. \end{cases} \tag{3.34}$$

Let $h = b_1 x_3 - a_1 \sin(x_1)$, $g = a_2 x_1 + x_2$. Then

$$\begin{cases} \dot{x}_2 &=& -a_2 x_2 + h, \\ \dot{g} &=& h, \\ \dot{h} &=& -b_1 a_3 g - a_4 h. \end{cases} \tag{3.35}$$

This system is exponentially stable. Next, through simple, lengthy and useless calculations, one can establish that Assumption 3.6 is satisfied with a positive definite quadratic function for γ and a function κ identically equal to a positive constant.

Since Assumptions 3.4 to 3.6 are satisfied, Theorem 3.7 applies and provides us with the following interval observer:

$$\begin{cases} \dot{\hat{x}}_1^+ &=& \hat{x}_2^+ + y - \hat{x}_1^+, \\ \dot{\hat{x}}_2^+ &=& b_1 \hat{x}_3^+ - a_1 \sin(y) - a_2 \hat{x}_2^+, \\ \dot{\hat{x}}_3^+ &=& b_0 u_s(y, \hat{x}^+) - a_3 \hat{x}_2^+ - a_4 \hat{x}_3^+, \\ \dot{\hat{x}}_1^- &=& \hat{x}_2^- + y - \hat{x}_1^-, \\ \dot{\hat{x}}_2^- &=& b_1 \hat{x}_3^- - a_1 \sin(y) - a_2 \hat{x}_2^-, \\ \dot{\hat{x}}_3^- &=& b_0 u_s(y, \hat{x}^+) - a_3 \hat{x}_2^- - a_4 \hat{x}_3^-, \end{cases} \tag{3.36}$$

where $a_1 = \frac{N}{M}$ with $N = \frac{mL_0G}{2K_\tau} + \frac{M_0L_0G}{K_\tau}$ and $b_0 = \frac{1}{L} = \frac{1}{0.025} = 40$, with the initial conditions

$$\hat{x}_0^+ = SR^+ x_0^+ - SR^- x_0^-, \quad \hat{x}_0^- = SR^+ x_0^- - SR^- x_0^+ \tag{3.37}$$

where $SR^+ = \begin{bmatrix} 1 & \frac{35}{2} & 70 \\ 0 & -2 & -12 \\ 0 & \frac{1}{2} & 3 \end{bmatrix}$, $SR^- = \begin{bmatrix} 0 & \frac{35}{2} & 70 \\ 0 & -3 & -12 \\ 0 & \frac{1}{2} & 2 \end{bmatrix}$ and the bounds

$$x^+ = S^+R\hat{x}^+ - S^-R\hat{x}^-, \quad x^- = S^+R\hat{x}^- - S^-R\hat{x}^+ \tag{3.38}$$

where $S^+R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $S^-R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Figure 3.1 below illustrates the result in the case where there is no disturbances. A trajectory with $t_0 = 0$, $x_0 = (20, 10, 1)^\top$, $x_0^+ = \left(\frac{41}{2}, \frac{11}{2}, \frac{3}{2}\right)^\top$, $x_0^- = \left(\frac{39}{2}, \frac{9}{2}, \frac{1}{2}\right)^\top$, $\hat{x}_0^+ = \left(108, -\frac{9}{2}, 4\right)^\top$, $\hat{x}_0^- = \left(-68, \frac{49}{2}, -2\right)^\top$ as initial condition is simulated. We see clearly that the bounds and the global asymptotically stability are ensured.

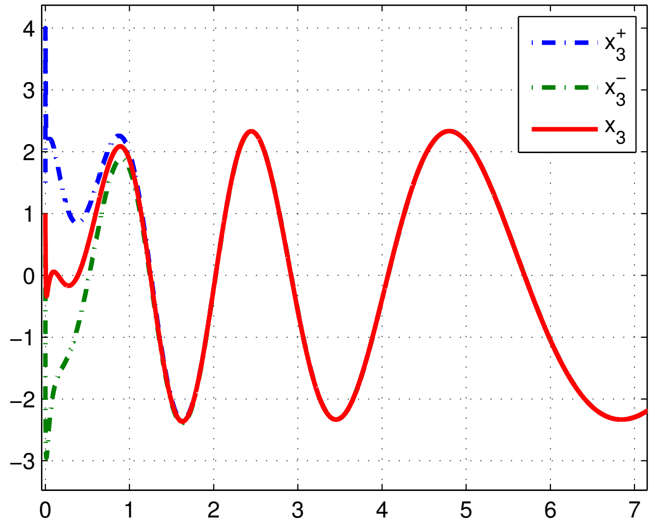

FIGURE 3.1: Evolution of x_3, x_3^+ and x_3^- without uncertainties $\delta(t)$.

Now we consider (3.31) in the presence of additive disturbances. For our simulations, we choose $\delta(t) = \left(\frac{1}{9}\sin(t), \frac{1}{9}\sin(t), \frac{1}{9}\sin(t)\right)^\top$ which are bounded by $\delta^+(t) = \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right)^\top$ and $\delta^-(t) = -\delta^+(t)$. Then, from (3.13), we deduce that the corresponding interval observer is:

$$\left\{\begin{array}{lcl} \dot{\hat{x}}_1^+ & = & \hat{x}_2^+ + y - \hat{x}_1^+ + \frac{176}{9}, \\ \dot{\hat{x}}_2^+ & = & b_1\hat{x}_3^+ - a_1\sin(y) - a_2\hat{x}_2^+ - \frac{29}{9}, \\ \dot{\hat{x}}_3^+ & = & b_0 u_s(y, \hat{x}^+) - a_3\hat{x}_2^+ - a_4\hat{x}_3^+ + \frac{2}{3}, \\ \dot{\hat{x}}_1^- & = & \hat{x}_2^- + y - \hat{x}_1^- - \frac{176}{9}, \\ \dot{\hat{x}}_2^- & = & b_1\hat{x}_3^- - a_1\sin(y) - a_2\hat{x}_2^- + \frac{29}{9}, \\ \dot{\hat{x}}_3^- & = & b_0 u_s(y, \hat{x}^+) - a_3\hat{x}_2^- - a_4\hat{x}_3^- - \frac{2}{3}, \end{array}\right. \tag{3.39}$$

Figure 3.2 below shows that in the case where the system is affected by additive disturbances, the interval observer still provided the solutions with bounds, which do not converge to the solution of the studied system. The initial conditions we selected are the same as those chosen in the undisturbed case of Figure 3.1.

3.5 Conclusion

For a family of systems affine in the unmeasured variables, we have presented a new technique of construction of interval observers, which makes it possible to cope with the presence of

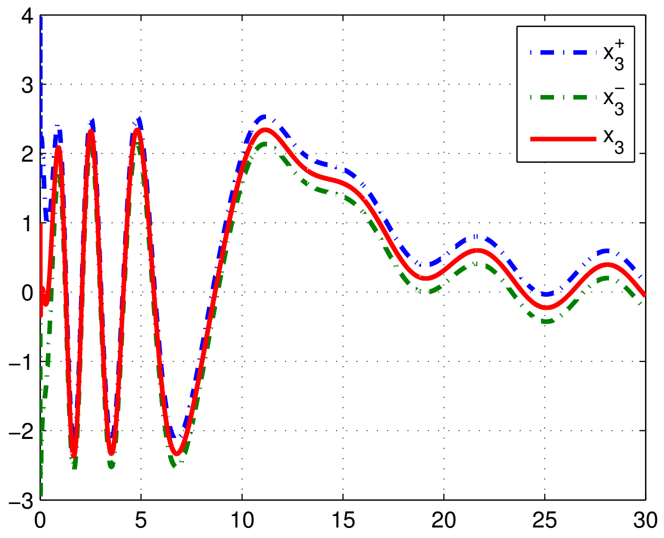

FIGURE 3.2: Evolution of x_3, x_3^+ and x_3^- with uncertainties $\delta(t)$.

additive disturbances. The dynamic part of interval observers is simply composed of two copies of a classical observer and therefore they can be used when it comes to stabilize the systems through a dynamic output feedback. Extensions to systems with special structures, in the spirit of what is done in [91] and in [43] and extensions to time-varying systems, systems with delay and systems with discrete measurements are expected.

Chapter 4

Interval Observers For Discrete–time Systems

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4.1 Introduction

In this chapter, interval observers design for discrete-time systems are taken into account. The technique has been developed in several contexts : in particular some works are devoted to families of linear systems [27, 88, 89, 98, 99] and others are devoted to nonlinear systems [2, 3, 105, 116, 117]. A common feature of these results is that they apply only to continuous-time systems. On the other hand, discrete-time systems are very important from a theoretical as well as an applied point of view and the problem of constructing observers or dynamic output feedbacks for them has been extensively studied (see for instance [19, 68, 134], [67, Chapt. 6]). The interest of the discrete-time systems partially stems from the fact that discretization techniques transform continuous-time systems into discrete-time systems. Besides, systems with sampled data often lead to discrete-time systems, as explained for instance in [11]. These systems are frequently affected by disturbances, which motivates the development of robust state estimation techniques, like the one based on interval observers.

This motivates the present work and the recent contribution [38], which, to the best of our knowledge, is the only existing work that is devoted to the design of interval observers for discrete-time systems.

Before comparing the results of [38], with those of the present chapter, we will summarize our contributions.

First, we will consider a nonlinear system of the form

$$x_{k+1} = \mathcal{F}(y_k, x_k, w_k) \ , \ k \in \mathbb{N}, \tag{4.1}$$

with $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$ is the output, where $w_k \in \mathbb{R}^\ell$ represents a disturbance and construct time-invariant interval observers, under a stability condition on the function $\mathcal{F}$ of a new type. Next, we will focus our attention on the fundamental family of the linear time-invariant discrete-time systems

$$x_{k+1} = \mathcal{A}x_k \ , \ k \in \mathbb{N}, \tag{4.2}$$

with $x_k \in \mathbb{R}^n$ when the spectral radius of $\mathcal{A}$ is smaller than 1 i.e. when $\mathcal{A}$ is Schur stable. We will show that in some cases, the technique of construction of time-invariant interval observers developed for (4.1) does not lead to interval observers for (4.2). To overcome this limitation, we will show how *time-varying interval observers* can be constructed for a family of linear systems with outputs and disturbances, which encompasses the family of systems (4.2). The construction we will propose relies on time-varying changes of coordinates that transform linear discrete-time systems into nonnegative discrete-time systems. We will design the change of coordinates by using the fact that any real matrix can be transformed into a matrix of the Jordan canonical form (see [110, Section 1.8]) and next by finding suitable changes of coordinates for elementary Jordan blocks. Surprisingly, although the changes of coordinates we will apply are time-varying, the transformed systems are autonomous. However, the interval observers we will construct are time-varying because they involve a time-varying change of coordinates, and thus they give lower and upper bounds for the state of the studied system that depend on the time.

The Section 4.4 of this chapter that is devoted to linear systems owes a great deal to [89], which presents constructions of interval observers for continuous-time linear systems by using extensively time-varying changes of coordinates. However, there are fundamental differences between the main results of [89] and those of the Section 4.4 because it turns out that a continuous-time system $\dot{x} = \mathcal{A}x$ is positive if and only if the matrix $\mathcal{A}$ is cooperative whereas a discrete-time system $x_{k+1} = \mathcal{A}x_k$ is positive if and only if no entry of $\mathcal{A}$ is negative. Consequently, the time-varying changes of coordinates we will use to obtain nonnegative linear systems cannot be deduced from the time-varying changes of coordinates used in [89] to transform a Hurwitz matrix into a Hurwitz and cooperative matrix.

Finally, let us observe that the paper [38] differs from our work presented in this chapter for the following reasons: (i) The first main result of [38] is devoted to time-invariant linear systems with outputs and relies on an observability condition which guarantees the existence of time-invariant changes of coordinates leading to equations for which interval observers can be readily constructed. (ii) The second main result of [38] is devoted to time-varying linear systems.

The chapter is organized as follows. Basic definitions and results are recalled in Section 4.2. In Section 4.3, we state and prove results of construction of interval observers for nonlinear systems. In Section 4.4, we state and prove that any time-invariant exponentially stable linear discrete-time system can be transformed into a block diagonal system with nonnegative and exponentially stable subsystems. A construction of time-varying interval observers for linear systems with outputs is proposed in Section 4.5. Section 4.6 provides some illustrative examples. Concluding remarks are drawn in Section 4.7.

4.2 Notation, definitions, basic result

We recalls now some notation, definitions and basis result which will be used throughout the next sections of the present chapter.

- The notation will be simplified whenever no confusion can arise from the context.
- Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0.
- All the inequalities must be understood componentwise (see Definition 1.27 of Chapter 1).
- $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$.
- For a matrix $A \in \mathbb{R}^{r\times s}$, $A^+ = \max(A, 0)$, $A^- = \max(-A, 0)$. Thus, $A = A^+ - A^-$.
- For a real number m, we define $\varpi(m) = \max\{m, 0\}$, $\mho(m) = \min\{m, 0\}$.
- A matrix $A \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* if every entry of A is nonnegative.
- A sequence (u_i) is *nonnegative* if for all integer k, u_k is nonnegative.
- The discrete-time dynamical system (4.1) is *nonnegative* if for every nonnegative initial condition x_0 and nonnegative sequence (w_k), the corresponding solution x_k is nonnegative for all $k \in \mathbb{N}$.
- diag$\{B_1, ..., B_j\}$ denotes the block diagonal matrix $\begin{bmatrix} B_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_j \end{bmatrix}$.

A general definition of interval observer for discrete-time nonlinear time-varying systems which is slightly different from Definition 1.48 is introduced :

Definition 4.1. Consider a time-varying system

$$x_{k+1} = f_1\left(k, x_k, w_k\right),\ k \in \mathbb{N}, \tag{4.3}$$

with an output $y_k = m(x_k, w_k) \in \mathbb{R}^p$, with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$, and where f_1 and m are two functions. The uncertainties w_k are such that there exists a sequence $\overline{w}_k = (w_k^+, w_k^-) \in \mathbb{R}^{2\ell}$ such that, for all $k \in \mathbb{N}$,

$$w_k^- \leq w_k \leq w_k^+. \tag{4.4}$$

Moreover, the initial condition at the instant k_0, $x_{k_0} \in \mathbb{R}^n$ is assumed to be bounded by two known bounds:

$$x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+. \tag{4.5}$$

Then, the dynamical system

$$z_{k+1} = f_2\left(k, z_k, y_k, \overline{w}_k\right),\ k \in \mathbb{N}, \tag{4.6}$$

associated with the initial condition $z_{k_0} = g(k_0, x_{k_0}^+, x_{k_0}^-) \in \mathbb{R}^{n_z}$ and bounds for the solution x_k for all $k > k_0$:

$$x_k^+ = h^+(k, z_k)\ ,\ x_k^- = h^-(k, z_k), \tag{4.7}$$

where f_2, g, h^+ and h^- are functions, is called an <u>interval observer</u> for (4.3) if

(i) any solution (x_k, z_k) of (4.3)-(4.6) with $\overline{w}_k = 0$ for all $k \in \mathbb{N}$ is such that $\lim\limits_{k\to+\infty} |h^+(k, z_k) - h^-(k, z_k)| = 0$.

(ii) for any vectors $x_{k_0}, x_{k_0}^-$ and $x_{k_0}^+$ in $\mathbb{R}^n$ satisfying (4.5), the solutions of (4.3), (4.6) with respectively x_{k_0}, z_{k_0} as initial condition at $k = k_0$, denoted respectively x_k and z_k satisfy, for all $k > k_0$, the inequalities

$$x_k^- = h^-(k, z_k) \leq x_k \leq h^+(k, z_k) = x_k^+. \tag{4.8}$$

Finally, note that Theorem 1.39 is instrumental for establishing our results in the present chapter.

4.3 Time–invariant interval observers

4.3.1 Interval observers for nonlinear systems

In this section we construct interval observers for nonlinear systems of the form

$$\begin{aligned} x_{k+1} &= \mathcal{F}(y_k, x_k, w_k)\ , k \in \mathbb{N}, \\ y_k &= \mathcal{H}(x_k), \end{aligned} \tag{4.9}$$

with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$ and where $y_k \in \mathbb{R}^p$ is the output. We introduce an assumption:

Assumption 4.2. *There exists a function $\mathcal{F}_c : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^n$ such that*

$$\mathcal{F}(y, x, w) = \mathcal{F}_c(y, x, x, w, w)\ ,\ \forall (y, x, w) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^\ell \tag{4.10}$$

and, for all fixed y, b, d, the function $(a, c) \to \mathcal{F}_c(y, a, b, c, d)$ is nondecreasing with respect to each of its variables and, for all fixed y, a, c, the function $(b, d) \to \mathcal{F}_c(y, a, b, c, d)$ is nonincreasing with respect to each of its variables. Moreover, for any sequence y_k which converges in norm to zero, all the solutions of the system

$$\left\{ \begin{aligned} a_{k+1} &= \mathcal{F}_c(y_k, a_k, b_k, 0, 0), \\ b_{k+1} &= \mathcal{F}_c(y_k, b_k, a_k, 0, 0), \end{aligned} \right. \tag{4.11}$$

converge to the origin.

Remark 4.3. From Lemma A.4 in Appendix A, one deduces easily that if the function $\mathcal{F}$ in (4.9) is of class C^1, then there exists a function $\mathcal{F}_c : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^n$ such that $\mathcal{F}(y, x, w) = \mathcal{F}_c(y, x, x, w, w)$ which possesses the monotonous properties required in Assumption 4.2. So the restrictive aspect of Assumption 4.2 is the stability property of the system (4.11). In [104], the result of Lemma A.4 is proved for the family of the global Lipschitz functions.

We state and prove the following result.

Theorem 4.4. *Assume that the system (4.9) satisfies Assumption 4.2. Let the sequence (w_k) be bounded by two known sequences (w_k^+), (w_k^-): for all integer $k \geq 0$,*

$$w_k^- \leq w_k \leq w_k^+. \tag{4.12}$$

Then the system

$$\left\{ \begin{aligned} z_{k+1}^+ &= \mathcal{F}_c(y_k, z_k^+, z_k^-, w_k^+, w_k^-), \\ z_{k+1}^- &= \mathcal{F}_c(y_k, z_k^-, z_k^+, w_k^-, w_k^+), \end{aligned} \right. \tag{4.13}$$

associated with the initial conditions

$$z_{k_0}^+ = x_{k_0}^+ \ , \ z_{k_0}^- = x_{k_0}^-, \tag{4.14}$$

and the bounds for the solutions x_k

$$x_k^+ = z_k^+ \ , \ x_k^- = z_k^-, \tag{4.15}$$

is an interval observer for system (4.9).

Remark 4.5. An extension of Theorem 4.4 to the case where the system (4.9) time-varying can be obtained. For the sake of brevity, we omit it.

Proof. Let us consider vectors x_{k_0}, $x_{k_0}^+$, $x_{k_0}^-$, $z_{k_0}^+$, $z_{k_0}^-$ in $\mathbb{R}^n$ such that

$$z_{k_0}^- = x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+ = z_{k_0}^+. \tag{4.16}$$

Next, let us consider the solutions (x_k), (z_k^+), (z_k^-) of the systems (4.9), (4.13) with initial conditions x_{k_0}, $z_{k_0}^+$, $z_{k_0}^-$. Using the equality (4.10), we obtain, for all integer $k \geq k_0$,

$$\begin{array}{l} z_{k+1}^+ - x_{k+1} = \mathcal{F}_c(y_k, z_k^+, z_k^-, w_k^+, w_k^-) - \mathcal{F}_c(y_k, x_k, x_k, w_k, w_k), \\ x_{k+1} - z_{k+1}^- = \mathcal{F}_c(y_k, x_k, x_k, w_k, w_k) - \mathcal{F}_c(y_k, z_k^-, z_k^+, w_k^-, w_k^+). \end{array} \tag{4.17}$$

Now, we prove by induction that for all $k \geq k_0$, $z_k^+ - x_k \geq 0$, $x_k - z_k^- \geq 0$. According to (4.16), the property is satisfied at the instant k_0. Assume that it is satisfied at the step $k \geq k_0$. Then, the monotonicity properties of $\mathcal{F}_c$ and the fact that for all integer j, $w_j^+ - w_j \geq 0$, $w_j - w_j^- \geq 0$, imply that $\mathcal{F}_c(y_k, z_k^+, z_k^-, w_k^+, w_k^-) - \mathcal{F}_c(y_k, x_k, x_k, w_k, w_k) \geq 0$, $\mathcal{F}_c(y_k, x_k, x_k, w_k, w_k) - \mathcal{F}_c(y_k, z_k^-, z_k^+, w_k^-, w_k^+) \geq 0$. It follows that $z_{k+1}^+ - x_{k+1} \geq 0$ and $x_{k+1} - z_{k+1}^- \geq 0$. Consequently, the induction assumption is satisfied at the step $k+1$. Finally, we conclude by observing that the global asymptotic stability of the system (4.11) implies that the solutions of the system (4.13), when the sequences (w_k^+) and (w_k^-) are identically equal to zero are such that $\lim\limits_{k \to +\infty} |z_k^+ - z_k^-| = 0$. □

We show now that if a system can be transformed through a change of coordinates into a system that satisfies Assumption 4.2, then again an interval observer can be constructed.

Theorem 4.6. *Consider the system*

$$\begin{array}{rcl} s_{k+1} & = & \mathcal{G}(r_k, s_k, w_k) \ , k \in \mathbb{N}, \\ r_k & = & \mathcal{J}(s_k) \ , \end{array} \tag{4.18}$$

with $s_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$ *and where* $r_k \in \mathbb{R}^p$ *is the output. Let the sequence* (w_k) *be bounded by two known sequences* (w_k^+), (w_k^-)*: for all integer* $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{4.19}$$

Assume that there is a diffeomorphism $\theta : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\theta(0) = 0$ *and the change of coordinates* $x_k = \theta(s_k)$ *transforms (4.18) into a system*

$$x_{k+1} = \mathcal{F}(y_k, x_k, w_k) \ , k \in \mathbb{N}, \tag{4.20}$$

with $y_k = \mathcal{J}(\theta^{-1}(x_k))$ *and*

$$\mathcal{F}(y_k, x_k, w_k) = \theta(\mathcal{G}(y_k, \theta^{-1}(x_k), w_k)),$$

that satisfies Assumption 4.2. Let $\theta_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\rho_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be two functions such that, for all* $x \in \mathbb{R}^n$,

$$\theta(x) = \theta_c(x, x) \ , \ \theta^{-1}(x) = \rho_c(x, x) \tag{4.21}$$

and both θ_c *and* ρ_c *are nondecreasing with respect to each of their* n *first variables and nonincreasing with respect to their* n *last variables.*

Then the system

$$\left\{ \begin{array}{lcl} z_{k+1}^+ & = & \mathcal{F}_c(r_k, z_k^+, z_k^-, w_k^+, w_k^-), \\ z_{k+1}^- & = & \mathcal{F}_c(r_k, z_k^-, z_k^+, w_k^-, w_k^+), \end{array} \right. \tag{4.22}$$

where $\mathcal{F}_c$ *is the function provided by Assumption 4.2, associated with the initial conditions*

$$z_{k_0}^+ = \theta_c(s_{k_0}^+, s_{k_0}^-) \ , \ z_{k_0}^- = \theta_c(s_{k_0}^-, s_{k_0}^+) \tag{4.23}$$

and the bounds for the solutions s_k

$$s_k^+ = \rho_c(z_k^+, z_k^-) \ , \ s_k^- = \rho_c(z_k^-, z_k^+), \tag{4.24}$$

is an interval observer for the system (4.18).

Proof. To begin with, we observe that the existence of the functions θ_c, ρ_c satisfying the conditions of Theorem 4.6 is a consequence of Lemma A.4 in A. Now, let us consider the solutions (s_k), (z_k^+), (z_k^-) of the systems (4.18), (4.22) with initial conditions $(s_{k_0}, s_{k_0}^+, s_{k_0}^-) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, $(z_{k_0}^+, z_{k_0}^-) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$s_{k_0}^- \leq s_{k_0} \leq s_{k_0}^+ \tag{4.25}$$

and the equalities (4.23) are satisfied.

From (4.25) and the monotonicity properties of θ_c, it follows that the inequalities

$$\theta_c(s_{k_0}^-, s_{k_0}^+) \leq \theta_c(s_{k_0}, s_{k_0}) \leq \theta_c(s_{k_0}^+, s_{k_0}^-) \tag{4.26}$$

are satisfied. From (4.23) and (4.21), it follows that

$$z_{k_0}^- \leq \theta(s_{k_0}) \leq z_{k_0}^+. \tag{4.27}$$

On the other hand, we know that, for all $k \in \mathbb{N}$,

$$\begin{array}{rcl} z_{k+1}^+ & = & \mathcal{F}_c(r_k, z_k^+, z_k^-, w_k^+, w_k^-), \\ \theta(s_{k+1}) & = & \mathcal{F}(r_k, \theta(s_k), w_k) = \mathcal{F}_c(r_k, \theta(s_k), \theta(s_k), w_k, w_k), \\ z_{k+1}^- & = & \mathcal{F}_c(r_k, z_k^-, z_k^+, w_k^-, w_k^+). \end{array} \tag{4.28}$$

Arguing as we did to prove Theorem 4.4, we deduce that, for all integer $k \geq k_0$, the inequalities

$$z_k^- \leq \theta(s_k) \leq z_k^+ \tag{4.29}$$

are satisfied. From the monotonous properties of ρ_c and the inequalities (4.29), we deduce that

$$\rho_c(z_k^-, z_k^+) \leq \rho_c(\theta(s_k), \theta(s_k)) \leq \rho_c(z_k^+, z_k^-). \tag{4.30}$$

Using (4.21), we obtain that, for all integer $k \geq k_0$,

$$\rho_c(z_k^-, z_k^+) \leq \theta^{-1}(\theta(s_k)) \leq \rho_c(z_k^+, z_k^-). \tag{4.31}$$

Thus the inequalities

$$s_k^- \leq s_k \leq s_k^+ \tag{4.32}$$

are satisfied for all integer $k \geq k_0$. Using $\rho_c(0,0) = \theta^{-1}(0) = 0$, we can conclude the proof. □

4.3.2 Interval observers for linear systems

In this section we analyze the consequences of the results of Section 4.3.1 when particularized to the family of the linear time-invariant systems.

As an direct consequence of Theorem 4.4, we have the following result:

Corollary 4.7. *Consider the system*

$$x_{k+1} = \mathcal{A}x_k + w_k \ , k \in \mathbb{N}, \tag{4.33}$$

with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^n$, *where* $\mathcal{A} \in \mathbb{R}^{n\times n}$ *is a constant matrix. Assume that the matrix*

$$\mathcal{A}^* = \begin{bmatrix} \mathcal{A}^+ & -\mathcal{A}^- \\ -\mathcal{A}^- & \mathcal{A}^+ \end{bmatrix} \tag{4.34}$$

is Schur stable. Let (w_k) *be a sequence bounded by two known sequences* (w_k^+), (w_k^-)*: for all integer* $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{4.35}$$

Then the system

$$\begin{cases} z_{k+1}^+ &= \mathcal{A}^+ z_k^+ - \mathcal{A}^- z_k^- + w_k^+, \\ z_{k+1}^- &= \mathcal{A}^+ z_k^- - \mathcal{A}^- z_k^+ + w_k^-, \end{cases} \tag{4.36}$$

associated with the initial conditions

$$z_{k_0}^+ = x_{k_0}^+ \ , \ z_{k_0}^- = x_{k_0}^- \tag{4.37}$$

and the bounds for the solutions x_k

$$x_k^+ = z_k^+ \ , \ x_k^- = z_k^-, \tag{4.38}$$

is an interval observer for the system (4.33).

Proof. We observe that, for all $x \in \mathbb{R}^n$, $\mathcal{A}x = \mathcal{L}(x,x)$ with $\mathcal{L}(a,b) = \mathcal{A}^+a - \mathcal{A}^-b$. Since the function $\mathcal{L}$ is nondecreasing with respect to the variables a_i and nonincreasing with respect to the variables b_i and since the system (4.36) is exponentially stable when (w_k^+) and (w_k^-) are identically equal to zero, we can conclude by applying Theorem 4.4. □

From Corollary 4.7, a question arises. If the matrix $\mathcal{A}$ is Schur stable, is the corresponding matrix $\mathcal{A}^*$ necessarily Schur stable? If the answer to the question was positive, then by Corollary 4.7 it would be possible to construct interval observers for any exponentially stable linear discrete-time system. Unfortunately, the answer is negative. For instance,

$$\mathcal{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{4.39}$$

is a Schur matrix, but the spectral radius of the associated matrix $\mathcal{A}^*$ is larger than 1.

Then, from Theorem 4.6, another question arises. If a linear time-invariant discrete-time system (4.33) is Schur stable, is it always possible to apply Theorem 4.6 with a linear change of coordinates θ ? If the answer was positive, then one might always transform an exponentially stable linear discrete-time system into a system for which an interval observer could be designed. But we conjecture that the answer is negative. The reason why we conjecture this is the following lemma, which establishes that some Schur stable systems cannot be transformed through a linear time-invariant change of coordinates into a system which satisfies the conditions of Corollary 4.7.

Lemma 4.8. *Let*

$$\mathcal{A} = \begin{bmatrix} -\omega & \omega \\ -\omega & -\omega \end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{4.40}$$

with $\omega \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. *The spectral radius of* $\mathcal{A}$ *is* $\sqrt{2}\omega < 1$. *For any invertible matrix*

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{4.41}$$

the matrix

$$\mathcal{A}_L = L\mathcal{A}L^{-1} \tag{4.42}$$

is such that the spectral radius of the matrix

$$\mathcal{A}_L^* = \begin{bmatrix} \mathcal{A}_L^+ & -\mathcal{A}_L^- \\ -\mathcal{A}_L^- & \mathcal{A}_L^+ \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{4.43}$$

is larger than 1.

Proof. To begin with, we observe that the eigenvalues of $\mathcal{A}$ are $-\omega + i\omega$ and $-\omega - i\omega$. Therefore the spectral radius of $\mathcal{A}$ is $\sqrt{2}\omega$.

Using the fact that, for all vector $V = (v^\top, -v^\top)^\top \in \mathbb{R}^4$, $\mathcal{A}_L^* V = \begin{pmatrix} (\mathcal{A}_L^+ + \mathcal{A}_L^-)v \\ -(\mathcal{A}_L^+ + \mathcal{A}_L^-)v \end{pmatrix}$, we deduce that the spectral radius of $\mathcal{A}_L^*$ is not smaller than the spectral radius of the matrix $\mathcal{A}_L^P = \mathcal{A}_L^+ + \mathcal{A}_L^-$. Therefore if we can prove that the spectral radius of $\mathcal{A}_L^P$ is larger than 1, the proof is completed. To analyze the spectral radius of $\mathcal{A}_L^P$, we observe that simple calculations lead to the equality

$$\mathcal{A}_L \quad = \quad \omega \begin{bmatrix} \frac{-l_{12}l_{22}-l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} - 1 & \frac{l_{12}^2+l_{11}^2}{l_{11}l_{22}-l_{12}l_{21}} \\ \frac{-l_{22}^2-l_{21}^2}{l_{11}l_{22}-l_{12}l_{21}} & \frac{l_{22}l_{12}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} - 1 \end{bmatrix}. \tag{4.44}$$

Consequently,

$$\mathcal{A}_L^P = \omega \begin{bmatrix} \left|\frac{l_{12}l_{22}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} + 1\right| & \left|\frac{l_{12}^2+l_{11}^2}{l_{11}l_{22}-l_{12}l_{21}}\right| \\ \left|\frac{l_{22}^2+l_{21}^2}{l_{11}l_{22}-l_{12}l_{21}}\right| & \left|\frac{l_{22}l_{12}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} - 1\right| \end{bmatrix}. \tag{4.45}$$

One can prove easily that for all real number m, the inequality $2 \leq |1+m| + |1-m|$ is satisfied. We deduce that tr$(\mathcal{A}_L^P) \geq 2\omega > 1$. On the other hand,

$$\begin{array}{rcl} \det \mathcal{A}_L^P & = & \omega^2 \left(\left| \left(\frac{l_{12}l_{22}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} \right)^2 - 1 \right| - \frac{(l_{22}^2+l_{21}^2)(l_{12}^2+l_{11}^2)}{(l_{11}l_{22}-l_{12}l_{21})^2} \right) \\ & = & \frac{\omega^2 \left[\left| (l_{12}l_{22}+l_{11}l_{21})^2 - (l_{11}l_{22}-l_{12}l_{21})^2 \right| - (l_{22}^2+l_{21}^2)(l_{12}^2+l_{11}^2) \right]}{(l_{11}l_{22}-l_{12}l_{21})^2} \\ & = & \frac{\omega^2 \left[\left| (l_{12}l_{22}+l_{11}l_{21})^2 - (l_{11}l_{22}-l_{12}l_{21})^2 \right| - [l_{22}^2l_{12}^2+l_{21}^2l_{12}^2+l_{22}^2l_{11}^2+l_{21}^2l_{11}^2] \right]}{(l_{11}l_{22}-l_{12}l_{21})^2}. \end{array} \tag{4.46}$$

Now, observing that

$$(l_{12}l_{22}+l_{11}l_{21})^2 = l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + 2l_{12}l_{22}l_{11}l_{21} \leq l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + l_{12}^2l_{21}^2 + l_{22}^2l_{11}^2$$

and

$$(l_{11}l_{22}-l_{12}l_{21})^2 = l_{11}^2l_{22}^2 + l_{12}^2l_{21}^2 - 2l_{11}l_{22}l_{12}l_{21} \leq l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + l_{12}^2l_{21}^2 + l_{22}^2l_{11}^2.$$

We deduce that

$$\left| (l_{12}l_{22}+l_{11}l_{21})^2 - (l_{11}l_{22}-l_{12}l_{21})^2 \right| \leq l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + l_{12}^2l_{21}^2 + l_{22}^2l_{11}^2,$$

which implies that $\det \mathcal{A}_L^P \leq 0$. This inequality in combination with the inequality tr$(\mathcal{A}_L^P) > 1$ implies that at least one of the eigenvalues of $\mathcal{A}_L^P$ is a real number larger than 1. □

4.4 Transformations of linear systems into nonnegative systems

The previous section motivates the main results of the present and the next section. In this section, we establish that any discrete-time exponentially stable time-invariant linear system can be transformed into a nonnegative and exponentially stable time-invariant system through a linear time-varying change of coordinates. In Section 4.5, we will use this result to construct interval observers for linear systems.

Theorem 4.9. *Consider the system*

$$x_{k+1} = \mathcal{A}x_k \;, k \in \mathbb{N}, \tag{4.47}$$

with $x_k \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a constant Schur stable matrix. Then there exists a time-varying change of coordinates $h_k = \mathcal{R}_k x_k$, where $(\mathcal{R}_k)$ is a sequence of invertible matrices in $\mathbb{R}^{n\times n}$ such that there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$, which transforms (4.47) into a positive and exponentially stable linear system.

The proof idea: firstly, we recall that any real matrix admits a real Jordan canonical form. Secondly, we transform systems in Jordan canonical form into positive systems.

Proof. This can be found in Appendix A. □

4.5 Time-varying interval observers

The result of this section shows how Theorem 4.9 can be used when it comes to constructing interval observers for linear systems with output.

Theorem 4.10. *Consider the system*

$$\begin{aligned} x_{k+1} &= \alpha x_k + w_k \ , k \in \mathbb{N}, \\ y_k &= C x_k + v_k, \end{aligned} \tag{4.48}$$

with $x_k \in \mathbb{R}^n$, the output $y_k \in \mathbb{R}^q$, $w_k \in \mathbb{R}^n$, $v_k \in \mathbb{R}^q$, where $\alpha \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{q\times n}$ are matrices such that there exists $K \in \mathbb{R}^{n\times q}$ such that the matrix $\mathcal{A} = \alpha + KC$ is Schur stable. Let the sequence (w_k) and (v_k) be bounded by known sequences (w_k^+), (w_k^-), (v_k^+), (v_k^-): for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+ \ , \ v_k^- \leq v_k \leq v_k^+. \tag{4.49}$$

Then there exist a sequence of invertible real matrices $(\mathcal{R}_k)$ and a real number $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$ and

$$\mathcal{R}_{k+1}\mathcal{A}\mathcal{R}_k^{-1} = \mathcal{E}, \tag{4.50}$$

where $\mathcal{E} \in \mathbb{R}^{n\times n}$ is a nonnegative Schur stable matrix. Let $\mathcal{S}_k = \mathcal{R}_k^{-1}$ for all $k \in \mathbb{N}$. Then the system

$$\begin{cases} z_{k+1}^+ = \mathcal{E}z_k^+ - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^- + (\mathcal{R}_{k+1}K)^+ v_k^+ - (\mathcal{R}_{k+1}K)^- v_k^-, \\ z_{k+1}^- = \mathcal{E}z_k^- - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+ w_k^- - \mathcal{R}_{k+1}^- w_k^+ + (\mathcal{R}_{k+1}K)^+ v_k^- - (\mathcal{R}_{k+1}K)^- v_k^+, \end{cases} \tag{4.51}$$

associated with the initial conditions

$$z_{k_0}^+ = \mathcal{R}_{k_0}^+ x_{k_0}^+ - \mathcal{R}_{k_0}^- x_{k_0}^- \ , \ z_{k_0}^- = \mathcal{R}_{k_0}^+ x_{k_0}^- - \mathcal{R}_{k_0}^- x_{k_0}^+ \tag{4.52}$$

and the bounds

$$x_k^+ = \mathcal{S}_k^+ z_k^+ - \mathcal{S}_k^- z_k^- \ , \ x_k^- = \mathcal{S}_k^+ z_k^- - \mathcal{S}_k^- z_k^+ \tag{4.53}$$

is an interval observer for system (4.48).

Proof. The sequence $(\mathcal{R}_k)$ and the matrix $\mathcal{E}$ are provided by Theorem 4.9. Next, we consider vectors $x_{k_0}, x_{k_0}^+, x_{k_0}^-, z_{k_0}^+, z_{k_0}^-$ in $\mathbb{R}^n$ such that

$$x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+ \tag{4.54}$$

and

$$z_{k_0}^+ = \mathcal{R}_{k_0}^+ x_{k_0}^+ - \mathcal{R}_{k_0}^- x_{k_0}^- \ , \ z_{k_0}^- = -\mathcal{R}_{k_0}^- x_{k_0}^+ + \mathcal{R}_{k_0}^+ x_{k_0}^-. \tag{4.55}$$

Since the matrices $\mathcal{R}_k^+$ and $\mathcal{R}_k^-$ are nonnegative matrices, then by Lemma A.5 in Appendix A, it follows that

$$z_{k_0}^- \leq \mathcal{R}_k x_{k_0} \leq z_{k_0}^+. \tag{4.56}$$

Next, let us consider the solutions (x_k), (z_k^+), (z_k^-) of the systems

$$\left\{\begin{array}{lcl} x_{k+1} & = & \mathcal{A}x_k - Ky_k + w_k + Kv_k, \\ z_{k+1}^+ & = & \mathcal{E}z_k^+ - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^- + (\mathcal{R}_{k+1}K)^+ v_k^+ \\ & & -(\mathcal{R}_{k+1}K)^- v_k^-, \\ z_{k+1}^- & = & \mathcal{E}z_k^- - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+ w_k^- - \mathcal{R}_{k+1}^- w_k^+ + (\mathcal{R}_{k+1}K)^+ v_k^- \\ & & -(\mathcal{R}_{k+1}K)^- v_k^+, \end{array}\right. \tag{4.57}$$

with initial conditions x_{k_0}, $z_{k_0}^+$, $z_{k_0}^-$ such that (4.54) and (4.55) are satisfied. Then

$$\begin{array}{rcl} \mathcal{R}_{k+1}x_{k+1} & = & \mathcal{R}_{k+1}[\mathcal{A}x_k - Ky_k + w_k + Kv_k] \\ & = & \mathcal{E}\mathcal{R}_k x_k - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}w_k + \mathcal{R}_{k+1}Kv_k. \end{array}$$

It follows that

$$\begin{array}{rcl} z_{k+1}^+ - \mathcal{R}_{k+1}x_{k+1} & = & \mathcal{E}[z_k^+ - \mathcal{R}_k x_k] + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^- - (\mathcal{R}_{k+1}^+ - \mathcal{R}_{k+1}^-)w_k \\ & & +(\mathcal{R}_{k+1}K)^+ v_k^+ - (\mathcal{R}_{k+1}K)^- v_k^- - ((\mathcal{R}_{k+1}K)^+ - (\mathcal{R}_{k+1}K)^+)v_k \\ & = & \mathcal{E}[z_k^+ - \mathcal{R}_k x_k] + p_k, \end{array}$$

with

$$p_k = \mathcal{R}_{k+1}^+(w_k^+ - w_k) + \mathcal{R}_{k+1}^-(w_k - w_k^-) + (\mathcal{R}_{k+1}K)^+(v_k^+ - v_k) + (\mathcal{R}_{k+1}K)^-(v_k - v_k^-)$$

and, similarly,

$$\mathcal{R}_{k+1}x_{k+1} - z_{k+1}^- \quad = \quad \mathcal{E}[\mathcal{R}_k x_k - z_k^-] + q_k,$$

with

$$q_k = \mathcal{R}_{k+1}^+(w_k - w_k^-) + \mathcal{R}_{k+1}^-(w_k^+ - w_k) + (\mathcal{R}_{k+1}K)^+(v_k - v_k^-) + (\mathcal{R}_{k+1}K)^-(v_k^+ - v_k).$$

Since the matrix $\mathcal{E}$ is nonnegative, for all integer k, $p_k \geq 0$, $q_k \geq 0$ and according to (4.56), $\mathcal{R}_{k_0}x_{k_0} - z^-_{k_0} \geq 0$, $z^+_{k_0} - \mathcal{R}_{k_0}x_{k_0} \geq 0$, we deduce from Theorem 1.39 that, for all $k \geq k_0$,

$$0 \leq z^+_k - \mathcal{R}_k x_k \ , \ 0 \leq \mathcal{R}_k x_k - z^-_k \, .$$

Therefore from Lemma A.5 in Appendix A, it follows that, for all $k \geq k_0$,

$$\mathcal{S}^+_k z^-_k - \mathcal{S}^-_k z^+_k \leq \mathcal{S}_k \mathcal{R}_k x_k \leq \mathcal{S}^+_k z^+_k - \mathcal{S}^-_k z^-_k \, .$$

Since $\mathcal{R}^{-1}_k = \mathcal{S}_k$, we deduce that, for all $k \geq k_0$,

$$\mathcal{S}^+_k z^-_k - \mathcal{S}^-_k z^+_k \leq x_k \leq \mathcal{S}^+_k z^+_k - \mathcal{S}^-_k z^-_k \, .$$

Finally, we deduce from (4.57) that, in the absence of w^+_k and w^-_k,

$$z^+_{k+1} - z^-_{k+1} = \mathcal{E}(z^+_k - z^-_k).$$

Since $\mathcal{E}$ is Schur stable, we can conclude. □

4.6 Illustrative examples

In this section, we illustrate Theorem 4.4 and Theorem 4.10 through two examples.

4.6.1 Nonlinear system

We apply Theorem 4.4 to the nonlinear system without output

$$x_{k+1} = \frac{1}{4}\sin(x_k) + x_k w_k \ , k \in \mathbb{N}, \tag{4.58}$$

with $x_k \in \mathbb{R}$, where $w_k \in \mathbb{R}$ represents disturbances. Let $\mathcal{F}(x, w) = \frac{1}{4}\sin(x) + xw$.

To verify the conditions in Assumption 4.2, we select the function $\mathcal{F}_c : \mathbb{R}^4 \to \mathbb{R}$

$$\mathcal{F}_c(a, b, c, d) = \frac{1}{4}a + \frac{1}{4}(\sin(b) - b), + \varpi(a)\varpi(c) + \varpi(b)\mho(c) + \mho(a)\varpi(d) + \mho(b)\mho(d), \tag{4.59}$$

with the functions ϖ and $\mho$ defined in the Section 4.2, which is such that

$$\mathcal{F}(x, w) = \mathcal{F}_c(x, x, w, w), \forall x \in \mathbb{R} \ , \ w \in \mathbb{R}$$

and is nondecreasing with respect to the variables a, c and nonincreasing with respect to the variables b, d. Next, we analyze the stability properties of the system

$$\left\{ \begin{array}{rcl} a_{k+1} & = & \frac{1}{4}a_k + \frac{1}{4}(\sin(b_k) - b_k), \\ b_{k+1} & = & \frac{1}{4}b_k + \frac{1}{4}(\sin(a_k) - a_k), \end{array} \right. \tag{4.60}$$

by using the positive definite quadratic function

$$V(a,b) = a^2 + b^2 \ .$$

From

$$V\left(a_{k+1}, b_{k+1}\right) = \left[\frac{1}{4}a_k + \frac{1}{4}(\sin(b_k) - b_k)\right]^2 + \left[\frac{1}{4}b_k + \frac{1}{4}(\sin(a_k) - a_k)\right]^2, \tag{4.61}$$

we deduce that

$$\begin{array}{rcl} V\left(a_{k+1}, b_{k+1}\right) & \leq & \frac{1}{8}\left[a_k^2 + (\sin(b_k) - b_k)^2\right] + \frac{1}{8}\left[b_k^2 + (\sin(a_k) - a_k)^2\right] \\ & \leq & \frac{1}{8}\left[a_k^2 + 4b_k^2\right] + \frac{1}{8}\left[b_k^2 + 4a_k^2\right] \\ & \leq & \frac{5}{8}V\left(a_k, b_k\right). \end{array} \tag{4.62}$$

Since V is a positive definite quadratic function, we conclude that the system (4.60) admits the origin as a globally exponentially stable equilibrium point.

According to Theorem 1, system (4.58) admits the following interval observer:

$$\begin{array}{rcl} z_{k+1}^+ & = & \frac{1}{4}z_k^+ + \frac{1}{4}(\sin(z_k^-) - z_k^-) + \varpi(z_k^+)\varpi(w_k^+) + \varpi(z_k^-)\mho(w_k^+) + \mho(z_k^+)\varpi(w_k^-) \\ & & +\mho(z_k^-)\mho(w_k^-)\ , \\ z_{k+1}^- & = & \frac{1}{4}z_k^- + \frac{1}{4}(\sin(z_k^+) - z_k^+) + \varpi(z_k^-)\varpi(w_k^-) + \varpi(z_k^+)\mho(w_k^-) + \mho(z_k^-)\varpi(w_k^+) \\ & & +\mho(z_k^+)\mho(w_k^+)\ , \\ z_{k_0}^+ & = & x_{k_0}^+\ ,\ z_{k_0}^- \ = \ x_{k_0}^-\ , \\ x_k^+ & = & z_k^+\ ,\ x_k^- \ = \ z_k^-\ . \end{array} \tag{4.63}$$

We present the following simulations.

Figure 4.1 gives the solution x and the bounds provided by the interval observer in the absence of disturbances.

Figure 4.2 gives the solution x and the bounds provided by the interval observer in the presence of disturbances:

$$w_k = \frac{4}{3}\cos(k)^2,$$

with the bounds

$$w_k^- = \frac{2}{9}\cos(k)^2,\ w_k^+ = \frac{1}{3}[2\cos(k)^2 + 1].$$

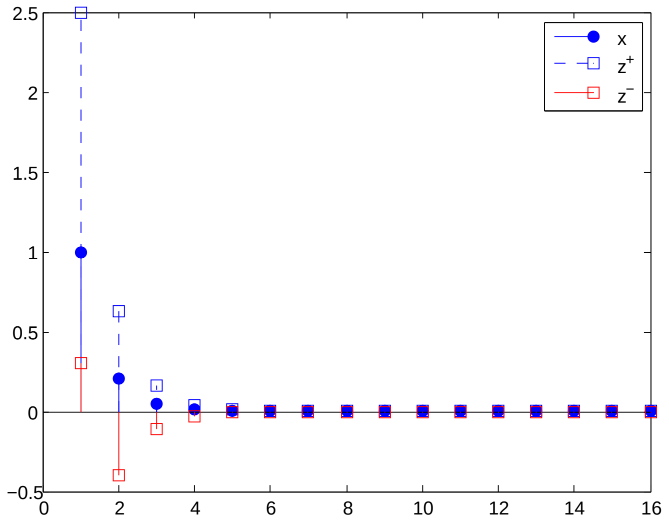

FIGURE 4.1: Solution in the absence of disturbances.

In both cases the selected initial conditions are

$$x(1) = 1,\ z^{+}(1) = \frac{5}{2},\ z^{-}(1) = \frac{3}{10}.$$

The simulations confirm the mathematical result: in the absence of disturbance, there is convergence of the two bounds of the solution and in the presence of disturbance, the bounds do not necessarily converge to the same value.

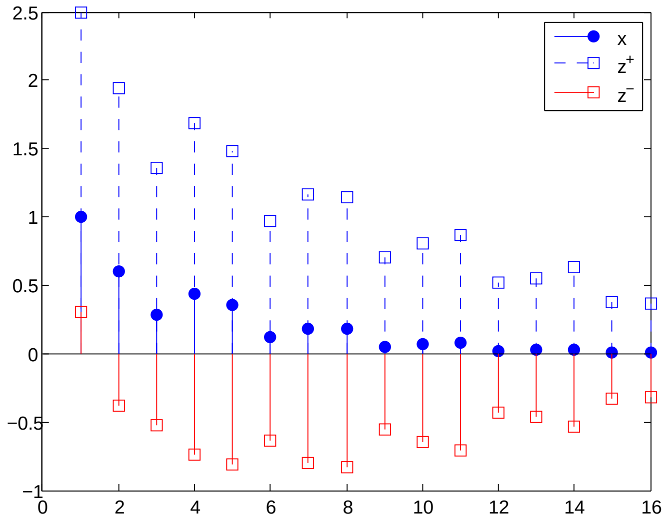

FIGURE 4.2: Solution in the presence of disturbances.

4.6.2 Linear system

We illustrate the construction presented in Section 4.5 by constructing step by step a time-varying interval observer for the linear system without output

$$x_{k+1} = \mathcal{A}x_k + w_k \ , k \in \mathbb{N} \ , \tag{4.64}$$

with

$$\mathcal{A} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} . \tag{4.65}$$

Since the eigenvalues of $\mathcal{A}$ are $-\frac{1}{2}$ and $-\frac{1}{2} \pm \frac{i}{2}$, the spectral radius of $\mathcal{A}$ is $\frac{\sqrt{2}}{2} < 1$. Therefore Theorem 4.10 applies to (4.64). We follow now the proof of this theorem.

Step 1 : Transformation of $\mathcal{A}$ into a matrix of Jordan form.

The equality

$$\mathcal{P}\mathcal{A}\mathcal{P}^{-1} = \mathcal{J} \ , \tag{4.66}$$

with

$$\mathcal{J} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathcal{P} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{4.67}$$

is satisfied. The Jordan matrix $\mathcal{J}$ is not nonnegative.

Step 2 : Construction of a diagonalizing time-varying change of coordinates.

For the particular case we consider, the matrix $\overline{\mathcal{Q}}_k$ in (A.48) is

$$\overline{\mathcal{Q}}_k = \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & \cos\left(\frac{5\pi}{4}k\right) & \sin\left(\frac{5\pi}{4}k\right) \\ 0 & -\sin\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) \end{bmatrix} , \tag{4.68}$$

the matrix $\mathcal{R}_k = \overline{\mathcal{Q}}_k\mathcal{P}$ is

$$\mathcal{R}_k = \begin{bmatrix} (-1)^k & 0 & 0 \\ -\cos\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) & \sin\left(\frac{5\pi}{4}k\right) \\ \sin\left(\frac{5\pi}{4}k\right) & -\sin\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) \end{bmatrix} \tag{4.69}$$

and

$$\mathcal{S}_k = \mathcal{R}_k^{-1} = \begin{bmatrix} (-1)^k & 0 & 0 \\ (-1)^k & \cos\left(\frac{5\pi}{4}k\right) & -\sin\left(\frac{5\pi}{4}k\right) \\ 0 & \sin\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) \end{bmatrix} . \tag{4.70}$$

One can check readily that, for all $k \in \mathbb{N}$,

$$|\mathcal{R}_k| + |\mathcal{S}_k| < 6 \tag{4.71}$$

and

$$\mathcal{E} = \mathcal{R}_{k+1}\mathcal{A}\mathcal{R}_k^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} . \tag{4.72}$$

The matrix $\mathcal{E}$ is a nonnegative matrix whose spectral radius is smaller than 1.

Step 3 : Interval observer.

According to Theorem 4.10, the system (4.64) admits the following interval observer

$$\begin{array}{rcl} z_{k+1}^+ &=& \mathcal{E}z_k^+ + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^- , \\ z_{k+1}^- &=& \mathcal{E}z_k^- + \mathcal{R}_{k+1}^+ w_k^- - \mathcal{R}_{k+1}^- w_k^+ , \\ z_{k_0}^+ &=& \mathcal{R}_{k_0}^+ x_{k_0}^+ - \mathcal{R}_{k_0}^- x_{k_0}^- , \\ z_{k_0}^- &=& \mathcal{R}_{k_0}^+ x_{k_0}^- - \mathcal{R}_{k_0}^- x_{k_0}^+ , \\ x_k^+ &=& \mathcal{S}_k^+ z_k^+ - \mathcal{S}_k^- z_k^- , \\ x_k^- &=& \mathcal{S}_k^+ z_k^- - \mathcal{S}_k^- z_k^+ , \end{array} \tag{4.73}$$

with $\mathcal{E}$ defined in (4.72),

$$\mathcal{R}_k^+ = \begin{bmatrix} \varpi\left((-1)^k\right) & 0 & 0 \\ \varpi\left(-\cos\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\sin\left(\frac{5\pi}{4}k\right)\right) \\ \varpi\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(-\sin\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{4.74}$$

$$\mathcal{S}_k^+ = \begin{bmatrix} \varpi\left((-1)^k\right) & 0 & 0 \\ \varpi\left((-1)^k\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(-\sin\left(\frac{5\pi}{4}k\right)\right) \\ 0 & \varpi\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{4.75}$$

$$\mathcal{R}_k^- = -\begin{bmatrix} \mho\left((-1)^k\right) & 0 & 0 \\ \mho\left(-\cos\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\sin\left(\frac{5\pi}{4}k\right)\right) \\ \mho\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \mho\left(-\sin\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{4.76}$$

and

$$\mathcal{S}_k^- = -\begin{bmatrix} \mho\left((-1)^k\right) & 0 & 0 \\ \mho\left((-1)^k\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \mho\left(-\sin\left(\frac{5\pi}{4}k\right)\right) \\ 0 & \mho\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{4.77}$$

We present simulations.

Figure 4.3 shows the second component x and the bounds provided by the interval observer in the absence of disturbances.

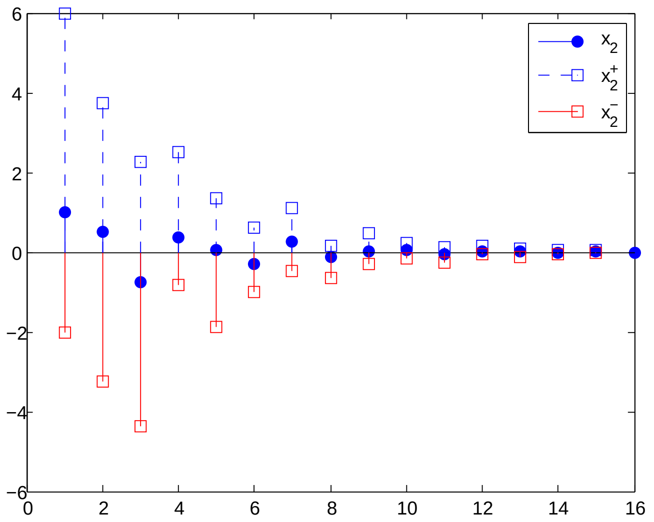

FIGURE 4.3: Solution in the absence of disturbances.

Figure 4.4 shows the second component x and the bounds provided by the interval observer in the presence of disturbances:

$$w_k = \begin{pmatrix} 0 \\ 2\sin(3k) \\ 0 \end{pmatrix},$$

with the bounds

$$w_k^+ = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \; w_k^- = \begin{pmatrix} 0 \\ 2\sin(3k) - 1 \\ 0 \end{pmatrix}.$$

In both cases the selected initial conditions are

$$x(1) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \; x^+(1) = \begin{pmatrix} 1.5 \\ 6 \\ 4 \end{pmatrix}, \; x^-(1) = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

The behavior of the other components of the solutions is similar. The simulations confirm the theoretic result: in the absence of disturbance, there is convergence of the two bounds of the solution and in the presence of disturbance, the bounds do not necessarily converge to the same value.

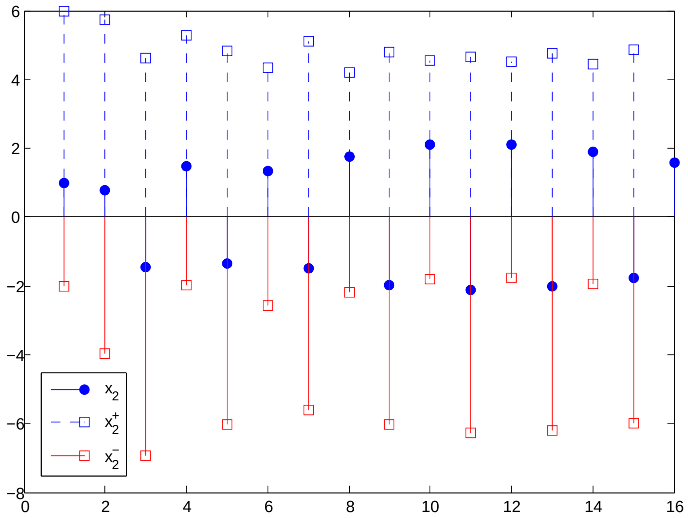

FIGURE 4.4: Solution in the presence of disturbances.

4.7 Conclusion

We have proposed a technique of construction of time-invariant interval observers for a family of nonlinear discrete-time time-invariant systems and a technique of construction of time-varying interval observers for discrete-time linear time-invariant systems. Much remains to be done. A discrete-time version of the contribution [99], which is devoted to linear systems with delays, can be expected. Extensions of the results of the present work to families of discrete-time nonlinear systems with delays can be expected too.

Chapter 5

Robust Interval Observers and Stabilization Design for Discrete–time Systems With Input and Output

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5.1 Introduction

The present chapter complements [38] and our work in Chapter 4. In Chapter [38], interval observers are constructed for families of time-varying discrete-time systems without inputs and in Chapter 4, interval observers for two important families of discrete-time systems are proposed. The first is composed of time-invariant nonlinear systems which possess specific stability and monotonicity properties. The second is the general family of the linear time-invariant exponentially stable systems. We established that these systems can be transformed into *nonnegative*[9] and *exponentially stable time-invariant* systems through linear, possibly time-varying, changes of coordinates. Using this key result, interval observers

9. See Section 1.5.1 of Chapter 1 for the definition of nonnegative system.

for linear systems without input and an output have been constructed, under an appropriate detectability assumption. Such constructions are based on some dynamic extension, which is nonnegative when the output is identically equal to zero.

We consider, in this chapter, a nonlinear system of the form:

$$\begin{cases} x_{k+1} &= [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}u_k + \Phi(y_k) + \nu_k, \\ y_k &= \mathcal{C}x_k, \end{cases}$$

with $k \in \mathbb{N}$, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times q}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$, where $\mathcal{A}_d : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is an unknown bounded nonlinear function, ν_k is an additive disturbance and $\Phi : \mathbb{R}^p \to \mathbb{R}^n$ is a known nonlinear function. We give some sufficient conditions ensuring that the system can be exponentially stabilized by a dynamic output feedback depending on y_k and the values provided by the bounds of the interval observer. Similar to Chapter 3, the proposed interval observer consists of two copies of Luenberger observers ([83]) with additional terms taking into account the presence of the uncertainties and of Φ endowed with appropriate outputs (the 'bounds' of the interval observer). This result may sound surprising because such observers or their associated error equations do not possess the property of being nonnegative systems, although this property is usually used when constructing interval observers (see, for example, [3, 38, 50, 95, 98] and the discussions therein). In fact, the notion of nonnegative system will be used as well, but only indirectly to select appropriate initial conditions and upper and lower bounds for the interval observer. It is worth noticing that the idea of taking advantage of interval observers to design stabilizing control laws is not new: in particular, it is used in [38] to stabilize nonlinear systems. However, to the best of the authors' knowledge, there do not exist any similar results in the literature to the one presented in this contribution, even for continuous time systems.

As presented in Chapter 3, the main advantage of the proposed approach is that it makes it possible to let classical observers play simultaneously the role of observers and interval observers and therefore the introduction of extra dynamics with some nonnegativity property is not explicitly needed. Moreover, since the choice of initial conditions and bounding outputs for the interval observer is not unique, this technique may be used to construct a bundle of interval observers, as done for instance in [15], without having to introduce extra dynamics. Thus, better estimates can be obtained without having to consider interval observers of dimension larger than twice the dimension of the studied system. Incidentally, it is worth mentioning that a single Luenberger observer cannot be the dynamics of an interval observer : usually the dimension of the interval observer is necessarily strictly larger than the dimension of the studied system since the upper and lower estimate of the state interval are generated by an observer.

The chapter is organized as follows. Notation, definitions and prerequisites are given in Section 5.2. In Section 5.3 a general construction of interval observer is presented. An illustrative example is proposed in Section 5.4. Concluding remarks are drawn in Section 5.5 and end the chapter.

5.2 Notation, definitions and prerequisites

Similar to previous chapters, let us resume some notations, definitions and prerequisites which will be used in the sequel.

- The notation will be simplified whenever no confusion can arise from the context.
- Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0.
- The Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimension are denoted $|\cdot|$.
- All the inequalities must be understood *componentwise* (see Definition 1.27 of Chapter 1).
- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive (resp. negative) semidefinite if for all vectors $v \in \mathbb{R}^n$, $v^\top Av \geq 0$ (resp. $v^\top Av \leq 0$). Then we denote $A \succeq 0$ (resp. $A \preceq 0$).
- For two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension, $\max\{A, B\}$ is the matrix where each entry is $m_{ij} = \max\{a_{ij}, b_{ij}\}$.
- For a matrix $A \in \mathbb{R}^{r\times s}$, $A^+ = \max\{A, 0\}$, $A^- = \max\{-A, 0\}$.
- A matrix $A \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* if $A^+ = A$ (or one can see Definition 1.32).
- A sequence (u_i) is *nonnegative* if for all integer k, u_k is nonnegative.
- A system $x_{k+1} = f(k, x_k)$ is *nonnegative* if for all integer k_0 and any initial condition $x_{k_0} \geq 0$, the solution x_k satisfies $x_k \geq 0$ for all integer $k \geq k_0$. Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then the inequality:

$$|x + y|^2 \leq 2|x|^2 + 2|y|^2 \tag{5.1}$$

  holds.

Due to the features of the systems considered in the present work, a slightly different definition of interval observer for discrete-time nonlinear time-varying systems than the one introduced in Chapter 4 is proposed. A definition of framers is also given. These two notions have been introduced, with slightly different features, in several papers (see, for instance, ([99], [51] to cite only a few).

Definition. *Consider a discrete-time system:*

$$x_{k+1} = f_1\left(k, x_k, \nu_k\right), \tag{5.2}$$

with $x_k \in \mathbb{R}^n$, $\nu_k \in \mathbb{R}^d$, an output $y_k = m(x_k) \in \mathbb{R}^p$, and where f_1 and m are two nonlinear functions. The initial condition at the instant $k_0 \in \mathbb{N}$, $x_{k_0} \in \mathbb{R}^n$ is assumed to be bounded by two known bounds:

$$x_{s,k_0} \leq x_{k_0} \leq x_{l,k_0} \tag{5.3}$$

and the disturbances ν_k are supposed to be upper and lower bounded by two known sequences ν_k^+, ν_k^-, i.e. for all $k \in \mathbb{N}$, $\nu_k^- \leq \nu_k \leq \nu_k^+$. Then, the dynamical system:

$$z_{k+1} = f_2\left(k, z_k, y_k, \nu_k^+, \nu_k^-\right), \tag{5.4}$$

associated with the initial condition

$$z_{k_0} = g(k_0, x_{l,k_0}, x_{s,k_0}) \in \mathbb{R}^{n_z}$$

and bounds for the solution x_k:

$$x_k^+ = h^+(k, z_k)\ ,\ x_k^- = h^-(k, z_k), \tag{5.5}$$

where f_2, g, h^+ and h^- are nonlinear functions, is called
(i) a <u>framer</u> for (5.2) if for any vectors x_{k_0}, x_{s,k_0} and x_{l,k_0} in $\mathbb{R}^n$ satisfying (5.3), the solutions of (5.2)-(5.4) with respectively x_{k_0}, $z_{k_0} = g(k_0, x_{l,k_0}, x_{s,k_0})$ as initial condition at $k = k_0$, denoted respectively x_k and z_k satisfy, for all $k \geq k_0$, the inequalities

$$x_k^- = h^-(k, z_k) \leq x_k \leq h^+(k, z_k) = x_k^+, \tag{5.6}$$

(ii) <u>a robust interval observer</u> for (5.2) if, in addition, there exists a function γ of class $\mathcal{K}$[10] such that, in the particular case where there is a constant $\overline{\nu} > 0$ such that for all $|\nu_k^+| + |\nu_k^-| \leq \overline{\nu}$ for all $k \in \mathbb{N}$, then any solution (x_k, z_k) of (5.2)-(5.4) is such that there exists $k_c \in \mathbb{N}$ such that, for all integer $k \geq k_c$, $|h^+(k, z_k) - h^-(k, z_k)| \leq \gamma(\overline{\nu})$.

5.3 Interval observer

In the sequel, a family of nonlinear systems is considered. It will be shown that, despite the presence of uncertainties, one can construct framers which are interval observers when they are in closed-loop with stabilizing output feedbacks that depend on the values of the bounds

10. A function $\alpha : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ if $\alpha(0) = 0$ and it is increasing.

of the framers. More precisely, we consider:

$$\begin{cases} x_{k+1} &= [\mathcal{A}+\mathcal{A}_d(x_k)]x_k+\mathcal{B}u_k+\Phi(y_k)+\nu_k, \\ y_k &= \mathcal{C}x_k, \end{cases} \tag{5.7}$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^q$ is the input and $y_k \in \mathbb{R}^p$ is the output, where Φ is a nonlinear function, where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times q}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$, $\mathcal{A}_d(x) \in \mathbb{R}^{n\times n}$ are unknown terms bounded in norm by a known constant and where ν_k are disturbances such that for two known sequences ν_k^+, ν_k^- the inequalities:

$$\nu_k^- \leq \nu_k \leq \nu_k^+ \tag{5.8}$$

are satisfied for all $k \in \mathbb{N}$.

Some assumptions are introduced.

Assumption 5.1. *There exists an invertible matrix $\mathcal{R} \in \mathbb{R}^{n\times n}$ such that*

$$\mathcal{R}\mathcal{A}\mathcal{S} = \mathcal{E}, \tag{5.9}$$

where $\mathcal{E} \in \mathbb{R}^{n\times n}$ is a nonnegative Schur stable matrix and $\mathcal{S} = \mathcal{R}^{-1}$.

Notice for later use that Assumption 5.1 guarantees the existence of a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{A}^\top Q\mathcal{A} - Q \preceq -I. \tag{5.10}$$

Assumption 5.2. *There exist a positive definite and radially unbounded continuous function $\mathfrak{U}$ and a continuous feedback θ such that, along the system*

$$\xi_{k+1} = [\mathcal{A}+\mathcal{A}_d(\xi_k)]\xi_k + \mathcal{B}\theta(\mathcal{C}\xi_k, \xi_k+d_k) + \Phi(\mathcal{C}\xi_k) + s_k, \tag{5.11}$$

where d_k and s_k are arbitrary sequences, the function $\mathfrak{U}$ satisfies, for all $k \in \mathbb{N}$,

$$\mathfrak{U}(\xi_{k+1}) - \mathfrak{U}(\xi_k) \leq -|\xi_k|^2 + \mathfrak{c}|d_k|^2 + \mathfrak{g}|s_k|^2, \tag{5.12}$$

with $\mathfrak{c} > 0$, $\mathfrak{g} > 0$. Moreover, there are two constants $0 < \mathfrak{u}_s < \mathfrak{u}_l$ such that, for all $\xi \in \mathbb{R}^n$,

$$\mathfrak{u}_s|\xi|^2 \leq \mathfrak{U}(\xi) \leq \mathfrak{u}_l|\xi|^2. \tag{5.13}$$

Assumption 5.3. *There are continuous functions $\mathfrak{P}$, $\mathfrak{Q}$ and a known constant matrix $\mathfrak{K} \geq 0$ such that, for all $x \in \mathbb{R}^n$,*

$$\mathcal{R}\mathcal{A}_d(x)\mathcal{S} = \mathfrak{P}(x) - \mathfrak{Q}(x), \tag{5.14}$$

$$0 \leq \mathfrak{P}(x) \leq \mathfrak{K}\,, \quad 0 \leq \mathfrak{Q}(x) \leq \mathfrak{K}, \tag{5.15}$$

and

$$|\mathfrak{K}| \leq \frac{1}{2|\mathcal{S}||\mathcal{R}|} \min\left\{\frac{1}{2\sqrt{\mathfrak{q}_2}}, \frac{\sqrt{3}}{\sqrt{5\mathfrak{c}(\mathfrak{q}_1 + 2\mathfrak{q}_2)}}\right\}, \tag{5.16}$$

with

$$\mathfrak{q}_1 = 2(2|Q\mathcal{A}|^2 + |Q|)\;,\; \mathfrak{q}_2 = 12|Q\mathcal{A}|^2 + 6|Q|. \tag{5.17}$$

Discussion of Theorem 5.4.

- Assumption 5.1 ensures the existence of a time-invariant change of coordinates that transforms $\mathcal{A}$ into a nonnegative Schur stable matrix. For the sake of simplicity, only this case is considered, although the general case can be handled by using the time-varying change of coordinates provided in [95].

- Assumption 5.1 implies that the matrix $\mathcal{A}$ is Schur stable. It does not follow that the linear approximation of the system we consider is asymptotically stable. In fact, Assumptions 5.1, 5.2, 5.3 do not even imply that, in the absence of $\mathcal{A}_d$, the system is globally or even locally exponentially stabilizable by a static output feedback. The example in Section 5.4 explicitly illustrates this fact. Besides, it is worth mentioning that any system $\xi_{k+1} = A\xi_k + Bu_k$ with output $y_k = C\xi_k$ such that (A, B) is stabilizable and (A, C) is observable satisfies Assumptions 5.1 to 5.3. Indeed, the observability of (A, C) ensures that there exists a matrix L such that $A + LC$ is Schur stable with distinct eigenvalues, which implies that Assumption 5.1 is satisfied (with $\mathcal{A} = A + LC$) and the stabilizability of (A, B) ensures that Assumption 5.2 is satisfied with a linear stabilizing feedback and finally, in the absence of disturbance, Assumption 5.3 is always satisfied.

- Assumption 5.2 is a stabilizability assumption by state feedback for (5.7). When (5.7) is a linear system, this assumption is always satisfied under the standard stabilizability assumption. It is also the case if Φ is of class C^1 and such that $\Phi(0) = 0$, the pair $\left(\mathcal{A} + \frac{\partial\Phi}{\partial y}(0)\mathcal{C}, \mathcal{B}\right)$ is stabilizable, the pair (A, C) is detectable and there exists a function Ω such that, for all $y \in \mathbb{R}^p$, $\Phi(y) - \frac{\partial\Phi}{\partial y}(0)y = \mathcal{B}\Omega(y)$. The requirement (5.13) is often satisfied in practice for systems of the type (5.11) and this assumption can be relaxed, but with the cost of an increased amount of complexity. So, for the sake of simplicity, this requirement is imposed.

- Assumption 5.3 is a restriction imposed on the unknown terms. Such a restriction is used to establish the stability of (5.7)-(5.18) in closed-loop with (5.21). The system (5.18), associated to (5.19) and (5.20) is a framer for the system (5.7) for any input. But it is an interval observer only when the system (5.7) is in closed-loop with suitably chosen stabilizing feedbacks. Of course, feedbacks different from (5.21) can be used: for instance one can choose $u(\hat{x}_k^-, y_k) = \theta(y_k, \hat{x}_k^-)$ or any convex combination of $\theta(y_k, \hat{x}_k^+)$ and $\theta(y_k, \hat{x}_k^-)$. When $\mathcal{A}_d(x)$ is bounded in norm, finding functions $\mathfrak{P}(x)$, $\mathfrak{Q}(x)$ such that Assumption 5.3 is satisfied is an easy task. Indeed, assume that all the entries of $\mathcal{R}\mathcal{A}_d(x)\mathcal{S}$ are bounded in norm by constants $\mathfrak{p}_{i,j} \geq 0$. Let $\mathfrak{P}(x)$ be the matrix whose entries are $\mathfrak{p}_{i,j}$. Then $\mathcal{R}\mathcal{A}_d(x)\mathcal{S} = \mathfrak{P}(x) - \mathfrak{Q}(x)$ with $\mathfrak{Q}(x) = \mathfrak{P}(x) - \mathcal{R}\mathcal{A}_d(x)\mathcal{S}$. Moreover, $\mathfrak{P}(x) \geq 0$, $\mathfrak{Q}(x) \geq 0$ and $\mathfrak{P}(x) \leq \mathfrak{K}$, $\mathfrak{Q}(x) \leq \mathfrak{K}$ with $\mathfrak{K} = 2\mathfrak{P}(x)$. Conversely, when $\mathfrak{P}$ and $\mathfrak{Q}$ are bounded, necessarily $\mathcal{A}_d$ is bounded.

We have the following result :

Theorem 5.4. *Let the system (5.7) satisfy Assumptions 5.1 to 5.3. Then the system :*

$$\left\{\begin{array}{rcl} \hat{x}_{k+1}^+ & = & \mathcal{A}\hat{x}_k^+ + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}\mathfrak{K}[\max\{0, \mathcal{R}\hat{x}_k^+\} - \min\{0, \mathcal{R}\hat{x}_k^-\}] \\ & & +\mathcal{S}[\mathcal{R}^+\nu_k^+ - \mathcal{R}^-\nu_k^-], \\ \hat{x}_{k+1}^- & = & \mathcal{A}\hat{x}_k^- + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}\mathfrak{K}[\min\{0, \mathcal{R}\hat{x}_k^-\} - \max\{0, \mathcal{R}\hat{x}_k^+\}] \\ & & +\mathcal{S}[\mathcal{R}^+\nu_k^- - \mathcal{R}^-\nu_k^+], \end{array}\right. \tag{5.18}$$

associated with the initial conditions

$$\begin{array}{rcl} \hat{x}_{k_0}^+ & = & \mathcal{S}\left[\mathcal{R}^+ x_{l,k_0} - \mathcal{R}^- x_{s,k_0}\right], \\ \hat{x}_{k_0}^- & = & \mathcal{S}\left[\mathcal{R}^+ x_{s,k_0} - \mathcal{R}^- x_{l,k_0}\right], \end{array} \tag{5.19}$$

the bounds for the solutions x_k

$$\begin{array}{rcl} x_k^+ & = & \mathcal{S}^+\mathcal{R}\hat{x}_k^+ - \mathcal{S}^-\mathcal{R}\hat{x}_k^-, \\ x_k^- & = & \mathcal{S}^+\mathcal{R}\hat{x}_k^- - \mathcal{S}^-\mathcal{R}\hat{x}_k^+, \end{array} \tag{5.20}$$

is a robust interval observer for the system (5.7) when in closed-loop with the feedback

$$u(\hat{x}_k^+, y_k) = \theta(y_k, \hat{x}_k^+). \tag{5.21}$$

Proof. The proof splits up into two parts. The first establishes that the system (5.18) with suitable initial conditions and bounds is a framer for the system (5.7) for any sequence of inputs u_k. The second is devoted to the stability analysis of the systems (5.7)-(5.18) in closed-loop with the dynamic output feedback (5.21).

1. *Property of framer.*

Let us prove that (5.18) is a framer for the system (5.7) with the initial conditions (5.19) and the bounds for the solutions x_k given in (5.20). For an initial instant $k_0 \in \mathbb{N}$, consider vectors $x_{k_0}, x_{l,k_0}, x_{s,k_0}$ in $\mathbb{R}^n$ such that $x_{s,k_0} \leq x_{k_0} \leq x_{l,k_0}$. Then, by Lemma A.5,

$$\mathcal{R}\hat{x}^-_{k_0} \leq \mathcal{R}x_{k_0} \leq \mathcal{R}\hat{x}^+_{k_0}, \tag{5.22}$$

where $\hat{x}^+_{k_0}$, $\hat{x}^-_{k_0}$ are the vectors defined in (5.19). Next, observe that the solutions x_k, $\hat{x}^+_k$, $\hat{x}^-_k$ of the systems (5.7) and (5.18) with the initial conditions x_{k_0}, $\hat{x}^+_{k_0}$, $\hat{x}^-_{k_0}$ selected above satisfy

$$\left\{\begin{array}{rcl} x_{k+1} &=& [\mathcal{A}+\mathcal{A}_d(x_k)]x_k + \mathcal{B}u_k + \Phi(y_k) + \nu_k, \\ \hat{x}^+_{k+1} &=& \mathcal{A}\hat{x}^+_k + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}\rho_1(\nu^+_k,\nu^-_k) + \mathcal{S}\mathfrak{K}[\max\{0,\mathcal{R}\hat{x}^+_k\} - \min\{0,\mathcal{R}\hat{x}^-_k\}], \\ \hat{x}^-_{k+1} &=& \mathcal{A}\hat{x}^-_k + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}\rho_2(\nu^+_k,\nu^-_k) + \mathcal{S}\mathfrak{K}[\min\{0,\mathcal{R}\hat{x}^-_k\} - \max\{0,\mathcal{R}\hat{x}^+_k\}], \end{array}\right. \tag{5.23}$$

with $\rho_1(\nu^+_k,\nu^-_k) = \mathcal{R}^+\nu^+_k - \mathcal{R}^-\nu^-_k$ and $\rho_2(\nu^+_k,\nu^-_k) = \mathcal{R}^+\nu^-_k - \mathcal{R}^-\nu^+_k$. It follows that

$$\left\{\begin{array}{rcl} \mathcal{R}x_{k+1} &=& \mathcal{R}\mathcal{A}x_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + \mathcal{R}\mathcal{A}_d(x_k)x_k + \mathcal{R}\nu_k\ , \\ \mathcal{R}\hat{x}^+_{k+1} &=& \mathcal{R}\mathcal{A}\hat{x}^+_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + \mathfrak{K}[\max\{0,\mathcal{R}\hat{x}^+_k\} - \min\{0,\mathcal{R}\hat{x}^-_k\}] \\ && +\rho_1(\nu^+_k,\nu^-_k), \\ \mathcal{R}\hat{x}^-_{k+1} &=& \mathcal{R}\mathcal{A}\hat{x}^-_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + \mathfrak{K}[\min\{0,\mathcal{R}\hat{x}^-_k\} - \max\{0,\mathcal{R}\hat{x}^+_k\}] \\ && +\rho_2(\nu^+_k,\nu^-_k). \end{array}\right. \tag{5.24}$$

Since Assumption 5.1 guarantees that $\mathcal{R}\mathcal{A} = \mathcal{E}\mathcal{R}$ with the notation $z_k = \mathcal{R}x_k$, $\hat{z}^+_k = \mathcal{R}\hat{x}^+_k$, $\hat{z}^-_k = \mathcal{R}\hat{x}^-_k$, we obtain:

$$\left\{\begin{array}{rcl} z_{k+1} &=& \mathcal{E}z_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + \mathcal{R}\mathcal{A}_d(x_k)\mathcal{S}z_k + \mathcal{R}\nu_k, \\ \hat{z}^+_{k+1} &=& \mathcal{E}\hat{z}^+_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + \mathfrak{K}[\max\{0,\hat{z}^+_k\} - \min\{0,\hat{z}^-_k\}] + \rho_1(\nu^+_k,\nu^-_k), \\ \hat{z}^-_{k+1} &=& \mathcal{E}\mathcal{R}\hat{x}^-_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + \mathfrak{K}[\min\{0,\hat{z}^-_k\} - \max\{0,\hat{z}^+_k\}] + \rho_2(\nu^+_k,\nu^-_k). \end{array}\right. \tag{5.25}$$

According to Assumption 5.3, the z_k-subsystem rewrites as

$$z_{k+1} \quad = \quad \mathcal{E}z_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) + [\mathfrak{P}(x_k) - \mathfrak{Q}(x_k)]z_k + \mathcal{R}\nu_k. \tag{5.26}$$

Let $w^+_k = \hat{z}^+_k - z_k$, $w^-_k = z_k - \hat{z}^-_k$. Then, from (5.25) and (5.26), it follows that

$$\left\{\begin{array}{rcl} w^+_{k+1} &=& \mathcal{E}w^+_k + \mathfrak{K}[\max\{0,\hat{z}^+_k\} - \min\{0,\hat{z}^-_k\}] - \mathfrak{P}(x_k)z_k + \mathfrak{Q}(x_k)z_k \\ && +\mathcal{R}^+\nu^+_k - \mathcal{R}^-\nu^-_k - \mathcal{R}\nu_k, \\ w^-_{k+1} &=& \mathcal{E}w^-_k - \mathfrak{K}[\min\{0,\hat{z}^-_k\} - \max\{0,\hat{z}^+_k\}] + \mathfrak{P}(x_k)z_k - \mathfrak{Q}(x_k)z_k \\ && +\mathcal{R}\nu_k - \mathcal{R}^+\nu^-_k + \mathcal{R}^-\nu^+_k. \end{array}\right.$$

Next, by reorganizing the terms, we obtain

$$\left\{\begin{array}{rcl} w_{k+1}^{+} & = & \mathcal{E}w_k^{+} + \mathfrak{K}\max\{0,\hat{z}_k^{+}\} - \mathfrak{P}(x_k)z_k + \mathfrak{Q}(x_k)z_k - \mathfrak{K}\min\{0,\hat{z}_k^{-}\} \\ & & +\mathcal{R}^{+}(\nu_k^{+}-\nu_k) + \mathcal{R}^{-}(\nu_k-\nu_k^{-}), \\ w_{k+1}^{-} & = & \mathcal{E}w_k^{-} + \mathfrak{P}(x_k)z_k - \mathfrak{K}\min\{0,\hat{z}_k^{-}\} + \mathfrak{K}\max\{0,\hat{z}_k^{+}\} - \mathfrak{Q}(x_k)z_k \\ & & +\mathcal{R}^{+}(\nu_k-\nu_k^{-}) + \mathcal{R}^{-}(\nu_k^{+}-\nu_k). \end{array}\right. \tag{5.27}$$

Now, we prove by induction that for all $k \geq k_0$,

$$\hat{z}_k^{-} \leq z_k \leq \hat{z}_k^{+}. \tag{5.28}$$

According to (5.22), the property is satisfied at the instant k_0. Assume that there exists $j > k_0$ such that, for all $i \in \{k_0, ..., j-1\}$, $\hat{z}_i^{-} \leq z_i \leq \hat{z}_i^{+}$. Since, for all $x \in \mathbb{R}^n$, $0 \leq \mathfrak{P}(x) \leq \mathfrak{K}$, it follows that, for all $i \in \{k_0, ..., j-1\}$,

$$\mathfrak{P}(x_i)\max\{0,z_i\} \leq \mathfrak{K}\max\{0,z_i\} \leq \mathfrak{K}\max\{0,\hat{z}_i^{+}\},$$

$$-\mathfrak{P}(x_i)\min\{0,z_i\} \leq -\mathfrak{K}\min\{0,z_i\} \leq -\mathfrak{K}\min\{0,\hat{z}_i^{-}\}.$$

Therefore $\mathfrak{K}\min\{0,\hat{z}_i^{-}\} \leq \mathfrak{P}(x_i)z_i \leq \mathfrak{K}\max\{0,\hat{z}_i^{+}\}$. Similarly, $\mathfrak{K}\min\{0,\hat{z}_i^{-}\} \leq \mathfrak{Q}(x_i)z_i \leq \mathfrak{K}\max\{0,\hat{z}_i^{+}\}$. Consequently,

$$\mathfrak{K}\max\{0,\hat{z}_i^{+}\} - \mathfrak{P}(x_i)z_i + \mathfrak{Q}(x_i)z_i - \mathfrak{K}\min\{0,\hat{z}_i^{-}\} \geq 0 \tag{5.29}$$

and

$$\mathfrak{P}(x_i)z_i - \mathfrak{K}\min\{0,\hat{z}_i^{-}\} + \mathfrak{K}\max\{0,\hat{z}_i^{+}\} - \mathfrak{Q}(x_i)z_i \geq 0. \tag{5.30}$$

From (5.29), (5.30), the inequalities $\nu_k - \nu_k^{-} \geq 0$, $\nu_k^{+} - \nu_k \geq 0$ and (5.27), we deduce that $w_j^{+} \geq 0$ and $w_j^{-} \geq 0$ and therefore

$$\hat{z}_j^{-} \leq z_j \leq \hat{z}_j^{+}.$$

Therefore the induction assumption is satisfied at the step $j+1$. We deduce that, for all $k \geq k_0$, the inequalities $\mathcal{R}\hat{x}_k^{-} \leq \mathcal{R}x_k \leq \mathcal{R}\hat{x}_k^{+}$ hold.

From Lemma A.5, it follows that, for all $k \geq k_0$,

$$\mathcal{S}^{+}\mathcal{R}\hat{x}_k^{-} - \mathcal{S}^{-}\mathcal{R}\hat{x}_k^{-} \leq x_k \leq \mathcal{S}^{+}\mathcal{R}\hat{x}_k^{+} - \mathcal{S}^{-}\mathcal{R}\hat{x}_k^{-}. \tag{5.31}$$

Thus, for all integer $k \geq k_0$,

$$x_k^{-} \;\leq\; x_k \;\leq\; x_k^{+}. \tag{5.32}$$

2. *Stability of the system (5.7) - (5.18) - (5.21).*

In the absence of additive disturbances, the closed loop system is:

$$\left\{\begin{array}{lcl}\hat{x}^+_{k+1} & = & \mathcal{A}\hat{x}^+_k + \mathcal{S}\mathfrak{K}[\max\{0,\mathcal{R}\hat{x}^+_k\} - \min\{0,\mathcal{R}\hat{x}^-_k\}] + \mathcal{B}\theta(y_k,\hat{x}^+_k) + \Phi(y_k),\\ x_{k+1} & = & [\mathcal{A}+\mathcal{A}_d(x_k)]x_k + \mathcal{B}\theta(y_k,\hat{x}^+_k) + \Phi(y_k),\\ \hat{x}^-_{k+1} & = & \mathcal{A}\hat{x}^-_k + \mathcal{S}\mathfrak{K}[\min\{0,\mathcal{R}\hat{x}^-_k\} - \max\{0,\mathcal{R}\hat{x}^+_k\}] + \mathcal{B}\theta(y_k,\hat{x}^+_k) + \Phi(y_k).\end{array}\right. \tag{5.33}$$

Let $p_k = \hat{x}^+_k - x_k$ and $q_k = \hat{x}^+_k - \hat{x}^-_k$. Then

$$\left\{\begin{array}{lcl}q_{k+1} & = & \mathcal{A}q_k + 2\mathfrak{F}(r_k),\\ p_{k+1} & = & \mathcal{A}p_k + \mathfrak{G}(r_k),\\ x_{k+1} & = & [\mathcal{A}+\mathcal{A}_d(x_k)]x_k + \mathcal{B}\theta(y_k,x_k+p_k) + \Phi(y_k),\end{array}\right. \tag{5.34}$$

with $r_k = (p_k,q_k,x_k)$, $\mathfrak{F}(r_k) = \mathcal{S}\mathfrak{K}[\max\{0,\mathcal{R}(p_k+x_k)\} - \min\{0,\mathcal{R}(p_k+x_k-q_k)\}]$, $\mathfrak{G}(r_k) = -\mathcal{A}_d(x_k)x_k + \mathfrak{F}(r_k)$. To establish the asymptotic stability of the system (5.34), consider first the positive definite quadratic function

$$\mathcal{V}(p,q) = p^\top Qp + q^\top Qq. \tag{5.35}$$

From Lemma A.6, it follows that

$$\begin{array}{lcl}\Delta\mathcal{V}_k & \leq & -|p_k|^2 - |q_k|^2 + \mathfrak{G}(r_k)^\top Q\mathfrak{G}(r_k) + 2p_k^\top\mathcal{A}^\top Q\mathfrak{G}(r_k) + 4\mathfrak{F}(r_k)^\top Q\mathfrak{F}(r_k)\\ & & +4q_k^\top\mathcal{A}^\top Q\mathfrak{F}(r_k),\end{array} \tag{5.36}$$

with the simplifying notation $\Delta\mathcal{V}_k = \mathcal{V}(p_{k+1},q_{k+1}) - \mathcal{V}(p_k,q_k)$. The triangle inequality gives:

$$\begin{array}{lcl}\Delta\mathcal{V}_k & \leq & -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + \mathfrak{G}(r_k)^\top Q\mathfrak{G}(r_k) + 2|Q\mathcal{A}|^2|\mathfrak{G}(r_k))|^2 + 4\mathfrak{F}(r_k)^\top Q\mathfrak{F}(r_k)\\ & & +8|Q\mathcal{A}|^2|\mathfrak{F}(r_k)|^2.\end{array} \tag{5.37}$$

It follows that:

$$\begin{array}{lcl}\Delta\mathcal{V}_k & \leq & -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + (2|Q\mathcal{A}|^2+|Q|)|\mathcal{A}_d(x_k)x_k - \mathfrak{F}(r_k)|^2\\ & & +4(2|Q\mathcal{A}|^2+|Q|)|\mathfrak{F}(r_k)|^2\\ & \leq & -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + \mathfrak{q}_1|\mathcal{A}_d(x_k)|^2|x_k|^2 + \mathfrak{q}_2|\mathfrak{F}(r_k)|^2,\end{array} \tag{5.38}$$

with $\mathfrak{q}_1$, $\mathfrak{q}_2$ defined in (5.17). Now, observe that Assumption 5.3 implies that, for all $x \in \mathbb{R}^n$, $|\mathfrak{P}(x) - \mathfrak{Q}(x)| \leq |\mathfrak{K}|$. Finally, we deduce that

$$|\mathcal{A}_d(x)| \leq \mathfrak{q}_3|\mathfrak{K}|, \tag{5.39}$$

with $\mathfrak{q}_3 = |\mathcal{S}||\mathcal{R}|$. Besides, for all $r \in \mathbb{R}^{3n}$

$$|\mathfrak{F}(r)| \leq |\mathcal{S}||\mathcal{R}||\mathfrak{K}|[|p+x|+|q|]. \tag{5.40}$$

Therefore

$$|\mathfrak{F}(r_k)|^2 \leq 2\mathfrak{q}_3^2|\mathfrak{K}|^2[(|p_k|+|q_k|)^2+|x_k|^2]. \tag{5.41}$$

It follows that

$$\begin{array}{rcl}\Delta\mathcal{V}_k & \leq & -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + \mathfrak{q}_1\mathfrak{q}_3^2|\mathfrak{K}|^2|x_k|^2 + 2\mathfrak{q}_2\mathfrak{q}_3^2|\mathfrak{K}|^2[(|p_k|+|q_k|)^2+|x_k|^2] \\ & \leq & -\frac{1}{2}|p_k|^2 - \frac{1}{2}|q_k|^2 + \mathfrak{q}_4|\mathfrak{K}|^2|x_k|^2 + 4\mathfrak{q}_2\mathfrak{q}_3^2|\mathfrak{K}|^2(|p_k|^2+|q_k|^2),\end{array} \tag{5.42}$$

with $\mathfrak{q}_4 = \mathfrak{q}_1\mathfrak{q}_3^2 + 2\mathfrak{q}_2\mathfrak{q}_3^2$. Since the inequality $|\mathfrak{K}| \leq \frac{1}{2|\mathcal{S}||\mathcal{R}|}\frac{1}{2\sqrt{\mathfrak{q}_2}}$ in Assumption 5.3 implies $|\mathfrak{K}|^2 \leq \frac{1}{16\mathfrak{q}_2\mathfrak{q}_3^2}$, the inequality

$$\Delta\mathcal{V}_k \quad \leq \quad -\tfrac{1}{4}|p_k|^2 - \tfrac{1}{4}|q_k|^2 + \mathfrak{q}_4|\mathfrak{K}|^2|x_k|^2 \tag{5.43}$$

holds. Now, observe that x_k satisfies:

$$x_{k+1} = [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}\theta(\mathcal{C}x_k, x_k+p_k) + \Phi(\mathcal{C}x_k).$$

From Assumption 5.2, it straightforwardly follows that

$$\mathfrak{U}(x_{k+1}) - \mathfrak{U}(x_k) \quad \leq \quad -|x_k|^2 + \mathfrak{c}|p_k|^2. \tag{5.44}$$

The inequalities (5.43) and (5.44) lead us to consider

$$\mathcal{W}(r) = \mathfrak{U}(x) + 5\mathfrak{c}\mathcal{V}(p,q). \tag{5.45}$$

This function is positive definite and radially unbounded and

$$\Delta\mathcal{W}_k \leq \left(-1 + 5\mathfrak{c}\mathfrak{q}_4|\mathfrak{K}|^2\right)|x_k|^2 - \frac{\mathfrak{c}}{4}|p_k|^2 - \frac{5\mathfrak{c}}{4}|q_k|^2, \tag{5.46}$$

with $\Delta\mathcal{W}_k = \mathcal{W}(r_{k+1}) - \mathcal{W}(r_k)$. Finally, observe that the inequality $\mathfrak{K} \leq \frac{1}{2|\mathcal{S}||\mathcal{R}|}\frac{\sqrt{3}}{\sqrt{5\mathfrak{c}(\mathfrak{q}_1+2\mathfrak{q}_2)}}$ in Assumption 5.3 implies that

$$|\mathfrak{K}|^2 \leq \frac{3}{20\mathfrak{c}\mathfrak{q}_4}, \tag{5.47}$$

which leads to the inequality

$$\Delta\mathcal{W}_k \leq -\frac{1}{4}|x_k|^2 - \frac{\mathfrak{c}}{4}|p_k|^2 - \frac{5\mathfrak{c}}{4}|q_k|^2. \tag{5.48}$$

Now, bearing in mind Assumption 5.2, we deduce easily that, when the disturbances are present, there is a constant $\mathfrak{n} > 0$ such that

$$\Delta\mathcal{W}_k \leq -\tfrac{1}{4}|x_k|^2 - \tfrac{\mathfrak{c}}{4}|p_k|^2 - \tfrac{5\mathfrak{c}}{4}|q_k|^2 + \mathfrak{n}(|x_k| + |p_k| + |q_k|)(|\nu_k^+| + |\nu_k^-|) + \mathfrak{g}|\nu_k|^2. \tag{5.49}$$

The triangle inequality implies that

$$\Delta\mathcal{W}_k \leq -\tfrac{1}{8}|x_k|^2 - \tfrac{\mathfrak{c}}{8}|p_k|^2 - \tfrac{\mathfrak{c}}{4}|q_k|^2 + 3\left[2 + \tfrac{3}{\mathfrak{c}}\right]\mathfrak{n}^2(|\nu_k^+| + |\nu_k^-|)^2 + \mathfrak{g}|\nu_k|^2. \tag{5.50}$$

The inequality (5.13) implies that there is a constant $\mathfrak{w}$ such that

$$\frac{1}{8}|x_k|^2 + \frac{\mathfrak{c}}{8}|p_k|^2 + \frac{\mathfrak{c}}{4}|q_k|^2 \geq \mathfrak{w}\mathcal{W}(r_k)$$

and since $|\nu_k| \leq |\nu_k^+| + |\nu_k^-|$, there is a constant $\mathfrak{y} > 0$ such that

$$\Delta\mathcal{W}_k \leq -\mathfrak{w}\mathcal{W}(r_k) + \mathfrak{y}(|\nu_k^+| + |\nu_k^-|)^2. \tag{5.51}$$

Assume that the sequence $|\nu_k^+| + |\nu_k^-|$ is bounded by a constant $\overline{\nu} > 0$. By using (5.51) one can prove that there is an integer k_0 such that, for all integer $k \geq k_0$, $\mathcal{W}(r_k) \leq 2\frac{\mathfrak{y}}{\mathfrak{w}}\overline{\nu}^2$. This inequality and (5.13) allow us to conclude. □

5.4 Illustrative example

In this section, Theorem 5.4 is illustrated through a system which is the model of some electromechanical system described in [30] after discretization [11]:

$$\left\{\begin{array}{rcl} x_{1,k+1} &=& x_{1,k} + hx_{2,k}, \\ x_{2,k+1} &=& x_{2,k} + hb_1x_{3,k} - h(a_1 + \varsigma_k)\sin(x_{1,k}) - ha_2x_{2,k}, \\ x_{3,k+1} &=& x_{3,k} + hb_0u_k - ha_3x_{2,k} - ha_4x_{3,k}, \\ y_k &=& x_{1,k}\ . \end{array}\right. \tag{5.52}$$

We let $b_0 = 40, b_1 = 15, a_1 = 35, a_2 = \frac{1}{4}, a_3 = 36$, and $a_4 = 200$. These values are close to the numerical values given in [30]. To illustrate the robustness of the proposed interval observers, the sequence (ς_j) is supposed to be unknown and such that for all $j \in \mathbb{N}$, $|\varsigma_j| \leq \overline{\varsigma}$,

11. Section 3.4 of Chapter 3 presents an interval observer design for continuous-time version of this model.

where $\bar{\varsigma}$ is a known constant. The system rewrites as:

$$\begin{cases} x_{1,k+1} &= hx_{2,k} + y_k, \\ x_{2,k+1} &= g_2 x_{2,k} + hb_1 x_{3,k} - ha_1 \sin(y_k) + \nu_k, \\ x_{3,k+1} &= -ha_3 x_{2,k} + g_3 x_{3,k} + hb_0 u_k, \end{cases} \tag{5.53}$$

with $g_2 = 1 - ha_2$, $g_3 = 1 - 200h$ and $\nu_k = -h\varsigma_k \sin(x_{1,k})$. Using the notation of Section 5.3 and selecting $h = \frac{1}{200}$, the following choices can be made: $\mathcal{A}_d(x_k) = 0$,

$$\mathcal{A} = \begin{bmatrix} 0 & \frac{1}{200} & 0 \\ 0 & \frac{799}{800} & \frac{3}{40} \\ 0 & -\frac{9}{50} & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5} \end{bmatrix}, \ \Phi(y) = \begin{bmatrix} y \\ -\frac{35}{200}\sin(y) \\ 0 \end{bmatrix}.$$

To check that Assumptions 5.1 to 5.3 are satisfied, introduce:

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{129} & -\frac{10}{129} \\ 0 & \frac{130}{129} & \frac{10}{129} \end{bmatrix}, \ \mathcal{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -13 & -\frac{1}{10} \end{bmatrix}. \tag{5.54}$$

Then $\mathcal{S} = \mathcal{R}^{-1}$ and

$$\mathcal{R}\mathcal{A}\mathcal{S} = \mathcal{E} \quad \text{with} \quad \mathcal{E} = \begin{bmatrix} 0 & \frac{1}{200} & \frac{1}{200} \\ 0 & \frac{1421}{103200} & \frac{647}{103200} \\ 0 & \frac{103}{10320} & \frac{2033}{2064} \end{bmatrix}. \tag{5.55}$$

The inequality $\mathcal{A}^\top Q\mathcal{A} - Q \preceq -I$ with the symmetric positive definite matrix $Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 50 & 4 \\ 0 & 4 & 5 \end{bmatrix}$ is satisfied and $\mathcal{E}$ is a nonnegative Schur stable matrix. Therefore Assumption 5.1 is satisfied.

The next step consists in determining a stabilizing control law for

$$\begin{cases} \xi_{1,k+1} &= \xi_{1,k} + h\xi_{2,k}, \\ \xi_{2,k+1} &= \xi_{2,k} + hb_1\xi_{3,k} - ha_1\sin(\xi_{1,k}) - ha_2\xi_{2,k} + s_{2,k}, \\ \xi_{3,k+1} &= \xi_{3,k} + hb_0\theta(\xi_{1,k}, \xi_k + d_k) - ha_3\xi_{2,k} - ha_4\xi_{3,k}, \end{cases} \tag{5.56}$$

so that Assumption 5.2 is satisfied. The coordinates:

$$\alpha_k = \frac{1}{4}\xi_{1,k} + \xi_{2,k}\ , \ \beta_k = 15\xi_{3,k} - 35\sin(\xi_{1,k}) \tag{5.57}$$

lead to

$$\left\{\begin{array}{rcl} \xi_{2,k+1} & = & -\frac{799}{800}\xi_{2,k} + \frac{1}{200}\beta_k + s_{2,k}, \\ \alpha_{k+1} & = & \alpha_k + \frac{1}{200}\beta_k + s_{2,k}, \\ \beta_{k+1} & = & 3\theta(\xi_{1,k}, \xi_k + d_k) - \frac{27}{10}\xi_{2,k} - 35\sin\left(\xi_{1,k} + \frac{1}{200}\xi_{2,k}\right). \end{array}\right. \tag{5.58}$$

Let

$$\theta(y,\xi) \quad = \quad \frac{9}{10}\xi_2 + \frac{35\sin\left(y+\frac{1}{200}\xi_2\right)}{3} - \frac{199}{600}\left(\frac{1}{4}y + \xi_2\right). \tag{5.59}$$

Then, the system (5.58) in closed-loop with $\theta(\xi_{1,k}, \xi_k + d_k)$ is

$$\left\{\begin{array}{rcl} \xi_{2,k+1} & = & -\frac{799}{800}\xi_{2,k} + \frac{1}{200}\beta_k + s_{2,k}, \\ \alpha_{k+1} & = & \alpha_k + \frac{1}{200}\beta_k + s_{2,k}, \\ \beta_{k+1} & = & -\frac{199}{200}\alpha_k + R(\xi_k, d_k), \end{array}\right. \tag{5.60}$$

with

$$\begin{array}{rcl} R(\xi_k, d_k) & = & \frac{341}{200}d_k + 35\sin\left(\xi_{1,k} + \frac{\xi_{2,k}}{200} + \frac{d_k}{200}\right) \\ & & -35\sin\left(\xi_{1,k} + \frac{\xi_{2,k}}{200}\right). \end{array}$$

Consider the quadratic function

$$\mathfrak{W}(\alpha,\beta) = (\alpha+\beta)^2 + 2\left(\frac{199}{20}\alpha + \frac{1}{20}\beta\right)^2. \tag{5.61}$$

Then, with the simplifying notation $\Delta\mathfrak{W}_k = \mathfrak{W}(\alpha_{k+1}, \beta_{k+1}) - \mathfrak{W}(\alpha_k, \beta_k)$, one can prove through simple calculations that there are constants $\mathfrak{f}_1 > 0$, $\mathfrak{f}_2 > 0$ such that $\Delta\mathfrak{W}_k \leq -\mathfrak{f}_1\alpha_k^2 - \mathfrak{f}_2\beta_k^2$. Since the function sinus is globally Lipschitz, we deduce easily that there is a constant $\mathfrak{f}_3 > 0$ such that, when s_k and d_k are present, the inequality:

$$\Delta\mathfrak{W}_k \leq -\mathfrak{f}_1\alpha_k^2 - \mathfrak{f}_2\beta_k^2 + \mathfrak{f}_3[d_k^2 + s_k^2] \tag{5.62}$$

is satisfied. On the other hand, using the simplifying notation $\Delta\Xi_k = \xi_{2,k+1}^2 - \xi_{2,k}^2$, simple calculations give

$$\Delta\Xi_k \quad \leq \quad -\frac{799}{640000}\xi_{2,k}^2 + \frac{639201}{32000000}\beta_k^2. \tag{5.63}$$

It follows that there are constants $\mathfrak{f}_4 > 0$, $\mathfrak{f}_5 > 0$ such that

$$\mathfrak{R}(\alpha,\beta,\xi_2) = \mathfrak{f}_4\xi_2^2 + \mathfrak{W}(\alpha,\beta) \tag{5.64}$$

satisfies

$$\Delta\mathfrak{R}_k \quad \leq \quad -\mathfrak{f}_5\xi_{2,k}^2 - \mathfrak{f}_1\alpha_k^2 - \mathfrak{f}_2\beta_k^2 + \mathfrak{f}_3[d_k^2 + s_k^2], \tag{5.65}$$

with $\Delta\mathfrak{R}_k = \mathfrak{R}(\alpha_{k+1}, \beta_{k+1}, \xi_{2,k+1}) - \mathfrak{R}(\alpha_k, \beta_k, \xi_{2,k})$. Then, one can easily prove that there exists $\mathfrak{f}_6 > 0$ such that Assumption 5.2 is satisfied with $\mathfrak{U}(\xi) = \mathfrak{f}_6\mathfrak{R}(a_2\xi_1 + \xi_2, b_1\xi_3 - a_1\sin(\xi_1), \xi_2)$.

Now, observe that Assumption 5.3 is satisfied since $\mathcal{A}_d = 0$. It follows that Theorem 5.4 applies to (5.53). Consequently, the dynamic extension:

$$\begin{cases} \hat{x}_{k+1}^+ &= \mathcal{A}\hat{x}_k^+ + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}[\mathcal{R}^+\nu_k^+ - \mathcal{R}^-\nu_k^-], \\ \hat{x}_{k+1}^- &= \mathcal{A}\hat{x}_k^- + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}[\mathcal{R}^+\nu_k^- - \mathcal{R}^-\nu_k^+], \end{cases} \tag{5.66}$$

with $\nu_k^+ = -\nu_k^- = \frac{\bar{\varsigma}}{200}[0 \quad 1 \quad 0]^\top$, associated with the initial conditions

$$\hat{x}_{k_0}^+ = \mathfrak{M}_1 x_{k_0}^+ + \mathfrak{M}_2 x_{k_0}^- \ , \ \hat{x}_{k_0}^- = \mathfrak{M}_1 x_{k_0}^- + \mathfrak{M}_2 x_{k_0}^+, \tag{5.67}$$

with $\mathfrak{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{130}{129} & \frac{10}{129} \\ 0 & -\frac{13}{129} & -\frac{1}{129} \end{bmatrix}$, $\mathfrak{M}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{129} & -\frac{10}{129} \\ 0 & \frac{13}{129} & \frac{130}{129} \end{bmatrix}$ and the bounds

$$x_k^+ = \mathfrak{N}_1\hat{x}_k^+ + \mathfrak{N}_2\hat{x}_k^- \ , \ x_k^- = \mathfrak{N}_1\hat{x}_k^- + \mathfrak{N}_2\hat{x}_k^+, \tag{5.68}$$

with $\mathfrak{N}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathfrak{N}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a robust interval observer for the system (5.53) when in closed-loop with

$$u(\hat{x}_k^+, y_k) = \frac{341}{600}\hat{x}_2^+ + \frac{35}{3}\sin\left(y + \frac{1}{200}\hat{x}_2^+\right) - \frac{199}{2400}y.$$

Figure 5.1 illustrates the result in the case where there is no disturbances ($\varsigma_j = 0$). A trajectory with $k_0 = 0, x_{k_0} = [20, 10, 5]^\top, x_{k_0}^+ = [23, 13, 8]^\top, x_{k_0}^- = [17, 7, 2]^\top, \hat{x}_{k_0}^+ = [23, \frac{581}{43}, \frac{58}{43}]^\top, \hat{x}_{k_0}^- = [17, \frac{279}{43}, \frac{372}{43}]^\top$ as initial conditions is simulated. Next, Figure 5.2 shows the solution with the same initial conditions in the case where the system is affected by disturbances $\varsigma_j = 8$. Due to the presence of this disturbance of large size, the distance between the bounds is almost the same for all time instants.

5.5 Conclusion

A new technique of construction of stable interval observers for discrete-time nonlinear time-invariant systems with uncertainties has been developed. A key advantage of this approach is the simplicity of the dynamics of the proposed interval observer: basically, it is composed of two copies of a classical Luenberger observer with extra terms whose presence is due to the uncertain terms. Much remains to be done. Other types of robustness properties, such as robustness with respect to noises in the measurements, may be the subject of further studies.

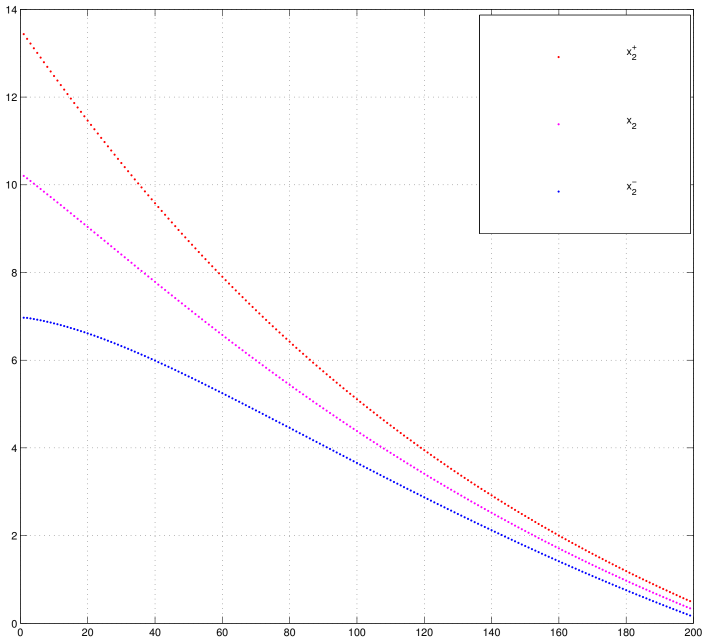

FIGURE 5.1: Evolution of the state component x_2 and its bounds x_2^+, x_2^- without uncertainty.

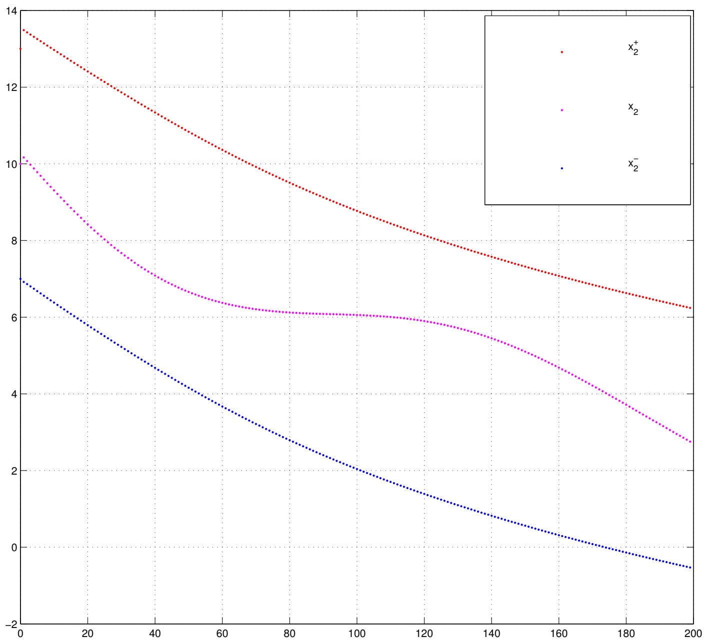

FIGURE 5.2: Evolution of the state component x_2 and its bounds x_2^+, x_2^- with the uncertainties.

Extensions of the results to families of discrete-time time-varying systems with delays and their adaptation to the continuous-time case can be expected too.

Chapter 6

Construction of Interval Observers for Continuous-time Systems with Discrete Measurements

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6.1 Introduction

After considering interval observers for both discrete-time systems and continuous-time systems with continuous measurements in previous chapters, the problem of interval observers design for continuous-time systems with discrete measurements is now taken into account. It is a well-known fact that the access to the state variables of a system is often difficult. It turns out that in practice, for continuous-time models, the measures are often delivered at discrete instants only. For this reason, for many decades, many researchers have addressed the problem of determining observers for continuous-time systems with discrete-time measurements, see e.g. [31, 34, 54, 61, 107], and a few recent contributions are devoted to the problem of constructing interval observers in the same context: [50, 93, 98]. In the present chapter, we revisit this problem. We consider continuous-time linear systems with input, output and additive disturbances under the assumptions of stabilizability and detectability. For any system in this family, we design a continuous-discrete interval observer.

It consists of *two standard observers* and of a framer as subsystems. We shall prove that a consequence of this is that the proposed interval observer possesses a strong stability property: the difference between its two bounds satisfies an inequality of ISS type [12] and converges exponentially to zero in the absence of disturbances. Moreover, the system can be globally asymptotically stabilized by using the information provided by the interval observer and this stability depends entirely on the stability properties achieved by the state feedback. It is worth mentioning that the idea of stabilizing systems through the information provided by interval observers is not new. It has been used in previous chapters and particular in the contributions [44, 96, 112].

The estimators we propose are significantly different from those presented in [98], which have, as dynamic part, continuous-time systems which are time-varying even when the studied system is time-invariant, while those we construct are continuous-discrete systems that are time-invariant when the studied system is. They are also distinct from the continuous-discrete framers presented in [50] because their asymptotic stability is not guaranteed. The design we propose owes a great deal to the papers [5, 31], where the state variables are estimated through an observer which (i) is a simple copy of the system when no new measurement is available (ii) makes an impulsive correction of the estimate when a new measurement is available.

It is worth pointing out that the estimators we propose are not derived directly from the interval observers constructed for continuous-time systems in [89] and for discrete-time systems in [38] or Chapters 4, although some of the key ideas of these works are used along our construction. Although, in these papers and in Chapter 4, the dimension of the proposed interval observers is twice the dimension of the system to be observed, those constructed in the present work is four times this dimension. The two subsystems which give estimates of the solutions at each instant where a new measurement is available as well as their associated error equations, are not a nonnegative system (see Section 1.5.1 for the definition of nonnegative system). This feature may sound astonishing since most of the designs of framers available in the literature rely on this property. In fact, as in Chapter 3 and Chapter 5, we will use the notion of nonnegative system only *indirectly*, to select for the interval observer appropriate upper and lower bounds for the solutions of the studied system at the instants of the impulsive corrections. This feature of our design is fundamental: it is the reason why the interval observers we propose are given by rather simple equations. The present chapter complements the paper [93]. In particular, by contrast with [93], we consider the case where the output is affected by a disturbance and, to slightly simplify the design, we choose a new framer to estimate the state variable between the instants of discontinuity. Moreover, the assumptions we introduce make it possible to asymptotically stabilize the system by bounded

12. See Section 1.3 of Chapter 1 for the notion of ISS.

feedbacks when the studied system possesses a property of robust stabilizability by bounded state feedback.

The chapter is organized as follows. Some notation and definitions are proposed in Section 6.2. A continuous-discrete interval observer is presented in Section 6.3. Section 6.4 is devoted to an illustrative example. Conclusions are drawn in Section 6.5.

6.2 Notation, definitions

Before going through the next section, let us introduce some notation and definitions for the convenience of the reader.

- $|\cdot|$ denotes the Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions.
- Any $k \times n$ matrix, whose entries are all 0 is denoted 0.
- I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.
- The inequalities must be understood componentwise (see Definition 1.27 of Chapter 1).
- $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$.
- For any matrix $M \in \mathbb{R}^{r\times s}$, we let $M^+ = \max(M, 0)$, $M^- = M^+ - M$.
- A matrix $M \in \mathbb{R}^{r\times s}$ is said to be nonnegative if all its entries are nonnegative (see Definition 1.32).
- A matrix $M \in \mathbb{R}^{r\times r}$ is said to be Metzler or cooperative if each off-diagonal entry of this matrix is nonnegative (see Definition 1.34).
- A square matrix $M \in \mathbb{R}^{n\times n}$ is said to be positive definite if for all non-zero vectors $v \in \mathbb{R}^n$, the inequality $v^\top M v > 0$ is satisfied and we denote $M \succ 0$.
- Let $\nu > 0$ be a constant and the sequence t_i be defined by

$$t_0 = 0\ ,\ t_i = \nu i\ ,\ \forall i \in \mathbb{N}. \tag{6.1}$$

- We shall use the simplifying notation: $\mathfrak{I}_k = [t_k, t_{k+1})$.
- Let $m : [0, +\infty) \to \mathbb{R}^l$ be a function that is continuous over each interval $\mathfrak{I}_i$ and such that $\lim\limits_{t\to t_i t<t_i} m(t)$ exists. Then, for all integer $k \in \mathbb{N}$, we let $m_k^{\mathfrak{s}} = \lim\limits_{t\to t_k t<t_k} m(t)$ and $m_k = m(t_k)$.

6.3 Continuous-discrete interval observer design

In this section, we construct a continuous-discrete interval observer. To this end, we consider the sequence of real numbers t_i defined in (6.1) and the linear system with output defined, over every interval $\mathfrak{I}_i$, by

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + \delta_1(t) \ , \\ y(t) & = & Cx(t_i) + \delta_{2,i} \ , \ \forall t \in \mathfrak{I}_i \ , \end{array} \tag{6.2}$$

where $x \in \mathbb{R}^n$ are the state variables, $u : [0, +\infty) \to \mathbb{R}^p$ is the input, $y \in \mathbb{R}^q$ is the output, $\delta_1 : [0, +\infty) \to \mathbb{R}^n$ is a disturbance, which is supposed to be piecewise continuous function, $\delta_{2,i} \in \mathbb{R}^q$ is a disturbance and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$ are constant matrices.

We introduce three assumptions.

Assumption 6.1. *There is a matrix $L \in \mathbb{R}^{n\times q}$ such that the spectral radius of the matrix*

$$G = Je^{\nu A}, \tag{6.3}$$

with

$$J = I_n - LC, \tag{6.4}$$

is smaller than 1, i.e. G is Schur stable. Moreover there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix $\mathfrak{G} = PGP^{-1}$ is nonnegative.

Assumption 6.2. *There is a Lipschitz continuous feedback u_s satisfying $u_s(0) = 0$ such that, for every function $d : [0, +\infty) \to \mathbb{R}^n$ for which there are two constants $s_1 > 0$, $s_2 > 0$ such that*

$$|d(t)| \leq s_1 e^{-s_2 t} \quad , \quad \forall t \geq 0, \tag{6.5}$$

all the solutions of the system

$$\dot{z}(t) = Az(t) + Bu_s(z(t) + d(t)) \tag{6.6}$$

converge asymptotically to the origin.

Assumption 6.3. *The unknown disturbance δ_1 is piecewise continuous and such that, for all $t \geq 0$,*

$$\underline{\delta}_1(t) \leq \delta_1(t) \leq \overline{\delta}_1(t) \ , \tag{6.7}$$

where $\overline{\delta}_1 : [0, +\infty) \to \mathbb{R}^n$ and $\underline{\delta}_1 : [0, +\infty) \to \mathbb{R}^n$ are known continuous functions. The unknown disturbance δ_2 is such that, for all integer $i \in \mathbb{N}$,

$$\underline{\delta}_{2,i} \leq \delta_{2,i} \leq \overline{\delta}_{2,i}, \tag{6.8}$$

where $\overline{\delta}_{2,i}$, $\underline{\delta}_{2,i}$ are known constants.

Let us introduce some notation. Let $D_A \in \mathbb{R}^{n\times n}$ denote the diagonal matrix such that all the diagonal entries of $A - D_A$ are equal to zero. Let $M_A = D_A + (A - D_A)^+$, $P_A = (A - D_A)^-$,

$$T_A = \begin{bmatrix} M_A & P_A \\ P_A & M_A \end{bmatrix}, \quad Q_A = \begin{bmatrix} M_A & -P_A \\ -P_A & M_A \end{bmatrix}, \tag{6.9}$$

$$N = P^{-1}, \quad W = PL, \quad Z_N = \begin{bmatrix} N^+ & -N^- \\ -N^- & N^+ \end{bmatrix} \tag{6.10}$$

and, for all $j \in \mathbb{N}$, $j \geq 1$,

$$\begin{aligned} R_{a,j}(\Delta_*) &= \int_{t_{j-1}}^{t_j} \left[\Im_j^+(\ell)\overline{\delta}_1(\ell) - \Im_j^-(\ell)\underline{\delta}_1(\ell)\right] d\ell \,, \\ R_{b,j}(\Delta_*) &= \int_{t_{j-1}}^{t_j} \left[\Im_j^+(\ell)\underline{\delta}_1(\ell) - \Im_j^-(\ell)\overline{\delta}_1(\ell)\right] d\ell \,, \end{aligned} \tag{6.11}$$

with $\Delta_* = (\overline{\delta}_1, \underline{\delta}_1)$, $\Im_j(\ell) = PJe^{(t_j-\ell)A}$ where J is the matrix defined in (6.4). Let $\Delta_\diamond = (\overline{\delta}_1, \underline{\delta}_1, \overline{\delta}_2, \underline{\delta}_2)$ and, for all $k \in \mathbb{N}$, $k \geq 1$,

$$\begin{aligned} S_{1,k}(\Delta_\diamond) &= N[R_{a,k}(\Delta_*) - W^+\underline{\delta}_{2,k} + W^-\overline{\delta}_{2,k}], \\ S_{2,k}(\Delta_\diamond) &= N[R_{b,k}(\Delta_*) - W^+\overline{\delta}_{2,k} + W^-\underline{\delta}_{2,k}]. \end{aligned} \tag{6.12}$$

Notice for later use that there is a constant $s_3 > 0$ such that, for $i = 1, 2$, and all $k \in \mathbb{N}$, $k \geq 1$,

$$|S_{i,k}(\Delta_\diamond)| \leq s_3 \left[\sup_{\ell\in[t_{k-1},t_k]} |\Delta_*(\ell)| + |\overline{\delta}_{2,k}| + |\underline{\delta}_{2,k}|\right] . \tag{6.13}$$

Discussion of Theorem 6.4.

• If the discrete-time system $g_{k+1} = e^{\nu A}g_k$ with the output Cg_k is detectable, there is a matrix $L_1 \in \mathbb{R}^{n\times q}$ such that the matrix $e^{\nu A} + L_1C$ is Schur stable. Then necessarily the matrix $e^{-\nu A}[e^{\nu A} + L_1C]e^{\nu A}$ is Schur stable. It follows that the matrix $[I_n - LC]e^{\nu A}$ with $L = -e^{-\nu A}L_1$ is Schur stable, which implies that the first part of Assumption 6.1 is satisfied. Detectability of the pair $(e^{\nu A}, C)$ is a mild condition and, for some pairs (A, C), as for instance $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, [1 \quad 0]\right)$, it is satisfied for all $\nu > 0$. We deduce that our assumptions imply less stringent requirements on the size of ν than the assumptions in [98]. Moreover, it is worth noticing that if the continuous-time system $\dot{h} = Ah$ with the output Ch is detectable, then there exists $\nu_* > 0$ such that for all $\nu \in (0, \nu_*]$, the discrete-time system $g_{k+1} = e^{\nu A}g_k$ with the output Cg_k is detectable.

• For the sake of simplicity, the matrix P in Assumption 6.1 is constant. However, we conjecture that this assumption can be relaxed by replacing the matrix P by a sequence of matrices that can be deduced from the results in [95]. Then the corresponding interval observer (6.14)-(6.15) would be time-varying. If the pair $(e^{\nu A}, C)$ is observable, distinct eigenvalues for the matrix $[I_n - LC]e^{\nu A}$ can be selected and then one can determine a constant matrix P for which Assumption 6.1 is satisfied.

• The x_a and x_b-subsystems of (6.14) are classical continuous-discrete observers for the system (6.2), which belong to the family of continuous-discrete observers used in [5]. Therefore the interval observer inherits the performances of the classical continuous-discrete observers.

• The goal of the x_a and x_b-subsystems of (6.14) is to provide with bounds for the solution x at the discrete instants t_k, while the goal of the $\overline{x}$ and $\underline{x}$-subsystems is to provide with bounds for the solution x over the intervals (t_k, t_{k+1}). An alternative choice of dynamics giving bounds over the intervals (t_k, t_{k+1}) is proposed in [93].

• Other choices of stabilizing feedback than (6.19) can be made. Among them, there is in particular $u(x_b) = u_s(x_b)$. Observe that when each eigenvalue of A has a nonpositive real part, then Assumption 6.2 is satisfied with bounded feedbacks of arbitrary size (see [130]). Observe also that Assumption 6.2 implies that the pair (A, B) is stabilizable, which implies that there is a matrix $K \in \mathbb{R}^{p\times m}$ such that $A + BK$ is Hurwitz. It follows that Assumption 6.2 is satisfied with the linear feedback $u_s(z) = Kz$. From this linear property and (6.18), one can easily deduce that this specific choice of feedback results in a closed-loop system possessing an ISS property with respect to δ_1 and δ_2.

We are ready to state and prove the following result:

Theorem 6.4. *Assume that the system (6.2) satisfies Assumptions 6.1 to 6.3. Then the system defined, for all $k \in \mathbb{N}$, by*

$$\left\{\begin{array}{rcl}
\begin{pmatrix} \dot{x}_a(t) \\ \dot{x}_b(t) \end{pmatrix} & = & \begin{pmatrix} Ax_a(t) + Bu(t) \\ Ax_b(t) + Bu(t) \end{pmatrix}, \ \forall t \in \mathfrak{I}_k, \\
\begin{pmatrix} x_{a,k} \\ x_{b,k} \end{pmatrix} & = & \begin{pmatrix} x^{\mathfrak{s}}_{a,k} \\ x^{\mathfrak{s}}_{b,k} \end{pmatrix} + \begin{pmatrix} L(y_k - Cx^{\mathfrak{s}}_{a,k}) \\ L(y_k - Cx^{\mathfrak{s}}_{b,k}) \end{pmatrix} + \begin{pmatrix} S_{1,k}(\Delta_\diamond) \\ S_{2,k}(\Delta_\diamond) \end{pmatrix} \text{ when } k \geq 1, \\
\begin{pmatrix} \dot{\overline{x}}(t) \\ \dot{\underline{x}}(t) \end{pmatrix} & = & Q_A \begin{pmatrix} \overline{x}(t) \\ \underline{x}(t) \end{pmatrix} + \begin{pmatrix} Bu(t) + \overline{\delta}_1(t) \\ Bu(t) + \underline{\delta}_1(t) \end{pmatrix}, \ \forall t \in \mathfrak{I}_k, \\
\begin{pmatrix} \overline{x}_k \\ \underline{x}_k \end{pmatrix} & = & Z_N \begin{pmatrix} Px_{a,k} \\ Px_{b,k} \end{pmatrix}, \text{ when } k \geq 1,
\end{array}\right. \tag{6.14}$$

with the initial conditions

$$\begin{pmatrix} x_a(0) \\ x_b(0) \\ x_{a,0} \\ x_{b,0} \\ \overline{x}_0 \\ \underline{x}_0 \end{pmatrix} = \begin{pmatrix} N[P^+\overline{x}_0 - P^-\underline{x}_0] \\ N[P^+\underline{x}_0 - P^-\overline{x}_0] \\ N[P^+\overline{x}_0 - P^-\underline{x}_0] \\ N[P^+\underline{x}_0 - P^-\overline{x}_0] \\ \overline{x}_0 \\ \underline{x}_0 \end{pmatrix} \tag{6.15}$$

and the bounds $\overline{x}$, $\underline{x}$ is an interval observer for the system (6.2): i.e. when $u(t)$ is a piecewise continuous function defined over $[0, +\infty)$ and bounded over every compact set and

$$\underline{x}_0 \leq x(0) \leq \overline{x}_0 \tag{6.16}$$

then, for all $t \geq 0$,

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{6.17}$$

and there are constants $c_i > 0$, $i = 1$ to 4 such that, for all $k \in \mathbb{N}$, $k \geq 1$, $t \in \mathfrak{I}_k$,

$$|\underline{x}(t) - \overline{x}(t)| \leq c_2 e^{-c_1 t}|\underline{x}_0 - \overline{x}_0| + c_3 \sup_{\ell \in [t_0, t_k]} |\Delta_*(\ell)| + c_4 \sup_{i \in \{1,\ldots,k\}} \left(|\overline{\delta}_{2,i}| + |\underline{\delta}_{2,i}|\right). \tag{6.18}$$

Moreover all the solutions of the system (6.14)-(6.2) in closed-loop with the feedback

$$u(x_a) = u_s(x_a), \tag{6.19}$$

where $u_s$is the feedback given by Assumption 6.2, converge to the origin when $\overline{\delta}_1 = \underline{\delta}_1 = 0$, $\overline{\delta}_2 = \underline{\delta}_2 = 0$.

Proof. First step: existence and uniqueness of the solutions.

We consider a solution of (6.2)-(6.14) with the initial conditions selected in (6.15) under the assumption that $u(t)$ is piecewise continuous, defined over $[0, +\infty)$ and bounded over every compact interval. One can show that such initial conditions generate one and only one solution of (6.14) as follows. From (6.15), it follows that the x_a and x_b-subsystems admit one and only one solution over $[t_0, t_1)$. Moreover, $x_{a,1}^{\mathfrak{s}}$ and $x_{b,1}^{\mathfrak{s}}$ exist because u is piecewise continuous. On the other hand, from (6.15) we can also deduce that $\overline{x}$ and $\underline{x}$-subsystems admit one and only one solution over $[t_0, t_1)$. Next, arguing similarly over each interval $\mathfrak{I}_k$, one establishes the existence of one and only one solution over $[0, +\infty)$.
Second step: framer.

In this part of the proof, we show that (6.14) is a framer for (6.2).

Since $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, $P^+ \geq 0$ and $P^- \geq 0$, by Lemma A.5 in Appendix A, it follows that

$$P^+\underline{x}_0 - P^-\overline{x}_0 \leq Px(0) \leq P^+\overline{x}_0 - P^-\underline{x}_0 \,. \tag{6.20}$$

Using (6.15), we obtain

$$Px_{b,0} \leq Px(0) \leq Px_{a,0} \,. \tag{6.21}$$

Now, to ease the analysis, we define two variables

$$e_a = x_a - x \;,\; e_b = x - x_b \,. \tag{6.22}$$

They satisfy, for all $k \in \mathbb{N}$,

$$\begin{cases} \dot{e}_a &= Ae_a - \delta_1(t),\ \forall t \in \mathfrak{I}_k \\ e_{a,k} &= e^{\mathfrak{s}}_{a,k} - LCe^{\mathfrak{s}}_{a,k} + L\delta_{2,k} + S_{1,k}(\Delta_\diamond), k \geq 1 \\ \dot{e}_b &= Ae_b + \delta_1(t),\ \forall t \in \mathfrak{I}_k \\ e_{b,k} &= e^{\mathfrak{s}}_{b,k} - LCe^{\mathfrak{s}}_{b,k} - L\delta_{2,k} - S_{2,k}(\Delta_\diamond), k \geq 1. \end{cases} \tag{6.23}$$

By integrating, for any $k \in \mathbb{N}$ any $t \in \mathfrak{I}_k$ over $[t_k, t)$, we deduce that, for all $k \in \mathbb{N}$, for all $t \in \mathfrak{I}_k$,

$$e_a(t) = e^{A(t-t_k)}e_{a,k} - \int_{t_k}^{t} e^{(t-\ell)A}\delta_1(\ell)d\ell, \tag{6.24}$$

which implies that

$$e^{\mathfrak{s}}_{a,k+1} = e^{\nu A}e_{a,k} - \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell.$$

Similarly, one can prove that

$$e^{\mathfrak{s}}_{b,k+1} = e^{\nu A}e_{b,k} + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell\ .$$

These equalities and (6.23) give, for all $k \in \mathbb{N}$,

$$\begin{cases} e_{a,k+1} &= Ge_{a,k} - J\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell + S_{1,k+1}(\Delta_\diamond) + L\delta_{2,k+1}, \\ e_{b,k+1} &= Ge_{b,k} + J\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell - S_{2,k+1}(\Delta_\diamond) - L\delta_{2,k+1}\ , \end{cases} \tag{6.25}$$

where G is the matrix defined in (6.3).

From the definitions of $\Im_j$, N, W, $S_{1,k}$, $S_{2,k}$, and $\mathfrak{G}$ in 6.11, 6.10, 6.12 and in Assumption 6.1, we deduce that, for all $k \in \mathbb{N}$,

$$\begin{cases} Pe_{a,k+1} &= \mathfrak{G}Pe_{a,k} + R_{a,k+1}(\Delta_*) - \int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell \\ &\quad + W\delta_{2,k+1} - W^+\underline{\delta}_{2,k+1} + W^-\overline{\delta}_{2,k+1}\ , \\ Pe_{b,k+1} &= \mathfrak{G}Pe_{b,k} - R_{b,k+1}(\Delta_*) + \int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell \\ &\quad - W\delta_{2,k+1} + W^+\overline{\delta}_{2,k+1} - W^-\underline{\delta}_{2,k+1}\ . \end{cases} \tag{6.26}$$

Using the inequalities

$$\begin{aligned} (\Im_{k+1}(\ell))^+\,\underline{\delta}_1(\ell) &\leq (\Im_{k+1}(\ell))^+\,\delta_1(\ell), \\ (\Im_{k+1}(\ell))^+\,\delta_1(\ell) &\leq (\Im_{k+1}(\ell))^+\,\overline{\delta}_1(\ell), \\ -\,(\Im_{k+1}(\ell))^-\,\overline{\delta}_1(\ell) &\leq -\,(\Im_{k+1}(\ell))^-\,\delta_1(\ell) \end{aligned}$$

and

$$-\left(\Im_{k+1}(\ell)\right)^{-}\delta_1(\ell) \leq -\left(\Im_{k+1}(\ell)\right)^{-}\underline{\delta}_1(\ell),$$

we deduce that

$$R_{a,k+1}(\Delta_*) - \int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell \geq 0 \tag{6.27}$$

and

$$\int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell - R_{b,k+1}(\Delta_*) \geq 0 \ . \tag{6.28}$$

Moreover, using the Lemma A.5 in Appendix A, we deduce that

$$W\delta_{2,k+1} - W^{+}\underline{\delta}_{2,k+1} + W^{-}\overline{\delta}_{2,k+1} \geq 0 \tag{6.29}$$

and

$$-W\delta_{2,k+1} + W^{+}\overline{\delta}_{2,k+1} - W^{-}\underline{\delta}_{2,k+1} \geq 0. \tag{6.30}$$

From (6.26), the inequalities (6.27), (6.28), (6.29), (6.30), the inequality $\mathfrak{G} \geq 0$ and the fact that the inequalities in (6.21) are equivalent to

$$0 \leq Pe_{a,0} \ , \ 0 \leq Pe_{b,0}, \tag{6.31}$$

it follows that, for all $k \in \mathbb{N}$,

$$0 \leq Pe_{b,k} \ , \ 0 \leq Pe_{a,k} \ , \tag{6.32}$$

we deduce that, for all $k \in \mathbb{N}$,

$$Px_{b,k} \leq Px_k \leq Px_{a,k} \ . \tag{6.33}$$

Since $N^{+} \geq 0$ and $N^{-} \geq 0$ and by Lemma A.5 in Appendix A, we have, for all $k \in \mathbb{N}$,

$$N^{+}Px_{b,k} - N^{-}Px_{a,k} \leq NPx_k \leq N^{+}Px_{a,k} - N^{-}Px_{b,k} \tag{6.34}$$

We deduce that, for all $k \in \mathbb{N}$,

$$\underline{x}_k \leq x_k \leq \overline{x}_k \ . \tag{6.35}$$

Next, let us analyze x, $\overline{x}$, and $\underline{x}$ over an interval $\mathfrak{I}_k$.
Since $A = M_A - P_A$, we have, for all $t \in \mathfrak{I}_k$,

$$\left\{\begin{array}{rcl} \dot{\overline{x}} - \dot{x} & = & M_A(\overline{x} - x) + P_A(x - \underline{x}) + \overline{\delta}_1(t) - \delta_1(t), \\ \dot{x} - \dot{\underline{x}} & = & P_A(\overline{x} - x) + M_A(x - \underline{x}) + \delta_1(t) - \underline{\delta}_1(t). \end{array}\right. \tag{6.36}$$

Since, according to (6.35), for all $k \in \mathbb{N}$, $\overline{x}(t_k) - x(t_k) \geq 0$, $x(t_k) - \underline{x}(t_k) \geq 0$ and for all $t \geq 0$, $\overline{\delta}_1(t) - \delta_1(t) \geq 0$, $\delta_1(t) - \underline{\delta}_1(t) \geq 0$, we deduce that

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{6.37}$$

for all $t \in \mathfrak{I}_k$ because the matrix T_A defined in (6.9) is cooperative. This allows us to conclude.

Third step: stability analysis of the framer.

Let $\tilde{e} = x_a - x_b$. Then, for all $k \in \mathbb{N}$,

$$\left\{ \begin{array}{rcl} \dot{\tilde{e}}(t) & = & A\tilde{e}(t) \; , \forall t \in \mathfrak{I}_k, \\ \tilde{e}_k & = & J\tilde{e}_k^{\mathfrak{s}} + S_{3,k}(\Delta_\diamond), \text{ when } k \geq 1, \end{array} \right. \tag{6.38}$$

with

$$S_{3,k}(\Delta_\diamond) = S_{1,k}(\Delta_\diamond) - S_{2,k}(\Delta_\diamond). \tag{6.39}$$

By adding the two equations in (6.25), for all $k \geq 0$,

$$\tilde{e}_{k+1} \quad = \quad G\tilde{e}_k + S_{3,k+1}(\Delta_\diamond). \tag{6.40}$$

From Assumption 6.1, we deduce that there exist real numbers $r_i > 0$, $i = 1$ to 3 such that, for all $k \in \mathbb{N}$, $k \geq 1$,

$$|\tilde{e}_k| \leq r_1 e^{-kr_2}|\tilde{e}_0| + r_3 \sup_{j\in\{1,\ldots,k\}} |S_{3,j}(\Delta_\diamond)| \; . \tag{6.41}$$

By integrating the differential equation in (6.38), we deduce that for all $t \in \mathfrak{I}_k$,

$$|\tilde{e}(t)| \leq e^{\nu|A|}|\tilde{e}(t_k)| \leq r_1 e^{\nu|A|}e^{-kr_2}|\tilde{e}_0| + e^{\nu|A|}r_3 \sup_{j\in\{1,\ldots,k\}} |S_{3,j}(\Delta_\diamond)|. \tag{6.42}$$

Now, observe that (6.14) implies that, when $k \geq 1$,

$$\left(\begin{array}{c} \overline{x}_k \\ \underline{x}_k \end{array} \right) = Z_N \left(\begin{array}{c} Px_{b,k} \\ Px_{b,k} \end{array} \right) + Z_N \left(\begin{array}{c} P\tilde{e}_k \\ 0 \end{array} \right). \tag{6.43}$$

It follows that, for all $k \geq 1$,

$$\overline{x}_k - \underline{x}_k = (N^+ + N^-)P\tilde{e}_k. \tag{6.44}$$

We deduce that there is a constant $r_4 > 0$ such that, for all $t \in \mathfrak{I}_k$,

$$|\overline{x}(t) - \underline{x}(t)| \leq r_4|\tilde{e}_k|. \tag{6.45}$$

Bearing in mind (6.41), we deduce that there are constants $r_5 > 0$, $r_6 > 0$ such that

$$|\overline{x}(t) - \underline{x}(t)| \leq r_5 e^{-kr_2}|\tilde{e}_0| + r_6 \sup_{j\in\{1,\dots,k\}} |S_{3,j}(\Delta_\diamond)|. \tag{6.46}$$

Bearing in mind (6.13), we deduce that (6.18) is satisfied.

Fourth step: stability analysis of the closed-loop system.

Let us establish that all the solutions of the system (6.14)-(6.2) in closed-loop with the feedback in (6.19) converge to the origin when $\overline{\delta}_1 = \underline{\delta}_1 = 0$, $\overline{\delta}_2 = \underline{\delta}_2 = 0$.

Observe that, for all $t \geq 0$,

$$\dot{x}(t) = Ax(t) + u_s(x(t) + e_a(t)). \tag{6.47}$$

We deduce from (6.25) that there are constants $r_7 > 0$ and $r_8 > 0$ such that, for all $k \in \mathbb{N}$,

$$|e_a(t_k)| \leq r_7 e^{-r_8 k}|\tilde{e}_0|. \tag{6.48}$$

We deduce that there is a constant $r_9 > 0$ such that for all $t \geq 0$,

$$|e_a(t)| \leq r_9 e^{-r_8 t}|\tilde{e}_0|. \tag{6.49}$$

This inequality, (6.47) and Assumption 6.2 allow us to conclude. □

6.4 Example

Throughout this section, we use the notation of Section 6.3 and denote by v_i the ith component of the vector v. To illustrate Theorem 6.4, we consider the two dimensional system of the family (6.2) with the matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = [1 \quad 0], \tag{6.50}$$

with disturbances admitting the bounds $\overline{\delta}_1 = c[1 \quad 1]^\top$, $\underline{\delta}_1 = -\overline{\delta}_1$, $\overline{\delta}_{2,k} = c$, $\underline{\delta}_{2,k} = -\overline{\delta}_{2,k}$, where c is a constant.

Next, we construct a continuous-discrete interval observer. Let us choose

$$L = [1 \quad 1]^\top. \tag{6.51}$$

Then $J = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$, $e^{\nu A} = \begin{bmatrix} \cos(\nu) & \sin(\nu) \\ -\sin(\nu) & \cos(\nu) \end{bmatrix}$, which implies that the choice $\nu = \frac{\pi}{4}$ gives the matrix

$$G = -\begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}, \tag{6.52}$$

which is Schur stable, but is not nonnegative. One can check readily that Assumption 6.1 is satisfied with $P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ since $\mathfrak{G} = PGP^{-1} = -G$. Since Assumption 6.2 is satisfied with, for instance,

$$u_s(x) = -\frac{5}{2}x_2, \tag{6.53}$$

both Assumptions 1 and 2 are satisfied. Therefore, Theorem 6.4 applies. Then, through simple calculations, we obtain that the system defined, for all $k \in \mathbb{N}$, by

$$\left\{\begin{array}{rcl}
\dot{x}_a(t) & = & Ax_a(t) + Bu(t),\ \forall t \in \mathfrak{I}_k \\
\dot{x}_b(t) & = & Ax_b(t) + Bu(t),\ \forall t \in \mathfrak{I}_k \\
x_{a,k} & = & \begin{pmatrix} y_k + c \\ y_k + x^{\mathfrak{s}}_{2,a,k} - x^{\mathfrak{s}}_{1,a,k} - (1+\sqrt{2})c \end{pmatrix},\ \text{when} \quad k \geq 1 \\
x_{b,k} & = & \begin{pmatrix} y_k - c \\ y_k + x^{\mathfrak{s}}_{2,b,k} - x^{\mathfrak{s}}_{1,b,k} + (1+\sqrt{2})c \end{pmatrix},\ \text{when} \quad k \geq 1 \\
\dot{\overline{x}}(t) & = & \begin{pmatrix} \overline{x}_2(t) + c \\ -\underline{x}_1(t) + c + u(t) \end{pmatrix},\ \forall t \in \mathfrak{I}_k \\
\dot{\underline{x}}(t) & = & \begin{pmatrix} \underline{x}_2(t) - c \\ -\overline{x}_1(t) - c + u(t) \end{pmatrix},\ \forall t \in \mathfrak{I}_k \\
\overline{x}_k & = & \begin{pmatrix} x_{1,a,k} \\ x_{2,b,k} \end{pmatrix},\quad \underline{x}_k = \begin{pmatrix} x_{1,b,k} \\ x_{2,a,k} \end{pmatrix}
\end{array}\right. \tag{6.54}$$

with $y_k = x_1(t_k)$ and the initial conditions

$$x_a(0) = x_{a,0} = \begin{pmatrix} \overline{x}_{1,0} \\ \underline{x}_{2,0} \end{pmatrix}, \tag{6.55}$$

$$x_b(0) = x_{b,0} = \begin{pmatrix} \underline{x}_{1,0} \\ \overline{x}_{2,0} \end{pmatrix} \tag{6.56}$$

such that

$$\underline{x}_0 \leq x(0) \leq \overline{x}_0 \tag{6.57}$$

and the bounds $\overline{x}$, $\underline{x}$ is an interval observer for the considered system.

We present simulations for the system (6.50)-(6.54) with u_s defined in (6.53) as feedback.

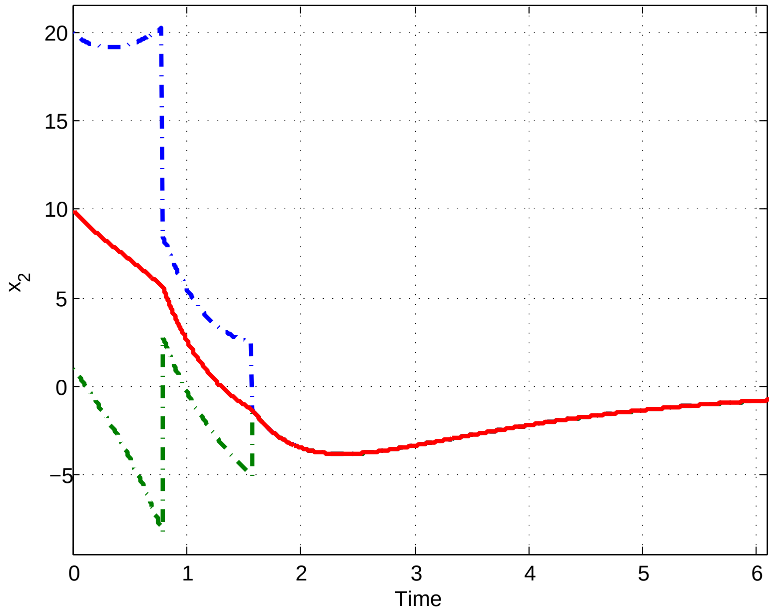

FIGURE 6.1: Evolution of the state component x_2 and its bounds without uncertainty.

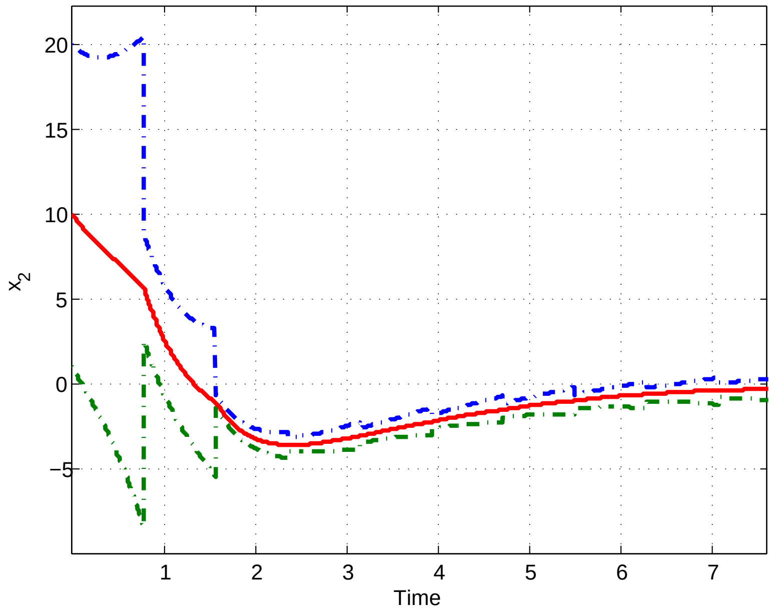

FIGURE 6.2: Evolution of the state component x_2 and its bounds with the uncertainties.

6.5 Conclusion

We have developed a new technique of construction of continuous-discrete interval observers for continuous-time systems with discrete measurements and disturbances in the measurements and the dynamics.

Many extensions of this result are possible. We plan to investigate the case where the sequence of the differences between two consecutive instants at which the measurements are available is not constant and to design reduced order interval observers in the spirit of what is done in [40].

Moreover, nonlinear systems with globally Lipschitz nonlinearities or triangular structures, time-varying systems and systems with delay (notably in the input) may be considered.

Chapter 7

Conclusions and Perspectives

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7.1 Contributions of this dissertation

This thesis presents new results in the field of state estimation based on the theory of positive systems. It is composed of two separate parts. The first one studies the problem of positive observer design for positive systems. The second one which deals with robust state estimation through the design of interval observers, is at the core of our work.

We began our thesis by proposing the design of a nonlinear positive observer for discrete-time positive time-varying linear systems based on the use of generalized polar coordinates in the positive orthant. The idea underlying the method is that first, the direction of the true state is correctly estimated in the projective space thanks to the Birkhoff theorem and then very mild assumptions on the output map allow to reconstruct the norm of the state.

Later, the thesis was continued by studying the so-called interval observers for different families of dynamic systems in continuous-time, in discrete-time and also in a context "continuous-discrete" (i.e. a class of continuous-time systems with discrete-time measurements). Thanks to interval observers, one can construct control laws which stabilize the considered systems. Precisely, five families of systems were mentioned in this thesis :

- A family of nonlinear continuous-time systems affine in the unmeasured part of state variables.
- A family of nonlinear discrete-time systems which possess specific stability and monotonicity properties.

- A fundamental family of linear time-invariant discrete-time systems.
- A family of nonlinear discrete-time systems with input and output.
- A family of continuous-time systems with discrete measurements.

Here we outline the key contributions of the dissertation as follows.

First, for a family of continuous-time nonlinear systems with input, output and uncertain terms affine in the unmeasured state components, a new interval observer design has been proposed. The main feature of the constructed interval observer is that it is composed of two copies of a classical observer whose corresponding error equations are, in general, not cooperative. The interval observer is globally asymptotically stable under an appropriate choice of dynamic output feedback which uses the values of the output and the bounds provided by the interval observer itself.

Secondly, discrete-time systems are very important and play a large role from a theoretical as well as an applied point of view (note the fact that discretization techniques transform continuous-time systems into discrete-time systems). Therefore, this kind of systems became the next subject of our study. First, time-invariant interval observers have been proposed for a family of nonlinear discrete-time systems which possesses specific stability and monotonicity properties. Second, it has been shown that, for any time-invariant exponentially stable discrete-time linear system with additive disturbances, time-varying exponentially stable discrete-time interval observers can be constructed. The latter result relies on the design of time-varying changes of coordinates which transform a linear system into a nonnegative one.

In order to complement the previous results, a new interval observer has been designed for a family of nonlinear discrete-time systems with input, output and uncertain terms. Its main feature is that, as presented in the case of nonlinear continuous-time systems, it is composed of two copies of classical observers. This interval observer has applied in the presence of unknown bounded nonlinear terms and additive disturbances and has also been used to achieve asymptotic stability through an appropriate choice of dynamic output feedback.

After constructing interval observers for both continuous-time and discrete-time systems, a question about interval observer design for continuous-time systems with discrete measurements has naturally arisen and the last chapter of the dissertation was devoted to solving this problem. Here we have considered continuous-time systems with input, output and additive disturbances in the particular case where the measurements are only available at discrete instants and have disturbances. For these systems, continuous-discrete interval observers that are asymptotically stable in the absence of disturbances have been constructed. These interval observers are composed of two copies of the studied system and of a framer, accompanied with appropriate outputs which give, componentwise, upper and lower bounds for the solutions of the studied system.

7.2 Future works

Positive observer

In the last ten years or so, the design of positive observers for positive systems (i.e. observers such that the estimated state respects positivity) has attracted an ever growing attention. Indeed, the requirement that the observer estimates be positive at any time seems desirable since it allows a physical interpretation at all times, that is, even when the estimation error is not small. The Chapter 2 gives some ideas how the Hilbert metric is useful to analyse contraction and convergence properties of positive maps. For future research, one can extend these results to the nonlinear case and to more general cones. For example, the cone of hermitian positive definite matrices on which the Hilbert metric is presented as follow :

$$d(X,Y) = \log\left(\frac{\lambda_{\max}\left(XY^{-1}\right)}{\lambda_{\min}\left(XY^{-1}\right)}\right) .$$

Interval observer

In spite of the results already available in the literature, interval observers are fertile ground for further studies. One of many research directions can be mentioned here, for example systems with delay. Extensions of the results presented in Chapter 4 to families of linear discrete-time systems with a point-wise delay or with delay in the input, families of discrete-time nonlinear systems with delay can be expected.

Moreover, we can also extend the results of Chapter 3 to systems with other special structures, such as nonlinear systems with globally Lipschitz nonlinearities or triangular systems.

Families of time-varying systems may be considered as well. Extensions of the results in Chapter 5 to discrete-time time-varying systems may be the subject of further research.

Furthermore, design of interval observers for continuous-time systems with discrete measurements is another beautiful problem to solve. For example, extensions of the results in Chapter 6 to the case where the sequence of the differences between two consecutive instants at which the measurements are available is not constant can be investigated. Besides, design reduced order interval observers in the spirit of what is done in [40] can be expected too.

Appendix A

Technical Lemmas

We prove the lemmas needed in several chapters.

Lemma A.1. *Let*

$$\mathcal{M} = \begin{bmatrix} -\kappa & \omega \\ -\omega & -\kappa \end{bmatrix} \tag{A.1}$$

where κ and ω are two real numbers such that $\kappa^2 + \omega^2 > 0$. Let α be any real number such that

$$\sin(\alpha) = -\tfrac{\omega}{\sqrt{\kappa^2+\omega^2}} \ , \ \cos(\alpha) = -\tfrac{\kappa}{\sqrt{\kappa^2+\omega^2}} \tag{A.2}$$

and let, for all $j \in \mathbb{N}$,

$$\mathcal{L}_j = \begin{bmatrix} \cos(\alpha j) & \sin(\alpha j) \\ -\sin(\alpha j) & \cos(\alpha j) \end{bmatrix}. \tag{A.3}$$

Then, for all $k \in \mathbb{N}$, the equality

$$\mathcal{L}_{k+1}\mathcal{M}\mathcal{L}_k^{-1} = \sqrt{\kappa^2 + \omega^2} I_2 \tag{A.4}$$

is satisfied.

Proof. The equalities

$$\begin{array}{rcl} \sin(\alpha(k+1)) & = & \sin(\alpha k)\cos(\alpha) + \cos(\alpha k)\sin(\alpha) \ , \\ \cos(\alpha(k+1)) & = & \cos(\alpha k)\cos(\alpha) - \sin(\alpha k)\sin(\alpha) \ , \end{array} \tag{A.5}$$

imply that

$$\begin{aligned} \mathcal{L}_{k+1} &= \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} -\kappa\cos(\alpha k)+\omega\sin(\alpha k) & -\kappa\sin(\alpha k)-\omega\cos(\alpha k) \\ \kappa\sin(\alpha k)+\omega\cos(\alpha k) & -\kappa\cos(\alpha k)+\omega\sin(\alpha k) \end{bmatrix} \\ &= \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} \cos(\alpha k) & \sin(\alpha k) \\ -\sin(\alpha k) & \cos(\alpha k) \end{bmatrix} \begin{bmatrix} -\kappa & -\omega \\ \omega & -\kappa \end{bmatrix} . \end{aligned} \tag{A.6}$$

Therefore

$$\mathcal{L}_{k+1} = \frac{1}{\sqrt{\kappa^2+\omega^2}} \mathcal{L}_k \mathcal{M}^\top . \tag{A.7}$$

From the equality

$$\mathcal{M}^\top \mathcal{M} = \left(\kappa^2+\omega^2\right) I_2 \tag{A.8}$$

we deduce that

$$\mathcal{L}_{k+1}\mathcal{M} = \sqrt{\kappa^2+\omega^2}\mathcal{L}_k . \tag{A.9}$$

This allows us to conclude. □

Lemma A.2. *We consider the system*

$$a_{k+1} = \mathcal{J} a_k \ , k \in \mathbb{N} \tag{A.10}$$

with

$$\mathcal{J} = \begin{bmatrix} \mathcal{M} & I_2 & 0 & \dots & 0 \\ 0 & \mathcal{M} & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \mathcal{M} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}, \tag{A.11}$$

with $\mathcal{M}$ defined in (A.1). Then there exists a constant α such that the time-varying change of coordinates

$$b_k = \overline{\mathcal{L}}_k a_k \tag{A.12}$$

with

$$\overline{\mathcal{L}}_k = \mathit{diag}\{\mathcal{L}_{k-p+1}, \mathcal{L}_{k-p+2}, ..., \mathcal{L}_{k-1}, \mathcal{L}_k\} \in \mathbb{R}^{2p\times 2p} \tag{A.13}$$

with, for all integer j, $\mathcal{L}_j$ defined in (A.3), transforms the system (A.10) into the system

$$b_{k+1} = \sqrt{\kappa^2+\omega^2} b_k + \mathcal{K} b_k, \tag{A.14}$$

with

$$\mathcal{K} = \begin{bmatrix} 0 & I_2 & 0 & \dots & 0 \\ 0 & 0 & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2p \times 2p}. \tag{A.15}$$

Proof. Let α be any real number such that

$$\sin(\alpha) \quad = \quad -\tfrac{\omega}{\sqrt{\kappa^2+\omega^2}} \ , \ \cos(\alpha) \quad = \quad -\tfrac{\kappa}{\sqrt{\kappa^2+\omega^2}} \ . \tag{A.16}$$

According to the definition of b_k in (A.12) and (A.10),

$$b_{k+1} = \overline{\mathcal{L}}_{k+1}\mathcal{J}a_k = \overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^{-1}b_k = \overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^\top b_k \ . \tag{A.17}$$

One can check readily that

$$\mathcal{J}\overline{\mathcal{L}}_k^\top = \begin{bmatrix} \mathcal{M}\mathcal{L}_{k-p+1}^\top & \mathcal{L}_{k-p+2}^\top & 0 & \dots & 0 \\ 0 & \mathcal{M}\mathcal{L}_{k-p+2}^\top & \mathcal{L}_{k-p+3}^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \mathcal{L}_k^\top \\ 0 & \dots & \dots & 0 & \mathcal{M}\mathcal{L}_k^\top \end{bmatrix} . \tag{A.18}$$

Now, from (A.9), we deduce that

$$\overline{\mathcal{L}}_{k+1} = \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} \mathcal{L}_{k-p+1}\mathcal{M}^\top & 0 & \dots & \dots & 0 \\ 0 & \mathcal{L}_{k-p+2}\mathcal{M}^\top & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & \mathcal{L}_{k-1}\mathcal{M}^\top & 0 \\ 0 & \dots & \dots & 0 & \mathcal{L}_k\mathcal{M}^\top \end{bmatrix} . \tag{A.19}$$

It follows that

$$\overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^\top = \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} \varphi_{k,1} & \nu_{k,1} & 0 & \dots & 0 \\ 0 & \varphi_{k,2} & \nu_{k,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \nu_{k,p-1} \\ 0 & \dots & \dots & 0 & \varphi_{k,p} \end{bmatrix} \tag{A.20}$$

with

$$\varphi_{k,i} = \mathcal{L}_{k-p+i}\mathcal{M}^\top\mathcal{M}\mathcal{L}_{k-p+i}^\top \tag{A.21}$$

and

$$\nu_{k,i} = \mathcal{L}_{k-p+i}\mathcal{M}^\top \mathcal{L}_{k-p+i+1}^\top. \tag{A.22}$$

Since $\mathcal{M}^\top \mathcal{M} = (\kappa^2 + \omega^2)I_2$ and $\mathcal{L}_i\mathcal{L}_i^\top = I_2$, we deduce that

$$\varphi_{k,i} = (\kappa^2 + \omega^2)\mathcal{L}_{k-p+i}\mathcal{L}_{k-p+i}^\top = (\kappa^2 + \omega^2)I_2 \ . \tag{A.23}$$

Now, observe that (A.9) implies that

$$\mathcal{L}_i\mathcal{M}^\top \mathcal{L}_{i+1}^\top = \mathcal{L}_i\mathcal{M}^\top \frac{1}{\sqrt{\kappa^2 + \omega^2}}\mathcal{M}\mathcal{L}_i^\top \ . \tag{A.24}$$

Using again the equalities $\mathcal{M}^\top \mathcal{M} = (\kappa^2 + \omega^2)I_2$ and $\mathcal{L}_i\mathcal{L}_i^\top = I_2$, we obtain

$$\mathcal{L}_i\mathcal{M}^\top \mathcal{L}_{i+1}^\top = \sqrt{\kappa^2 + \omega^2}\mathcal{L}_i\mathcal{L}_i^\top = \sqrt{\kappa^2 + \omega^2}I_2 \ . \tag{A.25}$$

We deduce that

$$\overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^\top = \begin{bmatrix} \sqrt{\kappa^2+\omega^2}I_2 & I_2 & 0 & \dots & 0 \\ 0 & \sqrt{\kappa^2+\omega^2}I_2 & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \sqrt{\kappa^2+\omega^2}I_2 \end{bmatrix} . \tag{A.26}$$

This allows to conclude. □

Lemma A.3. *We consider the system*

$$a_{k+1} = \mathcal{J}_a a_k \ , k \in \mathbb{N} \tag{A.27}$$

with

$$\mathcal{J}_a = \begin{bmatrix} -\mu & 1 & \dots & 0 \\ 0 & -\mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -\mu \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{A.28}$$

where μ is a positive real number. Then, the time-varying change of coordinates

$$b_k = \mathcal{G}_k a_k \tag{A.29}$$

with

$$\mathcal{G}_k = \mathit{diag}\{(-1)^k, (-1)^{k+1}, ..., (-1)^{k+n-1}\} \in \mathbb{R}^{n\times n} \tag{A.30}$$

gives

$$b_{k+1} = \mathcal{J}_b b_k \ , k \in \mathbb{N} \tag{A.31}$$

with

$$\mathcal{J}_b = \begin{bmatrix} \mu & 1 & \dots & 0 \\ 0 & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \mu \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{A.32}$$

Proof. According to (A.29) and (A.27),

$$b_{k+1} \quad = \quad \mathcal{G}_{k+1}\mathcal{J}_a a_k \ = \ \mathcal{G}_{k+1}\mathcal{J}_a \mathcal{G}_k^{-1} b_k \ . \tag{A.33}$$

Since $\mathcal{G}_k^{-1} = \mathcal{G}_k$ and $\mathcal{G}_{k+1} = -\mathcal{G}_k$, the sequence b_k satisfies

$$b_{k+1} = -\mathcal{G}_k \mathcal{J}_a \mathcal{G}_k b_k \ . \tag{A.34}$$

Now, we have

$$\mathcal{J}_a \mathcal{G}_k = \begin{bmatrix} \mu(-1)^{k+1} & (-1)^{k+1} & \dots & 0 \\ 0 & \mu(-1)^{k+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & (-1)^{k+n-1} \\ 0 & \dots & 0 & \mu(-1)^{k+n} \end{bmatrix} . \tag{A.35}$$

Therefore

$$\mathcal{G}_k \mathcal{J}_a \mathcal{G}_k \quad = \quad - \begin{bmatrix} \mu & 1 & \dots & 0 \\ 0 & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \mu \end{bmatrix} . \tag{A.36}$$

This equality and (A.34) allow to conclude. □

Proof of Theorem 4.9. *Step 1: Jordan canonical forms.*

From [110, Section 1.8], we deduce that for some integers $r \in \{0, 1, ..., n\}$, $s \in \{0, 1, ..., n - 1\}$ there exists a linear time-invariant change of coordinates

$$g_k = \mathcal{P} x_k, \tag{A.37}$$

which transforms (4.47) into

$$g_{k+1} = \overline{\mathcal{J}} g_k, \tag{A.38}$$

with

$$\overline{\mathcal{J}} = \text{diag}\{\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_s\} \in \mathbb{R}^{n\times n}, \tag{A.39}$$

where the matrices $\mathcal{J}_i$ are partitioned into two groups: the first r matrices are associated with the r real eigenvalues of multiplicity n_i of $\mathcal{A}$ and the others are associated with the imaginary eigenvalues of multiplicity m_i of $\mathcal{A}$. Therefore $n = \sum_{i=1}^{r} n_i + \sum_{r+1}^{s} 2m_i$ and, for $i = 1$ to r,

$$\mathcal{J}_i = \begin{bmatrix} -\mu_i & 1 & \dots & 0 \\ 0 & -\mu_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -\mu_i \end{bmatrix} \in \mathbb{R}^{n_i\times n_i}, \tag{A.40}$$

where the μ_i's are real numbers and, for $i = r+1$ to s,

$$\mathcal{J}_i = \begin{bmatrix} \mathcal{M}_i & I_2 & 0 & \dots & 0 \\ 0 & \mathcal{M}_i & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \mathcal{M}_i \end{bmatrix} \in \mathbb{R}^{2m_i\times 2m_i}, \tag{A.41}$$

with

$$\mathcal{M}_i = \begin{bmatrix} -\kappa_i & \omega_i \\ -\omega_i & -\kappa_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{A.42}$$

and

$$I_2 = \text{diag}\{1, 1\} \in \mathbb{R}^{2\times 2}, \tag{A.43}$$

where the ω_i's are non-zero real numbers.

Step 2: time-varying change of coordinates. We consider the system (A.38). From Lemmas A.2 and A.3 in Appendix A, we deduce that, for any system

$$a_{k+1} = \mathcal{J}_i a_k, \tag{A.44}$$

there exist a nonnegative matrix $\mathcal{H}_i$ with a spectral radius smaller than 1 and a sequence $(\mathcal{Q}_{k,i})$ of invertible matrices bounded in norm by 1 and with inverses bounded in norm by 1 such that the change of coordinates

$$b_k = \mathcal{Q}_{k,i} a_k \tag{A.45}$$

gives

$$b_{k+1} = \mathcal{H}_i b_k. \tag{A.46}$$

Next, we consider the change of coordinates

$$h_k = \overline{\mathcal{Q}}_k g_k, \tag{A.47}$$

with

$$\overline{\mathcal{Q}}_k = \mathrm{diag}\{\mathcal{Q}_{k,1}, \mathcal{Q}_{k,2}, ..., \mathcal{Q}_{k,s}\} \in \mathbb{R}^{n\times n}. \tag{A.48}$$

Then

$$h_{k+1} = \overline{\mathcal{Q}}_{k+1}\overline{\mathcal{J}}g_k = \overline{\mathcal{Q}}_{k+1}\overline{\mathcal{J}}\,\overline{\mathcal{Q}}_k^{-1}h_k = \overline{\mathcal{H}}h_k \tag{A.49}$$

with

$$\overline{\mathcal{H}} = \mathrm{diag}\{\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_s\} \in \mathbb{R}^{n\times n}.$$

Finally, we conclude by observing that the change of coordinates

$$h_k = \mathcal{R}_k x_k, \tag{A.50}$$

with $\mathcal{R}_k = \overline{\mathcal{Q}}_k\mathcal{P}$ gives the nonnegative exponentially stable time-invariant system

$$h_{k+1} = \overline{\mathcal{H}}h_k$$

and that $\mathcal{R}_k^{-1} = \mathcal{P}^{-1}\overline{\mathcal{Q}}_k^{-1}$, which implies that the sequences $(\mathcal{R}_k)$ and $(\mathcal{R}_k^{-1})$ are bounded in norm. □

Lemma A.4. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function of class C^1. Then there exists a function $f_c : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables such that, for all $\mathfrak{x} \in \mathbb{R}^m$, the equality*

$$f_c(\mathfrak{x}, \mathfrak{x}) = f(\mathfrak{x}) \tag{A.51}$$

is satisfied.

Proof. The function

$$\Gamma(s) = \sup_{|z|\leq\sqrt{s}} F(z) + 1 + s, \tag{A.52}$$

where $F : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$F(\mathfrak{x}) = |\mathfrak{x}||f(\mathfrak{x})| + \left|\frac{\partial f}{\partial \mathfrak{x}}(\mathfrak{x})\right|, \tag{A.53}$$

is continuous, positive and nondecreasing over $[0, +\infty)$. Then the function

$$\xi(s) = \frac{1}{2}\int_s^{+\infty} \frac{1}{e^m\Gamma(m)}dm \tag{A.54}$$

is well-defined, decreasing, continuously differentiable, such that, for all $s \geq 0$, $\xi(s) > 0$ and

$$\xi(s) \leq \frac{1}{2\Gamma(s)} \int_s^{+\infty} \frac{1}{e^m} dm \leq \frac{1}{2\Gamma(s)}, \tag{A.55}$$

$$|\xi'(s)| \leq \frac{1}{2\Gamma(s)}. \tag{A.56}$$

Now, we observe that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$f(\mathfrak{x}) = f_1(\mathfrak{x}) + f_2(\mathfrak{x}) \tag{A.57}$$

with

$$f_1(\mathfrak{x}) = \frac{4\sum_{j=1}^{m} \mathfrak{x}_j}{\xi(|\mathfrak{x}|^2)}, \; f_2(\mathfrak{x}) = \frac{f_3(\mathfrak{x})}{\xi(|\mathfrak{x}|^2)}, \; f_3(\mathfrak{x}) = \xi(|\mathfrak{x}|^2) f(\mathfrak{x}) - 4\sum_{j=1}^{m} \mathfrak{x}_j. \tag{A.58}$$

Moreover, (A.54) and (A.55) imply that, for all $i \in \{1, ..., m\}$ and for all $\mathfrak{x} \in \mathbb{R}^m$,

$$\begin{aligned}
\frac{\partial f_3}{\partial \mathfrak{x}_i}(\mathfrak{x}) &= 2\xi'(|\mathfrak{x}|^2)\mathfrak{x}_i f(\mathfrak{x}) + \xi(|\mathfrak{x}|^2)\frac{\partial f}{\partial \mathfrak{x}_i}(\mathfrak{x}) - 4 \\
&\leq 2|\xi'(|\mathfrak{x}|^2)||\mathfrak{x}_i||f(\mathfrak{x})| + \xi(|\mathfrak{x}|^2)\left|\frac{\partial f}{\partial \mathfrak{x}_i}(\mathfrak{x})\right| - 4 \\
&\leq \frac{1}{\Gamma(|\mathfrak{x}|^2)}|\mathfrak{x}_i||f(\mathfrak{x})| + \frac{1}{2\Gamma(|\mathfrak{x}|^2)}\left|\frac{\partial f}{\partial \mathfrak{x}_i}(\mathfrak{x})\right| - 4 \\
&\leq \frac{F(\mathfrak{x})}{\Gamma(|\mathfrak{x}|^2)} - 4 \\
&\leq -3,
\end{aligned} \tag{A.59}$$

where the last inequality is deduced from the definition of Γ. Therefore f_3 is nonincreasing with respect to each of its variables.

Now, we define two functions:

$$\phi_1(a,b) = \frac{4\sum_{j=1}^{m} \varpi(a_j)}{\xi\left(\sum_{j=1}^{m} \varpi(a_j)^2 + \sum_{j=1}^{m} \mho(b_j)^2\right)} + \frac{4\sum_{j=1}^{m} \mho(a_j)}{\xi\left(\sum_{j=1}^{m} \varpi(b_j)^2 + \sum_{j=1}^{m} \mho(a_j)^2\right)}, \tag{A.60}$$

$$\phi_2(a,b) = \frac{\varpi(f_3(b))}{\xi\left(\sum_{j=1}^{m} \varpi(a_j)^2 + \sum_{j=1}^{m} \mho(b_j)^2\right)} + \frac{\mho(f_3(b))}{\xi\left(\sum_{j=1}^{m} \varpi(b_j)^2 + \sum_{j=1}^{m} \mho(a_j)^2\right)}. \tag{A.61}$$

Using the fact that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$|\mathfrak{x}|^2 = \sum_{j=1}^{m} \mathfrak{x}_j^2 = \sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2 \tag{A.62}$$

we deduce that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$f_1(\mathfrak{x}) = \frac{4\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)}{\xi\left(\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2\right)} + \frac{4\sum_{j=1}^{m} \mho(\mathfrak{x}_j)}{\xi\left(\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2\right)}. \tag{A.63}$$

Therefore, for all $\mathfrak{x} \in \mathbb{R}^m$, $\phi_1(\mathfrak{x}, \mathfrak{x}) = f_1(\mathfrak{x})$ and ϕ_1 is nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables. We also observe that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$f_2(\mathfrak{x}) = \frac{\varpi(f_3(\mathfrak{x}))}{\xi\left(\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2\right)} + \frac{\mho(f_3(\mathfrak{x}))}{\xi\left(\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2\right)}, \tag{A.64}$$

which implies that, for all $\mathfrak{x} \in \mathbb{R}^m$, $\phi_2(\mathfrak{x}, \mathfrak{x}) = f_2(\mathfrak{x})$. Since f_3 is nonincreasing with respect to each of its variables, then ϕ_2 is nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables.

We are ready to conclude. We observe that

$$f(\mathfrak{x}) = f_1(\mathfrak{x}) + f_2(\mathfrak{x}) = f_c(\mathfrak{x}, \mathfrak{x}) \tag{A.65}$$

with $f_c(a, b) = \phi_1(a, b) + \phi_2(a, b)$. Thus the function f_c is nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables. $\square$

Lemma A.5. *Let $\xi_1 \in \mathbb{R}^g$, $\xi_2 \in \mathbb{R}^g$, $\xi_3 \in \mathbb{R}^g$ be vectors such that the inequalities*

$$\xi_1 \le \xi_2 \le \xi_3 \tag{A.66}$$

are satisfied. Let $M \in \mathbb{R}^{g \times g}$ be a constant matrix. Then the inequalities

$$M^+\xi_1 - M^-\xi_3 \le M\xi_2 \le M^+\xi_3 - M^-\xi_1 \tag{A.67}$$

are satisfied.

Proof. We have :

$$\xi_1 \le \xi_2 \le \xi_3$$

Since the matrices $M^+ = \max(M, 0)$ and $M^- = \max(-M, 0)$ are nonnegative matrices, it follows that

$$\begin{aligned} M^+\xi_1 \le M^+\xi_2 \le M^+\xi_3 \\ -M^-\xi_3 \le -M^-\xi_2 \le -M^-\xi_1 \end{aligned}$$

We deduce that

$$M^+\xi_1 - M^-\xi_3 \le (M^+ - M^-)\xi_2 \le M^+\xi_3 - M^-\xi_1 \ .$$

These inequalities rewrites :

$$M^+\xi_1 - M^-\xi_3 \le M\xi_2 \le M^+\xi_3 - M^-\xi_1 \ .$$

□

Lemma A.6. *Let $\xi_{k+1} = \mathcal{A}\xi_k + \mu_k$, where $\xi_k \in \mathbb{R}^n$, $\mu_k \in \mathbb{R}^n$, $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a matrix satisfying Assumption 1. Then $U(\xi) = \xi^\top Q\xi$ satisfies*

$$U(\xi_{k+1}) - U(\xi_k) \le -|\xi_k|^2 + 2\xi_k^\top \mathcal{A}^\top Q\mu_k + \mu_k^\top Q\mu_k. \tag{A.68}$$

Proof. The equality

$$U(\xi_{k+1}) - U(\xi_k) \ = \ \xi_k^\top[\mathcal{A}^\top Q\mathcal{A} - \mathcal{A}]\xi_k + 2\xi_k^\top \mathcal{A}^\top Q\mu_k + \mu_k^\top Q\mu_k \tag{A.69}$$

and the inequality (5.10) lead to (A.68). □

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