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ÉCOLE DOCTORALE 142 :
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THÈSE DE DOCTORAT

MATHÉMATIQUES

par

Shen LIN

Marche aléatoire indexée par un arbre
et marche aléatoire sur un arbre

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## Marche aléatoire indexée par un arbre et marche aléatoire sur un arbre

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Résumé

L'objet de cette thèse est d'étudier plusieurs modèles probabilistes reliant les marches aléatoires et les arbres aléatoires issus de processus de branchement critiques.

Dans la première partie, nous nous intéressons au modèle de marche aléatoire à valeurs dans un réseau euclidien et indexée par un arbre de Galton–Watson critique conditionné par la taille. Sous certaines hypothèses sur la loi de reproduction critique et la loi de saut centrée, nous obtenons, dans toutes les dimensions, la vitesse de croissance asymptotique du nombre de points visités par cette marche, lorsque la taille de l'arbre tend vers l'infini. Ces résultats nous permettent aussi de décrire le comportement asymptotique du nombre de points visités par une marche aléatoire branchante, quand la taille de la population initiale tend vers l'infini. Nous traitons également en parallèle certains cas où la marche aléatoire possède une dérive constante non nulle.

Dans la deuxième partie, nous nous concentrons sur les propriétés fractales de la mesure harmonique des grands arbres de Galton–Watson critiques. On comprend par mesure harmonique la distribution de sortie, hors d'une boule centrée à la racine de l'arbre, d'une marche aléatoire simple sur cet arbre. Lorsque la loi de reproduction critique appartient au domaine d'attraction d'une loi stable, nous prouvons que la masse de la mesure harmonique est asymptotiquement concentrée sur une partie de la frontière, cette partie ayant une taille négligeable par rapport à celle de la frontière. En supposant que la loi de reproduction critique a une variance finie, nous arrivons à évaluer la masse de la mesure harmonique portée par un sommet de la frontière choisi uniformément au hasard.

Mots clés : arbre de Galton–Watson critique, marche aléatoire indexée par un arbre, marche aléatoire branchante, nombre de points visités, mesure ISE, serpent brownien, super-mouvement brownien, marche aléatoire sur un arbre, mesure harmonique, dimension de Hausdorff, mesure invariante.

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## Tree-indexed random walk and random walk on trees

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Abstract

The aim of this Ph. D. thesis is to study several probabilistic models linking the random walks and the random trees arising from critical branching processes.

In the first part, we consider the model of random walk taking values in a Euclidean lattice and indexed by a critical Galton–Watson tree conditioned by the total progeny. Under some assumptions on the critical offspring distribution and the centered jump distribution, we obtain, in all dimensions, the asymptotic growth rate of the range of this random walk, when the size of the tree tends to infinity. These results also allow us to describe the asymptotic behavior of the range of a branching random walk, when the size of the initial population goes to infinity. In parallel, we treat likewise some cases where the random walk has a non-zero constant drift.

In the second part, we focus on the fractal properties of the harmonic measure on large critical Galton–Watson trees. By harmonic measure, we mean the exit distribution from a ball centered at the root of the tree by simple random walk on this tree. If the critical offspring distribution is in the domain of attraction of a stable distribution, we prove that the mass of the harmonic measure is asymptotically concentrated on a boundary subset of negligible size with respect to that of the boundary. Assuming that the critical offspring distribution has a finite variance, we are able to calculate the mass of the harmonic measure carried by a random vertex uniformly chosen from the boundary.

Keywords: critical Galton–Watson tree, tree-indexed random walk, branching random walk, range, ISE, Brownian snake, super-Brownian motion, random walk on trees, harmonic measure, Hausdorff dimension, invariant measure.

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INTRODUCTION

1.1 Arbres généalogiques aléatoires

1.1.1 Arbres de Galton–Watson

Un arbre discret est un graphe connexe sans cycle, et un arbre de Galton–Watson est un arbre discret aléatoire qui décrit la généalogie d’une population gouvernée par un processus de Galton–Watson.

Commençons par présenter le formalisme introduit par Neveu [79] pour les arbres *planaires* (enracinés et ordonnés). Soit \mathcal{U} l’ensemble des mots d’entiers défini par

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

où, par convention, $\mathbb{N} = \{1, 2, \dots\}$, et $\mathbb{N}^0 = \{\emptyset\}$ est réduit à un seul élément que l’on interprète comme la racine ou l’ancêtre de la population. Un élément de \mathcal{U} est une suite $u = (u_1, u_2, \dots, u_n)$, et l’on pose $|u| = n$ de sorte que $|u|$ représente la génération de u . En particulier, $|\emptyset| = 0$. Nous noterons \prec l’ordre lexicographique sur \mathcal{U} , tel que $\emptyset \prec 1 \prec (1, 1) \prec 2$ par exemple. Si $u = (u_1, u_2, \dots, u_n) \in \mathcal{U} \setminus \{\emptyset\}$, alors on appelle $\check{u} := (u_1, u_2, \dots, u_{n-1})$ le parent de u . Lorsque $u = (u_1, \dots, u_m)$ et $v = (v_1, \dots, v_n)$ sont des éléments de \mathcal{U} , on définit la concaténation de u et v par $uv = (u_1, \dots, u_m, v_1, \dots, v_n)$. En particulier, $u\emptyset = \emptyset u = u$.

Définition 1. Un *arbre planaire* τ est un sous-ensemble (fini ou infini) de \mathcal{U} vérifiant les trois propriétés suivantes :

- (i) $\emptyset \in \tau$;
- (ii) si $u \in \tau \setminus \{\emptyset\}$, alors $\check{u} \in \tau$;
- (iii) pour tout $u \in \tau$, il existe un entier $k_u(\tau) \geq 0$ tel que, pour chaque $j \in \mathbb{N}$, $uj \in \tau$ si et seulement si $1 \leq j \leq k_u(\tau)$.

Pour $u \in \tau$, nous appellerons $k_u(\tau)$ le nombre d’enfants de u . On note \mathcal{T} l’ensemble des arbres planaires, et \mathcal{T}_f le sous-ensemble contenant tous les arbres planaires finis. Si $\tau \in \mathcal{T}_f$, on appelle son nombre total de sommets (noté $\#\tau$) la taille de τ , et $h(\tau)$ désignera la hauteur de τ , c’est-à-dire

$$h(\tau) := \sup\{|u| : u \in \tau\}.$$

Pour un arbre τ infini, on pose par convention $\#\tau = h(\tau) = \infty$. Si $\tau \in \mathcal{T}$ et $u \in \tau$, on note $\tau[u] := \{w \in \mathcal{U} : uw \in \tau\}$ le sous-arbre de τ issu de u . Par la suite, on considère un arbre $\tau \in \mathcal{T}$

toujours comme un graphe dont les sommets sont les éléments de τ et les arêtes sont les couples $\{\check{u}, u\}$ pour tout $u \in \tau \setminus \{\emptyset\}$.

Tout arbre planaire fini est codé par une fonction continue appelée la *fonction de contour*. Pour définir la fonction de contour $C(\tau)$ d'un arbre planaire fini τ , imaginons que τ soit plongé sur le demi-plan supérieur de sorte que toutes ses arêtes soient de longueur unité, et considérons une particule qui explore l'arbre en partant de la racine à vitesse unité. Elle se déplace tout autour de τ dans le sens des aiguilles d'une montre. Chaque arête est visitée deux fois par la particule si bien qu'il lui faut un temps $2(\#\tau - 1)$ pour parcourir entièrement τ . Pour chaque entier $t \in [0, 2(\#\tau - 1)]$, on définit $C_t(\tau)$ comme la hauteur du sommet visité par la particule à l'instant t . Puis on interpole linéairement $C(\tau)$ sur l'intervalle $[0, 2(\#\tau - 1)]$, et l'on pose $C_t(\tau) = 0$ pour tout $t > 2(\#\tau - 1)$.

On peut aussi coder un arbre planaire fini d'une façon légèrement différente. Si $\tau \in \mathcal{T}_f$, énumérons les sommets de τ dans l'ordre lexicographique $u_0 = \emptyset \prec u_1 \prec \dots \prec u_{\#\tau-1}$. Pour chaque $n \in \{0, 1, \dots, \#\tau - 1\}$, on définit $H_n(\tau)$ comme la hauteur du sommet u_n . On pose $H_m(\tau) = 0$ pour $m \geq \#\tau$, et l'on prolonge alors $H(\tau)$ à \mathbb{R}_+ par interpolation linéaire. La fonction $H(\tau) = (H_t(\tau), t \geq 0)$ est appelée *fonction de hauteur* de τ .

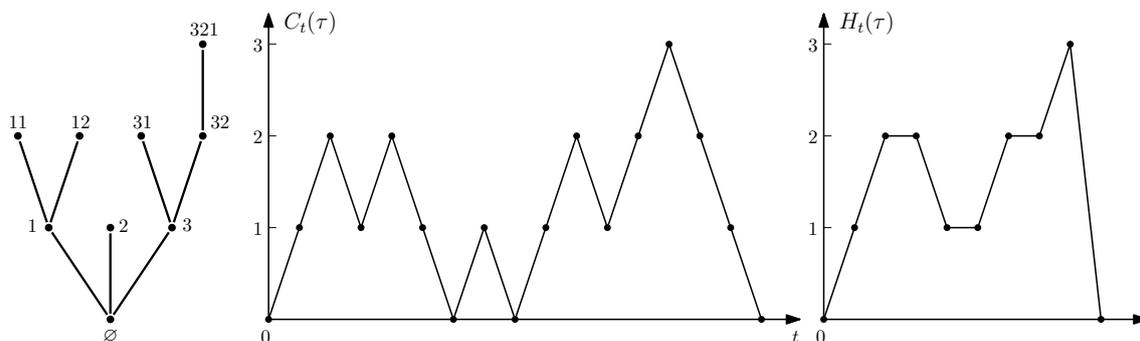


Figure 1.1 – Un arbre planaire τ , sa fonction de contour $C(\tau)$, et sa fonction de hauteur $H(\tau)$

Nous rappelons à présent la définition d'un arbre de Galton–Watson. Soit μ une mesure de probabilité sur \mathbb{Z}_+ telle que $\mu(1) < 1$. Un *arbre de Galton–Watson de loi de reproduction μ* (abrégé en μ -GW arbre) est un arbre planaire aléatoire tel que chaque sommet de l'arbre ait un nombre d'enfants de loi μ , et tel que les nombres d'enfants de sommets différents soient des variables aléatoires indépendantes. On peut vérifier que la loi Π_μ d'un μ -GW arbre est l'unique mesure de probabilité sur \mathcal{T} vérifiant les deux conditions suivantes :

- (i) $\Pi_\mu(k_{\emptyset}(\tau) = k) = \mu(k)$ pour tout $k \geq 0$;
- (ii) pour tout $k \geq 1$ tel que $\mu(k) > 0$, sous la mesure $\Pi_\mu(d\tau \mid k_{\emptyset}(\tau) = k)$, les sous-arbres $\tau[1], \dots, \tau[k]$ sont i.i.d. de loi Π_μ .

Il est bien connu qu'un arbre aléatoire de loi Π_μ est p.s. fini si et seulement si la moyenne de μ est inférieure ou égale à 1. Lorsque $\sum_{k \geq 0} k\mu(k) = 1$, Π_μ est donc une loi de probabilité sur \mathcal{T}_f , et l'on dira que μ (et, par extension, un μ -GW arbre) est *critique*.

1.1.2 Arbres de Galton–Watson conditionnés

Fixons une loi de reproduction μ . Sans mention contraire, nous supposons dans cette introduction que le pgcd du support de μ est égal à 1. Pour tout entier n suffisamment grand tel que $\Pi_\mu(\#\tau = n) > 0$, on note \mathbb{T}_n un arbre de Galton–Watson conditionné à avoir n sommets, c'est-à-dire un arbre aléatoire de loi $\Pi_\mu(d\tau | \#\tau = n)$.

Plusieurs classes combinatoires d'arbres aléatoires à n sommets peuvent être réalisées comme des arbres de Galton–Watson conditionnés à avoir n sommets pour des lois de reproduction particulières. Par exemple,

- lorsque μ est la loi géométrique de paramètre $1/2$, i.e. $\mu(k) = 2^{-(k+1)}$ pour tout entier $k \geq 0$, \mathbb{T}_n est uniformément distribué sur l'ensemble des arbres planaires à n sommets ;
- lorsque $\mu(0) = \mu(2) = 1/2$, \mathbb{T}_n est uniformément distribué sur l'ensemble des arbres binaires à n sommets ;
- lorsque μ est la loi de Poisson de paramètre 1, i.e. $\mu(k) = e^{-1}/k!$ pour tout entier $k \geq 0$, \mathbb{T}_n est uniformément distribué sur l'ensemble des arbres étiquetés non ordonnés à n sommets (souvent appelés « arbres de Cayley »).

Nous renvoyons à Le Gall [58, Section 1.5] pour plus d'explications et de précisions.

Remarque 1. Soit $\gamma > 0$ un paramètre fixé tel que $Z_\gamma = \sum_{k \geq 0} \mu(k)\gamma^k < \infty$. On définit une nouvelle loi de probabilité $\mu_{(\gamma)}$ en posant, pour tout $k \geq 0$,

$$\mu_{(\gamma)}(k) := \frac{\mu(k)\gamma^k}{Z_\gamma}.$$

On dit alors que les deux lois μ et $\mu_{(\gamma)}$ appartiennent à la même *famille exponentielle*. D'après Kennedy [42, Section 2], un μ -GW arbre conditionné à avoir n sommets a la même loi qu'un $\mu_{(\gamma)}$ -GW arbre conditionné à avoir n sommets. Donc, s'il existe $\gamma > 0$ tel que $Z_\gamma < \infty$ et $\mu_{(\gamma)}$ soit critique, alors étudier un arbre de Galton–Watson non critique conditionné revient à étudier un arbre de Galton–Watson critique conditionné. Dans le cas où cette réduction est impossible, on dit que μ est *non générique*, et nous renvoyons à [37, 38, 40, 46] pour l'étude des phénomènes différents qui se produisent dans ce cas.

Remarque 2. Quand μ est non critique, on peut démontrer la décroissance exponentielle de la probabilité $\Pi_\mu(\#\tau = n)$ lorsque n tend vers l'infini, alors que si μ est critique, la même probabilité décroît à une vitesse polynomiale. Selon Aldous [4], le cas critique est en ce sens plus naturel pour un modèle d'arbre aléatoire à n sommets.

De façon similaire, pour chaque entier $n \geq 1$ tel que $\Pi_\mu(h(\tau) \geq n) > 0$, on note $\mathbb{T}^{(n)}$ un arbre de Galton–Watson conditionné à avoir une hauteur supérieure à n , c'est-à-dire un arbre aléatoire de loi $\Pi_\mu(d\tau | h(\tau) \geq n)$.

Dans toute cette thèse, nous nous concentrons sur le cas où μ est critique.

1.1.3 Théorèmes de convergence pour les fonctions de codage

Soit μ une loi de reproduction critique et σ_μ^2 la variance de μ .

Théorème 1 (Aldous [5]). *Supposons que $\sigma_\mu = (\text{var } \mu)^{1/2} \in (0, \infty)$. Alors on a la convergence en loi*

$$\left(\frac{\sigma_\mu}{2\sqrt{n}} C_{2nt}(\mathbb{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(\text{loi})} (\mathbf{e}_t)_{0 \leq t \leq 1},$$

qui a lieu dans l'espace des fonctions continues $\mathcal{C}([0, 1], \mathbb{R})$ muni de la topologie uniforme. Dans la limite, $\mathbf{e} = (\mathbf{e}_t)_{0 \leq t \leq 1}$ désigne une excursion brownienne normalisée (i.e. de durée égale à 1), voir e.g. le livre de Revuz et Yor [82, Chapitre XII] pour sa définition.

Une application directe de ce théorème concerne la hauteur $h(\mathbb{T}_n)$ de l'arbre \mathbb{T}_n . En remarquant que $h(\mathbb{T}_n)$ est le maximum de $C(\mathbb{T}_n)$, on en déduit que $h(\mathbb{T}_n)/\sqrt{n}$ converge en loi, lorsque $n \rightarrow \infty$, vers $\frac{2}{\sigma_\mu} \max \mathbf{e}$.

Une généralisation du théorème d'Aldous au cas où μ est dans le domaine d'attraction d'une loi stable a été obtenue par Duquesne [25]. Rappelons que, si $\alpha \in (1, 2]$, on dit que la loi de reproduction critique μ appartient au domaine d'attraction d'une loi stable d'indice α s'il existe une suite croissante $(a_n)_{n \geq 1}$ telle que $a_n \rightarrow \infty$ et

$$\frac{Z[n] - n}{a_n} \xrightarrow[n \rightarrow \infty]{(\text{loi})} Y,$$

où $Z[n]$ est une somme de n variables aléatoires indépendantes de même loi μ , et la transformée de Laplace de la variable limite Y est donnée par $E[e^{-\lambda Y}] = e^{-\lambda^\alpha}$. Selon les résultats classiques de [31, Chapitres XIII et XVII], μ appartient au domaine d'attraction d'une loi stable d'indice α si et seulement si $\mu(1) < 1$ et

$$\sum_{k \geq 0} \mu(k)r^k = r + (1-r)^\alpha L(1-r) \quad \text{pour tout } r \in [0, 1), \quad (1.1)$$

où la fonction $L(x)$ est à *variation lente* lorsque $x \rightarrow 0^+$, i.e. L est une fonction réelle mesurable, positive dans un voisinage de 0, telle que

$$\lim_{x \rightarrow 0^+} \frac{L(tx)}{L(x)} = 1 \quad \text{pour tout } t > 0.$$

Lorsque μ a une variance finie et non nulle, la condition (1.1) est satisfaite avec $\alpha = 2$.

Théorème 2 (Duquesne [25]). *Soit μ une loi de reproduction critique dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Il existe une suite de nombres réels strictement positifs $A_n \rightarrow \infty$ telle que la convergence conjointe*

$$\left(\frac{A_n}{n} C_{2nt}(\mathbb{T}_n), \frac{A_n}{n} H_{nt}(\mathbb{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(\text{loi})} (H_t^{\text{exc}}, H_t^{\text{exc}})_{0 \leq t \leq 1}$$

a lieu dans $\mathcal{C}([0, 1], \mathbb{R})^2$. Dans la limite, $(H_t^{\text{exc}})_{0 \leq t \leq 1}$ désigne une excursion normalisée du processus de hauteur stable d'indice α , voir [25, Section 3] pour sa définition.

De plus, on peut montrer que la suite $(A_n/n^{1/\alpha})_{n \geq 1}$ est à variation lente, de sorte que A_n est grosso modo de l'ordre de $n^{1/\alpha}$. Il est aussi possible d'expliciter A_n en terme de la loi de reproduction μ , cf. [47, Theorem 1.10]. Lorsque $\alpha = 2$, le processus H^{exc} a la même loi que $\sqrt{2}\mathbf{e}$.

1.1.4 Arbres continus aléatoires

Les résultats présentés ci-dessus qui concernent les limites d'échelle des fonctions codant \mathbb{T}_n peuvent être interprétés comme des convergences des arbres discrets aléatoires vers certains « arbres continus aléatoires ».

Par définition, un *arbre réel* est un espace métrique (\mathcal{T}, d) tel que pour chaque couple de points $u, v \in \mathcal{T}$, il existe à reparamétrisation près un unique chemin injectif continu reliant u à v , et de plus ce chemin $[u, v]$ est isométrique au segment $[0, d(u, v)]$. L'ensemble des arbres continus que nous considérerons par la suite est l'ensemble des arbres réels compacts et enracinés, c'est-à-dire ayant un point $\rho \in \mathcal{T}$ distingué appelé racine. Deux arbres réels compacts et enracinés \mathcal{T} et \mathcal{T}' sont dits équivalents s'il existe une bijection isométrique entre \mathcal{T} et \mathcal{T}' préservant la racine. On note \mathbb{T} l'ensemble des classes d'équivalence d'arbres réels compacts enracinés.

Si (E, δ) est un espace métrique, rappelons que δ_{H} est la distance de Hausdorff entre les compacts de E . L'espace \mathbb{T} est muni de la distance de Gromov–Hausdorff d_{GH} définie de la façon suivante : si (\mathcal{T}, ρ, d) et $(\mathcal{T}', \rho', d')$ sont deux arbres réels compacts de racines respectives ρ et ρ' , on pose

$$d_{\text{GH}}((\mathcal{T}, \rho, d), (\mathcal{T}', \rho', d')) := \inf \{ \delta_{\text{H}}(\phi(\mathcal{T}), \phi'(\mathcal{T}')) \vee \delta(\phi(\rho), \phi'(\rho')) \},$$

où l'infimum est pris sur toutes les injections isométriques $\phi: \mathcal{T} \rightarrow E$ et $\phi': \mathcal{T}' \rightarrow E$ dans un même espace métrique (E, δ) . Il est clair que $d_{\text{GH}}((\mathcal{T}, \rho, d), (\mathcal{T}', \rho', d'))$ ne dépend que des classes d'équivalence de (\mathcal{T}, ρ, d) et $(\mathcal{T}', \rho', d')$. Evans, Pitman et Winter [30, Theorem 2] ont montré que l'espace métrique $(\mathbb{T}, d_{\text{GH}})$ ainsi obtenu est complet et séparable.

On peut construire un arbre réel par un codage similaire au codage d'un arbre planaire par sa fonction de contour. Plus précisément, si $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ est une fonction continue (non identiquement nulle) à support compact telle que $f(0) = 0$, pour $s, t \geq 0$ on définit

$$d_f(s, t) := f(s) + f(t) - 2 \min_{r \in [s \wedge t, s \vee t]} f(r).$$

Il est facile de vérifier que d_f est une pseudo-distance sur \mathbb{R}_+ . On peut alors introduire une relation d'équivalence sur \mathbb{R}_+ en disant $s \sim_f t$ si $d_f(s, t) = 0$. L'arbre continu codé par f est l'espace métrique quotient

$$\mathcal{T}_f := [0, +\infty) / \sim_f$$

muni de la distance d_f et enraciné en la classe d'équivalence de 0. On note $p_f: [0, +\infty) \rightarrow \mathcal{T}_f$ la projection canonique. Pour tout $s \in \mathbb{R}_+$, $p_f(s)$ est donc un sommet de \mathcal{T}_f à distance $f(s)$ de la racine.

Remarquons qu'un arbre planaire fini τ peut être vu comme un arbre réel compact enraciné $\mathcal{T}(\tau)$ si l'on remplace les arêtes de τ par des segments de longueur 1. Par ailleurs, si $C(\tau)$ est la fonction de contour de τ , on vérifie que l'arbre continu $\mathcal{T}(\tau)$ coïncide avec l'arbre continu $\mathcal{T}_{C(\tau)}$ construit à partir de $C(\tau)$ à isométrie près. Signalons toutefois que l'arbre planaire τ ne peut être reconstruit à partir de l'arbre réel $\mathcal{T}(\tau)$, car il n'y a pas d'ordre entre les enfants d'un même individu dans $\mathcal{T}(\tau)$. Par la suite, on utilisera la même notation τ pour désigner $\mathcal{T}(\tau)$ s'il n'y a pas de risque d'ambiguïté dans le contexte.

1.1.5 Limites d'échelle d'arbres de Galton–Watson conditionnés

D'après [58, Lemma 2.4], si f et f' sont deux fonctions (excursions) du type considéré précédemment, alors $d_{\text{GH}}(\mathcal{T}_f, \mathcal{T}_{f'}) \leq 2\|f - f'\|_{\infty}$. En utilisant cette propriété, on peut reformuler les théorèmes d'Aldous et de Duquesne en termes de convergence (au sens de Gromov–Hausdorff) des arbres de Galton–Watson conditionnés proprement renormalisés .

D'abord, si (\mathcal{T}, d) est un arbre réel et si $r > 0$, notons $r\mathcal{T}$ l'arbre réel obtenu de \mathcal{T} en multipliant par r la distance d . Rappelons que \mathbb{T}_n désigne un μ -GW arbre conditionné à avoir

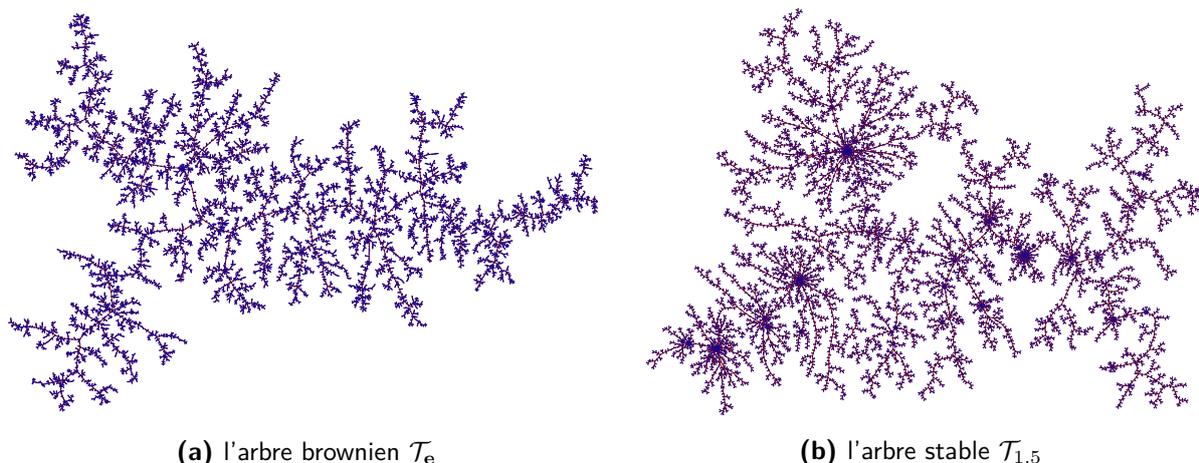


Figure 1.2 – Simulations approchées des arbres continus aléatoires (© I. Kortchemski)

n sommets. Quand la loi de reproduction μ est critique et de variance σ_μ^2 finie non nulle, le Théorème 1 entraîne que

$$\frac{\sigma_\mu}{2\sqrt{n}} \mathbb{T}_n \xrightarrow[n \rightarrow \infty]{(\text{loi})} \mathcal{T}_e,$$

la convergence ayant lieu en loi dans l'espace $(\mathbb{T}, d_{\text{GH}})$. Dans la limite, l'arbre continu aléatoire \mathcal{T}_e codé par l'excursion brownienne normalisée \mathbf{e} est appelé *arbre brownien continu* (abrégé en CRT pour « Continuum Random Tree »). Il a été introduit par Aldous [3, 4, 5]* au début des années 1990.

Plus généralement, si la loi de reproduction critique μ appartient au domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$, le Théorème 2, dont nous gardons les notations, implique alors la convergence en loi

$$\frac{A_n}{n} \mathbb{T}_n \xrightarrow[n \rightarrow \infty]{(\text{loi})} \mathcal{T}_{H^{\text{exc}}}$$

dans l'espace $(\mathbb{T}, d_{\text{GH}})$. On appelle l'objet limite $\mathcal{T}_{H^{\text{exc}}}$ *l'arbre de Lévy stable de paramètre α* , qui a été introduit par Le Gall et Le Jan [60], ainsi que par Duquesne et Le Gall [26]†. Pour rendre la notation plus claire, on écrit désormais \mathcal{T}_α pour un arbre de Lévy stable de paramètre α . En particulier, \mathcal{T}_2 a la même loi que $\mathcal{T}_{\sqrt{2}\mathbf{e}}$.

Soit \mathcal{T} un arbre continu. Le degré d'un point $u \in \mathcal{T}$ est par définition le nombre de composantes connexes de $\mathcal{T} \setminus \{u\}$. Un point de degré 1 est appelé une feuille, et un point de degré supérieur ou égal à 3 est un point de branchement. Il est bien connu que l'arbre brownien est p.s. un arbre binaire au sens où les points de branchement sont de degré 3. En revanche, les points de branchement de \mathcal{T}_α sont p.s. tous de degré infini quand $\alpha \in (1, 2)$.

Notons que Curien et Haas [19] ont prouvé que les arbres stables ($\mathcal{T}_\alpha, 1 < \alpha \leq 2$) peuvent être construits sur un même espace de probabilité en étant emboîtés les uns dans les autres (à des facteurs multiplicatifs près).

*. Remarquons que dans cette série d'articles, Aldous a plutôt pris $\mathcal{T}_{2\mathbf{e}}$ pour la définition du CRT.

†. La définition de $\mathcal{T}_{H^{\text{exc}}}$ en tant qu'arbre se trouve dans un article ultérieur de Duquesne et Le Gall [27].

1.2 Autour de la marche aléatoire classique sur \mathbb{Z}^d

1.2.1 Théorème de Dvoretzky et Erdős

Soit θ une mesure de probabilité sur le réseau euclidien \mathbb{Z}^d . Si Y_1, Y_2, \dots est une suite de variables aléatoires indépendantes de même loi θ , on définit la marche aléatoire $X = (X_n)_{n \geq 0}$ issue de l'origine $0 \in \mathbb{Z}^d$ avec la loi de saut θ en posant $X_0 = 0$ et $X_n = Y_1 + \dots + Y_n$ pour tout entier $n \geq 1$. On note $R_n := \#\{X_0, X_1, \dots, X_n\}$ le nombre de points visités par la marche jusqu'à l'instant n .

L'étude du comportement asymptotique de R_n lorsque $n \rightarrow \infty$ a fait l'objet de nombreux travaux, à commencer par le travail pionnier [29] de Dvoretzky et Erdős en 1951.

Théorème 3 (Dvoretzky et Erdős [29]). *Supposons que la loi de saut θ sur \mathbb{Z}^d est uniforme sur les $2d$ voisins de l'origine.*

- Si $d = 2$, alors

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{\text{p.s.}} \pi. \quad (1.2)$$

- Si $d \geq 3$, on note $q_d > 0$ la probabilité que la marche aléatoire X ne retourne plus à son point de départ, alors

$$\frac{1}{n} R_n \xrightarrow[n \rightarrow \infty]{\text{p.s.}} q_d. \quad (1.3)$$

En dimension $d = 1$, grâce au théorème de Donsker, selon lequel la marche aléatoire simple renormalisée $(n^{-1/2} X_{[nt]})_{t \geq 0}$ converge vers un mouvement brownien réel standard $(B_t)_{t \geq 0}$, on a immédiatement

$$\frac{R_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{(\text{loi})} \sup_{0 \leq t \leq 1} B_t - \inf_{0 \leq t \leq 1} B_t.$$

L'argument de Dvoretzky et Erdős pour le théorème précédent consiste à estimer les moments d'ordre 1 et d'ordre 2 de R_n . Cette méthode se généralise au cas où θ est centrée et de variance finie. En particulier, si $d = 2$, leur résultat (1.2) est lié au fait que la marche aléatoire simple sur \mathbb{Z}^2 ne retourne pas à l'origine avant l'instant n avec une probabilité $(\pi + o(1))/\log n$ lorsque $n \rightarrow \infty$.

Quand $d \geq 3$, nous pouvons obtenir la convergence (1.3) de R_n/n comme une conséquence simple du *théorème ergodique sous-additif de Kingman* [45]. Pour tous $m, n \geq 0$, le nombre de sites visités entre 0 et $n + m$ par la marche X est inférieur au nombre de sites visités entre 0 et n plus le nombre de sites visités entre n et $n + m$. Cela se traduit comme la relation $R_{n+m} \leq R_n + R_m \circ \tau_n$, où τ_n est l'opérateur de shift sur les trajectoires : $X_k \circ \tau_n = X_{n+k} - X_n$. En utilisant le théorème ergodique sous-additif de Kingman, on a donc

$$\frac{1}{n} R_n \xrightarrow[n \rightarrow \infty]{\text{p.s.}} q_d,$$

où $q_d = \lim_{n \rightarrow \infty} \frac{1}{n} E[R_n] = P(X_1 \neq 0, X_2 \neq 0, \dots)$. Cette approche due à Kesten, Spitzer et Whitman [85] s'applique à toutes les marches aléatoires, et l'on voit que la constante limite $q_d > 0$ si et seulement si la marche aléatoire X est *transitoire* (i.e. $d \geq 3$ dans le cadre de la marche aléatoire simple).

Nous renvoyons le lecteur à Jain et Pruitt [33, 35] pour un théorème de fluctuation du nombre de points visités R_n en dimension $d \geq 3$. L'analogue en dimension $d = 2$ se trouve dans Le Gall [54]. Lorsque la marche aléatoire est dans le domaine d'attraction d'une loi stable, voir un panorama des théorèmes limites pour R_n dans Le Gall et Rosen [64].

1.2.2 Principe d'invariance des temps locaux de la marche aléatoire

Soit $(X_n)_{n \geq 0}$ une marche aléatoire simple sur \mathbb{Z} avec $X_0 = 0$. Pour tous $n \geq 1$, $p \in \mathbb{Z}$, notons

$$\lambda_n^p := \#\{0 \leq k \leq n : X_k = p\}$$

qui représente le nombre de visites du point p par la marche X avant l'instant $n + 1$.

Le principe d'invariance de Donsker suggère fortement l'existence des analogues browniens comme limite en loi des nombres de visites de la marche aléatoire X en différents points convenablement renormalisés. Cela nous conduit à la notion de temps local du mouvement brownien.

Théorème 4. *Soit $(B_t)_{t \geq 0}$ un mouvement brownien réel issu de 0. Il existe un processus aléatoire $(L_t^a, a \in \mathbb{R}, t \geq 0)$ à valeurs dans \mathbb{R}_+ , unique à indistinguabilité près, tel que*

- (i) *p.s., l'application $(t, a) \mapsto L_t^a$ est continue, et croissante en la variable t ;*
- (ii) *p.s., pour toute fonction $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ mesurable, pour tout $t \geq 0$,*

$$\int_0^t \varphi(B_s) ds = \int da \varphi(a) L_t^a.$$

La variable L_t^a est appelée temps local au niveau a , à l'instant t du mouvement brownien B .

Remarquons que le temps local L_t^a peut être vu comme la densité de temps d'occupation en a , et défini comme la limite

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{a-\varepsilon < B_s < a+\varepsilon\}} ds.$$

Pour plus de renseignements sur le temps local brownien, voir e.g. le chapitre VI du livre de Revez et Yor [82].

En utilisant les marches aléatoires plongées dans un mouvement brownien, on peut établir le résultat classique suivant. Voir par exemple [78, Theorem 6.19] pour une version plus forte.

Théorème 5. *Soit un entier $k \geq 1$. Pour tous $a_1, \dots, a_k \in \mathbb{R}$, on a*

$$\left(\frac{1}{\sqrt{n}} \lambda_n^{\lfloor n^{1/2} a_1 \rfloor}, \dots, \frac{1}{\sqrt{n}} \lambda_n^{\lfloor n^{1/2} a_k \rfloor} \right) \xrightarrow[n \rightarrow \infty]{(\text{loi})} (L_1^{a_1}, \dots, L_1^{a_k}). \quad (1.4)$$

Cette convergence a lieu conjointement avec la convergence en loi de $(n^{-1/2} X_{\lfloor nt \rfloor})_{t \geq 0}$ vers le mouvement brownien $(B_t)_{t \geq 0}$. On peut même avoir une version fonctionnelle de la convergence (1.4), i.e.

$$\left(\frac{1}{\sqrt{n}} \lambda_{\lfloor nt \rfloor}^{\lfloor n^{1/2} a_1 \rfloor}, \dots, \frac{1}{\sqrt{n}} \lambda_{\lfloor nt \rfloor}^{\lfloor n^{1/2} a_k \rfloor} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(\text{loi})} (L_t^{a_1}, \dots, L_t^{a_k})_{t \geq 0}.$$

1.3 Marches aléatoires sur \mathbb{Z}^d indexées par un arbre

1.3.1 Arbres de Galton–Watson spatiaux

Un *arbre spatial discret* (d -dimensionnel) est un couple $(\tau, (z_u)_{u \in \tau})$ où $\tau \in \mathcal{T}_f$ et $z_u \in \mathbb{Z}^d$ pour tout $u \in \tau$. On note \mathcal{T}_f^* l'ensemble des arbres spatiaux discrets.

Soit θ une mesure de probabilité sur \mathbb{Z}^d . On supposera toujours que son support n'appartient pas à un sous-groupe strict de \mathbb{Z}^d . Étant donné un arbre $\tau \in \mathcal{T}_f$, on définit la *marche aléatoire*

$(Z_\tau(u), u \in \tau)$ sur \mathbb{Z}^d indexée par τ et de loi de saut θ de la façon suivante : d'abord, pour la racine \emptyset , on pose $Z_\tau(\emptyset) = 0$. Puis, à chaque arête e de l'arbre τ , nous attribuons de manière indépendante une variable aléatoire Y_e à valeurs dans \mathbb{Z}^d de loi θ . La position spatiale $Z_\tau(u)$ d'un sommet u différent de la racine est définie comme la somme des Y_e sur toutes les arêtes e le long du chemin simple de la racine à u dans l'arbre τ .

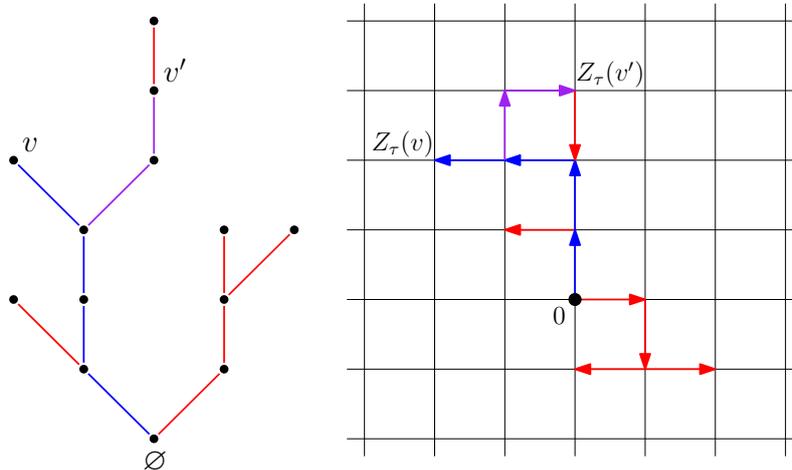


Figure 1.3 – Un arbre planaire τ et une marche aléatoire Z_τ sur \mathbb{Z}^2 indexée par τ

Soit μ une loi de reproduction critique. On écrit maintenant $\Pi_{\mu,\theta}^*$ la mesure de probabilité sur \mathcal{T}_f^* sous laquelle τ est distribué selon Π_μ et, conditionnellement à τ , les positions spatiales $(z_u)_{u \in \tau}$ sont distribuées comme une marche aléatoire indexée par τ et de loi de saut θ . On dit que la mesure $\Pi_{\mu,\theta}^*$ est la loi d'un *arbre de Galton–Watson spatial* issu de 0, de loi de reproduction μ et de loi de déplacement spatial θ .

Rappelons que \mathbb{T}_n désigne un μ -GW arbre conditionné à avoir n sommets. Notons $Z_{\mathbb{T}_n}$ une marche aléatoire sur \mathbb{Z}^d indexée par \mathbb{T}_n . Le couple $(\mathbb{T}_n, Z_{\mathbb{T}_n}) \in \mathcal{T}_f^*$ est alors distribué selon la loi de probabilité conditionnelle $\Pi_{\mu,\theta}^*(\cdot | \#\tau = n)$.

1.3.2 Mouvement brownien indexé par \mathcal{T}_e et serpent brownien

En s'inspirant du théorème d'Aldous, on cherche à définir le mouvement brownien indexé par l'arbre brownien \mathcal{T}_e , qui apparaîtra comme la limite d'échelle de la marche aléatoire $Z_{\mathbb{T}_n}$.

Considérons d'abord un arbre réel déterministe (\mathcal{T}, d) enraciné en ρ . Si $u, v \in \mathcal{T}$, on note $u \wedge v$ le plus proche ancêtre commun à u et v , c'est-à-dire le sommet de \mathcal{T} vérifiant l'égalité $[\rho, u \wedge v] = [\rho, u] \cap [\rho, v]$. Le *mouvement brownien indexé par \mathcal{T}* est défini comme un processus gaussien centré $(\mathcal{Z}_u)_{u \in \mathcal{T}}$ à valeurs dans \mathbb{R}^d , dont la distribution est caractérisée par les propriétés $\mathcal{Z}_\rho = 0$ et

$$\text{cov}(\mathcal{Z}_u, \mathcal{Z}_v) = d(\rho, u \wedge v) \cdot \text{Id}$$

pour tous les $u, v \in \mathcal{T}$, où Id désigne la matrice identité de taille d . Intuitivement, \mathcal{Z}_u correspond à la valeur au temps $d(\rho, u)$ d'un mouvement brownien standard d -dimensionnel, et \mathcal{Z}_v également. Mais \mathcal{Z}_v partage le même mouvement brownien avec \mathcal{Z}_u entre temps 0 et $d(\rho, u \wedge v)$, et il utilise ensuite un mouvement brownien indépendant entre les temps $d(\rho, u \wedge v)$ et $d(\rho, v)$.

Si l'arbre continu $\mathcal{T} = \mathcal{T}_f$ est codé par une fonction f höldérienne, le processus $(\mathcal{Z}_u)_{u \in \mathcal{T}_f}$ a une modification continue. Cela s'applique en particulier p.s. à chaque réalisation de l'excursion

brownienne normalisée \mathbf{e} . On peut donc définir un processus gaussien centré $\mathcal{Z}_{\mathcal{T}_\mathbf{e}}$ à trajectoires continues qui, conditionnellement à $\mathcal{T}_\mathbf{e}$, est un mouvement brownien indexé par l'arbre brownien.

Intimement lié à $\mathcal{Z}_{\mathcal{T}_\mathbf{e}}$, le *serpent brownien* est un objet très productif, introduit par Le Gall. Nous nous contentons ici d'une introduction concise, et nous renvoyons le lecteur à la monographie [57] pour une présentation détaillée. Commençons par quelques définitions. Une trajectoire arrêtée dans \mathbb{R}^d est une application continue $w: [0, \zeta_{(w)}] \rightarrow \mathbb{R}^d$, où le nombre $\zeta_{(w)}$ est appelé temps de vie de la trajectoire w . De plus, on note $\hat{w} = w(\zeta_{(w)})$ le point terminal de w . Lorsque $\zeta_{(w)} = 0$, la trajectoire w est identifiée à son point de départ $w(0) \in \mathbb{R}^d$. Notons \mathcal{W} l'ensemble de toutes les trajectoires arrêtées.

Définition 2. Le *serpent brownien standard* $(W_s)_{s \geq 0}$ issu de 0 est un processus à valeurs dans $\mathcal{C}(\mathbb{R}_+, \mathcal{W})$ dont la loi est caractérisée de façon suivante :

- (i) le processus du temps de vie $(\zeta_s := \zeta_{(W_s)}, s \geq 0)$ suit la loi d'un mouvement brownien réfléchi dans \mathbb{R}_+ issu de 0.
- (ii) $W_0 = 0$ p.s., et, conditionnellement à $(\zeta_s)_{s \geq 0}$, $(W_s)_{s \geq 0}$ est un processus de Markov inhomogène, dont le noyau de transition est spécifié comme suit. Pour tous $0 \leq s < s'$,
 - $W_{s'}(t) = W_s(t)$ pour tout $0 \leq t \leq m_\zeta(s, s') := \min\{\zeta_r : s \leq r \leq s'\}$;
 - $(W_{s'}(m_\zeta(s, s') + t) - W_{s'}(m_\zeta(s, s'))))_{0 \leq t \leq \zeta_{s'} - m_\zeta(s, s')}$ est indépendant de W_s , et suit la loi d'un mouvement brownien dans \mathbb{R}^d partant de 0.

En utilisant la théorie des excursions, on peut définir le serpent brownien $(W_s^{(1)})_{0 \leq s \leq 1}$ dirigé par une excursion brownienne normalisée \mathbf{e} et issu de 0, qui vérifie des propriétés similaires aux (i) et (ii) ci-dessus, avec la seule différence que le processus du temps de vie

$$(\zeta_s^{(1)} := \zeta_{(W_s^{(1)})}, s \geq 0)$$

est maintenant donné par l'excursion brownienne normalisée \mathbf{e} . Notons $(\widehat{W}_s^{(1)})_{0 \leq s \leq 1}$ le processus de la "tête" du serpent brownien $W^{(1)}$, et observons que $\widehat{W}_s^{(1)} = \widehat{W}_t^{(1)}$ si $d_\mathbf{e}(s, t) = 0$. Alors le mouvement brownien $\mathcal{Z}_{\mathcal{T}_\mathbf{e}}$ indexé par l'arbre brownien peut être construit à partir de $(\widehat{W}_s^{(1)})_{0 \leq s \leq 1}$ modulo la relation d'équivalence $\sim_\mathbf{e}$.

La mesure d'occupation \mathcal{I} du serpent brownien $W^{(1)}$ dirigé par \mathbf{e} est une mesure de probabilité définie sur \mathbb{R}^d par la formule

$$\langle \mathcal{I}, \varphi \rangle = \int_0^1 ds \varphi(\widehat{W}_s^{(1)})$$

pour toute fonction φ mesurable positive. Cette mesure aléatoire \mathcal{I} s'appelle l'*ISE* (abrégée pour « Integrated Super-Brownian Excursion »), et a été initialement introduite par Aldous dans [6] avec une normalisation légèrement différente. La mesure ISE en grande dimension intervient dans divers résultats asymptotiques en mécanique statistique ou en combinatoire. Voir e.g. l'article de vulgarisation [84] de Slade pour plus de références.

Rappelons que l'on peut coder un arbre planaire fini τ par sa fonction de contour C . De même, on peut définir la *fonction de contour spatiale* V d'un arbre spatial $(\tau, (z_u)_{u \in \tau}) \in \mathcal{S}_f^*$ en posant pour tout entier $t \in [0, 2(\#\tau - 1)]$, $V_t((z_u)_{u \in \tau}) = z_v$ où v est le sommet visité par C au temps t , puis en interpolant linéairement V sur l'intervalle $[0, 2(\#\tau - 1)]$. Le couple (C, V) code alors l'arbre spatial $(\tau, (z_u)_{u \in \tau})$.

Revenons maintenant à la marche aléatoire $Z_{\mathbb{T}_n}$ indexée par \mathbb{T}_n , et supposons que

Hypothèse **(H)** :

- (i) la loi de reproduction critique μ est différente de la masse de Dirac en 1, et il existe un réel $\eta > 0$ tel que $\sum_{k \geq 0} e^{\eta k} \mu(k) < \infty$.
- (ii) la loi de saut θ est une mesure de probabilité sur \mathbb{R}^d , de moyenne nulle et vérifiant la condition de moments suivante :

$$\lim_{r \rightarrow +\infty} r^4 \theta(\{x \in \mathbb{Z}^d : |x| > r\}) = 0.$$

Le prochain résultat est une extension multi-dimensionnelle du théorème 2 de l'article de Janson et Marckert [39].

Théorème 6. Soit $\sigma_\mu = (\text{var } \mu)^{1/2}$ et M_θ la matrice de covariance de θ . On écrit $M_\theta^{1/2}$ pour l'unique matrice symétrique définie positive telle que $M_\theta = (M_\theta^{1/2})^2$, et l'on note $M_\theta^{-1/2}$ son inverse. Sous l'hypothèse **(H)**, on a alors la convergence conjointe

$$\left(\frac{\sigma_\mu}{2} \frac{C_{2nt}(\mathbb{T}_n)}{\sqrt{n}}, M_\theta^{-1/2} \left(\frac{\sigma_\mu}{2} \right)^{1/2} \frac{V_{2nt}(Z_{\mathbb{T}_n})}{n^{1/4}} \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(\text{loi})} (\mathbf{e}_t, \widehat{W}_t^{(1)})_{0 \leq t \leq 1}.$$

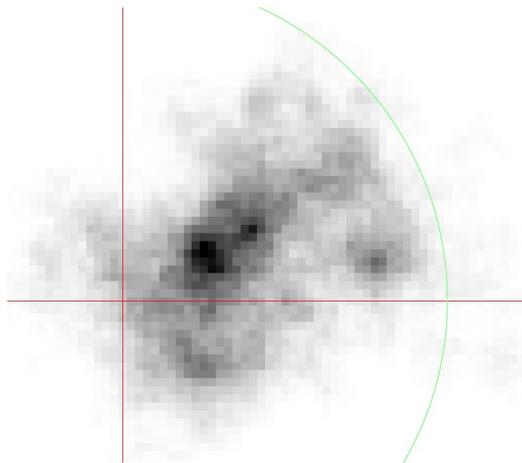


Figure 1.4 – Simulation de la mesure aléatoire $\sum_{u \in \mathbb{T}_n} \delta_{Z_{\mathbb{T}_n}(u)}$, où l'endroit sombre représente une masse supérieure, et l'arc vert a pour rayon $n^{1/4}$ (Image tirée de [84] © G. Slade).

Pour tout $u \in \mathbb{T}_n$, on note $\tilde{Z}_{\mathbb{T}_n}(u) = M_\theta^{-1/2} (\sigma_\mu/2)^{1/2} \frac{Z_{\mathbb{T}_n}(u)}{n^{1/4}}$. Soit \mathcal{I}_n la mesure de probabilité empirique associée à $\tilde{Z}_{\mathbb{T}_n}$, i.e.

$$\mathcal{I}_n := \frac{1}{n} \sum_{u \in \mathbb{T}_n} \delta_{\tilde{Z}_{\mathbb{T}_n}(u)}.$$

Sous l'hypothèse **(H)**, le théorème précédent entraîne la convergence en loi de \mathcal{I}_n vers la mesure ISE \mathcal{I} , lorsque $n \rightarrow \infty$. Cette convergence a lieu dans l'espace $\mathcal{M}_1(\mathbb{R}^d)$ des mesures de probabilité sur \mathbb{R}^d muni de la topologie faible.

1.3.3 Le nombre de points visités par la marche aléatoire $Z_{\mathbb{T}_n}$

Les résultats présentés dans cette section sont détaillés dans les chapitres 2 et 3. Ces chapitres sont basés sur nos deux articles [61, 62] écrits en collaboration avec J.-F. Le Gall.

Notons $\mathcal{R}_n := \#\{Z_{\mathbb{T}_n}(u) : u \in \mathbb{T}_n\} \leq n$ le nombre de points visités par la marche aléatoire $Z_{\mathbb{T}_n}$ indexée par un arbre de Galton–Watson conditionné \mathbb{T}_n de taille n .

Rappelons que si la loi de reproduction μ est géométrique de paramètre $1/2$, le μ -GW arbre conditionné \mathbb{T}_n est uniformément distribué sur l'ensemble des arbres planaires à n sommets. Dans ce cas particulier, nous avons établi, en analogie avec le Théorème 3 de Dvoretzky et Erdős, le théorème suivant, qui concerne le comportement asymptotique de \mathcal{R}_n pour notre modèle de la marche aléatoire $Z_{\mathbb{T}_n}$.

Théorème ([61, Theorem 1]).

Soit θ une mesure de probabilité sur \mathbb{Z}^d symétrique et de support fini. Pour chaque entier $n \geq 1$, soit \mathbb{T}_n un arbre planaire uniforme de taille n . Sachant \mathbb{T}_n , on prend $Z_{\mathbb{T}_n}$ une marche aléatoire dans \mathbb{Z}^d de loi de saut θ indexée par \mathbb{T}_n , et l'on note \mathcal{R}_n son nombre de points visités. Alors,

- si $d \geq 5$,

$$\frac{1}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(P)} c_\theta,$$

où $c_\theta > 0$ est une constante dépendant de θ , et $\xrightarrow{(P)}$ indique la convergence en probabilité;

- si $d = 4$,

$$\frac{\log n}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi^2 \sigma^4, \quad (1.5)$$

où $\sigma^2 = (\det M_\theta)^{1/4}$, M_θ désignant la matrice de covariance de θ ;

- si $d \leq 3$,

$$n^{-d/4} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(loi)} 2^{d/4} (\det M_\theta)^{1/2} \lambda_d(\text{supp}(\mathcal{I})), \quad (1.6)$$

où $\lambda_d(\text{supp}(\mathcal{I})) > 0$ représente la mesure de Lebesgue du support de la mesure aléatoire ISE \mathcal{I} .

Remarquons que la dimension critique pour le comportement de \mathcal{R}_n est maintenant $d = 4$, au lieu de $d = 2$ pour la marche aléatoire ordinaire. D'après le Théorème 6, l'ensemble des positions spatiales de $Z_{\mathbb{T}_n}$ est inclus dans une boule centrée à l'origine d'un rayon d'ordre $n^{1/4}$. Donc \mathcal{R}_n est au plus de l'ordre de $n^{d/4}$ en dimension $d \leq 4$. Signalons néanmoins que le Théorème 6 n'implique pas directement la convergence (1.6) lorsque $d \leq 3$.

Nous voudrions souligner le fait que, dans le théorème ci-dessus, les asymptotiques de \mathcal{R}_n en différents régimes restent valides pour un cadre beaucoup plus général. D'abord, en appliquant le théorème ergodique sous-additif de Kingman d'une façon non triviale, on peut démontrer la croissance linéaire de \mathcal{R}_n dans le contexte suivant :

Théorème ([61, Theorem 7]).

Soit μ une loi de reproduction critique dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Pour tout entier n suffisamment grand, soit \mathbb{T}_n un μ -GW arbre conditionné à avoir n sommets, et soit $Z_{\mathbb{T}_n}$ une marche aléatoire dans \mathbb{Z}^d de loi de saut θ indexée par \mathbb{T}_n . Si \mathcal{R}_n désigne le nombre de points visités par $Z_{\mathbb{T}_n}$, on a

$$\frac{1}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(P)} c_{\mu, \theta}, \quad (1.7)$$

où $c_{\mu, \theta} \in [0, 1]$ est une constante dépendante de μ et de θ .

De plus, nous avons trouvé une condition suffisante pour que la constante limite $c_{\mu, \theta} > 0$:

Proposition ([61, Proposition 5]).

Supposons en plus que la loi de saut θ est centrée et a des moments d'ordre $(d-1) \vee 2$ finis. Alors

$$c_{\mu, \theta} > 0 \quad \text{lorsque } d > \frac{2\alpha}{\alpha - 1}.$$

En particulier, si la loi de reproduction μ est de variance finie, alors $c_{\mu, \theta} > 0$ lorsque $d \geq 5$.

Lorsque $d = 4$, la convergence (1.5) est en fait établie en supposant que la loi de saut θ est *symétrique* et a des *moments exponentiels finis* ([61, Theorem 14]). Cependant, nous devons nous restreindre au cas où μ est géométrique de paramètre $1/2$ (et donc \mathbb{T}_n est uniforme sur les arbres planaires de taille n), parce que nos arguments s'appuient fortement sur l'utilisation d'une chaîne de Markov qui permet d'engendrer les arbres spatiaux discrets aléatoires appropriés. Ce processus à valeurs dans l'espace des trajectoires semi-infinies dans \mathbb{Z}^4 est un analogue discret du serpent brownien, que l'on appelle le *serpent discret*. Une estimation cruciale ([61, Proposition 8]) indique que la probabilité que la "tête" du serpent discret ne revienne pas à l'origine avant l'instant n se comporte comme $(4\pi^2\sigma^4 + o(1))/\log n$ quand $n \rightarrow \infty$. Cette estimation est analogue à l'asymptotique pour la probabilité que la marche aléatoire ordinaire dans le plan ne revienne plus à son point de départ avant l'instant n , mais la preuve se révèle beaucoup plus compliquée dans notre cadre.

Lorsque $d \leq 3$, nous avons prouvé le résultat suivant :

Théorème ([62, Theorem 5]).

Supposons que $d \leq 3$. Pour tout entier n suffisamment grand, soit \mathbb{T}_n un μ -GW arbre conditionné à avoir n sommets, et soit $Z_{\mathbb{T}_n}$ une marche aléatoire dans \mathbb{Z}^d de loi de saut θ indexée par \mathbb{T}_n . Soit $\sigma_\mu = (\text{var } \mu)^{1/2}$ et M_θ la matrice de covariance de θ . On note \mathcal{R}_n le nombre de points visités par $Z_{\mathbb{T}_n}$. Sous l'hypothèse **(H)**, on a

$$n^{-d/4} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(\text{loi})} \left(\frac{2}{\sigma_\mu}\right)^{d/2} (\det M_\theta)^{1/2} \lambda_d(\text{supp}(\mathcal{I})), \quad (1.8)$$

où $\lambda_d(\text{supp}(\mathcal{I}))$ est la même variable aléatoire figurant dans la convergence (1.6).

Notons que le support de la mesure aléatoire ISE \mathcal{I} a une mesure de Lebesgue strictement positive si et seulement si $d \leq 3$. Le cœur de nos arguments pour le théorème ci-dessus est un principe d'invariance des temps locaux de la marche aléatoire $Z_{\mathbb{T}_n}$. Pour chaque $a \in \mathbb{Z}^d$, soit

$$L_n(a) := \sum_{u \in \mathbb{T}_n} \mathbf{1}_{\{Z_{\mathbb{T}_n}(u)=a\}}$$

le nombre de visites en site a par la marche $Z_{\mathbb{T}_n}$. En utilisant les résultats de Sugitani [86], on peut établir l'existence p.s. d'une densité $(\ell_x, x \in \mathbb{R}^d)$ continue sur \mathbb{R}^d pour la mesure ISE \mathcal{I} , par rapport à la mesure de Lebesgue ([62, Proposition 1]), et, de plus, pour tout $x \in \mathbb{R}^d \setminus \{0\}$, il y a l'égalité p.s. $\{\ell_x > 0\} = \{x \in \text{supp}(\mathcal{I})\}$ ([62, Proposition 2]). Observons que

$$\mathcal{R}_n = \sum_{a \in \mathbb{Z}^d} \mathbf{1}_{\{L_n(a) > 0\}}.$$

Le théorème suivant nous permettra de démontrer la convergence (1.8) par un calcul de moment. Avant de l'énoncer, on introduit la notation $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor) \in \mathbb{Z}^d$ pour tout $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, et la constante

$$c_d := \frac{1}{(\det M_\theta)^{1/2}} \left(\frac{\sigma_\mu}{2}\right)^{\frac{d}{2}}.$$

Théorème ([62, Theorem 4]).

Supposons $d \leq 3$ et soit un entier $k \geq 1$. Pour tous $a_1, \dots, a_k \in \mathbb{R}^d \setminus \{0\}$, sous l'hypothèse **(H)**, on a

$$\left(n^{\frac{d}{4}-1} L_n \left(\lfloor \sqrt{\frac{2}{\sigma_\mu}} M_\theta^{\frac{1}{2}} n^{\frac{1}{4}} a_1 \rfloor \right), \dots, n^{\frac{d}{4}-1} L_n \left(\lfloor \sqrt{\frac{2}{\sigma_\mu}} M_\theta^{\frac{1}{2}} n^{\frac{1}{4}} a_k \rfloor \right) \right) \xrightarrow[n \rightarrow \infty]{(\text{loi})} (c_d \ell_{a_1}, \dots, c_d \ell_{a_k}).$$

On s'attend à ce que le résultat ci-dessus reste valable même si l'on enlève la condition $a_i \neq 0$, et que l'on puisse avoir une convergence fonctionnelle au lieu de celle au sens des marginales de dimension finie. En dimension $d = 1$, ce renforcement a été obtenu par Bousquet-Mélou et Janson [15, Theorem 3.6] dans un cas particulier, puis par Devroye et Janson [24, Theorem 1.1] dans un cadre plus général.

1.3.4 Le nombre de points visités par la marche aléatoire branchante

Les résultats présentés dans cette section sont détaillés dans les chapitres 2 et 3. Ces chapitres sont basés sur nos deux articles [61, 62] écrits en collaboration avec J.-F. Le Gall.

Au lieu de considérer une marche aléatoire indexée par un μ -GW arbre conditionné à un nombre fixe de sommets, il est aussi intéressant d'étudier la situation où la structure généalogique est codée par un nombre fini de μ -GW arbres *non conditionnés* et *indépendants*.

Considérons p arbres de Galton–Watson spatiaux

$$(\tau^{(1)}, (Z^{(1)}(u))_{u \in \tau^{(1)}}), \dots, (\tau^{(p)}, (Z^{(p)}(u))_{u \in \tau^{(p)}})$$

issus de 0, indépendants et identiquement distribués selon $\Pi_{\mu, \theta}^*$. Pour chaque entier $n \geq 0$, on définit la mesure ponctuelle aléatoire

$$\mathcal{X}_n^{[p]} := \sum_{j=1}^p \left(\sum_{u \in \tau^{(j)}, |u|=n} \delta_{Z^{(j)}(u)} \right),$$

qui représente la somme des mesures de Dirac aux points où se situent les sommets de génération n . On appelle $\mathcal{X}^{[p]} = (\mathcal{X}_n^{[p]}, n \geq 0)$ la *marche aléatoire branchante* sur \mathbb{Z}^d avec la configuration initiale $\mathcal{X}_0^{[p]} = p \delta_0$. Le nombre de points visités $R(\mathcal{X}^{[p]})$ est alors

$$R(\mathcal{X}^{[p]}) := \#\{a \in \mathbb{Z}^d : a = Z^{(j)}(u) \text{ pour un } j \in \{1, \dots, p\} \text{ et } u \in \tau^{(j)}\}.$$

On définit la fonction de hauteur $(H_k^{[p]}, k \geq 0)$ de la forêt $\tau^{(1)}, \dots, \tau^{(p)}$ comme la concaténation des fonctions de hauteur de chaque arbre $(H_i(\tau^{(j)}), 0 \leq i \leq \#\tau^{(j)} - 1)$, et l'on pose $H_k^{[p]} = 0$ pour $k \geq \#\tau^{(1)} + \dots + \#\tau^{(p)}$. De la même manière, on définit la fonction $(Z_k^{[p]}, k \geq 0)$ par la concaténation des fonctions $(Z_i^{(j)}, 0 \leq i \leq \#\tau^{(j)} - 1)$. Enfin, on prolonge les deux fonctions $H^{[p]}$ et $Z^{[p]}$ à \mathbb{R}_+ par interpolation linéaire. Nous pouvons maintenant énoncer notre analogue du Théorème 6.

Proposition ([62, Proposition 6]).

Sous l'hypothèse **(H)**, on a

$$\left(\left(\frac{\sigma_\mu}{2p} H_{p^2 s}^{[p]}, \sqrt{\frac{\sigma_\mu}{2p}} Z_{p^2 s}^{[p]} \right)_{s \geq 0}, p^{-2} (\#\tau^{(1)} + \dots + \#\tau^{(p)}) \right) \xrightarrow[p \rightarrow \infty]{(\text{loi})} ((\zeta_{s \wedge \xi}, M_\theta^{1/2} \widehat{W}_{s \wedge \xi})_{s \geq 0}, \xi),$$

où $(W_s)_{s \geq 0}$ est le serpent brownien standard issu de 0, et ξ désigne le premier temps d'atteinte de $2/\sigma_\mu$ par le temps local en 0 du processus du temps de vie $(\zeta_s)_{s \geq 0}$, la convergence des processus ayant lieu pour la topologie de la convergence uniforme sur tout compact.

Lorsque $d \leq 3$, on déduit une estimation importante pour la probabilité d'atteinte d'un point éloigné pour la marche aléatoire indexée par un μ -GW arbre.

Théorème ([62, Theorem 7]).

Sous la mesure de probabilité $\Pi_{\mu,\theta}^*$, notons $\mathbf{R} := \{z_u : u \in \tau\}$ l'ensemble des points visités par la marche aléatoire indexée par τ . Si $d \leq 3$, en supposant l'hypothèse **(H)**, on a

$$\lim_{|a| \rightarrow \infty} |M_\theta^{-1/2} a|^2 \Pi_{\mu,\theta}^*(a \in \mathbf{R}) = \frac{2(4-d)}{\sigma_\mu^2}.$$

Grâce au théorème précédent, nous avons obtenu un résultat analogue à la convergence (1.8) pour la marche aléatoire branchante en petites dimensions.

Théorème ([62, Theorem 8]).

Supposons que $d \leq 3$. Sous l'hypothèse **(H)**, on a

$$p^{-d/2} \mathbf{R}(\mathcal{X}^{[p]}) \xrightarrow[p \rightarrow \infty]{(\text{loi})} \left(\frac{2}{\sigma_\mu}\right)^d (\det M_\theta)^{1/2} \lambda_d \left(\bigcup_{t \geq 0} \text{supp } X_t \right),$$

où λ_d désigne la mesure de Lebesgue sur \mathbb{R}^d , $(X_t)_{t \geq 0}$ est un super-mouvement brownien en dimension d issu de δ_0 avec le mécanisme de branchement $\psi(u) = 2u^2$, et $\text{supp } X_t$ signifie le support topologique de X_t .

Le super-mouvement brownien $(X_t)_{t \geq 0}$ ci-dessus est un processus de Markov homogène à valeurs dans l'espace des mesures finies sur \mathbb{R}^d . Nous renvoyons le lecteur à Le Gall [57] pour voir ses liens avec le serpent brownien. Notons à nouveau que $\cup_{t \geq 0} \text{supp } X_t$ a une mesure de Lebesgue strictement positive si et seulement si $d \leq 3$.

En grande dimension ($d \geq 5$), on a aussi l'analogue de la convergence (1.7).

Théorème ([61, Proposition 20]).

Soit $d \geq 5$. On suppose que la loi de reproduction critique μ est de variance $\sigma_\mu^2 > 0$ finie, et la loi de saut θ est centrée avec des moments d'ordre $d-1$ finis. Avec la même constante $c_{\mu,\theta} > 0$ apparaissant dans (1.7), on a

$$\frac{1}{p^2} \mathbf{R}(\mathcal{X}^{[p]}) \xrightarrow[p \rightarrow \infty]{(\text{loi})} \frac{c_{\mu,\theta}}{\sigma_\mu^2} J,$$

où la variable aléatoire $J > 0$ a pour densité $(2\pi s^3)^{-1/2} \exp(-\frac{1}{2s})$ sur $(0, \infty)$.

La dimension critique reste encore $d = 4$. Comme précédemment, on se restreint à la loi de reproduction géométrique.

Théorème ([61, Proposition 21]).

Soit $d = 4$. On suppose que μ est la loi de reproduction critique géométrique, et θ est symétrique avec des moments exponentiels finis. Alors,

$$\frac{\log p}{p^2} \mathbf{R}(\mathcal{X}^{[p]}) \xrightarrow[p \rightarrow \infty]{(\text{loi})} 4\pi^2 \det(M_\theta)^{1/2} J,$$

où J est la même variable aléatoire que dans le théorème précédent.

Remarques. (1) Par un calcul des moments d'ordre 1 et 2, on sait que si $d \geq 5$, il existe des constantes $C_1, C_2 > 0$ dépendant de d, μ et θ telles que

$$C_1 |a|^{2-d} \leq \Pi_{\mu, \theta}^*(a \in \mathbf{R}) \leq C_2 |a|^{2-d} \quad \text{lorsque } |a| \rightarrow \infty.$$

Si $d = 4$, lorsque $|a| \rightarrow \infty$, on s'attend à ce que

$$\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \approx \frac{C}{|a|^2 \log |a|}.$$

Notons que la minoration pourrait être donnée par la même méthode de calcul des moments, tandis que la majoration reste ouverte.

(2) Dans l'article [61], nos résultats sur le comportement asymptotique de $\mathbf{R}(\mathcal{X}^{[p]})$ quand $d \geq 5$ ou $d = 4$ sont en effet plus forts que ceux que nous avons présentés ci-dessus, au sens où ils sont corrects pour une configuration initiale *quelconque* de la marche aléatoire branchante, à condition que $\langle \mathcal{X}_0^{[p]}, 1 \rangle = p$. Cependant, ce n'est plus le cas lorsque $d \leq 3$ (cf. les discussions dans la Section 3.5.3).

(3) Signalons que, en dimensions 2 et 3, Lalley et Zheng [49, Theorem 1] ont obtenu un principe d'invariance des temps locaux de la marche aléatoire branchante quand la loi de reproduction μ est Poisson de paramètre 1.

1.3.5 Et si l'on ajoute une dérive constante ?

Les résultats présentés dans cette section sont détaillés dans le chapitre 4. Ce chapitre est basé sur l'article [67] en préparation.

Revenons à la marche aléatoire $Z_{\mathbb{T}_n}$ indexée par \mathbb{T}_n . Dans la Section 1.3.3, où la loi de saut θ considérée est centrée, on a vu que la dimension 4 est critique pour le comportement asymptotique de \mathcal{R}_n (le nombre de points visités par $Z_{\mathbb{T}_n}$). On peut maintenant se demander comment \mathcal{R}_n se comporte si l'on ajoute une dérive constante à chaque pas de $Z_{\mathbb{T}_n}$. Dans cet objectif, soit $D \in \mathbb{Z}^d \setminus \{0\}$ la dérive fixée, et l'on définit une nouvelle mesure de probabilité $\tilde{\theta}$ sur \mathbb{Z}^d comme la mesure-image de θ par la translation $x \mapsto x + D$. Donc si l'on remplace la loi de saut θ par $\tilde{\theta}$, chaque déplacement spatial de $Z_{\mathbb{T}_n}$ sera composé d'un saut aléatoire selon θ et d'une dérive D déterministe.

D'abord, les arguments généraux développés pour la croissance linéaire de \mathcal{R}_n au cas centré s'adaptent bien au cas avec dérive, et l'on obtient le théorème suivant :

Théorème ([67, Theorem 1]).

Soit μ une loi de reproduction critique dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Pour tout entier n suffisamment grand, soit \mathbb{T}_n un μ -GW arbre conditionné à avoir n sommets, et soit $Z_{\mathbb{T}_n}$ une marche aléatoire dans \mathbb{Z}^d avec la loi de saut $\tilde{\theta}$ indexée par \mathbb{T}_n . Si \mathcal{R}_n désigne le nombre de points visités par $Z_{\mathbb{T}_n}$, il existe alors une constante $c_{\mu, \tilde{\theta}} \in [0, 1]$ dépendante de μ et de θ telle que

$$\frac{1}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(P)} c_{\mu, \tilde{\theta}}.$$

Supposons en plus que θ est centrée et a des moments d'ordre $2d \vee 3$ finis. Alors

$$c_{\mu, \tilde{\theta}} > 0 \quad \text{lorsque } d > \frac{\alpha + 1}{\alpha - 1}.$$

En particulier, si la loi de reproduction μ est de variance finie, alors $c_{\mu, \tilde{\theta}} > 0$ lorsque $d \geq 4$.

Lorsque $d \leq 2$, en supposant que la loi de reproduction critique μ est de variance finie et que la loi de saut $\tilde{\theta}$ est assez "bonne", nous nous attendons à ce que \mathcal{R}_n se comporte comme $n^{(d+1)/4}$ quand n tend vers l'infini. Nous renvoyons le lecteur à la Section 4.4 pour voir les discussions correspondantes.

Nous pensons que la dimension critique devient 3 pour le comportement asymptotique du nombre de points visités \mathcal{R}_n dans le cas avec dérive. Malheureusement, nous sommes incapables de traiter la dimension 3 dans un cadre général. Néanmoins, nous parvenons à traiter le cas très particulier où les deux premières coordonnées évoluent comme une marche aléatoire centrée sur \mathbb{Z}^2 , alors que la troisième augmente d'une unité à chaque pas. Cela conduit au résultat suivant.

Théorème ([67, Theorem 4]).

Soit $Z_{\mathbb{T}_n}$ une marche aléatoire sur \mathbb{Z}^2 indexée par un arbre uniforme \mathbb{T}_n avec n sommets. Supposons que sa loi de saut θ est symétrique et a des moments exponentiels finis. On note M_θ sa matrice de covariance. Pour chaque entier $k \geq 0$, notons $\Omega_k^{(n)}$ le nombre de points visités dans \mathbb{Z}^2 par les sommets en génération k de \mathbb{T}_n , i.e.

$$\Omega_k^{(n)} := \#\{Z_{\mathbb{T}_n}(u) : u \in \mathbb{T}_n, |u| = k\}.$$

Par convention $\#\emptyset = 0$. Pour la somme finie $R_n := \sum_{k \geq 0} \Omega_k^{(n)}$, on a la convergence

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi(\det M_\theta)^{1/2}.$$

Notre théorème ci-dessus est étroitement lié au travail récent de Lalley et Zheng [50], qui ont étudié diverses statistiques d'occupation d'une marche aléatoire sur \mathbb{Z}^d , $d \geq 2$ indexée par $\mathbb{T}^{(n)}$. Rappelons que $\mathbb{T}^{(n)}$ est un μ -GW arbre conditionné à atteindre la hauteur n . Soit maintenant

$d = 2$. On note Ω_n le nombre de points dans \mathbb{Z}^2 visités par les sommets de $\mathbb{T}^{(n)}$ en génération n . Sous l’hypothèse que la loi de reproduction μ est critique de variance finie, et que la loi de saut θ est uniforme sur les 5 sites de distance ≤ 1 de l’origine, leur résultat [50, Theorem 7] montre que la famille de variables aléatoires $(\frac{\log n}{n}\Omega_n, n \geq 1)$ est tendue. Rappelons que la hauteur de l’arbre \mathbb{T}_n est de l’ordre de \sqrt{n} . Alors R_n peut être interprété comme une somme de \sqrt{n} quantités, toutes de l’ordre de $\frac{\sqrt{n}}{\log n}$. Cet argument heuristique fournit le même ordre de grandeur de R_n que celui indiqué dans le théorème précédent.

1.4 La mesure harmonique sur un arbre de Galton–Watson critique

Dans la théorie des fonctions harmoniques, et notamment la résolution du problème de Dirichlet classique, la mesure harmonique est un concept mathématique extrêmement important. D’un point de vue probabiliste, la mesure harmonique sur le bord d’un domaine borné de \mathbb{R}^d ($d \geq 2$) est la loi de sortie d’un mouvement brownien démarré à l’intérieur du domaine. Dans le cas d’un graphe discret, la mesure harmonique sur le bord d’un ensemble fini se définit de la même façon, mais le mouvement brownien est remplacé par la marche aléatoire simple.

Il a été observé dans plusieurs contextes que la mesure harmonique sur un ensemble fractal est portée par un sous-ensemble de dimension strictement plus petite. En particulier, le fameux théorème de Makarov [74] montre que la mesure harmonique sur la frontière d’un domaine simplement connexe de \mathbb{R}^2 est toujours portée par un ensemble de dimension de Hausdorff 1, quelle que soit la dimension de Hausdorff de la frontière. Voir l’article de Bourgain [14] pour des analogues en dimension supérieure. Dans le cas discret de \mathbb{Z}^d , des résultats similaires se trouvent dans [52] et [13]. Ce phénomène de la « chute de dimension » a également été observé par Lyons, Pemantle et Peres [71, 72] dans le cas de la mesure harmonique à l’infini pour des marches aléatoires sur des arbres de Galton–Watson sur-critiques. Par la suite, nous allons présenter des résultats similaires dans le cas des arbres de Galton–Watson critiques.

Soit μ une loi de reproduction critique telle que $\mu(1) < 1$. Sous la probabilité \mathbb{P} , on note $\mathbb{T}^{(n)}$ un μ -GW arbre conditionné à avoir une hauteur supérieure à n , et $\mathbb{T}_n^{(n)}$ l’ensemble des sommets à hauteur n de l’arbre $\mathbb{T}^{(n)}$. Conditionnellement à $\mathbb{T}^{(n)}$, considérons une marche aléatoire simple sur $\mathbb{T}^{(n)}$ partant de la racine de l’arbre jusqu’à son premier temps d’atteinte de $\mathbb{T}_n^{(n)}$. La loi du premier point à hauteur n de $\mathbb{T}^{(n)}$ visité par la marche est donc une mesure de probabilité μ_n sur $\mathbb{T}_n^{(n)}$, appelée la mesure harmonique au niveau n .

1.4.1 La mesure harmonique sur la frontière

Les résultats présentés dans cette section sont détaillés dans le chapitre 5. Ce chapitre est basé sur l’article [66] publié.

On suppose dans cette partie que la loi de reproduction critique μ appartient au domaine d’attraction d’une loi stable d’indice $\alpha \in (1, 2]$. Alors il est connu que, modulo une fonction multiplicative à variation lente,

$$\#\mathbb{T}_n^{(n)} \approx n^{\frac{1}{\alpha-1}} \quad \text{quand } n \rightarrow \infty.$$

Dans le cas particulier où μ a une variance finie, le théorème classique de Yaglom montre que $n^{-1}\#\mathbb{T}_n^{(n)}$ converge en loi vers une variable exponentielle, alors que récemment, Curien et

Le Gall [20] ont prouvé qu'il existe une constante $\beta \approx 0,78$ qui ne dépend pas de μ et telle que « presque toute » la masse de μ_n est concentrée sur approximativement n^β sommets de $\mathbb{T}_n^{(n)}$. Le théorème suivant généralise à toutes les valeurs de $\alpha \in (1, 2]$ le résultat qu'ils ont obtenu dans le cas de variance finie ($\alpha = 2$).

Théorème ([66, Theorem 1]).

Supposons que la loi de reproduction critique μ appartient au domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2]$. Il existe alors une constante $\beta_\alpha \in (0, \frac{1}{\alpha-1})$, qui ne dépend que de α , telle que, pour tout $\delta > 0$, nous avons la convergence en probabilité

$$\mu_n\left(\{v \in \mathbb{T}_n^{(n)} : n^{-\beta_\alpha - \delta} \leq \mu_n(v) \leq n^{-\beta_\alpha + \delta}\}\right) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1.$$

Ainsi, pour tout $\varepsilon \in (0, 1)$, il existe avec \mathbb{P} -probabilité tendant vers 1 quand $n \rightarrow \infty$, un sous-ensemble $A_{n,\varepsilon}$ de $\mathbb{T}_n^{(n)}$ tel que $\#A_{n,\varepsilon} \leq n^{\beta_\alpha + \delta}$ et $\mu_n(A_{n,\varepsilon}) \geq 1 - \varepsilon$. Inversement, la plus grande μ_n -mesure d'un ensemble de cardinal inférieur à $n^{\beta_\alpha - \delta}$ tend en probabilité vers 0 quand $n \rightarrow \infty$.

On observe que la distribution d'atteinte μ_n au niveau n pour la marche aléatoire simple sur $\mathbb{T}^{(n)}$ reste la même si l'on élague $\mathbb{T}^{(n)}$ des branches n'atteignant pas le niveau n . Donc, il suffit de considérer la marche aléatoire simple sur l'arbre réduit \mathbb{T}^{*n} obtenu en ne gardant que les sommets de $\mathbb{T}^{(n)}$ ayant un descendant au niveau n . Si l'on multiplie la distance de graphe sur \mathbb{T}^{*n} par un facteur n^{-1} , les arbres réduits renormalisés $n^{-1}\mathbb{T}^{*n}$ convergent en loi, au sens de Gromov–Hausdorff, vers un arbre réel compact aléatoire $\Delta^{(\alpha)}$ que nous décrivons maintenant.

Pour chaque $\alpha \in (1, 2]$, on définit une loi de reproduction θ_α de la façon suivante : si $\alpha = 2$, on pose $\theta_2 = \delta_2$ la masse de Dirac en 2. Si $\alpha < 2$, on pose

$$\begin{aligned} \theta_\alpha(0) &= \theta_\alpha(1) = 0, \\ \theta_\alpha(k) &= \frac{\alpha(2-\alpha)(3-\alpha)\cdots(k-1-\alpha)}{k!}, \quad \forall k \geq 2, \end{aligned}$$

Soit U_\emptyset une variable aléatoire uniformément distribuée sur $[0, 1]$, et K_\emptyset une variable aléatoire distribuée selon θ_α , indépendante de U_\emptyset . Pour construire $\Delta^{(\alpha)}$, on commence avec un segment orienté de longueur U_\emptyset , dont l'origine sera la racine de l'arbre. À l'autre extrémité de ce segment, on attache les origines de K_\emptyset segments orientés de longueur $U_1, U_2, \dots, U_{K_\emptyset}$, où, conditionnellement à U_\emptyset et K_\emptyset , $U_1, U_2, \dots, U_{K_\emptyset}$ sont indépendantes et uniformément distribuées sur $[0, 1 - U_\emptyset]$. Cela termine la première étape de la construction. Dans la deuxième étape, on prend une nouvelle variable aléatoire indépendante K_1 distribuée selon θ_α , et au premier de ces K_\emptyset segments on attache K_1 nouveaux segments, dont les longueurs sont à nouveau indépendantes et uniformément distribuées sur $[0, 1 - U_\emptyset - U_1]$, conditionnellement à toutes les variables aléatoires apparues avant. Pour les $K_\emptyset - 1$ autres segments, on répète cette procédure de façon indépendante. On continue alors la construction par récurrence. Après une infinité d'étapes, on obtient à la fin un arbre réel non-compact, dont le complété est l'arbre réel compact $\Delta^{(\alpha)}$. On l'appellera l'*arbre réduit stable de paramètre α* . Notons que tous les nombres de descendants K_\emptyset, K_1, \dots impliqués dans la construction de $\Delta^{(2)}$ sont p.s. égaux à 2, ce qui correspond au mécanisme de branchement binaire. En revanche, ce n'est plus le cas lorsque $1 < \alpha < 2$. Voir la Figure 1.5 pour une illustration.

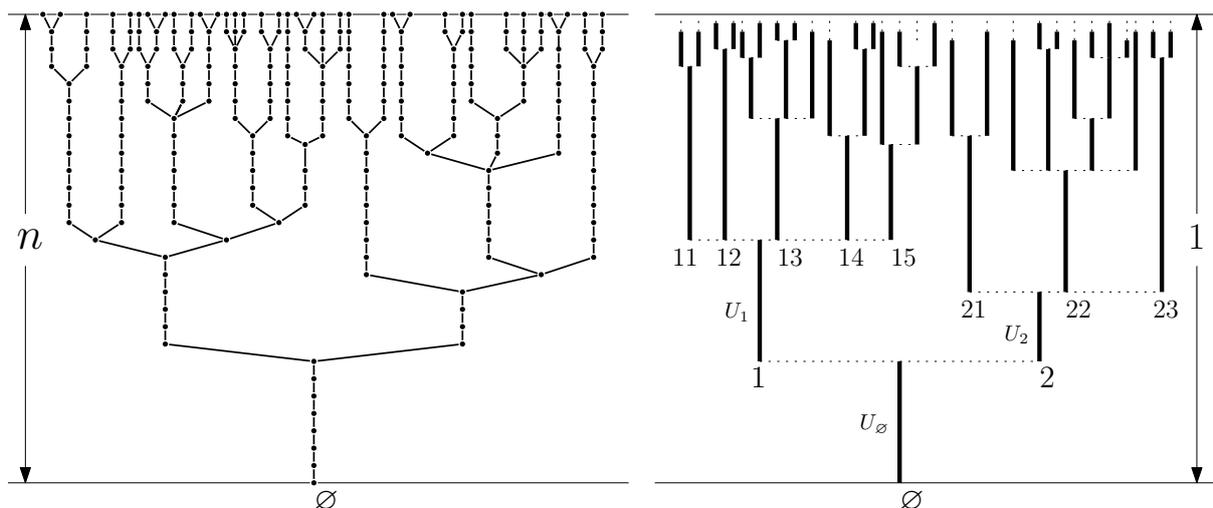


Figure 1.5 – À gauche, un grand arbre de Galton–Watson réduit T^{*n} ; à droite, la limite d'échelle $\Delta^{(\alpha)}$ l'arbre réduit stable de paramètre $1 < \alpha < 2$

La frontière $\partial\Delta^{(\alpha)}$ de $\Delta^{(\alpha)}$ est définie comme l'ensemble de tous les points à hauteur 1. Conditionnellement à $\Delta^{(\alpha)}$, on peut définir le mouvement brownien sur $\Delta^{(\alpha)}$ issu de la racine jusqu'au premier temps d'atteinte de $\partial\Delta^{(\alpha)}$. Il se comporte comme un mouvement brownien linéaire tant qu'il reste à l'intérieur d'un segment. Il est réfléchi à la racine de l'arbre, et lorsqu'il arrive à un point de branchement, il choisit chacun des segments adjacents avec des probabilités égales. La distribution aléatoire du point d'atteinte de $\partial\Delta^{(\alpha)}$ par le mouvement brownien est donc la mesure harmonique sur $\partial\Delta^{(\alpha)}$, notée μ_α . Le résultat suivant dans ce cadre continu est un ingrédient clé dans la preuve du théorème précédent.

Théorème ([66, Theorem 2]).

Pour chaque $\alpha \in (1, 2]$, avec la même constante $\beta_\alpha \in (0, \frac{1}{\alpha-1})$ que précédemment, on a \mathbb{P} p.s. $\mu_\alpha(dx)$ p.p.,

$$\lim_{r \downarrow 0} \frac{\log \mu_\alpha(\mathcal{B}(x, r))}{\log r} = \beta_\alpha,$$

où $\mathcal{B}(x, r)$ est la boule fermée de centre x et de rayon r dans $\Delta^{(\alpha)}$. En particulier, la dimension de Hausdorff de μ_α est \mathbb{P} p.s. égale à β_α (alors que la dimension de Hausdorff de $\partial\Delta^{(\alpha)}$ est égale à $\frac{1}{\alpha-1}$).

La constante β_α apparaissant dans les deux théorèmes précédents peut être exprimée en termes de la conductance de $\Delta^{(\alpha)}$. Si l'on voit l'arbre aléatoire $\Delta^{(\alpha)}$ comme un réseau de résistances avec un ohm par unité de longueur, la conductance équivalente entre la racine et la frontière $\partial\Delta^{(\alpha)}$ est une variable aléatoire continue que nous noterons $\mathcal{C}^{(\alpha)}$. D'un point de vue probabiliste, elle est la masse, sous la mesure d'excursion en dehors de la racine pour le mouvement brownien sur l'arbre, des excursions qui atteignent la hauteur 1. D'après la définition de

$\Delta^{(\alpha)}$ et l'interprétation du réseau électrique ci-dessus, $\mathcal{C}^{(\alpha)}$ satisfait l'équation récursive en loi

$$\mathcal{C}^{(\alpha)} \stackrel{(\text{loi})}{=} \left(U + \frac{1-U}{\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \dots + \mathcal{C}_{N_\alpha}^{(\alpha)}} \right)^{-1}, \quad (1.9)$$

où $(\mathcal{C}_i^{(\alpha)})_{i \geq 1}$ sont des copies i.i.d. de $\mathcal{C}^{(\alpha)}$, la variable aléatoire N_α est distribuée selon θ_α , et U est uniformément distribuée sur $[0, 1]$. Toutes ces variables aléatoires impliquées sont supposées indépendantes. Nous montrons que, pour tout $\alpha \in (1, 2]$, la loi γ_α de la conductance $\mathcal{C}^{(\alpha)}$ est caractérisée dans la famille de toutes les mesures de probabilité sur $[1, \infty)$ par l'équation (1.9).

Théorème ([66, Theorem 3 & Proposition 4]).

- Pour tout $\alpha \in (1, 2]$, la constante β_α apparaissant dans les deux théorèmes précédents est donnée par

$$\beta_\alpha = \frac{1}{2} \left(\frac{(\int \gamma_\alpha(ds)s)^2}{\iint \gamma_\alpha(ds)\gamma_\alpha(dt)\frac{st}{s+t-1}} - 1 \right).$$

- Il existe une constante $M > 0$ telle que pour tout $\alpha \in (1, 2]$, $\beta_\alpha < M$.

Notons que la dimension de la frontière $\partial\Delta^{(\alpha)}$ tend vers l'infini quand $\alpha \downarrow 1$. Néanmoins, la dimension de la mesure harmonique μ_α reste toujours bornée. Il est aussi intéressant de remarquer que, pour tout $\alpha \in (1, 2]$, la constante β_α s'exprime par la même fonction de la distribution γ_α , bien que les arbres réduits stables $(\Delta^{(\alpha)}, \alpha \in (1, 2])$ aient des mécanismes de branchement différents.

1.4.2 La mesure harmonique près d'un point typique de la frontière

Les résultats présentés dans cette section sont détaillés dans le chapitre 6. Ce chapitre est basé sur l'article [68] en préparation.

Nous allons étudier le comportement local de la mesure harmonique près d'un point typique sur la frontière $\mathbb{T}_n^{(n)}$. Supposons dans cette section que la loi de reproduction critique μ est de variance finie. Nous écrivons donc $\beta = \beta_2$, $\Delta = \Delta^{(2)}$ et $\mathcal{C} = \mathcal{C}^{(2)}$ pour alléger la notation. Cependant, afin de ne pas confondre avec la loi de reproduction μ , nous continuerons d'écrire μ_2 la mesure harmonique sur $\partial\Delta$.

Notre théorème principal dans le cadre discret est le suivant :

Théorème ([68, Theorem 1]).

Soit Ω_n un sommet uniformément choisi au hasard de $\mathbb{T}_n^{(n)}$. Il existe alors une constante universelle^a $\lambda \in (1, \infty)$, qui ne dépend pas de la loi de reproduction μ , telle que, pour tout $\delta > 0$, nous avons

$$\lim_{n \rightarrow \infty} \mathbb{P}(n^{-\lambda-\delta} \leq \mu_n(\Omega_n) \leq n^{-\lambda+\delta}) = 1.$$

^a. Les simulations numériques montrent que $\lambda \approx 1.21$.

Pour énoncer le résultat analogue dans le continu, on introduit un nouvel arbre réel (non-compact) binaire Γ , tel que chaque point à hauteur $y \in [0, \infty)$ dans cet arbre correspond, de façon bijective, à un point dans $\Delta \setminus \partial\Delta$ à hauteur $1 - e^{-y} \in [0, 1)$. Cet arbre continu aléatoire s'appelle l'*arbre de Yule*, car il décrit la généalogie du processus de Yule classique, où chaque individu a un temps de vie exponentielle de paramètre 1, ces temps de vie sont indépendants les uns des autres, et de plus chaque individu a exactement deux enfants. Par définition, la frontière à l'infini $\partial\Gamma$ de Γ est l'ensemble des géodésiques infinies partant de la racine. Grâce au branchement binaire, $\partial\Gamma$ peut s'identifier avec $\{1, 2\}^{\mathbb{N}}$. Pour tout $r > 0$, soit Γ_r l'ensemble des points de Γ à hauteur r . Par un argument de martingale, on peut définir

$$\mathcal{W} := \lim_{r \rightarrow \infty} e^{-r} \#\Gamma_r,$$

et il est bien connu que \mathcal{W} suit une loi exponentielle de paramètre 1. Pour tout $x \in \Gamma$, soit $H(x)$ la hauteur de x , et $\Gamma[x]$ le sous-arbre de descendants de x dans Γ . On définit de manière similaire

$$\mathcal{W}_x := \lim_{r \rightarrow \infty} e^{-r} \#\Gamma_r[x].$$

La *mesure uniforme* $\bar{\omega}$ sur $\partial\Gamma$ est une mesure de probabilité (aléatoire) sur $\partial\Gamma$ qui vérifie les deux propriétés suivantes :

- (1) p.s. $\bar{\omega}$ n'a pas d'atome, et son support topologique est $\partial\Gamma$;
- (2) pour tout $x \in \Gamma$ et pour chaque géodésique infinie \mathbf{v} passant par x ,

$$\bar{\omega}(\mathbf{B}(\mathbf{v}, H(x))) = e^{-H(x)} \frac{\mathcal{W}_x}{\mathcal{W}},$$

où $\mathbf{B}(\mathbf{v}, H(x))$ désigne l'ensemble de tous les géodésiques infinies dans Γ qui coïncident avec \mathbf{v} jusqu'à hauteur $H(x)$.

Puisque les frontières $\partial\Delta$ et $\partial\Gamma$ peuvent être identifiées par la bijection expliquée ci-dessus, on note ω la mesure de probabilité (aléatoire) sur $\partial\Delta$ induite par $\bar{\omega}$, qui sera appelée la *mesure uniforme* sur $\partial\Delta$.

Théorème ([68, Theorem 2]).

Avec la même constante λ que précédemment, on a \mathbb{P} p.s. $\omega(dx)$ p.p.,

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\log \mu_2(\mathcal{B}(\mathbf{x}, r))}{\log r} &= \lambda, \\ \lim_{r \downarrow 0} \frac{\log \omega(\mathcal{B}(\mathbf{x}, r))}{\log r} &= 1, \end{aligned}$$

où $\mathcal{B}(x, r)$ est la boule fermée de centre x et de rayon r dans Δ .

Comme corollaire direct, on obtient que \mathbb{P} p.s., la mesure harmonique μ_2 est singulière par rapport à la mesure uniforme ω .

Pour démontrer les deux derniers théorèmes, notons d'abord que, choisir uniformément au hasard un sommet de $\mathbb{T}_n^{(n)}$ est, dans un certain sens, équivalent à biaiser la loi de $\mathbb{T}^{(n)}$ par

$\#\mathcal{T}_n^{(n)}$. Cette observation nous amène à introduire l'*arbre de Galton–Watson biaisé par la taille*, noté $\widehat{\mathcal{T}}$, cf. [43, 70]. Écrivons $\widehat{\mu}$ la loi biaisée de μ , définie par $\widehat{\mu}(k) = k\mu(k)$ pour tout $k \geq 1$ (remarquons que $\widehat{\mu}$ est bien une mesure de probabilité car μ est critique). Soit $(\widehat{N}_k)_{k \geq 1}$ une suite de variables aléatoires i.i.d. de même loi $\widehat{\mu}$. Elles sont toutes égales ou supérieures à 1. La racine \emptyset de l'arbre $\widehat{\mathcal{T}}$ donne naissance à \widehat{N}_1 enfants. Prenons un des ses enfants uniformément au hasard, que l'on note \mathbf{v}_1 . Les autres engendrent indépendamment les uns des autres des μ -GW arbres ordinaires. L'individu \mathbf{v}_1 , quant à lui, donne naissance à \widehat{N}_2 enfants. De même, on prend un de ces nouveaux enfants au hasard, que l'on note \mathbf{v}_2 . Les autres sont les racines de μ -GW arbres ordinaires. On peut répéter cette procédure une infinité de fois, et l'arbre infini aléatoire obtenu est l'arbre biaisé $\widehat{\mathcal{T}}$ (voir la Figure 1.6). Il est clair par construction que $\widehat{\mathcal{T}}$ a un unique chemin infini $(\emptyset, \mathbf{v}_1, \mathbf{v}_2, \dots)$ partant de la racine, que l'on appelle l'épine dorsale.

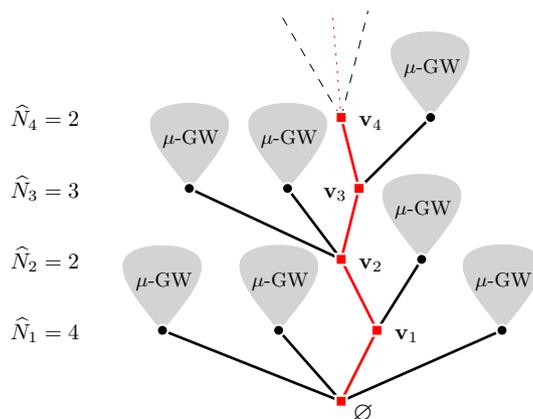


Figure 1.6 – Un arbre de Galton–Watson biaisé par la taille $\widehat{\mathcal{T}}$

De même, afin d'obtenir une meilleure compréhension d'un point distingué choisi au hasard selon la mesure uniforme ω sur $\partial\Delta$, on construit un *arbre réduit continu biaisé par la taille* $\widehat{\Delta}$ de la façon suivante : d'abord, la racine \emptyset de $\widehat{\Delta}$ a une ligne de descendants distinguée qui est un segment de longueur 1. Soit V_\emptyset une variable aléatoire à valeurs dans $[0, 1]$ de loi $2(1-x)dx$. À la hauteur V_\emptyset de la ligne de descendants distinguée, on greffe un sous-arbre qui est une copie indépendante de Δ normalisée par le facteur $(1-V_\emptyset)$. Ensuite, on se donne une copie indépendante V_1 de V_\emptyset , et à la hauteur $V_\emptyset + (1-V_\emptyset)V_1$ de la ligne de descendants distinguée, on greffe une autre copie indépendante de Δ , mais cette fois normalisée par le facteur $(1-V_\emptyset)(1-V_1)$. Notons que, pour chacune des greffes, on choisit le coté gauche ou le coté droite de la ligne de descendants distinguée avec équiprobabilité. Nous continuons cette procédure pour greffer une infinité de sous-arbres sur la ligne de descendants distinguée, avec la hauteur de la position de greffe tendant vers 1, et nous obtenons à la fin une réalisation de $\widehat{\Delta}$.

La constante λ apparaissant dans les deux théorèmes précédents peut aussi être exprimée en termes de la conductance de $\widehat{\Delta}$. Si l'on regarde l'arbre $\widehat{\Delta}$ comme un réseau électrique avec un ohm de résistance par unité de longueur, la conductance équivalente entre la racine et la hauteur 1 est une variable aléatoire \widehat{C} à valeurs dans $[1, \infty)$. On peut montrer que la distribution de \widehat{C} est caractérisée par l'équation réursive en loi

$$\widehat{C} \stackrel{\text{(loi)}}{=} \left(V + \frac{1-V}{\widehat{C} + \widehat{C}} \right)^{-1}, \quad (1.10)$$

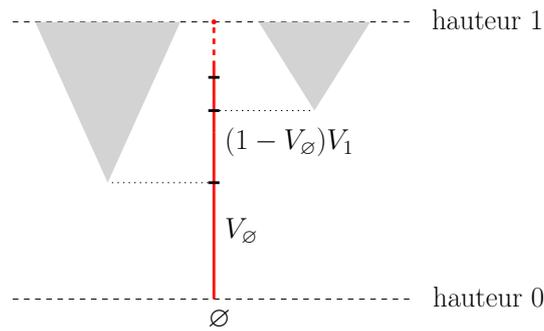


Figure 1.7 – Un arbre réduit continu biaisé par la taille $\widehat{\Delta}$

où la loi de la variable aléatoire V a densité $2(1 - x)$ sur $[0, 1]$, et $V, \mathcal{C}, \widehat{\mathcal{C}}$ sont supposées être indépendantes. On en déduit que la conductance $\widehat{\mathcal{C}}$ a des moments finis de tous les ordres, et, elle admet une densité \widehat{f} continue sur $[1, \infty)$, qui atteint un maximum global à $3/2$.

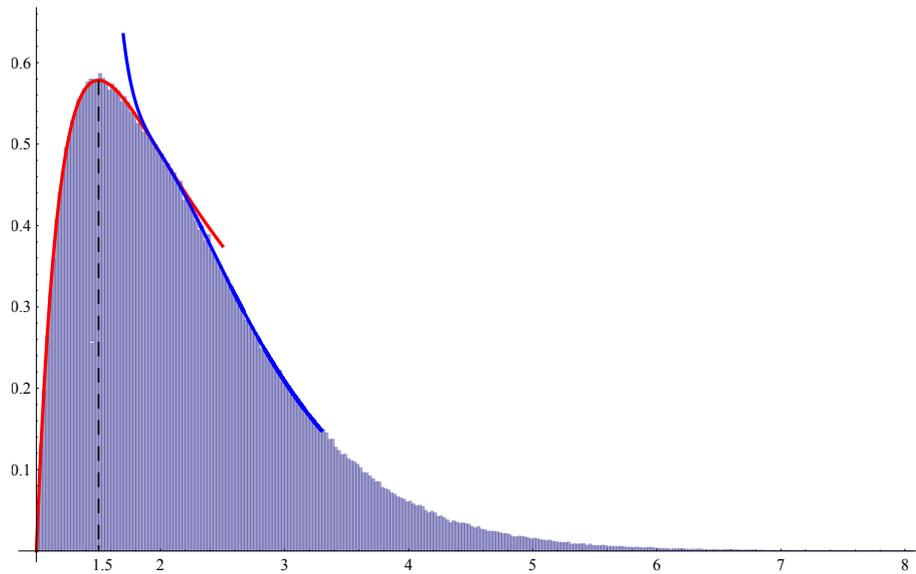


Figure 1.8 – Un histogramme de la distribution de $\widehat{\mathcal{C}}$ sur $(1, \infty)$ obtenu à partir des simulations basées sur (1.10). Les courbes rouges et bleues correspondent respectivement aux formules explicites pour la densité \widehat{f} sur $[1, 2]$ et $[2, 3]$.

Nous terminons avec la belle formule suivante pour la constante λ .

Proposition ([68, Proposition 4]).

La constante λ apparaissant dans les deux théorèmes précédents est donnée par

$$\lambda = \mathbb{E}[\widehat{\mathcal{C}}] - 1.$$

Bien entendu, on s'attend à ce que nos résultats dans cette section se généralisent au cas où la loi de reproduction μ appartienne au domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2)$. Cela fera l'objet d'un futur projet de recherche.

PREMIÈRE PARTIE

LA MARCHE ALÉATOIRE INDEXÉE PAR
UN ARBRE

THE RANGE OF TREE-INDEXED RANDOM WALK IN HIGH DIMENSIONS

LES RÉSULTATS DE CE CHAPITRE SONT ISSUS DE L'ARTICLE [61], ÉCRIT EN COLLABORATION AVEC JEAN-FRANÇOIS LE GALL ET ACCEPTÉ POUR PUBLICATION DANS LE *Journal of the Institute of Mathematics of Jussieu*.

We provide asymptotics for the range R_n of a random walk on the d -dimensional lattice indexed by a random tree with n vertices. Using Kingman's subadditive ergodic theorem, we prove under general assumptions that $n^{-1}R_n$ converges to a constant, and we give conditions ensuring that the limiting constant is strictly positive. On the other hand, in dimension 4 and in the case of a symmetric random walk with exponential moments, we prove that R_n grows like $n/\log n$. We apply our results to asymptotics for the range of branching random walk when the initial size of the population tends to infinity.

2.1 Introduction

The main goal of this work is to derive asymptotics for the number of distinct sites of the lattice visited by a tree-indexed random walk. Asymptotics for the range of an ordinary random walk on the d -dimensional lattice \mathbb{Z}^d have been studied extensively since the pioneering work of Dvoretzky and Erdős [29]. Consider for simplicity the case of a simple random walk on \mathbb{Z}^d , and, for every integer $n \geq 1$, let R_n be the number of distinct sites of \mathbb{Z}^d visited by the random walk up to time n . When $d \geq 3$, let $q_d > 0$ be the probability that the random walk never returns to its starting point. Then,

- if $d \geq 3$,

$$\frac{1}{n} R_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} q_d,$$

- if $d = 2$,

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \pi,$$

- if $d = 1$,

$$n^{-1/2} R_n \xrightarrow[n \rightarrow \infty]{(d)} \sup_{0 \leq t \leq 1} B_t - \inf_{0 \leq t \leq 1} B_t,$$

where $\xrightarrow{(d)}$ indicates convergence in distribution and $(B_t)_{t \geq 0}$ is a standard linear Brownian motion. The cases $d \geq 3$ and $d = 2$ were obtained in [29], whereas the case $d = 1$ is a very

easy consequence of Donsker's invariance theorem (see e.g. [34]). The preceding asymptotics have been extended to much more general random walks. In particular, for any random walk in \mathbb{Z}^d , an application of Kingman's subadditive ergodic theorem [45] shows that the quantity R_n/n converges a.s. to the probability that the random walk does not return to its starting point (which is positive if the random walk is transient). See also [34] for the almost sure convergence of the (suitably normalized) range of an arbitrary recurrent random walk in the plane, [33] for a central limit theorem for the range of transient random walk, [54] for a non-Gaussian central limit theorem in the plane and [64] for a general study of the range of random walks in the domain of attraction of a stable distribution.

In the present work, we discuss similar asymptotics for tree-indexed random walk. We consider (discrete) plane trees, which are rooted ordered trees that can be viewed as describing the genealogy of a population starting with one ancestor or root, which is usually denoted by the symbol \emptyset . Given such a tree \mathcal{T} and a probability measure θ on \mathbb{Z}^d , we can consider the random walk with jump distribution θ indexed by the tree \mathcal{T} . This means that we assign a (random) spatial location $Z_{\mathcal{T}}(u) \in \mathbb{Z}^d$ to every vertex u of \mathcal{T} , in the following way. First, the spatial location $Z_{\mathcal{T}}(\emptyset)$ of the root is the origin of \mathbb{Z}^d . Then, we assign independently to every edge e of the tree \mathcal{T} a random variable X_e distributed according to θ , and we let the spatial location $Z_{\mathcal{T}}(u)$ of the vertex u be the sum of the quantities X_e over all edges e belonging to the simple path from \emptyset to u in the tree. The number of distinct spatial locations is called the range of the tree-indexed random walk $Z_{\mathcal{T}}$.

Let us state a particular case of our results.

Theorem 2.1. *Let θ be a probability distribution on \mathbb{Z}^d , which is symmetric and has finite support. Assume that θ is not supported on a strict subgroup of \mathbb{Z}^d . For every integer $n \geq 1$, let \mathcal{T}_n be a random tree uniformly distributed over all plane trees with n vertices. Conditionally given \mathcal{T}_n , let $Z_{\mathcal{T}_n}$ be a random walk with jump distribution θ indexed by \mathcal{T}_n , and let \mathcal{R}_n stand for the range of $Z_{\mathcal{T}_n}$. Then,*

- if $d \geq 5$,

$$\frac{1}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(P)} c_{\theta},$$

where $c_{\theta} > 0$ is a constant depending on θ , and $\xrightarrow{(P)}$ indicates convergence in probability;

- if $d = 4$,

$$\frac{\log n}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{L^2} 8 \pi^2 \sigma^4,$$

where $\sigma^2 = (\det M_{\theta})^{1/4}$, with M_{θ} denoting the covariance matrix of θ ;

- if $d \leq 3$,

$$n^{-d/4} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{(d)} a_{\theta} \lambda_d(\text{supp}(\mathcal{I})),$$

where $a_{\theta} = 2^{d/4}(\det M_{\theta})^{1/2}$ is a constant depending on θ , and $\lambda_d(\text{supp}(\mathcal{I}))$ stands for the Lebesgue measure of the support of the random measure on \mathbb{R}^d known as ISE (Integrated Super-Brownian Excursion).

Notice the obvious analogy with the results for the range of (ordinary) random walk that were recalled above. At an intuitive level, \mathcal{R}_n is likely to be smaller than the range R_n of ordinary random walk, because one expects many more self-intersections in the tree-indexed case. This is reflected in the fact that the ‘‘critical dimension’’ is now $d = 4$ instead of $d = 2$. In the same way

as $d = 2$ is critical for the recurrence of random walk on \mathbb{Z}^d , one may say that $d = 4$ is critical for the recurrence of tree-indexed random walk, in the sense that for random walk indexed by a “typical” large tree of size n , the number of returns to the origin will grow logarithmically with n . Furthermore, one may notice that the set of all spatial locations of \mathcal{T}_n is contained in the ball of radius $Cn^{1/4}$ centered at the origin, with a probability close to 1 if the constant C is sufficiently large (see Janson and Marckert [39] or Kesten [44] in a slightly different setting), so that the range \mathcal{R}_n is at most of order $n^{d/4}$ in dimension $d \leq 3$. We finally mention that the limiting constant c_θ in dimension $d \geq 5$ can again be interpreted as a probability of no return to the origin for random walk indexed by a certain infinite random tree: See Section 2.2 below for more details.

Let us emphasize that asymptotics of the type of Theorem 2.1 hold in a much more general setting. Firstly, it is enough to assume that the jump distribution θ is centered and has sufficiently high moments (a little more is needed when $d = 4$). Our argument to get the case $d \geq 5$ of Theorem 2.1 relies on an application of Kingman’s subadditive ergodic theorem, which gives the convergence of $\frac{1}{n}\mathcal{R}_n$ to a (possibly vanishing) constant in any dimension d , without any moment assumption on θ . Secondly, in all cases except the critical dimension $d = 4$, we can handle more general random trees. Our methods apply to Galton–Watson trees with an offspring distribution having mean one and finite variance, which are conditioned to have exactly n vertices. In the special case where the offspring distribution is geometric with parameter $1/2$, we recover uniformly distributed plane trees, but the setting of conditioned Galton–Watson trees includes other important “combinatorial trees” such as binary trees or Cayley trees (see e.g. [58]). Some of our results even hold for an offspring distribution with infinite variance in the domain of attraction of a stable distribution.

In the present work, we deal with the cases $d \geq 5$ and $d = 4$ of Theorem 2.1, and the extensions that have just been described. The companion paper [61] addresses the “subcritical” case $d \leq 3$, which involves different methods and is closely related to the invariance principles connecting branching random walk with super-Brownian motion.

Let us turn to a more precise description of our main results and of our methods. In Section 2.2 below, we discuss the convergence of $\frac{1}{n}\mathcal{R}_n$ in a general setting. The basic ingredient of the proof is the introduction of a suitable probability measure on a certain set of infinite trees. Roughly speaking, for any offspring distribution μ with mean one, we construct a random infinite tree consisting of an infinite “spine” and, for each node of the spine, of a random number of Galton–Watson trees with offspring distribution μ that branch off the spine at this node. This construction is related to the infinite size-biased Galton–Watson tree (see [70] and references therein), with the difference that we consider only subtrees branching off the right side of the spine. For a more precise description, see subsection 2.2.3. The law of our infinite tree turns out to be invariant under a shift transformation, which basically involves re-rooting the tree at the first vertex (in lexicographical order) that does not belong to the spine. If we consider a random walk (with an arbitrary jump distribution θ) indexed by this infinite tree, the number of distinct locations of the random walk at the first n vertices of the infinite tree yields a subadditive process R_n , to which we can apply Kingman’s theorem in order to get the almost sure convergence of $\frac{1}{n}R_n$ to a constant (Theorem 2.4). One then needs to discuss the positivity of the limiting constant, and this leads to conditions depending both on the offspring distribution μ and on the jump distribution θ . More precisely, we give a criterion (Proposition 2.5) involving the Green function of the random walk and the generating function of μ , which ensures that the limiting constant is positive. In the case when μ has finite variance and if the jump distribution θ is centered (with sufficiently high moments), this criterion is satisfied if $d \geq 5$. The preceding

line of reasoning is of course very similar to the classical application of Kingman’s theorem to the range of ordinary random walk. In the present setting however, additional ingredients are needed to transfer the asymptotics from the case of the infinite random tree to a single Galton–Watson tree conditioned to have n vertices. At this point, we need to assume that the offspring distribution μ has finite variance or is in the domain of attraction of a stable distribution, so that we can use known results [26] on the scaling limit of the height process associated with a sequence of Galton–Watson trees with offspring distribution μ : Applying these results to the sequence of trees that branch off the spine of the infinite tree yields information about the “large” trees in the sequence, which is essentially what we need to cover the case of a single Galton–Watson tree conditioned to be large (Theorem 2.7). The case $d \geq 5$ of Theorem 2.1 follows as a special case of the results in Section 2.2.

Section 2.3, which is the most technical part of the paper, is devoted to the proof of a generalized version of the case $d = 4$ of Theorem 2.1 (Theorem 2.14). We restrict our attention to the case when the offspring distribution is geometric with parameter $1/2$, and we assume that the jump distribution θ is symmetric with small exponential moments. While the symmetry assumption can presumably be weakened without too much additional work, the existence of exponential moments is used at a crucial point of our proof where we rely on the multidimensional extension of the celebrated Komlós-Major-Tusnády strong invariance principle. Our approach is based on the path-valued Markov chain called the discrete snake. In our setting, this process, which we denote by $(W_n)_{n \geq 0}$, takes values in the space of all infinite paths $w : (-\infty, \zeta] \cap \mathbb{Z} \rightarrow \mathbb{Z}^d$, where $\zeta = \zeta(w) \in \mathbb{Z}$ is called the lifetime of w . If ζ_n denotes the lifetime of W_n , the process $(\zeta_n)_{n \geq 0}$ evolves like simple random walk on \mathbb{Z} . Furthermore, if $\zeta_{n+1} = \zeta_n - 1$, the path W_{n+1} is obtained by restricting W_n to the interval $(-\infty, \zeta_n - 1] \cap \mathbb{Z}$, whereas if $\zeta_{n+1} = \zeta_n + 1$, the path W_{n+1} is obtained by adding to W_n one step distributed according to θ . We assume that the initial value W_0 is just a path (indexed by negative times) of the random walk with jump distribution θ started from the origin. Then the values of the discrete snake generate a random walk indexed by an infinite random tree, which corresponds, in the particular case of the geometric offspring distribution, to the construction developed in Section 2.2. Note however that, in contrast with Section 2.2, the Markovian properties of the discrete snake play a very important role in Section 2.3. A key estimate (Proposition 2.8) states that the probability that the “head of the discrete snake” (that is the process $(W_k(\zeta_k))_{k \geq 0}$) does not return to the origin before time n behaves like $c/\log n$ for a certain constant c . This is analogous to the well-known asymptotics for the probability that simple random walk in \mathbb{Z}^2 does not come back to its starting point before time n , but the proof, which is developed in subsection 2.3.2, turns out to be much more involved in our setting. The main result of Section 2.3 (Theorem 2.14) gives the case $d = 4$ of Theorem 2.1 under slightly more general assumptions. It would be of interest to extend this result to more general offspring distributions, but this would require a different approach.

Section 2.4 applies the preceding results to asymptotics for the range of a branching random walk in \mathbb{Z}^d , $d \geq 4$, when the size of the initial population tends to infinity. This study is related to the recent work of Lalley and Zheng [50], who discuss the number of distinct sites occupied by a nearest neighbor branching random walk in \mathbb{Z}^d at a fixed time. Note that the genealogical structures of descendants of the different initial particles are described by independent Galton–Watson trees, which makes it possible to apply our results about the range of tree-indexed random walk. Still one needs to verify that points that are visited by the descendants of two distinct initial particles give a negligible contribution in the limit. The analogous problem for low dimensions $d \leq 3$ is addressed in [61].

Let us finally mention that there are certain analogs of our results in a continuous setting. In particular, Delmas [22] considers the Lebesgue measure of the tubular neighborhood of radius ε of the range of a super-Brownian measure in \mathbb{R}^d , $d \geq 4$, and proves that this measure behaves like ε^{d-4} when $d \geq 5$ and like $|\log \varepsilon|^{-1}$ when $d = 4$. This is analogous to our results for branching random walk in Section 2.4. An exact analog of the cases $d \geq 4$ in Theorem 2.1 is obtained by considering the Lebesgue measure of the tubular neighborhood of radius ε of the support of the so-called integrated super-Brownian excursion (ISE). Asymptotics for this quantity when $\varepsilon \rightarrow 0$ are derived in [22, Corollary 2.4]. We do not know of any analog of Proposition 2.8 in the continuous setting. Abraham and Werner [1] discuss a very similar problem for the Brownian snake and super-Brownian motion in \mathbb{R}^d , but only when $d \leq 3$.

Notation. We use the notation $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ for $a, b \in \mathbb{Z}$, with $a \leq b$. Similarly, $\llbracket -\infty, a \rrbracket := (-\infty, a] \cap \mathbb{Z}$ for $a \in \mathbb{Z}$. For any finite set A , $\#A$ denotes the cardinality of A .

2.2 Linear growth of the range

2.2.1 Finite trees

We use the standard formalism for plane trees. We set

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. If $u = (u_1, \dots, u_n) \in \mathcal{U}$, we set $|u| = n$ (in particular $|\emptyset| = 0$). We write \prec for the lexicographical order on \mathcal{U} , so that $\emptyset \prec 1 \prec (1, 1) \prec 2$ for instance.

If $u, v \in \mathcal{U}$, uv stands for the concatenation of u and v . In particular $\emptyset u = u\emptyset = u$. The genealogical (partial) order \ll is then defined by saying that $u \ll v$ if and only if $v = uw$ for some $w \in \mathcal{U}$.

A plane tree (also called rooted ordered tree) \mathcal{T} is a finite subset of \mathcal{U} such that the following holds:

- (i) $\emptyset \in \mathcal{T}$.
- (ii) If $u = (u_1, \dots, u_n) \in \mathcal{T} \setminus \{\emptyset\}$ then $\hat{u} := (u_1, \dots, u_{n-1}) \in \mathcal{T}$.
- (iii) For every $u = (u_1, \dots, u_n) \in \mathcal{T}$, there exists an integer $k_u(\mathcal{T}) \geq 0$ such that, for every $j \in \mathbb{N}$, $(u_1, \dots, u_n, j) \in \mathcal{T}$ if and only if $1 \leq j \leq k_u(\mathcal{T})$.

The notions of a child and a parent of a vertex of \mathcal{T} are defined in an obvious way. The quantity $k_u(\mathcal{T})$ in (iii) is the number of children of u in \mathcal{T} . If $u \in \mathcal{T}$, we write $[\mathcal{T}]_u = \{v \in \mathcal{U} : uv \in \mathcal{T}\}$, which corresponds to the subtree of descendants of u in \mathcal{T} . We denote the set of all plane trees by \mathbb{T}_f .

Throughout this work, we consider a probability measure μ on \mathbb{Z}_+ , which is critical in the sense that

$$\sum_{k=0}^{\infty} k \mu(k) = 1.$$

We exclude the degenerate case where $\mu(1) = 1$. The law of the Galton–Watson tree with offspring distribution μ is a probability measure on the space \mathbb{T}_f , which we denote by Π_μ (see e.g. [58, Section 1]).

We also let θ be a probability measure on \mathbb{Z}^d , which is adapted in the sense that it is not supported on a strict subgroup of \mathbb{Z}^d .

A (d -dimensional) spatial tree is a pair $(\mathcal{T}, (z_u)_{u \in \mathcal{T}})$ where $\mathcal{T} \in \mathbb{T}_f$ and $z_u \in \mathbb{Z}^d$ for every $u \in \mathcal{T}$. Let \mathbb{T}_f^* be the set of all spatial trees. We write $\Pi_{\mu, \theta}^*$ for the probability distribution on \mathbb{T}_f^* under which \mathcal{T} is distributed according to Π_μ and, conditionally on \mathcal{T} , the “spatial locations” $(z_u)_{u \in \mathcal{T}}$ are distributed as random walk indexed by \mathcal{T} , with jump distribution θ , and started from 0 at the root \emptyset (see the definition given in Section 1). We then set

$$a_{\mu, \theta} := \Pi_{\mu, \theta}^*(z_u \neq 0, \forall u \in \mathcal{T} \setminus \{\emptyset\}),$$

and, for every $y \in \mathbb{Z}^d$,

$$h_{\mu, \theta}(y) := \Pi_{\mu, \theta}^*(z_u \neq -y, \forall u \in \mathcal{T}).$$

Notice that $a_{\mu, \theta} > 0$, simply because with positive probability a tree distributed according to Π_μ consists only of the root.

2.2.2 Infinite trees

We now introduce a certain class of infinite trees. Each tree in this class will consist of an infinite ray or spine starting from the root, and finite subtrees branching off every node of this infinite ray. We label the vertices of the infinite ray by nonpositive integers $0, -1, -2, \dots$. The reason for labelling the vertices of the spine by negative integers comes from the fact that -1 is viewed as the parent of 0 , -2 as the parent of -1 , and so on.

More precisely, we consider the set

$$\mathcal{V} := \mathbb{Z}_- \times \mathcal{U}$$

where $\mathbb{Z}_- = \{0, -1, -2, \dots\}$. For every $j \in \mathbb{Z}_-$, we identify the element (j, \emptyset) of \mathcal{V} with the integer j , and we thus view \mathbb{Z}_- as a subset of \mathcal{V} . We define the lexicographical order on \mathcal{V} as follows. If $j, j' \in \mathbb{Z}_-$, we have $j \prec j'$ if and only if $j \leq j'$. If $u \in \mathcal{U} \setminus \{\emptyset\}$, we have always $j' \prec (j, u)$. If $u, u' \in \mathcal{U} \setminus \{\emptyset\}$, we have $(j, u) \prec (j', u')$ if either $j > j'$, or $j = j'$ and $u \prec u'$. The genealogical (partial) order \ll on \mathcal{V} is defined in an obvious way: in agreement with the preceding heuristic interpretation, the property $j \ll j'$ for $j, j' \in \mathbb{Z}_-$ holds if and only if $j \leq j'$.

Let \mathcal{T} be a subset of \mathcal{V} such that $\mathbb{Z}_- \subset \mathcal{T}$. For every $j \in \mathbb{Z}_-$, we set

$$\mathcal{T}_j := \{u \in \mathcal{U} : (j, u) \in \mathcal{T}\}.$$

We say that \mathcal{T} is an *infinite tree* if, for every $j \in \mathbb{Z}_-$, \mathcal{T}_j is a (finite) plane tree, and furthermore $\mathcal{T} \setminus \mathbb{Z}_-$ is infinite. We write \mathbb{T} for the set of all infinite trees. By convention, the root of an infinite tree \mathcal{T} is the vertex 0 . Clearly, \mathcal{T} is determined by the collection $(\mathcal{T}_j)_{j \in \mathbb{Z}_-}$. Note that the lexicographical order of vertices corresponds to the order of visit when one “moves around” the tree in clockwise order, starting from the “bottom” of the spine and assuming that the “subtrees” \mathcal{T}_j are drawn on the right side of the spine, as in Fig. 1.

We next define a shift transformation τ on the space \mathbb{T} . Starting from an infinite tree \mathcal{T} , its image $\tau(\mathcal{T}) = \mathcal{T}'$ is obtained informally as follows. We look for the first vertex (in lexicographical order) of $\mathcal{T} \setminus \mathbb{Z}_-$. Call this vertex v . We then “re-root” the tree \mathcal{T} at v and, in the case when v is not a child of 0 (or equivalently if $\mathcal{T}_0 = \{\emptyset\}$), we remove the vertices of the spine that are strict descendants of the parent of v .

For a more formal definition, let $k \in \mathbb{Z}_-$ be the unique integer such that $v \in \mathcal{T}_k$ (necessarily, $v = (k, 1)$). Then, \mathcal{T}' is determined by requiring that:

- $\mathcal{T}'_j = \mathcal{T}_{j+k+1}$ if $j \leq -2$;

- $\mathcal{T}'_0 = [\mathcal{T}_k]_1$;
- \mathcal{T}'_{-1} is the unique plane tree such that there exists a bijection from $\mathcal{T}_k \setminus \{u \in \mathcal{T}_k : 1 \ll u\}$ onto \mathcal{T}'_{-1} that preserves both the lexicographical order and the genealogical order.

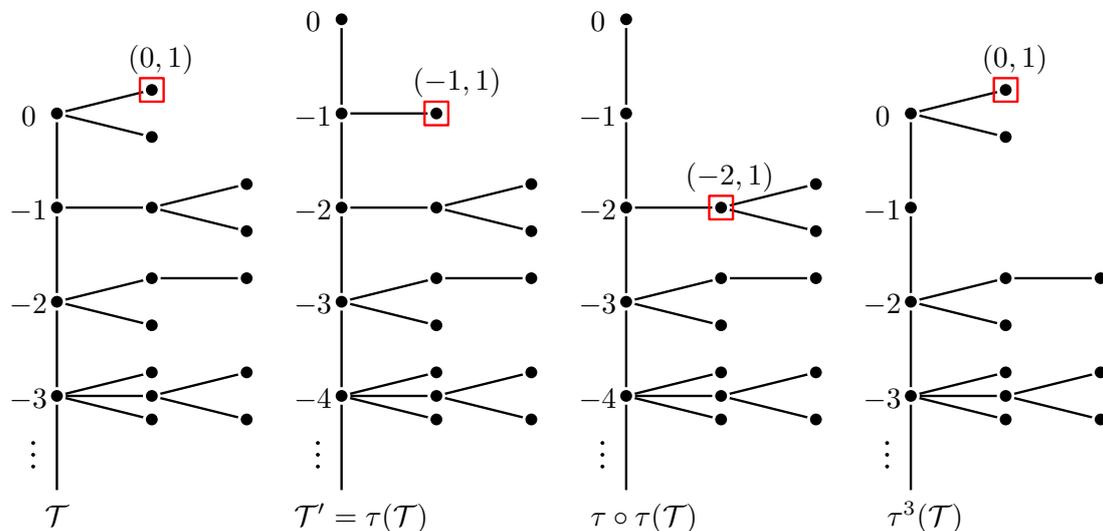


Figure 2.1: The first 3 iterations of the shift transformation on an infinite tree \mathcal{T} . At each step, the marked vertex will become the new root after the shift.

Fig. 2.1 explains the construction of \mathcal{T}' better than the formal definition.

2.2.3 The invariant measure on infinite trees

Let \mathbf{P}_μ be the probability measure on \mathbb{T} that is determined by the following conditions. Under $\mathbf{P}_\mu(d\mathcal{T})$,

- the trees $\mathcal{T}_0, \mathcal{T}_{-1}, \mathcal{T}_{-2}, \dots$ are independent;
- \mathcal{T}_0 is distributed according to Π_μ ;
- for every integer $j \leq -1$,

$$\mathbf{P}_\mu(k_\emptyset(\mathcal{T}_j) = n) = \mu([n+1, \infty)),$$

for every $n \geq 0$; furthermore, conditionally on $k_\emptyset(\mathcal{T}_j) = n$, the trees $[\mathcal{T}_j]_1, [\mathcal{T}_j]_2, \dots, [\mathcal{T}_j]_n$ are independent and distributed according to Π_μ .

Notice that $\sum_{n \geq 0} \mu([n+1, \infty)) = 1$ due to the criticality of the probability measure μ . The reason for introducing the probability measure \mathbf{P}_μ comes from the next proposition.

Proposition 2.2. *The probability measure \mathbf{P}_μ is invariant under the shift τ .*

Proof. Suppose that \mathcal{T} is distributed according to \mathbf{P}_μ and set $\mathcal{T}' = \tau(\mathcal{T})$ as above. We need to verify that \mathcal{T}' is also distributed according to \mathbf{P}_μ , or equivalently that the trees $\mathcal{T}'_0, \mathcal{T}'_{-1}, \dots$ satisfy the same properties as $\mathcal{T}_0, \mathcal{T}_{-1}, \dots$ above. The key point is to calculate the distribution of $(k_\emptyset(\mathcal{T}'_j), j \leq 0)$. Fix an integer $p \geq 1$, and let $n_0, n_1, \dots, n_p \in \mathbb{Z}_+$. Also let k be the element of \mathbb{Z}_- determined as in the definition of $\mathcal{T}' = \tau(\mathcal{T})$ at the end of subsection 2.2.2. The event

$$\{k = 0\} \cap \{k_\emptyset(\mathcal{T}'_0) = n_0, k_\emptyset(\mathcal{T}'_{-1}) = n_1, \dots, k_\emptyset(\mathcal{T}'_{-p}) = n_p\}$$

holds if and only if we have

$$k_{\emptyset}(\mathcal{T}_0) = n_1 + 1, k_1(\mathcal{T}_0) = n_0, k_{\emptyset}(\mathcal{T}_{-1}) = n_2, \dots, k_{\emptyset}(\mathcal{T}_{-p+1}) = n_p,$$

which occurs with probability

$$\mu(n_1 + 1)\mu(n_0)\mu([n_2 + 1, \infty)) \dots \mu([n_p + 1, \infty)).$$

Let $\ell \in \mathbb{Z}_- \setminus \{0\}$. Similarly, the event

$$\{k = \ell\} \cap \{k_{\emptyset}(\mathcal{T}'_0) = n_0, k_{\emptyset}(\mathcal{T}'_{-1}) = n_1, \dots, k_{\emptyset}(\mathcal{T}'_{-p}) = n_p\}$$

holds if and only if we have

$$k_{\emptyset}(\mathcal{T}_0) = 0, \dots, k_{\emptyset}(\mathcal{T}_{\ell+1}) = 0, k_{\emptyset}(\mathcal{T}_{\ell}) = n_1 + 1, k_1(\mathcal{T}_{\ell}) = n_0, k_{\emptyset}(\mathcal{T}_{\ell-1}) = n_2, \dots, k_{\emptyset}(\mathcal{T}_{\ell-p+1}) = n_p,$$

which occurs with probability

$$\mu(0)\mu([1, \infty))^{-\ell-1}\mu([n_1 + 2, \infty))\mu(n_0)\mu([n_2 + 1, \infty)) \dots \mu([n_p + 1, \infty)).$$

Summarizing, we see that the event

$$\{k_{\emptyset}(\mathcal{T}'_0) = n_0, k_{\emptyset}(\mathcal{T}'_{-1}) = n_1, \dots, k_{\emptyset}(\mathcal{T}'_{-p}) = n_p\}$$

has probability

$$\begin{aligned} & \mu(n_0)\mu(n_1 + 1)\mu([n_2 + 1, \infty)) \dots \mu([n_p + 1, \infty)) \\ & + \mu(n_0)\mu(0) \left(\sum_{\ell=-1}^{-\infty} \mu([1, \infty))^{-\ell-1} \right) \mu([n_1 + 2, \infty))\mu([n_2 + 1, \infty)) \dots \mu([n_p + 1, \infty)) \\ & = \mu(n_0)\mu([n_1 + 1, \infty))\mu([n_2 + 1, \infty)) \dots \mu([n_p + 1, \infty)) \end{aligned}$$

as desired. An immediate generalization of the preceding argument shows that, if \mathbf{t}_0 and $\mathbf{t}_{j,i}$, $1 \leq j \leq p$, $1 \leq i \leq n_j$ are given plane trees, the event

$$\{k_{\emptyset}(\mathcal{T}'_{-1}) = n_1, \dots, k_{\emptyset}(\mathcal{T}'_{-p}) = n_p\} \cap \{\mathcal{T}'_0 = \mathbf{t}_0\} \cap \left(\bigcap_{j=1}^p \left(\bigcap_{i=1}^{n_j} \{[\mathcal{T}'_{-j}]_i = \mathbf{t}_{j,i}\} \right) \right)$$

has probability

$$\mu([n_1 + 1, \infty))\mu([n_2 + 1, \infty)) \dots \mu([n_p + 1, \infty)) \times \Pi_{\mu}(\mathbf{t}_0) \times \prod_{j=1}^p \left(\prod_{i=1}^{n_j} \Pi_{\mu}(\mathbf{t}_{j,i}) \right).$$

This completes the proof. □

2.2.4 Random walk indexed by the infinite tree

Let $\mathcal{T} \in \mathbb{T}$. The definition of random walk indexed by \mathcal{T} requires some extra care because we need to specify the orientation of edges: The (oriented) edges of \mathcal{T} are all pairs (x, y) of elements of \mathcal{T} such that there exists $j \in \mathbb{Z}_-$ such that

- either $x = (j, u)$, $y = (j, v)$, where $u, v \in \mathcal{T}_j$ and u is the parent of v ;

– or $x = j - 1$, $y = j$.

See Fig. 2.2. We write $\mathcal{E}(\mathcal{T})$ for the collection of all oriented edges of \mathcal{T} . The random walk indexed by \mathcal{T} is a collection $(Z_{\mathcal{T}}(u))_{u \in \mathcal{T}}$ of random variables with values in \mathbb{Z}^d , such that $Z_{\mathcal{T}}(0) = 0$ and the random variables $(Z_{\mathcal{T}}(y) - Z_{\mathcal{T}}(x))_{(x,y) \in \mathcal{E}(\mathcal{T})}$ are independent and distributed according to θ . Let $P_{(\mathcal{T})}$ stand for the distribution of the collection $(Z_{\mathcal{T}}(u))_{u \in \mathcal{T}}$.

Let \mathbb{T}^* be the set of all pairs $(\mathcal{T}, (z_u)_{u \in \mathcal{T}})$ where $\mathcal{T} \in \mathbb{T}$ and $z_u \in \mathbb{Z}^d$ for every $u \in \mathcal{T}$. We define a probability measure $\mathbf{P}_{\mu, \theta}^*$ on \mathbb{T}^* by declaring that $\mathbf{P}_{\mu, \theta}^*$ is the law of the random pair $(\mathcal{T}, (\mathcal{Z}_u)_{u \in \mathcal{T}})$ where \mathcal{T} is distributed according to \mathbf{P}_{μ} and conditionally on $\mathcal{T} = \mathcal{T}$, $(\mathcal{Z}_u)_{u \in \mathcal{T}}$ is distributed according to $P_{(\mathcal{T})}$.

We next define a shift transformation τ^* on \mathbb{T}^* . For $(\mathcal{T}, (z_u)_{u \in \mathcal{T}}) \in \mathbb{T}^*$, we set $\tau^*(\mathcal{T}, (z_u)_{u \in \mathcal{T}}) = (\mathcal{T}', (z'_u)_{u \in \mathcal{T}'})$, where $\mathcal{T}' = \tau(\mathcal{T})$ and the spatial locations of vertices of \mathcal{T}' (which may be viewed as a subset of \mathcal{T}) are obtained by shifting all original locations z_u so that the location of the root of \mathcal{T}' is again 0. More precisely, if $k \in \mathbb{Z}_-$ is defined as above in the definition of $\mathcal{T}' = \tau(\mathcal{T})$, there is a unique bijection $\phi_{\mathcal{T}}$ from \mathcal{T}' onto $\mathcal{T} \setminus \{k + 1, k + 2, \dots, 0\}$ that maps 0 to $(k, 1)$ and preserves both the lexicographical order and the genealogical order, and we set

$$z'_u = z_{\phi_{\mathcal{T}}(u)} - z_{\phi_{\mathcal{T}}(0)}$$

for every $u \in \mathcal{T}'$.

Proposition 2.3. *The probability measure $\mathbf{P}_{\mu, \theta}^*$ is invariant under τ^* .*

This is an easy consequence of Proposition 2.2 and the way the spatial positions are constructed. We leave the details to the reader.

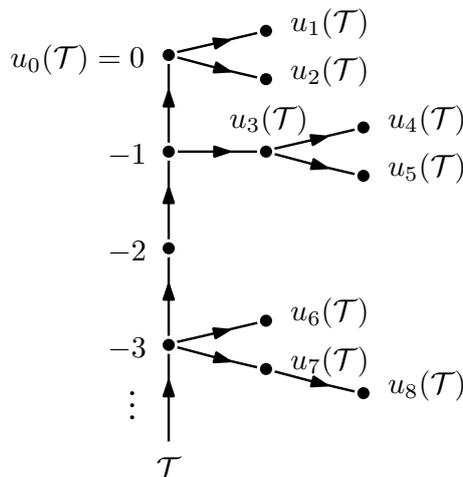


Figure 2.2: The orientation of edges of \mathcal{T} , and the sequence $u_0(\mathcal{T}), u_1(\mathcal{T}), u_2(\mathcal{T}), \dots$

Let $\mathcal{T}^* = (\mathcal{T}, (z_u)_{u \in \mathcal{T}}) \in \mathbb{T}^*$. We define a sequence $(u_i(\mathcal{T}))_{i \geq 0}$ of elements of \mathcal{T} as follows. First, $u_0(\mathcal{T}) = 0$ is the root of \mathcal{T} . Then $u_1(\mathcal{T}), u_2(\mathcal{T}), \dots$ are all elements of $\mathcal{T} \setminus \mathbb{Z}_-$ listed in lexicographical order (see Fig. 2.2). We set, for every integer $n \geq 1$,

$$R_n(\mathcal{T}^*) := \#\{z_{u_0(\mathcal{T})}, z_{u_1(\mathcal{T})}, \dots, z_{u_{n-1}(\mathcal{T})}\}.$$

We let $S = (S_k)_{k \geq 0}$ be a random walk in \mathbb{Z}^d with jump distribution θ , which starts from x under the probability measure P_x , for every $x \in \mathbb{Z}^d$.

Recall the notation $a_{\mu, \theta}$ and $h_{\mu, \theta}$ introduced at the end of subsection 2.2.1.

Theorem 2.4. *We have*

$$\frac{R_n}{n} \xrightarrow[n \rightarrow \infty]{} c_{\mu, \theta}, \quad \mathbf{P}_{\mu, \theta}^* \text{-a.s.}$$

where the limiting constant $c_{\mu, \theta} \in [0, 1]$ may be defined as

$$c_{\mu, \theta} = a_{\mu, \theta} E_0 \left[\prod_{j=1}^{\infty} \Phi_{\mu, \theta}(-S_j) \right],$$

with

$$\Phi_{\mu, \theta}(x) = \sum_{k=0}^{\infty} \mu([k+1, \infty)) \left(\sum_{y \in \mathbb{Z}^d} \theta(y) h_{\mu, \theta}(x+y) \right)^k,$$

for every $x \in \mathbb{Z}^d$.

Proof. Set $\tau_n^* = (\tau^*)^n$ for every integer $n \geq 1$. We claim that, for every $n, m \geq 1$,

$$R_{n+m} \leq R_n + R_m \circ \tau_n^*.$$

Indeed, $R_n(\mathcal{T}^*)$ is the number of distinct elements among $z_{u_0}(\mathcal{T}), z_{u_1}(\mathcal{T}), \dots, z_{u_{n-1}}(\mathcal{T})$, and similarly $R_{n+m}(\mathcal{T}^*)$ is the number of distinct elements among $z_{u_0}(\mathcal{T}), z_{u_1}(\mathcal{T}), \dots, z_{u_{n+m-1}}(\mathcal{T})$. On the other hand, from the construction of the shift transformation, it is fairly easy to verify that $R_m \circ \tau_n^*(\mathcal{T}^*)$ is the number of distinct elements among $z_{u_n}(\mathcal{T}), z_{u_{n+1}}(\mathcal{T}), \dots, z_{u_{n+m-1}}(\mathcal{T})$. The bound of the preceding display follows immediately.

Since $1 \leq R_n \leq n$, we can then apply Kingman's subadditive ergodic theorem to the sequence $(R_n)_{n \geq 1}$, and we get that R_n/n converges almost surely. The fact that the limit is constant is immediate from a simple zero-one law argument (we could also verify that τ^* is ergodic). Furthermore, the limiting constant $c_{\mu, \theta}$ is recovered by

$$c_{\mu, \theta} = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_{\mu, \theta}^*[R_n].$$

However, with the preceding notation,

$$\begin{aligned} \mathbf{E}_{\mu, \theta}^*[R_n] &= \mathbf{E}_{\mu, \theta}^* \left[\sum_{i=0}^{n-1} \mathbf{1}_{\{z_{u_j} \neq z_{u_i}, \forall j \in [i+1, n-1]\}} \right] \\ &= \sum_{i=0}^{n-1} \mathbf{P}_{\mu, \theta}^*(z_{u_j} \neq z_{u_i}, \forall j \in [i+1, n-1]) \\ &= \sum_{i=0}^{n-1} \mathbf{P}_{\mu, \theta}^*(z_{u_j} \neq 0, \forall j \in [1, n-i-1]) \end{aligned}$$

using the shift invariance in the last equality. It now follows that

$$c_{\mu, \theta} = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_{\mu, \theta}^*[R_n] = \mathbf{P}_{\mu, \theta}^*(z_{u_j} \neq 0, \forall j \geq 1),$$

and the right-hand side is easily computed in the form given in the theorem, using the definition of $\mathbf{P}_{\mu, \theta}^*$. \square

Theorem 2.4 does not give much information when the limiting constant $c_{\mu,\theta}$ is equal to 0. In the next proposition, we give sufficient conditions that ensure $c_{\mu,\theta} > 0$. We let g_μ denote the generating function of μ ,

$$g_\mu(r) := \sum_{k=0}^{\infty} \mu(k) r^k, \quad 0 \leq r \leq 1.$$

In the remaining part of this subsection, we assume that the random walk S is transient (it is not hard to see that $c_{\mu,\theta} = 0$ if S is recurrent). We denote the Green function of S by G_θ , that is

$$G_\theta(x) := E_0 \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{S_k=x\}} \right], \quad x \in \mathbb{Z}^d.$$

Proposition 2.5. (i) *The property $c_{\mu,\theta} > 0$ holds if*

$$\prod_{j=1}^{\infty} \left(\frac{1 - g_\mu((1 - G_\theta(S_j))_+)}{G_\theta(S_j)} \right) > 0, \quad P_0\text{-a.s.}$$

(ii) *Suppose that the random walk S is centered and has finite moments of order $(d-1) \vee 2$. Then,*

- *if μ has finite variance, then $c_{\mu,\theta} > 0$ if $d \geq 5$.*
- *if μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$, then $c_{\mu,\theta} > 0$ if $d > \frac{2\alpha}{\alpha-1}$.*

Proof. (i) We have already noticed that $a_{\mu,\theta} > 0$. We then observe that, for every $r \in [0, 1)$,

$$\sum_{k=0}^{\infty} \mu([k+1, \infty)) r^k = \frac{1 - g_\mu(r)}{1 - r}. \quad (2.1)$$

Next we can get a lower bound on the function $h_{\mu,\theta}(y)$ by saying that the probability for tree-indexed random walk to visit the point $-y$ is bounded above by the expected value of the number of vertices at which the random walk sits at $-y$. Since μ is critical, it follows that

$$h_{\mu,\theta}(y) \geq 1 - G_\theta(-y)$$

for every $y \in \mathbb{Z}^d$. Hence, for every $x \in \mathbb{Z}^d$,

$$\sum_{y \in \mathbb{Z}^d} \theta(y) h_{\mu,\theta}(x+y) \geq 1 - \sum_{y \in \mathbb{Z}^d} \theta(y) G_\theta(-x-y).$$

However,

$$\sum_{y \in \mathbb{Z}^d} \theta(y) G_\theta(-x-y) = \sum_{y \in \mathbb{Z}^d} \theta(y) E_{x+y} \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{S_k=0\}} \right] = E_x \left[\sum_{k=1}^{\infty} \mathbf{1}_{\{S_k=0\}} \right] \leq G_\theta(-x).$$

Consequently, using (2.1), we have, for all x such that $G_\theta(-x) > 0$,

$$\Phi_{\mu,\theta}(x) \geq \frac{1 - g_\mu((1 - G_\theta(-x))_+)}{G_\theta(-x)}.$$

The assertion in (i) follows, noting that $G_\theta(S_j) > 0$ for every $j \geq 0$, P_0 -a.s.

(ii) If S is centered with finite moments of order $(d - 1) \vee 2$, then a standard bound for the Green function (see e.g. [54, Théorème 3.5]) gives the existence of a constant C_θ such that, for every $x \in \mathbb{Z}^d$,

$$G_\theta(x) \leq C_\theta |x|^{2-d} \quad (2.2)$$

(recall that we assume that S is transient, so that necessarily $d \geq 3$ here).

Suppose first that μ has a finite variance σ_μ^2 . Then,

$$g_\mu(1 - s) = 1 - s + \frac{\sigma_\mu^2}{2} s^2 + o(s^2)$$

as $s \rightarrow 0$. Consequently,

$$\frac{1 - g_\mu(1 - s)}{s} = 1 - \frac{\sigma_\mu^2}{2} s + o(s)$$

as $s \rightarrow 0$. By taking $s = G_\theta(S_j)$, we see that the condition in (i) will be satisfied if

$$\sum_{j=1}^{\infty} G_\theta(S_j) < \infty, \quad P_0\text{-a.s.}$$

However, using the local limit theorem and the preceding bound for G_θ , it is an easy matter to verify that the property

$$E_0 \left[\sum_{j=1}^{\infty} G_\theta(S_j) \right] < \infty$$

holds if $d \geq 5$. This gives the desired result when μ has a finite variance.

Suppose now that μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$. Then the generating function of μ must satisfy the property

$$g_\mu(1 - s) = 1 - s + s^\alpha L(s)$$

where L is slowly varying as $s \downarrow 0$ (see e.g. the discussion in [26, p.60]). By the same argument as above, we see that the condition in (i) will be satisfied if

$$\sum_{j=1}^{\infty} G_\theta(S_j)^{\alpha-1} L(G_\theta(S_j)) < \infty, \quad P_0\text{-a.s.}$$

and this holds if

$$(\alpha - 1)(d - 2)/2 > 1,$$

which completes the proof. \square

Remarks. 1. The moment assumption in (ii) can be weakened a little: According to [51], the bound (2.2) holds provided the random walk S (is centered and) has moments of order $(d - 2 + \varepsilon) \vee 2$ for some $\varepsilon > 0$. However moments of order $d - 2$ would not be sufficient for this bound.

2. Suppose that the random walk S satisfies the conditions in part (ii) of the proposition. If μ has finite variance, it is not hard to verify that $c_{\mu,\theta} = 0$ if $d \leq 4$. Let us briefly sketch the argument. It is enough to consider the case $d = 4$. Under the probability measure $\Pi_{\mu,\theta}^*$, write N_x for the number of vertices whose spatial location is equal to x . Then, if $x \neq 0$,

$$\Pi_{\mu,\theta}^*[N_x] = G_\theta(x) \geq C'_\theta |x|^{-2}$$

for some constant $C'_\theta > 0$. On the other hand, standard arguments for Galton–Watson trees show that there exists a constant K_μ such that

$$\Pi_{\mu,\theta}^*[(N_x)^2] \leq K_\mu \sum_{z \in \mathbb{Z}^4} G_\theta(z) G_\theta(x-z)^2.$$

Using (2.2) and simple calculations, we obtain the existence of a constant $K'_{\mu,\theta}$ such that, for every $x \in \mathbb{Z}^4$ with $|x| \geq 2$,

$$\Pi_{\mu,\theta}^*[(N_x)^2] \leq K'_{\mu,\theta} |x|^{-2} \log |x|.$$

Hence, for every $x \in \mathbb{Z}^4$ with $|x| \geq 2$,

$$1 - h_{\mu,\theta}(x) = \Pi_{\mu,\theta}^*(N_{-x} \geq 1) \geq \frac{(\Pi_{\mu,\theta}^*[N_{-x}])^2}{\Pi_{\mu,\theta}^*[(N_{-x})^2]} \geq (C'_\theta)^2 (K'_{\mu,\theta})^{-1} |x|^{-2} (\log |x|)^{-1}.$$

The property $c_{\mu,\theta} = 0$ now follows easily. In the next section, we will see (in a particular case) that the proper normalization factor for R_n is $(\log n)/n$ when $d = 4$.

3. The paper [67] considers certain random walks with drift on \mathbb{Z}^d , for which the property $c_{\mu,\theta} > 0$ holds if $d \geq 4$ although the bound (2.2) no longer holds.

4. It is an interesting question whether the condition $d > \frac{2\alpha}{\alpha-1}$ is also sharp when μ is in the domain of attraction of a stable distribution of index α . We will not discuss this problem here as our main interest lies in the case when μ has finite variance.

2.2.5 Conditioned trees

Our goal is now to obtain an analog of the convergence of Theorem 2.4 for random walk indexed by a single Galton–Watson tree conditioned to be large. Recall from subsection 2.2.1 the notation \mathbb{T}_f^* for the set of all spatial trees. If $\mathcal{T}^* = (\mathcal{T}, (z_u)_{u \in \mathcal{T}})$ is a spatial tree with at least n vertices, we keep the same notation $R_n(\mathcal{T}^*)$ for the number of distinct points in the sequence $z_{u_0}, z_{u_1}, \dots, z_{u_{\#\mathcal{T}-1}}$, where $u_0, u_1, \dots, u_{\#\mathcal{T}-1}$ are the vertices of \mathcal{T} listed in lexicographical order. Also recall from subsection 2.2.1 the definition of the probability measure $\Pi_{\mu,\theta}^*$ on \mathbb{T}_f^* .

Proposition 2.6. *Assume that μ has finite variance σ_μ^2 , or that μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$. For every $n \geq 1$, let $\mathcal{T}_{(>n)}^*$ be a random spatial tree distributed according to the probability measure $\Pi_{\mu,\theta}^*(\cdot | \#\mathcal{T} > n)$. Then, for every $a \in (0, 1]$,*

$$\frac{1}{n} R_{\lfloor an \rfloor}(\mathcal{T}_{(>n)}^*) \xrightarrow[n \rightarrow \infty]{} c_{\mu,\theta} a$$

in probability.

Proof. We first consider the case when μ has finite variance σ_μ^2 . Let $\mathcal{T}^* = (\mathcal{T}, (\mathcal{L}_u)_{u \in \mathcal{T}})$ be a \mathbb{T}^* -valued random variable distributed according to $\mathbf{P}_{\mu,\theta}^*$ under the probability measure P . Recall the notation \mathcal{T}_j , for $j \in \mathbb{Z}_-$, introduced in subsection 2.2.2. By construction, the “subtrees”

$$\mathcal{T}_0, [\mathcal{T}_{-1}]_1, [\mathcal{T}_{-1}]_2, \dots, [\mathcal{T}_{-1}]_{k_\emptyset(\mathcal{T}_{-1})}, [\mathcal{T}_{-2}]_1, [\mathcal{T}_{-2}]_2, \dots, [\mathcal{T}_{-2}]_{k_\emptyset(\mathcal{T}_{-2})}, [\mathcal{T}_{-3}]_1, \dots$$

then form an infinite sequence of independent random trees distributed according to Π_μ . To simplify notation we denote this sequence by $\mathcal{T}_{(0)}, \mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \dots$. We then introduce the height

process $(H_k)_{k \geq 0}$ associated with this sequence of trees (see [58, Section 1]). This means that, for every $j \geq 0$, we first enumerate the vertices of \mathcal{T}_j in lexicographical order, then we concatenate the finite sequences obtained in this way to get an infinite sequence $(v_k)_{k \geq 0}$ of elements in \mathcal{U} , and we finally set $H_k := |v_k|$ for every $k \geq 0$. Note that the infinite sequence of vertices $(v_k)_{k \geq 0}$ thus obtained is essentially the same as the sequence $(u_k(\mathcal{T}))_{k \geq 0}$ introduced in subsection 2.2.4.

Then (see e.g. [58, Theorem 1.8]), we have the convergence in distribution

$$\left(\frac{1}{\sqrt{n}} H_{\lfloor nt \rfloor} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left(\frac{2}{\sigma_\mu} |\beta_t| \right)_{t \geq 0}, \quad (2.3)$$

where $(\beta_t)_{t \geq 0}$ denotes a standard linear Brownian motion. Next, for every integer $n \geq 1$, set

$$k_n := \inf \{ k \geq 0 : \#\mathcal{T}_{(k)} > n \}.$$

Clearly, the tree $\mathcal{T}_{(k_n)}$ is distributed according to $\Pi_\mu(\cdot \mid \#\mathcal{T} > n)$. Also set

$$d_{k_n} := \sum_{0 \leq j < k_n} \#\mathcal{T}_{(j)}.$$

Using the convergence (2.3), it is not hard to prove (see e.g. the proof of Theorem 5.1 in [59]) that

$$\frac{1}{n} d_{k_n} \xrightarrow[n \rightarrow \infty]{(d)} D_1, \quad (2.4)$$

where D_1 denotes the initial time of the first excursion of β away from 0 with duration greater than 1.

By Theorem 2.4 and an obvious monotonicity argument, we have for every integer $K > 0$,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq K} \left| \frac{1}{n} R_{\lfloor nt \rfloor}(\mathcal{T}^*) - c_{\mu, \theta} t \right| = 0, \quad \text{a.s.}$$

and it follows that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} R_{(d_{k_n} + \lfloor an \rfloor) \wedge Kn}(\mathcal{T}^*) - \frac{1}{n} R_{d_{k_n} \wedge Kn}(\mathcal{T}^*) - c_{\mu, \theta} \left(\left(\frac{d_{k_n}}{n} + a \right) \wedge K - \frac{d_{k_n}}{n} \wedge K \right) \right| = 0, \quad \text{a.s.}$$

Since K can be chosen arbitrarily large, we deduce from the last convergence and (2.4) that we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(R_{d_{k_n} + \lfloor an \rfloor}(\mathcal{T}^*) - R_{d_{k_n}}(\mathcal{T}^*) \right) = c_{\mu, \theta} a$$

in probability.

Let $\mathcal{T}_{(k_n)}^*$ stand for the spatial tree obtained from $\mathcal{T}_{(k_n)}$ by keeping the spatial positions induced by \mathcal{T}^* . Then, by construction, we have

$$R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*) \geq R_{d_{k_n} + \lfloor an \rfloor}(\mathcal{T}^*) - R_{d_{k_n}}(\mathcal{T}^*).$$

Therefore, using the preceding convergence in probability, we obtain that, for every fixed $\varepsilon > 0$,

$$P\left(R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*) \geq (c_{\mu, \theta} a - \varepsilon)n \right) \xrightarrow[n \rightarrow \infty]{} 1. \quad (2.5)$$

We claim that we have also

$$P\left(R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*) \leq (c_{\mu, \theta} a + \varepsilon)n \right) \xrightarrow[n \rightarrow \infty]{} 1. \quad (2.6)$$

To see this, we argue by contradiction and suppose that for all n belonging to a sequence $(n_j)_{j \geq 1}$ converging to infinity, we have

$$P\left(R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*) > (c_{\mu, \theta} a + \varepsilon)n\right) \geq \delta$$

for some $\delta > 0$ independent of n . We suppose that $c_{\mu, \theta} > 0$ (the case when $c_{\mu, \theta} = 0$ is easier). We observe that, for every fixed n , the tree $\mathcal{T}_{(k_n)}$ and the quantity $R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*)$ are independent of the random variable d_{k_n} . Notice that $\mathcal{T}_{(k_n)}^*$ is *not* independent of d_{k_n} , because the value of d_{k_n} clearly influences the distribution of the spatial location of the root of $\mathcal{T}_{(k_n)}$. However, if we simultaneously translate all spatial locations of $\mathcal{T}_{(k_n)}^*$ so that the new location of the root is 0, the new locations become independent of d_{k_n} , and the translation does not affect $R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*)$. On the other hand, from the convergence in distribution (2.4), we can find $\delta' > 0$ such that, for every sufficiently large n ,

$$P\left(d_{k_n} \leq \frac{\varepsilon}{2c_{\mu, \theta}} n\right) \geq \delta'.$$

Using the preceding independence property, we conclude that, for every sufficiently large n in the sequence $(n_j)_{j \geq 1}$,

$$P\left(R_{\lfloor (\varepsilon n / 2c_{\mu, \theta}) + an \rfloor}(\mathcal{T}^*) \geq (c_{\mu, \theta} a + \varepsilon)n\right) \geq P\left(d_{k_n} \leq \frac{\varepsilon}{2c_{\mu, \theta}} n\right) P\left(R_{\lfloor an \rfloor}(\mathcal{T}_{(k_n)}^*) > (c_{\mu, \theta} a + \varepsilon)n\right) \geq \delta\delta'.$$

However Theorem 2.4 implies that

$$\frac{1}{n} R_{\lfloor (\varepsilon n / 2c_{\mu, \theta}) + an \rfloor}(\mathcal{T}^*) \xrightarrow[n \rightarrow \infty]{} c_{\mu, \theta} a + \frac{\varepsilon}{2}, \quad \text{a.s.}$$

and so we arrive at a contradiction, which completes the proof of (2.6).

By construction, the tree $\mathcal{T}_{(k_n)}$ is distributed according to $\Pi_\mu(\cdot \mid \#\mathcal{T} > n)$, and if we shift all spatial locations of $\mathcal{T}_{(k_n)}^*$ so that the new location of the root is 0, we get a random spatial tree distributed according to $\Pi_{\mu, \theta}^*(\cdot \mid \#\mathcal{T} > n)$. The convergence of the proposition thus follows from (2.5) and (2.6).

The proof in the case when μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$ is essentially the same, noting that Theorems 2.3.1 and 2.3.2 in [26] give an analog of the convergence (2.3), where the role of reflected Brownian motion is played by the so-called height process associated with the stable Lévy process with index α . We omit the details. \square

We now would like to get a statement analogous to Proposition 2.6 for a tree conditioned to have a fixed number of vertices. This will follow from Proposition 2.6 by an absolute continuity argument. Before stating the result, we need to introduce some notation. Let \mathcal{G} be the smallest subgroup of \mathbb{Z} that contains the support of μ . Plainly, the cardinality of the vertex set of a tree distributed according to Π_μ belongs to $1 + \mathcal{G}$. On the other hand, for every sufficiently large integer $p \in 1 + \mathcal{G}$, we have $\Pi_\mu(\#\mathcal{T} = p) > 0$, so that the definition of $\Pi_\mu(\cdot \mid \#\mathcal{T} = p)$ makes sense.

If $\mathcal{T}^* = (\mathcal{T}, (z_u)_{u \in \mathcal{T}})$ is a spatial tree, we write $\mathcal{R}(\mathcal{T}^*)$ for the number of distinct elements in $\{z_u : u \in \mathcal{T}\}$.

Theorem 2.7. *Assume that μ has finite variance σ_μ^2 , or that μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$. For every sufficiently large integer $n \in \mathcal{G}$, let $\mathcal{T}_{(n)}^*$*

be a random spatial tree distributed according to the probability measure $\Pi_{\mu,\theta}^*(\cdot \mid \#\mathcal{T} = n + 1)$. Then,

$$\frac{1}{n} \mathcal{R}(\mathcal{T}_{(n)}^*) \xrightarrow{n \rightarrow \infty, n \in \mathcal{G}} c_{\mu,\theta}$$

in probability.

Proof. We assume in the proof that $\mathcal{G} = \mathbb{Z}$. Only minor modifications are needed to deal with the general case.

We first consider the case when μ has finite variance σ_μ^2 . The arguments needed to derive Theorem 2.7 from Proposition 2.6 are then similar to the proof of Theorem 6.1 in [59]. The basic idea is as follows. For every $a \in (0, 1)$, the law under $\Pi_\mu(\cdot \mid \#\mathcal{T} = n + 1)$ of the subtree obtained by keeping only the first $\lfloor an \rfloor$ vertices of \mathcal{T} is absolutely continuous with respect to the law under $\Pi_\mu(\cdot \mid \#\mathcal{T} > n)$ of the same subtree, with a density that is bounded independently of n . A similar property holds for spatial trees, and so we can use the convergence of Proposition 2.6, for a tree distributed according to $\Pi_{\mu,\theta}^*(\cdot \mid \#\mathcal{T} > n)$, to get a similar convergence for a tree distributed according to $\Pi_{\mu,\theta}^*(\cdot \mid \#\mathcal{T} = n + 1)$. Let us give some details for the sake of completeness.

As previously, we write $u_0(\mathcal{T}), u_1(\mathcal{T}), \dots, u_{\#\mathcal{T}-1}(\mathcal{T})$ for the vertices of a plane tree \mathcal{T} listed in lexicographical order. The Lukasiewicz path of \mathcal{T} is then the finite sequence $(X_\ell(\mathcal{T}), 0 \leq \ell \leq \#\mathcal{T})$, which is defined inductively by

$$X_0(\mathcal{T}) = 0, \quad X_{\ell+1}(\mathcal{T}) - X_\ell(\mathcal{T}) = k_{u_\ell(\mathcal{T})}(\mathcal{T}) - 1, \quad \text{for every } 0 \leq \ell < \#\mathcal{T},$$

where we recall that, for every $u \in \mathcal{T}$, $k_u(\mathcal{T})$ is the number of children of u in \mathcal{T} . The tree \mathcal{T} is determined by its Lukasiewicz path. A key result (see e.g. [58, Section 1]) states that under $\Pi_\mu(d\mathcal{T})$, the Lukasiewicz path is distributed as a random walk on \mathbb{Z} with jump distribution ν determined by $\nu(j) = \mu(j + 1)$ for every $j \geq -1$, which starts from 0 and is stopped at the first time when it hits -1 (in particular, the law of $\#\mathcal{T}$ under $\Pi_\mu(d\mathcal{T})$ coincides with the law of the latter hitting time). For notational convenience, we let $(Y_k)_{k \geq 0}$ be a random walk on \mathbb{Z} with jump distribution ν , which starts from j under the probability measure $P_{(j)}$, and we set

$$T := \inf\{k \geq 0 : Y_k = -1\}.$$

Next take n large enough so that $\Pi_\mu(\#\mathcal{T} = n + 1) > 0$. Fix $a \in (0, 1)$, and consider a tree \mathcal{T} such that $\#\mathcal{T} > n$. Then, the collection of vertices $u_0(\mathcal{T}), \dots, u_{\lfloor an \rfloor}(\mathcal{T})$ forms a subtree of \mathcal{T} (because in the lexicographical order the parent of a vertex comes before this vertex), and we denote this subtree by $\rho_{\lfloor an \rfloor}(\mathcal{T})$. It is elementary to verify that $\rho_{\lfloor an \rfloor}(\mathcal{T})$ is determined by the sequence $(X_\ell(\mathcal{T}), 0 \leq \ell \leq \lfloor an \rfloor)$. Let f be a bounded function on $\mathbb{Z}^{\lfloor an \rfloor + 1}$. Using the Markov property at time $\lfloor an \rfloor$ for the random walk with jump distribution ν , one verifies that

$$\begin{aligned} & \Pi_\mu \left[f((X_k)_{0 \leq k \leq \lfloor an \rfloor}) \mid \#\mathcal{T} = n + 1 \right] \\ &= \frac{P_{(0)}(T > n)}{P_{(0)}(T = n + 1)} \Pi_\mu \left[f((X_k)_{0 \leq k \leq \lfloor an \rfloor}) \frac{\psi_n(X_{\lfloor an \rfloor})}{\psi'_n(X_{\lfloor an \rfloor})} \mid \#\mathcal{T} > n \right] \end{aligned} \quad (2.7)$$

where, for every integer $j \geq 0$,

$$\psi_n(j) = P_{(j)}(T = n + 1 - \lfloor an \rfloor), \quad \psi'_n(j) = P_{(j)}(T > n - \lfloor an \rfloor).$$

See [59, pp.742-743] for details of the derivation of (2.7). We now let n tend to infinity. Using Kemperman's formula (see e.g. Pitman [81, p.122]) and a standard local limit theorem, one easily checks that, for every $c > 0$,

$$\lim_{n \rightarrow \infty} \left(\sup_{j \geq c\sqrt{n}} \left| \frac{P_{(0)}(T > n)}{P_{(0)}(T = n + 1)} \frac{\psi_n(j)}{\psi'_n(j)} - \Gamma_a \left(\frac{j}{\sigma_\mu \sqrt{n}} \right) \right| \right) = 0, \quad (2.8)$$

where for every $x \geq 0$,

$$\Gamma_a(x) = \frac{2(2\pi(1-a)^3)^{-1/2} \exp(-x^2/2(1-a))}{\int_{1-a}^{\infty} ds (2\pi s^3)^{-1/2} \exp(-x^2/2s)}.$$

See again [59, pp.742-743] for details. Note that the function Γ_a is bounded over \mathbb{R}_+ . Furthermore, from the local limit theorem again, it is easy to verify that

$$\lim_{c \downarrow 0} \limsup_{n \rightarrow \infty} \Pi_\mu(X_{\lfloor an \rfloor} \leq c\sqrt{n} \mid \#\mathcal{T} = n + 1) = 0, \quad \lim_{c \downarrow 0} \limsup_{n \rightarrow \infty} \Pi_\mu(X_{\lfloor an \rfloor} \leq c\sqrt{n} \mid \#\mathcal{T} > n) = 0. \quad (2.9)$$

(We take this opportunity to point out that the analogous statement in [59, p.743] is written incorrectly.) By combining (2.7), (2.8) and (2.9), we obtain that, for any uniformly bounded sequence of functions $(f_n)_{n \geq 1}$ on $\mathbb{Z}^{\lfloor an \rfloor + 1}$, we have

$$\lim_{n \rightarrow \infty} \left| \Pi_\mu \left[f_n((X_k)_{0 \leq k \leq \lfloor an \rfloor}) \mid \#\mathcal{T} = n + 1 \right] - \Pi_\mu \left[f_n((X_k)_{0 \leq k \leq \lfloor an \rfloor}) \Gamma_a \left(\frac{X_{\lfloor an \rfloor}}{\sigma_\mu \sqrt{n}} \right) \mid \#\mathcal{T} > n \right] \right| = 0. \quad (2.10)$$

This convergence applies in particular to the case when, for every n , $f_n((X_k)_{0 \leq k \leq \lfloor an \rfloor})$ is a function of the tree $\rho_{\lfloor an \rfloor}(\mathcal{T})$. If we now replace Π_μ by $\Pi_{\mu, \theta}^*$, the same convergence still holds, and we can even allow the function of the tree $\rho_{\lfloor an \rfloor}(\mathcal{T})$ to depend also on the spatial locations of the vertices of $\rho_{\lfloor an \rfloor}(\mathcal{T})$ (the point is that the conditional distribution of these spatial locations given the tree \mathcal{T} only depends on the subtree $\rho_{\lfloor an \rfloor}(\mathcal{T})$). Consequently, if $\varepsilon > 0$ is fixed, we have

$$\lim_{n \rightarrow \infty} \left| \Pi_{\mu, \theta}^* \left[\mathbf{1}_{\{|R_{\lfloor an \rfloor} - c_{\mu, \theta} an| > \varepsilon n\}} \mid \#\mathcal{T} = n + 1 \right] - \Pi_{\mu, \theta}^* \left[\mathbf{1}_{\{|R_{\lfloor an \rfloor} - c_{\mu, \theta} an| > \varepsilon n\}} \Gamma_a \left(\frac{X_{\lfloor an \rfloor}}{\sigma_\mu \sqrt{n}} \right) \mid \#\mathcal{T} > n \right] \right| = 0.$$

Recalling that the function Γ_a is bounded, and using Proposition 2.6, we now obtain that

$$\lim_{n \rightarrow \infty} \Pi_{\mu, \theta}^* \left(|R_{\lfloor an \rfloor} - c_{\mu, \theta} an| > \varepsilon n \mid \#\mathcal{T} = n + 1 \right) = 0.$$

Since $0 \leq \mathcal{R}(\mathcal{T}^*) - R_{\lfloor an \rfloor}(\mathcal{T}^*) \leq n + 1 - \lfloor an \rfloor$, $\Pi_{\mu, \theta}^*(\cdot \mid \#\mathcal{T} = n + 1)$ -a.s., and a can be chosen arbitrarily close to 1, the convergence in Theorem 2.7 follows.

Very similar arguments can be used in the case when μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$. We now refer to the proof of Lemma 3.3 in [48] for the exact analogs of the properties (2.7) – (2.10) used in the finite variance case. We leave the details to the reader. \square

The case $d \geq 5$ of Theorem 2.1 follows from Theorem 2.7 and Proposition 2.5, noting that when μ is the critical geometric distribution, a tree distributed according to $\Pi_\mu(\cdot \mid \#\mathcal{T} = n)$ is uniformly distributed over the set of all plane trees with n vertices (see e.g. [58, Section 1.5]).

2.3 The critical dimension

In this section, we discuss the dimension $d = 4$, which is critical in the case of random walks that are centered and have sufficiently high moments. We restrict our attention to the case when the offspring distribution is geometric with parameter $1/2$. Our main tool is the discrete snake, which is a path-valued Markov chain that can be used to generate the spatial positions of the tree-indexed random walk.

2.3.1 Limit theorems

We now let θ be a symmetric probability distribution on \mathbb{Z}^4 . We assume that θ has small exponential moments and is not supported on a strict subgroup of \mathbb{Z}^4 . As previously, we write $S = (S_k)_{k \geq 0}$ for the random walk in \mathbb{Z}^4 with jump distribution θ , and we now assume that S starts from 0 under the probability measure P . We will also assume for simplicity that the covariance matrix M_θ of θ is of the form $\sigma^2 \text{Id}$, where Id is the four-dimensional identity matrix and $\sigma > 0$. This isotropy condition can be removed, and the reader will easily check that all subsequent arguments remain valid for a non-isotropic random walk: the role of σ^2 is then played by $(\det M_\theta)^{1/4}$.

We first introduce the free discrete snake associated with θ . This is a Markov chain with values in the space \mathcal{W} that we now define. The space \mathcal{W} is the set of all semi-infinite discrete paths $w = (w(k))_{k \in \llbracket -\infty, \zeta \rrbracket}$ with values in \mathbb{Z}^4 . Here $\zeta = \zeta(w) \in \mathbb{Z}$ is called the lifetime of w . We often write $\widehat{w} = w(\zeta(w))$ for the endpoint of w .

If $w \in \mathcal{W}$, we let \bar{w} stand for the new path obtained by “erasing” the endpoint of w , namely $\zeta(\bar{w}) = \zeta(w) - 1$ and $\bar{w}(k) = w(k)$ for every $k \in \llbracket -\infty, \zeta(w) - 1 \rrbracket$. If $x \in \mathbb{Z}^4$, we let $w \oplus x$ be the path obtained from w by “adding” the point x to w , namely $\zeta(w \oplus x) = \zeta(w) + 1$, $(w \oplus x)(k) = w(k)$ for every $k \in \llbracket -\infty, \zeta(w) \rrbracket$ and $(w \oplus x)(\zeta(w) + 1) = x$.

The free discrete snake is the Markov chain $(W_n)_{n \geq 0}$ in \mathcal{W} whose transition kernel is defined by

$$Q(w, dw') = \frac{1}{2} \delta_{\bar{w}}(dw') + \frac{1}{2} \sum_{x \in \mathbb{Z}^4} \theta(x) \delta_{w \oplus (\widehat{w} + x)}(dw').$$

We will write $\zeta_n = \zeta(W_n)$ to simplify notation. It will also be convenient to write W_n^* for the path W_n shifted so that its endpoint is 0: $W_n^*(k) = W_n(k) - \widehat{W}_n$ for every $k \in \llbracket -\infty, \zeta_n \rrbracket$.

If $w \in \mathcal{W}$, $\mathbb{P}_{(w)}$ will denote the probability measure under which the discrete snake W starts from w . For every integer $m \in \mathbb{Z}$, we also write \mathbb{P}_m for a probability measure under which $\zeta_0 = m$ a.s. and the initial value W_0 of the discrete snake is distributed as $(-S_{m-k})_{k \in \llbracket -\infty, m \rrbracket}$ (since S is symmetric we could omit the minus sign here). We write \mathbb{P} for \mathbb{P}_0 . As usual, the expectation under \mathbb{P}_m , resp. under \mathbb{P} , is denoted by \mathbb{E}_m , resp. by \mathbb{E} . Note that $(\zeta_n)_{n \geq 0}$ is a simple random walk on \mathbb{Z} started from m under \mathbb{P}_m . We will use the notation

$$\tau_p := \inf\{n \geq 0 : \zeta_n = \zeta_0 - p\}$$

for every integer $p \geq 0$.

Furthermore, from the form of the transition kernel of the discrete snake, it is easy to verify that for every $n \geq 0$, for every integer $\ell \in \mathbb{Z}$ such that $\mathbb{P}_m(\zeta_n = \ell) > 0$, the conditional distribution of W_n^* under $\mathbb{P}_m(\cdot \mid \zeta_n = \ell)$, coincides with the distribution of W_0 under \mathbb{P}_ℓ .

Proposition 2.8. *We have*

$$\lim_{n \rightarrow \infty} (\log n) \mathbb{P}(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, n \rrbracket) = 4\pi^2 \sigma^4.$$

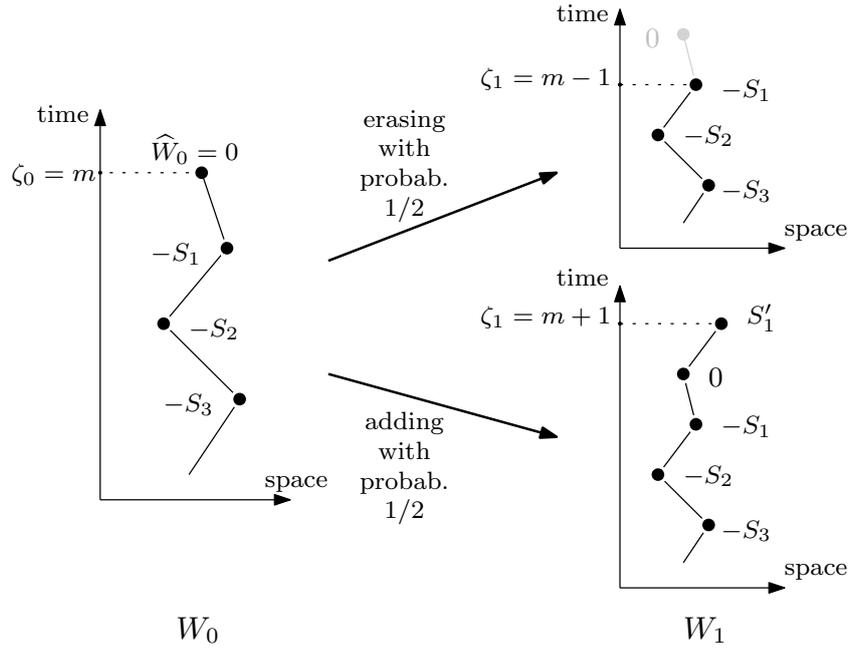


Figure 2.3: The discrete snake under \mathbb{P}_m . In this illustration, S'_1 is an independent copy of S_1 .

Furthermore,

$$\lim_{p \rightarrow \infty} (\log p) \mathbb{P}(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, \tau_p \rrbracket) = 2\pi^2 \sigma^4.$$

The proof of Proposition 2.8 is given in subsection 2.3.2 below. Our first theorem is concerned with the range of the free snake.

Theorem 2.9. Set $R_n := \#\{\widehat{W}_0, \widehat{W}_1, \dots, \widehat{W}_n\}$ for every integer $n \geq 0$. We have

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2(\mathbb{P})} 4\pi^2 \sigma^4.$$

Proof. We first observe that

$$\mathbb{E}[R_n] = \mathbb{E}\left[\sum_{i=0}^n \mathbf{1}_{\{\widehat{W}_j \neq \widehat{W}_i, \forall j \in \llbracket i+1, n \rrbracket\}}\right] = \sum_{i=0}^n \mathbb{P}(\widehat{W}_j \neq \widehat{W}_i, \forall j \in \llbracket i+1, n \rrbracket).$$

Then, by applying the Markov property of the free snake, we have

$$\begin{aligned} \mathbb{E}[R_n] &= \sum_{i=0}^n \mathbb{E}\left[\mathbb{P}_{(W_i)}(\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, n-i \rrbracket)\right] \\ &= \sum_{i=0}^n \mathbb{E}\left[\mathbb{P}_{(W_i^*)}(\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, n-i \rrbracket)\right] \\ &= \sum_{i=0}^n \mathbb{P}(\widehat{W}_j \neq 0, \forall j \in \llbracket 1, n-i \rrbracket), \end{aligned}$$

where the second equality is easy by translation invariance, and the last one is a simple consequence of the remark before the statement of Proposition 2.8. Using now the result of Proposition 2.8, we get

$$\lim_{n \rightarrow \infty} \frac{\log n}{n} \mathbb{E}[R_n] = 4\pi^2\sigma^4. \quad (2.11)$$

Let us turn to the second moment. We have similarly

$$\begin{aligned} \mathbb{E}[(R_n)^2] &= \mathbb{E}\left[\sum_{i=0}^n \sum_{j=0}^n \mathbf{1}_{\{\widehat{W}_k \neq \widehat{W}_i, \forall k \in \llbracket i+1, n \rrbracket; \widehat{W}_\ell \neq \widehat{W}_j, \forall \ell \in \llbracket j+1, n \rrbracket\}}\right] \\ &= 2 \sum_{0 \leq i < j \leq n} \mathbb{P}\left(\widehat{W}_k \neq \widehat{W}_i, \forall k \in \llbracket i+1, n \rrbracket; \widehat{W}_\ell \neq \widehat{W}_j, \forall \ell \in \llbracket j+1, n \rrbracket\right) + \mathbb{E}[R_n] \\ &= 2 \sum_{0 \leq i < j \leq n} \mathbb{E}\left[\mathbb{P}_{(W_i)}(\widehat{W}_k \neq \widehat{W}_0, \forall k \in \llbracket 1, n-i \rrbracket; \widehat{W}_\ell \neq \widehat{W}_{j-i}, \forall \ell \in \llbracket j-i+1, n-i \rrbracket)\right] \\ &\quad + \mathbb{E}[R_n] \\ &= 2 \sum_{0 \leq i < j \leq n} \mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, n-i \rrbracket; \widehat{W}_\ell \neq \widehat{W}_{j-i}, \forall \ell \in \llbracket j-i+1, n-i \rrbracket\right) + \mathbb{E}[R_n], \end{aligned}$$

where the last equality again follows from the observation preceding Proposition 2.8. Let us fix $\alpha \in (0, 1/4)$ and define

$$\sigma_n := \inf \{k \geq 0 : \zeta_k \leq -n^{\frac{1}{2}-\alpha}\}.$$

By standard estimates, we have

$$\lim_{n \rightarrow \infty} (\log n)^2 \mathbb{P}(\sigma_n \leq n^{1-3\alpha} \text{ or } \sigma_n \geq n^{1-\alpha}) = 0.$$

Thus, using also (2.11),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(\frac{\log n}{n}\right)^2 \mathbb{E}[(R_n)^2] &= \limsup_{n \rightarrow \infty} 2 \left(\frac{\log n}{n}\right)^2 \sum_{0 \leq i < j \leq n} \mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, n-i \rrbracket; \right. \\ &\quad \left. \widehat{W}_\ell \neq \widehat{W}_{j-i}, \forall \ell \in \llbracket j-i+1, n-i \rrbracket; n^{1-3\alpha} \leq \sigma_n \leq n^{1-\alpha}\right). \end{aligned}$$

Clearly, in order to study the limsup in the right-hand side, we may restrict the sum to indices i and j such that $j-i > n^{1-\alpha}$. However, if $0 \leq i < j \leq n$ are fixed such that $j-i > n^{1-\alpha}$,

$$\begin{aligned} &\mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, n-i \rrbracket; \widehat{W}_\ell \neq \widehat{W}_{j-i}, \forall \ell \in \llbracket j-i+1, n-i \rrbracket; n^{1-3\alpha} \leq \sigma_n \leq n^{1-\alpha}\right) \\ &\leq \mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, \sigma_n \rrbracket; \widehat{W}_\ell \neq \widehat{W}_{j-i}, \forall \ell \in \llbracket j-i+1, n-i \rrbracket; n^{1-3\alpha} \leq \sigma_n \leq n^{1-\alpha}\right) \\ &= \mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, \sigma_n \rrbracket; n^{1-3\alpha} \leq \sigma_n \leq n^{1-\alpha}\right) \mathbb{P}\left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n-j \rrbracket\right). \end{aligned}$$

To derive the last equality, we use the strong Markov property at time σ_n and then, after conditioning on $\sigma_n = m$, the Markov property at time $j-i-m$ for the free snake shifted at time σ_n and the observation preceding Proposition 2.8. Now obviously,

$$\mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, \sigma_n \rrbracket; n^{1-3\alpha} \leq \sigma_n \leq n^{1-\alpha}\right) \leq \mathbb{P}\left(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, \lfloor n^{1-3\alpha} \rfloor \rrbracket\right),$$

and it follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left(\frac{\log n}{n} \right)^2 \mathbb{E}[(R_n)^2] \\ & \leq \limsup_{n \rightarrow \infty} 2 \left(\frac{\log n}{n} \right)^2 \sum_{\substack{0 \leq i < j \leq n \\ j-i > n^{1-\alpha}}} \mathbb{P}(\widehat{W}_k \neq 0, \forall k \in \llbracket 1, \lfloor n^{1-3\alpha} \rfloor \rrbracket) \mathbb{P}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n-j \rrbracket) \\ & = \frac{1}{1-3\alpha} (4\pi^2 \sigma^4)^2 \end{aligned}$$

by Proposition 2.8. Since α can be chosen arbitrarily small, we get

$$\limsup_{n \rightarrow \infty} \left(\frac{\log n}{n} \right)^2 \mathbb{E}[(R_n)^2] \leq (4\pi^2 \sigma^4)^2. \quad (2.12)$$

Theorem 2.9 is an immediate consequence of (2.11) and (2.12). \square

We now aim to prove a result similar to Theorem 2.9 for the “excursion” of the discrete snake. We set

$$T := \inf\{k \geq 0 : \zeta_k = -1\}.$$

For every integer $n \geq 1$, we let $W^{(n)} = (W_k^{(n)})_{0 \leq k \leq 2n}$ be a process defined under \mathbb{P} , whose distribution coincides with the conditional distribution of $(W_k)_{0 \leq k \leq 2n}$ knowing that $T = 2n + 1$. To simplify notation, we write $\zeta_k^{(n)} = \zeta(W_k^{(n)})$. Note that $(\zeta_k^{(n)})_{0 \leq k \leq 2n}$ is the contour function, also called depth-first walk, of a Galton–Watson tree with geometric offspring distribution of parameter $1/2$, conditioned to have $n + 1$ vertices (see e.g. [81, Chapter 6]). We have already noticed that the latter tree is uniformly distributed over plane trees with $n + 1$ vertices. From the form of the transition mechanism of the discrete snake, it then follows that $\{\widehat{W}_k^{(n)}, 0 \leq k \leq 2n\}$ is distributed as the set of all spatial locations of a random walk with jump distribution θ indexed by a uniform random plane tree with $n + 1$ vertices.

We will need two simple estimates that we gather in the next lemma.

Lemma 2.10. (i) *Let $r \geq 1$ be an integer. There exists a constant $C(r)$ such that, for every integers $n \geq 1$ and $m \geq 0$,*

$$\mathbb{E} \left[(\#\{k \in \llbracket 0, 2n \rrbracket : \zeta_k^{(n)} = m\})^r \right] \leq C(r) (m + 1)^r.$$

(ii) *Let $\varepsilon > 0$. Then, for every $r > 0$,*

$$\mathbb{P} \left(\sup_{0 \leq k \leq 2n} \zeta_k^{(n)} > n^{\frac{1}{2} + \varepsilon} \right) = O(n^{-r})$$

as $n \rightarrow \infty$.

Part (i) of the lemma can be deduced from Theorem 1.13 in Janson [36] using the connection between $\zeta^{(n)}$ and the critical geometric Galton–Watson tree (it is also possible to give a direct argument), while Part (ii) is standard. Notice that Part (i) of Lemma 2.10 implies

$$\mathbb{E} \left[(\#\{k \in \llbracket 0, 2n \rrbracket : \zeta_k^{(n)} \leq n^{\frac{1}{2} - \frac{\alpha}{2}}\})^2 \right] = o\left(\left(\frac{n}{\log n}\right)^2\right)$$

as $n \rightarrow \infty$.

We will make a repeated use of Kemperman's formula for simple random walk (see [81, p.122] for a more general version): For every choice of the integers m, k such that $k > m \geq 0$,

$$\mathbb{P}_m(T = k) = \frac{m+1}{k} \mathbb{P}_m(\zeta_k = -1) = \frac{m+1}{k} \mathbb{P}_0(\zeta_k = m+1). \quad (2.13)$$

Together with this formula, we will use the local limit theorem for simple random walk on \mathbb{Z} , which we state in the form found in Lawler and Limic [53, Proposition 2.5.3, Corollary 2.5.4]: As $k \rightarrow \infty$,

$$\mathbb{P}_0(\zeta_k = m) = \sqrt{\frac{2}{\pi k}} \exp\left(-\frac{m^2}{2k}\right) \exp\left(O\left(\frac{1}{k} + \frac{m^4}{k^3}\right)\right) \quad (2.14)$$

uniformly over integers m such that $|m| \leq k$ and $k+m$ is even.

We fix $\alpha \in (0, 1/4)$ and to simplify notation, we write $p_n = \lfloor n^{\frac{1}{2}-\alpha} \rfloor$ for every integer $n \geq 1$. Recall the notation $\tau_p = \inf\{n \geq 0 : \zeta_n = \zeta_0 - p\}$.

Lemma 2.11. *If $\eta > 0$ is sufficiently small, we have*

$$\lim_{n \rightarrow \infty} \left(\sup_{\substack{n^{1-\eta} \leq k \leq 2n \\ n^{\frac{1}{2}-\frac{\alpha}{2}} \leq m \leq n^{\frac{1}{2}+\eta}}} \left| (\log n) \mathbb{P}_m(\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket \mid T = k) - \frac{4\pi^2 \sigma^4}{1-2\alpha} \right| \right) = 0,$$

where in the supremum we consider only integers m and k such that $k+m$ is odd.

Proof. We first explain how to choose η . We set $q_n = \lfloor n^{1-\frac{3\alpha}{2}} \rfloor$ and note that

$$\mathbb{P}_0(\zeta_{q_n} > n^{\frac{1}{2}-\frac{\alpha}{2}}) \leq \mathbb{P}_0(\zeta_{q_n} > q_n^{\frac{1}{2}+c(\alpha)})$$

where $c(\alpha) = \frac{\alpha}{4-6\alpha} > 0$. By a standard bound, the latter probability is bounded (for n large) by $\exp(-n^\gamma)$, where the constant $\gamma = \gamma(\alpha) > 0$ only depends on α . We fix $\eta > 0$ such that $3\eta < \gamma$ and $\eta \in (0, \alpha/8)$.

To simplify notation, we then set

$$\Delta_n := \{(m, k) : n^{\frac{1}{2}-\frac{\alpha}{2}} \leq m \leq n^{\frac{1}{2}+\eta}, n^{1-\eta} \leq k \leq 2n \text{ and } k+m \text{ is odd}\}.$$

Since $p_n \sim n^{-\alpha/4} \sqrt{q_n}$, standard estimates give, for every $\delta \in (0, \frac{\alpha}{4})$,

$$\lim_{n \rightarrow \infty} n^\delta \mathbb{P}_0(\tau_{p_n} \geq q_n) = 0. \quad (2.15)$$

We claim that we have also, for every $\delta \in (0, \frac{\alpha}{4})$,

$$\lim_{n \rightarrow \infty} n^\delta \sup_{(m,k) \in \Delta_n} \mathbb{P}_m(\tau_{p_n} \geq q_n \mid T = k) = 0. \quad (2.16)$$

Let us postpone the proof of (2.16) and derive the estimate of the lemma.

Let us consider $(m, k) \in \Delta_n$. We have

$$\begin{aligned} & \mathbb{P}_m\left(\{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\} \cap \{\tau_{p_n} \leq q_n\} \cap \{T = k\}\right) \\ &= \mathbb{E}_m\left[\mathbf{1}_{\{\tau_{p_n} \leq q_n\}} \mathbf{1}_{\{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbb{P}_{m-p_n}(T = k - \ell)_{\ell=\tau_{p_n}}\right], \end{aligned} \quad (2.17)$$

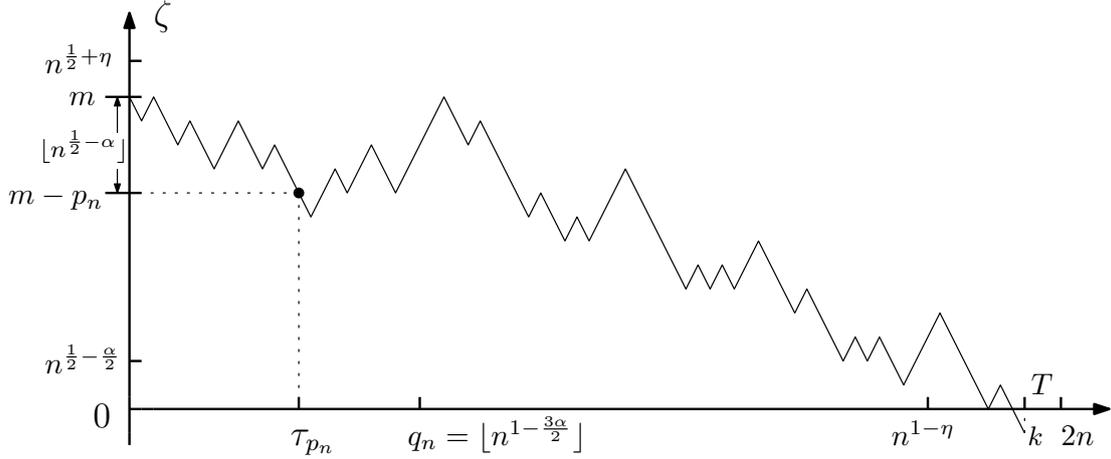


Figure 2.4: Illustration of the proof of Lemma 2.11

where we have used the strong Markov property at τ_{p_n} . We now would like to say that the quantity $\mathbb{P}_{m-p_n}(T = k - \ell)$, evaluated at $\ell = \tau_{p_n}$, does not differ too much from $\mathbb{P}_m(T = k)$ under our conditions on m, k and τ_{p_n} (see Fig. 4 for an illustration). Let k' be an integer such that $k - q_n \leq k' \leq k$ and $k' + m - p_n$ is odd. By Kemperman's formula,

$$\mathbb{P}_{m-p_n}(T = k') = \frac{m - p_n + 1}{k'} \mathbb{P}_0(\zeta_{k'} = m - p_n + 1) \quad (2.18)$$

and by (2.14),

$$\mathbb{P}_0(\zeta_{k'} = m - p_n + 1) = \sqrt{\frac{2}{\pi k'}} \exp\left(-\frac{(m - p_n + 1)^2}{2k'}\right) \exp\left(O\left(\frac{1}{k'} + \frac{(m - p_n + 1)^4}{k'^3}\right)\right). \quad (2.19)$$

Next observe that

$$\begin{aligned} \left| \frac{(m - p_n + 1)^2}{2k'} - \frac{(m + 1)^2}{2k} \right| &\leq \frac{(m + 1)^2}{2} \left(\frac{1}{k'} - \frac{1}{k} \right) + \frac{(m + 1)^2 - (m - p_n + 1)^2}{2k'} \\ &\leq \frac{q_n(m + 1)^2}{kk'} + \frac{p_n(m + 1)}{k'}, \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$, uniformly in m, k, k' . Comparing the estimate for $\mathbb{P}_{m-p_n}(T = k')$ that follows from (2.18) and (2.19) with the similar estimate for $\mathbb{P}_m(T = k)$ that follows from (2.13) and (2.14), we get

$$\lim_{n \rightarrow \infty} \left(\sup_{m, k, k'} \left| \frac{\mathbb{P}_{m-p_n}(T = k')}{\mathbb{P}_m(T = k)} - 1 \right| \right) = 0,$$

where the supremum is over all choices of (m, k, k') such that $(m, k) \in \Delta_n$ and k' satisfies the preceding conditions. Using (2.17), we obtain that, for any fixed $\delta > 0$, we have for all sufficiently large n , for every $(m, k) \in \Delta_n$,

$$\begin{aligned} (1 - \delta) \mathbb{P}_m(\{\tau_{p_n} \leq q_n\} \cap \{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\}) \\ \leq \mathbb{P}_m(\{\tau_{p_n} \leq q_n\} \cap \{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\} \mid T = k) \\ \leq (1 + \delta) \mathbb{P}_m(\{\tau_{p_n} \leq q_n\} \cap \{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\}). \end{aligned}$$

The quantity $\mathbb{P}_m(\{\tau_{p_n} \leq q_n\} \cap \{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\})$ does not depend on $m \in \mathbb{Z}$, and $(\log n) \mathbb{P}_0(\tau_{p_n} > q_n)$ tends to 0 by (2.15). Using Proposition 2.8, we have thus

$$\lim_{n \rightarrow \infty} (\log n) \mathbb{P}_0(\{\tau_{p_n} \leq q_n\} \cap \{\widehat{W}_j \neq \widehat{W}_0, \forall j \in \llbracket 1, \tau_{p_n} \rrbracket\}) = \frac{4\pi^2 \sigma^4}{1 - 2\alpha}.$$

The estimate of the lemma follows from the preceding considerations and (2.16).

It remains to prove (2.16). If $(m, k) \in \Delta_n$, we have

$$\mathbb{P}_m(\tau_{p_n} \geq q_n \mid T = k) = \frac{\mathbb{P}_m(\{\tau_{p_n} \geq q_n\} \cap \{T = k\})}{\mathbb{P}_m(T = k)}.$$

Recall formula (2.13) for $\mathbb{P}_m(T = k)$ and also note that by (2.14),

$$\mathbb{P}_0(\zeta_k = m + 1) = \sqrt{\frac{2}{\pi k}} \exp\left(-\frac{(m+1)^2}{2k}\right) \exp\left(O\left(\frac{1}{k} + \frac{m^4}{k^3}\right)\right), \quad (2.20)$$

when $n \rightarrow \infty$, uniformly in $(m, k) \in \Delta_n$. Notice that $\frac{1}{k} + \frac{m^4}{k^3} \rightarrow 0$ as $n \rightarrow \infty$, uniformly in $(m, k) \in \Delta_n$, and that $\frac{m^2}{2k} \leq n^{3\eta}$ if $(m, k) \in \Delta_n$. By our choice of η , it follows that

$$\mathbb{P}_m(\zeta_{q_n} > m + n^{\frac{1}{2} - \frac{\alpha}{2}} \mid T = k) \leq \frac{\mathbb{P}_m(\zeta_{q_n} > m + n^{\frac{1}{2} - \frac{\alpha}{2}})}{\mathbb{P}_m(T = k)} = \frac{k}{m+1} \frac{\mathbb{P}_0(\zeta_{q_n} > n^{\frac{1}{2} - \frac{\alpha}{2}})}{\mathbb{P}_0(\zeta_k = m+1)} = O\left(\frac{1}{n}\right) \quad (2.21)$$

as $n \rightarrow \infty$, uniformly in m and k .

On the other hand, by applying the Markov property at time q_n , we have

$$\mathbb{P}_m(\{\tau_{p_n} \geq q_n\} \cap \{\zeta_{q_n} \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}\} \cap \{T = k\}) = \mathbb{E}_m \left[\mathbf{1}_{\{\tau_{p_n} \geq q_n\} \cap \{\zeta_{q_n} \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}\}} \mathbb{P}_{\zeta_{q_n}}(T = k - q_n) \right].$$

On the event $\{\tau_{p_n} \geq q_n\} \cap \{\zeta_{q_n} \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}\}$ we have $m - p_n \leq \zeta_{q_n} \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}$, \mathbb{P}_m a.s. If $m - p_n \leq m' \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}$ and $m' + k - q_n$ is odd, using again Kemperman's formula, we have

$$\mathbb{P}_{m'}(T = k - q_n) = \frac{m' + 1}{k - q_n} \mathbb{P}_{m'}(\zeta_{k - q_n} = -1) = \frac{m' + 1}{k - q_n} \mathbb{P}_0(\zeta_{k - q_n} = m' + 1).$$

Furthermore, from (2.14),

$$\mathbb{P}_0(\zeta_{k - q_n} = m' + 1) = \sqrt{\frac{2}{\pi(k - q_n)}} \exp\left(-\frac{(m' + 1)^2}{2(k - q_n)}\right) \exp\left(O\left(\frac{1}{k} + \frac{m'^4}{k^3}\right)\right). \quad (2.22)$$

Now observe that

$$-\frac{(m' + 1)^2}{2(k - q_n)} + \frac{(m + 1)^2}{2k} \leq -\frac{(m' + 1)^2 - (m + 1)^2}{2k} = -\frac{(m' - m)(m' + m + 2)}{2k}$$

and the right-hand side tends to 0 as $n \rightarrow \infty$, uniformly in $(m, k) \in \Delta_n$ and m' such that $m - p_n \leq m' \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}$. By comparing (2.20) and (2.22), noting that $m' \leq 2m$ under our assumptions, we get

$$\limsup_{n \rightarrow \infty} \left(\sup_{m, k, m'} \frac{\mathbb{P}_{m'}(T = k - q_n)}{\mathbb{P}_m(T = k)} \right) \leq 2.$$

It follows that, for all n sufficiently large, we have, for every $(m, k) \in \Delta_n$,

$$\frac{\mathbb{P}_m(\{\tau_{p_n} \geq q_n\} \cap \{\zeta_{q_n} \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}\} \cap \{T = k\})}{\mathbb{P}_m(T = k)} \leq 3 \mathbb{P}_m(\tau_{p_n} \geq q_n) = 3 \mathbb{P}_0(\tau_{p_n} \geq q_n).$$

Recalling (2.15), we have thus proved that, for every $\delta \in (0, \frac{\alpha}{4})$,

$$\lim_{n \rightarrow \infty} n^\delta \sup_{(m, k) \in \Delta_n} \mathbb{P}_m(\{\tau_{p_n} \geq q_n\} \cap \{\zeta_{q_n} \leq m + n^{\frac{1}{2} - \frac{\alpha}{2}}\} \mid T = k) = 0,$$

and by combining this with (2.21), we get the desired estimate (2.16). \square

We set, for every $n \geq 1$,

$$R_n^\bullet := \#\{\widehat{W}_0^{(n)}, \widehat{W}_1^{(n)}, \dots, \widehat{W}_{2n}^{(n)}\}.$$

Proposition 2.12. *We have*

$$\limsup_{n \rightarrow \infty} \left(\frac{\log n}{n}\right)^2 \mathbb{E}[(R_n^\bullet)^2] \leq (8\pi^2 \sigma^4)^2.$$

Proof. We note that

$$R_n^\bullet = \sum_{i=1}^{2n} \mathbf{1}_{\{\widehat{W}_\ell^{(n)} \neq \widehat{W}_i^{(n)}, \forall \ell \in \llbracket i+1, 2n \rrbracket\}}$$

and therefore

$$\mathbb{E}[(R_n^\bullet)^2] = \sum_{i, j=1}^{2n} \mathbb{P}(A_n(i, j)), \quad (2.23)$$

where $A_n(i, j)$ is defined by

$$A_n(i, j) := \{\widehat{W}_\ell^{(n)} \neq \widehat{W}_i^{(n)}, \forall \ell \in \llbracket i+1, 2n \rrbracket\} \cap \{\widehat{W}_\ell^{(n)} \neq \widehat{W}_j^{(n)}, \forall \ell \in \llbracket j+1, 2n \rrbracket\}.$$

We fix $\alpha \in (0, 1/4)$, and define p_n and q_n for every $n \geq 1$ as above. We also fix $\eta > 0$ so that the conclusion of Lemma 2.11 holds.

In view of proving the proposition, we will use formula (2.23). In this formula, we can restrict our attention to values of i and j such that $j - i > n^{1 - \frac{\alpha}{2}}$ and $j < 2n - n^{1 - \eta}$ (or the same with i and j interchanged). Also, when bounding $\mathbb{P}(A_n(i, j))$, we may impose the additional constraint that $n^{\frac{1}{2} - \frac{\alpha}{2}} \leq \zeta_i^{(n)} \leq n^{\frac{1}{2} + \eta}$ and $n^{\frac{1}{2} - \frac{\alpha}{2}} \leq \zeta_j^{(n)} \leq n^{\frac{1}{2} + \eta}$: Indeed, Lemma 2.10 readily shows that the event where either of these constraints is not satisfied will give a negligible contribution to the sum in (2.23).

Let us fix $i, j \in \llbracket 1, 2n \rrbracket$ such that $j - i > n^{1 - \frac{\alpha}{2}}$ and $j < 2n - n^{1 - \eta}$. By using the definition of $W^{(n)}$ as a conditioned process and applying the Markov property at time i , we have

$$\begin{aligned} & \mathbb{P}(A_n(i, j) \cap \{n^{\frac{1}{2} - \frac{\alpha}{2}} \leq \zeta_i^{(n)} \leq n^{\frac{1}{2} + \eta}\} \cap \{n^{\frac{1}{2} - \frac{\alpha}{2}} \leq \zeta_j^{(n)} \leq n^{\frac{1}{2} + \eta}\}) \\ &= \frac{\mathbb{E}\left[\mathbf{1}_{\{n^{\frac{1}{2} - \frac{\alpha}{2}} \leq \zeta_i \leq n^{\frac{1}{2} + \eta}\}} \mathbf{1}_{\{T > i\}} \mathbb{E}_{\zeta_i} \left[\mathbf{1}_{\{n^{\frac{1}{2} - \frac{\alpha}{2}} \leq \zeta_{j-i} \leq n^{\frac{1}{2} + \eta}\}} \mathbf{1}_{A'_n(i, j)} \mathbf{1}_{\{T = 2n + 1 - i\}} \right]\right]}{\mathbb{P}(T = 2n + 1)}, \end{aligned} \quad (2.24)$$

where

$$A'_n(i, j) := \{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, 2n - i \rrbracket\} \cap \{\widehat{W}_\ell \neq \widehat{W}_{j-i}, \forall \ell \in \llbracket j - i + 1, 2n - i \rrbracket\}.$$

Setting $r = j - i$, we are thus led to bound

$$\mathbb{E}_m \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, k-1 \rrbracket\}} \mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_r, \forall \ell \in \llbracket r+1, k-1 \rrbracket\}} \mathbf{1}_{\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}} \mathbf{1}_{\{T=k\}} \right], \quad (2.25)$$

where $n^{\frac{1}{2}-\frac{\alpha}{2}} \leq m \leq n^{\frac{1}{2}+\eta}$, $r > n^{1-\frac{\alpha}{2}}$ and $r + n^{1-\eta} < k \leq 2n$ (and moreover $k + m$ needs to be odd). Recall the notation τ_{p_n} , and set

$$\tau_{p_n}^{(r)} := \inf\{\ell \geq r : \zeta_\ell = \zeta_r - p_n\}.$$

Thanks to (2.16), we can also introduce the constraint $\tau_{p_n} \leq q_n$ inside the expectation in (2.25), up to an error that is bounded above by $\mathbb{P}_m(T = k) o(n^{-\delta})$ for some $\delta > 0$ (here the term $o(n^{-\delta})$ is uniform in m, r, k satisfying the preceding conditions). Furthermore, we get an upper bound by replacing the interval $\llbracket 1, k-1 \rrbracket$, resp. $\llbracket r+1, k-1 \rrbracket$, by $\llbracket 1, \tau_{p_n} \rrbracket$, resp. $\llbracket r+1, \tau_{p_n}^{(r)} \rrbracket$. Next, using the Markov property at time r , and noting that $r > q_n$, we have

$$\begin{aligned} & \mathbb{E}_m \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{\tau_{p_n} \leq q_n\}} \mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_r, \forall \ell \in \llbracket r+1, \tau_{p_n}^{(r)} \rrbracket\}} \mathbf{1}_{\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}} \mathbf{1}_{\{T=k\}} \right] \\ &= \mathbb{E}_m \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{\tau_{p_n} \leq q_n\}} \mathbf{1}_{\{T > r\}} \mathbf{1}_{\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}} \right. \\ & \quad \left. \times \mathbb{E}_{(W_r)} \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{T=k-r\}} \right] \right]. \end{aligned}$$

See Fig. 5 for an illustration.

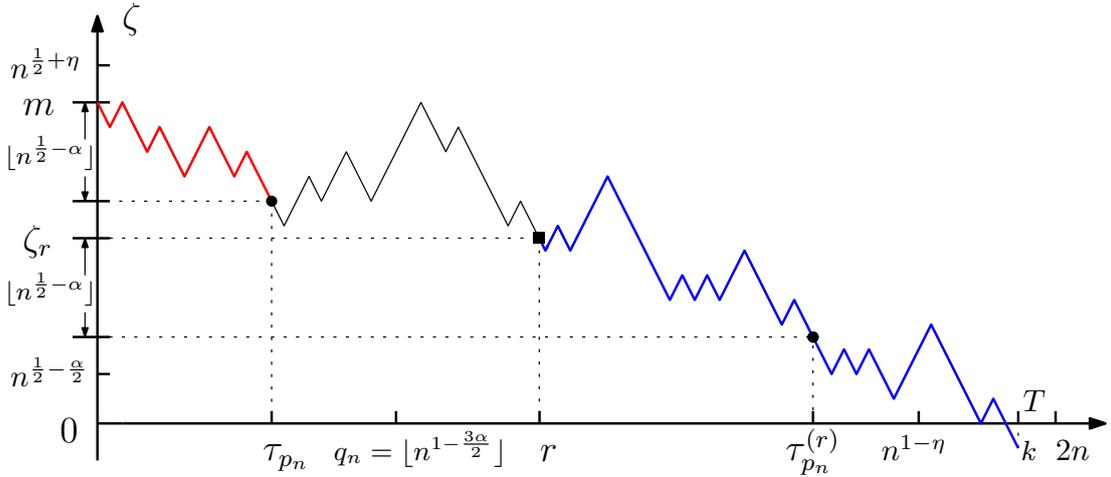


Figure 2.5: Illustration of the proof of Proposition 2.12

Then the key observation is the following. Let $z_0 = m, z_1, \dots, z_r$ be a simple random walk trajectory over $\llbracket 0, r \rrbracket$ such that $0 \leq \min\{z_\ell : 0 \leq \ell \leq r\} \leq m - p_n$. Then under \mathbb{P}_m , conditionally on the event $\{\zeta_1 = z_1, \dots, \zeta_r = z_r\}$, the path $(W_r(z_r) - W_r(z_r - \ell))_{\ell \geq 0}$ is *independent* of the event $\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}$, and distributed as $(S_\ell)_{\ell \geq 0}$. This property easily follows from the construction of the discrete snake.

Thanks to the latter observation, we may rewrite the right-hand side of the last display, after conditioning with respect to ζ_1, \dots, ζ_r , in the form

$$\mathbb{E}_m \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{\tau_{p_n} \leq q_n\}} \mathbf{1}_{\{T > r\}} \mathbf{1}_{\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}} \mathbb{E}_{\zeta_r} \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{T=k-r\}} \right] \right]. \quad (2.26)$$

Recall that $k - r > n^{1-\eta}$, and let $\varepsilon > 0$. It follows from Lemma 2.11 that, for n large enough, on the event $\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}$, the quantity

$$\mathbb{E}_{\zeta_r} \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{T=k-r\}} \right]$$

is bounded above by

$$\left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right) (\log n)^{-1} \mathbb{P}_{\zeta_r}(T = k - r).$$

Hence the quantity (2.26) is also bounded by

$$\begin{aligned} & \left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right) (\log n)^{-1} \mathbb{E}_m \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{\tau_{p_n} \leq q_n\}} \mathbf{1}_{\{T > r\}} \mathbf{1}_{\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}} \mathbb{P}_{\zeta_r}(T = k - r) \right] \\ &= \left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right) (\log n)^{-1} \mathbb{E}_m \left[\mathbf{1}_{\{\widehat{W}_\ell \neq \widehat{W}_0, \forall \ell \in \llbracket 1, \tau_{p_n} \rrbracket\}} \mathbf{1}_{\{\tau_{p_n} \leq q_n\}} \mathbf{1}_{\{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_r \leq n^{\frac{1}{2}+\eta}\}} \mathbf{1}_{\{T=k\}} \right], \end{aligned}$$

again by the Markov property at time r . Finally, another application of Lemma 2.11 shows that the quantity in the last display is bounded above for n large by

$$\left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right)^2 (\log n)^{-2} \mathbb{P}_m(T = k).$$

Summarizing, we see that the quantity (2.25) is bounded above for n large by

$$\left(\left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right)^2 (\log n)^{-2} + o(n^{-\delta}) \right) \mathbb{P}_m(T = k).$$

Finally, from (2.24), we have for n large

$$\begin{aligned} & \mathbb{P}(A_n(i, j) \cap \{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_i^{(n)} \leq n^{\frac{1}{2}+\eta}\} \cap \{n^{\frac{1}{2}-\frac{\alpha}{2}} \leq \zeta_j^{(n)} \leq n^{\frac{1}{2}+\eta}\}) \\ & \leq \left(\left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right)^2 (\log n)^{-2} + o(n^{-\delta}) \right) \frac{\mathbb{E} \left[\mathbf{1}_{\{T > i\}} \mathbb{P}_{\zeta_i}(T = 2n + 1 - i) \right]}{\mathbb{P}(T = 2n + 1)} \\ & = \left(\frac{4\pi^2\sigma^4}{1-2\alpha} + \varepsilon \right)^2 (\log n)^{-2} + o(n^{-\delta}), \end{aligned}$$

where the term $o(n^{-\delta})$ is uniform in i and j satisfying the preceding conditions. The statement of the proposition follows by summing this bound over i and j . \square

Lemma 2.13. *We have*

$$\liminf_{n \rightarrow \infty} \frac{\log n}{n} \mathbb{E}[R_n^\bullet] \geq 8\pi^2\sigma^4.$$

Proof. Let $\delta > 0$ and $\varepsilon \in (0, \frac{1}{2})$. To simplify notation we write $n(\varepsilon) = \lfloor 2(1 - 2\varepsilon)n \rfloor$ in this proof. We fix $0 < a < b$ such that, if $(\mathbf{e}_t)_{0 \leq t \leq 1}$ denotes a normalized Brownian excursion defined under the probability measure P , we have

$$P(\mathbf{e}_\varepsilon \notin (a, b)) = P(\mathbf{e}_{1-\varepsilon} \notin (a, b)) < \delta.$$

Since we know that the sequence of processes $((2n)^{-1/2} \zeta_{\lfloor 2nt \rfloor}^{(n)})_{0 \leq t \leq 1}$ converges in distribution to $(\mathbf{e}_t)_{0 \leq t \leq 1}$, it follows that, for every sufficiently large n ,

$$\mathbb{P} \left(\zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)} \notin [a\sqrt{2n}, b\sqrt{2n}] \text{ or } \zeta_{\lfloor 2n\varepsilon \rfloor + n(\varepsilon)}^{(n)} \notin [a\sqrt{2n}, b\sqrt{2n}] \right) \leq \delta. \quad (2.27)$$

Let $\mu_\varepsilon^{(n)}$ denote the law of $\zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)}$. If F_n is a nonnegative function on $\mathbb{Z}^{n(\varepsilon)+1}$, the Markov property gives

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)} \leq b\sqrt{2n}\}} \mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{\lfloor 2n\varepsilon \rfloor + n(\varepsilon)}^{(n)} \leq b\sqrt{2n}\}} F_n((\zeta_{\lfloor 2n\varepsilon \rfloor + k}^{(n)})_{0 \leq k \leq n(\varepsilon)}) \right] \\ &= \mathbb{E} \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)} \leq b\sqrt{2n}\}} \mathbb{E}_{\zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)}} \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{n(\varepsilon)} \leq b\sqrt{2n}\}} F_n((\zeta_k)_{0 \leq k \leq n(\varepsilon)}) \mid T = 2n + 1 - \lfloor 2n\varepsilon \rfloor \right] \right] \\ &= \sum_{a\sqrt{2n} \leq m \leq b\sqrt{2n}} \mu_\varepsilon^{(n)}(m) \frac{\mathbb{E}_m \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{n(\varepsilon)} \leq b\sqrt{2n}\}} F_n((\zeta_k)_{0 \leq k \leq n(\varepsilon)}) \mathbf{1}_{\{T = 2n + 1 - \lfloor 2n\varepsilon \rfloor\}} \right]}{\mathbb{P}_m(T = 2n + 1 - \lfloor 2n\varepsilon \rfloor)} \\ &= \sum_{a\sqrt{2n} \leq m \leq b\sqrt{2n}} \mu_\varepsilon^{(n)}(m) \frac{\mathbb{E}_m \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{n(\varepsilon)} \leq b\sqrt{2n}\}} F_n((\zeta_k)_{0 \leq k \leq n(\varepsilon)}) \mathbf{1}_{\{T > n(\varepsilon)\}} \mathbb{P}_{\zeta_{n(\varepsilon)}}(T = \tilde{n}(\varepsilon)) \right]}{\mathbb{P}_m(T = 2n + 1 - \lfloor 2n\varepsilon \rfloor)} \end{aligned}$$

where $\tilde{n}(\varepsilon) := 2n + 1 - \lfloor 2n\varepsilon \rfloor - n(\varepsilon)$.

Let $m, m' \in [a\sqrt{2n}, b\sqrt{2n}]$ be such that $m + \lfloor 2n\varepsilon \rfloor$ and $m' + \lfloor 2n\varepsilon \rfloor + n(\varepsilon)$ are even. By Kemperman's formula (2.13),

$$\frac{\mathbb{P}_{m'}(T = \tilde{n}(\varepsilon))}{\mathbb{P}_m(T = 2n + 1 - \lfloor 2n\varepsilon \rfloor)} = \frac{2n + 1 - \lfloor 2n\varepsilon \rfloor}{\tilde{n}(\varepsilon)} \frac{m' + 1}{m + 1} \frac{\mathbb{P}_0(\zeta_{\tilde{n}(\varepsilon)} = m' + 1)}{\mathbb{P}_0(\zeta_{2n+1-\lfloor 2n\varepsilon \rfloor} = m + 1)}$$

and using (2.14), we easily obtain that there exists a finite constant $C(\varepsilon, a, b)$ such that, for every sufficiently large n , and every m, m' satisfying the above conditions,

$$\frac{\mathbb{P}_{m'}(T = \tilde{n}(\varepsilon))}{\mathbb{P}_m(T = 2n + 1 - \lfloor 2n\varepsilon \rfloor)} \leq C(\varepsilon, a, b).$$

We thus obtain that, for every large enough n ,

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)} \leq b\sqrt{2n}\}} \mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{\lfloor 2n\varepsilon \rfloor + n(\varepsilon)}^{(n)} \leq b\sqrt{2n}\}} F_n((\zeta_{\lfloor 2n\varepsilon \rfloor + k}^{(n)})_{0 \leq k \leq n(\varepsilon)}) \right] \\ & \leq C(\varepsilon, a, b) \sum_{a\sqrt{2n} \leq m \leq b\sqrt{2n}} \mu_\varepsilon^{(n)}(m) \mathbb{E}_m \left[\mathbf{1}_{\{a\sqrt{2n} \leq \zeta_{n(\varepsilon)} \leq b\sqrt{2n}\}} F_n((\zeta_k)_{0 \leq k \leq n(\varepsilon)}) \mathbf{1}_{\{T > n(\varepsilon)\}} \right] \\ & \leq C(\varepsilon, a, b) \sum_{a\sqrt{2n} \leq m \leq b\sqrt{2n}} \mu_\varepsilon^{(n)}(m) \mathbb{E}_m \left[F_n((\zeta_k)_{0 \leq k \leq n(\varepsilon)}) \right]. \end{aligned}$$

Let G_n be a nonnegative measurable function on $\mathcal{W}^{n(\varepsilon)+1}$. The preceding bound remains valid if we replace $F_n((\zeta_{\lfloor 2n\varepsilon \rfloor + k}^{(n)})_{0 \leq k \leq n(\varepsilon)})$ by $G_n((W_{\lfloor 2n\varepsilon \rfloor + k}^{(n)})_{0 \leq k \leq n(\varepsilon)})$ in the left-hand side and $F_n((\zeta_k)_{0 \leq k \leq n(\varepsilon)})$ by $G_n((W_k)_{0 \leq k \leq n(\varepsilon)})$ in the right-hand side (just use the fact that the conditional distribution of $W^{(n)}$ given $\zeta^{(n)}$ is the same as the conditional distribution of W given ζ). In particular, if we let $G_n(w_0, w_1, \dots, w_{n(\varepsilon)})$ be the indicator function of the set where

$$\left| \frac{\log n(\varepsilon)}{n(\varepsilon)} \#\{\hat{w}_0, \hat{w}_1, \dots, \hat{w}_{n(\varepsilon)}\} - 4\pi^2\sigma^4 \right| > \delta,$$

we obtain that

$$\begin{aligned} & \mathbb{P} \left(\zeta_{\lfloor 2n\varepsilon \rfloor}^{(n)} \in [a\sqrt{2n}, b\sqrt{2n}], \zeta_{\lfloor 2n\varepsilon \rfloor + n(\varepsilon)}^{(n)} \in [a\sqrt{2n}, b\sqrt{2n}], \left| \frac{\log n(\varepsilon)}{n(\varepsilon)} R_n^{\bullet, \varepsilon} - 4\pi^2\sigma^4 \right| > \delta \right) \\ & \leq C(\varepsilon, a, b) \mathbb{P} \left(\left| \frac{\log n(\varepsilon)}{n(\varepsilon)} R_{n(\varepsilon)} - 4\pi^2\sigma^4 \right| > \delta \right), \end{aligned} \tag{2.28}$$

where

$$R_n^{\bullet,\varepsilon} := \#\{\widehat{W}_{[2n\varepsilon]}^{(n)}, \widehat{W}_{[2n\varepsilon]+1}^{(n)}, \dots, \widehat{W}_{[2n\varepsilon]+n(\varepsilon)}^{(n)}\}.$$

Here we used the (obvious) fact that the distribution of R_n under \mathbb{P}_m does not depend on m .

By Theorem 2.9, the right-hand side of (2.28) tends to 0 as $n \rightarrow \infty$. Using also (2.27), we obtain that

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\left|\frac{\log n(\varepsilon)}{n(\varepsilon)} R_n^{\bullet,\varepsilon} - 4\pi^2\sigma^4\right| > \delta\right) \leq \delta.$$

Since $R_n^\bullet \geq R_n^{\bullet,\varepsilon}$ and since both δ and ε can be chosen arbitrarily small, the statement of the lemma follows. \square

Theorem 2.14. *We have*

$$\frac{\log n}{n} R_n^\bullet \xrightarrow[n \rightarrow \infty]{L^2(\mathbb{P})} 8\pi^2\sigma^4.$$

Proof. By combining Proposition 2.12 and Lemma 2.13, we get that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{E}\left[\left(\frac{\log n}{n} R_n^\bullet - 8\pi^2\sigma^4\right)^2\right] \\ & \leq \left(\limsup_{n \rightarrow \infty} \mathbb{E}\left[\left(\frac{\log n}{n} R_n^\bullet\right)^2\right]\right) - 16\pi^2\sigma^4 \left(\liminf_{n \rightarrow \infty} \mathbb{E}\left[\frac{\log n}{n} R_n^\bullet\right]\right) + (8\pi^2\sigma^4)^2 \leq 0, \end{aligned}$$

which gives the desired result. \square

Theorem 2.14 and the remarks before Lemma 2.10 give the case $d = 4$ of Theorem 2.1.

2.3.2 Proof of the main estimate

In this subsection, we prove Proposition 2.8, which was a key ingredient of the results of the previous subsection. We first recall some basic facts. For every $x \in \mathbb{Z}^4$ and $k \geq 0$, we set

$$p_k(x) = P(S_k = x)$$

and we now denote the Green function of the random walk S by

$$G(x) = \sum_{k=0}^{\infty} p_k(x)$$

($G = G_\theta$ in the notation of Section 2). A standard estimate (see e.g. [53, Chapter 4]) states that

$$\lim_{x \rightarrow \infty} |x|^2 G(x) = \frac{1}{2\pi^2\sigma^2}. \quad (2.29)$$

Let \mathbf{p} be the period of the random walk S . Since S is assumed to be symmetric, we have $\mathbf{p} = 1$ or 2 . Then from the local limit theorem (see e.g. [53, Chapter 2]), we have

$$\lim_{j \rightarrow \infty, j \in \mathbf{p}\mathbb{Z}} j^2 p_j(0) = \frac{\mathbf{P}}{4\pi^2\sigma^4}. \quad (2.30)$$

We state our first lemma.

Lemma 2.15. *We have $\mathbb{P}(\widehat{W}_k = 0) = 0$ if $k \notin \mathbf{p}\mathbb{Z}$, and*

$$\lim_{k \rightarrow \infty, k \in \mathbf{p}\mathbb{Z}} k \mathbb{P}(\widehat{W}_k = 0) = \frac{\mathbf{P}}{4\pi^2\sigma^4}.$$

Proof. The first assertion is clear since ζ_k is odd when k is odd, \mathbb{P} a.s. Then, for every integer $k \geq 0$, set

$$\underline{\zeta}_k = \min_{0 \leq j \leq k} \zeta_j,$$

and

$$X_k = \zeta_k - 2\underline{\zeta}_k.$$

From the construction of the discrete snake, and the fact that S is symmetric, the conditional distribution of \widehat{W}_k knowing that $X_k = m$ is the law of S_m . Consequently,

$$\mathbb{P}(\widehat{W}_k = 0) = \sum_{m=0}^{\infty} \mathbb{P}(X_k = m) p_m(0). \quad (2.31)$$

Asymptotics for $P(S_m = 0) = p_m(0)$ are given by (2.30). We then need to evaluate $\mathbb{P}(X_k = m)$. Set $\widetilde{X}_k = 1 + X_k$ for every $k \geq 0$. The discrete version of Pitman's theorem (see [80, Lemma 3.1]) shows that, under the probability measure \mathbb{P} , $(\widetilde{X}_k)_{k \geq 0}$ is a Markov chain on $\{1, 2, \dots\}$ with transition kernel Q given by $Q(1, 2) = 1$ and for every $j \geq 2$,

$$Q(j, j+1) = \frac{1}{2} \frac{j+1}{j}, \quad Q(j, j-1) = \frac{1}{2} \frac{j-1}{j}.$$

This Markov chain is also the discrete h -transform of simple random walk on \mathbb{Z}_+ (killed upon hitting 0) corresponding to $h(j) = j$. Let $(Y_k)_{k \geq 0}$ stand for a simple random walk on \mathbb{Z} that starts from ℓ under the probability measure P_ℓ , and let $H_0 = \inf\{n \geq 0 : Y_n = 0\}$. It follows from the preceding observations that, for every integer $k \geq 1$ and every $m \geq 1$ such that $1 \leq m \leq k+1$ and $k+m$ is odd,

$$\begin{aligned} \mathbb{P}(\widetilde{X}_k = m) &= m P_1(Y_k = m, H_0 > k) \\ &= m \left(P_0(Y_k = m-1) - P_0(Y_k = m+1) \right) \\ &= m \times 2^{-k} \left(\binom{k}{\frac{k+m-1}{2}} - \binom{k}{\frac{k+m+1}{2}} \right) \\ &= \frac{2m^2}{k+m+1} P_0(Y_k = m-1) \end{aligned}$$

Hence, for every $m \geq 0$,

$$\mathbb{P}(X_k = m) = \frac{2(m+1)^2}{k+m+2} P_0(Y_k = m). \quad (2.32)$$

From (2.31) and (2.32), we get

$$\mathbb{P}(\widehat{W}_k = 0) = \sum_{m=0}^k \frac{2}{k+m+2} \left((m+1)^2 p_m(0) \right) P_0(Y_k = m),$$

and the second assertion of the lemma follows using (2.30). \square

In the next lemma, for every integer $k \geq 0$, we use the notation \widetilde{W}_k for the time-shifted path $\widetilde{W}_k = (\widetilde{W}_k(j))_{j \leq 0}$, where $\widetilde{W}_k(j) := W_k(\zeta_k + j)$, for every $j \leq 0$.

Lemma 2.16. *Let $k \geq 1$ such that $\mathbb{P}(\widehat{W}_k = 0) > 0$. Under the conditional probability measure $\mathbb{P}(\cdot \mid \widehat{W}_k = 0)$, the two pairs (W_0, \widetilde{W}_k) and (\widetilde{W}_k, W_0) have the same distribution.*

Proof. Write $\pi_k(i, j)$, $i, j \geq 0$ for the joint distribution under \mathbb{P} of the pair

$$\left(- \min_{0 \leq \ell \leq k} \zeta_\ell, \zeta_k - \min_{0 \leq \ell \leq k} \zeta_\ell \right).$$

By an easy time-reversal argument, we have $\pi_k(i, j) = \pi_k(j, i)$ for every $i, j \geq 0$. On the other hand, under \mathbb{P} , conditionally on

$$\left(- \min_{0 \leq \ell \leq k} \zeta_\ell, \zeta_k - \min_{0 \leq \ell \leq k} \zeta_\ell \right) = (i, j)$$

we have $W_0(-i - \ell) = W_k(-i - \ell) = \widetilde{W}_k(-j - \ell)$ for every $\ell \geq 0$, and the two random paths

$$(W_0(-i + \ell) - W_0(-i))_{0 \leq \ell \leq i}$$

and

$$(\widetilde{W}_k(-j + \ell) - \widetilde{W}_k(-j))_{0 \leq \ell \leq j} = (W_k(-i + \ell) - W_0(-i))_{0 \leq \ell \leq j}$$

are independent and distributed as the random walk S stopped respectively at time i and at time j . Note that the event $\{\widehat{W}_k = 0\}$ occurs if and only if the latter two paths have the same endpoint. The statement of the lemma easily follows from the preceding observations and the property $\pi_k(i, j) = \pi_k(j, i)$. \square

Let us fix $\eta \in (0, 1/4)$. Thanks to Lemma 2.15, we may choose $\delta > 0$ small enough so that, for every sufficiently large n ,

$$\sum_{k=\lfloor (1-\delta)n \rfloor}^n \mathbb{P}(\widehat{W}_k = 0) < \eta.$$

We then observe that

$$\begin{aligned} 1 &= \sum_{k=0}^n \mathbb{P}\left(\widehat{W}_k = 0; \widehat{W}_\ell \neq 0, \forall \ell \in \llbracket k+1, n \rrbracket\right) \\ &= \sum_{k=0}^n \mathbb{E}\left[\mathbf{1}_{\{\widehat{W}_k=0\}} \mathbb{P}_{(W_k)}\left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n-k \rrbracket\right)\right] \\ &= \sum_{k=0}^n \mathbb{E}\left[\mathbf{1}_{\{\widehat{W}_k=0\}} \mathbb{P}_{(W_0)}\left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n-k \rrbracket\right)\right]. \end{aligned}$$

In the second equality, we applied the Markov property of the discrete snake at time k , and in the third one we used Lemma 2.16.

From the last equalities and our choice of δ , it follows that, for n large,

$$\mathbb{E}\left[\left(\sum_{k=0}^{\lfloor (1-\delta)n \rfloor} \mathbf{1}_{\{\widehat{W}_k=0\}}\right) \mathbb{P}_{(W_0)}\left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, \lfloor \delta n \rfloor \rrbracket\right)\right] \geq 1 - \eta.$$

Next fix $\varepsilon \in (0, 1/2)$ and write $n(\varepsilon) = \lfloor n^{\frac{1}{2}+\varepsilon} \rfloor$ to simplify notation. For every integer $p \geq 1$, there exists a constant $C_{p,\varepsilon}$ such that, for every $n \geq 1$,

$$\mathbb{P}(\tau_{n(\varepsilon)} \leq n) \leq C_{p,\varepsilon} n^{-p}.$$

Hence, we also get, for every sufficiently large n ,

$$\mathbb{E} \left[\left(\sum_{k=0}^{\tau_{n(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right) \mathbb{P}_{(W_0)} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, \lfloor \delta n \rfloor \rrbracket \right) \right] \geq 1 - 2\eta.$$

By conditioning with respect to W_0 , we see that the left-hand side of the preceding display is equal to

$$\mathbb{E} \left[\mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_{n(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right] \mathbb{P}_{(W_0)} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, \lfloor \delta n \rfloor \rrbracket \right) \right].$$

We now note that, for every integer $m \geq 1$,

$$\mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_m-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right] = 2 \sum_{j=0}^{m-1} G(-W_0(-j)) \quad (2.33)$$

(we could write $G(W_0(-j))$ instead of $G(-W_0(-j))$ because S is symmetric, but the preceding formula would hold also in the non-symmetric case). To derive formula (2.33), first consider the case $m = 1$. By a standard property of simple random walk, we have for every integer $i \geq 0$,

$$\mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_1-1} \mathbf{1}_{\{\zeta_k=i\}} \right] = 2.$$

Then using the conditional distribution of W given the lifetime process ζ , we obtain

$$\begin{aligned} \mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_1-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right] &= \sum_{i=0}^{\infty} \mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_1-1} \mathbf{1}_{\{\zeta_k=i\}} \mathbf{1}_{\{\widehat{W}_k=0\}} \right] \\ &= \sum_{i=0}^{\infty} \mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_1-1} \mathbf{1}_{\{\zeta_k=i\}} \right] p_i(-W_0(0)) \\ &= 2G(-W_0(0)). \end{aligned}$$

(Of course here $W_0(0) = 0$, but the previous calculation holds independently of the value of $W_0(0)$.) The same argument shows that, for every $j \in \llbracket 1, m-1 \rrbracket$,

$$\mathbb{E}_{(W_0)} \left[\sum_{k=\tau_j}^{\tau_{j+1}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right] = 2G(-W_0(-j))$$

and formula (2.33) follows.

From (2.33) and the preceding considerations, we get that, for all sufficiently large n ,

$$2 \mathbb{E} \left[\left(\sum_{j=0}^{n(\varepsilon)-1} G(-W_0(-j)) \right) \mathbb{P}_{(W_0)} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, \lfloor \delta n \rfloor \rrbracket \right) \right] \geq 1 - 2\eta. \quad (2.34)$$

Now recall that, under the probability measure \mathbb{P} , $(-W_0(-j))_{j \geq 0}$ has the same distribution as $(S_j)_{j \geq 0}$. At this point we need two other lemmas.

Lemma 2.17. *For every integer $p \geq 1$, there exists a constant $C(p)$ such that, for every $n \geq 2$,*

$$E \left[\left(\sum_{j=0}^n G(S_j) \right)^p \right] \leq C(p) (\log n)^p.$$

Proof. We first observe that

$$E\left[\left(\sum_{j=0}^n G(S_j)\right)^p\right] = E\left[\left(\sum_{j=0}^n G(S_j) \mathbf{1}_{\{|S_j| \leq n\}}\right)^p\right] + o(1)$$

as $n \rightarrow \infty$, because the event where $\sup\{|S_j| : 0 \leq j \leq n\} > n$ has a probability which decreases to 0 faster than any negative power of n . For every integer $k \geq 1$ and $x \in \mathbb{Z}^4$, set

$$G_k(x) = \sum_{i=0}^k p_i(x).$$

Using (2.29) and the standard local limit theorem (see e.g. [53, Chapter 2]) one easily verifies that, for every sufficiently large n , for all $x \in \mathbb{Z}^4$ such that $|x| \leq n$, the bound $G_{n^3}(x) \geq \frac{1}{2}G(x)$ holds. Thanks to this observation, it is enough to bound

$$E\left[\left(\sum_{j=0}^n G_{n^3}(S_j) \mathbf{1}_{\{|S_j| \leq n\}}\right)^p\right].$$

However, if S' stands for another random walk with the same distribution as S but independent of S , we have

$$\sum_{j=0}^n G_{n^3}(S_j) = E\left[\sum_{j=0}^n \sum_{i=0}^{n^3} \mathbf{1}_{\{S_j=S'_i\}} \mid S\right],$$

and by Lemma 1 in Marcus and Rosen [76], we know that there exists a constant $C'(p)$ such that, for every $n \geq 2$,

$$E\left[\left(\sum_{j=0}^n \sum_{i=0}^{n^3} \mathbf{1}_{\{S_j=S'_i\}}\right)^p\right] \leq C'(p) (\log n)^p.$$

The desired bound follows since the conditional expectation is a contraction in L^p . \square

Lemma 2.18. *For every $\alpha > 0$, there exists a constant C_α such that, for every integer $m \geq 2$, we have*

$$P\left(\left|\sum_{k=0}^m G(S_k) - \frac{1}{4\pi^2\sigma^4} \log m\right| \geq \alpha \log m\right) \leq C_\alpha (\log m)^{-3/2}.$$

We postpone the proof of Lemma 2.18 and complete the proof of Proposition 2.8. An application of Hölder's inequality gives for $p \geq 2$,

$$\begin{aligned} & E\left[\left(\sum_{j=0}^m G(S_j)\right) \mathbf{1}_{\{\sum_{j=0}^m G(S_j) \geq (\frac{1}{4\pi^2\sigma^4} + \alpha) \log m\}}\right] \\ & \leq E\left[\left(\sum_{j=0}^m G(S_j)\right)^p\right]^{1/p} P\left(\sum_{j=0}^m G(S_j) \geq (\frac{1}{4\pi^2\sigma^4} + \alpha) \log m\right)^{1/q} \\ & \leq C(p)^{1/p} \log m \times P\left(\left|\sum_{j=0}^m G(S_j) - \frac{1}{4\pi^2\sigma^4} \log m\right| \geq \alpha \log m\right)^{1/q} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and we used Lemma 2.17. Choosing $p \geq 4$ and using Lemma 2.18, we obtain that

$$\lim_{m \rightarrow \infty} E\left[\left(\sum_{j=0}^m G(S_j)\right) \mathbf{1}_{\{\sum_{j=0}^m G(S_j) \geq (\frac{1}{4\pi^2\sigma^4} + \alpha) \log m\}}\right] = 0.$$

From (2.34) and the fact that $(-W_0(-j))_{j \geq 0}$ has the same distribution as $(S_j)_{j \geq 0}$, we then get, for every sufficiently large n ,

$$\begin{aligned} & 2 \left(\frac{1}{4\pi^2\sigma^4} + \alpha \right) (\log n(\varepsilon)) \mathbb{P} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, \lfloor \delta n \rfloor \rrbracket \right) \\ & \geq 1 - 2\eta - 2 \mathbb{E} \left[\left(\sum_{j=0}^{n(\varepsilon)-1} G(-W_0(-j)) \right) \mathbf{1}_{\left\{ \sum_{j=0}^{n(\varepsilon)-1} G(-W_0(-j)) \geq \left(\frac{1}{4\pi^2\sigma^4} + \alpha \right) \log n(\varepsilon) \right\}} \right] \\ & \geq 1 - 3\eta. \end{aligned}$$

Since $\log n(\varepsilon) \leq (\frac{1}{2} + \varepsilon) \log n$, the preceding bound implies that

$$\liminf_{n \rightarrow \infty} (\log n) \mathbb{P} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, \lfloor \delta n \rfloor \rrbracket \right) \geq \frac{1 - 3\eta}{1 + 2\varepsilon} \left(\frac{1}{4\pi^2\sigma^4} + \alpha \right)^{-1}.$$

Now note that the ratio $\log \lfloor \delta n \rfloor / \log n$ tends to 1 as $n \rightarrow \infty$, and that η , ε and α can be chosen arbitrarily small. We conclude that

$$\liminf_{n \rightarrow \infty} (\log n) \mathbb{P} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket \right) \geq 4\pi^2\sigma^4.$$

The proof of the analogous result for the limsup behavior is similar. In the same way as we proceeded above, we arrive at the bound

$$\mathbb{E} \left[\left(\sum_{k=0}^n \mathbf{1}_{\{\widehat{W}_k=0\}} \right) \mathbb{P}_{(W_0)} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket \right) \right] \leq 1.$$

At this point, we would like to replace the sum from $k = 0$ to n by a sum from $k = 0$ to $\tau_{n'(\varepsilon)-1}$, where $n'(\varepsilon) = \lfloor n^{\frac{1}{2}-\varepsilon} \rfloor$ for some fixed $\varepsilon \in (0, 1/2)$. Simple arguments give the existence of a constant C'_ε such that, for every integer $n \geq 1$,

$$\mathbb{P}(\tau_{n'(\varepsilon)} \geq n) \leq C'_\varepsilon n^{-\varepsilon/2}.$$

We can then write

$$1 \geq \mathbb{E} \left[\left(\sum_{k=0}^{\tau_{n'(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right) \mathbb{P}_{(W_0)} \left(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket \right) \right] - \mathbb{E} \left[\mathbf{1}_{\{\tau_{n'(\varepsilon)} \geq n\}} \left(\sum_{k=0}^{\tau_{n'(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right) \right],$$

and by the Cauchy–Schwarz inequality, we have

$$\mathbb{E} \left[\mathbf{1}_{\{\tau_{n'(\varepsilon)} \geq n\}} \left(\sum_{k=0}^{\tau_{n'(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right) \right] \leq (C'_\varepsilon n^{-\varepsilon/2})^{1/2} \times \mathbb{E} \left[\left(\sum_{k=0}^{\tau_{n'(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right)^2 \right]^{1/2}. \quad (2.35)$$

To bound the expectation in the right-hand side, one can verify that, for every integer $m \geq 1$,

$$\mathbb{E} \left[\left(\sum_{k=0}^{m-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right)^2 \middle| W_0 \right] \leq 4 \left(\sum_{j=0}^{m-1} G(-W_0(-j)) \right)^2 + 4 \sum_{j=0}^{m-1} \Phi(-W_0(-j))$$

where, for every $x \in \mathbb{Z}^4$,

$$\Phi(x) := \sum_{y \in \mathbb{Z}^4} G(y) G(x-y)^2.$$

The proof of the latter bound is similar to that of (2.33) above, and we leave the details to the reader. One then checks from (2.29) that there exists a constant \tilde{C} such that

$$\Phi(x) \leq \tilde{C} (|x| \vee 1)^{-2} (1 + \log(|x| \vee 1)), \quad \text{for every } x \in \mathbb{Z}^4.$$

It easily follows that

$$\mathbb{E} \left[\left(\sum_{k=0}^{\tau_m-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right)^2 \right] = O((\log m)^2)$$

as $m \rightarrow \infty$. Consequently the right-hand side of (2.35) tends to 0 as $n \rightarrow \infty$ and if $\eta > 0$ is fixed, we have, for all n sufficiently large,

$$\mathbb{E} \left[\left(\sum_{k=0}^{\tau_{n'(\varepsilon)}-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right) \mathbb{P}_{(W_0)}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket) \right] \leq 1 + \eta.$$

Just as we obtained (2.34), we deduce from the latter bound that

$$2 \mathbb{E} \left[\left(\sum_{j=0}^{n'(\varepsilon)-1} G(-W_0(-j)) \right) \mathbb{P}_{(W_0)}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket) \right] \leq 1 + \eta. \quad (2.36)$$

Then fix $\alpha \in (0, (4\pi^2\sigma^4)^{-1})$. It follows from (2.36) that

$$\begin{aligned} & 2 \left(\frac{1}{4\pi^2\sigma^4} - \alpha \right) (\log n'(\varepsilon)) \mathbb{E} \left[\mathbf{1}_{\left\{ \sum_{j=0}^{n'(\varepsilon)-1} G(-W_0(-j)) \geq \left(\frac{1}{4\pi^2\sigma^4} - \alpha \right) \log n'(\varepsilon) \right\}} \mathbb{P}_{(W_0)}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket) \right] \\ & \leq 2 \mathbb{E} \left[\left(\sum_{j=0}^{n'(\varepsilon)-1} G(-W_0(-j)) \right) \mathbb{P}_{(W_0)}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket) \right] \\ & \leq 1 + \eta. \end{aligned}$$

On the other hand,

$$(\log n'(\varepsilon)) \mathbb{P} \left(\sum_{j=0}^{n'(\varepsilon)-1} G(-W_0(-j)) < \left(\frac{1}{4\pi^2\sigma^4} - \alpha \right) \log n'(\varepsilon) \right) \xrightarrow[n \rightarrow \infty]{} 0$$

by Lemma 2.18. By combining the last two displays, we get

$$\limsup_{n \rightarrow \infty} 2 \left(\frac{1}{4\pi^2\sigma^4} - \alpha \right) (\log n'(\varepsilon)) \mathbb{P}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket) \leq 1 + \eta.$$

Since η , ε and α can be chosen arbitrarily small, we get

$$\limsup_{n \rightarrow \infty} (\log n) \mathbb{P}(\widehat{W}_\ell \neq 0, \forall \ell \in \llbracket 1, n \rrbracket) \leq 4\pi^2\sigma^4,$$

which completes the proof of the first assertion of Proposition 2.8. The second assertion is an easy consequence of the first one, noting that, for every $\varepsilon > 0$, both $\mathbb{P}(\tau_p \geq p^{2+\varepsilon})$ and $\mathbb{P}(\tau_p \leq p^{2-\varepsilon})$ are $o((\log p)^{-1})$ as $p \rightarrow \infty$. \square

Proof of Lemma 2.18. The general strategy of the proof is to derive an analogous result for Brownian motion in \mathbb{R}^4 , and then to use a strong invariance principle to transfer this result to the random walk S .

We let $B = (B_t)_{t \geq 0}$ be a four-dimensional Brownian motion started from 0 and set $\rho_t = |B_t|$ for every $t \geq 0$, so that $(\rho_t)_{t \geq 0}$ is a four-dimensional Bessel process started from 0. Here is the Brownian motion version of Lemma 2.18.

Lemma 2.19. *Let $\varepsilon > 0$. There exist two constants $C(\varepsilon)$ and $\beta(\varepsilon) > 0$ such that, for every $t > r \geq 1$,*

$$P\left(\left|\int_r^t \frac{ds}{\rho_s^2} - \frac{1}{2} \log\left(\frac{t}{r}\right)\right| > \varepsilon \log\left(\frac{t}{r}\right)\right) \leq C(\varepsilon) \left(\frac{t}{r}\right)^{-\beta(\varepsilon)}.$$

Let us postpone the proof of Lemma 2.19. We fix $\alpha > 0$ and consider an integer $n \geq 1$. By an extension due to Zaitsev [89] of the celebrated Komlós–Major–Tusnády strong invariance principle, we can construct on the same probability space the finite sequence (S_1, \dots, S_n) and the Brownian motion $(B_t)_{t \geq 0}$, in such a way that, for some constants $c > 0, c' > 0$ and $K > 0$ that do not depend on n , we have

$$E\left[\exp\left(c \max_{1 \leq k \leq n} |S_k - \sigma B_k|\right)\right] \leq K \exp(c' \log n).$$

It readily follows that we can find constants $C > 0$ and $a > 0$ (again independent of n) such that

$$P\left(\max_{1 \leq k \leq n} |S_k - \sigma B_k| > C \log n\right) \leq K n^{-a}.$$

Let $A > 2$ be a constant. Then

$$P\left(\inf_{t \geq (\log n)^4} \sigma |B_t| \leq AC \log n\right) = P\left(\inf_{t \geq 1} \sigma |B_t| \leq \frac{AC}{\log n}\right) = O((\log n)^{-2})$$

by an easy estimate. On the event

$$E_n := \left\{ \max_{1 \leq k \leq n} |S_k - \sigma B_k| \leq C \log n \right\} \cap \left\{ \inf_{t \geq (\log n)^4} \sigma |B_t| > AC \log n \right\}$$

we have, for every integer k such that $(\log n)^4 \leq k \leq n$,

$$|S_k| \geq \sigma |B_k| - C \log n \geq (1 - \eta) \sigma |B_k|$$

and

$$|S_k| \leq \sigma |B_k| + C \log n \leq (1 + \eta) \sigma |B_k|$$

where $\eta = 1/A$. We now fix A so that $\eta \in (0, \frac{1}{5})$ and $5\eta < \pi^2 \sigma^4 \alpha / 2$.

Recalling our estimate (2.29), we also see that (provided n is large enough) we have on the event E_n , for every integer k such that $(\log n)^4 \leq k \leq n$,

$$(1 - 3\eta) \frac{1}{2\pi^2 \sigma^4} |B_k|^{-2} \leq G(S_k) \leq (1 + 3\eta) \frac{1}{2\pi^2 \sigma^4} |B_k|^{-2}.$$

Consequently, we have on the event E_n ,

$$(1 - 3\eta) \frac{1}{2\pi^2 \sigma^4} \sum_{k=\lceil (\log n)^4 \rceil}^n |B_k|^{-2} \leq \sum_{k=\lceil (\log n)^4 \rceil}^n G(S_k) \leq (1 + 3\eta) \frac{1}{2\pi^2 \sigma^4} \sum_{k=\lceil (\log n)^4 \rceil}^n |B_k|^{-2}.$$

The next step is to observe that

$$\sum_{k=\lceil (\log n)^4 \rceil}^n |B_k|^{-2}$$

is close to

$$\int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2}$$

up to a set of small probability. Indeed simple estimates show that, for any choice of $\kappa > 0$, we have

$$\sup_{0 \leq k \leq n} \sup_{k \leq s \leq k+1} |B_s - B_k| \leq \kappa \log n$$

outside of a set of probability $O(n^{-1})$. By choosing κ suitably, we then see that on the event

$$\tilde{E}_n := E_n \cap \left\{ \sup_{0 \leq k \leq n} \sup_{k \leq s \leq k+1} |B_s - B_k| \leq \kappa \log n \right\}$$

we have

$$(1 - \eta) \int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2} \leq \sum_{k=\lceil (\log n)^4 \rceil}^n |B_k|^{-2} \leq (1 + \eta) \int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2},$$

and consequently

$$(1 - 5\eta) \frac{1}{2\pi^2\sigma^4} \int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2} \leq \sum_{k=\lceil (\log n)^4 \rceil}^n G(S_k) \leq (1 + 5\eta) \frac{1}{2\pi^2\sigma^4} \int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2}. \quad (2.37)$$

We also need to bound the quantity

$$\sum_{k=0}^{\lceil (\log n)^4 \rceil - 1} G(S_k).$$

However, from Lemma 2.17 with $p = 2$, we immediately get that, for every integer $m \geq 2$ and every $h > 0$,

$$P\left(\sum_{k=0}^m G(S_k) \geq h\right) \leq \frac{C(2)(\log m)^2}{h^2}. \quad (2.38)$$

Finally,

$$\begin{aligned} & P\left(\left|\sum_{k=0}^n G(S_k) - \frac{1}{4\pi^2\sigma^4} \log n\right| \geq \alpha \log n\right) \\ & \leq P\left(\sum_{k=0}^{\lceil (\log n)^4 \rceil} G(S_k) \geq \frac{\alpha}{2} \log n\right) + P\left(\left|\sum_{k=\lceil (\log n)^4 \rceil}^n G(S_k) - \frac{1}{4\pi^2\sigma^4} \log n\right| \geq \frac{\alpha}{2} \log n\right). \end{aligned}$$

The first term in the right-hand side is $O((\log n)^{-3/2})$ by (2.38). On the other hand, by (2.37), the second term is bounded by

$$P(\tilde{E}_n^c) + P\left(\left|\int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2} - \frac{1}{2} \log n\right| \geq \alpha' \log n\right)$$

where $\alpha' = (\frac{1}{2}\pi^2\sigma^4\alpha) \wedge \frac{1}{4}$ is a constant independent of n , which satisfies

$$(1 + 5\eta)\left(\frac{1}{2} + \alpha'\right) \frac{1}{2\pi^2\sigma^4} < \frac{1}{4\pi^2\sigma^4} + \frac{\alpha}{2} \quad \text{and} \quad (1 - 5\eta)\left(\frac{1}{2} - \alpha'\right) \frac{1}{2\pi^2\sigma^4} > \frac{1}{4\pi^2\sigma^4} - \frac{\alpha}{2}.$$

(Here we use our choice of η such that $5\eta < \pi^2\sigma^4\alpha/2$.) From preceding estimates, we have $P(\tilde{E}_n^c) = O((\log n)^{-2})$. On the other hand, Lemma 2.19 implies that

$$P\left(\left|\int_{\lceil (\log n)^4 \rceil}^{n+1} \frac{ds}{|B_s|^2} - \frac{1}{2} \log n\right| \geq \alpha' \log n\right) = O(n^{-b})$$

for some $b > 0$. This completes the proof of Lemma 2.18. \square

Proof of Lemma 2.19. By a scaling argument, it is enough to consider the case $r = 1$, and we consider only that case. For every integer $k \geq 0$, set

$$\gamma_k := \inf \{t \geq 0 : \rho_t = e^k\}$$

and

$$X_k := \int_{\gamma_k}^{\gamma_{k+1}} \frac{ds}{\rho_s^2}.$$

A scaling argument shows that the variables X_k , $k \geq 0$ are identically distributed. Moreover, the strong Markov property of the Bessel process implies that the variables X_k , $k \geq 0$ are independent. Furthermore, the absolute continuity relations between Bessel processes can be used to verify that these variables have small exponential moments. More precisely, using the explicit form of the density of the law over the time interval $[0, t]$ of the four-dimensional Bessel process started at 1 with respect to Wiener measure (see question 3 in Exercise XI.1.22 of Revuz and Yor [82]), it is an easy exercise of martingale theory to verify that

$$E[e^{3X_0/8}] = E\left[\exp\left(\frac{3}{8} \int_{\gamma_0}^{\gamma_1} \frac{ds}{\rho_s^2}\right)\right] = \sqrt{e} < \infty.$$

Set

$$c_0 = E[X_0] = E[X_k]$$

for every $k \geq 0$. We can apply Cramér's large deviation theorem to the sequence $(X_k)_{k \geq 0}$. It follows that, for every $\delta > 0$, there exists a constant $b(\delta) > 0$ such that for every sufficiently large n ,

$$P\left(\left|\int_{\gamma_0}^{\gamma_n} \frac{ds}{\rho_s^2} - c_0 n\right| > \delta n\right) \leq \exp(-b(\delta)n). \quad (2.39)$$

On the other hand, it is easy to verify that the variable

$$\int_1^{\gamma_0} \frac{ds}{\rho_s^2}$$

has exponential moments. Just use the above-mentioned argument involving the density of the law of the Bessel process to verify that

$$E\left[\exp\left(\frac{3}{8} \int_1^{\gamma_0} \frac{ds}{\rho_s^2}\right)\right] < \infty$$

(deal separately with the cases $1 < \gamma_0$ and $\gamma_0 < 1$). It then follows that, for every $\delta > 0$, and for all sufficiently large n ,

$$P\left(\int_1^{\gamma_0} \frac{ds}{\rho_s^2} > \delta n\right) \leq \exp(-b'(\delta)n)$$

with some constant $b'(\delta) > 0$. The same bound holds for the variable

$$\int_{e^{2m}}^{\gamma_m} \frac{ds}{\rho_s^2},$$

for any integer $m \geq 0$, since this variable has the same law as

$$\int_1^{\gamma_0} \frac{ds}{\rho_s^2}$$

by scaling.

By combining the latter facts with (2.39), we obtain that, for every $\delta > 0$, there exists a constant $\tilde{b}(\delta) > 0$ such that, for every sufficiently large n ,

$$P\left(\left|\int_1^{e^{2n}} \frac{ds}{\rho_s^2} - c_0 n\right| > \delta n\right) \leq \exp(-\tilde{b}(\delta)n). \quad (2.40)$$

At this stage, we can identify the constant c_0 , since the preceding arguments also show that

$$c_0 = \lim_{n \rightarrow \infty} \frac{1}{n} E\left[\int_1^{e^{2n}} \frac{ds}{\rho_s^2}\right] = 1$$

by a direct calculation of $E[(\rho_s)^{-2}] = (2s)^{-1}$. Once we know that $c_0 = 1$, the statement of Lemma 2.19 follows from (2.40) by elementary considerations: For every $t \geq 1$, choose n such that $e^{2n} \leq t < e^{2(n+1)}$ and observe that

$$\left\{\int_1^t \frac{ds}{\rho_s^2} - \frac{1}{2} \log t > \varepsilon \log t\right\} \subseteq \left\{\int_1^{e^{2(n+1)}} \frac{ds}{\rho_s^2} - n > 2\varepsilon n\right\},$$

whereas

$$\left\{\int_1^t \frac{ds}{\rho_s^2} - \frac{1}{2} \log t < -\varepsilon \log t\right\} \subseteq \left\{\int_1^{e^{2n}} \frac{ds}{\rho_s^2} - n - 1 < -2\varepsilon n\right\}.$$

This completes the proof. \square

2.4 The range of branching random walk

In this last section, we apply the preceding results to asymptotics for the range of branching random walk in \mathbb{Z}^d , $d \geq 4$. We assume that the offspring distribution μ is critical and has finite variance $\sigma_\mu^2 > 0$, and that the jump distribution θ is centered and has finite moments of order $d - 1$ (and as usual that θ is not supported on a strict subgroup of \mathbb{Z}^d).

Let $M_p(\mathbb{Z}^d)$ stand for the set of all finite point measures on \mathbb{Z}^d . Let $\mathcal{Z} = (\mathcal{Z}_n)_{n \geq 0}$ denote the (discrete time) branching random walk with jump distribution θ and offspring distribution μ . This is the Markov chain with values in $M_p(\mathbb{Z}^d)$, whose transition kernel \mathbf{Q} can be described as follows. If

$$\omega = \sum_{i=1}^p \delta_{x_i} \in M_p(\mathbb{Z}^d),$$

$\mathbf{Q}(\omega, \cdot)$ is the distribution of

$$\sum_{i=1}^p \sum_{j=1}^{\xi_i} \delta_{x_i + Y_{i,j}},$$

where ξ_1, \dots, ξ_p are independent and distributed according to μ and, conditionally on (ξ_1, \dots, ξ_p) , the random variables $Y_{i,j}$, $1 \leq i \leq p$, $1 \leq j \leq \xi_i$, are independent and distributed according to θ . More informally, each particle alive at time n is replaced at time $n + 1$ by a number of offspring distributed according to μ , and the spatial position of each of these offspring is obtained by adding a jump distributed according to θ to the position of its parent.

The range of \mathcal{Z} is then defined by

$$\mathbf{R}(\mathcal{Z}) := \#\{x \in \mathbb{Z}^d : \exists n \geq 0, \mathcal{Z}_n(x) \geq 1\}.$$

We also write $\mathbf{N}(\mathcal{Z})$ for the total progeny of \mathcal{Z} ,

$$\mathbf{N}(\mathcal{Z}) := \sum_{n=0}^{\infty} \langle \mathcal{Z}_n, 1 \rangle$$

where $\langle \mathcal{Z}_n, 1 \rangle$ is the total mass of \mathcal{Z}_n . It is well known (and easy to prove using the Lukasiewicz path introduced in the proof of Theorem 2.7) that $\mathbf{N}(\mathcal{Z})$ has the distribution of the hitting time of $-\langle \mathcal{Z}_0, 1 \rangle$ by a random walk on \mathbb{Z} with jump distribution $\nu(k) = \mu(k+1)$, for $k = -1, 0, 1, \dots$, started from 0.

Proposition 2.20. *Suppose that $d \geq 5$. For every integer $p \geq 1$, let $\mathcal{Z}^{(p)}$ be a branching random walk with jump distribution θ and offspring distribution μ , such that $\langle \mathcal{Z}_0^{(p)}, 1 \rangle = p$. Then,*

$$\lim_{p \rightarrow \infty} \frac{\mathbf{R}(\mathcal{Z}^{(p)})}{\mathbf{N}(\mathcal{Z}^{(p)})} = c_{\mu, \theta} \quad \text{in probability,}$$

where $c_{\mu, \theta} > 0$ is the constant in Theorem 2.4. Consequently,

$$\frac{1}{p^2} \mathbf{R}(\mathcal{Z}^{(p)}) \xrightarrow[p \rightarrow \infty]{(d)} \frac{c_{\mu, \theta}}{\sigma_{\mu}^2} J,$$

where the positive random variable J has density $(2\pi s^3)^{-1/2} \exp(-\frac{1}{2s})$ on $(0, \infty)$.

Proof. We may and will assume that there exists a sequence $\mathcal{T}^1, \mathcal{T}^2, \dots$ of independent random trees distributed according to Π_{μ} , such that, for every $p \geq 1$, the genealogy of $\mathcal{Z}^{(p)}$ is coded by $\mathcal{T}^1, \mathcal{T}^2, \dots, \mathcal{T}^p$, meaning that \mathcal{T}^i is the genealogical tree of the descendants of the i -th initial particle of $\mathcal{Z}^{(p)}$, for every $p \geq 1$ and $i \in \{1, \dots, p\}$. Notice that we have then

$$\mathbf{N}(\mathcal{Z}^{(p)}) = \#\mathcal{T}^1 + \dots + \#\mathcal{T}^p.$$

For every $i \in \{1, \dots, p\}$, we will write $\mathcal{S}_i^{(p)}$ for the set of all spatial locations occupied by the particles of $\mathcal{Z}^{(p)}$ that are descendants of the i -th initial particle. Note that the location of the i -th initial particle may depend on p . Clearly, we have

$$\mathbf{R}(\mathcal{Z}^{(p)}) \leq \#\mathcal{S}_1^{(p)} + \dots + \#\mathcal{S}_p^{(p)}. \quad (2.41)$$

Let $(H_k)_{k \geq 0}$ be the height process associated with the sequence $\mathcal{T}^1, \mathcal{T}^2, \dots$ (see the proof of Proposition 2.6). Then, as an easy consequence of (2.3), we have the joint convergence in distribution

$$\left(\left(\frac{1}{p} H_{\lfloor p^2 t \rfloor \wedge \mathbf{N}(\mathcal{Z}^{(p)})} \right)_{t \geq 0}, \frac{1}{p^2} \mathbf{N}(\mathcal{Z}^{(p)}) \right) \xrightarrow[p \rightarrow \infty]{(d)} \left(\left(\frac{2}{\sigma_{\mu}} |\beta_{t \wedge J_{1/\sigma_{\mu}}}| \right)_{t \geq 0}, J_{1/\sigma_{\mu}} \right), \quad (2.42)$$

where β is a standard linear Brownian motion, and for every $s \geq 0$, $J_s = \inf\{t \geq 0 : L_t^0(\beta) > s\}$, where $(L_t^0(\beta))_{t \geq 0}$ is the local time process of β at level 0. See [58, Section 1.4] for details of the derivation of (2.42).

Fix $\varepsilon > 0$. For $\alpha \in (0, 1)$, let $i_{p,1}, i_{p,2}, \dots, i_{p,m_p}$ be all indices $i \in \{1, \dots, p\}$ such that $\#\mathcal{T}^i \geq \alpha p^2$. It follows from (2.42) that, if α has been chosen sufficiently small, the bound

$$\mathbf{N}(\mathcal{Z}^{(p)}) - (\#\mathcal{T}^{i_{p,1}} + \dots + \#\mathcal{T}^{i_{p,m_p}}) = \sum_{i \in \{1, \dots, p\} \setminus \{i_{p,1}, \dots, i_{p,m_p}\}} \#\mathcal{T}^i < \varepsilon p^2 \quad (2.43)$$

will hold with probability arbitrarily close to 1, uniformly for all sufficiently large p . On the other hand, it also follows from (2.42) that m_p converges in distribution as $p \rightarrow \infty$ to a Poisson distribution with parameter $\sigma_\mu^{-1} \sqrt{2/\pi\alpha}$ (here the quantity $\sqrt{2/\pi\alpha}$ is the mass that the Itô excursion measure assigns to excursions of length greater than α). In particular, by choosing α even smaller if necessary, we have $P(m_p \geq 1) > 1 - \varepsilon$ for all p large enough. We now fix $\alpha > 0$ so that the preceding properties hold for all p large enough.

Next we observe that, conditionally on m_p , the trees $\mathcal{T}^{i_{p,1}}, \dots, \mathcal{T}^{i_{p,m_p}}$ are independent and distributed according to $\Pi_\mu(\cdot \mid \#\mathcal{T} \geq \alpha p^2)$. From Theorem 2.7, we now get that

$$P\left(\left|\frac{\#\mathcal{S}_{i_{p,1}}^{(p)} + \dots + \#\mathcal{S}_{i_{p,m_p}}^{(p)}}{\#\mathcal{T}^{i_{p,1}} + \dots + \#\mathcal{T}^{i_{p,m_p}}} - c_{\mu,\theta}\right| > \varepsilon \mid m_p \geq 1\right) \xrightarrow{p \rightarrow \infty} 0. \quad (2.44)$$

Then, on the one hand, we have from (2.41),

$$R(\mathcal{Z}^{(p)}) \leq \#\mathcal{S}_{i_{p,1}}^{(p)} + \dots + \#\mathcal{S}_{i_{p,m_p}}^{(p)} + \sum_{i \in \{1, \dots, p\} \setminus \{i_{p,1}, \dots, i_{p,m_p}\}} \#\mathcal{T}^i,$$

and on the other hand,

$$R(\mathcal{Z}^{(p)}) \geq \#\mathcal{S}_{i_{p,1}}^{(p)} + \dots + \#\mathcal{S}_{i_{p,m_p}}^{(p)} - \sum_{1 \leq k < \ell \leq m_p} \#(\mathcal{S}_{i_{p,k}}^{(p)} \cap \mathcal{S}_{i_{p,\ell}}^{(p)}).$$

Taking into account the bound (2.43) and the fact that $p^{-2}\mathbf{N}(\mathcal{Z}^{(p)})$ converges in distribution to a positive random variable, we see that the first assertion of the proposition will follow from the last two bounds and (2.44), provided we can verify that

$$\frac{1}{p^2} \sum_{1 \leq k < \ell \leq m_p} \#(\mathcal{S}_{i_{p,k}}^{(p)} \cap \mathcal{S}_{i_{p,\ell}}^{(p)}) \xrightarrow{p \rightarrow \infty} 0. \quad (2.45)$$

Recall that m_p converges in distribution to a finite random variable. In order to establish (2.45), it is enough to verify that, if $\mathcal{S}^{(p),1}$, respectively $\mathcal{S}^{(p),2}$, is the set of points visited by a random walk indexed by a tree distributed according to $\Pi_\mu(\cdot \mid \#\mathcal{T} \geq \alpha p^2)$, with the spatial location of the root equal to x_1 , resp. to x_2 , and if $\mathcal{S}^{(p),1}$ and $\mathcal{S}^{(p),2}$ are independent, we have

$$\frac{1}{p^2} E\left[\#(\mathcal{S}^{(p),1} \cap \mathcal{S}^{(p),2})\right] \xrightarrow{p \rightarrow \infty} 0.$$

However,

$$E\left[\#(\mathcal{S}^{(p),1} \cap \mathcal{S}^{(p),2})\right] = \sum_{y \in \mathbb{Z}^d} P(y \in \mathcal{S}^{(p),1}) P(y \in \mathcal{S}^{(p),2}) \leq \sum_{y \in \mathbb{Z}^d} P(y \in \mathcal{S}^{(p),1})^2,$$

using the Cauchy–Schwarz inequality and translation invariance, which also allows us to take $x_1 = 0$. By a first moment argument, we have then

$$P(y \in \mathcal{S}^{(p),1}) \leq \frac{G_\theta(y)}{\Pi_\mu(\#\mathcal{T} \geq \alpha p^2)} \wedge 1 \leq (c_\mu^{-1} \sqrt{\alpha} p G_\theta(y)) \wedge 1,$$

where the constant $c_\mu > 0$ depends only on μ . Here we used the classical bound

$$\Pi_\mu(\#\mathcal{T} \geq k) \geq c_\mu k^{-1/2}, \quad k \geq 1,$$

which follows from the fact that the distribution of $\#\mathcal{T}$ under Π_μ coincides with the law of the first hitting time of -1 by a random walk on \mathbb{Z} with jump distribution ν started from 0 (see the proof of Theorem 2.7). Finally, we have

$$\frac{1}{p^2} E[\#(\mathcal{S}^{(p),1} \cap \mathcal{S}^{(p),2})] \leq \sum_{y \in \mathbb{Z}^d} (c_{(\mu)}^{-2} \alpha G_\theta(y)^2) \wedge \frac{1}{p^2}$$

and the right-hand side tends to 0 as $p \rightarrow \infty$ by dominated convergence, noting that

$$\sum_{y \in \mathbb{Z}^d} G_\theta(y)^2 < \infty$$

by (2.2). This completes the proof of the first assertion of the proposition.

The second assertion follows from the first one and the convergence in distribution of $p^{-2}\mathbf{N}(\mathcal{Z}^{(p)})$ to J_{1/σ_μ} . Just note that J_{1/σ_μ} has the same law as $\sigma_\mu^{-2}J_1$ by scaling, and that J_1 is distributed as the first hitting of 1 by a standard linear Brownian motion, whose density is as stated in the proposition. \square

We now state the result corresponding to Proposition 2.20 in the critical dimension $d = 4$. As previously, we must restrict our attention to the geometric offspring distribution.

Proposition 2.21. *Suppose that $d = 4$, and that μ is the critical geometric offspring distribution. Also assume that θ is symmetric and has small exponential moments, and set $\sigma^2 = (\det M_\theta)^{1/4}$. For every integer $p \geq 1$, let $\mathcal{Z}^{(p)}$ be a branching random walk with jump distribution θ and offspring distribution μ , such that $\langle \mathcal{Z}_0^{(p)}, 1 \rangle = p$. Then,*

$$\lim_{p \rightarrow \infty} \frac{(\log p) \mathbf{R}(\mathcal{Z}^{(p)})}{\mathbf{N}(\mathcal{Z}^{(p)})} = 8\pi^2 \sigma^4 \quad \text{in probability.}$$

Consequently,

$$\frac{\log p}{p^2} \mathbf{R}(\mathcal{Z}^{(p)}) \xrightarrow[p \rightarrow \infty]{(d)} 4\pi^2 \sigma^4 J,$$

where J is as in Proposition 2.20.

The proof of Proposition 2.21 goes along the same lines as that of Proposition 2.20, using now Theorem 2.14 instead of Theorem 2.7. A few minor modifications are needed, but we will leave the details to the reader.

THE RANGE OF TREE-INDEXED RANDOM WALK IN LOW DIMENSIONS

LES RÉSULTATS DE CE CHAPITRE SONT ISSUS DE L'ARTICLE [62], ÉCRIT EN COLLABORATION AVEC JEAN-FRANÇOIS LE GALL ET ACCEPTÉ POUR PUBLICATION DANS *The Annals of Probability*.

We study the range R_n of a random walk on the d -dimensional lattice \mathbb{Z}^d indexed by a random tree with n vertices. Under the assumption that the random walk is centered and has finite fourth moments, we prove in dimension $d \leq 3$ that $n^{-d/4}R_n$ converges in distribution to the Lebesgue measure of the support of the integrated super-Brownian excursion (ISE). An auxiliary result shows that the suitably rescaled local times of the tree-indexed random walk converge in distribution to the density process of ISE. We obtain similar results for the range of critical branching random walk in \mathbb{Z}^d , $d \leq 3$. As an intermediate estimate, we get exact asymptotics for the probability that a critical branching random walk starting with a single particle at the origin hits a distant point. The results of the present article complement those derived in higher dimensions in our earlier work.

3.1 Introduction

In the present paper, we continue our study of asymptotics for the number of distinct sites of the lattice visited by a tree-indexed random walk. We consider (discrete) plane trees, which are rooted ordered trees that can be viewed as describing the genealogy of a population starting with one ancestor or root, which is denoted by the symbol \emptyset . Given such a tree \mathcal{T} and a probability measure θ on \mathbb{Z}^d , we can consider the random walk with jump distribution θ indexed by the tree \mathcal{T} . This means that we assign a (random) spatial location $Z_{\mathcal{T}}(u) \in \mathbb{Z}^d$ to every vertex u of \mathcal{T} , in the following way. First, the spatial location $Z_{\mathcal{T}}(\emptyset)$ of the root is the origin of \mathbb{Z}^d . Then we assign independently to every edge e of the tree \mathcal{T} a random variable Y_e distributed according to θ , and we let the spatial location $Z_{\mathcal{T}}(u)$ of the vertex u be the sum of the quantities Y_e over all edges e belonging to the simple path from \emptyset to u in the tree. The number of distinct spatial locations is called the range of the tree-indexed random walk $Z_{\mathcal{T}}$.

In our previous work [61], we stated the following result. Let θ be a probability distribution on \mathbb{Z}^d , which is symmetric with finite support and is not supported on a strict subgroup of \mathbb{Z}^d , and for every integer $n \geq 1$, let \mathcal{T}_n° be a random tree uniformly distributed over all plane trees with n vertices. Conditionally given \mathcal{T}_n° , let $Z_{\mathcal{T}_n^\circ}$ be a random walk with jump distribution θ indexed by \mathcal{T}_n° , and let R_n stand for the range of $Z_{\mathcal{T}_n^\circ}$. Then,

- if $d \geq 5$,

$$\frac{1}{n} R_n \xrightarrow[n \rightarrow \infty]{(P)} c_\theta,$$

where $c_\theta > 0$ is a constant depending on θ , and $\xrightarrow{(P)}$ indicates convergence in probability;

- if $d = 4$,

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2} 8 \pi^2 \sigma^4,$$

where $\sigma^2 = (\det M_\theta)^{1/4}$, with M_θ denoting the covariance matrix of θ ;

- if $d \leq 3$,

$$n^{-d/4} R_n \xrightarrow[n \rightarrow \infty]{(d)} c_\theta \lambda_d(\text{supp}(\mathcal{I})), \quad (3.1)$$

where $c_\theta = 2^{d/4} (\det M_\theta)^{1/2}$ is a constant depending on θ , and $\lambda_d(\text{supp}(\mathcal{I}))$ stands for the Lebesgue measure of the support of the random measure on \mathbb{R}^d known as Integrated Super-Brownian Excursion or ISE (see Subsection 2.3 below for a definition of ISE in terms of the Brownian snake, and note that our normalization is slightly different from the one in [6]).

Only the cases $d \geq 5$ and $d = 4$ were proved in [61], in fact in a greater generality than stated above, especially when $d \geq 5$. In the present work, we concentrate on the case $d \leq 3$ and we prove a general version of the convergence (3.1), where instead of considering a uniformly distributed plane tree with n vertices we deal with a Galton–Watson tree with offspring distribution μ conditioned to have n vertices.

Let us specify the assumptions that will be in force throughout this article. We always assume that $d \leq 3$ and

- μ is a nondegenerate critical offspring distribution on \mathbb{Z}_+ , such that, for some $\lambda > 0$,

$$\sum_{k=0}^{\infty} e^{\lambda k} \mu(k) < \infty,$$

and we set $\rho := (\text{var } \mu)^{1/2} > 0$;

- θ is a probability measure on \mathbb{Z}^d , which is not supported on a strict subgroup of \mathbb{Z}^d ; θ is such that

$$\lim_{r \rightarrow +\infty} r^4 \theta(\{x \in \mathbb{Z}^d : |x| > r\}) = 0, \quad (3.2)$$

and θ has zero mean; we set $\sigma := (\det M_\theta)^{1/2d} > 0$, where M_θ denotes the covariance matrix of θ .

Note that (3.2) holds if θ has finite fourth moments.

For every $n \geq 1$ such that this makes sense, let \mathcal{T}_n be a Galton–Watson tree with offspring distribution μ conditioned to have n vertices. Note that the case when \mathcal{T}_n is uniformly distributed over plane trees with n vertices is recovered when μ is the geometric distribution with parameter $1/2$ (see, e.g., Section 2.2 in [63]). Let $Z_{\mathcal{T}_n}$ denote the random walk with jump distribution θ indexed by \mathcal{T}_n , and let R_n be the range of $Z_{\mathcal{T}_n}$. Theorem 3.5 below shows that the convergence (3.1) holds, provided that c_θ is replaced by the constant $2^{d/2} \sigma^d \rho^{-d/2}$.

An interesting auxiliary result is an invariance principle for “local times” of our tree-indexed random walk. For every $a \in \mathbb{Z}^d$, let

$$L_n(a) = \sum_{u \in \mathcal{T}_n} \mathbf{1}_{\{Z_{\mathcal{T}_n}(u)=a\}}$$

be the number of visits of a by the tree-indexed random walk $Z_{\mathcal{T}_n}$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, set $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$. Then Theorem 3.4 shows that the process

$$\left(n^{\frac{d}{4}-1} L_n(\lfloor n^{1/4} x \rfloor) \right)_{x \in \mathbb{R}^d \setminus \{0\}}$$

converges as $n \rightarrow \infty$, in the sense of weak convergence of finite-dimensional marginals, to the density process of ISE (up to scaling constants and a linear transformation of the variable x). Notice that the latter density process exists because $d \leq 3$, by results due to Sugitani [86]. In dimension $d = 1$, this invariance principle has been obtained earlier in a stronger (functional) form by Bousquet-Mélou and Janson [15, Theorem 3.6] in a particular case, and then by Devroye and Janson [24, Theorem 1.1] in a more general setting. Such a strengthening might also be possible when $d = 2$ or 3 , but we have chosen not to investigate this question here as it is not relevant to our main applications. In dimensions 2 and 3, Lalley and Zheng [49, Theorem 1] also give a closely related result for local times of critical branching random walk in the case of a Poisson offspring distribution and for a particular choice of θ .

Our tree-indexed random walk can be viewed as a branching random walk starting with a single initial particle and conditioned to have a fixed total progeny. Therefore, it is not surprising that our main results have analogs for branching random walks, as it was already the case in dimension $d \geq 4$ (see Propositions 20 and 21 in [61]). For every integer $p \geq 1$, consider a (discrete time) branching random walk starting initially with p particles located at the origin of \mathbb{Z}^d , such that the offspring number of each particle is distributed according to μ , and each newly born particle jumps from the location of its parent according to the jump distribution θ . Let $\mathcal{V}^{[p]}$ stand for the set of all sites of \mathbb{Z}^d visited by this branching random walk. Then Theorem 3.8 shows that, similarly as in (3.1), the asymptotic distribution of $p^{-d/2} \#\mathcal{V}^{[p]}$ is the Lebesgue measure of the range of a super-Brownian motion starting from δ_0 (note again that this Lebesgue measure is positive because $d \leq 3$, see [21] or [86]). In a related direction, we mention the article of Lalley and Zheng [50], which gives estimates for the number of occupied sites *at a given time* by a critical nearest neighbor branching random walk in \mathbb{Z}^d .

Our proof of Theorem 3.8 depends on an asymptotic estimate for the hitting probability of a distant point by branching random walk, which seems to be new and of independent interest. To be specific, consider the set $\mathcal{V}^{[1]}$ of all sites visited by the branching random walk starting with a single particle at the origin. Consider for simplicity the isotropic case where $M_\theta = \sigma^2 \text{Id}$, where Id is the identity matrix. Then Theorem 3.7 shows that

$$\lim_{|a| \rightarrow \infty} |a|^2 P(a \in \mathcal{V}^{[1]}) = \frac{2(4-d)\sigma^2}{\rho^2}.$$

See Subsection 5.1 for a discussion of similar estimates in higher dimensions.

Not surprisingly, our proofs depend on the known relations between tree-indexed random walk (or branching random walk) and the Brownian snake (or super-Brownian motion). In particular, we make extensive use of a result of Janson and Marckert [39] showing that the “discrete snake” coding our tree-indexed random walk $Z_{\mathcal{T}_n}$ converges in distribution in a strong (functional) sense to the Brownian snake driven by a normalized Brownian excursion. It follows from this convergence that the set of all sites visited by the tree-indexed random walk converges in distribution (modulo a suitable rescaling) to the support of ISE, in the sense of the Hausdorff distance between compact sets. But, of course, this is not sufficient to derive asymptotics for the *number* of visited sites.

Our assumptions on μ and θ are similar to those in [39]. We have not striven for the greatest generality, and it is plausible that these assumptions can be relaxed. See, in particular, [39] for a discussion of the necessity of the existence of exponential moments for the offspring distribution μ . It might also be possible to replace our condition (3.2) on θ by a second moment assumption, but this would require different methods as the results of [39] show that the strong convergence of discrete snakes to the Brownian snake no longer holds without (3.2).

The paper is organized as follows. Section 3.2 presents our main notation and gives some preliminary results about the Brownian snake. Section 3.3 is devoted to our main result about the range of tree-indexed random walk in dimension $d \leq 3$. Section 3.4 discusses similar results for branching random walk, and Section 3.5 presents a few complements and open questions.

3.2 Preliminaries on trees and the Brownian snake

3.2.1 Finite trees

We use the standard formalism for plane trees. We set

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. If $u = (u_1, \dots, u_n) \in \mathcal{U}$, we set $|u| = n$ (in particular $|\emptyset| = 0$). We write \prec for the lexicographical order on \mathcal{U} , so that $\emptyset \prec 1 \prec (1, 1) \prec 2$ for instance.

A plane tree (or rooted ordered tree) \mathcal{T} is a finite subset of \mathcal{U} such that:

- (i) $\emptyset \in \mathcal{T}$;
- (ii) If $u = (u_1, \dots, u_n) \in \mathcal{T} \setminus \{\emptyset\}$ then $\check{u} := (u_1, \dots, u_{n-1}) \in \mathcal{T}$;
- (iii) For every $u = (u_1, \dots, u_n) \in \mathcal{T}$, there exists an integer $k_u(\mathcal{T}) \geq 0$ such that, for every $j \in \mathbb{N}$, $(u_1, \dots, u_n, j) \in \mathcal{T}$ if and only if $1 \leq j \leq k_u(\mathcal{T})$.

The notions of a descendant or of an ancestor of a vertex of \mathcal{T} are defined in an obvious way. If $u, v \in \mathcal{T}$, we will write $u \wedge v \in \mathcal{T}$ for the most recent common ancestor of u and v . We denote the set of all plane trees by \mathbb{T}_f .

Let \mathcal{T} be a tree with $p = \#\mathcal{T}$ vertices and let $\emptyset = v_0 \prec v_1 \prec \dots \prec v_{p-1}$ be the vertices of \mathcal{T} listed in lexicographical order. We define the height function $(H_i)_{0 \leq i \leq p}$ of \mathcal{T} by setting $H_i = |v_i|$ for every $0 \leq i \leq p-1$, and $H_p = 0$ by convention.

Recall that we have fixed a probability measure μ on \mathbb{Z}_+ satisfying the assumptions given in Section 1, and that $\rho^2 = \text{var } \mu$. The law of the Galton–Watson tree with offspring distribution μ is a probability measure on the space \mathbb{T}_f , which is denoted by Π_μ (see, e.g., [58, Section 1]).

We will need some information about the law of the total progeny $\#\mathcal{T}$ under Π_μ . It is well known (see, e.g., [58, Corollary 1.6]) that this law is the same as the law of the first hitting time of -1 by a random walk on \mathbb{Z} with jump distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, \dots$ started from 0. Combining this with Kemperman’s formula (see, e.g., [81, p.122]) and using a standard local limit theorem, one gets

$$\lim_{k \rightarrow \infty} k^{1/2} \Pi_\mu(\#\mathcal{T} \geq k) = \frac{2}{\rho\sqrt{2\pi}}. \quad (3.3)$$

Suppose that μ is not supported on a strict subgroup of \mathbb{Z} , so that the random walk with jump distribution ν is aperiodic. The preceding asymptotics can then be strengthened in the form

$$\lim_{k \rightarrow \infty} k^{3/2} \Pi_\mu(\#\mathcal{T} = k) = \frac{1}{\rho\sqrt{2\pi}}. \quad (3.4)$$

3.2.2 Tree-indexed random walk

A (d -dimensional) spatial tree is a pair $(\mathcal{T}, (z_u)_{u \in \mathcal{T}})$ where $\mathcal{T} \in \mathbb{T}_f$ and $z_u \in \mathbb{Z}^d$ for every $u \in \mathcal{T}$. We let \mathbb{T}_f^* be the set of all spatial trees.

Recall that θ is a probability measure on \mathbb{Z}^d satisfying the assumptions listed in the introduction. We write $\Pi_{\mu, \theta}^*$ for the probability distribution on \mathbb{T}_f^* under which \mathcal{T} is distributed according to Π_μ and, conditionally on \mathcal{T} , the ‘‘spatial locations’’ $(z_u)_{u \in \mathcal{T}}$ are distributed as random walk indexed by \mathcal{T} , with jump distribution θ , and started from 0 at the root \emptyset : This means that, under the probability measure $\Pi_{\mu, \theta}^*$, we have $z_\emptyset = 0$ a.s. and, conditionally on \mathcal{T} , the quantities $(z_u - z_{\dot{u}}, u \in \mathcal{T} \setminus \{\emptyset\})$ are independent and distributed according to θ .

3.2.3 The Brownian snake

We refer to [57] for the basic facts about the Brownian snake that we will use. The Brownian snake $(W_s)_{s \geq 0}$ is a Markov process taking values in the space \mathcal{W} of all (d -dimensional) stopped paths: Here a stopped path w is just a continuous mapping $w : [0, \zeta(w)] \rightarrow \mathbb{R}^d$, where the number $\zeta(w) \geq 0$, which depends on w , is called the lifetime of w . A stopped path w with zero lifetime will be identified with its starting point $w(0) \in \mathbb{R}^d$. The endpoint $w(\zeta(w))$ of a stopped path w is denoted by \hat{w} .

It will be convenient to argue on the canonical space $C(\mathbb{R}_+, \mathcal{W})$ of all continuous mappings from \mathbb{R}_+ into \mathcal{W} , and to let $(W_s)_{s \geq 0}$ be the canonical process on this space. We write $\zeta_s := \zeta(W_s)$ for the lifetime of W_s . If $x \in \mathbb{R}^d$, the law of the Brownian snake starting from x is the probability measure \mathbb{P}_x on $C(\mathbb{R}_+, \mathcal{W})$ that is characterized as follows:

- (i) The distribution of $(\zeta_s)_{s \geq 0}$ under \mathbb{P}_x is the law of a reflected linear Brownian motion on \mathbb{R}_+ started from 0.
- (ii) We have $W_0 = x$, \mathbb{P}_x a.s. Furthermore, under \mathbb{P}_x and conditionally on $(\zeta_s)_{s \geq 0}$, the process $(W_s)_{s \geq 0}$ is (time-inhomogeneous) Markov with transition kernels specified as follows. If $0 \leq s < s'$,
 - $W_{s'}(t) = W_s(t)$ for every $0 \leq t \leq m_\zeta(s, s') := \min\{\zeta_r : s \leq r \leq s'\}$;
 - $(W_{s'}(m_\zeta(s, s') + t) - W_{s'}(m_\zeta(s, s')))_{0 \leq t \leq \zeta_{s'} - m_\zeta(s, s')}$ is a standard Brownian motion in \mathbb{R}^d independent of W_s .

We will refer to the process $(W_s)_{s \geq 0}$ under \mathbb{P}_0 as the standard Brownian snake.

We will also be interested in (infinite) excursion measures of the Brownian snake, which we denote by \mathbb{N}_x , $x \in \mathbb{R}^d$. For every $x \in \mathbb{R}^d$, the distribution of the process $(W_s)_{s \geq 0}$ under \mathbb{N}_x is characterized by properties analogous to (i) and (ii) above, with the only difference that in (i) the law of reflected linear Brownian motion is replaced by the Itô measure of positive excursions of linear Brownian motion, normalized in such a way that $\mathbb{N}_x(\sup\{\zeta_s : s \geq 0\} > \varepsilon) = (2\varepsilon)^{-1}$, for every $\varepsilon > 0$.

We write $\gamma := \sup\{s \geq 0 : \zeta_s > 0\}$, which corresponds to the duration of the excursion under \mathbb{N}_x . A special role will be played by the probability measures $\mathbb{N}_x^{(r)} := \mathbb{N}_x(\cdot \mid \gamma = r)$, which are

defined for every $x \in \mathbb{R}^d$ and every $r > 0$. Under $\mathbb{N}_x^{(r)}$, the “lifetime process” $(\zeta_s)_{0 \leq s \leq r}$ is a Brownian excursion with duration r . From the analogous decomposition for the Itô measure of Brownian excursions, we have

$$\mathbb{N}_0 = \int_0^\infty \frac{dr}{2\sqrt{2\pi r^3}} \mathbb{N}_0^{(r)}. \quad (3.5)$$

The total occupation measure of the Brownian snake is the finite measure \mathcal{Z} on \mathbb{R}^d defined under \mathbb{N}_x , or under $\mathbb{N}_x^{(r)}$, by the formula

$$\langle \mathcal{Z}, \varphi \rangle = \int_0^\gamma ds \varphi(\widehat{W}_s),$$

for any nonnegative measurable function φ on \mathbb{R}^d .

Under $\mathbb{N}_x^{(1)}$, \mathcal{Z} is a random probability measure, which in the case $x = 0$ is called ISE for integrated super-Brownian excursion (the measure \mathcal{I} in (3.1) is thus distributed as \mathcal{Z} under $\mathbb{N}_0^{(1)}$). Note that our normalization of ISE is slightly different from the one originally proposed by Aldous [6].

The following result will be derived from known properties of super-Brownian motion via the connection between the Brownian snake and superprocesses.

Proposition 3.1. *Both \mathbb{N}_x a.e. and $\mathbb{N}_x^{(1)}$ a.s., the random measure \mathcal{Z} has a continuous density on \mathbb{R}^d , which will be denoted by $(\ell^y, y \in \mathbb{R}^d)$.*

REMARK. When $d = 1$, this result, under the measure $\mathbb{N}_0^{(1)}$, can be found in [15, Theorem 2.1].

Proof. By translation invariance, it is enough to consider the case $x = 0$. We rely on the Brownian snake construction of super-Brownian motion to deduce the statement of the proposition from Sugitani’s results [86]. Let $(W^i)_{i \in I}$ be a Poisson point measure on $C(\mathbb{R}_+, \mathcal{W})$ with intensity \mathbb{N}_0 . With every $i \in I$, we associate the occupation measure \mathcal{Z}^i of W^i . Then Theorem IV.4 in [57] shows that there exists a super-Brownian motion $(X_t)_{t \geq 0}$ with branching mechanism $\psi(u) = 2u^2$ and initial value $X_0 = \delta_0$, such that

$$\int_0^\infty dt X_t = \sum_{i \in I} \mathcal{Z}^i.$$

As a consequence of [86, Theorems 2 and 3], the random measure $\int_0^\infty dt X_t$ has a.s. a continuous density on $\mathbb{R}^d \setminus \{0\}$. On the other hand, let $B(0, \varepsilon)$ denote the closed ball of radius ε centered at 0 in \mathbb{R}^d . Then, for every $\varepsilon > 0$, the event

$$\mathcal{A}_\varepsilon := \{\#\{i \in I: \mathcal{Z}^i(B(0, \varepsilon)^c) > 0\} = 1\}$$

has positive probability (see, e.g., [57, Proposition V.9]). On the event \mathcal{A}_ε , write i_0 for the unique index in I such that $\mathcal{Z}^{i_0}(B(0, \varepsilon)^c) > 0$. Then, still on the event \mathcal{A}_ε , the measures $\int_0^\infty dt X_t$ and \mathcal{Z}^{i_0} coincide on $B(0, \varepsilon)^c$. The conditional distribution of W^{i_0} knowing \mathcal{A}_ε is $\mathbb{N}_0(\cdot \mid \mathcal{Z}(B(0, \varepsilon)^c) > 0)$, and we conclude that \mathcal{Z} has a continuous density on $B(0, \varepsilon)^c$, $\mathbb{N}_0(\cdot \mid \mathcal{Z}(B(0, \varepsilon)^c) > 0)$ a.s. As this holds for any $\varepsilon > 0$, we obtain that, \mathbb{N}_0 a.e., the random measure \mathcal{Z} has a continuous density on $\mathbb{R}^d \setminus \{0\}$. Via a scaling argument, the same property holds $\mathbb{N}_0^{(1)}$ a.s. This argument does not exclude the possibility that \mathcal{Z} might have a singularity at 0, but we can use the rerooting invariance property (see [6, Section 3.2] or [65, Section 2.3]) to complete the proof. According to

this property, if under the measure $\mathbb{N}_0^{(1)}$ we pick a random point distributed according to \mathcal{Z} and then shift \mathcal{Z} so that this random point becomes the origin of \mathbb{R}^d , the resulting random measure has the same distribution as \mathcal{Z} . Consequently, we obtain that $\mathbb{N}_0^{(1)}$ a.s., $\mathcal{Z}(dx)$ a.e., the measure \mathcal{Z} has a continuous density on $\mathbb{R}^d \setminus \{x\}$. It easily follows that \mathcal{Z} has a continuous density on \mathbb{R}^d , $\mathbb{N}_0^{(1)}$ a.s., and by scaling again the same property holds under \mathbb{N}_0 . \square

Let us introduce the random closed set

$$\mathcal{R} := \{\widehat{W}_s : 0 \leq s \leq \gamma\}.$$

Note that, by construction, \mathcal{Z} is supported on \mathcal{R} , and it follows that, for every $y \in \mathbb{R}^d \setminus \{x\}$,

$$\{\ell^y > 0\} \subset \{y \in \mathcal{R}\}, \quad \mathbb{N}_x \text{ a.e. or } \mathbb{N}_x^{(1)} \text{ a.s.} \quad (3.6)$$

Proposition 3.2. *For every $y \in \mathbb{R}^d \setminus \{x\}$,*

$$\{\ell^y > 0\} = \{y \in \mathcal{R}\}, \quad \mathbb{N}_x \text{ a.e. and } \mathbb{N}_x^{(1)} \text{ a.s.}$$

Proof. Fix $y \in \mathbb{R}^d$, and consider the function $u(x) := \mathbb{N}_x(\ell^y > 0)$, for every $x \in \mathbb{R}^d \setminus \{y\}$. By simple scaling and rotational invariance arguments (see the proof of Proposition V.9 (i) in [57] for a similar argument), we have

$$u(x) = C_d |x - y|^{-2}$$

with a certain constant $C_d > 0$ depending only on d . On the other hand, an easy application of the special Markov property [56] shows that, for every $r > 0$, and every $x \in B(y, r)^c$, we have

$$u(x) = \mathbb{N}_x \left[1 - \exp \left(- \int X^{B(y, r)^c} (dz) u(z) \right) \right]$$

where $X^{B(y, r)^c}$ stands for the exit measure of the Brownian snake from the open set $B(y, r)^c$. Theorem V.4 in [57] now shows that the function u must solve the partial differential equation $\Delta u = 4u^2$ in $\mathbb{R}^d \setminus \{y\}$. It easily follows that $C_d = 2 - d/2$.

The preceding line of reasoning also applies to the function $v(x) := \mathbb{N}_x(y \in \mathcal{R})$ (see [57, page 91]), and shows that we have $v(x) = (2 - d/2)|x - y|^{-2} = u(x)$ for every $x \in \mathbb{R}^d \setminus \{y\}$ – note that this formula for v can also be derived from [21, Theorem 1.3] and the connection between the Brownian snake and super-Brownian motion. Recalling (3.6), this is enough to conclude that

$$\{\ell^y > 0\} = \{y \in \mathcal{R}\}, \quad \mathbb{N}_x \text{ a.e.} \quad (3.7)$$

for every $x \in \mathbb{R}^d \setminus \{y\}$.

We now want to obtain that the equality in (3.7) also holds $\mathbb{N}_x^{(1)}$ a.s. Note that, for every fixed x , we could use a scaling argument to get that $\{\ell^y > 0\} = \{y \in \mathcal{R}\}$, $\mathbb{N}_x^{(1)}$ a.s., for λ_d a.e. $y \in \mathbb{R}^d$, where we recall that λ_d stands for Lebesgue measure on \mathbb{R}^d . In order to get the more precise assertion of the proposition, we use a different method.

By translation invariance, we may assume that $x = 0$ and we fix $y \in \mathbb{R}^d \setminus \{0\}$. We set $T_y := \inf\{s \geq 0 : \widehat{W}_s = y\}$. Also, for every $s > 0$, we set

$$\tilde{\ell}_s^y := \liminf_{\varepsilon \rightarrow 0} (\lambda_d(B(y, \varepsilon)))^{-1} \int_0^s dr \mathbf{1}_{\{|\widehat{W}_{r-y}| \leq \varepsilon\}},$$

Note that $\tilde{\ell}_\gamma^y = \ell^y$, \mathbb{N}_0 a.e. and $\mathbb{N}_0^{(1)}$ a.e. We then claim that, for every $s > 0$,

$$\{T_y \leq s\} = \{\tilde{\ell}_s^y > 0\}, \quad \mathbb{N}_0 \text{ a.e.} \quad (3.8)$$

The inclusion $\{\tilde{\ell}_s^y > 0\} \subset \{T_y \leq s\}$ is obvious. In order to prove the reverse inclusion, we argue by contradiction and assume that

$$\mathbb{N}_0(T_y \leq s, \tilde{\ell}_s^y = 0) > 0.$$

Note that $\mathbb{N}_0(T_y = s) = 0$ (because $\mathbb{N}_0(\widehat{W}_s = y) = 0$), and so we have also $\mathbb{N}_0(T_y < s, \tilde{\ell}_s^y = 0) > 0$. For every $\eta > 0$, let

$$T_y^{(\eta)} := \inf \{r \geq T_y : \zeta_r \leq (\zeta_{T_y} - \eta)^+\}.$$

Notice that, by the properties of the Brownian snake, the path $W_{T_y^{(\eta)}}$ is just W_{T_y} stopped at time $(\zeta_{T_y} - \eta)^+$.

From the strong Markov property at time T_y , we easily get that $T_y^{(\eta)} \downarrow T_y$ as $\eta \downarrow 0$, \mathbb{N}_0 a.e. on $\{T_y < \infty\}$. Hence, on the event $\{T_y < s\}$, we have also $T_y^{(\eta)} < s$ for η small enough, \mathbb{N}_0 a.e. Therefore, we can find $\eta > 0$ such that

$$\mathbb{N}_0(T_y < s, \tilde{\ell}_{T_y^{(\eta)}}^y = 0) > 0.$$

However, using the strong Markov property at time $T_y^{(\eta)}$, and Lemma V.5 and Proposition V.9(i) in [57], we immediately see that, conditionally on the past up to time $T_y^{(\eta)}$, the event $\{\widehat{W}_r \neq y, \forall r \geq T_y^{(\eta)}\}$ occurs with positive probability. Hence, we get

$$\mathbb{N}_0(T_y < s, \tilde{\ell}_\gamma^y = 0) > 0.$$

Since $\tilde{\ell}_\gamma^y = \ell^y$, this contradicts (3.7), and this contradiction completes the proof of our claim (3.8).

Finally, we observe that, for every $s \in (0, 1)$, the law of $(W_r)_{0 \leq r \leq s}$ under $\mathbb{N}_0^{(1)}$ is absolutely continuous with respect to the law of the same process under \mathbb{N}_0 (this is a straightforward consequence of the similar property for the Itô excursion measure and the law of the normalized Brownian excursion, see, e.g., [82, Chapter XII]). Hence, (3.8) also gives, for every $s \in (0, 1)$,

$$\{T_y \leq s\} = \{\tilde{\ell}_s^y > 0\}, \quad \mathbb{N}_0^{(1)} \text{ a.s.},$$

and the fact that the equality in (3.7) also holds $\mathbb{N}_0^{(1)}$ a.s. readily follows. \square

3.3 Asymptotics for the range of tree-indexed random walk

Throughout this section, we consider only integers $n \geq 1$ such that $\Pi_\mu(\#\mathcal{T} = n) > 0$ (and when we let $n \rightarrow \infty$, we mean along such values). For every such integer n , let $(\mathcal{T}_n, (Z^n(u))_{u \in \mathcal{T}_n})$ be distributed according to $\Pi_{\mu, \theta}^*(\cdot | \#\mathcal{T} = n)$. Then \mathcal{T}_n is a Galton–Watson tree with offspring distribution μ conditioned to have n vertices, and conditionally on \mathcal{T}_n , $(Z^n(u))_{u \in \mathcal{T}_n}$ is a random walk with jump distribution θ indexed by \mathcal{T}_n .

We set, for every $t > 0$ and $x \in \mathbb{R}^d$,

$$p_t(x) := \frac{1}{(2\pi t)^{d/2} \sqrt{\det M_\theta}} \exp\left(-\frac{x \cdot M_\theta^{-1} x}{2t}\right),$$

where $x \cdot y$ stands for the usual scalar product in \mathbb{R}^d .

For every $a \in \mathbb{Z}^d$, we also set

$$L_n(a) := \sum_{u \in \mathcal{T}_n} \mathbf{1}_{\{Z^n(u)=a\}}.$$

Lemma 3.3. *For every $\varepsilon > 0$, there exists a constant C_ε such that, for every n and every $b \in \mathbb{Z}^d$ with $|b| \geq \varepsilon n^{1/4}$,*

$$E[(L_n(b))^2] \leq C_\varepsilon n^{2-\frac{d}{2}}.$$

Furthermore, for every $x, y \in \mathbb{R}^d \setminus \{0\}$, and for every choice of the sequences (x_n) and (y_n) in \mathbb{Z}^d such that $n^{-1/4}x_n \rightarrow x$ and $n^{-1/4}y_n \rightarrow y$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} n^{\frac{d}{2}-2} E[L_n(x_n) L_n(y_n)] = \varphi(x, y),$$

where

$$\varphi(x, y) := \rho^4 \int_{(\mathbb{R}_+)^3} dr_1 dr_2 dr_3 (r_1 + r_2 + r_3) e^{-\rho^2(r_1+r_2+r_3)^2/2} \int_{\mathbb{R}^d} dz p_{r_1}(z) p_{r_2}(x-z) p_{r_3}(y-z).$$

The function φ is continuous on $(\mathbb{R}^d \setminus \{0\})^2$.

REMARK. The function φ is in fact continuous on $(\mathbb{R}^d)^2$. See the appendix at the end of the chapter for a proof.

Proof. We first establish the second assertion of the lemma. We let $u_0^n, u_1^n, \dots, u_{n-1}^n$ be the vertices of \mathcal{T}_n listed in lexicographical order. By definition,

$$L_n(x_n) = \sum_{i=0}^{n-1} \mathbf{1}_{\{Z^n(u_i^n)=x_n\}},$$

so that

$$E[L_n(x_n) L_n(y_n)] = E \left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \mathbf{1}_{\{Z^n(u_i^n)=x_n, Z^n(u_j^n)=y_n\}} \right].$$

Let H^n be the height function of the tree \mathcal{T}_n , so that $H_i^n = |u_i^n|$ for every $i \in \{0, 1, \dots, n-1\}$. If $i, j \in \{0, 1, \dots, n-1\}$, we also use the notation $\check{H}_{i,j}^n = |u_i^n \wedge u_j^n|$ for the generation of the most recent common ancestor to u_i^n and u_j^n , and note that

$$\left| \check{H}_{i,j}^n - \min_{i \wedge j \leq k \leq i \vee j} H_k^n \right| \leq 1. \quad (3.9)$$

Write $\pi_k = \theta^{*k}$ for the transition kernels of the random walk with jump distribution θ . By conditioning with respect to the tree \mathcal{T}_n , we get

$$\begin{aligned} E[L_n(x_n) L_n(y_n)] &= E \left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{a \in \mathbb{Z}^d} \pi_{\check{H}_{i,j}^n}^n(a) \pi_{H_i^n - \check{H}_{i,j}^n}^n(x_n - a) \pi_{H_j^n - \check{H}_{i,j}^n}^n(y_n - a) \right] \\ &= n^2 E \left[\int_0^1 \int_0^1 ds dt \Phi_{x_n, y_n}^n(H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n) \right], \end{aligned} \quad (3.10)$$

where we have set, for every integers $k, \ell, m \geq 0$ such that $k \wedge \ell \geq m$,

$$\Phi_{x_n, y_n}^n(k, \ell, m) := \sum_{a \in \mathbb{Z}^d} \pi_m(a) \pi_{k-m}(x_n - a) \pi_{\ell-m}(y_n - a).$$

In the remaining part of the proof, we assume that θ is aperiodic (meaning that the subgroup generated by $\{k \geq 0: \pi_k(0) > 0\}$ is \mathbb{Z}). Only minor modifications are needed to treat the general case. We can then use the local limit theorem, in a form that can be obtained by combining Theorems 2.3.9 and 2.3.10 in [53]. There exists a sequence δ_n converging to 0 such that, for every $n \geq 1$,

$$\sup_{a \in \mathbb{Z}^d} \left(\left(1 + \frac{|a|^2}{n}\right) n^{d/2} |\pi_n(a) - p_n(a)| \right) \leq \delta_n. \quad (3.11)$$

Let $(k_n), (\ell_n), (m_n)$ be three sequences of positive integers such that $n^{-1/2}k_n \rightarrow u$, $n^{-1/2}\ell_n \rightarrow v$ and $n^{-1/2}m_n \rightarrow w$, where $0 < w < u \wedge v$. Write

$$n^{d/2} \Phi_{x_n, y_n}^n(k_n, \ell_n, m_n) = n^{3d/4} \int_{\mathbb{R}^d} dz \pi_{m_n}(\lfloor zn^{1/4} \rfloor) \pi_{k_n - m_n}(x_n - \lfloor zn^{1/4} \rfloor) \pi_{\ell_n - m_n}(y_n - \lfloor zn^{1/4} \rfloor),$$

and note that, for every fixed $z \in \mathbb{R}^d$,

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{d/4} \pi_{m_n}(\lfloor zn^{1/4} \rfloor) &= p_w(z), \\ \lim_{n \rightarrow \infty} n^{d/4} \pi_{k_n - m_n}(x_n - \lfloor zn^{1/4} \rfloor) &= p_{u-w}(x - z), \\ \lim_{n \rightarrow \infty} n^{d/4} \pi_{\ell_n - m_n}(y_n - \lfloor zn^{1/4} \rfloor) &= p_{v-w}(y - z), \end{aligned}$$

by (3.11). These convergences even hold uniformly in z . It then follows that

$$\lim_{n \rightarrow \infty} n^{d/2} \Phi_{x_n, y_n}^n(k_n, \ell_n, m_n) = \int_{\mathbb{R}^d} dz p_w(z) p_{u-w}(x - z) p_{v-w}(y - z) =: \Psi_{x,y}(u, v, w). \quad (3.12)$$

Indeed, using (3.11) again, we have, for every $K > 2(|x| \vee |y|) + 2$ and every sufficiently large n ,

$$\begin{aligned} & n^{3d/4} \int_{\{|z| \geq K+1\}} dz \pi_{m_n}(\lfloor zn^{1/4} \rfloor) \pi_{k_n - m_n}(x_n - \lfloor zn^{1/4} \rfloor) \pi_{\ell_n - m_n}(y_n - \lfloor zn^{1/4} \rfloor) \\ & \leq C \int_{\{|z| \geq K+1\}} dz \left(\frac{1}{(|z| - 1)^2} \right)^3, \end{aligned}$$

with a constant C independent of n and K . The right-hand side of the last display tends to 0 as K tends to infinity. Together with the previously mentioned uniform convergence, this suffices to justify (3.12).

By [58, Theorem 1.15], we have

$$\left(\frac{\rho}{2} n^{-1/2} H_{[nt]}^n \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t)_{0 \leq t \leq 1},$$

where $(\mathbf{e}_t)_{0 \leq t \leq 1}$ is a normalized Brownian excursion, and we recall that ρ^2 is the variance of μ . The latter convergence holds in the sense of the weak convergence of laws on the Skorokhod space $\mathbb{D}([0, 1], \mathbb{R}_+)$ of càdlàg functions from $[0, 1]$ into \mathbb{R}_+ . Using the Skorokhod representation theorem, we may and will assume that this convergence holds almost surely, uniformly in $t \in [0, 1]$. Recalling (3.9), it follows that we have also

$$\frac{\rho}{2} n^{-1/2} \check{H}_{[ns], [nt]}^n \xrightarrow[n \rightarrow \infty]{} \min_{s \wedge t \leq r \leq s \vee t} \mathbf{e}_r =: m_{\mathbf{e}}(s, t),$$

uniformly in $s, t \in [0, 1]$, a.s.

As a consequence of (3.12) and the preceding observations, we have, for every $s, t \in (0, 1)$ with $s \neq t$,

$$\lim_{n \rightarrow \infty} n^{d/2} \Phi_{x_n, y_n}^n (H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n) = \Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right), \quad \text{a.s.} \quad (3.13)$$

We claim that we can deduce from (3.10) and (3.13) that

$$\lim_{n \rightarrow \infty} n^{\frac{d}{2}-2} E[L_n(x_n) L_n(y_n)] = E \left[\int_0^1 \int_0^1 ds dt \Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right) \right]. \quad (3.14)$$

Note that the right-hand side of (3.14) coincides with the function $\varphi(x, y)$ in the lemma. To see this, we can use Theorem III.6 of [57] to verify that the joint density of the triple

$$(m_{\mathbf{e}}(s, t), \mathbf{e}_s - m_{\mathbf{e}}(s, t), \mathbf{e}_t - m_{\mathbf{e}}(s, t))$$

when s and t are chosen uniformly over $[0, 1]$, independently and independently of \mathbf{e} , is

$$16(r_1 + r_2 + r_3) \exp(-2(r_1 + r_2 + r_3)^2).$$

So the proof of the second assertion will be complete if we can justify (3.14). By Fatou's lemma, (3.10) and (3.13), we have first

$$\liminf_{n \rightarrow \infty} n^{\frac{d}{2}-2} E[L_n(x_n) L_n(y_n)] \geq E \left[\int_0^1 \int_0^1 ds dt \Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right) \right].$$

Furthermore, dominated convergence shows that, for every $K > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} E \left[\int_0^1 \int_0^1 ds dt \left(n^{d/2} \Phi_{x_n, y_n}^n (H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n) \wedge K \right) \right] \\ &= E \left[\int_0^1 \int_0^1 ds dt \left(\Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right) \wedge K \right) \right]. \end{aligned}$$

Write $\Gamma_n(s, t) = n^{d/2} \Phi_{x_n, y_n}^n (H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n)$ to simplify notation. In view of the preceding comments, it will be enough to verify that

$$\lim_{K \rightarrow \infty} \left(\limsup_{n \rightarrow \infty} E \left[\int_0^1 \int_0^1 ds dt \Gamma_n(s, t) \mathbf{1}_{\{\Gamma_n(s, t) > K\}} \right] \right) = 0. \quad (3.15)$$

To this end, we will make use of the bound

$$\sup_{k \geq 0} \pi_k(x) \leq M (|x|^{-d} \wedge 1), \quad (3.16)$$

which holds for every $x \in \mathbb{Z}^d$ with a constant M independent of x . This bound can be obtained easily by combining (3.11) and Proposition 2.4.6 in [53]. Then let $k, \ell, m \geq 0$ be integers such that $k \wedge \ell \geq m$, and recall that

$$\Phi_{x_n, y_n}^n(k, \ell, m) = \sum_{a \in \mathbb{Z}^d} \pi_m(a) \pi_{k-m}(x_n - a) \pi_{\ell-m}(y_n - a).$$

Fix $\varepsilon > 0$ such that $|x| \wedge |y| > 2\varepsilon$. Consider first the contribution to the sum in the right-hand side coming from values of a such that $|a| \leq \varepsilon n^{1/4}$. For such values of a (and assuming that n is large enough), the estimate (3.16) allows us to bound both $\pi_{k-m}(x_n - a)$ and $\pi_{\ell-m}(y_n - a)$ by $M\varepsilon^{-d}n^{-d/4}$. On the other hand, if $|a| \geq \varepsilon n^{1/4}$, we can bound $\pi_m(a)$ by $M\varepsilon^{-d}n^{-d/4}$, whereas (3.11) shows that the sum

$$\sum_{|a| \geq \varepsilon n^{1/4}} \pi_{k-m}(x_n - a) \pi_{\ell-m}(y_n - a)$$

is bounded above by $c_1((k-m)^{-d/2} \wedge (\ell-m)^{-d/2} \wedge 1)$ for some constant c_1 . Summarizing, we get the bound

$$\begin{aligned} \Phi_{x_n, y_n}^n(k, \ell, m) &\leq M^2 \varepsilon^{-2d} n^{-d/2} + c_1 M \varepsilon^{-d} n^{-d/4} ((k-m)^{-d/2} \wedge (\ell-m)^{-d/2} \wedge 1) \\ &\leq c_{1,\varepsilon} n^{-d/2} + c_{2,\varepsilon} n^{-d/4} ((k+\ell-2m)^{-d/2} \wedge 1), \end{aligned}$$

where $c_{1,\varepsilon}$ and $c_{2,\varepsilon}$ are constants that do not depend on n, k, ℓ, m . Then observe that, for every $s, t \in (0, 1)$,

$$H_{[ns]}^n + H_{[nt]}^n - 2\check{H}_{[ns],[nt]}^n = d_n(u_{[ns]}^n, u_{[nt]}^n),$$

where d_n denotes the usual graph distance on \mathcal{T}_n . From the preceding bound, we thus get

$$\Gamma_n(s, t) \leq c_{1,\varepsilon} + c_{2,\varepsilon} n^{d/4} (d_n(u_{[ns]}^n, u_{[nt]}^n)^{-d/2} \wedge 1).$$

It follows that, for every $K > 0$,

$$\begin{aligned} &\int_0^1 \int_0^1 ds dt \Gamma_n(s, t) \mathbf{1}_{\{\Gamma_n(s,t) > c_{1,\varepsilon} + c_{2,\varepsilon} K\}} \\ &\leq \int_0^1 \int_0^1 ds dt (c_{1,\varepsilon} + c_{2,\varepsilon} n^{d/4} (d_n(u_{[ns]}^n, u_{[nt]}^n)^{-d/2} \wedge 1)) \mathbf{1}_{\{n^{d/4} d_n(u_{[ns]}^n, u_{[nt]}^n)^{-d/2} > K\}} \\ &= n^{-2} \sum_{u,v \in \mathcal{T}_n} (c_{1,\varepsilon} + c_{2,\varepsilon} n^{d/4} (d_n(u, v)^{-d/2} \wedge 1)) \mathbf{1}_{\{d_n(u,v) < K^{-2/d} n^{1/2}\}}. \end{aligned}$$

By an estimate found in Theorem 1.3 of [24], there exists a constant c_0 that only depends on μ , such that, for every integer $k \geq 1$,

$$E[\#\{(u, v) \in \mathcal{T}_n \times \mathcal{T}_n : d_n(u, v) = k\}] \leq c_0 k n. \quad (3.17)$$

It then follows that

$$\begin{aligned} &E \left[\int_0^1 \int_0^1 ds dt \Gamma_n(s, t) \mathbf{1}_{\{\Gamma_n(s,t) > c_{1,\varepsilon} + c_{2,\varepsilon} K\}} \right] \\ &\leq n^{-1} (c_{1,\varepsilon} + c_{2,\varepsilon} n^{d/4}) + c_0 n^{-1} \sum_{k=1}^{\lfloor K^{-2/d} n^{1/2} \rfloor} k (c_{1,\varepsilon} + c_{2,\varepsilon} n^{d/4} k^{-d/2}). \end{aligned}$$

It is now elementary to verify that the right-hand side of the preceding display has a limit $g(K)$ when $n \rightarrow \infty$, and that $g(K)$ tends to 0 as $K \rightarrow \infty$ (note that we use the fact that $d \leq 3$). This completes the proof of (3.15) and of the second assertion of the lemma.

The proof of the first assertion is similar and easier. We first note that

$$E[L_n(b)^2] = E \left[\sum_{u,v \in \mathcal{T}_n} \Phi_{b,b}^n(|u|, |v|, |u \wedge v|) \right]$$

where the function $\Phi_{b,b}^n$ is defined as above. Then, assuming that $|b| \geq 2\varepsilon n^{1/4}$, the same arguments as in the first part of the proof give the bound

$$\Phi_{b,b}^n(|u|, |v|, |u \wedge v|) \leq c_{1,\varepsilon} n^{-d/2} + c_{2,\varepsilon} n^{-d/4} (d_n(u, v)^{-d/2} \wedge 1).$$

By summing over all choices of u and v , it follows that

$$\begin{aligned} E[L_n(b)^2] &\leq c_{1,\varepsilon} n^{2-d/2} + c_{2,\varepsilon} n^{-d/4} \left(n + E \left[\sum_{u,v \in \mathcal{T}_n, 1 \leq d_n(u,v) \leq \sqrt{n}} d_n(u, v)^{-d/2} \right] + n^2 \times n^{-d/4} \right) \\ &\leq (c_{1,\varepsilon} + 2c_{2,\varepsilon}) n^{2-d/2} + c_{2,\varepsilon} n^{-d/4} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k^{-d/2} E[\#\{(u, v) \in \mathcal{T}_n : d_n(u, v) = k\}], \end{aligned}$$

and the bound stated in the first assertion easily follows from (3.17).

Let us finally establish the continuity of φ . We fix $\varepsilon > 0$ and verify that φ is continuous on the set $\{|x| \geq 2\varepsilon, |y| \geq 2\varepsilon\}$. We split the integral in dz in two parts:

- The integral over $|z| \leq \varepsilon$. Write $\varphi_{1,\varepsilon}(x, y)$ for the contribution of this integral. We observe that, if $|z| \leq \varepsilon$, the function $x \mapsto p_{r_2}(x - z)$ is Lipschitz uniformly in z and in r_2 on the set $\{|x| \geq 2\varepsilon\}$, and a similar property holds for the function $y \mapsto p_{r_3}(y - z)$. It follows that $\varphi_{1,\varepsilon}$ is a Lipschitz function of (x, y) on the set $\{|x| \geq 2\varepsilon, |y| \geq 2\varepsilon\}$.
- The integral over $|z| > \varepsilon$. Write $\varphi_{2,\varepsilon}(x, y)$ for the contribution of this integral. Note that if $(u_n, v_n)_{n \geq 1}$ is a sequence in $\mathbb{R}^d \times \mathbb{R}^d$ such that $|u_n| \wedge |v_n| \geq 2\varepsilon$ for every n , and (u_n, v_n) converges to (x, y) as $n \rightarrow \infty$, we have, for every fixed $r_1, r_2, r_3 > 0$,

$$\int_{\{|z| > \varepsilon\}} dz p_{r_1}(z) p_{r_2}(u_n - z) p_{r_3}(v_n - z) \xrightarrow{n \rightarrow \infty} \int_{\{|z| > \varepsilon\}} dz p_{r_1}(z) p_{r_2}(x - z) p_{r_3}(y - z).$$

We can then use dominated convergence, since there exist constants c_ε and \tilde{c}_ε that depend only on ε , such that

$$\int_{\{|z| > \varepsilon\}} dz p_{r_1}(z) p_{r_2}(u_n - z) p_{r_3}(v_n - z) \leq c_\varepsilon p_{r_2+r_3}(u_n - v_n) \leq \tilde{c}_\varepsilon (r_2 + r_3)^{-d/2},$$

and the right-hand side is integrable for the measure $(r_1+r_2+r_3)e^{-\rho^2(r_1+r_2+r_3)^2/2} dr_1 dr_2 dr_3$. It follows that $\varphi_{2,\varepsilon}$ is also continuous on the set $\{|x| \geq 2\varepsilon, |y| \geq 2\varepsilon\}$.

The preceding considerations complete the proof. \square

In what follows, we use the notation $W^{(1)} = (W_s^{(1)})_{0 \leq s \leq 1}$ for a process distributed according to $\mathbb{N}_0^{(1)}$. We recall a result of Janson and Marckert [39] that will play an important role below. As in the proof of Lemma 3.3, we let $u_0^n, u_1^n, \dots, u_{n-1}^n$ be the vertices of \mathcal{T}_n listed in lexicographical order. For every $j \in \{0, 1, \dots, n-1\}$ write $Z_j^n = Z^n(u_j^n)$ for the spatial location of u_j^n , and $Z_n^n = 0$ by convention. Recalling our assumption (3.2), we get from [39, Theorem 2] that

$$\left(\sqrt{\frac{\rho}{2}} n^{-1/4} Z_{[nt]}^n \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (M_\theta^{1/2} \widehat{W}_t^{(1)})_{0 \leq t \leq 1}, \quad (3.18)$$

where as usual $M_\theta^{1/2}$ is the unique positive definite symmetric matrix such that $M_\theta = (M_\theta^{1/2})^2$, and the convergence holds in distribution in the Skorokhod space $\mathbb{D}([0, 1], \mathbb{R}^d)$. Note that there

are two minor differences between [39] and the present setting. First, [39] considers one-dimensional labels, whereas our spatial locations take values in \mathbb{Z}^d . However, we can simply project $Z_n(u)$ on the coordinate axes to get tightness in the convergence (3.18) from the results of [39], and convergence of finite-dimensional marginals is easy just as in [39, Proof of Theorem 1]. Second, the “discrete snake” of [39] lists the labels encountered when exploring the tree \mathcal{T}_n in depth first traversal (or contour order), whereas we are here enumerating the vertices in lexicographical order. Nevertheless, the very same arguments that are used to relate the contour process and the height function of a random tree (see [75] or [58, Section 1.6]) show that asymptotics for the discrete snakes of [39] imply similar asymptotics for the labels listed in lexicographical order of vertices.

In the next theorem, the notation $(l^x, x \in \mathbb{R}^d)$ stands for the collection of local times of $W^{(1)}$, which are defined as the continuous density of the occupation measure of $W^{(1)}$ as in Proposition 3.1. We define a constant $c > 0$ by setting

$$c := \frac{1}{\sigma} \sqrt{\frac{\rho}{2}}, \quad (3.19)$$

where $\sigma^2 = (\det M_\theta)^{1/d}$ as previously. We also use the notation $M_\theta^{-1/2} = (M_\theta^{1/2})^{-1}$.

Theorem 3.4. *Let $x^1, \dots, x^p \in \mathbb{R}^d \setminus \{0\}$, and let $(x_n^1), \dots, (x_n^p)$ be sequences in \mathbb{Z}^d such that $\sqrt{\frac{\rho}{2}} n^{-1/4} M_\theta^{-1/2} x_n^j \rightarrow x^j$ as $n \rightarrow \infty$, for every $1 \leq j \leq p$. Then,*

$$(n^{\frac{d}{4}-1} L_n(x_n^1), \dots, n^{\frac{d}{4}-1} L_n(x_n^p)) \xrightarrow[n \rightarrow \infty]{(d)} (c^d l^{x^1}, \dots, c^d l^{x^p}),$$

where the constant c is given by (3.19).

REMARKS. (i) As mentioned in the introduction, this result should be compared with Theorem 1 in [49], which deals with local times of branching random walk in \mathbb{Z}^d for $d = 2$ or 3 . See also [15, Theorem 3.6] and [24, Theorem 1.1] for stronger versions of the convergence in Theorem 3.4 when $d = 1$.

(ii) It is likely that the result of Lemma 3.3 still holds when $x = 0$ or $y = 0$, and then the condition $x^i \neq 0$ in the preceding theorem could be removed, using also the remark after Lemma 3.3. Proving this reinforcement of Lemma 3.3 would however require additional technicalities. Since this extension is not needed in the proof of our main results, we will not address this problem here.

Proof. To simplify the presentation, we give the details of the proof only in the isotropic case where $M_\theta = \sigma^2 \text{Id}$ (the non-isotropic case is treated in exactly the same manner at the cost of a somewhat heavier notation). Our condition on the sequences (x_n^j) then just says that $c n^{-1/4} x_n^j \rightarrow x^j$ as $n \rightarrow \infty$.

By the Skorokhod representation theorem, we may and will assume that the convergence (3.18) holds a.s. To obtain the result of the theorem, it is then enough to verify that, if $x \in \mathbb{R}^d \setminus \{0\}$ and (x_n) is a sequence in \mathbb{Z}^d such that $c n^{-1/4} x_n \rightarrow x$ as $n \rightarrow \infty$, we have

$$n^{\frac{d}{4}-1} L_n(x_n) \xrightarrow[n \rightarrow \infty]{(P)} c^d l^x. \quad (3.20)$$

To this end, fix x and the sequence (x_n) , and for every $\varepsilon \in (0, |x|)$, let g_ε be a nonnegative continuous function on \mathbb{R}^d , with compact support contained in the open ball of radius ε centered at x , and such that

$$\int_{\mathbb{R}^d} g_\varepsilon(y) dy = 1.$$

It follows from (3.18) (which we assume to hold a.s.) that, for every fixed $\varepsilon \in (0, |x|)$,

$$\int_0^1 g_\varepsilon(c n^{-1/4} Z_{[nt]}^n) dt \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \int_0^1 g_\varepsilon(\widehat{W}_t^{(1)}) dt.$$

Furthermore,

$$\int_0^1 g_\varepsilon(\widehat{W}_t^{(1)}) dt = \int_{\mathbb{R}^d} g_\varepsilon(y) l^y dy \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} l^x,$$

by the continuity of local times. Let $\delta > 0$. By combining the last two convergences, we can find $\varepsilon_1 \in (0, |x|)$ such that, for every $\varepsilon \in (0, \varepsilon_1)$, there exists an integer $n_1(\varepsilon)$ so that for every $n \geq n_1(\varepsilon)$,

$$P\left(\left|\int_0^1 g_\varepsilon(c n^{-1/4} Z_{[nt]}^n) dt - l^x\right| > \delta\right) < \delta. \quad (3.21)$$

However, we have

$$\begin{aligned} \int_0^1 g_\varepsilon(c n^{-1/4} Z_{[nt]}^n) dt &= \frac{1}{n} \sum_{a \in \mathbb{Z}^d} g_\varepsilon(c n^{-1/4} a) L_n(a) \\ &= n^{\frac{d}{4}-1} \int_{\mathbb{R}^d} g_\varepsilon(c n^{-1/4} \lfloor n^{1/4} y \rfloor) L_n(\lfloor n^{1/4} y \rfloor) dy. \end{aligned}$$

Set

$$\eta_n(\varepsilon) := \int_{\mathbb{R}^d} g_\varepsilon(c n^{-1/4} \lfloor n^{1/4} y \rfloor) dy$$

and note that

$$\eta_n(\varepsilon) \xrightarrow[n \rightarrow \infty]{} \int_{\mathbb{R}^d} g_\varepsilon(cy) dy = c^{-d}.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} &E\left[\left(\int_0^1 g_\varepsilon(c n^{-1/4} Z_{[nt]}^n) dt - \eta_n(\varepsilon) n^{\frac{d}{4}-1} L_n(x_n)\right)^2\right] \\ &= E\left[\left(n^{\frac{d}{4}-1} \int_{\mathbb{R}^d} g_\varepsilon(c n^{-1/4} \lfloor n^{1/4} y \rfloor) (L_n(\lfloor n^{1/4} y \rfloor) - L_n(x_n)) dy\right)^2\right] \\ &\leq \eta_n(\varepsilon) \times n^{\frac{d}{2}-2} \int_{\mathbb{R}^d} dy g_\varepsilon(c n^{-1/4} \lfloor n^{1/4} y \rfloor) E[(L_n(\lfloor n^{1/4} y \rfloor) - L_n(x_n))^2]. \end{aligned}$$

Using the first assertion of Lemma 3.3, one easily gets that, for every fixed $\varepsilon \in (0, |x|)$,

$$n^{\frac{d}{2}-2} \int_{\mathbb{R}^d} dy |g_\varepsilon(c n^{-1/4} \lfloor n^{1/4} y \rfloor) - g_\varepsilon(cy)| E[(L_n(\lfloor n^{1/4} y \rfloor) - L_n(x_n))^2] \xrightarrow[n \rightarrow \infty]{} 0.$$

On the other hand, by the second assertion of the lemma,

$$n^{\frac{d}{2}-2} \int_{\mathbb{R}^d} dy g_\varepsilon(cy) E[(L_n(\lfloor n^{1/4} y \rfloor) - L_n(x_n))^2] \xrightarrow[n \rightarrow \infty]{} \int_{\mathbb{R}^d} dy g_\varepsilon(cy) \left(\varphi(y, y) - 2\varphi\left(\frac{x}{c}, y\right) + \varphi\left(\frac{x}{c}, \frac{x}{c}\right)\right).$$

If γ_ε stands for the limit in the last display, the continuity of φ ensures that γ_ε tends to 0 as $\varepsilon \rightarrow 0$.

From the preceding considerations, we have

$$\limsup_{n \rightarrow \infty} E\left[\left(\int_0^1 g_\varepsilon(c n^{-1/4} Z_{[nt]}^n) dt - \eta_n(\varepsilon) n^{\frac{d}{4}-1} L_n(x_n)\right)^2\right] \leq c^{-d} \gamma_\varepsilon.$$

Hence we can find $\varepsilon_2 \in (0, |x|)$ small enough so that, for every $\varepsilon \in (0, \varepsilon_2)$, there exists an integer $n_2(\varepsilon)$ such that, for every $n \geq n_2(\varepsilon)$,

$$P\left(\left|\int_0^1 g_\varepsilon(cn^{-1/4}Z_{[nt]}^n) dt - \eta_n(\varepsilon)n^{\frac{d}{4}-1}L_n(x_n)\right| > \delta\right) < \delta. \quad (3.22)$$

By combining (3.21) and (3.22), we see that, for every $\varepsilon \in (0, \varepsilon_1 \wedge \varepsilon_2)$ and $n \geq n_1(\varepsilon) \vee n_2(\varepsilon)$,

$$P\left(|\eta_n(\varepsilon)n^{\frac{d}{4}-1}L_n(x_n) - l^x| > 2\delta\right) < 2\delta.$$

Our claim (3.20) easily follows, since $\eta_n(\varepsilon)$ tends to c^{-d} as $n \rightarrow \infty$. \square

Set $R_n = \#\{Z^n(u) : u \in \mathcal{T}_n\}$. Recall the constant c from (3.19), and also recall that λ_d denotes Lebesgue measure on \mathbb{R}^d .

Theorem 3.5. *We have*

$$n^{-d/4} R_n \xrightarrow[n \rightarrow \infty]{(d)} c^{-d} \lambda_d(\mathcal{S})$$

where \mathcal{S} stands for the support of ISE.

Proof. Again, for the sake of simplicity, we give details only in the isotropic case $M_\theta = \sigma^2 \text{Id}$. From the definition of ISE, we may take $\mathcal{S} = \{\widehat{W}_t^{(1)} : 0 \leq t \leq 1\}$. We then set, for every $\varepsilon > 0$,

$$\mathcal{S}_\varepsilon := \{x \in \mathbb{R}^d : \text{dist}(x, \mathcal{S}) \leq \varepsilon\}.$$

As in the preceding proof, we may and will assume that the convergence (3.18) holds almost surely. It then follows that, for every $\varepsilon > 0$,

$$P(\{cn^{-1/4}Z^n(u) : u \in \mathcal{T}_n\} \subset \mathcal{S}_\varepsilon) \xrightarrow[n \rightarrow \infty]{} 1.$$

Fix $K > 0$, and let $B(0, K)$ stand for the closed ball of radius K centered at 0 in \mathbb{R}^d . Also set $\mathcal{S}_\varepsilon^{(K)} := \mathcal{S}_\varepsilon \cap B(0, K + \varepsilon)$. It follows that we have also

$$P\left(\left(\{Z^n(u) : u \in \mathcal{T}_n\} \cap B(0, c^{-1}n^{1/4}K)\right) \subset c^{-1}n^{1/4}\mathcal{S}_\varepsilon^{(K)}\right) \xrightarrow[n \rightarrow \infty]{} 1.$$

Applying the latter convergence with ε replaced by $\varepsilon/2$, we get

$$P\left(\#\left(\{Z^n(u) : u \in \mathcal{T}_n\} \cap B(0, c^{-1}n^{1/4}K)\right) \leq c^{-d}n^{d/4}\lambda_d(\mathcal{S}_{\varepsilon/2}^{(K)})\right) \xrightarrow[n \rightarrow \infty]{} 1.$$

Write $R_n^{(K)} := \#\left(\{Z^n(u) : u \in \mathcal{T}_n\} \cap B(0, c^{-1}n^{1/4}K)\right)$. Since $\lambda_d(\mathcal{S}_\varepsilon^{(K)}) \downarrow \lambda_d(\mathcal{S} \cap B(0, K))$ as $\varepsilon \downarrow 0$, we obtain that, for every $\delta > 0$,

$$P\left(n^{-d/4}R_n^{(K)} \leq c^{-d}\lambda_d(\mathcal{S} \cap B(0, K)) + \delta\right) \xrightarrow[n \rightarrow \infty]{} 1,$$

and therefore, since the variables $n^{-d/4}R_n^{(K)}$ are uniformly bounded,

$$\lim_{n \rightarrow \infty} E\left[\left(n^{-d/4}R_n^{(K)} - c^{-d}\lambda_d(\mathcal{S} \cap B(0, K))\right)^+\right] = 0. \quad (3.23)$$

On the other hand, we claim that we have also

$$\liminf_{n \rightarrow \infty} E \left[n^{-d/4} R_n^{(K)} \right] \geq c^{-d} E \left[\lambda_d(\mathcal{S} \cap B(0, K)) \right]. \quad (3.24)$$

To see this, observe that

$$\begin{aligned} E[R_n^{(K)}] &= \sum_{a \in \mathbb{Z}^d \cap B(0, c^{-1}n^{1/4}K)} P(L_n(a) > 0) \\ &= \int_{B(0, c^{-1}n^{1/4}K)} dx P(L_n(\lfloor x \rfloor) > 0) + O(n^{(d-1)/4}) \\ &= n^{d/4} \int_{B(0, c^{-1}K)} dy P(L_n(\lfloor n^{1/4}y \rfloor) > 0) + O(n^{(d-1)/4}) \end{aligned}$$

as $n \rightarrow \infty$. By Theorem 3.4, for every $y \neq 0$,

$$\liminf_{n \rightarrow \infty} P(L_n(\lfloor n^{1/4}y \rfloor) > 0) \geq P(l^{cy} > 0) = P(cy \in \mathcal{S}),$$

where the equality is derived from Proposition 3.2. Fatou's lemma then gives

$$\liminf_{n \rightarrow \infty} n^{-d/4} E[R_n^{(K)}] \geq \int_{B(0, c^{-1}K)} dy P(cy \in \mathcal{S}) = c^{-d} E[\lambda_d(\mathcal{S} \cap B(0, K))],$$

which completes the proof of (3.24).

Using the trivial identity $|x| = 2x^+ - x$ for every real x , we deduce from (3.23) and (3.24) that

$$\lim_{n \rightarrow \infty} E \left[\left| n^{-d/4} R_n^{(K)} - c^{-d} \lambda_d(\mathcal{S} \cap B(0, K)) \right| \right] = 0.$$

However, we see from (3.18) that, for every $\delta > 0$, we can choose K sufficiently large so that we have both $P(\mathcal{S} \subset B(0, K)) \geq 1 - \delta$ and $P(R_n^{(K)} = R_n) \geq 1 - \delta$ for every integer n . It then follows from the previous convergence that $n^{-d/4} R_n$ converges in probability to $c^{-d} \lambda_d(\mathcal{S})$ as $n \rightarrow \infty$, and this completes the proof of Theorem 3.5. \square

3.4 Branching random walk

We will now discuss similar results for branching random walk in \mathbb{Z}^d . We consider a system of particles in \mathbb{Z}^d that evolves in discrete time in the following way. At time $n = 0$, there are p particles all located at the origin of \mathbb{Z}^d (we will comment on more general initial configurations in Subsection 3.5.3). A particle located at the site $a \in \mathbb{Z}^d$ at time n gives rise at time $n + 1$ to a random number of offspring distributed according to μ , and their locations are obtained by adding to a (independently for each offspring) a spatial displacement distributed according to θ .

In a more formal way, we consider p independent random spatial trees

$$(\mathcal{T}^{(1)}, (Z^{(1)}(u))_{u \in \mathcal{T}^{(1)}}), \dots, (\mathcal{T}^{(p)}, (Z^{(p)}(u))_{u \in \mathcal{T}^{(p)}})$$

distributed according to $\Pi_{\mu, \theta}^*$, and, for every integer $n \geq 0$, we consider the random point measure

$$X_n^{[p]} := \sum_{j=1}^p \left(\sum_{u \in \mathcal{T}^{(j)}, |u|=n} \delta_{Z^{(j)}(u)} \right),$$

which corresponds to the sum of the Dirac point masses at the positions of all particles alive at time n .

The set $\mathcal{V}^{[p]}$ of all sites visited by the particles is the union over all $n \geq 0$ of the supports of $X_n^{[p]}$, or equivalently

$$\mathcal{V}^{[p]} = \{a \in \mathbb{Z}^d : a = Z^{(j)}(u) \text{ for some } j \in \{1, \dots, p\} \text{ and } u \in \mathcal{T}^{(j)}\}.$$

In a way similar to Theorem 3.5, we are interested in limit theorems for $\#\mathcal{V}^{[p]}$ when $p \rightarrow \infty$. To this end, we will first state an analog of the convergence (3.18). For every $j \in \{1, \dots, p\}$, let

$$\emptyset = u_0^{(j)} \prec u_1^{(j)} \prec \dots \prec u_{\#\mathcal{T}^{(j)}-1}^{(j)}$$

be the vertices of $\mathcal{T}^{(j)}$ listed in lexicographical order, and set $H_i^{(j)} = |u_i^{(j)}|$ and $Z_i^{(j)} = Z^{(j)}(u_i^{(j)})$, for $0 \leq i \leq \#\mathcal{T}^{(j)} - 1$. Define the height function $(H_k^{[p]}, k \geq 0)$ of $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(p)}$ by concatenating the discrete functions $(H_i^{(j)}, 0 \leq i \leq \#\mathcal{T}^{(j)} - 1)$, and setting $H_k^{[p]} = 0$ for $k \geq \#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)}$. Similarly, define the function $(Z_k^{[p]}, k \geq 0)$ by concatenating the discrete functions $(Z_i^{(j)}, 0 \leq i \leq \#\mathcal{T}^{(j)} - 1)$, and setting $Z_k^{[p]} = 0$ for $k \geq \#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)}$. Finally, we use linear interpolation to define $H_t^{[p]}$ and $Z_t^{[p]}$ for every real $t \geq 0$. We can now state our analog of (3.18).

Proposition 3.6. *We have*

$$\left(\left(\frac{\rho}{2} p^{-1} H_{p^2 s}^{[p]}, \sqrt{\frac{\rho}{2}} p^{-\frac{1}{2}} Z_{p^2 s}^{[p]} \right)_{s \geq 0}, p^{-2} (\#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)}) \right) \xrightarrow[p \rightarrow \infty]{(d)} ((\zeta_{s \wedge \tau}, M_\theta^{1/2} \widehat{W}_{s \wedge \tau})_{s \geq 0}, \tau),$$

where $(W_s)_{s \geq 0}$ is a standard Brownian snake, τ denotes the first hitting time of $2/\rho$ by the local time at 0 of the lifetime process $(\zeta_s)_{s \geq 0}$, and the convergence of processes holds in the sense of the topology of uniform convergence on compact sets.

The joint convergence of the processes $\frac{\rho}{2} p^{-1} H_{p^2 s}^{[p]}$ and of the random variables $p^{-2} (\#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)})$ is a consequence of [58, Theorem 1.8], see in particular (7) and (9) in [58] (note that the local times of the process $(\zeta_s)_{s \geq 0}$ are chosen to be right-continuous in the space variable, so that our local time at 0 is twice the local time that appears in the display (7) in [58]). Given the latter joint convergence, the desired statement can be obtained by following the arguments of the proof of Theorem 2 in [39]. The fact that we are dealing with unconditioned trees makes things easier than in [39] and we omit the details.

We now state an intermediate result, which is of independent interest. Under the probability measure $\Pi_{\mu, \theta}^*$, we let $\mathbf{R} := \{z_u : u \in \mathcal{T}\}$ be the set of all points visited by the tree-indexed random walk.

Theorem 3.7. *We have*

$$\lim_{|a| \rightarrow \infty} |M_\theta^{-1/2} a|^2 \Pi_{\mu, \theta}^*(a \in \mathbf{R}) = \frac{2(4-d)}{\rho^2}.$$

Proof. We start by proving the upper bound

$$\limsup_{|a| \rightarrow \infty} |M_\theta^{-1/2} a|^2 \Pi_{\mu, \theta}^*(a \in \mathbf{R}) \leq \frac{2(4-d)}{\rho^2}.$$

By an easy compactness argument, it is enough to prove that, if (a_k) is a sequence in \mathbb{Z}^d such that $|a_k| \rightarrow \infty$ and $a_k/|a_k| \rightarrow x$, with $x \in \mathbb{R}^d$ and $|x| = 1$, then

$$\limsup_{k \rightarrow \infty} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) \leq \frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}. \quad (3.25)$$

Set $p_k = |a_k|^2 \in \mathbb{Z}_+$ to simplify notation. We note that

$$P(a_k \in \mathcal{V}^{[p_k]}) = 1 - \left(1 - \Pi_{\mu, \theta}^*(a_k \in \mathbf{R})\right)^{p_k}. \quad (3.26)$$

On the other hand, it follows from our definitions that

$$P(a_k \in \mathcal{V}^{[p_k]}) \leq P\left(\exists s \geq 0: \frac{1}{\sqrt{p_k}} Z_s^{(p_k)} = \frac{a_k}{|a_k|}\right).$$

We can then use Proposition 3.6 to get

$$\begin{aligned} \limsup_{k \rightarrow \infty} P(a_k \in \mathcal{V}^{[p_k]}) &\leq \mathbb{P}_0\left(\exists s \in [0, \tau]: M_\theta^{1/2} \widehat{W}_s = \sqrt{\frac{\rho}{2}} x\right) \\ &= 1 - \exp\left(-\frac{2}{\rho} \mathbb{N}_0\left(\sqrt{\frac{\rho}{2}} M_\theta^{-1/2} x \in \mathcal{R}\right)\right) \\ &= 1 - \exp\left(-\frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}\right). \end{aligned}$$

The second line follows from excursion theory for the Brownian snake, and the third one uses the formula for $\mathbb{N}_0(y \in \mathcal{R})$, which has been recalled already in the proof of Proposition 3.2. By combining the bound of the last display with (3.26), we get our claim (3.25), and this completes the proof of the upper bound.

Let us turn to the proof of the lower bound. As in the proof of the upper bound, it is enough to consider a sequence (a_k) in \mathbb{Z}^d such that $|a_k| \rightarrow \infty$ and $a_k/|a_k| \rightarrow x$, with $x \in \mathbb{R}^d$ and $|x| = 1$, and then to verify that

$$\liminf_{k \rightarrow \infty} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) \geq \frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}. \quad (3.27)$$

As previously, we set $p_k = |a_k|^2$. We fix $0 < \varepsilon < M$, and we introduce the function g_μ defined on \mathbb{Z}_+ by $g_\mu(j) = \Pi_\mu(\#\mathcal{T} = j)$. Then

$$\begin{aligned} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) &\geq p_k^3 \int_\varepsilon^M dr \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}, \#\mathcal{T} = \lfloor p_k^2 r \rfloor) \\ &= p_k^3 \int_\varepsilon^M dr g_\mu(\lfloor p_k^2 r \rfloor) P(L_{\lfloor p_k^2 r \rfloor}(a_k) > 0), \end{aligned}$$

where we use the same notation as in Lemma 3.3: $L_n(b)$ denotes the number of visits of site b by a random walk indexed by a tree distributed according to $\Pi_\mu(\cdot \mid \#\mathcal{T} = n)$. Note that Theorem 3.4 gives, for every $r \in [\varepsilon, M]$,

$$\liminf_{k \rightarrow \infty} P(L_{\lfloor p_k^2 r \rfloor}(a_k) > 0) \geq P(l^{r^{-1/4} z} > 0),$$

where we write $z = \sqrt{\frac{\rho}{2}} M_\theta^{-1/2} x$ to simplify notation. To complete the argument, we consider for simplicity the aperiodic case where μ is not supported on a strict subgroup of \mathbb{Z} (the reader will easily be able to extend our method to the general case, using (3.3) instead of (3.4)). By (3.4), we have for every $r \in [\varepsilon, M]$,

$$\lim_{k \rightarrow \infty} p_k^3 g_\mu(\lfloor p_k^2 r \rfloor) = \frac{1}{\rho \sqrt{2\pi r^3}}.$$

Using this together with the preceding display, and applying Fatou's lemma, we obtain

$$\liminf_{k \rightarrow \infty} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) \geq \int_\varepsilon^M \frac{dr}{\rho \sqrt{2\pi r^3}} P(l^{r^{-1/4}z} > 0). \quad (3.28)$$

A scaling argument shows that

$$P(l^{r^{-1/4}z} > 0) = \mathbb{N}_0^{(1)}(\ell^{r^{-1/4}z} > 0) = \mathbb{N}_0^{(r)}(\ell^z > 0).$$

Using this remark and formula (3.5), we see that the right-hand side of (3.28) can be rewritten as $\frac{2}{\rho} \mathbb{N}_0(\mathbf{1}_{\{\varepsilon < \gamma < M\}} \mathbf{1}_{\{\ell^z > 0\}})$. By choosing ε small enough and M large enough, the latter quantity can be made arbitrarily close to

$$\frac{2}{\rho} \mathbb{N}_0(\ell^z > 0) = \frac{2}{\rho} \left(2 - \frac{d}{2}\right) |z|^{-2} = \frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}.$$

This completes the proof of the lower bound and of Theorem 3.7. \square

Recall our notation $\mathcal{V}^{[p]}$ for the set of all sites visited by the branching random walk starting with p initial particles located at the origin.

Theorem 3.8. *We have*

$$p^{-d/2} \#\mathcal{V}^{[p]} \xrightarrow[p \rightarrow \infty]{(d)} \left(\frac{2\sigma}{\rho}\right)^d \lambda_d \left(\bigcup_{t \geq 0} \text{supp } X_t \right),$$

where $(X_t)_{t \geq 0}$ is a d -dimensional super-Brownian motion with branching mechanism $\psi(u) = 2u^2$ started from δ_0 , and $\text{supp } X_t$ denotes the topological support of X_t .

Proof. Via the Skorokhod representation theorem, we may and will assume that the convergence in Proposition 3.6 holds a.s., and we will then prove that the convergence of the theorem holds in probability. If $\varepsilon > 0$ is fixed, the (a.s.) convergence in Proposition 3.6 implies that, a.s. for all large enough p , we have

$$\sqrt{\frac{\rho}{2}} p^{-1/2} \mathcal{V}^{[p]} \subset \mathcal{U}_\varepsilon(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\}),$$

where, for any compact subset \mathcal{K} of \mathbb{R}^d , $\mathcal{U}_\varepsilon(\mathcal{K})$ denotes the set of all points whose distance from \mathcal{K} is strictly less than ε . It follows that we have a.s.

$$\limsup_{p \rightarrow \infty} p^{-d/2} \#\mathcal{V}^{[p]} \leq \left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\mathcal{U}_{2\varepsilon}(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\})).$$

Since ε was arbitrary, we also get a.s.

$$\limsup_{p \rightarrow \infty} p^{-d/2} \# \mathcal{V}^{[p]} \leq \left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\}). \quad (3.29)$$

To get an estimate in the reverse direction, we argue in a way very similar to the proof of Theorem 3.5. We fix $K > 0$, and note that a minor modification of the preceding arguments also gives a.s.

$$\limsup_{p \rightarrow \infty} p^{-d/2} \#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K)) \leq \left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')),$$

where $K' = \sqrt{\frac{\rho}{2}}K$. Since the variables $p^{-d/2} \#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K))$ are uniformly bounded, it follows that

$$\lim_{p \rightarrow \infty} E \left[\left(p^{-d/2} \#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K)) - \left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right)^+ \right] = 0. \quad (3.30)$$

On the other hand,

$$\begin{aligned} p^{-d/2} E[\#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K))] &= p^{-d/2} \sum_{a \in \mathbb{Z}^d \cap B(0, p^{1/2}K)} P(a \in \mathcal{V}^{[p]}) \\ &= p^{-d/2} \sum_{a \in \mathbb{Z}^d \cap B(0, p^{1/2}K)} \left(1 - (1 - \Pi_{\mu, \theta}^*(a \in \mathbf{R}))^p\right) \\ &\xrightarrow{p \rightarrow \infty} \int_{B(0, K)} dx \left(1 - \exp\left(-\frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}\right)\right), \end{aligned}$$

where the last line is an easy consequence of Theorem 3.7. Furthermore,

$$\begin{aligned} E \left[\left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right] \\ &= \left(\frac{2}{\rho}\right)^{d/2} \int_{B(0, K')} dy \left(1 - \exp\left(-\frac{2}{\rho} \mathbb{N}_0(M_\theta^{-1/2} y \in \mathcal{R})\right)\right) \\ &= \left(\frac{2}{\rho}\right)^{d/2} \int_{B(0, K')} dy \left(1 - \exp\left(-\frac{4-d}{\rho |M_\theta^{-1/2} y|^2}\right)\right) \\ &= \int_{B(0, K)} dx \left(1 - \exp\left(-\frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}\right)\right). \end{aligned}$$

From the last two displays, we get

$$\lim_{p \rightarrow \infty} E \left[p^{-d/2} \#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K)) \right] = E \left[\left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right]. \quad (3.31)$$

From (3.30) and (3.31), we have

$$\lim_{p \rightarrow \infty} E \left[\left| p^{-d/2} \#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K)) - \left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right| \right] = 0.$$

Since, by choosing K large enough, $P(\mathcal{V}^{[p]} \subset B(0, p^{1/2}K))$ can be made arbitrarily close to 1, uniformly in p , we have proved that

$$p^{-d/2} \# \mathcal{V}^{[p]} \xrightarrow[p \rightarrow \infty]{(P)} \left(\frac{2}{\rho}\right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\}) = \left(\frac{2\sigma^2}{\rho}\right)^{d/2} \lambda_d(\{\widehat{W}_s : 0 \leq s \leq \tau\}). \quad (3.32)$$

The relations between the Brownian snake and super-Brownian motion [57, Theorem IV.4] show that the quantity $\lambda_d(\{\widehat{W}_s : 0 \leq s \leq \tau\})$ is the Lebesgue measure of the range of a super-Brownian motion (with branching mechanism $2u^2$) started from $(2/\rho)\delta_0$. Finally, simple scaling arguments show that the limit can be expressed in the form given in the theorem. \square

3.5 Open problems and questions

3.5.1 The probability of visiting a distant point

Theorem 3.7 gives the asymptotic behavior of the probability that a branching random walk starting with a single particle at the origin visits a distant point $a \in \mathbb{Z}^d$. It would be of interest to have a similar result in dimension $d \geq 4$, assuming that θ is centered and has sufficiently high moments. When $d \geq 5$, a simple calculation of the first and second moments of the number of visits of a (see, e.g., the remarks following Proposition 5 in [61]) gives the bounds

$$C_1 |a|^{2-d} \leq \Pi_{\mu, \theta}^*(a \in \mathbf{R}) \leq C_2 |a|^{2-d}$$

with positive constants C_1 and C_2 depending on d, μ and θ . When $d = 4$, one expects that

$$\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \approx \frac{C}{|a|^2 \log |a|}.$$

Calculations of moments give $\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \geq c_1 (|a|^2 \log |a|)^{-1}$, but proving the reverse bound $\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \leq c_2 (|a|^2 \log |a|)^{-1}$ with some constant c_2 seems a nontrivial problem. This problem, in the particular case of the geometric offspring distribution, and some related questions are discussed in Section 3.2 of [9].

3.5.2 The range in dimension four

With our previous notation R_n for the range of a random walk indexed by a random tree distributed according to $\Pi_\mu(\cdot | \#\mathcal{T} = n)$, Theorem 14 in [61] states that in dimension $d = 4$,

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi^2 \sigma^4,$$

provided μ is the geometric distribution with parameter $1/2$, and θ is symmetric and has exponential moments. It would be of interest to extend this result to more general offspring distributions. It seems difficult to adapt the methods of [61] to a more general case, so new arguments would be needed. In particular, finding the exact asymptotics of $\Pi_{\mu, \theta}^*(a \in \mathbf{R})$ (see the previous subsection) in dimension $d = 4$ would certainly be helpful.

3.5.3 Branching random walk with a general initial configuration

One may ask whether a result such as Theorem 3.8 remains valid for more general initial configurations of the branching particle system: Compare with Propositions 20 and 21 in [61], which deal with the case $d \geq 4$ and require no assumption on the initial configurations. In the present setting, Theorem 3.8 remains valid for instance if we assume that the initial positions of the particles stay within a bounded set independently of p . On the other hand, one might consider the case where we only assume that the image of $p^{-1}X_0^{[p]}$ under the mapping $a \mapsto p^{-1/2}a$

converges weakly to a finite measure ξ on \mathbb{R}^d . This condition ensures the convergence of the (rescaled) measure-valued processes $X^{[p]}$ to a super-Brownian motion Y with initial value $Y_0 = \xi$, and it is natural to expect that we have, with a suitable constant C ,

$$p^{-d/2} \#\mathcal{V}^{[p]} \xrightarrow[p \rightarrow \infty]{(d)} C \lambda_d \left(\bigcup_{t \geq 0} \text{supp } Y_t \right). \quad (3.33)$$

For trivial reasons, (3.33) will not hold in dimension $d = 1$. Indeed, for $\frac{1}{2} < \alpha < 1$, we may let the initial configuration consist of $p - \lfloor p^\alpha \rfloor$ particles uniformly spread over $\{1, 2, \dots, \sqrt{p}\}$ and $\lfloor p^\alpha \rfloor$ other particles located at distinct points outside $\{1, 2, \dots, \sqrt{p}\}$. Then the preceding assumptions hold (ξ is the Lebesgue measure on $[0, 1]$), but (3.33) obviously fails since $\#\mathcal{V}^{[p]} \geq \lfloor p^\alpha \rfloor$. In dimension 2, (3.33) fails again, for more subtle reasons: One can construct examples where the descendants of certain initial particles that play no role in the convergence of the initial configurations contribute to the asymptotics of $\#\mathcal{V}^{[p]}$ in a significant manner. Still, it seems likely that some version of (3.33) holds under more stringent conditions on the initial configurations (in dimension 3 at least, the union in the right-hand side of (3.33) should exclude $t = 0$, as can be seen from simple examples).

3.6 Appendix

Recall that

$$\varphi(x, y) = \rho^4 \int_{(\mathbb{R}_+)^3} dr_1 dr_2 dr_3 (r_1 + r_2 + r_3) e^{-\rho^2(r_1+r_2+r_3)^2/2} \int_{\mathbb{R}^d} dz p_{r_1}(z) p_{r_2}(x-z) p_{r_3}(y-z).$$

Here we prove the continuity of φ on $(\mathbb{R}^d)^2$.

For notational ease, we define the norm J on \mathbb{R}^d by

$$J(x)^2 = x \cdot M_\theta^{-1} x, \quad \forall x \in \mathbb{R}^d.$$

First observe that for $r_1, r_2, r_3 > 0$,

$$\begin{aligned} & (2\pi)^{\frac{3d}{2}} (\det M_\theta)^{\frac{3}{2}} (r_1 r_2 r_3)^{\frac{d}{2}} \int_{\mathbb{R}^d} dz p_{r_1}(z) p_{r_2}(x-z) p_{r_3}(y-z) \\ &= \int_{\mathbb{R}^d} dz \exp \left(-\frac{J(z)^2}{2r_1} - \frac{J(x-z)^2}{2r_2} - \frac{J(y-z)^2}{2r_3} \right) \\ &= \int_{\mathbb{R}^d} dz \exp \left(-\frac{\Delta_{123}}{2r_1 r_2 r_3} J \left(\frac{r_1 r_3}{\Delta_{123}} x + \frac{r_1 r_2}{\Delta_{123}} y - z \right)^2 - \frac{r_1 + r_3}{2\Delta_{123}} J(x)^2 - \frac{r_1 + r_2}{2\Delta_{123}} J(y)^2 + \frac{r_1}{\Delta_{123}} x \cdot M_\theta^{-1} y \right), \end{aligned}$$

where $\Delta_{123} := r_1 r_2 + r_1 r_3 + r_2 r_3$. Since for every $x, y \in \mathbb{R}^d$,

$$-\frac{r_1 + r_3}{2} J(x)^2 - \frac{r_1 + r_2}{2} J(y)^2 + r_1 x \cdot M_\theta^{-1} y \leq 0,$$

we get the following estimate

$$\begin{aligned} & \int_{\mathbb{R}^d} dz p_{r_1}(z) p_{r_2}(x-z) p_{r_3}(y-z) \\ & \leq \frac{1}{(2\pi)^{\frac{3d}{2}} (\det M_\theta)^{\frac{3}{2}} (r_1 r_2 r_3)^{\frac{d}{2}}} \int_{\mathbb{R}^d} dz \exp \left(-\frac{\Delta_{123}}{2r_1 r_2 r_3} J \left(\frac{r_1 r_3}{\Delta_{123}} x + \frac{r_1 r_2}{\Delta_{123}} y - z \right)^2 \right) \\ & = \frac{1}{(2\pi)^d \cdot (\det M_\theta) \cdot (\Delta_{123})^{d/2}}, \end{aligned}$$

and it follows that for every $x, y \in \mathbb{R}^d$,

$$\varphi(x, y) \leq \frac{\rho^4}{(2\pi)^d \cdot (\det M_\theta)} \int_{(\mathbb{R}_+)^3} dr_1 dr_2 dr_3 \frac{r_1 + r_2 + r_3}{(r_1 r_2 + r_1 r_3 + r_2 r_3)^{d/2}} \exp\left(-\frac{\rho^2(r_1 + r_2 + r_3)^2}{2}\right).$$

We claim that the integral

$$\int_{(\mathbb{R}_+)^3} dr_1 dr_2 dr_3 \frac{r_1 + r_2 + r_3}{(r_1 r_2 + r_1 r_3 + r_2 r_3)^{d/2}} \exp\left(-\frac{\rho^2(r_1 + r_2 + r_3)^2}{2}\right)$$

converges when $1 \leq d \leq 3$. In fact, due to the exponential term in the integrand, it is enough to show the convergence of

$$I := \int_{0 < r_1, r_2, r_3 < 1} dr_1 dr_2 dr_3 \frac{r_1 + r_2 + r_3}{(r_1 r_2 + r_1 r_3 + r_2 r_3)^{d/2}}. \quad (3.34)$$

Using the spherical coordinate

$$\begin{cases} r_1 = r \sin \theta \cos \psi, \\ r_2 = r \sin \theta \sin \psi, \\ r_3 = r \cos \theta, \end{cases}$$

we see that

$$\begin{aligned} I &= \int_0^1 r^2 dr \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\psi \frac{r(\sin \theta \cos \psi + \sin \theta \sin \psi + \cos \theta)}{r^d (\sin^2 \theta \sin \psi \cos \psi + \sin \theta \cos \theta \cos \psi + \sin \theta \cos \theta \sin \psi)^{d/2}} \\ &= \int_0^1 r^{3-d} dr \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\psi \frac{\sin \theta \cos \psi + \sin \theta \sin \psi + \cos \theta}{(\sin \theta)^{\frac{d}{2}-1} (\sin \theta \sin \psi \cos \psi + \cos \theta \cos \psi + \cos \theta \sin \psi)^{\frac{d}{2}}} \\ &= \frac{1}{4-d} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\psi \frac{\sin \theta \cos \psi + \sin \theta \sin \psi + \cos \theta}{(\sin \theta)^{\frac{d}{2}-1} (\sin \theta \sin \psi \cos \psi + \cos \theta \cos \psi + \cos \theta \sin \psi)^{\frac{d}{2}}}, \end{aligned}$$

as $1 \leq d \leq 3$. Observe that for any fixed $\varepsilon \in (0, \frac{\pi}{4})$, the denominator

$$(\sin \theta)^{\frac{d}{2}-1} (\sin \theta \sin \psi \cos \psi + \cos \theta \cos \psi + \cos \theta \sin \psi)^{\frac{d}{2}}, \quad \theta, \psi \in (\varepsilon, \frac{\pi}{2} - \varepsilon)$$

is uniformly bounded away from 0. Also notice that the integrand in the original definition (3.34) of the integral I is symmetric with respect to r_1, r_2 and r_3 . So we only need to verify the convergence of

$$II := \int_0^\varepsilon d\theta \int_0^\varepsilon d\psi \frac{\sin \theta \cos \psi + \sin \theta \sin \psi + \cos \theta}{(\sin \theta)^{\frac{d}{2}-1} (\sin \theta \sin \psi \cos \psi + \cos \theta \cos \psi + \cos \theta \sin \psi)^{\frac{d}{2}}}.$$

As $\cos \theta \cos \psi \geq 1/2$ for every $\theta, \psi \in (0, \varepsilon)$, we have

$$II \leq \int_0^\varepsilon d\theta \int_0^\varepsilon d\psi \frac{3}{(\sin \theta)^{\frac{d}{2}-1} (\frac{1}{2})^{\frac{d}{2}}} = 3\varepsilon \cdot 2^{d/2} \cdot \int_0^\varepsilon d\theta (\sin \theta)^{1-\frac{d}{2}} < \infty,$$

which finishes the proof of our claim.

Therefore, we have shown for every $x, y \in \mathbb{R}^d$, $1 \leq d \leq 3$ that

$$\varphi(x, y) \leq \frac{\rho^4}{(2\pi)^d \cdot (\det M_\theta)} \int_{(\mathbb{R}_+)^3} dr_1 dr_2 dr_3 \frac{r_1 + r_2 + r_3}{(r_1 r_2 + r_1 r_3 + r_2 r_3)^{d/2}} \exp\left(-\frac{\rho^2(r_1 + r_2 + r_3)^2}{2}\right) < \infty,$$

and the continuity of φ on $(\mathbb{R}^d)^2$ follows immediately as a simple application of the dominated convergence theorem.

THE RANGE OF TREE-INDEXED RANDOM WALK WITH DRIFT

LES RÉSULTATS DE CE CHAPITRE SONT ISSUS DE L'ARTICLE [67] EN PRÉPARATION.

We study the range R_n of a random walk on the d -dimensional lattice \mathbb{Z}^d indexed by a random tree with n edges, under the assumption that the jump distribution is not centered.

4.1 Introduction

We continue our study of the tree-indexed random walk in the d -dimensional lattice \mathbb{Z}^d , but this time its jump distribution is no longer centered. Some partial results are presented in this note concerning the asymptotics for the number of distinct sites in \mathbb{Z}^d visited by this non-centered tree-indexed random walk.

We let θ be a centered probability distribution on \mathbb{Z}^d , which has finite first moments and is centered. We also assume that θ is not supported on a strict subgroup of \mathbb{Z}^d . We write $S = (S_k)_{k \geq 0}$ for a random walk in \mathbb{Z}^d with jump distribution θ . It is assumed that the random walk S starts from x under the probability measure P_x , for every $x \in \mathbb{R}^d$. We fix $D \in \mathbb{Z}^d \setminus \{0\}$ and define a new probability distribution $\tilde{\theta}$ on \mathbb{Z}^d as the image of θ under the translation $x \in \mathbb{Z}^d \mapsto x + D$. If we set for every $k \geq 0$ that

$$\tilde{S}_k := S_k + kD,$$

then $\tilde{S} = (\tilde{S}_k)_{k \geq 0}$ is a random walk in \mathbb{Z}^d with constant drift D , and its jump distribution is $\tilde{\theta}$.

Now we consider the random walk with jump distribution $\tilde{\theta}$ indexed by a plane tree \mathcal{T} . This means that we assign a (random) spatial location $Z_{\mathcal{T}}(u) \in \mathbb{Z}^d$ to every vertex u of \mathcal{T} , in the following way. First, the spatial location $Z_{\mathcal{T}}(\emptyset)$ of the root \emptyset is the origin of \mathbb{Z}^d . Then, we assign independently to every edge e of the tree \mathcal{T} a random variable X_e distributed according to $\tilde{\theta}$, and we let the spatial location $Z_{\mathcal{T}}(u)$ of the vertex u be the sum of the quantities X_e over all edges e belonging to the simple path from \emptyset to u in the tree \mathcal{T} . Due to our definition of $\tilde{\theta}$, this random walk $Z_{\mathcal{T}}$ indexed by \mathcal{T} has constant drift D . The number of distinct spatial locations is called the range of the tree-indexed random walk $Z_{\mathcal{T}}$.

Let μ be a probability measure on \mathbb{Z}_+ , which is critical in the sense that it has mean one. As in [61, 62], we always consider the critical Galton–Watson tree with offspring distribution μ as a random plane tree. The degenerate case where $\mu(1) = 1$ is excluded. Let \mathcal{G} be the smallest subgroup of \mathbb{Z} that contains the support of μ . Then for every sufficiently large integer $n \in \mathcal{G}$, we may define \mathcal{T}_n as a μ -Galton–Watson tree conditioned to have exactly $n + 1$ vertices.

Conditionally given \mathcal{T}_n , let $Z_{\mathcal{T}_n}$ be a random walk with jump distribution $\tilde{\theta}$ indexed by \mathcal{T}_n , and let \mathcal{R}_n stand for the range of $Z_{\mathcal{T}_n}$. In this work, we show that the asymptotics of \mathcal{R}_n as $n \rightarrow \infty$ exhibit different behaviors depending on the dimension d , and we compare these results with the corresponding ones shown in [61, 62] for the case where the jump distribution is centered.

4.2 Linear growth of the range

In this section, we use the general results established in [61, Section 2] to show the following theorem.

Theorem 4.1. *Assume that μ has finite variance, or that μ belongs to the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$. For every sufficiently large integer $n \in \mathcal{G}$, let \mathcal{T}_n be the μ -Galton–Watson tree conditioned to have $n + 1$ vertices. Conditionally given \mathcal{T}_n , let $Z_{\mathcal{T}_n}$ be a random walk with jump distribution $\tilde{\theta}$ indexed by \mathcal{T}_n , and let \mathcal{R}_n denote the range of $Z_{\mathcal{T}_n}$. Then there exists a constant $c_{\mu, \tilde{\theta}} \in [0, 1]$ depending on μ and $\tilde{\theta}$ such that*

$$\frac{1}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty, n \in \mathcal{G}]{} c_{\mu, \tilde{\theta}} \quad \text{in probability.}$$

Moreover, suppose in addition that θ is centered and has finite moments of order $2d \vee 3$.

- if μ has finite variance, then $c_{\mu, \tilde{\theta}} > 0$ if $d \geq 4$.
- if μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$, then $c_{\mu, \tilde{\theta}} > 0$ if $d > \frac{\alpha+1}{\alpha-1}$.

Proof. The first assertion about the convergence in probability of $n^{-1}\mathcal{R}_n$ follows directly from Theorems 4 and 7 in [61]. In particular, we get from Theorem 4 in [61] that the limiting constant $c_{\mu, \tilde{\theta}}$ can be expressed as

$$c_{\mu, \tilde{\theta}} = a_{\mu, \tilde{\theta}} E_0 \left[\prod_{j=1}^{\infty} \Phi_{\mu, \tilde{\theta}}(-\tilde{S}_j) \right], \quad (4.1)$$

in which $a_{\mu, \tilde{\theta}}$ is a positive constant depending on μ and $\tilde{\theta}$, and for every $x \in \mathbb{Z}^d$,

$$\Phi_{\mu, \tilde{\theta}}(x) := \sum_{k=0}^{\infty} \mu([k+1, \infty)) \left(\sum_{y \in \mathbb{Z}^d} \tilde{\theta}(y) h_{\mu, \tilde{\theta}}(x+y) \right)^k,$$

where $h_{\mu, \tilde{\theta}}(x)$ is the probability that a random walk with jump distribution $\tilde{\theta}$ indexed by a μ -Galton–Watson tree never visits $-x$. We denote the Green function of \tilde{S} by $G_{\tilde{\theta}}$, that is

$$G_{\tilde{\theta}}(x) := E_0 \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{\tilde{S}_k=x\}} \right], \quad \forall x \in \mathbb{Z}^d.$$

By [61, Proposition 5 (i)], we know that the property $c_{\mu, \tilde{\theta}} > 0$ holds if P_0 -a.s.

$$\prod_{j=1}^{\infty} \left(\frac{1 - g_{\mu}((1 - G_{\tilde{\theta}}(\tilde{S}_j))_+)}{G_{\tilde{\theta}}(\tilde{S}_j)} \right) > 0, \quad (4.2)$$

where g_{μ} stands for the generating function of μ . At this point, we need the following lemma whose proof is postponed to the end of this section.

Lemma 4.2. *If the random walk S is centered with finite moments of order $2d \vee 3$, there exists a constant $C_{\tilde{\theta}}$ such that, for every $x \in \mathbb{Z}^d$,*

$$G_{\tilde{\theta}}(x) \leq C_{\tilde{\theta}} |x|^{\frac{1-d}{2}}. \quad (4.3)$$

Suppose that μ has a finite variance. Then using the Taylor series expansion of g_μ at 1, we know that the condition (4.2) will be satisfied if

$$\sum_{j=1}^{\infty} G_{\tilde{\theta}}(\tilde{S}_j) < \infty, \quad P_0\text{-a.s.} \quad (4.4)$$

However, for any $\varepsilon > 0$, the law of large numbers shows that P_0 -a.s. for all sufficiently large n ,

$$|\tilde{S}_n| \geq n^{1-\varepsilon}|D|.$$

Combining the latter display with Lemma 4.2, one easily checks that the property (4.4) holds if $\frac{d-1}{2} > 1$, i.e. $d \geq 4$.

The case where μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$ can be treated similarly (see the proof of Proposition 5 (ii) in [61]), and we omit the details. \square

Proof of Lemma 4.2. We set $\delta = |D|^{-1}$ within this proof, and we assume for technical convenience that the random walk S is aperiodic. Only minor modifications are needed to treat the general case. We begin by writing for every $x \in \mathbb{Z}^d \setminus \{0\}$ that

$$G_{\tilde{\theta}}(x) = \sum_{k \geq 2\delta|x|} P_0(\tilde{S}_k = x) + \sum_{k < 2\delta|x|} P_0(\tilde{S}_k = x).$$

Since $k \geq 2\delta|x|$ implies that $|x - kD| \geq \frac{1}{2}k|D|$, we have

$$P_0(\tilde{S}_k = x) = P_0(S_k = x - kD) \leq P_0\left(|S_k| \geq \frac{1}{2}k|D|\right).$$

Provided the random walk S is centered with finite moments of order $d + 1$, the Rosenthal inequality implies the existence of a constant $C < \infty$ such that

$$E_0[|S_k|^{d+1}] \leq Ck^{\frac{d+1}{2}} \quad \text{for all } k \geq 1.$$

Thus,

$$P_0\left(|S_k| \geq \frac{1}{2}k|D|\right) \leq \left(\frac{1}{2}k|D|\right)^{-(d+1)} E_0[|S_k|^{d+1}] \leq C\left(\frac{1}{2}|D|\right)^{-(d+1)} k^{-\frac{d+1}{2}},$$

and it follows that

$$\sum_{k \geq 2\delta|x|} P_0(\tilde{S}_k = x) = O(|x|^{\frac{1-d}{2}}) \quad \text{as } |x| \rightarrow \infty.$$

Next, we continue the decomposition

$$\sum_{k < 2\delta|x|} P_0(\tilde{S}_k = x) = \left(\sum_{k < |x|^{\frac{1}{2}}} + \sum_{|x|^{\frac{1}{2}} \leq k \leq \delta|x|/2} + \sum_{\delta|x|/2 < k < 2\delta|x|} \right) P_0(S_k = x - kD),$$

and we will bound each sum in the right-hand side. Firstly, Proposition 2.4.6 in [53] states that, if the random walk S is centered with finite moments of order $d \vee 2$, then there exists $c < \infty$ such that for all n and x ,

$$P_0(S_n = x) \leq \frac{c}{n^{d/2}} \left(\frac{\sqrt{n}}{|x|} \right)^{d \vee 2}.$$

Using this estimate, one easily shows that

$$\sum_{k < |x|^{\frac{1}{2}}} P_0(S_k = x - kD) = o(|x|^{\frac{1-d}{2}}) \quad \text{as } |x| \rightarrow \infty.$$

Then for the second sum, we observe that $|x - kD| \geq k|D|$ if $k \leq \delta|x|/2$. As the random walk S is assumed to have finite moments of order $2d$, we can similarly apply the Rosenthal inequality to show that

$$\sum_{|x|^{\frac{1}{2}} \leq k \leq \delta|x|/2} P_0(S_k = x - kD) = O(|x|^{\frac{1-d}{2}}) \quad \text{as } |x| \rightarrow \infty.$$

Finally, let us complete the proof by showing that

$$\sum_{\delta|x|/2 < k < 2\delta|x|} P_0(S_k = x - kD) = O(|x|^{\frac{1-d}{2}}) \quad \text{as } |x| \rightarrow \infty. \quad (4.5)$$

To this end, we will use the local limit theorem for the random walk S . We write $\pi_k(x) := P_0(S_k = x)$ for every $k \geq 1$ and every $x \in \mathbb{Z}^d$. We set also

$$p_k(x) := \frac{1}{(2\pi k)^{d/2} \sqrt{\det M_\theta}} \exp\left(-\frac{x \cdot M_\theta^{-1} x}{2k}\right),$$

where M_θ denotes the covariance matrix of θ , and $x \cdot y$ stands for the usual scalar product in \mathbb{R}^d . Using the definition of p_k , it is not difficult to find constants $c_1, c_2 > 0$ such that

$$\sum_{\delta|x|/2 < k < 2\delta|x|} p_k(x - kD) \leq \frac{c_1}{|x|^{\frac{d}{2}}} \sum_{j=0}^{\infty} \exp\left(-\frac{c_2 j^2}{|x|}\right),$$

and the right-hand side of the last display is bounded by $|x|^{\frac{1-d}{2}}$ up to some multiplicative constant. On the other hand, since S is centered with finite moments of order 3, according to [53, Theorem 2.3.10], there exists $c_3 < \infty$ such that

$$|\pi_k(x - kD) - p_k(x - kD)| \leq \frac{c_3}{|x|} k^{\frac{1-d}{2}},$$

for all k, x satisfying that $|x - kD| \geq |x|^{1/2}$. Besides, we can use [53, Theorem 2.3.9] to show that

$$\sum_k' |\pi_k(x - kD) - p_k(x - kD)| = o(|x|^{\frac{1-d}{2}}),$$

where the sum is taken over all k such that $\delta|x|/2 < k < 2\delta|x|$ and $|x - kD| < |x|^{1/2}$. By combining these two results, one easily verifies that

$$\sum_{\delta|x|/2 < k < 2\delta|x|} |\pi_k(x - kD) - p_k(x - kD)| = O(|x|^{\frac{1-d}{2}}) \quad \text{as } |x| \rightarrow \infty.$$

The bound (4.5) thus follows from the identity

$$\sum_{\delta|x|/2 < k < 2\delta|x|} P_0(S_k = x - kD) = \sum_{\delta|x|/2 < k < 2\delta|x|} p_k(x - kD) + \sum_{\delta|x|/2 < k < 2\delta|x|} (\pi_k(x - kD) - p_k(x - kD)).$$

Therefore, we have finished the proof of Lemma 4.2. \square

Remark. We expect that the estimate (4.3) in Lemma 4.2 should hold under a weaker moment assumption on θ . More generally, it seems that deriving sharp estimates is still an open problem for the Green function of a general random walk on \mathbb{Z}^d with non-zero drift.

4.3 The critical dimension $d = 3$

For the tree-indexed random walk with non-zero drift, the critical dimension is expected to be 3, in the same sense as dimension 4 is shown to be critical for the *centered* tree-indexed random walk in [61, Section 3]. Unfortunately, we are so far unable to show this statement in its full generality. In the following discussions, we will restrict our attention to the case when the offspring distribution μ is geometric with parameter $1/2$ (the associated conditioned Galton–Watson tree \mathcal{T}_n is then uniformly distributed over all plane trees with $n + 1$ vertices), and we will consider a special case of jump distribution $\tilde{\theta}$ that we now describe.

Let θ be a symmetric probability distribution on \mathbb{Z}^3 . Assume that θ is supported on the subgroup spanned by $(0, 1, 0)$ and $(0, 0, 1)$, and that θ has small exponential moments. Then we take $D = (1, 0, 0)$ and define $\tilde{\theta}$ as the image of θ under the translation $x \in \mathbb{Z}^3 \mapsto x + D$. As previously, we write $S = (S_k)_{k \geq 0}$ for the random walk in \mathbb{Z}^3 with jump distribution θ , and we now assume that S starts from the origin 0 under the probability measure P . Due to our assumption on θ , the random walk S actually moves on the 2-dimensional hyperplane spanned by $(0, 1, 0)$ and $(0, 0, 1)$, which is isomorphic to \mathbb{Z}^2 . By abuse of notation, we let M_θ be the covariance matrix of the probability distribution on \mathbb{Z}^2 induced by θ , and we will suppose for simplicity that M_θ is of the form $\sigma^2 \text{Id}$, where Id is the two-dimensional identity matrix and $\sigma > 0$. This isotropy condition can of course be removed, and the reader will easily check that all subsequent arguments remain valid for a non-isotropic random walk: the role of σ^2 is then played by $(\det M_\theta)^{1/2}$. The following result is the analog of the case $d = 4$ of [61, Theorem 1].

Theorem 4.3. *Let \mathcal{R}_n be the range of the random walk $Z_{\mathcal{T}_n}$ in \mathbb{Z}^3 indexed by the uniform plane tree \mathcal{T}_n with $n + 1$ vertices, with jump distribution $\tilde{\theta}$. Under the above assumptions on θ and D , the range \mathcal{R}_n satisfies that*

$$\frac{\log n}{n} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi\sigma^2.$$

The preceding theorem is closely related to Theorem 7 in a recent work of Lalley and Zheng [50], where they considered a nearest-neighbor critical branching random walk in \mathbb{Z}^2 initiated by a single particle at the origin, assuming the critical offspring distribution to be of finite variance. We let Ω_n be the number of distinct sites occupied by this two-dimensional branching random walk at a *fixed* generation n , and denote by G_n the event that the associated critical branching process survives up to generation n . It is shown in [50, Theorem 7] that the conditional distributions of $\frac{\log n}{n} \Omega_n$ given the event G_n are tight. Following the same notation used in [50], one simply writes this result as

$$\Omega_n = O_p(n/\log n) \quad \text{given } G_n. \tag{4.6}$$

Combining (4.6) with [50, Theorem 6], we see that $n/\log n$ is the correct order of magnitude for Ω_n conditionally on G_n . If F_n denotes the event that the critical branching process has exactly n descendants, one expects that the size of Ω_n conditionally on F_n should be of the same order as the size of Ω_n conditionally on G_n .

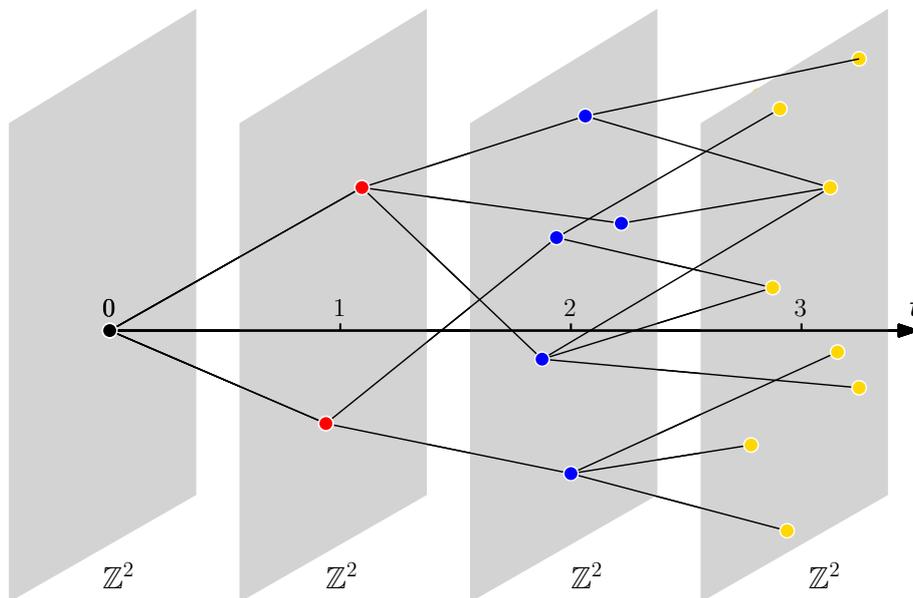


Figure 4.1: Following the construction of $Z_{\mathcal{T}_n}$, the vertices of \mathcal{T}_n are embedded in $\mathbb{Z}^2 \times \mathbb{Z}_+$, and they are colored differently according to their generation.

Now return to the tree-indexed random walk $Z_{\mathcal{T}_n}$ considered in the setting of Theorem 4.3. Since $D = (1, 0, 0)$, for every positive integer k smaller than the height of \mathcal{T}_n , we have

$$\{Z_{\mathcal{T}_n}(u) : u \in \mathcal{T}_n, |u| = k\} \subset \{(k, y, z) \in \mathbb{Z}^3 : (y, z) \in \mathbb{Z}^2\}.$$

See Fig. 4.1 for an illustration. In this situation, the range of $Z_{\mathcal{T}_n}$ turns out to be the sum of the numbers of occupied sites at each generation k , over all integers k from 0 to the maximal generation of \mathcal{T}_n . Theorem 4.3 above can therefore be reformulated as follows.

Theorem 4.4. *Let $\{Z_{\mathcal{T}_n}(u) : u \in \mathcal{T}_n\}$ be a two-dimensional random walk indexed by the uniform plane tree \mathcal{T}_n with $n + 1$ vertices. We assume that its jump distribution θ is symmetric, not supported on a strict subgroup of \mathbb{Z}^2 , and has small exponential moments. For every integer $k \geq 0$, we write $\Omega_k^{(n)}$ for the number of distinct sites in \mathbb{Z}^2 occupied by the k -th generation of \mathcal{T}_n , i.e.*

$$\Omega_k^{(n)} := \#\{Z_{\mathcal{T}_n}(u) : u \in \mathcal{T}_n, |u| = k\}.$$

By convention, $\Omega_k^{(n)} = 0$ if the set $\{u \in \mathcal{T}_n : |u| = k\}$ is empty. We define $R_n := \sum_{k \geq 0} \Omega_k^{(n)}$, then

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi(\det M_\theta)^{1/2},$$

where M_θ stands for the covariance matrix of θ .

Since the tree \mathcal{T}_n has height of order \sqrt{n} , one may intuitively think of R_n as a sum of \sqrt{n} quantities, all being of order $\frac{\sqrt{n}}{\log n}$ as suggested by (4.6). In view of Theorem 4.4, this heuristic argument indeed gives the correct asymptotic growth rate of R_n . We will establish Theorem 4.3 (and thus Theorem 4.4 as well) in the next two subsections.

4.3.1 Proof of Theorem 4.3

Our main tool is the free discrete snake $(W_n)_{n \geq 0}$ in \mathbb{Z}^3 associated with the jump distribution $\tilde{\theta}$. For its definition and the related notation, we refer the reader to [61, Section 3.1]. In particular, under the probability measure \mathbb{P} , the initial value W_0 of the discrete snake is distributed as $(-\tilde{S}_{-k})_{k \leq 0}$, where $\tilde{S} = (\tilde{S}_k)_{k \geq 0}$ defined as $\tilde{S}_k = S_k + kD$ is a random walk with jump distribution $\tilde{\theta}$. As an analog of Theorem 9 in [61], we have the following result concerning the range of the free snake.

Theorem 4.5. *Set $R_n := \#\{\widehat{W}_0, \widehat{W}_1, \dots, \widehat{W}_n\}$ for every integer $n \geq 0$. We have*

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2(\mathbb{P})} 4\pi\sigma^2.$$

The key ingredient for proving the preceding theorem is the next proposition, which is the analog of Proposition 8 in [61].

Proposition 4.6. *We have*

$$\lim_{n \rightarrow \infty} (\log n) \mathbb{P}(0 \notin \{\widehat{W}_1, \widehat{W}_2, \dots, \widehat{W}_n\}) = 4\pi\sigma^2. \quad (4.7)$$

Once Proposition 4.6 is established, Theorem 4.5 can be shown in the same way as Theorem 9 in [61]. Furthermore, we can verify that Theorem 4.3, viewed as a similar result to Theorem 4.5 for the “excursion” of the discrete snake, can be derived from the latter by the same method developed in [61] (see the arguments between Lemma 10 and Theorem 14 therein). So it is enough to prove the key estimate (4.7) in the rest of this section. To this end, we shall follow the outline of Section 3.2 in [61], during which necessary modifications of the proof will be explained in details.

Our first lemma is the analog of Lemma 15 in [61].

Lemma 4.7. *There exists a constant $M > 0$ such that*

$$\limsup_{k \rightarrow \infty} k \mathbb{P}(\widehat{W}_k = 0) \leq M.$$

Proof. For every integer $k \geq 0$, we set

$$\underline{\zeta}_k = \min_{0 \leq j \leq k} \zeta_j,$$

and

$$X_k = \zeta_k - 2\underline{\zeta}_k.$$

Recall that $(\zeta_k)_{k \geq 0}$ is a simple random walk on \mathbb{Z} started from 0 under \mathbb{P} . A discrete analog of Pitman’s theorem (see e.g. [55]) shows that for every integer $m \geq 0$ such that $\mathbb{P}(X_k = m) > 0$ and $0 \leq j \leq m$,

$$\mathbb{P}(\zeta_k - \underline{\zeta}_k = j \mid X_k = m) = \frac{1}{m+1}.$$

From the construction of the discrete snake, the conditional distribution of \widehat{W}_k knowing that $\zeta_k - \underline{\zeta}_k = j$ and $X_k = m$ is the law of $\widetilde{S}_j^{(1)} - \widetilde{S}_{m-j}^{(2)}$, where $\widetilde{S}^{(1)}$ and $\widetilde{S}^{(2)}$ are two independent copies of \widetilde{S} under the probability measure P . Using the fact that S is symmetric, we have

$$P(\widetilde{S}_j^{(1)} - \widetilde{S}_{m-j}^{(2)} = 0) = P(S_m + (2j - m)D = 0),$$

and accordingly,

$$\mathbb{P}(\widehat{W}_k = 0) = \sum_{m=0}^{\infty} \mathbb{P}(X_k = m) \left(\sum_{j=0}^m \mathbb{P}(\zeta_k - \underline{\zeta}_k = j \mid X_k = m) P(S_m + (2j - m)D = 0) \right).$$

Then applying formula (32) in [61] for the probability $\mathbb{P}(X_k = m)$, we obtain

$$\mathbb{P}(\widehat{W}_k = 0) = \sum_{m=0}^k \frac{2}{k + m + 2} \left((m + 1) \sum_{j=0}^m P(S_m + (2j - m)D = 0) \right) P_0(Y_k = m), \quad (4.8)$$

where $(Y_k)_{k \geq 0}$ stands for a simple random walk on \mathbb{Z} that starts from 0 under the probability measure P_0 . However, observe that $P(S_m + (2j - m)D = 0) = P(S_m = 0) \mathbf{1}_{\{m=2j\}}$ and

$$(m + 1) \sum_{j=0}^m P(S_m + (2j - m)D = 0) = (m + 1) P(S_m = 0) \mathbf{1}_{\{m \text{ is even}\}}.$$

Since the symmetric random walk S is essentially two-dimensional, the local limit theorem implies that we can find a constant $M > 0$ such that

$$\limsup_{m \rightarrow \infty} (m + 1) \sum_{j=0}^m P(S_m + (2j - m)D = 0) \leq M. \quad (4.9)$$

The result of the lemma therefore follows by combining (4.8) and (4.9). \square

Owing to the previous lemma, for any $\eta \in (0, 1/4)$, we can choose $\delta > 0$ small enough so that, for every sufficiently large n ,

$$\sum_{k=\lfloor (1-\delta)n \rfloor}^n \mathbb{P}(\widehat{W}_k = 0) < \eta.$$

At the same time, one can check that the statement of Lemma 16 in [61] remains valid in the present setting, and equation (33) in [61] now becomes

$$\mathbb{E}_{(W_0)} \left[\sum_{k=0}^{\tau_m-1} \mathbf{1}_{\{\widehat{W}_k=0\}} \right] = 2 \sum_{j=0}^{m-1} G_{\bar{\theta}}(-W_0(-j)),$$

where $\tau_m = \inf\{k \geq 0 : S_k = S_0 - m\}$ and $G_{\bar{\theta}}$ is the Green function of the non-centered random walk \widetilde{S} , namely

$$G_{\bar{\theta}}(x) = \sum_{k=0}^{\infty} P(\widetilde{S}_k = x) = \sum_{k=0}^{\infty} P(S_k = x - kD), \quad \forall x \in \mathbb{Z}^3.$$

Recall that under \mathbb{P} , the random path $(-W_0(-j))_{j \geq 0}$ has the same distribution as $(\widetilde{S}_j)_{j \geq 0}$. The following lemma is the analog of Lemma 17 in [61].

Lemma 4.8. *For every integer $p \geq 1$, there exists a constant $\tilde{C}(p)$ such that, for every $n \geq 2$,*

$$E\left[\left(\sum_{j=0}^n G_{\tilde{\theta}}(\tilde{S}_j)\right)^p\right] \leq \tilde{C}(p) (\log n)^p.$$

Proof. If $x = (k, \bar{x})$ with $k \geq 0$ and $\bar{x} \in \mathbb{Z}^2$, we have $G_{\tilde{\theta}}(x) = P(S_k = (0, \bar{x}))$. Otherwise, $G_{\tilde{\theta}}(x) = 0$. Moreover, due to our assumptions on θ , we know that for every $\ell \geq 1$ there exists a finite constant $C_\ell > 0$ such that

$$P(S_k = (0, \bar{x})) \leq C_\ell \left(\frac{1}{k} \wedge \left(\frac{k}{|\bar{x}|^2}\right)^\ell\right). \quad (4.10)$$

So in particular, one recovers the estimate (4.3) in the case $d = 3$, namely

$$G_{\tilde{\theta}}(x) \leq \frac{C_{\tilde{\theta}}}{|x|}, \quad \forall x \in \mathbb{Z}^3 \setminus \{0\}. \quad (4.11)$$

Notice that P -a.s. $|\tilde{S}_j| \geq j|D|$ for every integer $j \geq 0$. The required moment estimate thus follows immediately from (4.11). \square

Using (4.10) and (4.11), one can check that there exists a positive constant \tilde{C} depending on $\tilde{\theta}$ such that for every $x \in \mathbb{Z}^3$,

$$\Phi_{\tilde{\theta}}(x) := \sum_{y \in \mathbb{Z}^3} G_{\tilde{\theta}}(y) G_{\tilde{\theta}}(x - y)^2 \leq \frac{\tilde{C}(1 + \log(|x| \vee 1))}{|x| \vee 1},$$

which entails that

$$\mathbb{E}\left[\left(\sum_{k=0}^{\tau_m-1} \mathbf{1}_{\{\widehat{W}_k=0\}}\right)^2\right] = O((\log m)^2) \quad \text{as } m \rightarrow \infty.$$

For the derivation of the last display, we refer the reader to the arguments between displays (35) and (36) in [61].

We state our last lemma, which is the analog of Lemma 18 in [61].

Lemma 4.9. *For every $\alpha > 0$, there exists a constant C_α such that, for every integer $m \geq 2$, we have*

$$P\left(\left|\sum_{k=0}^m G_{\tilde{\theta}}(\tilde{S}_k) - \frac{1}{4\pi\sigma^2} \log m\right| \geq \alpha \log m\right) \leq C_\alpha (\log m)^{-3/2}. \quad (4.12)$$

Notice that, with all the ingredients prepared above, the key estimate (4.7) can be proved in a similar manner as Proposition 8 in [61]. Hence, in order to complete the proof of Proposition 4.6 (and that of Theorem 4.3), it suffices to prove Lemma 4.9. This will be our objective in the next subsection.

4.3.2 Proof of Lemma 4.9

Within this subsection, we slightly abuse our notation by using the same letter to denote the symmetric probability distribution on \mathbb{Z}^2 induced by θ . Correspondingly, the random walk S with jump distribution θ is directly viewed as a random walk on \mathbb{Z}^2 . It is assumed as usual that S starts from the origin 0 under the probability measure P . For every integer $k \geq 0$, we write its k -step transition probability

$$\pi_k(x) := P(S_k = x) \quad \text{for every } x \in \mathbb{Z}^2.$$

For technical convenience, we assume additionally that the random walk S is aperiodic. The arguments below can easily be extended to treat the periodic case.

Recall that under P , the projection of \tilde{S} on the one-dimensional subspace $\mathbb{Z} \times \{0\}^2$ is simply the deterministic walk $(|D|k)_{k \geq 0}$, while its projection on the orthogonal two-dimensional subspace $\{0\} \times \mathbb{Z}^2$ is the random walk S that we described above. Further considerations show that under P , $(G_{\tilde{\theta}}(\tilde{S}_k))_{k \geq 0}$ is identically distributed as $(\pi_k(S_k))_{k \geq 0}$. The estimate (4.12) is thus equivalent to

$$P\left(\left|\sum_{k=0}^m \pi_k(S_k) - \frac{1}{4\pi\sigma^2} \log m\right| \geq \alpha \log m\right) \leq C_\alpha (\log m)^{-3/2}. \quad (4.13)$$

Our strategy for proving the latter estimate is to first derive an analogous result for Brownian motion in \mathbb{R}^2 , and then to use a strong invariance principle to transfer this result to the random walk S .

We let $B = (B_t)_{t \geq 0}$ be a planar Brownian motion started from 0, also defined under the probability measure P . We write

$$p_t(x) := \frac{1}{2\pi t} \exp\left(-\frac{|x|^2}{2t}\right), \quad \forall x \in \mathbb{R}^2, t > 0,$$

for its transition density function. The Brownian motion version of (4.13) is given as follows.

Lemma 4.10. *Let $\varepsilon > 0$. There exist two positive constants $C(\varepsilon)$ and $\beta(\varepsilon)$ such that, for every $t > r \geq 1$,*

$$P\left(\left|\int_r^t p_s(B_s) ds - \frac{1}{4\pi} \log\left(\frac{t}{r}\right)\right| > \varepsilon \log\left(\frac{t}{r}\right)\right) \leq C(\varepsilon) \left(\frac{t}{r}\right)^{-\beta(\varepsilon)}.$$

Proof. By a scaling argument, it is enough to consider the case $t > r = 1$, and we will only treat that case. For every $u \geq 0$, we set

$$X_u := e^{-u/2} B_{e^u},$$

and then

$$\begin{aligned} P\left(\left|\int_1^t p_s(B_s) ds - \frac{1}{4\pi} \log t\right| > \varepsilon \log t\right) &= P\left(\left|\int_1^t \frac{1}{2\pi s} e^{-\frac{|B_s|^2}{2s}} ds - \frac{1}{4\pi} \log t\right| > \varepsilon \log t\right) \\ &= P\left(\left|\int_0^{\log t} \frac{1}{2\pi} e^{-\frac{|X_u|^2}{2}} du - \frac{1}{4\pi} \log t\right| > \varepsilon \log t\right). \end{aligned} \quad (4.14)$$

Note that $(X_u)_{u \geq 0}$ is a two-dimensional stationary Ornstein–Uhlenbeck process. It is a Harris recurrent regular Feller process with the invariant probability measure $\mathcal{N}(0, \text{Id})$. By the

ergodic theory of Feller processes (see e.g. Theorem 20.21 in [41] or Section X.3 in [82]), we have the pathwise convergence

$$\frac{1}{t} \int_0^t \exp\left(-\frac{|X_u|^2}{2}\right) du \xrightarrow[t \rightarrow \infty]{\text{a.s.}} E\left[\exp\left(-\frac{|X_0|^2}{2}\right)\right] = \frac{1}{2}.$$

On the other hand, let $\mathcal{M}_1(\mathbb{R}^2)$ be the space of all (Borel) probability measures on \mathbb{R}^2 equipped with the weak topology. For every $t > 0$, the empirical measure

$$L_t := \frac{1}{t} \int_0^t \delta_{X_u} du$$

is a well-defined $\mathcal{M}_1(\mathbb{R}^2)$ -valued random variable. Since the stationary Gaussian process $(X_u)_{u \geq 0}$ has a continuous spectral density vanishing at ∞ , we can apply Theorem 2.1 in [16] to see that the family of the laws of $(L_t)_{t \geq 0}$ satisfies the large deviation principle in $\mathcal{M}_1(\mathbb{R}^2)$ with a good convex rate function. As the mapping

$$\mu \in \mathcal{M}_1(\mathbb{R}^2) \mapsto \int \exp\left(-\frac{|x|^2}{2}\right) \mu(dx) \in \mathbb{R}$$

is continuous, it follows by the contraction principle (Theorem 4.2.1 in [23]) that the family of the laws of

$$Y_t := \frac{1}{t} \int_0^t \exp\left(-\frac{|X_u|^2}{2}\right) du, \quad t > 0$$

also satisfies the large deviation principle with a good rate function I . Moreover, noting that $Y_t \in (0, 1)$ for any $t > 0$, we can use [23, Theorem 4.5.10] to check that $x = 1/2$ is the unique point such that $I(x) = 0$. Hence, for every $\varepsilon > 0$, there exist two constants $C(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ such that, for every $t > 0$,

$$P\left(\left|\frac{1}{t} \int_0^t \exp\left(-\frac{|X_u|^2}{2}\right) du - \frac{1}{2}\right| > \varepsilon\right) \leq C(\varepsilon) \exp(-\beta(\varepsilon)t). \quad (4.15)$$

The statement of Lemma 4.10 therefore follows readily from (4.14) and (4.15). \square

Recall our assumptions that θ has small exponential moments, and that its covariance matrix is equal to $\sigma^2 \text{Id}$. By an extension due to Zaitsev [89] of the celebrated Komlós–Major–Tusnády strong invariance principle, we can construct on the same probability space the finite sequence (S_1, \dots, S_m) and the Brownian motion $(B_t)_{t \geq 0}$, in such a way that, for some constants $c > 0$, $c' > 0$ and $K > 0$ that do not depend on m , we have

$$E\left[\exp\left(c \max_{1 \leq k \leq m} |S_k - \sigma B_k|\right)\right] \leq K \exp(c' \log m).$$

It follows that we can find constants $C > 0$ and $a > 0$ (again independent of m) such that

$$P\left(\max_{1 \leq k \leq m} |S_k - \sigma B_k| > C \log m\right) \leq K m^{-a}. \quad (4.16)$$

On the other hand, from the standard estimate of Gaussian tail probability

$$\int_x^\infty \exp\left(-\frac{u^2}{2}\right) du < \int_x^\infty \frac{u}{x} \exp\left(-\frac{u^2}{2}\right) du = \frac{1}{x} \exp\left(-\frac{x^2}{2}\right), \quad \forall x > 0,$$

we know that for any integer $p \geq 1$,

$$P\left(\exists k \in \mathbb{N} \cap [(\log m)^{10}, m] \text{ such that } |\sigma B_k| \geq \frac{k^{2/3}}{2}\right) = O(m^{-p}),$$

as m tends to infinity. Together with (4.16), this implies that

$$P\left(\exists k \in \mathbb{N} \cap [(\log m)^{10}, m] \text{ such that } |S_k| \geq k^{2/3}\right) = O(m^{-a}).$$

So for the event

$$E_m := \left\{ \max_{1 \leq k \leq m} |S_k - \sigma B_k| \leq C \log m \right\} \cap \left\{ |S_k| \vee |\sigma B_k| \leq k^{2/3}, \forall k \in [(\log m)^{10}, m] \right\},$$

we have $P(E_m^c) = O((\log m)^{-2})$ according to the preceding estimates.

Since θ is supposed to have an exponential moment, one has the following local limit theorem for the θ -random walk S on \mathbb{Z}^2 , which can be found in Lawler and Limic [53, Theorem 2.3.11]. There exists a constant $\rho > 0$ such that for all $n \geq 0$ and all $x \in \mathbb{Z}^2$ with $|x| < \rho n$,

$$\pi_n(x) = p_{\sigma^2 n}(x) \exp\left(O\left(\frac{1}{n} + \frac{|x|^4}{n^3}\right)\right).$$

Thus for any $\eta > 0$, there exists $M(\eta) \in \mathbb{N}$ such that for every integer $m \geq M(\eta)$, on the event E_m we have

$$(1 - \eta) \sum_{k=\lceil (\log m)^{10} \rceil}^m p_{\sigma^2 k}(S_k) \leq \sum_{k=\lceil (\log m)^{10} \rceil}^m \pi_k(S_k) \leq (1 + \eta) \sum_{k=\lceil (\log m)^{10} \rceil}^m p_{\sigma^2 k}(S_k). \quad (4.17)$$

We next observe that on the event E_m ,

$$\begin{aligned} \left| \sum_{k=\lceil (\log m)^{10} \rceil}^m (p_{\sigma^2 k}(S_k) - p_{\sigma^2 k}(\sigma B_k)) \right| &\leq \sum_{k=\lceil (\log m)^{10} \rceil}^m \frac{1}{2\pi\sigma^2 k} \left| \exp\left(-\frac{|S_k|^2}{2\sigma^2 k}\right) - \exp\left(-\frac{|\sigma B_k|^2}{2\sigma^2 k}\right) \right| \\ &\leq \sum_{k=\lceil (\log m)^{10} \rceil}^m \frac{1}{2\pi\sigma^2 k} \left| \frac{|S_k|^2 - |\sigma B_k|^2}{2\sigma^2 k} \right| \\ &\leq \sum_{k=\lceil (\log m)^{10} \rceil}^m \frac{2\sigma C |B_k| \log m + C^2 (\log m)^2}{4\pi\sigma^4 k^2} \\ &= o((\log m)^{-2}), \quad \text{as } m \rightarrow \infty. \end{aligned} \quad (4.18)$$

Similarly, we can show that the sum

$$\sum_{k=\lceil (\log m)^{10} \rceil}^m p_{\sigma^2 k}(\sigma B_k)$$

is close to the integral

$$\int_{\lceil (\log m)^{10} \rceil}^{m+1} p_{\sigma^2 s}(\sigma B_s) ds = \frac{1}{\sigma^2} \int_{\lceil (\log m)^{10} \rceil}^{m+1} p_s(B_s) ds$$

up to a set of small probability. Indeed, outside of a set of probability $O(m^{-1})$, we have

$$\sup_{0 \leq k \leq m} \sup_{k \leq s \leq k+1} |B_s - B_k| \leq \log m.$$

One then verifies that on the event

$$\tilde{E}_m := E_m \cap \left\{ \sup_{0 \leq k \leq m} \sup_{k \leq s \leq k+1} |B_s - B_k| \leq \log m \right\},$$

it holds that

$$\left| \sum_{k=\lceil (\log m)^{10} \rceil}^m p_{\sigma^2 k}(\sigma B_k) - \frac{1}{\sigma^2} \int_{\lceil (\log m)^{10} \rceil}^{m+1} p_s(B_s) ds \right| = o((\log m)^{-2}). \quad (4.19)$$

Furthermore, by symmetry of the jump distribution θ , we notice that for every integer $k \geq 0$,

$$E[\pi_k(S_k)] = \pi_{2k}(0).$$

An application of the local limit theorem yields immediately that

$$E \left[\sum_{k=0}^m \pi_k(S_k) \right] = \frac{1}{4\pi\sigma^2} \log m + O(1) \quad \text{as } m \rightarrow \infty.$$

By analogous arguments (cf. the proof of [50, Lemma 27] in a slightly different setting), we have

$$E \left[\left(\sum_{k=0}^m \pi_k(S_k) \right)^2 \right] = \left(\frac{1}{4\pi\sigma^2} \log m \right)^2 + o((\log m)^2) \quad \text{as } m \rightarrow \infty.$$

In particular, there exists $C' > 0$ such that for every integer $m \geq 2$ and every $h > 0$,

$$P \left(\sum_{k=0}^m \pi_k(S_k) \geq h \right) \leq \frac{C'(\log m)^2}{h^2}. \quad (4.20)$$

Finally, for every $\alpha > 0$,

$$\begin{aligned} & P \left(\left| \sum_{k=0}^m \pi_k(S_k) - \frac{1}{4\pi\sigma^2} \log m \right| \geq \alpha \log m \right) \\ & \leq P \left(\sum_{k=0}^{\lceil (\log m)^{10} \rceil} \pi_k(S_k) \geq \frac{\alpha}{2} \log m \right) + P \left(\left| \sum_{k=\lceil (\log m)^{10} \rceil}^m \pi_k(S_k) - \frac{1}{4\pi\sigma^2} \log m \right| \geq \frac{\alpha}{2} \log m \right). \end{aligned}$$

The first term in the right-hand side is $o((\log m)^{-3/2})$ by (4.20). On the other hand, by (4.17), (4.18) and (4.19), the second term is bounded by

$$P(\tilde{E}_m^c) + P \left(\left| \frac{1}{\sigma^2} \int_{\lceil (\log m)^{10} \rceil}^{m+1} p_s(B_s) ds - \frac{1}{4\pi\sigma^2} \log m \right| \geq \alpha' \log m \right)$$

with some constant $\alpha' > 0$ independent of m , provided that m is sufficiently large (to be precise, we need to have chosen a small $\eta > 0$ depending on α for this argument to work). We have seen that $P(\tilde{E}_m^c) = O((\log m)^{-2})$. Meanwhile, Lemma 4.10 implies that

$$P \left(\left| \frac{1}{\sigma^2} \int_{\lceil (\log m)^{10} \rceil}^{m+1} p_s(B_s) ds - \frac{1}{4\pi\sigma^2} \log m \right| \geq \alpha' \log m \right) = O(m^{-b})$$

for some $b > 0$. The proof of Lemma 4.9 is therefore completed.

4.4 The lower dimensions $d \leq 2$

In this section, we briefly discuss dimensions $d = 1$ and $d = 2$, without giving the details of the arguments. In contrast with the previous setting, we will consider in this section a general probability distribution $\tilde{\theta}$ on \mathbb{Z}^d , whose mean $D \in \mathbb{R}^d$ is different from 0. As usual, for $n \in \mathcal{G}$, we denote by $Z_{\mathcal{T}_n}$ a random walk with jump distribution $\tilde{\theta}$ indexed by a μ -Galton–Watson tree \mathcal{T}_n conditioned to have exactly $n + 1$ vertices.

4.4.1 The one-dimensional case

We assume that

- μ is a nondegenerate critical offspring distribution on \mathbb{Z}_+ , such that, for some $\lambda > 0$,

$$\sum_{k=0}^{\infty} e^{\lambda k} \mu(k) < \infty,$$

and we set $\sigma_\mu = (\text{var } \mu)^{1/2} > 0$;

- $\tilde{\theta}$ is a probability measure on \mathbb{Z} , which is not supported on a strict subgroup of \mathbb{Z} . The measure $\tilde{\theta}$ satisfies

$$\lim_{r \rightarrow +\infty} r^2 \tilde{\theta}(\{x \in \mathbb{Z} : |x| > r\}) = 0,$$

and its mean $D \neq 0$.

Without loss of generality, we may suppose by symmetry that $D > 0$.

Under the preceding assumptions, Theorem 8 in [39] implies that

$$\left(\frac{1}{\sqrt{n}} \max_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u), \frac{1}{\sqrt{n}} \min_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u) \right) \xrightarrow[n \rightarrow \infty, n \in \mathcal{G}]{(d)} \left(\frac{2D}{\sigma_\mu} \max_{0 \leq s \leq 1} \mathbf{e}_s, 0 \right),$$

where in the limit $\mathbf{e} = (\mathbf{e}_s)_{0 \leq s \leq 1}$ denotes a normalized Brownian excursion. As an immediate corollary, we have

$$\frac{1}{\sqrt{n}} \left(\max_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u) - \min_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u) \right) \xrightarrow[n \rightarrow \infty, n \in \mathcal{G}]{(d)} \frac{2D}{\sigma_\mu} \max_{0 \leq s \leq 1} \mathbf{e}_s. \quad (4.21)$$

The distribution of the limit variable $\max_{[0,1]} \mathbf{e}_s$ was calculated by Chung [18] in the following form. For every $x > 0$,

$$P\left(\max_{0 \leq s \leq 1} \mathbf{e}_s > x \right) = 2 \sum_{k=1}^{\infty} (4k^2 x^2 - 1) \exp(-2k^2 x^2).$$

See e.g. [10, Proposition 2.1] for further information on this distribution.

If the jump distribution θ is supported on the set $\{0, \pm 1\}$, then the range \mathcal{R}_n of $Z_{\mathcal{T}_n}$ satisfies the equality

$$\mathcal{R}_n = \max_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u) - \min_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u),$$

and from (4.21) we obtain that

$$\frac{\mathcal{R}_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty, n \in \mathcal{G}]{(d)} \frac{2D}{\sigma_\mu} \max_{0 \leq s \leq 1} \mathbf{e}_s. \quad (4.22)$$

In general we have only

$$\mathcal{R}_n \leq \max_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u) - \min_{u \in \mathcal{T}_n} Z_{\mathcal{T}_n}(u),$$

and the convergence (4.21) merely entails $\mathcal{R}_n = O_p(\sqrt{n})$, by which we mean that the distributions of $(\mathcal{R}_n/\sqrt{n}, n \in \mathcal{G})$ are tight. In fact, one can extend the convergence (4.22) to more general jump distributions. We omit here the details.

4.4.2 The two-dimensional case

For the tree-indexed random walk $Z_{\mathcal{T}_n}$ on \mathbb{Z}^2 with jump distribution $\tilde{\theta}$, it is not difficult to see that its range $\mathcal{R}_n = O_p(n^{3/4})$, provided we assume some conditions on μ and θ similar to those in the previous subsection. To further describe the possible distributional limit of $n^{-3/4}\mathcal{R}_n$, we introduce the one-dimensional Brownian snake $W^{(1)} = (W_s^{(1)})_{0 \leq s \leq 1}$, which starts from 0 and is driven by the normalized Brownian excursion \mathbf{e} . As explained in [62], it is a random process in \mathbb{R} distributed according to the normalized excursion measure $\mathbb{N}_0^{(1)}$ of Brownian snake. Given a positive constant $c > 0$, for every $r \geq 0$, we let $X_r^{(c)}$ be the random measure on \mathbb{R} defined by the formula

$$\langle X_r^{(c)}, f \rangle = \int_0^1 dL_s^r(\mathbf{e}) f(cW_s^{(1)})$$

for any nonnegative measurable function f on \mathbb{R} , where $(L_s^r(\mathbf{e}), s \geq 0)$ stands for the local time of \mathbf{e} at level r . From this definition, the support of the measure $X_r^{(c)}$ is given as

$$\text{supp}(X_r^{(c)}) = \{cW_s^{(1)} : 0 \leq s \leq 1 \text{ such that } \mathbf{e}_s = r\}.$$

We conjecture that under some mild assumptions on μ and $\tilde{\theta}$, there exist two positive constants c_1 and c_2 , both depending on μ and $\tilde{\theta}$, such that

$$\frac{\mathcal{R}_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty, n \in \mathcal{G}]{(d)} \lambda_2(\{(c_1 r, x) \in \mathbb{R}^2 : r \geq 0 \text{ and } x \in \text{supp}(X_r^{(c_2)})\}),$$

where λ_2 denotes the Lebesgue measure on \mathbb{R}^2 .

DEUXIÈME PARTIE

LA MARCHE ALÉATOIRE SUR UN ARBRE
ET LA MESURE HARMONIQUE

THE HARMONIC MEASURE OF BALLS IN CRITICAL GALTON–WATSON TREES

LES RÉSULTATS DE CE CHAPITRE SONT ISSUS DE L'ARTICLE [66] PUBLIÉ DANS *The Electronic Journal of Probability* **19** (2014), NO. 97, 1–35.

We study properties of the harmonic measure of balls in large critical Galton–Watson trees whose offspring distribution is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2]$. Here the harmonic measure refers to the hitting distribution of height n by simple random walk on the critical Galton–Watson tree conditioned on non-extinction at generation n . For a ball of radius n centered at the root, we prove that, although the size of the boundary is roughly of order $n^{\frac{1}{\alpha-1}}$, most of the harmonic measure is supported on a boundary subset of size approximately equal to $n^{\beta\alpha}$, where the constant $\beta_\alpha \in (0, \frac{1}{\alpha-1})$ depends only on the index α . Using an explicit expression of β_α , we are able to show the uniform boundedness of $(\beta_\alpha, 1 < \alpha \leq 2)$. These are generalizations of results in a recent paper of Curien and Le Gall [20].

5.1 Introduction

Recently, Curien and Le Gall have studied in [20] the properties of harmonic measure on generation n of a critical Galton–Watson tree, whose offspring distribution has finite variance and which is conditioned to have height greater than n . They have shown the existence of a universal constant $\beta < 1$ such that, with high probability, most of the harmonic measure on generation n of the tree is concentrated on a set of approximately n^β vertices, although the number of vertices at generation n is of order n . Their approach is based on the study of a similar continuous model, where it is established that the Hausdorff dimension of the (continuous) harmonic measure is almost surely equal to β .

In this paper, we continue the above work by extending their results to the critical Galton–Watson trees whose offspring distribution has infinite variance. To be more precise, let ρ be a non-degenerate probability measure on \mathbb{Z}_+ with mean one, and we assume throughout this paper that ρ is in the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$, which means that

$$\sum_{k \geq 0} \rho(k)r^k = r + (1-r)^\alpha L(1-r) \quad \text{for any } r \in [0, 1), \quad (5.1)$$

where the function $L(x)$ is slowly varying as $x \rightarrow 0^+$. We point out that the finite variance condition for ρ is sufficient for the previous statement to hold with $\alpha = 2$. When $\alpha \in (1, 2)$, by results of [31, Chapters XIII and XVII], the condition (5.1) is satisfied if and only if the tail

probability

$$\sum_{k \geq x} \rho(k) = \rho([x, +\infty))$$

varies regularly with exponent $-\alpha$ as $x \rightarrow +\infty$. See e.g. [12] for the definition of regularly varying functions.

Under the probability measure \mathbb{P} , for every integer $n \geq 0$, we let $\mathsf{T}^{(n)}$ be a Galton–Watson tree with offspring distribution ρ , conditioned on non-extinction at generation n . Conditionally given the tree $\mathsf{T}^{(n)}$, we consider simple random walk on $\mathsf{T}^{(n)}$ starting from the root. The probability distribution of the first hitting point of generation n by random walk will be called the harmonic measure μ_n , which is supported on the set $\mathsf{T}_n^{(n)}$ of all vertices of $\mathsf{T}^{(n)}$ at generation n .

Let $q_n > 0$ be the probability that a critical Galton–Watson tree $\mathsf{T}^{(0)}$ survives up to generation n . It is shown in [83] that, as $n \rightarrow \infty$, the probability q_n decreases as $n^{-\frac{1}{\alpha-1}}$ up to multiplication by a slowly varying function, and $q_n \# \mathsf{T}_n^{(n)}$ converges in distribution to a non-trivial limit distribution on \mathbb{R}_+ , whose Laplace transform can be written explicitly in terms of the parameter α . The following theorem generalizes the result [20, Theorem 1] in the finite variance case ($\alpha = 2$) to all $\alpha \in (1, 2]$.

Theorem 5.1. *If the offspring distribution ρ has mean one and belongs to the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$, there exists a constant $\beta_\alpha \in (0, \frac{1}{\alpha-1})$, which only depends on α , such that for every $\delta > 0$, we have the convergence in \mathbb{P} -probability*

$$\mu_n \left(\{v \in \mathsf{T}_n^{(n)} : n^{-\beta_\alpha - \delta} \leq \mu_n(v) \leq n^{-\beta_\alpha + \delta}\} \right) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1. \quad (5.2)$$

Consequently, for every $\varepsilon \in (0, 1)$, there exists, with \mathbb{P} -probability tending to 1 as $n \rightarrow \infty$, a subset $A_{n,\varepsilon}$ of $\mathsf{T}_n^{(n)}$ such that $\#A_{n,\varepsilon} \leq n^{\beta_\alpha + \delta}$ and $\mu_n(A_{n,\varepsilon}) \geq 1 - \varepsilon$. Conversely, the maximal μ_n -measure of a set of cardinality bounded by $n^{\beta_\alpha - \delta}$ tends to 0 as $n \rightarrow \infty$, in \mathbb{P} -probability.

The last two assertions of the preceding theorem are easy consequences of the convergence (5.2), as explained in [20].

We observe that the hitting distribution μ_n of generation n by simple random walk on $\mathsf{T}^{(n)}$ is unaffected if we remove the branches of $\mathsf{T}^{(n)}$ that do not reach height n . Thus in order to establish the preceding result, we may consider simple random walk on T^{*n} , the reduced tree associated with $\mathsf{T}^{(n)}$, which consists of all vertices of $\mathsf{T}^{(n)}$ that have at least one descendant at generation n .

When the critical offspring distribution ρ has infinite variance, scaling limits of the discrete reduced trees T^{*n} have been studied in [87] and [88]. If we scale the graph distances by the factor n^{-1} , the discrete reduced trees $n^{-1}\mathsf{T}^{*n}$ converge to a random compact rooted \mathbb{R} -tree $\Delta^{(\alpha)}$ that we now describe. For every $\alpha \in (1, 2]$, we define the α -offspring distribution θ_α as follows. For $\alpha = 2$, we let $\theta_2 = \delta_2$ be the Dirac measure at 2. If $\alpha < 2$, θ_α is the probability measure on \mathbb{Z}_+ given by

$$\begin{aligned} \theta_\alpha(0) &= \theta_\alpha(1) = 0, \\ \theta_\alpha(k) &= \frac{\alpha \Gamma(k - \alpha)}{k! \Gamma(2 - \alpha)} = \frac{\alpha(2 - \alpha)(3 - \alpha) \cdots (k - 1 - \alpha)}{k!}, \quad \forall k \geq 2, \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. We let U_\emptyset be a random variable uniformly distributed over $[0, 1]$, and let K_\emptyset be a random variable distributed according to θ_α , independent of U_\emptyset . To construct $\Delta^{(\alpha)}$, one starts with an oriented line segment of length U_\emptyset , whose origin will be the

root of the tree. We call K_\emptyset the offspring number of the root \emptyset . Correspondingly, at the other end of the first line segment, we attach the origins of K_\emptyset oriented line segments with respective lengths $U_1, U_2, \dots, U_{K_\emptyset}$, such that, conditionally given U_\emptyset and K_\emptyset , the variables $U_1, U_2, \dots, U_{K_\emptyset}$ are independent and uniformly distributed over $[0, 1 - U_\emptyset]$. This finishes the first step of the construction. In the second step, for the first of these K_\emptyset line segments, we independently sample a new offspring number K_1 distributed as θ_α , and attach K_1 new line segments whose lengths are again independent and uniformly distributed over $[0, 1 - U_\emptyset - U_1]$, conditionally on all the random variables appeared before. For the other $K_\emptyset - 1$ line segments, we repeat this procedure independently. We continue in this way and after an infinite number of steps we get a random non-compact rooted \mathbb{R} -tree, whose completion is the random compact rooted \mathbb{R} -tree $\Delta^{(\alpha)}$. See Fig. 5.1 in Section 5.2.1 for an illustration. We will call $\Delta^{(\alpha)}$ the reduced stable tree of parameter α . Notice that all the offspring numbers involved in the construction of $\Delta^{(2)}$ are a.s. equal to 2, which correspond to the binary branching mechanism. In contrast, this is no longer the case when $1 < \alpha < 2$.

We denote by \mathbf{d} the intrinsic metric on $\Delta^{(\alpha)}$. By definition, the boundary $\partial\Delta^{(\alpha)}$ consists of all points of $\Delta^{(\alpha)}$ at height 1. As the continuous analogue of simple random walk, we can define Brownian motion on $\Delta^{(\alpha)}$ starting from the root and up to the first hitting time of $\partial\Delta^{(\alpha)}$. It behaves like linear Brownian motion as long as it stays inside a line segment of $\Delta^{(\alpha)}$. It is reflected at the root of $\Delta^{(\alpha)}$ and when it arrives at a branching point, it chooses each of the adjacent line segments with equal probabilities. We define the (continuous) harmonic measure μ_α as the (quenched) distribution of the first hitting point of $\partial\Delta^{(\alpha)}$ by Brownian motion.

Theorem 5.2. *For every index $\alpha \in (1, 2]$, with the same constant β_α as in Theorem 5.1, we have \mathbb{P} -a.s. $\mu_\alpha(dx)$ -a.e.,*

$$\lim_{r \downarrow 0} \frac{\log \mu_\alpha(\mathcal{B}_\mathbf{d}(x, r))}{\log r} = \beta_\alpha, \quad (5.3)$$

where $\mathcal{B}_\mathbf{d}(x, r)$ stands for the closed ball of radius r centered at x in the metric space $(\Delta^{(\alpha)}, \mathbf{d})$. Consequently, the Hausdorff dimension of μ_α is \mathbb{P} -a.s. equal to β_α .

According to Lemma 4.1 in [71], the last assertion of the preceding theorem follows directly from (5.3). As another direct consequence of (5.3), we have that \mathbb{P} -a.s. for $\mu_\alpha(dx)$ -a.e. $x \in \partial\Delta^{(\alpha)}$, $\mu_\alpha(\mathcal{B}_\mathbf{d}(x, r)) \rightarrow 0$ as $r \downarrow 0$, which is equivalent to non-atomicity of μ_α .

Since it has been proved in [28, Theorem 1.5] that the Hausdorff dimension of $\partial\Delta^{(\alpha)}$ with respect to \mathbf{d} is a.s. equal to $\frac{1}{\alpha-1}$, the previous theorem implies that the harmonic measure has a.s. strictly smaller Hausdorff dimension than that of the whole boundary of the reduced stable tree. This phenomenon of dimension drop has been shown in [20, Theorem 2] for the special case of binary branching $\alpha = 2$.

We prove Theorem 5.2 in Section 5.2.5, where our approach is different and shorter than the one developed in [20] for the special case $\alpha = 2$.

Notice that the Hausdorff dimension of the boundary $\partial\Delta^{(\alpha)}$ increases to infinity when $\alpha \downarrow 1$. However, it is an interesting fact that the Hausdorff dimension of the harmonic measure remains bounded when $\alpha \downarrow 1$.

Theorem 5.3. *There exists a constant $C > 0$ such that for any $\alpha \in (1, 2]$, we have $\beta_\alpha < C$.*

Our proof of Theorem 5.3 relies on the fact that the constant β_α in Theorems 5.1 and 5.2 can be expressed in terms of the conductance of $\Delta^{(\alpha)}$. Informally, if we think of the random tree $\Delta^{(\alpha)}$ as a network of resistors with unit resistance per unit length, the effective conductance between the root and the boundary $\partial\Delta^{(\alpha)}$ is a random variable which we denote by $\mathcal{C}^{(\alpha)}$. From

a probabilistic point of view, it is the mass under the Brownian excursion measure for the excursion paths away from the root that hit height 1. Following the definition of $\Delta^{(\alpha)}$ and the above electric network interpretation, the distribution of $\mathcal{C}^{(\alpha)}$ satisfies the recursive distributional equation

$$\mathcal{C}^{(\alpha)} \stackrel{(d)}{=} \left(U + \frac{1-U}{\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \dots + \mathcal{C}_{N_\alpha}^{(\alpha)}} \right)^{-1}, \quad (5.4)$$

where $(\mathcal{C}_i^{(\alpha)})_{i \geq 1}$ are i.i.d. copies of $\mathcal{C}^{(\alpha)}$, the integer-valued random variable N_α is distributed according to θ_α , and U is uniformly distributed over $[0, 1]$. All these random variables are supposed to be independent.

Proposition 5.4. *For any $\alpha \in (1, 2]$, the distribution γ_α of the conductance $\mathcal{C}^{(\alpha)}$ is characterized in the class of all probability measures on $[1, \infty)$ by the distributional equation (5.4). The constant β_α appearing in Theorems 5.1 and 5.2 is given by*

$$\beta_\alpha = \frac{1}{2} \left(\frac{(\int \gamma_\alpha(ds)s)^2}{\iint \gamma_\alpha(ds)\gamma_\alpha(dt) \frac{st}{s+t-1}} - 1 \right). \quad (5.5)$$

Interestingly, formula (5.5) expresses the exponent β_α as the same function of the distribution γ_α , for all $\alpha \in (1, 2]$. In the course of the proof, we obtain two other formulas for β_α (see (5.23) and (5.24) below), but they both depend on α in a more complicated way, which also involves the distribution θ_α .

The paper is organized as follows. In Section 5.2 below, we study the continuous model of Brownian motion on $\Delta^{(\alpha)}$. A formal definition of the reduced stable tree $\Delta^{(\alpha)}$ is given in Section 5.2.1. In Section 5.2.2 we explain how to relate $\Delta^{(\alpha)}$ to an infinite supercritical continuous-time Galton–Watson tree $\Gamma^{(\alpha)}$, and we reformulate Theorem 5.2 in terms of Brownian motion with drift 1/2 on $\Gamma^{(\alpha)}$. Properties of the law of the random conductance $\mathcal{C}^{(\alpha)}$, including the first assertion of Proposition 5.4, are discussed in Section 5.2.3, and Section 5.2.4 gives the coupling argument that allows one to derive Theorem 5.3 from formula (5.5). Section 5.2.5 is devoted to the proofs of Theorem 5.2 and of formula (5.5). We emphasize that our approach to Theorem 5.2 is different from the one used in [20] when $\alpha = 2$. In fact we use an invariant measure for the environment seen by Brownian motion on $\Gamma^{(\alpha)}$ at the last passage time of a node of the n -th generation, instead of the last passage time at a height h as in [20]. We then apply the ergodic theory on Galton–Watson trees, which is a powerful tool initially developed in [71].

In Section 5.3 we proceed to the discrete setting concerning simple random walk on the discrete reduced tree \mathbb{T}^{*n} . Let us emphasize that, when the critical offspring distribution ρ is in the domain of attraction of a stable distribution of index $\alpha \in (1, 2)$, the convergence of discrete reduced trees is less simple than in the special case $\alpha = 2$ where we have a.s. a binary branching structure. See Proposition 5.14 for a precise statement in our more general setting. Apart from this ingredient, we need several estimates for the discrete reduced tree \mathbb{T}^{*n} to derive Theorem 5.1 from Theorem 5.2. For example, Lemma 5.13 gives a bound for the size of level sets in \mathbb{T}^{*n} , and Lemma 5.19 presents a moment estimate for the (discrete) conductance $\mathcal{C}_n(\mathbb{T}^{*n})$ between generations 0 and n in \mathbb{T}^{*n} . Although the result analogous to Lemma 5.19 in [20] is a second moment estimate, we only manage to give a moment estimate of order strictly smaller than α if the critical offspring distribution ρ satisfies (5.1) with $\alpha \in (1, 2]$. Nevertheless, this is sufficient for our proof of Theorem 5.1, which is adapted from the one given in [20].

Comments and several open questions are gathered in Section 5.4. Following the work of Aïdékon [2], we obtain a candidate for the speed of Brownian motion with drift 1/2 on the infinite tree $\Gamma^{(\alpha)}$, expressed by (5.39) in terms of the continuous conductance $\mathcal{C}^{(\alpha)}$. Nonetheless, the monotonicity properties of this quantity remains open. It would also be of interest to know whether or not the Hausdorff dimension β_α of the continuous harmonic measure μ_α is monotone with respect to $\alpha \in (1, 2]$.

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5.2 The continuous setting

5.2.1 The reduced stable tree

We set

$$\mathcal{V} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where by convention $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. If $v = (v_1, \dots, v_n) \in \mathcal{V}$, we set $|v| = n$ (in particular, $|\emptyset| = 0$), and if $n \geq 1$, we define the parent of v as $\hat{v} = (v_1, \dots, v_{n-1})$ and then say that v is a child of \hat{v} . For two elements $v = (v_1, \dots, v_n)$ and $v' = (v'_1, \dots, v'_m)$ belonging to \mathcal{V} , their concatenation is $vv' := (v_1, \dots, v_n, v'_1, \dots, v'_m)$. The notions of a descendant and an ancestor of an element of \mathcal{V} are defined in the obvious way, with the convention that every $v \in \mathcal{V}$ is both an ancestor and a descendant of itself. If $v, w \in \mathcal{V}$, $v \wedge w$ is the unique element of \mathcal{V} such that it is a common ancestor of v and w , and $|v \wedge w|$ is maximal.

An infinite subset Π of \mathcal{V} is called an infinite discrete tree if there exists a collection of positive integers $k_v = k_v(\Pi) \in \mathbb{N}$ for every $v \in \mathcal{V}$ such that

$$\Pi = \{\emptyset\} \cup \{(v_1, \dots, v_n) \in \mathcal{V} : v_j \leq k_{(v_1, \dots, v_{j-1})} \text{ for every } 1 \leq j \leq n\}.$$

Recall the definition of the α -offspring distribution θ_α for $\alpha \in (1, 2]$. It will also be convenient to consider the case $\alpha = 1$, where we define θ_1 as the probability measure on \mathbb{Z}_+ given by

$$\begin{aligned} \theta_1(0) &= \theta_1(1) = 0, \\ \theta_1(k) &= \frac{1}{k(k-1)}, \quad \forall k \geq 2. \end{aligned}$$

If $\alpha \in (1, 2]$, the generating function of θ_α is given (see e.g. [26, p.74]) as

$$\sum_{k \geq 0} \theta_\alpha(k) r^k = \frac{(1-r)^\alpha - 1 + \alpha r}{\alpha - 1}, \quad \forall r \in (0, 1], \quad (5.6)$$

while for $\alpha = 1$,

$$\sum_{k \geq 0} \theta_1(k) r^k = r + (1-r) \log(1-r), \quad \forall r \in (0, 1]. \quad (5.7)$$

Notice that for $\alpha \in (1, 2]$, the mean of θ_α is given by

$$m_\alpha = \frac{\alpha}{\alpha - 1} \in [2, \infty),$$

whereas θ_1 has infinite mean.

For fixed $\alpha \in [1, 2]$, we introduce a collection $(K_\alpha(v))_{v \in \mathcal{V}}$ of independent random variables distributed according to θ_α under the probability measure \mathbb{P} , and define a random infinite discrete tree

$$\Pi^{(\alpha)} := \{\emptyset\} \cup \{(v_1, \dots, v_n) \in \mathcal{V} : v_j \leq K_\alpha((v_1, \dots, v_{j-1})) \text{ for every } 1 \leq j \leq n\}.$$

We point out that $\Pi^{(2)}$ is an infinite binary tree.

Let $(U_v)_{v \in \mathcal{V}}$ be another collection, independent of $(K_\alpha(v))_{v \in \mathcal{V}}$, consisting of independent real random variables uniformly distributed over $[0, 1]$ under the same probability measure \mathbb{P} . We set now

$$Y_\emptyset = U_\emptyset$$

and then by induction, for every $v \in \Pi^{(\alpha)} \setminus \{\emptyset\}$,

$$Y_v = Y_{\hat{v}} + U_v(1 - Y_{\hat{v}}).$$

Note that a.s. $0 \leq Y_v < 1$ for every $v \in \Pi^{(\alpha)}$. Consider then the set

$$\Delta_0^{(\alpha)} := (\{\emptyset\} \times [0, Y_\emptyset]) \cup \left(\bigcup_{v \in \Pi^{(\alpha)} \setminus \{\emptyset\}} \{v\} \times (Y_{\hat{v}}, Y_v] \right).$$

There is a straightforward way to define a metric \mathbf{d} on $\Delta_0^{(\alpha)}$, so that $(\Delta_0^{(\alpha)}, \mathbf{d})$ is a (noncompact) \mathbb{R} -tree and, for every $x = (v, r) \in \Delta_0^{(\alpha)}$, we have $\mathbf{d}((\emptyset, 0), x) = r$. To be specific, let $x = (v, r) \in \Delta_0^{(\alpha)}$ and $y = (w, r') \in \Delta_0^{(\alpha)}$:

- If v is a descendant (or an ancestor) of w , we set $\mathbf{d}(x, y) = |r - r'|$.
- Otherwise, $\mathbf{d}(x, y) = \mathbf{d}((v \wedge w, Y_{v \wedge w}), x) + \mathbf{d}((v \wedge w, Y_{v \wedge w}), y) = (r - Y_{v \wedge w}) + (r' - Y_{v \wedge w})$.

See Fig. 5.1 for an illustration of the tree $\Delta_0^{(\alpha)}$ when $\alpha < 2$.

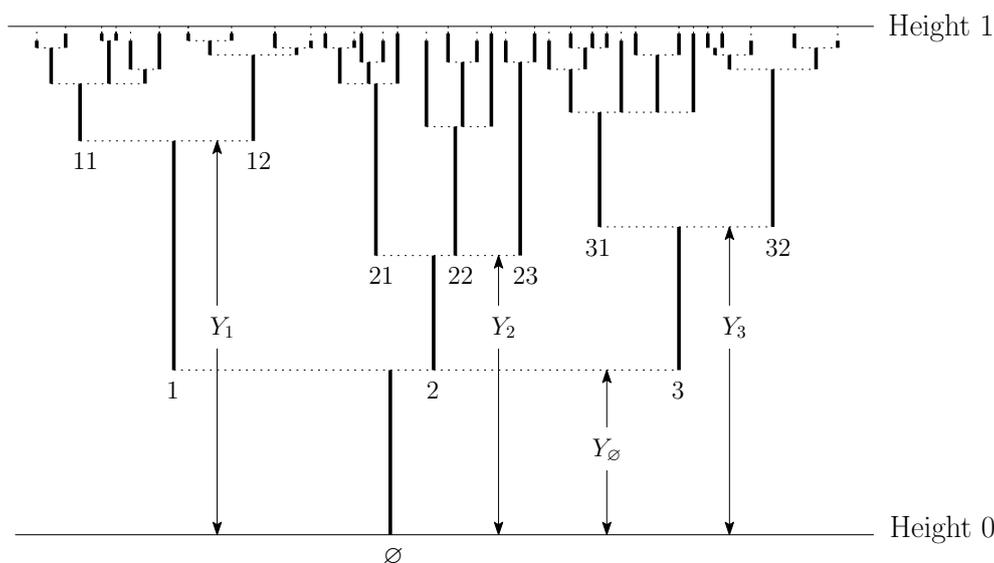


Figure 5.1: The random tree $\Delta_0^{(\alpha)}$ when $1 \leq \alpha < 2$

We let $\Delta^{(\alpha)}$ be the completion of $\Delta_0^{(\alpha)}$ with respect to the metric \mathbf{d} . Then

$$\Delta^{(\alpha)} = \Delta_0^{(\alpha)} \cup \partial\Delta^{(\alpha)}$$

where by definition $\partial\Delta^{(\alpha)} := \{x \in \Delta^{(\alpha)} : \mathbf{d}((\emptyset, 0), x) = 1\}$, which can be identified with a random subset of $\mathbb{N}^{\mathbb{N}}$. It is immediate to see that $(\Delta^{(\alpha)}, \mathbf{d})$ is an a.s. compact \mathbb{R} -tree, which we will call the reduced stable tree of index α .

The point $(\emptyset, 0)$ is called the root of $\Delta^{(\alpha)}$. For every $x \in \Delta^{(\alpha)}$, we set $H(x) = \mathbf{d}((\emptyset, 0), x)$ and call $H(x)$ the height of x . We can define a (non-strict) genealogical order on $\Delta^{(\alpha)}$ by setting $x \prec y$ if and only if x belongs to the geodesic path from the root to y .

For every $\varepsilon \in (0, 1)$, we set

$$\Delta_\varepsilon^{(\alpha)} := \{x \in \Delta^{(\alpha)} : H(x) \leq 1 - \varepsilon\},$$

which is also an a.s. compact \mathbb{R} -tree for the metric \mathbf{d} . The leaves of $\Delta_\varepsilon^{(\alpha)}$ are the points of the form $(v, 1 - \varepsilon)$ for all $v \in \mathcal{V}$ such that $Y_{\hat{v}} < 1 - \varepsilon \leq Y_v$. The branching points of $\Delta_\varepsilon^{(\alpha)}$ are the points of the form (v, Y_v) for all $v \in \mathcal{V}$ such that $Y_v < 1 - \varepsilon$.

Now conditionally on $\Delta^{(\alpha)}$, we can define Brownian motion on $\Delta_\varepsilon^{(\alpha)}$ starting from the root. Informally, this process behaves like linear Brownian motion as long as it stays on an “open interval” of the form $\{v\} \times (Y_{\hat{v}}, Y_v \wedge (1 - \varepsilon))$, and it is reflected at the root $(\emptyset, 0)$ and at the leaves of $\Delta_\varepsilon^{(\alpha)}$. When it arrives at a branching point of the tree, it chooses each of the possible line segments ending at this point with equal probabilities. By taking a sequence $\varepsilon_n = 2^{-n}$, $n \geq 1$ and then letting n go to infinity, we can construct under the same probability measure P a Brownian motion B on $\Delta^{(\alpha)}$ starting from the root, which is defined up to its first hitting time T of $\partial\Delta^{(\alpha)}$. We refer the reader to [20, Section 2.1] for the details of this construction. The harmonic measure μ_α is then the distribution of B_{T-} under P , which is a (random) probability measure on $\partial\Delta^{(\alpha)} \subseteq \mathbb{N}^{\mathbb{N}}$.

5.2.2 The continuous-time Galton–Watson tree

In this subsection, we introduce a new tree which shares the same branching structure as $\Delta^{(\alpha)}$, such that each point of $\Delta^{(\alpha)}$ at height $s \in [0, 1)$ corresponds to a point of the new tree at height $-\log(1 - s) \in [0, \infty)$ in a bijective way. As it turns out, this new random tree is a continuous-time Galton–Watson tree.

To define it, we take $\alpha \in [1, 2]$ and start with the same random infinite tree $\Pi^{(\alpha)}$ introduced in Section 5.2.1. Consider now a collection $(V_v)_{v \in \mathcal{V}}$ of independent real random variables exponentially distributed with mean 1 under the probability measure \mathbb{P} . We set

$$Z_\emptyset = V_\emptyset$$

and then by induction, for every $v \in \Pi^{(\alpha)} \setminus \{\emptyset\}$,

$$Z_v = Z_{\hat{v}} + V_v.$$

The continuous-time Galton–Watson tree (hereafter to be called CTGW tree for short) of stable index α is the set

$$\Gamma^{(\alpha)} := (\{\emptyset\} \times [0, Z_\emptyset]) \cup \left(\bigcup_{v \in \Pi^{(\alpha)} \setminus \{\emptyset\}} \{v\} \times (Z_{\hat{v}}, Z_v] \right),$$

which is equipped with the metric d defined in the same way as \mathbf{d} in the preceding subsection. For this metric, $\Gamma^{(\alpha)}$ is a.s. a non-compact \mathbb{R} -tree. For every $x = (v, r) \in \Gamma^{(\alpha)}$, we keep the notation $H(x) = r = d((\emptyset, 0), x)$ for the height of the point x .

Observe that if U is uniformly distributed over $[0, 1]$, the random variable $-\log(1 - U)$ is exponentially distributed with mean 1. Hence we may and will suppose that the collection $(V_v)_{v \in \mathcal{V}}$ is constructed from the collection $(U_v)_{v \in \mathcal{V}}$ in the previous subsection via the formula $V_v = -\log(1 - U_v)$ for every $v \in \mathcal{V}$. Then, the mapping Ψ defined on $\Delta_0^{(\alpha)}$ by

$$\Psi(v, r) := (v, -\log(1 - r)) \quad \text{for every } (v, r) \in \Delta_0^{(\alpha)},$$

is a homeomorphism from $\Delta_0^{(\alpha)}$ onto $\Gamma^{(\alpha)}$.

By stochastic analysis, we can write for every $t \in [0, T]$,

$$\Psi(B_t) = W \left(\int_0^t (1 - H(B_s))^{-2} ds \right) \quad (5.8)$$

where $(W(t))_{t \geq 0}$ is Brownian motion with constant drift $1/2$ towards infinity on the CTGW tree $\Gamma^{(\alpha)}$ (this process is defined in a similar way as Brownian motion on $\Delta_\varepsilon^{(\alpha)}$, except that it behaves like Brownian motion with drift $1/2$ on every “open interval” of the tree). Note that again W is defined under the probability measure P . Since all the offspring numbers involved in the CTGW tree $\Gamma^{(\alpha)}$ are larger than 2, it is easy to see that the Brownian motion W is transient. From now on, when we speak about Brownian motion on the CTGW tree or on other similar trees, we will always mean Brownian motion with drift $1/2$ towards infinity.

By definition, the boundary of $\Gamma^{(\alpha)}$ is the set of all infinite geodesics in $\Gamma^{(\alpha)}$ starting from the root $(\emptyset, 0)$ (these are called geodesic rays), and it can be canonically embedded into $\mathbb{N}^{\mathbb{N}}$. Due to the transience of Brownian motion on $\Gamma^{(\alpha)}$, there is an a.s. unique geodesic ray denoted by W_∞ that is visited by $(W(t))_{t \geq 0}$ at arbitrarily large times. We say that W_∞ is the exit ray of Brownian motion on $\Gamma^{(\alpha)}$. The distribution of W_∞ under P yields a probability measure ν_α on $\mathbb{N}^{\mathbb{N}}$. Thanks to (5.8), we have in fact $\nu_\alpha = \mu_\alpha$, provided we think of both μ_α and ν_α as (random) probability measures on $\mathbb{N}^{\mathbb{N}}$. The statement of Theorem 5.2 is then reduced to checking that for every $1 < \alpha \leq 2$, \mathbb{P} -a.s., $\nu_\alpha(dy)$ -a.e.

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log \nu_\alpha(\mathcal{B}(y, r)) = -\beta_\alpha, \quad (5.9)$$

where $\mathcal{B}(y, r)$ denotes the set of all geodesic rays that coincide with y up to height r .

Infinite continuous trees. To prove (5.9), we will apply the tools of ergodic theory to certain transformations on a space of finite-degree rooted infinite continuous trees that we now describe. We let \mathbb{T} be the set of all pairs $(\Pi, (z_v)_{v \in \Pi})$ that satisfy the following conditions:

- (1) Π is an infinite discrete tree, in the sense of Section 5.2.1.
- (2) We have
 - (i) $z_v \in [0, \infty)$ for all $v \in \Pi$;
 - (ii) $z_{\hat{v}} < z_v$ for every $v \in \Pi \setminus \{\emptyset\}$;
 - (iii) for every $\mathbf{v} \in \Pi_\infty := \{(v_1, v_2, \dots, v_n, \dots) \in \mathbb{N}^{\mathbb{N}} : (v_1, v_2, \dots, v_n) \in \Pi, \forall n \geq 1\}$,

$$\lim_{n \rightarrow \infty} z_{(v_1, \dots, v_n)} = +\infty.$$

In the preceding definition, we allow the possibility that $z_\emptyset = 0$. Notice that property (iii) implies that $\#\{v \in \Pi: z_v \leq r\} < \infty$ for every $r > 0$.

We equip \mathbb{T} with the σ -field generated by the coordinate mappings. If $(\Pi, (z_v)_{v \in \Pi}) \in \mathbb{T}$, we can consider the associated “tree”

$$\mathcal{T} := (\{\emptyset\} \times [0, z_\emptyset]) \cup \left(\bigcup_{v \in \Pi \setminus \{\emptyset\}} \{v\} \times (z_v, z_v] \right),$$

equipped with the distance defined as above. The set Π_∞ is identified with the collection of all geodesic rays in Π , and will be viewed as the boundary of the tree \mathcal{T} . We keep the notation $H(x) = r$ for the height of a point $x = (v, r) \in \mathcal{T}$. The genealogical order on \mathcal{T} is defined as previously and again is denoted by \prec . If $\mathbf{u} = (u_1, u_2, \dots) \in \Pi_\infty$, and $x = (v, r) \in \mathcal{T}$, we write $x \prec \mathbf{u}$ if $v = (u_1, u_2, \dots, u_k)$ for some integer $k \geq 0$.

We will often abuse notation and say that we consider a tree $\mathcal{T} \in \mathbb{T}$: This means that we are given a pair $(\Pi, (z_v)_{v \in \Pi})$ satisfying the above properties, and we consider the associated tree \mathcal{T} . In particular, \mathcal{T} has an order structure (in addition to the genealogical partial order) given by the lexicographical order on Π . Elements of \mathbb{T} will be called infinite continuous trees. Clearly, for every stable index $\alpha \in [1, 2]$, the CTGW tree $\Gamma^{(\alpha)}$ can be viewed as a random variable with values in \mathbb{T} , and we write $\Theta_\alpha(d\mathcal{T})$ for its distribution.

Let us fix $\mathcal{T} = (\Pi, (z_v)_{v \in \Pi}) \in \mathbb{T}$. Under our previous notation, k_\emptyset is the number of offspring at the first branching point of \mathcal{T} . We denote by $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \dots, \mathcal{T}_{(k_\emptyset)}$ the subtrees of \mathcal{T} obtained at the first branching point. To be more precise, for every $1 \leq i \leq k_\emptyset$, we define the shifted discrete tree $\Pi[i] = \{v \in \mathcal{V}: iv \in \Pi\}$, and $\mathcal{T}_{(i)}$ is the infinite continuous tree corresponding to the pair

$$\left(\Pi[i], (z_{iv} - z_\emptyset)_{v \in \Pi[i]} \right).$$

Under $\Theta_\alpha(d\mathcal{T})$, we know by definition that k_\emptyset is distributed according to θ_α . Moreover, conditionally on k_\emptyset , the branching property of the CTGW tree states that the subtrees $\mathcal{T}_{(1)}, \dots, \mathcal{T}_{(k_\emptyset)}$ are i.i.d. following the same law Θ_α .

If $r > 0$, the level set of $\mathcal{T} \in \mathbb{T}$ at height r is

$$\mathcal{T}_r = \{x \in \mathcal{T}: H(x) = r\}.$$

For $\alpha \in (1, 2]$, we have the classical result

$$\mathbb{E}[\#\Gamma_r^{(\alpha)}] = \exp\left(\frac{r}{\alpha - 1}\right) = \exp((m_\alpha - 1)r),$$

which can be derived from the following identity (see e.g. Theorem 2.7.1 in [26]) stating that for every $u > 0$,

$$\mathbb{E}[\exp(-u \#\Gamma_r^{(\alpha)})] = 1 - [1 - e^{-r}(1 - (1 - e^{-u})^{1-\alpha})]^{1-\alpha}.$$

5.2.3 The continuous conductance

Recall that, for $\alpha \in [1, 2]$, the random variable $\mathcal{C}^{(\alpha)}$ is defined as the conductance between the root and the set $\partial\Delta^{(\alpha)}$ in the continuous tree $\Delta^{(\alpha)}$ viewed as an electric network. One can also give a more probabilistic definition of the conductance. If \mathcal{T} is a (deterministic) infinite continuous tree, the conductance $\mathcal{C}(\mathcal{T})$ between the root and the boundary $\partial\mathcal{T}$ can be defined in terms of excursion measures of Brownian motion with drift $1/2$ on \mathcal{T} . Under this definition, we can set $\mathcal{C}^{(\alpha)} = \mathcal{C}(\Gamma^{(\alpha)}) \in [1, \infty)$. For details, we refer the reader to Section 2.3 in [20].

In this subsection, we will prove for $\alpha \in (1, 2]$ that the law of $\mathcal{C}^{(\alpha)}$ is characterized by the distributional identity (5.4) in the class of all probability measures on $[1, \infty)$, and discuss some of the properties of this law. For $u \in (0, 1)$, $n \in \mathbb{N}$ and $(x_i)_{i \geq 1} \in [1, \infty)^{\mathbb{N}}$, we define

$$G(u, n, (x_i)_{i \geq 1}) := \left(u + \frac{1-u}{x_1 + x_2 + \cdots + x_n} \right)^{-1},$$

so that (5.4) can be rewritten as

$$\mathcal{C}^{(\alpha)} \stackrel{(d)}{=} G(U, N_\alpha, (\mathcal{C}_i^{(\alpha)})_{i \geq 1}) \quad (5.10)$$

where $U, N_\alpha, (\mathcal{C}_i^{(\alpha)})_{i \geq 1}$ are as in (5.4). Note that (5.10) also holds for $\alpha = 1$. Let \mathcal{M} be the set of all probability measures on $[1, \infty]$ and let $\Phi_\alpha: \mathcal{M} \rightarrow \mathcal{M}$ map a distribution σ to

$$\Phi_\alpha(\sigma) = \text{Law}(G(U, N_\alpha, (X_i)_{i \geq 1}))$$

where $(X_i)_{i \geq 1}$ are independent and identically distributed according to σ , while U, N_α are as in (5.4). We suppose in addition that U, N_α and $(X_i)_{i \geq 1}$ are independent.

We write γ_α for the distribution of $\mathcal{C}^{(\alpha)}$, and define for all $\ell \geq 0$ the Laplace transform

$$\varphi_\alpha(\ell) := \mathbb{E}[\exp(-\ell \mathcal{C}^{(\alpha)}/2)] = \int_1^\infty e^{-\ell r/2} \gamma_\alpha(dr).$$

Proposition 5.5. *Let us fix the stable index $\alpha \in (1, 2]$. The law γ_α of $\mathcal{C}^{(\alpha)}$ is the unique fixed point of the mapping Φ_α on \mathcal{M} , and we have $\Phi_\alpha^k(\sigma) \rightarrow \gamma_\alpha$ weakly as $k \rightarrow \infty$, for every $\sigma \in \mathcal{M}$. Furthermore,*

1. *If $\alpha = 2$, all moments of γ_2 are finite, and γ_2 has a continuous density over $[1, \infty)$. The Laplace transform φ_2 solves the differential equation*

$$2\ell \varphi''(\ell) + \ell \varphi'(\ell) + \varphi^2(\ell) - \varphi(\ell) = 0.$$

2. *If $\alpha \in (1, 2)$, only the first and the second moments of γ_α are finite. The distribution γ_α has a continuous density over $[1, \infty)$, and the Laplace transform φ_α solves the differential equation*

$$2\ell \varphi''(\ell) + \ell \varphi'(\ell) + \frac{(1 - \varphi(\ell))^\alpha + \varphi(\ell) - 1}{\alpha - 1} = 0. \quad (5.11)$$

Proof. The case $\alpha = 2$ has been derived in [20, Proposition 6] and is listed above for the sake of completeness. We will prove the corresponding assertion for $\alpha \in (1, 2)$ by similar methods.

Firstly, the stochastic partial order \preceq on \mathcal{M} is defined by saying that $\sigma \preceq \sigma'$ if and only if there exists a coupling (X, Y) of σ and σ' such that a.s. $X \leq Y$. It is clear that for any $\alpha \in [1, 2]$, the mapping Φ_α is increasing for the stochastic partial order.

We endow the set \mathcal{M}_1 of all probability measures on $[1, \infty]$ that have a finite first moment with the 1-Wasserstein metric

$$d_1(\sigma, \sigma') := \inf \{E[|X - Y|] : (X, Y) \text{ coupling of } (\sigma, \sigma')\}.$$

The metric space (\mathcal{M}_1, d_1) is Polish and its topology is finer than the weak topology on \mathcal{M}_1 . From the easy bound

$$G(u, n, (x_i)_{i \geq 1}) \leq x_1 + x_2 + \cdots + x_n$$

and the fact that $\mathbb{E}N_\alpha < \infty$ when $\alpha \neq 1$, we immediately see that Φ_α maps \mathcal{M}_1 into \mathcal{M}_1 when $\alpha > 1$. We then observe that the mapping Φ_α is strictly contractant on (\mathcal{M}_1, d_1) . To see this, let $(X_i, Y_i)_{i \geq 1}$ be independent copies of a coupling between $\sigma, \sigma' \in \mathcal{M}_1$ under the probability measure \mathbb{P} . As in (5.10), let U be uniformly distributed over $[0, 1]$ and N_α be distributed according to θ_α . Assume that U, N_α and $(X_i, Y_i)_{i \geq 1}$ are independent under \mathbb{P} . Then the two variables $G(U, N_\alpha, (X_i)_{i \geq 1})$ and $G(U, N_\alpha, (Y_i)_{i \geq 1})$ give a coupling of $\Phi_\alpha(\sigma)$ and $\Phi_\alpha(\sigma')$. Using the fact that $X_i, Y_i \geq 1$, we have

$$\begin{aligned} & |G(U, N_\alpha, (X_i)_{i \geq 1}) - G(U, N_\alpha, (Y_i)_{i \geq 1})| \\ &= \left| \left(U + \frac{1-U}{X_1 + X_2 + \dots + X_{N_\alpha}} \right)^{-1} - \left(U + \frac{1-U}{Y_1 + Y_2 + \dots + Y_{N_\alpha}} \right)^{-1} \right| \\ &= \left| \frac{(X_1 + X_2 + \dots + X_{N_\alpha} - Y_1 - Y_2 - \dots - Y_{N_\alpha})(1-U)}{(U(X_1 + X_2 + \dots + X_{N_\alpha}) + 1 - U)(U(Y_1 + Y_2 + \dots + Y_{N_\alpha}) + 1 - U)} \right| \\ &\leq (|X_1 - Y_1| + |X_2 - Y_2| + \dots + |X_{N_\alpha} - Y_{N_\alpha}|) \frac{1-U}{(1 + (N_\alpha - 1)U)^2}. \end{aligned}$$

Notice that for any integer $k \geq 2$,

$$\mathbb{E} \left[\frac{k(1-U)}{(1 + (k-1)U)^2} \right] = 1 + \frac{k-1-k \log k}{(k-1)^2}.$$

Taking expected values and minimizing over the choice of the coupling between σ and σ' , we get

$$\begin{aligned} d_1(\Phi_\alpha(\sigma), \Phi_\alpha(\sigma')) &\leq \mathbb{E} \left[\frac{N_\alpha(1-U)}{(1 + (N_\alpha - 1)U)^2} \right] d_1(\sigma, \sigma') \\ &= \left(1 + \mathbb{E} \left[\frac{N_\alpha - 1 - N_\alpha \log N_\alpha}{(N_\alpha - 1)^2} \right] \right) d_1(\sigma, \sigma') = c_\alpha d_1(\sigma, \sigma'), \end{aligned}$$

with $c_\alpha < 1$. So for $\alpha \in (1, 2]$, the mapping Φ_α is contractant on \mathcal{M}_1 and by completeness it has a unique fixed point $\tilde{\gamma}_\alpha$ in \mathcal{M}_1 . Furthermore, for every $\sigma \in \mathcal{M}_1$, we have $\Phi_\alpha^k(\sigma) \rightarrow \tilde{\gamma}_\alpha$ for the metric d_1 , hence also weakly, as $k \rightarrow \infty$.

Since we know from (5.10) that γ_α is also a fixed point of Φ_α , the equality $\gamma_\alpha = \tilde{\gamma}_\alpha$ will follow if we can verify that $\tilde{\gamma}_\alpha$ is the unique fixed point of Φ_α in \mathcal{M} . To this end, it will be enough to show that we have $\Phi_\alpha^k(\sigma) \rightarrow \tilde{\gamma}_\alpha$ as $k \rightarrow \infty$, for every $\sigma \in \mathcal{M}$.

For any $\alpha \in [1, 2]$, we apply Φ_α to the Dirac measure δ_∞ at ∞ to see

$$\begin{aligned} \Phi_\alpha(\delta_\infty) &= \text{Law}(U^{-1}), \\ \Phi_\alpha^2(\delta_\infty) &= \text{Law} \left(\left(U + \frac{1-U}{U_1^{-1} + U_2^{-1} + \dots + U_{N_\alpha}^{-1}} \right)^{-1} \right), \end{aligned}$$

where we introduce a new sequence $(U_i)_{i \geq 1}$ consisting of i.i.d. copies of U , independent of N_α and U under \mathbb{P} . Thus the first moment of $\Phi_\alpha^2(\delta_\infty)$ is given by

$$\begin{aligned} & \sum_{k \geq 2} \theta_\alpha(k) \int_0^1 du \int_0^1 du_1 \dots \int_0^1 du_k \left(u + \frac{1-u}{u_1^{-1} + u_2^{-1} + \dots + u_k^{-1}} \right)^{-1} \\ &= \sum_{k \geq 2} \theta_\alpha(k) \int_0^1 du_1 \dots \int_0^1 du_k \frac{1}{1 - (u_1^{-1} + u_2^{-1} + \dots + u_k^{-1})^{-1}} \log \left(\frac{1}{u_1} + \frac{1}{u_2} + \dots + \frac{1}{u_k} \right) \\ &\leq 2 \sum_{k \geq 2} \theta_\alpha(k) \int_0^1 du_1 \dots \int_0^1 du_k \log \left(\frac{1}{u_1} + \frac{1}{u_2} + \dots + \frac{1}{u_k} \right), \end{aligned}$$

in which the integrals can be bounded as follows,

$$\begin{aligned}
& \int_0^1 du_1 \cdots \int_0^1 du_k \log \left(\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_k} \right) \\
&= k! \int_{0 < u_1 < u_2 < \cdots < u_k < 1} du_1 du_2 \cdots du_k \log \left(\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_k} \right) \\
&= k! \int_{0 < u_2 < u_3 < \cdots < u_k < 1} du_2 du_3 \cdots du_k \left[u_2 \log \left(\frac{2}{u_2} + \frac{1}{u_3} + \cdots + \frac{1}{u_k} \right) + \frac{\log \left(2 + \frac{u_2}{u_3} + \cdots + \frac{u_2}{u_k} \right)}{u_2^{-1} + u_3^{-1} + \cdots + u_k^{-1}} \right] \\
&\leq k! \int_{0 < u_2 < u_3 < \cdots < u_k < 1} du_2 du_3 \cdots du_k \left[u_2 \log \frac{k}{u_2} + \frac{\log k}{k-1} \right] \\
&= \log k + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{k \log k}{k-1} \leq \left(2 + \frac{k}{k-1} \right) \log k.
\end{aligned}$$

Using Stirling's formula, we know that $\theta_\alpha(k) = O(k^{-(1+\alpha)})$ as $k \rightarrow +\infty$. As

$$\sum_{k \geq 2} \left(2 + \frac{k}{k-1} \right) \frac{\log k}{k^{1+\alpha}} < +\infty$$

for all $\alpha \in [1, 2]$, we get $\Phi_\alpha^2(\delta_\infty) \in \mathcal{M}_1$. By monotonicity, we have also $\Phi_\alpha^2(\sigma) \in \mathcal{M}_1$ for every $\sigma \in \mathcal{M}$, and from the preceding results we get $\Phi_\alpha^k(\sigma) \rightarrow \tilde{\gamma}_\alpha$ for every $\sigma \in \mathcal{M}$. This implies that $\gamma_\alpha = \tilde{\gamma}_\alpha$ is the unique fixed point of Φ_α in \mathcal{M} .

For every $t \in \mathbb{R}$ we set $F_\alpha(t) = \gamma_\alpha([t, \infty])$. For every integer $k \geq 2$, we write $F_\alpha^{(k)}(t) = \mathbb{P}(\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_k^{(\alpha)} \geq t)$, where $(\mathcal{C}_k^{(\alpha)})_{k \geq 1}$ are independent and identically distributed according to γ_α . Then we have, for every $t > 1$,

$$\begin{aligned}
F_\alpha(t) &= \mathbb{P} \left(U + \frac{1-U}{\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)}} \leq t^{-1} \right) \\
&= \mathbb{P} \left(U < t^{-1} \text{ and } \frac{t-Ut}{1-Ut} \leq \mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)} \right) \\
&= \mathbb{E} \left[\int_0^{1/t} du F_\alpha^{(N_\alpha)} \left(\frac{t-ut}{1-ut} \right) \right] \\
&= \frac{t-1}{t} \int_t^\infty \frac{dx}{(x-1)^2} \mathbb{E} [F_\alpha^{(N_\alpha)}(x)]. \tag{5.12}
\end{aligned}$$

By definition, we have $F_\alpha^{(k)}(t) = 1$ for every $t \in [1, 2]$ and $k \geq 2$. It follows from (5.12) that

$$F_\alpha(t) = \frac{D^{(\alpha)}}{t} + 1 - D^{(\alpha)}, \quad \forall t \in [1, 2], \tag{5.13}$$

where

$$D^{(\alpha)} = 2 - \int_2^\infty \frac{dx}{(x-1)^2} \mathbb{E} [F_\alpha^{(N_\alpha)}(x)] \in [1, 2].$$

We observe that the right-hand side of (5.12) is a continuous function of $t \in (1, \infty)$, so that F_α is continuous on $[1, \infty)$ (the right-continuity at 1 is obvious from (5.13)). Thus γ_α has no atom and it follows that all functions $F_\alpha^{(k)}$, $k \geq 2$ are continuous on $[1, \infty)$. By dominated convergence the function $x \mapsto \mathbb{E}[F_\alpha^{(N_\alpha)}(x)]$ is also continuous on $[1, \infty)$. Using (5.12) again we obtain that F_α

is continuously differentiable on $[1, \infty)$ and consequently γ_α has a continuous density $f_\alpha = -F'_\alpha$ with respect to the Lebesgue measure on $[1, \infty)$.

Let us finally derive the differential equation (5.11). To this end, we first differentiate (5.12) with respect to t to get that the linear differential equation

$$t(t-1)F'_\alpha(t) - F_\alpha(t) = -\mathbb{E}[F_\alpha^{(N_\alpha)}(t)] \quad (5.14)$$

holds for $t \in [1, \infty)$. Then let $g: [1, \infty) \rightarrow \mathbb{R}_+$ be a monotone continuously differentiable function. From the definition of F_α and Fubini's theorem, we have

$$\int_1^\infty dt g'(t)F_\alpha(t) = \mathbb{E}[g(\mathcal{C}^{(\alpha)})] - g(1)$$

and similarly

$$\int_1^\infty dt g'(t)\mathbb{E}[F_\alpha^{(N_\alpha)}(t)] = \mathbb{E}[g(\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \dots + \mathcal{C}_{N_\alpha}^{(\alpha)})] - g(1).$$

We then multiply both sides of (5.14) by $g'(t)$ and integrate for t running from 1 to ∞ to get

$$\mathbb{E}[\mathcal{C}_1^{(\alpha)}(\mathcal{C}_1^{(\alpha)} - 1)g'(\mathcal{C}_1^{(\alpha)})] + \mathbb{E}[g(\mathcal{C}_1^{(\alpha)})] = \mathbb{E}[g(\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \dots + \mathcal{C}_{N_\alpha}^{(\alpha)})]. \quad (5.15)$$

When $g(x) = \exp(-x\ell/2)$ for $\ell > 0$, we readily obtain (5.11) by using the generating function of N_α given in (5.6). Finally, taking $g(x) = x$ in (5.15), we get

$$\mathbb{E}[(\mathcal{C}^{(\alpha)})^2] = \mathbb{E}[N_\alpha]\mathbb{E}[\mathcal{C}^{(\alpha)}] = \frac{\alpha}{\alpha-1}\mathbb{E}[\mathcal{C}^{(\alpha)}].$$

Nevertheless, by taking $g(x) = x^2$ in (5.15), we see that the third moment of $\mathcal{C}^{(\alpha)}$ is infinite since $\mathbb{E}[(N_\alpha)^2] = \infty$. \square

The arguments of the preceding proof also yield the following lemma in the case $\alpha = 1$.

Lemma 5.6. *The conductance $\mathcal{C}^{(1)}$ of the tree $\Delta^{(1)}$ satisfies the bound*

$$\mathbb{E}[\mathcal{C}^{(1)}] \leq 2 \sum_{k \geq 2} \left(2 + \frac{k}{k-1}\right) \frac{\log k}{k(k-1)} < +\infty. \quad (5.16)$$

Additionally, the Laplace transform φ_1 of the law of $\mathcal{C}^{(1)}$ solves the differential equation

$$2\ell\varphi''(\ell) + \ell\varphi'(\ell) + (1 - \varphi(\ell))\log(1 - \varphi(\ell)) = 0.$$

Proof. The law of $\mathcal{C}^{(1)}$ is a fixed point of the mapping Φ_1 defined via (5.10) with $\alpha = 1$. By the same monotonicity argument that we used above, it follows that the first moment of $\mathcal{C}^{(1)}$ is bounded above by the first moment of $\Phi_1^2(\delta_\infty)$, and the calculation of this first moment in the previous proof leads to the right-hand side of (5.16).

As an analogue to (5.15), we have

$$\mathbb{E}[\mathcal{C}_1^{(1)}(\mathcal{C}_1^{(1)} - 1)g'(\mathcal{C}_1^{(1)})] + \mathbb{E}[g(\mathcal{C}_1^{(1)})] = \mathbb{E}[g(\mathcal{C}_1^{(1)} + \mathcal{C}_2^{(1)} + \dots + \mathcal{C}_{N_1}^{(1)})].$$

By taking $g(x) = \exp(-x\ell/2)$ and using (5.7), one can then derive the differential equation satisfied by φ_1 . \square

5.2.4 The reduced stable trees are nested

In this short subsection, we introduce a coupling argument to explain how Theorem 5.3 follows from the identity (5.5) in Proposition 5.4.

Recall the definition of the α -offspring distribution θ_α . From the obvious fact

$$1 - \sum_{i=2}^{k-1} \frac{\alpha}{i - \alpha} < 0, \quad \forall \alpha \in (1, 2), k \geq 3,$$

one deduces that for all $k \geq 3$,

$$\frac{d}{d\alpha} \theta_\alpha(k) < 0, \quad \forall \alpha \in (1, 2).$$

This implies that for every $k \geq 3$, $\theta_\alpha([2, k])$ is a strictly increasing function of $\alpha \in (1, 2)$. Using the inverse transform sampling, we can construct on a common probability space a sequence of random variables $(N_\alpha, \alpha \in [1, 2])$ such that a.s.

$$N_{\alpha_2} \geq N_{\alpha_1} \quad \text{for all } 1 \leq \alpha_2 \leq \alpha_1 \leq 2.$$

Then following the same procedure explained in Section 5.2.1, we can construct simultaneously all reduced stable trees as a nested family. More precisely, there exists a family of compact \mathbb{R} -trees $(\bar{\Delta}^{(\alpha)}, \alpha \in [1, 2])$ such that

$$\begin{aligned} \bar{\Delta}^{(\alpha)} &\stackrel{(d)}{=} \Delta^{(\alpha)} \quad \text{for all } 1 \leq \alpha \leq 2; \\ \bar{\Delta}^{(\alpha_1)} &\subseteq \bar{\Delta}^{(\alpha_2)} \quad \text{for all } 1 \leq \alpha_2 \leq \alpha_1 \leq 2. \end{aligned}$$

Consequently, the family of conductances $(\bar{\mathcal{C}}^{(\alpha)}, \alpha \in [1, 2])$ associated with $(\bar{\Delta}^{(\alpha)}, \alpha \in [1, 2])$ is decreasing with respect to α . In particular, the mean $\mathbb{E}[\mathcal{C}^{(\alpha)}]$ is decreasing with respect to α , and it follows from (5.16) that $(\mathbb{E}[\mathcal{C}^{(\alpha)}], \alpha \in [1, 2])$ is uniformly bounded by the constant

$$C_0 := 2 \sum_{k \geq 2} \left(2 + \frac{k}{k-1}\right) \frac{\log k}{k(k-1)} < +\infty.$$

Proof of Theorem 5.3. For any $\alpha \in (1, 2]$, γ_α is a probability measure on $[1, \infty)$ and

$$\begin{aligned} \iint \gamma_\alpha(ds) \gamma_\alpha(dt) \frac{st}{s+t-1} &\geq \iint \gamma_\alpha(ds) \gamma_\alpha(dt) \frac{st}{s+t} \\ &\geq \iint \gamma_\alpha(ds) \gamma_\alpha(dt) \frac{st}{2(s \vee t)} \\ &= \frac{1}{2} \iint \gamma_\alpha(ds) \gamma_\alpha(dt) (s \wedge t) \geq \frac{1}{2}. \end{aligned}$$

So we derive from (5.5) that

$$\beta_\alpha \leq \frac{1}{2} \left(2(\mathbb{E}[\mathcal{C}^{(\alpha)}])^2 - 1\right) \leq \frac{1}{2} (2C_0^2 - 1) < \infty. \quad \square$$

5.2.5 Proof of Theorem 5.2

The proof of Theorem 5.2 given below will follow the approach sketched in [20, Section 5.1]. We will first establish the flow property of harmonic measure (Lemma 5.7), and then find an explicit invariant measure for the environment seen by Brownian motion on the CTGW tree $\Gamma^{(\alpha)}$ at the last visit of a vertex of the n -th generation (Proposition 5.8). After that, we will rely on arguments of ergodic theory to complete the proof of Theorem 5.2 and that of Proposition 5.4.

Throughout this subsection, we fix the stable index $\alpha \in (1, 2]$ once and for all. For notational ease, we will omit the superscripts and subscripts concerning α in all the proofs involved. Recall that \mathbb{P} stands for the probability measure under which the CTGW tree $\Gamma^{(\alpha)}$ is defined, whereas Brownian motion with drift $1/2$ on the CTGW tree is defined under the probability measure P .

The flow property of harmonic measure

We fix an infinite continuous tree $\mathcal{T} \in \mathbb{T}$, and write as before $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \dots, \mathcal{T}_{(k_\emptyset)}$ for the subtrees of \mathcal{T} at the first branching point. Here we slightly abuse notation by writing $W = (W(t))_{t \geq 0}$ for Brownian motion with drift $1/2$ on \mathcal{T} started from the root. As in Section 5.2.2, W_∞ stands for the exit ray of W , and the distribution of W_∞ on the boundary of \mathcal{T} is the harmonic measure of \mathcal{T} , denoted as $\nu_{\mathcal{T}}$. Let K be the index such that W_∞ “belongs to” $\mathcal{T}_{(K)}$ and we write W'_∞ for the ray of $\mathcal{T}_{(K)}$ obtained by shifting W_∞ at the first branching point of \mathcal{T} .

Lemma 5.7. *Let $j \in \{1, 2, \dots, k_\emptyset\}$. Conditionally on $\{K = j\}$, the law of W'_∞ is the harmonic measure of $\mathcal{T}_{(j)}$.*

The proof is similar to that of [20, Lemma 7] and is therefore omitted.

The invariant measure and ergodicity

We introduce the set

$$\mathbb{T}^* \subseteq \mathbb{T} \times \mathbb{N}^{\mathbb{N}}$$

of all pairs consisting of a tree $\mathcal{T} \in \mathbb{T}$ and a distinguished geodesic ray \mathbf{v} in \mathcal{T} . Given a distinguished geodesic ray $\mathbf{v} = (v_1, v_2, \dots)$ in \mathcal{T} , we let $S(\mathcal{T}, \mathbf{v})$ be obtained by shifting $(\mathcal{T}, \mathbf{v})$ at the first branching point of \mathcal{T} , that is

$$S(\mathcal{T}, \mathbf{v}) = (\mathcal{T}_{(v_1)}, \tilde{\mathbf{v}}),$$

where $\tilde{\mathbf{v}} = (v_2, v_3, \dots)$ and $\mathcal{T}_{(v_1)}$ is the subtree of \mathcal{T} rooted at the first branching point that is chosen by \mathbf{v} .

Under the probability measure $\mathbb{P} \otimes P$, we can view $(\Gamma^{(\alpha)}, W_\infty)$ as a random variable with values in \mathbb{T}^* . We write $\Theta_\alpha^*(d\mathcal{T}d\mathbf{v})$ for the distribution of $(\Gamma^{(\alpha)}, W_\infty)$. The next proposition gives an invariant measure absolutely continuous with respect to Θ_α^* under the shift S .

Proposition 5.8. *For every $r \geq 1$, set*

$$\kappa_\alpha(r) := \sum_{k=2}^{\infty} k \theta_\alpha(k) \int \gamma_\alpha(dt_1) \int \gamma_\alpha(dt_2) \cdots \int \gamma_\alpha(dt_k) \frac{rt_1}{r + t_1 + t_2 + \cdots + t_k - 1}.$$

The finite measure $\kappa_\alpha(\mathcal{C}(\mathbb{T}))\Theta_\alpha^(d\mathcal{T}d\mathbf{v})$ is invariant under S .*

Remark. The preceding formula for κ_α is suggested by the analogous formula in [20, Proposition 25] for $\alpha = 2$.

Proof. First notice that the function κ is bounded, since for every $r \geq 1$,

$$\kappa(r) \leq \sum_{k=2}^{\infty} k\theta(k) \int t_1^\gamma(dt_1) < \infty.$$

Let us fix $\mathcal{T} \in \mathbb{T}$, then for any $1 \leq i \leq k_\emptyset$ and any bounded measurable function g on $\mathbb{N}^{\mathbb{N}}$, the flow property of harmonic measure gives that

$$\int \nu_{\mathcal{T}}(d\mathbf{v}) \mathbf{1}_{\{v_1=i\}} g(\tilde{\mathbf{v}}) = \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k_\emptyset)})} \int \nu_{\mathcal{T}_{(i)}}(d\mathbf{u}) g(\mathbf{u}).$$

Recall that $\Theta^*(d\mathcal{T} d\mathbf{v}) = \Theta(d\mathcal{T})\nu_{\mathcal{T}}(d\mathbf{v})$ by construction. Let F be a bounded measurable function on \mathbb{T}^* . Using the preceding display, we have

$$\begin{aligned} & \int F \circ S(\mathcal{T}, \mathbf{v}) \kappa(\mathcal{C}(\mathcal{T})) \Theta^*(d\mathcal{T} d\mathbf{v}) \\ &= \sum_{k=2}^{\infty} \theta(k) \sum_{i=1}^k \int F(\mathcal{T}_{(i)}, \mathbf{u}) \kappa(\mathcal{C}(\mathcal{T})) \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k)})} \Theta(d\mathcal{T} | k_\emptyset = k) \nu_{\mathcal{T}_{(i)}}(d\mathbf{u}). \end{aligned} \quad (5.17)$$

Observe that under $\Theta(d\mathcal{T} | k_\emptyset = k)$, the subtrees $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \dots, \mathcal{T}_{(k)}$ are independent and distributed according to Θ , and furthermore,

$$\mathcal{C}(\mathcal{T}) = \left(U + \frac{1 - U}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k)})} \right)^{-1},$$

where U is uniformly distributed over $[0, 1]$ and independent of $(\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \dots, \mathcal{T}_{(k)})$. Using these observations, together with a simple symmetry argument, we get that the integral (5.17) is given by

$$\begin{aligned} & \sum_{k=2}^{\infty} k\theta(k) \int_0^1 dx \int \Theta(d\mathcal{T}_1) \cdots \int \Theta(d\mathcal{T}_k) \int \nu_{\mathcal{T}_1}(d\mathbf{u}) F(\mathcal{T}_1, \mathbf{u}) \\ & \quad \times \frac{\mathcal{C}(\mathcal{T}_1)}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)} \kappa\left(\left(x + \frac{1 - x}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)} \right)^{-1} \right) \\ &= \int \Theta^*(d\mathcal{T}_1 d\mathbf{u}) F(\mathcal{T}_1, \mathbf{u}) \left[\sum_{k=2}^{\infty} k\theta(k) \int_0^1 dx \int \Theta(d\mathcal{T}_2) \cdots \int \Theta(d\mathcal{T}_k) \right. \\ & \quad \left. \times \frac{\mathcal{C}(\mathcal{T}_1)}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)} \kappa\left(\left(x + \frac{1 - x}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)} \right)^{-1} \right) \right]. \end{aligned}$$

The proof is thus reduced to checking that, for every $r \geq 1$, $\kappa(r)$ is equal to

$$\sum_{k=2}^{\infty} k\theta(k) \int_0^1 dx \int \Theta(d\mathcal{T}_2) \cdots \int \Theta(d\mathcal{T}_k) \frac{r}{r + \mathcal{C}(\mathcal{T}_2) + \cdots + \mathcal{C}(\mathcal{T}_k)} \kappa\left(\left(x + \frac{1 - x}{r + \mathcal{C}(\mathcal{T}_2) + \cdots + \mathcal{C}(\mathcal{T}_k)} \right)^{-1} \right). \quad (5.18)$$

To this end, we will reformulate the last expression in the following way. Under the probability measure \mathbb{P} , we introduce an i.i.d. sequence $(C_i)_{i \geq 0}$ distributed according to γ , and a random

variable N distributed according to θ . In addition, under the same probability measure \mathbb{P} , let U be uniformly distributed over $[0, 1]$, $(\tilde{\mathcal{C}}_i)_{i \geq 0}$ be an independent copy of $(\mathcal{C}_i)_{i \geq 0}$, and \tilde{N} be an independent copy of N . We assume that all these random variables are independent. Note that by definition, for every $r \geq 1$,

$$\kappa(r) = \mathbb{E} \left[\frac{r \tilde{N} \tilde{\mathcal{C}}_1}{r + \tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}} - 1} \right].$$

It follows that (5.18) can be written as

$$\begin{aligned} & \sum_{k=2}^{\infty} k \theta(k) \mathbb{E} \left[\frac{r}{r + \mathcal{C}_2 + \cdots + \mathcal{C}_k} \frac{\left(U + \frac{1-U}{r + \mathcal{C}_2 + \cdots + \mathcal{C}_k} \right)^{-1} \tilde{N} \tilde{\mathcal{C}}_1}{\left(U + \frac{1-U}{r + \mathcal{C}_2 + \cdots + \mathcal{C}_k} \right)^{-1} + \tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}} - 1} \right] \\ &= r \sum_{k=2}^{\infty} k \theta(k) \mathbb{E} \left[\frac{\tilde{N} \tilde{\mathcal{C}}_1}{(r + \mathcal{C}_2 + \cdots + \mathcal{C}_k) \left(1 + (\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}} - 1) \left(U + \frac{1-U}{r + \mathcal{C}_2 + \cdots + \mathcal{C}_k} \right) \right)} \right] \\ &= r \sum_{k=2}^{\infty} k \theta(k) \mathbb{E} \left[\frac{\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}}}{(\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}} - 1) \left(U(r + \mathcal{C}_2 + \cdots + \mathcal{C}_k) + 1 - U \right) + r + \mathcal{C}_2 + \cdots + \mathcal{C}_k} \right] \\ &= r \sum_{k=2}^{\infty} k \theta(k) \mathbb{E} \left[\frac{\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}}}{(\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}}) \left(U(r + \mathcal{C}_2 + \cdots + \mathcal{C}_k - 1) + 1 \right) + (r + \mathcal{C}_2 + \cdots + \mathcal{C}_k - 1)(1 - U)} \right] \\ &= r \sum_{k=2}^{\infty} k \theta(k) \mathbb{E} \left[\frac{1}{(r + \mathcal{C}_2 + \cdots + \mathcal{C}_k - 1) \left(U + \frac{1-U}{\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}}} \right) + 1} \right] \\ &= r \sum_{k=2}^{\infty} k \theta(k) \mathbb{E} \left[\frac{\tilde{\mathcal{C}}}{r + \tilde{\mathcal{C}} + \mathcal{C}_2 + \cdots + \mathcal{C}_k - 1} \right] = \mathbb{E} \left[\frac{r N \tilde{\mathcal{C}}}{r + \tilde{\mathcal{C}} + \mathcal{C}_2 + \cdots + \mathcal{C}_N - 1} \right], \end{aligned}$$

where

$$\tilde{\mathcal{C}} := \left(U + \frac{1-U}{\tilde{\mathcal{C}}_1 + \cdots + \tilde{\mathcal{C}}_{\tilde{N}}} \right)^{-1}$$

is independent of $(\mathcal{C}_i)_{i \geq 0}$ and N . By (5.4), the random variable $\tilde{\mathcal{C}}$ is also distributed according to γ . So the right-hand side of the last long display is equal to $\kappa(r)$, which completes the proof of the proposition. \square

We normalize κ_α by setting

$$\hat{\kappa}_\alpha(r) = \frac{\kappa_\alpha(r)}{\int \kappa_\alpha(\mathcal{C}(\mathcal{T})) \Theta_\alpha^*(d\mathcal{T} d\mathbf{v})} = \frac{\kappa_\alpha(r)}{\int \kappa_\alpha(\mathcal{C}(\mathcal{T})) \Theta_\alpha(d\mathcal{T})}$$

for every $r \geq 1$. Then $\hat{\kappa}_\alpha(\mathcal{C}(\mathcal{T})) \Theta_\alpha^*(d\mathcal{T} d\mathbf{v})$ is a probability measure on \mathbb{T}^* invariant under the shift S . To simplify notation, we set $\Upsilon_\alpha^*(d\mathcal{T} d\mathbf{v}) := \hat{\kappa}_\alpha(\mathcal{C}(\mathcal{T})) \Theta_\alpha^*(d\mathcal{T} d\mathbf{v})$. Let π_1 be the canonical projection from \mathbb{T}^* onto \mathbb{T} . The image of Υ_α^* under this projection is the probability measure $\Upsilon_\alpha(d\mathcal{T}) := \hat{\kappa}_\alpha(\mathcal{C}(\mathcal{T})) \Theta_\alpha(d\mathcal{T})$.

Proposition 5.9. *The shift S acting on the probability space $(\mathbb{T}^*, \Upsilon_\alpha^*)$ is ergodic.*

Proof. Our arguments proceed in a similar way as in the proof of [20, Proposition 13]. We define a transition kernel $\mathbf{p}(\mathcal{T}, d\mathcal{T}')$ on \mathbb{T} by setting

$$\mathbf{p}(\mathcal{T}, d\mathcal{T}') = \sum_{i=1}^{k_\emptyset} \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k_\emptyset)})} \delta_{\mathcal{T}_{(i)}}(d\mathcal{T}').$$

Informally, under the probability measure $\mathbf{p}(\mathcal{T}, d\mathcal{T}')$, we choose one of the subtrees of \mathcal{T} obtained at the first branching point, with probability equal to its harmonic measure.

For every integer $n \geq 1$, we denote by S^n the mapping on \mathbb{T}^* obtained by iterating n times the shift S , and then we consider the process $(Z_n)_{n \geq 0}$ on the probability space $(\mathbb{T}^*, \Upsilon^*)$ with values in \mathbb{T} , defined by $Z_0(\mathcal{T}, \mathbf{v}) = \mathcal{T}$ and

$$Z_n(\mathcal{T}, \mathbf{v}) = \pi_1(S^n(\mathcal{T}, \mathbf{v}))$$

for every $n \geq 1$. According to Proposition 5.8 and the flow property of harmonic measure, the process $(Z_n)_{n \geq 0}$ is a Markov chain with transition kernel \mathbf{p} under its stationary measure $\Upsilon(d\mathcal{T})$.

We write \mathbb{T}^∞ for the set of all infinite sequences $(\mathcal{T}^0, \mathcal{T}^1, \dots)$ of elements in \mathbb{T} , and let $\widehat{\mathbb{T}}^\infty$ be the set of all infinite sequences $(\mathcal{T}^0, \mathcal{T}^1, \dots)$ in \mathbb{T}^∞ , such that, for every integer $j \geq 1$, \mathcal{T}^j is one of the subtrees of \mathcal{T}^{j-1} above the first branching point of \mathcal{T}^{j-1} . Note that $\widehat{\mathbb{T}}^\infty$ is a measurable subset of \mathbb{T}^∞ and that $(Z_n(\mathcal{T}, \mathbf{v}))_{n \geq 0} \in \widehat{\mathbb{T}}^\infty$ for every $(\mathcal{T}, \mathbf{v}) \in \mathbb{T}^*$. If $(\mathcal{T}^0, \mathcal{T}^1, \dots) \in \widehat{\mathbb{T}}^\infty$, there exists a geodesic ray \mathbf{v} in \mathcal{T}^0 such that $\mathcal{T}^j = S^j(\mathcal{T}^0, \mathbf{v})$ for every $j \geq 1$, and we set $\phi(\mathcal{T}^0, \mathcal{T}^1, \dots) := (\mathcal{T}^0, \mathbf{v})$. Notice that \mathbf{v} is a priori not unique, but to make the previous definition rigorous we can take the smallest possible \mathbf{v} in lexicographical ordering (of course for the random trees that we consider later this uniqueness problem does not arise). In this way, we define a measurable mapping ϕ from $\widehat{\mathbb{T}}^\infty$ into \mathbb{T}^* such that

$$\phi(Z_0(\mathcal{T}, \mathbf{v}), Z_1(\mathcal{T}, \mathbf{v}), \dots) = (\mathcal{T}, \mathbf{v}), \quad \Upsilon^*\text{-a.s.} \quad (5.19)$$

Now given a measurable subset A of \mathbb{T}^* such that $S^{-1}(A) = A$, we aim at proving that $\Upsilon^*(A) \in \{0, 1\}$. To this end, we consider the pre-image $B = \phi^{-1}(A)$, which is a measurable subset of $\widehat{\mathbb{T}}^\infty \subset \mathbb{T}^\infty$. Due to the previous constructions, B is shift-invariant for the Markov chain Z in the sense that

$$\{(Z_0, Z_1, \dots) \in B\} = \{(Z_1, Z_2, \dots) \in B\}, \quad \text{a.s.}$$

Using Proposition 16.2 in [73], we then obtain a measurable subset D of \mathbb{T} , such that

$$\mathbf{1}_B(Z_0, Z_1, \dots) = \mathbf{1}_D(Z_0) \quad \text{a.s.},$$

and moreover $\mathbf{p}(\mathcal{T}, D) = \mathbf{1}_D(\mathcal{T})$, $\Upsilon(d\mathcal{T})$ -a.s. It follows thus from (5.19) that Υ^* -a.s. we have $(\mathcal{T}, \mathbf{v}) \in A$ if and only if $\mathcal{T} \in D$.

However from the property $\mathbf{p}(\mathcal{T}, D) = \mathbf{1}_D(\mathcal{T})$, $\Upsilon(d\mathcal{T})$ -a.s., one can verify that $\Upsilon(D) \in \{0, 1\}$. First note that this property also implies that $\mathbf{p}(\mathcal{T}, D) = \mathbf{1}_D(\mathcal{T})$, $\Theta(d\mathcal{T})$ -a.s. Hence, $\Theta(d\mathcal{T})$ -a.s., the tree \mathcal{T} belongs to D if and only if each of its subtrees above the first branching point belongs to D (it is clear that that the measure $\mathbf{p}(\mathcal{T}, \cdot)$ assigns a positive mass to each of these subtrees). Then, the branching property of the CTGW tree shows that

$$\Theta(D) = \sum_{k=2}^{\infty} \theta(k) \Theta(D)^k$$

which is only possible if $\Theta(D) = 0$ or 1 , or equivalently if $\Upsilon(D) = 0$ or 1 . Therefore $\Upsilon^*(A)$ is either 0 or 1 , which completes the proof. \square

Proof of Theorem 5.2

Having established Proposition 5.8 and Proposition 5.9, we can now apply the ergodic theorem to the two functionals on \mathbb{T}^* defined as follows. First let $J_n(\mathcal{T}, \mathbf{v})$ denote the height of the n -th branching point on the geodesic ray \mathbf{v} . One immediately verifies that, for every $n \geq 1$,

$$J_n = \sum_{i=0}^{n-1} J_1 \circ S^i.$$

If $M = \int \kappa(\mathcal{C}(\mathcal{T}))\Theta^*(d\mathcal{T}d\mathbf{v})$, it follows from the ergodic theorem that Θ^* -a.s.,

$$\frac{1}{n}J_n \xrightarrow[n \rightarrow \infty]{} M^{-1} \int J_1(\mathcal{T}, \mathbf{v})\kappa(\mathcal{C}(\mathcal{T}))\Theta^*(d\mathcal{T}d\mathbf{v}). \quad (5.20)$$

Note that the limit can be written as

$$M^{-1}\mathbb{E}\left[|\log(1-U)|\kappa\left(\left(U + \frac{1-U}{\mathcal{C}_1 + \dots + \mathcal{C}_N}\right)^{-1}\right)\right]$$

with the notation used in the proof of Proposition 5.8.

Secondly, let $\mathbf{x}_{n,\mathbf{v}}$ denote the $n+1$ -st branching point on the geodesic ray \mathbf{v} . If $\mathbf{v} = (v_1, v_2, \dots)$, then $\mathbf{x}_{n,\mathbf{v}} = ((v_1, \dots, v_n), J_{n+1}(\mathcal{T}, \mathbf{v}))$ with the notation of Section 5.2.2. We set for every $n \geq 1$,

$$F_n(\mathcal{T}, \mathbf{v}) := \log \nu_{\mathcal{T}}(\{\mathbf{u} \in \partial\mathcal{T} : \mathbf{x}_{n,\mathbf{v}} \prec \mathbf{u}\}).$$

By the flow property of harmonic measure (Lemma 5.7), we have

$$F_n = \sum_{i=0}^{n-1} F_1 \circ S^i,$$

and by the ergodic theorem, Θ^* -a.s.,

$$\frac{1}{n}F_n \xrightarrow[n \rightarrow \infty]{} M^{-1} \int F_1(\mathcal{T}, \mathbf{v})\kappa(\mathcal{C}(\mathcal{T}))\Theta^*(d\mathcal{T}d\mathbf{v}), \quad (5.21)$$

where the limit can be written as

$$M^{-1}\mathbb{E}\left[\frac{N\mathcal{C}_1}{\mathcal{C}_1 + \dots + \mathcal{C}_N} \log\left(\frac{\mathcal{C}_1}{\mathcal{C}_1 + \dots + \mathcal{C}_N}\right) \kappa\left(\left(U + \frac{1-U}{\mathcal{C}_1 + \dots + \mathcal{C}_N}\right)^{-1}\right)\right].$$

By combining (5.20) and (5.21), we obtain that the convergence (5.9) holds with limit

$$-\beta = \frac{\mathbb{E}\left[\frac{N\mathcal{C}_1}{\mathcal{C}_1 + \dots + \mathcal{C}_N} \log\left(\frac{\mathcal{C}_1}{\mathcal{C}_1 + \dots + \mathcal{C}_N}\right) \kappa\left(\left(U + \frac{1-U}{\mathcal{C}_1 + \dots + \mathcal{C}_N}\right)^{-1}\right)\right]}{\mathbb{E}\left[|\log(1-U)|\kappa\left(\left(U + \frac{1-U}{\mathcal{C}_1 + \dots + \mathcal{C}_N}\right)^{-1}\right)\right]}.$$

Proposition 5.10. *We have $\beta < \frac{1}{\alpha-1}$.*

Proof. We use the notation

$$\mathcal{W}(\mathcal{T}) = \lim_{r \rightarrow \infty} e^{-\frac{r}{\alpha-1}} \#\mathcal{T}_r,$$

which exists $\Theta(d\mathcal{T})$ -a.s. by a martingale argument. Since $\sum \theta(k)k \log k < \infty$, the Kesten–Stigum theorem (for CTGW trees, see e.g. [7, Theorem III.7.2]) implies that the previous convergence

holds in the L^1 -sense and $\int \mathcal{W}(\mathcal{T})\Theta(d\mathcal{T}) = 1$. Moreover, $\Theta(\mathcal{W}(\mathcal{T}) = 0) = 0$ and the Laplace transform

$$\int e^{-u\mathcal{W}(\mathcal{T})}\Theta(d\mathcal{T}) = 1 - \frac{u}{(1+u^{\alpha-1})^{\frac{1}{\alpha-1}}} \quad \text{for any } u \in (0, \infty)$$

can be obtained by applying Theorem III.8.3 in [7] together with (5.6). In particular, it follows from a Tauberian theorem (cf. [31, Chapter XIII.5]) that $\int |\log \mathcal{W}(\mathcal{T})|\Theta(d\mathcal{T}) < \infty$.

Let $\mathcal{T}_{(1)}, \dots, \mathcal{T}_{(k_\emptyset)}$ be the subtrees of \mathcal{T} at the first branching point, and let $J(\mathcal{T}) = J_1(\mathcal{T}, \mathbf{v})$ be the height of the first branching point. Then, $\Theta(d\mathcal{T})$ -a.s.

$$\mathcal{W}(\mathcal{T}) = e^{-\frac{J(\mathcal{T})}{\alpha-1}} (\mathcal{W}(\mathcal{T}_{(1)}) + \dots + \mathcal{W}(\mathcal{T}_{(k_\emptyset)})),$$

so that we can define a probability measure $w_{\mathcal{T}}$ on $\{1, 2, \dots, k_\emptyset\}$ by setting

$$w_{\mathcal{T}}(i) = \frac{e^{-\frac{J(\mathcal{T})}{\alpha-1}} \mathcal{W}(\mathcal{T}_{(i)})}{\mathcal{W}(\mathcal{T})}, \quad 1 \leq i \leq k_\emptyset.$$

On the other hand, for $1 \leq i \leq k_\emptyset$, let $\nu_{\mathcal{T}}^*(i)$ denote the mass assigned by the harmonic measure $\nu_{\mathcal{T}}$ to the rays “contained” in $\mathcal{T}_{(i)}$, that is,

$$\nu_{\mathcal{T}}^*(i) = \int \mathbf{1}_{\{v_1=i\}} \nu_{\mathcal{T}}(d\mathbf{v}) = \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \dots + \mathcal{C}(\mathcal{T}_{(k_\emptyset)})}.$$

By a concavity argument,

$$\sum_{i=1}^{k_\emptyset} \nu_{\mathcal{T}}^*(i) \log \frac{w_{\mathcal{T}}(i)}{\nu_{\mathcal{T}}^*(i)} \leq 0, \quad (5.22)$$

and the inequality is strict with positive Θ -probability.

Recall that $\Upsilon(d\mathcal{T}) = M^{-1}\kappa(\mathcal{C}(\mathcal{T}))\Theta(d\mathcal{T})$ is the image of the probability measure $\Upsilon^*(d\mathcal{T}d\mathbf{v})$ under the canonical projection π_1 from \mathbb{T}^* to \mathbb{T} . According to the discussion before Proposition 5.10, we can write

$$\beta = \left(\int \Upsilon(d\mathcal{T})J(\mathcal{T}) \right)^{-1} \int \Upsilon(d\mathcal{T}) \sum_{i=1}^{k_\emptyset} \nu_{\mathcal{T}}^*(i) \log \frac{1}{\nu_{\mathcal{T}}^*(i)},$$

which by (5.22) is strictly smaller than

$$\left(\int \Upsilon(d\mathcal{T})J(\mathcal{T}) \right)^{-1} \int \Upsilon(d\mathcal{T}) \sum_{i=1}^{k_\emptyset} \nu_{\mathcal{T}}^*(i) \log \frac{1}{w_{\mathcal{T}}(i)}.$$

However, it follows from the definition of $w_{\mathcal{T}}$ that

$$\begin{aligned} \int \Upsilon(d\mathcal{T}) \sum_{i=1}^{k_\emptyset} \nu_{\mathcal{T}}^*(i) \log \frac{1}{w_{\mathcal{T}}(i)} &= \frac{1}{\alpha-1} \int \Upsilon(d\mathcal{T})J(\mathcal{T}) + \int \Upsilon(d\mathcal{T}) \sum_{i=1}^{k_\emptyset} \nu_{\mathcal{T}}^*(i) \log \frac{\mathcal{W}(\mathcal{T})}{\mathcal{W}(\mathcal{T}_{(i)})} \\ &= \frac{1}{\alpha-1} \int \Upsilon(d\mathcal{T})J(\mathcal{T}) + \int \Upsilon^*(d\mathcal{T}d\mathbf{v}) \log \frac{\mathcal{W} \circ \pi_1(\mathcal{T}, \mathbf{v})}{\mathcal{W} \circ \pi_1(S(\mathcal{T}, \mathbf{v}))} \\ &= \frac{1}{\alpha-1} \int \Upsilon(d\mathcal{T})J(\mathcal{T}), \end{aligned}$$

where in the last equality we used the fact that Υ^* is invariant under the shift S , and that $\log \mathcal{W}(\mathcal{T})$ is integrable under $\Theta(d\mathcal{T})$ hence also under Υ^* . Therefore, we have shown $\beta < \frac{1}{\alpha-1}$ and the proof of Theorem 5.2 is completed. \square

Proof of Proposition 5.4

We have seen above that

$$\beta = \frac{\mathbb{E}\left[\frac{N\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_N} \log\left(\frac{\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_N}\right) \kappa\left(\left(U + \frac{1-U}{\mathcal{C}_1+\dots+\mathcal{C}_N}\right)^{-1}\right)\right]}{\mathbb{E}\left[\log(1-U) \kappa\left(\left(U + \frac{1-U}{\mathcal{C}_1+\dots+\mathcal{C}_N}\right)^{-1}\right)\right]} \quad (5.23)$$

On account of Proposition 5.5, the proof of Proposition 5.4 will be completed if we can verify that the preceding expression for β is consistent with formula (5.5). In the following calculations, we will keep using the same notation introduced in the proof of Proposition 5.8.

Firstly, the numerator of the right-hand side of (5.23) is equal to

$$\begin{aligned} & \mathbb{E}\left[\frac{N\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_N} \log\left(\frac{\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_N}\right) \frac{(U + \frac{1-U}{\mathcal{C}_1+\dots+\mathcal{C}_N})^{-1}(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}})}{(U + \frac{1-U}{\mathcal{C}_1+\dots+\mathcal{C}_N})^{-1} + \tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1}\right] \\ &= \mathbb{E}\left[\frac{N\mathcal{C}_1(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}) \log\frac{\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_N}}{\mathcal{C}_1+\dots+\mathcal{C}_N + \tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1 + U(\mathcal{C}_1+\dots+\mathcal{C}_N-1)(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1)}\right]. \end{aligned}$$

For every integer $k \geq 2$, we define for $x \in (1, \infty)$ the function

$$G_{c_1, \dots, c_k, u}(x) := \frac{xc_1 \log \frac{c_1}{c_1+\dots+c_k}}{c_1+\dots+c_k+x-1+(c_1+\dots+c_k-1)(x-1)u},$$

where $u \in (0, 1)$ and $c_1, \dots, c_k \in (1, \infty)$. We can apply (5.15) to get

$$\begin{aligned} & \mathbb{E}[G_{c_1, \dots, c_k, U}(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}) \mid \mathcal{C}_1, \dots, \mathcal{C}_k, U] \\ &= \mathbb{E}\left[\frac{\mathcal{C}_0^2 \mathcal{C}_1 (\mathcal{C}_1+\dots+\mathcal{C}_k) \log \frac{\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_k}}{(\mathcal{C}_0+\mathcal{C}_1+\dots+\mathcal{C}_k-1+(\mathcal{C}_0-1)(\mathcal{C}_1+\dots+\mathcal{C}_k-1)U)^2} \mid \mathcal{C}_1, \dots, \mathcal{C}_k, U\right]. \end{aligned}$$

With help of the last display, the numerator of the right-hand side of (5.23) becomes

$$\mathbb{E}\left[\frac{N\mathcal{C}_0^2 \mathcal{C}_1 (\mathcal{C}_1+\dots+\mathcal{C}_N) \log \frac{\mathcal{C}_1}{\mathcal{C}_1+\dots+\mathcal{C}_N}}{(\mathcal{C}_0+\mathcal{C}_1+\dots+\mathcal{C}_N-1+(\mathcal{C}_0-1)(\mathcal{C}_1+\dots+\mathcal{C}_N-1)U)^2}\right].$$

We now integrate with respect to U and recall that for $a, b, c > 0$, $\int_0^1 du \frac{a}{(b+cu)^2} = \frac{a}{b(b+c)}$. So the numerator of the right-hand side of (5.23) coincides with

$$\mathbb{E}\left[\frac{N\mathcal{C}_0 \mathcal{C}_1 \log \frac{\mathcal{C}_1}{\mathcal{C}_1+\mathcal{C}_2+\dots+\mathcal{C}_N}}{\mathcal{C}_0+\mathcal{C}_1+\dots+\mathcal{C}_N-1}\right].$$

On the other hand, the denominator of the right-hand side of (5.23) is equal to

$$\begin{aligned} & \mathbb{E}\left[\frac{(\mathcal{C}_1+\dots+\mathcal{C}_N)(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}) \log(1-U)}{\mathcal{C}_1+\dots+\mathcal{C}_N + (\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1)(\mathcal{C}_1+\dots+\mathcal{C}_N)U + (\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1)(1-U)}\right] \\ &= \mathbb{E}\left[\frac{(\mathcal{C}_1+\dots+\mathcal{C}_N)(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}) \log(1-U)}{\mathcal{C}_1+\dots+\mathcal{C}_N + \tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1 + (\mathcal{C}_1+\dots+\mathcal{C}_N-1)(\tilde{\mathcal{C}}_1+\dots+\tilde{\mathcal{C}}_{\tilde{N}}-1)U}\right] \\ &= \mathbb{E}\left[\frac{\mathcal{C}_0^2 (\mathcal{C}_1+\dots+\mathcal{C}_N)^2 \log(1-U)}{(\mathcal{C}_0+\mathcal{C}_1+\dots+\mathcal{C}_N-1+(\mathcal{C}_0-1)(\mathcal{C}_1+\dots+\mathcal{C}_N-1)U)^2}\right] \\ &= -\mathbb{E}\left[\frac{\mathcal{C}_0^2 \mathcal{C}_1^2 (-1+\mathcal{C}_0+\mathcal{C}_1-2\mathcal{C}_0\mathcal{C}_1+(\mathcal{C}_0-1)(\mathcal{C}_1-1)U) \log(1-U)}{(\mathcal{C}_0+\mathcal{C}_1-1+(\mathcal{C}_0-1)(\mathcal{C}_1-1)U)^3}\right], \end{aligned}$$

where we have repeatedly used (5.15) in the last two equalities, the first time to replace $\tilde{\mathcal{C}}_1 + \dots + \tilde{\mathcal{C}}_N$ by \mathcal{C}_0 , the second time to replace $\mathcal{C}_1 + \dots + \mathcal{C}_N$ by \mathcal{C}_1 . In order to integrate with respect to U , we appeal to the identity that for $a, b, c > 0$,

$$\int_0^1 du \frac{(a+bu) \log(1-u)}{(c+bu)^3} = \frac{b(c-a) + (2b+c+a)c \log \frac{c}{b+c}}{2bc(b+c)^2}.$$

Applying this formula, we see that the denominator of the right-hand side of (5.23) coincides with

$$-\mathbb{E}\left[\frac{\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0 + \mathcal{C}_1 - 1}\right].$$

We have thus obtained the following formula

$$\beta = \frac{\mathbb{E}\left[\frac{N\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_N - 1} \log \frac{\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N}{\mathcal{C}_1}\right]}{\mathbb{E}\left[\frac{\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0 + \mathcal{C}_1 - 1}\right]}. \quad (5.24)$$

By a symmetry argument, the numerator of the right-hand side of (5.24) is equal to

$$\begin{aligned} & \mathbb{E}\left[\frac{N\mathcal{C}_0\mathcal{C}_1 \log(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)}{\mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_N - 1}\right] - \mathbb{E}\left[\frac{N\mathcal{C}_0\mathcal{C}_1 \log(\mathcal{C}_1)}{\mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_N - 1}\right] \\ &= \mathbb{E}\left[\frac{\mathcal{C}_0(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N) \log(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)}{\mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_N - 1}\right] - \mathbb{E}\left[\frac{\mathcal{C}_0(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N) \log(\mathcal{C}_0)}{\mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_N - 1}\right] \\ &= \mathbb{E}[f(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)] - \mathbb{E}[g(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)], \end{aligned} \quad (5.25)$$

where we have set, for every $x \geq 1$,

$$f(x) = \mathbb{E}\left[\frac{\mathcal{C}_0 x}{\mathcal{C}_0 + x - 1} \log x\right] \quad \text{and} \quad g(x) = \mathbb{E}\left[\frac{\mathcal{C}_0 x}{\mathcal{C}_0 + x - 1} \log \mathcal{C}_0\right].$$

We can replace $\mathbb{E}[f(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)]$ by $\mathbb{E}[f(\mathcal{C}_1)] + \mathbb{E}[\mathcal{C}_1(\mathcal{C}_1 - 1)f'(\mathcal{C}_1)]$ using (5.15), and similarly for g , to obtain

$$\mathbb{E}[f(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)] - \mathbb{E}[g(\mathcal{C}_1 + \mathcal{C}_2 + \dots + \mathcal{C}_N)] = \frac{1}{2} \left(\mathbb{E}[\mathcal{C}_0]^2 - \mathbb{E}\left[\frac{\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0 + \mathcal{C}_1 - 1}\right] \right).$$

Plugging this into (5.25) yields the required formula (5.5), and hence finishes the proof of Proposition 5.4.

5.2.6 A second approach to Theorem 5.2

In this section, we outline a different approach to Theorem 5.2, which contains certain intermediate results of independent interest. This approach involves an invariant measure for the environment seen by Brownian motion on the CTGW tree $\Gamma^{(\alpha)}$ at the last visit of a fixed height. This is similar to Section 3 of [20], and for this reason we will leave the proofs to Section 5.5.

We fix the index $\alpha \in (1, 2]$, and we first introduce some additional notation. For $\mathcal{T} \in \mathbb{T}$ and $r > 0$, if $x \in \mathcal{T}_r$, let $\mathcal{T}[x]$ denote the subtree of descendants of x in \mathcal{T} . To define it formally, we write v_x for the unique element of \mathcal{V} such that $x = (v_x, r)$, and define the shifted discrete tree $\Pi[v_x] = \{v \in \mathcal{V} : v_x v \in \Pi\}$. Then $\mathcal{T}[x]$ is the infinite continuous tree corresponding to the pair

$$\left(\Pi[v_x], (z_{v_x v} - r)_{v \in \Pi[v_x]}\right).$$

For a fixed $r > 0$, we know that $\Gamma^{(\alpha)}$ has a.s. no branching point at height r . As there is a unique point $x \in \Gamma_r^{(\alpha)}$ such that $x \prec W_\infty$, we write $\Gamma^{(\alpha)}\langle r \rangle = \Gamma^{(\alpha)}[x]$ for the subtree above level r selected by harmonic measure.

To describe the distribution of $\Gamma^{(\alpha)}\langle r \rangle$, recall that for every $x \geq 0$,

$$\varphi_\alpha(x) = \mathbb{E}[\exp(-x\mathcal{C}^{(\alpha)}/2)] = \Theta_\alpha(\exp(-x\mathcal{C}(\mathcal{T})/2)).$$

Proposition 5.11. *The distribution under $\mathbb{P} \otimes P$ of the subtree $\Gamma^{(\alpha)}\langle r \rangle$ above level r selected by harmonic measure is*

$$\Phi_r^{(\alpha)}(\mathcal{C}(\mathcal{T})) \Theta_\alpha(d\mathcal{T}),$$

where, for every $c > 0$,

$$\Phi_r^{(\alpha)}(c) := E_{(c)} \left[\exp - \int_0^r ds \left(m_\alpha (1 - \varphi_\alpha(X_s))^{\alpha-1} - \frac{1}{\alpha-1} \right) \right].$$

Here $X = (X_s)_{0 \leq s \leq r}$ stands for the solution of the stochastic differential equation

$$dX_s = 2\sqrt{X_s} d\eta_s + (2 - X_s) ds$$

that starts under the probability measure $P_{(c)}$ with an exponential distribution of parameter $c/2$. In the previous SDE, $(\eta_s)_{s \geq 0}$ denotes a standard linear Brownian motion.

Now we define shifts $(\tau_r)_{r \geq 0}$ on \mathbb{T}^* in the following way. For $r = 0$, τ_0 is the identity mapping of \mathbb{T}^* . For $r > 0$ and $(\mathcal{T}, \mathbf{v}) \in \mathbb{T}^*$, we write $\mathbf{v} = (v_1, v_2, \dots)$ and $\mathbf{v}_n = (v_1, \dots, v_n)$ for every $n \geq 0$ (by convention, $\mathbf{v}_0 = \emptyset$). Also let $x_{r,\mathbf{v}}$ be the unique element of \mathcal{T}_r such that $x_{r,\mathbf{v}} \prec \mathbf{v}$. Then we set

$$\tau_r(\mathcal{T}, \mathbf{v}) = \left(\mathcal{T}[x_{r,\mathbf{v}}], (v_{k+1}, v_{k+2}, \dots) \right),$$

where $k = \min\{n \geq 0: z_{\mathbf{v}_n} \geq r\}$. Informally, $\tau_r(\mathcal{T}, \mathbf{v})$ is obtained by taking the subtree of \mathcal{T} consisting of descendants of the vertex at height r on the distinguished geodesic ray, and keeping in this subtree the “same” geodesic ray. It is straightforward to verify that $\tau_r \circ \tau_s = \tau_{r+s}$ for every $r, s \geq 0$.

The next proposition gives an invariant measure absolutely continuous with respect to Θ_α^* under the shifts τ_r . To simplify notation, we set first

$$C_1(\alpha) := 2 \int_0^\infty ds \varphi'_\alpha(s)^2 e^{s/2} = \iint \gamma_\alpha(d\ell) \gamma_\alpha(d\ell') \frac{\ell\ell'}{\ell + \ell' - 1}.$$

Proposition 5.12. *For every $c > 0$,*

$$\lim_{r \rightarrow +\infty} \Phi_r^{(\alpha)}(c) = \Phi_\infty^{(\alpha)}(c) := \frac{1}{C_1(\alpha)} \int \gamma_\alpha(ds) \frac{cs}{c+s-1}.$$

The probability measure Λ_α^* on \mathbb{T}^* defined as

$$\Lambda_\alpha^*(d\mathcal{T}d\mathbf{v}) := \Phi_\infty^{(\alpha)}(\mathcal{C}(\mathcal{T})) \Theta_\alpha^*(d\mathcal{T}d\mathbf{v})$$

is invariant under the shifts τ_r , $r \geq 0$.

Furthermore, one can easily adapt the proof of Proposition 13 in [20] to show that for every $r > 0$, the shift τ_r acting on the probability space $(\mathbb{T}^*, \Lambda_\alpha^*)$ is ergodic. Applying Birkhoff’s ergodic theorem to a suitable functional (see Section 3.4 of [20]) leads to the convergence (5.3) in Theorem 5.2, with β_α given by formula (5.24). See Section 5.5 for more details.

5.3 The discrete setting

5.3.1 Galton–Watson trees

Let us first introduce discrete (finite) rooted ordered trees, which are also called plane trees in combinatorics. A plane tree \mathbf{t} is a finite subset of \mathcal{V} such that the following holds:

- (i) $\emptyset \in \mathbf{t}$.
- (ii) If $u = (u_1, \dots, u_n) \in \mathbf{t} \setminus \{\emptyset\}$, then $\hat{u} = (u_1, \dots, u_{n-1}) \in \mathbf{t}$.
- (iii) For every $u = (u_1, \dots, u_n) \in \mathbf{t}$, there exists an integer $k_u(\mathbf{t}) \geq 0$ such that, for every $j \in \mathbb{N}$, $(u_1, \dots, u_n, j) \in \mathbf{t}$ if and only if $1 \leq j \leq k_u(\mathbf{t})$.

In this section we will say tree instead of plane tree for short. The same notation and terminology introduced at the beginning of Section 5.2.1 will be used in this section: $|u|$ is the generation of u , uv denotes the concatenation of u and v , \prec stands for the (non-strict) genealogical order and $u \wedge v$ is the maximal element of $\{w \in \mathcal{V} : w \prec u \text{ and } w \prec v\}$. A vertex with no child is called a leaf.

The height of a tree \mathbf{t} is

$$h(\mathbf{t}) := \max\{|v| : v \in \mathbf{t}\}.$$

We write \mathcal{T} for the set of all trees, and \mathcal{T}_n for the set of all trees with height n .

We view a tree \mathbf{t} as a graph whose vertices are the elements of \mathbf{t} and whose edges are the pairs $\{\hat{u}, u\}$ for all $u \in \mathbf{t} \setminus \{\emptyset\}$. The set \mathbf{t} is equipped with the distance

$$d(u, v) := \frac{1}{2}(|u| + |v| - 2|u \wedge v|).$$

Notice that this is half the usual graph distance. We will write $B_{\mathbf{t}}(v, r)$, or simply $B(v, r)$ if there is no ambiguity, for the closed ball of radius r centered at v , with respect to the distance d in the tree \mathbf{t} .

The set of all vertices of \mathbf{t} at generation n is denoted by

$$\mathbf{t}_n := \{v \in \mathbf{t} : |v| = n\}.$$

If $v \in \mathbf{t}$, the subtree of descendants of v is

$$\tilde{\mathbf{t}}[v] := \{v' \in \mathbf{t} : v \prec v'\}.$$

Note that $\tilde{\mathbf{t}}[v]$ is not a tree under the previous definition, but we can turn it into a tree by relabeling its vertices as

$$\mathbf{t}[v] := \{w \in \mathcal{V} : vw \in \mathbf{t}\}.$$

If $v \in \mathbf{t}$, then for every $i \in \{0, 1, \dots, |v|\}$ we write $\langle v \rangle_i$ for the ancestor of v at generation i . Suppose that $|v| = n$. Then $B_{\mathbf{t}}(v, i) \cap \mathbf{t}_n = \tilde{\mathbf{t}}[\langle v \rangle_{n-i}] \cap \mathbf{t}_n$, for every $i \in \{0, 1, \dots, n\}$. This simple observation will be used repeatedly below.

Let ρ be a non-trivial probability measure on \mathbb{Z}_+ with mean one, which belongs to the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$. Therefore property (5.1) at the beginning of the Introduction holds. For every integer $n \geq 0$, we let $\mathbb{T}^{(n)}$ be a Galton–Watson tree with offspring distribution ρ , conditioned on non-extinction at generation n , viewed as a random subset of \mathcal{V} (see e.g. [58] for a precise definition of Galton–Watson trees). In particular, $\mathbb{T}^{(0)}$ is just a Galton–Watson tree with offspring distribution ρ . We suppose that the random trees $\mathbb{T}^{(n)}$ are defined under the probability measure \mathbb{P} .

We let \mathbb{T}^{*n} be the reduced tree associated with $\mathbb{T}^{(n)}$, which consists of all vertices of $\mathbb{T}^{(n)}$ that have (at least) one descendant at generation n . Note that $|v| \leq n$ for every $v \in \mathbb{T}^{*n}$. A priori \mathbb{T}^{*n} is not a tree in the sense of the preceding definition. However we can relabel the vertices of \mathbb{T}^{*n} , preserving both the lexicographical order and the genealogical order, so that \mathbb{T}^{*n} becomes a tree in the sense of our definitions. We will always assume that this relabeling has been done.

Conditionally on $\mathbb{T}^{(n)}$, the hitting distribution of generation n is the same for simple random walk on $\mathbb{T}^{(n)}$ and that on the reduced tree \mathbb{T}^{*n} . In view of studying properties of this hitting distribution, we can consider directly a simple random walk on \mathbb{T}^{*n} starting from the root \emptyset , which we denote by $Z^n = (Z_k^n)_{k \geq 0}$. This random walk is defined under the probability measure P . Let

$$H_n := \inf\{k \geq 0: |Z_k^n| = n\}$$

be the first hitting time of generation n by Z^n , and set $\Sigma_n = Z_{H_n}^n$ to be the hitting point. The discrete harmonic measure μ_n is the law of Σ_n under P , which is a (random) probability measure on the level set \mathbb{T}_n^{*n} .

Set $q_n = \mathbb{P}(h(\mathbb{T}^{(0)}) \geq n)$. If L is the slowly varying function appearing in (5.1), it has been established in [83, Lemma 2] that

$$q_n^{\alpha-1} L(q_n) \sim \frac{1}{(\alpha-1)n} \quad \text{as } n \rightarrow \infty. \quad (5.26)$$

By the asymptotic inversion property of slowly varying functions (see e.g. [12, Section 1.5.7]), it follows that

$$q_n \sim n^{-\frac{1}{\alpha-1}} \ell(n) \quad \text{as } n \rightarrow \infty, \quad (5.27)$$

for a function ℓ slowly varying at ∞ . Moreover, it is shown in [83, Theorem 1] that, as $n \rightarrow \infty$, $q_n \#\mathbb{T}_n^{*n}$ converges in distribution to the positive random variable $\mathcal{W}(\Gamma^{(\alpha)})$ introduced in the proof of Proposition 5.10.

We will need to estimate the size of level sets in \mathbb{T}^{*n} . The following lemma is an analogue of Lemma 15 in [20].

Lemma 5.13. *For every $r \geq 1$, there exists a constant $C = C(r, \rho)$ depending on r and the offspring distribution ρ such that, for every integer $n \geq 2$ and every integer $p \in [1, n/2]$,*

$$\mathbb{E}[(\log \#\mathbb{T}_{n-p}^{*n})^r]^{\frac{1}{r}} \leq C \log \frac{n}{p} \quad \text{and} \quad \mathbb{E}[(\log \#\mathbb{T}_n^{*n})^r]^{\frac{1}{r}} \leq C \log n.$$

Proof. We can find $a = a(r) > 0$ such that the function $x \mapsto (\log(a+x))^r$ is concave over $[1, \infty)$. Then as in the proof of [20, Lemma 15],

$$\mathbb{E}[(\log \#\mathbb{T}_{n-p}^{*n})^r]^{\frac{1}{r}} \leq \mathbb{E}[(\log(a + \#\mathbb{T}_{n-p}^{*n}))^r]^{\frac{1}{r}} \leq \log(a + \mathbb{E}[\#\mathbb{T}_{n-p}^{*n}]) = \log(a + \frac{qp}{q_n}).$$

Using Potter's bounds on slowly varying function (see e.g. [12, Theorem 1.5.6]), one can deduce from (5.27) that there exists a constant $C' = C'(\rho) > 0$ such that for every $n \geq 2$ and every $p \in [1, n/2]$,

$$\log\left(\frac{qp}{q_n}\right) \leq C' \log\left(\frac{n}{p}\right),$$

from which the first bound of the lemma easily follows. The second estimate can be shown in a similar way. \square

The goal of this section is to prove Theorem 5.1. We will assume in the rest of this section that the critical offspring distribution ρ satisfies (5.1) with a fixed $\alpha \in (1, 2]$. Accordingly, we will omit the superscripts and subscripts concerning α if there is no ambiguity.

5.3.2 Convergence of discrete reduced trees

We first define truncations of the discrete reduced tree \mathbb{T}^{*n} . For every $s \in [0, n]$, we set

$$R_s(\mathbb{T}^{*n}) := \{v \in \mathbb{T}^{*n} : |v| \leq n - \lfloor s \rfloor\}.$$

Recall from Section 5.2.1 the definition of the continuous reduced tree Δ of index α . For every $\varepsilon \in (0, 1)$, we have set $\Delta_\varepsilon = \{x \in \Delta : H(x) \leq 1 - \varepsilon\}$. We will implicitly use the fact that, for every fixed ε , there is a.s. no branching point of Δ at height $1 - \varepsilon$. The skeleton of Δ_ε is defined as the following plane tree

$$\text{Sk}(\Delta_\varepsilon) := \{\emptyset\} \cup \{v \in \Pi \setminus \{\emptyset\} : Y_{\hat{v}} \leq 1 - \varepsilon\} = \{\emptyset\} \cup \{v \in \Pi \setminus \{\emptyset\} : (\hat{v}, Y_{\hat{v}}) \in \Delta_\varepsilon\}.$$

A vertex v of $\text{Sk}(\Delta_\varepsilon)$ is a leaf of $\text{Sk}(\Delta_\varepsilon)$ if and only if $Y_v > 1 - \varepsilon$.

Let \mathbf{t} be a tree. We write $\mathcal{S}(\mathbf{t})$ for the set of all vertices of \mathbf{t} whose number of children is different from 1. Then we can find a unique tree $[\mathbf{t}] \in \mathcal{T}$ such that there exists a bijection from $[\mathbf{t}]$ onto $\mathcal{S}(\mathbf{t})$ that preserves the genealogical order and the lexicographical order of vertices. Denote the inverse of this canonical bijection by $u \in \mathcal{S}(\mathbf{t}) \mapsto [u] \in [\mathbf{t}]$. In a less formal way, $[\mathbf{t}]$ is just the tree obtained from \mathbf{t} by removing all vertices that have exactly one child.

Proposition 5.14. *We can construct the reduced trees \mathbb{T}^{*n} and the (continuous) reduced stable tree Δ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that the following assertions hold for every fixed $\varepsilon \in (0, 1)$ with \mathbb{P} -probability one.*

- (1) *For every sufficiently large integer n , there exists an injective mapping $\Psi_n^\varepsilon : u \mapsto w_u^{n,\varepsilon}$ from $\text{Sk}(\Delta_\varepsilon)$ into $\mathcal{S}(R_{\varepsilon n}(\mathbb{T}^{*n}))$ satisfying the following properties.*
 - (1.a) *The mapping Ψ_n^ε preserves both the lexicographical order and the genealogical order.*
 - (1.b) *If u is a leaf of $\text{Sk}(\Delta_\varepsilon)$, $[w_u^{n,\varepsilon}]$ is a leaf of $[R_{\varepsilon n}(\mathbb{T}^{*n})]$ and $|w_u^{n,\varepsilon}| = n - \lfloor \varepsilon n \rfloor$. The restricted mapping*

$$\Psi_n^\varepsilon \upharpoonright_{\text{Leaves}} : \text{Leaves of } \text{Sk}(\Delta_\varepsilon) \longrightarrow \{v \in \mathcal{S}(R_{\varepsilon n}(\mathbb{T}^{*n})) : [v] \text{ is a leaf of } [R_{\varepsilon n}(\mathbb{T}^{*n})]\}$$

is bijective.

- (1.c) *For every vertex u of $\text{Sk}(\Delta_\varepsilon)$,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} |w_u^{n,\varepsilon}| &= Y_u \wedge (1 - \varepsilon), \\ \lim_{n \rightarrow \infty} \frac{1}{n} |\bar{w}_u^{n,\varepsilon}| &= Y_{\hat{u}}, \end{aligned}$$

*where \hat{u} denotes the parent of u in $\text{Sk}(\Delta_\varepsilon)$, and $\bar{w}_u^{n,\varepsilon}$ stands for the vertex in $\mathcal{S}(R_{\varepsilon n}(\mathbb{T}^{*n}))$ such that $[\bar{w}_u^{n,\varepsilon}]$ is the parent of $[w_u^{n,\varepsilon}]$ in $[R_{\varepsilon n}(\mathbb{T}^{*n})]$. (Notice that $\bar{w}_u^{n,\varepsilon}$ does not necessarily coincide with $w_{\hat{u}}^{n,\varepsilon}$.)*

- (2) *The mapping Ψ_n^ε is asymptotically unique in the sense that, if $\tilde{\Psi}_n^\varepsilon$ is another mapping such that the preceding properties hold, then for n sufficiently large,*

$$\Psi_n^\varepsilon(u) = \tilde{\Psi}_n^\varepsilon(u) \quad \text{for every } u \in \text{Sk}(\Delta_\varepsilon).$$

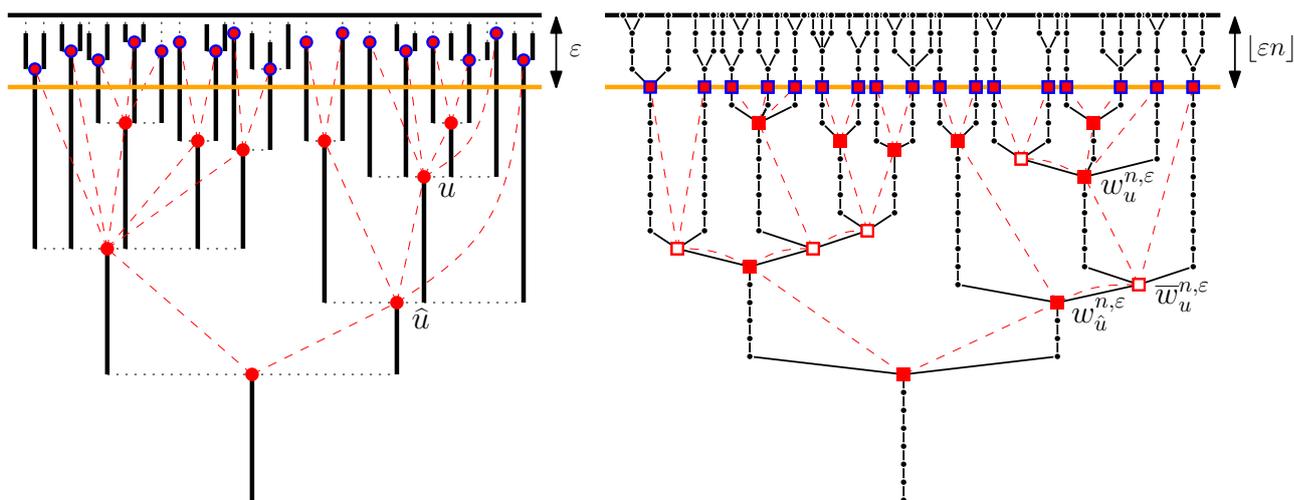


Figure 5.2: On the left, the tree Δ , its truncation Δ_ε and its skeleton $\text{Sk}(\Delta_\varepsilon)$. On the right, a large reduced tree \mathbb{T}^{*n} of height n , its truncation $R_{\varepsilon n}(\mathbb{T}^{*n})$ and the associated tree $[R_{\varepsilon n}(\mathbb{T}^{*n})]$. The vertices depicted as filled red disks on the left correspond to the vertices depicted as filled red squares on the right, via the mapping Ψ_n^ε .

Proposition 5.14 (see Fig. 5.2 for an illustration) essentially results from the convergence in distribution of the rescaled contour functions associated with the trees $\mathbb{T}^{(n)}$ towards the excursion of the stable height process with height greater than 1 (see [26, Section 2.5]). By using the Skorokhod representation theorem, one may assume that the trees $\mathbb{T}^{(n)}$ and the excursion of the stable height process are constructed so that the latter convergence holds almost surely. The various assertions of Proposition 5.14 then easily follow (cf. [26, Section 2.6]), using the relation between the excursion of the stable height process with height greater than 1 and the limiting reduced tree Δ , which can be found in [26, Section 2.7].

Remark 1. Let us take $0 < \delta < \varepsilon$. If u is not a leaf of $\text{Sk}(\Delta_\varepsilon)$, we must have $w_u^{n,\varepsilon} = w_u^{n,\delta}$ for sufficiently large n . On the other hand, if u is a leaf of $\text{Sk}(\Delta_\varepsilon)$, then for large n , $w_u^{n,\varepsilon}$ must be an ancestor of $w_u^{n,\delta}$.

Remark 2. We expect that a result more precise than Proposition 5.14 should hold. For all sufficiently large n , the mapping Ψ_n^ε should be a bijection, and the equality $\bar{w}_u^{n,\varepsilon} = w_u^{n,\varepsilon}$ should hold for all $u \in \text{Sk}(\Delta_\varepsilon)$ (in other words, there should be no white square in the right part of Fig. 5.2). However this refinement does not easily follow from the results of [26], and we will omit it since it is not needed for our purposes.

5.3.3 Convergence of harmonic measures

Recall that μ is the continuous harmonic measure on the boundary $\partial\Delta$ of the reduced stable tree, and that μ_n is the discrete harmonic measure on \mathbb{T}_n^{*n} . For every $x \in \partial\Delta_\varepsilon$, we set

$$\mu^\varepsilon(x) = \mu(\{y \in \partial\Delta : x \prec y\}) = P(x \prec B_{T_-}).$$

Similarly, we define a probability measure μ_n^ε on $\mathbb{T}_{n-[\varepsilon n]}^{*n}$ by setting

$$\mu_n^\varepsilon(u) = \mu_n(\{v \in \mathbb{T}_n^{*n} : u \prec v\}),$$

for every $u \in \mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}$. Clearly, μ_n^ε is the distribution of $\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}$.

Proposition 5.15. *Suppose that the reduced trees \mathbb{T}^{*n} and the (continuous) tree Δ have been constructed so that the properties of Proposition 5.14 hold, and recall the notation $(w_u^{n,\varepsilon})_{u \in \text{Sk}(\Delta_\varepsilon)}$ introduced therein. Then \mathbb{P} -a.s. for every $x = (u, 1 - \varepsilon) \in \partial\Delta_\varepsilon$,*

$$\lim_{n \rightarrow \infty} \mu_n^\varepsilon(w_u^{n,\varepsilon}) = \mu^\varepsilon(x).$$

Proof. Let $\delta \in (0, \varepsilon)$ and set $T_\delta = \inf\{t \geq 0 : H(B_t) = 1 - \delta\} < T$. Define a probability measure $\mu^{\varepsilon,(\delta)}$ on $\partial\Delta_\varepsilon$ by setting for every $x \in \partial\Delta_\varepsilon$,

$$\mu^{\varepsilon,(\delta)}(x) = P(x \prec B_{T_\delta}).$$

Similarly, we write $\mu_n^{(\delta)}$ for the distribution of the hitting point of generation $n - \lfloor \delta n \rfloor$ by random walk on \mathbb{T}^{*n} started from \emptyset . Then we define a probability measure $\mu_n^{\varepsilon,(\delta)}$ on $\mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}$ by setting

$$\mu_n^{\varepsilon,(\delta)}(v) = \mu_n^{(\delta)}(\{w \in \mathbb{T}_{n-\lfloor \delta n \rfloor}^{*n} : v \prec w\}),$$

for every $v \in \mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}$.

As in the proof of [20, Proposition 18], we have \mathbb{P} -a.s.

$$\begin{aligned} \lim_{\delta \rightarrow 0} \left(\sup_{x \in \partial\Delta_\varepsilon} |\mu^{\varepsilon,(\delta)}(x) - \mu^\varepsilon(x)| \right) &= 0, \\ \lim_{\delta \rightarrow 0} \left(\limsup_{n \rightarrow \infty} \left(\sup_{v \in \mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}} |\mu_n^{\varepsilon,(\delta)}(v) - \mu_n^\varepsilon(v)| \right) \right) &= 0. \end{aligned}$$

So the convergence of the proposition will follow if we can verify that for every fixed $\delta \in (0, \varepsilon)$, we have \mathbb{P} -a.s. for every $x = (u, 1 - \varepsilon) \in \partial\Delta_\varepsilon$,

$$\lim_{n \rightarrow \infty} \mu_n^{\varepsilon,(\delta)}(w_u^{n,\varepsilon}) = \mu^{\varepsilon,(\delta)}(x). \quad (5.28)$$

To this end, we may and will assume that the reduced trees \mathbb{T}^{*n} and the (continuous) tree Δ have been constructed so that the properties of Proposition 5.14 hold simultaneously for ε and for δ .

Firstly, by considering the successive passage times of Brownian motion stopped at time T_δ in the set $\{(u, Y_u \wedge (1 - \delta)) : u \in \text{Sk}(\Delta_\delta)\}$, we get a Markov chain $X^{(\delta)}$, which is absorbed in the set $\{(v, 1 - \delta) : v \text{ is a leaf of } \text{Sk}(\Delta_\delta)\}$, and whose transition kernels are explicitly described in terms of the quantities $Y_u, u \in \text{Sk}(\Delta_\delta)$ by series and parallel circuits calculation.

Secondly, let n be sufficiently large so that assertions (1) and (2) of Proposition 5.14 hold with ε as well as with δ , and consider simple random walk on \mathbb{T}^{*n} started from \emptyset and stopped at the first hitting time of generation $n - \lfloor \delta n \rfloor$. By considering the successive passage times of this random walk in the set $\mathcal{S}(R_{\delta n}(\mathbb{T}^{*n}))$, we again get a Markov chain $X^{(\delta),n}$, which is absorbed in the set

$$\{v \in \mathcal{S}(R_{\delta n}(\mathbb{T}^{*n})) : [v] \text{ is a leaf of } [R_{\delta n}(\mathbb{T}^{*n})]\}.$$

By property (1.b) of Proposition 5.14, this set is exactly $\{w_v^{n,\delta} : v \text{ is a leaf of } \text{Sk}(\Delta_\delta)\}$. As previously, the transition kernels of this Markov chain $X^{(\delta),n}$ can be written explicitly in terms of the quantities $|v|, v \in \mathcal{S}(R_{\delta n}(\mathbb{T}^{*n}))$.

Recall that by Proposition 5.14,

$$\Psi_n^\delta(\text{Sk}(\Delta_\delta)) = \{w_u^{n,\delta} : u \in \text{Sk}(\Delta_\delta)\}$$

is a subset of $\mathcal{S}(R_{\delta n}(\mathbb{T}^{*n}))$, and that the mapping Ψ_n^δ is injective. If we let $\tilde{X}^{(\delta),n}$ be the Markov chain restricted to the subset $\Psi_n^\delta(\text{Sk}(\Delta_\delta))$, then after identifying both sets $\{(u, Y_u \wedge (1-\delta)) : u \in \text{Sk}(\Delta_\delta)\}$ and $\Psi_n^\delta(\text{Sk}(\Delta_\delta))$ with $\text{Sk}(\Delta_\delta)$, we can view both $X^{(\delta)}$ and $\tilde{X}^{(\delta),n}$ as Markov chains with values in the set $\text{Sk}(\Delta_\delta)$. Using property (1.c) of Proposition 5.14, we see that the transition kernels of $\tilde{X}^{(\delta),n}$ converge to those of $X^{(\delta)}$.

Write $X_\infty^{(\delta)}$ for the absorption point of $X^{(\delta)}$, and similarly write $X_\infty^{(\delta),n}$ for that of $X^{(\delta),n}$. Notice that $X_\infty^{(\delta),n}$ is also the absorption point of the restricted Markov chain $\tilde{X}^{(\delta),n}$. We thus obtain that the distribution of $X_\infty^{(\delta),n}$ converges to that of $X_\infty^{(\delta)}$ (recall that both $X_\infty^{(\delta),n}$ and $X_\infty^{(\delta)}$ are viewed as taking values in the set of leaves of $\text{Sk}(\Delta_\delta)$). Consequently, for every $u \in \mathcal{V}$ such that $x = (u, 1-\varepsilon) \in \partial\Delta_\varepsilon$, we have

$$\lim_{n \rightarrow \infty} P(u \prec X_\infty^{(\delta),n}) = P(u \prec X_\infty^{(\delta)}).$$

However, from our definitions, we have

$$P(u \prec X_\infty^{(\delta)}) = \mu^{\varepsilon,(\delta)}(x),$$

and, for n sufficiently large, since $w_u^{n,\varepsilon}$ coincides with the ancestor of $w_u^{n,\delta}$ at generation $n - \lfloor \varepsilon n \rfloor$ (see Remark 1 after Proposition 5.14),

$$P(u \prec X_\infty^{(\delta),n}) = \mu_n^{\varepsilon,(\delta)}(w_u^{n,\varepsilon}).$$

This completes the proof of (5.28) and of the proposition. \square

Recall that β is the Hausdorff dimension of the continuous harmonic measure μ .

Proposition 5.16. *Let $r \geq 1$ and $\xi \in (0, 1)$. We can find $\varepsilon_0 \in (0, 1/2)$ such that the following holds. For every $\varepsilon \in (0, \varepsilon_0)$, there exists $n_0 \geq 0$ such that for every $n \geq n_0$,*

$$\mathbb{E} \otimes E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) - \beta \log \varepsilon \right|^r \right] \leq \xi |\log \varepsilon|^r.$$

Proof. Recall our notation $\mathcal{B}_d(x, r)$ for the closed ball of radius r centered at $x \in \Delta$. Fix $\eta \in (0, 1)$. Since B_{T_-} is distributed according to μ , it follows from Theorem 5.2 that there exists $\varepsilon_0 \in (0, 1/2)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have

$$\mathbb{P} \otimes P \left(\left| \log \mu(\mathcal{B}_d(B_{T_-}, 2\varepsilon)) - \beta \log \varepsilon \right| > (\eta/2) |\log \varepsilon| \right) < \eta/2. \quad (5.29)$$

Let us fix $\varepsilon \in (0, \varepsilon_0)$ and assume that the reduced trees \mathbb{T}^{*n} and the (continuous) tree Δ have been constructed so that the properties of Proposition 5.14 hold. We now claim that, under $\mathbb{P} \otimes P$,

$$\mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) \xrightarrow[n \rightarrow \infty]{(d)} \mu(\mathcal{B}_d(B_{T_-}, 2\varepsilon)). \quad (5.30)$$

To see this, let f be a continuous function on $[0, 1]$. Since the distribution of $\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}$ under P is μ_n^ε , we have

$$\mathbb{E} \otimes E \left[f(\mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor})) \right] = \mathbb{E} \left[\sum_{v \in \mathbb{T}^{*n}_{n-\lfloor \varepsilon n \rfloor}} \mu_n^\varepsilon(v) f(\mu_n^\varepsilon(v)) \right].$$

By property (1.b) of Proposition 5.14, we know that \mathbb{P} -a.s. for n sufficiently large,

$$\sum_{v \in \mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}} \mu_n^\varepsilon(v) f(\mu_n^\varepsilon(v)) = \sum_{x=(u,1-\varepsilon) \in \partial \Delta_\varepsilon} \mu_n^\varepsilon(w_u^{n,\varepsilon}) f(\mu_n^\varepsilon(w_u^{n,\varepsilon})),$$

and by Proposition 5.15 the latter quantities converge as $n \rightarrow \infty$ towards

$$\sum_{x \in \partial \Delta_\varepsilon} \mu^\varepsilon(x) f(\mu^\varepsilon(x)) = E[f(\mu(\mathcal{B}_a(B_{T_-}, 2\varepsilon)))] ,$$

which establishes the convergence (5.30) as claimed.

By (5.29) and (5.30), we can find $n_0 = n_0(\varepsilon) \geq \varepsilon^{-1}$ such that for $n \geq n_0$,

$$\mathbb{P} \otimes P \left(\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) - \beta \log \varepsilon \right| > \eta |\log \varepsilon| \right) < \eta.$$

Using the Cauchy–Schwarz inequality, we have then

$$\begin{aligned} & \mathbb{E} \otimes E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) - \beta \log \varepsilon \right|^r \right] \\ & \leq \eta^r |\log \varepsilon|^r + \eta^{\frac{1}{2}} \mathbb{E} \otimes E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) - \beta \log \varepsilon \right|^{2r} \right]^{1/2} \\ & \leq \eta^r |\log \varepsilon|^r + 2^r \eta^{\frac{1}{2}} |\beta \log \varepsilon|^r + 2^r \eta^{\frac{1}{2}} \mathbb{E} \otimes E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) \right|^{2r} \right]^{1/2}. \end{aligned} \quad (5.31)$$

Since $r \geq 1$, the function

$$g(x) := (x \wedge e^{-2r}) |\log(x \wedge e^{-2r})|^{2r}$$

is nondecreasing and concave over $[0, 1]$. Thus, we obtain

$$\begin{aligned} E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) \right|^{2r} \right] &= \sum_{v \in \mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}} \mu_n^\varepsilon(v) |\log \mu_n^\varepsilon(v)|^{2r} \\ &\leq \sum_{v \in \mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}} g(\mu_n^\varepsilon(v)) + (2r)^{2r} \\ &\leq \#\mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n} \times g\left(\left(\#\mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n}\right)^{-1}\right) + (2r)^{2r} \\ &\leq \left| \log \#\mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n} \right|^{2r} + 2(2r)^{2r}. \end{aligned}$$

We now use Lemma 5.13 to see

$$\mathbb{E} \otimes E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) \right|^{2r} \right] \leq \mathbb{E} \left[\left| \log \#\mathbb{T}_{n-\lfloor \varepsilon n \rfloor}^{*n} \right|^{2r} \right] + 2(2r)^{2r} \leq C^{2r} \left(\log \frac{n}{\lfloor \varepsilon n \rfloor} \right)^{2r} + 2(2r)^{2r}.$$

By combining the last estimate with (5.31), we get that, for every $n \geq n_0(\varepsilon)$,

$$\mathbb{E} \otimes E \left[\left| \log \mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) - \beta \log \varepsilon \right|^r \right] \leq (\eta^r + 2^r \eta^{\frac{1}{2}} \beta^r) |\log \varepsilon|^r + 2^{r+1} \eta^{\frac{1}{2}} ((2r)^r + C^r |\log \varepsilon|^r).$$

The statement of the proposition follows since η was arbitrary. \square

5.3.4 The flow property of discrete harmonic measure

We briefly recall the flow property of the discrete harmonic measure μ_n presented in [20, Section 4.3.1]. Let $\mathbf{t} \in \mathcal{T}_n$ be a plane tree of height n and $Z^{(\mathbf{t})} = (Z_k^{(\mathbf{t})})_{k \geq 0}$ be simple random walk on \mathbf{t} starting from \emptyset . We set

$$H_n^{(\mathbf{t})} := \inf\{k \geq 0: |Z_k^{(\mathbf{t})}| = n\} \quad \text{and} \quad \Sigma_n^{(\mathbf{t})} := Z_{H_n^{(\mathbf{t})}}^{(\mathbf{t})}.$$

We write $\mu_n^{(\mathbf{t})}$ for the distribution of $\Sigma_n^{(\mathbf{t})}$, considered as a measure on \mathbf{t} supported by \mathbf{t}_n .

For $0 \leq p \leq n$, we set

$$L_p^{(\mathbf{t})} := \sup\{k \leq H_n^{(\mathbf{t})}: |Z_k^{(\mathbf{t})}| = p\}.$$

Clearly $\Sigma_n^{(\mathbf{t})} \in \tilde{\mathfrak{t}}[Z_{L_p^{(\mathbf{t})}}^{(\mathbf{t})}]$, and therefore $Z_{L_p^{(\mathbf{t})}}^{(\mathbf{t})} = \langle \Sigma_n^{(\mathbf{t})} \rangle_p$.

Lemma 5.17 (Lemma 20 in [20]). *Let $p \in \{0, 1, \dots, n-1\}$ and $z \in \mathbf{t}_p$. Then, conditionally on $\langle \Sigma_n^{(\mathbf{t})} \rangle_p = z$, the process*

$$\left(Z_{(L_p^{(\mathbf{t})} + k) \wedge H_n^{(\mathbf{t})}}^{(\mathbf{t})} \right)_{k \geq 0}$$

is distributed as simple random walk on $\tilde{\mathfrak{t}}[z]$ starting from z and conditioned to hit $\tilde{\mathfrak{t}}[z] \cap \mathbf{t}_n$ before returning to z , and stopped at this hitting time. Consequently, for every integer $q \in \{0, 1, \dots, n-p\}$, the conditional distribution of

$$\frac{\mu_n^{(\mathbf{t})}(B_{\mathbf{t}}(\Sigma_n^{(\mathbf{t})}, q))}{\mu_n^{(\mathbf{t})}(B_{\mathbf{t}}(\Sigma_n^{(\mathbf{t})}, n-p))}$$

knowing that $\langle \Sigma_n^{(\mathbf{t})} \rangle_p = z$ is equal to the distribution of

$$\mu_{n-p}^{(\mathbf{t}[z])}(B_{\mathbf{t}[z]}(\Sigma_{n-p}^{(\mathbf{t}[z])}, q)).$$

5.3.5 The subtree selected by the discrete harmonic measure

We begin by introducing the conductance of discrete trees. Let i be a positive integer and let $\mathbf{t} \in \mathcal{T}$ be a tree such that $h(\mathbf{t}) \geq i$. Consider the new graph \mathbf{t}' obtained by adding to the graph \mathbf{t} an edge between the root \emptyset and an extra vertex ∂ . We denote by $\mathcal{C}_i(\mathbf{t})$ the effective conductance between ∂ and generation i of \mathbf{t} in the graph \mathbf{t}' . In probabilistic terms, it is equal to the probability that simple random walk on \mathbf{t}' starting from \emptyset hits generation i of \mathbf{t} before hitting the vertex ∂ .

Recall that for $i \in \{1, \dots, n-1\}$, $\tilde{\mathbb{T}}^{*n}[\langle \Sigma_n \rangle_{n-i}]$ is the subtree of \mathbb{T}^{*n} above generation $n-i$ that is selected by harmonic measure, and $\mathbb{T}^{*n}[\langle \Sigma_n \rangle_{n-i}]$ is the tree obtained by relabeling the vertices of $\tilde{\mathbb{T}}^{*n}[\langle \Sigma_n \rangle_{n-i}]$ as explained above.

Lemma 5.18. *For every integer $i \in \{1, \dots, n-1\}$ and every nonnegative function F on \mathcal{T} ,*

$$\mathbb{E} \otimes E[F(\mathbb{T}^{*n}[\langle \Sigma_n \rangle_{n-i}])] \leq (i+1) \mathbb{E}[\mathcal{C}_i(\mathbb{T}^{*i}) F(\mathbb{T}^{*i})].$$

This lemma is proved in [20] under the assumption that ρ has finite variance. Actually the proof uses only the branching property of Galton–Watson trees and remains valid under our assumptions on ρ .

Meanwhile, we have the following moment estimate for the conductance $\mathcal{C}_n(\mathbb{T}^{*n})$.

Lemma 5.19. *For every $r \in (0, \alpha)$, there exists a constant $K = K(r, \rho) \geq 1$ depending on r and the offspring distribution ρ such that, for every integer $n \geq 1$,*

$$\mathbb{E}[\mathcal{C}_n(\mathbb{T}^{*n})^r] \leq \frac{K}{(n+1)^r}.$$

Proof. We can assume $n \geq 2$, and set $j = \lfloor n/2 \rfloor \geq 1$. An application of the Nash–Williams inequality [73, Chapter 2] gives

$$\mathcal{C}_n(\mathbb{T}^{*n}) \leq \frac{\#\mathbb{T}_j^{*n}}{j}.$$

On the other hand,

$$\begin{aligned} \mathbb{E}[(\#\mathbb{T}_j^{*n})^r] &= \mathbb{E}[(\#\{v \in \mathbb{T}_j^{(0)} : h(\mathbb{T}^{(0)}[v]) \geq n-j\})^r \mid h(\mathbb{T}^{(0)}) \geq n] \\ &= q_n^{-1} \mathbb{E}[(\#\{v \in \mathbb{T}_j^{(0)} : h(\mathbb{T}^{(0)}[v]) \geq n-j\})^r]. \end{aligned}$$

Notice that given $\#\mathbb{T}_j^{(0)} = k$, the conditional distribution of $\#\{v \in \mathbb{T}_j^{(0)} : h(\mathbb{T}^{(0)}[v]) \geq n-j\}$ is the binomial distribution $\mathcal{B}(k, q_{n-j})$. Using Jensen’s inequality, we get

$$\begin{aligned} \mathbb{E}[(\#\{v \in \mathbb{T}_j^{(0)} : h(\mathbb{T}^{(0)}[v]) \geq n-j\})^r] &\leq \mathbb{E}\left[\mathbb{E}\left[(\#\{v \in \mathbb{T}_j^{(0)} : h(\mathbb{T}^{(0)}[v]) \geq n-j\})^2 \mid \#\mathbb{T}_j^{(0)}\right]^{\frac{r}{2}}\right] \\ &= \mathbb{E}\left[\left(q_{n-j}^2(\#\mathbb{T}_j^{(0)})^2 + (q_{n-j} - q_{n-j}^2)\#\mathbb{T}_j^{(0)}\right)^{\frac{r}{2}}\right] \\ &\leq q_{n-j}^r \mathbb{E}[(\#\mathbb{T}_j^{(0)})^r] + (q_{n-j} - q_{n-j}^2)^{\frac{r}{2}} \mathbb{E}[(\#\mathbb{T}_j^{(0)})^{\frac{r}{2}}]. \end{aligned}$$

At this point, we need the following result proved in [32, Lemma 11] for the unconditioned Galton–Watson tree. For any $\gamma \in (0, \alpha)$, there is a finite constant $C(\gamma)$ such that for every $m \geq 1$,

$$\mathbb{E}[(\#\mathbb{T}_m^{(0)})^\gamma] \leq C(\gamma) q_m^{1-\gamma}. \quad (5.32)$$

The original statement of the latter bound in [32] was given for any $\gamma \in [1, \alpha)$, while the case $\gamma \in (0, 1)$ follows from the (trivial) case $\gamma = 1$ by applying the Hölder inequality to

$$\mathbb{E}[\mathbf{1}_{\{\mathbb{T}_m^{(0)} \neq \emptyset\}} (\#\mathbb{T}_m^{(0)})^\gamma]$$

(we can in fact take $C(\gamma) = 1$ for any $\gamma \in (0, 1)$).

With the help of (5.32), we conclude that

$$\mathbb{E}[\mathcal{C}_n(\mathbb{T}^{*n})^r] \leq j^{-r} q_n^{-1} \left(C(r) q_{n-j}^r q_j^{1-r} + C(r/2) (q_{n-j} - q_{n-j}^2)^{\frac{r}{2}} q_j^{1-\frac{r}{2}} \right),$$

and the statement of the lemma readily follows from (5.27). \square

5.3.6 Proof of Theorem 5.1

Following [20], we will show

$$\mathbb{E} \otimes E[|\log \mu_n(\Sigma_n) + \beta \log n|] = o(\log n) \quad \text{as } n \rightarrow \infty, \quad (5.33)$$

which is sufficient for establishing Theorem 5.1. The proof given below is adapted from [20, Section 4.3.2]. For later convenience, we introduce the notation

$$\bar{\alpha} := \frac{\alpha + 1}{2} \in \left(1, \frac{3}{2}\right)$$

and its Hölder conjugate

$$\alpha^* := \frac{\bar{\alpha}}{\bar{\alpha} - 1} \in (3, \infty).$$

Fix $\xi > 0$. Let $\varepsilon > 0$ and $n_0 \geq 1$ be such that the conclusion of Proposition 5.16 holds for every $n \geq n_0$ with the exponent $r = \alpha^*$. Without loss of generality, we may and will assume that $\varepsilon = 1/N$, for some integer $N \geq 4$, which is fixed throughout the proof. We also fix a constant $\gamma > 0$, such that $\gamma \log N < 1/2$.

Let $n > N$ be sufficiently large so that $N^{\lfloor \gamma \log n \rfloor} \geq n_0$. Then we let $\ell \geq 1$ be the unique integer such that $N^\ell < n \leq N^{\ell+1}$, and write

$$\log \mu_n(\Sigma_n) = \log \frac{\mu_n(\Sigma_n)}{\mu_n(B(\Sigma_n, N))} + \sum_{j=2}^{\ell} \log \frac{\mu_n(B(\Sigma_n, N^{j-1}))}{\mu_n(B(\Sigma_n, N^j))} + \log \mu_n(B(\Sigma_n, N^\ell)). \quad (5.34)$$

To simplify notation, we set

$$\begin{aligned} A_1^n &:= \log \frac{\mu_n(\Sigma_n)}{\mu_n(B(\Sigma_n, N))} + \beta \log N, \\ A_j^n &:= \log \frac{\mu_n(B(\Sigma_n, N^{j-1}))}{\mu_n(B(\Sigma_n, N^j))} + \beta \log N \quad \text{for every } j \in \{2, \dots, \ell\}, \\ A_{\ell+1}^n &:= \log \mu_n(B(\Sigma_n, N^\ell)) + \beta \log(n/N^\ell). \end{aligned}$$

From (5.34), we see that

$$\mathbb{E} \otimes E[|\log \mu_n(\Sigma_n) + \beta \log n|] = \mathbb{E} \otimes E\left[\left|\sum_{j=1}^{\ell+1} A_j^n\right|\right] \leq \sum_{i=1}^{\ell+1} \mathbb{E} \otimes E[|A_j^n|]. \quad (5.35)$$

We will bound each term in the sum of the right-hand side.

FIRST STEP: A PRIORI BOUNDS. We verify that, for every $j \in \{1, 2, \dots, \ell + 1\}$,

$$\mathbb{E} \otimes E[|A_j^n|] \leq (CK^{1/\bar{\alpha}} + \beta) \log N, \quad (5.36)$$

where $C = C(\alpha^*, \rho)$ is the constant in Lemma 5.13 for the exponent $r = \alpha^*$, and $K = K(\bar{\alpha}, \rho)$ is the constant in Lemma 5.19 for the exponent $r = \bar{\alpha}$.

Suppose first that $2 \leq j \leq \ell$. Using the second assertion of Lemma 5.17, with $p = n - N^j$ and $q = N^{j-1}$, we obtain that, for every $z \in \mathbb{T}_{n-N^j}^{*n}$, the conditional distribution of A_j^n under P , knowing that $\langle \Sigma_n \rangle_{n-N^j} = z$, is the same as the distribution of

$$\log \mu_{N^j}^{(\mathbb{T}^{*n}[z])}(B(\Sigma_{N^j}^{(\mathbb{T}^{*n}[z])}, N^{j-1})) + \beta \log N.$$

Recalling that $\mu_{N^j}^{(\mathbb{T}^{*n}[z])}$ is the distribution of $\Sigma_{N^j}^{(\mathbb{T}^{*n}[z])}$ under P , we get

$$\begin{aligned} E[|A_j^n| | \langle \Sigma_n \rangle_{n-N^j} = z] &\leq E\left[|\log \mu_{N^j}^{(\mathbb{T}^{*n}[z])}(B(\Sigma_{N^j}^{(\mathbb{T}^{*n}[z])}, N^{j-1}))|\right] + \beta \log N \\ &= G_j(\mathbb{T}^{*n}[z]) + \beta \log N, \end{aligned} \quad (5.37)$$

where for any tree $\mathfrak{t} \in \mathcal{T}_{N^j}$,

$$G_j(\mathfrak{t}) := \int \mu_{N^j}^{(\mathfrak{t})}(dy) |\log \mu_{N^j}^{(\mathfrak{t})}(B_{\mathfrak{t}}(y, N^{j-1}))| = \sum_{z \in \mathfrak{t}_{N^j-N^{j-1}}} \mu_{N^j}^{(\mathfrak{t})}(\tilde{\mathfrak{t}}[z]) |\log \mu_{N^j}^{(\mathfrak{t})}(\tilde{\mathfrak{t}}[z])|.$$

As explained in [20], we have the entropy bound $G_j(\mathbf{t}) \leq \log \#\mathbf{t}_{N^j - N^{j-1}}$ for any tree $\mathbf{t} \in \mathcal{T}_{N^j}$. So we get from (5.37) that

$$\begin{aligned} \mathbb{E} \otimes E[|A_j^n|] &\leq \mathbb{E} \otimes E[\log \#\mathbf{T}_{N^j - N^{j-1}}^{*n}[\langle \Sigma_n \rangle_{n - N^j}]] + \beta \log N \\ &\leq (N^j + 1) \mathbb{E} \left[\mathcal{C}_{N^j}(\mathbf{T}^{*N^j}) \log \#\mathbf{T}_{N^j - N^{j-1}}^{*N^j} \right] + \beta \log N \\ &\leq (N^j + 1) \mathbb{E} \left[(\mathcal{C}_{N^j}(\mathbf{T}^{*N^j}))^{\bar{\alpha}} \right]^{1/\bar{\alpha}} \mathbb{E} \left[(\log \#\mathbf{T}_{N^j - N^{j-1}}^{*N^j})^{\alpha^*} \right]^{1/\alpha^*} + \beta \log N \\ &\leq K^{1/\bar{\alpha}} \mathbb{E} \left[(\log \#\mathbf{T}_{N^j - N^{j-1}}^{*N^j})^{\alpha^*} \right]^{1/\alpha^*} + \beta \log N, \end{aligned}$$

using successively Lemma 5.18, the Hölder inequality and Lemma 5.19. Finally, Lemma 5.13 gives

$$\mathbb{E} \left[(\log \#\mathbf{T}_{N^j - N^{j-1}}^{*N^j})^{\alpha^*} \right]^{1/\alpha^*} \leq C \log N,$$

and this completes the proof of (5.36) when $2 \leq j \leq \ell$. The cases $j = 1$ and $j = \ell + 1$ can be treated in a similar manner. For details we refer the reader to [20, Section 4.3.2].

SECOND STEP: REFINED BOUNDS. Let us prove that, if $\lfloor \gamma \log n \rfloor \leq j \leq \ell$,

$$\mathbb{E} \otimes E[|A_j^n|] \leq K^{1/\bar{\alpha}} \xi^{1/\alpha^*} \log N. \quad (5.38)$$

Recall that for $j \in \{\lfloor \gamma \log n \rfloor, \dots, \ell\}$ we have $N^j \geq n_0$. From (5.37), we have

$$E[|A_j^n|] = E[F_j(\mathbf{T}^{*n}[\langle \Sigma_n \rangle_{n - N^j}])],$$

where, if $\mathbf{t} \in \mathcal{T}_{N^j}$,

$$F_j(\mathbf{t}) := |\beta \log N - G_j(\mathbf{t})| = \left| \int \mu_{N^j}^{(\mathbf{t})}(dy) \left(\log \mu_{N^j}^{(\mathbf{t})}(B_{\mathbf{t}}(y, N^{j-1})) + \beta \log N \right) \right|.$$

Using Lemma 5.18 as in the first step, we have

$$\mathbb{E} \otimes E[|A_j^n|] = \mathbb{E} \otimes E[F_j(\mathbf{T}^{*n}[\langle \Sigma_n \rangle_{n - N^j}])] \leq (N^j + 1) \mathbb{E} \left[\mathcal{C}_{N^j}(\mathbf{T}^{*N^j}) F_j(\mathbf{T}^{*N^j}) \right].$$

We then apply the Hölder inequality together with the bound of Lemma 5.19 for $r = \bar{\alpha}$ to get

$$\begin{aligned} \mathbb{E} \otimes E[|A_j^n|] &\leq K^{1/\bar{\alpha}} \mathbb{E} [F_j(\mathbf{T}^{*N^j})^{\alpha^*}]^{1/\alpha^*} \\ &\leq K^{1/\bar{\alpha}} \mathbb{E} \left[\left(\int \mu_{N^j}(dy) \left| \log \mu_{N^j}(B(y, N^{j-1})) + \beta \log N \right| \right)^{\alpha^*} \right]^{1/\alpha^*} \\ &\leq K^{1/\bar{\alpha}} \mathbb{E} \left[\int \mu_{N^j}(dy) \left| \log \mu_{N^j}(B(y, N^{j-1})) + \beta \log N \right|^{\alpha^*} \right]^{1/\alpha^*} \\ &= K^{1/\bar{\alpha}} \cdot \mathbb{E} \otimes E \left[\left| \log \mu_{N^j}^{1/N}(\langle \Sigma_{N^j} \rangle_{N^j - N^{j-1}}) + \beta \log N \right|^{\alpha^*} \right]^{1/\alpha^*}, \end{aligned}$$

where the last equality follows from the definition of the measure μ_n^ε at the beginning of Section 5.3.3. Now recall that $1/N = \varepsilon$ and note that $N^j - N^{j-1} = N^j - \varepsilon N^j$. Since we have $N^j \geq n_0$, we can apply Proposition 5.16 with $r = \alpha^*$ and get that the right-hand side of the preceding display is bounded above by $K^{1/\bar{\alpha}} \xi^{1/\alpha^*} \log N$, which finishes the proof of (5.38).

By combining (5.36) and (5.38), and using (5.35), we arrive at the bound

$$\begin{aligned} \mathbb{E} \otimes E[|\log \mu_n(\Sigma_n) + \beta \log n|] &\leq \lfloor \gamma \log n \rfloor (K^{1/\bar{\alpha}} C + \beta) \log N + \ell K^{1/\bar{\alpha}} \xi^{1/\alpha^*} \log N \\ &\leq (\gamma (K^{1/\bar{\alpha}} C + \beta) \log N + K^{1/\bar{\alpha}} \xi^{1/\alpha^*}) \log n, \end{aligned}$$

which holds for every sufficiently large n . By choosing ξ and then γ arbitrarily small, we see that our claim (5.33) follows from the last bound, and this completes the proof of Theorem 5.1.

5.4 Comments and questions

Following [20, Section 5.2], let us consider the supercritical offspring distribution $\theta_\alpha^{(n)}$ of index $\alpha \in (1, 2]$, defined as $\theta_\alpha^{(n)}(1) = 1 - \frac{1}{n}$ and

$$\theta_\alpha^{(n)}(k) = \frac{1}{n} \theta_\alpha(k) \quad \text{for every } k \geq 2.$$

We let $\mathsf{T}_\alpha^{(n)}$ be an infinite Galton–Watson tree with offspring distribution $\theta_\alpha^{(n)}$, then $n^{-1}\mathsf{T}_\alpha^{(n)}$ viewed as a metric space with the graph distance rescaled by the factor n^{-1} , converges in distribution in an appropriate sense (e.g. for the local Gromov–Hausdorff topology) to the CTGW tree $\Gamma^{(\alpha)}$, as $n \rightarrow \infty$.

Consider then the biased random walk $(Z_k^{(n)})_{k \geq 0}$ on $\mathsf{T}_\alpha^{(n)}$ with bias parameter $\lambda^{(n)} = 1 - \frac{1}{n}$ towards the root (see [72] or [2] for a precise definition of this process). Then heuristically the rescaled process

$$\left(n^{-1} Z_{\lfloor n^2 t \rfloor}^{(n)} \right)_{t \geq 0}$$

will converge in distribution in some sense, as $n \rightarrow \infty$, to Brownian motion $(W(t))_{t \geq 0}$ with drift $1/2$ on the CTGW tree $\Gamma^{(\alpha)}$. Furthermore, the rescaled “conductance” $n\mathcal{C}(\mathsf{T}_\alpha^{(n)}, \lambda^{(n)})$ converges in distribution to the continuous conductance $\mathcal{C}^{(\alpha)} = \mathcal{C}(\Gamma^{(\alpha)})$.

Following this informal passage to the limit, we can find a candidate for the limit of $n\mathbf{V}_\alpha^{(n)}$ as $n \rightarrow \infty$, where $\mathbf{V}_\alpha^{(n)}$ stands for the speed of the biased random walk $Z^{(n)}$ on $\mathsf{T}_\alpha^{(n)}$. One can either directly employ an explicit formula of $\mathbf{V}_\alpha^{(n)}$ stated in [2, Theorem 1.1], or use the invariant measure for the environment seen from the random walker ([2, Theorem 4.1]) to calculate the speed as the proportion of last-exit points. Both methods give rise to the following quantity which should be interpreted as the speed of Brownian motion W with drift $1/2$ on $\Gamma^{(\alpha)}$,

$$\mathbf{V}_\alpha := \frac{\mathbb{E} \left[\frac{\mathcal{C}_0^{(\alpha)} \mathcal{C}_1^{(\alpha)}}{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1} \right]}{\mathbb{E} \left[\frac{2\mathcal{C}_0^{(\alpha)}}{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1} \right]}, \quad (5.39)$$

where $\mathcal{C}_0^{(\alpha)}$ and $\mathcal{C}_1^{(\alpha)}$ are two independent copies of $\mathcal{C}^{(\alpha)}$ under the probability measure \mathbb{P} .

Since the conductance $\mathcal{C}^{(\alpha)}$ is a.s. strictly larger than 1, we see immediately from (5.39) that $\mathbf{V}_\alpha < \frac{1}{2}$ for any $\alpha \in (1, 2]$. On the other hand, according to the coupling explained in Section 5.2.4, the denominator of the right-hand side of (5.39)

$$\mathbb{E} \left[\frac{2\mathcal{C}_0^{(\alpha)}}{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1} \right] = \mathbb{E} \left[\frac{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)}}{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1} \right] = 1 + \mathbb{E} \left[\frac{1}{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1} \right]$$

is increasing with respect to α .

Question 1. *If we apply the coupling explained in Section 5.2.4, does the derivative $\frac{d}{d\alpha}\mathcal{C}^{(\alpha)}$ of the conductance with respect to α exist almost surely?*

An affirmative answer to Question 1 would allow us to take the derivative of the numerator in (5.39) with respect to α , and to see that

$$\frac{d}{d\alpha} \mathbb{E} \left[\frac{\mathcal{C}_0^{(\alpha)} \mathcal{C}_1^{(\alpha)}}{\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1} \right] = \mathbb{E} \left[\frac{\mathcal{C}_0^{(\alpha)} (\mathcal{C}_0^{(\alpha)} - 1) \frac{d}{d\alpha} \mathcal{C}_1^{(\alpha)} + \mathcal{C}_1^{(\alpha)} (\mathcal{C}_1^{(\alpha)} - 1) \frac{d}{d\alpha} \mathcal{C}_0^{(\alpha)}}{(\mathcal{C}_0^{(\alpha)} + \mathcal{C}_1^{(\alpha)} - 1)^2} \right] \leq 0$$

because a.s. $\frac{d}{d\alpha}C^{(\alpha)} \leq 0$. Hence, the numerator in the right-hand side of (5.39) would be decreasing with respect to α , and so would be the speed V_α .

Question 2. *Does the speed V_α decrease with respect to α ?*

A similar question was raised in [8], concerning the monotonicity of the speed with respect to the offspring distribution for biased random walk on Galton–Watson trees with no leaves. It has been proved in [77] that this monotonicity holds for high values of bias.

We also want to ask the same question of monotonicity for the Hausdorff dimension of the continuous harmonic measure μ_α .

Question 3. *Does the Hausdorff dimension β_α decrease with respect to α ?*

Finally, it is interesting to figure out the dimension of the harmonic measure on $\partial\Delta^{(1)}$. Due to the fact that θ_1 has infinite mean, it may require different methods to treat the case $\alpha = 1$ in the continuous setting.

5.5 Appendix: proofs postponed from Section 5.2.6

Before starting the proofs, we state first a useful “spine” decomposition of the CTGW tree $\Gamma^{(\alpha)}$ for $\alpha \in (1, 2]$, which is a reformulation of the standard results about the size-biased Galton–Watson trees, see e.g. [17]. Recall that $m_\alpha = \frac{\alpha}{\alpha-1}$ is the mean of the α -offspring distribution θ_α . The size-biased α -offspring distribution $\hat{\theta}_\alpha$ is then defined as

$$\hat{\theta}_\alpha(k) = \frac{(k+1)\theta_\alpha(k+1)}{m_\alpha} \quad \text{for every } k \geq 1. \tag{5.40}$$

We take $\mathcal{T} = (\Pi, (z_v)_{v \in \Pi}) \in \mathbb{T}$. If $\llbracket 0, x \rrbracket$ denotes the geodesic segment in \mathcal{T} between the root and x , we can define the subtrees of \mathcal{T} branching off $\llbracket 0, x \rrbracket$. To this end, set $n_x = |v_x|$ and let $v_{x,0} = \emptyset, v_{x,1}, \dots, v_{x,n_x} = v_x$ be the successive ancestors of v_x from generation 0 to generation n_x . For every $1 \leq i \leq n_x$ set $r_{x,i} = z_{v_{x,i-1}}$, and write $k_{x,i} = k_{v_{x,i-1}} - 1$ as the number of siblings of $v_{x,i}$ in Π . Then, for every $1 \leq i \leq n_x$ and $1 \leq j \leq k_{x,i}$, the j -th subtree branching off the ancestral line $\llbracket 0, x \rrbracket$ at $v_{x,i-1}$, which is denoted by $\mathcal{T}_{x,i,j}$, corresponds to the pair

$$\left(\Pi[\tilde{v}_{x,i,j}], (z_{\tilde{v}_{x,i,j}v} - r_{x,i})_{v \in \Pi[\tilde{v}_{x,i,j}]} \right),$$

where $\tilde{v}_{x,i,j}$ is the j -th child of $v_{x,i-1}$ different from $v_{x,i}$. To simplify notation, we introduce the point measure

$$\xi_{r,x}(\mathcal{T}) = \sum_{i=1}^{n_x} \sum_{j=1}^{k_{x,i}} \delta_{(r_{x,i}, \mathcal{T}_{x,i,j})},$$

which belongs to the set $\mathcal{M}_p(\mathbb{R}_+ \times \mathbb{T})$ of all finite point measures on $\mathbb{R}_+ \times \mathbb{T}$.

Lemma 5.20. *Fix $\alpha \in (1, 2]$. Let F be a nonnegative measurable function on \mathbb{T} , and let H be a nonnegative measurable function on $\mathcal{M}_p(\mathbb{R}_+ \times \mathbb{T})$. For $r > 0$,*

$$\mathbb{E} \left[\sum_{x \in \Gamma_r^{(\alpha)}} F(\Gamma^{(\alpha)}[x]) H(\xi_{r,x}(\Gamma^{(\alpha)})) \right] = e^{(m_\alpha-1)r} \mathbb{E}[F(\Gamma^{(\alpha)})] \times \mathbb{E} \left[H \left(\sum_{i \in I} \sum_{j=1}^{k_i} \delta_{(s_i, \mathcal{T}_{i,j})} \right) \right],$$

where we assume that, under the probability measure \mathbb{P} ,

$$\mathcal{N}_\alpha := \sum_{i \in I} \delta_{(s_i, k_i, (\mathcal{T}_{i,j}, j \geq 1))}$$

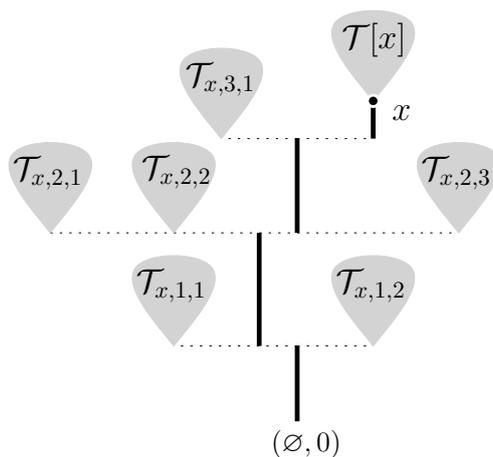


Figure 5.3: Schematic representation of the spine decomposition

is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{N} \times \mathbb{T}^{\mathbb{N}}$ with intensity

$$m_\alpha \mathbf{1}_{[0,r]}(s) ds \widehat{\theta}_\alpha(dk) \prod_{j=1}^{\infty} \Theta_\alpha(d\mathcal{T}_j).$$

From now on we fix the stable index $\alpha \in (1, 2]$. Unless otherwise specified, we will omit the superscripts and subscripts concerning α in the following proofs.

5.5.1 Proof of Proposition 5.11

Let F be a nonnegative measurable function on \mathbb{T} , and consider the quantity

$$I_r := \mathbb{E} \otimes E[F(\Gamma\langle r \rangle)] = \mathbb{E} \otimes E \left[\sum_{x \in \Gamma_r} F(\Gamma[x]) \mathbf{1}_{\{x \prec W_\infty\}} \right], \quad (5.41)$$

where the notation $\mathbb{E} \otimes E$ means that we consider the expectation first under the probability measure P (under which the Brownian motion W is defined) and then under \mathbb{P} .

Let us fix $x \in \Gamma_r$ and $R > r$. We write $\widetilde{\Gamma}[x] := \{y \in \Gamma : x \prec y\}$ to denote the subset of Γ composed of all descendants of x in Γ . Define

$$\Gamma^{x,R} := \{y \in \Gamma \setminus \widetilde{\Gamma}[x] : H(y) \leq R\} \cup \widetilde{\Gamma}[x].$$

Let $W^{x,R}$ be Brownian motion with drift $1/2$ on $\Gamma^{x,R}$. We assume that $W^{x,R}$ is reflected both at the root and at the leaves of $\Gamma^{x,R}$, which are the points y of $\Gamma \setminus \widetilde{\Gamma}[x]$ such that $H(y) = R$. Write $(\ell_t^{x,R})_{t \geq 0}$ for the local time process of $W^{x,R}$ at x . From excursion theory, $\ell_\infty^{x,R}$ has an exponential distribution with parameter $\mathcal{C}(\Gamma[x])/2$. For details, we refer the reader to [20, Section 3.1].

We then consider for every $a \in [0, r]$ the local time process $(L_t^{a,R})_{t \geq 0}$ of $W^{x,R}$ at the unique point of $\llbracket 0, x \rrbracket$ at distance a from the root. Note in particular that $L_t^{r,R} = \ell_t^{x,R}$. As a consequence of a classical Ray-Knight theorem, conditionally on $\ell_\infty^{x,R} = \ell$, the process $(L_\infty^{r-a,R})_{0 \leq a \leq r}$ is distributed as the process $(X_a)_{0 \leq a \leq r}$ which solves the stochastic differential equation

$$\begin{cases} dX_a = 2\sqrt{X_a} d\eta_a + (2 - X_a) da \\ X_0 = \ell \end{cases} \quad (5.42)$$

where $(\eta_a)_{a \geq 0}$ is a standard linear Brownian motion. In what follows, we will write P_ℓ for the probability measure under which the process X starts from ℓ , and $P_{(c)}$ for the probability measure under which the process X starts with an exponential distribution with parameter $c/2$.

Now write $\{x_i, 1 \leq i \leq n_x\}$ for the branching points of $\Gamma^{x,R}$ (or equivalently of Γ) that belong to $\llbracket 0, x \rrbracket$, and set $a_i = H(x_i)$ for $1 \leq i \leq n_x$. We denote by k_i the number of subtrees branching off $\llbracket 0, x \rrbracket$ at x_i , and write

$$(\Gamma_{i,j}^{x,R})_{1 \leq j \leq k_i}$$

for the finite subtrees of $\Gamma^{x,R}$ that branch off $\llbracket 0, x \rrbracket$ at x_i . See Fig. 5.4 for an illustration.

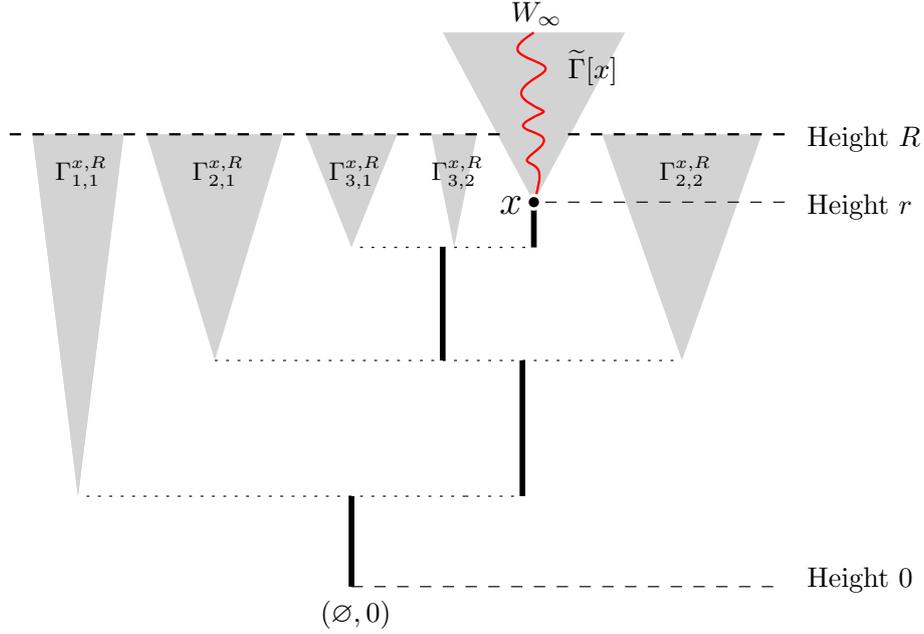


Figure 5.4: The infinite tree $\Gamma^{x,R}$ and the finite subtrees that branch off $\llbracket 0, x \rrbracket$

Let $A_{x,R}$ be the event that $W^{x,R}$ never hits the leaves of $\Gamma^{x,R}$, or equivalently that $W^{x,R}$ escapes to infinity in $\tilde{\Gamma}[x]$ before hitting any leaf of $\Gamma^{x,R}$. Excursion theory shows that

$$P(A_{x,R} \mid (L_\infty^{a,R})_{0 \leq a \leq r}) = \exp \left(- \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{i,j}^{x,R}) L_\infty^{a_i,R} \right),$$

where $\mathcal{C}(\Gamma_{i,j}^{x,R})$ refers to the conductance of $\Gamma_{i,j}^{x,R}$ between its root x_i and the set of its leaves (this conductance is defined by an easy adaptation of the definition given at the beginning of Section 5.2.3).

From the preceding observations, we have thus

$$\begin{aligned} P(A_{x,R}) &= E \left[\exp \left(- \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{i,j}^{x,R}) L_\infty^{a_i,R} \right) \right] \\ &= E_{(\mathcal{C}(\Gamma[x]))} \left[\exp \left(- \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{i,j}^{x,R}) X_{r-a_i} \right) \right]. \end{aligned} \tag{5.43}$$

At this point, we let R tend to infinity. It is easy to verify that $P(A_{x,R})$ increases to $P(A_x)$, where $A_x := \{x \prec W_\infty\}$. Furthermore, for every $i \in \{1, \dots, n_x\}$ and $j \in \{1, \dots, k_i\}$, $\mathcal{C}(\Gamma_{i,j}^{x,R})$ decreases to $\mathcal{C}(\Gamma_{x,i,j})$, where $\Gamma_{x,i,j}$ is the j -th subtree of Γ branching off $\llbracket 0, x \rrbracket$ at x_i . Consequently, we obtain

$$P(x \prec W_\infty) = E_{(\mathcal{C}(\Gamma[x]))} \left[\exp \left(-\frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{x,i,j}) X_{r-a_i} \right) \right].$$

We can now return to the computation of the quantity I_r defined in (5.41).

$$\begin{aligned} I_r &= \mathbb{E} \left[\sum_{x \in \Gamma_r} F(\Gamma[x]) P(x \prec W_\infty) \right] \\ &= \mathbb{E} \left[\sum_{x \in \Gamma_r} F(\Gamma[x]) E_{(\mathcal{C}(\Gamma[x]))} \left[\exp \left(-\frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{x,i,j}) X_{r-a_i} \right) \right] \right]. \end{aligned}$$

Note that the quantity inside the sum over $x \in \Gamma_r$ is a function of $\Gamma[x]$ and of the subtrees of Γ branching off the segment $\llbracket 0, x \rrbracket$. We can thus apply Lemma 5.20 to get

$$I_r = e^{(m-1)r} \int \Theta(d\mathcal{T}) F(\mathcal{T}) E_{(\mathcal{C}(\mathcal{T}))} \left[\mathbf{E} \left[\exp \left(-\frac{1}{2} \int \mathcal{N}(ds dk \prod_{j=1}^{\infty} d\mathcal{T}_j) \sum_{j=1}^k \mathcal{C}(\mathcal{T}_j) X_{r-s} \right) \right] \right],$$

where the constant $m = \frac{\alpha}{\alpha-1}$ is the mean of α -offspring distribution θ . Under the probability measure \mathbf{P} , the random measure \mathcal{N} is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{N} \times \mathbb{T}^{\mathbb{N}}$ with intensity

$$m \mathbf{1}_{[0,r]}(s) ds \hat{\theta}(dk) \prod_{j=1}^{\infty} \Theta(d\mathcal{T}_j),$$

where the size-biased offspring distribution $\hat{\theta}$ is defined by (5.40).

Now we can use the exponential formula for Poisson measures to arrive at

$$\begin{aligned} I_r &= e^{(m-1)r} \int \Theta(d\mathcal{T}) F(\mathcal{T}) E_{(\mathcal{C}(\mathcal{T}))} \left[\exp \left(-m \int_0^r ds \sum_{k=1}^{\infty} \tilde{\theta}(k) (1 - \varphi(X_s)^k) \right) \right] \\ &= \int \Theta(d\mathcal{T}) F(\mathcal{T}) E_{(\mathcal{C}(\mathcal{T}))} \left[\exp - \int_0^r ds \left(1 - m \sum_{k=1}^{\infty} \tilde{\theta}(k) \varphi(X_s)^k \right) \right], \end{aligned} \quad (5.44)$$

where we recall that for every $x \geq 0$,

$$\varphi(x) = \mathbb{E}[\exp(-x \mathcal{C}/2)] = \Theta(\exp(-x \mathcal{C}(\mathcal{T})/2))$$

is the Laplace transform (evaluated at $x/2$) of the distribution of the conductance $\mathcal{C}(\Gamma)$. Observe that for any $r \in (0, 1)$, the identity

$$\sum_{k=1}^{\infty} \hat{\theta}(k) r^k = 1 - (1-r)^{\alpha-1}$$

follows by differentiating (5.6). Applying this to (5.44), we have thus proved Proposition 5.11.

5.5.2 Proof of Proposition 5.12

In order to study the asymptotic behavior of Φ_r when r tends to $+\infty$, we first observe that, in terms of the law $\gamma(ds)$ of $\mathcal{C}(\Gamma)$, we have

$$\varphi(\ell) = \int_{[1,\infty)} e^{-\ell s/2} \gamma(ds), \quad \varphi'(\ell) = -\frac{1}{2} \int_{[1,\infty)} s e^{-\ell s/2} \gamma(ds).$$

It follows that $\varphi(\ell) \leq e^{-\ell/2}$ and $|\varphi'(\ell)| \leq \frac{1}{2}(\int s \gamma(ds))e^{-\ell/2}$. By differentiating (5.11), we have

$$2\ell \varphi'''(\ell) + (2 + \ell)\varphi''(\ell) + \frac{\alpha}{\alpha - 1}\varphi'(\ell)(1 - (1 - \varphi(\ell))^{\alpha-1}) = 0. \quad (5.45)$$

Lemma 5.21. *For every $\ell \geq 0$,*

$$\lim_{r \rightarrow \infty} E_\ell \left[\exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha - 1} \right) \right] = - \frac{\varphi'(\ell)e^{\ell/2}}{\int_0^\infty ds \varphi'(s)^2 e^{s/2}}.$$

Additionally, there exists a constant $A < \infty$ such that, for every $\ell \geq 0$ and $r > 0$,

$$E_\ell \left[\exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha - 1} \right) \right] \leq A.$$

Proof. Firstly, under $\int \Theta(d\mathcal{T}) P_{(\mathcal{C}(\mathcal{T}))}$, the density of X_0 is

$$q(\ell) = \int_{[1,\infty)} \gamma(ds) \frac{s}{2} e^{-s\ell/2} = -\varphi'(\ell). \quad (5.46)$$

So from Lemma 5.11, we have

$$\begin{aligned} 1 &= \int \Theta(d\mathcal{T}) E_{(\mathcal{C}(\mathcal{T}))} \left[\exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha - 1} \right) \right] \\ &= - \int d\ell \varphi'(\ell) E_\ell \left[\exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha - 1} \right) \right]. \end{aligned}$$

We can generalize the last identity via a minor extension of the calculations of the preceding subsection. Let L_∞^0 be the total local time accumulated by the process W at the root of Γ . Fix $r > 0$ and take a nonnegative measurable function F on \mathbb{T} . Let h be a bounded nonnegative continuous function on $(0, \infty)$. As an analogue of I_r in the preceding subsection, we set

$$I_r^h := \mathbb{E} \otimes E \left[h(L_\infty^0) \sum_{x \in \Gamma_r} F(\Gamma[x]) \mathbf{1}_{\{x < W_\infty\}} \right].$$

The same calculations that led to (5.43) give, for every $x \in \Gamma_r$ and $R > r$,

$$\begin{aligned} E[h(L_\infty^{0,R}) \mathbf{1}_{A_{x,R}}] &= E \left[h(L_\infty^{0,R}) \exp \left(-\frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{i,j}^{x,R}) L_\infty^{a_i,R} \right) \right] \\ &= E_{(\mathcal{C}(\Gamma[x]))} \left[h(X_r) \exp \left(-\frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{i,j}^{x,R}) X_{r-a_i} \right) \right]. \end{aligned}$$

When $R \rightarrow \infty$, $L_\infty^{0,R}$ converges to L_∞^0 , and so we get

$$E[h(L_\infty^0) \mathbf{1}_{\{x < W_\infty\}}] = E_{(\mathcal{C}(\Gamma[x]))} \left[h(X_r) \exp \left(-\frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{k_i} \mathcal{C}(\Gamma_{i,j}^{x,R}) X_{r-a_i} \right) \right].$$

We then sum over $x \in \Gamma_r$ and integrate with respect to \mathbb{P} . By the same manipulations as in the previous proof, we arrive at

$$I_r^h = \int \Theta(d\mathcal{T}) F(\mathcal{T}) E_{(\mathcal{C}(\mathcal{T}))} \left[h(X_r) \exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1} \right) \right]. \quad (5.47)$$

Note that if $F = 1$,

$$I_r^h = \mathbb{E} \otimes E[h(L_\infty^0)] = - \int_0^\infty d\ell \varphi'(\ell) h(\ell)$$

since given $\Gamma = \mathcal{T}$ the local time L_∞^0 follows an exponential distribution with parameter $\mathcal{C}(\mathcal{T})/2$, and we use the same calculation as in (5.46). Hence the case $F = 1$ of (5.47) gives

$$\int_0^\infty d\ell \varphi'(\ell) E_\ell \left[h(X_r) \exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1} \right) \right] = \int_0^\infty d\ell \varphi'(\ell) h(\ell). \quad (5.48)$$

By a standard truncation argument, this identity also holds if h is unbounded.

Lemma 5.22. *The process*

$$M_t := -\varphi'(X_t) \exp \left(\frac{X_t}{2} - \int_0^t ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1} \right) \right), \quad t \geq 0$$

is a martingale under P_ℓ , for every $\ell \geq 0$.

Proof of Lemma 5.22. From the stochastic differential equation (5.42), an application of Itô's formula shows that the finite variation part of the semimartingale $-M_t$ is

$$\int_0^t \left(2X_s \varphi'''(X_s) + (2 + X_s) \varphi''(X_s) + \varphi'(X_s) \left(\frac{\alpha}{\alpha-1} - m(1 - \varphi(X_s))^{\alpha-1} \right) \right) Y_s ds, \quad (5.49)$$

where for any $s \geq 0$,

$$Y_s := \exp \left(\frac{X_s}{2} - \int_0^s du \left(m(1 - \varphi(X_u))^{\alpha-1} - \frac{1}{\alpha-1} \right) \right).$$

Recall that $m = \frac{\alpha}{\alpha-1}$, and hence (5.49) vanishes thanks to (5.45), whereupon M is a local martingale. Furthermore, we have already noticed that, for every $\ell \geq 0$, $|\varphi'(\ell)| \leq C e^{-\ell/2}$, where $C := \frac{1}{2} \int s \gamma(ds)$. It follows that $|M|$ is bounded by $C \exp(\frac{t}{\alpha-1})$ over the time interval $[0, t]$, and thus M is a (true) martingale. \square

We return to the proof of Lemma 5.21. Let $\ell \geq 0$ and $t > 0$. On the probability space where X is defined, we introduce a new probability measure Q_ℓ^t by setting

$$Q_\ell^t := \frac{M_t}{M_0} \cdot P_\ell.$$

The fact that Q_ℓ^t is a probability measure follows from the martingale property derived in Lemma 5.22. By definition of M_t , we have P_ℓ -a.s.

$$\frac{M_t}{M_0} = \frac{\varphi'(X_t)}{\varphi'(\ell)} \exp \left(\frac{X_t - \ell}{2} - \int_0^t ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1} \right) \right),$$

so that the martingale part of $\log \frac{M_t}{M_0}$ is

$$\int_0^t \sqrt{X_s} d\eta_s + 2 \int_0^t \frac{\varphi''(X_s)}{\varphi'(X_s)} \sqrt{X_s} d\eta_s,$$

where η is the linear Brownian motion in (5.42). An application of Girsanov's theorem shows that under Q_ℓ^t , the process

$$\tilde{\eta}_s := \eta_s - \int_0^s \sqrt{X_u} \left(1 + \frac{2\varphi''(X_u)}{\varphi'(X_u)}\right) du, \quad 0 \leq s \leq t,$$

is a linear Brownian motion over the time interval $[0, t]$. Furthermore, on the same time interval $[0, t]$, the process X satisfies the stochastic differential equation

$$dX_s = 2\sqrt{X_s} d\tilde{\eta}_s + 2X_s \left(1 + \frac{2\varphi''(X_s)}{\varphi'(X_s)}\right) ds + (2 - X_s) ds,$$

or equivalently, using (5.11),

$$dX_s = 2\sqrt{X_s} d\tilde{\eta}_s + \left(2 - X_s + \frac{2}{\alpha - 1} \frac{1 - \varphi(X_s) - (1 - \varphi(X_s))^\alpha}{\varphi'(X_s)}\right) ds. \quad (5.50)$$

Notice that the function

$$\ell \mapsto \frac{1 - \varphi - (1 - \varphi)^\alpha}{\varphi'}(\ell)$$

is continuously differentiable over $[0, \infty)$, takes negative values on $(0, \infty)$ and vanishes at 0. Pathwise uniqueness, and therefore also weak uniqueness, holds for (5.50) by an application of the classical Yamada-Watanabe criterion. The preceding considerations show that, under the probability measure Q_ℓ^t and on the time interval $[0, t]$, the process X is distributed as the diffusion process on $[0, \infty)$ started from ℓ , with generator

$$\mathcal{L} = 2r \frac{d^2}{dr^2} + \left(2 - r + \frac{2}{\alpha - 1} \frac{1 - \varphi - (1 - \varphi)^\alpha}{\varphi'}(r)\right) \frac{d}{dr}.$$

Write \tilde{X} for this diffusion process, and assume that \tilde{X} starts from ℓ under the probability measure P_ℓ . Note that 0 is an entrance point for \tilde{X} , but independently of its starting point, \tilde{X} does not visit 0 at a positive time. By comparing the solutions of (5.42) and (5.50), we know that \tilde{X} is recurrent on $(0, \infty)$.

We next observe that, by (5.48) and a few lines of calculations, the finite measure λ on $(0, \infty)$ defined by

$$\lambda(d\ell) := \varphi'(\ell)^2 e^{\ell/2} d\ell$$

is invariant for \tilde{X} . We normalize λ by setting

$$\hat{\lambda} = \frac{\lambda}{\lambda((0, \infty))}.$$

It is then easy to prove that the distribution of \tilde{X}_t under P_ℓ converges weakly to $\hat{\lambda}$ as $t \rightarrow \infty$, for any $\ell \geq 0$. Consequently, for any bounded continuous function g on $[0, \infty)$, and every $\ell \geq 0$,

$$E_\ell[g(\tilde{X}_t)] \xrightarrow[t \rightarrow \infty]{} \int g d\hat{\lambda}. \quad (5.51)$$

By the same argument as in [20, Section 3.2], the preceding convergence remains true if g is a continuous, increasing and nonnegative function such that $\int g d\hat{\lambda} < \infty$.

We can thus apply (5.51) to the function

$$g(\ell) = -\frac{1}{\varphi'(\ell)} e^{-\ell/2},$$

which satisfies the desired properties and in particular $\int g d\lambda = -\int \varphi'(\ell) d\ell = 1$. For this function g ,

$$E_\ell[g(\tilde{X}_t)] = Q_\ell^t[g(X_t)] = -\frac{e^{-\ell/2}}{\varphi'(\ell)} E_\ell\left[\exp - \int_0^t ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1}\right)\right].$$

It follows from (5.51) that, for every $\ell \geq 0$,

$$\lim_{t \rightarrow \infty} -\frac{e^{-\ell/2}}{\varphi'(\ell)} E_\ell\left[\exp - \int_0^t ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1}\right)\right] = \int g d\hat{\lambda} = \frac{1}{\lambda((0, \infty))} = \frac{1}{\int_0^\infty ds \varphi'(s)^2 e^{s/2}},$$

which gives the first assertion of the lemma. The second assertion of Lemma 5.21 can be shown in the same way as in [20]. \square

By definition, we have

$$\Phi_r(c) = \frac{c}{2} \int_0^\infty d\ell e^{-c\ell/2} E_\ell\left[\exp - \int_0^r ds \left(m(1 - \varphi(X_s))^{\alpha-1} - \frac{1}{\alpha-1}\right)\right].$$

From Lemma 5.21 and an application of the dominated convergence theorem, we get

$$\lim_{r \rightarrow +\infty} \Phi_r(c) = c \int_0^\infty d\ell e^{-c\ell/2} \times \left(-\frac{\varphi'(\ell)e^{\ell/2}}{C_1}\right),$$

where

$$C_1 := 2 \int_0^\infty ds \varphi'(s)^2 e^{s/2} = \int \int \gamma(d\ell) \gamma(d\ell') \frac{\ell\ell'}{\ell + \ell' - 1}.$$

By a straightforward calculation, the preceding limit is identified with $\Phi_\infty(c)$ defined in the statement of Proposition 5.12.

Finally, with all the ingredients prepared above in this appendix, we can show the invariance of $\Lambda^*(d\mathcal{T}d\mathbf{v})$ under the shifts $(\tau_r, r \geq 0)$ in the same way as in [20, Proposition 12], and the proof of Proposition 5.12 is therefore completed.

5.5.3 Another derivation of formula (5.24)

Recall that $\nu_{\mathcal{T}}$ stands for the harmonic measure of a tree $\mathcal{T} \in \mathbb{T}$. For every $r > 0$, we consider the nonnegative measurable function G_r defined on \mathbb{T}^* by the formula

$$G_r(\mathcal{T}, \mathbf{v}) := -\log \nu_{\mathcal{T}}(\mathcal{B}_{\mathcal{T}}(\mathbf{v}, r)),$$

where $\mathcal{B}_{\mathcal{T}}(\mathbf{v}, r)$ denotes the set of all geodesic rays of \mathcal{T} that coincide with the ray \mathbf{v} over the interval $[0, r]$. The flow property of harmonic measure (cf. Lemma 7 in [20]) implies that, for every $r, s > 0$, we have

$$G_{r+s} = G_r + G_s \circ \tau_r.$$

Since the shift τ_r acting on $(\mathbb{T}^*, \Lambda^*)$ is ergodic, the Birkhoff ergodic theorem implies that Λ^* -a.s.

$$\lim_{s \rightarrow \infty} \frac{G_s}{s} = \Lambda^*(G_1).$$

Recall that Λ^* has a strictly positive density with respect to Θ^* . So the latter convergence also holds Θ^* -a.s., which gives the convergence (5.9) with $\beta = \Lambda^*(G_1) > 0$.

For $\varepsilon > 0$, we define a nonnegative function H_ε on \mathbb{T}^* by setting

$$H_\varepsilon(\mathcal{T}, \mathbf{v}) := \begin{cases} 0 & \text{if } z_\emptyset \geq \varepsilon, \\ -\log \nu_{\mathcal{T}}(\{\mathbf{v}' \in \mathbb{N}^{\mathbb{N}} : \mathbf{v}_1 \prec \mathbf{v}'\}) & \text{if } z_\emptyset < \varepsilon, \end{cases}$$

where we write $\mathcal{T} = (\Pi, (z_v)_{v \in \Pi})$ as in Section 5.2.2. Clearly, $H_\varepsilon(\mathcal{T}, \mathbf{v}) \leq G_\varepsilon(\mathcal{T}, \mathbf{v})$, and $H_\varepsilon(\mathcal{T}, \mathbf{v}) = G_\varepsilon(\mathcal{T}, \mathbf{v})$ if $z_{\mathbf{v}_1} \geq \varepsilon$. More generally, $H_\varepsilon \circ \tau_r(\mathcal{T}, \mathbf{v}) = G_\varepsilon \circ \tau_r(\mathcal{T}, \mathbf{v})$ if there is at most one index $i \geq 0$ such that $r \leq z_{\mathbf{v}_i} < r + \varepsilon$. It follows from these remarks that, for every integer $n \geq 1$,

$$G_1 \geq \sum_{k=0}^{n-1} H_{1/n} \circ \tau_{k/n}, \quad (5.52)$$

and for every $(\mathcal{T}, \mathbf{v}) \in \mathbb{T}^*$,

$$G_1(\mathcal{T}, \mathbf{v}) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} H_{1/n} \circ \tau_{k/n}(\mathcal{T}, \mathbf{v}). \quad (5.53)$$

Let us then investigate the behavior of $\Lambda^*(H_\varepsilon)$ when $\varepsilon \rightarrow 0$. By considering the subtrees $\mathcal{T}_{(1)}, \dots, \mathcal{T}_{(k_\emptyset)}$ of \mathcal{T} obtained at the first branching point, we can write

$$\Lambda^*(H_\varepsilon) = - \int \Theta(d\mathcal{T}) \Phi_\infty(\mathcal{C}(\mathcal{T})) \mathbf{1}_{\{z_\emptyset < \varepsilon\}} \sum_{i=1}^{k_\emptyset} \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\sum_{j=1}^{k_\emptyset} \mathcal{C}(\mathcal{T}_{(j)})} \log \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\sum_{j=1}^{k_\emptyset} \mathcal{C}(\mathcal{T}_{(j)})}. \quad (5.54)$$

Recall the branching property of the CTGW tree, and notice that

$$\mathcal{C}(\mathcal{T}) = \frac{\sum_{j=1}^{k_\emptyset} \mathcal{C}(\mathcal{T}_{(j)})}{e^{-z_\emptyset} + (1 - e^{-z_\emptyset})(\sum_{j=1}^{k_\emptyset} \mathcal{C}(\mathcal{T}_{(j)}))}.$$

Substituting this into (5.54), we see that $\Lambda^*(H_\varepsilon)$ can be expanded as

$$\begin{aligned} & - \sum_{k=2}^{\infty} \theta(k) \int \Theta(d\mathcal{T}_1) \int \Theta(d\mathcal{T}_2) \cdots \int \Theta(d\mathcal{T}_k) \sum_{i=1}^k \frac{\mathcal{C}(\mathcal{T}_i)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \log \frac{\mathcal{C}(\mathcal{T}_i)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \\ & \quad \times \int_0^\varepsilon dz e^{-z} \Phi_\infty\left(\frac{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)}{e^{-z} + (1 - e^{-z})(\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j))}\right) \\ & = - \sum_{k=2}^{\infty} k \theta(k) \int \Theta(d\mathcal{T}_1) \int \Theta(d\mathcal{T}_2) \cdots \int \Theta(d\mathcal{T}_k) \frac{\mathcal{C}(\mathcal{T}_1)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \log \frac{\mathcal{C}(\mathcal{T}_1)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \\ & \quad \times \int_0^\varepsilon dz e^{-z} \Phi_\infty\left(\frac{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)}{e^{-z} + (1 - e^{-z})(\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j))}\right) \end{aligned}$$

by a symmetry argument. Since Φ_∞ is a bounded continuous function, and

$$\left| \frac{\mathcal{C}(\mathcal{T}_1)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \log \frac{\mathcal{C}(\mathcal{T}_1)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \right| \leq 1,$$

we can let $\varepsilon \rightarrow 0$ in the preceding expression and get

$$\lim_{\varepsilon \rightarrow 0} \frac{\Lambda^*(H_\varepsilon)}{\varepsilon} = - \sum_{k=2}^{\infty} k\theta(k) \int \Theta(d\mathcal{T}_1) \cdots \int \Theta(d\mathcal{T}_k) \frac{\mathcal{C}(\mathcal{T}_1)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \log \frac{\mathcal{C}(\mathcal{T}_1)}{\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j)} \Phi_\infty \left(\sum_{j=1}^k \mathcal{C}(\mathcal{T}_j) \right). \quad (5.55)$$

Note that we used the fact that θ has a finite first moment. Since the limit in the preceding display is finite, we can use (5.53) and Fatou's lemma to get that $\Lambda^*(G_1) < \infty$, and then use (5.52) (to justify dominated convergence) and (5.53) again to obtain that

$$\beta = \Lambda^*(G_1) = \lim_{n \rightarrow \infty} n \Lambda^*(H_{1/n})$$

coincides with the right-hand side of (5.55). Using the expression of Φ_∞ , we can therefore reformulate β as in formula (5.24).

TYPICAL BEHAVIOR OF THE HARMONIC MEASURE

LES RÉSULTATS DE CE CHAPITRE SONT ISSUS DE L'ARTICLE [68] EN PRÉPARATION.

We study the typical behavior of the harmonic measure of balls in large critical Galton–Watson trees whose offspring distribution has finite variance. Here the harmonic measure refers to the hitting distribution of height n by simple random walk on the critical Galton–Watson tree conditioned to have height greater than n . For a ball of radius n centered at the root, we prove that, with high probability, the mass of the harmonic measure carried by a random vertex uniformly chosen from the boundary is approximately equal to $n^{-\lambda}$, where $\lambda \in (1, \infty)$ is a universal constant which does not depend on the offspring distribution.

6.1 Introduction

We let θ be a non-degenerate probability measure on \mathbb{Z}_+ , and we assume that θ has mean one and finite variance $\sigma^2 > 0$. Under the probability \mathbb{P} , for every integer $n \geq 0$, let $\mathsf{T}^{(n)}$ be a Galton–Watson tree with offspring distribution θ , conditioned on non-extinction at generation n . We denote by $\mathsf{T}_n^{(n)}$ the set of all vertices of $\mathsf{T}^{(n)}$ at generation n . Then the classical Yaglom’s theorem states that $n^{-1} \#\mathsf{T}_n^{(n)}$ converges in distribution to an exponential distribution with parameter $2/\sigma^2$.

Conditionally on the tree $\mathsf{T}^{(n)}$, we consider a simple random walk on $\mathsf{T}^{(n)}$ starting from the root. Let Σ_n be the first hitting point of generation n by the random walk. The distribution μ_n of Σ_n , which is called the harmonic measure of $\mathsf{T}^{(n)}$, is a random probability measure supported on the level set $\mathsf{T}_n^{(n)}$.

Theorem 6.1. *Let Ω_n be a random vertex uniformly chosen from $\mathsf{T}_n^{(n)}$. There exists a universal constant $\lambda \in (1, \infty)$, which does not depend on the offspring distribution θ , such that for every $\delta > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(n^{-\lambda-\delta} \leq \mu_n(\Omega_n) \leq n^{-\lambda+\delta}) = 1. \tag{6.1}$$

Numerical simulations show that $\lambda \approx 1.21$.

The convergence (6.1) can be rewritten as

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{\#\mathsf{T}_n^{(n)}} \sum_{v \in \mathsf{T}_n^{(n)}} \mathbf{1}_{\{\mu_n(v) > n^{-\lambda+\delta} \text{ or } \mu_n(v) < n^{-\lambda-\delta}\}} \right] = 0.$$

Using the theorem of Yaglom, it is easy to see that the preceding convergence is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[\sum_{v \in \mathbb{T}_n^{(n)}} \mathbf{1}_{\{\mu_n(v) > n^{-\lambda+\delta} \text{ or } \mu_n(v) < n^{-\lambda-\delta}\}} \right] = 0. \quad (6.2)$$

We take $\delta \in (0, \lambda - 1)$, and define $A_n := \{v \in \mathbb{T}_n^{(n)} : \mu_n(v) > n^{-\lambda+\delta}\}$. The convergence (6.2) implies that for every $\varepsilon > 0$,

$$\mathbb{P} \left(\frac{\#A_n}{n} > \varepsilon \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

On the other hand, by the definition of A_n , for any vertex $v \in \mathbb{T}_n^{(n)} \setminus A_n$, we have $\mu_n(v) \leq n^{-\lambda+\delta}$, and it follows that $\mu_n(\mathbb{T}_n^{(n)} \setminus A_n) \leq n^{-\lambda+\delta} \# \mathbb{T}_n^{(n)}$. Using again Yaglom's theorem, we get that

$$\mathbb{P} \left(\mu_n(\mathbb{T}_n^{(n)} \setminus A_n) > \varepsilon \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Therefore, it holds with probability tending to 1 as $n \rightarrow \infty$ that, up to a mass of size ε , the harmonic measure μ_n is supported on a subset of $\mathbb{T}_n^{(n)}$ of cardinality smaller than εn . This simple consequence of Theorem 6.1 has already been observed in a recent paper of Curien and Le Gall [20], where they have shown the existence of a universal constant $\beta \in (0, 1)$ independent of the offspring distribution θ , such that for, every $\delta > 0$, we have the convergence in \mathbb{P} -probability

$$\mu_n(\{v \in \mathbb{T}_n^{(n)} : n^{-\beta-\delta} \leq \mu_n(v) \leq n^{-\beta+\delta}\}) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1. \quad (6.3)$$

As pointed out by Curien and Le Gall in [20], for studying the harmonic measure μ_n on $\mathbb{T}_n^{(n)}$, we can directly consider simple random walk on the reduced tree \mathbb{T}^{*n} , which consists of all vertices of $\mathbb{T}^{(n)}$ that have at least one descendant at generation n . Moreover, if we scale the graph distances by n^{-1} , the rescaled discrete reduced trees $n^{-1}\mathbb{T}^{*n}$ converge to a random compact rooted \mathbb{R} -tree Δ , whose structure is described as follows. We take a random variable U_\emptyset uniformly distributed over $[0, 1]$, and start with an oriented line segment of length U_\emptyset , whose origin will be the root of Δ . At the other end of this initial line segment, we attach the initial point of two new line segments with respective lengths U_1 and U_2 , in such a way that, conditionally given U_\emptyset , the variables U_1 and U_2 are independent and uniformly distributed over $[0, 1 - U_\emptyset]$. At the other end of the first of these 2 line segments, we attach two line segments whose lengths are independent and uniformly distributed over $[0, 1 - U_\emptyset - U_1]$, again conditionally on U_\emptyset and U_1 . We repeat this procedure independently for the second line segment with U_1 replaced by U_2 . We continue this construction by induction, and after an infinite number of steps we obtain a random non-compact rooted \mathbb{R} -tree Δ_0 with binary branching (see Figure 6.1), whose completion with respect to its intrinsic metric \mathbf{d} is the continuous reduced tree Δ . We assume that Δ is also defined under the probability measure \mathbb{P} . Its boundary $\partial\Delta$ is defined as the set of all points of Δ at height 1 (i.e. at distance 1 from the root). We can define Brownian motion on Δ starting from the root and up to the first hitting time of $\partial\Delta$. The (continuous) harmonic measure μ on $\partial\Delta$ is the (quenched) distribution of the first hitting point of $\partial\Delta$ by Brownian motion.

We then define another (non-compact) random \mathbb{R} -tree Γ with binary branching, such that each point of Γ at height $y \in [0, \infty)$ corresponds to a point of Δ_0 at height $1 - e^{-y} \in [0, 1)$. The resulting new tree Γ is the Yule tree which describes the genealogy of the classical Yule process, where individuals have independent exponential lifetimes with parameter 1 and each

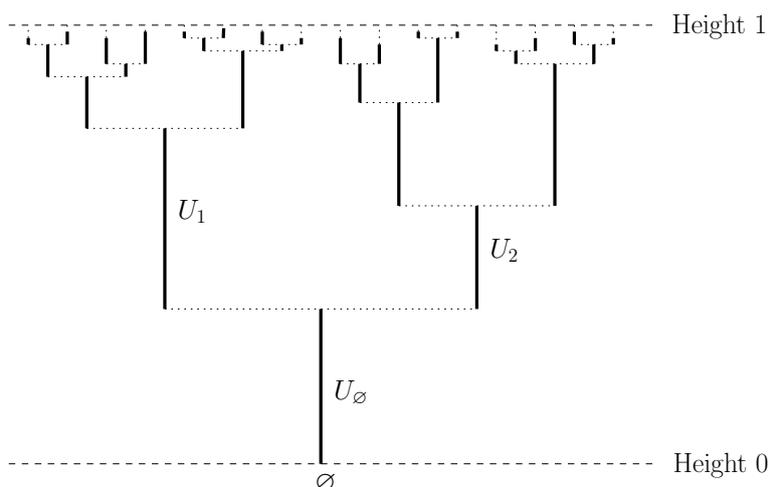


Figure 6.1: The random tree Δ_0

individual has exactly two offspring. By definition, the boundary $\partial\Gamma$ of Γ is the set of all infinite geodesics in Γ starting from the root (these are called geodesic rays). Due to the binary branching mechanism, both $\partial\Delta$ and $\partial\Gamma$ can be canonically identified with $\{1, 2\}^{\mathbb{N}}$.

For every $r > 0$, we write Γ_r for the level set of Γ at height r . By a martingale argument, we can define

$$\mathcal{W} := \lim_{r \rightarrow \infty} e^{-r} \#\Gamma_r,$$

and it is well known that \mathcal{W} follows an exponential distribution of parameter 1. For any $x \in \Gamma$, we write $\Gamma[x]$ for the subtree of descendants of x in Γ , and we similarly define

$$\mathcal{W}_x := \lim_{r \rightarrow \infty} e^{-r} \#\Gamma_r[x].$$

It is immediate to see that for every $r > 0$,

$$\sum_{x \in \Gamma_r} e^{-r} \mathcal{W}_x = \mathcal{W}.$$

For every $x \in \Gamma$, let $H(x)$ denote the height of x . The uniform measure $\bar{\omega}$ on $\partial\Gamma$ is defined as the unique random probability measure on $\partial\Gamma$ satisfying that, for every $x \in \Gamma$ and for every geodesic ray $\mathbf{v} \in \partial\Gamma$ passing through x ,

$$\bar{\omega}(\mathcal{B}(\mathbf{v}, H(x))) = e^{-H(x)} \frac{\mathcal{W}_x}{\mathcal{W}},$$

where $\mathcal{B}(\mathbf{v}, H(x))$ stands for the set of all geodesic rays in Γ that coincide with \mathbf{v} up to height $H(x)$. In earlier work, $\bar{\omega}$ is also named as the branching measure on the boundary of Γ . Recall that $\partial\Delta$ can be identified with $\partial\Gamma$ as explained above. We let ω be the random probability measure on $\partial\Delta$ induced by $\bar{\omega}$, which will be referred to as the uniform measure on $\partial\Delta$.

Theorem 6.2. *With the same constant λ as in Theorem 6.1, we have \mathbb{P} -a.s. $\omega(d\mathbf{v})$ -a.e.*

$$\lim_{r \downarrow 0} \frac{\log \mu(\mathcal{B}_d(\mathbf{v}, r))}{\log r} = \lambda, \tag{6.4}$$

$$\lim_{r \downarrow 0} \frac{\log \omega(\mathcal{B}_d(\mathbf{v}, r))}{\log r} = 1, \tag{6.5}$$

where $\mathcal{B}_{\mathbf{d}}(\mathbf{v}, r)$ stands for the closed ball of radius r centered at \mathbf{v} in the metric space (Δ, \mathbf{d}) .

Remark. The Hausdorff measure of $\partial\Delta$ with respect to \mathbf{d} is a.s. equal to 1. An exact Hausdorff measure function can be found in Duquesne and Le Gall [28, Theorem 1.3].

Corollary 6.3. \mathbb{P} -a.s. the two measures μ and ω on the boundary of Δ are mutually singular.

Proof. With the same constant β as in (6.3), it is shown in Theorem 3 of [20] that \mathbb{P} -a.s. $\mu(d\mathbf{v})$ -a.e.,

$$\lim_{r \downarrow 0} \frac{\log \mu(\mathcal{B}_{\mathbf{d}}(\mathbf{v}, r))}{\log r} = \beta.$$

If we define

$$B = \left\{ \mathbf{v} \in \partial\Delta : \lim_{r \downarrow 0} \frac{\log \mu(\mathcal{B}_{\mathbf{d}}(\mathbf{v}, r))}{\log r} = \beta \right\},$$

then \mathbb{P} -a.s. $\mu(B) = 1$. However, since $\beta < 1 < \lambda$, we get from (6.4) that \mathbb{P} -a.s. $\omega(B) = 0$, which finishes the proof. \square

Similar results for supercritical infinite Galton–Watson trees can be found in Theorem 3 of [69] and in Theorem 6.3 of [71], where the uniform measure and the visibility measure (defined as the law of the geodesic ray chosen by *forward* simple random walk) on the boundary of the infinite tree are considered.

In order to get a better understanding of the distinguished geodesic ray in the Yule tree Γ chosen randomly according to the uniform measure $\bar{\omega}(d\mathbf{v})$, we follow the ideas of [70] to construct a size-biased version $\hat{\Gamma}$ of Γ , which is the genealogical tree of the following branching process. Initially, there is one particle having an exponential lifetime with parameter 2, and it reproduces two offspring simultaneously when it dies. We choose one of them uniformly at random and the chosen one will continue as the initial ancestor, while the other offspring will independently evolve as the classical Yule process. The size-biased Yule tree $\hat{\Gamma}$ thus defined is an infinite random \mathbb{R} -tree with binary branching. We can apply to $\hat{\Gamma}$ the same transformation that relates Γ and Δ_0 (essentially, every point of Γ at height y corresponds to a point of Δ_0 at height $1 - e^{-y}$), and we get a bounded (yet non-compact) rooted \mathbb{R} -tree $\hat{\Delta}_0$, which can be interpreted as the size-biased version of Δ_0 . The completion of $\hat{\Delta}_0$ with respect to its intrinsic metric is denoted as $\hat{\Delta}$, and we call $\hat{\Delta}$ the size-biased reduced tree. Its boundary $\partial\hat{\Delta}$ is similarly defined as the set of all points in $\hat{\Delta}$ at height 1. We assume that $\hat{\Delta}$ is defined under the probability measure \mathbb{P} .

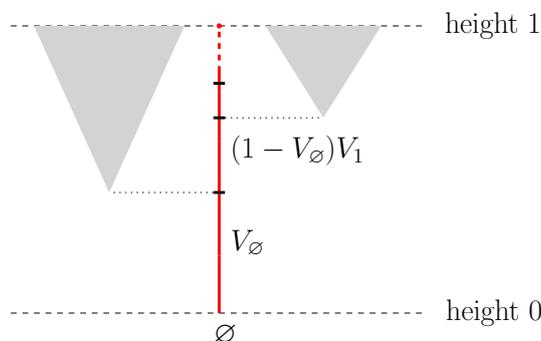


Figure 6.2: Schematic representation of the size-biased reduced tree $\hat{\Delta}$

Due to the previous description of $\widehat{\Gamma}$, one can construct $\widehat{\Delta}$ directly as follows. At first, the root \emptyset of $\widehat{\Delta}$ has a distinguished descendant line of length 1. Let V_\emptyset be a random variable taking values in $[0, 1]$ with density $2(1-x)$, and we graft to the distinguished descendant line at height V_\emptyset a subtree which is an independent copy of Δ scaled by the factor $(1-V_\emptyset)$. In the second step, we take V_1 as an independent copy of V_\emptyset and graft to the distinguished descendant line at height $V_\emptyset + (1-V_\emptyset)V_1$ another independent copy of Δ scaled by the factor $(1-V_\emptyset)(1-V_1)$. Note that for each grafting, we choose the left-hand side or the right-hand side of the distinguished descendant line with equal probabilities. We continue this procedure to graft more subtrees to the distinguished descendant line, with the height of the grafting position increasing to 1. After an infinite number of steps we obtain a realization of $\widehat{\Delta}$. See Figure 6.2 for an illustration.

The constant λ appearing in Theorems 6.1 and 6.2 can be expressed in terms of the (continuous) conductance of $\widehat{\Delta}$. Informally, if we think of the random trees Δ and $\widehat{\Delta}$ as electric networks of resistors with unit resistance per unit length, the effective conductances between the root and the boundary in Δ and $\widehat{\Delta}$ are continuous random variables denoted respectively as \mathcal{C} and $\widehat{\mathcal{C}}$. From a probabilistic point of view, each of these conductances can be obtained as the mass under the Brownian excursion measure in the corresponding tree for the excursion paths away from the root that hit height 1. It is easy to see that both \mathcal{C} and $\widehat{\mathcal{C}}$ take values in $[1, \infty)$. Following the above construction of $\widehat{\Delta}$ and the electric network interpretation, the distribution of $\widehat{\mathcal{C}}$ satisfies the recursive distributional equation

$$\widehat{\mathcal{C}} \stackrel{(d)}{=} \left(V + \frac{1-V}{\widehat{\mathcal{C}} + \mathcal{C}} \right)^{-1}, \quad (6.6)$$

where in the right-hand side V, \mathcal{C} and $\widehat{\mathcal{C}}$ are independent, and the distribution of the random variable V has density $2(1-x)$ over $[0, 1]$. We prove that the law $\widehat{\gamma}$ of $\widehat{\mathcal{C}}$ has finite moments of all orders, and it has a continuous density \widehat{f} over $[1, \infty)$, which reaches its global maximum at $3/2$. The density function \widehat{f} exhibits a singular behavior analogous to that of the density function of \mathcal{C} (see [20, Section 2.3]). Although \widehat{f} is twice continuously differentiable on the interval $(1, 3)$, it is shown that \widehat{f} is not third-order differentiable at the point 2. A similar singular behavior is expected at all integer points $n \geq 2$. See Figure 6.3.

Proposition 6.4. *The distribution $\widehat{\gamma}$ of the conductance $\widehat{\mathcal{C}}$ is characterized in the class of all probability measures on $[1, \infty)$ by the distributional equation (6.6). The constant λ appearing in Theorems 6.1 and 6.2 is given by*

$$\lambda = \mathbb{E}[\widehat{\mathcal{C}}] - 1 \in (1, \infty). \quad (6.7)$$

The remainder of this paper is structured as follows. We start the next section by defining formally the continuous random trees Δ and Γ . The notation of the random variables involved will be slightly different from the one used in this Introduction. The distribution $\widehat{\gamma}$ of the conductance $\widehat{\mathcal{C}}$ is studied in Section 6.2.4, and the proof of Theorem 6.2 and of formula (6.7) is given in Section 6.2.5. The size-biased continuous random trees $\widehat{\Delta}$ and $\widehat{\Gamma}$ are introduced respectively in Section 6.2.6 and in Section 6.2.7. Then, we gather some preliminaries for the discrete setting in Section 6.3. In the end, Section 6.4 is devoted to the proof of Theorem 6.1.

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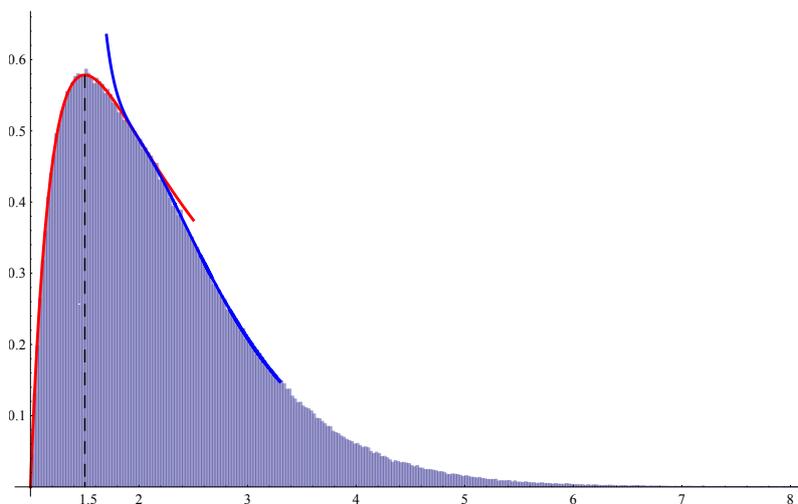


Figure 6.3: A histogram of the distribution $\hat{\gamma}$ over $(1, \infty)$ obtained from the simulations based on (6.6). The red and the blue curves correspond respectively to the explicit formulae for the density of $\hat{\gamma}$ over $[1, 2]$ and $[2, 3]$.

6.2 The continuous setting

6.2.1 The reduced tree Δ

We set

$$\mathcal{V} := \bigcup_{n=0}^{\infty} \{1, 2\}^n,$$

where $\{1, 2\}^0 = \{\emptyset\}$. If $v = (v_1, \dots, v_n) \in \mathcal{V}$, we set $|v| = n$ (in particular $|\emptyset| = 0$). If $v \neq \emptyset$, we define the parent of v as $\bar{v} = (v_1, \dots, v_{n-1})$, and we then say that v is a child of \bar{v} . If both $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_n)$ belong to \mathcal{V} , their concatenation is $uv := (u_1, \dots, u_m, v_1, \dots, v_n)$. The notions of a descendant and an ancestor of an element of \mathcal{V} are defined in the obvious way, with the convention that a vertex $v \in \mathcal{V}$ is both an ancestor and a descendant of itself. If $v, w \in \mathcal{V}$, $v \wedge w$ is the unique element of \mathcal{V} that is an ancestor of both v and w and such that $|v \wedge w|$ is maximal.

We then consider a collection $(U_v)_{v \in \mathcal{V}}$ of independent real random variables uniformly distributed over $[0, 1]$ under the probability measure \mathbb{P} . We set $Y_\emptyset = U_\emptyset$ and then by induction, for every $v \in \{1, 2\}^n$ with $n \geq 1$,

$$Y_v = Y_{\bar{v}} + U_v(1 - Y_{\bar{v}}).$$

Note that a.s., $0 \leq Y_v < 1$ for every $v \in \mathcal{V}$. Consider then the set

$$\Delta_0 := (\{\emptyset\} \times [0, Y_\emptyset]) \cup \left(\bigcup_{v \in \mathcal{V} \setminus \{\emptyset\}} \{v\} \times (Y_{\bar{v}}, Y_v] \right).$$

We can define a natural metric \mathbf{d} on Δ_0 , so that (Δ_0, \mathbf{d}) is a (noncompact) \mathbb{R} -tree and, for every $x = (v, r) \in \Delta_0$, $\mathbf{d}((\emptyset, 0), x) = r$. To be specific, let $x = (v, r) \in \Delta_0$ and $y = (w, r') \in \Delta_0$:

- If v is a descendant of w or w is a descendant of v , we set $\mathbf{d}(x, y) = |r - r'|$.

- Otherwise, $\mathbf{d}(x, y) = \mathbf{d}((v \wedge w, Y_{v \wedge w}), x) + \mathbf{d}((v \wedge w, Y_{v \wedge w}), y) = (r - Y_{v \wedge w}) + (r' - Y_{v \wedge w})$.

See Figure 6.4 for an illustration of the tree Δ_0 .

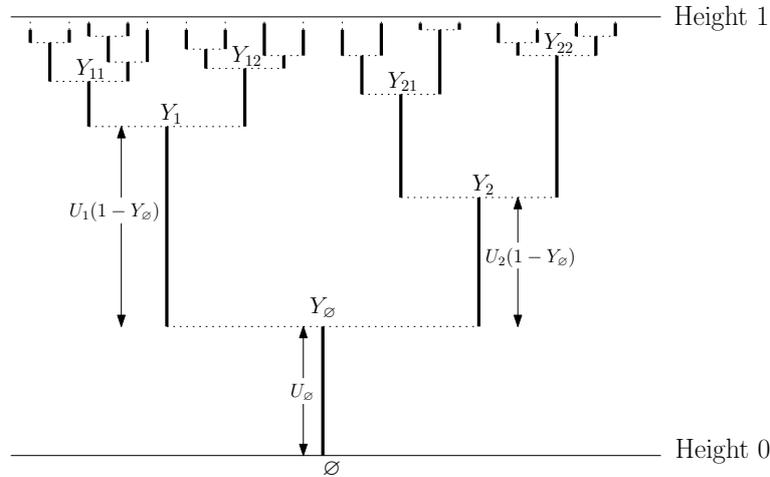


Figure 6.4: The random tree Δ_0

We let Δ be the completion of Δ_0 with respect to the metric \mathbf{d} . Then $\Delta = \Delta_0 \cup \partial\Delta$, and the boundary $\partial\Delta := \{x \in \Delta : \mathbf{d}((\emptyset, 0), x) = 1\}$ is canonically identified with $\{1, 2\}^{\mathbb{N}}$. Note that (Δ, \mathbf{d}) is a compact \mathbb{R} -tree.

The point $(\emptyset, 0)$ is called the root of Δ . For every $x \in \Delta$, we set $H(x) = \mathbf{d}((\emptyset, 0), x)$ and call it the height of x . We can define a genealogical order on Δ by setting $x \prec y$ if and only if x belongs to the geodesic path from the root to y .

For every $\varepsilon \in (0, 1)$, we set $\Delta_\varepsilon := \{x \in \Delta : H(x) \leq 1 - \varepsilon\}$, which is also a compact \mathbb{R} -tree for the metric \mathbf{d} . The leaves of Δ_ε are the points of the form $(v, 1 - \varepsilon)$ for all $v \in \mathcal{V}$ such that $Y_{\bar{v}} < 1 - \varepsilon \leq Y_v$. The branching points of Δ_ε are the points of the form (v, Y_v) for all $v \in \mathcal{V}$ such that $Y_v < 1 - \varepsilon$.

Conditionally on Δ , we can take any $\varepsilon \in (0, 1)$ and define Brownian motion on Δ_ε starting from the root. Informally, this process behaves like linear Brownian motion as long as it stays on an “open interval” of the form $\{v\} \times (Y_{\bar{v}}, Y_v \wedge (1 - \varepsilon))$, and it is reflected at the root $(\emptyset, 0)$ and at the leaves of Δ_ε . When it arrives at a branching point of the tree, it chooses each of three possible line segments ending at this point with equal probabilities. By taking a sequence $(\varepsilon_n = 2^{-n})_{n \geq 1}$ and then letting n go to infinity, we can construct under the same probability measure P a Brownian motion $(B_t)_{t \geq 0}$ on Δ starting from the root up to its first hitting time τ of $\partial\Delta$. We refer the reader to [20] for the details of this construction. The harmonic measure μ is the distribution of $B_{\tau-}$ under P , which is a (random) probability measure on $\partial\Delta = \{1, 2\}^{\mathbb{N}}$.

6.2.2 The Yule tree Γ

To define the Yule tree, consider a collection $(I_v)_{v \in \mathcal{V}}$ of independent real random variables exponentially distributed with mean 1 under the probability measure \mathbb{P} . We set $Z_\emptyset = I_\emptyset$ and then by induction, for every $v \in \{1, 2\}^n$, with $n \geq 1$, $Z_v = Z_{\bar{v}} + I_v$. The Yule tree is the set

$$\Gamma := (\{\emptyset\} \times [0, Z_\emptyset]) \cup \left(\bigcup_{v \in \mathcal{V} \setminus \{\emptyset\}} \{v\} \times (Z_{\bar{v}}, Z_v] \right),$$

which is equipped with the metric d defined in the same way as \mathbf{d} in the preceding section. For every $x = (v, r) \in \Gamma$, $d((\emptyset, 0), x) = r$ and we keep the notation $H(x) = r$ for the height of the point x .

Observe that if U is uniformly distributed over $[0, 1]$, the random variable $-\log(1 - U)$ is exponentially distributed with mean 1. Hence we may and will suppose that the collection $(I_v)_{v \in \mathcal{V}}$ is constructed from $(U_v)_{v \in \mathcal{V}}$ in the previous section via the formula $I_v = -\log(1 - U_v)$, for every $v \in \mathcal{V}$. Then, the mapping Ψ defined on Δ_0 by $\Psi(v, r) = (v, -\log(1 - r))$, for every $(v, r) \in \Delta_0$, is a homeomorphism from Δ_0 onto Γ .

Using stochastic calculus, we can write, for every $t \in [0, \tau)$,

$$\Psi(B_t) = W\left(\int_0^t (1 - H(B_s))^{-2} ds\right) \tag{6.8}$$

where $(W(t))_{t \geq 0}$ is Brownian motion with constant drift $1/2$ towards infinity on the Yule tree (this process is defined in a similar way as Brownian motion on Δ , except that it behaves like Brownian motion with drift $1/2$ on every “open interval” of the tree). Note that W is also defined under the probability measure P . From now on, Brownian motion on the Yule tree Γ or on other similar trees will always refer to Brownian motion with drift $1/2$ towards infinity.

By definition, the boundary $\partial\Gamma$ is the set of all geodesic rays in Γ starting from the root $(\emptyset, 0)$. From the transience of Brownian motion on Γ , there is a.s. a unique geodesic ray denoted by W_∞ that is visited by $(W(t), t \geq 0)$ at arbitrarily large times. The distribution of W_∞ under P yields a probability measure ν on the boundary $\partial\Gamma$. Thanks to (6.8), we have in fact $\nu = \mu$, provided we identify $\partial\Delta$ and $\partial\Gamma$ with $\{1, 2\}^\mathbb{N}$ and view both μ and ν as (random) probability measures on $\{1, 2\}^\mathbb{N}$.

Yule-type trees. We define \mathcal{T} to be the set of all collections $(z_v)_{v \in \mathcal{V}}$ of positive real numbers such that the following properties hold:

- (i) $z_{\bar{v}} < z_v$ for every $v \in \mathcal{V} \setminus \{\emptyset\}$;
- (ii) for every $\mathbf{v} = (v_1, v_2, \dots) \in \{1, 2\}^\mathbb{N}$, $\lim_{n \rightarrow \infty} z_{(v_1, \dots, v_n)} = +\infty$.

Notice that we allow the possibility that $z_\emptyset = 0$. We equip \mathcal{T} with the σ -field generated by the coordinate mappings. If $(z_v)_{v \in \mathcal{V}} \in \mathcal{T}$, we consider the associated “tree”

$$T := (\{\emptyset\} \times [0, z_\emptyset]) \cup \left(\bigcup_{v \in \mathcal{V} \setminus \{\emptyset\}} \{v\} \times (z_{\bar{v}}, z_v] \right),$$

which is equipped with the distance d similarly defined as above. If $x = (v, r) \in T$ we still write $H(x) = r$ for the height of x . The genealogical (partial) order on T is defined as previously and will again be denoted by \prec . The set of all geodesic rays in T is called the boundary ∂T , which is naturally identified with $\{1, 2\}^\mathbb{N}$. If $\mathbf{u} = (u_1, u_2, \dots, u_n, \dots) \in \{1, 2\}^\mathbb{N}$, and $x = (v, r) \in T$, we write $x \prec \mathbf{u}$ if $v = (u_1, u_2, \dots, u_k)$ for some integer $k \geq 0$.

We will often say that we consider a tree $T \in \mathcal{T}$: this means that we are given a collection $(z_v)_{v \in \mathcal{V}}$ satisfying the above properties, and we consider the associated tree T . In particular, T has an order structure given by the lexicographical order on \mathcal{V} . Elements of \mathcal{T} will be called Yule-type trees. The Yule tree Γ can be viewed as a random variable taking values in \mathcal{T} , and we write $\Theta(dT)$ for its distribution.

Let us fix $T \in \mathcal{T}$. If $r > 0$, the level set at height r is $T_r := \{x \in T : H(x) = r\}$. If $x \in T_r$, we can then consider the subtree $T[x]$ of descendants of x in T . Formally, we view $T[x]$ as an

element of \mathcal{T} : We write v_x for the unique element of \mathcal{V} such that $x = (v_x, r)$, and define $T[x]$ as the Yule-type tree corresponding to the collection $(z_{v_x v} - r)_{v \in \mathcal{V}}$.

As we have seen in the Introduction, the limit $\mathcal{W}(T) = \lim_{r \rightarrow \infty} e^{-r} \#T_r$ exists $\Theta(dT)$ -a.s. For every $x \in T$, we similarly set $\mathcal{W}(T[x]) = \lim_{r \rightarrow \infty} e^{-r} \#T_r[x]$. If $\mathbf{v} \in \partial T$ is a geodesic ray passing through x , let $\mathcal{B}_d(\mathbf{v}, H(x))$ denote the set of geodesic rays in T that coincide with \mathbf{v} up to height $H(x)$. Then $\Theta(dT)$ -a.s., the uniform measure $\bar{\omega}_T$ on ∂T is defined as the unique probability measure on ∂T satisfying that

$$\bar{\omega}_T(\mathcal{B}_d(\mathbf{v}, H(x))) = \frac{e^{-H(x)} \mathcal{W}(T[x])}{\mathcal{W}(T)}, \quad \forall x \in T \text{ and } \mathbf{v} \in \partial T \text{ such that } x \prec \mathbf{v}.$$

On the other hand, for a fixed Yule-type tree $T \in \mathcal{T}$, we define the harmonic measure μ_T on ∂T as the distribution of the first hitting point of ∂T by Brownian motion on T (with drift 1/2 towards infinity).

6.2.3 The invariant measure

We write

$$\mathcal{T}^* := \mathcal{T} \times \{1, 2\}^{\mathbb{N}}$$

for the set of all pairs consisting of a tree $T \in \mathcal{T}$ and a distinguished geodesic ray \mathbf{v} . Then following [20, Section 5.1], we define a shift transformation on \mathcal{T}^* by shifting (T, \mathbf{v}) at the first branching point of T . More precisely, if T corresponds to the collection $(z_v)_{v \in \mathcal{V}}$, we write $T_{(1)}$ and $T_{(2)}$ for the two subtrees of T obtained at the first branching point, which means, for $i \in \{1, 2\}$, $T_{(i)}$ is the tree corresponding to the collection $(z_{iv} - z_\emptyset)_{v \in \mathcal{V}}$. For any geodesic ray $\mathbf{v} = (v_1, v_2, \dots)$ in the tree T , we set $S(T, \mathbf{v}) := (T_{(v_1)}, \tilde{\mathbf{v}})$, where $\tilde{\mathbf{v}} = (v_2, v_3, \dots)$.

The following proposition is the analogue of Proposition 6.1 in [71] for the Yule tree.

Proposition 6.5. *The probability measure $\mathcal{W}(T)\Theta(dT)\bar{\omega}_T(d\mathbf{v})$ is invariant under S .*

Proof. Under $\Theta(dT)$, if T corresponds to the collection $(z_v)_{v \in \mathcal{V}}$, then z_\emptyset is exponentially distributed with mean 1. Conditionally on z_\emptyset , the branching property of the Yule tree states that $T_{(1)}$ and $T_{(2)}$ are i.i.d. of the same law Θ .

Let F be a bounded measurable function on \mathcal{T}^* . By the definition of the shift S and then the preceding observation, we have

$$\begin{aligned} \int F \circ S(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}) &= \sum_{i=1}^2 \int F(T_{(i)}, \mathbf{u}) (e^{-z_\emptyset} \mathcal{W}(T_{(i)})) \Theta(dT) \bar{\omega}_{T_{(i)}}(d\mathbf{u}) \\ &= 2 \left(\int_0^\infty e^{-2z} dz \right) \times \int F(T, \mathbf{u}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{u}) \\ &= \int F(T, \mathbf{u}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{u}), \end{aligned}$$

which completes the proof of the statement. \square

6.2.4 The continuous conductances

For a fixed Yule-type tree T , we consider the excursion measure of Brownian motion (with drift 1/2) on T away from the root, and define the conductance $\mathcal{C}(T)$ as the mass assigned

by this excursion measure to the set of trajectories that never return to the root. Note that $1 \leq \mathcal{C}(T) < \infty$ for any $T \in \mathcal{T}$. For more details we refer the reader to [20, Section 2.3].

To simplify notation, we introduce under the probability measure \mathbb{P} a pair of random variables $(\mathcal{W}, \mathcal{C})$ that is distributed as $(\mathcal{W}(T), \mathcal{C}(T))$ under $\Theta(dT)$. In addition, we let $\widehat{\mathcal{C}}$ be a random variable defined under \mathbb{P} that is distributed as $\mathcal{C}(T)$ under the probability measure $\mathcal{W}(T)\Theta(dT)$.

Let T be a Yule-type tree corresponding to the collection $(z_v)_{v \in \mathcal{V}}$. Recall that $T_{(1)}$ and $T_{(2)}$ stand for the two subtrees of T obtained at the first branching point. From the identity $\mathcal{W}(T) = e^{-z_\emptyset} (\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)}))$ for every $T \in \mathcal{T}$, it follows that the distribution of \mathcal{W} satisfies the recursive equation

$$\mathcal{W} \stackrel{(d)}{=} (1 - U)(\mathcal{W}_1 + \mathcal{W}_2),$$

in which \mathcal{W}_1 and \mathcal{W}_2 are independent copies of \mathcal{W} , and, U is uniformly distributed over $[0, 1]$ and independent of $(\mathcal{W}_1, \mathcal{W}_2)$. Moreover, the preceding equation holds jointly with a similar distributional identity for the conductance \mathcal{C} (see equation (2) in [20]). To sum up, we have

$$(\mathcal{W}, \mathcal{C}) \stackrel{(d)}{=} \left((1 - U)(\mathcal{W}_1 + \mathcal{W}_2), \left(U + \frac{1 - U}{\mathcal{C}_1 + \mathcal{C}_2} \right)^{-1} \right), \quad (6.9)$$

where U is as above, while $(\mathcal{W}_i, \mathcal{C}_i)_{i \in \{1, 2\}}$ are two independent copies of $(\mathcal{W}, \mathcal{C})$, and are independent of U .

Lemma 6.6. *The random variable $\widehat{\mathcal{C}}$ satisfies the distributional identity (6.6).*

Proof. By definition, the distribution of $\widehat{\mathcal{C}}$ is determined by

$$\mathbb{E}[g(\widehat{\mathcal{C}})] = \mathbb{E}[\mathcal{W} g(\mathcal{C})] \quad (6.10)$$

for every nonnegative measurable function g . Using (6.9) and symmetry, we have

$$\begin{aligned} \mathbb{E}[g(\widehat{\mathcal{C}})] &= \mathbb{E} \left[(1 - U)(\mathcal{W}_1 + \mathcal{W}_2) g \left(\left(U + \frac{1 - U}{\mathcal{C}_1 + \mathcal{C}_2} \right)^{-1} \right) \right] \\ &= \mathbb{E} \left[2(1 - U)\mathcal{W}_1 g \left(\left(U + \frac{1 - U}{\mathcal{C}_1 + \mathcal{C}_2} \right)^{-1} \right) \right]. \end{aligned}$$

Recall that the random variable V has density function $2(1 - x)$ over $[0, 1]$. The statement of the lemma therefore follows by applying (6.10) in reverse order. \square

The law γ of the conductance \mathcal{C} has been discussed at length in [20, Proposition 6]. By similar arguments, we can study the law $\widehat{\gamma}$ of $\widehat{\mathcal{C}}$. These properties are collected in the next proposition.

For every $v \in (0, 1)$, $x \geq 0$ and $c \geq 1$, we define

$$G(v, x, c) := \left(v + \frac{1 - v}{x + c} \right)^{-1}.$$

Let \mathcal{M} be the set of all probability measures on $[0, \infty]$ and let $\widehat{\Phi}: \mathcal{M} \rightarrow \mathcal{M}$ map a probability measure σ to

$$\widehat{\Phi}(\sigma) = \text{Law}(G(V, X, \mathcal{C})) \quad (6.11)$$

where V and \mathcal{C} are as in (6.6), while X is distributed according to σ , and is independent of the pair (V, \mathcal{C}) .

Proposition 6.7. (1) The distributional equation (6.6) characterizes the law $\hat{\gamma}$ of $\hat{\mathcal{C}}$ in the sense that, $\hat{\gamma}$ is the unique fixed point of the mapping $\hat{\Phi}$ on \mathcal{M} , and for every $\sigma \in \mathcal{M}$, the k -th iterate $\hat{\Phi}^k(\sigma)$ converges to $\hat{\gamma}$ weakly as $k \rightarrow \infty$.

(2) The law $\hat{\gamma}$ has a continuous density over $[1, \infty)$, and all its moments are finite.

(3) For any monotone continuously differentiable function $g: [1, \infty) \rightarrow \mathbb{R}_+$, we have

$$\mathbb{E}[\hat{\mathcal{C}}(\hat{\mathcal{C}} - 1)g'(\hat{\mathcal{C}})] + 2\mathbb{E}[g(\hat{\mathcal{C}})] = 2\mathbb{E}[g(\hat{\mathcal{C}} + \mathcal{C})], \quad (6.12)$$

where $\hat{\mathcal{C}}$ and \mathcal{C} are always assumed to be independent under the probability measure \mathbb{P} .

(4) We define, for all $\ell \geq 0$, the Laplace transforms $\varphi(\ell) = \mathbb{E}[\exp(-\ell\mathcal{C}/2)]$ and

$$\hat{\varphi}(\ell) = \mathbb{E}[\exp(-\ell\hat{\mathcal{C}}/2)] = \int_1^\infty e^{-\ell r/2} \hat{\gamma}(dr).$$

Then $\hat{\varphi}$ solves the linear differential equation

$$2\ell\phi''(\ell) + \ell\phi'(\ell) - 2(1 - \varphi(\ell))\phi(\ell) = 0.$$

The proof of this proposition is very similar to that of the analogous results in [20, Proposition 6]. We skip the details.

Remark 1. Using assertion (1) in Proposition 6.7, one can approximate the law $\hat{\gamma}$ of $\hat{\mathcal{C}}$ by iterating the mapping $\hat{\Phi}$. An application of the Monte-Carlo method gives $\mathbb{E}[\hat{\mathcal{C}}] \approx 2.21$.

Remark 2. Following the preceding proposition, we discuss here some smoothness properties of the density of $\hat{\gamma}$. For every $t \geq 1$, we set $\hat{F}(t) = \mathbb{P}(\hat{\mathcal{C}} \geq t)$, and we get from (6.6) that

$$\hat{F}(t) = 2 \left(\frac{t-1}{t} \right)^2 \int_t^\infty dx \frac{x}{(x-1)^3} \mathbb{P}(\hat{\mathcal{C}} + \mathcal{C} \geq x). \quad (6.13)$$

Since $\mathbb{P}(\hat{\mathcal{C}} + \mathcal{C} \geq t) = 1$ for every $t \in [1, 2]$, we obtain from the last display that

$$\hat{F}(t) = \frac{4t-2}{t^2} A_0 - 2A_0 + 1, \quad \forall t \in [1, 2], \quad (6.14)$$

where

$$A_0 := 2 - \int_2^\infty dx \frac{x}{(x-1)^3} \mathbb{P}(\hat{\mathcal{C}} + \mathcal{C} \geq x) \in \left(\frac{1}{2}, 2 \right).$$

Let $\hat{f} = -\hat{F}'$ be the density of the law $\hat{\gamma}$. Then it follows from (6.14) that for all $t \in [1, 2]$,

$$\hat{f}(t) = 4A_0 \times \frac{t-1}{t^3} \quad \text{and} \quad \hat{f}'(t) = 4A_0 \times \frac{3-2t}{t^4}.$$

In particular, we have $\hat{f}(1) = 0$, $\hat{f}(2) = A_0/2$ and $\hat{f}'(\frac{3}{2}) = 0$. A numerical approximation of $\hat{f}(2)$ gives that $A_0 \approx 0.976$.

For the density f of the law of \mathcal{C} , it is shown in [20, Section 2.3] that there exists a constant $K_0 \in (1, 2)$ such that $f(t) = K_0 t^{-2}$ for $t \in [1, 2]$. The explicit forms of f and \hat{f} over $[1, 2]$ can be used to calculate by convolution the probability $\mathbb{P}(\hat{\mathcal{C}} + \mathcal{C} \geq t)$ for $t \in [2, 3]$. The values of \hat{F} over $[2, 3]$ are thus determined via the ordinary differential equation

$$t(t-1)\hat{F}'(t) - 2\hat{F}(t) = -2\mathbb{P}(\hat{\mathcal{C}} + \mathcal{C} \geq t), \quad (6.15)$$

which is a direct consequence of (6.13). By solving this differential equation, we are able to get an explicit, yet complicated, expression of \widehat{F} over $[2, 3]$, in terms of the two (unknown) parameters A_0 and K_0 (numerical approximations of $f(1)$ show that $K_0 \approx 1.477$). One can then verify that the density \widehat{f} is continuously differentiable on $(1, 3)$. Furthermore, it holds that

$$\widehat{f}''(2-) = \widehat{f}''(2+) = 0,$$

and that \widehat{f} is twice continuously differentiable on $(1, 3)$. However, \widehat{f} is not third-order differentiable at the point 2, as one has

$$\widehat{f}'''(2-) = \frac{3A_0}{4}, \quad \text{while} \quad \widehat{f}'''(2+) = \frac{3A_0}{4} - 4A_0K_0.$$

This is similar to the singular behavior of the density f pointed out in [20, Section 2.3], where it is shown that $f'''(2-) \neq f'''(2+)$. One may conjecture that the density \widehat{f} of $\widehat{\gamma}$ is twice continuously differentiable on the whole interval $(1, \infty)$, but not third-order differentiable at all integers $n \geq 2$.

We finally note that $3/2$ is the global maximum point for the density \widehat{f} . In fact, we have seen that \widehat{f} reaches its maximum at $3/2$ over the interval $[1, 2]$. Meanwhile, it is elementary to verify, by differentiating (6.15), that the function \widehat{f} is strictly decreasing over $[3/2, \infty)$.

6.2.5 Proof of Theorem 6.2 and of Proposition 6.4

We have seen in Proposition 6.5 that the probability measure $\mathcal{W}(T)\Theta(dT)\bar{\omega}_T(d\mathbf{v})$ on \mathcal{T}^* is invariant under the shift S . Recall that $\mathcal{W}(T)$ follows an exponential distribution of mean 1 under $\Theta(dT)$. Taking into account that $\mathcal{W}(T) > 0$, $\Theta(dT)$ -a.s., we can then verify, in a similar way as in [66, Section 2.5], that the shift S acting on the probability space $(\mathcal{T}^*, \mathcal{W}(T)\Theta(dT)\bar{\omega}_T(d\mathbf{v}))$ is ergodic. We shall now apply Birkhoff's ergodic theorem to the three functionals defined below.

Firstly, let $H_n(T, \mathbf{v})$ denote the height of the n -th branching point on the geodesic ray \mathbf{v} . One immediately verifies that, for every $n \geq 1$,

$$H_n = \sum_{i=0}^{n-1} H_1 \circ S^i,$$

where S^i stands for the i -th iterate of the shift S . It follows thus from the ergodic theorem that $\mathcal{W}(T)\Theta(dT)\bar{\omega}_T(d\mathbf{v})$ -a.s.,

$$\frac{1}{n}H_n \xrightarrow[n \rightarrow \infty]{} \int H_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}). \quad (6.16)$$

Since the density $\mathcal{W}(T)$ is strictly positive, the latter convergence also holds $\Theta(dT)\bar{\omega}_T(d\mathbf{v})$ -a.s. By definition of H_1 and then the branching property of the Yule tree,

$$\begin{aligned} \int H_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}) &= \int z_\emptyset \mathcal{W}(T) \Theta(dT) \\ &= \int z_\emptyset e^{-z_\emptyset} (\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})) \Theta(dT) \\ &= 2 \left(\int_0^\infty z e^{-2z} dz \right) \times \int \mathcal{W}(T) \Theta(dT) = \frac{1}{2}. \end{aligned} \quad (6.17)$$

Secondly, for a fixed geodesic ray $\mathbf{v} = (v_1, v_2, \dots) \in \{1, 2\}^{\mathbb{N}}$, we let $\mathbf{x}_{n, \mathbf{v}}$ denote the $n+1$ -st branching point on the geodesic ray \mathbf{v} , i.e. $\mathbf{x}_{n, \mathbf{v}} = ((v_1, \dots, v_n), H_{n+1}(T, \mathbf{v}))$. We set, for every

$n \geq 1$, the functional $F_n(T, \mathbf{v}) := \log \bar{\omega}_T(\{\mathbf{u} \in \partial T : \mathbf{x}_{n, \mathbf{v}} \prec \mathbf{u}\})$. In particular,

$$F_1(T, \mathbf{v}) = \log \frac{\mathcal{W}(T_{(v_1)})}{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})}.$$

By definition of $\bar{\omega}_T$, one can check that

$$F_n = \sum_{i=0}^{n-1} F_1 \circ S^i.$$

Using the ergodic theorem again, we have $\Theta(dT) \bar{\omega}_T(d\mathbf{v})$ -a.s.,

$$\frac{1}{n} F_n \xrightarrow[n \rightarrow \infty]{} \int F_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}),$$

in which the limit can be calculated as follows:

$$\begin{aligned} & \int F_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}) \\ &= \sum_{i=1}^2 \int e^{-z\varnothing} \mathcal{W}(T_{(i)}) \log \frac{\mathcal{W}(T_{(i)})}{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})} \Theta(dT) \\ &= \sum_{i=1}^2 \int e^{-z\varnothing} \mathcal{W}(T_{(i)}) \log \mathcal{W}(T_{(i)}) \Theta(dT) - \int \left(\sum_{i=1}^2 e^{-z\varnothing} \mathcal{W}(T_{(i)}) \log (e^{z\varnothing} \mathcal{W}(T)) \right) \Theta(dT) \\ &= 2 \left(\int_0^\infty e^{-2z} dz \right) \times \int \mathcal{W}(T) \log \mathcal{W}(T) \Theta(dT) - \int \mathcal{W}(T) \log (e^{z\varnothing} \mathcal{W}(T)) \Theta(dT) \\ &= - \int z\varnothing \mathcal{W}(T) \Theta(dT). \end{aligned} \tag{6.18}$$

Note that we used the fact that $\int \mathcal{W}(T) |\log \mathcal{W}(T)| \Theta(dT) < \infty$ to derive the last equality. In view of (6.17), we see that $\Theta(dT) \bar{\omega}_T(d\mathbf{v})$ -a.s., F_n/n converges to $-\frac{1}{2}$ whereas H_n/n converges to $\frac{1}{2}$. By considering the ratio F_n/H_n and taking $n \rightarrow \infty$, we get that $\Theta(dT)$ -a.s. $\bar{\omega}_T(d\mathbf{v})$ -a.e.,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log \bar{\omega}_T(\mathcal{B}_d(\mathbf{v}, r)) = -1,$$

from which the convergence (6.5) readily follows.

Thirdly, we turn to the harmonic measure μ_T and set, for every $n \geq 1$, the functional $G_n(T, \mathbf{v}) := \log \mu_T(\{\mathbf{u} \in \partial T : \mathbf{x}_{n, \mathbf{v}} \prec \mathbf{u}\})$. In particular,

$$G_1(T, \mathbf{v}) = \log \frac{\mathcal{C}(T_{(v_1)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})}.$$

The flow property of the harmonic measure μ_T (see Lemma 7 in [66]) yields that

$$G_n = \sum_{i=0}^{n-1} G_1 \circ S^i.$$

Similarly we have the $\Theta(dT) \bar{\omega}_T(d\mathbf{v})$ -almost sure convergence

$$\frac{1}{n} G_n \xrightarrow[n \rightarrow \infty]{} \int G_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}), \tag{6.19}$$

and we calculate the limit as follows:

$$\begin{aligned}
& \int G_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}) \\
&= \sum_{i=1}^2 \int e^{-z\varnothing} \mathcal{W}(T_{(i)}) \log \frac{\mathcal{C}(T_{(i)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})} \Theta(dT) \\
&= 2 \left(\int_0^\infty e^{-2z} dz \right) \times \int \mathcal{W}(T_{(1)}) \log \frac{\mathcal{C}(T_{(1)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})} \Theta(dT) \\
&= \int \mathcal{W}(T_{(1)}) \log \frac{\mathcal{C}(T_{(1)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})} \Theta(dT).
\end{aligned}$$

Putting the convergence (6.19) together with (6.16), we see that $\Theta(dT)$ -a.s. $\bar{\omega}_T(d\mathbf{v})$ -a.e.,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log \mu_T(\mathcal{B}_d(\mathbf{v}, r)) = 2 \int \mathcal{W}(T_{(1)}) \log \frac{\mathcal{C}(T_{(1)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})} \Theta(dT).$$

Using the branching property of the Yule tree and recalling the notation in Section 6.2.4, we have therefore \mathbb{P} -a.s. $\omega(d\mathbf{v})$ -a.e. that

$$\lim_{r \downarrow 0} \frac{\log \mu(\mathcal{B}_d(\mathbf{v}, r))}{\log r} = 2 \mathbb{E} \left[\log \left(\frac{\widehat{\mathcal{C}} + \mathcal{C}}{\widehat{\mathcal{C}}} \right) \right],$$

where $\widehat{\mathcal{C}}$ and \mathcal{C} are supposed to be independent under the probability measure \mathbb{P} . However, by taking $g(x) = \log(x)$ in (6.12), we see that

$$\mathbb{E}[\widehat{\mathcal{C}}] - 1 = 2 \mathbb{E} \left[\log \left(\frac{\widehat{\mathcal{C}} + \mathcal{C}}{\widehat{\mathcal{C}}} \right) \right]. \tag{6.20}$$

By defining $\lambda := \mathbb{E}[\widehat{\mathcal{C}}] - 1$, the proof of the convergence (6.4) is hence completed.

Finally, in view of Proposition 6.7, it only remains to verify that $\lambda > 1$. In fact, we know from the display following (6.19) that

$$\lambda = 2 \int e^{-z\varnothing} \left(\sum_{i=1}^2 \mathcal{W}(T_{(i)}) \log \frac{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})}{\mathcal{C}(T_{(i)})} \right) \Theta(dT).$$

By concavity of the logarithm,

$$\sum_{i=1}^2 \frac{\mathcal{W}(T_{(i)})}{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})} \log \left(\frac{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})}{\mathcal{W}(T_{(i)})} \cdot \frac{\mathcal{C}(T_{(i)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})} \right) \leq 0,$$

which entails that

$$\sum_{i=1}^2 \mathcal{W}(T_{(i)}) \log \frac{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})}{\mathcal{C}(T_{(i)})} \geq \sum_{i=1}^2 \mathcal{W}(T_{(i)}) \log \frac{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})}{\mathcal{W}(T_{(i)})}.$$

Notice that the previous inequality is strict if and only if for $i \in \{1, 2\}$,

$$\frac{\mathcal{W}(T_{(i)})}{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})} \neq \frac{\mathcal{C}(T_{(i)})}{\mathcal{C}(T_{(1)}) + \mathcal{C}(T_{(2)})}.$$

Since the latter property holds with positive probability under $\Theta(dT)$, we have

$$\begin{aligned} \lambda &> 2 \int e^{-z_\emptyset} \left(\sum_{i=1}^2 \mathcal{W}(T_{(i)}) \log \frac{\mathcal{W}(T_{(1)}) + \mathcal{W}(T_{(2)})}{\mathcal{W}(T_{(i)})} \right) \Theta(dT) \\ &= -2 \int F_1(T, \mathbf{v}) \mathcal{W}(T) \Theta(dT) \bar{\omega}_T(d\mathbf{v}). \end{aligned}$$

By (6.18) and (6.17), the right-hand side of the last display is equal to 1. We have therefore finished the proof of Theorem 6.2 and of Proposition 6.4.

6.2.6 The size-biased Yule tree $\hat{\Gamma}$

Let $(\hat{\Gamma}, \hat{\mathbf{v}}) \in \mathcal{T}^*$ be a random variable distributed according to $\mathcal{W}(T)\Theta(dT)\bar{\omega}_T(d\mathbf{v})$. We now give a direct construction of this random variable under the probability measure \mathbb{P} . In the following description, all the random variables involved are supposed to be defined under \mathbb{P} .

First, we introduce a sequence $(a_k)_{k \geq 1}$ of i.i.d. random variables uniformly distributed over the set $\{1, 2\}$, and another sequence $(J_k)_{k \geq 1}$ of i.i.d. real random variables exponentially distributed with mean $1/2$. Let $(\Gamma^{(k)})_{k \geq 1}$ be a collection of independent Yule trees, each of which corresponding respectively to the collection $(Z_v^{(k)})_{v \in \mathcal{V}}$ with the notation introduced in Section 6.2.2. We assume that $(a_k)_{k \geq 1}$, $(J_k)_{k \geq 1}$ and $(\Gamma^{(k)})_{k \geq 1}$ are independent.

For every integer $n \geq 1$, we set $\mathbf{v}_n = (a_1, a_2, \dots, a_n) \in \{1, 2\}^n$ and $Z_{\mathbf{v}_n} = \sum_{k=1}^n J_k$. We write $\tilde{\mathbf{v}}_n = (a_1, a_2, \dots, a_{n-1}, 3 - a_n) \in \{1, 2\}^n$ for the unique sibling of \mathbf{v}_n in \mathcal{V} , and define the subtree $\Gamma\langle \tilde{\mathbf{v}}_n \rangle$ grafted at $\tilde{\mathbf{v}}_n$ as

$$\Gamma\langle \tilde{\mathbf{v}}_n \rangle := \left(\{\tilde{\mathbf{v}}_n\} \times (Z_{\mathbf{v}_n}, Z_{\mathbf{v}_n} + Z_\emptyset^{(n)}) \right) \cup \left(\bigcup_{v \in \mathcal{V} \setminus \{\emptyset\}} \{\tilde{\mathbf{v}}_n v\} \times (Z_{\mathbf{v}_n} + Z_v^{(n)}, Z_{\mathbf{v}_n} + Z_v^{(n)}) \right).$$

Finally, let $\hat{\Gamma}$ be the following Yule-type tree

$$\hat{\Gamma} := (\{\emptyset\} \times [0, Z_{\mathbf{v}_1}]) \cup \left(\bigcup_{n \geq 1} \{\mathbf{v}_n\} \times (Z_{\mathbf{v}_n}, Z_{\mathbf{v}_{n+1}}) \right) \cup \left(\bigcup_{n \geq 1} \Gamma\langle \tilde{\mathbf{v}}_n \rangle \right).$$

We will call $\hat{\Gamma}$ the size-biased Yule tree. See Figure 6.5 for an illustration.

Lemma 6.8. *The pair $(\hat{\Gamma}, \hat{\mathbf{v}} = (a_1, a_2, \dots)) \in \mathcal{T} \times \{1, 2\}^{\mathbb{N}}$ constructed above follows the required distribution $\mathcal{W}(T)\Theta(dT)\bar{\omega}_T(d\mathbf{v})$.*

The proof of this lemma is based on similar calculations carried out in the previous section. We leave the details to the reader.

6.2.7 The size-biased reduced tree $\hat{\Delta}$

Recall the bijection $\Psi: (v, r) \in \Delta_0 \mapsto (v, -\log(1 - r)) \in \Gamma$ introduced in Section 6.2.2. We now apply the inverse mapping $\Psi^{-1}(v, s) = (v, 1 - e^{-s})$ to the size-biased Yule tree $\hat{\Gamma}$.

We keep the notation of the preceding section. For every integer $n \geq 1$, we set $V_n = 1 - \exp(-J_n)$, and then by induction,

$$\hat{Y}_{\mathbf{v}_n} = \hat{Y}_{\mathbf{v}_{n-1}} + (1 - \hat{Y}_{\mathbf{v}_{n-1}})V_n.$$

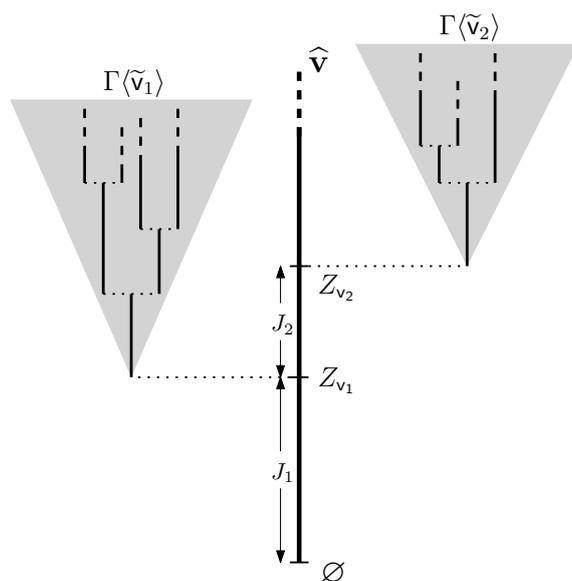


Figure 6.5: Schematic representation of the size-biased Yule tree $\hat{\Gamma}$

Notice that $(V_n)_{n \geq 1}$ are i.i.d. random variables with density function $2(1-x)\mathbf{1}_{x \in [0,1]}$, and that for every $n \geq 1$, $\hat{Y}_{v_n} = 1 - \exp(-Z_{v_n})$. Thus,

$$\Psi^{-1}(\hat{\Gamma}) = (\{\emptyset\} \times [0, \hat{Y}_{v_1}]) \cup \left(\bigcup_{n \geq 1} \{v_n\} \times (\hat{Y}_{v_n}, \hat{Y}_{v_{n+1}}] \right) \cup \left(\bigcup_{n \geq 1} \Psi^{-1}(\Gamma(\tilde{v}_n)) \right).$$

We point out that, independently for every $n \geq 1$, $\Psi^{-1}(\Gamma(\tilde{v}_n))$ is a rescaled copy of the precompact reduced tree Δ_0 with the scaling factor $(1 - \hat{Y}_{v_n})$. From now on we will denote $\Psi^{-1}(\hat{\Gamma})$ by $\hat{\Delta}_0$. See Figure 6.6 for an illustration of $\hat{\Delta}_0$.

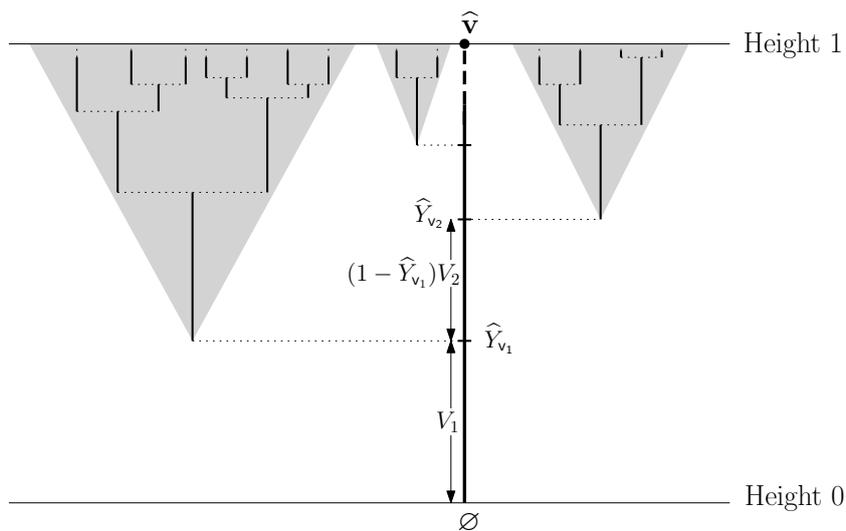


Figure 6.6: Schematic representation of the random tree $\hat{\Delta}_0$

As for Δ_0 , one can define the intrinsic metric \mathbf{d} on $\widehat{\Delta}_0$ such that $(\widehat{\Delta}_0, \mathbf{d})$ is a noncompact \mathbb{R} -tree, and for every $x = (v, r) \in \widehat{\Delta}_0$, we have $\mathbf{d}((\emptyset, 0), x) = r$. Similarly as in Section 6.2.1, we then let $\widehat{\Delta}$ be the completion of $\widehat{\Delta}_0$ with respect to \mathbf{d} , so that $(\widehat{\Delta}, \mathbf{d})$ is a compact \mathbb{R} -tree. In fact, $\widehat{\Delta} = \widehat{\Delta}_0 \cup \partial\widehat{\Delta}$, and the boundary $\partial\widehat{\Delta} := \{x \in \widehat{\Delta} : \mathbf{d}((\emptyset, 0), x) = 1\}$ is canonically identified with $\{1, 2\}^{\mathbb{N}}$. We will call $\widehat{\Delta}$ the size-biased reduced tree. We keep the same notation $H(x) = \mathbf{d}((\emptyset, 0), x)$ for the height of $x \in \widehat{\Delta}$. For every $\varepsilon \in (0, 1)$, we set

$$\widehat{\Delta}_\varepsilon := \{x \in \Delta : H(x) \leq 1 - \varepsilon\}.$$

One can think of both Δ and $\widehat{\Delta}$ as electric networks of ideal resistors with unit resistance per unit length, and define $\mathcal{C}(\Delta)$ (resp. $\mathcal{C}(\widehat{\Delta})$) be the effective conductance between the root and the set $\partial\Delta$ (resp. $\partial\widehat{\Delta}$) in the corresponding network. As explained in [20, Section 2.3], $\mathcal{C}(\Delta)$ is identically distributed as the random variable \mathcal{C} introduced in Section 6.2.4. Analogously, $\mathcal{C}(\widehat{\Delta})$ has the same distribution as $\widehat{\mathcal{C}}$ according to Lemma 6.8. We will thus call $\widehat{\mathcal{C}}$ the continuous conductance of the size-biased reduced tree $\widehat{\Delta}$.

6.3 The discrete setting

6.3.1 Notation for discrete trees

We set

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. If $u = (u_1, \dots, u_n) \in \mathcal{U}$, $|u| = n$ is the generation (or height) of u . In particular, $|\emptyset| = 0$.

A (rooted ordered) tree \mathcal{T} is a subset of \mathcal{U} such that the following holds:

- (i) $\emptyset \in \mathcal{T}$;
- (ii) If $u = (u_1, \dots, u_n) \in \mathcal{T} \setminus \{\emptyset\}$, then $\bar{u} := (u_1, \dots, u_{n-1}) \in \mathcal{T}$;
- (iii) For every $u = (u_1, \dots, u_n) \in \mathcal{T}$, there exists an integer $k_u(\mathcal{T}) \geq 0$ such that, for every $j \in \mathbb{N}$, $(u_1, \dots, u_n, j) \in \mathcal{T}$ if and only if $1 \leq j \leq k_u(\mathcal{T})$.

The notions of a child and a parent of a vertex of \mathcal{T} are defined in an obvious way. We write \prec for the genealogical order on \mathcal{T} . The quantity $k_u(\mathcal{T})$ in (iii) is called the number of children of u in \mathcal{T} . We always view a tree \mathcal{T} as a graph whose vertices are the elements of \mathcal{T} and whose edges are the pairs $\{\bar{u}, u\}$ for all $u \in \mathcal{T} \setminus \{\emptyset\}$.

If \mathcal{T} is finite, we call it a plane tree. The set of all plane trees is denoted by \mathbb{T}_f . For an infinite tree \mathcal{T} , we say it has a single infinite line of descent if there exists a unique sequence of positive integers $(u_n)_{n \geq 1}$ such that $(u_1, u_2, \dots, u_n) \in \mathcal{T}$ for all $n \geq 1$. We denote by \mathbb{T}_∞ the set of all infinite trees that have a single infinite line of descent.

The height of a tree \mathcal{T} is written as $h(\mathcal{T}) := \sup\{|u| : u \in \mathcal{T}\}$. The set of all vertices of \mathcal{T} at generation n is denoted by $\mathcal{T}_n := \{u \in \mathcal{T} : |u| = n\}$. If $u \in \mathcal{T}$, the subtree of descendants of u is $\widetilde{\mathcal{T}}[u] := \{u' \in \mathcal{T} : u \prec u'\}$. Note that $\widetilde{\mathcal{T}}[u]$ is not a tree under our definition, but we can relabel its vertices to turn it into a tree, by setting $\mathcal{T}[u] := \{w \in \mathcal{U} : uw \in \mathcal{T}\}$.

Let \mathcal{T} be a tree of height larger than n , and consider a simple random walk $X = (X_k)_{k \geq 0}$ on \mathcal{T} starting from the root \emptyset . This random walk is defined under the probability measure $P^{\mathcal{T}}$. We write $\tau_n := \inf\{k \geq 0 : |X_k| = n\}$ for the first hitting time of generation n by X , and we define the discrete harmonic measure $\mu_n^{\mathcal{T}}$ supported on \mathcal{T}_n as the law of X_{τ_n} under $P^{\mathcal{T}}$.

Critical Galton–Watson trees. Let θ be a non-degenerate probability measure on \mathbb{Z}_+ , and assume that θ has mean one and finite variance $\sigma^2 > 0$. For every integer $n \geq 0$, we let $\mathbb{T}^{(n)}$ be a Galton–Watson tree with offspring distribution θ , conditioned on non-extinction at generation n , viewed as a random element in \mathbb{T}_f . In particular, $\mathbb{T}^{(0)}$ is just a Galton–Watson tree with offspring distribution θ . From our assumption that θ is critical (i.e. of mean 1), it follows that for all $n \geq 0$, $\mathbb{T}^{(n)}$ is finite. We suppose that the random trees $\mathbb{T}^{(n)}$ are defined under the probability measure \mathbb{P} .

Let \mathbb{T}^{*n} be the reduced tree associated with $\mathbb{T}^{(n)}$, which is the random tree composed of all vertices of $\mathbb{T}^{(n)}$ that have descendants at generation n . It is always implicitly assumed that we have relabeled the vertices of \mathbb{T}^{*n} , preserving both the lexicographical order and the genealogical order, so that \mathbb{T}^{*n} becomes a plane tree in the sense of our preceding definition.

For every $n \geq 1$ we set $q_n := \mathbb{P}(\#\mathbb{T}_n^{(0)} > 0)$. By a standard result (see e.g. Theorem 9.1 of [7, Chapter 1]) on the non-extinction probability up to generation n , we have

$$q_n \sim \frac{2}{n\sigma^2}, \quad \text{as } n \rightarrow \infty. \tag{6.21}$$

Size-biased Galton–Watson tree. We introduce a random variable \widehat{N} distributed according to the size-biased distribution of θ , that is, for every $k \geq 0$, $\mathbb{P}(\widehat{N} = k) = k\theta(k)$. Let $(\widehat{N}_k)_{k \geq 1}$ be a sequence of independent copies of \widehat{N} defined under \mathbb{P} . Now we follow [43] and [70] to construct a size-biased Galton–Watson tree $\widehat{\mathbb{T}}$ defined under \mathbb{P} . First, the root \emptyset of $\widehat{\mathbb{T}}$ is given number \widehat{N}_1 of children. Choose one of these children uniformly at random, say \mathbf{v}_1 . It has a number \widehat{N}_2 of children, whereas the other children of the root have independently ordinary θ -Galton–Watson descendant trees. Again, among the children of \mathbf{v}_1 we choose one uniformly at random, call it \mathbf{v}_2 , and give the others independent θ -Galton–Watson descendant trees. Meanwhile the vertex \mathbf{v}_2 has a number \widehat{N}_3 of children. Since a.s. $\widehat{N} \geq 1$, we can repeat this procedure infinitely many times. The resulting random infinite tree $\widehat{\mathbb{T}}$ is called a size-biased Galton–Watson tree (see Figure 6.7). It is clear by the construction that $\widehat{\mathbb{T}}$ is a random element in \mathbb{T}_∞ and that its unique infinite line of descent is $(\mathbf{v}_1, \mathbf{v}_2, \dots)$, which we will call the spine of $\widehat{\mathbb{T}}$.

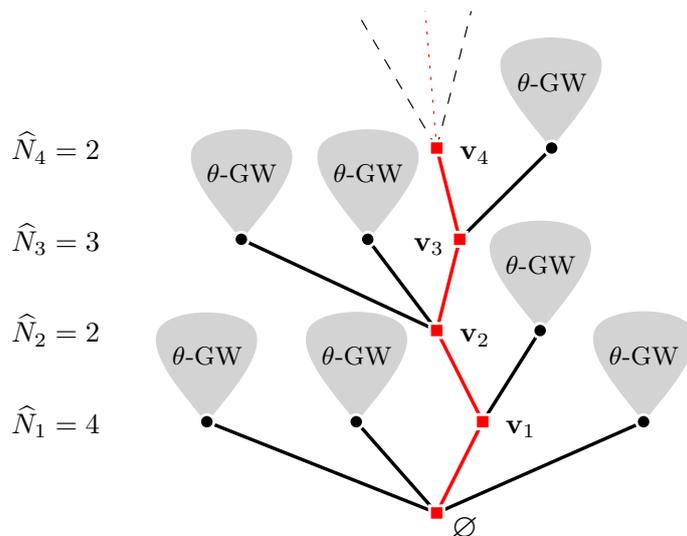


Figure 6.7: Schematic representation of a size-biased Galton–Watson tree $\widehat{\mathbb{T}}$

Let $[\widehat{\mathbb{T}}]^{(n)}$ be the plane tree obtained from $\widehat{\mathbb{T}}$ by keeping only its first n generations, i.e.,

$$[\widehat{\mathbb{T}}]^{(n)} := \{v \in \widehat{\mathbb{T}} : |v| \leq n\}.$$

It is shown in [43] and [70] that $[\widehat{\mathbb{T}}]^{(n)}$ is distributed according to the law of $\mathbb{T}^{(n)}$ size-biased by $\#\mathbb{T}_n^{(n)}$. Moreover, conditionally given the first n levels of $\widehat{\mathbb{T}}$, the vertex \mathbf{v}_n on the spine is uniformly distributed on the n -th level of $\widehat{\mathbb{T}}$. Besides, notice that

$$\mathbb{E}[\#\mathbb{T}_n^{(n)}] = \frac{\mathbb{E}[\#\mathbb{T}_n^{(0)}]}{q_n} = \frac{1}{q_n}.$$

All these observations are summarized in the following proposition.

Proposition 6.9. *Let $F(\mathcal{T}, v)$ be a nonnegative measurable function defined on $\mathbb{T}_f \times \mathcal{U}$. Then for every integer $n \geq 1$,*

$$\mathbb{E} \left[\sum_{v \in \mathbb{T}_n^{(n)}} F(\mathbb{T}^{(n)}, v) \right] = \frac{1}{q_n} \mathbb{E} [F([\widehat{\mathbb{T}}]^{(n)}, \mathbf{v}_n)].$$

For every integer $n \geq 1$, let $[\widehat{\mathbb{T}}]^n$ be the plane tree obtained from $\widehat{\mathbb{T}}$ by erasing the (infinite) tree of descendants of the vertex \mathbf{v}_n , and keeping all other vertices. By convention, the vertex \mathbf{v}_n is included in $[\widehat{\mathbb{T}}]^n$. Notice that in general $[\widehat{\mathbb{T}}]^n \neq [\widehat{\mathbb{T}}]^{(n)}$, since the height of $[\widehat{\mathbb{T}}]^n$ can be strictly greater than n .

At last, we let $[\widehat{\mathbb{T}}]^{*n}$ be the reduced tree associated with the plane tree $[\widehat{\mathbb{T}}]^n$ up to generation n , which consists of all vertices of $[\widehat{\mathbb{T}}]^n$ that have (at least) one descendant at generation n . We implicitly assume that the relabeling has been done to turn $[\widehat{\mathbb{T}}]^{*n}$ into a tree. It is easy to check that $[\widehat{\mathbb{T}}]^{*n}$ is also the reduced tree associated with $[\widehat{\mathbb{T}}]^{(n)}$ up to generation n .

6.3.2 Convergence of discrete reduced trees

We briefly recall in this section the result on the convergence of discrete reduced trees \mathbb{T}^{*n} . For every real number $s \in [0, n]$, we write the truncation of the tree \mathbb{T}^{*n} at level $n - \lfloor s \rfloor$ as

$$R_s(\mathbb{T}^{*n}) := \{v \in \mathbb{T}^{*n} : |v| \leq n - \lfloor s \rfloor\}.$$

For every $\varepsilon \in (0, 1)$, we have set $\Delta_\varepsilon = \{x \in \Delta : H(x) \leq 1 - \varepsilon\}$. We know that, for every fixed ε , there is a.s. no branching point of Δ at height $1 - \varepsilon$. The skeleton of Δ_ε is defined as the following plane tree

$$\text{Sk}(\Delta_\varepsilon) := \{\emptyset\} \cup \{v \in \mathcal{V} \setminus \{\emptyset\} : Y_{\bar{v}} \leq 1 - \varepsilon\} = \{\emptyset\} \cup \{v \in \mathcal{V} \setminus \{\emptyset\} : (\bar{v}, Y_{\bar{v}}) \in \Delta_\varepsilon\}.$$

Consider then the set $\mathbb{T}_{f, \text{bin}}$ of all plane trees in which every vertex has either 0, 1 or 2 children. For $\mathcal{T} \in \mathbb{T}_{f, \text{bin}}$ we write $\mathcal{S}(\mathcal{T})$ for the set of all vertices of \mathcal{T} having 0 or 2 children. Then there is a unique plane tree $\langle \mathcal{T} \rangle$ such that one can find a canonical bijection $u \mapsto w_u$ from $\langle \mathcal{T} \rangle$ onto $\mathcal{S}(\mathcal{T})$ that preserves the genealogical order and the lexicographical order of vertices.

The following result is Proposition 16 in [20].

Proposition 6.10. *We can construct the reduced trees \mathbb{T}^{*n} and the (continuous) tree Δ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that the following properties hold for every fixed $\varepsilon \in (0, 1)$ with \mathbb{P} -probability one.*

- (i) For every sufficiently large integer n , we have $R_{\varepsilon n}(\mathbb{T}^{*n}) \in \mathbb{T}_{f,bin}$ and $\langle R_{\varepsilon n}(\mathbb{T}^{*n}) \rangle = \text{Sk}(\Delta_\varepsilon)$.
- (ii) For every sufficiently large n , such that the properties stated in (i) hold, and for every $u \in \text{Sk}(\Delta_\varepsilon)$, let $w_u^{n,\varepsilon}$ denote the vertex of $\mathcal{S}(R_{\varepsilon n}(\mathbb{T}^{*n}))$ corresponding to u via the canonical bijection from $\langle R_{\varepsilon n}(\mathbb{T}^{*n}) \rangle$ onto $\mathcal{S}(R_{\varepsilon n}(\mathbb{T}^{*n}))$. Then we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |w_u^{n,\varepsilon}| = Y_u \wedge (1 - \varepsilon).$$

6.3.3 Convergence of discrete conductances

Let $\mathcal{T} \in \mathbb{T}$ be a tree of height larger than n , and consider the new graph \mathcal{T}' obtained by adding to the graph \mathcal{T} an edge between the root \emptyset and an extra vertex ∂ . We define as before a simple random walk X on \mathcal{T}' , starting from the root \emptyset , under the probability measure $P^{\mathcal{T}'}$. Let τ_∂ be the first hitting time of ∂ by X , and for every integer $1 \leq i \leq n$, let τ_i be the first hitting time of generation i (of the tree \mathcal{T}) by X . We write

$$\mathcal{C}_i(\mathcal{T}) := P^{\mathcal{T}'}(\tau_i < \tau_\partial).$$

This notation is justified by the fact that $\mathcal{C}_i(\mathcal{T})$ can be interpreted as the effective conductance between ∂ and generation i of \mathcal{T} in the graph \mathcal{T}' , see e.g. [73, Chapter 2].

Recall the notation that $\mathcal{C}(\Delta)$ stands for the conductance between the root and the set $\partial\Delta$ in the reduced tree Δ . Analogously, for every $\varepsilon \in (0, 1)$, $\mathcal{C}(\Delta_\varepsilon)$ denotes the conductance between the root and the set $\{x \in \Delta : H(x) = 1 - \varepsilon\}$ in Δ . The following proposition is stated in [20]. We provide here a detailed proof.

Proposition 6.11. *Suppose that the reduced trees \mathbb{T}^{*n} and the tree Δ are constructed so that the properties stated in Proposition 6.10 hold. Then*

$$n \mathcal{C}_n(\mathbb{T}^{*n}) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mathcal{C}(\Delta).$$

Proof. By definition, for every $\varepsilon \in (0, 1)$,

$$\mathcal{C}_{n - \lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}) = P^{(\mathbb{T}^{*n})'}(\tau_{n - \lfloor \varepsilon n \rfloor} < \tau_\partial) \geq P^{(\mathbb{T}^{*n})'}(\tau_n < \tau_\partial) = \mathcal{C}_n(\mathbb{T}^{*n}).$$

Note that there is probability at least $1 - \frac{\lfloor \varepsilon n \rfloor}{n+1}$ that, after hitting the generation $n - \lfloor \varepsilon n \rfloor$, the simple random walk on $(\mathbb{T}^{*n})'$ will hit the generation n before moving down to the extra vertex ∂ . Hence it follows from the strong Markov property of simple random walk that

$$0 \leq \mathcal{C}_{n - \lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}) - \mathcal{C}_n(\mathbb{T}^{*n}) \leq \frac{\lfloor \varepsilon n \rfloor}{n+1} \mathcal{C}_{n - \lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}).$$

By similar probabilistic arguments, we also have

$$0 \leq \mathcal{C}(\Delta_\varepsilon) - \mathcal{C}(\Delta) \leq \varepsilon \mathcal{C}(\Delta_\varepsilon),$$

which entails particularly that $\mathcal{C}(\Delta_\varepsilon) \leq 2\mathcal{C}(\Delta)$ if $\varepsilon < 1/2$.

Let n be sufficiently large so that assertions (i) and (ii) of Proposition 6.10 hold with $\varepsilon \in (0, \frac{1}{2})$. By calculating the conductances using the series law and parallel law, we see that a.s.

$$\lim_{n \rightarrow \infty} |n \mathcal{C}_{n - \lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}) - \mathcal{C}(\Delta_\varepsilon)| = 0.$$

Then it follows from

$$\begin{aligned} |nC_n(\mathbb{T}^{*n}) - \mathcal{C}(\Delta)| &\leq |nC_n(\mathbb{T}^{*n}) - n\mathcal{C}_{n-\lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n})| + |n\mathcal{C}_{n-\lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}) - \mathcal{C}(\Delta_\varepsilon)| + |\mathcal{C}(\Delta_\varepsilon) - \mathcal{C}(\Delta)| \\ &\leq \lfloor \varepsilon n \rfloor \mathcal{C}_{n-\lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}) + |n\mathcal{C}_{n-\lfloor \varepsilon n \rfloor}(\mathbb{T}^{*n}) - \mathcal{C}(\Delta_\varepsilon)| + \varepsilon \mathcal{C}(\Delta_\varepsilon) \end{aligned}$$

that

$$\limsup_{n \rightarrow \infty} |nC_n(\mathbb{T}^{*n}) - \mathcal{C}(\Delta)| \leq 2\varepsilon \mathcal{C}(\Delta_\varepsilon) \leq 4\varepsilon \mathcal{C}(\Delta).$$

Letting $\varepsilon \rightarrow 0$, we conclude that $|nC_n(\mathbb{T}^{*n}) - \mathcal{C}(\Delta)| \rightarrow 0$ as $n \rightarrow \infty$. \square

Similarly, we write $\mathcal{C}(\widehat{\Delta}_\varepsilon)$ for the conductance between the root and the set

$$\{x \in \widehat{\Delta} : H(x) = 1 - \varepsilon\}$$

in $\widehat{\Delta}$. By the same reasoning as in the previous proof, we point out that, for every $\varepsilon \in (0, 1/2)$,

$$0 \leq \mathcal{C}(\widehat{\Delta}_\varepsilon) - \mathcal{C}(\widehat{\Delta}) \leq 2\varepsilon \mathcal{C}(\widehat{\Delta}). \quad (6.22)$$

For future reference, we state the following result, which is Lemma 22 in [20].

Lemma 6.12. *There exists a constant $K \geq 1$ such that, for every integer $n \geq 1$,*

$$\mathbb{E} \left[(nC_n(\mathbb{T}^{*n}))^2 \right] \leq K.$$

6.3.4 Backward size-biased Galton–Watson tree

We introduce in this section a new infinite random tree $\check{\mathbb{T}}$, which is a variant of the size-biased Galton–Watson tree $\widehat{\mathbb{T}}$.

First, the random tree $\check{\mathbb{T}}$ has a unique infinite ray of vertices $(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \dots)$, which will be referred to as its spine. We declare that, for every $n \geq 0$, the vertex \mathbf{u}_n is at generation $-n$. This gives a genealogical order on the spine of $\check{\mathbb{T}}$: \mathbf{u}_1 is viewed as the parent of \mathbf{u}_0 , \mathbf{u}_2 is viewed as the parent of \mathbf{u}_1 , and so on.

Next, we describe the finite subtrees in $\check{\mathbb{T}}$ branching off every node of the spine. To this end, we recall that \widehat{N} follows the size-biased distribution of θ , and we denote by \mathbf{L} a random variable which, conditionally on \widehat{N} , is uniformly distributed on the set $\{0, 1, \dots, \widehat{N} - 1\}$. Let $(\mathbf{L}_n, \widehat{N}_n)_{n \geq 1}$ be a sequence of i.i.d. copies of the pair $(\mathbf{L}, \widehat{N})$, and set $\mathbf{R}_n = \widehat{N}_n - \mathbf{L}_n - 1$ for every $n \geq 1$. To every vertex \mathbf{u}_n we give a number \mathbf{L}_n of children to the left of the spine and a number \mathbf{R}_n of children to the right of the spine. Each of these children (there are $\widehat{N}_n - 1$ in total) will independently have an ordinary θ -Galton–Watson descendant tree (later we will say that these Galton–Watson trees are grafted at \mathbf{u}_n), and we also assume the independence of these Galton–Watson trees among all $n \geq 1$. This finishes the construction of $\check{\mathbb{T}}$. See Figure 6.8 for an illustration. We remark that $\check{\mathbb{T}}$ is not a tree in the sense of Section 6.3.1. However, due to its obvious tree structure, we will call $\check{\mathbb{T}}$ the backward size-biased Galton–Watson tree.

The genealogical (partial) order on $\check{\mathbb{T}}$ is defined in the following way. We simply keep the genealogical orders inherited from the grafted Galton–Watson trees and combine them with the genealogical order on the spine. For instance, \mathbf{u}_2 is an ancestor of any vertex in the subtrees grafted at \mathbf{u}_1 . We can also define in a consistent manner the notion of generation for every vertex in $\check{\mathbb{T}}$. In fact, for any vertex v not on the spine, there is a unique vertex \mathbf{u}_m on the spine such that v belongs to a finite subtree grafted at \mathbf{u}_m , then we say that the generation of v in $\check{\mathbb{T}}$ is equal to $-m + 1$ plus the initial generation of v inside the corresponding grafted plane tree.

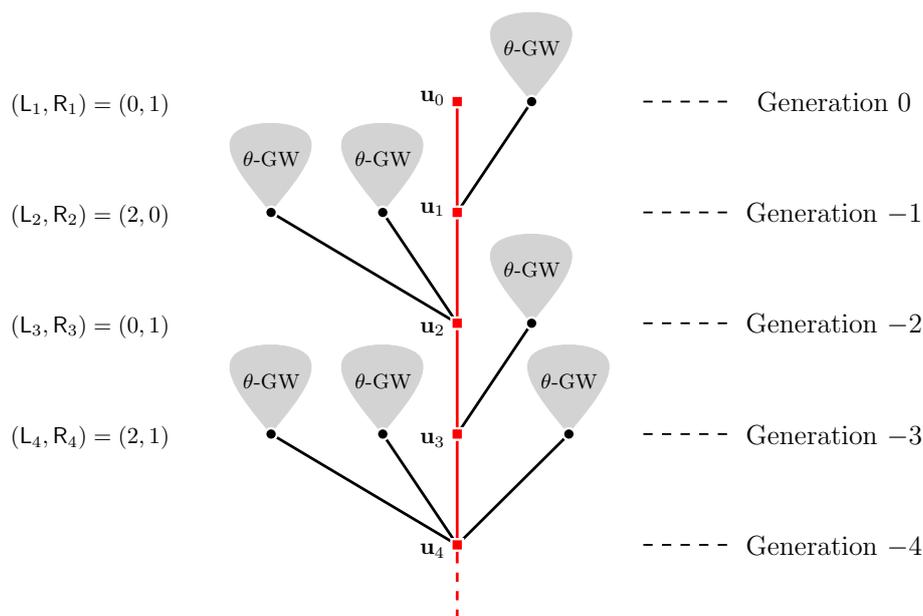


Figure 6.8: Schematic representation of the backward size-biased Galton–Watson tree $\check{\mathbb{T}}$

For every $n \geq 1$, let $[\check{\mathbb{T}}]^n$ be the plane tree obtained from $\check{\mathbb{T}}$ by only keeping the finite tree above the vertex \mathbf{u}_n . We take \mathbf{u}_n as the root of $[\check{\mathbb{T}}]^n$, and the lexicographical order on the set of vertices of $[\check{\mathbb{T}}]^n$ corresponds to the order of visit when one “moves around” the finite tree $[\check{\mathbb{T}}]^n$ in clockwise order, starting from the root \mathbf{u}_n . A key observation is that, viewed as a random plane tree, $[\check{\mathbb{T}}]^n$ has the same distribution as the random tree $[\hat{\mathbb{T}}]^n$ defined in Section 6.3.1. Moreover, the root \mathbf{u}_n of $[\check{\mathbb{T}}]^n$ corresponds to the root \emptyset of $[\hat{\mathbb{T}}]^n$, and the vertex \mathbf{u}_0 in $[\check{\mathbb{T}}]^n$ corresponds to the vertex \mathbf{v}_n in $[\hat{\mathbb{T}}]^n$.

6.4 Proof of Theorem 6.1

Recall that $\lambda = \mathbb{E}[\hat{\mathcal{C}}] - 1$ is the constant greater than 1 that appears in the convergence (6.4). Let $\delta > 0$. By applying Proposition 6.9 to the indicator function

$$F(\mathcal{T}, v) = \mathbf{1}\{n^{-\lambda-\delta} \leq P^{\mathcal{T}}(X_{\tau_n} = v) \leq n^{-\lambda+\delta}\}^c, \quad \text{for } \mathcal{T} \in \mathbb{T}_f \text{ and } v \in \mathcal{T},$$

we see that

$$\mathbb{E} \left[\sum_{v \in \mathbb{T}_n^{(n)}} \mathbf{1}\{n^{-\lambda-\delta} \leq P^{\mathbb{T}^{(n)}}(X_{\tau_n} = v) \leq n^{-\lambda+\delta}\}^c \right] = \frac{1}{q_n} \mathbb{P} \left(\{n^{-\lambda-\delta} \leq P^{[\hat{\mathbb{T}}]^{(n)}}(X_{\tau_n} = \mathbf{v}_n) \leq n^{-\lambda+\delta}\}^c \right).$$

Notice that in the left-hand side of the last display, $P^{\mathbb{T}^{(n)}}(X_{\tau_n} = v)$ is by definition the harmonic measure $\mu_n(v)$ at vertex v . In view of (6.2) and (6.21), the proof of convergence (6.1) is thus reduced to showing that for every $\delta > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(n^{-\lambda-\delta} \leq P^{[\hat{\mathbb{T}}]^{(n)}}(X_{\tau_n} = \mathbf{v}_n) \leq n^{-\lambda+\delta} \right) = 1. \tag{6.23}$$

Since the hitting distribution of generation n is the same for simple random walk on $[\widehat{\mathbb{T}}]^{(n)}$ and on its reduced tree $[\widehat{\mathbb{T}}]^{*n}$, we have the equality

$$P^{[\widehat{\mathbb{T}}]^{(n)}}(X_{\tau_n} = \mathbf{v}_n) = P^{[\widehat{\mathbb{T}}]^{*n}}(X_{\tau_n} = \mathbf{v}_n) = P^{[\widehat{\mathbb{T}}]^n}(X_{\tau_n} = \mathbf{v}_n)$$

under the probability measure \mathbb{P} . Furthermore, according to the final remark in Section 6.3.4, the (random) probability $P^{[\widehat{\mathbb{T}}]^n}(X_{\tau_n} = \mathbf{v}_n)$ is distributed under \mathbb{P} as

$$P^{[\check{\mathbb{T}}]^n}(X_{\tau_n} = \mathbf{u}_0),$$

by which we mean the probability that a simple random walk on $[\check{\mathbb{T}}]^n$ starting from the root \mathbf{u}_n hits level n (of $[\check{\mathbb{T}}]^n$) for the first time at \mathbf{u}_0 . So the convergence (6.23) is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(n^{-\lambda-\delta} \leq P^{[\check{\mathbb{T}}]^n}(X_{\tau_n} = \mathbf{u}_0) \leq n^{-\lambda+\delta}\right) = 1. \quad (6.24)$$

In order to show the latter convergence, we denote by $-M_1, -M_2, \dots$ the generations of the vertices on the spine of $\check{\mathbb{T}}$ where there is (at least) one grafted plane tree that reaches the generation 0, i.e. has a descendant of generation 0. This sequence of negative integers $(-M_k)_{k \geq 1}$ is listed in the strict decreasing order, and we set by convention $M_0 = 0$. For every $k \geq 1$, we also set $L_k := M_k - M_{k-1} \geq 1$.

For every $n \geq 1$, let $k_n := k_n(\check{\mathbb{T}})$ be the index such that $M_{k_n} \leq n < M_{k_n+1}$.

Lemma 6.13. *We have \mathbb{P} -a.s.*

$$\lim_{n \rightarrow \infty} \frac{k_n}{2 \log n} = 1.$$

Proof. Recall that for every $j \geq 1$, there are $\widehat{N}_j - 1$ Galton–Watson trees grafted at \mathbf{u}_j in $\check{\mathbb{T}}$. Consider the event that at least one of those plane trees grafted at \mathbf{u}_j reaches generation 0, and let ϵ_j be the corresponding indicator function. Then,

$$\mathbb{P}(\epsilon_j = 0) = \mathbb{E}\left[(1 - q_{j-1})^{\widehat{N}_j - 1}\right] = \mathbb{E}\left[(1 - q_{j-1})^{\widehat{N} - 1}\right].$$

Let g_θ be the generating function of θ , i.e.

$$g_\theta(r) := \sum_{k \geq 0} \theta(k) r^k, \quad 0 \leq r \leq 1.$$

Since θ has a finite variance σ^2 ,

$$g_\theta(1 - s) = 1 - s + \frac{\sigma^2}{2} s^2 + o(s^2) \quad \text{as } s \rightarrow 0.$$

As the mean of $\widehat{N} - 1$ is σ^2 , we have $\mathbb{E}[(1 - s)^{\widehat{N} - 1}] = 1 - \sigma^2 s + o(s)$ as $s \rightarrow 0$, which, together with (6.21), yields that

$$\mathbb{P}(\epsilon_j = 0) = 1 - \frac{2}{j} + o(j^{-1}) \quad \text{as } j \rightarrow \infty. \quad (6.25)$$

Notice that by definition, $k_n = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n$. Hence,

$$\mathbb{E}[k_n] = \sum_{j=1}^n (1 - \mathbb{P}(\epsilon_j = 0)) \sim 2 \log n \quad \text{as } n \rightarrow \infty.$$

Since $\epsilon_1, \dots, \epsilon_n$ are independent, and we have also $\text{var}(k_n) = O(\log n)$, the L^2 -convergence of $k_n/(2 \log n)$ follows immediately. The a.s. convergence is then obtained by standard monotonicity and Borel-Cantelli arguments. \square

We introduce some additional notation before stating the next proposition. For every $j \geq 0$, we write $P_j^{\check{\Gamma}}$ for the (quenched) probability measure under which we consider a simple random walk $X = (X_k)_{k \geq 0}$ on $\check{\Gamma}$ starting from the vertex \mathbf{u}_j . Under $P_j^{\check{\Gamma}}$, we denote by S_0 the hitting time of generation 0 by the simple random walk X , and for every $i \geq 0$, $\Pi_i := \inf\{k \geq 0: X_k = \mathbf{u}_i\}$ denotes the hitting time of vertex \mathbf{u}_i .

Proposition 6.14. *For every $\delta > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have*

$$\mathbb{P}\left(P^{[\check{\Gamma}]^n}(X_{\tau_n} = \mathbf{u}_0) \geq n^{-\lambda+\delta}\right) \leq 8\mathbb{P}\left(P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \geq n^{-\lambda+\delta}/2\right),$$

and

$$\mathbb{P}\left(P^{[\check{\Gamma}]^n}(X_{\tau_n} = \mathbf{u}_0) \leq n^{-\lambda-\delta}\right) \leq 8\mathbb{P}\left(P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \leq n^{-\lambda-\delta}\right).$$

Proof. We keep the notation used in the proof of Lemma 6.13, and observe that

$$\begin{aligned} \mathbb{P}(M_{k_n+1} - n > n) &= \mathbb{P}(\epsilon_{n+j} = 0 \text{ for all } 1 \leq j \leq n) \\ &= \prod_{j=1}^n \mathbb{P}(\epsilon_{n+j} = 0) \xrightarrow{n \rightarrow \infty} \frac{1}{4} \end{aligned}$$

by (6.25). One can thus find an integer n_0 such that for every $n \geq n_0$,

$$\mathbb{P}(M_{k_n+1} > 2n) \geq \frac{1}{8}. \quad (6.26)$$

Since $M_{k_n+1} - n$ is independent of the finite tree above the vertex \mathbf{u}_n in $\check{\Gamma}$,

$$\mathbb{P}(M_{k_n+1} > 2n, P^{[\check{\Gamma}]^n}(X_{\tau_n} = \mathbf{u}_0) \geq n^{-\lambda+\delta}) = \mathbb{P}(M_{k_n+1} > 2n) \times \mathbb{P}(P^{[\check{\Gamma}]^n}(X_{\tau_n} = \mathbf{u}_0) \geq n^{-\lambda+\delta}). \quad (6.27)$$

On the other hand, it is crucial to note that under the probability measure \mathbb{P} , the probability $P^{[\check{\Gamma}]^n}(X_{\tau_n} = \mathbf{u}_0)$ has the same distribution as the conditional probability

$$P_n^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0 \mid S_0 < \Pi_{M_{k_n+1}}),$$

which can be calculated as

$$\begin{aligned} P_n^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0 \mid S_0 < \Pi_{M_{k_n+1}}) &= \frac{P_n^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}})}{P_n^{\check{\Gamma}}(S_0 < \Pi_{M_{k_n+1}})} \\ &= \frac{P_n^{\check{\Gamma}}(\Pi_{M_{k_n}} < \Pi_{M_{k_n+1}}) \times P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}})}{P_n^{\check{\Gamma}}(S_0 < \Pi_{M_{k_n+1}})} \end{aligned}$$

by the strong Markov property of the random walk. On the event $\{M_{k_n+1} > 2n\}$, simple considerations show that $P_n^{\check{\Gamma}}(S_0 < \Pi_{M_{k_n+1}}) \geq 1/2$. Hence, we have

$$\begin{aligned} \mathbb{P}(M_{k_n+1} > 2n, P^{[\check{\Gamma}]^n}(X_{\tau_n} = \mathbf{u}_0) \geq n^{-\lambda+\delta}) &= \mathbb{P}(M_{k_n+1} > 2n, P_n^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0 \mid S_0 < \Pi_{M_{k_n+1}}) \geq n^{-\lambda+\delta}) \\ &\leq \mathbb{P}(M_{k_n+1} > 2n, P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \geq n^{-\lambda+\delta}/2) \\ &\leq \mathbb{P}(P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \geq n^{-\lambda+\delta}/2). \end{aligned}$$

The last display, together with (6.26) and (6.27), yields the first inequality in the statement of the proposition.

We can argue in a similar manner for the second inequality stated in the proposition. Its proof is even simpler because it suffices to use the bound

$$P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \leq P_n^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0 | S_0 < \Pi_{M_{k_n+1}}),$$

instead of the estimate $P_n^{\check{\Gamma}}(S_0 < \Pi_{M_{k_n+1}}) \geq 1/2$ used above. \square

According to (6.24) and the preceding result, we can therefore derive Theorem 6.1 from the following proposition.

Proposition 6.15. *For every $\delta > 0$, it holds that*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(n^{-\lambda-\delta} \leq P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \leq n^{-\lambda+\delta}\right) = 1. \quad (6.28)$$

6.4.1 Proof of Proposition 6.15

Under the probability measure \mathbb{P} , for every $k \geq 1$, we set

$$p_k = p_k(\check{\Gamma}) := P_{M_k}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k+1}}).$$

By the definition of M_k , there exists at least one plane tree grafted to \mathbf{u}_{M_k} that reaches generation 0. The root of this subtree is necessarily a child of \mathbf{u}_{M_k} distinct from \mathbf{u}_{M_k-1} . If such a subtree is unique, we let $c_k = c_k(\check{\Gamma})$ be the probability that a simple random walk starting from its root reaches generation 0 before visiting \mathbf{u}_{M_k} . If there is more than one such grafted trees, we take c_k to be the sum of the corresponding probabilities. This definition is justified by the fact that c_k can be interpreted as the effective conductance between \mathbf{u}_{M_k} and generation 0 in the graph that consists only of the vertex \mathbf{u}_{M_k} and all the subtrees grafted to it.

We also set, for every $k \geq 1$,

$$h_k = h_k(\check{\Gamma}) := P_{M_{k-1}}^{\check{\Gamma}}(S_0 < \Pi_{M_k}),$$

which is the probability that a simple random walk starting from $\mathbf{u}_{M_{k-1}}$ reaches generation 0 before visiting \mathbf{u}_{M_k} . With the notation of the beginning of Section 6.3.3, it is clear that

$$h_k = \mathcal{C}_{M_{k-1}}([\check{\Gamma}]^{M_k-1}).$$

We write $\ell_k = 1/L_k = (M_k - M_{k-1})^{-1}$ for all $k \geq 1$. Then simple considerations show that

$$p_1 = \frac{\ell_1}{\ell_1 + c_1 + \ell_2},$$

and, for all $k \geq 2$,

$$p_k = \frac{\ell_k}{\ell_k + c_k + \ell_{k+1}} \left(p_{k-1} + \frac{\ell_k}{\ell_k + c_{k-1} + h_{k-1}} p_k \right). \quad (6.29)$$

To establish the last formula, we consider the excursions of simple random walk outside of vertex \mathbf{u}_{M_k} , which are independent of the same law. Under this excursion law, the random walk makes its first jump with equal probability towards one of its neighbors, which are $\mathbf{u}_{M_{k-1}}$, $\mathbf{u}_{M_{k+1}}$ and

the children of \mathbf{u}_{M_k} distinct from $\mathbf{u}_{M_{k-1}}$. The respective probabilities for an excursion to visit $\mathbf{u}_{M_{k-1}}$, to visit $\mathbf{u}_{M_{k+1}}$, and to reach generation 0 in one of the subtrees grafted at \mathbf{u}_{M_k} , are proportional respectively to $1/L_k = \ell_k$, to $1/L_{k+1} = \ell_{k+1}$ and to c_k . So the probability for the random walk starting from \mathbf{u}_{M_k} to visit $\mathbf{u}_{M_{k-1}}$ before hitting $\mathbf{u}_{M_{k+1}}$ or reaching generation 0 is

$$\frac{\ell_k}{\ell_k + \ell_{k+1} + c_k}.$$

Next, conditionally on the latter event, the strong Markov property leads us to consider a simple random walk that starts from $\mathbf{u}_{M_{k-1}}$. With probability p_{k-1} it reaches generation 0 by hitting the vertex \mathbf{u}_0 before moving down to \mathbf{u}_{M_k} . However, we must also add the probability that this random walk goes back down to \mathbf{u}_{M_k} before reaching generation 0, which is equal to

$$\frac{\ell_k}{\ell_k + c_{k-1} + h_{k-1}},$$

multiplied by the probability p_k that once returning to \mathbf{u}_{M_k} the random walk will eventually hit generation 0 at \mathbf{u}_0 before moving down to $\mathbf{u}_{M_{k+1}}$.

We derive from (6.29) that

$$p_{k-1} = p_k \left(\frac{\ell_k + c_k + \ell_{k+1}}{\ell_k} - \frac{\ell_k}{\ell_k + c_{k-1} + h_{k-1}} \right),$$

from which it follows that

$$p_1 = p_k \times \prod_{j=2}^k \left(1 + \frac{c_j + \ell_{j+1}}{\ell_j} - \frac{\ell_j}{\ell_j + c_{j-1} + h_{j-1}} \right).$$

We define thus, for every $j \geq 2$,

$$Q_j = Q_j(\check{\Gamma}) := \log \left(1 + \frac{c_j + \ell_{j+1}}{\ell_j} - \frac{\ell_j}{\ell_j + c_{j-1} + h_{j-1}} \right).$$

Lemma 6.16. *We have*

$$\frac{1}{k} \sum_{j=2}^k Q_j \xrightarrow[k \rightarrow \infty]{L^2(\mathbb{P})} \frac{\lambda}{2}. \quad (6.30)$$

The proof of this key lemma is postponed to the next section. Now for any $\delta > 0$, let us consider the event

$$\left\{ (\lambda - \delta) \log n \leq \sum_{j=2}^{k_n} Q_j \leq (\lambda + \delta) \log n \right\}.$$

Using Lemma 6.13 and Lemma 6.16, we see that the last event holds with \mathbb{P} -probability tending to 1 as $n \rightarrow \infty$. As

$$p_{k_n} = p_1 \exp \left(- \sum_{j=2}^{k_n} Q_j \right),$$

we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(p_1 n^{-\lambda-\delta} \leq p_{k_n} \leq p_1 n^{-\lambda+\delta} \right) = 1.$$

Recalling the definition of p_k , we conclude that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(p_1 n^{-\lambda-\delta} \leq P_{M_{k_n}}^{\check{\Gamma}}(X_{S_0} = \mathbf{u}_0, S_0 < \Pi_{M_{k_n+1}}) \leq p_1 n^{-\lambda+\delta} \right) = 1.$$

Since δ is arbitrary, the required convergence (6.28) readily follows from the last display, and this completes the proof of Proposition 6.15 and Theorem 6.1. It remains to prove Lemma 6.16.

6.4.2 Proof of Lemma 6.16

From the definition of Q_k , we can write for every $k \geq 2$

$$Q_k = \log \left(1 + \frac{M_k c_k + \frac{M_k}{L_{k+1}}}{\frac{M_k}{L_k}} - \frac{\frac{M_k}{L_k}}{\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}}(M_{k-1}c_{k-1} + M_{k-1}h_{k-1})} \right).$$

Lemma 6.17. *We have*

$$\left(\frac{L_{k+1}}{M_k}, \frac{L_k}{M_{k-1}}, M_k c_k, M_{k-1} c_{k-1}, M_{k-1} h_{k-1} \right) \xrightarrow[k \rightarrow \infty]{(d)} (\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}', \widehat{\mathcal{C}}),$$

where in the limit

– \mathcal{R} and \mathcal{R}' are two positive random variables with the same distribution given by

$$\mathbb{P}(\mathcal{R} > x) = (1+x)^{-2} \text{ for all } x \geq 0;$$

– \mathcal{C} and \mathcal{C}' are distributed according to the law γ ;

– $\widehat{\mathcal{C}}$ is distributed according to the law $\widehat{\gamma}$.

Furthermore, we suppose that $\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}', \widehat{\mathcal{C}}$ are all defined under the probability measure \mathbb{P} , and they are independent.

Proof. We first observe that $(M_{k-1}, L_k)_{k \geq 1}$ is a Markov chain on $\mathbb{Z}_+ \times \mathbb{N}$ whose initial distribution and transition probabilities are given as follows. Initially $M_0 = 0$ and for every integer $\ell \geq 1$,

$$\mathbb{P}(L_1 > \ell) = \prod_{j=1}^{\ell} \mathbb{P}(\epsilon_j = 0),$$

where $\epsilon_j = 0$ means as previously that none of the Galton–Watson trees grafted at \mathbf{u}_j reaches generation 0. Then for every $k \geq 1$, $M_k = M_{k-1} + L_k$ a.s. and conditionally on $\{M_k = m\}$,

$$\mathbb{P}(L_{k+1} > \ell \mid M_k = m) = \prod_{j=1}^{\ell} \mathbb{P}(\epsilon_{m+j} = 0) =: F(m, \ell), \quad \text{for every } \ell \geq 1. \quad (6.31)$$

Using (6.25), it is elementary to verify that for every $x > 0$,

$$F(m, \lfloor xm \rfloor) \xrightarrow{m \rightarrow \infty} \frac{1}{(1+x)^2}.$$

For every $k \geq 1$, let $\mathcal{F}_k = \sigma(L_1, L_2, \dots, L_k)$ be the σ -field generated by $(L_i, 1 \leq i \leq k)$, so that $(\mathcal{F}_k)_{k \geq 1}$ is the natural filtration associated with the Markov chain $(M_{k-1}, L_k)_{k \geq 1}$. As $M_k \geq k$, it is clear that $M_k \rightarrow \infty$ as $k \rightarrow \infty$. By dominated convergence and the last display, we get that for every $x, y > 0$,

$$\begin{aligned} \mathbb{P}\left(\frac{L_k}{M_{k-1}} > x, \frac{L_{k+1}}{M_k} > y\right) &= \mathbb{E}\left[\mathbf{1}\left\{\frac{L_k}{M_{k-1}} > x\right\} \mathbb{P}\left(\frac{L_{k+1}}{M_k} > y \mid \mathcal{F}_k\right)\right] \\ &= \mathbb{E}\left[\mathbf{1}\left\{\frac{L_k}{M_{k-1}} > x\right\} \mathbb{P}\left(\frac{L_{k+1}}{M_k} > y \mid M_k\right)\right] \\ &\xrightarrow{k \rightarrow \infty} \frac{1}{(1+x)^2(1+y)^2}, \end{aligned}$$

which entails that

$$\left(\frac{L_k}{M_{k-1}}, \frac{L_{k+1}}{M_k} \right) \xrightarrow[k \rightarrow \infty]{(d)} (\mathcal{R}', \mathcal{R}). \tag{6.32}$$

Note that conditionally on M_k and on the number of subtrees grafted at \mathbf{u}_k that hit generation 0, these subtrees are Galton–Watson trees conditioned to have height greater than $M_k - 1$. Furthermore, the property $\mathbb{E}[\widehat{N}] = \sum k^2 \theta(k) < \infty$ and the estimate (6.21) entail that a.s. for all sufficiently large k , there is a unique subtree grafted at \mathbf{u}_{M_k} that reaches generation 0. Hence by Proposition 6.11, we obtain the convergence

$$(M_{k-1}c_{k-1}, M_k c_k) \xrightarrow[k \rightarrow \infty]{(d)} (\mathcal{C}', \mathcal{C}), \tag{6.33}$$

which holds jointly with (6.32), provided we let $(\mathcal{C}', \mathcal{C})$ be independent of $(\mathcal{R}', \mathcal{R})$.

Let $J \geq 2$ be a fixed integer. We can generalize the preceding arguments to show that the $2J$ -tuple

$$\left(\frac{L_{k-1}}{M_{k-2}}, \frac{L_{k-2}}{M_{k-3}}, \dots, \frac{L_{k-J}}{M_{k-J-1}}, M_{k-2}c_{k-2}, M_{k-3}c_{k-3}, \dots, M_{k-J-1}c_{k-J-1} \right)$$

converges in distribution as $k \rightarrow \infty$ to $(\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_J, \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_J)$. These random variables appearing in the limit are all independent, and $(\mathcal{R}_j)_{1 \leq j \leq J}$, respectively $(\mathcal{C}_j)_{1 \leq j \leq J}$, have the same distribution as \mathcal{R} , resp. as \mathcal{C} . If we set $V_j = \frac{\mathcal{R}_j}{1+\mathcal{R}_j}$ for all $1 \leq j \leq J$, then $(V_j)_{1 \leq j \leq J}$ are i.i.d. with the same law of density $2(1-x)$ on $[0, 1]$, and the previous convergence can be reformulated as

$$\left(\frac{L_{k-j}}{M_{k-j}}, M_{k-j-1}c_{k-j-1} \right)_{1 \leq j \leq J} \xrightarrow[k \rightarrow \infty]{(d)} (V_j, \mathcal{C}_j)_{1 \leq j \leq J}. \tag{6.34}$$

For all integers $k \geq 2$ and $1 \leq j < k$, we define

$$h_k^{(j)} = h_k^{(j)}(\check{\Upsilon}) := P_{M_{k-1}}^{\check{\Upsilon}}(S_0 < \Pi_{M_k} \wedge \Pi_{M_{k-j-1}}),$$

which is the probability that a simple random walk starting from $\mathbf{u}_{M_{k-1}}$ reaches generation 0 before visiting \mathbf{u}_{M_k} or $\mathbf{u}_{M_{k-j-1}}$. We set by convention $h_k^{(0)} = 0$. From the interpretation of $h_{k-1}^{(J)}$ as a conductance, we obtain the formula

$$h_{k-1}^{(J)} = \left(L_{k-1} + (c_{k-2} + h_{k-2}^{(J-1)})^{-1} \right)^{-1}$$

by the series law and parallel law. It follows that

$$M_{k-1}h_{k-1}^{(J)} = \left(\frac{L_{k-1}}{M_{k-1}} + \frac{1 - \frac{L_{k-1}}{M_{k-1}}}{M_{k-2}c_{k-2} + M_{k-2}h_{k-2}^{(J-1)}} \right)^{-1}.$$

The same calculation can be repeated for $M_{k-2}h_{k-2}^{(J-1)}, M_{k-3}h_{k-3}^{(J-2)}$, etc. By using (6.34) and the fact that $h_{k-J-1}^{(0)} = 0$, we see that the law of $M_{k-1}h_{k-1}^{(J)}$ converges weakly to $\widehat{\Phi}^J(\delta_0)$ as $k \rightarrow \infty$, where $\widehat{\Phi}$ is the mapping defined in (6.11). Moreover, according to assertion (1) in Proposition 6.7, $\widehat{\Phi}^J(\delta_0) \rightarrow \widehat{\gamma} = \text{Law}(\widehat{\mathcal{C}})$ weakly as $J \rightarrow \infty$.

On the other hand,

$$h_k - h_k^{(J)} = P_{M_{k-1}}^{\check{\Upsilon}}(\Pi_{M_{k-J-1}} < S_0 < \Pi_{M_k}) \geq 0.$$

Note that there is probability at least $1 - \frac{M_{k-J-1}}{M_k}$ that, after hitting the vertex $\mathbf{u}_{M_{k-J-1}}$, the simple random walk on $\check{\mathbb{T}}$ will reach generation 0 before hitting \mathbf{u}_{M_k} . Hence, by the strong Markov property of simple random walk, we have

$$h_k - h_k^{(J)} \leq P_{M_{k-1}}^{\check{\mathbb{T}}}(\Pi_{M_{k-J-1}} < S_0 \wedge \Pi_{M_k}) \times \frac{M_{k-J-1}}{M_k}.$$

Since by similar reasoning,

$$P_{M_{k-1}}^{\check{\mathbb{T}}}(\Pi_{M_{k-J-1}} < S_0 \wedge \Pi_{M_k}) \leq \frac{1}{M_k - M_{k-J} + 1},$$

it follows that

$$M_k h_k - M_k h_k^{(J)} \leq \frac{M_{k-J-1}}{M_k - M_{k-J} + 1}.$$

Thus, for any $\eta > 0$,

$$\mathbb{P}(|M_{k-1} h_{k-1} - M_{k-1} h_{k-1}^{(J)}| \geq \eta) \leq \mathbb{P}\left(\frac{M_{k-J-1} - 1}{M_{k-1} - M_{k-J-1} + 1} \geq \eta\right).$$

But due to the previous discussions, it is clear that

$$\lim_{J \rightarrow \infty} \limsup_{k \rightarrow \infty} \mathbb{P}\left(\frac{M_{k-J-1} - 1}{M_{k-1} - M_{k-J-1} + 1} \geq \eta\right) = 0.$$

So we obtain, for any $\eta > 0$, that

$$\lim_{J \rightarrow \infty} \limsup_{k \rightarrow \infty} \mathbb{P}(|M_{k-1} h_{k-1} - M_{k-1} h_{k-1}^{(J)}| \geq \eta) = 0.$$

Finally, by applying [11, Theorem 3.2] we get that

$$M_{k-1} h_{k-1} \xrightarrow[k \rightarrow \infty]{(d)} \widehat{\mathcal{C}}. \quad (6.35)$$

Notice that h_{k-1} only depends on M_{k-1} and the finite tree strictly above the vertex $\mathbf{u}_{M_{k-1}}$ in $\check{\mathbb{T}}$. So conditionally on M_{k-1} , the latter quantity h_{k-1} is independent of $(L_k, L_{k+1}, c_{k-1}, c_k)$. In consequence, the convergence (6.35) holds jointly with (6.32) and (6.33), if we take $\widehat{\mathcal{C}}$ to be independent of $(\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}')$. The proof of Lemma 6.17 is therefore complete. \square

Lemma 6.18. *It holds that*

$$\sup_{k \geq 1} \mathbb{E}[(M_k h_k)^2] < \infty.$$

Proof. From the interpretation of h_k as a conductance, we know by the series law and parallel law that

$$h_k = \left(L_k + (c_{k-1} + h_{k-1})^{-1}\right)^{-1}.$$

It follows that

$$M_k h_k = \frac{M_{k-1} c_{k-1} + M_{k-1} h_{k-1}}{\frac{L_k}{M_k} (M_{k-1} c_{k-1} + M_{k-1} h_{k-1}) + 1 - \frac{L_k}{M_k}}.$$

From the interpretation of c_{k-1} and h_{k-1} as conductances, it is easy to see that $M_{k-1}c_{k-1} \geq 1$ and $M_{k-1}h_{k-1} \geq 1$. Hence,

$$M_k h_k \leq \frac{M_{k-1}c_{k-1} + M_{k-1}h_{k-1}}{\frac{2L_k}{M_k} + 1 - \frac{L_k}{M_k}} \leq M_{k-1}c_{k-1} + \frac{M_{k-1}h_{k-1}}{1 + \frac{L_k}{M_k}}.$$

For any $\eta > 0$, one can take $C(\eta) > 1 + \frac{1}{\eta}$ so that $(a+b)^2 \leq C(\eta)a^2 + (1+\eta)b^2$ for every $a, b > 0$. Applying this inequality to the last display, we obtain

$$\mathbb{E}[(M_k h_k)^2] \leq C(\eta)\mathbb{E}[(M_{k-1}c_{k-1})^2] + (1+\eta)\mathbb{E}\left[\left(\frac{M_{k-1}h_{k-1}}{1 + \frac{L_k}{M_k}}\right)^2\right]. \quad (6.36)$$

Notice that

$$\mathbb{E}\left[\left(\frac{M_{k-1}h_{k-1}}{1 + \frac{L_k}{M_k}}\right)^2\right] = \mathbb{E}\left[\sum_{\ell=1}^{\infty} \mathbb{P}(L_k = \ell | M_{k-1}) \frac{(M_{k-1}h_{k-1})^2}{(1 + \frac{\ell}{M_{k-1} + \ell})^2}\right].$$

According to (6.31), there exists a constant $c > 0$ such that, a.s. for all integers $k \geq 2$, we have $\mathbb{P}(L_k \geq M_{k-1} | M_{k-1}) \geq c$. Thus,

$$\begin{aligned} & \sum_{\ell \geq 1} \mathbb{P}(L_k = \ell | M_{k-1}) \frac{(M_{k-1}h_{k-1})^2}{(1 + \frac{\ell}{M_{k-1} + \ell})^2} \\ & \leq \left(\frac{2}{3}\right)^2 \sum_{\ell \geq M_{k-1}} \mathbb{P}(L_k = \ell | M_{k-1}) (M_{k-1}h_{k-1})^2 + \sum_{\ell < M_{k-1}} \mathbb{P}(L_k = \ell | M_{k-1}) (M_{k-1}h_{k-1})^2 \\ & \leq \left(1 - \frac{5}{9}c\right) (M_{k-1}h_{k-1})^2, \end{aligned}$$

and it follows that

$$\mathbb{E}\left[\left(\frac{M_{k-1}h_{k-1}}{1 + \frac{L_k}{M_k}}\right)^2\right] \leq \left(1 - \frac{5}{9}c\right) \mathbb{E}[(M_{k-1}h_{k-1})^2].$$

Together with (6.36), the last display entails that

$$\mathbb{E}[(M_k h_k)^2] \leq C(\eta)\mathbb{E}[(M_{k-1}c_{k-1})^2] + (1+\eta)\left(1 - \frac{5}{9}c\right)\mathbb{E}[(M_{k-1}h_{k-1})^2].$$

Recall that by Lemma 6.12, $\mathbb{E}[(M_{k-1}c_{k-1})^2]$ is uniformly bounded with respect to k . So by choosing η sufficiently small so that $(1+\eta)\left(1 - \frac{5}{9}c\right) < 1$, we see that there exist positive constants $C < \infty$ and $\rho < 1$, both independent of k , such that for all $k \geq 2$,

$$\mathbb{E}[(M_k h_k)^2] \leq C + \rho \mathbb{E}[(M_{k-1}h_{k-1})^2].$$

The sequence $(\mathbb{E}[(M_k h_k)^2])_{k \geq 1}$ is therefore bounded. \square

With the notation of Lemma 6.17, we set

$$Q_\infty := \log \left(1 + \frac{C + \frac{1}{\mathcal{R}}}{\frac{1+\mathcal{R}'}{\mathcal{R}'}} - \frac{1}{1 + \mathcal{R}'(C' + \widehat{C})} \right).$$

Lemma 6.19. (i) We have $\lim_{k \rightarrow \infty} \mathbb{E}[Q_k] = \mathbb{E}[Q_\infty]$.

(ii) We have the equality $\mathbb{E}[Q_\infty] = \lambda/2$.

(iii) It holds that

$$\sup_{i,j \geq 1} \mathbb{E}[|Q_i Q_j|] < \infty \quad \text{and} \quad \sup_{i,j \geq 1} \mathbb{E}[(Q_i Q_j)^2] < \infty.$$

Proof. (i) On the one hand, since $\frac{M_k}{L_k} > 1$,

$$Q_k \leq \log \left(1 + \frac{M_k c_k + \frac{M_k}{L_{k+1}}}{\frac{M_k}{L_k}} \right) \leq \log \left(1 + M_k c_k + \frac{M_k}{L_{k+1}} \right).$$

On the other hand,

$$\begin{aligned} Q_k &\geq \log \left(1 - \frac{\frac{M_k}{L_k}}{\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}}(M_{k-1}c_{k-1} + M_{k-1}h_{k-1})} \right) \\ &= -\log \left(1 + \frac{\frac{M_k}{L_k}}{\frac{M_k}{M_{k-1}}(M_{k-1}c_{k-1} + M_{k-1}h_{k-1})} \right) = -\log \left(1 + \frac{\frac{M_{k-1}}{L_k}}{M_{k-1}c_{k-1} + M_{k-1}h_{k-1}} \right). \end{aligned}$$

Using the facts that $M_{k-1}c_{k-1} \geq 1$ and $M_{k-1}h_{k-1} \geq 1$, we arrive at

$$|Q_k| \leq \max \left\{ \log \left(1 + M_k c_k + \frac{M_k}{L_{k+1}} \right), \log \left(1 + \frac{M_{k-1}}{2L_k} \right) \right\}.$$

One can find $A > 0$ such that $\log(1+x) \leq A + x^{1/2}$ for every $x > 0$. It follows that

$$|Q_k| \leq A + \left(M_k c_k + \frac{M_k}{L_{k+1}} \right)^{\frac{1}{2}} + \left(\frac{M_{k-1}}{L_k} \right)^{\frac{1}{2}} \leq A + (M_k c_k)^{\frac{1}{2}} + \left(\frac{M_k}{L_{k+1}} \right)^{\frac{1}{2}} + \left(\frac{M_{k-1}}{L_k} \right)^{\frac{1}{2}}. \quad (6.37)$$

Recall the convergence in distribution $M_k c_k \xrightarrow{(d)} \mathcal{C}$ shown in Lemma 6.17. By Lemma 6.12, it follows that

$$\mathbb{E}[M_k c_k] \xrightarrow[k \rightarrow \infty]{} \mathbb{E}[\mathcal{C}]. \quad (6.38)$$

In particular, $\sup_k \mathbb{E}[M_k c_k] < \infty$. Meanwhile, using (6.25) and (6.31), it is not difficult to verify that there exists a positive constant K such that for every $x > 0$,

$$\sup_{k \geq 1} \mathbb{P} \left(\frac{M_k}{L_{k+1}} > x \right) \leq \frac{K}{1+x}.$$

So using the formula

$$\mathbb{E} \left[\left(\frac{M_k}{L_{k+1}} \right)^\alpha \right] = \int_0^\infty \alpha x^{\alpha-1} \mathbb{P} \left(\frac{M_k}{L_{k+1}} > x \right) dx,$$

we get the existence of a constant $\alpha \in (\frac{1}{2}, 1)$ such that

$$\sup_{k \geq 1} \mathbb{E} \left[\left(\frac{M_k}{L_{k+1}} \right)^\alpha \right] < \infty. \quad (6.39)$$

Hence, it follows from (6.37) that $(Q_k)_{k \geq 2}$ is bounded in L^p with some $p > 1$. The sequence $(Q_k)_{k \geq 2}$ is thus uniformly integrable. However, according to Lemma 6.17, Q_k converges in distribution to Q_∞ . Therefore, $Q_k \rightarrow Q_\infty$ in L^1 and we have

$$\lim_{k \rightarrow \infty} \mathbb{E}[Q_k] = \mathbb{E}[Q_\infty].$$

(ii) Recall that $V = \frac{\mathcal{R}}{1+\mathcal{R}}$ and $V' = \frac{\mathcal{R}'}{1+\mathcal{R}'}$ are independent with the same law of density $2(1-x)$ on $[0, 1]$. Noting that

$$\begin{aligned} \log\left(1 + \frac{\mathcal{C} + \frac{1}{\mathcal{R}}}{\frac{1+\mathcal{R}'}{\mathcal{R}'}} - \frac{1}{1 + \mathcal{R}'(\mathcal{C}' + \widehat{\mathcal{C}})}\right) &= \log\left(\left(\mathcal{C} + \frac{1}{\mathcal{R}}\right)V' + \frac{V'(\mathcal{C}' + \widehat{\mathcal{C}})}{1 - V' + V'(\mathcal{C}' + \widehat{\mathcal{C}})}\right) \\ &= \log V' + \log\left(\mathcal{C} + \frac{1}{\mathcal{R}} + \left(V' + \frac{1 - V'}{\mathcal{C}' + \widehat{\mathcal{C}}}\right)^{-1}\right), \end{aligned}$$

we can use the distributional identity (6.6) to obtain

$$\mathbb{E}[Q_\infty] = \mathbb{E}[\log V'] + \mathbb{E}\left[\log\left(\mathcal{C} + \frac{1}{\mathcal{R}} + \widehat{\mathcal{C}}\right)\right].$$

Since $\mathcal{R} = \frac{V}{1-V}$, it follows that

$$\log\left(\mathcal{C} + \frac{1}{\mathcal{R}} + \widehat{\mathcal{C}}\right) = \log(1 - V + V(\mathcal{C} + \widehat{\mathcal{C}})) - \log V,$$

which yields $\mathbb{E}[Q_\infty] = \mathbb{E}[\log(1 - V + V(\mathcal{C} + \widehat{\mathcal{C}}))]$. To complete the proof of assertion (ii), we apply (6.6) again to see that

$$\mathbb{E}[\log(1 - V + V(\mathcal{C} + \widehat{\mathcal{C}}))] = \mathbb{E}[\log(\widehat{\mathcal{C}} + \mathcal{C})] - \mathbb{E}[\log(\widehat{\mathcal{C}})] = \mathbb{E}\left[\log\left(\frac{\widehat{\mathcal{C}} + \mathcal{C}}{\widehat{\mathcal{C}}}\right)\right],$$

which is equal to $\lambda/2$ according to (6.20).

(iii) There exists a constant $\tilde{A} > 0$ such that $\log(1+x) \leq \tilde{A} + x^{1/4}$ for every $x > 0$. It follows then from the same arguments as in the proof of assertion (i) that

$$\begin{aligned} |Q_i Q_j| &\leq \left(\log\left(1 + M_i c_i + \frac{M_i}{L_{i+1}}\right) + \log\left(1 + \frac{M_{i-1}}{2L_i}\right)\right) \left(\log\left(1 + M_j c_j + \frac{M_j}{L_{j+1}}\right) + \log\left(1 + \frac{M_{j-1}}{2L_j}\right)\right) \\ &\leq \left(2\tilde{A} + \left(M_i c_i + \frac{M_i}{L_{i+1}}\right)^{\frac{1}{4}} + \left(\frac{M_{i-1}}{L_i}\right)^{\frac{1}{4}}\right) \left(2\tilde{A} + \left(M_j c_j + \frac{M_j}{L_{j+1}}\right)^{\frac{1}{4}} + \left(\frac{M_{j-1}}{L_j}\right)^{\frac{1}{4}}\right). \end{aligned}$$

In order to prove that $\sup_{i,j} \mathbb{E}[|Q_i Q_j|] < \infty$, it is enough to develop the product in the last line of the preceding display, and to show that the expectation of each term is uniformly bounded with respect to i and j . In fact, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\left[\left(\frac{M_{i-1}}{L_i}\right)^{\frac{1}{4}} \left(\frac{M_{j-1}}{L_j}\right)^{\frac{1}{4}}\right] \leq \mathbb{E}\left[\left(\frac{M_{i-1}}{L_i}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\frac{M_{j-1}}{L_j}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

and the right-hand side is uniformly bounded according to (6.39). Moreover, as

$$\mathbb{E}\left[\left(M_i c_i + \frac{M_i}{L_{i+1}}\right)^{\frac{1}{4}} \left(M_j c_j + \frac{M_j}{L_{j+1}}\right)^{\frac{1}{4}}\right] \leq \mathbb{E}\left[\left(\left(M_i c_i\right)^{\frac{1}{4}} + \left(\frac{M_i}{L_{i+1}}\right)^{\frac{1}{4}}\right) \left(\left(M_j c_j\right)^{\frac{1}{4}} + \left(\frac{M_j}{L_{j+1}}\right)^{\frac{1}{4}}\right)\right],$$

we can develop the right-hand side and similarly use (6.38) and (6.39) to show that it is uniformly bounded. All the other terms can be treated in an analogous way. By similar arguments, one can also prove that $\sup_{i,j} \mathbb{E}[(Q_i Q_j)^2] < \infty$. This finishes the proof of assertion (iii). \square

Using assertions (i) and (ii) of Lemma 6.19, we have thus

$$\lim_{k \rightarrow \infty} \mathbb{E}\left[\frac{1}{k} \sum_{j=2}^k Q_j\right] = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=2}^k \mathbb{E}[Q_j] = \mathbb{E}[Q_\infty] = \frac{\lambda}{2}. \quad (6.40)$$

Lemma 6.20. *We have*

$$\limsup_{k \rightarrow \infty} \mathbb{E} \left[\left(\frac{1}{k} \sum_{j=2}^k Q_j \right)^2 \right] \leq (\mathbb{E}[Q_\infty])^2 = \frac{\lambda^2}{4}. \quad (6.41)$$

Proof. We first note that for every $\delta \in (0, 1/2)$,

$$\left| \mathbb{E} \left[\sum_{2 \leq i, j \leq k} Q_i Q_j \right] - \mathbb{E} \left[\sum_{\substack{\delta k \leq i, j \leq k \\ |i-j| > \delta k}} Q_i Q_j \right] \right| \leq 4\delta k^2 \times \sup_{i,j} \mathbb{E}[|Q_i Q_j|].$$

In view of Lemma 6.19 (iii), the estimate (6.41) will be proved if we can show for arbitrarily small $\delta \in (0, 1/2)$ that

$$\limsup_{k \rightarrow \infty} \frac{1}{k^2} \mathbb{E} \left[\sum_{\substack{\delta k \leq i, j \leq k \\ |i-j| > \delta k}} Q_i Q_j \right] \leq (\mathbb{E}[Q_\infty])^2. \quad (6.42)$$

We thus fix $\delta \in (0, 1/2)$ in the following arguments. By symmetry, we can further restrict our attention to the indices i and j such that $\delta k \leq i, j \leq k$ and $j - i > \delta k$.

Notice that by the Cauchy–Schwarz inequality,

$$\begin{aligned} \left| \frac{1}{k^2} \mathbb{E} \left[\sum_{\substack{\delta k \leq i, j \leq k \\ j-i > \delta k}} Q_i Q_j \mathbf{1}_{\{M_{i+1} \geq \varepsilon M_{j-1}\}} \right] \right| &\leq \frac{1}{k^2} \sum_{\substack{\delta k \leq i, j \leq k \\ j-i > \delta k}} \mathbb{P}(M_{i+1} \geq \varepsilon M_{j-1})^{\frac{1}{2}} \mathbb{E}[(Q_i Q_j)^2]^{\frac{1}{2}} \\ &\leq \sup_{\substack{\delta k \leq i, j \leq k \\ j-i > \delta k}} \mathbb{P}(M_{i+1} \geq \varepsilon M_{j-1})^{\frac{1}{2}} \times \sup_{i,j} \mathbb{E}[(Q_i Q_j)^2]^{\frac{1}{2}}. \end{aligned} \quad (6.43)$$

However, observe that Lemma 6.13 can be reformulated as

$$\frac{\log M_k}{k} \xrightarrow[k \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \frac{1}{2},$$

and it follows that for all $\varepsilon \in (0, 1)$,

$$\limsup_{k \rightarrow \infty} \mathbb{P}(\{\text{there exist } i, j \in [\delta k, k] \text{ with } j - i > \delta k \text{ such that } M_{i+1} \geq \varepsilon M_{j-1}\}) = 0. \quad (6.44)$$

Together with Lemma 6.19 (iii), the latter display implies that the right-hand side of (6.43) converges to 0 as $k \rightarrow \infty$. The proof of (6.42) is thus reduced to showing that for fixed δ ,

$$\limsup_{\varepsilon \rightarrow 0} \left(\limsup_{k \rightarrow \infty} \frac{2}{k^2} \mathbb{E} \left[\sum_{\substack{\delta k \leq i, j \leq k \\ j-i > \delta k}} Q_i Q_j \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \right] \right) \leq (\mathbb{E}[Q_\infty])^2. \quad (6.45)$$

To this end, we take $\varepsilon \in (0, 1/2)$ and define, for every $k \geq 2$,

$$h_k^\varepsilon = h_k^\varepsilon(\check{\Gamma}) := \mathcal{C}_{M_k - 1 - \lfloor \varepsilon M_k \rfloor}([\check{\Gamma}]^{*(M_k - 1)})$$

where $[\check{\mathbb{T}}]^{*(M_k-1)}$ stands for the reduced tree associated with $[\check{\mathbb{T}}]^{M_k-1}$ up to height $M_k - 1$. In other words, h_k^ε is the probability that a simple random walk on $\check{\mathbb{T}}$ starting from \mathbf{u}_{M_k-1} hits a point of generation $-\lfloor \varepsilon M_k \rfloor$ that has a descendant at generation 0 before hitting \mathbf{u}_{M_k} . Comparing with the definition of h_k , it is clear that $h_k^\varepsilon \geq h_k$. On the other hand, by similar arguments as in the proof of Proposition 6.11, we obtain

$$h_k^\varepsilon - h_k \leq \frac{\lfloor \varepsilon M_k \rfloor}{M_k} h_k^\varepsilon,$$

which entails that

$$M_k h_k^\varepsilon - M_k h_k \leq \lfloor \varepsilon M_k \rfloor h_k^\varepsilon \leq 2\varepsilon M_k h_k. \quad (6.46)$$

We set

$$Q_k^\varepsilon = Q_k^\varepsilon(\check{\mathbb{T}}) := \log \left(1 + \frac{M_k c_k + \frac{M_k}{L_{k+1}}}{\frac{M_k}{L_k}} - \frac{\frac{M_k}{L_k}}{\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}} (M_{k-1} c_{k-1} + M_{k-1} h_{k-1}^\varepsilon)} \right) \geq Q_k.$$

Using the elementary inequality $0 \leq \log x - \log y \leq \frac{x-y}{y}$ for $x \geq y > 0$, we see that

$$\begin{aligned} Q_k^\varepsilon - Q_k &\leq \frac{\frac{M_k}{L_k}}{M_k c_k + \frac{M_k}{L_{k+1}}} \left(\frac{\frac{M_k}{L_k}}{\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}} (M_{k-1} c_{k-1} + M_{k-1} h_{k-1})} - \frac{\frac{M_k}{L_k}}{\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}} (M_{k-1} c_{k-1} + M_{k-1} h_{k-1}^\varepsilon)} \right) \\ &\leq \frac{\left(\frac{M_k}{L_k}\right)^2}{M_k c_k + \frac{M_k}{L_{k+1}}} \cdot \frac{\frac{M_k}{M_{k-1}} (M_{k-1} h_{k-1}^\varepsilon - M_{k-1} h_{k-1})}{\left(\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}} (M_{k-1} c_{k-1} + M_{k-1} h_{k-1})\right)^2}. \end{aligned}$$

Taking account of the easy facts that $M_k c_k \geq 1$ and $M_k = M_{k-1} + L_k$, we obtain

$$Q_k^\varepsilon - Q_k \leq \frac{\left(\frac{M_k}{L_k}\right)^2 \frac{M_k}{M_{k-1}} (M_{k-1} h_{k-1}^\varepsilon - M_{k-1} h_{k-1})}{\left(\frac{M_k}{L_k} + \frac{M_k}{M_{k-1}}\right)^2} \leq M_{k-1} h_{k-1}^\varepsilon - M_{k-1} h_{k-1},$$

which, together with (6.46), implies that

$$Q_k^\varepsilon - Q_k \leq 2\varepsilon M_{k-1} h_{k-1}.$$

This allows us to approximate $\mathbb{E}[Q_i Q_j \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}}]$ by $\mathbb{E}[Q_i Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}}]$, because

$$\begin{aligned} \left| \mathbb{E}[Q_i Q_j \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}}] - \mathbb{E}[Q_i Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}}] \right| &\leq \mathbb{E}[|Q_i (Q_j^\varepsilon - Q_j)|] \\ &\leq \mathbb{E}[(Q_j^\varepsilon - Q_j)^2]^{\frac{1}{2}} \times \mathbb{E}[(Q_i)^2]^{\frac{1}{2}} \\ &\leq 2\varepsilon \mathbb{E}[(M_{j-1} h_{j-1})^2]^{\frac{1}{2}} \times \mathbb{E}[(Q_i)^2]^{\frac{1}{2}}, \end{aligned}$$

and the right-hand side converges to 0 uniformly with respect to i, j and k when $\varepsilon \rightarrow 0$, according to Lemma 6.18 and assertion (iii) of Lemma 6.19.

Let us consider indices i, j such that $\delta k \leq i, j \leq k$ and $j - i > \delta k$, and let $\tilde{\mathcal{F}}_i$ be the σ -field generated by the variable M_{i+1} and the finite part of $\check{\mathbb{T}}$ above the vertex $\mathbf{u}_{M_{i+1}}$. Informally, one can think of it as the σ -field generated by $[\check{\mathbb{T}}]^{M_{i+1}}$. Thus, Q_i is $\tilde{\mathcal{F}}_i$ -measurable and

$$\mathbb{E}[Q_i Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}}] = \mathbb{E}\left[Q_i \mathbb{E}\left[Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \mid \tilde{\mathcal{F}}_i\right]\right]. \quad (6.47)$$

At this point, we observe that

$$\mathbb{E}\left[Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \mid \tilde{\mathcal{F}}_i\right] = \mathbb{E}\left[Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \mid M_{i+1}\right]. \quad (6.48)$$

On the other hand, one can generalize the proof of Lemma 6.17 to show that

$$\left(\frac{L_{j+1}}{M_j}, \frac{L_j}{M_{j-1}}, M_j c_j, M_{j-1} c_{j-1}, M_{j-1} h_{j-1}^\varepsilon\right) \xrightarrow{j \rightarrow \infty}^{(d)} (\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}', \widehat{\mathcal{C}}_\varepsilon), \quad (6.49)$$

where in the limit, the first four random variables $\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}'$ are the same as in Lemma 6.17, whereas $\widehat{\mathcal{C}}_\varepsilon$ is distributed as $\mathcal{C}(\widehat{\Delta}_\varepsilon)$. It is assumed in addition that $\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}', \widehat{\mathcal{C}}_\varepsilon$ are independent under \mathbb{P} . Furthermore, it is not hard to verify that the convergence (6.49) is still valid if, instead of the distribution of

$$\left(\frac{L_{j+1}}{M_j}, \frac{L_j}{M_{j-1}}, M_j c_j, M_{j-1} c_{j-1}, M_{j-1} h_{j-1}^\varepsilon\right),$$

we consider the conditional distribution of the same random 5-tuple given M_{i+1} , and let i and j tend to infinity satisfying that $j - i > \delta j$.

We define thus

$$Q_\infty^\varepsilon := \log \left(1 + \frac{\mathcal{C} + \frac{1}{\mathcal{R}}}{\frac{1 + \mathcal{R}'}{\mathcal{R}'}} - \frac{1}{1 + \mathcal{R}'(\mathcal{C}' + \widehat{\mathcal{C}}_\varepsilon)}\right).$$

By the same arguments used for assertions (i) and (iii) of Lemma 6.19, we obtain

$$\lim_{\substack{i, j \rightarrow \infty \\ j - i > \delta j}} \left(\sup_{\ell: \mathbb{P}(M_{i+1} = \ell) > 0} \left| \mathbb{E}[Q_j^\varepsilon \mid M_{i+1} = \ell] - \mathbb{E}[Q_\infty^\varepsilon] \right| \right) = 0,$$

and we see that

$$\sup_{\ell: \mathbb{P}(M_{i+1} = \ell) > 0} \mathbb{E}[(Q_j^\varepsilon)^2 \mid M_{i+1} = \ell]$$

is uniformly bounded for all i, j satisfying that $j - i > \delta j$. In view of (6.44) and (6.48), it follows that a.s.

$$\lim_{\substack{i, j \rightarrow \infty \\ j - i > \delta j}} \left| \mathbb{E}[Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \mid \tilde{\mathcal{F}}_i] - \mathbb{E}[Q_\infty^\varepsilon] \right| = 0.$$

Hence, we get from (6.47) and Lemma 6.19 (i) that

$$\lim_{k \rightarrow \infty} \left(\sup_{\substack{i, j \in [\delta k, k] \\ j - i > \delta k}} \mathbb{E}[Q_i Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}}] \right) \leq \mathbb{E}[Q_\infty] \mathbb{E}[Q_\infty^\varepsilon].$$

Finally, it remains to estimate the difference between $\mathbb{E}[Q_\infty^\varepsilon]$ and $\mathbb{E}[Q_\infty]$. To do this, we use a coupling argument by defining both $\widehat{\mathcal{C}}_\varepsilon$ and $\widehat{\mathcal{C}}$ from a common reduced tree $\widehat{\Delta}$, independent of $(\mathcal{R}, \mathcal{R}', \mathcal{C}, \mathcal{C}')$, so that $\widehat{\mathcal{C}} = \mathcal{C}(\widehat{\Delta})$ and $\widehat{\mathcal{C}}_\varepsilon = \mathcal{C}(\widehat{\Delta}_\varepsilon)$. Since $\mathcal{C} \geq 1$ and $\widehat{\mathcal{C}}_\varepsilon \geq \widehat{\mathcal{C}} \geq 1$, one can proceed in the same way as we did for bounding $Q_k^\varepsilon - Q_k$, and arrive at

$$0 \leq Q_\infty^\varepsilon - Q_\infty \leq \frac{1 + \frac{1}{\mathcal{R}'}}{\mathcal{C} + \frac{1}{\mathcal{R}}} \frac{\mathcal{R}'(\widehat{\mathcal{C}}_\varepsilon - \widehat{\mathcal{C}})}{(1 + \mathcal{R}'(\mathcal{C}' + \widehat{\mathcal{C}}))^2} \leq \left(1 + \frac{1}{\mathcal{R}'}\right) \frac{\mathcal{R}'(\widehat{\mathcal{C}}_\varepsilon - \widehat{\mathcal{C}})}{(1 + \mathcal{R}')^2} \leq \widehat{\mathcal{C}}_\varepsilon - \widehat{\mathcal{C}}.$$

Taking account of (6.22), the last display gives $|\mathbb{E}[Q_\infty^\varepsilon] - \mathbb{E}[Q_\infty]| \leq 2\varepsilon \mathbb{E}[\widehat{\mathcal{C}}]$, and the right-hand side converges to 0 as $\varepsilon \rightarrow 0$.

According to the previous discussions, we can conclude that

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \left(\limsup_{k \rightarrow \infty} \frac{2}{k^2} \mathbb{E} \left[\sum_{\substack{\delta k \leq i, j \leq k \\ j-i > \delta k}} Q_i Q_j \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \right] \right) \\
&= \limsup_{\varepsilon \rightarrow 0} \left(\limsup_{k \rightarrow \infty} \frac{2}{k^2} \mathbb{E} \left[\sum_{\substack{\delta k \leq i, j \leq k \\ j-i > \delta k}} Q_i Q_j^\varepsilon \mathbf{1}_{\{M_{i+1} < \varepsilon M_{j-1}\}} \right] \right) \\
&\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{E}[Q_\infty] \mathbb{E}[Q_\infty^\varepsilon] = (\mathbb{E}[Q_\infty])^2,
\end{aligned}$$

which finishes the proof of (6.45). The proof of Lemma 6.20 is therefore completed. \square

Proof of Lemma 6.16. By combining (6.40) and (6.41), we have

$$\limsup_{k \rightarrow \infty} \mathbb{E} \left[\left(\frac{1}{k} \sum_{j=2}^k Q_j - \frac{\lambda}{2} \right)^2 \right] \leq \left(\limsup_{k \rightarrow \infty} \mathbb{E} \left[\left(\frac{1}{k} \sum_{j=2}^k Q_j \right)^2 \right] \right) - \lambda \lim_{k \rightarrow \infty} \mathbb{E} \left[\frac{1}{k} \sum_{j=2}^k Q_j \right] + \frac{\lambda^2}{4} \leq 0,$$

which gives the desired result.

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