# Geometry and flatnessof control systems of minimal differential weight

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par

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# Géométrie et platitude des systèmes de contrôle de poids différentiel minimal

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# RÉSUMÉ


Au cours de nos travaux, nous avons étudié et résolu les problèmes suivants :

1. **Platitude des systèmes de contrôle à deux entrées linéarisables dynamiquement via une pré-intégration :**

   Nous avons donné une caractérisation géométrique complète des systèmes affines par rapport aux contrôles, à deux entrées, définis sur un espace d'état de dimension $n$, linéarisables dynamiquement via une pré-intégration d'un contrôle adéquate. Ils forment une classe particulière de systèmes plats : ils sont de poids différentiel $n + 3$. Nous avons décrit les formes normales, compatibles avec les sorties plates minimales, et présenté un système d'EDP à résoudre afin de trouver toutes les sorties plates minimales. Nous avons illustré nos résultats en analysant deux exemples : le moteur à induction et le réacteur chimique.

2. **Platitude des systèmes multi-entrées linéarisables dynamiquement via une pré-intégration :**

   Nous avons généralisé les résultats concernant les systèmes de contrôle à deux entrées, plats de poids différentiel $n + 3$, où $n$ est la dimension de l'espace d'état. Nous avons donné une caractérisation géométrique complète des systèmes multi-entrées, affines par rapport aux contrôles, linéarisables dynamiquement via une pré-intégration d'un contrôle bien choisi. Ils forment une classe particulière de systèmes plats : ils ont un poids différentiel de $n + m + 1$, où $m$ est le nombre de contrôles. Nous avons présenté des formes normales compatibles avec les sorties plates minimales et décrit toutes les sorties plates minimales. Nous avons appliqué nos résultats à deux exemples : le quadrirotor et le réacteur chimique.

3. **Caractérisation des systèmes multi-entrées statiquement équivalents à une forme triangulaire compatible avec la forme multi-chaînée et leur platitude $x$-maximale :**

   Nous avons étudié la platitude des systèmes affines par rapport aux contrôles, avec $m + 1$ entrées, pour $m \geq 1$, définis sur un espace d'état de dimension $n = km + 1$. Tout d'abord, nous avons donné une description géométrique complète des systèmes multi-entrées statiquement équivalents à une forme triangulaire compatible avec la forme chaînée, si $m = 1$, ou avec la forme multi-chaînée, si $m \geq 2$. Ensuite, la platitude de ces systèmes a été analysée et résolue. Nous avons discuté les singularités dans l'espace de contrôle et déterminé toutes les sorties plates, si $m = 1$, et toutes les sorties plates minimales, si $m \geq 2$. Nous avons

appliqué ces résultats au système mécanique d'une pièce roulant sans glissement sur une table en mouvement. Nous avons répondu à la question suivante : quelle doit être le dynamique de la table pour que ce système mécanique soit équivalent à la forme triangulaire compatible avec la forme chaînée ?

Indépendamment des points abordés précédemment dans ce chapitre, nous avons introduit le concept de platitude $x$-maximale. Un système de contrôle est $x$-maximalement plat si le nombre d'états gagnés à chaque dérivation successive des sorties plates est le plus grand possible. Premièrement, nous avons montré qu'un système linéaire par rapport aux contrôles est $x$-maximalement plat si et seulement s'il est statiquement équivalent à la forme multi-chaînée. Deuxièmement, nous avons généralisé ce résultat aux systèmes affines par rapport aux contrôles dont le sous-système linéaire est statiquement équivalent à la forme multi-chaînée. Nous avons prouvé qu'ils sont $x$-maximalement plats si et seulement si la dérive présente une forme triangulaire compatible avec la forme multi-chaînée. Nous avons montré également que cette dernière condition n'est pas nécessaire pour la $x$- platitude des système affines dont le sous-système linéaire est statiquement équivalente à la forme multi-chaînée.

# INTRODUCTION

Dans cette thèse, nous nous intéressons aux systèmes de contrôle nonlinéaires. Le contrôle de tels systèmes représente un domaine très actif de recherche en mathématiques appliquées, ainsi qu'en automatique. Un *système de contrôle nonlinéaire* est donné par une équation de la forme :

$$\Xi : \dot{x} = F(x, u),$$

où $x$ est l'état du système défini sur un ouvert $X$ de $\mathbb{R}^n$ (ou plus généralement sur une variété différentielle $X$, de dimension $n$), appelé espace d'état. Les valeurs du contrôle $u$ (appelé également l'entrée ou la commande) sont dans un sousensemble $U$ de $\mathbb{R}^m$, appelé espace du contrôle ; dans les problèmes abordés dans ce mémoire $U$ est un ouvert de $\mathbb{R}^m$, très souvent $\mathbb{R}^m$ entier. Le point désigne la dérivée par rapport à une variable indépendante, notée généralement par $t$ et qui représente le temps. Un système de contrôle nonlinéaire est donc un système d'équations nonlinéaires décrivant l'évolution temporelle des variables d'état du système sous l'action d'un nombre fini de variables indépendantes (les contrôles) qui peuvent être choisies librement afin de réaliser certains objectifs.

Les systèmes que nous étudions dans cette thèse sont principalement des *systèmes affines par rapport aux contrôles*. Ces systèmes admettent la forme suivante :

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u,$$

où $g = (g_1, \cdots, g_m)$ et $u = (u_1, \cdots, u_m)^\top$. Si la dérive $f$ est identiquement nulle, i.e., le système est de la forme suivante :

$$\Sigma_{lin} : \dot{x} = \sum_{i=1}^{m} g_i(x)u_i = g(x)u,$$

alors le système sera appelé linéaire par rapport aux contrôles.

## Équivalence des systèmes par bouclage statique

Un problème important en théorie du contrôle est de savoir si deux systèmes se ressemblent. Plus précisément, on souhaiterait savoir si les deux systèmes appartiennent à la même classe pour une certaine relation d'équivalence. En général, une telle relation d'équivalence est définie par une classe des transformations sur les systèmes, deux systèmes étant équivalents s'ils peuvent être transformés l'un en l'autre par une transformation de la classe.

Deux systèmes sont équivalents dans l'espace d'état, s'ils sont liés par un difféomorphisme (dans l'espace d'état). En conséquence, leurs trajectoires (correspondant aux mêmes contrôles) seront liées par ce même difféomorphisme. Lorsque nous considérons l'équivalence dans l'espace d'état, le contrôle reste inchangé. Cependant le rôle du contrôle est crucial dans l'étude des systèmes de contrôle (qu'ils soient linéaires ou non) et nous souhaitons le prendre en compte dans les relations d'équivalence. L'équivalence par bouclage augmente la classe des transformations considérées précédemment (transformations dans l'espace d'état) en permettant également la transformation des contrôles.

Considérons deux systèmes $\Xi : \dot{x} = F(x,u), x \in X, u \in U$, et $\tilde{\Xi} : \dot{\tilde{x}} = \tilde{F}(\tilde{x},\tilde{u}), \tilde{x} \in \tilde{X}$, $\tilde{u} \in \tilde{U}$. Les systèmes $\Xi$ et $\tilde{\Xi}$ sont *équivalents par bouclage statique* (ou statiquement équivalents), s'il existe un difféomorphisme $\chi : X \times U \mapsto \tilde{X} \times \tilde{U}$ de la forme

$$(\tilde{x},\tilde{u}) = \chi(x,u) = (\phi(x), \psi(x,u))$$

qui transforme le système $\Xi$ en $\tilde{\Xi}$, i.e.,

$$D\phi(x)F(x,u) = \tilde{F}(\phi(x), \psi(x,u)).$$

Remarquons que le difféomorphisme $\chi$ est triangulaire : en effet, $\phi$ dépend uniquement de l'état et joue le rôle d'un changement de coordonnées sur $X$, alors que $\psi$, appelé le bouclage, change les coordonnées dans l'espace du contrôle d'une manière dépendante de l'état. Les ensembles des trajectoires des deux systèmes coïncident, cependant ils sont différemment paramétrés par rapport aux contrôles $u$ et $\tilde{u}$.

Pour les systèmes de la forme $\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u$, afin de préserver la forme affine du système, nous restreignons la classe des bouclages aux bouclages affines

$$\tilde{u} = \psi(x,u) = \tilde{\alpha}(x) + \tilde{\beta}(x)u,$$

où $\tilde{\alpha} = (\tilde{\alpha}_1, \cdots, \tilde{\alpha}_m)^\top$ et $\tilde{\beta}(x)$ est une matrice de taille $m \times m$, inversible, et $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_m)^\top$. Notons par $u = \alpha(x) + \beta(x)\tilde{u}$ la transformation inverse et soit $\tilde{\Sigma}$ un autre système de contrôle défini par

$$\tilde{\Sigma} \ : \ \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \sum_{i=1}^{m} \tilde{g}_i(\tilde{x})\tilde{u}_i = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u},$$

où $\tilde{x} \in \tilde{X}$ et $\tilde{u} \in \tilde{U}$. Les systèmes $\Sigma$ et $\tilde{\Sigma}$ sont équivalents par bouclage statique si et seulement si

$$\tilde{f} = \phi_*(f + g\alpha) \text{ et } \tilde{g} = \phi_*(g\beta).$$

Si les transformations précédentes sont définies localement, autour des points $x_0 \in X$ et $\tilde{x}_0 \in \tilde{X}$ fixés, alors $\Sigma$ et $\tilde{\Sigma}$ sont dits localement équivalents par bouclage statique.

Pour les systèmes linéaires par rapport aux contrôles, i.e., de la forme $\Sigma_{lin} : \dot{x} = \sum_{i=1}^{m} g_i(x)u_i = g(x)u$, l'équivalence par bouclage statique coïncide avec l' équivalence des distributions engendrées par les champs de vecteurs $g_i$ et $\tilde{g}_i$.

La linéarisation par bouclage statique est un sous-problème de l'équivalence par bouclage statique et consiste à transformer un système nonlinéaire sous la forme la plus simple possible, c'est à dire sous la forme d'un système linéaire. Si nous sommes en mesure de compenser les nonlinéarités par le bouclage, alors le système transformé possède toutes les propriétés d'un système linéaire. Ainsi, nous pouvons résoudre des problèmes, très compliqués en général, qui deviennent plus simples pour les systèmes linéaires. La linéarisation par bouclage statique est donc un outils très important et puissant dans l'étude des systèmes nonlinéaires. Du point de vue mathématique, si nous souhaitons classifier les systèmes de contrôle nonlinéaires, un des problèmes les plus naturels est alors de caractériser les systèmes nonlinéaires qui sont statiquement équivalents à un système linéaire.

Un système $\Sigma$ est *statiquement linéarisable* (ou linéarisable par bouclage statique) s'il est équivalent par bouclage statique à un système contrôlable de la forme

$$\Lambda : \dot{z} = Az + Bv,$$

où $A$ et $B$ sont des matrices constantes de taille $n \times n$ et $n \times m$. Le problème de la linéarisation statique d'un système avec une seule entrée a été formulé et résolu par Brockett [4] (pour le bouclage restreint $u = \alpha + \tilde{u}$). Ensuite, Jakubczyk et Respondek [23] et, indépendamment, Hunt et Su [19], voir aussi [20], ont donné les conditions nécessaires et suffisantes suivantes, résolvant ainsi le problème de la linéarisation par bouclage statique d'un système affine avec un nombre arbitraire de contrôles. Considérons les distributions suivantes, associées au système $\Sigma$,

$$\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i], \text{ où } \mathcal{D}^0 = \text{span}\{g_1, \cdots, g_m\}.$$

Le système $\Sigma$ est localement linéarisable si et seulement si pour tout $i \geq 0$, les distributions $\mathcal{D}^i$ sont de rang constant, *involutives* et $\mathcal{D}^{n-1} = TX$.

## Platitude

La notion de platitude a été introduite en théorie du contrôle dans les années 1990 par Fliess, Lévine, Martin et Rouchon [13, 14] (voir aussi [2, 21, 32, 54]) et a attiré beaucoup d'attention grâce à ces multiple applications dans les problèmes de suivi et de planification de trajectoires [15, 22, 36, 53, 55, 58, 66]. Toutes les solutions d'un système plat peuvent être paramétrées par un nombre fini de fonctions et de leurs dérivées. Ceci représente la propriété fondamentale des systèmes plats.

Considérons un entier $p \geq -1$, nous lui associons $X^p = X \times U \times \mathbb{R}^{mp}$ et $\bar{u}^p = (u, \dot{u}, \ldots, u^{(p)})$. Si $p = -1$, alors $X^{-1}$ désigne simplement l'espace d'état $X$ et $\bar{u}^{-1}$ est vide.

**Definition 0.0.1.** Le système $\Xi$ est *plat* en $(x_0, \bar{u}_0^p) \in X^p$, où $p \geq -1$, s'il existe un voisinage $\mathcal{O}^p$ de $(x_0, \bar{u}_0^p)$ et $m$ fonctions lisses $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(p)})$, $1 \leq i \leq m$, définies dans $\mathcal{O}^p$, satisfaisant la propriété suivante : il existe un entier $s$ et des fonctions lisses $\gamma_i$, $1 \leq i \leq n$, et $\delta_j$, $1 \leq j \leq m$, tels que

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \text{ et } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

le long de chaque trajectoire $x(t)$ définie par un contrôle $u(t)$ tel que $(x(t), u(t), \cdots, u^{(p)}(t)) \in \mathcal{O}^p$, où $\varphi$ dénote le $m$-tuple $(\varphi_1, \ldots, \varphi_m)$ et est appelé *sortie plate*.

La platitude est étroitement liée à la linéarisation par bouclage statique ou dynamique. Les systèmes statiquement linéarisables sont clairement plats. En général, les systèmes plats ne sont pas statiquement linéarisables, cependant ils peuvent être vus comme la généralisation des ceux-ci. En effet, un système est plat si et seulement s'il est linéarisable par bouclage dynamique inversible et endogène. [13, 14, 32, 55]. Nous expliquons par la suite ces différentes notions.

Le système $\Xi : \dot{x} = F(x,u)$ est linéarisable dynamiquement si et seulement s'il existe un pré-compensateur inversible et endogène de la forme

$$\Theta : \begin{cases} \dot{y} &= G(x,y,v),\ y \in Y \subset \mathbb{R}^r,\ v \in V \subset \mathbb{R}^m \\ u &= \psi(x,y,v) \end{cases}$$

tel que le système pré-compensé

$$\Xi \circ \Theta : \begin{cases} \dot{x} &= F(x,\psi(x,y,v)) \\ \dot{y} &= G(x,y,v) \end{cases}$$

soit linéarisable statiquement. Un pré-compensateur $\Theta$ est endogène si l'état $y$ du pré-compensateur est une fonction de l'état d'origine $x$, du contrôle d'origine $u$ et ses dérivées, i.e., s'il existe une fonction $\mu$ et un entier $\rho$, suffisamment grand, tels que $y = \mu(x,u,\ldots,u^{(\rho)})$. Un pré-compensateur est inversible si on peut exprimer le contrôle du pré-compensateur $v$ comme une fonction de l'état du pré-compensateur $y$, de l'état d'origine $x$, du contrôle d'origine $u$ et de ses dérivées, i.e., $v = \bar{v}(x,y,u,\ldots,u^{(\rho)})$, ce qui, dans le cas d'un pré-compensateur endogène, donne $v = \nu(x,u,\ldots,u^{(\rho)})$.

Remarquons quelques propriétés des systèmes linéarisables par bouclage dynamique inversible et endogène. Tout d'abord, constatons que la dimension de l'état n'est pas préservée par bouclage dynamique endogène. En revanche la dimension de l'espace du contrôle est conservée. Deuxièmement, l'hypothèse pour $\phi$ d'être un difféomorphisme (hypothèse demandée dans le cas de la linéarisation statique) n'est plus requise. Finalement, les trajectoires de $\Xi$ sont en bijection avec celles d'un système trivial, i.e., $m$ fonctions libres $\varphi_1,\cdots,\varphi_m$ (les sorties plates) et sans dynamique.

Si $u^{(r)}$, avec $r \le p$, est la dérivée la plus élevée du contrôle impliquée dans les expressions de $\varphi_i$, alors le système est appelé $(x,u,\cdots,u^{(r)})$-plat. Dans le cas particulier, $\varphi_i = \varphi_i(x)$, pour $1 \le i \le m$, le système est appelé $x$-plat.

Le nombre minimal de dérivées de $\varphi_i$ utilisées pour exprimer $x$ et $u$ est appelé le poids différentiel de la sortie plate $\varphi$ (voir [58]) et est formalisé comme suit.

Par définition, pour toute sortie plate $\varphi$ de $\Xi$, il existe des entiers $s_1,\ldots,s_m$ tels que

$$\begin{aligned} x &= \gamma(\varphi_1,\dot{\varphi}_1,\ldots,\varphi_1^{(s_1)},\ldots,\varphi_m,\dot{\varphi}_m,\ldots,\varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1,\dot{\varphi}_1,\ldots,\varphi_1^{(s_1)},\ldots,\varphi_m,\dot{\varphi}_m,\ldots,\varphi_m^{(s_m)}). \end{aligned}$$

De plus, nous pouvons choisir $(s_1,\ldots,s_m)$ tels que (voir [58]) si pour un autre $m$-uplet $(\tilde{s}_1,\ldots,\tilde{s}_m)$ nous avons

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1,\dot{\varphi}_1,\ldots,\varphi_1^{(\tilde{s}_1)},\ldots,\varphi_m,\dot{\varphi}_m,\ldots,\varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1,\dot{\varphi}_1,\ldots,\varphi_1^{(\tilde{s}_1)},\ldots,\varphi_m,\dot{\varphi}_m,\ldots,\varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

alors $s_i \leq \tilde{s}_i$, pour $1 \leq i \leq m$. Nous appelons $\sum_{i=1}^{m}(s_i+1) = m + \sum_{i=1}^{m} s_i$ le poids différentiel de $\varphi$. Une sortie plate de $\Xi$ est appelée *minimale* si son poids est le plus petit parmi toutes les sorties plates de $\Xi$. Le *poids différentiel* d'un système plat $\Xi$ est égal au poids d'une sortie plate minimale de $\Xi$ et permet de déterminer la plus petite dimension possible d'un pré-compensateur linéarisant dynamiquement le système. En effet, la dimension $r$ d'un tel pré-compensateur satisfait $r \geq \sum_{i=1}^{m} s_i - n$. On voit et on dit que le poids différentiel mesure la plus petite dimension possible d'un pré-compensateur linéarisant dynamiquement le système.

Premièrement, le but de cette thèse est de donner une caractérisation complète des systèmes de contrôle qui ne sont pas linéarisables statiquement, mais qui le deviennent après l'application d'un bouclage dynamique aussi simple que possible. Ce sont les systèmes plats qui se rapprochent le plus des systèmes linéarisables statiquement et ils forment une classe particulière de systèmes plats : ils sont de poids différentiel $n+m+1$. D'un côté, nous souhaiterions donner des conditions nécessaires et suffisantes *vérifiables* (par exemple, des conditions de type involutivité) et d'un autre côté, nous voudrions décrire et comprendre la géométrie de cette classe de systèmes (présenter des formes normales, donner la description de sorties plates, etc.). Dans un premier temps, nous donnons des conditions nécessaires et suffisantes pour qu'un système devienne statiquement linéarisable après la prolongation d'un contrôle bien choisi (ou de manière équivalente, pour qu'il soit plat de poids différentiel $n+m+1$). Les conditions présentées sont vérifiables et leur vérification nécessite uniquement des dérivations et des opérations algébriques, sans nécessiter la résolution d'EDP ou mettre le système sous une forme normale. Ensuite, nous présentons les formes normales, donnons la description de sorties plates et en déduisons un système d'EDP à résoudre afin de calculer les sorties plates. La platitude a donc deux niveaux de difficulté : le premier consiste à donner une caractérisation géométrique des systèmes plats (et nos résultats donnent des conditions nécessaires et suffisantes vérifiables- sans résoudre des EDP- pour caractériser les systèmes plats de poids différentiel $n+m+1$) alors que le second correspond au calcul des sorties plates et pour cela nous sommes obligés de résoudre des EDP.

Deuxièmement, nous souhaiterions généraliser la platitude des systèmes linéaires par rapport aux contrôles avec deux entrées, problème résolu par Martin et Rouchon [33], au cas affine : nous donnons la caractérisation et analysons la platitude des systèmes statiquement équivalents à une forme triangulaire compatible avec la forme chaînée. Puis, nous étendons ces résultats aux systèmes statiquement équivalents à une forme triangulaire compatible avec la forme multi-chaînée.

Troisièmement, nous introduisons le concept de platitude $x$-maximale (la propriété selon laquelle chaque dérivée successive de $\varphi$ permet de gagner le nombre maximal de fonctions (composantes) de l'état $x$). Nous montrons que dans la classe des systèmes linéaires par rapport aux contrôles, un système est $x$-maximalement plat si et seulement s'il est statiquement équivalent à la forme multi-chaînée. Puis nous généralisons ce résultat en montrant que dans la classe des systèmes affines dont le sous-système linéaire est statiquement équivalent à la forme multi-chaînée, les seuls systèmes $x$-maximalement plats sont les systèmes statiquement équivalents à la forme triangulaire compatible avec la forme multi-chaînée.

Par la suite, nous présenterons chapitre par chapitre les résultats obtenus dans cette thèse.

**Chapitre 1. Platitude des systèmes de contrôle à deux entrées linéarisables dynamiquement via une pré-intégration**

Les résultats de ce chapitre on été présentés à NOLCOS 2013, [45], et ont été soumis au *European Journal of Control*, [44].

Dans ce chapitre nous étudions la platitude des systèmes affines par rapport aux contrôles, à deux entrées, définis sur un espace d'état de dimension $n$, linéarisables dynamiquement via une pré-intégration d'un contrôle bien choisi. Ce sont les systèmes plats qui se rapprochent le plus des systèmes linéarisables statiquement.

Les systèmes linéarisables par bouclage statique sont plats. Effectivement, ils sont équivalents par bouclage statique à la forme canonique de Brunovský :

$$(Br)\left\{\begin{array}{lcl} \dot z_1^i & = & z_2^i \\ & \vdots & \\ \dot z_{\rho_i-1}^i & = & z_{\rho_i}^i \\ \dot z_{\rho_i}^i & = & v_i \end{array}\right.$$

où $1 \leq i \leq m$ et $\sum_{i=1}^m \rho_i = n$ (voir [5]) et sont plats avec $\varphi = (z_1^1, \cdots, z_1^m)$ une sortie plate minimale (de poids différentiel $n+m$). Une façon équivalente de décrire les systèmes statiquement linéarisables est le fait qu'ils sont plats de poids différentiel $n+m$. Par conséquent, le poids différentiel d'un système plat, qui n'est pas linéarisable statiquement, est strictement supérieur à $n+m$ et mesure la plus petite dimension possible d'un pré-compensateur linéarisant dynamiquement le système.

En général, les systèmes plats ne sont pas linéarisables par bouclage statique, à l'exception des systèmes avec une seule entrée, pour lesquels la platitude se réduit à la linéarisation par bouclage statique. Les systèmes plats peuvent être vus comme la généralisation de systèmes linéaires. Notamment, ils sont linéarisables par bouclage dynamique, inversible et endogène (voir [13, 14, 32, 55]). Notre objectif est de décrire complètement les plus simples systèmes plats qui ne sont pas linéarisables statiquement : les systèmes affines par rapport aux contrôles, à deux entrées, linéarisables dynamiquement via une pré-intégration d'un contrôle. Ils forment une classe particulière de systèmes plats : ils sont de poids différentiel $n+3$. Dans ce chapitre, nous donnons une caractérisation géométrique complète de cette classe de systèmes.

Considérons le système de contrôle

$$\Sigma \;:\; \dot x = f(x) + u_1 g_1(x) + u_2 g_2(x), \tag{1}$$

où $x \in X$ et $u = (u_1, u_2)^\top \in \mathbb{R}^2$. Nous lui associons les distributions suivantes $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$, où $\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}$. Supposons que $\Sigma$ n'est pas linéarisable statiquement. Cela se produit s'il existe un entier $k$ tel que la distribution $\mathcal{D}^k$ est non involutive. Par la suite, $k$ désigne le plus petit entier vérifiant cette propriété.

**Proposition 0.0.1.** *Les conditions suivantes sont équivalentes :*

(i) *$\Sigma$ est plat au point $(x_0, \bar{u}_0^p)$, de poids différentiel $n+3$;*

(ii) *$\Sigma$ est x-plat au point $(x_0, u_0)$, de poids différentiel $n+3$;*

(iii) *Il existe, localement, autour de $x_0$, une transformation inversible $u = \alpha(x) + \beta(x)\tilde{u}$ ramenant $\Sigma$ sous la forme $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2\tilde{g}_2(x)$, telle que la prolongation*

$$\tilde{\Sigma}^{(1,0)} : \left\{ \begin{array}{lcl} \dot{x} & = & \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{g}_2(x) \\ \dot{y}_1 & = & v_1 \end{array} \right.$$

*soit linéarisable statiquement, où $y_1 = \tilde{u}_1$, $v_2 = \tilde{u}_2$, $\tilde{f} = f + \alpha g$ et $\tilde{g} = g\beta$, avec $g = (g_1, g_2)$ et $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$.*

Notre résultat principal est donné par les deux théorèmes suivants correspondants au cas $k \geq 1$ (Théorème 0.0.1) et au cas $k = 0$ (Théorème 0.0.2). Pour les deux théorèmes, nous supposons $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$. Nous notons par $\overline{\mathcal{D}}^k$ l'adhérence involutive de $\mathcal{D}^k$.

**Theorem 0.0.1.** *Supposons $k \geq 1$ et $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$. Le système $\Sigma$, donné par (1), est x-plat au point $x_0 \in X$, de poids différentiel $n+3$, si et seulement si les conditions suivantes sont satisfaites :*

*(A1)* $\operatorname{rg} \overline{\mathcal{D}}^k = 2k+3$;

*(A2)* $\operatorname{rg} (\overline{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$, *impliquant l'existence d'un champ de vecteurs non nul $g_c \in \mathcal{D}^0$ tel que $ad_f^{k+1} g_c \in \overline{\mathcal{D}}^k$;*

*(A3) Les distributions $\mathcal{H}^i$, pour $i \geq k$, sont involutives, où $\mathcal{H}^k = \mathcal{D}^{k-1} + \operatorname{span} \{ad_f^k g_c\}$ et $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, pour $i \geq k$ ;*

*(A4) Il existe $\rho$ tel que $\mathcal{H}^\rho = TX$.*

Le Théorème 0.0.1 donne des conditions nécessaires et suffisantes pour qu'un système $\Sigma$ soit plat de poids différentiel $n+3$, et donc pour qu'il devienne statiquement linéarisable après la prolongation d'un contrôle bien choisi. La propriété structurelle fondamentale de ces systèmes est l'existence de la sous-distribution involutive $\mathcal{H}^k$, de corang un dans $\mathcal{D}^k$. Le théorème précédent nous permet également de définir le contrôle à prolonger afin d'obtenir un système $\tilde{\Sigma}^{(1,0)}$ statiquement linéarisable. Le champ de vecteurs non nul $g_c \in \mathcal{D}^0$ peut être exprimé comme $g_c = \beta_1 g_1 + \beta_2 g_2$, où $\beta_1$ et $\beta_2$ sont des fonctions qui ne s'annulent pas simultanément. On en déduit que le contrôle à prolonger afin de linéariser dynamiquement le système est donné par $u_p(t) = \beta_2(x(t))u_1(t) - \beta_1(x(t))u_2(t)$.

Si $k = 0$, un résultat similaire peut être formulé, mais dans ce cas, la distribution $\mathcal{H}^1$ n'est pas définie de la même manière que $\mathcal{H}^{k+1}$, mais par $\mathcal{H}^1 = \mathcal{G}^1$, où $\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$ (comparer les conditions $(A3)$ et $(A3)'$). De plus, les systèmes plats avec $k = 0$ possèdent des singularités dans l'espace du contrôle (voir Section 1.3 pour la définition de l'ensemble des contrôles singuliers $U_{sing}$, i.e., les contrôles pour lesquels le système cesse d'être plat).

**Theorem 0.0.2.** *Supposons $k = 0$ et $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] \neq TX$. Le système $\Sigma$, donné par (1), est x-plat au point $(x_0, u_0) \in X \times \mathbb{R}^2$, de poids différentiel $n+3$, si et seulement si les conditions suivantes sont satisfaites*

*(A1)'* $\operatorname{rk} \overline{\mathcal{D}}^0 = 3$ *est involutive;*

*(A2)'* $\operatorname{rg}(\overline{\mathcal{D}}^0 + [f, \mathcal{D}^0] = 4$*, impliquant l'existence d'un champ de vecteurs non nul $g_c \in \mathcal{D}^0$ tel que $ad_f g_c \in \mathcal{G}^1$;*

*(A3)' Les distributions $\mathcal{H}^i$, pour $i \geq 1$, sont involutives, où $\mathcal{H}^1 = \mathcal{G}^1$ et $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, pour $i \geq 2$;*

*(A4)' Il existe $\rho$ tel que $\mathcal{H}^\rho = TX$;*

*(CR)* $u_0 \notin U_{sing}(x_0)$.

Si $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$, nous distinguons deux cas (correspondants à la façon dont $\mathcal{D}^k$ perd son involutivité) : d'une part $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ et $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$, d'autre part $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Dans le premier cas, nous montrons que le système est plat de poids différentiel $n+3$ sans aucune condition additionnelle. Dans le deuxième cas, le système doit vérifier quelques conditions supplémentaires similaires à celle du Théorème 0.0.1. Si $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$, la condition $(A2)$, nous permettant de définir le champ de vecteurs $g_c$ (et, en conséquence, la sous-distribution involutive $\mathcal{H}^k$), n'a pas de sens. Par conséquent, nous devons définir $\mathcal{H}^k$ d'une autre manière. Pour cela nous utiliserons la distribution caractéristique de $\mathcal{D}^k$ (voir Théorème 1.3.4, Section 1.3).

Nous caractérisons ensuite toutes les sorties plates minimales des systèmes de poids différentiel $n+3$. Soit $\mu$ le plus grand entier tel que corang de l'inclusion $\mathcal{H}^{\mu-1} \subset \mathcal{H}^\mu$ soit deux et $\rho$ le plus petit entier tel que $\mathcal{H}^\rho = TX$.

**Proposition 0.0.2.** *Considérons un système de contrôle $\Sigma$, de la forme (1), x-plat en $x_0$ (en $(x_0, u_0)$, si $k = 0$), de poids différentiel $n+3$.*

*(i) Supposons $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ ou $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ et $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Une paire de fonctions lisses $(\varphi_1,\ \varphi_2)$, définies dans un voisinage de $x_0$, est une sortie x-plate minimale au point $x_0$ si et seulement si (à une permutation près)*

$$\begin{array}{rcl} d\varphi_1 & \perp & \mathcal{H}^{\rho-1} \\ d\varphi_2 & \perp & \mathcal{H}^{\mu-1} \end{array}$$

*et $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$ (à une permutation de $\varphi_1$ et $\varphi_2$ près).*

*De plus, la pair $(\varphi_1,\ \varphi_2)$ est unique, à un difféomorphisme près, i.e., si $(\tilde{\varphi}_1,\ \tilde{\varphi}_2)$ est une autre sortie plate minimale, alors il existe des fonctions lisses $h_1$ et $h_2$ inversibles ($h_2$ par rapport à son premier argument), telles que*

$$\begin{array}{rcl} \tilde{\varphi}_1 & = & h_1(\varphi_1) \\ \tilde{\varphi}_2 & = & h_2(\varphi_2, \varphi_1, L_f\varphi_1, \ldots, L_f^{\rho-\mu}\varphi_1). \end{array}$$

*Si $\rho = \mu$, alors $\tilde{\varphi}_i = h_i(\varphi_1, \varphi_2)$, $1 \leq i \leq 2$, et $h = (h_1, h_2)$ est un difféomorphisme.*

(ii) *Supposons* $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ *et* $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$. *Une paire de fonctions lisses* $(\varphi_1, \varphi_2)$ *définies dans un voisinage de* $x_0$ *est une sortie x-plate minimale au point* $x_0$ *si et seulement si* $(d\varphi_1 \wedge d\varphi_2)(x_0) \neq 0$ *et la distribution involutive* $\mathcal{L} = (\mathrm{span}\,\{d\varphi_1, d\varphi_2\})^{\perp}$ *satisfait*

$$\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k.$$

*De plus, pour toute fonction* $\varphi_1$*, satisfaisant*

$$d\varphi_1 \perp \mathcal{D}^{k-1} \text{ et } (L_{ad_f^k g_1}\varphi_1, L_{ad_f^k g_2}\varphi_1)(x_0) \neq (0,0),$$

*il existe* $\varphi_2$ *tel que la pair* $(\varphi_1, \varphi_2)$ *soit une sortie x-plate minimale. Étant donné une telle fonction* $\varphi_1$*, le choix de* $\varphi_2$ *est unique, à un difféomorphisme près, c'est à dire, si* $(\varphi_1, \tilde{\varphi}_2)$ *est une autre sortie plate minimale, alors il existe une application lisse h, inversible par rapport à son deuxième argument, telle que*

$$\tilde{\varphi}_2 = h(\varphi_1, \varphi_2).$$

Tout d'abord, la proposition précédente nous permet de vérifier si une paire de fonctions $(\varphi_1, \varphi_2)$ est une sortie $x$-plate minimale d'un système de poids différentiel $n+3$. De plus, elle répond à la question : y-a-t-il beaucoup de paires $(\varphi_1, \varphi_2)$ qui sont des sorties $x$-plates minimales ? Finalement, elle nous permet de construire explicitement un système d'équations aux dérivées partielles du premier ordre à résoudre afin de trouver toutes les sorties plates minimales (voir Section 1.5).

Enfin, nous décrivons les formes normales compatibles avec les sorties plates minimales et appliquons nos résultats à deux exemples : le moteur à induction et le réacteur chimique.

**Chapitre 2. Platitude des systèmes multi-entrées linéarisables dynamiquement via une pré-intégration**

Les résultats de ce chapitre on été présentés à CDC 2013, [46], et ont été soumis au *SIAM Journal on Control and Optimization*, [43].

Ce chapitre est dédié à la généralisation des résultats décrits dans le chapitre précédent. Nous étudions les systèmes multi-entrées, affines par rapport aux contrôles, définis sur un espace d'état de dimension $n$, linéarisables dynamiquement via une pré-intégration d'un contrôle bien choisi. Ils forment la classe des systèmes plats les plus simples, qui ne sont pas linéarisables statiquement. Les systèmes statiquement linéarisables sont plats et une façon équivalente de les décrire est la suivante : ils sont plats de poids différentiel $n+m$. Par conséquent, pour tout système plat, qui n'est pas statiquement linéarisable, le nombre minimal de dérivées des sorties plates utilisées pour exprimer toutes les variables d'état et du contrôle est strictement supérieur à $n+m$. Les systèmes plats, qui se rapprochent le plus des systèmes linéarisables statiquement sont les systèmes linéarisables dynamiquement via une pré-intégration d'un contrôle. Ils forment la classe particulière des systèmes que nous caractérisons dans ce chapitre : les systèmes plats de poids différentiel $n+m+1$.

Considérons un systèmes de contrôle

$$\Sigma \,:\, \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \tag{2}$$

où $x \in X$ et $u = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$. Supposons que $\Sigma$ n'est pas linéarisable statiquement. Cela se produit s'il existe un entier $k$ tel que le distribution $\mathcal{D}^k$ soit non involutive. Par la suite, $k$ désigne le plus petit entier vérifiant cette propriété.

**Proposition 0.0.3.** *Les conditions suivantes sont équivalentes :*

*(i)* $\Sigma$ *est plat au point* $(x_0, \bar{u}_0^p)$, *de poids différentiel* $n+m+1$;

*(ii)* $\Sigma$ *est x-plat au point* $(x_0, u_0)$, *de poids différentiel* $n+m+1$;

*(iii) Il existe, localement, autour de* $x_0$, *une transformation inversible* $u = \alpha(x) + \beta(x)\tilde{u}$ *ramenant* $\Sigma$ *sous la forme* $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \sum_{i=1}^{m} \tilde{u}_i \tilde{g}_i(x)$, *telle que la prolongation*

$$\tilde{\Sigma}^{(1,0,\ldots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^{m} v_i \tilde{g}_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

*est linéarisable statiquement, où* $y_1 = \tilde{u}_1$, $v_i = \tilde{u}_i$, *pour* $2 \le i \le m$, $\tilde{f} = f + \alpha g$ *et* $\tilde{g} = g\beta$, *avec* $g = (g_1, \cdots, g_m)$ *et* $\tilde{g} = (\tilde{g}_1, \cdots, \tilde{g}_m)$.

Notre résultat principal est donné par les deux théorèmes suivants correspondant au cas $k \ge 1$ (Théorème 0.0.3) et au cas $k = 0$ (Théorème 0.0.4). Par la suite, nous supposons $\operatorname{rg} \mathcal{D}^k - \operatorname{rg} \mathcal{D}^{k-1} \ge 2$ (voir Section 2.7 où nous montrons que ce cette condition est nécessaire pour la linéarisation dynamique via une pré-intégration, et donc pour la platitude de poids différentiel $n+m+1$). Pour les deux théorèmes, nous supposons $\operatorname{corg}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \ge 2$, le cas de ce corang égal à 1 sera discuté plus tard.

**Theorem 0.0.3.** *Supposons* $k \ge 1$ *et* $\operatorname{corg}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \ge 2$. *Le système* $\Sigma$, *donné par (2), est x-plat au point* $x_0$, *de poids différentiel* $n+m+1$, *si et seulement si les conditions suivantes sont satisfaites :*

*(A1) Il existe une sous-distribution involutive* $\mathcal{H}^k$ *de corang un dans* $\mathcal{D}^k$ *;*

*(A2) Les distributions* $\mathcal{H}^i$, *pour* $i \ge k+1$, *sont involutives, où* $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$;

*(A3) Il existe* $\rho$ *tel que* $\mathcal{H}^\rho = TX$.

L'existence de la sous-distribution involutive $\mathcal{H}^k$ de corang un dans $\mathcal{D}^k$ est la propriété structurelle fondamentale de ces systèmes. Afin de vérifier si les conditions du Théorème 0.0.3 sont satisfaites, il faut prouver que $\mathcal{D}^k$ admet une sous-distribution involutive $\mathcal{H}^k$ de corang un. Nous expliquons en Section 2.3 comment vérifier l'existence de la sous-distribution involutive $\mathcal{H}^k$ et comment la calculer explicitement, si elle existe. La condition $\operatorname{corg}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \ge 2$ implique l'unicité de $\mathcal{H}^k$.

Le théorème précédent nous permet également de définir le contrôle à prolonger, qui est défini à une fonction multiplicative près, pour que le système prolongé associé $\tilde{\Sigma}^{(1,0,\ldots,0)}$ soit statiquement linéarisable. Nous montrons en Section 2.7 que la sous-distribution $\mathcal{H}^k$ permet d'identifier une unique sous-distribution involutive $\mathcal{H}$ de corang un dans $\mathcal{D}^0$ telle que $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k\mathcal{H}$. C'est la sous-distribution $\mathcal{H}$ qui nous permet ensuite de définir le contrôle à prolonger. Nous expliquons cela dans la Section 2.3.

Si $k = 0$, un résultat similaire peut être formulé, mais dans ce cas, la distribution $\mathcal{H}^1$ n'est pas définie de la même façon que $\mathcal{H}^{k+1}$, mais comme $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$, où $\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$ (comparer $(A2)$ et $(A2)'$). De plus, les systèmes plats avec $k = 0$ possèdent des singularités dans l'espace du contrôle (voir Section 2.3 pour la définition de $U_{sing}$, l'ensemble des contrôles singuliers, pour lesquels le système cesse d'être plat).

**Theorem 0.0.4.** *Supposons $k = 0$ et* $\operatorname{corg}(\mathcal{D}^0 \subset [\mathcal{D}^0, \mathcal{D}^0]) \geq 2$. *Le système $\Sigma$, donné par (2), est x-plat au point $(x_0, u_0)$, de poids différentiel $n + m + 1$, si et seulement si les conditions suivantes sont satisfaites :*

*(A1)′ Il existe une sous-distribution involutive $\mathcal{H}^0$ de corang un dans $\mathcal{D}^0$;*

*(A2)′ Les distributions $\mathcal{H}^i$, pour $i \geq 1$, sont involutives, où $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$ et $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, pour $i \geq 2$;*

*(A3)′ Il existe $\rho$ tel que $\mathcal{H}^\rho = TX$.*

*(CR) $u_0 \notin U_{sing}(x_0)$.*

Considérons maintenant le cas $\operatorname{corg}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$. Si la distribution $\mathcal{D}^k$ contient une sous-distribution involutive de corang un, celle-ci n'est jamais unique. La perte d'involutivité de $\mathcal{D}^k$ peut se réaliser de deux manières différentes : d'une part $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, d'autre part $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ et il existe $1 \leq i, j \leq m$ tels que $[ad_f^k g_i, ad_f^k g_j] \notin \mathcal{D}^k$. Si $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, alors nous définissons la sous-distribution involutive $\mathcal{H}^k$ de façon unique, en utilisant la distribution caractéristique $\mathcal{C}^k$ de $\mathcal{D}^k$ (voir le Théorème 0.0.5 ci-dessous).

Considérerons une distribution $\mathcal{D}$. Un champ de vecteur $c \in \mathcal{D}$ est dit caractéristique pour $\mathcal{D}$ si $[c, \mathcal{D}] \subset \mathcal{D}$. La distribution caractéristique $\mathcal{C}$ de $\mathcal{D}$ est la distribution générée par tous les champs caractéristiques. L'involutivité de la distribution caractéristique $\mathcal{C}$ est une conséquence directe de l'identité de Jacobi.

Si $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ et il existe $1 \leq i, j \leq m$ tels que $[ad_f^k g_i, ad_f^k g_j] \notin \mathcal{D}^k$, alors toute sous-distribution involutive $\mathcal{H}^k$ de corang un dans $\mathcal{D}^k$ peut être utilisée pour définir le contrôle à prolonger (distributions différentes donnant des contrôles différents) afin d'obtenir un système prolongé $\tilde{\Sigma}^{(1,0,\ldots,0)}$ statiquement linéarisable.

**Theorem 0.0.5.** *Supposons* $\operatorname{corg}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ *et $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Le système $\Sigma$, donné par (2), est x-plat au point $(x_0, u_0)$, de poids différentiel $n + m + 1$, si et seulement si les conditions suivantes sont satisfaites :*

*(C1)* $\mathrm{rg}\,\mathcal{C}^{\mathrm{k}} = \mathrm{rg}\,\mathcal{D}^{\mathrm{k}} - 2$*, où* $\mathcal{C}^k$ *est la distribution caractéristique de* $\mathcal{D}^k$ *;*

*(C2)* $\mathrm{rg}\,(\mathcal{C}^{\mathrm{k}} \cap \mathcal{D}^{\mathrm{k}-1}) = \mathrm{rg}\,\mathcal{D}^{\mathrm{k}-1} - 1$*;*

*(C3) Les distributions* $\mathcal{H}^i$*, pour* $i \geq k$*, sont involutives, où* $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$ *et* $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$*;*

*(C4) Il existe* $\rho$ *tel que* $\mathcal{H}^\rho = TX$*.*

Ensuite, nous introduisons deux formes normales, compatibles avec les sorties plates minimales. Ces résulats ont été présentés à ECC 2014, voir [47].

Finalement, nous caractérisons toutes les sorties plates minimales des systèmes plats de poids différentiel $n + m + 1$. Notre résultat (voir Proposition 2.5.1, Section 2.5) nous permet de vérifier si un $m$-uplet de fonctions $(\varphi_1, \cdots, \varphi_m)$ est une sortie $x$-plate minimale de poids différentiel $n + m + 1$ et répond à la question : y-a-t-il beaucoup des $m$-uplets $(\varphi_1, \cdots, \varphi_m)$ qui sont des sorties $x$-plates minimales ? De plus, il nous permet de construire explicitement un système d'équations aux dérivées partielles du premier ordre, à résoudre afin de trouver toutes les sorties plates minimales. Finalement, nous illustrons nos résultats via deux exemples : le quadrirotor et le réacteur chimique.

**Chapitre 3. Caractérisation des systèmes multi-entrées statiquement équivalents à une forme triangulaire compatible avec la forme multi-chaînée et leur platitude $x$-maximale**

La première partie de ce chapitre est consacrée à la platitude d'une classe particulière de systèmes affines par rapport aux contrôles, avec $m + 1$ entrées, où $m \geq 1$, définis sur un espace d'état de dimension $n = km + 1$, $k \geq 1$. Les résultats de cette partie ont été réalisés en collaboration avec ShunJie Li (Zhejiang University) et ont été soumis au *International Journal of Control* [27].

La platitude des systèmes linéaires par rapport aux contrôles, à deux entrées, i.e., de la forme

$$\Sigma_{lin} : \dot{x} = u_0 g_0(x) + u_1 g_1(x),$$

défini sur un ouvert $X$ de $\mathbb{R}^n$, a été résolue par Martin and Rouchon [34] (voir aussi [8, 29,33,41,61]). Ils ont montré que, sur un ouvert dense $X'$ de $X$, le système $\Sigma_{lin}$ est plat si et seulement si la distribution $\mathcal{G} = \mathrm{span}\,\{g_0, g_1\}$ est une structure de Goursat ou, de manière équivalente, si et seulement si le système est localement équivalent par bouclage statique à la forme chaînée. Giaro, Kumpera et Ruiz [17] sont les premiers à avoir remarqué l'existence de points singuliers dans le problème de la transformation d'une distribution de rang deux sous la forme normale de Goursat. Puis, Murray [40] a donné une condition de régularité permettant de transformer un système $\Sigma_{lin}$ sous la forme chaînée autour d'un point arbitraire $x^*$. Li and Respondek [29] on montré qu'un système dont la distribution associée est une structure de Goursat est $x$-plat seulement aux points où la condition de régularité est satisfaite. Ils ont également décrit toutes les sorties plates.

Dans ce chapitre, nous généralisons ces résultats : nous caractérisation les systèmes affines statiquement équivalents à la forme triangulaire suivante :

$$TCh_1^k : \begin{cases} \dot{z}_0 = v_0 & \dot{z}_1 = f_1(z_0, z_1, z_2) \quad +z_2 v_0 \\ & \dot{z}_2 = f_2(z_0, z_1, z_2, z_3) + z_3 v_0 \\ & \quad \vdots \\ & \dot{z}_{k-1} = f_2(z_0, \cdots, z_k) \quad +z_k v_0 \\ & \dot{z}_k = v_1 \end{cases}$$

Remarquons que dans le système de coordonnées $z$, dans lequel les champs $g_1$ et $g_2$ sont sous la forme chaînée, la dérive $f$ a une forme triangulaire très spéciale. C'est la raison pour laquelle nous appelons $TCh_1^k$ la forme triangulaire compatible avec la forme chaînée.

Ensuite, nous étendons ces résultats aux systèmes statiquement équivalents à une forme triangulaire compatible avec la forme multi-chaînée. Nous caractérisation les systèmes affines avec $m+1$ entrées, où $m \geq 2$, statiquement équivalents à la forme normale obtenue en remplaçant, dans $TCh_1^k$, chaque état $z_i$ par le vecteur $z^i = (z_1^i, \cdots, z_m^i)$, les fonctions lisses $f_i$ par $f^i = (f_1^i, \cdots, f_m^i)$ et le contrôle $v_1$ par le vecteur $(v_1, \cdots, v_m)$. Cette forme sera notée par $TCh_m^k$. La caractérisation des systèmes statiquement équivalents à la forme multi-chaînée a été étudiée et résolue dans [59] (voir aussi [39, 50, 63, 68]). Il est immédiat que ces systèmes sont $x$-plats et toutes leurs sorties plates minimales ont été décrites dans [58].

Considérons le système affine

$$\Sigma : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x), \tag{3}$$

défini sur une variété $X$, de dimension $n = km + 1$, où $m \geq 1$. Nous lui associons la distribution $\mathcal{G} = \text{span}\,\{g_0, \cdots, g_m\}$ et la suite de distributions définie par

$$\mathcal{G}^0 = \mathcal{G} \text{ et } \mathcal{G}^{i+1} = \mathcal{G}^i + [\mathcal{G}^i, \mathcal{G}^i],\ i \geq 0.$$

On note par $\mathcal{C}^i$ la distribution caractéristique de $\mathcal{G}^i$. Nous rappelons ci-dessous la définition de la distribution caractéristique.

Considérerons une distribution $\mathcal{D}$. Un champ de vecteur $c \in \mathcal{D}$ est dit caractéristique pour $\mathcal{D}$ si $[c, \mathcal{D}] \subset \mathcal{D}$. La distribution caractéristique $\mathcal{C}$ de $\mathcal{D}$ est la distribution générée par tous les champs caractéristiques. L'involutivité de la distribution caractéristique $\mathcal{C}$ est une conséquence directe de l'identité de Jacobi.

Notre résultat principal est donné par les deux théorèmes suivants correspondant au cas $m = 1$ (Théorème 0.0.6), respectivement au cas $m \geq 2$ (Théorème 0.0.7).

**Theorem 0.0.6.** *Considérons le système $\Sigma$, donné par (3), avec $m = 1$, et fixons $x^* \in X$. Le système $\Sigma$ est statiquement équivalent, autour de $x^*$, à la forme triangulaire $TCh_1^k$ si et seulement si les conditions suivantes sont satisfaites :*

*(Ch1)* $\mathcal{G}^{k-1} = TX;$

*(Ch2)* $\mathcal{G}^{k-3}$ *est de rang constant* $k-1$*, contient* $\mathcal{C}^{k-2}$*, la sous-distribution caractéristique de* $\mathcal{G}^{k-2}$*, et le corang de* $\mathcal{C}^{k-2}$ *dans* $\mathcal{G}^{k-3}$ *est constant, égal à un;*

*(Ch3)* $\mathcal{G}^0(x^*)$ *n'est pas contenue dans* $\mathcal{C}^{k-2}(x^*)$*;*

*(Comp)* $[f, \mathcal{C}^i] \subset \mathcal{G}^i$*, pour* $1 \leq i \leq k-2$*, où* $\mathcal{C}^i$ *est la distribution caractéristique de* $\mathcal{G}^i$*.*

Les conditions *(Ch1)-(Ch3)* caractérisent la forme chaînée (voir [59]) alors que la condition de compatibilité *(Comp)* prend en compte la dérive $f$ et donne les conditions de compatibilité pour que $f$ soit sous la forme triangulaire, dans le bon système de coordonnées, i.e., dans le système des coordonnées $z$ dans lequel les champs contrôlés $g_i$ sont sous la forme chaînée.

Nous traitons maintenant le cas $m \geq 2$. Afin de simplifier l'écriture, nous utilisons la notation suivante : $\bar{z}^i = (z_1^1, \cdots z_m^1, z_1^2, \cdots z_m^2, \cdots, z_1^i, \cdots z_m^i)$, pour $2 \leq i \leq k$.

Le Théorème 0.0.7 donne des conditions nécessaires et suffisantes pour qu'un système $\Sigma$, avec $m \geq 2$, soit statiquement équivalent à la forme triangulaire suivante :

$$TCh_m^k : \left\{ \begin{array}{llllll} \dot{z}_0 = v_0 & \dot{z}_1^1 = f_1^1(z_0, \bar{z}^2) & +z_1^2 v_0 & \cdots & \dot{z}_m^1 = f_m^1(z_0, \bar{z}^2) & +z_m^2 v_0 \\ & \dot{z}_1^2 = f_1^2(z_0, \bar{z}^3) & +z_1^3 v_0 & \cdots & \dot{z}_m^2 = f_m^2(z_0, \bar{z}^3) & +z_m^3 v_0 \\ & \vdots & & & \vdots & \\ & \dot{z}_1^{k-1} = f_1^{k-1}(z_0, \bar{z}^k) & +z_1^k v_0 & \cdots & \dot{z}_m^{k-1} = f_m^{k-1}(z_0, \bar{z}^k) & +z_m^k v_0 \\ & \dot{z}_1^k = v_1 & & \cdots & \dot{z}_m^k = v_m & \end{array} \right.$$

**Theorem 0.0.7.** *Considérons le système* $\Sigma$*, donné par (3) avec* $m \geq 2$*, et fixons* $x^* \in X$*. Le système* $\Sigma$ *est statiquement équivalent, autour de* $x^*$*, à la forme triangulaire* $TCh_m^k$ *si et seulement si les conditions suivantes sont satisfaites :*

*(m-Ch1)* $\mathcal{G}^{k-1} = TX$*;*

*(m-Ch2)* $\mathcal{G}^{k-2}$ *est de rang constant* $(k-1)m+1$ *et contient une sous-distribution involutive* $\mathcal{L}$*, de corang constant, égal à un, dans* $\mathcal{G}^{k-2}$*;*

*(m-Ch3)* $\mathcal{G}^0(x^*)$ *n'est pas contenue dans* $\mathcal{L}(x^*)$*;*

*(m-Comp)* $[f, \mathcal{C}^i] \subset \mathcal{G}^i$*, pour* $1 \leq i \leq k-2$*, où* $\mathcal{C}^i$ *est la distribution caractéristique de* $\mathcal{G}^i$*.*

Les conditions *(m-Ch1)-(m-Ch3)* caractérisent la forme multi-chaînée et *(m-Comp)* prend en compte la dérive $f$ et donne les conditions de compatibilité pour que $f$ soit sous la forme triangulaire souhaitée dans le bon système de coordonnées.

La caractérisation de la forme chaînée diffère de celle de la forme multi-chaînée (comparer les conditions *(Ch1)-(Ch3)* et *(m-Ch1)-(C-mCh3)*), mais les conditions de compatibilité sont les mêmes (comparer *(Comp)* et *(m-Comp)*). La sous-distribution involutive $\mathcal{L}$, qui est cruciale pour la forme multi-chaînée, n'est pas présente dans les conditions de compatibilité, cependant elle joue un rôle très important dans le calcul des sorties plates minimales et des singularités (voir Section 3.1.4).

Ensuite, nous discutons la platitude des systèmes de contrôle statiquement équivalents à $TCh_1^k$, si $m = 1$, ou à $TCh_m^k$, si $m \geq 2$, et déterminons toutes les sorties $x$-plates, si $m = 1$, et toutes les sorties $x$-plates minimales, si $m \geq 2$ (voir Theorems 3.1.3 et 3.1.4, Section 3.1.4). Les systèmes équivalents à $TCh_1^k$ ou à $TCh_m^k$ sont $x$-plats et manifestent des singularités (dépendantes de l'état) dans l'espace de contrôle. L'ensemble des contrôles singuliers (pour lesquels le système cesse d'être plat) est défini de manière invariante à l'aide de la dérive $f$ et des distributions caractéristiques $\mathcal{C}^i$, ainsi que de la sous-distribution involutive $\mathcal{L}$, si $m \geq 2$.

Nous montrons que la description des sorties plates des systèmes statiquement équivalents à $TCh_1^k$ (respectivement des sorties plates minimales des systèmes statiquement équivalents à $TCh_m^k$) coïncide avec celle des sorties plates de la forme chaînée (respectivement avec celle des sorties plates minimales pour la forme multichaînée). A un cas particulier près, la dérive (sauf pour ce cas particulier) ne joue donc aucun rôle dans la caractérisation des sorties plates, mais elle intervient dans la définition des contrôles singuliers. La Proposition 3.1.2 (respectivement la Proposition 3.1.4), Section 3.1.4, nous permet d'en déduire explicitement un système d'EDP à résoudre afin de trouver toutes les sorties plates (respectivement toutes les sorties plates minimales).

En fin, nous souhaiterions appliquer ces résultats à un système mécanique : une pièce qui roule sans glissement sur une table en mouvement. Nous nous sommes posés la question suivante : quand ce système est-il statiquement équivalent à la forme triangulaire compatible avec la forme chaînée ? Nous avons montré que le système peut se mettre sous la forme $TCh_1^k$ si et seulement si la dynamique de la table est décrite par les équations suivantes :

$$\begin{cases} \dot{x} &= \quad cy + e \\ \dot{y} &= -cx + f \end{cases}$$

où $c$, $e$ et $f$ sont des constantes réelles.

Dans la deuxième partie de ce chapitre, nous introduisons le concept de platitude $x$-maximale. Ces résultats ont été réalisés en colaboration avec ShunJie Li et ont été présentés à MTNS 2014, [42].

Considérons un système $\Xi : \dot{x} = F(x, u)$, où $x \in X \subset \mathbb{R}^n$ et $u \in U \subset \mathbb{R}^m$, plat au point $(x^*, \bar{u}^{p*}) \in X^p$. Soit $(\varphi_1, \ldots, \varphi_m)$ une sortie plate. Étant donné que pour tout $l \geq 0$, toutes les dérivées successives des sorties plates $\varphi_i^{(j)}$, $1 \leq i \leq m$, $0 \leq j \leq l$, sont indépendantes, à la dérivation suivante nous obtenons $m$ nouvelles fonctions indépendantes

$$\varphi_i^{(l+1)} = \varphi_i^{(l+1)}(x, u, \dot{u}, \ldots, u^{(p+l+1)}),\ 1 \leq i \leq m.$$

Nous nous intéressons au problème suivant : combien des nouvelles fonctions dépendantes de l'état uniquement, obtenons-nous après chaque dérivation successive ? Un système de contrôle est $x$-maximalement plat si le nombre de nouvelles fonctions d'états indépendantes exprimées à chaque dérivation successive des sorties plates est le plus grand possible. Afin de formaliser ceci, pour deux codistributions $\mathcal{E}$ et $\mathcal{F}$, nous définissons leur intersection ponctuelle $\mathcal{E} \cap \mathcal{F}$ par $(\mathcal{E} \cap \mathcal{F})(x) = \mathcal{E}(x) \cap \mathcal{F}(x)$, pour $x \in X$, et nous introduisons les notations :

$$\begin{aligned}\Phi^j &= \operatorname{span}\{d\varphi_i, \cdots, d\varphi_i^{(j)}, 1 \le i \le m\},\\ \mathcal{A}^j &= \Phi^j \cap T^*X\\ &= \operatorname{span}\{d\varphi_i, \cdots, d\varphi_i^{(j)}, 1 \le i \le m\} \cap T^*X,\end{aligned}$$

et définissons $a^j(\xi) = \dim \mathcal{A}^j(\xi)$, où $\xi = (x, u, \dot{u}, \ddot{u}, \cdots)$. Le vecteur $(a^0(\xi), a^1(\xi), \cdots, a^\rho(\xi))$ sera appelé $x$-vecteur de croissance de la suite des codistributions $\Phi^0 \subset \Phi^1 \subset \cdots \subset \Phi^\rho$, où $\rho$ est le plus petit entier tel que $\mathcal{A}^\rho = T^*X$.

**Definition 0.0.2.** Un système $\Xi$ plat en $(x^*, \bar{u}^{p*}) \in X^p$, pour $p \geq -1$, est appelé *$x$-maximalement plat* en $(x^*, \bar{u}^{p*})$ s'il existe une sortie plate en $(x^*, \bar{u}^{p*})$ pour laquelle les codistributions $\mathcal{A}^j$ ne dépendent pas du contrôle où des dérivées du contrôle et, si dans un voisinage de $x^*$, la suite $(a^0(x), a^1(x), \cdots, a^\rho(x))$ est constante et la plus grande possible parmi tous les systèmes plats avec $\dim U = m$ et $\dim X = n$.

Tout d'abord, remarquer que les systèmes $x$-maximalement plats sont simplement les systèmes statiquement linéarisables avec les indices de contrôlabilité $\rho_i = \frac{n}{m}$, pour $1 \leq i \leq m$, (Proposition 3.2.1, Section 3.2.2). En effet, pour ces systèmes, le nombre de nouveaux états gagnés à chaque dérivation successive des sorties plates est $m$, le plus grand possible. En général, un système plat n'est pas statiquement linéarisables, néanmoins, nous pouvons nous intéresser à la platitude $x$-maximale d'une classe particulière de systèmes. Par la suite, nous supposons que le nombre de contrôles est $m + 1$ (et pas $m$). Nous verrons que, effectivement, un contrôle joue un rôle particulier.

Un système plat $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$, linéaire par rapport aux contrôles, définie sur un espace d'états de dimension $n = km + 1$, n'est jamais statiquement linéarisable (sauf s'il a autant d'états que des contrôles). Par conséquent, il ne peut pas admettre un $x$-vecteur de croissance $(m + 1, 2(m + 1), 3(m + 1), \cdots)$. Le $x$-vecteur de croissance peut commencer par $m + 1$ (si le système est $x$-plat), mais, étant donné que le système est linéaire par rapport aux contrôles, les dérivées $\dot{\varphi}_i$, pour $0 \leq i \leq m$, font nécessairement intervenir le contrôle. Donc le nombre maximal de nouvelles fonctions dépendantes uniquement de l'état que les dérivées $\dot{\varphi}_i$, pour $0 \leq i \leq m$, peuvent fournir est au plus $m$. Ainsi, la deuxième composante du $x$-vecteur de croissance peur être au plus $2m + 1$. Le $x$-vecteur de croissance maximal est donc $(m + 1, 2m + 1, 3m + 1, \cdots, km + 1)$ et il est réalisé par les systèmes statiquement équivalents à la forme multi-chaînée, voir Proposition 3.2.2, Section 3.2.2.

Une question naturelle se pose : à quelles conditions la platitude $x$-maximale de $\Sigma_{lin}$ est-elle préservée si nous perturbons le système $\Sigma_{lin}$ (statiquement équivalent à la forme multi-chaînée) en ajoutant une dérive $f$ et obtenant de cette manière un système affine par rapport aux contrôles $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$ ? Autrement dit, quelles sont les conditions satisfaites par la dérive $f$ afin que le $x$-vecteur de croissance associé au système $\Sigma_{aff}$ (dont le sous-système linéaire $\Sigma_{lin}$ est statiquement équivalent à la forme multi-chaînée) soit $(m + 1, 2m + 1, 3m + 1, \cdots, km + 1)$ ?

Le résultat principal de la deuxième partie de ce chapitre, donné par le Théorème 0.0.8, répond à la question précédente et généralise ainsi la Proposition 3.2.2. Avant d'énoncer le théorème, introduisons quelques notations. Un bouclage statique, inversible, $u = \alpha(x) + \beta(x)\tilde{u}$ transforme le système $\Sigma_{aff}$ sous la forme $\tilde{\Sigma}_{aff} : \dot{x} = \tilde{f}(x) + \sum_{i=0}^{m} \tilde{u}_i \tilde{g}_i(x)$, où $\tilde{f} = f + g\alpha$ et $\tilde{g} = g\beta$, avec $g = (g_0, \cdots, g_m)$ et $\tilde{g} = (\tilde{g}_0, \cdots, \tilde{g}_m)$. A $\tilde{\Sigma}_{aff}$, nous associons la $(k-1)$-prolongation

$$\tilde{\Sigma}_{aff}^{(k-1,0,\ldots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_0(x) + \sum_{i=1}^{m} u_i^p \tilde{g}_i(x) \\ \dot{y}_1 &= y_2 \\ &\vdots \\ \dot{y}_{k-2} &= y_{k-1} \\ \dot{y}_{k-1} &= u_0^p \end{cases}$$

avec $y_1 = \tilde{u}_0$, $u_i^p = \tilde{u}_i$, pour $1 \leq i \leq m$, obtenue en prolongeant $k-1$ fois le contrôle $\tilde{u}_0$ comme $u_0^p = \tilde{u}_0^{(k-1)}$. La dérive et les champs de vecteurs contrôlés du système prolongé $\tilde{\Sigma}_{lin}^{(k-1,0,\ldots,0)}$ seront notés par $f_p$ et, respectivement, par $g_{pi}$, où $0 \leq i \leq m$. Les distributions du système prolongé seront notées en utilisant le sous-index $p$, i.e., $\mathcal{D}_p^0 = \text{span}\{g_{p0}, \cdots, g_{pm}\}$ et $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [f_p, \mathcal{D}_p^i]$.

Le résultat suivant est valide pour les deux cas, $m = 1$ et $m \geq 2$, et caractérise la forme triangulaire compatible avec la forme multi-chainée du point de vue de la platitude $x$-maximale (si $m = 1$ la forme multi-chainée désigne simplement la forme chainée). L'ensemble des contrôles singuliers sera noté par $U_{aff}^{sing}$.

**Theorem 0.0.8.** *Considérons la classe $\mathfrak{C}$ des système affine par rapport aux contrôles $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$ dont le sous-système linéaire $\Sigma_{lin} : \dot{x} = \sum_{i=0}^{m} u_i g_i(x)$ est statiquement équivalent à la forme multi-chainée. Pour $\Sigma_{aff} \in \mathfrak{C}$, les conditions suivantes sont équivalentes:*

*(Aff 1)* *$\Sigma_{aff}$ est x-maximalement plat en $(x^*, \bar{u}^{r*})$, pour $r \geq -1$, dans la classe $\mathfrak{C}$;*

*(Aff 2)* *$\Sigma_{aff}$ est x-maximalement x-plat en $(x^*, u^*)$ dans la classe $\mathfrak{C}$;*

*(Aff 3)* *$\Sigma_{aff}$ admet une sortie plate en $(x^*, u^*)$ dont le x-vecteur de croissance est constant, égal à $(m+1, 2m+1, 3m+1, \cdots, km+1)$ et les codistributions $\mathcal{A}^j$, pour $0 \leq j \leq k-1$, ne dépendent pas du contrôle ni de ses dérivées;*

*(Aff 4)* *$\Sigma_{aff}$ est localement, autour de $x^*$, statiquement équivalent à la forme triangulaire compatible avec la forme multi-chaînée $TCh_m^k$ et $u^* \notin U_{aff}^{sing}(x^*)$;*

*(Aff 5)* *Le système $\Sigma_{aff}$ satisfait, autour de $(x^*, u^*)$, avec $u^*(x^*) \notin U_{aff}^{sing}(x^*)$, les conditions suivantes :*

*(m-Ch1)'* *$\mathcal{G}^{k-1} = TX$;*

*(m-Ch2)'* *$\mathcal{G}^{k-2}$ est de rang constant $(k-1)m+1$ et, si $m \geq 2$, contient une sous-distribution involutive $\mathcal{L}$ de corang constant un dans $\mathcal{G}^{k-2}$;*

*(m-Ch3)'* *$\mathcal{G}^0(x^*)$ n'est pas contenue dans $\mathcal{L}(x^*)$, si $m \geq 2$ (ou n'est pas contenue dans $\mathcal{C}^{k-2}(x^*)$, si $m = 1$);*

*(m-Comp)* *$[f, \mathcal{C}^i] \subset \mathcal{G}^i$, pour $1 \leq i \leq k-2$, où $\mathcal{C}^i$ est la distribution caractéristique de $\mathcal{G}^i$.*

*(Aff 6)* *Il existe, autour de $x^*$, un bouclage statique inversible $u = \alpha(x) + \beta(x)\tilde{u}$, qui transforme le système $\Sigma_{aff}$ sous la forme $\tilde{\Sigma}_{aff}$ : $\dot{x} = \tilde{f}(x) + \sum_{i=0}^{m} \tilde{u}_i \tilde{g}_i(x)$, telle que les distributions $\mathcal{D}_p^i$ associées à la $(k-1)$-prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\ldots,0)}$ satisfassent : pour tout $0 \leq i \leq k-2$, les distributions $\mathcal{D}_p^i \cap TX$ ne dépendent pas de $y$, sont involutives, de rang constant $m(i+1)$ et $\mathcal{D}_p^{k-1} \cap TX = TX$.*

Nous ne prétendons pas qu'un système $\Sigma_{aff}$ satisfaisant une des conditions ci-dessus soit $x$-maximalement plat. De toute évidence, les systèmes $x$-maximalement plats sont les systèmes linéarisables statiquement. Le théorème précédent décrit les systèmes $x$-maximalement plats parmi les systèmes de la classe $\mathfrak{C}$ des systèmes affines dont le sous-système linéaire est statiquement équivalent à la forme multi-chaînée. De même que pour les systèmes linéaires, le $x$-vecteur de croissance commence par $m+1$, et, étant donné que le sous-système linéaire est statiquement équivalent à la forme multi-chaînée, sa deuxième composante peut être au plus $2m+1$. Les conditions *(m-Ch1)'-(m-Ch3)'* et *(m-Comp)* regroupent les deux cas, $m = 1$ et $m \geq 2$, et donnent les conditions nécessaires et suffisantes pour que le système soit statiquement équivalent à la forme triangulaire $TCh_m^k$.

Supposons maintenant que $\Sigma_{aff}$ soit $x$-plat et que son sous-système linéaire soit statiquement équivalent à la forme multi-chaînée. Nous souhaiterions savoir si $\Sigma_{aff}$ satisfait les conditions du Théorème 0.0.8. Autrement dit, un système $x$-plat, affine par rapport aux contrôles, dont le sous-système linéaire est statiquement équivalent à la forme multi-chaînée, est-il nécessairement équivalent à la forme triangulaire $TCh_m^k$ ? La réponse à cette question est négative comme le démontre l'exemple présenté en Section 3.2.4.

# 1 Flatness of Two-Input Control-Affine Systems Linearizable via One-fold Prolongation

**Abstract**

We study flatness of two-input control-affine systems, defined on an $n$-dimensional state-space. We give a complete geometric characterization of systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control. They form a particular class of flat systems: they are of differential weight equal to $n+3$. We give normal forms compatible with the minimal flat outputs and provide a system of first order PDE's to be solved in order to find all minimal flat outputs. We illustrate our results by two examples: the induction motor and the polymerization reactor.

## 1.1 Introduction

In this paper, we study flatness of nonlinear control systems of the form

$$\Xi : \ \dot{x} = F(x, u),$$

where $x$ is the state defined on a open subset $X$ of $\mathbb{R}^n$ and $u$ is the control taking values in an open subset $U$ of $\mathbb{R}^m$ (more generally, an $n$-dimensional manifold $X$ and an $m$-dimensional manifold $U$, respectively). The dynamics $F$ are smooth and the word smooth will always mean $\mathcal{C}^\infty$-smooth.

The notion of flatness has been introduced in control theory in the 1990's by Fliess, Lévine, Martin and Rouchon [13, 14] (see also [21, 22, 32, 54]) and has attracted a lot of attention because of its multiple applications in the problem of trajectory tracking and motion planning (see, e.g. [15, 26, 36, 52, 55, 58, 62]).

The fundamental property of flat systems is that all their solutions may be parametrized by $m$ functions and their time-derivatives, $m$ being the number of controls. More precisely, the system $\Xi : \ \dot{x} = F(x, u)$ is *flat* if we can find $m$ functions, $\varphi_i(x, u, \ldots, u^{(r)})$, for some $r \geq 0$, called *flat outputs*, such that

$$x = \gamma(\varphi, \ldots, \varphi^{(s)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(s)}), \tag{1.1}$$

for a certain integer $s$ and for all solutions of $\Xi$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$. Therefore all state and control variables can be determined from the flat outputs without integration and all trajectories of the system can be completely parameterized.

It is well known that systems linearizable via invertible static feedback are flat. Their description (1.1) uses the minimal possible, which is $n + m$, number of time-derivatives of the components of flat outputs $\varphi_i$. For any flat system, that is not static feedback linearizable, the minimal number of derivatives needed to express $x$ and $u$ (which will be called the differential weight) is thus bigger than $n + m$ and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system. Any single input-system is flat if and only if it is static feedback linearizable (and thus of differential weight n+1), see [9, 54]. Therefore the simplest systems for which the differential weight is bigger than $n + m$ are systems with two controls linearizable via one-dimensional precompensator, thus of differential weight $n + 3$. They form the class that we are studying in the paper: our goal is to give a geometric characterization of two-input control-affine systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control.

The paper is organized as follows. In Section 1.2, we recall the definition of flatness and define the notion of differential weight of a flat system. In Section 1.3, we give our main results. We characterize two-input control-affine systems linearizable via one-fold prolongation of a suitably chosen control, that is flat systems, of differential weight $n + 3$. We present in Section 1.4 normal forms compatible with the minimal flat outputs and give a system of first order PDE's to be solved in order to find all minimal flat outputs in Section 1.5. We illustrate our results by two examples in Section 1.6 and provide proofs in Section 1.7.

## 1.2 Flatness

Flat systems form a class of control systems, whose set of all trajectories can be parametrized by a finite number of functions and their time-derivatives. Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set $U^{-1}$ is empty and $\bar{u}^{-1}$ in an empty sequence.

**Definition 1.2.1.** The system $\Xi : \dot{x} = F(x, u)$ is *flat* at $(x_0, \bar{u}_0^l) \in X \times U^l$, for $l \geq -1$, if there exists a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and $m$ smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property: there exist an integer $s$ and smooth functions $\gamma_i$, $1 \leq i \leq n$, and $\delta_j$, $1 \leq j \leq m$, such that

$$\begin{aligned} x_i &= \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \\ u_j &= \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \end{aligned}$$

along any trajectory $x(t)$ given by a control $u(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

When necessary to indicate the number of derivatives of $u$ on which the flat outputs $\varphi_i$ depend, we will say that the system $\Xi$ is $(x, u, \cdots, u^{(r)})$-flat if $u^{(r)}$ is the highest derivative on which $\varphi_i$ depend and in the particular case $\varphi_i = \varphi_i(x)$, we will say that the system is $x$-flat.

In general, $r$ is smaller than the integer $l$ needed to define the neighborhood $\mathcal{O}^l$ which, in turn, is smaller than the number of derivatives of $\varphi_i$ that are involved (in our study $r = -1$ and $l = -1$ or 0). The minimal number of derivatives of components of a flat output, needed to express $x$ and $u$, will be called the differential weight of that flat output and is formalized as follows.

By definition, for any flat output $\varphi$ of a flat system $\Xi$ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{array}{rcl} x & = & \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u & = & \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}). \end{array}$$

Moreover, we can choose $(s_1, \ldots, s_m)$ such that (see [58]) if for any other $m$-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ we have

$$\begin{array}{rcl} x & = & \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u & = & \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{array}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^m (s_i + 1) = m + \sum_{i=1}^m s_i$ the differential weight of $\varphi$. A flat output of $\Xi$ is called *minimal* if its differential weight is the lowest among all flat outputs of $\Xi$. We define the *differential weight* of a flat system to be equal to the differential weight of any of its minimal flat outputs.

Consider a control-affine system of the form

$$\Sigma \ : \ \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x), \tag{1.2}$$

where $f$ and $g_1, \ldots, g_m$ are smooth vector fields on $X$. The system $\Sigma$ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \alpha(x) + \beta(x)v$, to a linear controllable system

$$\Lambda : \dot{z} = Az + Bv.$$

The problem of static feedback linearization was solved by Jakubczyk and Respondek [23] and Hunt and Su [19] who gave geometric necessary and sufficient conditions. The following theorem recalls their result and, furthermore, gives an equivalent way of describing static feedback linearizable systems from the point of view of differential weight.

Define inductively the sequence of distributions $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$, where $\mathcal{D}^0 = \text{span}\{g_1, \cdots, g_m\}$.

**Theorem 1.2.1.** *The following conditions are equivalent:*

*(i)* $\Sigma$ *is locally static feedback linearizable, around* $x_0 \in X$;

*(ii)* $\Sigma$ *is locally static feedback equivalent, around* $x_0 \in X$, *to the Brunovský canonical form*

$$(Br) \quad \begin{array}{rcl} \dot{z}^i_j & = & z^i_{j+1} \\ \dot{z}^i_{\rho_i} & = & v_i \end{array}$$

*where* $1 \leq i \leq m$, $1 \leq j \leq \rho_i - 1$, *and* $\sum_{i=1}^m \rho_i = n$;

*(iii) For any $i \geq 0$, the distributions $\mathcal{D}^i$ are of constant rank, around $x_0 \in X$, involutive and $\mathcal{D}^{n-1} = TX$;*

*(iv) $\Sigma$ is flat at $x_0 \in X$, of differential weight $n + m$.*

The geometry of static feedback linearizable systems is given by the following sequence of nested involutive distributions:

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{n-1} = TX.$$

It is well known that a feedback linearizable system is static feedback equivalent to the Brunovský canonical form, see [5], and is clearly flat with $\varphi = (\varphi_1, \cdots, \varphi_m) = (z_1^1, \cdots, z_1^m)$ being a minimal flat output (of differential weight $n + m$). Therefore, for static feedback linearizable systems, the representation of all states and controls uses the minimal possible, which is $n + m$, number of time-derivatives of $\varphi_i$ and an equivalent way of describing them is that they are flat systems of differential weight $n + m$.

In general, a flat system is not linearizable by invertible static feedback, with the exception of the single-input case where flatness reduces to static feedback linearization, see [9]. Flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see [13, 14, 32, 54]. Our goal in this paper is to describe the simplest flat systems that are not static feedback linearizable: two-inputs control-affine systems that become static feedback linearizable after one-fold prolongation, which is the simplest dynamic feedback. They are flat systems of differential weight equal to $n + 3$. In this paper, we will completely characterize them and show how their geometry differs but also how it reminds that given by the involutive distributions $\mathcal{D}^i$ for static feedback linearizable systems. We will also give normal forms compatible with the minimal flat outputs (thus generalizing the Brunovský normal form) and provide a system of first order PDE's to find all minimal flat outputs.

## 1.3 Main results

Throughout, we will consider two-input control-affine systems of the form

$$\Sigma \; : \; \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x), \tag{1.3}$$

where $x \in X$, $u = (u_1, u_2)^t \in \mathbb{R}^2$ and $f$, $g_1$, and $g_2$ are $\mathcal{C}^\infty$-smooth vector fields on $X$ and that are not static feedback linearizable.

We make the following assumption:

(Assumption 1) From now on, unless stated otherwise, we assume that all ranks involved are constant in a neighborhood of a given $x_0 \in X$.

**Remark 1.3.1.** All results presented here are valid on an open and dense subset of either $X$ or $X \times U$ and hold locally, around a given point of that set.

Flat systems of differential weight $n+3$ form a particular class of dynamic feedback linearizable systems, namely, they become static feedback linearizable after one prolongation of a suitably chosen control. More precisely, we have the following result:

**Proposition 1.3.1.** *Consider a two-input control-system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$. The following conditions are equivalent:*

*(i) $\Sigma$ is flat at $(x_0, \bar{u}_0^l)$, with the differential weight $n+3$, for a certain $l \geq 1$;*

*(ii) $\Sigma$ is x-flat at $(x_0, u_0)$, with the differential weight $n+3$;*

*(iii) There exists, around $x_0$, an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing the system $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 \tilde{g}_2(x)$, such that the prolongation*

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + v_2 \tilde{g}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

*is locally static feedback linearizable, where $y_1 = \tilde{u}_1$, $v_2 = \tilde{u}_2$, $\tilde{f} = f + \alpha g$ and $\tilde{g} = g\beta$, where $g = (g_1, g_2)$ and $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$.*

A system $\Sigma$ satisfying $(iii)$ will be called dynamically linearizable via invertible one-fold prolongation. Notice that $\tilde{\Sigma}^{(1,0)}$ is, indeed, obtained by prolonging the first control $\tilde{u}_1$ one time as $v_1 = \dot{\tilde{u}}_1$ and not prolonging $\tilde{u}_2$ (which explains the notation). The above result asserts that for systems of the differential weight $n+3$, flatness and $x$-flatness coincide and that, moreover, they are equivalent to linearizability via the simplest dynamic feedback, namely one-fold prolongation.

Recall that the system $\Sigma$ is assumed not static feedback linearizable. This occurs if there exists an integer $k$ such that $\mathcal{D}^k$ is not involutive. Suppose that $k$ is the smallest integer satisfying that property. Moreover, the condition $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} = 2$ is for dynamic linearizability via one-fold prolongation and thus for flatness of differential weight $n+m+1$, as asserts Proposition 1.7.1, in Section 1.7. Therefore throughout we will suppose the following:

(Assumption 2) *$k$ is the smallest integer such that $\mathcal{D}^k$ is not involutive and* $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} = 2$.

Our main result describing flat systems of differential weight $n+3$ is given by two following theorems corresponding to the first noninvolutive distribution $\mathcal{D}^k$ being either $\mathcal{D}^0$, i.e., $k=0$ (Theorem 1.3.2) or $\mathcal{D}^k$, for $k \geq 1$ (Theorem 1.3.1). For both theorems, we assume that $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$. The particular case $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ (met in applications, see Example 5.1) will be discussed at the end of this section (in Theorem 1.3.4). We will denote by $\overline{\mathcal{D}}^k$ the involutive closure of $\mathcal{D}^k$, i.e., the smallest involutive distribution containing $\mathcal{D}^k$.

**Theorem 1.3.1.** *Assume $k \geq 1$ and $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$. Consider the two-input control system $\Sigma$, given by (1.3). The system $\Sigma$ is flat at $x_0$, of differential weight $n+3$, if and only if the following conditions hold:*

*(A1)* $\mathrm{rk}\,\overline{\mathcal{D}}^k = 2k+3$;

*(A2)* $\mathrm{rk}\,(\overline{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$, *implying the existence of a non-zero vector field* $g_c \in \mathcal{D}^0$ *such that* $ad_f^{k+1} g_c \in \overline{\mathcal{D}}^k$;

*(A3)* *The distributions* $\mathcal{H}^i$, *for* $i \geq k$, *are involutive, where* $\mathcal{H}^k = \mathcal{D}^{k-1} + \mathrm{span}\,\{ad_f^k g_c\}$ *and* $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, *for* $i \geq k$;

*(A4)* *There exists* $\rho$ *such that* $\mathcal{H}^\rho = TX$.

The geometry of the systems described by the previous theorem can be summarized by the following sequence of inclusions:

$$\begin{array}{ccccccccccccc}
\mathcal{D}^0 \underset{2}{\subset} \cdots \underset{2}{\subset} \mathcal{D}^{k-1} & \underset{2}{\subset} & \mathcal{D}^k & \underset{1}{\subset} & \overline{\mathcal{D}}^k & & & & & & & & \\
 & & {}_1\cup & & \| & & & & & & & & \\
 & & \mathcal{H}^k & \underset{2}{\subset} & \mathcal{H}^{k+1} & \underset{2}{\subset} & \cdots & \underset{2}{\subset} & \mathcal{H}^\mu & \underset{1}{\subset} & \mathcal{H}^{\mu+1} & \underset{1}{\subset} \cdots \underset{1}{\subset} & \mathcal{H}^\rho = TX
\end{array}$$

where all the distributions, except $\mathcal{D}^k$, are involutive and the integers beneath the inclusion symbol "$\subset$" indicate coranks. According to condition *(A1)*, only one Lie bracket can stick out from the noninvolutive distribution $\mathcal{D}^k$, thus the loss of involutivity of $\mathcal{D}^k$ is minimal. Moreover, if we take the brackets of $\mathcal{D}^k$ with $f$, we gain only one new direction, see $(A2)$, implying the existence of a distinguished vector field $g_c$ in $\mathcal{D}^0$ that allows us to define the subdistribution $\mathcal{H}^k$. Notice that the existence of the corank one involutive subdistribution $\mathcal{H}^k$ in $\mathcal{D}^k$ is the main structural property of flat systems of differential weight $n+3$. Indeed, $\mathcal{H}^k$ takes the role of the noninvolutive distribution $\mathcal{D}^k$ and moreover, its successive brackets with the drift are again involutive (replacing the distributions $\mathcal{D}^{k+i}$).

It is easy to check that $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$. Indeed, by definition, $\mathcal{H}^{k+1} = \mathcal{D}^k + \mathrm{span}\,\{ad_f^{k+1} g_c\}$ and is involutive. Moreover $\mathrm{rk}\,\mathcal{H}^{k+1} = 2k+3$, otherwise we obtain $\mathcal{H}^{k+1} = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive. Since $\mathcal{D}^k \subset \mathcal{H}^{k+1}$ and $\mathrm{rk}\,\mathcal{H}^{k+1} = 2k+3$, it follows that $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$. Thus the direction completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k$ has to be colinear with $ad_f^{k+1} g_c$ modulo $\mathcal{D}^k$.

**Example 1.3.1.** The following examples shows that the existence of the involutive subdistribution $\mathcal{H}^k$ in $\mathcal{D}^k$ plays indeed a crucial role. If we do not assume its existence and define the sequence of distributions by $\mathcal{H}^{k+1} = \overline{\mathcal{D}}^k$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $i \geq k+1$, then the result is not true anymore as shown by the following two-input control system:

$$(S): \left\{ \begin{array}{rclrcl} \dot{x}_1 & = & x_2 & \dot{z}_1 & = & \frac{1}{2} z_3^2 + z_2 \\ \dot{x}_2 & = & x_3 & \dot{z}_2 & = & z_3 \\ \dot{x}_3 & = & u_1 & \dot{z}_3 & = & u_2 \end{array} \right.$$

which is not dynamically linearizable via one-fold prolongation. A way to see it is that the system is composed by two independent single-input subsystems, the first one linear and the second one that cannot be dynamically linearizable. Another way to see it is that $(S)$ is in fact the prolongation (obtained by prolonging twice the first control) of the following control system

$$(\tilde{S}):\left\{\begin{array}{lrcl} \dot{x}_1=\tilde{u}_1 & \dot{z}_1 & = & \frac{1}{2}z_3^2+z_2 \\ & \dot{z}_2 & = & z_3 \\ & \dot{z}_3 & = & \tilde{u}_2 \end{array}\right.$$

which has been shown in [55] (Theorem 3.1, case 2) not to be linearizable by endogenous dynamic feedback. Hence, $(\tilde{S})$ is not flat and we deduce that $(S)$ is not flat as either, and in particular, not flat of differential weight $n+3$. The first noninvolutive distribution is $\mathcal{D}^1 = \operatorname{span}\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_1}\}$, so $k=1$ and we clearly have $\overline{\mathcal{D}}^1 = \operatorname{span}\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\}$. Defining the sequence $\mathcal{H}$ as above, we have $\mathcal{H}^2 = \overline{\mathcal{D}}^1$ and $\mathcal{H}^3 = TX$. Moreover, $\operatorname{rk}(\overline{\mathcal{D}}^1 + [f,\mathcal{D}^1]) = 6$ and $ad_f g_2 = \frac{\partial}{\partial z_1} \in \overline{\mathcal{D}}^1$. Thus the distinguished vector field defined by item $(A2)$ is $g_2 = \frac{\partial}{\partial z_3}$ and the distribution $\mathcal{H}^1$ is given by $\mathcal{H}^1 = \operatorname{span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_1}\}$ and is clearly noninvolutive. So all conditions, except the involutivity of $\mathcal{H}^1$, are verified, however, the system is not flat of differential weight $n+3$, proving that if we skip the assumption of the involutivity of $\mathcal{H}^1$, the result does not hold anymore.

Theorem 1.3.1 enables us to define the control (given up to a multiplicative function) to be prolonged in order to obtain a locally static feedback linearizable $\tilde{\Sigma}^{(1,0)}$. The vector field $g_c \in \mathcal{D}^0$ (see $(A2)$) can be expressed as $g_c = \beta_1 g_1 + \beta_2 g_2$, for some smooth functions (not vanishing simultaneously) on $X$. We define the to-be-prolonged control as $u_p(t) = \beta_2(x(t))u_1(t) - \beta_1(x(t))u_2(t)$ and it is the control that needs to be preintegrated in order to dynamically linearize the system, that is, we put $v_1 = \frac{d}{dt}(\beta_2 u_1 - \beta_1 u_2) = \frac{d}{dt}\tilde{u}_1$.

If $k=0$, i.e., the first noninvolutive distribution is $\mathcal{D}^0 = \mathcal{G}^0$, then a similar result holds, but in the chain of involutive subdistributions $\mathcal{H}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots$ (playing the role of $\mathcal{H}^k \subset \mathcal{H}^{k+1} \subset \mathcal{H}^{k+2} \subset \cdots$), with $\mathcal{H}^0 = \operatorname{span}\{g_c\}$, the distribution $\mathcal{H}^1$ is not defined as $\mathcal{H}^{k+1}$ but as $\mathcal{H}^1 = \mathcal{G}^1 = \mathcal{G}^0 + [\mathcal{G}^0, \mathcal{G}^0]$ (compare $(A3)$ and $(A3)'$). Moreover, flat systems with $k=0$ exhibit a singularity in the control space (created by one-fold prolongation of the to-be-prolonged control) which is defined by

$$U_{sing}(x) = \{u \in \mathbb{R}^2 : \ (g_1 \wedge g_c \wedge [f + u_1 g_1 + u_2 g_c, g_c])(x) = 0\},$$

and excluded by condition $(RC)$, where the vector fields $g_1$ and $g_c$ are such that $\mathcal{D}^0 = \operatorname{span}\{g_1, g_c\}$. Notice that the set of singular controls is non empty due to condition $(A2)'$.

**Theorem 1.3.2.** *Assume $k=0$ and $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$. Consider the two-input control system $\Sigma$, given by (1.3). $\Sigma$ is flat at $(x_0, u_0)$, of differential weight $n+3$, if and only if the following conditions hold:*

*(A1)′* $\operatorname{rk}\overline{\mathcal{D}}^0 = 3$ *;*

*(A2)′* $\operatorname{rk}(\overline{\mathcal{D}}^0 + [f,\mathcal{D}^0]) = 4$, *implying the existence of a non-zero vector field $g_c \in \mathcal{D}^0$ such that $ad_f g_c \in \mathcal{G}^1$;*

*(A3)′* *The distributions* $\mathcal{H}^i$*, for* $i \geq 1$*, are involutive, where* $\mathcal{H}^1 = \mathcal{G}^1$ *and* $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$*, for* $i \geq 2$*;*

*(A4)′* *There exists* $\rho$ *such that* $\mathcal{H}^\rho = TX$*;*

*(RC)* $u_0 \notin U_{sing}(x_0)$.

Recall that we have assumed that the rank of all distributions involved is constant in a neighborhood of $x_0$. Thus item $(A1)'$ implies that we actually have $\overline{\mathcal{D}}^0 = \mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$. A similar result can be formulated for the singular case when both vector fields $ad_f g_1$ and $ad_f g_2$ vanish modulo $\overline{\mathcal{D}}^0 = \mathcal{G}^1$ at $x_0$ and the direction completing $\mathcal{H}^1 = \mathcal{G}^1$ to $\mathcal{H}^2 = \mathcal{G}^1 + [f, \mathcal{G}^1]$ is given by $ad_f[g_1, g_2]$. In this case, item $(A2)'$ should be replaced by $\mathrm{rk}\,(\overline{\mathcal{D}}^0 + [f, \overline{\mathcal{D}}^0]) = 4$ and $\mathrm{rk}\,(\overline{\mathcal{D}}^0 + [f, \mathcal{D}^0])(x_0) = 3$. The existence of a non-zero vector field $g_c \in \mathcal{D}^0$ such that $ad_f g_c \in \mathcal{G}^1$ is no longer redundant and we have to add it explicitly in the conditions of the theorem.

The conditions of both theorems are verifiable, i.e., given a two-input control-affine system, we can easily verify whether it is flat with the differential weight $n+3$ and verification involves derivations and algebraic operations only, without solving PDE's or bringing the system into a normal form.

The cases $k = 0$ and $k \geq 1$ are similar, but they have slightly different geometries. Even if at first sight, it seems not possible to merge them (due to the different definitions of the distributions $\mathcal{H}^1$ and $\mathcal{H}^{k+1}$ and to the existence of singularities in the control space for $k = 0$), the following result enables us to unify them. Theorem 1.3.3 is based on the observation that in both cases, we actually have $\mathcal{H}^{k+1} = \overline{\mathcal{D}}^k$ (by definition of $\mathcal{H}^1$, for $k = 0$, and as a direct consequence of the definition of $\mathcal{H}^{k+1}$, for $k \geq 1$, see the comment after Theorem 1.3.1).

**Theorem 1.3.3.** *Assume* $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$*. Consider the two-input control system* $\Sigma$*, given by (1.3).* $\Sigma$ *is x-flat at* $(x_0, u_0)$*, of differential weight* $n+3$*, if and only if*

*(A1)″* $\mathrm{rk}\,\overline{\mathcal{D}}^k = 2k+3$;

*(A2)″* $\mathrm{rk}\,(\overline{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$, *implying the existence of a non-zero vector field* $g_c \in \mathcal{D}^0$ *such that* $ad_f^{k+1} g_c \in \overline{\mathcal{D}}^k$*;*

*(A3)″* *The distribution* $\mathcal{H}^k = \mathcal{D}^{k-1} + \mathrm{span}\,\{ad_f^k g_c\}$ *is involutive, where* $\mathcal{D}^{k-1}$ *is empty if* $k = 0$*;*

*(A4)″* *The distributions* $\mathcal{H}^i$*, for* $i \geq k+1$*, are involutive, where* $\mathcal{H}^{k+1} = \overline{\mathcal{D}}^k$ *and* $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$*, for* $i \geq k+1$ *;*

*(A5)″* *There exists* $\rho$ *such that* $\mathcal{H}^\rho = TX$*.*

*(A6)″* $\mathrm{rk}\,(\mathcal{D}^k + \mathrm{span}\,\{ad_{f+gu}^{k+1} g_c\})(x_0, u_0) = 2k+3$, *where* $f + gu = f + u_1 g_1 + u_2 g_c$*.*

If $k = 0$, condition $(A3)''$ is clearly verified and item $(A6)''$ immediately implies that $(g_1 \wedge g_c \wedge [f + u_1 g_1 + u_2 g_c, g_c])(x_0, u_0) \neq 0$, thus $u_0 \notin U_{sing}(x_0)$. If $k \geq 1$, it can be easily shown that $(A6)''$ does not depend on the control and that we have $ad_f^{k+1} g_c(x_0) \notin \mathcal{D}^k(x_0)$. From this and since $ad_f^{k+1} g_c \in \overline{\mathcal{D}}^k$ and $\operatorname{rk} \overline{\mathcal{D}}^k = 2k+3$, we deduce that $\overline{\mathcal{D}}^k = \mathcal{D}^k + \operatorname{span}\{ad_f^{k+1} g_c\} = \mathcal{H}^k + [f, \mathcal{H}^k]$, giving the condition that $\mathcal{H}^{k+1} = \mathcal{H}^k + [f, \mathcal{H}^k]$, which at first glance, seems missing in the statement of Theorem 1.3.3.

Let us now consider the case $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$. The involutivity of $\mathcal{D}^k$ can be lost in two different ways: either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$ or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. As asserts Theorem 1.3.4 below, in the first case, the system is flat of differential weight $n+3$ without any additional condition whereas in the second case, the system $\Sigma$ has to verify some additional conditions analogous to those of Theorem 1.3.1. Since the condition $(A2)$, enabling us to compute the involutive subdistribution $\mathcal{H}^k$, has no sense in that case, we have to define $\mathcal{H}^k$ in another way. To this end, we introduce the characteristic distribution of $\mathcal{D}^k$, defined as follows. For a distribution $\mathcal{D}$, we call $c \in \mathcal{D}$ a characteristic vector field of $\mathcal{D}$ if $[c, \mathcal{D}] \subset \mathcal{D}$. The characteristic distribution $\mathcal{C}$ of $\mathcal{D}$ is the distribution spanned by all its characteristic vector fields. It follows directly from the Jacobi identity that the characteristic distribution is always involutive.

In the case $k = 0$ and $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$, the singular controls are not defined by $U_{sing}(x)$ but as

$$U'_{sing}(x) = \{u \in \mathbb{R}^2 \ : \ \dim \operatorname{span}\{g_1, g_2, ad_f g_1 + u_2[g_2, g_1], ad_f g_2 + u_1[g_1, g_2]\}(x) = 3.$$

**Theorem 1.3.4.** *Assume $k \geq 0$ and $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$. Then*

*(i) either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and then $\Sigma$ is flat at any $x_0 \in X$ such that $\mathcal{D}^{k+1}(x_0) = T_{x_0}X$ (flat at any $(x_0, u_0) \in X \times \mathbb{R}^2$, such that $u_0 \notin U'_{sing}(x_0)$, if $k = 0$). Moreover, if $\Sigma$ is flat, it is flat of differential weight $n+3$.*

*(ii) or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, then $k \geq 1$ and $\Sigma$ is flat of differential weight $n+3$ at $x_0 \in X$ if and only if $\Sigma$ satisfies around $x_0$ the following conditions*

*(C1)* $\operatorname{rk} \mathcal{C}^k = 2k$, *where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$ ;*

*(C2)* $\operatorname{rk}(\mathcal{C}^k + \mathcal{D}^{k-1}) = 2k+1$;

*(C3) The distribution $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$ is involutive;*

*(C4) $\mathcal{H}^{k+1} = TX$, where $\mathcal{H}^{k+1} = \mathcal{H}^k + [f, \mathcal{H}^k]$.*

The assumptions of Theorem 1.3.4 (i), i.e., $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, imply that the state-space is of dimension $n = 2k+3$ and that the rank of $\mathcal{D}^{k+1}$ (which is feedback invariant in this case) is maximal and equal to $n = 2k+3$ on an open and dense subset of X. Although, $\mathcal{D}^{k+1}(x_0) = T_{x_0}X$ on an open and dense subset of $X$, in order to prove flatness at $x_0 \in X$, we have to suppose that that condition is satisfied at $x_0$ and not only on an open and dense subset.

It can be shown that in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ (no matter whether $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ or not), the involutive subdistribution $\mathcal{H}^k$ can always be defined as above, i.e., the definition of $\mathcal{H}^k$ given by item $(A3)$ of Theorem 1.3.1 and that provided by conditions $(C1)-(C3)$ of the above theorem are equivalent if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. In other words, under $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, Theorem 1.3.4 holds with no extra assumptions. This is not valid anymore if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$; indeed, in that case $\mathcal{C}^k = \mathcal{D}^{k-1}$, condition $(C2)$ is not verified and $(C3)$ would give $\mathcal{H}^k = \mathcal{D}^{k-1}$.

## 1.4 Normal forms

The following proposition gives two different (although static feedback equivalent) normal forms for the class of two-input flat systems of differential weight $n+3$ (below $U_{sing}$ is to be replaced by $U'_{sing}$, if $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] = TX$). For $i = 1, 2$, denote $\bar{z}^i_j = (z^i_1, \cdots, z^i_j)$.

The integers $\rho_i$ and $\mu_i$ that show up in the normal forms are related to $\rho$ and $\mu$ defined via the nested sequence of distributions $\mathcal{D}^i$ and $\mathcal{H}^i$. Let $\mu$ be the smallest integer such that corank $(\mathcal{H}^\mu \subset \mathcal{H}^{\mu+1})$ is one and $\rho$ is the smallest integer such that $\mathcal{H}^\rho = TX$. It follows that $\mu \leq \rho$ and $\rho + \mu + 1 = n$. Define two pairs of indices $(\rho_1, \rho_2)$ and $(\mu_1, \mu_2)$ by $\rho = \max(\rho_1, \rho_2) = k + \max(\mu_1, \mu_2)$ and $\mu = \min(\rho_1, \rho_2) = k + \min(\mu_1, \mu_2)$. We have $\rho_1 + \rho_2 + 1 = n$ and $\mu_1 + \mu_2 + 2k + 1 = n$, implying $\rho_i \geq k+1$ and $\mu_i \geq 1$. It follows that $\mu \geq k+1$ and the equality $\rho = k+1$ holds if and only if $\mu_1 = \mu_2 = 1$ corresponding to $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$.

**Proposition 1.4.1.** *Consider a flat two-input control-affine system $\Sigma$, given by (1.3). The following conditions are equivalent:*

*(i)* *$\Sigma$ is flat at $x_0$ (at $(x_0, u_0)$, such that $u_0 = (u_{10}, u_{20}) \notin U_{sing}(x_0)$, if $k = 0$) of differential weight $n+3$;*

*(ii)* *$\Sigma$ is locally, around $x_0$, static feedback equivalent to the following normal form in a neighborhood $Z$ of $z_0 \in \mathbb{R}^n$:*

$$(NF1): \begin{cases} \dot{z}^1_1 = z^1_2 & \dot{z}^2_1 = z^2_2 \\ \quad \vdots & \quad \vdots \\ \dot{z}^1_{\rho_1-1} = z^1_{\rho_1} & \dot{z}^2_{\rho_2-1} = z^2_{\rho_2} \\ \dot{z}^1_{\rho_1} = \tilde{u}_1 & \dot{z}^2_{\rho_2} = a(z) + b(z)\tilde{u}_1 \\ & \dot{z}^2_{\rho_2+1} = \tilde{u}_2 \end{cases}$$

*where either $k \geq 1$ and then $a = z^2_{\rho_2+1}$, $b = b(\bar{z}^1_{\rho_1-k+1}, \bar{z}^2_{\rho_2-k+1})$ and $(\frac{\partial b}{\partial z^1_{\rho_1-k+1}}, \frac{\partial b}{\partial z^2_{\rho_2-k+1}})(z_0) \neq (0,0)$ or $k = 0$ and then $b = z^2_{\rho_2+1}$ and $a = a(z)$ is any function and, moreover, $\frac{\partial a}{\partial z^2_{\rho_2+1}}(z_0) + \tilde{u}_{10} \neq 0$.*

*(iii)* *$\Sigma$ is locally, around $x_0$, static feedback equivalent to the following normal form in a*

*neighborhood $W$ of $w_0 \in \mathbb{R}^n$:*

$$
(NF2) : \begin{cases}
\dot{w}_1^1 = w_2^1 & \dot{w}_1^2 = w_2^2 \\
\vdots & \vdots \\
\dot{w}_{\mu_1-1}^1 = w_{\mu_1}^1 & \dot{w}_{\mu_2-1}^2 = w_{\mu_2}^2 \\
\dot{w}_{\mu_1}^1 = w_{\mu_1+1}^1 & \dot{w}_{\mu_2}^2 = d(\bar{w}_{\mu_1+1}^1, \bar{w}_{\mu_2+1}^2) \\
\dot{w}_{\mu_1+1}^1 = w_{\mu_1+2}^1 & \dot{w}_{\mu_2+1}^2 = w_{\mu_2+2}^2 \\
\vdots & \vdots \\
\dot{w}_{\mu_1+k}^1 = \tilde{u}_1 & \dot{w}_{\mu_2+k}^2 = w_{\mu_2+k+1}^2 \\
 & \dot{w}_{\mu_2+k+1}^2 = \tilde{u}_2
\end{cases}
$$

*where $d$ is either of the form $d = c(\bar{w}_{\mu_1}^1, \bar{w}_{\mu_2+1}^2) + w_{\mu_2+1}^2 w_{\mu_1+1}^1$, with $(\frac{\partial c}{\partial w_{\mu_2+1}^2} + w_{\mu_1+1}^1)(w_0) \neq 0$, or $d = d(\bar{w}_{\mu_1+1}^1, \bar{w}_{\mu_2+1}^2)$ such that $\frac{\partial d}{\partial w_{\mu_2+1}^2}(w_0) \neq 0$ and $\frac{\partial^2 d}{\partial (w_{\mu_1+1}^1)^2}(w_0) \neq 0$; if $k = 0$, we put $w_{\mu_1+1}^1 = \tilde{u}_1$ and only the case $d = c(\bar{w}_{\mu_1}^1, \bar{w}_{\mu_2+1}^2) + w_{\mu_2+1}^2 \tilde{u}_1$ is possible.*

*Moreover, the minimal x-flat outputs and the normal forms (NF1) (resp. (NF2)) are compatible: if $(\varphi_1, \varphi_2)$ is a minimal x-flat output at $x_0$, then there exists an invertible static feedback transformation bringing the system $\Sigma$ into (NF1) with $\varphi_1 = z_1^1$ and $\varphi_2 = z_1^2$ (resp. into (NF2) with $\varphi_1 = w_1^1$ and $\varphi_2 = w_1^2$).*

*Remarks*. Each of the above normals forms has its importance and we below discuss them.

1. Both normal forms are the closest possible to Brunovský canonical form. In fact, only one nonlinearity is present, which is due to the fact that the noninvolutivity of $\mathcal{D}^k$ is minimal: $\mathcal{D}^k$ is squeezed between two involutive distributions $\mathcal{H}^k$ and $\mathcal{H}^{k+1}$ and both inclusions are of corank one (see the sequence of inclusions summarizing the geometry of flat systems of differential weight $n+3$), so only one direction of $\overline{\mathcal{D}}^k$ sticks out of $\mathcal{D}^k$.

2. The normal form (NF1) (resp. (NF2)) is valid around $z_0 \in \mathbb{R}^n$ (resp. $w_0 \in \mathbb{R}^n$), which may be zero or not. Therefore both forms can be used around any point (equilibrium or not).

3. It is immediate to see that (NF1) and (NF2) are flat with $\varphi = (z_1^1, z_1^2)$ (resp. $\varphi = (w_1^1, w_1^2)$) being minimal flat outputs and a simple computation shows that their differential weight is, indeed, $n+3$.

4. It is clear that (NF1) becomes locally static feedback linearizable after a one-fold prolongation of $\tilde{u}_1$. Moreover, if we replace $\tilde{u}_1$ by $\hat{u}_1 = \beta(z)\tilde{u}_1$, with $\beta(z) \neq 0$, and we prolong $\hat{u}_1$ instead of $\tilde{u}_1$, the prolonged system is also locally static feedback linearizable.

5. The normal forms apply to both cases $k \geq 1$ and $k = 0$, independently of whether $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$, the latter corresponding to $\mu_1 = \mu_2 = 1$.

6. The nonivolutive distribution $\mathcal{D}^k$ is easier to be analyzed with the help of (NF2). Firstly, the integer $k$ is explicit. Secondly, we see that the involutivity of $\mathcal{D}^k$ can be lost in two different ways, either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$, corresponding to the two possible definitions of the function $d$ (see item *(iii)*).

7. Notice that for $k = 0$, (NF1) and (NF2) coincide. It is clear from them that in the case $k = 0$ (and only in that case!), the precompensator creates singularities in the control space (depending on the state). Indeed, the controls $\tilde{u}_0$ satisfying the condition $\frac{\partial a}{\partial z^2_{\rho_2+1}}(z_0) + \tilde{u}_{10} = 0$ are singular for (NF1) (resp. $\frac{\partial c}{\partial w^2_{\mu_2+1}}(w_0) + \tilde{u}_{10} = 0$ for (NF2), the functions $a = c$ being the same), see Section 1.4.1.

A natural question appears: to what extent is the form (NF1) canonical? In other words, when two systems, both brought into (NF1) determined, respectively, by two functions $a$ and $\hat{a}$, if $k = 0$, or by $b$ and $\hat{b}$, if $k \geq 1$, are static feedback equivalent? In order to answer this question, we define the notion of structure preserving diffeomorphism.

**Definition 1.4.1.** A diffeomorphism $\psi : Z \mapsto \hat{Z}$ is called (NF1)-structure preserving (shortly, SP-diffeomorphism) if there exists a (local) feedback transformation $\hat{u} = \alpha(z) + \beta(z)\tilde{u}$ such that $(\hat{z}, \hat{u}) = (\psi, \alpha + \beta\tilde{u})$ maps (NF1) into

$$
(\hat{NF1}) \begin{cases} \dot{\hat{z}}^i_j & = \hat{z}^i_{j+1},\ 1 \leq i \leq 2,\ 1 \leq j \leq \rho_i - 1, \\ \dot{\hat{z}}^1_{\rho_1} & = \hat{u}_1 \\ \dot{\hat{z}}^2_{\rho_2} & = \hat{a}(\hat{z}) + \hat{b}(\hat{z})\hat{u}_1 \\ \dot{\hat{z}}^2_{\rho_2+1} & = \hat{u}_2 \end{cases}
$$

where $\hat{a}$ and $\hat{b}$ satisfy the same conditions as the functions $a$ and $b$ (see Theorem 1.4.1(ii)).

We will indicate the drift and the control vector fields of the normal form by the subindex NF, i.e., $(NF1) : \dot{z} = f_{NF}(z) + \tilde{u}_1 g^1_{NF}(z) + \tilde{u}_1 g^2_{NF}(z)$. Moreover, we can almost always suppose that $\rho_1 \geq \rho_2$. Indeed, if $\rho_1 \leq \rho_2$, then around any point at which the function $b$ does not vanish, we can apply the invertible static feedback $\bar{u}_1 = a + b\tilde{u}_1$ and $\bar{u}_2 = \tilde{u}_2$ transforming the system into a new form for which the chain of pure integrators is of length $\rho_1 \geq \rho_2$. The following proposition, in which we assume $\rho_1 \geq \rho_2$, gives necessary and sufficient conditions for a diffeomorphism $\psi$ to preserve the structure of (NF1).

**Proposition 1.4.2.** *Assume $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$.*

*(i) The diffeomorphism $\hat{z} = \psi(z)$ is a SP-diffeomorphism, preserving (NF1), if and only if $\hat{z}^i_j = L^{j-1}_{f_{NF}}\varphi_i$, for $1 \leq i \leq 2$, $1 \leq j \leq \rho_i$, and $\hat{z}^2_{\rho_2+1} = L^{\rho_2}_{f_{NF}}\varphi_2$, if $k \geq 1$, or $\hat{z}^2_{\rho_2+1} = L_{g^1_{NF}} L^{\rho_2-1}_{f_{NF}}\varphi_2$, if $k = 0$, where $(\varphi_1, \varphi_2)$ is a minimal x-flat output of (NF1).*

*(ii) (NF1) and $(\hat{NF1})$, given, respectively, by $a$ and $\hat{a}$, if $k = 0$, or $b$ and $\hat{b}$, if $k \geq 1$, are feedback equivalent if and only if there exist two smooth functions $\varphi_1(z^1_1)$ and*

$\varphi_2(z_1^1, \cdots, z_{\rho_1-\rho_2+1}^1, z_1^2)$ *such that* $a(z) = \hat{a}(\psi(z))$, *if* $k = 0$, *or* $b(z) = \hat{b}(\psi(z))$, *if* $k \geq 1$, *where* $\hat{z}_j^i = \psi_j^i = L_{f_{NF}}^{j-1}\varphi_i$, *for* $1 \leq i \leq 2$, $1 \leq j \leq \rho_i$, *and* $\hat{z}_{\rho_2+1}^2 = L_{f_{NF}}^{\rho_2}\varphi_2$, *if* $k \geq 1$, *or* $\hat{z}_{\rho_2+1}^2 = L_{g_{NF}^1} L_{f_{NF}}^{\rho_2-1}\varphi_2$, *if* $k = 0$.

According to item $(i)$, minimal flat outputs determine all structure preserving diffeomorphisms which have a very particular form. So, to compute them we, first, have to calculate minimal flat outputs. In Section 1.5 below, we answer the question of whether a given pair of smooth functions on $X$ is a minimal flat output and provide a system of PDS's to be solved in order to find all minimal flat outputs: if $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the pair $(\varphi_1, \varphi_2)$ is a minimal flat output at $x_0$ if and only if (after permuting $\varphi_1$ and $\varphi_2$, if necessary) $d\varphi_1 \perp \mathcal{H}^{\rho-1}$, $d\varphi_2 \perp \mathcal{H}^{\mu-1}$ and $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$ (where $\rho$ and $\mu$ are defined just before Proposition 1.4.1).

For the particular case $\rho_1 \leq \rho_2$ and $b(z_0) = 0$, as well as for the normal form (NF2), a similar analysis can be done. We do not present those cases here.

### 1.4.1 Flatness singularities in the control space

For locally static feedback linearizable systems, even if flat outputs are defined locally around a given $x_0$, they are always global with respect to the control, so we never face singularities in the control space. For flat systems of differential weight $n+3$, a prolongation may create singularities in the control space and this is always the case if $k = 0$. Indeed, we have seen that if the first noninvolutive distribution is $\mathcal{D}^0$, then the system ceases to be flat for some singular controls. Let us now further analyze the set of singular controls. Recall that an invariant description of the singular controls is given by

$$U_{sing}(x) = \{u \in \mathbb{R}^2 : \ (g_1 \wedge g_c \wedge [f + u_1 g_1 + u_2 g_c, g_c])(x) = 0\},$$

where $\mathcal{D}^0 = \operatorname{span}\{g_1, g_c\}$, if $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] \neq TX$, or by

$$U'_{sing}(x) = \{u \in \mathbb{R}^2 \ : \ \dim \operatorname{span}\{g_1, g_2, ad_f g_1 + u_2[g_2, g_1], ad_f g_2 + u_1[g_1, g_2]\}(x) = 3,$$

if $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] = TX$.

The set of singular controls involves both the drift and controlled vector fields. As for the characterization of flatness of differential weight $n+3$, the vector field $g_c$ plays a crucial role in describing singularities and a direct calculation of singular controls can be performed using $U_{sing}$, if $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] \neq TX$ (resp. $U'_{sing}$, if $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] = TX$). For $k \geq 1$, there are no singular controls (like for the static feedback linearizable case). An explanation of this is that the normal form (NF2) can be seen as a prolongation of a subsystem which is static feedback linearizable. Indeed, the normal form (NF2), with $k \geq 1$, is the prolongation of the subsystem given by the first $\mu_1$ equations of the $w^1$-chain and the first $\mu_2$ equations of the $w^2$-chain and for which $w_{\mu_1+1}^1$ and $w_{\mu_2+1}^2$ are the new controls. To obtain (NF2), the first control has to be prolonged $k$ times, while the second one $k+1$ times. The reduced system is static feedback linearizable, so without singularities in the control space. Consequently, (NF2) does not exhibit singularities either.

## 1.5 Calculating flat outputs

In this subsection, firstly, we answer the question whether a given pair of smooth functions on $X$ forms a flat output and, secondly, provide a system of first order PDS's to be solved in order to find all minimal flat outputs. In particular, we will discus uniqueness of flat outputs for flat systems of differential weight $n+3$. Recall the definition of integers $\mu$ and $\rho$ given just before the statement of Proposition 1.4.1.

**Proposition 1.5.1.** *Consider the control system $\Sigma$, given by (1.3), that is flat at $x_0$ (at $(x_0, u_0)$, if $k=0$), of differential weight $n+3$.*

*(i) Assume $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Then a pair $(\varphi_1, \varphi_2)$ of smooth functions on a neighborhood of $x_0$ is a minimal x-flat output at $x_0$ if and only if (after permuting $\varphi_1$ and $\varphi_2$, if necessary)*

*(Fo1) $d\varphi_1 \perp \mathcal{H}^{\rho-1}$ and $d\varphi_2 \perp \mathcal{H}^{\mu-1}$;*

*(Fo2) $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$.*

*Moreover, the pair $(\varphi_1, \varphi_2)$ is unique, up to a nonlinear reparametrization depending on $L_f\varphi_1, \ldots, L_f^{\rho-\mu}\varphi_1$, i.e., if $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ is another minimal x-flat output, then there exist smooth maps $h_1$ and $h_2$, smoothly invertible ($h_2$ with respect to its first argument), such that*

$$\begin{aligned} \tilde{\varphi}_1 &= h_1(\varphi_1) \\ \tilde{\varphi}_2 &= h_2(\varphi_2, \varphi_1, L_f\varphi_1, \ldots, L_f^{\rho-\mu}\varphi_1). \end{aligned}$$

*If $\rho = \mu$, then $\tilde{\varphi}_i = h_i(\varphi_1, \varphi_2)$, $1 \leq i \leq 2$, and $h = (h_1, h_2)$ is a diffeomorphism.*

*(ii) Assume $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$. Then a pair $(\varphi_1,\ \varphi_2)$ of smooth functions on a neighborhood of $x_0$ is a minimal x-flat output at $x_0$ if and only if*

*(Fo1)' $(d\varphi_1 \wedge d\varphi_2)(x_0) \neq 0$;*

*(Fo2)' The involutive distribution $\mathcal{L} = (\mathrm{span}\,\{d\varphi_1, d\varphi_2\})^\perp$ satisfies*

$$\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k,$$

*implying the existence of a nonzero vector field $g_c \in \mathcal{D}^0$ such that $\mathcal{L} = \mathcal{D}^{k-1} + \mathrm{span}\,\{ad_f^k g_c\}$;*

*(Fo3)' $(L_{ad_f^{k+1}g_c}\varphi_1, L_{ad_f^{k+1}g_c}\varphi_2)(x_0) \neq (0,0)$.*

*Moreover, for any function $\varphi_1$, satisfying $d\varphi_1 \perp \mathcal{D}^{k-1}$ and $(L_{ad_f^k g_1}\varphi_1, L_{ad_f^k g_2}\varphi_1)(x_0) \neq (0,0)$, there exists $\varphi_2$ such that the pair $(\varphi_1, \varphi_2)$ is a minimal x-flat output; given any such $\varphi_1$, the choice of $\varphi_2$ is unique, up to a diffeomorphism, that is, if $(\varphi_1, \tilde{\varphi}_2)$ is another minimal x-flat output, then there exists a smooth map $h$, smoothly invertible with respect to the second argument such that*

$$\tilde{\varphi}_2 = h(\varphi_1, \varphi_2).$$

In the case $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, there are as many flat outputs as functions of three variables. Indeed, the distribution $\mathcal{D}^{k-1}$ is involutive and of corank three. According to item $(ii)$, $\varphi_1$ can be chosen as any function of three independent functions, whose differentials span $(\mathcal{D}^{k-1})^\perp$ and then there exists a unique $\varphi_2$ (up to a diffeomorphism) completing it to a minimal $x$-flat output. This reminds very much non-uniqueness of flat outputs of two-control driftless systems [29].

As an immediate corollary of Proposition 1.5.1, we obtain a system of PDE's whose solutions give all minimal $x$-flat outputs. In the case $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the vector field $g_c$ is well defined (and is given up to a multiplicative function). So chose $g_1$ and $g_c$ such that $\mathcal{D}^0 = \text{span}\{g_1, g_c\}$ and for any $1 \leq j \leq \mu - 1$ and $1 \leq j' \leq \mu$, denote $v_j = ad_f^{j-1} g_1$, $v_{\mu+j'} = ad_f^{j'-1} g_c$, $v_{2\mu} = ad_f^{\mu-1} g_1$, $v_{2\mu+1} = ad_f^{\mu} g_c$ and (only if $\mu_1 \neq \mu_2$) complete them, for $1 \leq i \leq \rho - \mu$, by $v_{2\mu+1+i} = ad_f^{\mu+i-1} g_1$, if $\mu_1 > \mu_2$, or by $v_{2\mu+1+i} = ad_f^{\mu+i} g_c$, if $\mu_2 > \mu_1$. We thus have defined $n-1$ vector fields $v_1, \ldots, v_{n-1}$ satisfying $\mathcal{H}^{\mu-1} = \text{span}\{v_1, \ldots, v_{2\mu-1}\}$ and $\mathcal{H}^{\rho-1} = \text{span}\{v_1, \ldots, v_{n-1}\}$. In this case the result follows immediately and is stated as item $(i)$ of proposition below. If $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, then for $1 \leq j \leq k = \mu - 1$, denote $w_j = ad_f^{j-1} g_1$ and $w_{k+j} = ad_f^{j-1} g_2$. Clearly, $\mathcal{D}^{k-1} = \text{span}\{w_1, \ldots, w_{2k}\}$ but we have to construct one more vector field $w$, as described in item $(ii)$ below.

**Proposition 1.5.2.** *Consider the control system $\Sigma$ that is flat at $x_0$ (at $(x_0, u_0)$, if $k = 0$), of differential weight $n + 3$.*

*(i) Assume $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Then a pair $(\varphi_1, \varphi_2)$ of smooth functions on a neighborhood of $x_0$ is a minimal $x$-flat output at $x_0$ if and only if (after permuting $\varphi_1$ and $\varphi_2$, if necessary) they satisfy*

$$\begin{aligned} L_{v_j}\varphi_1 &= 0, \quad 1 \leq j \leq n-1, \\ L_{v_j}\varphi_2 &= 0, \quad 1 \leq j \leq 2\mu - 1, \end{aligned}$$

*and $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$.*

*(ii) Assume $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$. Then a pair $(\varphi_1, \varphi_2)$ of smooth functions on a neighborhood of $x_0$ is a minimal $x$-flat output at $x_0$ if and only if (after permuting $\varphi_1$ and $\varphi_2$, if necessary) $\varphi_1$ is any function satisfying*

$$L_{w_j}\varphi_1 = 0, \quad 1 \leq j \leq 2k,$$

*and $(L_{ad_f^k g_1}\varphi_1, L_{ad_f^k g_2}\varphi_1)(x_0) \neq (0,0)$ and, for any $\varphi_1$ as above, $\varphi_2$ is given by*

$$\begin{aligned} L_{w_j}\varphi_2 &= 0, \quad 1 \leq j \leq 2k, \\ L_w\varphi_2 &= 0, \end{aligned}$$

*where $w = (L_{ad_f^k g_2}\varphi_1) ad_f^k g_1 - (L_{ad_f^k g_1}\varphi_1) ad_f^k g_2$ and $(d\varphi_1 \wedge d\varphi_2)(x_0) \neq 0$.*

Clearly, the distribution $\mathcal{L}$ spanned by $w$ and $\mathcal{D}^{k-1}$ is of corank two and, as can be proved, involutive thus implying that for any $\varphi_1$ we can solve the system of equations for $\varphi_2$. Different choices of $\varphi_1$ lead, in general, to different involutive distributions $\mathcal{L}$ and thus to different functions $\varphi_2$ and, as we have mentioned, there are as many choices as nondegenerate functions of three variables.

## 1.6 Examples

### 1.6.1 Induction motor: first model with $\theta$, the mechanical position

Consider the induction motor (called direct-quadrature model in [10]), see [12, 35], described by the following control system with 2 inputs and 6 states:

$$\Sigma_{IM_6}: \begin{cases} \dot{\theta} &= \omega \\ \dot{\omega} &= \mu\psi_d i_q - \frac{\tau_L}{J} \\ \dot{\psi}_d &= -\eta\psi_d + \eta M i_d \\ \dot{\rho} &= n_p\omega + \frac{\eta M i_q}{\psi_d} \\ \dot{i}_d &= -\gamma i_d + \frac{\eta M \psi_d}{\sigma L_R L_S} + n_p\omega i_q + \frac{\eta M i_q^2}{\psi_d} + \frac{u_d}{\sigma L_S} \\ \dot{i}_q &= -\gamma i_q - \frac{M n_p \omega\psi_d}{\sigma L_R L_S} - n_p\omega i_d - \frac{\eta M i_d i_q}{\psi_d} + \frac{u_q}{\sigma L_S} \end{cases}$$

where $u_d$, $u_q$ are the inputs (the stator voltages), $i_d$ and $i_q$ are the stator currents, $\psi_d$ and $\rho$ are two well-chosen functions of the rotor fluxes (see [10] for their precise expression), $\theta$ is the mechanical position of the rotor and $\omega$ is the rotor speed. All other parameters of the motor (the inductances $L_S$ and $L_R$, the coefficient of mutual inductance $X$, the rotor moment of inertia $J$, the load-torque $\tau_L$, etc.) can be supposed constant and known.

After applying a suitable static feedback transformation (which has also a physical interpretation, see [10] for more details) the model of the induction motor is transformed into the following form:

$$\tilde{\Sigma}_{IM_6}: \begin{cases} \dot{\theta} &= \omega \\ \dot{\omega} &= \mu\psi_d i_q - \frac{\tau_L}{J} \\ \dot{\psi}_d &= -\eta\psi_d + \eta M i_d \\ \dot{\rho} &= n_p\omega + \frac{\eta M i_q}{\psi_d} \\ \dot{i}_d &= \tilde{u}_d \\ \dot{i}_q &= \tilde{u}_q. \end{cases}$$

This system is not static feedback linearizablee, however it becomes static feedback linearizable via one-fold invertible prolongation, thus it is flat, a property that has been already observed and applied in [12, 35].

Indeed, the distribution

$$\begin{aligned} \mathcal{D}^1 &= \operatorname{span}\left\{\tfrac{\partial}{\partial i_d}, \tfrac{\partial}{\partial i_q}, \eta M\tfrac{\partial}{\partial \psi_d}, \mu\psi_d\tfrac{\partial}{\partial\omega} + \tfrac{\eta M}{\psi_d}\tfrac{\partial}{\partial\rho}\right\} \\ &= \operatorname{span}\left\{\tfrac{\partial}{\partial i_d}, \tfrac{\partial}{\partial i_q}, \tfrac{\partial}{\partial \psi_d}, \tfrac{\partial}{\partial\omega} + \tfrac{\eta M}{\mu\psi_d^2}\tfrac{\partial}{\partial\rho}\right\} \end{aligned}$$

is not involutive and $\mathcal{D}^1 + [\mathcal{D}^1, \mathcal{D}^1] = \overline{\mathcal{D}}^1 = \text{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}, \frac{\partial}{\partial \omega}, \frac{\partial}{\partial \rho}\} \neq TX$. It is easy to see that $ad_f^2 \frac{\partial}{\partial i_d} \in \overline{\mathcal{D}}^1$, thus $\frac{\partial}{\partial i_d}$ plays the role of the distinguished vector field $g_c$ defined by condition $(A2)$ of Theorem 1.3.1. We can now construct the sequence:

$$\mathcal{H}^1 = \text{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}\} \underset{2}{\subsetneq} \mathcal{H}^2 = \text{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}, \frac{\partial}{\partial \omega}, \frac{\partial}{\partial \rho}\} \underset{1}{\subsetneq} \mathcal{H}^3 = TX,$$

where all distributions $\mathcal{H}^i$ are clearly involutive. It follows that all conditions of Theorem 1.3.1 are verified, therefore the system is flat of differential weight $n + 3 = 9$ and it becomes static feedback linearizable after a one-fold prolongation of $\tilde{u}_q$.

Let us now compute its minimal flat outputs $(\varphi_1, \varphi_2)$. Since $k = 1$ and $\mathcal{D}^1 + [\mathcal{D}^1, \mathcal{D}^1] \neq TX$, we are in the first case of Proposition 1.5.1 with $\rho = 2$ and $\mu = 1$, so $(\varphi_1, \varphi_2)$ should satisfy $d\varphi_1 \perp \mathcal{H}^2$, $d\varphi_2 \perp \mathcal{H}^1$ and the regularity condition. It follows that $(\varphi_1, \varphi_2) = (\theta, \rho)$ and the pair $(\varphi_1, \varphi_2)$ is unique up to a diffeomorphism.

### 1.6.2 Induction motor: second model, without $\theta$, the mechanical position

Let us now consider the following model (see [10]), obtained from the first one, for which we do not take into account $\theta$, the mechanical position of the rotor.

$$\Sigma_{IM_5} : \begin{cases} \dot{\omega} &= \mu \psi_d i_q - \frac{\tau_L}{J} \\ \dot{\psi}_d &= -\eta \psi_d + \eta M i_d \\ \dot{\rho} &= n_p \omega + \frac{\eta M i_q}{\psi_d} \\ \dot{i}_d &= -\gamma i_d + \frac{\eta M \psi_d}{\sigma L_R L_S} + n_p \omega i_q + \frac{\eta M i_q^2}{\psi_d} + \frac{u_d}{\sigma L_S} \\ \dot{i}_q &= -\gamma i_q - \frac{M n_p \omega \psi_d}{\sigma L_R L_S} - n_p \omega i_d - \frac{\eta M i_d i_q}{\psi_d} + \frac{u_q}{\sigma L_S} \end{cases}$$

After applying a suitable static feedback transformation, the system is transformed into the following form:

$$\tilde{\Sigma}_{IM_5} : \begin{cases} \dot{\omega} &= \mu \psi_d i_q - \frac{\tau_L}{J} \\ \dot{\psi}_d &= -\eta \psi_d + \eta M i_d \\ \dot{\rho} &= n_p \omega + \frac{\eta M i_q}{\psi_d} \\ \dot{i}_d &= \tilde{u}_d \\ \dot{i}_q &= \tilde{u}_q. \end{cases}$$

We will next compare the two different models of the induction motor. In particular, we will see how omitting $\theta$ as a state variable changes properties of flatness and we will show the surprising fact that, contrary to the first case, for the second one the flat outputs are no longer unique.

As for the first model (with $\theta$, the mechanical position), the system without $\theta$ is not static feedback linearizable, however it becomes static feedback linearizable via one-fold invertible prolongation, thus it is flat of differential weight $n + 3 = 8$. Indeed, as above, the distribution

$$\mathcal{D}^1 = \text{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \eta M \frac{\partial}{\partial \psi_d}, \mu\psi_d \frac{\partial}{\partial \omega} + \frac{\eta M}{\psi_d}\frac{\partial}{\partial \rho}\} = \text{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}, \frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho}\}$$

is not involutive, but now $\mathcal{D}^1 + [\mathcal{D}^1, \mathcal{D}^1] = \overline{\mathcal{D}}^1 = TX$ and $[\mathcal{D}^0, \mathcal{D}^1] \subset \mathcal{D}^1$. Thus we are in the first case of Theorem 1.3.4, with $k = 1$, and the system is flat without additional conditions.

According to Propositions 1.5.1(ii) and 1.5.2(ii), the system admits many flat outputs (their choice being parameterized by a function of three well defined variables) and let us calculate some of them. Recall that a pair of two independent functions $(\varphi_1, \varphi_2)$ is a minimal $x$-flat output if and only if the involutive distribution $\mathcal{L} = (\text{span}\,\{d\varphi_1, d\varphi_2\})^\perp$ satisfies $\mathcal{D}^0 \subset \mathcal{L} \subset \mathcal{D}^1$. Hence the distribution $\mathcal{L}$ has to be of the form $\mathcal{L} = \text{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, h\}$, where $h$ is any vector field of the form $h = \alpha\frac{\partial}{\partial \psi_d} + \beta(\frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho})$ such that $\mathcal{L}$ is involutive and for any smooth functions $\alpha, \beta$ satisfying $(\alpha, \beta) \neq (0, 0)$.

Let us first take $\mathcal{L} = \text{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}\}$. The associated flat outputs are independent functions of $\omega$, $\rho$ and we can take $(\varphi_1, \varphi_2) = (\omega, \rho)$.

Using the same procedure, let us now give some less intuitive minimal flat outputs. Choose $\mathcal{L} = \text{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho}\}$. Any two independent functions $\varphi_1$ and $\varphi_2$ depending on $\omega, \psi_d, \rho$ whose differentials annihilate $\mathcal{L}$, that is, satisfying $\frac{\partial \varphi_i}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial \varphi_i}{\partial \rho} \equiv 0$, for $1 \leq i \leq 2$, can be taken as minimal flat outputs. Solving those equations, we get $\varphi_i = \varphi_i(\psi_d, \frac{\eta M}{\mu\psi_d^2}\omega - \rho)$. We can choose, for instance, $(\varphi_1, \varphi_2) = (\psi_d, \frac{\eta M}{\mu\psi_d^2}\omega - \rho)$.

Finally, let $\mathcal{L} = \text{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d} + \frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho}\}$. The functions $\varphi_1$ and $\varphi_2$ depend on $\omega$, $\psi_d$, $\rho$ and satisfy $\frac{\partial \varphi_i}{\partial \psi_d} + \frac{\partial \varphi_i}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial \varphi_i}{\partial \rho} \equiv 0$, for $1 \leq i \leq 2$. Solving those equations, we obtain $\varphi_i = \varphi_i(\rho + \frac{\eta M}{\mu\psi_d}, \psi_d - \omega)$. We can choose $(\varphi_1, \varphi_2) = (\rho + \frac{\eta M}{\mu\psi_d}, \psi_d - \omega)$.

Notice that while for the first model (with $\theta$, the mechanical position), the minimal flat outputs are unique, for the reduced one there are many minimal flat outputs (the choice being parameterized by a function of three well defined variables $\omega$, $\rho$, $\psi_d$). Recall that for the first case, the minimal flat outputs are $(\theta, \rho)$ and can been seen as the counterparts of $(\varphi_1, \varphi_2) = (\omega, \rho)$ for the second model (since $\dot{\theta} = \omega$).

### 1.6.3 Polymerization reactor

Consider the reactor [36, 60]:

$$\Sigma_{PR}: \begin{cases} \dot{C}_m &= \frac{C_{mm_s}}{\tau} - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{C_m}{\tau} + R_m(C_m, C_i, C_s, T) \\ \dot{C}_i &= -k_i(T)C_i + u_2\frac{C_{ii_s}}{V} - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{C_i}{\tau} \\ \dot{C}_s &= u_2\frac{C_{si_s}}{V} + \frac{C_{sm_s}}{\tau} - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{C_s}{\tau} \\ \dot{\mu} &= -M_mR_m(C_m, C_i, C_s, T) - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{\mu}{\tau} \\ \dot{T} &= \theta(C_m, C_i, C_s, \mu, T) + \alpha_1 T_j \\ \dot{T}_j &= f_6(T, T_j) + \alpha_4 u_1 \end{cases}$$

where $u_1, u_2$ are the control inputs and $C_{mm_s}$, $C_{ii_s}$, $C_{si_s}$, $C_{sm_s}$, $M_m$, $\bar{\epsilon}$, $\tau$, $V$, $\alpha_1$, $\alpha_4$ are constant positive physical parameters. The functions $R_m$, $k_i$, $\theta$ and $f_6$ are not well-known and can be considered arbitrary: they derive from experimental data and semi-empirical considerations and involve kinetic laws, heat transfer coefficients and reaction enthalpies.

After applying the change of coordinates

$$\begin{aligned} \tilde{C}_m &= \mu + M_mC_m \\ \tilde{C}_i &= C_i - \frac{C_{ii_s}}{C_{si_s}}C_s \\ \tilde{C}_s &= -k_i(T)C_i - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})(\frac{C_i}{\tau} - \frac{C_{ii_s}}{C_{si_s}}\frac{C_s}{\tau}) - \frac{C_{ii_s}}{C_{si_s}}\frac{C_{sm_s}}{\tau}, \\ \tilde{\mu} &= \frac{1}{\tau}(M_mC_{mm_s} - (1+\bar{\epsilon})\mu - M_mC_m), \\ \tilde{T} &= T \\ \tilde{T}_j &= \theta(C_m, C_i, C_s, \mu, T) + \alpha_1 T_j \end{aligned}$$

and a suitable static feedback transformation, we obtain:

$$\tilde{\Sigma}_{PR}: \begin{cases} \dot{\tilde{C}}_i = \tilde{C}_s & \dot{\tilde{C}}_m = \tilde{\mu} \\ \dot{\tilde{C}}_s = \tilde{u}_1 & \dot{\tilde{\mu}} = b(\tilde{C}_m, \tilde{C}_i, \tilde{C}_s, \tilde{\mu}, \tilde{T}) \\ & \dot{\tilde{T}} = \tilde{T}_j \\ & \dot{\tilde{T}}_j = \tilde{u}_2 \end{cases}$$

where $b$ is a smooth function depending explicitly on $\tilde{T} = T$.

If $(\frac{\partial^2 b}{\partial\tilde{T}\partial\tilde{C}_s}, \frac{\partial^2 b}{\partial\tilde{C}_s^2}) \neq (0,0)$, then the distribution $\mathcal{D}^1 = \mathrm{span}\,\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{C}_i} + \frac{\partial b}{\partial\tilde{C}_s}\frac{\partial}{\partial\tilde{\mu}}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial\tilde{T}}\}$ is noninvolutive, $\mathrm{rk}\,\overline{\mathcal{D}}^1 = 5$ and $\overline{\mathcal{D}}^1 \neq TX$. Consequently, we are in the case of Theorem 1.3.1 with $k = 1$.

Let us suppose that $\frac{\partial^2 b}{\partial\tilde{C}_s^2} \neq 0$. Therefore, $[\mathcal{D}^0, \mathcal{D}^1] \not\subset \mathcal{D}^1$ and the corank one involutive subdistribution $\mathcal{H}^1$ can be computed in two different ways (see the condition $(A3)$ of Theorem 1.3.1 and the comment following Theorem 1.3.4). We will calculate $\mathcal{H}^1$ by applying the procedure given by Theorem 1.3.1 (see [46] where we apply Theorem 1.3.4 to construct $\mathcal{H}^1$). The distribution

$$\overline{\mathcal{D}}^1 + [f, \mathcal{D}^1] = \mathrm{span}\,\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{C}_i}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial T}, \frac{\partial}{\partial\tilde{\mu}}, \frac{\partial b}{\partial\tilde{C}_s}\frac{\partial}{\partial\tilde{C}_m}\}$$

is of rank 6 (provided that $\frac{\partial b}{\partial \tilde{C}_s}$ does not vanish) and $\tilde{g}_2 = \frac{\partial}{\partial \tilde{T}_j}$ satisfies $ad_f \tilde{g}_2 \in \overline{\mathcal{D}}^1$. Therefore, item $(A2)$ of Theorem 1.3.1 is verified and $\tilde{g}_2$ plays the role of $g_c$.

Thus the corank one subdistribution $\mathcal{H}^1$ is given by

$$\mathcal{H}^1 = \mathcal{D}^0 + \text{span}\,\{ad_f \tilde{g}_2\} = \text{span}\,\{\frac{\partial}{\partial \tilde{C}_s}, \frac{\partial}{\partial \tilde{T}_j}, \frac{\partial}{\partial T}\}$$

and is clearly involutive. We have

$$\mathcal{H}^2 = \mathcal{H}^1 + [f, \mathcal{H}^1] = \text{span}\,\{\frac{\partial}{\partial \tilde{C}_s}, \frac{\partial}{\partial \tilde{C}_i}, \frac{\partial}{\partial \tilde{T}_j}, \frac{\partial}{\partial T}, \frac{\partial}{\partial \mu}\}$$

involutive and $\mathcal{H}^3 = TX$. The system $\tilde{\Sigma}_{PR}$ satisfies all conditions of Theorem 1.3.1, hence the corresponding prolongation (obtained by prolonging $\tilde{u}_1$)

$$\tilde{\Sigma}_{PR}^{(1,0)} \begin{cases} \dot{\tilde{C}}_i = \tilde{C}_s & \dot{\tilde{C}}_m = \tilde{\mu} \\ \dot{\tilde{C}}_s = y & \dot{\tilde{\mu}} = b(\tilde{C}_m, \tilde{C}_i, \tilde{C}_s, \tilde{\mu}, \tilde{T}) \\ \dot{y} = v_1 & \dot{\tilde{T}} = \tilde{T}_j \\ & \dot{\tilde{T}}_j = v_2 \end{cases}$$

where $y = \tilde{u}_1$ and $v_2 = \tilde{u}_2$, is locally static feedback linearizable. Indeed, all its linearizability distributions $\mathcal{D}_p^i$, for the prolonged system $\tilde{\Sigma}_{PR}^{(1,0)}$, for $i \geq 0$, are involutive, of constant rank and $\text{rk}\,\mathcal{D}_p^3 = 7$. Therefore, the prolonged system can be brought into the Brunovský canonical form with $\tilde{C}_m = M_m C_m + \mu$, $\tilde{C}_i = C_i - \frac{C_{ii_s}}{C_{si_s}} C_s$ playing the role of top variables.

Let us now compute the minimal flat outputs $(\varphi_1, \varphi_2)$ of $\tilde{\Sigma}_{PR}$. We are in the first case of Proposition 1.5.1, with $\rho = 3$ and $\mu = 2$. Since the differential of $\varphi_1$ annihilates $\mathcal{H}^2$, it follows that $\varphi_1 = \varphi_1(\tilde{C}_m)$ with $\frac{\partial \varphi_1}{\partial \tilde{C}_m} \neq 0$. The differential of $\varphi_2$ annihilates $\mathcal{H}^1$ and satisfies $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \neq 0$. This yields $\varphi_2 = \varphi_2(\tilde{C}_m, \tilde{C}_i, \tilde{\mu})$ with $\frac{\partial \varphi_2}{\partial \tilde{C}_i} \neq 0$. Hence, a choice of minimal flat outputs is $(\varphi_1, \varphi_2) = (\tilde{C}_m, \tilde{C}_i)$. This is conform with the fact that $\tilde{C}_m$ and $\tilde{C}_i$ are the top variables of the Brunovský canonical form (see the above remark).

## 1.7 Proofs

### 1.7.1 Notations and useful results

Consider a control system of the form $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 h_2(x)$. By $\Sigma^{(1,0)}$ we will denote the system $\Sigma$ with one-fold prolongation of the first control, that is

$$\Sigma^{(1,0)} : \begin{cases} \dot{x} = f(x) + y_1 g_1(x) + +v_2 h_2(x) \\ \dot{y}_1 = v_1 \end{cases}$$

with $y_1 = u_1$ and $v_2 = u_2$. Throughout this section,

$$F = \sum_{i=1}^{n} (f_i + y_1 g_{1i}) \frac{\partial}{\partial x_i}$$

stands for the drift and

$$G_1 = \frac{\partial}{\partial y_1}, \quad H_2 = \sum_{i=1}^{n} h_{2i} \frac{\partial}{\partial x_i}$$

denote the control vector fields of the prolonged system.

To $\Sigma^{(1,0)}$, we associate the distributions $\mathcal{D}_p^0 = \operatorname{span}\{G_1, H_2\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [F, \mathcal{D}_p^i]$, for $i \geq 0$, (the subindex $p$ referring to the prolonged system $\Sigma^{(1,0)}$).

We start by stating and proving two propositions needed in the proofs of our main results, but also having an independent interest.

**Proposition 1.7.1.** *Consider a two-input control-affine system $\Sigma$, given by (1.3), defined on a n-dimensional manifold $X$, dynamically linearizable via invertible one-fold prolongation and let $\mathcal{D}^k$ be the first noninvolutive distribution. The following conditions are satisfied:*

*(i)* *$\mathcal{D}^k$ is feedback invariant;*

*(ii)* *If $k \geq 1$, then* $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} = 2$;

*(iii)* *If $\overline{\mathcal{D}}^k = TX$, then $n = 2k+3$.*

*Proof.* We first prove $(i)$. It is well known that the involutive distributions $\mathcal{D}^i$, for $i \leq k-1$, are feedback invariant. Let us show that this is also the case for the first noninvolutive distribution $\mathcal{D}^k$. By definition $\mathcal{D}^k = \mathcal{D}^{k-1} + \operatorname{span}\{ad_f^k g_1, ad_f^k g_2\}$. We first show that $\mathcal{D}^k$ is invariant under the transformations of type $\tilde{f} = f + \alpha_1 g_1 + \alpha_2 g_2$, where $\alpha_1$ and $\alpha_2$ are smooth functions. We have:

$$ad_{\tilde{f}} g_i = [f + \alpha_1 g_1 + \alpha_2 g_2, g_i] = ad_f g_i \operatorname{mod} \mathcal{D}^0$$

and by induction, we get $ad_{\tilde{f}}^{k-1} g_i = ad_f^{k-1} g_i \operatorname{mod} \mathcal{D}^{k-2}$. From this, we deduce

$$ad_{\tilde{f}}^k g_i = [f + \alpha_1 g_1 + \alpha_2 g_2, ad_{\tilde{f}}^{k-1} g_i \operatorname{mod} \mathcal{D}^{k-2}] = ad_f^k g_i \operatorname{mod} \mathcal{D}^{k-1}$$

which yields

$$\tilde{\mathcal{D}}^k = \tilde{\mathcal{D}}^{k-1} + \operatorname{span}\{ad_{\tilde{f}}^k g_1, ad_{\tilde{f}}^k g_2\} = \mathcal{D}^{k-1} + \operatorname{span}\{ad_f^k g_1, ad_f^k g_2\} = \mathcal{D}^k.$$

Let us now study the transformations involving the controlled vector fields, i.e., of the type $\tilde{g}_1 = \beta_{11} g_1 + \beta_{21} g_2$ and $\tilde{g}_2 = \beta_{12} g_1 + \beta_{22} g_2$, where $\beta = (\beta_{ij}(x))$ is an invertible matrix. We have:

$$\begin{aligned} ad_f \tilde{g}_1 &= \beta_{11} ad_f g_1 + \beta_{21} ad_f g_2 \operatorname{mod} \mathcal{D}^0, \\ ad_f \tilde{g}_2 &= \beta_{12} ad_f g_1 + \beta_{22} ad_f g_2 \operatorname{mod} \mathcal{D}^0, \end{aligned}$$

and by induction, we get

$$\begin{aligned} ad_f^{k-1}\tilde{g}_1 &= \beta_{11}ad_f^{k-1}g_1 + \beta_{21}ad_f^{k-1}g_2 \bmod \mathcal{D}^{k-2}, \\ ad_f^{k-1}\tilde{g}_2 &= \beta_{12}ad_f^{k-1}g_1 + \beta_{22}ad_f^{k-1}g_2 \bmod \mathcal{D}^{k-2}. \end{aligned}$$

It follows that

$$\begin{aligned} ad_f^{k}\tilde{g}_1 &= [f, \beta_{11}ad_f^{k-1}g_1 + \beta_{21}ad_f^{k-1}g_2 \bmod \mathcal{D}^{k-2}] = \beta_{11}ad_f^{k}g_1 + \beta_{21}ad_f^{k}g_2 \bmod \mathcal{D}^{k-1}, \\ ad_f^{k}\tilde{g}_2 &= [f, \beta_{12}ad_f^{k-1}g_1 + \beta_{22}ad_f^{k-1}g_2 \bmod \mathcal{D}^{k-2}] = \beta_{12}ad_f^{k}g_1 + \beta_{22}ad_f^{k}g_2 \bmod \mathcal{D}^{k-1}. \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{D}}^k &= \tilde{\mathcal{D}}^{k-1} + \mathrm{span}\,\{ad_f^k\tilde{g}_1, ad_f^k\tilde{g}_2\} \\ &= \mathcal{D}^{k-1} + \mathrm{span}\,\{\beta_{11}ad_f^kg_1 + \beta_{21}ad_f^kg_2, \beta_{12}ad_f^kg_1 + \beta_{22}ad_f^kg_2\} \\ &= \mathcal{D}^{k-1} + \mathrm{span}\,\{ad_f^kg_1, ad_f^kg_2\} \\ &= \mathcal{D}^k. \end{aligned}$$

The distribution $\mathcal{D}^k$ is thus invariant under the considered classes of transformations. We have shown that $\mathcal{D}^k$ is feedback invariant.

We know turn to item $(ii)$. Assume $\mathrm{rk}\,\mathcal{D}^k - \mathrm{rk}\,\mathcal{D}^{k-1} = 1$ and let $l$ be the smallest integer such that $\mathrm{rk}\,\mathcal{D}^l - \mathrm{rk}\,\mathcal{D}^{l-1} = 1$. It is clear that $1 \le l \le k$. Since $\Sigma$ is dynamically linearizable via invertible one-fold prolongation, there exists an invertible static feedback transformation, $u(x) = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2\tilde{h}_2(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{h}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_2 = \tilde{u}_2$, is locally static feedback linearizable. For simplicity of notation, we will drop the tilde, but we will keep distinguishing $g_1$ from $h_2$ (which could also be denoted $g_2$) whose control is not preintegrated.

Since $\Sigma^{(1,0)}$ is locally static feedback linearizable, for any $i \ge 0$ the distributions $\mathcal{D}_p^i$ are involutive, of constant rank, and there exists an integer $\rho$ such that $\mathrm{rk}\,\mathcal{D}_p^\rho = n+1$. We have

$$\begin{aligned} \mathcal{D}_p^0 &= \mathrm{span}\,\{\tfrac{\partial}{\partial y_1}, h_2\}, \\ \mathcal{D}_p^1 &= \mathrm{span}\,\{\tfrac{\partial}{\partial y_1}, g_1, h_2, ad_fh_2 + y_1[g_1, h_2]\}. \end{aligned}$$

Since $k \ge 1$, the distribution $\mathcal{D}^0 = \mathrm{span}\,\{g_1, h_2\}$ is involutive, thus $[g_1, h_2] \in \mathcal{D}^0$ and $\mathcal{D}_p^1 = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, h_2, ad_fh_2\}$. It is easy to prove (by an induction argument) that, for $1 \le i \le l-1$,

$$\mathcal{D}_p^i = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1}g_1, h_2, \cdots, ad_f^ih_2\},$$

and thus

$$\mathcal{D}_p^l = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{l-1}g_1, h_2, \cdots, ad_f^lh_2\}.$$

We distinguish two sub cases: $ad_f^l h_2 \in \mathcal{D}^{l-1} = \text{span}\{g_1, \cdots, ad_f^{l-1} g_1, h_2, \cdots, ad_f^{l-1} h_2\}$ (and in this case $\text{rk}\,\mathcal{D}_p^l = 2l+1$) and $ad_f^l h_2 \notin \mathcal{D}^{l-1}$ (and in this case $\text{rk}\,\mathcal{D}_p^l = 2l+2$).

Let us first assume $ad_f^l h_2 \in \mathcal{D}^{l-1}$. We have:

$$\mathcal{D}_p^j = \text{ span}\{\frac{\partial}{\partial y_1}\} + \mathcal{D}^{j-1}, \text{ for } j \geq l,$$

and the involutivity of $\mathcal{D}_p^j$ implies that of $\mathcal{D}^{j-1}$. For $j = k+1$, it follows that $\mathcal{D}^k$ is involutive, which contradicts the fact that $\mathcal{D}^k$ was supposed nonivolutive.

Let us now assume $ad_f^l h_2 \notin \mathcal{D}^{l-1}$. Since $\text{rk}\,\mathcal{D}^l = 2l+1$, we deduce that $\mathcal{D}^l = \text{span}\{g_1, \cdots, ad_f^{l-1} g_1, h_2, \cdots, ad_f^l h_2\}$. Moreover, we have

$$\mathcal{D}_p^j = \text{ span}\{\frac{\partial}{\partial y_1}\} + \mathcal{D}^j, \text{ for } j \geq l$$

and the involutivity of $\mathcal{D}_p^j$ implies that of $\mathcal{D}^j$. For $j = k$, it follows that $\mathcal{D}^k$ is involutive, which contradicts the assumption of noninvolutivity of $\mathcal{D}^k$. Therefore, $\text{rk}\,\mathcal{D}^k - \text{rk}\,\mathcal{D}^{k-1} = 2$. Consequently, $\text{rk}\,\mathcal{D}^k = 2k+2$.

Finally, we prove item $(iii)$. Suppose $\overline{\mathcal{D}}^k = TX$. Due to $(ii)$, $\text{rk}\,\mathcal{D}^k = 2k+2$ (if $k = 0$, this is still true, since the controlled vector fields are assumed independent). For the prolonged system $\Sigma^{(1,0)}$, we have

$$\mathcal{D}_p^{k+1} = \text{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^k g_1, h_2, \cdots, ad_f^{k+1} h_2\}.$$

The distribution $\mathcal{E} = \mathcal{D}_p^{k+1} \cap TX = \text{span}\{g_1, \cdots, ad_f^k g_1, h_2, \cdots, ad_f^{k+1} h_2\}$ is involutive (as intersection of involutive distributions) and its rank is $2k+3$, otherwise we obtain $\mathcal{E} = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive. Since $\mathcal{D}^k \subset \mathcal{E}$ and $\mathcal{E} = 2k+3$, we deduce $\overline{\mathcal{D}}^k = \mathcal{E}$. On the other hand, $\overline{\mathcal{D}}^k = TX$ and from this, it follows immediately that $n = 2k+3$.

**Proposition 1.7.2.** *Consider a two-input control system $\Sigma$, given by (1.3), and let $\mathcal{D}^k$ be the first noninvolutive distribution. Assume $k \geq 1$ and $\mathcal{D}^k$ satisfies the conditions $(A1)-(A2)$ of Theorem 1.3.1. If the distribution $\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span}\{ad_f^k g_c\}$ is involutive, where $g_c$ is defined by item $(A2)$, then all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + \text{span}\{ad_f^i g_c\}$, for $1 \leq i \leq k-1$, are involutive.*

*Proof.* Let us first show that under the conditions $(A1)-(A2)$ of Theorem 1.3.1, there exists a non-zero vector field $g_c \in \mathcal{D}^0$ such that $ad_f^{k+1} g_c \in \overline{\mathcal{D}}^k$. Due to $(A1)$ and $(A2)$, we have $\text{rk}\,\overline{\mathcal{D}}^k = 2k+3$ and $\text{rk}\,(\overline{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$, thus we can always assume (permute $g_1$ and $g_2$, if necessary) that $ad_f^{k+1} g_1 \notin \overline{\mathcal{D}}^k$. Hence there exists a smooth function $\alpha$, defined in a neighborhood of $x_0$, such that

$ad_f^{k+1}g_2 = \alpha ad_f^{k+1}g_1 \bmod \overline{\mathcal{D}}^k$. It follows that $ad_f^{k+1}g_2 = ad_f^{k+1}(\alpha g_1) \bmod \overline{\mathcal{D}}^k$, which gives $ad_f^{k+1}(g_2 - \alpha g_1) = 0 \bmod \overline{\mathcal{D}}^k$. The vector field $g_c = g_2 - \alpha g_1$ is clearly nonzero (since $g_1$ and $g_2$ are independent everywhere on $X$) and satisfies $ad_f^{k+1}g_c \in \overline{\mathcal{D}}^k$.

We can now show the involutivity of the distributions $\mathcal{H}^i$. Assume that $\mathcal{H}^{k-1} = \mathcal{D}^{k-2} + \text{span}\,\{ad_f^{k-1}g_c\}$ is not involutive. Since $\mathcal{D}^{k-2} \subset \mathcal{H}^{k-1} \subset \mathcal{D}^{k-1}$, where $\mathcal{D}^{k-2}$ and $\mathcal{D}^{k-1}$ are involutive and both inclusions are of corank one, it follows $\text{rk}\,\overline{\mathcal{H}}^{k-1} = 2k$ and the new direction completing $\mathcal{H}^{k-1}$ to its involutive closure is given by a vector field of the form $[ad_f^l g_i, ad_f^{k-1}g_c]$, with $1 \le i \le 2$ and $0 \le l \le k-2$, and is necessarily collinear with $ad_f^{k-1}g_1$ modulo $\mathcal{H}^{k-1}$. Hence, there exists a smooth function $\beta$, defined in a neighborhood of $x_0$, not vanishing at $x_0$, such that $[ad_f^l g_i, ad_f^{k-1}g_c] = \beta ad_f^{k-1}g_1 \bmod \mathcal{H}^{k-1}$. From this, applying the Jacobi identity and the involutivity of $\mathcal{H}^k$, it follows

$$\begin{aligned}[ad_f^l g_i, ad_f^k g_c] &= [f, [ad_f^l g_i, ad_f^{k-1}g_c]] - [ad_f^{l+1}g_i, ad_f^{k-1}g_c] \\ &= [f, \beta ad_f^{k-1}g_1] \bmod \mathcal{H}^k \\ &= \beta ad_f^k g_1 \bmod \mathcal{H}^k.\end{aligned}$$

On the other hand, $[ad_f^l g_i, ad_f^k g_c] \in \mathcal{H}^k$, and consequently $ad_f^k g_1 \in \mathcal{H}^k$, which contradicts our assumption, otherwise $\mathcal{D}^k = \mathcal{H}^k$ and $\mathcal{D}^k$ would be involutive. Therefore, $\mathcal{H}^{k-1}$ is involutive. Following the same line, we prove that the involutivity of $\mathcal{H}^i$ implies that of $\mathcal{H}^{i-1}$, for $1 \le i \le k-1$.

### 1.7.2 Proof of Proposition 1.3.1

We will show the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Consider a flat control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$, of differential weight $n+3$, and let

$$\varphi_1 = \varphi_1(x, \bar{u}_1^p, \bar{u}_2^r) \text{ and } \varphi_2 = \varphi_2(x, \bar{u}_1^s, \bar{u}_2^q)$$

be its minimal flat outputs, defined in $\mathcal{O}^l$, a neighborhood of $(x_0, \bar{u}_0^l)$, with $p$, $r$, $s$, $q \ge -1$ and at least one of them non negative, and such that $\varphi_1$ (respectively $\varphi_2$) depends explicitly on $u_1^{(p)}$ and $u_2^{(r)}$ (respectively on $u_1^{(s)}$ and $u_2^{(q)}$). We can always suppose (after permuting $\varphi_1$ and $\varphi_2$ or $u_1$ and $u_2$, if necessary) that $p$ is the highest control derivative on which the flat outputs may depend, i.e., $p$ is the maximum of $p$, $r$, $s$ and $q$.

We will denote by $k_1$ (respectively $k_2$) the order of the highest derivative of $\varphi_1$ (respectively $\varphi_2$) involved in the expression of $x$ and $u$ i.e.,

$$x = \gamma(\varphi_1, \cdots, \varphi_1^{(k_1)}, \varphi_2, \cdots, \varphi_2^{(k_2)}) \text{ and } u = \delta(\varphi_1, \cdots, \varphi_1^{(k_1)}, \varphi_2, \cdots, \varphi_2^{(k_2)}).$$

Throughout we will use the following notation $\bar{\varphi}_i^{k_i} = (\varphi_i, \cdots, \varphi_i^{(k_i)})$. Since the differential weight equals $n+3$, we have $k_1 + k_2 = n+1$.

There are two cases to be considered: the first corresponds to $p \geq 1$, i.e., $\varphi_1$ involves at least a control derivative; the second deals with $p = 0$, i.e., $\varphi_1 = \varphi_1(x, u_1, u_2)$.

Let us first suppose $p \geq 1$. Since $\varphi_1^{(l)}$ depends explicitly on $u_1^{(p+l)}$, for $l \geq 0$, it follows that $x$ and $u$ and $\varphi_1^{(l)}$ are independent for any $l \geq 0$, and consequently to express all states $x$ and controls $u$ in function of the flat outputs and their time derivative, at least, $\varphi_2, \cdots, \varphi_2^{(n+1)}$ are used. Therefore, $k_2 \geq n+1$ and the differential weight of the system is greater than $n + k_1 + 3$. Hence, the only possible case is $k_1 = 0$, i.e., $\gamma$ and $\delta$ involve only $\varphi_1$ and not its time-derivatives. Thus assume $k_1 = 0$. If $s \geq 1$ or $q \geq 1$, we repeat the same procedure and we find that $k_2$ should also be zero, which is impossible. It follows that $s \leq 0$ and $l \leq 0$, i.e., $\varphi_2 = \varphi_2(x, u_1, u_2)$. If $\varphi_2$ depends explicitly on $u_1$ (respectively on $u_2$), we can apply the invertible feedback $v_1 = \varphi_2(x, u_1, u_2)$ and $v_2 = u_2$ (respectively $v_2 = \varphi_2(x, u_1, u_2)$ and $v_1 = u_1$), then all states and the remaining control should be expressed only in function of $\varphi_1$ and its time derivatives, which is impossible, since $k_1 = 0$. Therefore, $\varphi_2 = \varphi_2(x)$.

Let $\rho$ be the relative degree of $\varphi_2$, i.e. $\varphi_2^{(\rho)}$ is the first derivative of $\varphi_2$ involving explicitly the control. More precisely, the relative degree $\rho_i$ of a component $\varphi_i = \varphi_i(x)$ of a flat output $(\varphi_1, \varphi_2)$, defined on a neighborhood $\mathcal{X}$ of $x_0$, is the smallest integer such that

$$L_{g_j} L_f^q \varphi_i \equiv 0,\ 1 \leq j \leq 2, 0 \leq q \leq \rho_i - 2,$$

$$L_{g_j} L_f^{\rho_i - 1} \varphi_i(x) \neq 0, \text{ for some } 1 \leq j \leq 2 \text{ and for some } x \in \mathcal{X}.$$

By introducing $z_i = L_f^{i-1}\varphi_2$, for $1 \leq i \leq \rho$, we get:

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \text{ for } 1 \leq i \leq \rho - 1, \\ \dot{z}_\rho &= L_f^\rho \varphi_2 + u_1 L_{g_1} L_f^{\rho-1} \varphi_2 + u_2 L_{g_2} L_f^{\rho-1} \varphi_2. \end{aligned}$$

and according to (Assumption 1), we have $(L_{g_1} L_f^{\rho-1}\varphi_2, L_{g_2} L_f^{\rho-1}\varphi_2)(z_0) \neq (0,0)$ and assume
$L_{g_1} L_f^{\rho-1}\varphi_2 \neq 0$ (otherwise permute $g_1$ and $g_2$). Applying $v_1 = L_f^\rho \varphi_2 + u_1 L_{g_1} L_f^{\rho-1}\varphi_2 + u_2 L_{g_2} L_f^{\rho-1}\varphi_2$ and $v_2 = u_2$, we obtain $\varphi_2^{(\rho)} = v_1$. If $\rho < n$, at least one state and a control should be expressed using $\varphi_1$ and its time derivatives, which is impossible. It follows that $\rho = n$, but in this case, the system $\Sigma$ would be static feedback equivalent to the linear single-input system

$$\begin{cases} \dot{z}_i = z_{i+1}, \text{ for } 1 \leq i \leq n-1, \\ \dot{z}_n = v_1 \end{cases}$$

which gives a contradiction.

We have thus proved that we cannot have $p \geq 1$ and then, the only possible case is $p = 0$, i.e., $\varphi_i = \varphi_i(x, u_1, u_2)$, for $1 \leq i \leq 2$, with $\frac{\partial \varphi_1}{\partial u_1} \neq 0$. It is immediate that $\operatorname{rk} \frac{\partial \varphi}{\partial u} = 1$, where $\frac{\partial \varphi}{\partial u}$ denotes the matrix $(\frac{\partial \varphi_i}{\partial u_j})$, $1 \leq i, j \leq 2$, otherwise apply the invertible static feedback $\tilde{u}_1 = \varphi_1(x, u_1, u_2)$ and $\tilde{u}_2 = \varphi_2(x, u_1, u_2)$, which transforms the system $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = f(x) + \psi_1(x, \tilde{u}_1, \tilde{u}_2)g_1(x) + \psi_2(x, \tilde{u}_1, \tilde{u}_2)g_2(x)$, where $(\tilde{u}_1, \tilde{u}_2)$ is a flat output. It is clear that the state coordinates $x$ cannot be represented in terms of flat outputs, contradicting the flatness assumption.

Apply the invertible static feedback $\tilde{u}_1 = \varphi_1(x, u_1, u_2)$ and $\tilde{u}_2 = u_2$ which brings the system into the form $\tilde{\Sigma} : \dot{x} = f(x) + \psi(x, \tilde{u}_1, \tilde{u}_2)g_1(x) + \tilde{u}_2 g_2(x)$ with $\varphi_1 = \tilde{u}_1$ and $\varphi_2 = \varphi_2(x, \tilde{u}_1)$. Since $\varphi_1 = \tilde{u}_1$, all states and the control $\tilde{u}_2$ have to be expressed with the help of $\varphi_2$, it follows that $\varphi_2$ involves at least $n+1$ derivatives, i.e., $k_2 \geq n$, and since $k_1 + k_2 = n + 1$, we deduce $k_1 \leq 1$.

If $\varphi_2$ depends explicitly on $\tilde{u}_1 = \varphi_1$, then $\varphi_2^{(n)}$ would explicitly depend on $\varphi_1^{(n)}$ and the differential weight would be at least $2n + 2$, which contradicts our assumption. We deduce $\varphi_2 = \varphi_2(x)$.

Now we proceed as above: introduce $z_i = L_f^{i-1}\varphi_2$, for $1 \leq i \leq \rho$, where $\rho$ is the relative degree of $\varphi_2$, and complete them to a coordinate system $(z_1, \cdots, z_\rho, \cdots, z_n)$. We have

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \text{ for } 1 \leq i \leq \rho - 1, \\ \dot{z}_\rho &= L_f^\rho \varphi_2 + \psi(x, \tilde{u}_1, \tilde{u}_2)L_{g_1}L_f^{\rho-1}\varphi_2 + \tilde{u}_2 L_{g_2}L_f^{\rho-1}\varphi_2. \end{aligned}$$

If $\varphi_2^{(\rho)}$ depends explicitly on $\tilde{u}_2$, apply the invertible static feedback $v_2 = L_f^\rho\varphi_2 + \psi(x, \tilde{u}_1, \tilde{u}_2)L_{g_1}L_f^{\rho-1}\varphi_2 + \tilde{u}_2 L_{g_2}L_f^{\rho-1}\varphi_2$ and $v_1 = \tilde{u}_1$. We obtain $\varphi_1 = v_1$, $\varphi_2^{(\rho)} = v_2$. If $\rho < n$, at least one state would not be represented as function of $\varphi_i^{(j)}$, contradicting the flatness assumption. If $\rho = n$, then $\Sigma$ would be static feedback equivalent to a linear single-input control system, which is a impossible. It follows that $\varphi_2^{(\rho)} = \dot{z}_\rho = a(z, \tilde{u}_1)$, where $a$ is a smooth function depending explicitly on $\tilde{u}_1$.

Moreover, we should be able to express all the remaining coordinates $z_{\rho+1}, \cdots, z_n$ and the control $\tilde{u}_2$ with the help of $\varphi_2^{(\rho+i)}$, for $0 \leq i \leq k_2 - \rho$. Recall that $k_2 \geq n$ and notice that each derivative $\varphi_2^{(\rho+i)}$ involves explicitly $\varphi_1^{(i)}$. In the best case, $k_2 = n$ and then the highest derivative of $\varphi_1$ involved is that of order $n - \rho$, i.e., $z = \gamma(\bar{\varphi}_1{}^{n-\rho}, \bar{\varphi}_2{}^n)$ and $\tilde{u} = \delta_2(\bar{\varphi}_1{}^{n-\rho}, \bar{\varphi}_2{}^n)$.

The above expressions involve $2n - \rho + 2$ derivatives of the minimal flat outputs. Since the differential weight of the system is $n + 3$, we deduce that $\rho = n - 1$ and the system $\Sigma$ can be written as follows:

$$\tilde{\Sigma} : \begin{cases} \dot{z}_1 &= z_2 \\ & \vdots \\ \dot{z}_{n-2} &= z_{n-1} \\ \dot{z}_{n-1} &= a(z, \tilde{u}_1) \\ \dot{z}_n &= b(z, \tilde{u}_1, \tilde{u}_2) \end{cases}$$

with $a$ (respectively $b$) depending explicitly on $\tilde{u}_1$ (respectively $\tilde{u}_2$) and $(\varphi_1, \varphi_2) = (\tilde{u}_1, z_1)$ being a minimal flat output. It is easy to see that $\tilde{\Sigma}$ is in fact static feedback linearizable (apply the invertible static feedback $v_1 = a(z, \tilde{u}_1)$ and $v_2 = b(z, \tilde{u}_1, \tilde{u}_2)$), thus of differential weight $n + 2$ (with $(\tilde{\varphi}_1, \tilde{\varphi}_2) = (z_1, z_n)$ an $x$-flat output of differential weight $n + 2$), contradicting the minimality of $(\varphi_1, \varphi_2) = (\tilde{u}_1, z_1)$.

This finishes the proof of the second case $p = 0$. For both cases, $p \geq 1$ and $p = 0$, we have found a contradiction with our assumptions. It follows that the minimal flat outputs $\varphi_1$, $\varphi_2$ depend only on $x$ and thus the system is $x$-flat.

$(ii) \Rightarrow (iii)$. Let us consider an $x$-flat control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ and let $(\varphi_1, \varphi_2)$ be a minimal $x$-flat output, defined in a neighborhood $\mathcal{X}$ of $x_0$, whose differential weight is $n+3$. We will denote by $k_1$ (respectively $k_2$) the order of the highest derivative of $\varphi_1$ (respectively $\varphi_2$) involved in the expression of $x$ and $u$ i.e., $x = \gamma(\bar{\varphi_1}^{k_1}, \bar{\varphi_2}^{k_2})$ and $u = \delta(\bar{\varphi_1}^{k_1}, \bar{\varphi_2}^{k_2})$, where $\bar{\varphi_i}^{k_i} = (\varphi_i, \dot{\varphi}_i, \cdots, \varphi_i{}^{(k_i)})$. We clearly have $k_1 + k_2 = n + 1$.

Let $\mu$ and $\rho$, the relative degree of $\varphi_1$ and $\varphi_2$. We thus have $L_{g_j} L_f^p \varphi_1 \equiv 0$, for $0 \leq p \leq \mu - 2$, and $L_{g_j} L_f^q \varphi_2 \equiv 0$, for $0 \leq q \leq \rho - 2$. It is well known that $d\varphi_1^{(i)}$ and $d\varphi_2^{(j)}$ are independent at $x_0$, for any $i$, $j \geq 0$, thus we can put $w_i = L_f^{i-1}\varphi_1$, $z_j = L_f^{j-1}\varphi_2$, for $1 \leq i \leq \mu$ and $1 \leq j \leq \rho$, and complete them to a coordinate system $\xi = (w_1, \cdots, w_\mu, z_1, \cdots, z_\rho, z_{\rho+1}, \cdots, z_{\rho+\nu})$, where $n = \mu + \rho + \nu$.

Consider the decoupling matrix $D = (D_{ij})_{1 \leq i,j \leq 2}$ given by

$$D = \begin{pmatrix} L_{g_1} L_f^{\mu-1}\varphi_1 & L_{g_2} L_f^{\mu-1}\varphi_1 \\ L_{g_1} L_f^{\rho-1}\varphi_2 & L_{g_2} L_f^{\rho-1}\varphi_2 \end{pmatrix}$$

By definition of the relative degree, we have $1 \leq \operatorname{rk} D(x) \leq 2$ and according to (Assumption 1), $\operatorname{rk} D(x)$ is constant in a neighborhood of $x_0$. It is easy to see that $\operatorname{rk} D(x) = 1$. Indeed, if $\operatorname{rk} D(x) = 2$, the flatness assumption would imply that the system $\Sigma$ is locally static feedback linearizable thus of differential weight $n+2$, contradicting the fact that $\Sigma$ is flat of differential weight $n+3$. Therefore, we can always assume $\operatorname{rk} D(x) = 1$, $\forall x \in \mathcal{X}'$, where $\mathcal{X}'$ is an open dense subset of $\mathcal{X}$, and $L_{g_1} L_f^{\mu-1}\varphi_1(x_0) \neq 0$ (if not, permute $g_1$ and $g_2$). Applying the invertible static feedback transformation

$$\begin{aligned} \tilde{u}_1 &= L_f^{\mu}\varphi_1 + L_{g_1} L_f^{\mu-1}\varphi_1 u_1 + L_{g_2} L_f^{\mu-1}\varphi_1 u_2 \\ \tilde{u}_2 &= u_2, \end{aligned}$$

we get:

$$\begin{array}{rclrcl} \dot{w}_1 &=& w_2 & \dot{z}_1 &=& z_2 \\ &\vdots& & &\vdots& \\ \dot{w}_{\mu-1} &=& w_\mu & \dot{z}_{\rho-1} &=& z_\rho \\ \dot{w}_\mu &=& \tilde{u}_1 & \dot{z}_\rho &=& d(w, z, \tilde{u}_1) \end{array}$$

with $(\varphi_1, \varphi_2) = (w_1, z_1)$ and $d$ a smooth function, affine with respect to $\tilde{u}_1$ and depending explicitly on $\tilde{u}_1$. Since $\varphi_1^{(\mu)} = \tilde{u}_1$ and $\varphi_2^{(\rho)} = d(z, \tilde{u}_1)$, we have to express all the remaining coordinates $z_{\rho+1}, \cdots, z_{\rho+\nu}$ and the control $\tilde{u}_2$ with the help of $\varphi_2^{(\rho+i)}$, for $0 \leq i \leq k_2 - \rho$, but each derivative $\varphi_2^{(\rho+i)}$ involves explicitly $\varphi_1^{(\mu+i)}$. In the best case, $k_2 = \rho + \nu$ and then the highest derivative of $\varphi_1$ involved is that of order $\mu + \nu - \rho = n - \rho$, i.e., we have $z = \gamma(\bar{\varphi_1}^{n-\rho}, \bar{\varphi_2}^{\rho+\nu})$ and $\tilde{u} = \delta_2(\bar{\varphi_1}^{n-\rho}, \bar{\varphi_2}^{\rho+\nu})$.

The above expressions involve $n+\nu+2$ derivatives of the minimal flat outputs. Since the differential weight of the system is $n+3$, we deduce that $\nu = 1$. Hence, after applying a static invertible feedback transformation (leaving $\tilde{u}_1$ unchanged an that we continue to denote by $\tilde{u}$), the system $\Sigma$ can be written as follows:

$$\begin{array}{lllllll}
\dot{w}_1 & = & w_2 & \dot{z}_1 & = & z_2 \\
 & \vdots & & & \vdots & \\
\dot{w}_{\mu-1} & = & w_\mu & \dot{z}_{\rho-1} & = & z_\rho \\
\dot{w}_\mu & = & \tilde{u}_1 & \dot{z}_\rho & = & a(w,z)+b(w,z)\tilde{u}_1 \\
 & & & \dot{z}_{\rho+1} & = & \tilde{u}_2
\end{array}$$

with $\varphi_1 = w_1$ and $\varphi_2 = z_1$ being a minimal flat output. Flatness implies $(\frac{\partial a}{\partial z_{\rho+1}} + \frac{\partial b}{\partial z_{\rho+1}}\tilde{u}_{10})(\xi_0) \neq 0$, where $\xi_0 = (w_0, z_0)$.

If $\mathcal{D}^0$ is involutive, then $\frac{\partial b}{\partial z_{\rho+1}} = 0$. It follows that $\frac{\partial a}{\partial z_{\rho+1}}(\xi_0) \neq 0$ and we can introduce new coordinates $z_i^1 = w_i$, for $1 \leq i \leq \mu$, $z_j^2 = z_j$, for $1 \leq j \leq \rho$, and $z_{\rho+1} = a$ and apply a suitable invertible static feedback, to get

$$\tilde{\Sigma} : \left\{ \begin{array}{lllllll}
\dot{z}_1^1 & = & z_2^1 & \dot{z}_1^2 & = & z_2^2 \\
 & \vdots & & & \vdots & \\
\dot{z}_{\mu-1}^1 & = & z_\mu^1 & \dot{z}_{\rho-1}^2 & = & z_\rho^2 \\
\dot{z}_\mu^1 & = & v_1 & \dot{z}_\rho^2 & = & z_{\rho+1}^2 + b(\bar{z}_\mu^1, \bar{z}_\rho^2)v_1 \\
 & & & \dot{z}_{\rho+1} & = & v_2
\end{array} \right.$$

where $(\varphi_1, \varphi_2) = (z_1^1, z_1^2)$ and $\bar{z}_j^i = (z_1^i, \cdots, z_j^i)$. Notice that, in this case, the system is $x$-flat for any $u_0 \in \mathbb{R}^2$, so we do not face singularities in the control space.

If $\mathcal{D}^0$ is noninvolutive, then $\frac{\partial b}{\partial z_{\rho+1}}(\xi_0) \neq 0$ and we can introduce new coordinates $z_i^1 = w_i$, for $1 \leq i \leq \mu$, $z_j^2 = z_j$, for $1 \leq j \leq \rho$, and $z_{\rho+1} = b$ and apply a suitable invertible static feedback, to get

$$\tilde{\Sigma} : \left\{ \begin{array}{lllllll}
\dot{z}_1^1 & = & z_2^1 & \dot{z}_1^2 & = & z_2^2 \\
 & \vdots & & & \vdots & \\
\dot{z}_{\mu-1}^1 & = & z_\mu^1 & \dot{z}_{\rho-1}^2 & = & z_\rho^2 \\
\dot{z}_\mu^1 & = & v_1 & \dot{z}_\rho^2 & = & a(\bar{z}_\mu^1, \bar{z}_{\rho+1}^2) + z_{\rho+1}^2 v_1 \\
 & & & \dot{z}_{\rho+1} & = & v_2
\end{array} \right.$$

where $(\varphi_1, \varphi_2) = (z_1^1, z_1^2)$. Contrary of the previous case, now, there exist singular controls: the system is $x$-flat at $(z_0, v_0)$ such that $(\frac{\partial a}{\partial z_{\rho+1}^2} + v_{10})(z_0) \neq 0$.

It is immediate that, for both cases, the prolongation $\tilde{\Sigma}^{(1,0)}$, obtained by prolonging $v_1$ is locally static feedback linearizable.

$(iii) \Rightarrow (i)$. Consider a control systems $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ dynamically linearizable via one-fold prolongation, i.e., there exists an invertible feedback transformation, $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} \ : \ \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2\tilde{h}_2(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{h}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

where $y_1 = \tilde{u}_1$ and $v_2 = \tilde{u}_2$, is locally static feedback linearizable.

$\tilde{\Sigma}^{(1,0)}$ is equivalent via a diffeomorphism $z = \phi(x, y_1)$ and an invertible transformation, $v = \alpha(x, y_1) + \beta(x, y_1)\bar{v}$, to the Brunovský canonical form:

$$\begin{array}{rcl} \dot{z}_i^j &=& z_i^j, \ 1 \leq j \leq \rho_i - 1, \\ \dot{z}_i^{\rho_i} &=& \bar{v}_i \end{array}$$

where $1 \leq i \leq 2$ and $\rho_1 + \rho_2 = n + 1$, for which $\varphi = (z_1^1, z_2^1)$ is a minimal flat output of differential weight $n + 3$. Since $z_i^j = \varphi_i^{(j-1)}$, the original variables can be expressed as $(x, u_1)^t = \gamma(\bar{\varphi}_1^{\rho_1 - 1}, \bar{\varphi}_2^{\rho_2 - 1})$ and $u_2 = \delta_2(\bar{\varphi}_1^{\rho_1}, \bar{\varphi}_2^{\rho_2})$. We deduce that $\varphi = (\varphi_1(x, u_1), \varphi_1(x, u_1))$ is a minimal flat output of $\Sigma$ of differential weight $n + 3$.

### 1.7.3 Proof of Theorem 1.3.1

*Necessity.* Consider a control system $\Sigma : \dot{x} = f(x) + u_1g_1(x) + u_2g_2(x)$ and assume that it is flat of differential weight $n + 3$. According to Proposition 1.3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} \ : \ \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2\tilde{h}_2(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{h}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_2 = \tilde{u}_2$ is locally static feedback linearizable. For simplicity of notation, we will drop the tilde, but we will keep distinguishing $g_1$ from $h_2$ (which could also be denoted $g_2$) whose control is not preintegrated.

Recall that to $\Sigma^{(1,0)}$, we associate the distributions $\mathcal{D}_p^0 = \mathrm{span}\,\{G_1, H_2\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [F, \mathcal{D}_p^i]$, for $i \geq 0$, where $F$ (respectively $G_1$ and $H_2$) denotes the drift (respectively the control vector fields) of the prolonged system.

Since $\Sigma^{(1,0)}$ is locally static feedback linearizable, for any $i \geq 0$ the distributions $\mathcal{D}_p^i$ are involutive, of constant rank, and there exists an integer $\rho$ such that $\mathrm{rk}\,\mathcal{D}_p^\rho = n + 1$. We have

$$\begin{array}{rcl} \mathcal{D}_p^0 &=& \mathrm{span}\,\{\frac{\partial}{\partial y_1}, h_2\}, \\ \mathcal{D}_p^1 &=& \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, h_2, ad_f h_2 + y_1[g_1, h_2]\}. \end{array}$$

Since $k \geq 1$, the distribution $\mathcal{D}^0 = \mathrm{span}\,\{g_1, h_2\}$ is involutive, thus $[g_1, h_2] \in \mathcal{D}^0$ and $\mathcal{D}_p^1 = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, h_2, ad_f h_2\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq k$,

$$\mathcal{D}_p^i = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1}g_1, h_2, \cdots, ad_f^i h_2\}.$$

Since the intersection of involutive distributions is an involutive distribution, it follows that $\mathcal{D}_p^i \bigcap TX = \operatorname{span}\{g_1, \cdots, ad_f^{i-1}g_1, h_2, \cdots, ad_f^i h_2\}$ is involutive, for $1 \leq i \leq k$. We deduce that

$$\mathcal{H}^k = \operatorname{span}\{g_1, \cdots, ad_f^{k-1}g_1, h_2, \cdots, ad_f^k h_2\}$$

is involutive. It is immediate that $\mathcal{D}^{k-1} \subset \mathcal{H}^k \subset \mathcal{D}^k$, where both inclusions are of corank one, otherwise $\mathcal{H}^k = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive or $\mathcal{H}^k = \mathcal{D}^{k-1}$ and $\operatorname{rk}\mathcal{D}^k - \operatorname{rk}\mathcal{D}^{k-1}$ would be equal to one, which contradicts our hypotheses. The involutivity of

$$\mathcal{D}_p^{k+1} = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{k-1}g_1, ad_f^k g_1, h_2, \cdots, ad_f^k h_2, ad_f^{k+1}h_2\}$$

implies that of $\mathcal{D}^k + \operatorname{span}\{ad_f^{k+1}h_2\}$. It yields $\overline{\mathcal{D}}^k = \mathcal{D}^k + \operatorname{span}\{ad_f^{k+1}h_2\}$ and $\operatorname{rk}\overline{\mathcal{D}}^k = 2k+3$, where $\overline{\mathcal{D}}^k$ is the involutive closure of $\mathcal{D}^k$. This gives $(A1)$.

Recall that $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, for $i \geq k+1$. We thus have

$$\mathcal{D}_p^{k+1} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^k + [f, \mathcal{H}^k] = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^{k+1}$$

and, by an induction argument,

$$\mathcal{D}_p^{k+i} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^{k+i},\ i \geq 2.$$

Consequently, the involutivity of $\mathcal{D}_p^{k+i}$ implies that of $\mathcal{H}^{k+i}$, for $i \geq 1$. Moreover, $\operatorname{rk}\mathcal{D}_p^\rho = n+1$, implying that $\operatorname{rk}\mathcal{H}^\rho = n$, i.e., $\mathcal{H}^\rho = TX$, which proves $(A3)$ and $(A4)$.

It remains to show that $\operatorname{rk}(\overline{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$. We have $\mathcal{D}_p^{k+1} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \overline{\mathcal{D}}^k$. Assume $ad_f^{k+1}g_1 \in \overline{\mathcal{D}}^k$, if not, the rank in question is, indeed, $2k+4$. Hence for any vector field $\xi \in \mathcal{D}^k$, we have $[f, \xi] \in \overline{\mathcal{D}}^k$. By successive applications of the Jacobi identity, it follows immediately that $\overline{\mathcal{D}}^k + [f, \overline{\mathcal{D}}^k] = \overline{\mathcal{D}}^k$.

Therefore for the prolonged system we obtain

$$\mathcal{D}_p^{k+2} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \overline{\mathcal{D}}^k + [f, \overline{\mathcal{D}}^k] = \mathcal{D}_p^{k+1}$$

thus contradicting the existence of $\rho$ such that $\operatorname{rk}\mathcal{D}_p^\rho = n+1$ (recall that $\overline{\mathcal{D}}^k = \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$) and implying that $\Sigma^{(1,0)}$ is not static feedback linearizable. By Proposition 1.3.1, the system $\Sigma$ would not be $x$-flat and thus $\operatorname{rk}(\overline{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$ and $(A2)$ holds.

*Sufficiency.* Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ satisfying $(A1) - (A4)$. Transform $\Sigma$ via a static feedback into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 g_c(x)$, where $g_c$ is defined by condition $(A2)$. For simplicity of notation, we will drop the tilde and we will keep distinguishing $g_1$ from $g_c$. By Proposition 1.7.2, the involutivity of $\mathcal{H}^i = \mathcal{D}^{i-1} + \text{span}\,\{ad_f^i g_c\}$ follows for $1 \leq i \leq k-1$. It is immediate to see that the prolongation

$$\Sigma^{(1,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + v_2 g_c(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_2 = u_2$ is locally static feedback linearizable. Indeed, the linearizability distributions $\mathcal{D}_p^i$, associated to $\Sigma^{(1,0)}$, are of the form

$$\mathcal{D}_p^i \;=\; \text{span}\,\{\tfrac{\partial}{\partial y_1}\} + \mathcal{H}^i,\ i \geq 1.$$

The involutivity of $\mathcal{H}^i$ implies that of $\mathcal{D}_p^i$. Moreover, $\text{rk}\,\mathcal{H}^\rho = n$, thus $\text{rk}\,D_p^\rho = n+1$ and $\Sigma^{(1,0)}$ is locally static feedback linearizable. By Proposition 1.3.1, the system $\Sigma$ is flat of differential weight $n+3$.

### 1.7.4 Proof of Theorem 1.3.2

*Necessity.* Consider the control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ and assume that it is flat of differential weight $n+3$. According to Proposition 1.3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 \tilde{h}_2(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + v_2 \tilde{h}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_2 = \tilde{u}_2$ is locally static feedback linearizable, around $(x_0, y_0)$. For simplicity of notation, we will drop the tilde, but we will keep distinguishing $g_1$ from $h_2$ (which could also be denoted $g_2$) whose control is not preintegrated.

Since $\Sigma^{(1,0)}$ is locally static feedback linearizable, for any $i \geq 0$ the distributions $\mathcal{D}_p^i$ are involutive, of constant rank, and there exists an integer $\rho$ such that $\text{rk}\,\mathcal{D}_p^\rho = n+1$. We have

$$\begin{aligned} \mathcal{D}_p^0 &= \text{span}\,\{\tfrac{\partial}{\partial y_1}, h_2\}, \\ \mathcal{D}_p^1 &= \text{span}\,\{\tfrac{\partial}{\partial y_1}, g_1, h_2, ad_f h_2 + y_1[g_1, h_2]\}. \end{aligned}$$

Since $k = 0$, the distribution $\mathcal{D}^0 = \text{span}\,\{g_1, h_2\}$ is noninvolutive, thus $[g_1, h_2] \notin \mathcal{D}^0$ and $\mathcal{D}_p^1 = \text{span}\,\{\frac{\partial}{\partial y_1}, g_1, h_2, [g_1, h_2]\}$. We clearly have $ad_f h_2 \in \mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$, consequently, a non zero vector field $g_c \in \mathcal{D}^0$ such that $ad_f g_c \in \mathcal{G}^1$, whose existence is claimed in $(A2)'$, can be taken as $g_c = h_2$. $\mathcal{D}_p^1$ has constant rank around $(x_0, y_{10})$, it follows that $\text{rk}\,(\text{span}\,\{g_1, h_2, ad_f h_2 + y_1[g_1, h_2]\}(x_0, y_{10})) = 3$. This yields $(g_1 \wedge g_c \wedge [f + u_{10} g_1 + u_{20} g_c, g_c])(x_0) \neq 0$ and proves $(RC)$.

The involutivity of $\mathcal{D}_p^1$ implies that of $\mathcal{H}^1 = \mathcal{G}^1$ and gives $(A1)'$. The rest of the proof follows the same line as that of Theorem 1.3.1.

*Sufficiency.* Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ satisfying $(A1)' - (A4)'$ and $(RC)$. Transform $\Sigma$ via a static feedback into the form $\tilde{\Sigma} : \dot{x} =$

$\tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2 g_c(x)$, where $g_c$ is defined by condition $(A2)'$. For simplicity of notation, we drop the tilde, but we keep distinguishing $g_1$ from $g_c$. It is immediate to see that the prolongation

$$\Sigma^{(1,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + v_2 g_c(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_2 = u_2$ is locally static feedback linearizable. Indeed, we have $\mathcal{D}_p^0 = \text{span}\,\{\frac{\partial}{\partial y_1}, g_c\}$, which is clearly involutive, and

$$\mathcal{D}_p^1 = \text{span}\,\{\frac{\partial}{\partial y_1}, g_1, g_c, ad_f g_c + y_1[g_1, g_c]\}.$$

Since $u_0 \notin U_{sing}(x_0)$, where $U_{sing}(x) = \{u \in \mathbb{R}^2 : (g_1 \wedge g_c \wedge [f + u_1 g_1 + u_2 g_c, g_c])(x) = 0\}$, it follows that rk (span $\{g_1, g_c, ad_f g_c + y_{10}[g_1, g_c]\}(x_0)) = 3$. Now, recall that $ad_f g_c \in \mathcal{G}^1 = \text{span}\,\{g_1, g_c, [g_1, g_c]\}$. These observations yield

$$\text{span}\,\{g_1, g_c, ad_f g_c + y_1[g_1, g_c]\} = \mathcal{G}^1,$$

around $(x_0, y_{10})$. Therefore, the distribution $\mathcal{D}_p^1$ is given by

$$\mathcal{D}_p^1 = \text{span}\,\{\frac{\partial}{\partial y_1}\} + \mathcal{G}^1$$

and is involutive. Recall that $\mathcal{H}^1 = \mathcal{G}^1$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, thus the linearizability distributions $\mathcal{D}_p^i$, associated to $\Sigma^{(1,0)}$, are of the form

$$\mathcal{D}_p^i = \text{span}\,\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^i,\ i \geq 1.$$

The involutivity of $\mathcal{H}^i$ implies that of $\mathcal{D}_p^i$. Moreover, rk $\mathcal{H}^\rho = n$, thus rk $D_p^\rho = n + 1$ and $\Sigma^{(1,0)}$ is locally static feedback linearizable. By Proposition 1.3.1, the system $\Sigma$ is flat of differential weight $n + 3$.

### 1.7.5 Proof of Theorem 1.3.4

We start by proving the first item $(i)$. Consider a two-input control-affine system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$, defined on a $n$-dimensional manifold $X$. Assume $k \geq 1$. Under the assumptions $\mathcal{D}^i$ involutive and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, we necessarily have rk $\mathcal{D}^k = 2k + 2$ (otherwise, we would have $\mathcal{D}^k = \mathcal{D}^{k-1} + \text{span}\,\{v\}$, where $v$ is either $ad_f^k g_1$ or $ad_f^k g_2$, and the noninvolutivity of $\mathcal{D}^k$ would imply $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$). We deduce rk $\mathcal{D}^i = 2(i+1)$, for $1 \leq i \leq k-1$, and since $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$, it follows that $n = 2k + 3$. It is clear that there exists local coordinates $z = (z^0, z_1^1, \cdots, z_{k+1}^1, z_2^1, \cdots, z_2^{k+1})$ in which $\Sigma$ (after applying a suitable invertible feedback) takes the form:

$$\tilde{\Sigma} : \begin{cases} \dot{z}^0 &= \alpha(z^0, \bar{z}_2^1, \bar{z}_2^2) & & \\ \dot{z}_1^1 &= z_2^1 & \dot{z}_1^2 &= z_2^2 \\ &\vdots & &\vdots \\ \dot{z}_k^1 &= z_{k+1}^1 & \dot{z}_k^2 &= z_{k+1}^2 \\ \dot{z}_{k+1}^1 &= v_1 & \dot{z}_{k+1}^2 &= v_2 \end{cases}$$

with $\alpha$ a smooth function. We have

$$\mathcal{D}^{k-1} = \text{span}\,\{\frac{\partial}{\partial z_2^1}, \cdots, \frac{\partial}{\partial z_{k+1}^1}, \frac{\partial}{\partial z_2^2}, \cdots, \frac{\partial}{\partial z_{k+1}^2}\}$$

$$\mathcal{D}^{k} = \mathcal{D}^{k-1} + \text{span}\,\{\frac{\partial}{\partial z_1^1} + \frac{\partial \alpha}{\partial z_2^1}\frac{\partial}{\partial z^0}, \frac{\partial}{\partial z_1^2} + \frac{\partial \alpha}{\partial z_2^2}\frac{\partial}{\partial z^0}\}.$$

Since $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, it follows

$$\frac{\partial^2 \alpha}{\partial (z_2^1)^2} = \frac{\partial^2 \alpha}{\partial (z_2^2)^2} = \frac{\partial^2 \alpha}{\partial z_2^1 \partial z_2^2} = 0$$

and thus $\alpha$ is an affine function of $z_2^1$ and $z_2^2$ and can be written as

$$\alpha(z^0, \bar{z}_2^1, \bar{z}_2^2) = a(z^0, z_1^1, z_1^2) z_2^1 + b(z^0, z_1^1, z_1^2) z_2^2 + c(z^0, z_1^1, z_1^2)$$

where $a$, $b$, $c$ are smooth and verify $(\frac{\partial b}{\partial z_1^1} + a\frac{\partial b}{\partial z^0} - \frac{\partial a}{\partial z_1^2} - b\frac{\partial a}{\partial z^0})(z_0) \neq 0$. The last condition is due to the fact that $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$.

From $\mathcal{D}^{k+1}(x_0) = T_{x_0}X$, we deduce $\operatorname{rk} \mathcal{D}^{k+1}(z_0) = 2k+3$. Suppose $ad_f^{k+1} g_1(z_0) \notin \mathcal{D}^k(z_0)$, otherwise permute $v_1$ and $v_2$. This condition is invariant with respect to invertible feedback transformations of the form $v_1 = \beta_{11}\tilde{v}_1 + \beta_{12}\tilde{v}_2$, $v_2 = \beta_{22}\tilde{v}_2$.

Since the vector field $\xi = \frac{\partial}{\partial z^0} a(z^0, z_1^1, z_1^2) + \frac{\partial}{\partial z_1^1}$ is non zero at any $z_0$, there exists a smooth function $\psi : \mathbb{R}^3 \mapsto \mathbb{R}$, depending on $z^0, z_1^1, z_1^2$, such that $\frac{\partial \psi}{\partial z^0}(z_0) \neq 0$ and $L_\xi \psi = \frac{\partial \psi}{\partial z^0} a + \frac{\partial \psi}{\partial z_1^1} = 0$. We put $w_1^1 = \psi(z^0, z_1^1, z_1^2)$ and obtain

$$\tilde{\Sigma} : \begin{cases} \begin{array}{lclclcl} \dot{w}_1^1 & = & \tilde{b}(w_1^1, z_1^1, z_1^2) z_2^2 + \tilde{c}(w_1^1, z_1^1, z_1^2) & & & & \\ \dot{z}_1^1 & = & z_2^1 & & \dot{z}_1^2 & = & z_2^2 \\ & \vdots & & & & \vdots & \\ \dot{z}_k^1 & = & z_{k+1}^1 & & \dot{z}_k^2 & = & z_{k+1}^2 \\ \dot{z}_{k+1}^1 & = & v_1 & & \dot{z}_{k+1}^2 & = & v_2 \end{array} \end{cases}$$

Since $[ad_f^k g_1, ad_f^k g_2] = [\frac{\partial}{\partial z_1^1}, \frac{\partial}{\partial z_1^2} + \tilde{b}\frac{\partial}{\partial w_1^1}] = \frac{\partial \tilde{b}}{\partial z_1^1}\frac{\partial}{\partial w_1^1} \notin \mathcal{D}^k$, we get $\frac{\partial \tilde{b}}{\partial z_1^1}(w_0, z_0) \neq 0$ and applying the invertible change of coordinates

$$\begin{array}{lcl} w_i^1 & = & L_f^{i-2}\tilde{b},\ 2 \leq i \leq k+2, \\ w_i^2 & = & z_i^2,\ 2 \leq i \leq k+1 \end{array}$$

and a suitable invertible static feedback transformation (leaving $v_2$ unchanged and thus preserving the fact that $ad_f^{k+1} g_1(z_0) \notin \mathcal{D}^k(z_0)$) we get

$$(NF^*) : \begin{cases} \begin{array}{lclclcl} \dot{w}_1^1 & = & w_2^1 w_2^2 + \tilde{c}(\bar{w}_2^1, w_1^2) & & & & \\ \dot{w}_2^1 & = & w_3^1 & & \dot{w}_1^2 & = & w_2^2 \\ & \vdots & & & & \vdots & \\ \dot{w}_{k+1}^1 & = & w_{k+2}^1 & & \dot{w}_k^2 & = & w_{k+1}^2 \\ \dot{w}_{k+2}^1 & = & \tilde{v}_1 & & \dot{w}_{k+1}^2 & = & \tilde{v}_2 \end{array} \end{cases}$$

with $(w_2^2 + \frac{\partial \tilde{c}}{\partial w_2^1})(w_0) \neq 0$, which is clearly flat of differential weight $n+3 = 2k+6$ at $w_0$ and we can take $(\varphi_1, \varphi_2) = (w_1^1, w_2^2)$ as a minimal $x$-flat output.

If $k = 0$ and $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] = TX$, the same arguments apply with $\mathcal{D}^{k+1}$ replaced by span $\{g_1, g_2, ad_f g_1 + u_2[g_2, g_1], ad_f g_2 + u_1[g_1, g_2]\}$ and $ad_f^{k+1} g_1$ by $ad_f g_1 + u_2[g_2, g_1]$. We do not develop this case here.

Let us now show $(ii)$.

*Necessity.* Let us consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ and assume that it is flat of differential weight $n+3$. According to Proposition 1.3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 \tilde{h}_2(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + v_2 \tilde{h}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_2 = \tilde{u}_2$, is locally static feedback linearizable. For simplicity of notation, we will drop the tilde, but we will keep distinguishing $g_1$ from $h_2$ whose control is not preintegrated.

Since $\Sigma^{(1,0)}$ is locally static feedback linearizable, for any $i \geq 0$ the distributions $\mathcal{D}_p^i$ are involutive, of constant rank, and there exists an integer $\rho$ such that $\operatorname{rk} \mathcal{D}_p^\rho = n+1$. We have

$$\begin{aligned} \mathcal{D}_p^0 &= \operatorname{span}\{\tfrac{\partial}{\partial y_1}, h_2\}, \\ \mathcal{D}_p^1 &= \operatorname{span}\{\tfrac{\partial}{\partial y_1}, g_1, h_2, ad_f h_2 + y_1[g_1, h_2]\}. \end{aligned}$$

Since $k \geq 1$, the distribution $\mathcal{D}^0 = \operatorname{span}\{g_1, h_2\}$ is involutive, thus $[g_1, h_2] \in \mathcal{D}^0$ and $\mathcal{D}_p^1 = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, h_2, ad_f h_2\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq k$,

$$\mathcal{D}_p^i = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1} g_1, h_2, \cdots, ad_f^i h_2\}.$$

Since the intersection of involutive distributions is an involutive distribution, it follows that $\mathcal{D}_p^i \cap TX = \operatorname{span}\{g_1, \cdots, ad_f^{i-1} g_1, h_2, \cdots, ad_f^i h_2\}$ is involutive, for $1 \leq i \leq k$. We deduce that the distribution

$$\mathcal{E} = \operatorname{span}\{g_1, \cdots, ad_f^{k-1} g_1, h_2, \cdots, ad_f^k h_2\}$$

is involutive. Next we will prove that $\mathcal{E} = \mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$.

It is immediate that $\mathcal{D}^{k-1} \subset \mathcal{E} \subset \mathcal{D}^k$, where both inclusions are of corank one, otherwise $\mathcal{E} = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive, which contradicts our hypotheses. Applying Jacobi identity, it can be proved that $[ad_f^{k-1} h_2, ad_f^k g_1] \in \mathcal{D}^k$, which gives immediately $[ad_f^{k-1} h_2, \mathcal{D}^k] \in \mathcal{D}^k$, i.e., $ad_f^{k-1} h_2 \in \mathcal{C}^k$, where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$. Moreover, since $\mathcal{D}^k = \mathcal{E} + \operatorname{span}\{ad_f^k g_1\}$ is noninvolutive and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, we deduce $[ad_f^{k-1} g_1, ad_f^k g_1] \notin \mathcal{D}^k$.

The involutivity of

$$\mathcal{D}_p^{k+1} = \text{span}\,\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{k-1}g_1, ad_f^k g_1, h_2, \cdots, ad_f^k h_2, ad_f^{k+1}h_2\}$$

implies that of $\mathcal{D}^k + \text{span}\,\{ad_f^{k+1}h_2\}$. It yields $\overline{\mathcal{D}}^k = \mathcal{D}^k + \text{span}\,\{ad_f^{k+1}h_2\}$ and $\text{rk}\,\overline{\mathcal{D}}^k = 2k+3$, where $\overline{\mathcal{D}}^k$ is the involutive closure of $\mathcal{D}^k$. Therefore, $\text{rk}\,\mathcal{C}^k = 2k$ (this gives (C1)) and the new direction $[ad_f^{k-1}g_1, ad_f^k g_1]$ completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k$ has to be collinear with $ad_f^{k+1}h_2$. Hence there exists a smooth function $\alpha$ such that $[ad_f^k h_2, ad_f^k g_1] = \alpha[ad_f^{k-1}g_1, ad_f^k g_1] \bmod \mathcal{D}^k$. It follows $[ad_f^k h_2 - \alpha ad_f^{k-1}g_1, ad_f^k g_1] = 0 \bmod \mathcal{D}^k$. It is easy to show that

$$\mathcal{C}^k = \mathcal{D}^{k-2} + \text{span}\,\{ad_f^{k-1}h_2, ad_f^k h_2 - \alpha ad_f^{k-1}g_1\},$$

which yields $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1} = \text{span}\,\{g_1, \cdots, ad_f^{k-1}g_1, h_2, \cdots, ad_f^k h_2\}$ and $\text{rk}\,(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = 2k-1$, showing (C2). Now recall that the involutive subdistribution $\mathcal{E}$ is given by $\mathcal{E} = \text{span}\,\{g_1, \cdots, ad_f^{k-1}g_1,\ h_2, \cdots, ad_f^k h_2\}$, it follows immediately that we actually have $\mathcal{E} = \mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, implying the involutivity of $\mathcal{H}^k$ and proving $(C3)$. Moreover, since $\overline{\mathcal{D}}^k = TX = \mathcal{D}^k + \text{span}\,\{ad_f^{k+1}h_2\}$, we deduce that $\mathcal{H}^{k+1} = TX$, which shows $(C4)$.

*Sufficiency.* Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ defined on a $n$-dimensional manifold $X$, and satisfying $(C1) - (C4)$. Since $\text{rk}\,\mathcal{C}^k = 2k$ and $\mathcal{D}^k$ is not involutive and its rank is at most $2k+2$, we deduce that $\text{rk}\,\mathcal{D}^k = 2k+2$ and we actually have $\mathcal{D}^k = \text{span}\,\{v_1, v_2\} + \mathcal{C}^k$. Since $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = \overline{\mathcal{D}}^k = TX$, we have $n = 2k+3$.

We will prove that conditions $(C1) - (C2)$ enable us to define a nonzero vector field $g_c \in \mathcal{D}^0$ such that the involutive subdistribution $\mathcal{H}^k = \mathcal{D}^{k-1} + \mathcal{C}^k$ can be written as

$$\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span}\,\{ad_f^k g_c\}.$$

In order to define $g_c$, notice that clearly $\mathcal{D}^{k-2} \subset \mathcal{C}^k$ and since $\text{rk}\,(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = 2k-1$, we have

$$\mathcal{C}^k \cap \mathcal{D}^{k-1} = \mathcal{D}^{k-2} + \text{span}\,\{v\},$$

with $v$ of the form $v = \alpha_1 ad_f^{k-1}g_1 + \alpha_2 ad_f^{k-1}g_2$, where $\alpha_1$ and $\alpha_2$ are smooth functions not vanishing simultaneously. It follows $v = ad_f^{k-1}(\alpha_1 g_1 + \alpha_2 g_2) \bmod \mathcal{D}^{k-2}$ and we put $g_c = \alpha_1 g_1 + \alpha_2 g_2$. We can always suppose $\alpha_2(x_0) \neq 0$ (otherwise permute $g_1$ and $g_2$). Since $ad_f^{k-1}g_c \in \mathcal{C}^k$, we have $[ad_f^{k-1}g_c, \mathcal{D}^k] \subset \mathcal{D}^k$ and it can be shown, by the involutivity of $\mathcal{D}^{k-1}$ and applying the Jacobi identity, that $[ad_f^{k-1}g_1, ad_f^k g_c] \in \mathcal{D}^k$. Therefore, the new direction completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k = TX$ is given by $[ad_f^{k-1}g_1, ad_f^k g_1]$ and there exists a smooth function $\alpha$ such that $[ad_f^k g_c, ad_f^k g_1] = \alpha[ad_f^{k-1}g_1, ad_f^k g_1] \bmod \mathcal{D}^k$. This gives $[ad_f^k g_c - \alpha ad_f^{k-1}g_1, ad_f^k g_1] = 0 \bmod \mathcal{D}^k$ and it can be easily verified that

$$\mathcal{C}^k = \mathcal{D}^{k-2} + \text{span}\,\{ad_f^{k-1}g_c, ad_f^k g_c - \alpha ad_f^{k-1}g_1\},$$

which gives, as claimed,

$$\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span}\,\{ad_f^k g_c\}.$$

The involutivity of $\mathcal{H}^k$ implies that of all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + \text{span}\,\{ad_f^i g_c\}$, for $1 \leq i \leq k-1$. The proof of this statement follows by the same line as the proof of Proposition 1.7.2.

We are now in position to show that the control $\Sigma$ is dynamically linearizable via one-fold prolongation. Transform $\Sigma$ via a static feedback into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2 g_c(x)$, where $g_c$ is defined as above. Applying the same arguments as in the proof of Theorem 1.3.1, it is immediate to see that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{g}_c(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_2 = \tilde{u}_2$ is locally static feedback linearizable and by Proposition 1.3.1, the system is flat of differential weight $n+3$.

### 1.7.6 Proof of Proposition 1.4.1

We will prove the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Consider a flat control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$, of differential weight $n+3$ and denote by $(\varphi_1, \varphi_2)$ its minimal flat outputs, defined in a neighborhood of $x_0$. By Proposition 1.3.1, it is $x$-flat and according to the proof of the implication $(ii) \Rightarrow (iii)$ of Proposition 1.3.1, the system $\Sigma$ can be transformed by a change of coordinates and an invertible static feedback into the form

$$\tilde{\Sigma} : \begin{cases} \dot{z}_1^1 = z_2^1 & \dot{z}_1^2 = z_2^2 \\ \quad\vdots & \quad\vdots \\ \dot{z}_{\rho_1-1}^1 = z_{\rho_1}^1 & \dot{z}_{\rho_2-1}^2 = z_{\rho_2}^2 \\ \dot{z}_{\rho_1}^1 = \tilde{u}_1 & \dot{z}_{\rho_2}^2 = a(z) + b(z)\tilde{u}_1 \\ & \dot{z}_{\rho_2+1}^2 = \tilde{u}_2 \end{cases}$$

where $\rho_1 + \rho_2 + 1 = n$, $(\varphi_1, \varphi_2) = (z_1^1, z_1^2)$ and $\frac{\partial a}{\partial z_{\rho_2+1}^2}(z_0) + \frac{\partial b}{\partial z_{\rho_2+1}^2}(z_0)\tilde{u}_{10} \neq 0$.

If the first noninvolutive distribution is $\mathcal{D}^0$, then $b = z_{\rho_2+1}^2$ and $a = a(z)$ is a function satisfying $\frac{\partial a}{\partial z_{\rho_2+1}^2}(z_0) + \tilde{u}_{10} \neq 0$. If the first noninvolutive distribution is $\mathcal{D}^k$, with $k \geq 1$, then we put $a = z_{\rho_2+1}^2$ and $b = b(\bar{z}_{\rho_1}^1, \bar{z}_{\rho_2}^2)$.

Moreover, the involutivity of $\mathcal{D}^i$, for $0 \leq i \leq k-1$, implies $b = b(\bar{z}_{\rho_1-k+1}^1, \bar{z}_{\rho_2-k+1}^2)$. The involutivity of $\mathcal{D}^k$ can be lost in two different ways: either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$ and in this case

$$\frac{\partial b}{\partial z_{\rho_1-k+1}^1} + b\frac{\partial b}{\partial z_{\rho_2-k+1}^2} \equiv 0 \text{ and } \frac{\partial b}{\partial z_{\rho_2-k+1}^2}(z_0) \neq 0$$

or $[\mathcal{D}^{k-1},\mathcal{D}^k]\not\subset\mathcal{D}^k$ and in this case

$$(\frac{\partial b}{\partial z^1_{\rho_1-k+1}}+b\frac{\partial b}{\partial z^2_{\rho_2-k+1}})(z_0)\neq 0.$$

We have obtained the normal form $(NF1)$.

$(ii)\Rightarrow(iii)$. If $k=0$, then $(NF1)$ and $(NF2)$ coincide. We can thus suppose $k\geq 1$. Since $\mathcal{D}^k$ is noninvolutive, of rank $2k+2$, it follows immediately that $\rho_1,\rho_2\geq k+1$, thus there exits two integers $\mu_1\geq 1$ and $\mu_2\geq 1$ such that $\rho_1=\mu_1+k$ and $\rho_2=\mu_2+k$.

By a direct calculation, we can check that the involutive distribution $\mathcal{D}^{k-1}$ is annihilated by $\mu_1+\mu_2$ differentials $dz^1_i$, $1\leq i\leq\mu_1$, and $dz^2_j$, $1\leq j\leq\mu_2$. Since rank $\mathcal{D}^{k-1}=2k$ and $n=\mu_1+\mu_2+2k+1$, there exists a function $\psi=\psi(\bar z^1_{\mu_1+1},\bar z^2_{\mu_2+1})$ such that $d\psi$ is independent at $z_0$ of $dz^1_i$, $dz^2_j$ and annihilates $\mathcal{D}^{k-1}$. It follows $L_{ad_f^{k-1}g_2}\psi=\frac{\partial\psi}{\partial z^1_{\mu_1+1}}+b(\bar z^1_{\mu_1+1},\bar z^2_{\mu_2+1})\frac{\partial\psi}{\partial z^2_{\mu_2+1}}=0$ and since $d\psi$ and $dz^1_i$, $dz^2_j$ are independent at $z_0$, we deduce $\frac{\partial\psi}{\partial z^2_{\mu_2+1}}(z_0)\neq 0$. Define $\hat z^2_{\mu_2+1+j}=L^j_f\psi$, for $0\leq j\leq k$, a valid change of coordinates and, after applying a suitable invertible static feedback transformation, bring the system into the form:

$$\begin{cases}
\dot z^1_1 = z^1_2 & \dot z^2_1 = z^2_2\\
\quad\vdots & \quad\vdots\\
\dot z^1_{\mu_1-1} = z^1_{\mu_1} & \dot z^2_{\mu_2-1} = z^2_{\mu_2}\\
\dot z^1_{\mu_1} = z^1_{\mu_1+1} & \dot z^2_{\mu_2} = d(\bar z^1_{\mu_1+1},\bar z^2_{\mu_2},\hat z^2_{\mu_2+1})\\
\dot z^1_{\mu_1+1} = z^1_{\mu_1+2} & \dot{\hat z}^2_{\mu_2+1} = \hat z^2_{\mu_2+2}\\
\quad\vdots & \quad\vdots\\
\dot z^1_{\mu_1+k} = \hat v_1 & \dot{\hat z}^2_{\mu_2+k} = \hat z^2_{\mu_2+k+1}\\
 & \dot{\hat z}^2_{\mu_2+k+1} = \hat v_2
\end{cases}$$

with $\frac{\partial d}{\partial\hat z^2_{\mu_2+1}}(z_0)\neq 0$ and $(\varphi_1,\varphi_2)=(z^1_1,z^2_1)$.

We will next analyze the conditions satisfied by the function $d$. To simplify the notation, we will write $z$ (respectively $v$) instead of $\hat z$ (respectively $\hat v$). We have

$$\begin{aligned}
\mathcal{D}^{k-1} &= \text{span}\,\{\frac{\partial}{\partial z^1_{\mu_1+1}},\cdots,\frac{\partial}{\partial z^1_{\mu_1+k}},\frac{\partial}{\partial z^2_{\mu_2+2}},\cdots,\frac{\partial}{\partial z^2_{\mu_2+k+1}}\}\ \text{and}\\
\mathcal{D}^k &= \mathcal{D}^{k-1}+\text{span}\,\{\frac{\partial}{\partial z^1_{\mu_1}}+\frac{\partial d}{\partial z^1_{\mu_1}}\frac{\partial}{\partial z^2_{\mu_2}},\frac{\partial}{\partial z^2_{\mu_2+1}}\}.
\end{aligned}$$

If $[\mathcal{D}^{k-1},\mathcal{D}^k]\not\subset\mathcal{D}^k$, then it follows immediately that $\frac{\partial^2 d}{\partial(z^1_{\mu_1+1})^2}(z_0)\neq 0$. We have obtained the normal form $(NF2)$ for the case $[\mathcal{D}^{k-1},\mathcal{D}^k]\not\subset\mathcal{D}^k$. On the other hand, if $[\mathcal{D}^{k-1},\mathcal{D}^k]\subset\mathcal{D}^k$ and $[ad^k_fg_1,ad^k_fg_2]\notin\mathcal{D}^k$, we obtain $\frac{\partial^2 d}{\partial(z^1_{\mu_1+1})^2}=0$. Thus

$$d(\bar z^1_{\mu_1+1},\bar z^2_{\mu_2+1})=d_1(\bar z^1_{\mu_1},\bar z^2_{\mu_2+1})+d_2(\bar z^1_{\mu_1},\bar z^2_{\mu_2+1})z^1_{\mu_1+1}.$$

Since $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$,

$$\frac{\partial d_2}{\partial z^2_{\mu_2+1}}(z_0) \neq 0.$$

It is easy to see that $L_{g_i} L_f^j d_2 = 0$, for $1 \leq i \leq 2$ and $0 \leq j \leq k-1$. Moreover, the functions $L_f^j d_2$ are independent, for $0 \leq j \leq k$, then the following change of coordinates $\tilde{z}^2_{\mu_2+1+j} = L_f^j d_2$, for $0 \leq j \leq k$, is valid and, after applying a suitable invertible static feedback transformation, it brings the system into the form:

$$\begin{cases}
\begin{array}{llllll}
\dot{z}^1_1 & = & z^1_2 & \dot{z}^2_1 & = & z^2_2 \\
 & \vdots & & & \vdots & \\
\dot{z}^1_{\mu_1-1} & = & z^1_{\mu_1} & \dot{z}^2_{\mu_2-1} & = & z^2_{\mu_2} \\
\dot{z}^1_{\mu_1} & = & z^1_{\mu_1+1} & \dot{z}^2_{\mu_2} & = & \tilde{d}(\bar{z}^1_{\mu_1+1}, \bar{z}^2_{\mu_2}, \tilde{z}^2_{\mu_2+1}) + \tilde{z}^2_{\mu_2+1} z^1_{\mu_1+1} \\
\dot{z}^1_{\mu_1+1} & = & z^1_{\mu_1+2} & \dot{\tilde{z}}^2_{\mu_2+1} & = & \tilde{z}^2_{\mu_2+2} \\
 & \vdots & & & \vdots & \\
\dot{z}^1_{\mu_1+k} & = & \tilde{v}_1 & \dot{\tilde{z}}^2_{\mu_2+k} & = & \tilde{z}^2_{\mu_2+k+1} \\
 & & & \dot{\tilde{z}}^2_{\mu_2+k+1} & = & \tilde{v}_2
\end{array}
\end{cases}$$

with $(\varphi_1, \varphi_2) = (z^1_1, z^2_1)$. This is the normal form $(NF2)$ corresponding to the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$.

Note that we have also proved that the minimal $x$-flat outputs and the normal forms $(NF1)$ (resp. $(NF2)$) are compatible. Indeeed, if $(\varphi_1, \varphi_2)$ is a minimal $x$-flat output at $x_0$, then there exists an invertible static feedback transformation bringing the system $\Sigma$ into $(NF1)$ with $\varphi_1 = z^1_1$ and $\varphi_2 = z^2_1$ (resp. into $(NF2)$ with $\varphi_1 = w^1_1$ and $\varphi_2 = w^2_1$).

$(iii) \Rightarrow (i)$. Consider a control system $\Sigma$ static feedback equivalent to the normal form $(NF2)$. It is clear that $\Sigma$ is flat, with $(\varphi_1, \varphi_2) = (w^1_1, w^2_1)$ minimal flat outputs of differential weight $n+3$.

### 1.7.7 Proof of Proposition 1.5.1

Consider the control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ that is $x$-flat at $x_0$ (at $(x_0, u_0)$, if $k = 0$), of differential weight $n+3$.

We start by proving item $(i)$ corresponding to $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ or $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$.

*Necessity.* Let the pair $(\varphi_1, \varphi_2)$ be a minimal flat output, defined on a neighborhood $\mathcal{X}$ of $x_0$. According to Proposition 1.4.1 and its proof, there exists a valid local change of coordinates in which the system, after applying a suitable feedback, takes the form $(NF2)$, with $(\varphi_1, \varphi_2) = (w^1_1, w^2_1)$. Recall that $\rho$ and $\mu$ are defined as $\rho = k + \max(\mu_1, \mu_2)$, $\mu = k + \min(\mu_1, \mu_2)$. It is easy to check that (after permuting $\varphi_1$ and $\varphi_2$, if necessary)

$$\begin{array}{lll} d\varphi_1 & \perp & \mathcal{H}^{\rho-1} \\ d\varphi_2 & \perp & \mathcal{H}^{\mu-1} \end{array}$$

and $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$ are valid on $\mathcal{X}$. The distributions $\mathcal{H}^i$ are feedback invariant which proves necessity of the conditions.

*Sufficiency.* Since $\Sigma$ is $x$-flat with of differential weight $n+3$, it follows, by Proposition 1.4.1, that it is locally static feedback equivalent to $(NF2)$. Bring $\Sigma$ into the form $(NF2)$, around $w_0$. In coordinates $w$ we have

$$\begin{aligned} \mathcal{D}^i &= \operatorname{span}\{\frac{\partial}{\partial w^1_{\mu_1+k-i}}, \cdots, \frac{\partial}{\partial w^1_{\mu_1+k}}, \frac{\partial}{\partial w^2_{\mu_2+k+1-i}}, \cdots, \frac{\partial}{\partial w^2_{\mu_2+k+1}}\}, \text{ for } 0 \leq i \leq k-1, \\ \mathcal{D}^k &= \mathcal{D}^{k-1} + \operatorname{span}\{\frac{\partial}{\partial w^1_{\mu_1}} + \frac{\partial d}{\partial w^1_{\mu_1}}\frac{\partial}{\partial w^2_{\mu_2}}, \frac{\partial}{\partial w^2_{\mu_2+1}}\}, \end{aligned}$$

where $d$ satisfies $\frac{\partial d}{\partial w^2_{\mu_2+1}}(w_0) \neq 0$.

First notice that the new direction completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$ is necessarily $\frac{\partial}{\partial w^2_{\mu_2}}$ and is collinear with $ad_f^{k+1}g_2$. It follows that $g_2$ plays the role of $g_c$ defined by item $(A2)$ of Theorem 1.3.1, if $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$. Moreover, if $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, then we have $[ad_f^{k-1}g_1, ad_f^k g_1] \notin \mathcal{D}^k$ and $[ad_f^{k-1}g_2, \mathcal{D}^k] \subset \mathcal{D}^k$, therefore the vector field $g_2$ is such that the distribution $\mathcal{H}^k$, defined by conditions$(C1)-(C3)$ of Theorem 1.3.4 (ii), is given by $\mathcal{H}^k = \mathcal{D}^{k-1} + \operatorname{span}\{ad_f^k g_2\}$.

Let us suppose $\mu_1 < \mu_2$. The same reasoning applies if $\mu_1 > \mu_2$ or $\mu_1 = n$ and we do not develop these cases here. We have

$$\begin{aligned} \mathcal{H}^{k+\mu_2-1} = \mathcal{H}^{\rho-1} &= \operatorname{span}\{\frac{\partial}{\partial w^1_1}, \cdots, \frac{\partial}{\partial w^1_{\mu_1+k}}, \frac{\partial}{\partial w^2_2}, \cdots, \frac{\partial}{\partial w^2_{\mu_2+k+1}}\}, \\ \mathcal{H}^{k+\mu_1-1} = \mathcal{H}^{\mu-1} &= \operatorname{span}\{\frac{\partial}{\partial w^1_2}, \cdots, \frac{\partial}{\partial w^1_{\mu_1+k}}, \frac{\partial}{\partial w^2_{\mu_2-p+2}}, \cdots, \frac{\partial}{\partial w^2_{\mu_2+k+1}}\}. \end{aligned}$$

Since $d\varphi_1 \perp \mathcal{H}^{\rho-1}$, it follows that we choose $\varphi_1$ as a function depending on $w^2_1$ only and satisfying $\frac{\partial\varphi_1}{\partial w^2_1}(w_0) \neq 0$. We deduce that $L_f^j\varphi_1$, for $1 \leq j \leq \mu_2 - 1$, are independent functions and that $L_f^j\varphi_1$ depend on $w^2_1, w^2_2, \cdots, w^2_{j+1}$.

Since $d\varphi_2 \perp \mathcal{H}^{\mu-1}$ and $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(w_0) \neq 0$, where $\rho - \mu = \mu_2 - \mu_1$, we choose $\varphi_2 = \varphi_2(w^1_1, \bar{w}^2_{\mu_2-\mu_1+1})$ and $\frac{\partial\varphi_2}{\partial w^1_1}(w_0) \neq 0$. We deduce that $L_f^j\varphi_2$, for $1 \leq j \leq \mu_1 + k - 1$, are independent functions of $w^1_1, \cdots, w^1_{j+1}, w^2_1, \cdots w^2_{\mu_2-\mu_1+j+1}$.

We apply the following change of coordinates

$$\begin{aligned} \tilde{w}^1_j &= L_f^{j-1}\varphi_1, \ 1 \leq j \leq \mu_1 + k, \\ \tilde{w}^2_j &= L_f^{j-1}\varphi_2, \ 1 \leq j \leq \mu_2, \end{aligned}$$

and an invertible static feedback transformation, to get

$$\begin{cases} \dot{\tilde{w}}_1^1 = \tilde{w}_2^1 & \dot{\tilde{w}}_1^2 = \tilde{w}_2^2 \\ \quad\vdots & \quad\vdots \\ \dot{\tilde{w}}_{\mu_1-1}^1 = \tilde{w}_{\mu_1}^1 & \dot{\tilde{w}}_{\mu_2-1}^2 = \tilde{w}_{\mu_2}^2 \\ \dot{\tilde{w}}_{\mu_1}^1 = \tilde{w}_{\mu_1+1}^1 & \dot{\tilde{w}}_{\mu_2}^2 = \tilde{d}(\bar{w}_{\mu_1+1}^1, \bar{w}_{\mu_2+1}^2) \\ \dot{\tilde{w}}_{\mu_1+1}^1 = \tilde{w}_{\mu_1+2}^1 & \dot{w}_{\mu_2+1}^2 = w_{\mu_2+2}^2 \\ \quad\vdots & \quad\vdots \\ \dot{\tilde{w}}_{\mu_1+k}^1 = v_1 & \dot{w}_{\mu_2+k}^2 = w_{\mu_2+k+1}^2 \\ & \dot{w}_{\mu_2+k+1}^2 = v_2 \end{cases}$$

This is the normal form $(NF2)$, with $\varphi_1 = \tilde{w}_1^1$ and $\varphi_2 = \tilde{w}_1^2$ as minimal $x$-flat outputs. It follows that $(\varphi_1, \varphi_2)$ is also a minimal flat output of the original system $\Sigma$.

Let us now show that the pair $(\varphi_1, \varphi_2)$ of minimal flat outputs is unique, up to a diffeomorphism. We have already proved that the minimal $x$-flat outputs and the normal form $(NF2)$ are compatible, i.e., if $(\varphi_1, \ \varphi_2)$ is a minimal $x$-flat output at $x_0$, we can introduce new coordinates in which the original system $\Sigma$ takes, via an invertible static feedback transformation, the form $(NF2)$, with $\varphi_1$ and $\varphi_2$ playing the role of the top variables $w_1^1$ and $w_1^2$. Let $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ be another minimal flat output. Clearly, $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ is also a minimal flat output of $(NF2)$. We have just proven that

$$\begin{array}{lcl} d\tilde{\varphi}_1 & \perp & \mathcal{H}^{\rho-1} \\ d\tilde{\varphi}_2 & \perp & \mathcal{H}^{\mu-1}. \end{array}$$

The distribution $\mathcal{H}^{\rho-1}$ is of corank one in $TX$ so $d\varphi_1$ and $d\tilde{\varphi}_1$ are collinear and thus there exists a function $h_1$ such that $\tilde{\varphi}_1 = h_1(\varphi_1)$, where $h_1'(\cdot) \neq 0$. We have

$$\begin{aligned} (\mathcal{H}^{\mu-1})^\perp &= \operatorname{span}\{d\varphi_2, d\varphi_1, dL_f\varphi_1, \cdots, dL_f^{\rho-\mu}\varphi_1\} \\ &= \operatorname{span}\{d\tilde{\varphi}_2, d\tilde{\varphi}_1, dL_f\tilde{\varphi}_1, \cdots, dL_f^{\rho-\mu}\tilde{\varphi}_1\} \\ &= \operatorname{span}\{d\tilde{\varphi}_2, d\varphi_1, dL_f\varphi_1, \cdots, dL_f^{\rho-\mu}\varphi_1\}, \end{aligned}$$

implying that $d\varphi_2$ and $d\tilde{\varphi}_2$ are collinear modulo $d\varphi_1, dL_f\varphi_1, \cdots, dL_f^{\rho-\mu}\varphi_1$, thus there exists a function $h_2$, invertible with respect to $\varphi_2$, such that $\tilde{\varphi}_2 = h_2(\varphi_2, \varphi_1, L_f\varphi_1, \cdots, L_f^{\rho-\mu}\varphi_1)$.

We now turn to item $(ii)$ corresponding to the case $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$.

*Necessity.* Let the pair $(\varphi_1, \varphi_2)$ be a minimal flat output of $\Sigma$, defined on a neighborhood $\mathcal{X}$ of $x_0$. According to Proposition 1.4.1 and its proof, there exists a valid local change of coordinates in which the system, after applying a suitable feedback, takes the form

$$(NF2): \begin{cases} \dot{w}_1^1 = w_2^1 & \dot{w}_1^2 = d(\bar{w}_1^1, \bar{w}_2^2) + w_2^1 w_2^2 \\ \dot{w}_2^1 = w_3^1 & \dot{w}_2^2 = w_3^2 \\ \quad\vdots & \quad\vdots \\ \dot{w}_{k+1}^1 = \tilde{u}_1 & \dot{w}_{k+1}^2 = w_{k+2}^2 \\ & \dot{w}_{k+2}^2 = \tilde{u}_2 \end{cases}$$

where $(\frac{\partial d}{\partial w_2^2} + w_2^1)(w_0) \neq 0$ and $(\varphi_1, \varphi_2) = (w_1^1, w_1^2)$ (permute $\varphi_1$ and $\varphi_2$, if necessary). It is clear that $(d\varphi_1 \wedge \varphi_2)(w_0) \neq 0$. Moreover, we have

$$\mathcal{L} = (\text{span}\,\{d\varphi_1, \varphi_2\})^\perp = \text{span}\,\{\frac{\partial}{\partial w_{k+1}^1}, \cdots, \frac{\partial}{\partial w_2^1}, \frac{\partial}{\partial w_{k+2}^2}, \cdots, \frac{\partial}{\partial w_2^2}\},$$

which is clearly involutive and satisfies $\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k$, where both inclusions are of corank one. The vector field $g_2 = \frac{\partial}{\partial w_{k+2}^2}$ is such that $\mathcal{L} = \mathcal{D}^{k-1} + \text{span}\,\{ad_f^k g_2\}$ and since $(\frac{\partial d}{\partial w_2^2} + w_2^1)(w_0) \neq 0$, we have $L_{ad_f^k g_2}\varphi_2(w_0) \neq 0$.

*Sufficiency.* Since $\Sigma$ is $x$-flat at $x_0$, with the differential weight $n+3$, it follows, by Proposition 1.4.1 that it can be locally transformed into the form

$$(NF2): \begin{cases} \dot{w}_1^1 = w_2^1 & \dot{w}_1^2 = a(\bar{w}_1^1, \bar{w}_2^2) + w_2^1 w_2^2 \\ \dot{w}_2^1 = w_3^1 & \dot{w}_2^2 = w_3^2 \\ \quad\vdots & \quad\vdots \\ \dot{w}_{k+1}^1 = \tilde{u}_1 & \dot{w}_{k+1}^2 = w_{k+2}^2 \\ & \dot{w}_{k+2}^2 = \tilde{u}_2 \end{cases}$$

Applying the following change of coordinates:

$$\begin{aligned} w &= w_1^2 - \tfrac{1}{2} w_1^1 w_2^2 \\ z_i^1 &= w_i^1 \\ z_i^2 &= w_{i+1}^2 \end{aligned}$$

for $1 \leq i \leq k+1$, we get the following symmetric form

$$(\tilde{NF2}): \begin{cases} \dot{w} = a(w, z_1^1, z_1^2) + \tfrac{1}{2} z_2^1 z_1^2 - \tfrac{1}{2} z_1^1 z_2^2 & \\ \dot{z}_1^1 = z_2^1 & \dot{z}_1^2 = z_2^2 \\ \quad\vdots & \quad\vdots \\ \dot{z}_k^1 = z_{k+1}^1 & \dot{z}_k^2 = z_{k+1}^2 \\ \dot{z}_{k+1}^1 = \tilde{u}_1 & \dot{z}_{k+1}^2 = \tilde{u}_2 \end{cases}$$

for which the vector fields $g_1$ and $g_2$ play the same role (this will be useful for computing the distribution $\mathcal{L}$).

Since the involutive distribution $\mathcal{L} = (\text{span}\,\{d\varphi_1, \varphi_2\})^\perp$ satisfies $\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k$, it is immediate that both inclusions are of corank one (otherwise either $\mathcal{L} = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive or $\mathcal{L} = \mathcal{D}^{k-1}$ and $\text{rk}\,\mathcal{D}^k - \text{rk}\,\mathcal{D}^{k-1}$ would be equal to one). It follows that $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{D}^{k-1} + \text{span}\,\{\alpha_1 ad_f^k g_1 + \alpha_2 ad_f^k g_2\},$$

where $\alpha_1, \alpha_2$ are two smooth functions non vanishing simultaneously. Since $g_1, g_2$ play the same role for the form $(\tilde{NF2})$, there is no loss in generality in assuming $\alpha_2(w_0, z_0) \neq 0$. Thus,

$$\mathcal{L} = \mathcal{D}^{k-1} + \text{span}\,\{ad_f^k(g_2 + \alpha g_1)\} = \mathcal{D}^{k-1} + \text{span}\,\{ad_f^k g_c\}.$$

where $\alpha = \frac{\alpha_1}{\alpha_2}$ and $g_c = g_2 + \alpha g_1$.

Applying the invertible static feedback transformation $\tilde{u}_1 = v_1 + \alpha v_2$ and $\tilde{u}_2 = v_2$, we get

$$(N\tilde{F}2) : \begin{cases} \dot{w} &= a(w, z_1^1, z_1^2) + \frac{1}{2}z_2^1 z_1^2 - \frac{1}{2}z_1^1 z_2^2 & & \\ \dot{z}_1^1 &= z_2^1 & \dot{z}_1^2 &= z_2^2 \\ &\vdots & &\vdots \\ \dot{z}_k^1 &= z_{k+1}^1 & \dot{z}_k^2 &= z_{k+1}^2 \\ \dot{z}_{k+1}^1 &= v_1 + \alpha(w, z)v_2 & \dot{z}_{k+1}^2 &= v_2 \end{cases}$$

for which $g_2 = g_c = \frac{\partial}{\partial z_{k+1}^2} + \alpha(x, z, w)\frac{\partial}{\partial z_{k+1}^1}$, where $g_c$ is the vector field defined above. By an induction argument, it can be shown that

$$\mathcal{D}^i = \text{span}\,\{\frac{\partial}{\partial z_{k+1-i}^1}, \cdots, \frac{\partial}{\partial z_{k+1}^1}, \frac{\partial}{\partial z_{k+1-i}^2}, \cdots, \frac{\partial}{\partial z_{k+1}^2}\},$$

for $1 \leq i \leq k-1$, and

$$ad_f^k g_1 = (-1)^k(\frac{\partial}{\partial z_1^1} + \frac{1}{2}z_1^2\frac{\partial}{\partial w}),$$

$$ad_f^k g_c = (-1)^k(\frac{\partial}{\partial z_1^2} - \frac{1}{2}z_1^1\frac{\partial}{\partial w} + \alpha(\frac{\partial}{\partial z_1^1} + \frac{1}{2}z_1^2\frac{\partial}{\partial w}))\,\text{mod}\,\mathcal{D}^{k-1},$$

then

$$\mathcal{L} = \text{span}\,\{\frac{\partial}{\partial z_2^1}, \cdots, \frac{\partial}{\partial z_{k+1}^1}, \frac{\partial}{\partial z_2^2}, \cdots, \frac{\partial}{\partial z_{k+1}^2}, \frac{\partial}{\partial z_1^2} - \frac{1}{2}z_1^1\frac{\partial}{\partial w} + \alpha(\frac{\partial}{\partial z_1^1} + \frac{1}{2}z_1^2\frac{\partial}{\partial w})\}.$$

Since $\mathcal{L} = (\text{span}\,\{d\varphi_1, d\varphi_2\})^\perp$ is involutive, we deduce that $\alpha$, $\varphi_1$ and $\varphi_2$ are functions of $z_1^1, z_1^2$ and $w$, only. Moreover, $\varphi_i$ satisfies

$$\frac{\partial \varphi_i}{\partial z_1^2} - \frac{1}{2}z_1^1\frac{\partial \varphi_i}{\partial w} + \alpha(\frac{\partial \varphi_i}{\partial z_1^1} + \frac{1}{2}z_1^2\frac{\partial \varphi_i}{\partial w}) = 0,\ 1 \leq i \leq 2.$$

Since $d\varphi_1 \wedge d\varphi_2(w_0, z_0) \neq 0$ and $TX = \mathcal{L} + \text{span}\,\{ad_f^k g_1, [ad_f^k g_1, ad_f^k g_2]\}$, it follows

$$\text{rk}\begin{pmatrix} L_{ad_f^k g_1}\varphi_1 & L_{[ad_f^k g_1, ad_f^k g_2]}\varphi_1 \\ L_{ad_f^k g_1}\varphi_2 & L_{[ad_f^k g_1, ad_f^k g_2]}\varphi_2 \end{pmatrix}(w_0, z_0) = 2.$$

We have $[ad_f^k g_1, ad_f^k g_2] = \frac{\partial}{\partial w}\,\text{mod}\,\text{span}\,\{ad_f^k g_1\}$. Therefore, the above rank becomes

$$\text{rk}\begin{pmatrix} \frac{\partial \varphi_1}{\partial z_1^1} + \frac{1}{2}z_1^2\frac{\partial \varphi_1}{\partial w} & \frac{\partial \varphi_1}{\partial w} \\ \frac{\partial \varphi_2}{\partial z_1^1} + \frac{1}{2}z_1^2\frac{\partial \varphi_2}{\partial w} & \frac{\partial \varphi_2}{\partial w} \end{pmatrix}(w_0, z_0) = 2.$$

This implies

$$\text{rk}\begin{pmatrix} \frac{\partial \varphi_1}{\partial z_1^1} & \frac{\partial \varphi_1}{\partial w} \\ \frac{\partial \varphi_2}{\partial z_1^1} & \frac{\partial \varphi_2}{\partial w} \end{pmatrix}(w_0, z_0) = 2,$$

thus we can introduce new coordinates $\tilde{w} = \varphi_1(z_1^1, z_1^2, w)$ and $\tilde{z}_1^1 = \varphi_2(z_1^1, z_1^2, w)$:

$$\begin{cases} \dot{\tilde{w}} &= (z_2^1 - \alpha z_2^2)(\frac{\partial \varphi_1}{\partial z_1^1} + \frac{1}{2} z_1^2 \frac{\partial \varphi_1}{\partial w}) + a \frac{\partial \varphi_1}{\partial w} & & \\ \dot{\tilde{z}}_1^1 &= (z_2^1 - \alpha z_2^2)(\frac{\partial \varphi_2}{\partial z_1^1} + \frac{1}{2} z_1^2 \frac{\partial \varphi_2}{\partial w}) + a \frac{\partial \varphi_2}{\partial w} & \dot{z}_1^2 &= z_2^2 \\ \dot{z}_2^1 &= z_3^1 & \dot{z}_2^2 &= z_3^2 \\ &\vdots & &\vdots \\ \dot{z}_k^1 &= z_{k+1}^1 & \dot{z}_k^2 &= z_{k+1}^2 \\ \dot{z}_{k+1}^1 &= v_1 + \alpha(x, z, w) v_2 & \dot{z}_{k+1}^2 &= v_2 \end{cases}$$

with $(\varphi_1, \varphi_2) = (\tilde{w}, \tilde{z}_1^1)$. Since $(L_{ad_f^k g_1} \varphi_1, L_{ad_f^k g_1} \varphi_2)(w_0, z_0) \neq (0, 0)$, we have $(\frac{\partial \varphi_1}{\partial z_1^1} + \frac{1}{2} z_1^2 \frac{\partial \varphi_1}{\partial w}, \frac{\partial \varphi_2}{\partial z_1^1} + \frac{1}{2} z_1^2 \frac{\partial \varphi_2}{\partial w})(\xi_0) \neq (0, 0)$, where $\xi_0 = (\tilde{w}_0, z_0)$, and we can assume, without lose of generality that $\frac{\partial \varphi_2}{\partial z_1^1} + \frac{1}{2} z_1^2 \frac{\partial \varphi_2}{\partial w}(\xi_0) \neq 0$. Therefore, the following change of coordinates $\tilde{z}_i^1 = L_f^{i-1} \varphi_2$, for $2 \leq i \leq k+1$ is valid and brings the system into the form

$$\begin{cases} \dot{\tilde{w}} &= a(\tilde{z}_1^1, z_1^2, \tilde{w}) \tilde{z}_2^1 + b(\tilde{z}_1^1, z_1^2, \tilde{w}) & & \\ \dot{\tilde{z}}_1^1 &= \tilde{z}_2^1 & \dot{z}_1^2 &= z_2^2 \\ &\vdots & &\vdots \\ \dot{\tilde{z}}_k^1 &= \tilde{z}_{k+1}^1 & \dot{z}_k^2 &= z_{k+1}^2 \\ \dot{\tilde{z}}_{k+1}^1 &= \tilde{v}_1 & \dot{z}_{k+1}^2 &= \tilde{v}_2 \end{cases}$$

with $(\varphi_1, \varphi_2) = (\tilde{w}, \tilde{z}_1^1)$. In these coordinates, we have

$$\mathcal{D}^k = \text{span}\{\frac{\partial}{\partial z_1^1} + a \frac{\partial}{\partial w}, \frac{\partial}{\partial z_2^1}, \cdots, \frac{\partial}{\partial z_{k+1}^1}, \frac{\partial}{\partial z_1^2}, \cdots, \frac{\partial}{\partial z_{k+1}^2}\}.$$

Since $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$, it follows that $\frac{\partial a}{\partial z_1^2}(w_0, z_0) \neq 0$ and we put $\tilde{z}_i^2 = L_f^{i-1} a$, for $2 \leq i \leq k+1$, and apply a suitable invertible feedback transformation, to get

$$\begin{cases} \dot{\tilde{w}} &= \tilde{z}_1^2 \tilde{z}_2^1 + b(\tilde{z}_1^1, \tilde{z}_1^2, \tilde{w}) & & \\ \dot{\tilde{z}}_1^1 &= \tilde{z}_2^1 & \dot{\tilde{z}}_1^2 &= \tilde{z}_2^2 \\ &\vdots & &\vdots \\ \dot{\tilde{z}}_k^1 &= \tilde{z}_{k+1}^1 & \dot{\tilde{z}}_k^2 &= \tilde{z}_{k+1}^2 \\ \dot{\tilde{z}}_{k+1}^1 &= \hat{v}_1 & \dot{\tilde{z}}_{k+1}^2 &= \hat{v}_2 \end{cases}$$

with $(\varphi_1, \varphi_2) = (\tilde{w}, \tilde{z}_1^1)$. We have $\mathcal{L} = \mathcal{D}^{k-1} + \text{span}\{\frac{\partial}{\partial \tilde{z}_1^2}\} = \mathcal{D}^{k-1} + \text{span}\{ad_f^k g_2\}$, so $g_2$ plays the role of the vector field $g_{2c}$ and from $(L_{ad_f^{k+1} g_{2c}} \varphi_1, L_{ad_f^{k+1} g_{2c}} \varphi_2)(w_0, z_0) \neq (0, 0)$, it follows that $(\tilde{z}_2^1 + \frac{\partial b}{\partial \tilde{z}_1^2})(w_0, z_0) \neq 0$. This is the normal form $(NF2)$ with $(\varphi_1, \varphi_2) = (\tilde{w}, \tilde{z}_1^1)$ a minimal $x$-flat output of differential weight $n+3$.

It remains to study the uniqueness of $(\varphi_1, \varphi_2)$. The results of Proposition 1.5.2 show that for a given arbitrary function $\varphi_1$ satisfying

$$d\varphi_1 \perp \mathcal{D}^{k-1} \text{ and } (L_{ad_k^k g_1} \varphi_1, L_{ad_k^k g_2} \varphi_1)(x_0) \neq 0,$$

there always exists a function $\varphi_2$, independent with $\varphi_1$ such that $(\varphi_1, \varphi_2)$ is a minimal flat output of $\Sigma$ at $x_0$ (respectively at $(x_0, u_0)$, if $k = 0$). We have already proved that the minimal $x$-flat outputs and the normal form $(NF2)$ are compatible, i.e., if $(\varphi_1, \varphi_2)$ is a minimal $x$-flat output at $x_0$, we can introduce new coordinates (permute $\varphi_1$ and $\varphi_2$, if necessary) $w_1^1 = \varphi_1$ and $w_1^2 = \varphi_2$ and complete them to a coordinate system in which the original system $\Sigma$ takes, via an invertible static feedback transformation, the form $(NF2)$ with $\varphi_1 = w_1^1$ and $\varphi_2 = w_1^2$.

Suppose that there exist another function $\tilde{\varphi}_2$ such that $(\varphi_1, \tilde{\varphi}_2) = (w_1^1, \tilde{\varphi}_2)$ is a minimal flat output of $\Sigma$. By Proposition 1.5.2, $\tilde{\varphi}_2$ must satisfy

$$d\tilde{\varphi}_2 \perp \mathcal{D}^{k-1} \text{ and } L_w \tilde{\varphi}_2 = 0,$$

where $w = (L_{ad_f^k g_2}\varphi_1) ad_f^k g_1 - (L_{ad_f^k g_1}\varphi_1) ad_f^k g_2 = -\frac{\partial}{\partial w_2^2}$. It follows that $\tilde{\varphi}_2 = h(w_1^1, w_1^2) = h(\varphi_1, \varphi_2)$, where $h$ is a smooth function such that $\frac{\partial h}{\partial w_1^2}(w_0) \neq 0$, i.e., $h$ is invertible with respect to $w_1^2 = \varphi_2$.

### 1.7.8 Proof of Proposition 1.5.2

*Proof of (i).* It is an immediate consequence of Proposition 1.5.1 (i).

*Proof of (ii). Necessity.* Assume that $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ is an $x$-flat control system of differential weight $n+3$ and such that $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$. Let $(\varphi_1, \varphi_2)$ be its minimal flat output defined in a neighborhood $\mathcal{X}$ of $x_0$. By Proposition 1.4.1 we can bring $\Sigma$ into the form

$$(NF2)\begin{cases} \dot{w}_1^1 = w_2^1 & \dot{w}_1^2 = a(\bar{w}_1^1, \bar{w}_2^2) + w_2^1 w_2^2 \\ \dot{w}_2^1 = w_3^1 & \dot{w}_2^2 = w_3^2 \\ \quad\vdots & \quad\vdots \\ \dot{w}_{k+1}^1 = \tilde{u}_1 & \dot{w}_{k+1}^2 = w_{k+2}^2 \\ & \dot{w}_{k+2}^2 = \tilde{u}_2 \end{cases}$$

with $\varphi_1 = w_1^1$ and $\varphi_2 = w_1^2$. By a direct computation, we get $L_{w_j}\varphi_i = 0$, for $1 \leq j \leq 2k$ and $1 \leq i \leq 2$, where $w_j = \frac{\partial}{\partial w_{k+2-j}^1}$, for $1 \leq j \leq k$, and $w_{k+j} = \frac{\partial}{\partial w_{k+3-j}^2}$, for $1 \leq j \leq k$, and $L_w \varphi_2 = 0$, where $w = -\frac{\partial}{\partial w_2^2}$. Thus proves the desired relations on $\mathcal{X}$.

It remains to prove that for $i = 1$ or $2$ we have $(L_{ad_f^k g_1}\varphi_i, L_{ad_f^k g_2}\varphi_i)(x_0) \neq (0,0)$. Bring $\Sigma$, locally around $x_0 \in \mathcal{X}$, into the form $(NF2)$, which is always possible by our assumption. Then the equations, that we have just proved on $\mathcal{X}$, implies that $d\varphi_i \perp \mathcal{D}^{k-1}$, for $1 \leq i \leq 2$. Assume $L_{ad_f^k g_1}\varphi_i(w_0) = L_{ad_f^k g_2}\varphi_i(w_0) = 0$, for $1 \leq i \leq 2$ (otherwise the condition that we want to show holds). It follows that

$$\mathrm{rk}\begin{pmatrix} L_{ad_f^k g_1}\varphi_1 & L_{ad_f^k g_2}\varphi_1 & L_{[ad_f^k g_1, ad_f^k g_2]}\varphi_1 \\ L_{ad_f^k g_1}\varphi_2 & L_{ad_f^k g_2}\varphi_2 & L_{[ad_f^k g_1, ad_f^k g_2]}\varphi_2 \end{pmatrix}(w_0) = 1$$

contradicting the independence of flat outputs. Indeed, since $(d\varphi_1 \wedge d\varphi_2)(w_0) \neq 0$ and $TX = \mathcal{D}^{k-1} + \mathrm{span}\,\{ad_f^k g_1, ad_f^k g_2, [ad_f^k g_1, ad_f^k g_2]\}$, the above rank must be 2.

*Sufficiency.* In order to find $\varphi_1$, we have to solve the following system of first order PDE's

$$\begin{array}{l} L_{w_j}\varphi_1 = 0, \ \ 1 \le j \le 2k, \\ (L_{ad_f^k g_1}\varphi_1, L_{ad_f^k g_2}\varphi_1)(x_0) \neq (0,0), \end{array}$$

which always possesses solutions, since $\mathcal{D}^{k-1} = \text{span}\,\{w_j,\ 1 \le j \le 2k\}$ is involutive. Moreover, since the corank of $\mathcal{D}^{k-1}$ in $TX$ is three, the space of solutions is that of functions of three variables.

To find the second component of the flat output, $\varphi_2$, we have to solve the following system of $n-2 = 2k+1$ equations

$$\begin{array}{rcl} L_{w_j}\varphi_2 & = & 0, \ \ 1 \le j \le 2k, \\ L_w\varphi_2 & = & 0, \end{array}$$

where $w = (L_{ad_f^k g_2}\varphi_1)ad_f^k g_1 - (L_{ad_f^k g_1}\varphi_1)ad_f^k g_2$. Notice that $\varphi_1$ solves this system and recall that we are looking for a solution $\varphi_2$ independent with $\varphi_1$. By Frobenious theorem, the above system has two independent solutions if and only if the distribution

$$\mathcal{L} = \text{span}\,\{w,\ w_j,\ 1 \le j \le 2k\} = \mathcal{D}^{k-1} + \text{span}\,\{w\}$$

is involutive.

Below, we will prove that $\mathcal{L}$ is, indeed, involutive. To this end, it is sufficient to show that $[ad_f^j g_i, w] \in \mathcal{L}$, for any $1 \le i \le 2$ and $0 \le j \le k-1$. Since $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, it follows that there exist smooth functions $\alpha_{il}^j$ and $\beta_{il}^j$ such that

$$[ad_f^j g_i, ad_f^k g_l] = \alpha_{il}^j ad_f^k g_1 + \beta_{il}^j ad_f^k g_2 \,\text{mod}\, \mathcal{D}^{k-1}$$

for any $1 \le i \le, l 2$ and $0 \le j \le k-1$, and thus

$$L_{[ad_f^j g_i, ad_f^k g_l]}\varphi_1 = \alpha_{il}^j L_{ad_f^k g_1}\varphi_1 + \beta_{il}^j L_{ad_f^k g_2}\varphi_1.$$

On the other hand,

$$L_{[ad_f^j g_i, ad_f^k g_l]}\varphi_1 = L_{ad_f^j g_i} L_{ad_f^k g_l}\varphi_1 - L_{ad_f^k g_l} L_{ad_f^j g_i}\varphi_1 = L_{ad_f^j g_i} L_{ad_f^k g_l}\varphi_1.$$

We have

$$\begin{array}{rcl} [ad_f^j g_i, w] & = & [ad_f^j g_i, (L_{ad_f^k g_2}\varphi_1)ad_f^k g_1 - (L_{ad_f^k g_1}\varphi_1)ad_f^k g_2] \\ & = & (L_{ad_f^k g_2}\varphi_1)[ad_f^j g_i, )ad_f^k g_1] - (L_{ad_f^k g_1}\varphi_1)[ad_f^j g_i, ad_f^k g_2] \\ & & +(L_{ad_f^j g_i} L_{ad_f^k g_2}\varphi_1)ad_f^k g_1 - (L_{ad_f^j g_i} L_{ad_f^k g_1}\varphi_1)ad_f^k g_2 \\ & = & (L_{ad_f^k g_2}\varphi_1)(\alpha_{i1}^j ad_f^k g_1 + \beta_{i1}^j ad_f^k g_2) - (L_{ad_f^k g_1}\varphi_1)(\alpha_{i2}^j ad_f^k g_1 + \beta_{i2}^j ad_f^k g_2) \\ & & +(\alpha_{i2}^j L_{ad_f^k g_1}\varphi_1 + \beta_{i2}^j L_{ad_f^k g_2}\varphi_1)ad_f^k g_1 \\ & & -(\alpha_{i1}^j L_{ad_f^k g_1}\varphi_1 + \beta_{i1}^j L_{ad_f^k g_2}\varphi_1)ad_f^k g_2 \,\text{mod}\, \mathcal{D}^{k-1} \\ & = & ((\alpha_{i1}^j + \beta_{i2}^j)L_{ad_f^k g_2}\varphi_1)ad_f^k g_1 - ((\alpha_{i1}^j + \beta_{i2}^j)L_{ad_f^k g_1}\varphi_1)ad_f^k g_2 \,\text{mod}\, \mathcal{D}^{k-1} \\ & = & (\alpha_{i1}^j + \beta_{i2}^j)w \,\text{mod}\, \mathcal{D}^{k-1} \\ & = & 0 \,\text{mod}\, \mathcal{L}, \end{array}$$

for $1 \leq i \leq 2$ and $0 \leq j \leq k-1$. Consequently, $\mathcal{L}$ is involutive, and the above system has two independent solutions $\varphi_1$ and $\varphi_2$. Moreover, the involutive distribution $\mathcal{L} = (\operatorname{span}\{d\varphi_1, d\varphi_2\})^{\perp}$ satisfies $\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k$, where both inclusions are of corank one, and by Proposition 1.5.1, $\varphi_1$ and $\varphi_2$ are minimal flat outputs.

# 2 Multi-input Control Affine Systems Linearizable via One-fold Prolongation and Their Flatness


**Abstract**

We study flatness of multi-input control-affine systems. We give a geometric characterization of systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control. They form a particular class of flat systems. Namely those of differential weight $n+m+1$, where $n$ is the dimension of the state-space and $m$ is the number of controls. We propose conditions (verifiable by differentiation and algebraic operations) describing that class, construct normal forms and provide a system of PDE's giving all minimal flat outputs. We illustrate our results by two examples: the quadrotor helicopter and a polymerization reactor.


## 2.1 Introduction

In this paper, we study flatness of nonlinear control systems of the form

$$\Xi : \ \dot{x} = \ F(x, u),$$

where $x$ is the state defined on a open subset $X$ of $\mathbb{R}^n$ and $u$ is the control taking values in an open subset $U$ of $\mathbb{R}^m$ (more generally, an $n$-dimensional manifold $X$ and an $m$-dimensional manifold $U$, respectively). The dynamics $F$ are smooth and the word smooth will always mean $\mathcal{C}^\infty$-smooth.

The notion of flatness has been introduced in control theory in the 1990's, by Fliess, Lévine, Martin and Rouchon [13,14], see also [21,22,32,54], and has attracted a lot of attention because of its multiple applications in the problem of constructive controllability and motion planning (see, e.g. [15,36,53,55,58,62,66]). Flat systems form a class of control systems whose set of trajectories can be parametrized by $m$ functions and their time-derivatives, $m$ being the number of controls. More precisely, the system $\Xi : \dot{x} = \ F(x, u)$ is *flat* if we can find $m$ functions, $\varphi_i(x, u, \ldots, u^{(r)})$, for some $r \geq 0$, such that

$$x = \gamma(\varphi, \ldots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(s)}), \tag{2.1}$$

for a certain integer $s$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ is called *a flat output*. Therefore the time-evolution of all state and control variables can be determined from that of flat outputs without integration and all trajectories of the system can be completely parameterized. A similar notion, of systems of undetermined differential equations integrable without integration, has been studied by Hilbert [18] and Cartan [8], see also [66], where connections between Cartan prolongations and flatness were studied.

Flatness is closely related to the notion of feedback linearization. It is well known that systems linearizable via invertible static feedback are flat. Their description (2.1) uses the minimal possible, which is $n + m$, number of time-derivatives of the components of the flat output $\varphi$. In general, a flat system is not linearizable by static feedback, with the exception of the single-input case where flatness reduces to static feedback linearization, see [9] and [54]. For any flat system, that is not static feedback linearizable, the minimal number of time-derivatives of $\varphi_i$ needed to express $x$ and $u$ (which is called the differential weight [58]) is thus bigger than $n + m$ and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system. Therefore the simplest systems for which the differential weight is bigger than $n + m$ are systems linearizable via one-dimensional precompensator, thus of differential weight $n + m + 1$. They form the class that we are studying in the paper: our goal is to give a geometric characterization of control-affine systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control.

The paper is organized as follows. In Section 2.2, we recall the definition of flatness and define the notion of differential weight of a flat system. In Section 2.3, we give our main results: we characterize control-affine systems that become static feedback linearizable after a one-fold prolongation. They form a particular class of flat systems, that is, flat systems of differential weight $n + m + 1$. We present their normal forms in Section 2.4 and describe all minimal flat outputs in Section 2.5. We illustrate our results by two examples in Section 2.6 and provide proofs in Section 2.7.

## 2.2 Flatness

The fundamental property of flat systems is that all their solutions may be parametrized by a finite number of functions and their time-derivatives. Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set $U^{-1}$ is empty and $\bar{u}^{-1}$ in an empty sequence.

**Definition 2.2.1.** The system $\Xi : \dot{x} = F(x, u)$ is *flat* at $(x_0, \bar{u}_0^l) \in X \times U^l$, for $l \geq -1$, if there exists a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and $m$ smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property: there exist an integer $s$ and smooth functions $\gamma_i$, $1 \leq i \leq n$, and $\delta_j$, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

along any trajectory $x(t)$ given by a control $u(t)$ that satisfies $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

Whenever necessary to specify the number of derivatives of $u$ on which the components of the flat outputs $\varphi$ depend, we will say that the system $\Xi$ is $(x, u, \cdots, u^{(r)})$-flat if the $r-th$-derivative is the highest involved. In the particular case $\varphi_i = \varphi_i(x)$, for $1 \leq i \leq m$, we will say that the system is $x$-flat.

In general, $r$ is smaller than the integer $l$ needed to define the neighborhood $\mathcal{O}^l$ which, in turn, is smaller than the numbers of derivatives of $\varphi$ that are involved. In our study, $r$ is always equal to -1, i.e., the flat outputs depend on $x$ only, and $l$ is -1 or 0.

The minimal number of derivatives of components of a flat output, needed to express $x$ and $u$, will be called the differential weight of that flat output and is formalized as follows. By definition, for any flat output $\varphi$ of $\Xi$ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}). \end{aligned}$$

Moreover, we can choose $(s_1, \ldots, s_m)$ such that (see [58]) if for any other $m$-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ we have

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^{m}(s_i + 1) = m + \sum_{i=1}^{m} s_i$ the differential weight of $\varphi$. A flat output of $\Xi$ is called *minimal* if its differential weight is the lowest among all flat outputs of $\Xi$. We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output.

Consider a control-affine system

$$\Sigma \; : \; \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \tag{2.2}$$

where $f$ and $g_1, \cdots, g_m$ are smooth vector fields on $X$. The system $\Sigma$ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \alpha(x) + \beta(x)v$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$.

The problem of static feedback linearization was solved by Jakubczyk and Respondek [23] and Hunt and Su [19] who gave geometric necessary and sufficient conditions. The following theorem recalls their result and, furthermore, gives an equivalent way of describing static feedback linearizable systems from the point of view of differential weight.

Define inductively the sequence of distributions $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$, where $\mathcal{D}^0$ is given by $\mathcal{D}^0 = \text{span}\,\{g_1, \cdots, g_m\}$.

**Theorem 2.2.1.** *The following conditions are equivalent:*

*(i)* $\Sigma$ *is locally static feedback linearizable, around* $x_0 \in X$*;*

*(ii)* *$\Sigma$ is locally static feedback equivalent, around $x_0 \in X$, to the Brunovský canonical form*

$$(Br) : \begin{cases} \dot{z}_i^j &= z_i^{j+1} \\ \dot{z}_i^{\rho_i} &= v_i \end{cases}$$

*where $1 \leq i \leq m$, $1 \leq j \leq \rho_i - 1$, and $\sum_{i=1}^m \rho_i = n$;*

*(iii)* *For any $i \geq 0$, the distributions $\mathcal{D}^i$ are of constant rank, around $x_0 \in X$, involutive and $\mathcal{D}^{n-1} = TX$;*

*(iv)* *$\Sigma$ is flat at $x_0 \in X$, of differential weight $n+m$.*

The geometry of static feedback linearizable systems is given by the following sequence of nested involutive distributions:

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{n-1} = TX.$$

It is well known that a feedback linearizable system is static feedback equivalent to the Brunovský canonical form, see [5], and is clearly flat with $\varphi = (\varphi_1, \cdots, \varphi_m) = (z_1^1, \cdots, z_m^1)$ is a minimal flat output (of differential weight $n+m$). Therefore, for static feedback linearizable systems, the representation of all states and controls uses the minimal possible, which is $n+m$, number of time-derivatives of $\varphi_i$ and an equivalent way of describing them is that they are flat systems of differential weight $n+m$.

In general, a flat system is not linearizable by static feedback, with the exception of the single-input case. Any single input-system is flat if and only if it is static feedback linearizable, see [9, 54], and thus of differential weight $n+1$. Flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see [13, 14, 32, 55]. Our goal is thus to describe the simplest flat systems that are not static feedback linearizable: control-affine systems that become static feedback linearizable after one-fold prolongation, which is the simplest dynamic feedback. They are flat systems of differential weight $n+m+1$, see Proposition 2.3.1 below. In this paper, we will completely characterize them and show how their geometry differs and how it reminds that given by the involutive distributions $\mathcal{D}^i$ for static feedback linearizable systems. We will also give normal forms compatible with the minimal flat outputs (thus generalizing the Brunovský normal form).

## 2.3 Main results

Throughout, we deal only with systems that are not static feedback linearizable. This occurs if there exists an integer $k$ such that $\mathcal{D}^k$ is not involutive. Suppose that $k$ is the smallest integer satisfying that property and assume $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} \geq 2$ (see Proposition 2.7.1, in Section 2.7, asserting that the latter is necessary for dynamic linearizability via one-fold prolongation and thus for flatness of differential weight $n+m+1$). We also assume $m \geq 3$. The case $m = 2$ is studied in details in [44,45] and will be briefly discussed at the end of this section.

We make the following assumption:

(Assumption 1) From now on, unless stated otherwise, we assume that all ranks involved are constant in a neighborhood of a given $x_0 \in X$.

**Remark 2.3.1.** All results presented here are valid on an open and dense subset of either $X$ or $X \times U$ and hold locally, around a given point of that set.

The proofs of all results of this section are given in Section 2.7 (except that of Proposition 2.3.1 which is presented in Appendix 2.A).

**Proposition 2.3.1.** *Consider a control system $\Xi : \dot{x} = F(x, u)$. The following conditions are equivalent:*

*(i) $\Xi$ is flat at $(x_0, \bar{u}_0^l)$, of differential weight $n + m + 1$, for a certain $l \geq 1$;*

*(ii) $\Xi$ is x-flat at $(x_0, u_0)$, of differential weight $n + m + 1$;*

*(iii) There exists, around $x_0$, an invertible static feedback transformation $u = \psi(x, \tilde{u})$ bringing the system $\Xi$ into $\tilde{\Xi} : \dot{x} = \tilde{F}(x, \tilde{u}) = F(x, \psi(x, u))$, such that the prolongation*

$$\tilde{\Xi}^{(1,0,\ldots,0)} : \begin{cases} \dot{x} &= \tilde{F}(x, y_1, v_2, \cdots, v_m) \\ \dot{y}_1 &= v_1 \end{cases}$$

*is locally static feedback linearizable, with $y_1 = \tilde{u}_1$, $v_i = \tilde{u}_i$, for $2 \leq i \leq m$.*

*Moreover, if $\Xi$ is actually a control-affine system of the form $\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$, then the equivalence $(i) \iff (ii) \iff (iii)$ holds with the general feedback $u = \psi(x, \tilde{u})$ being replaced by $u = \psi(x, \tilde{u}) = \alpha(x) + \beta(x)\tilde{u}$, the system $\tilde{\Xi}$ by $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \sum_{i=1}^{m} \tilde{u}_i \tilde{g}_i(x)$ and the prolongation $\tilde{\Xi}^{(1,0,\ldots,0)}$ by*

$$\tilde{\Sigma}^{(1,0,\ldots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^{m} v_i \tilde{g}_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

*with $y_1 = \tilde{u}_1$, $v_i = \tilde{u}_i$, for $2 \leq i \leq m$, $\tilde{f} = f + \alpha g$ and $\tilde{g} = g\beta$, where $g = (g_1, \cdots, g_m)$ and $\tilde{g} = (\tilde{g}_1, \cdots, \tilde{g}_m)$.*

To simplify the understanding of the paper, from now on, we will consider only the control-affine case. The generalization for the control-nonlinear systems is straightforward.

A system $\Sigma$ satisfying $(iii)$ will be called dynamically linearizable via invertible one-fold prolongation. Notice that $\tilde{\Sigma}^{(1,0,\ldots,0)}$ is, as indicated by the notation, obtained by prolonging the control $\tilde{u}_1$ as $v_1 = \dot{\tilde{u}}_1$ and keeping $v_i = \tilde{u}_i$, for $2 \leq i \leq m$. The above results asserts that for systems of differential weight $n + m + 1$, flatness and $x$-flatness coincide and that, moreover, these properties are equivalent to linearizability via the simplest dynamic feedback, namely one-fold preintegration.

Let $\mathcal{A}$ and $\mathcal{B}$ be two distributions of constant rank and $f$ a vector field. Denote $[\mathcal{A}, \mathcal{B}] = \{[a, b] : a \in \mathcal{A}, b \in \mathcal{B}\}$ and $[f, \mathcal{B}] = \{[f, b] : b \in \mathcal{B}\}$. Clearly, $[\mathcal{A}, \mathcal{B}] = [\mathcal{A}, \mathcal{B}] + \mathcal{A} + \mathcal{B}$ (because we take all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$ and not just generators) and although the right hand side is more detailed, we will use the left hand side that is more compact. We will use that notation throughout. If $\mathcal{A} \subset \mathcal{B}$, we will write $\operatorname{cork}(\mathcal{A} \subset \mathcal{B})$ to denote $\operatorname{rk}(\mathcal{B}/\mathcal{A})$. So, frequently used $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k])$ simply means $\operatorname{rk}([\mathcal{D}^k, \mathcal{D}^k]/\mathcal{D}^k) = \operatorname{rk}(([\mathcal{D}^k, \mathcal{D}^k] + \mathcal{D}^k)/\mathcal{D}^k)$.

Recall that *$k$ is the smallest integer such that $\mathcal{D}^k$ is not involutive.* The integer $k$ plays an important role in our study. Our main result describing flat systems of differential weight $n + m + 1$ is given by the two following theorems corresponding to the first noninvolutive distribution $\mathcal{D}^k$ being either $\mathcal{D}^0$, i.e., $k = 0$ (Theorem 2.3.2) or $\mathcal{D}^k$, for $k \geq 1$ (Theorem 2.3.1). For both theorems, we assume that $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$. The particular case $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ will be discussed at the end of this section (Theorem 2.3.4).

**Theorem 2.3.1.** *Assume $k \geq 1$ and* $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$. *A control system $\Sigma$ given by (2.2), is flat at $x_0$, of differential weight $n + m + 1$, if and only if it satisfies around $x_0$:*

*(A1) There exists an involutive distribution $\mathcal{H}^k \subset \mathcal{D}^k$, of corank one;*

*(A2) The distributions $\mathcal{H}^i$, for $i \geq k+1$, are involutive, where $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$;*

*(A3) There exists $\rho$ such that $\mathcal{H}^\rho = TX$.*

The geometry of systems described by the previous theorem can be summarized by the following sequence of inclusions:

$$
\begin{array}{ccccc}
\mathcal{D}^0 \subset \cdots \subset \mathcal{D}^{k-1} & \subset & \mathcal{D}^k & \subset & \overline{\mathcal{D}}^k \\
 & & 1\cup & & \cap \\
 & & \mathcal{H}^k & \subset & \mathcal{H}^{k+1} \subset \cdots \subset \mathcal{H}^\rho = TX
\end{array}
$$

where all distributions, except $\mathcal{D}^k$, are involutive, $\overline{\mathcal{D}}^k$ is the involutive closure of $\mathcal{D}^k$ and the inclusion $\mathcal{H}^k \subset \mathcal{D}^k$ is of corank one. The main structural condition is the existence of a corank one involutive subdistribution $\mathcal{H}^k$ in $\mathcal{D}^k$. Under the hypotheses $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, the subdistribution $\mathcal{H}^k$ is unique and can be explicitly calculated [6, 50]. Its construction will be described in Proposition 2.3.2, after stating Theorem 2.3.2. Moreover, under the assumption $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, the condition $(A1)$ implies (via the Jacobi identity) the inclusion $\mathcal{D}^{k-1} \subset \mathcal{H}^k$. The latter yields $\mathcal{D}^k \subset \mathcal{H}^{k+1}$ which gives $\overline{\mathcal{D}}^k \subset \mathcal{H}^{k+1}$ (since $\mathcal{H}^{k+1}$ is involutive by $(A2)$). It is clear that in the particular case $\overline{\mathcal{D}}^k = TX$, we have $\rho = k + 1$.

The previous theorem enables us to define, up to a multiplicative function, the control $u_p$, which is given up to a multiplicative function, to be prolonged in order to obtain $\tilde{\Sigma}^{(1,0,\ldots,0)}$ that is locally static feedback linearizable. According to Proposition 2.7.2(ii) in Section 2.7, to $\mathcal{H}^k$ we can associate a unique corank one subdistribution $\mathcal{H}$ in $\mathcal{D}^0$ such that $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k \mathcal{H}$. Since $\operatorname{rk} \mathcal{H} = m - 1$, we can find $m$ functions $\beta_1, \ldots, \beta_m$ (not vanishing simultaneously) such that $u_p(x) = u_1(x)\beta_1(x) + \cdots + u_m(x)\beta_m(x) = 0$ if and only if $\sum_{i=1}^m u_i(x) g_i(x) \in \mathcal{H}(x)$. The to-be-prolonged control $u_p$ (becoming $\tilde{u}_1$ after feedback) that needs to be preintegrated in order to dynamically linearize the system is $u_p = \tilde{u}_1 = u_1(x)\beta_1(x) + \cdots + u_m(x)\beta_m(x)$ and we put $v_1 = \frac{d}{dt} u_p = \frac{d}{dt} \tilde{u}_1$.

If $k = 0$, i.e., the first noninvolutive distribution is $\mathcal{D}^0 = \mathcal{G}^0$, then a similar result holds, but in the chain of involutive subdistributions $\mathcal{H}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots$ (playing the role of $\mathcal{H}^k \subset \mathcal{H}^{k+1} \subset \mathcal{H}^{k+2} \subset \cdots$), the distribution $\mathcal{H}^1$ is not defined as $\mathcal{H}^1 = \mathcal{H}^0 + [f, \mathcal{H}^0]$, but as $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$, where $\mathcal{G}^1 = \mathcal{G}^0 + [\mathcal{G}^0, \mathcal{G}^0] = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$, (compare $(A2)$ and $(A2)'$) and satisfies an additional nonsingularity condition $(RC)$. In fact, flat systems with $k = 0$ exhibit a singularity in the control space (created by one-fold prolongation of the to-be-prolonged control) defined by

$$U_{sing}(x) = \{u(x) \in \mathbb{R}^m : \mathrm{rk}\,\mathrm{span}\,\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^{m} u_i h_i, h_j], 2 \le j \le m\}(x) < \mathrm{rk}\,\mathcal{H}^1(x)\}$$

and excluded by *(RC)*, where $\mathcal{H}^0 = \mathrm{span}\,\{h_2, \dots, h_m\}$ and $\mathcal{D}^0 = \mathrm{span}\,\{g_1, h_2, \dots, h_m\}$.

**Theorem 2.3.2.** *Assume $k = 0$ and* $\mathrm{cork}\,(\mathcal{D}^0 \subset [\mathcal{D}^0, \mathcal{D}^0]) \ge 2$. *A system $\Sigma$ given by (2.2), is flat at $(x_0, u_0)$, of differential weight $n + m + 1$, if and only if it satisfies:*

*(A1)′ There exists an involutive distribution $\mathcal{H}^0 \subset \mathcal{D}^0$, of corank one;*

*(A2)′ The distributions $\mathcal{H}^i$, for $i \ge 1$, are involutive, where $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$ and $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, for $i \ge 2$;*

*(A3)′ There exists $\rho$ such that $\mathcal{H}^\rho = TX$;*

*(RC) $u_0 \notin U_{sing}(x_0)$.*

Similarly to Theorem 2.3.1, if $\overline{\mathcal{D}}^0 = TX$, then $\rho = 1$.

The cases $k = 0$ and $k \ge 1$ are similar, but they have slightly different geometries. Even if at first sight, it seems not possible to merge them (due to the different definitions of the distributions $\mathcal{H}^1$ and $\mathcal{H}^{k+1}$ and to the existence of singularities in the control space for $k = 0$), the following result enables us to unify them. Theorem 2.3.3 is based on the observation that in both cases, we actually have $\mathcal{H}^{k+1} = \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] + [f, \mathcal{H}^k]$ (by definition of $\mathcal{H}^1$, for $k = 0$, and as a direct consequence of the definition of $\mathcal{H}^{k+1}$, for $k \ge 1$, see the comments just after Theorem 2.3.1). According to Proposition 2.7.2(ii) in Section 2.7, to $\mathcal{H}^k$ we can associate a unique corank one subdistribution $\mathcal{H}$ in $\mathcal{D}^0$ such that $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k \mathcal{H}$. Let $g_1$ and $h_j$, for $2 \le j \le m$, be vector fields such that $\mathcal{H} = \mathrm{span}\,\{h_2, \dots, h_m\}$ and $\mathcal{D}^0 = \mathrm{span}\,\{g_1, h_2, \dots, h_m\}$.

**Theorem 2.3.3.** *Assume* $\mathrm{cork}\,(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \ge 2$. *A system $\Sigma$, given by (2.2), is flat at $(x_0, u_0)$, of differential weight $n + m + 1$, if and only if it satisfies*

*(A1)″ There exists an involutive distribution $\mathcal{H}^k \subset \mathcal{D}^k$, of corank one;*

*(A2)″ The distributions $\mathcal{H}^i$, for $i \ge k + 1$, are involutive, where $\mathcal{H}^{k+1} = \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] + [f, \mathcal{H}^k]$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $i \ge k + 1$;*

*(A3)″ There exists $\rho$ such that $\mathcal{H}^\rho = TX$;*

*(A4)″* $\mathrm{rk}\,(\mathcal{D}^k + [f + u_1 g_1 + \sum_{i=1}^{m} h_i, \mathcal{H}^k])(x_0, u_0) = \mathrm{rk}\,\mathcal{H}^{k+1}(x_0)$.

If $k = 0$, condition $(A4)''$ immediately gives $u_0 \notin U_{sing}(x_0)$. If $k \geq 1$, it can be easily shown that $(A4)''$ does not depend on the control and that the directions in $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]$ that are not in $\mathcal{H}^k$ are in fact spanned by the vector fields $ad_f^{k+1} h_j$, implying that $\overline{\mathcal{D}}^k \subset \mathcal{H}^{k+1} = \mathcal{H}^k + [f, \mathcal{H}^k]$. Theorem 2.3.3 is a direct consequence of Theorems 2.3.1 and Theorem 2.3.2 and we do not present its proof here.

In order to verify the conditions of Theorem 2.3.1 (respectively Theorems 2.3.2 and 2.3.3), we have to check whether the distribution $\mathcal{D}^k$ (respectively $\mathcal{D}^0$) contains an involutive subdistribution $\mathcal{H}^k$ (respectively $\mathcal{H}^0$) of corank one. Now we will explain how to do it. Consider a distribution $\mathcal{D}$ of rank $d$, defined on a manifold $X$ of dimension $n$ and define its annihilator $\mathcal{D}^\perp = \{\omega \in \Lambda^1(X) : <\omega, f>= 0, \forall f \in \mathcal{D}\}$, where $\Lambda^1(X)$ is the space of smooth differentials 1-forms on $X$. Let $\omega_1, \ldots, \omega_s$, where $s = n - d$, be differential 1-forms locally spanning the annihilator of $\mathcal{D}$, that is $\mathcal{D}^\perp = \mathcal{I} = \operatorname{span}\{\omega_1, \ldots, \omega_s\}$. The *Engel rank* of $\mathcal{D}$ equals 1 at $x$ if and only if $\mathcal{D}$ is non involutive and $(d\omega_i \wedge d\omega_j)(x) = 0 \bmod \mathcal{I}$, for any $1 \leq i, j \leq s$. For any $\omega \in \mathcal{I}$, we define $\mathcal{W}(\omega) = \{f \in \mathcal{D} : f \lrcorner d\omega \in \mathcal{D}^\perp\}$, where $\lrcorner$ is the interior product. The characteristic distribution $\mathcal{C} = \{f \in \mathcal{D} : [f, \mathcal{D}] \subset \mathcal{D}\}$ of $\mathcal{D}$ is given by

$$\mathcal{C} = \cap_{i=1}^s \mathcal{W}(\omega_i).$$

It follows directly from the Jacobi identity that the characteristic distribution is always involutive. Let $\operatorname{rk}(\mathcal{D} + [\mathcal{D}, \mathcal{D}]) = d + r$. Choose the differential forms $\omega_1, \ldots, \omega_r, \ldots, \omega_s$ such that $\mathcal{I} = \operatorname{span}\{\omega_1, \ldots, \omega_s\}$ and $\mathcal{I}^1 = \operatorname{span}\{\omega_{r+1}, \ldots, \omega_s\}$, where $\mathcal{I}^1$ is the annihilator of $\mathcal{D} + [\mathcal{D}, \mathcal{D}]$. Define the distribution

$$\mathcal{V} = \sum_{i=1}^{r} \mathcal{W}(\omega_i).$$

Although the distributions $\mathcal{W}(\omega_i)$ depend on the choice of $\omega_i$'s, the distribution $\mathcal{V}$ does not and we have the following result [50] based on [6].

**Proposition 2.3.2.** *Consider a distribution $\mathcal{D}$ of rank $d$ and let* $\operatorname{rk}(\mathcal{D} + [\mathcal{D}, \mathcal{D}]) = d + r$.

*(i) Assume $r \geq 3$. The distribution $\mathcal{D}$ contains an involutive subdistribution $\mathcal{H}$ of corank one if and only if it satisfies*

*(ISD1) The Engel rank of $\mathcal{D}$ equals one;*

*(ISD2) The characteristic distribution $\mathcal{C}$ of $\mathcal{D}$ has rank $d - r - 1$.*

*Moreover, that involutive subdistribution is unique and is given by $\mathcal{H} = \mathcal{V}$.*

*(ii) Assume $r = 2$. The distribution $\mathcal{D}$ contains a corank one subdistribution $\mathcal{L}$ satisfying $[\mathcal{L}, \mathcal{L}] \subset \mathcal{D}$ if and only it verifies (ISD1)-(ISD2). In that case, $\mathcal{L}$ is unique and given by $\mathcal{L} = \mathcal{V}$. Moreover, $\mathcal{L} = \mathcal{V}$ is the involutive distribution $\mathcal{H}$ of corank one in $\mathcal{D}$ if and only if $\mathcal{L} = \overline{\mathcal{L}}$.*

*(iii) Assume $r = 1$. The distribution $\mathcal{D}$ contains an involutive subdistribution of corank one if and only it satisfies the condition (ISD2). In the case $r = 1$, if an involutive subdistribution of corank one exists, it is never unique.*

The above conditions are easy to check and a unique involutive subdistribution of corank one can be constructed if $r \geq 2$. As a consequence, the conditions of Theorems 2.3.1, 2.3.2 and 2.3.3 are verifiable, i.e., given a control-affine system, we can verify whether it is flat with the differential weight $n+m+1$ and verification involves differentiation and algebraic operations only, without solving PDE's or bringing the system into a normal form.

Let us now consider the case $r = 1$, that is, $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$. If the distribution $\mathcal{D}^k$ contains a corank one involutive subdistribution, the latter is no longer unique (see (iii) of Proposition 2.3.2). The involutivity of $\mathcal{D}^k$ can be lost in two different ways: either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and there exist $1 \leq i, j \leq m$ such that $[ad_f^k g_i, ad_f^k g_j] \notin \mathcal{D}^k$. As asserts Theorem 2.3.4 below, in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the corank one involutive subdistribution $\mathcal{H}^k$ can be uniquely identified by another argument. Namely, $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, where $\mathcal{C}^k$ is the characteristic distribution (defined above) of $\mathcal{D}^k$, i.e., $\mathcal{C}^k = \{f \in \mathcal{D}^k : [f, \mathcal{D}^k] \subset \mathcal{D}^k\}$. The subdistribution $\mathcal{H}^k$ has to verify some additional conditions analogous to those of Theorem 2.3.1. If $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and there exist $1 \leq i, j \leq m$ such that $[ad_f^k g_i, ad_f^k g_j] \notin \mathcal{D}^k$, any corank one involutive subdistribution $\mathcal{H}^k$ may serve to define a control (different distributions yield different controls) whose prolongation gives a static feedback linearizable system.

**Theorem 2.3.4.** *Assume* $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ *and* $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. *A control system* $\Sigma$, *given by (2.2), is flat at* $x_0$, *of differential weight* $n+m+1$, *if and only if the following conditions are satisfied:*

*(C1)* $\operatorname{rk} \mathcal{C}^k = \operatorname{rk} \mathcal{D}^k - 2$, *where* $\mathcal{C}^k$ *is the characteristic distribution of* $\mathcal{D}^k$ *;*

*(C2)* $\operatorname{rk} (\mathcal{C}^k \cap \mathcal{D}^{k-1}) = \operatorname{rk} \mathcal{D}^{k-1} - 1$;

*(C3) The distributions* $\mathcal{H}^i$, *for* $i \geq k$, *are involutive, where* $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$ *and* $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$;

*(C4) There exists* $\rho$ *such that* $\mathcal{H}^\rho = TX$.

It is clear that the above result can be applied only for $k \geq 1$, otherwise $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ would not have any sens. It can be shown that in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ (no matter what is the value of $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k])$), the involutive subdistribution $\mathcal{H}^k$ can always be defined as above, i.e., the computation of $\mathcal{H}^k$ using the procedure given by Proposition 2.3.2 and that provided by conditions (C1) $-$ (C3) of the above theorem are equivalent if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. This is not valid anymore if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$; indeed, in that case, we have $\mathcal{D}^{k-1} \subset \mathcal{C}^k$, the condition (C2) is not verified and (C3) would give $\mathcal{H}^k = \mathcal{C}^k$. Notice that in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, the inclusion $\mathcal{C}^k \subset \mathcal{H}^k$ is always satisfied and is of corank one if additionally $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$, i.e., $\mathcal{H}^k = \mathcal{C}^k + \operatorname{span}\{g\}$, where $g$ is a vector field belonging to $\mathcal{D}^k$, but not to $\mathcal{D}^{k-1}$.

Let us now compare the above results with the case of two-input control-affine systems, i.e., $m = 2$, in which any corank one involutive subdistribution $\mathcal{H}^k$ of $\mathcal{D}^k$ satisfies $\operatorname{cork}(\mathcal{D}^k \subset \mathcal{H}^{k+1}) = 1$, therefore, $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$ and we necessarily have $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$. Thus, neither Theorem 2.3.1 (if $k \geq 1$) nor Theorem

2.3.2 (if $k = 0$) applies to the case $m = 2$. On the other hand, Theorem 2.3.4 covers the case $m = 2$ but only if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. In [45], we treat the case $m = 2$ in its full generality. Namely, we define (by another method) the involutive subdistribution $\mathcal{H}^k$ in all cases satisfying $\overline{\mathcal{D}}^k \neq TX$ (no mater whether $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$). Moreover, in the particular case $\overline{\mathcal{D}}^k = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the subdistribution $\mathcal{H}^k$ is defined as in Theorem 2.3.4. Finally, if $\overline{\mathcal{D}}^k = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, we have shown, in [45], that the system is flat of differential weight n+3 without any additional condition.

## 2.4 Normal forms

It is well known [19, 23] that any static feedback linearizable and controllable system is feedback equivalent to the Brunovský canonical form that consists of $m$ independent chains of integrators of length $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ (called controllability indices). We will prove that systems dynamically linearizable via one-fold prolongation can be brought into a normal form generalizing that of Brunovský. For multi-input control systems at most $m-1$ components (at most only one component for each chain) are replaced by arbitrary (nonlinear) functions involving a certain number of variables that depends on $k$ and on the length of each chain. Each normal form contains at least one linear chain. We will denote by $r$ the number of linear chains. We have $r \geq 1$.

Before presenting the normal forms, let us introduce some notations. Let $z_j = (z_{j1}^1, \cdots, z_{jd}^1)$ be a subset of coordinate functions and let $\rho_j = (\rho_{j1}, \cdots, \rho_{jd})$ be a multi-index. Then $z_j^{(\rho_j)} = c_j$ denotes the following system

$$
\begin{aligned}
\dot{z}_{ji}^q &= z_{ji}^{q+1}, 1 \leq q \leq \rho_{ji} - 1 \\
\dot{z}_{ji}^{\rho_{ji}} &= c_{ji}, 1 \leq i \leq d.
\end{aligned}
$$

composed by $d$ chains. We will consider two cases: $c_j = \tilde{u}_j$ and $c_j = a_j + b_j \tilde{u}_1$. In the first case, $z_j^{(\rho_j)} = \tilde{u}_j$ is just the Brunovský canonical form, the chains will be called linear and the components $\rho_{ji}$ of $\rho_j$ (which are simply the controllability indices ) will be called lengths of the linear chains. In the second case, $z_j^{(\rho_j)} = a_j + b_j \tilde{u}_1$ is followed by the derivation $\dot{z}_j^{\rho_j+1} = \tilde{u}_j$ which stands for $(\dot{z}_{j1}^{\rho_{j1}+1}, \cdots, \dot{z}_{jd}^{\rho_{jd}+1}) = (\tilde{u}_{j1}, \cdots, \tilde{u}_{jd})$, the chains are nonlinear (for each chain only one component, before the last one, can be nonlinear) and their lengths are $\rho_{j1} + 1$.

Throughout, $z_j^q$, where $q = (q_1, \cdots, q_d)$, stands for the subset of coordinates $z_j^q = (z_{j1}^{q_1}, \cdots, z_{jd}^{q_d})$ and $\bar{z}_j^q$ denotes $\bar{z}^q = (z_1^1, \cdots, z_{j1}^{q_1}, \cdots, z_{jd}^1, \cdots, z_{jd}^{q_d})$. If $q_i \leq 0$, then $z_i^1, \cdots, z_i^{q_i}$ is absent in $\bar{z}^q$. For an integer $s$, we will denote by $q + s$ the vector $q + s = (q_1 + s, \cdots, q_d + s)$. Let $\beta(z) = (\beta_1(z), \cdots, \beta_p(z))$ a $p$-tuple of smooth functions. We use the notion $\frac{\partial \beta}{\partial z_j^q}$ for the matrix given by $(\frac{\partial \beta_l}{\partial z_{ji}^q})$, $1 \leq l \leq p$, $1 \leq i \leq d$.

The following proposition gives two different (although static feedback equivalent) normal forms (NF1) and (NF2) for the class of two-input flat systems of differential weight $n+m+1$. Recall that the first non involutive distribution is $\mathcal{D}^k$. Before stating our result, let us discuss the notations used for each normal form.

For (NF1) we define four subsets of coordinates $z_j$, $1 \leq j \leq 4$, with the following properties:

(1) $\dim z_1 = \dim \tilde{u}_1 = 1$.

Thus, according to the above notation, we simply have $z_1 = z_{11}^1$. The $z_1$-chain is the special linear chain whose control $\tilde{u}_1$ has to be prolonged in order to obtain a static feedback linearizable prolongation. Its length $\rho_1 = \rho_{11}$ satisfies $\rho_{11} \geq k+1$.

(2) $\dim z_2 = \dim \tilde{u}_2 = r-1$.

According to the above notation, we have $z_2 = (z_{21}^1, \cdots, z_{2r-1}^1)$ to which we associate the lengths $\rho_2 = (\rho_{21}, \cdots, \rho_{2r-1})$. The $z_2$-chains denote the remaining linear chains. Their lengths $\rho_{2j}$, $1 \leq j \leq r-1$, are arbitrary. If $r=1$, i.e., $\dim z_2 = 0$, then there is only one linear chain given by $z_1$.

(3) $\dim z_3 = \dim \tilde{u}_3 = p-r$, where $r+1 \leq p \leq m$.

The $z_3$-chains correspond to the nonlinear chains whose lengths are at least $k+2$, i.e., $\rho_{3i} \geq k+1$, for $1 \leq i \leq p-r$.

(4) $\dim z_4 = \dim \tilde{u}_4 = m-p$.

The $z_4$-chains correspond to the nonlinear chains whose lengths are lower than $k+1$, i.e., $\rho_{4i} \leq k$, for $1 \leq i \leq m-p$. If $p=m$, there is no nonlinear chain of length lower than $k+1$.

If $k \geq 1$, we suppose, without loss of generality, that $\rho_{31} \geq \rho_{32} \geq \cdots \geq \rho_{3p-r} \geq k+1 > k \geq \rho_{41} \geq \rho_{42} \geq \cdots \geq \rho_{4m-p}$. The integers $\rho_{ji}$ satisfy $\sum_{j=1}^{4} \sum_{i=1}^{\dim z_j} \rho_{ji} + m - r = n$.

Similarly, for the normal form (NF2), we define four chains $w_j$, $1 \leq j \leq 4$, satisfying:

(1)′ $\dim w_1 = \dim \tilde{u}_1 = 1$.

The $w_1$-chain is the special linear chain whose control has to be prolonged in order to obtain a static feedback linearizable prolongation. Its length is denoted by $\mu_1 + k$, where $\mu_1 \geq 1$.

(2)′ $\dim w_2 = \dim \tilde{u}_2 = r-1$.

We have $w_2 = (w_{21}^1, \cdots, w_{2r-1}^1)$ to which we associate the lengths $\mu_2 + k = (\mu_{21}+k, \cdots, \mu_{2r-1}+k)$. The $w_2$-chains denote the remaining linear chains. Their lengths are arbitrary, i.e., the integers $\mu_{2i}$, $1 \leq i \leq r-1$, are such that $\mu_{2i}+k \geq 1$ and can be negative.

(3)′ $\dim w_3 = \dim w_4 = \dim \tilde{u}_3 = m-r$.

The length of the $w_3$-chains is denoted by $\mu_3$ and that of all $w_4$-chains equals $k+1$.

The integers $\mu_{ji}$ satisfy $\sum_{j=1}^{3}\sum_{i=1}^{\dim w_j}\mu_{ji} + rk + (m-r)(k+1) = n$.

**Proposition 2.4.1.** *Consider a control-affine system $\Sigma$ that is not static feedback linearizable. The following conditions are equivalent:*

*(i)* $\Sigma$ *is flat at* $x_0$ *(at* $(x_0, u_0)$*, such that* $u_0 \notin U_{sing}(x_0)$*, if* $k = 0$*) of differential weight* $n+m+1$*;*

*(ii)* $\Sigma$ *is locally, around* $x_0$*, static feedback equivalent to the following normal form in a neighborhood of* $z_0 \in \mathbb{R}^n$*:*

$$(NF1)\begin{cases} z_1^{(\rho_1)} = \tilde{u}_1 \quad z_2^{(\rho_2)} = \tilde{u}_2 \quad z_3^{(\rho_3)} = a_3(z) + b_3(z)\tilde{u}_1 \quad z_4^{(\rho_4)} = a_4(z) + b_4(z)\tilde{u}_1 \\ \qquad\qquad\qquad\qquad\qquad\; \dot{z}_3^{\rho_3+1} = \tilde{u}_3 \qquad\qquad\qquad\quad\; \dot{z}_4^{\rho_4+1} = \tilde{u}_4 \end{cases}$$

*(a) either* $k = 0$ *and then* $\operatorname{rk}\frac{\partial b}{\partial(z_2^{\rho_2}, z_3^{\rho_3+1}, z_4^{\rho_4+1})}(z_0) \geq 1$*, and* $\operatorname{rk}\frac{\partial(a+b\tilde{u}_1)}{\partial(z_3^{\rho_3+1}, z_4^{\rho_4+1})}(z_0, \tilde{u}_0) = m-r$*, where* $b = (b_3, b_4)$ *and* $a + b\tilde{u}_1 = (a_3(z) + b_3(z)\tilde{u}_1, a_4(z) + b_4(z)\tilde{u}_1)$*, implying that for all pairs of functions* $(a_{ji}, b_{ji})$*, we can always normalize one of them to* $z_{ji}^{\rho_{ji}+1}$*;*

*(b) or* $k \geq 1$ *and then* $a_3 = z_3^{\rho_3+1}$*,* $a_4 = z_4^{\rho_4+1}$*,* $b_3 = b_3(\bar{z}_1^{\rho_1-k+1}, \bar{z}_2^{\rho_1-k}, \bar{z}_3^{\rho_3-k+1}, \bar{z}_4^{\rho_4-k+1})$*, the i-component of* $b_4$*, for* $1 \leq i \leq m-p$*, is given by* $b_{4i} = b_{4i}(\bar{z}_1^{\rho_1-\rho_{4i}+1}, \bar{z}_2^{\rho_1-\rho_{4i}}, \bar{z}_3^{\rho_3-\rho_{4i}+1}, \bar{z}_4^{\rho_4-\rho_{4i}+1})$ *and* $\operatorname{rk}\frac{\partial b_3}{\partial(z_1^{\rho_1-k+1}, z_2^{\rho_1-k}, z_3^{\rho_3-k+1}, z_4^{\rho_4-k+1})}(z_0) \geq 1$*.*

*(iii)* $\Sigma$ *is locally, around* $x_0$*, static feedback equivalent to the following normal form in a neighborhood of* $w_0 \in \mathbb{R}^n$*:*

$$(NF2)\begin{cases} w_1^{(\mu_1+k)} = \tilde{u}_1 \quad w_2^{(\mu_2+k)} = \tilde{u}_2 \quad w_3^{(\mu_3)} = d(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2}, \bar{w}_3^{\mu_3}, w_4) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad\; w_4^{(k+1)} = \tilde{u}_3 \end{cases}$$

*where* $\operatorname{rk} D(w_0) = m-r$*, with* $D = \frac{\partial d}{\partial w_4}$*, and* $\operatorname{rk}\frac{\partial D}{\partial(w_1^{\mu_1+1}, w_2^{\mu_2}, w_4)}(w_0) \geq 1$*; if* $k = 0$*, we put* $w_1^{\mu_1+1} = \tilde{u}_1$*, the functions* $d_i$ *are of the form* $d_i = a_i(w) + b_i(w)\tilde{u}_1$ *and* $\operatorname{rk} D$ *is calculated at* $(w_0, \tilde{u}_0)$*.*

*Remarks.* Each of the above normals forms has its importance and we below discuss them.

1. The normal form (NF1) (resp. (NF2)) is valid around $z_0 \in \mathbb{R}^n$ (resp. $w_0 \in \mathbb{R}^n$), which may be zero or not. Therefore both forms can be used around any point (equilibrium or not).

2. It is easy to see that (NF1) (resp. (NF2)) is flat with the top variables $\varphi = (z_1, z_2, z_3, z_4)$ (resp. $\varphi = (w_1, w_2, w_3)$ being minimal flat outputs of differential weight $n+m+1$.

3. It is clear that (NF1) becomes locally static feedback linearizable after a one-fold prolongation of $\tilde{u}_1$, which is the to-be-prolonged control. Moreover, if we replace $\tilde{u}_1$ by $\hat{u}_1 = \beta(z)\tilde{u}_1$, with $\beta(z) \neq 0$, and we prolong $\hat{u}_1$ instead of $\tilde{u}_1$, the prolonged system is also locally static feedback linearizable.

4. The normal forms apply to all cases $k \geq 1$ or $k = 0$, independently of the value of cork $(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k])$.

5. Notice that if $p = m$, then the length of all nonlinear chains of (NF1) is at least $k+2$; if $r = 1$, then only one chain of (NF2) (given by $w_1$) is linear.

6. The nonivolutive distribution $\mathcal{D}^k$ is easier to be analyzed with the help of (NF2), since the integer $k$ appears explicitly.

7. It is clear from (NF1) (and from (NF2) as well) that in the case $k = 0$ (and only in that case!), the precompensator creates singularities in the control space (depending on state). Indeed, the controls $\tilde{u}_0$ satisfying rk $\frac{\partial(a+b\tilde{u}_1)}{\partial(z_3^{\rho_3+1}, z_4^{\rho_4+1})}(z_0, \tilde{u}_0) < m - r$ are singular for (NF1) (we have the same condition for (NF2) with $(z_3^{\rho_3+1}, z_4^{\rho_4+1})$ replaced by $w_4$). An invariant description of that set of singular controls is given by $U_{sing}$.

8. The minimal $x$-flat outputs and the normal forms (NF1) (resp. (NF2)) are compatible: if $\varphi$ is a minimal $x$-flat output at $x_0$, then there exists an invertible static feedback transformation bringing the system $\Sigma$ into (NF1) with $\varphi = (z_1, z_2, z_3, z_4)$ (resp. into (NF2) with $\varphi = (w_1, w_2, w_3)$).

## 2.5 Calculating flat outputs

The goal of this section is to answer the question whether a given $m$-tuple of smooth functions forms a minimal $x$-flat output.

Recall that if $k = 0$, we can construct the following sequence of inclusions of involutive distributions:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^{\rho-1} \subset \mathcal{H}^\rho = TX,$$

where $\mathcal{H}^0$ is an involutive corank one subdistribution of $\mathcal{D}^0$, the distribution $\mathcal{H}^1$ is defined by $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$, with $\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$, and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $1 \leq i \leq \rho - 1$, and $\rho$ is the smallest integer such that $\mathcal{H}^\rho = TX$.

If $k \geq 1$, according to Proposition 2.7.2 (in Section 2.7 below), we can construct as for the case $k = 0$, the following sequence of inclusions of involutive distributions:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^k \subset \cdots \subset \mathcal{H}^{\rho-1} \subset \mathcal{H}^\rho = TX,$$

where $\mathcal{H}^0$ is the involutive corank one subdistribution of $\mathcal{D}^0$ associated to $\mathcal{H}^k$ and $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$, for $1 \leq i \leq k-1$ (see Proposition 2.7.2 for details). For $i \geq 2$, we actually have $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$. We will denote by $r_i$ the corank of the inclusion $\mathcal{H}^{i-1} \subset \mathcal{H}^i$, for $i \geq 0$. We clearly have $m \geq r_1 \geq r_2 \geq \cdots \geq r_q \geq 1$.

We can now state our result describing all minimal $x$-flat outputs of differential weight $n + m + 1$. The following proposition answers the question whether a given $m$-tuple of smooth functions $(\varphi_1, \cdots, \varphi_{r_1}, \psi_{r_1+1}, \cdots, \psi_m)$ forms a minimal $x$-flat output and holds for both cases $k = 0$ and $k \geq 1$. If $r_1 = m$, then in the $m$-tuple $(\varphi_1, \cdots, \varphi_{r_1}, \psi_{r_1+1}, \cdots, \psi_m)$ the components $\psi_j$ are missing.

**Proposition 2.5.1.** *Consider the control system $\Sigma$, given by (2.2), that is flat at $x_0$ (at $(x_0, u_0)$, if $k = 0$), of differential weight $n + m + 1$. Then a m-tuplet $(\varphi_1, \cdots, \varphi_{r_1}, \psi_{r_1+1}, \cdots, \psi_m)$ of smooth functions defined on a neighborhood of $x_0$ is a minimal x-flat output at $x_0$ if and only if (after permuting them, if necessary):*

*(FO1)* $d\varphi_{r_{i+2}+1}, \cdots, d\varphi_{r_{i+1}} \perp \mathcal{H}^i$, *for* $0 \leq i \leq \rho - 1$, *with* $r_{\rho+1} = 0$;

*(FO2)* $dL\varphi_i^{(j_i)}$ *and* $d\psi_j$ *are independent at* $x_0$, *where* $r_{l+1} + 1 \leq i \leq r_l$, $0 \leq j_i \leq l - 1$, $1 \leq l \leq \rho$, $r_1 + 1 \leq j \leq m$;

## 2.6 Examples

### 2.6.1 Quadrotor helicopter

A quadrotor is a four rotor helicopter. Assume that a body frame is fixed at the center of gravity of the quadrotor, with the $z$-axis pointing up-wards. The body frame is related to the inertial frame by a position vector $(x, y, z)$ and 3 angles $(\theta, \psi, \varphi)$ representing pitch, roll and yaw, respectively. The equations of motion are given by the following control system (see [1,3]):

$$\Sigma_{QH} : \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1(\cos\varphi \sin\theta \cos\psi + \sin\varphi \sin\psi) \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= u_1(\sin\varphi \sin\theta \cos\psi - \cos\varphi \sin\psi) \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -g + u_1(\cos\varphi \cos\psi) \\ \dot{\theta} &= u_2 \\ \dot{\psi} &= u_3 \\ \dot{\varphi} &= u_4 \end{cases}$$

The control $u_1$ represents the total thrust on the body in the $z$-axis, $u_2$ and $u_3$ are the pitch and roll inputs and $u_4$ is the yawing moment. The quadrotor helicopter has been shown to be flat, with $(x_1, y_1, z_1, \varphi)$ a flat output (see [3]). The system is not static feedback linearizable, but it becomes static feedback linearizable after a one fold prolongation. To illustrate our results, fix $\xi_0 \in X$ such that $(\cos\theta \cos\psi \cos\varphi(\cos\varphi \sin\theta \cos\psi + \sin\varphi \sin\psi))(\xi_0) \neq 0$. Applying the invertible feedback transformation

$$\begin{aligned} &\tilde{u}_1 = u_1(\cos\varphi \sin\theta \cos\psi + \sin\varphi \sin\psi) \\ &\tilde{u}_i = u_i, \ 2 \leq i \leq 4, \end{aligned}$$

we get:

$$\tilde{\Sigma}_{QH} : \begin{cases} \dot{x}_1 = x_2 & \dot{y}_1 = y_2 \\ \dot{x}_2 = \tilde{u}_1 & \dot{y}_2 = \tilde{u}_1 a(\theta, \psi, \varphi) \\ & \dot{\theta} = \tilde{u}_2 \\ \dot{z}_1 = z_2 & \dot{\varphi} = \tilde{u}_4 \\ \dot{z}_2 = -g + \tilde{u}_1 b(\theta, \psi, \varphi) & \\ \dot{\psi} = \tilde{u}_3 & \end{cases}$$

where

$$a = \frac{\sin\varphi\sin\theta\cos\psi - \cos\varphi\sin\psi}{\cos\varphi\sin\theta\cos\psi+\sin\varphi\sin\psi} \quad \text{and} \quad b = \frac{\cos\varphi\cos\psi}{\cos\varphi\sin\theta\cos\psi+\sin\varphi\sin\psi}.$$

The distribution

$$\mathcal{D}^0 = \operatorname{span}\{\frac{\partial}{\partial x_2} + a\frac{\partial}{\partial y_2} + b\frac{\partial}{\partial z_2}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \varphi}\}$$

is not involutive. Indeed, the vector fields $g_i$, $1 \leq i \leq 4$, $[g_1, g_2]$ and $[g_1, g_3]$ are independent at $\xi_0$ (provided that $cos\theta_0 \cos\psi_0 \cos\varphi_0 \neq 0$, which is verified according to our assumption), thus we obtain

$$\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] = \operatorname{span}\{\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\}.$$

Here $k = 0$ and cork $(\mathcal{D}^0 \subset [\mathcal{D}^0, \mathcal{D}^0]) = 2$, consequently we are in the case of Theorem 2.3.2. It is immediate to identify the unique corank one involutive subdistribution of $\mathcal{D}^0$, that is $\mathcal{H}^0 = \operatorname{span}\{\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \varphi}\}$.

We have $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0] = \mathcal{G}^1$ (since $[f, g_i] = 0$, for $2 \leq i \leq 4$), which is clearly involutive, and $\mathcal{H}^2 = TX$. The system $\tilde{\Sigma}_{QH}$ satisfies all conditions of Theorem 2.3.2, hence the corresponding prolongation given by

$$\tilde{\Sigma}_{QH}^{(1,0,0,0)} : \left\{ \begin{array}{lllllll} \dot{x}_1 & = & x_2 & & \dot{y}_1 & = & y_2 \\ \dot{x}_2 & = & \tilde{u}_1 & & \dot{y}_2 & = & \tilde{u}_1 a(\theta, \psi, \varphi) \\ \dot{\tilde{u}}_1 & = & v_1 & & \dot{\theta} & = & v_2 \\ \dot{z}_1 & = & z_2 & & \dot{\varphi} & = & v_4 \\ \dot{z}_2 & = & -g + \tilde{u}_1 b(\theta, \psi, \varphi) & & & & \\ \dot{\psi} & = & v_3 & & & & \end{array} \right.$$

where $v_i = \tilde{u}_i$, for $2 \leq i \leq 4$, is locally static feedback linearizable. Indeed, applying the following change of coordinates $\tilde{\theta} = \tilde{u}_1 a(\theta, \psi, \varphi)$ and $\tilde{\psi} = -g + \tilde{u}_1 b(\theta, \psi, \varphi)$ (which is valid in a neighborhood of $\xi_0$ and for $\tilde{u}_{10} \neq 0$) and a suitable feedback transformation, we get

$$\tilde{\Sigma}_{QH}^{(1,0,0,0)} : \left\{ \begin{array}{lllllll} \dot{x}_1 & = & x_2 & & \dot{y}_1 & = & y_2 \\ \dot{x}_2 & = & w & & \dot{y}_2 & = & \tilde{\theta} \\ \dot{w} & = & \tilde{v}_1 & & \dot{\tilde{\theta}} & = & \tilde{v}_2 \\ \dot{z}_1 & = & z_2 & & \dot{\varphi} & = & \tilde{v}_4 \\ \dot{z}_2 & = & \tilde{\psi} & & & & \\ \dot{\tilde{\psi}} & = & \tilde{v}_3 & & & & \end{array} \right.$$

which is the Brunovský canonical form with $(x_1, y_1, z_1, \varphi)$ playing the role of the top variables. From this, it is obvious that $(x_1, y_1, z_1, \varphi)$ is a minimal flat output, i.e. of differential weight $n + m + 1 = 14$.

### 2.6.2 Polymerization reactor

The control system that we consider in this example has two-inputs. Recall that, according to the statement made at the end of the Section 2.3, Theorem 2.3.4 covers also the case $m = 2$, but only if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ (which is the case for the polymerization reactor), where $\mathcal{D}^k$ is the first noninvolutive distribution. That is illustrated by the following example which has been also treated in [45]. In that paper, the involutive subdistribution $\mathcal{H}^k$, that plays a crucial role in our analyses, was defined by another method.

Consider the reactor (see [36, 60]):

$$\Sigma_{PR}: \begin{cases} \dot{C}_m &= \frac{C_{mm_s}}{\tau} - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{C_m}{\tau} + R_m(C_m, C_i, C_s, T) \\ \dot{C}_i &= -k_i(T)C_i + u_2\frac{C_{ii_s}}{V} - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{C_i}{\tau} \\ \dot{C}_s &= u_2\frac{C_{si_s}}{V} + \frac{C_{sm_s}}{\tau} - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{C_s}{\tau} \\ \dot{\mu} &= -M_mR_m(C_m, C_i, C_s, T) - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})\frac{\mu}{\tau} \\ \dot{T} &= \theta(C_m, C_i, C_s, \mu, T) + \alpha_1 T_j \\ \dot{T}_j &= f_6(T, T_j) + \alpha_4 u_1 \end{cases}$$

where $u_1, u_2$ are the control inputs and $C_{mm_s}$, $C_{ii_s}$, $C_{si_s}$, $C_{sm_s}$, $M_m$, $\bar{\epsilon}$, $\tau$, $V$, $\alpha_1$, $\alpha_4$ are constant positive physical parameters. The functions $R_m$, $k_i$, $\theta$ and $f_6$ are not well-known and can be considered arbitrary: they derive from experimental data and semi-empirical considerations and involve kinetic laws, heat transfer coefficients and reaction enthalpies.

The system has been proved to be flat [36, 60], see also [44]. Below we will show how our results apply to it. After applying the change of coordinates

$$\begin{aligned} \tilde{C}_m &= \mu + M_mC_m \\ \tilde{C}_i &= C_i - \frac{C_{ii_s}}{C_{si_s}}C_s \\ \tilde{C}_s &= -k_i(T)C_i - (1+\bar{\epsilon}\frac{\mu}{\mu+M_mC_m})(\frac{C_i}{\tau} - \frac{C_{ii_s}}{C_{si_s}}\frac{C_s}{\tau}) - \frac{C_{ii_s}}{C_{si_s}}\frac{C_{sm_s}}{\tau}, \\ \tilde{\mu} &= \frac{1}{\tau}(M_mC_{mm_s} - (1+\bar{\epsilon})\mu - M_mC_m), \\ \tilde{T} &= T \\ \tilde{T}_j &= \theta(C_m, C_i, C_s, \mu, T) + \alpha_1 T_j \end{aligned}$$

and a suitable feedback transformation, we obtain:

$$\tilde{\Sigma}_{PR}: \begin{cases} \dot{\tilde{C}}_i = \tilde{C}_s & \dot{\tilde{C}}_m = \tilde{\mu} \\ \dot{\tilde{C}}_s = \tilde{u}_1 & \dot{\tilde{\mu}} = b(\tilde{C}_m, \tilde{C}_i, \tilde{C}_s, \tilde{\mu}, \tilde{T}) \\ & \dot{\tilde{T}} = \tilde{T}_j \\ & \dot{\tilde{T}}_j = \tilde{u}_2 \end{cases}$$

where $b$ is a smooth function depending explicitly on $T$.

If $(\frac{\partial^2 b}{\partial\tilde{T}\partial\tilde{C}_s}, \frac{\partial^2 b}{\partial\tilde{C}_s^2}) \neq (0,0)$, then the distribution $\mathcal{D}^1 = \text{span}\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{C}_i} + \frac{\partial b}{\partial\tilde{C}_s}\frac{\partial}{\partial\tilde{\mu}}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial T}\}$ is noninvolutive and cork $(\mathcal{D}^1 \subset [\mathcal{D}^1, \mathcal{D}^1]) = 1$. It follows, see Proposition 2.3.2(iii), that an involutive subdistribution of corank one in $\mathcal{D}^1$ cannot be unique. Let us suppose that $\frac{\partial^2 b}{\partial\tilde{C}_s^2} \neq 0$. Therefore, $[\mathcal{D}^0, \mathcal{D}^1] \not\subset \mathcal{D}^1$. Consequently, we are in the case of Theorem 2.3.4, with $m = 2$.

The characteristic distribution of $\mathcal{D}^1$ is:

$$\mathcal{C}^1 = \text{span}\,\{\frac{\partial}{\partial \tilde{T}_j}, \frac{\partial}{\partial T} - \frac{\partial^2 b}{\partial T \partial \tilde{C}_s}(\frac{\partial^2 b}{\partial \tilde{C}_s^2})^{-1}\frac{\partial}{\partial \tilde{C}_s}\}$$

and satisfies the conditions $(C1)$ and $(C2)$. Indeed, $\text{rk}\,\mathcal{C}^1 = 2$ ans $\text{rk}\,(\mathcal{C}^1 \cap \mathcal{D}^0) = 1$. The corank one subdistribution

$$\mathcal{H}^1 = \mathcal{C}^1 + \mathcal{D}^0 = \text{span}\,\{\frac{\partial}{\partial \tilde{C}_s}, \frac{\partial}{\partial \tilde{T}_j}, \frac{\partial}{\partial T}\}$$

is involutive. We have

$$\mathcal{H}^2 = \mathcal{H}^1 + [f, \mathcal{H}^1] = \text{span}\,\{\frac{\partial}{\partial \tilde{C}_s}, \frac{\partial}{\partial \tilde{C}_i}, \frac{\partial}{\partial \tilde{T}_j}, \frac{\partial}{\partial T}, \frac{\partial}{\partial \mu}\}$$

involutive and $\mathcal{H}^3 = TX$. The system $\tilde{\Sigma}_{PR}$ satisfies all conditions of Theorem 2.3.4, hence the corresponding prolongation given by $\tilde{u}_1 = y$, $\dot{y} = v_1$, and $\tilde{u}_2 = v_1$ is locally static feedback linearizable. Indeed, all associated distributions $\mathcal{D}_p^i$, for $i \geq 0$, associated to the prolongation $\tilde{\Sigma}_{PR}^{(1,0)}$, are involutive, of constant rank and $\text{rk}\,\mathcal{D}_p^3 = 7$. Therefore, the prolonged system can be brought into Brunovský canonical form with $\tilde{C}_m$, $\tilde{C}_i$ playing the role of top variables (and thus of minimal flat outputs, of differential weight $n+3$).

## 2.7 Proofs

### 2.7.1 Notations and Useful results

Consider a control system of the form

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x) + u_1 g_1(x) + \sum_{i=2}^{m} u_i h_i(x),$$

where the change of notation is to distinguish the first control (respectively the first vector field $g_1$) from the remaining controls $u_i$ (respectively remaining vector fields $g_i$), for $2 \leq i \leq m$. By $\Sigma^{(1,0,\cdots,0)}$ we will denote the system $\Sigma$ with one-fold prolongation, that is

$$\Sigma^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + \sum_{i=2}^{m} v_i h_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_i = u_2$, for $2 \leq i \leq m$. Throughout this section,

$$F = \sum_{i=1}^{n} (f_i + y_1 g_{1i}) \frac{\partial}{\partial x_i}$$

stands for the drift and

$$G_1 = \frac{\partial}{\partial y_1}, \quad H_j = \sum_{i=1}^{n} h_{ji} \frac{\partial}{\partial x_i}, \text{ for } 2 \leq j \leq m,$$

denote the control vector fields of the prolonged system.

To $\Sigma^{(1,0,\cdots,0)}$, we associate the distributions $\mathcal{D}_p^0 = \operatorname{span}\{G_1, H_2, \cdots H_m\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [F, \mathcal{D}_p^i]$, for $i \geq 0$, the subindex $p$ referring to the prolonged system $\Sigma^{(1,0,\cdots,0)}$.

In our proofs we will need the two following technical results. Consider a control system $\Sigma$, given by (2.2), and let $\mathcal{D}^k$ be the first noninvolutive distribution.

**Proposition 2.7.1.** *Assume that $\Sigma$ is dynamically linearizable via invertible one-fold prolongation. If $k \geq 1$, then* $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} \geq 2$.

*Proof.* Assume $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} = 1$ and let $l$ be the smallest integer such that $\operatorname{rk} \mathcal{D}^l - \operatorname{rk} \mathcal{D}^{l-1} = 1$. It is clear that $1 \leq l \leq k$. Since $\Sigma$ is dynamically linearizable via invertible one-fold prolongation, there exists an invertible static feedback transformation, $u(x) = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=2}^m \tilde{u}_i \tilde{h}_i(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{h}_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_i = \tilde{u}_i$, for $2 \leq i \leq m$, is locally static feedback linearizable. For simplicity of notation, we will drop the tildes, but we will keep distinguishing $g_1$ from $h_i$ (which could also be denoted $g_i$) whose controls are not preintegrated.

Since $\Sigma^{(1,0,\cdots,0)}$ is locally static feedback linearizable, for any $i \geq 0$ the distributions $\mathcal{D}_p^i$ are involutive, of constant rank, and there exists an integer $\rho$ such that $\operatorname{rk} \mathcal{D}_p^\rho = n + 1$. We have

$$\begin{aligned} \mathcal{D}_p^0 &= \operatorname{span}\{\tfrac{\partial}{\partial y_1}, h_j, 2 \leq j \leq m\}, \\ \mathcal{D}_p^1 &= \operatorname{span}\{\tfrac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j + y_1 [g_1, h_j], 2 \leq j \leq m\}. \end{aligned}$$

Since $k \geq 1$, the distribution $\mathcal{D}^0 = \operatorname{span}\{g_1, h_j, 2 \leq j \leq m\}$ is involutive, thus $[g_1, h_j] \in \mathcal{D}^0$, for $2 \leq j \leq m$, and $\mathcal{D}_p^1 = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j, 2 \leq j \leq m\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq l$,

$$\mathcal{D}_p^i = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1} g_1, h_j, \cdots, ad_f^i h_j, 2 \leq j \leq m\}.$$

We have $\mathcal{D}^{l-1} = \operatorname{span}\{g_1, \cdots, ad_f^{l-1} g_1, h_j, \cdots, ad_f^{l-1} h_j, 2 \leq j \leq m\}$ and by the definition of $l$ either $ad_f^l h_j \in \mathcal{D}^{l-1}$, for all $2 \leq j \leq m$, i.e., $ad_f^l g_1 \notin \mathcal{D}^{l-1}$, or there exists an integer $2 \leq s \leq m$ such that $ad_f^l h_s \notin \mathcal{D}^{l-1}$.

In the first case:

$$\mathcal{D}_p^j = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{D}^{j-1}, \text{ for } j \geq l,$$

The involutivity of the distribution $\mathcal{D}_p^j$, associated to the prolonged system, implies that of $\mathcal{D}^{j-1}$. For $j = k+1$, it contradicts the fact that $\mathcal{D}^k$ is noninvolutive.

In the second case, there exists an integer $2 \leq s \leq m$ such that $ad_f^l h_s \notin \mathcal{D}^{l-1}$. Since $\operatorname{rk}\mathcal{D}^l = \operatorname{rk}\mathcal{D}^{l-1} + 1$, we deduce that $\mathcal{D}^l = \operatorname{span}\{g_1, \cdots, ad_f^{l-1}g_1,\ h_j, \cdots, ad_f^{l-1}h_j,\ ad_f^l h_s,\ 2 \leq j \leq m\}$. Moreover, for $\Sigma^{(1,0,\cdots,0)}$, we have

$$\mathcal{D}_p^j = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{D}^j, \text{ for } j \geq l,$$

and the involutivity of $\mathcal{D}_p^j$ implies that of $\mathcal{D}^j$. For $j = k$, it follows that $\mathcal{D}^k$ is involutive, which contradicts the assumption of noninvolutivity of $\mathcal{D}^k$. Thus $l$, if it exists, satisfies $l \geq k+1$ and $\operatorname{rk}\mathcal{D}^k - \operatorname{rk}\mathcal{D}^{k-1} \geq 2$.

**Proposition 2.7.2.** *Assume $k \geq 1$ and suppose that $\mathcal{D}^k$ contains an involutive subdistribution $\mathcal{H}^k$, of corank one.*

*(i) If* $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, *then $\mathcal{H}^k$ satisfies $\mathcal{D}^{k-1} \subset \mathcal{H}^k$.*

*(ii) If $\mathcal{H}^k$ satisfies $\mathcal{D}^{k-1} \subset \mathcal{H}^k$, then there exists a distribution $\mathcal{H}$, uniquely associated to $\mathcal{H}^k$, such that $\mathcal{H} \subset \mathcal{D}^0$, of corank one, and $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k\mathcal{H}$. Moreover, all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i\mathcal{H}$, for $0 \leq i \leq k-1$, where $\mathcal{D}^{-1}$ is empty and $\mathcal{H}^0 = \mathcal{H}$, are involutive.*

*Remark.* Notice that for $1 \leq i \leq k-1$, we actually have $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$ and if we denote by $r_i$ the corank of the inclusion $\mathcal{H}^{i-1} \subset \mathcal{H}^i$, for $i \geq 0$, we clearly have $m \geq r_1 \geq r_2 \geq \cdots \geq r_k$.

*Proof of (i).* Since $\operatorname{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, according to Proposition 2.3.2, if the distribution $\mathcal{D}^k$ contains an involutive subdistribution $\mathcal{H}^k$, of corank one, then $\mathcal{H}^k$ is unique. Using Jacobi identity, it is easy to show that $\mathcal{D}^{k-2} \subset \mathcal{H}^k$. Suppose $\mathcal{D}^{k-1} \not\subset \mathcal{H}^k$, i.e., there exists a vector field $v \in \mathcal{D}^{k-1}$, of the form $v = \sum_{i=1}^m \alpha_i ad_f^{k-1} g_i \bmod \mathcal{D}^{k-2}$, satisfying

$$\mathcal{D}^k = \mathcal{H}^k + \operatorname{span}\{v\},$$

where $\alpha_i$ are smooth functions, not vanishing simultaneously and such that there exists an integer $i$ verifying $\alpha_i \neq 0$ and $ad_f^{k-1}g_i \notin \mathcal{D}^{k-2}$. The vector field $v$ can also be written as $v = ad_f^{k-1}(\sum_{i=1}^m \alpha_i g_i) \bmod \mathcal{D}^{k-2}$ and we put $g_c = \sum_{i=1}^m \alpha_i g_i$, i.e., $v = ad_f^{k-1} g_c \bmod \mathcal{D}^{k-2}$. Therefore,

$$\mathcal{D}^k = \mathcal{H}^k + \operatorname{span}\{ad_f^{k-1}g_c\}.$$

We can always assume, without restriction of generality, that $\alpha_1$ is nonzero and $ad_f^{k-1}g_1 \notin \mathcal{D}^{k-2}$ and since $g_c = \sum_{i=1}^m \alpha_i g_i$, we clearly have $\mathcal{D}^0 = \operatorname{span}\{g_1, g_2, \cdots, g_m\} = \operatorname{span}\{g_c, g_2, \cdots, g_m\}$. By abuse of notation, we will write $g_1$ instead of $g_c$, i.e.,

$$\mathcal{D}^k = \mathcal{H}^k + \operatorname{span}\{ad_f^{k-1}g_1\}.$$

From this, we deduce that the involutive subdistribution $\mathcal{H}^k$ is given by

$$\mathcal{H}^k = \operatorname{span}\{g_1, \cdots, ad_f^{k-2}g_1, ad_f^k g_1,\ g_j, \cdots, ad_f^k g_j,\ 2 \leq j \leq m\}.$$

Thus, the new directions, completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k$, where $\overline{\mathcal{D}}^k$ is the involutive closure of $\mathcal{D}^k$, are obtained with

$$[ad_f^k g_i, ad_f^{k-1} g_1]$$

for some $i$ such that $1 \leq i \leq m$, and since cork $(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, there are at least two integers $i$ satisfying this property. Suppose

$$[ad_f^k g_s, ad_f^{k-1} g_1] \notin \mathcal{D}^k,$$

where $s \neq 1$ (according to the above remark, such integer $s$ always exists). Applying Jacobi identity, we obtain

$$\begin{aligned}[ad_f^k g_s, ad_f^{k-1} g_1] = [[f, ad_f^{k-1} g_s], ad_f^{k-1} g_1] &= [[f, ad_f^{k-1} g_1], ad_f^{k-1} g_s] + [f, [ad_f^{k-1} g_1, ad_f^{k-1} g_s]] \\ &= [ad_f^k g_1, ad_f^{k-1} g_s] \operatorname{mod} \mathcal{D}^k\end{aligned}$$

and since the vector fields $ad_f^k g_1$ and $ad_f^{k-1} g_s$ belong to $\mathcal{H}^k$, which is involutive, $[ad_f^k g_1, ad_f^{k-1} g_s] \in \mathcal{H}^k$. It follows immediately that $[ad_f^k g_s, ad_f^{k-1} g_1] \in \mathcal{D}^k$, which contradicts our assumption. Therefore, the inclusion $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ holds.

*Proof of (ii).* Let us first show the existence of the distribution $\mathcal{H}$.

Denote cork $(\mathcal{D}^{k-1} \subset \mathcal{D}^k) = r$ and suppose that the vector fields $g_i \in \mathcal{D}^0$, for $1 \leq i \leq r$, satisfy

$$\mathcal{D}^k = \mathcal{D}^{k-1} + \operatorname{span}\{ad_f^k g_i, 1 \leq i \leq r\}.$$

Thus there exist smooth functions $\alpha_j^i$ such that

$$ad_f^k g_j = \sum_{i=1}^{r} \alpha_j^i ad_f^k g_i \operatorname{mod} \mathcal{D}^{k-1},$$

for $r+1 \leq j \leq m$. It follows

$$ad_f^k (g_j - \sum_{i=1}^{r} \alpha_j^i g_i) = 0 \operatorname{mod} \mathcal{D}^{k-1}.$$

Denote $h_j = g_j - \sum_{i=1}^{r} \alpha_j^i g_i$, for $r+1 \leq j \leq m$. We clearly have $\mathcal{D}^0 = \operatorname{span}\{g_1, \cdots, g_r, h_{r+1}, \cdots, h_m\}$, with $h_j$ such that $ad_f^k h_j \in \mathcal{D}^{k-1}$, for $r+1 \leq j \leq m$.

Since $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ and $\mathcal{H}^k \subset \mathcal{D}^k$, of corank one, there exist smooth functions $\lambda_j^i$, for $1 \leq i, j \leq r$, such that the $r \times r$-matrix $\Lambda = (\lambda_j^i)$ is invertible and the distributions $\mathcal{H}^k$ and $\mathcal{D}^k$ verify

$$\mathcal{H}^k = \mathcal{D}^{k-1} + \operatorname{span}\{\sum_{i=1}^{r} \lambda_j^i ad_f^k g_i, 2 \leq j \leq r\},$$

$$\mathcal{D}^k = \mathcal{H}^{k-1} + \operatorname{span}\{\sum_{i=1}^{r} \lambda_1^i ad_f^k g_i\}.$$

Denote $\tilde{g}_1 = \sum_{i=1}^{r} \lambda_1^i g_i$ and $h_j = \sum_{i=1}^{r} \lambda_j^i g_i$, for $2 \leq j \leq r$. We put

$$\mathcal{H} = \mathrm{span}\,\{h_j,\, 2 \leq j \leq m\},$$

which is clearly of corank one in $\mathcal{D}^0 = \mathrm{span}\,\{\tilde{g}_1, h_j,\, 2 \leq j \leq m\}$, and satisfies

$$\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k \mathcal{H}.$$

We will prove next the involutivity of all distributions $\mathcal{H}^i$, for $0 \leq i \leq k-1$. Assume that the distribution $\mathcal{H}^{k-1}$ given by

$$\mathcal{H}^{k-1} = \mathcal{D}^{k-2} + ad_f^{k-1}\mathcal{H} = \mathcal{D}^{k-2} + \mathrm{span}\,\{ad_f^{k-1} h_j,\, 2 \leq j \leq m\}$$

is not involutive. Since the inclusion $\mathcal{H}^{k-1} \subset \mathcal{D}^{k-1}$ is of corank one and $\mathcal{D}^{k-1}$ is involutive, it follows $\overline{\mathcal{H}}^{k-1} = \mathcal{D}^{k-1}$. Moreover, conditions $\mathcal{D}^{k-2} \subset \mathcal{H}^{k-1}$ and $\mathcal{D}^{k-2}$ involutive imply that the new direction completing $\mathcal{H}^{k-1}$ to its involutive closure is given by a vector field of the form $[ad_f^l h_i, ad_f^{k-1} h_j]$ or of the form $[ad_f^l \tilde{g}_1, ad_f^{k-1} h_j]$, where $2 \leq i, j \leq m$ and $0 \leq l \leq k-1$, and is necessarily collinear with $ad_f^{k-1}\tilde{g}_1 \bmod \mathcal{H}^{k-1}$.

Let us suppose that there exists two integers $2 \leq i, j \leq m$ such that $[ad_f^l h_i, ad_f^{k-1} h_j] \notin \mathcal{H}^{k-1}$. The same reasoning applies if $[ad_f^l \tilde{g}_1, ad_f^{k-1} h_j] \notin \mathcal{H}^{k-1}$.

Hence, there exists a non zero smooth function $\alpha$ such that

$$[ad_f^l h_i, ad_f^{k-1} h_j] = \alpha ad_f^{k-1} g_1 \bmod \mathcal{H}^{k-1}.$$

From this, applying Jacobi identity and the involutivity of $\mathcal{H}^k$, it follows

$$\begin{aligned}
[ad_f^l h_i, ad_f^k h_j] = [[ad_f^l h_i, [f, ad_f^{k-1} h_j]] &= [f, [ad_f^l h_i, ad_f^{k-1} h_j]] - [ad_f^{l+1} h_i, ad_f^{k-1} h_j] \\
&= [f, \alpha ad_f^{k-1} \tilde{g}_1] \bmod \mathcal{H}^k \\
&= \alpha ad_f^k \tilde{g}_1 \bmod \mathcal{H}^k.
\end{aligned}$$

On the other hand, $[ad_f^l h_i, ad_f^k h_j] \in \mathcal{H}^k$, and consequently $ad_f^k \tilde{g}_1 \in \mathcal{H}^k$, which contradicts our assumption, otherwise $\mathcal{D}^k = \mathcal{H}^k$ and $\mathcal{D}^k$ would be involutive. Therefore, $\mathcal{H}^{k-1}$ is involutive. Following the same line, the involutivity of $\mathcal{H}^i$ implies that of $\mathcal{H}^{i-1}$, for $1 \leq i \leq k-1$.

The following result is of independent interest and will be used to obtain the normal form (NF2), so we will give its proof in Section 2.7.5, where we show Proposition 2.4.1.

**Proposition 2.7.3.** *Assume that $\Sigma$ is dynamically linearizable via invertible one-fold prolongation. If $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$ and $\mathcal{H}^{k+1} \neq TX$, where $\overline{\mathcal{D}}^k$ is the involutive closure of $\mathcal{D}^k$ and $\mathcal{H}^{k+1}$ is defined by item $(A2)$ of Theorem 2.3.1 (resp. by item $(A2)'$ of Theorem 2.3.2, if $k = 0$), then the inclusion $\mathcal{H}^{k+1} \subset \mathcal{H}^{k+1} + \mathcal{D}^{k+1}$ is of corank one.*

### 2.7.2 Proof of Theorem 2.3.1

*Necessity.* Let us consider a flat control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$, of differential weight $n+m+1$. According to Proposition 2.3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} \ : \ \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \sum_{i=1}^{m} \tilde{u}_i\tilde{h}_i(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + \sum_{i=2}^{m} v_i\tilde{h}_i(x) \\ \dot{y}_1 &= v_1, \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \le j \le m$, is locally static feedback linearizable. For simplicity of notation, we will drop the tildes, we will keep distinguishing $g_1$ from $h_j$ (which could also be denoted $g_j$, $2 \le j \le m$) whose controls are not preintegrated. Since $\Sigma^{(1,0,\cdots,0)}$ is locally static feedback linearizable, $\mathcal{D}_p^i$ are involutive, of constant rank, for any $i \ge 0$, and there exists an integer $\rho$ such that $\operatorname{rk}\mathcal{D}_p^{\rho} = n+1$. We have

$$\mathcal{D}_p^0 = \operatorname{span}\{\tfrac{\partial}{\partial y_1}, h_j, 2 \le j \le m\},$$
$$\mathcal{D}_p^1 = \operatorname{span}\{\tfrac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j + y_1[g_1, h_j],\ 2 \le j \le m\}.$$

Since $k \ge 1$, the distribution $\mathcal{D}^0 = \operatorname{span}\{g_1, h_j,\ 2 \le j \le m\}$ is involutive, thus $[g_1, h_j] \in \mathcal{D}^0$ and hence $\mathcal{D}_p^1 = \operatorname{span}\{\tfrac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j, 2 \le j \le m\}$. It is easy to prove (by an induction argument) that, for $1 \le i \le k$,

$$\mathcal{D}_p^i = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1} g_1, h_j, \cdots, ad_f^i h_j,\ 2 \le j \le m\}.$$

Define

$$\mathcal{H}^k = \operatorname{span}\{g_1, \cdots, ad_f^{k-1} g_1, h_j, \cdots, ad_f^k h_j,\ 2 \le j \le m\}.$$

Since the intersection of involutive distributions is an involutive distribution, $\mathcal{H}^k = \mathcal{D}_p^i \cap TX = \operatorname{span}\{g_1, \cdots, ad_f^{i-1} g_1, h_j, \cdots, ad_f^i h_j,\ 2 \le j \le m\}$ is involutive, for $1 \le i \le k$. We deduce that $\mathcal{H}^k$ is involutive. It is immediate that $\mathcal{D}^{k-1} \subset \mathcal{H}^k \subset \mathcal{D}^k$, where the second inclusion is of corank one, otherwise $\mathcal{H}^k = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive or $\mathcal{H}^k = \mathcal{D}^{k-1}$ and $\operatorname{rk}\mathcal{D}^k - \operatorname{rk}\mathcal{D}^{k-1} = 1$, which contradicts our hypothesis. Recall that $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, for $i \ge k+1$. We have

$$\mathcal{D}_p^{k+1} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^k + [f, \mathcal{H}^k] = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^{k+1}$$

and by an induction argument

$$\mathcal{D}_p^{k+i} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^{k+i},\ i \ge 2.$$

Consequently, the involutivity of $\mathcal{D}_p^{k+i}$ implies that of $\mathcal{H}^{k+i}$, for $i \ge 1$. Moreover, $\operatorname{rk}\mathcal{D}_p^{\rho} = n+1$, proving that $\operatorname{rk}\mathcal{H}^{\rho} = n$, i.e., $\mathcal{H}^{\rho} = TX$.

*Sufficiency.* Consider a control system satisfying $(A1) - (A3)$ and let $\mathcal{H}^0 = \operatorname{span}\{h_j, 2 \le j \le m\}$ be the distribution defined by Proposition 2.7.2(ii). This system is static feedback equivalent to $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^{m} u_i h_i(x)$. By the same proposition, the involutivity of $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$ follows for $0 \le i \le k-1$. It is immediate to see that the prolongation

$$\Sigma^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_j = u_j$, for $2 \le j \le m$, is locally static feedback linearizable. Indeed, the linearizability distributions $\mathcal{D}_p^i$, associated to $\Sigma^{(1,0,\cdots,0)}$, are of the form

$$\mathcal{D}_p^i \;=\; \operatorname{span}\{\tfrac{\partial}{\partial y_1}\} + \mathcal{H}^i,\ i \ge 0,$$

and the involutivity of $\mathcal{H}^i$ implies that of $\mathcal{D}_p^i$, because $\mathcal{H}^i$ does not depend on $y_1$. Moreover, $\operatorname{rk} \mathcal{H}^\rho = n$, thus $\operatorname{rk} D_p^\rho = n+1$ and $\Sigma^{(1,0,\cdots,0)}$ is locally static feedback linearizable. By Proposition 2.3.1, the system $\Sigma$ is flat of differential weight $n+m+1$.

### 2.7.3 Proof of Theorem 2.3.2

*Necessity.* Let us consider a flat control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, of differential weight $n+m+1$. According to Proposition 2.3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=1}^m \tilde{u}_i \tilde{h}_i(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{h}_i(x) \\ \dot{y}_1 &= v_1, \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \le j \le m$, is locally static feedback linearizable, around $(x_0, y_0)$. For simplicity of notation, we will drop the tildes, we will keep distinguishing $g_1$ from $h_j$ (which could also be denoted $g_j$, $2 \le j \le m$) whose controls are not preintegrated.

Since $\Sigma^{(1,0,\cdots,0)}$ is locally static feedback linearizable, $\mathcal{D}_p^i$ is involutive, of constant rank, for any $i \ge 0$, and there exists an integer $\rho$ such that $\operatorname{rk} \mathcal{D}_p^\rho = n+1$. We have

$$\mathcal{D}_p^0 = \operatorname{span}\{\frac{\partial}{\partial y_1}, h_j,\ 2 \le j \le m\}$$

involutive. It follows immediately that

$$\mathcal{H}^0 = \operatorname{span}\{h_j,\ 2 \le j \le m\}$$

is involutive (as intersection of involutive distributions $\mathcal{H}^0 = \mathcal{D}_p^0 \cap TX$) and of corank one in $\mathcal{D}^0$. This shows $(A1)'$. The distribution

$$\mathcal{D}_p^1 = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j + y_1[g_1, h_j],\ 2 \le j \le m\}$$

is involutive and we deduce that $[g_1, h_j] \in \mathcal{D}_p^1$ and $ad_f h_j \in \mathcal{D}_p^1$. Thus

$$\mathcal{D}_p^1 = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j, [g_1, h_j],\ 2 \le j \le m\} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{G}^1 + [f, \mathcal{H}^0],$$

where $\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$.

The involutivity of $\mathcal{D}_p^1$ implies that of $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$, because $\mathcal{H}^1 = \mathcal{D}_p^1 \cap TX$ is the intersection of two involutive distributions. Moreover, $\mathcal{D}_p^1$ has constant rank around $(x_0, y_{10})$, it follows that

$$\mathrm{rk}\,(\mathrm{span}\,\{g_1, h_j, ad_f h_j + y_{10}[g_1, h_j],\, 2 \leq j \leq m\})(x_0) = \mathrm{rk}\,\mathcal{H}^1(x_0).$$

Recall that $k = 0$, i.e., $\mathcal{D}^0 = \mathrm{span}\,\{g_1, h_j,\, 2 \leq j \leq m\}$ is noninvolutive and that the rank of the right hand side could a priori drop at $y_{10}$. From this, it is immediate that $u_0 \notin U_{sing}(x_0)$, where $U_{sing}(x_0) = \{u_0 \in \mathbb{R}^m : \mathrm{rk}\,(\mathrm{span}\,\{g_1, h_j, [f + u_{10}g_1 + \sum_{i=2}^m u_{i0}h_i, h_j], 2 \leq j \leq m\})(x_0) < \mathrm{rk}\,\mathcal{H}^1(x_0)\}$, implying $(RC)$.

The rest of the proof follows the same line as that of Theorem 2.3.1.

*Sufficiency.* Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^m u_i h_i(x)$ satisfying $(A1)' - (A3)'$ and $(RC)$, where the corank one involutive subdistribution is given by $\mathcal{H}^0 = \mathrm{span}\,\{h_j, 2 \leq j \leq m\}$. It is immediate to see that the prolongation

$$\Sigma^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_i = u_2$, for $2 \leq i \leq m$, is locally static feedback linearizable, around $(x_0, y_0)$. Indeed, we have $\mathcal{D}_p^0 = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, h_j,\, 2 \leq j \leq m\}$, which is clearly involutive, and

$$\mathcal{D}_p^1 = \mathrm{span}\,\{\frac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j + y_1[g_1, h_j],\, 2 \leq j \leq m\}.$$

Since $u_0 \notin U_{sing}(x_0)$, we have

$$\mathrm{rk}\,(\mathrm{span}\,\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^m u_i h_i, h_j], 2 \leq j \leq m\})(x_0, u_0) = \mathrm{rk}\,\mathcal{H}^1(x_0),$$

where $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$. Moreover,

$$\mathrm{span}\,\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^m u_i h_i, h_j], 2 \leq j \leq m\}) \subset \mathcal{H}^1.$$

This yields

$$\mathrm{span}\,\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^m u_i h_i, h_j], 2 \leq j \leq m\} = \mathcal{H}^1,$$

around $(x_0, u_0)$, and the involutivity of $\mathcal{H}^0 = \mathrm{span}\,\{h_j,\, 2 \leq j \leq m\}$ implies

$$\mathcal{H}^1 = \mathrm{span}\,\{g_1, h_j, ad_f h_j + u_1[g_1, h_j], 2 \leq j \leq m\},$$

around $(x_0, u_0)$, and thus

$$\mathcal{D}_p^1 = \mathrm{span}\,\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^1.$$

It follows, by induction, that all linearizability distributions $\mathcal{D}_p^i$, associated to $\Sigma^{(1,0,\cdots,0)}$, are of the form

$$\mathcal{D}_p^i \;=\; \text{span}\,\{\tfrac{\partial}{\partial y_1}\} + \mathcal{H}^i,\; i \geq 1.$$

and the involutivity of $\mathcal{H}^i$ implies that of $\mathcal{D}_p^i$. Moreover, $\text{rk}\, H^\rho = n$, thus $\text{rk}\, D_p^\rho = n+1$ and $\Sigma^{(1,0,\cdots,0)}$ is locally static feedback linearizable. By Proposition 2.3.1, the system $\Sigma$ is flat of differential weight $n+m+1$.

### 2.7.4 Proof of Theorem 2.3.4

Before giving the proof of Theorem 2.3.4, notice that under the assumption $\mathcal{D}^i$ involutive, for all $0 \leq i \leq k-1$, we have $\mathcal{D}^{k-2} \subset \mathcal{C}^k$, where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$. We will use that property in our proof.

*Necessity.* Let us consider a flat control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, of differential weight $n+m+1$, and assume $\text{cork}\,(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. We clearly have $k \geq 1$, otherwise the condition $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ would not have any sens. According to Proposition 2.3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing $\Sigma$ into the form $\tilde{\Sigma} \,:\, \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=1}^m \tilde{u}_i \tilde{h}_i(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{h}_i(x) \\ \dot{y}_1 &= v_1, \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \leq j \leq m$, is locally static feedback linearizable, around $(x_0, y_0)$. For simplicity of notation, we will drop the tildes, we will keep distinguishing $g_1$ from $h_j$ (which could also be denoted $g_j$, $2 \leq j \leq m$) whose controls are not preintegrated.

Since $\Sigma^{(1,0,\cdots,0)}$ is locally static feedback linearizable, $\mathcal{D}_p^i$ is involutive, of constant rank, for any $i \geq 0$, and there exists an integer $\rho$ such that $\text{rk}\,\mathcal{D}_p^\rho = n+1$. Since $k \geq 1$, the distribution $\mathcal{D}^0 = \text{span}\,\{g_1, h_j,\, 2 \leq j \leq m\}$ is involutive, thus $[g_1, h_j] \in \mathcal{D}^0$ and hence $\mathcal{D}_p^1 = \text{span}\,\{\frac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j, 2 \leq j \leq m\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq k$,

$$\mathcal{D}_p^i = \text{span}\,\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1} g_1, h_j, \cdots, ad_f^i h_j,\, 2 \leq j \leq m\}.$$

Since the intersection of involutive distributions is an involutive distribution, $\mathcal{D}_p^i \cap TM \;=\; \text{span}\,\{g_1, \cdots, ad_f^{i-1} g_1, h_j, \cdots, ad_f^i h_j,\, 2 \leq j \leq m\}$ is involutive, for $1 \leq i \leq k$. We deduce that the distribution

$$\mathcal{E} = \text{span}\,\{g_1, \cdots, ad_f^{k-1} g_1, h_j, \cdots, ad_f^k h_j,\, 2 \leq j \leq m\}$$

is involutive. Next we will prove that $\mathcal{E}^k = \mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$.

It is immediate that $\mathcal{D}^{k-1} \subset \mathcal{E} \subset \mathcal{D}^k$, where the second inclusion is of corank one, otherwise $\mathcal{E} = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive, which contradicts our hypotheses.

Applying the Jacobi identity, it can be proved that $[ad_f^{k-1}h_j, ad_f^k g_1] \in \mathcal{D}^k$, for all $2 \leq j \leq m$, and since $\mathcal{E}^k$ is involutive, we immediately have $[ad_f^{k-1}h_j, \mathcal{D}^k] \in \mathcal{D}^k$, for $2 \leq j \leq m$. Thus $ad_f^{k-1}h_j \in \mathcal{C}^k$, for all $2 \leq j \leq m$, where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$. Moreover, since $\mathcal{D}^k = \mathcal{E}^k + \operatorname{span}\{ad_f^k g_1\}$ is noninvolutive and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, we deduce that the new direction completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k$ is given by $[ad_f^{k-1}g_1, ad_f^k g_1] \notin \mathcal{D}^k$. Hence there exists smooth functions $\alpha_j$ such that $[ad_f^k h_j, ad_f^k g_1] = \alpha_j [ad_f^{k-1}g_1, ad_f^k g_1] \operatorname{mod} \mathcal{D}^k$, for $2 \leq j \leq m$. It follows $[ad_f^k h_j - \alpha_j ad_f^{k-1} g_1, ad_f^k g_1] = 0 \operatorname{mod} \mathcal{D}^k$. It is easy to show that

$$\mathcal{C}^k = \mathcal{D}^{k-2} + \operatorname{span}\{ad_f^{k-1}h_j, ad_f^k h_j - \alpha_j ad_f^{k-1} g_1,\ 2 \leq j \leq m\}$$

which yields $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1} = \operatorname{span}\{g_1, \cdots, ad_f^{k-1}g_1, h_j, \cdots, ad_f^k h_j\ 2 \leq j \leq m\}$, $\operatorname{rk}\mathcal{C}^k = \operatorname{rk}\mathcal{D}^k - 2$ and $\operatorname{rk}(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = \operatorname{rk}\mathcal{D}^{k-1} - 1$.

The rest of the proof follows the same line as that of Theorem 2.3.1.

*Sufficiency.* Consider a control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$ satisfying (C1)-(C4). We start our proof with the observation that the conditions (C1)-(C2) enable us to define a distribution $\mathcal{H}$ such that $\mathcal{H} \subset \mathcal{D}^0$, of corank one, and $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k \mathcal{H}$.

To this aim, let us denote by $r$ the corank of $\mathcal{D}^{k-2} \subset \mathcal{D}^{k-1}$. Assume that the vector fields $g_i \in \mathcal{D}^0$, for $1 \leq i \leq r$, satisfy

$$\mathcal{D}^{k-1} = \mathcal{D}^{k-2} + \operatorname{span}\{ad_f^{k-1}g_i,\ 1 \leq i \leq r\}.$$

Using similar arguments to those used in the proof of Proposition 2.7.2(ii), we can defined $m-r$ vector fields $h_j$, for $r+1 \leq j \leq m$, such that $\mathcal{D}^0 = \operatorname{span}\{g_1, \cdots, g_r, h_{r+1}, \cdots, h_m\}$ and $ad_f^{k-1}h_j \in \mathcal{D}^{k-2}$, for $r+1 \leq j \leq m$.

It is clear that $\mathcal{D}^{k-2} \subset \mathcal{C}^k$ and since $\operatorname{rk}(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = \operatorname{rk}\mathcal{D}^{k-1} - 1$, we have

$$\mathcal{C}^k \cap \mathcal{D}^{k-1} = \mathcal{D}^{k-2} + \operatorname{span}\{c_j,\ 1 \leq j \leq r-1\},$$

where the vector fields $c_j$ are of the form

$$c_j = \sum_{i=1}^{r} \lambda_j^i ad_f^{k-1} g_i = ad_f^{k-1}\left(\sum_{i=1}^{r} \lambda_j^i g_i\right) \operatorname{mod} \mathcal{D}^{k-2},$$

with $\lambda_j^i$ smooth functions such that the matrix $\Lambda = (\lambda_j^i)$, for $1 \leq i \leq r$ and $1 \leq j \leq r-1$, is of full rank. Denote $h_{j+1} = \sum_{i=1}^r \lambda_j^i g_i$, for $1 \leq j \leq r-1$, and suppose, without loss of generality, that they are independent from $g_1$.

Since $ad_f^{k-1}h_j \in \mathcal{C}^k$, for $2 \leq j \leq m$, we have $[ad_f^{k-1}h_j, \mathcal{D}^k] \subset \mathcal{D}^k$. From this, it can be shown, applying the Jacobi identity, that $[ad_f^{k-1}g_1, ad_f^k h_j] \in \mathcal{D}^k$, for $2 \leq j \leq m$.

Therefore, the new direction completing $\mathcal{D}^k$ to $\overline{\mathcal{D}}^k = \mathcal{D}^k + [\mathcal{D}^{k-1}, \mathcal{D}^k]$ is given by $[ad_f^{k-1}g_1, ad_f^k g_1]$ and there exist smooth functions $\alpha_j$ such that $[ad_f^k h_j, ad_f^k g_1] = \alpha_j [ad_f^{k-1}g_1, ad_f^k g_1] \bmod \mathcal{D}^k$, for $2 \leq j \leq m$. This gives $[ad_f^k h_j - \alpha_j ad_f^{k-1}g_1, ad_f^k g_1] = 0 \bmod \mathcal{D}^k$ and it can be easily verified that the characteristic distribution $\mathcal{C}^k$ is given by

$$\mathcal{C}^k = \mathcal{D}^{k-2} + \text{span}\,\{ad_f^{k-1}h_j, ad_f^k h_j - \alpha_j ad_f^{k-1}g_1,\, 2 \leq j \leq m\}.$$

It follows immediately

$$\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span}\,\{ad_f^k h_j,\, 2 \leq j \leq m\} = \mathcal{D}^{k-1} + ad_f^k \mathcal{H},$$

where the corank one subdistribution $\mathcal{H}$ of $\mathcal{D}^0$ is given by

$$\mathcal{H} = \text{span}\,\{h_j,\, 2 \leq j \leq m\}.$$

The involutivity of $\mathcal{H}^k$ implies that of all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$, for $0 \leq i \leq k-1$, where $\mathcal{D}^{-1}$ is empty and $\mathcal{H}^0 = \mathcal{H}$. The proof of this statement follows by the same method as that used in the proof of Proposition 2.7.2(ii).

We are now in position to show that the control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$ is dynamically linearizable via one-fold prolongation. Transform $\Sigma$ via an invertible static feedback into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=2}^m \tilde{u}_i h_i(x)$, where the vector fields $h_i$ are defined as above. Applying the same arguments as in the proof of Theorem 2.3.1, it is immediate to see that the prolongation

$$\tilde{\Sigma}^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 &= v_1, \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \leq j \leq m$, is locally static feedback linearizable.

### 2.7.5 Proof of Proposition 2.4.1

Proposition 2.7.3 is used to obtain the normal form (NF2), so we will start with its proof.

*Proof of Proposition 2.7.3.* We give the proof of Proposition 2.7.3 only for $k \geq 1$. If $k = 0$, then the same arguments apply. Since $\Sigma$ is dynamically linearizable via invertible one-fold prolongation, it satifies conditions $(A1) - (A3)$ of Theorem 2.3.1. We will show that in the case $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$ and $\mathcal{H}^{k+1} \neq TX$, condition $(A3)$ implies that the inclusion $\mathcal{H}^{k+1} \subset \mathcal{H}^{k+1} + \mathcal{D}^{k+1}$ is of corank one. Since there exists an integer $\rho \geq k+2$ such that $\mathcal{H}^\rho = TX$ and $\mathcal{H}^{k+1} \neq TX$, we clearly have $\mathcal{H}^{k+1} \subsetneq \mathcal{H}^{k+2}$.

If $\overline{\mathcal{D}}^k = \mathcal{H}^{k+1}$, then it follows, from the definition of $\mathcal{H}^{k+1}$, that

$$\overline{\mathcal{D}}^k = \text{span}\,\{g_1, \cdots, ad_f^k g_1, h_j, \cdots, ad_f^{k+1} h_j, 2 \leq j \leq m\}$$

with $\mathcal{H}^{k+1} \neq TX$ and $\mathcal{H} = \text{span}\,\{h_j, 2 \leq j \leq m\}$ defined by Proposition 2.7.2(ii). Thus the distribution $\mathcal{D}_p^{k+1}$ associated to the prolonged system $\Sigma^{(1,0,\cdots,0)}$ is

$$\mathcal{D}_p^{k+1} = \text{span}\,\{\frac{\partial}{\partial y_1}\} + \overline{\mathcal{D}}^k.$$

Assume also $ad_f^{k+1} g_1 \in \mathcal{H}^{k+1}$, if not, the inclusion in question is, indeed, of corank one. Hence for any vector field $\xi \in \mathcal{D}^k$, we have $[f, \xi] \in \overline{\mathcal{D}}^k$. By successive application of the Jacobi identity, it follows immediately that $\overline{\mathcal{D}}^k + [f, \overline{\mathcal{D}}^k] = \overline{\mathcal{D}}^k$.

Therefore, for the prolonged system we obtain

$$\mathcal{D}_p^{k+2} = \text{span}\,\{\frac{\partial}{\partial y_1}\} + \overline{\mathcal{D}}^k + [f, \overline{\mathcal{D}}^k] = \mathcal{D}_p^{k+1},$$

and we deduce $\mathcal{H}^{k+1} = \mathcal{H}^{k+2}$, which gives a contradiction.

*Proof of Proposition 2.4.1.*

We will prove the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Consider an $x$-flat control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, of differential weight $n + m + 1$ and let $(\varphi_1, \cdots, \varphi_m)$ be a minimal flat output, defined in a neighborhood of $x_0$. It can be shown (see proof of Proposition 2.3.1 in Appendices 2.A) that the system $\Sigma$ can be transformed by a change of coordinates and an invertible static feedback, around $z_0$, into the form

$$\begin{array}{rclrcl}
\dot{z}_1^i & = & z_2^i & \dot{z}_1^j & = & z_2^j \\
 & \vdots & & & \vdots & \\
\dot{z}_{\rho_i - 1}^i & = & z_{\rho_i}^i & \dot{z}_{\rho_j - 1}^j & = & z_{\rho_j}^j \\
\dot{z}_{\rho_i}^i & = & \tilde{u}_i & \dot{z}_{\rho_j}^j & = & a_j(z) + b_j(z)\tilde{u}_1 \\
 & & & \dot{z}_{\rho_j + 1}^j & = & \tilde{u}_j
\end{array}$$

for $1 \leq i \leq r$ and $r + 1 \leq j \leq m$, where $\sum_{i=1}^m \rho_i + m - r = n$, $(\varphi_1, \cdots, \varphi_m) = (z_1^1, \cdots, z_1^m)$, the functions $a_j$ and $b_j$ are smooth such that $\mathcal{D}^k$ is noninvolutive and satisfying the following condition: $\text{rk}\, D(z_0, \tilde{u}_0) = m - r$, where $D$ stands for the Jacobi matrix $D_{jl} = \frac{\partial(a_j + \tilde{u}_1 b_j)}{\partial z_{\rho_l + 1}^l}$, for $r + 1 \leq j, l \leq m$. Since $\text{rk}\, D(z_0, \tilde{u}_0) = m - r$, it is immediate that we can always normalize $a_j$ ou $b_j$ to $z_{\rho_l + 1}^l$.

If $k = 0$, then there exist integers $s$ and $q$, where $2 \leq s \leq m$ and $r + 1 \leq q \leq m$, such that $\frac{\partial b_q}{\partial z_{\rho_s}^s} \neq 0$, if $2 \leq s \leq r$, or $\frac{\partial b_q}{\partial z_{\rho_s + 1}^s} \neq 0$, if $r + 1 \leq s \leq m$. This is the form (NF1), for $k = 0$.

Let us now consider the case $k \geq 1$. We have $\frac{\partial b_j}{\partial z_{\rho_i}^i} = 0$, for $2 \leq i \leq r$, and $\frac{\partial b_j}{\partial z_{\rho_l + 1}^l} = 0$, for $r + 1 \leq l \leq m$. It follows that $D_{jl} = \frac{\partial a_j}{\partial z_{\rho_l + 1}^l}$, for $r + 1 \leq j, l \leq m$, thus $\text{rk}\, D$ is calculated at $z_0$ only and equals $m - r$. Hence we can introduce local coordinates $\tilde{z}_{\rho_j + 1}^j = a_j(z)$ (to simplicity we will drop the tilde) and apply a suitable invertible feedback that brings the system into the form

$$\begin{array}{lcllcl}
\dot{z}_1^i & = & z_2^i & \dot{z}_1^j & = & z_2^j \\
 & \vdots & & & \vdots & \\
\dot{z}_{\rho_i-1}^i & = & z_{\rho_i}^i & \dot{z}_{\rho_j-1}^j & = & z_{\rho_j}^j \\
\dot{z}_{\rho_i}^i & = & \tilde{u}_i & \dot{z}_{\rho_j}^j & = & z_{\rho_j+1}^j + b_j(z)\tilde{u}_1 \\
 & & & \dot{z}_{\rho_j+1}^j & = & \tilde{u}_j
\end{array}$$

for $1 \leq i \leq r$ and $r+1 \leq j \leq m$, with $(\varphi_1, \cdots, \varphi_m) = (z_1^1, \cdots, z_1^m)$. Since $\operatorname{rk} \mathcal{D}^k - \operatorname{rk} \mathcal{D}^{k-1} \geq 2$ and $\mathcal{D}^k$ is noninvolutive, a direct computation shows that there exists at least one integer $r+1 \leq j \leq m$ such that $\rho_j \geq k+1$, if $\rho_1 \geq k+1$, or at least two integers $r+1 \leq j \leq m$ such that $\rho_j \geq k+1$, if $\rho_1 \leq k$.

Suppose, without loss of generality, $\rho_{r+1} \geq \rho_{r+2} \geq \cdots \geq \rho_p \geq k+1 \geq k \geq \rho_{p+1} \geq \cdots \geq \rho_m$, where $r+1 \leq p \leq m$, (if $p = m$, the length of all chains $z^j$ is greater than $k+2$). We next prove that we can always assume that $\rho_1 \geq k+1$. Indeed, if $\rho_1 \leq k$, the noninvolutive distribution $\mathcal{D}^k$ is given by

$$\mathcal{D}^k = \mathcal{D}^{k-1} + \operatorname{span}\{\frac{\partial}{\partial z_{\rho_i-k}^i}, \frac{\partial}{\partial z_{\rho_j-k+1}^j}, \sum_{j=r+1}^{p} b_j \frac{\partial}{\partial z_{\rho_j-k}^j}, r+1 \leq j \leq p, \rho_i \geq k+1, 2 \leq i \leq r\}$$

and since the rank of $\mathcal{D}^k$ is constant in a neighborhood of $z_0$, it follows that there exists at least one integer $r+1 \leq s \leq p$ such that $b_s(z_0) \neq 0$. We apply the invertible static feedback transformation $v_1 = z_{\rho_s+1}^s + b_s(z)\tilde{u}_1$ and $v_i = \tilde{u}_i$, for $2 \leq i \leq m$ and $i \neq s$, to get

$$\begin{array}{llllllll}
\dot{z}_1^1 & = z_2^1 & \dot{z}_1^i & = z_2^i & \dot{z}_1^s & = z_2^s & \dot{z}_1^j & = z_2^j \\
\vdots & & \vdots & & \vdots & & \vdots & \\
\dot{z}_{\rho_1}^1 & = -\frac{1}{b_s} z_{\rho_s+1}^s + \frac{1}{b_s} v_1 & \dot{z}_{\rho_i}^i & = v_i & \dot{z}_{\rho_s}^s & = v_1 & \dot{z}_{\rho_j}^j & = z_{\rho_j+1}^j - \frac{b_j}{b_s} z_{\rho_s+1}^s + \frac{b_j}{b_s} v_1 \\
 & & & & \dot{z}_{\rho_s+1}^s & = v_s & \dot{z}_{\rho_j+1}^j & = v_j
\end{array}$$

for $2 \leq i \leq r$, $r+1 \leq j \leq m$, with $j \neq s$ and $(\varphi_1, \cdots, \varphi_m) = (z_1^1, \cdots, z_1^m)$. The following change of coordinates

$$\begin{array}{ll}
\tilde{z}_l^1 & = z_l^s, \ 1 \leq l \leq \rho_s, \\
\tilde{z}_l^i & = z_l^i, \ 2 \leq i \leq r, 1 \leq l \leq \rho_i, \\
\tilde{z}_l^s & = z_l^1, \ 1 \leq l \leq \rho_1, \\
\tilde{z}_{\rho_s+1}^s & = -\frac{1}{b_s} z_{\rho_s+1}^s \\
\tilde{z}_l^j & = z_l^j, \ r+1 \leq j \leq m, j \neq s, 1 \leq l \leq \rho_j, \\
\tilde{z}_{\rho_j+1}^j & = z_{\rho_j+1}^j - \frac{b_j}{b_s} z_{\rho_s+1}^s, \ r+1 \leq j \leq m, j \neq s
\end{array}$$

is valid and after applying a suitable invertible static feedback, transforms the system into

$$\begin{array}{rclrcl}
\dot{\tilde{z}}_1^i & = & \tilde{z}_2^i & \dot{\tilde{z}}_1^j & = & \tilde{z}_2^j \\
& \vdots & & & \vdots & \\
\dot{\tilde{z}}_{\tilde{\rho}_i-1}^i & = & \tilde{z}_{\tilde{\rho}_i}^i & \dot{\tilde{z}}_{\tilde{\rho}_j-1}^j & = & \tilde{z}_{\tilde{\rho}_j}^j \\
\dot{\tilde{z}}_{\tilde{\rho}_i}^i & = & \tilde{v}_i & \dot{\tilde{z}}_{\tilde{\rho}_j}^j & = & \tilde{z}_{\tilde{\rho}_j+1}^j + b_j(\tilde{z})\tilde{v}_1 \\
& & & \dot{\tilde{z}}_{\tilde{\rho}_j+1}^j & = & \tilde{v}_j
\end{array}$$

for $1 \leq i \leq r$ and $r+1 \leq j \leq m$, with $\tilde{\rho}_1 = \rho_s \geq k+1$ and $(\varphi_1, \cdots, \varphi_m)$ playing the role of top variables. Hence we can always assume that $\rho_1 \geq k+1$. Recall that we assumed $\rho_{r+1} \geq \rho_{r+2} \geq \cdots \geq \rho_p \geq k+1 \geq k \geq \rho_{p+1} \geq \cdots \geq \rho_m$.

The involutivity of $\mathcal{D}^l$, for $0 \leq l \leq \rho_m - 2$, implies that all functions $b_j$, for $r+1 \leq j \leq m$ depend on $\bar{z}^i_{\rho_i-l-1}$, $\bar{z}^j_{\rho_j-l}$, for $2 \leq i \leq r$, $j = 1$ and $r+1 \leq j \leq m$. We have

$$\mathcal{D}^{\rho_m-1} = \operatorname{span}\Big\{ \frac{\partial}{\partial z^1_{\rho_1}}, \cdots, \frac{\partial}{\partial z^1_{\rho_1-\rho_m+2}}, \frac{\partial}{\partial z^i_{\rho_i}}, \cdots, \frac{\partial}{\partial z^i_{\rho_i-\rho_m+1}}, \frac{\partial}{\partial z^j_{\rho_j+1}}, \cdots, \frac{\partial}{\partial z^j_{\rho_j-\rho_m+2}},$$
$$\frac{\partial}{\partial z^1_{\rho_1-\rho_m+1}} + b_m \frac{\partial}{\partial z^m_1} + \textstyle\sum_{j=r+1}^{m-1} b_j \frac{\partial}{\partial z^j_{\rho_j-\rho_m+1}}, \ 2 \leq i \leq r,\ r+1 \leq j \leq m \Big\}.$$

Since $\mathcal{D}^{\rho_m-1}$ is involutive, it follows that $b_m$ is a function of $\bar{z}^i_{\rho_i-\rho_m}$, $\bar{z}^j_{\rho_j-\rho_m+1}$, for $2 \leq i \leq r$, $j = 1$ and $r+1 \leq j \leq m$. The only $z^m$-coordinate that can be involved in the expression of $b_m$ is $z^m_1$. Moreover, $b_m$ is no longer present in the expression of $\mathcal{D}^i$, for $i \geq \rho_m$. Therefore, the involutivity of $\mathcal{D}^i$, for $\rho_m \leq i \leq k$, does not imply any additional condition on $b_m$.

In the same way, by induction, it can be shown that all functions $b_s$, for $r+1 \leq s \leq p$, depend on $\bar{z}^i_{\rho_i-k}$, $\bar{z}^j_{\rho_j-k+1}$, for $2 \leq i \leq r$, $j = 1$ and $r+1 \leq j \leq m$, respectively all functions $b_t$, for $p+1 \leq t \leq m$, depend on $\bar{z}^i_{\rho_i-\rho_t}$, $\bar{z}^j_{\rho_j-\rho_t+1}$, for $2 \leq i \leq r$, $j = 1$ and $r+1 \leq j \leq t$.

The noninvolutivity of the distribution

$$\mathcal{D}^{k} = \operatorname{span}\Big\{ \frac{\partial}{\partial z^1_{\rho_1}}, \cdots, \frac{\partial}{\partial z^1_{\rho_1-k+1}}, \frac{\partial}{\partial z^i_{\rho_i}}, \cdots, \frac{\partial}{\partial z^i_{\rho_i-k}}, \frac{\partial}{\partial z^j_{\rho_j+1}}, \cdots, \frac{\partial}{\partial z^j_{\rho_j-k+1}},$$
$$\frac{\partial}{\partial z^1_{\rho_1-k}} + \textstyle\sum_{j=r+1}^{p} b_j \frac{\partial}{\partial z^j_{\rho_j-k}}, \ 2 \leq i \leq r,\ r+1 \leq j \leq m \Big\}.$$

yields the existence of some integers $s$, $i$ and $j$, such that $r+1 \leq s \leq p$, $2 \leq i \leq r$, $j = 1$ or $r+1 \leq j \leq m$, satisfying $\left(\frac{\partial b_s}{\partial z^i_{\rho_i-k}}, \frac{\partial b_s}{\partial z^j_{\rho_j-k+1}}\right) \neq (0,0)$.

$(ii) \Rightarrow (iii)$. If $k = 0$, then (NF1) and (NF2) coincide. We can thus suppose $k \geq 1$. Since $\rho_j \geq k+1$, for $j = 1$ and $r+1 \leq j \leq p$, there exist integers $\mu_j \geq 1$ such that $\rho_j = \mu_j + k$, for $j = 1$ and $r+1 \leq j \leq p$. We distinguish two cases: $k = 1$ and $k \geq 2$.

Let us first assume $k = 1$. Since $1 \leq \rho_l \leq k$, for $p+1 \leq l \leq m$, it follows that

$\rho_l = 1$. Using the above notations (NF1) is given by

$$\begin{array}{llll} \dot{z}_1^1 = z_2^1 & \dot{z}_1^i = z_2^i & \dot{z}_1^s = z_2^s & \\ \vdots & \vdots & \vdots & \\ \dot{z}_{\mu_1+1}^1 = \tilde{u}_1 & \dot{z}_{\rho_i}^i = \tilde{u}_i & \dot{z}_{\mu_s+1}^s = z_{\mu_s+2}^s + b_s\tilde{u}_1 & \dot{z}_1^l = z_2^l + b_l\tilde{u}_1 \\ & & \dot{z}_{\mu_s+2}^s = \tilde{u}_s & \dot{z}_2^l = v_l \end{array}$$

where all functions $b_s$, for $r+1 \le s \le m$, depend on $z_{\rho_i}^i$, $z_{\mu_j+1}^j$ and $z_1^l$, for $2 \le i \le r$, $j = 1$ and $r+1 \le j \le p$ and $p+1 \le l \le m$.

Since the vector field $g_1 = \frac{\partial}{\partial z_{\mu_1+1}^1} + \sum_{s=r+1}^m b_s \frac{\partial}{\partial z_{\rho_s}^s}$ is non zero, there exists smooth independent functions $\psi_s$, for $r+1 \le s \le m$, depending on $\bar{z}_{\rho_i-1}^i$, $\bar{z}_{\mu_j+1}^j$ and $z_1^l$, for $2 \le i \le r$, $j = 1$ and $r+1 \le j \le p$ and $p+1 \le l \le m$, such that $L_{g_1}\psi_s = 0$ and the matrix given by $(\frac{\partial\psi_s}{\partial z_{\mu_q+2}^q})$, for $r+1 \le s,q \le m$, is of full rank, where $\mu_q = 0$, for $p+1 \le q \le m$. We introduce new coordinates $\tilde{z}_{\mu_q+1}^q = \psi_q$, $\tilde{z}_{\mu_q+2}^q = L_f\psi_q$, for $r+1 \le q \le m$, where $\mu_q = 0$, for $p+1 \le q \le m$, and apply a suitable invertible static feedback bringing (NF1) into

$$\begin{array}{llll} \dot{z}_1^1 = z_2^1 & \dot{z}_1^i = z_2^i & \dot{z}_1^s = z_2^s & \\ \vdots & \vdots & \vdots & \\ \dot{z}_{\mu_1}^1 = z_{\mu_1+1}^1 & \dot{z}_{\rho_i-1}^i = z_{\rho_i}^i & \dot{z}_{\mu_s}^s = d_s(z) & \\ \dot{z}_{\mu_1+1}^1 = v_1 & \dot{z}_{\rho_i}^i = v_i & \dot{\tilde{z}}_{\mu_s+1}^s = \tilde{z}_{\mu_s+2}^s & \dot{\tilde{z}}_1^l = \tilde{z}_2^l \\ & & \dot{\tilde{z}}_{\mu_s+2}^s = \tilde{u}_s & \dot{\tilde{z}}_2^l = v_l \end{array}$$

all smooth functions $d_s$ involve only $\bar{z}_{\rho_i-1}^i$, $\bar{z}_{\mu_j+1}^j$ and $z_1^l$, for $2 \le i \le r$, $j = 1$ and $r+1 \le j \le p$ and $p+1 \le l \le m$, and are such that the matrix $(\frac{\partial d_s}{\partial z_{\mu_t+1}^t})$, for $r+1 \le s,t \le p$, is of full rank.

Since $\mathcal{D}^1 = \text{span}\{\frac{\partial}{\partial z_{\mu_1+1}^1}, \frac{\partial}{\partial z_{\rho_i-1}^i}, \frac{\partial}{\partial z_{\rho_i}^i}, \frac{\partial}{\partial \tilde{z}_{\mu_j+1}^j}, \frac{\partial}{\partial \tilde{z}_{\mu_j+2}^j}, \frac{\partial}{\partial \tilde{z}_1^l}, \frac{\partial}{\partial \tilde{z}_2^l}, \frac{\partial}{\partial z_{\mu_1}^1} + \sum_{s=r+1}^p \frac{\partial d_s}{\partial z_{\mu_1+1}^1}\frac{\partial}{\partial z_{\mu_s}^s}$, for $2 \le i \le r$, $r+1 \le j \le p$ and $p+1 \le l \le m\}$ is noninvolutive, it follows that there exist $s$, $i$, $j$ and $l$, with $r+1 \le j \le m$, $2 \le i \le r$, $r+1 \le s \le p$, $j = 1$ or $r+1 \le j \le m$, such that $(\frac{\partial^2 d_s}{\partial z_{\rho_i-1}^i \partial z_{\mu_1+1}^1}, \frac{\partial^2 d_s}{\partial \tilde{z}_{\mu_j+1}^j \partial z_{\mu_1+1}^1}, \frac{\partial^2 d_s}{\partial \tilde{z}_1^l \partial z_{\mu_1+1}^1}) \neq (0,0,0)$, where $\tilde{z}_{\mu_1+1}^1$ stands for $z_{\mu_1+1}^1$. This is the normal form (NF2) for $k = 1$.

Let us now suppose $k \ge 2$. Using the notations $\rho_j = \mu_j + k$, for $j = 1$ and $r+1 \le j \le p$, we have:

$$(NF1)\left\{\begin{array}{llll} \dot{z}_1^1 = z_2^1 & \dot{z}_1^i = z_2^i & \dot{z}_1^s = z_2^s & \dot{z}_1^l = z_2^l \\ \vdots & \vdots & \vdots & \vdots \\ \dot{z}_{\mu_1+k}^1 = \tilde{u}_1 & \dot{z}_{\rho_i}^i = \tilde{u}_i & \dot{z}_{\mu_s+k}^s = z_{\mu_s+k+1}^s + b_s\tilde{u}_1 & \dot{z}_{\rho_l}^l = z_{\rho_l+1}^l + b_l\tilde{u}_1 \\ & & \dot{z}_{\mu_s+k+1}^s = \tilde{u}_s & \dot{z}_{\rho_l+1}^l = \tilde{u}_l \end{array}\right.$$

where $\rho_{r+1} \ge \rho_{r+2} \ge \cdots \ge \rho_p \ge k+1 \ge k \ge \rho_{p+1} \ge \cdots \ge \rho_m$, for $r+1 \le p \le m$. All functions $b_s$, for $r+1 \le s \le p$, depend on $\bar{z}_{\rho_i-k}^i$, $\bar{z}_{\mu_j+1}^j$ and $\bar{z}_{\rho_l-k+1}^l$, for $2 \le i \le r$,

$p+1 \leq l \leq m$, $j=1$ and $r+1 \leq j \leq p$ and all functions $b_t$, for $p+1 \leq t \leq m$, depend on $\bar{z}^i_{\rho_i-\rho_t}$, $\bar{z}^j_{\mu_j+k-\rho_t+1}$ and $\bar{z}^l_{\rho_l-\rho_t+1}$, where $2 \leq i \leq r$, $p+1 \leq l \leq t$, $j=1$ and $r+1 \leq j \leq p$.

The proof of this case (for which we give only the main ideas) consists in redressing all involutive distributions $\mathcal{D}^i$, for $0 \leq i \leq k-1$. To this end, we first eliminate $\tilde{u}_1$ from the equations of $\dot{z}^s_{\mu_s+k}$ and $\dot{z}^l_{\rho_l}$, by introducing the following change of coordinates (which is clearly valid in a neighborhood of $z_0$):

$$\begin{aligned} \tilde{z}^s_{\mu_s+k} &= z^s_{\mu_s+k} - b_s z^1_{\mu_1+k} & \tilde{z}^l_{\rho_l} &= z^l_{\rho_l} - b_l z^1_{\mu_1+k} \\ \tilde{z}^s_{\mu_s+k+1} &= L_f \tilde{z}^s_{\mu_s+k} & \tilde{z}^l_{\rho_l+1} &= L_f \tilde{z}^l_{\rho_l} \end{aligned}$$

for $r+1 \leq s \leq p$ and $p+1 \leq l \leq m$. Applying a suitable invertible static feedback transformation, we get:

$$\begin{array}{llll} \dot{z}^1_1 = z^1_2 & \dot{z}^i_1 = z^i_2 & \dot{z}^s_1 = z^s_2 & \dot{z}^l_1 = z^l_2 \\ \quad\vdots & \quad\vdots & \quad\vdots & \quad\vdots \\ \dot{z}^1_{\mu_1+k-1} = z^1_{\mu_1+k} & \dot{z}^i_{\rho_i-1} = z^i_{\rho_i} & \dot{z}^s_{\mu_s+k-1} = \tilde{z}^s_{\mu_s+k} + b_s z^1_{\mu_1+k} & \dot{z}^l_{\rho_l-1} = \tilde{z}^l_{\rho_l} + b_l z^1_{\mu_1+k} \\ \dot{z}^1_{\mu_1+k} = v_1 & \dot{z}^i_{\rho_i} = v_i & \dot{\tilde{z}}^s_{\mu_s+k} = \tilde{z}^s_{\mu_s+k+1} & \dot{\tilde{z}}^l_{\rho_l} = \tilde{z}^l_{\rho_l+1} \\ & & \dot{\tilde{z}}^s_{\mu_s+k+1} = v_s & \dot{\tilde{z}}^l_{\rho_l+1} = v_l \end{array}$$

Next we eliminate $z^1_{\mu_1+k}$ from the equations of $\dot{z}^s_{\mu_s+k-1}$ and $\dot{z}^l_{\rho_l-1}$, by applying a similar change of coordinates (that we will also denote by $\tilde{z}$):

$$\begin{aligned} \tilde{z}^s_{\mu_s+k-1} &= z^s_{\mu_s+k-1} - b_s z^1_{\mu_1+k-1} & \tilde{z}^l_{\rho_l-1} &= z^l_{\rho_l-1} - b_l z^1_{\mu_1+k-1} \\ \tilde{z}^s_{\mu_s+k} &= L_f \tilde{z}^s_{\mu_s+k-1} & \tilde{z}^l_{\rho_l} &= L_f \tilde{z}^l_{\rho_l-1} \\ \tilde{z}^s_{\mu_s+k+1} &= L_f^2 \tilde{z}^s_{\mu_s+k-1} & \tilde{z}^l_{\rho_l+1} &= L_f^2 \tilde{z}^l_{\rho_l-1} \end{aligned}$$

for $r+1 \leq s \leq p$ and $p+1 \leq l \leq m$. Then we repeat this process $\rho_m - 3$ times transforming the $z^s$ and $z^l$-chains into

$$\begin{array}{ll} \dot{z}^s_1 = z^s_2 & \dot{z}^l_1 = z^l_2 \\ \quad\vdots & \quad\vdots \\ \dot{z}^s_{\mu_s+k-\rho_m+1} = \tilde{z}^s_{\mu_s+k-\rho_m+2} + b_s z^1_{\mu_1+k-\rho_m+2} & \dot{z}^l_{\rho_l-\rho_m+1} = \tilde{z}^l_{\rho_l-\rho_m+2} + b_l z^1_{\mu_1+k-\rho_m+2} \\ \dot{\tilde{z}}^s_{\mu_s+k-\rho_m+2} = \tilde{z}^s_{\mu_s+k-\rho_m+3} & \dot{\tilde{z}}^l_{\rho_l-\rho_m+2} = \tilde{z}^l_{\rho_l-\rho_m+3} \\ \quad\vdots & \quad\vdots \\ \dot{\tilde{z}}^s_{\mu_s+k} = \tilde{z}^s_{\mu_s+k+1} & \dot{\tilde{z}}^l_{\rho_l} = \tilde{z}^l_{\rho_l+1} \\ \dot{\tilde{z}}^s_{\mu_s+k+1} = \tilde{v}_s & \dot{\tilde{z}}^l_{\rho_l+1} = \tilde{v}_l \end{array}$$

with $z^1$-chain and $z^i$-chains remaining unchanged. The function $b_m$, associated to the $z^m$-chain, depends on $\bar{z}^i_{\rho_i-\rho_m}$, $\bar{z}^j_{\mu_j+k-\rho_m+1}$ and $\bar{z}^l_{\rho_l-\rho_m+1}$, for $2 \leq i \leq r$, $p+1 \leq l \leq m$, $j=1$ and $r+1 \leq j \leq p$. It is clear that for the $z^m$-chain, the nonlinearities have been "pushed" to the last possible level, the top equation being $\dot{z}^m_1 = \tilde{z}^m_2 + b_m z^1_{\mu_1+k-\rho_m+2}$.

Since the vector field $\xi = \frac{\partial}{\partial z^1_{\mu_1+k-\rho_m+1}} + \sum_{s=r+1}^{p} b_s \frac{\partial}{\partial z^s_{\mu_s+k-\rho_m+1}} + \sum_{l=p+1}^{m} b_l \frac{\partial}{\partial z^l_{\rho_l-\rho_m+1}}$ is non zero, there exists a smooth function $\psi_m$, depending on the same variables as $b_m$, such that $L_{\xi}\psi_m = 0$ and $\frac{\partial \psi_m}{\partial z_1^m}(z_0) \neq 0$. By introducing $\hat{z}_1^m = \psi_m$, $\hat{z}_2^m = L_f\psi_m, \cdots$, $\hat{z}_{\rho_m}^m = L_f^{\rho_m - 1}\psi_m$ and applying a suitable feedback transformation, we linearize the $z^m$-chain.

By repeating this argument for all $z^l$-chains, for $p+1 \leq l \leq m$, all distributions $\mathcal{D}^i$, for $0 \leq i \leq \rho_p$, can be redressed. We continue to redress the remaining involutive distributions $\mathcal{D}^i$, for $\rho_p + 1 \leq i \leq k-1$, by transforming the $z^s$-chains in a very similar way (that we will not detail here). Finally, we will get

$$
\begin{array}{llllllll}
\dot{z}_1^1 & = z_2^1 & \dot{z}_1^i & = z_2^i & \dot{z}_1^s & = z_2^s & \dot{\hat{z}}_1^l & = \hat{z}_2^l \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & \dot{z}_{\mu_s - 1}^s & = z_{\mu_s}^s & & \\
 & \vdots & & \vdots & \dot{z}_{\mu_s}^s & = d_s(z) & & \vdots \\
 & & & & \dot{\hat{z}}_{\mu_s+1}^s & = \hat{z}_{\mu_s+2}^s & & \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
\dot{z}_{\mu_1+k-1}^1 & = z_{\mu_1+k}^1 & \dot{z}_{\rho_i-1}^i & = z_{\rho_i}^i & \dot{\hat{z}}_{\mu_s+k-1}^s & = \hat{z}_{\mu_s+k}^s & \dot{\hat{z}}_{\rho_l-1}^l & = \hat{z}_{\rho_l}^l \\
\dot{z}_{\mu_1+k}^1 & = \hat{v}_1 & \dot{z}_{\rho_i}^i & = \hat{v}_i & \dot{\hat{z}}_{\mu_s+k}^s & = \hat{z}_{\mu_s+k+1}^s & \dot{\hat{z}}_{\rho_l}^l & = \hat{z}_{\rho_l+1}^l \\
 & & & & \dot{\hat{z}}_{\mu_s+k+1}^s & = \hat{v}_s & \dot{\hat{z}}_{\rho_l+1}^l & = \hat{v}_l
\end{array}
$$

where the functions $d_s$ involve only $\bar{z}_{\rho_i-k}^i, \bar{z}_{\mu_j+1}^j, \bar{z}_{\rho_l+1-k}^l$, for $2 \leq i \leq r$, $p+1 \leq l \leq m$, $j = 1$ and $r+1 \leq j \leq p$, and are such that the matrix given by $(\frac{\partial d_s}{\partial \hat{z}_{\mu_t+1}^t})$, for $r+1 \leq s, t \leq p$, is of full rank at $z_0$.

A simple computation shows that the noninvolutivity of $\mathcal{D}^k$ implies the existence of some integers $s$, $i$, $j$ and $l$, with $2 \leq i \leq r$, $r+1 \leq s \leq p$, $j=1$ and $r+1 \leq j \leq p$, $p+1 \leq l \leq m$, such that $(\frac{\partial^2 d_s}{\partial z_{\rho_i-k}^i \partial z_{\mu_1+1}^1}, \frac{\partial^2 d_s}{\partial \hat{z}_{\mu_j+1}^j \partial z_{\mu_1+1}^1}, \frac{\partial^2 d_s}{\partial \bar{z}_1^l \partial z_{\mu_1+1}^1}) \neq (0,0,0)$, where $\hat{z}_{\mu_1+1}^1$ stands for $z_{\mu_1+1}^1$.

We have obtained (NF2), for $k \geq 2$.

$(iii) \Rightarrow (ii)$. Consider a control system $\Sigma$ static feedback equivalent to the normal form (NF2). It is clear that the system is flat, with $\varphi = (w_1, w_2, w_3)$ a minimal flat output of differential weight $n+m+1$.

### 2.7.6 Proof of Proposition 2.5.1

Before giving the proof of Proposition 2.5.1 we will show that for both cases, $k=0$ and $k \geq 1$, we can identify an involutive distribution $\mathcal{E}$ in $\mathcal{C} \cap \mathcal{H}^0$ (where $\mathcal{C}$ is the characteristic distribution of $\mathcal{D}^0$) of corank $r_1 - 1$ in $\mathcal{H}^0$, with $r_1 = \mathrm{cork}\,(\mathcal{H}^0 \subset \mathcal{H}^1)$, such that $[f, \mathcal{E}] \in \mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$. For $k \geq 1$, we have $\mathcal{C} = \mathcal{D}^0$, $\mathcal{C} \cap \mathcal{H}^0 = \mathcal{H}^0$ and $\mathcal{G}^1 = \mathcal{D}^0$.

Let us first consider the case $k=0$. Recall that we can construct the following sequence of inclusions of involutive distributions:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^{\rho-1} \subset \mathcal{H}^{\rho} = TX,$$

where $\mathcal{H}^0$ is an involutive corank one subdistribution of $\mathcal{D}^0$, $\mathcal{H}^1$ is defined by $\mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$, with $\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$, and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $1 \leq i \leq \rho - 1$, and $\rho$ is the smallest integer such that $\mathcal{H}^\rho = TX$. We denote by $r_i$ the corank of the inclusion $\mathcal{H}^{i-1} \subset \mathcal{H}^i$, for $i \geq 0$. We clearly have $m \geq r_1 \geq r_2 \geq \cdots \geq r_q \geq 1$.

We will show that we can identify an involutive subdistribution $\mathcal{E}$ in $\mathcal{C}$, the characteristic distribution of $\mathcal{D}^0$, of corank $r_1 - 1$ in $\mathcal{H}^0$ and such that $[f, \mathcal{E}] \in \mathcal{G}^1$.

Let $\mathcal{D}^0 = \operatorname{span}\{g_1, h_2, \cdots, h_m\}$, where $\mathcal{H}^0 = \operatorname{span}\{h_2, \cdots, h_m\}$. Assume $\operatorname{rk} \mathcal{G}^1 = m + p - 1$, with $p - 1 \geq 2$ (this is due to the assumption cork $(\mathcal{D}^0 \subset [\mathcal{D}^0, \mathcal{D}^0]) \geq 2$). By permuting $h_j$, $2 \leq j \leq m$, we can suppose that the vector fields $[g_1, h_i]$, for $2 \leq i \leq p$, are independent and satisfy $[g_1, h_i] \notin \mathcal{D}^0$, for $2 \leq i \leq p$. For $1 \leq j \leq m - p$, there exist some smooth functions $\alpha_j^i$ such that

$$[g_1, h_{p+j}] = \sum_{i=2}^{p} \alpha_j^i [g_1, h_i] \bmod \mathcal{D}^0.$$

From this, we deduce

$$[g_1, h_{p+j} - \sum_{i=2}^{p} \alpha_j^i h_i] = 0 \bmod \mathcal{D}^0$$

and it is immediate that the vector fields $\tilde{h}_{p+j} = h_{p+j} - \sum_{i=2}^{p} \alpha_j^i h_i$, for $1 \leq j \leq m - p$, are characteristic for $\mathcal{D}^0$, i.e., $[\tilde{h}_{p+j}, \mathcal{D}^0] \subset \mathcal{D}^0$, and $\mathcal{C} = \operatorname{span}\{\tilde{h}_{p+1}, \cdots, \tilde{h}_m\}$. We have $\mathcal{C} \subset \mathcal{H}^0$ and the corank of this inclusion is $p - 1$. Let us now assume $\operatorname{rk}(\mathcal{G}^1 + [f, \mathcal{C}]) = m + p - 1 + q$, i.e., there are $q$ vector fields of the form $[f, c]$, $c \in \mathcal{C}$, independent modulo $\mathcal{G}^1$. Since $\mathcal{G}^1 + [f, \mathcal{C}] \subset \mathcal{H}^1 = \mathcal{G}^1 + [f, \mathcal{H}^0]$ and $\operatorname{rk} \mathcal{H}^1 = m - 1 + r_1$, we obviously have $p + q \leq r_1 \leq m$. Suppose that the vector fields $[g_1, h_{p+s}]$, for $1 \leq s \leq q$, are independent and $[g_1, h_{p+s}] \notin \mathcal{G}^1$, for $1 \leq s \leq q$. There exist smooth functions $\hat{\alpha}_l^s$ such that, for $1 \leq l \leq m - (p + q)$,

$$[f, h_{p+q+l}] = \sum_{s=1}^{q} \beta_l^s [f, h_{p+s}] \bmod \mathcal{G}^1,$$

implying

$$[f, h_{p+q+l} - \sum_{s=1}^{q} \beta_l^s h_{p+s}] = 0 \bmod \mathcal{G}^1.$$

Thus the vector fields $\tilde{h}_{p+q+l} = h_{p+q+l} - \sum_{s=1}^{q} \beta_l^s h_{p+s}$ are in $\mathcal{C}$ and verify $[f, \tilde{h}_{p+q+l}] \in \mathcal{G}^1$. Put $\mathcal{E} = \operatorname{span}\{\tilde{h}_{p+q+1}, \cdots, \tilde{h}_m\}$. The distribution $\mathcal{E}$ satisfies $[f, \mathcal{E}] \in \mathcal{G}^1$.

We can always assume, without loss of generality, that $\mathcal{H}^0 = \operatorname{span}\{h_2, \cdots, h_p, h_{p+1}, \cdots, h_{p+q}, h_{p+q+1}, \cdots, h_m\}$, where $\mathcal{C} = \operatorname{span}\{h_{p+1}, \cdots, h_m\}$ and $\mathcal{E} = \operatorname{span}\{h_{p+q+1}, \cdots, h_m\}$. If $p + q = m$, the distribution $\mathcal{E}$ is simply empty.

Now, we prove that the corank of $\mathcal{E}$ in $\mathcal{H}^0$ is $r_1 - 1$, i.e., we necessarily have $r_1 = p + q$. Recall that for flat systems of differential weight $n + m + 1$, the following regularity condition should be satisfied:

$$\operatorname{rk} \operatorname{span}\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^{m} u_i h_i, h_j], 2 \leq j \leq m\}(x_0, u_0) = \operatorname{rk} \mathcal{H}^1(x_0).$$

According to the above assumption and notations, this relation can be written as

$$\mathrm{rk\,span}\,\{g_1, h_j, ad_f h_i + u_{10}[g_1, h_i], ad_f h_s, 2 \le j \le m, 2 \le i \le p < s \le p+q\}(x_0) = m - 1 + r_1.$$

The rank of the left hand side of this expression is at most $m + p - 1 + q$ and since $p + q \le r_1$, it follows that the above equality holds if and only if $r_1 = p + q$.

Next we prove that the distribution $\mathcal{E} = \mathrm{span}\,\{h_{p+q+1}, \cdots, h_m\}$ is involutive. First, observe that, since $\mathcal{E} \subset \mathcal{C}$ and $[f, \mathcal{E}] \in \mathcal{G}^1$, we have $[f, [h_i, h_j]] = [[f, h_i], h_j] + [h_i, [f, h_j]] = 0 \bmod \mathcal{G}^1$, for all $p + q + 1 \le i, j \le m$.

Suppose that $\mathcal{E}$ is not involutive. Hence there exist at least two integers $p + q + 1 \le i, j \le m$ such that $[h_i, h_j] \notin \mathcal{E}$. Since $\mathcal{E}$ is contained in $\mathcal{C}$, which is involutive, it follows that there exist smooth functions $\alpha_l$, for $p \le l \le p + q$, not vanishing simultaneously, such that $[h_i, h_j] = \sum_{l=p+1}^{p+q} \alpha_l h_l + e$, where $e \in \mathcal{E}$. Then,

$$[f, [h_i, h_j]] = [f, \sum_{l=p+1}^{p+q} \alpha_l h_l + e] = \sum_{l=p+1}^{p+q} \alpha_l ad_f h_l \bmod \mathcal{G}^1.$$

Recall that $[f, [h_i, h_j]] = 0 \bmod \mathcal{G}^1$ and the vector fields $ad_f h_{p+1}, \cdots, ad_f h_{p+q}$ are independent modulo $\mathcal{G}^1$, which contradicts the existence of $l$ such that $\alpha_l \neq 0$. Therefore, the distribution $\mathcal{E}$ is indeed involutive.

To summarize, we have the following sequence of inclusions of involutive distributions:

$$\mathcal{E} \underset{q}{\subset} \mathcal{C} \underset{p-1}{\subset} \mathcal{H}^0 \underset{r_1}{\subset} \mathcal{H}^1 \underset{r_2}{\subset} \cdots \underset{r_{\rho-1}}{\subset} \mathcal{H}^{\rho-1} \underset{r_\rho}{\subset} \mathcal{H}^\rho = TX,$$

with $r_1 = p + q$.

Let us new consider the case $k \ge 1$. According to Proposition 2.7.2 (in Section 2.7.1 above), we can construct as for the case $k = 0$, the following sequence of inclusions of involutive distributions:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^k \subset \cdots \subset \mathcal{H}^{\rho-1} \subset \mathcal{H}^\rho = TX,$$

where $\mathcal{H}^0$ is the involutive corank one subdistribution of $\mathcal{D}^0$ associated to $\mathcal{H}^k$ (see Proposition 2.7.2 for details), $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$ and $\rho$ is the smallest integer such that $\mathcal{H}^\rho = TX$. We denote by $r_i$ the corank of the inclusion $\mathcal{H}^{i-1} \subset \mathcal{H}^i$, for $i \ge 0$. We clearly have $m \ge r_1 \ge r_2 \ge \cdots \ge r_q \ge 1$.

We show that we can identify an involutive distribution $\mathcal{E}$ in $\mathcal{H}^0$, of corank $r_1 - 1$, such that $[f, \mathcal{E}] \in \mathcal{D}^0$. Let $\mathcal{D}^0 = \mathrm{span}\,\{g_1, h_2, \cdots, h_m\}$, where $\mathcal{H}^0 = \mathrm{span}\,\{h_2, \cdots, h_m\}$. Then $\mathcal{H}^1 = \mathrm{span}\,\{g_1, h_j, ad_f h_j, 2 \le j \le m\}$. Since $\mathrm{cork}\,(\mathcal{H}^0 \subset \mathcal{H}^1) = r_1$ and $g_0 \notin \mathcal{H}^0$, we deduce that the vector fields $ad_f h_j$, for $2 \le j \le m$, add $r_1 - 1$ new directions. Assume $ad_f h_i \notin \mathcal{D}^0$, for $2 \le i \le r_1$, where all $ad_f h_i$, for $2 \le i \le r_1$, are independent. It follows that there exist smooth functions $\alpha_l^i$ such that

$$ad_f h_{r_1+l} = \sum_{i=2}^{r_1} \alpha_l^i ad_f h_i \bmod \mathcal{D}^0,$$

for $1 \leq l \leq m - r_1$. We deduce

$$[f, h_{r_1+l} - \sum_{i=2}^{r_1} \alpha_l^i h_i] = 0 \bmod \mathcal{D}^0,$$

for $1 \leq l \leq m - r_1$. We put $\tilde{h}_{r_1+l} = h_{r_1+l} - \sum_{i=2}^{r_1} \alpha_l^i h_i$, for $1 \leq l \leq m - r_1$, and define $\mathcal{E} = \mathrm{span}\,\{h_{r_1+1}, \cdots, h_m\}$. It is clear that $\mathcal{E}$ is of corank $r_1 - 1$ in $\mathcal{H}^0$ and satisfies $[f, \mathcal{E}] \in \mathcal{D}^0$. If $r_1 = m$, the distribution $\mathcal{E}$ is simply empty.

We can always assume, without loss of generality, that $\mathcal{H}^0 = \mathrm{span}\,\{h_2, \cdots, h_{r_1}, h_{r_1+1}, \cdots, h_m\}$, with $\mathcal{E} = \mathrm{span}\,\{h_{r_1+1}, \cdots, h_m\}$. For all $r_1 + 1 \leq i, j \leq m$, it is easy to check (by applying Jacobi identity) that $[f, [h_i, h_j]] \in \mathcal{D}^0$.

Let us prove that $\mathcal{E}$ is involutive. Suppose that there exist two integers $r_1 + 1 \leq i, j \leq m$ such that $[h_i, h_j] = \sum_{l=2}^{r_1} \alpha_l h_l \bmod \mathcal{E}$, with $\alpha_l$ smooth functions non vanishing simultaneously. Then,

$$[f, [h_i, h_j]] = [f, \sum_{l=2}^{r_1} \alpha_l h_l \bmod \mathcal{E}] = \sum_{l=2}^{r_1} \alpha_l ad_f h_l \bmod \mathcal{D}^0.$$

Since there exists $l$ such that $\alpha_l \neq 0$, it follows that $[f, [h_i, h_j]] \notin \mathcal{D}^0$, which contradicts the above observation. Therefore, the distribution $\mathcal{E}$ is involutive.

*Proof of Proposition 2.5.1*

We give the idea of the proof for the case $k \geq 1$. If $k = 0$ similar arguments apply. Consider a control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$ that is $x$-flat at $x_0$ of differential weight $n + m + 1$. Throughout we will use the notations introduced in Section 2.5.

*Necessity.* Let $(\varphi_1, \cdots, \varphi_{r_1}, \psi_{r_1+1}, \cdots, \psi_m)$ be a minimal flat output, defined on a neighborhood $\mathcal{X}$ of $x_0$. According to Proposition 2.4.1 and its proof, there exists a valid local change of coordinates in which the system, after applying a suitable feedback, takes the form $(NF1)$ on $\mathcal{X}'$, an open and dense subset of $\mathcal{X}$, with $\varphi_i$ and $\psi_j$ playing the role of top variables. Moreover, if $r_1 \leq m - 1$, there exist $m - r_1$ linear chains of length 1 and we can always suppose that the flat outputs corresponding to these chains are $\psi_j$, for $r_1 + 1 \leq j \leq m$. We easily deduce that the conditions *(FO1)-(FO2)* hold on $\mathcal{X}'$. Since all functions $\varphi_i$ and $\psi_j$, for $1 \leq i \leq r_1$, and $r_1 + 1 \leq j \leq m$, as well as all distributions involved in the above conditions are defined on $\mathcal{X}$, by continuity *(FO1)-(FO2)* are valid on $\mathcal{X}$.

*Sufficiency.* Bring the system $\Sigma$ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=2}^{r_1} \tilde{u}_i h_i(x) + \sum_{l=r_1+1}^{m} \tilde{u}_l h_l(x)$, with $\mathcal{H}^0 = \mathrm{span}\,\{h_2, \cdots, h_{r_1}, h_{r_1+1}, \cdots, h_m\}$ and $\mathcal{E} = \mathrm{span}\,\{h_{r_1+1}, \cdots, h_m\}$. To simplify notation, we will drop the tildes.

Let $\varphi_1, \cdots, \varphi_{r_1}, \psi_{r_1+1}, \cdots, \psi_m$ any functions satisfying conditions *(FO1)-(FO2)*. According to the definition of the sequence of distributions $\mathcal{H}^i$ and to the condition *(FO1)*, it can be shown that the differentials $d\varphi_i, \cdots, dL_f^{\rho_i - 1}\varphi_i$ are independent at $x_0$ and annihilate the distribution $\mathcal{H}^0$, for $r_{l+1} + 1 \leq i \leq r_l$, with $\rho_i = l$, $1 \leq l \leq \rho$ and $r_{\rho+1} = 0$. Consider some functions $\psi_2, \cdots, \psi_{r_1}$ such that their differentials are independent of $d\varphi_i, \cdots, dL_f^{\rho_i - 1}\varphi_i$ and such that $d\psi_j \perp \mathcal{E}$, for $2 \leq j \leq r_1$. Introduce

$z^i_j = L_f^{j-1}\varphi_i$, $1 \le j \le \rho_i$, for $r_{l+1}+1 \le i \le r_l$, with $\rho_i = l$, $1 \le l \le \rho$, $r_{\rho+1} = 0$, and $w_j = \psi_j$, for $2 \le j \le r_1$, and let $w_s$, for $r_1+1 \le s \le m$, be any functions completing them to a coordinate system. In these coordinate the system reads:

$$\begin{cases} \dot z^i_1 &= z^i_2 \\ \dot z^i_2 &= z^i_3 \\ &\vdots \\ \dot z^i_{\rho_i-1} &= z^i_{\rho_i} \\ \dot z^i_{\rho_i} &= L_f^{\rho_i}\varphi_i + u_1 L_{g_1} L_f^{\rho_i-1}\varphi_i \end{cases} \qquad \begin{array}{rcl} \dot w_p &=& L_f\psi_p + u_1 L_{g_1}\psi_p + \sum_{j=2}^{r_1} u_j L_{h_j}\psi_p \\ \dot w_s &=& a_s + u_1 b_s + \sum_{j=2}^{r_1} u_j c_{js} + \sum_{j=r_1+1}^{m} u_j d_{js} \end{array}$$

for $r_{l+1}+1 \le i \le r_l$, with $\rho_i = l$, $1 \le l \le \rho$, $r_{\rho+1} = 0$, and $2 \le p \le r_1$, $r_1+1 \le s \le m$. There exists an integer $i$ such that $L_{g_1}L_f^{\rho_i-1}\varphi_i(x_0) \neq 0$ and we suppose $i = 1$ (if not permute the functions $\varphi_i$). Moreover, the matrices $(L_{h_j}\psi_p)$, for $2 \le j, p \le r_1$, and $(d_{js})$, for $r_1+1 \le j, s \le m$, are of full rank at $x_0$. We apply the invertible static feedback transformation

$$\begin{array}{rcl} \tilde u_1 &=& L_f^{\rho}\varphi_1 + u_1 L_{g_1} L_f^{\rho-1}\varphi_1, \\ \tilde u_p &=& L_f\psi_p + u_1 L_{g_1}\psi_p + \sum_{j=2}^{r_1} u_j L_{h_j}\psi_p,\ 2 \le p \le r_1, \\ \tilde u_s &=& a_s + u_1 b_s + \sum_{j=2}^{r_1} u_j c_{js} + \sum_{j=r_1+1}^{m} u_j d_{js},\ r_1+1 \le s \le m, \end{array}$$

to get

$$\begin{cases} \dot z^1_1 &= z^1_2 & \dot z^i_1 &= z^i_2 \\ \dot z^1_2 &= z^1_3 & \dot z^i_2 &= z^i_3 \\ &\vdots & &\vdots \\ \dot z^1_{\rho_1-1} &= z^1_{\rho_1} & \dot z^i_{\rho_i-1} &= z^i_{\rho_i} \\ \dot z^1_{\rho_i} &= \tilde u_1 & \dot z^i_{\rho_i} &= a_i + b_i \tilde u_1 \end{cases} \qquad \begin{array}{rcl} \dot w_p &=& \tilde u_p \\ \dot w_s &=& \tilde u_s \end{array}$$

for $2 \le i \le r_1$, $2 \le p \le r_1$ and $r_1+1 \le s \le m$.

By assumption $k \ge 1$, i.e., the first noninvolutive distribution cannot be $\mathcal{D}^0$, so all functions $b_i$ depend on $z$ only. Recall that the distribution $\mathcal{E}$ is such that $[f, \mathcal{E}] \in \mathcal{D}^0$. In these coordinates we have $\mathcal{H}^0 = \text{span}\{\frac{\partial}{\partial w_2}, \cdots, \frac{\partial}{\partial w_m}\}$ and $\mathcal{E} = \text{span}\{\frac{\partial}{\partial w_{r_1+1}}, \cdots, \frac{\partial}{\partial w_m}\}$. It follows that $\frac{\partial a_i}{\partial w_s} = 0$, for $2 \le i \le r_1$, $r_1+1 \le s \le m$. Moreover, since cork $(\mathcal{H}^0 \subset \mathcal{H}^1) = r_1$, we deduce that the matrix $(\frac{\partial a_i}{\partial w_t})$, for $2 \le i, t \le r_1$, is of full rank at $x_0$. We introduce new coordinates $z^i_{\rho_i+1} = a_i$, for $2 \le i \le r_1$, and apply a suitable invertible static feedback transformation to get

$$\begin{cases} \dot z^1_1 &= z^1_2 & \dot z^i_1 &= z^i_2 \\ \dot z^1_2 &= z^1_3 & \dot z^i_2 &= z^i_3 \\ &\vdots & &\vdots \\ \dot z^1_{\rho_1-1} &= z^1_{\rho_1} & \dot z^i_{\rho_i-1} &= z^i_{\rho_i} \\ \dot z^1_{\rho_1} &= v_1 & \dot z^i_{\rho_i} &= z^i_{\rho_i+1} + b_i(\bar z^1_{\rho_1}, \cdots, \bar z^{r_1}_{\rho_{r_1}})v_1 \\ & & \dot z^i_{\rho_i+1} &= v_i \end{cases} \qquad \dot w_s = v_s$$

for $2 \le i \le r_1$ and $r_1+1 \le s \le m$. Now, by condition $(FO2)$, $d\psi_j$, for $r_1+1 \le j \le m$, and all $dz^i_j$ are independent. Hence, we can introduce new coordinates $\tilde w_s = \psi_s$

and apply a suitable invertible static feedback to get exactly the above form with $w$ replaced by $\tilde{w}$ and for which $z_1^i = \varphi_i$ and $\tilde{w}_s = \psi_s$ is a minimal $x$-flat output of differential weight $n+m+1$.

If $r_1 = m$, there are no functions $\psi_s$ and the same proof holds. We will find the same normal form as above, but without the linear chains of length 1 corresponding to $\tilde{w}_s$.

# Appendices

## 2.A. Proof of Proposition 2.3.1

We will show the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Consider a control system $\Xi : \dot{x} = F(x,u)$ and assume that $\Xi$ is flat at $(x_0, \bar{u}_0^l)$, of differential weight $n+m+1$. Let $\varphi = (\varphi_1, \cdots, \varphi_m)$ be a minimal flat output. We will denote by $s_i$ the order of the highest derivative of $\varphi_i$, for $1 \leq i \leq m$, involved in the expression of $x$ and $u$, i.e.,

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(s_1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(s_1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(s_m)}), \end{aligned}$$

where $\sum_{i=1}^m s_i + m = n+m+1$. We will use the notation $\text{d.w.}(\varphi) = n+m+1$.

Denote $\mathcal{X} = \text{span}\{dx_1, \cdots, dx_n\}$ and $\mathcal{U} = \text{span}\{du_1, \cdots, du_m\}$. Assume that there exists $\varphi_l = \varphi_l(x, u, \dot{u}, \cdots, u^{(r)})$, where $r \geq 1$.

The differential weight of $\varphi$ being $n+m+1$ implies that, clearly, $s_l = 0$. Indeed, if $s_l \geq 1$, then $d\varphi_l \wedge \cdots \wedge d\varphi_l^{(s_l)} \neq 0 \,\text{mod}\, (\mathcal{X} + \mathcal{U})$ and $\text{d.w.}(\varphi)$ would be $n+m+s_l+1 > n+m+1$. Denote $\eta = \varphi_l(x, u, \dot{u}, \cdots, u^{(r)})$. If there exists a flat output $\varphi_i$ such that $d\varphi_i \wedge d\eta \neq 0 \,\text{mod}\, (\mathcal{X} + \mathcal{U})$, then $\text{d.w.}(\varphi)$ would be at least $n+m+2$. We thus have $\varphi_i = \varphi_i(x, u, \eta)$, for $1 \leq i \leq m$, and we separate the components $\varphi_i$ that depend explicitly on $\eta$ by permuting $\varphi_i$ such that $\varphi_i = \varphi_i(x,u)$, for $1 \leq i \leq p$, and $\varphi_j = \varphi_j(x, u, \eta)$, for $p+1 \leq j \leq m$, where $\frac{\partial \varphi_j}{\partial \eta} \neq 0$. We assume, without loss of generality, that $l = m$, i.e., $\varphi_m = \eta$. Clearly, $s_j = 0$, for $p+1 \leq j \leq m$ (if not $d\varphi_j \wedge d\dot{\varphi}_j \neq 0 \,\text{mod}\, (\mathcal{X} + \mathcal{U})$ contradicting $\text{d.w.}(\varphi) = n+m+1$). Let $\rho_i$, for $1 \leq i \leq p$, be the smallest integer such that the derivative $\varphi_i^{(\rho_i)}$ depends explicitly on the control $u$. In particular, $\rho_i = 0$, if $\varphi_i$ depends explicitly on $u$. We have $\varphi_i^{(\rho_i)} = c_i(x,u)$ and denote $\text{rk}\,(\frac{\partial c_i}{\partial u_l}) = p_1 \leq p$, for $1 \leq i \leq p$ and $1 \leq l \leq m$. By a suitably static feedback, we get

$$\begin{aligned} (S1) \quad \varphi_i^{(\rho_i)} &= v_i, \ 1 \leq i \leq p_1, \\ (S2) \quad \varphi_i^{(\rho_i)} &= c_i(x, v_1, \cdots, v_{p_1}), \ p_1 \leq i \leq p. \end{aligned}$$

We will consider separately the cases $p_1 = p$ and $p_1 < p$.

If $p_1 = p$ denote $z_i^j = L_F^{(j-1)} \varphi_i$, $1 \leq j \leq \rho_i$, $1 \leq i \leq p$, ($z_i^j$ are absent if $\rho_i = 0$), and let $w$ be the complementary coordinates, $\dim z + \dim w = n$. The system in

$(z, w)$-coordinates reads

$$(S) \quad \begin{array}{rcl} \dot{z} &=& Az + Bv \\ \dot{w} &=& d(z, w, v) \end{array}$$

where $(A, B)$ is in Brunovský canonical form with $p - r$ chains, where $r = \mathrm{rk}\,(\frac{\partial \varphi_i}{\partial v_l})$, for $1 \leq i \leq p$, $1 \leq l \leq m$, i.e., $\mathrm{rk}\, B = p - r$. The $w$-part is nonempty since it has to involve the controls $v_{p+1}, \cdots, v_m$ absent in the $z$-part (recall that $p < m$) and thus $\dim w = q \geq m - p$.

Denote

$$\begin{array}{rcl} \Phi &=& \mathrm{span}\,\{d\varphi_i^{(j)}, 1 \leq i \leq p, 0 \leq j \leq s_i\}, \\ \mathcal{N} &=& \mathrm{span}\,\{d\varphi_i, p+1 \leq i \leq m\} \end{array}$$

(recall that, for $p + 1 \leq i \leq m$, $\varphi_i = \varphi_i(x, u, \eta)$ and $s_i = 0$). Clearly, $\mathrm{span}\,\{dw_1, \cdots, dw_q, dv_{p+1}, \cdots, dv_m\} \cap \Phi = 0$. Notice that $\mathrm{rk}\,\mathcal{N} = m - p$. By definition of flatness, we should have $\mathcal{X} + \mathcal{U} \subset \Phi + \mathcal{N}$, but there are at least $q \geq m - p$ variables among $w_1, \cdots, w_q, v_{p+1}, \cdots, v_m$ whose differentials are lost in $\Phi + \mathcal{N}$.

Now suppose $p_1 < p$. We have $d\eta \notin \mathcal{X} + \mathcal{U}$ and since $\mathrm{d.w.}(\varphi) = n + m + 1$, we deduce $\Phi + \mathcal{N} = \mathrm{span}\,\{d\eta\} + \mathcal{X} + \mathcal{U}$, so differentiating one more time $(S2)$, we conclude $\varphi_i^{(\rho_i+1)} \wedge d\eta = 0 \,\mathrm{mod}\,(\mathcal{X} + \mathcal{U})$, for $p_1 + 1 \leq i \leq p$. It follows that only one column of the matrix $(\frac{\partial c_i}{\partial v_l})$, $p_1 + 1 \leq i \leq p$, $1 \leq l \leq p_1$, is nonzero and we may assume that $\frac{\partial c_i}{\partial v_1} \neq 0$, so

$$\varphi_i^{(\rho_i)} = c_i(x, v_1), \;\; p_1 + 1 \leq i \leq p.$$

Since $\mathrm{d.w.}(\varphi) = n + m + 1$, it follows, firstly, that $\eta = \eta(x, v, \dot{v}_1)$ and, secondly, that

$$\mathrm{rk}\,(\frac{\partial \dot{c}_i}{\partial v_j}) = p - p_1, \; p_1 + 1 \leq i \leq p, \; p_1 + 1 \leq j \leq m.$$

Recall that $p < m$ so there are $m - p$ components (after a permutation) $v_{p+1}, \cdots, v_m$ such that $dv_{p+1} \wedge \cdots \wedge dv_m \neq 0 \,\mathrm{mod}\,(\mathcal{X} + \Phi)$, where $\Phi = \mathrm{span}\,\{d\varphi_i^{(j)}, 1 \leq i \leq p, 0 \leq j \leq s_i\}$. Define $z_i^j = L_F^{(j-1)}\varphi_i$, $1 \leq i \leq p$, $0 \leq j \leq \rho_i$, and put $\mathcal{Z} = \mathrm{span}\,\{dz_i^j\}$. Let $w_1, \cdots, w_q$ coordinate functions such that $dw_1 \wedge \cdots \wedge dw_q \neq 0 \,\mathrm{mod}\,(\mathcal{Z} + \mathrm{span}\,\{dc_i, p_1 + 1 \leq i \leq p\})$, where the exterior product is nonzero at one, and thus at almost any, value of $v_1$ (since controls enter independently into the system). Clearly, $\mathrm{span}\,\{dw_1, \cdots, dw_q, dv_{p+1}, \cdots, dv_m\} \cap \Phi = 0$. Since $s_i = 0$ for $\varphi_i$, $p + 1 \leq i \leq m$, it follows, like in the case $p_1 = p$, that there are at least $q \geq m - p$ variables among $w_1, \cdots, w_q, v_{p+1}, \cdots, v_m$ that cannot be expressed as functions of $\varphi_i^{(j)}, 1 \leq i \leq m, 0 \leq j \leq s_i$.

It remains to consider the case of $\Xi$ being $(x, u)$-flat. Let $\rho_i$ the relative degree of $\varphi_i$, that is, the smallest integer such that the derivative $\varphi_i^{(\rho_i)}$ depends explicitly on the control $u$ and let $r_1$ denote the rank of the decoupling matrix $r_1 = \mathrm{rk}\,(\frac{\partial \varphi_i^{(\rho_i)}}{\partial u_j})$, $1 \leq i, j \leq m$. Put $r_0 = \mathrm{rk}\,(\frac{\partial \varphi_i}{\partial u_j})$, $1 \leq i, j \leq m$. Clearly $r_0 \leq r_1$ and let $r_0 + (m - r_2)$

be the number of $\varphi_i$ whose relative degree $\rho_i$ is zero. After permuting the $\varphi_i$'s and applying a static invertible feedback $u = u(x, v)$, we get

$$\begin{array}{rcl} \varphi_i & = & v_i,\ 1 \le i \le r_0, \\ \varphi_i^{(\rho_i)} & = & v_i,\ r_0+1 \le i \le r_1,\ \text{where } \rho_i \ge 1, \\ \varphi_i^{(\rho_i)} & = & c_i(x, v_1, \cdots, v_{r_1}),\ r_1+1 \le i \le r_2, \\ \varphi_i & = & \eta_i(x, v_1, \cdots, v_{r_0}),\ r_2+1 \le i \le m. \end{array}$$

The system is $(x, u)$-flat, so $r_0 \ge 1$. Define $z_i^j = L_F^{(j-1)}\varphi_i$, $r_0+1 \le i \le r_2$, $0 \le j \le \rho_i$, which are functions on $X$ by the definition of the relative degree. We have

$$\begin{array}{rcl} \dot{z}_i^j & = & z_i^{j+1},\ 1 \le j \le \rho_i - 1, \\ \dot{z}_i^{\rho_i} & = & v_i,\ r_0+1 \le i \le r_1, \end{array}$$

$$\begin{array}{rcl} \dot{z}_i^j & = & z_i^{j+1},\ 1 \le j \le \rho_i - 1, \\ \dot{z}_i^{\rho_i} & = & c_i(x, v_1, \cdots, v_{r_1}),\ r_1+1 \le i \le r_2. \end{array}$$

By d.w.$(\varphi) = n + m + 1$, we can differentiate $\varphi_i^{(\rho_i)} = \dot{z}_i^{\rho_i}$ only one time to produce independent controls and, moreover all $c_i$ can depend on one (the same for all $c_i$) control, say, $v_l$. It follows that rk $(\frac{\partial \dot{c}_i}{\partial v_j}) = r_2 - r_1$, where $r_1+1 \le i \le r_2$, $r_0+1 \le j \le m$, (at one and thus at almost any value of $(v_l, \dot{v}_l)$). Then there exist functions $z_i^{\rho_i+1}$, for $r_1+1 \le i \le r_2$, independent of $z_i^j$, $0 \le j \le \rho_i$, such that rk $(\frac{\partial c_i}{\partial z_j^{\rho_j+1}}) = r_2 - r_1$, for $r_1+1 \le i, j \le r_2$. By applying a static invertible feedback, the overall system becomes

$$\begin{array}{rclrclrcl} \dot{z}_i^j & = & z_i^{j+1},\ 1 \le j \le \rho_i - 1, & \dot{z}_i^j & = & z_i^{j+1},\ 1 \le j \le \rho_i - 1, & \dot{w} & = & d(z, w, v) \\ \dot{z}_i^{\rho_i} & = & v_i,\ r_0+1 \le i \le r_1, & \dot{z}_i^{\rho_i} & = & c_i(z, w, v_l) & & & \\ & & & \dot{z}_i^{\rho_i+1} & = & v_i,\ r_1+1 \le i \le r_2, & & & \end{array}$$

where $w = (w_1, \cdots, w_q)$ are any functions completing $z_i^j$'s to a coordinate system. The system is supposed to be $(x, u)$-flat with a flat output

$$\varphi_i = v_i,\ 1 \le i \le r_0,\ \varphi_i = z_i^1,\ r_0+1 \le i \le r_2,\ \varphi_i = \eta_i(z, w, v_1, \cdots, v_{r_0}),\ r_2+1 \le i \le m.$$

Notice that the number of controls equals the number of the components of flat outputs and is $m \ge r_2$.

We will consider the cases depending on whether the control $v_l$ (whose derivation $\dot{v}_l$ is involved since $v_l$ is present in all $c_i(z, w, v_l)$) satisfies $1 \le l \le r_0$ or $r_0+1 \le l \le r_1$.

Consider that case $r_0+1 \le l \le r_1$ and notice that the first $r_0$ controls $v_1, \cdots, v_{r_0}$ and the last $m - r_2$ controls $v_{r_2+1}, \cdots, v_m$ (existing if $r_2 < m$) do not affect the $z$-subsystem, so they are present in the $w$-subsystem. Therefore, we have $\dim w = q \ge m - r_2 + r_0$.

Denote the set of indices $I = \{1, \cdots, r_0\} \cup \{r_2+1, \cdots, m\}$. Notice that $s_i = 0$, for $\varphi_i$ such that $i \in I$ (since $\varphi_i = v_i$, for $1 \le i \le r_0$, and $\varphi_i = \varphi_i(z, w, v_1, \cdots, v_{r_0})$, for $r_2+1 \le i \le m$). Therefore, for flatness we should have $\mathcal{X}+\mathcal{U} \subset \Phi + \mathcal{N}$, where $\Phi = \text{span}\,\{d\varphi_i^j, r_0+1 \le i \le r_2, 0 \le j \le \rho_i+1\}$ and $\mathcal{N} = \text{span}\,\{d\varphi_i, i \in I\}$. Clearly, $dw_1, \cdots, dw_q$ and $dv_i$, $i \in I$, are in $\mathcal{X}+\mathcal{U}$; are independent modulo $\Phi$ and thus there is $q + r_0 + (m-r_2)$ of them. Since the controls $v_i$, $i \in I$, are independent, $q$ cannot be smaller than the cardinality of $I$, which is $r_0+m-r_2$. So for flatness, we need

$$q + r_0 + m - r_2 \ge 2(r_0+m-r_2) \ge r_0+m-r_2,$$

($r_0+m-r_2$ being $\text{rk}\,\mathcal{N}$), which holds if $r_0+m-r_2 = 0$. This holds if and only if $r_2 = m$ and $r_0 = 0$, but the latter is impossible since $\varphi$ is an $(x,u)$-flat output implying $r_0 \ge 1$.

Now we will consider the case $1 \le l \le r_0$. Without loss of generality we may assume $l_0 = r_0$. So we rewrite $\dot{z}_i^{(\rho_i)} = c_i(z, w, v_{r_0})$, for $r_1+1 \le i \le r_2$. We will distinguishing those flat outputs $\varphi_i$, $i \ge r_2+1$, that depend on $(z,w)$ and $v_{r_0}$ only from those that depend also on other controls. After a permutation, we may assume

$$\begin{aligned} \varphi_i &= \varphi_i(z,w,v_{r_0}), \text{ for } r_2+1 \le i \le r_3, \\ \varphi_i &= \varphi_i(z,w,v_1,\cdots,v_{r_0-1},v_{r_0}), \text{ for } r_3+1 \le i \le m, \end{aligned}$$

where each $\varphi_i$, $i \ge r_3$, depend non trivially on at least one $v_j$, $1 \le j \le r_0-1$. It follows that $s_i = 0$, for $r_3+1 \le i \le m$, but $s_i = 1$, for $r_2+1 \le i \le r_3$ (since we can differentiate one time $v_{r_0} = v_l$).

For flatness, we should have $\mathcal{X}+\mathcal{U} \subset \Phi+\mathcal{N}$, where $\Phi = \text{span}\,\{d\varphi_{r_0}, d\dot{\varphi}_{r_0}\} + \text{span}\,\{d\varphi_i^{(j)}, r_0+1 \le i \le r_1, 0 \le j \le \rho_i\} + \text{span}\,\{d\varphi_i^{(j)}, r_1+1 \le i \le r_2, 0 \le j \le \rho_i+1\}$ and $\mathcal{N} = \text{span}\,\{d\varphi_i,\, d\dot{\varphi}_j,\, i \in I,\, r_2+1 \le j \le r_3\}$. Notice that the definition of both $\Phi$ and $\mathcal{N}$ is slightly different because now $1 \le l \le r_0$ implying that $d\varphi_{r_0}$ and $d\dot{\varphi}_{r_0}$are added to $\Phi$ and $\mathcal{N}$ contains also $d\dot{\varphi}_{r_2+1}, \cdots, d\dot{\varphi}_{r_3}$ for which $s_i = 1$.

Notice that the first $r_0-1$ controls $v_1, \cdots, v_{r_0-1}$ and the last $m-r_2$ controls $v_{r_2+1}, \cdots, v_m$ (existing if $r_2 < m$) do not affect the $z$-subsystem, so they have to affect the $w$-subsystem, implying that $\dim w = q \ge r_0 - 1 + m - r_2$.

Clearly, $dv_1, \cdots, dv_{r_0-1}$, $v_{r_2+1}, \cdots, v_m$ and $dw_1, \cdots, dw_q$ are in $\mathcal{X}+\mathcal{U}$; are independent modulo $\Phi$ and thus there are $q + r_0 - 1 + (m-r_2)$ of them. We have $\text{rk}\,\mathcal{N} = r_0 - 1 + (m-r_3) + 2(r_3-r_2)$. So for flatness, we need

$$q + r_0 - 1 + m - r_2 \ge 2(r_0-1+m-r_2) \ge r_0 - 1 + (m-r_3) + 2(r_3-r_2),$$

which is equivalent to

$$0 \ge -(r_0-1) - (m-r_3).$$

This is the case if and only if $r_0 = 1$ and $m = r_3$. This implies that system is of the form

$$\begin{aligned} &\dot{z}_i^j = z_i^{j+1},\ 1 \le j \le \rho_i - 1, && \dot{z}_i^j = z_i^{j+1},\ 1 \le j \le \rho_i - 1, && \dot{w} = v_i,\ r_2+1 \le i \le m, \\ &\dot{z}_i^{\rho_i} = v_i,\ 2 \le i \le r_1, && \dot{z}_i^{\rho_i} = c_i(z,w,v_1) && \\ & && \dot{z}_i^{\rho_i+1} = v_i,\ r_1+1 \le i \le r_2, && \end{aligned}$$

which is, indeed, $(x,u)$-flat of differential weight $n+m+1$, with the flat output being $\varphi_1 = v_1$, $\varphi_i = z_i^1, 2 \le i \le r_2$, $\varphi_i = \varphi_i(w, v_1)$, $r_2+1 \le i \le m$, where $\text{rk}\,(\frac{\partial \varphi_i}{\partial w_j}) = m - r_2$, $r_2+1 \le i, j \le m$.

Now we will show that it is also $x$-flat with the differential weight $n+m+1$. To this end, observe that (by the definition of the relative degree) $\frac{\partial c_i}{\partial v_1} \neq 0$, in particular $\frac{\partial c_{r_1+1}}{\partial v_1} \neq 0$. Apply the static feedback $\tilde{v}_1 = c_{r_1+1}(z,w,v_1)$ to get

$$\begin{aligned} \dot{z}_{r_1+1}^{\rho_{r_1}} &= \tilde{v}_1 \quad & \dot{z}_i^{\rho_i} &= \tilde{c}_i(z,w,v_1) \\ & & \dot{z}_i^{\rho_i+1} &= v_i,\ r_1+2 \leq i \leq r_2. \end{aligned}$$

By permuting $z_i^{\rho_i}$ , we have

$$\operatorname{rk} \frac{\partial \tilde{c}_i}{\partial z_j^{\rho_j}} = r_2 - r_1 - 1,\ r_1+2 \leq i \leq r_2, r_1+1 \leq j \leq r_2.$$

Now rename $\tilde{v}_1$ by $v_{r_1}$ and $v_{r_1}$ by $v_1$, as well as $z_{r_1+1}^{\rho_{r_1}}$ by $w_1$. We get the system

$$\begin{array}{lll} \dot{z}_i^j = z_i^{j+1},\ 1 \leq j \leq \rho_i - 1, & \dot{z}_i^j = z_i^{j+1},\ 1 \leq j \leq \rho_i - 1, & \dot{w}_1 = v_1 \\ \dot{z}_i^{\rho_i} = v_i,\ 2 \leq i \leq r_1+1, & \dot{z}_i^{\rho_i} = \tilde{c}_i(z,w,v_1) & \dot{w}_i = v_i,\ r_2+1 \leq i \leq m. \\ & \dot{z}_i^{\rho_i+1} = v_i,\ r_1+2 \leq i \leq r_2, & \end{array}$$

This system is $x$-flat, with the differential weight $n+m+1$ ,with $x$-flat outputs being $\varphi_1 = w_1$, $\varphi_i = z_i^1, 2 \leq i \leq r_2$, $\varphi_i = w_i, r_2+1 \leq i \leq m$.

$(ii) \Rightarrow (iii)$. Consider an $x$-flat control system $\Xi$ of differential weight $n+m+1$ and let $\varphi = (\varphi_1, \cdots, \varphi_m)$ be a minimal $x$-flat output. We will denote by $s_i$ the order of the highest derivative of $\varphi_i$, for $1 \leq i \leq m$, involved in the expression of $x$ and $u$, i.e.,

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(k_1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(k_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(k_1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(k_m)}), \end{aligned}$$

where d.w.$(\varphi) = \sum_{i=1}^m s_i + m = n+m+1$. Denote $\mathcal{X} = \operatorname{span}\{dx_1, \cdots, dx_n\}$ and $\mathcal{U} = \operatorname{span}\{du_1, \cdots, du_m\}$.

Let $\rho_i$ be the relative degree of $\varphi_i$, for $1 \leq i \leq m$, i.e., $\varphi_i^{(\rho_i)}$ is the lowest derivative involving explicitly the control. Define $z_i^j = L_f^{j-1}\varphi_i$, for $1 \leq i \leq m$ and $1 \leq j \leq \rho_i$, and let $w$ complete them to a coordinate system. Put $c_i = L_f^{\rho_i}\varphi_i$, where $c_i = (z,w,u)$. Form the decoupling matrix $D = (\frac{\partial c_i}{\partial u_j})$, for $1 \leq i,j \leq m$, and denote by $r$ its rank. Flatness of differential weight $n+m+1$ implies $1 \leq r \leq m-1$. Indeed, if the rank were $m$, by a suitable invertible static feedback we could transform the system into the form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1},\ 1 \leq j \leq \rho_i - 1, \\ \dot{z}_i^{\rho_i} &= v_i,\ 1 \leq i \leq m, \end{aligned}$$

and get a static feedback linearizable system, thus of differential weight $n+m$. Suppose that the $r$ first lines of $D$ are independent and apply the invertible static feedback transformation $v_i = c_i$, for $1 \leq i \leq r$, $v_i = u_i$, for $r+1 \leq i \leq m$. The $z$-subsystem reads

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, 1 \leq j \leq \rho_i - 1, & \dot{z}_i^j &= z_i^{j+1}, 1 \leq j \leq \rho_i - 1, \\ \dot{z}_i^{\rho_i} &= v_i, 1 \leq i \leq r, & \dot{z}_i^{\rho_i} &= c_i(z, w, v_1, \cdots, v_r), r+1 \leq i \leq m. \end{aligned}$$

Observe that, for $q \geq 1$, $c_i^{(q)}$ depends on $v_1^{(q)}, \cdots, v_r^{(q)}$. Recall that $\text{d.w.}(\varphi) = n+m+1$ and notice that, obviously, $dv_i^{(q)}$ are independent modulo $\mathcal{X} + \mathcal{U}$. It follows that in order to produce the remaining controls $v_{r+1}, \cdots, v_m$, we are allowed to differentiate $c_i$, for $r+1 \leq i \leq m$, only one time and, moreover, only one control among $v_1, \cdots, v_r$, say $v_1$, can be present in all $c_i$, for $r+1 \leq i \leq m$. Let $z_i^{\rho_i+1}$, for $r+1 \leq i \leq m$, be any functions completing $z_i^j$, for $1 \leq j \leq \rho_i$, $1 \leq i \leq m$, to a coordinate system (we replace $w$ by $z_i^{\rho_i+1}$). Applying a suitable invertible feedback (to controls $v_{r+1}, \cdots, v_m$) we get

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, 1 \leq j \leq \rho_i - 1, & \dot{z}_i^j &= z_i^{j+1}, 1 \leq j \leq \rho_i - 1, \\ \dot{z}_i^{\rho_i} &= v_i, 1 \leq i \leq r, & \dot{z}_i^{\rho_i} &= c_i(z, v_1) \\ & & \dot{z}_i^{\rho_i+1} &= v_i, r+1 \leq i \leq m. \end{aligned}$$

Obviously the system becomes static feedback linearizable via the preintegration $v_1 = y_1$, $\dot{y}_1 = \tilde{v}_1$, $\tilde{v}_i = v_i$, $2 \leq i \leq m$.

$(iii) \Rightarrow (i)$. Suppose that the first prolongation of $\Xi : \dot{x} = F(x,u)$, given by

$$\Xi^{(1,0,\cdots,0)} : \begin{cases} \dot{x} &= F(x, y_1, v_2, \cdots, v_m) \\ \dot{y}_1 &= v_1 \end{cases}$$

where $u_1 = y_1$ and $u_i == v_i$, $2 \leq i \leq m$, is locally static feedback linearizable. Hence, $\Sigma$ is flat.

$\Xi^{(1,0,\cdots,0)}$ is equivalent via a diffeomorphism $z = \phi(x, y_1)$ and an invertible transformation, $v = \psi(x, y_1, \tilde{v})$, to the Brunovský canonical form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, 1 \leq j \leq \rho_i - 1, \\ \dot{z}_i^{\rho_i} &= v_i, 1 \leq i \leq m, \end{aligned}$$

where $\sum_{i=1}^m \rho_i = n+1$, for which $\varphi = (z_1^1, \cdots, z_m^1)$ is a minimal flat output of differential weight $n+m+1$. It follows that $z = \gamma(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(\rho_1-1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(\rho_m-1)})$, thus for the original variables $x$ and the first component of $u$, we have $(x, u_1)^t = \phi^{-1} \circ \gamma(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(\rho_1-1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(\rho_m-1)})$. Moreover, $v = \delta(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(\rho_1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(\rho_m)})$ and we deduce that $u_i = \delta_i(\varphi_1, \dot{\varphi}_1, \cdots, \varphi_1^{(\rho_1)}, \cdots, \varphi_m, \dot{\varphi}_m, \cdots, \varphi_m^{(\rho_m)})$, for $2 \leq i \leq m$, yielding that $\varphi$ is a flat output of $\Xi$ of differential weight $n+m+1$.

Notice that if the original system $\Xi$ is the control-affine system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, then the distribution $\mathcal{F} = \text{Im}\frac{\partial F}{\partial u}$ does not depend on $u$ and thus the distribution $\tilde{\mathcal{F}} = \mathrm{Im}\frac{\partial \tilde{F}}{\partial v}$ of the system

$$\begin{array}{rclrcl} \dot{z}_i^j & = & z_i^{j+1},\ 1 \le j \le \rho_i - 1, & \dot{z}_i^j & = & z_i^{j+1},\ 1 \le j \le \rho_i - 1, \\ \dot{z}_i^{\rho_i} & = & v_i,\ 1 \le i \le r, & \dot{z}_i^{\rho_i} & = & c_i(z, v_1) \\ & & & \dot{z}_i^{\rho_i+1} & = & v_i,\ r+1 \le i \le m. \end{array}$$

does not depend on $v$ (see Lemma 2.7.1 below) implying that $c_i(z, v_1)$, for $r+1 \le i \le m$, are actually affine functions of $v_1$, that is, $c_i(z, v_1) = a_i(z) + b_i(z)v_1$.

**Lemma 2.7.1.** *Consider a control system $\Xi : \dot{x} = F(x, u)$, where $x \in X$ and $u \in U$, and define its associated distribution by $\mathcal{F}(x, u) = \mathrm{Im}\frac{\partial F}{\partial u}(x, u)$. The dependency or not of $\mathcal{F}$ on the control $u$ is invariant by invertible static feedback transformations.*

*Proof.* Apply the invertible static feedback transformation $u = \psi(x, v)$ that brings the system $\Xi$ into $\tilde{\Xi} : \dot{x} = \tilde{F}(x, v)$, with $\tilde{F}(x, v) = F(x, \psi(x, v))$. We have $\frac{\partial \tilde{F}}{\partial v} = \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial v}$ implying that $\tilde{\mathcal{F}}(x, v) = \mathrm{Im}\frac{\partial \tilde{F}}{\partial v}(x, v) = \mathrm{Im}(\frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial v})(x, v) = \mathrm{Im}\frac{\partial F}{\partial \psi}(x, v) = \mathrm{Im}\frac{\partial F}{\partial u}(x, u) = \mathcal{F}(x, u)$. Therefore $\mathcal{F}$ does not depend on $u$, i.e., $\mathcal{F} = \mathcal{F}(x)$, if and only if $\tilde{\mathcal{F}}$ does not depend on $v$ and $\tilde{\mathcal{F}}(x) = \mathcal{F}(x)$. $\square$

# 3 | Control-Affine Systems Compatible with the Multi-Chained Form and Their $x$-Maximal Flatness


**Abstract**

We study flatness of control-affine systems, with $m+1$ inputs, defined on a $(nm+1)$-dimensional state-space. In the first part of this paper, we give a complete geometric characterization of systems locally static feedback equivalent to a triangular form compatible with the chained form, for $m = 1$, respectively with the $m$-chained form, for $m \geq 2$. They are $x$-flat systems. We provide a system of first order PDE's to be solved in order to find all $x$-flat outputs, for $m = 1$, respectively all minimal $x$-flat outputs, for $m \geq 2$. We illustrate our results by by examples, in particular by an application to a mechanical system: the coin rolling without slipping on a moving table.

In the second part of the paper, we introduce the concept of $x$-maximal flatness. A control system is $x$-maximally flat if the number of new states gained by each successive derivation of the flat output is the largest possible. Firstly, we show that the only control-linear systems that are $x$-maximally flat are those that are static feedback equivalent to the $m$-chained form. Secondly, we generalize that result from control-linear systems to control-affine systems whose control-linear subsystem is static feedback equivalent to the $m$-chained form. We prove that they are $x$-maximally flat if and only if the drift exhibits a triangular form compatible with the $m$-chained form (and recently characterized in [65] and [27]). We also show that if we skip the assumption of the $x$-maximal flatness, the latter condition is not necessary for $x$-flatness of control-affine system whose associated control-linear subsystem is static feedback equivalent to the $m$-chained form.

## 3.1 Characterization of Control-Affine Systems Compatible with the Multi-Chained Form

**Abstract**

In the first part of the paper, we give a complete geometric characterization of systems locally static feedback equivalent to a triangular form compatible with the chained form, for $m = 1$, respectively with the $m$-chained form, for $m \geq 2$. They are $x$-flat systems. We provide a system of first order PDE's to be solved in order to find all $x$-flat outputs, for $m = 1$, respectively all minimal $x$-flat outputs, for $m \geq 2$. We illustrate our results by examples, in particular by an application to a mechanical system: the coin rolling without slipping on a moving table.

### 3.1.1 Introduction

The notion of flatness has been introduced in control theory in the 1990's by Fliess, Lévine, Martin and Rouchon ( [13,14], see also [21,22,32,54]) and has attracted a lot of attention because of its multiple applications in the problem of trajectory tracking, motion planning and constructive controllability (see, e.g. [15,26,36,52,55,58,62]).

The fundamental property of flat systems is that all their solutions may be parameterized by $m$ functions and their time-derivatives, $m$ being the number of controls. More precisely, consider a nonlinear control system

$$\Xi : \dot{x} = F(x, u)$$

where $x$ is the state defined on an open subset $X$ of $\mathbb{R}^n$, $u$ is the control taking values in an open subset $U$ of $\mathbb{R}^m$ (more generally, an $n$-dimensional manifold $X$ and an $m$-dimensional manifold $U$, respectively) and the dynamics $F$ are smooth (the word smooth will always mean $\mathcal{C}^\infty$-smooth). The system $\Xi$ is *flat* if we can find $m$ functions, $\varphi_i(x, u, \dots, u^{(r)})$, for some $r \geq 0$, called *flat outputs*, such that

$$x = \gamma(\varphi, \dots, \varphi^{(s)}) \text{ and } u = \delta(\varphi, \dots, \varphi^{(s)}), \tag{3.1}$$

for a certain integer $s$ and suitable maps $\gamma$ and $\delta$, where $\varphi = (\varphi_1, \dots, \varphi_m)$. Therefore all state and control variables can be determined from the flat outputs without integration and all trajectories of the system can be completely parameterized. In the particular case $\varphi_i = \varphi_i(x)$, for $1 \leq i \leq m$, we will say that the system is $x$-flat. The minimal number of derivatives of components of a flat output $\varphi$, needed to express $x$ and $u$, will be called the differential weight of $\varphi$ (see Section 3.1.2 for precise definitions).

The problem of flatness of driftless two-input control-linear systems of the form

$$\Sigma_{lin} : \dot{x} = u_0 g_0(x) + u_1 g_1(x),$$

defined on a open subset $X$ of $\mathbb{R}^n$, has been solved by Martin and Rouchon in [34] (see also [29,33] and a related result of Cartan [8]). According to their result, on an open and dense subset $X'$ of $X$, the system $\Sigma_{lin}$ is flat if and only if, its associated distribution $\mathcal{G} = \text{span}\{g_0, g_1\}$ can be locally brought into the Goursat normal form, or equivalently, the control system $\Sigma_{lin}$ is locally static feedback equivalent to the chained form:

$$Ch_1^k : \begin{cases} \dot{z}_0 = v_0 & \begin{aligned} \dot{z}_1 &= z_2 v_0 \\ \dot{z}_2 &= z_3 v_0 \\ &\vdots \\ \dot{z}_{k-1} &= z_k v_0 \\ \dot{z}_k &= v_1 \end{aligned} \end{cases}$$

where $n = k + 1$.

The first who noticed the existence of singular points in the problem of transforming a distribution of rank two into the Goursat normal form were Giaro, Kumpera and Ruiz [17]. Murray presented in [40] a regularity condition that guarantees the feedback equivalence of $\Sigma_{lin}$ to the chained form $Ch_1^k$ around an arbitrary point $x^*$. In [29], Li and Respondek studied and solved the following problem: can a driftless two-input system be locally flat at a singular point of $\mathcal{G}$? In other words, can $\Sigma_{lin}$ be flat without being locally equivalent to the chained form? Their result shows that a Goursat structure is $x$-flat only at regular points of $\mathcal{G}$. They also described all $x$-flat outputs and showed that they are parametrized by an arbitrary function of three variables canonically defined up to a diffemorphism.

In this paper we give a generalization of these results. Our goal is to characterize control-affine systems that are static feedback equivalent to the following triangular form

$$TCh_1^k : \begin{cases} \dot{z}_0 = v_0 & \begin{aligned} \dot{z}_1 &= f_1(z_0, z_1, z_2) &+ z_2 v_0 \\ \dot{z}_2 &= f_2(z_0, z_1, z_2, z_3) &+ z_3 v_0 \\ &\vdots \\ \dot{z}_{k-1} &= f_2(z_0, \cdots, z_k) &+ z_k v_0 \\ \dot{z}_k &= v_1 \end{aligned} \end{cases}$$

compatible with the chained form. Indeed, notice that in the $z$-coordinates the distribution spanned by the controlled vector fields is in the chained form (Goursat normal form) and the drift has a triangular structure.

We will completely characterize control-affine systems that are static feedback equivalent to $TCh_1^k$ and show how their geometry differs and how it reminds that of control-linear systems feedback equivalent to the chained form. Then, we will extend this result to the triangular form compatible with the $m$-chained form, i.e., we will characterize control-affine systems with $m+1$ inputs, where $m \geq 2$, that are static feedback equivalent to a normal form obtained by replacing $z_j$, in $TCh_1^k$, by the vector $z^j = (z_1^j, \cdots, z_m^j)$, the smooth functions $f_j$ by $f^j = (f_1^j, \cdots, f_m^j)$ and the control $v_1$ by the control vector $(v_1, \cdots, v_m)$. This form will be denoted by $TCh_m^k$. Its associated distribution $\mathcal{G} = \text{span}\{g_0, \cdots, g_m\}$, where $g_i$, for $0 \leq i \leq m$, are the controlled vector fields, is called a Cartan distribution (or a contact distribution) for curves [7,48,67]. The problem of characterizing control-linear systems that are locally static feedback equivalent to the $m$-chained form (or equivalently, that of characterizing Cartan distributions for curves) has been studied and solved ( [59], see also [39,49,50,63,68]). It is immediate that systems locally feedback equivalent to the $m$-chained form are flat and in [58], all their minimal flat outputs (i.e., those whose differential weight is the lowest among all flat outputs of the system) have been described.

It is easy to see that the normal form $TCh_1^k$ (respectively $TCh_m^k$) is $x$-flat at any point of $X \times \mathbb{R}^2$ (respectively $X \times \mathbb{R}^{m+1}$) satisfying some regularity conditions and we describe all its $x$-flat outputs (respectively all its minimal $x$-flat outputs). Their description reminds very much that of control-linear systems feedback equivalent to the chained form, for $m = 1$, respectively to the $m$-chained form, for $m \geq 2$, although new phenomena appear related to singularities in the state and control-space.

Since $TCh_1^k$ and$TCh_m^k$ are flat, the paper gives sufficient conditions for a system to be $x$-flat. We will also show that these conditions are not necessary for $x$-flatness of control-affine system whose associated distribution spanned by the controlled vector fields $\mathcal{G} = \text{span}\{g_0, \cdots, g_m\}$ is feedback equivalent to the $m$-chained form. Indeed, we show that there are $x$-flat control-affine systems for which there exist local coordinates in which the distribution spanned by the controlled vector fields has the $m$-chained structure but the drift is not triangular (see Example 3.1.5.1).

The triangular form $TCh_1^k$ was considered in [30], where its flatness was observed but its description was not addressed. A characterization of $TCh_1^k$ has been recently proven by Silveira [64] and by Silveira et al. [65], where a solution dual to ours (using an approach based on differential forms and codistributions rather than distributions) is given. Our aim is to treat in a homogeneous way the two-input case of $TCh_1^k$ and the multi-input case of $TCh_m^k$, using the formalism of vector fields and distributions, as well as to describe all flat outputs and their singularities (which are more natural to deal with in the language of vector fields).

The paper is organized as follows. In Section 3.1.2, we recall the definition of flatness and define the notion of differential weight of a flat system. In Section 3.1.3, we give our main results: we characterize control-affine systems static feedback equivalent to the triangular form $TCh_1^k$, for $m = 1$, and to $TCh_m^k$, for $m \geq 2$. We describe in Section 3.1.4 all minimal flat outputs including their singularities and we study also singular control values at which the system ceases to be flat. Moreover, we give also in that section a system of first order PDE's to be solved in order to find all $x$-flat outputs, for $m = 1$, and all minimal $x$-flat outputs, for $m \geq 2$. We illustrate our results by two examples in Section 3.1.5 and provide proofs in Section 3.1.6.

### 3.1.2 Flatness

Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set $U^{-1}$ is empty and $\bar{u}^{-1}$ is an empty sequence.

**Definition 3.1.1.** The system $\Xi : \dot{x} = F(x, u)$ is *flat* at $(x^*, \bar{u}^{*l}) \in X \times U^l$, for $l \geq -1$, if there exists a neighborhood $\mathcal{O}^l$ of $(x^*, \bar{u}^{*l})$ and $m$ smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property:

there exist an integer $s$ and smooth functions $\gamma_i$, $1 \leq i \leq n$, and $\delta_j$, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

along any trajectory $x(t)$ given by a control $u(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called *flat output*.

When necessary to indicate the number of derivatives of $u$ on which the flat outputs $\varphi_i$ depend, we will say that the system $\Xi$ is $(x, u, \cdots, u^{(r)})$-flat if $u^{(r)}$ is the highest derivative on which $\varphi_i$ depend and in the particular case $\varphi_i = \varphi_i(x)$, we will say that the system is $x$-flat. In general, $r$ is smaller than the integer $l$ needed to define the neighborhood $\mathcal{O}^l$ which, in turn, is smaller than the number of derivatives of $\varphi_i$ that are involved. In our study, $r$ is always equal to -1, i.e., the flat outputs depend on $x$ only, and $l$ is 0.

The minimal number of derivatives of components of a flat output $\varphi$, needed to express $x$ and $u$, will be called the differential weight of that flat output and will be formalized as follows. By definition, for any flat output $\varphi$ of $\Xi$ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}), \end{aligned}$$

Moreover, we can choose $(s_1, \ldots, s_m)$ such that (see [58]) if for any other $m$-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ we have

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$.

We will call $\sum_{i=1}^{m}(s_i + 1) = m + \sum_{i=1}^{m} s_i$ the differential weight of $\varphi$. A flat output of $\Xi$ is called *minimal* if its differential weight is the lowest among all flat outputs of $\Xi$. We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output.

### 3.1.3 Main results: characterization of the triangular form

From now on, we will denote the number of controls by $m + 1$ (and not by $m$) since, as we will see below, for all classes of systems that follow one control plays a particular role.

Consider the control-affine system

$$\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x), \tag{3.2}$$

defined on an open subset $X$ of $\mathbb{R}^n$, where $n = km + 1$ (or an $n$-dimensional manifold $X$), where $f$ and $g_0, \cdots, g_m$ are smooth vector fields on $X$ and the number of controls is $m + 1 \geq 2$.

To $\Sigma_{aff}$ we associate the following distribution $\mathcal{G} = \text{span}\{g_0, \cdots, g_m\}$. We define inductively the derived flag of $\mathcal{G}$ by

$$\mathcal{G}^0 = \mathcal{G} \text{ and } \mathcal{G}^{i+1} = \mathcal{G}^i + [\mathcal{G}^i, \mathcal{G}^i],\ i \geq 0.$$

Let $\mathcal{D}$ be a non involutive distribution of rank $d$, defined on $X$ and define its annihilator $\mathcal{D}^\perp = \{\omega \in \Lambda^1(X) : < \omega, f >= 0, \forall f \in \mathcal{D}\}$, where $\Lambda^1(X)$ stands for the collection of smooth differential 1-forms on $X$. A vector field $c \in \mathcal{D}$ is called characteristic for $\mathcal{D}$ if it satisfies $[c, \mathcal{D}] \subset \mathcal{D}$. The characteristic distribution of $\mathcal{D}$, denoted by $\mathcal{C}$, is the distribution spanned by all its characteristic vector fields, i.e.,

$$\mathcal{C} = \{c \in \mathcal{D} : [c, \mathcal{D}] \subset \mathcal{D}\}$$

and can be computed as follows. Let $\omega_1, \ldots, \omega_q$, where $q = n - d$, be differential 1-forms locally spanning the annihilator of $\mathcal{D}$, that is $\mathcal{D}^\perp = \text{span}\{\omega_1, \ldots, \omega_q\}$. For any $\omega \in \mathcal{D}^\perp$, we define $\mathcal{W}(\omega) = \{f \in \mathcal{D} : f \lrcorner d\omega \in \mathcal{D}^\perp\}$, where $\lrcorner$ is the interior product. The characteristic distribution of $\mathcal{D}$ is given by

$$\mathcal{C} = \bigcap_{i=1}^{q} \mathcal{W}(\omega_i).$$

It follows directly from the Jacobi identity that the characteristic distribution is always involutive.

Our main results describing control-affine systems locally static feedback equivalent to the triangular form compatible to the chained form and to the $m$-chained form, are given by the two following theorems corresponding to two-input control-affine systems, i.e., $m = 1$ (Theorem 3.1.1), and to control-affine systems with $m + 1$ inputs, for $m \geq 2$ (Theorem 3.1.2). Let us first consider the case $m = 1$, which has also been solved, using the formalism of differential forms and codistributions, by Silveira [64] and by Silveira et al. [65].

**Theorem 3.1.1.** *Consider a two-input control-affine system $\Sigma_{aff}$, given by (3.2), for $m = 1$, and fix $x^* \in X$, an open subset of $\mathbb{R}^{k+1}$. The system $\Sigma$ is locally, around $x^*$, static feedback equivalent to the triangular form $TCh_1^k$ if and only if the following conditions are satisfied:*

*(Ch1)* $\mathcal{G}^{k-1} = TX$;

*(Ch2)* $\mathcal{G}^{k-3}$ *is of constant rank* $k - 1$ *and, moreover, the characteristic distribution* $\mathcal{C}^{k-2}$ *of* $\mathcal{G}^{k-2}$ *is contained in* $\mathcal{G}^{k-3}$ *and has constant corank one in* $\mathcal{G}^{k-3}$;

*(Ch3)* $\mathcal{G}^0(x^*)$ *is not contained in* $\mathcal{C}^{k-2}(x^*)$;

*(Comp)* $[f, \mathcal{C}^i] \subset \mathcal{G}^i$, *for* $1 \leq i \leq k-2$, *where* $\mathcal{C}^i$ *is the characteristic distribution of* $\mathcal{G}^i$.

It was stated and proved in [59] that items *(Ch1)-(Ch3)* characterize, locally, the chained form (or equivalently the Goursat normal form). Therefore, they are equivalent to the well known conditions describing the chained form [40] (see also [25,34,37,38,51]):

*(Ch1)'* $\operatorname{rk} \mathcal{G}^i = i + 2$, *for* $0 \leq i \leq k-1$,

*(Ch2)'* $\operatorname{rk} \mathcal{G}^i(x^*) = \operatorname{rk} \mathcal{G}_i(x^*) = i + 2$, *for* $0 \leq i \leq k-1$, *where the distributions* $\mathcal{G}_i$ *form the Lie flag of* $\mathcal{G}$ *and are defined by* $\mathcal{G}_0 = \mathcal{G}$ *and* $\mathcal{G}_{i+1} = \mathcal{G}_i + [\mathcal{G}_0, \mathcal{G}_i]$, $i \geq 0$,

and assure the existence of a change of coordinates $z = \phi(x)$ and of an invertible static feedback transformation of the form $u = \beta\tilde{u}$, bringing the control vector fields $g_0$ and $g_1$ into the chained form.

Item $(Comp)$ takes into account the drift and gives the compatibility conditions for $f$ to have the desired triangular form in the right system of coordinates, i.e., in coordinates $z$ in which the controlled vector fields are in the chained form.

Since the distribution $\mathcal{G}$, associated to $\Sigma_{aff}$, satisfies $(Ch1)'$, all characteristic distributions $\mathcal{C}^i$ of $\mathcal{G}^i$ are well defined, for $1 \leq i \leq k-2$. Indeed, recall the following result due to Cartan [8]:

**Lemma 3.1.1.** (E. Cartan) *Consider a rank two distribution* $\mathcal{G}$ *defined on a manifold* $X$ *of dimension* $k+1$, *for* $k \geq 3$. *If* $\mathcal{G}$ *satisfies* $\operatorname{rk} \mathcal{G}^i = i+2$, *for* $0 \leq i \leq k-1$, *everywhere on* $X$, *then each distribution* $\mathcal{G}^i$, *for* $0 \leq i \leq k-3$, *contains a unique involutive subdistribution* $\mathcal{C}^{i+1}$ *that is characteristic for* $\mathcal{G}^{i+1}$ *and has constant corank one in* $\mathcal{G}^i$.

The conditions of the above theorem are verifiable, i.e., given a two-input control-affine system and an initial point $x^*$, we can verify whether it is locally static feedback equivalent, around $x^*$, to $TCh_1^k$ and verification (in terms of vector fields of the initial system) involves derivations and algebraic operations only, without solving PDE's.

Next, we consider the case $m \geq 2$ and extend the above result to a triangular form compatible with the $m$-chained form. An $(m+1)$-input driftless control system $\Sigma_{lin} : \dot{z} = \sum_{i=0}^{m} v_i g_i(z)$, defined on $\mathbb{R}^{km+1}$, is said to be in the $m$-chained form if it is represented by

$$Ch_m^k : \begin{cases} \dot{z}_0 = v_0 & \dot{z}_1^1 = z_1^2 v_0 & \cdots & \dot{z}_m^1 = z_m^2 v_0 \\ & \dot{z}_1^2 = z_1^3 v_0 & & \dot{z}_m^2 = z_m^3 v_0 \\ & \vdots & & \vdots \\ & \dot{z}_1^{k-1} = z_1^k v_0 & & \dot{z}_m^{k-1} = z_m^k v_0 \\ & \dot{z}_1^k = v_1 & \cdots & \dot{z}_m^k = v_m \end{cases}$$

Denote $\bar{z}^j = (z_1^1, \cdots, z_m^1, z_1^2, \cdots, z_m^2, \cdots, z_1^j, \cdots z_m^j)$, for $2 \leq j \leq k$. Our goal is to characterize the following triangular normal form

$$TCh_m^k : \begin{cases} \dot{z}_0 = v_0 & \dot{z}_1^1 = f_1^1(z_0, \bar{z}^2) + z_1^2 v_0 & \cdots & \dot{z}_m^1 = f_m^1(z_0, \bar{z}^2) + z_m^2 v_0 \\ & \dot{z}_1^2 = f_1^2(z_0, \bar{z}^3) + z_1^3 v_0 & & \dot{z}_m^2 = f_m^2(z_0, \bar{z}^3) + z_m^3 v_0 \\ & \vdots & & \vdots \\ & \dot{z}_1^{k-1} = f_1^{k-1}(z_0, \bar{z}^k) + z_1^k v_0 & \cdots & \dot{z}_m^{k-1} = f_m^{k-1}(z_0, \bar{z}^k) + z_m^k v_0 \\ & \dot{z}_1^k = v_1 & \cdots & \dot{z}_m^k = v_m \end{cases}$$

with $m+1$ inputs, $m \geq 2$. Theorem 3.1.2 below gives necessary and sufficient conditions for a control system to be locally static feedback equivalent to $TCh^k_m$.

**Theorem 3.1.2.** *Consider a control-affine system $\Sigma_{aff}$, given by (3.2), on an open subset $X$ of $\mathbb{R}^{km+1}$, for $m \geq 2$, and fix $x^* \in X$. The system $\Sigma_{aff}$ is locally, around $x^*$, static feedback equivalent to the triangular form $TCh^k_m$ if and only if the following conditions are satisfied:*

*(m-Ch1)* $\mathcal{G}^{k-1} = TX$;

*(m-Ch2)* $\mathcal{G}^{k-2}$ *is of constant rank* $(k-2)m+1$ *and contains an involutive subdistribution* $\mathcal{L}$ *that has constant corank one in* $\mathcal{G}^{k-2}$;

*(m-Ch3)* $\mathcal{G}^0(x^*)$ *is not contained in* $\mathcal{L}(x^*)$;

*(m-Comp)* $[f, \mathcal{C}^i] \subset \mathcal{G}^i$, *for* $1 \leq i \leq k-2$, *where* $\mathcal{C}^i$ *is the characteristic distribution of* $\mathcal{G}^i$.

In order to verify the conditions of Theorem 3.1.2, we have to check whether the distribution $\mathcal{G}^{k-2}$ contains an involutive subdistribution $\mathcal{L}$ of corank one. Checkable necessary and sufficient conditions for the existence of such an involutive subdistribution, together with a construction, follow from the work of Bryant [6] and are given explicitly in [50]. We present in Appendix 3.1.A the conditions for the existence and construction of $\mathcal{L}$. In our case, if such a distribution exists, it is always unique. As a consequence, all conditions of Theorem 3.1.2 are verifiable, i.e., given a control-affine system and an initial point $x^*$, we can verify whether it is locally static feedback equivalent, around $x^*$, to $TCh^k_m$ and verification involves derivations and algebraic operations only, without solving PDE's.

Conditions *(m-Ch1)-(m-Ch3)* characterize the $m$-chained form [59] (see also [49, 50]) and assure the existence of a change of coordinates $z = \phi(x)$ and of an invertible static feedback transformation of the form $u = \beta\tilde{u}$, bringing the control vector fields $g_i$ into the $m$-chained form. We define the diffeomorphism $\phi$ and the feedback transformation $\beta$ in Appendix 3.1.B. The diffemorphism $\phi$ defines also the coordinates in which the system takes the triangular form $TCh^k_m$.

Item *(m-Comp)* takes into account the drift and gives the compatibility conditions for $f$ to have the desired triangular form in the right system of coordinates, i.e., in $z$-coordinates in which the controlled vector fields are in the $m$-chained form. Formally it has the same form as $(Comp)$ in the case $m = 1$.

The characteristic distributions $\mathcal{C}^i$, for $1 \leq i \leq k-2$, are well defined and have corank one in $\mathcal{G}^{i-1}$. Indeed, recall the following result stated in [59]:

**Lemma 3.1.2.** *Assume that a distribution $\mathcal{G}$ defined on a manifold $X$ of dimension $km+1$ satisfies the conditions (m-Ch1)-(m-Ch3) of Theorem 3.1.2. Then $\mathcal{G}^i$ has constant rank $(i+1)m+1$, for $0 \leq i \leq k-2$, and contains an involutive subdistribution $\mathcal{L}^i$ of corank one in $\mathcal{G}^i$. Moreover $\mathcal{L}^i$ is the unique corank one subdistribution satisfying this property, for $0 \leq i \leq k-2$, and it coincides with the characteristic distribution $\mathcal{C}^{i+1}$ of $\mathcal{G}^{i+1}$, for $0 \leq i \leq k-3$.*

It has been shown in [56] (see also [59]) that all information about the distribution $\mathcal{G}$ is encoded completely in the existence of the last involutive subdistribution $\mathcal{L}^{k-2}$ (being, actually, the involutive distribution $\mathcal{L}$ of item *(m-Ch2)* of Theorem 3.1.2) which implies the existence of all involutive subdistributions $\mathcal{L}^i = \mathcal{C}^{i+1}$, for $0 \leq i \leq k-3$.

The characterization of the chained form (conditions *(Ch1)-(Ch3)* of Theorem 3.1.1) and that of the $m$-chained form (*(m-Ch1)-(C-mCh3)* of Theorem 3.1.2) are different, but compatibility conditions are the same, compare *(Comp)* and *(m-Comp)*. The involutive subdistribution $\mathcal{L}$, which is crucial for the $m$-chained form, is absent in the compatibility conditions, but plays a very important role in calculating minimal flat outputs and in describing singularities (see Section 3.1.4).

### 3.1.4 Flatness and flat outputs description

In this section, firstly, we discuss flatness of control systems static feedback equivalent to $TCh_1^k$, respectively to $TCh_m^k$. Secondly, we answer the question whether a given pair (respectively an $(m+1)$-tuple) of smooth functions on $X$ is an $x$-flat output for a system static feedback equivalent to $TCh_1^k$ (respectively a minimal $x$-flat output for a system static feedback equivalent to $TCh_m^k$) and, finally, provide a system of PDS's to be solved in order to find all these flat outputs. In particular, we will discuss their uniqueness, their singularities, and compare their description with that of flat outputs for the chained form (respectively for the $m$-chained form).

#### 3.1.4.1 Flatness of control systems static feedback equivalent to $TCh_1^k$

Let us first consider the case $m = 1$. It is clear that $TCh_1^k$ is $x$-flat, with $\varphi = (z_0, z_1)$ being a flat output around any point $(z^*, v^*)$ satisfying

$$\frac{\partial f_i}{\partial z_{i+1}}(z^*) + v_0^* \neq 0, \text{ for } 1 \leq i \leq k-1,$$

where $v^* = (v_0^*, v_1^*)$. Therefore control systems equivalent to $TCh_1^k$ are $x$-flat and exhibit a singularity in the control space (depending on the state) which we will describe in an invariant way as follows. For $\mathcal{C}^1 \subset \mathcal{C}^2 \subset \cdots \subset \mathcal{C}^{k-2}$, the sequence of characteristic distributions $\mathcal{C}^i$ of $\mathcal{G}^i$, for $1 \leq i \leq k-2$, see Lemma 3.1.1, choose vector fields $c_1, \ldots, c_{k-2}$ such that $\mathcal{C}^i = \text{span}\,\{c_1, \ldots, c_i\}$. For each $0 \leq i \leq k-3$, define

$$U^i_{sing}(x) = \left\{ u^i(x) = (u^i_0(x), u^i_1(x))^\top : [f + u^i_0 g_0 + u^i_1 g_1, \mathcal{C}^{i+1}] \subset \mathcal{G}^i \right\}.$$

The controls $u^i(x)$ exist, are smooth, and for any $0 \leq i \leq k-3$ define (for any fixed $x \in X$) a 1-dimensional affine subspace of $U = \mathbb{R}^2$. To see those three properties, notice that $[f, c_{i+1}]$, $[g_0, c_{i+1}]$, and $[g_1, c_{i+1}]$ span a distribution of rank one modulo $\mathcal{G}^i$ (since all three belong to $\mathcal{G}^{i+1}$ and corank$(\mathcal{G}^i \subset \mathcal{G}^{i+1}) = 1$) and either $[g_0, c_{i+1}]$ or $[g_1, c_{i+1}]$ (or both) does not vanish modulo $\mathcal{G}^i$. To calculate $U^i_{sing}(x)$ explicitly, assume that we have chosen $(g_0, g_1)$ such that $g_1 = c_1$. Then $[g_1, c_{i+1}] = [c_1, c_{i+1}] \in \mathcal{G}^i$ and

$[f, c_{i+1}] = \alpha[g_0, c_{i+1}] \bmod \mathcal{G}^i$, for some smooth function $\alpha$. We put $u_0^i(x) = -\alpha(x)$ and $u_1^i(x)$ arbitrary. It is clear that the definition of $(u_0^i(x), u_1^i(x))$ does not depend on the choice of $c_1, \ldots, c_{k-2}$ and is feedback invariant (independently of whether we have chosen $g_1 = c_1$ or not). Indeed, if $u^i(x) \in U^i_{sing}(x)$, then for the feedback modified system $\dot{x} = \tilde{f} + \tilde{g}\tilde{u}$, where $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$, it is the feedback modified control $\tilde{u}^i = \beta^{-1}(-\alpha + u^i)$ that, clearly, satisfies $\tilde{u}^i \in U^i_{sing}$.

Let $\mathcal{L}$ be any involutive distribution of corank two in $TX$ such that $\mathcal{L} \subset \mathcal{G}^{k-2}$. Fix $l \in \mathcal{L}$ such that $l \notin \mathcal{C}^{k-2}$ and put

$$U^{k-2}_{\mathcal{L}-sing}(x) = \left\{ u^{k-2}(x) = (u_0^{k-2}(x), u_1^{k-2}(x))^\top : [f + u_0^{k-2} g_0 + u_1^{k-2} g_1, l] \in \mathcal{G}^{k-2} \right\}.$$

If $\mathcal{G}^0(x^*) \not\subset \mathcal{L}(x^*)$, where $x^*$ is a nominal point around which we work, then the controls $u^{k-2}(x)$ exist, are smooth, and (for any fixed $x \in X$) form a 1-dimensional affine subset of $U = \mathbb{R}^2$ because $\mathcal{G}^{k-2}$ is of corank one in $TX$ and either $[g_0, l]$ or $[g_1, l]$ is not in $\mathcal{G}^{k-2}$. If $\mathcal{G}^0(x^*) \subset \mathcal{L}(x^*)$, then under the assumption, which we will always assume, $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$, where the functions $\varphi_0$ and $\varphi_1$ are such that $\mathcal{L}^\perp = \operatorname{span}\{d\varphi_0, d\varphi_1\}$, we have $u^* \notin U^{k-2}_{\mathcal{L}-sing}(x^*)$ and in $\mathcal{X}^* \times \mathbb{R}^2$, where $\mathcal{X}^*$ is a sufficiently small neighborhood of $x^*$, the set $U^{k-2}_{\mathcal{L}-sing}(x)$ consists of two connected components that define, for each fixed value $x \in \mathcal{X}^*$, $x \neq x^*$, an affine subspace of $U = \mathbb{R}^2$.

Clearly $U^{k-2}_{\mathcal{L}-sing}$ is feedback invariant and does not depend on the choice of $l \in \mathcal{L}$ but it depends on the distribution $\mathcal{L}$. Define

$$U^{k-2}_{sing} = \bigcap_{\mathcal{L}} U^{k-2}_{\mathcal{L}-sing}$$

where the intersection is taken over all $\mathcal{L}$ as above, that is, involutive distribution of corank two in $TX$, satisfying $\mathcal{L} \subset \mathcal{G}^{k-2}$ . Define

$$U_{sing} = \bigcup_{i=0}^{k-3} U^i_{sing} \cup U^{k-2}_{sing}$$

and

$$U_{\mathcal{L}-sing} = \bigcup_{i=0}^{k-3} U^i_{sing} \cup U^{k-2}_{\mathcal{L}-sing}.$$

We will use both sets in Theorem 3.1.3 describing controls singular for flatness and in Proposition 3.1.1 comparing flat outputs of the triangular form $TCh_1^k$ with those of the associated chained form $Ch_1^k$.

**Theorem 3.1.3.** *Consider a two-input control-affine system $\Sigma_{aff} : \dot{x} = f(x) + u_0 g_0(x) + u_1 g_1(x)$, defined on an open subset $X$ of $\mathbb{R}^{k+1}$, where $k + 1 \geq 4$. Assume that $\Sigma_{aff}$ is locally, around $x^* \in X$, static feedback equivalent to $TCh_1^k$. Then we have:*

*(F1) $\Sigma_{aff}$ is x-flat at any $(x^*, u^*) \in X \times \mathbb{R}^2$ such that $u^* \notin U_{sing}(x^*)$.*

*(F2) Let $\varphi_0$, $\varphi_1$ be two smooth functions defined in a neighborhood $\mathcal{X}$ of $x^*$ and $g$ be an arbitrary vector field in $\mathcal{G}$ such that $g(x^*) \notin \mathcal{C}^{k-2}(x^*)$. Then the following conditions are equivalent in $\mathcal{X}$:*

*(i)* *The pair* $(\varphi_0, \varphi_1)$ *is an x-flat output of* $\Sigma_{aff}$ *at* $(x^*, u^*) \in \mathcal{X}^* \times \mathbb{R}^2$, *where* $\mathcal{X}^*$ *is a neighborhood of* $x^*$ *;*

*(ii)* *The pair* $(\varphi_0, \varphi_1)$ *satisfies the following conditions:*

*(FO1)* $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$, *where* $\dot{\varphi}_i = L_{F_{aff}}\varphi_i$, *for* $i = 0, 1$ *and* $F_{aff} = f + u_0 g_0 + u_1 g_1$*;*

*(FO2)* $L_c\varphi_0 = L_c\varphi_1 = 0$ *and* $(L_g\varphi_0)(L_{[c,g]}\varphi_1) - (L_g\varphi_1)(L_{[c,g]}\varphi_0) = 0$, *for any* $c \in \mathcal{C}^{k-2}$*;*

*(FO3)* $u^* \notin U_{\mathcal{L}-sing}(x^*)$, *where* $\mathcal{L} = (\mathrm{span}\,\{d\varphi_0, d\varphi_1\})^{\perp}$.

*(iii)* *The pair* $(\varphi_0, \varphi_1)$ *satisfies the following conditions:*

*(FO1)′* $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$ , *where* $\dot{\varphi}_i = L_{F_{aff}}\varphi_i$, *for* $i = 0, 1$, *and* $F_{aff} = f + u_0 g_0 + u_1 g_1$*;*

*(FO2)′* $\mathcal{L} = (\mathrm{span}\,\{d\varphi_0, d\varphi_1\})^{\perp} \subset \mathcal{G}^{k-2}$*;*

*(FO3)′* $u^* \notin U_{\mathcal{L}-sing}(x^*)$.

Notice that since $\Sigma_{aff}$ is locally, around $x^*$, static feedback equivalent to $TCh_1^k$, its associated control-linear system $\Sigma_{lin} : \dot{x} = u_0 g_0(x) + u_1 g_1(x)$ is locally, around $x^*$, static feedback equivalent to the chained form $Ch_1^k$. The next result shows how the similarities and differences between two-input control-linear systems and control-affine systems locally equivalent to $TCh_1^k$ are reflected by their flatness. It turns out that flat outputs of $\Sigma_{lin}$ are flat outputs of $\Sigma_{aff}$ (independently of the choice of $f$ although singular control values depend on $f$) and most of flat outputs of $\Sigma_{aff}$ are flat outputs of the corresponding $\Sigma_{lin}$ but not all, as the following proposition explains. Define

$$U_{char}(x) = \left\{ u(x) = (u_0(x), u_1(x))^{\top} : (u_0 g_0 + u_1 g_1)(x) \in \mathcal{C}^1(x) \right\}.$$

**Proposition 3.1.1.** *Consider a two-input control-affine system* $\Sigma_{aff} : \dot{x} = f(x) + u_0 g_0(x) + u_1 g_1(x)$, *defined on an open subset* $X$ *of* $\mathbb{R}^{k+1}$, *where* $k + 1 \geq 4$, *and its associated control-linear system* $\Sigma_{lin} : \dot{x} = u_0 g_0(x) + u_1 g_1(x)$. *Assume that* $\Sigma_{aff}$ *is locally, around* $x^* \in X$, *static feedback equivalent to* $TCh_1^k$. *Then we have:*

*(F3)* $\Sigma_{lin}$ *is x-flat at any* $(x^*, u^*) \in X \times \mathbb{R}^2$ *such that* $u^* \notin U_{char}(x^*)$.

*(F4)* *A pair* $(\varphi_0, \varphi_1)$ *of smooth functions defined in a neighborhood* $\mathcal{X}$ *of* $x^*$ *is an x-flat output of* $\Sigma_{lin}$ *at* $(x^*, u^*) \in \mathcal{X}^* \times \mathbb{R}^2$ *such that* $\mathcal{X}^* \subset X$ *is an open neighborhood of* $x^*$ *and* $u^* \notin U_{char}(x^*)$ *if and only if it satisfies the conditions (FO1)-(FO2) or, equivalently, (FO1)′-(FO2)′ of Theorem 3.1.3, where* $\dot{\varphi}_i$, *for* $i = 0, 1$, *is understood as* $\dot{\varphi}_i = L_{F_{lin}}\varphi_i$ *and* $F_{lin} = u_0 g_0 + u_1 g_1$*;*

*(F5)* *If* $(\varphi_0, \varphi_1)$ *is a flat output of* $\Sigma_{lin}$ *at* $(x^*, u^*)$, *where* $u^* \notin U_{char}(x^*)$, *then* $(\varphi_0, \varphi_1)$ *is a flat output of* $\Sigma_{aff}$ *at*$(x^*, \tilde{u}^*)$, *where* $\tilde{u}^* \notin U_{\mathcal{L}-sing}(x^*)$ *with* $\mathcal{L} = (\mathrm{span}\,\{d\varphi_0, d\varphi_1\})^{\perp}$.

*(F6)* *Let* $g$ *be an arbitrary vector field in* $\mathcal{G}$ *such that* $g(x^*) \notin \mathcal{C}^{k-2}(x^*)$. *If* $(\varphi_0, \varphi_1)$ *is a flat output of* $\Sigma_{aff}$ *at* $(x^*, \tilde{u}^*)$, *where* $\tilde{u}^* \notin U_{\mathcal{L}-sing}(x^*)$, *with* $\mathcal{L} = (\mathrm{span}\,\{d\varphi_0, d\varphi_1\})^{\perp}$, *and satisfies* $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0, 0)$, *then* $(\varphi_0, \varphi_1)$ *is a flat output of* $\Sigma_{lin}$ *at* $(x^*, u^*)$, *where* $u^* \notin U_{char}(x^*)$.

For a pair of functions $(\varphi_0, \varphi_1)$, the conditions to be a flat output are, formally, the same for $\Sigma_{aff}$ and the associated control-linear system $\Sigma_{lin}$ and are given by (FO1)-(FO2) (or, equivalently, by (FO1)′-(FO2)′). Notice, however, that the vector field along which we differentiate changes from $F_{aff}$ into $F_{lin}$ and thus the conditions change as well. This implies that there is more flat outputs for $\Sigma_{aff}$ than for the associated $\Sigma_{lin}$. Actually, the condition (FO1) applied to $\Sigma_{lin}$ implies that $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0,0)$ (thus obtaining the same necessary and sufficient conditions as those given in [29] for two-input control-linear systems ), whereas (FO1) applied to $\Sigma_{aff}$ still admits systems for which $(L_g\varphi_0, L_g\varphi_1)(x^*) = (0,0)$ as the following example shows.

**Example 3.1.1.** Consider the control-affine system:

$$\dot z_0 = v_0 \qquad \begin{array}{rcl} \dot z_1 & = & z_0 + z_2 v_0 \\ \dot z_2 & = & z_3 v_0 \\ & \vdots & \\ \dot z_{k-1} & = & z_k v_0 \\ \dot z_k & = & v_1 \end{array}$$

which is in the triangular form compatible with the chained form $TCh_1^k$. We claim that it is $x$-flat with $(\varphi_0, \varphi_1) = (z_1 - z_0 z_2, z_2)$ as $x$-flat output around $z^* = 0$, although $(L_g\varphi_0, L_g\varphi_1)(0) = (0,0)$, for any vector field in $\mathcal{G}$ such that $g(z^*) \notin \mathcal{C}^{k-2}(z^*)$, provided that $v_0^* \neq 0$ and $(1 - z_3^* v_0^*) \neq 0$, the latter condition being always satisfied at $z^* = 0$, but not in a neighborhood.

Indeed, we have $\dot\varphi_0 = z_0 - z_0 z_3 v_0$, $\dot\varphi_1 = z_3 v_0$ and it follows that $\dot\varphi_0 = z_0(1 - \dot\varphi_1)$, from which we deduce $z_0 = \frac{\dot\varphi_0}{1-\dot\varphi_1}$, provided that $1 - \dot\varphi_1 = 1 - z_3^* v_0^* \neq 0$. By differentiating that relation, we get $v_0 = \dot z_0 = \frac{d}{dt}(\frac{\dot\varphi_0}{1-\dot\varphi_1}) = \delta_0(\bar\varphi_0^2, \bar\varphi_1^2)$, where $\bar\varphi_i^j = (\varphi_i, \dot\varphi_i, \cdots, \varphi_i^{(j)})$. From $\dot\varphi_1 = z_3 v_0$, we compute $z_3 = \frac{\dot\varphi_1}{v_0} = \gamma_3(\bar\varphi_0^2, \bar\varphi_1^2)$. Then, $\dot z_3$ gives $z_4 = \gamma_4(\bar\varphi_0^3, \bar\varphi_1^3)$ and so on. Finally we get $z_k = \gamma_k(\bar\varphi_0^{k-1}, \bar\varphi_1^{k-1})$ and $v_1 = \delta_1(\bar\varphi_0^k, \bar\varphi_1^k)$. Thus $(\varphi_0, \varphi_1) = (z_1 - z_0 z_2, z_2)$ is indeed an $x$-flat output of the system around $z^* = 0$ such that $z_3^* v_0^* \neq 1$.

Let us now consider the chained form $Ch_1^k$ and take $g = g_0$. We compute $L_g\varphi_0 = -z_0 z_3 v_0$, $L_g\varphi_1 = z_3 v_0$ and we clearly have $(L_g\varphi_0, L_g\varphi_1)(0) = (0,0)$. Since the condition $(L_g\varphi_0, L_g\varphi_1)(z^*) \neq (0,0)$ is necessary for $(\varphi_0, \varphi_1)$ to be an $x$-flat output for the chained form, see [29], we deduce that $(\varphi_0, \varphi_1) = (z_1 - z_0 z_2, z_2)$ is not an $x$-flat output at $z^* = 0$ for $Ch_1^k$. □

For control-linear systems $\Sigma_{lin}$, the choice of a flat output is not unique (different choices are parameterized by an arbitrary function of three variables whose differentials annihilate $\mathcal{C}^{k-2}$, as assures Proposition 3.1.2 below) but all flat outputs exhibit the same singularity in control space (see item $(F4)$ of Proposition 3.1.1), which is the control $u_c$, where $u_c \in U_{char}$ such that $u_{c,0} g_0 + u_{c,1} g_1 \in \mathcal{C}^1$ ( for any $x \in X$, it defines a one-dimensional linear subspace of $U = \mathbb{R}^2$). In the control-affine case, the nature of singularities changes substantially: each choice of a flat output creates its own singularities in the control space. More precisely, a flat output $(\varphi_0, \varphi_1)$ ceases to be a flat output for controls $u^*$ belonging to $U_{\mathcal{L}-sing}$ which is the union of $\bigcup_{i=0}^{k-3} U_{sing}^i$ (universal for all choices of $(\varphi_0, \varphi_1)$ and consisting, for each fixed $x \in X$, of the union of $k-2$ one-dimensional affine subspaces of $U = \mathbb{R}^2$) and of $U^{k-2}_{\mathcal{L}-sing}$, which is a one-dimensional affine subspace of $U = \mathbb{R}^2$ that depends on $(\varphi_0, \varphi_1)$ since $\mathcal{L} = (\mathrm{span}\,\{d\varphi_0, d\varphi_1\})^{\perp}$. All those $k-1$ affine subspaces are, in general, different although some of them may coincide and, indeed, in the control-linear case all of them coincide and reduce to the linear-space of $U = \mathbb{R}^2$ containing the characteristic controls $u_c$ that correspond to the characteristic distribution $\mathcal{C}^1$, that is, the corresponding trajectories remain tangent to $\mathcal{C}^1$. Moreover, if we apply an invertible feedback $u = \beta\tilde{u}$ (which always exists and can be explicitly calculated) such that $\mathcal{C}^1 = \mathrm{span}\,\{\tilde{g}_1\}$ and $\mathcal{G}^0 = \mathrm{span}\,\{\tilde{g}_0, \tilde{g}_1\}$, a control $\tilde{u}_c$ is characteristic, that is, singular for flatness of $\Sigma_{lin}$, if and only if the feedback modified control is $\tilde{u}_c = \beta^{(-1)}u_c = (0, \tilde{u}_{c,1})^T$.

Now it is clear that the control-affine system $\Sigma_{aff}$ is flat if we avoid the universal singular set $\bigcup_{i=0}^{k-3} U^i_{sing}$ as well as the set singular for all choices of flat outputs $(\varphi_0, \varphi_1)$, that is the set $\bigcap U^{k-2}_{\mathcal{L}-sing}$ (the intersection taken over all $\mathcal{L}$), which explains different statements for a fixed choice of $(\varphi_0, \varphi_1)$ in item $(F2)(ii)$ and an arbitrary choice of $(\varphi_0, \varphi_1)$ in item $(F1)$.

Notice that Theorem 3.1.3 is valid for any $k \geq 3$ (thus for a system defined on a manifold $X$ of dimension at least 4). In fact, in item $(ii)$, we use the characteristic distribution $\mathcal{C}^{k-2}$ of $\mathcal{G}^{k-2}$, but if dim $X$ = 3, i.e., $k = 2$, such a distribution does not exist and item $(ii)$ does not apply to that case. Item $(iii)$, however, is well defined even for dim $X$ = 3 and remains equivalent to $(i)$.

As an immediate corollary of Theorem 3.1.3, we obtain a system of first order PDE's, described by Proposition 3.1.2 below, whose solutions give all $x$-flat outputs. Like for systems equivalent to the chained form (see [29]), $x$-flat outputs for the systems feedback equivalent to the triangular form $TCh^k_1$ are far from being unique: since the distribution $\mathcal{C}^{k-2}$ is involutive and of corank three, there are as many functions $\varphi_0$ satisfying $L_c\varphi_0 = 0$, for any $c \in \mathcal{C}^{k-2}$, as functions of three variables. Indeed, according to the following proposition, $\varphi_0$ can be chosen as any function of the three independent functions, whose differentials annihilate $\mathcal{C}^{k-2}$, and if moreover, $< d\varphi_0, \mathcal{G}^0 > (x^*) \neq 0$, then there exists a unique $\varphi_1$ (up to a diffeomorphism) completing it to an $x$-flat output.

**Proposition 3.1.2.** *Consider a two-input control-affine system* $\Sigma_{aff} : \dot{x} = f(x) + u_0g_0(x) + u_1g_1(x)$, *defined on a manifold X, of dimension* $k+1 \geq 4$, *that is locally, around* $x^* \in X$, *static feedback equivalent to* $TCh^k_1$. *Let* $\mathcal{C}^{k-2} = \mathrm{span}\,\{c_1, \cdots, c_{k-2}\}$ *be the characteristic distribution of* $\mathcal{G}^{k-2}$ *such that* $c_{k-2}(x^*) \notin \mathcal{C}^{k-3}(x^*)$ *and* $g$ *be an arbitrary vector field in* $\mathcal{G}$ *such that* $g(x^*) \notin \mathcal{C}^{k-2}(x^*)$. *Then*

*(i) For any smooth function* $\varphi_0$ *such that*

$$\text{(Flat 1)}\quad L_{c_i}\varphi_0 = 0,\ 1 \leq i \leq k-2,\ \text{and}\ < d\varphi_0, \mathcal{G}^{k-2} > (x^*) \neq 0,$$

*the distribution* $\mathcal{L} = \mathcal{C}^{k-2} + \mathrm{span}\,\{v\}$ *is involutive, where* $v = (L_g\varphi_0)[c_{k-2}, g] - (L_{[c_{k-2},g]}\varphi_0)g$.

*(ii) A pair* $(\varphi_0, \varphi_1)$ *of smooth functions defined on a neighborhood of* $x^*$ *is an x-flat output at* $(x^*, u^*)$ *with* $u^* \notin U_{\mathcal{L}-sing}(x^*)$, *if and only if (after permuting* $\varphi_0$ *and* $\varphi_1$, *if*

*necessary)* $\varphi_0$ *is any function satisfying (Flat 1) and* $\varphi_1$ *satisfies*

$$\text{(Flat 2)} \quad \begin{cases} (d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0, \\ L_{c_i}\varphi_1 = 0, \text{ for } 1 \leq i \leq k-2, \\ L_v \varphi_1 = 0. \end{cases}$$

(iii) *If in (Flat 1), we replace* $< d\varphi_0, \mathcal{G}^{k-2} > (x^*) \neq 0$ *by* $< d\varphi_0, \mathcal{G}^0 > (x^*) \neq 0$*, then for any function* $\varphi_0$ *satisfying* $L_c \varphi_0 = 0$*, for any* $c \in \mathcal{C}^{k-2}$*, and* $< d\varphi_0, \mathcal{G}^0 > (x^*) \neq 0$*, there always exists* $\varphi_1$ *such that the pair* $(\varphi_0, \varphi_1)$ *is an x-flat output of* $\Sigma_{aff}$*; given any such* $\varphi_0$*, the choice of* $\varphi_1$ *is unique, up to a diffeomorphism, that is, if* $(\varphi_0, \tilde{\varphi}_1)$ *is another minimal x-flat output, then there exists a smooth map h, smoothly invertible with respect to the second argument, such that*

$$\tilde{\varphi}_1 = h(\varphi_0, \varphi_1).$$

*Remark.* Notice that for a function $\varphi_0$ satisfying $< d\varphi_0, \mathcal{G}^{k-2} > (x^*) \neq 0$ (and not the stronger condition $< d\varphi_0, \mathcal{G}^0 > (x^*) \neq 0$, or equivalently $L_g \varphi_0(x^*) \neq 0$, see Proposition 3.1.2(iii)), it can be impossible to find, among all solutions of $L_{c_i}\varphi_1 = L_v \varphi_1 = 0, 1 \leq i \leq k-2$, a function $\varphi_1$ satisfying $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$ and therefore item $(iii)$ does not hold, in general, under the weaker condition $< d\varphi_0, \mathcal{G}^{k-2} > (x^*) \neq 0$. This is, for example, the case of control-linear systems.

As expected, the system of PDE's allowing us to compute all $x$-flat outputs of a system locally static feedback equivalent to $TCh_1^k$ does not depend on the drift $f$ and it is the same as that provided in [29] for $x$-flat outputs in the case of control-linear $\Sigma_{lin}$ feedback equivalent to the chained form. For more details and the proof of Proposition 3.1.2 in the case $L_g \varphi_0(x^*) \neq 0$, we refer the reader to [29].

Finally, it turns out that almost all $x$-flat outputs are compatible with the triangular form $TCh_1^k$ (as are $x$-flat outputs of the chained form). In fact, for any given flat output $(\varphi_0, \varphi_1)$ of a system $\Sigma_{aff}$ feedback equivalent to $TCh_1^k$, verifying $(L_g \varphi_0, L_g \varphi_1)(x^*) \neq (0, 0)$, we can bring $\Sigma_{aff}$ into $TCh_1^k$ for which $\varphi_0$ and $\varphi_1$ serve as the two top variables, as the following proposition assures. The following result is technical and will be useful in our proofs, but it has its own interest.

**Proposition 3.1.3.** *Assume that* $\Sigma_{aff}$ *is locally, around* $x^*$*, static feedback equivalent to the triangular form* $TCh_1^k$ *and let* $(\varphi_0, \varphi_1)$ *be an x-flat output around* $(x^*, u^*)$*, such that* $(L_g \varphi_0, L_g \varphi_1)(x^*) \neq (0, 0)$*, where g is an arbitrary vector field in* $\mathcal{G}$ *such that* $g(x^*) \notin \mathcal{C}^{k-2}(x^*)$*. Then we can bring* $\Sigma_{aff}$ *to* $TCh_1^k$ *around* $z^*$ *such that* $z_0 = \varphi_0$ *and* $z_1 = \varphi_1$ *(after permuting* $\varphi_0$ *and* $\varphi_1$*, if necessary).*

*Remark.* The above proposition is valid around $z^*$ which is not necessary equal to 0. If we want to map $x^*$ into $z^* = 0$, then an affine transformation of flat outputs may be needed. More precisely, we can bring $\Sigma_{aff}$ to $TCh_1^k$ around $z^* = 0$ such that $z_0 = \varphi_0$ and $z_1 = \varphi_1 + k_0 \varphi_0$ (after permuting $\varphi_0$ and $\varphi_1$), where $k_0 \in \mathbb{R}$.

#### 3.1.4.2 Flatness of control systems static feedback equivalent to $TCh_m^k$

We now turn to the case $m \geq 2$. It is clear that $TCh_m^k$ is $x$-flat, with $\varphi = (z_0, z_1^1, \cdots, z_m^1)$ being a flat output, at any point $(z^*, v^*) \in \mathbb{R}^{km+1} \times \mathbb{R}^{m+1}$ satisfying

$$\operatorname{rk} F^l(z^*, v^*) = m, \text{ for } 1 \leq l \leq k-1,$$

where $F^l = (F_{ij}^l)$, for $1 \leq l \leq k-1$, is the $m \times m$ matrix given by

$$F_{ij}^l = \frac{\partial(f_j^l + z_j^{l+1} v_0)}{\partial z_i^{l+1}}, \text{ for } 1 \leq i, j \leq m.$$

Therefore, flat systems equivalent to $TCh_m^k$ exhibit singularities in the control space (depending on the state) defined in an invariant way by

$$U_{m-sing}(x) = \bigcup_{i=0}^{k-2} U_{m-sing}^i(x),$$

where

$$U_{m-sing}^i(x) = \{u(x) \in \mathbb{R}^2 : \operatorname{rk}(\mathcal{G}^i + [f + gu, \mathcal{L}^{i+1}])(x) < (i+2)m + 1\},$$

with $\mathcal{L}^{i+1} = \mathcal{C}^{i+1}$, for $0 \leq i \leq k-3$, where $\mathcal{C}^{i+1}$ is the characteristic distribution of $\mathcal{G}^{i+1}$, and $\mathcal{L}^{k-1} = \mathcal{L}$, the involutive subdistribution of $\mathcal{G}^{k-2}$ and $gu = \sum_{i=0}^m u_i g_i$. This singularity is excluded by item *(m-F1)* of the next theorem describing all minimal $x$-flat outputs of control-affine systems feedback equivalent to the triangular form $TCh_m^k$.

**Theorem 3.1.4.** *Consider a control-affine system* $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$, *with* $m \geq 2$, *defined on an open subset* $X$ *of* $\mathbb{R}^{km+1}$, *where* $k \geq 2$, *that is locally, around* $x^* \in X$, *static feedback equivalent to* $TCh_m^k$ *and its associated control-linear system* $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$.

*(m-F1)* $\Sigma_{aff}$ *is x-flat, of differential weight* $(k+1)(m+1)$, *at any* $(x^*, u^*) \in X \times \mathbb{R}^{m+1}$ *such that* $u^* \notin U_{m-sing}(x^*)$.

*(m-F2) If* $(\varphi_0, \cdots, \varphi_m)$ *is a minimal x-flat output of* $\Sigma_{aff}$ *at* $(x^*, u^*)$, *where* $u^* \notin U_{m-sing}(x^*)$, *then there exists an open neighborhood* $\mathcal{X}^*$ *of* $x^*$ *and coordinates* $(z_0, z_1^1, \cdots, z_m^1, \cdots, z_1^k, \cdots, z_m^k)$ *on* $\mathcal{X}^*$ *in which* $\Sigma_{aff}$ *is locally feedback equivalent to the triangular form* $TCh_m^k$, *such that* $\varphi_0 = z_0$ *and* $\varphi_i = z_i^1$, *for* $1 \leq i \leq m$, *after permuting the components* $\varphi_i$ *of the flat output* $\varphi$, *if necessary.*

*(m-F3) Let* $\varphi_0, \varphi_1, \cdots, \varphi_m$ *be* $m+1$ *smooth functions defined in a neighborhood of* $x^*$. *The following conditions are equivalent:*

*(i) The* $(m+1)$*-tuple* $(\varphi_0, \varphi_1, \cdots, \varphi_m)$ *is a minimal x-flat output of* $\Sigma_{aff}$ *at* $(x^*, u^*)$, *where* $u^* \notin U_{m-sing}(x^*)$;

*(ii) The* $(m+1)$*-tuple* $(\varphi_0, \varphi_1, \cdots, \varphi_m)$ *is a minimal x-flat output of* $\Sigma_{lin}$ *at* $(x^*, \tilde{u}^*)$, *where* $\tilde{u}^*$ *is such that* $\sum_{i=0}^m \tilde{u}_i^* g_i(x^*) \notin \mathcal{C}^1(x^*)$, *where* $\mathcal{C}^1$ *is the characteristic distribution of* $\mathcal{G}^1$;

*(iii) The $(m+1)$-tuple $(\varphi_0, \varphi_1, \cdots, \varphi_m)$ satisfies the following conditions in a neighborhood of $x^*$:*

*(m-FO1)* $d\varphi_0 \wedge d\varphi_1 \wedge \cdots \wedge d\varphi_m(x^*) \neq 0$;

*(m-FO2)* $\mathcal{L} = (\mathrm{span}\,\{d\varphi_0, d\varphi_1, \cdots, d\varphi_m\})^\perp$, *where $\mathcal{L}$ denotes the involutive subdistribution of corank one in $\mathcal{G}^{k-2}$.*

*Moreover, the $(m+1)$-tuple $(\varphi_0, \varphi_1, \cdots, \varphi_m)$ is unique, up to a diffeomorphism, i.e., if $(\tilde{\varphi}_0, \tilde{\varphi}_1, \cdots, \tilde{\varphi}_m)$ is another minimal x-flat output, then there exist smooth maps $h_i$ such that $\tilde{\varphi}_i = h_i(\varphi_0, \varphi_1, \cdots, \varphi_m)$, $0 \leq i \leq m$, and $h = (h_0, h_1, \cdots, h_m)$ is a local diffeomorphism.*

Theorem 3.1.4 indicates how flatness of control-affine systems locally equivalent to $TCh_m^k$ reminds, but also how it differs from, that of control-linear systems locally equivalent to the $m$-chained form $Ch_m^k$.

While Theorem 3.1.3, associated to the case $m = 1$, allows us to compute all $x$-flat outputs of $TCh_1^k$, Theorem 3.1.4 describes all minimal $x$-flat outputs of $TCh_m^k$. Functions whose differentials annihilate $\mathcal{L}$ are clearly not the only $x$-flat outputs of $TCh_m^k$. They are, however, the only that possess the minimality property, i.e., when determining, with their help, all state and control variables, we use the minimal possible number of derivatives, which is $(k+1)(m+1)$, see the proof of Theorem 3.1.4. According to item $(ii)$, their description coincides with that of minimal $x$-flat outputs of $\Sigma_{lin}$. Indeed, conditions *(m-FO1)-(m-FO2)* are the same as those given in [58] for control-linear systems feedback equivalent to the $m$-chained form. The presence of the drift has no influence on characterizing minimal $x$-flat outputs, but, analogously to the case $m = 1$, it plays a role in describing singularities in the control space.

For control-affine systems, it is the drift $f$, the characteristic distributions $\mathcal{C}^i$, for $1 \leq i \leq k-2$, and the involutive subdistribution $\mathcal{L}$ of corank one in $\mathcal{G}^{k-2}$, that describe singularities in the control space. Although $\mathcal{L}$ is not involved in the compatibility conditions (see item *(m-Comp)* of Theorem 3.1.2), it plays an important role in determining the singular controls at which the system ceases to be flat.

The description of the set of singular controls $U_{m-sing}$ is also valid for driftless systems, i.e., for $f = 0$, but it is redundant. In fact, the set of singular controls $u_c$ for control-linear systems can be described using the first characteristic distribution $\mathcal{C}^1$ only: the singular controls $u_c$ are such that the corresponding trajectories are tangent to the characteristic distribution $\mathcal{C}^1$, that is, $u_c$ verifying $\sum_{i=0}^m u_{c,i}(x) g_i(x) \in \mathcal{C}^1(x)$. Clearly, they form, for any $x \in X$, an $m$-dimensional linear subspace of $U = \mathbb{R}^{m+1}$. If we apply an invertible feedback $u = \beta \tilde{u}$ such that $\mathcal{C}^1 = \mathrm{span}\,\{\tilde{g}_1, \cdots, \tilde{g}_m\}$ and $\mathcal{G}^0 = \mathrm{span}\,\{\tilde{g}_0\} + \mathcal{C}^1$, then the singular controls $\tilde{u}_c$ are of the form $\tilde{u}_c = (0, \tilde{u}_{c,1}, \cdots, \tilde{u}_{c,m})$.

Finally, it turns out that minimal $x$-flat outputs and the triangular form $TCh_m^k$ are compatible: in fact, for any $m+1$ smooth functions $\varphi_0, \varphi_1, \cdots, \varphi_m$ that form a minimal $x$-flat output of a system $\Sigma_{aff}$ feedback equivalent to $TCh_m^k$, we can bring $\Sigma_{aff}$ into the form $TCh_m^k$ for which $\varphi_0, \varphi_1, \cdots, \varphi_m$ play the role of the top variables, as item *(m-F2)* assures. An analogous result is also valid for minimal $x$-flat outputs and the $m$-chained form, see [28].

As an immediate corollary of Theorem 3.1.4, we get the following system of PDE's whose solutions give all minimal $x$-flat outputs for control-affine systems static feedback equivalent to $TCh_m^k$. Denote by $v_j$, for $1 \leq j \leq (k-1)m$, the vector fields spanning the distribution $\mathcal{L}$ (for their computation see Appendix 3.1.A).

**Proposition 3.1.4.** *Consider a control-affine system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$, with $m \geq 2$, defined on an open subset $X$ of $\mathbb{R}^{km+1}$, where $k \geq 2$, that is locally, around $x^* \in X$, static feedback equivalent to $TCh_m^k$. Let $\mathcal{L} = \operatorname{span}\{v_j,\ 1 \leq j \leq (k-1)m\}$ be the involutive subdistribution of corank one in $\mathcal{G}^{k-2}$. Then smooth functions $\varphi_0, \varphi_1, \cdots, \varphi_m$, defined in a neighborhood of $x^*$, form a minimal $x$-flat output at $(x^*, u^*)$, $u^* \notin U_{m-sing}(x^*)$ if and only if*

$$L_{v_j}\varphi_i = 0,\ 1 \leq j \leq (k-1)m,\ 0 \leq i \leq m,$$

*and $d\varphi_0 \wedge d\varphi_1 \wedge \cdots \wedge d\varphi_m(x^*) \neq 0$.*

## 3.1.5 Examples and applications

### 3.1.5.1 Example: $TCh_1^k$ is not necessary for flatness

In the previous section we have seen that systems locally static feedback equivalent to the triangular form $TCh_m^k$, $m = 1$ or $m \geq 2$, are $x$-flat and we have described all $x$-flat outputs. Therefore being static feedback equivalent to $TCh_m^k$, $m = 1$ or $m \geq 2$ is sufficient for $x$-flatness. A natural question arises: is static feedback equivalence to $TCh_m^k$ necessary for flatness, provided that the control-linear subsystem is static feedback equivalent to the chained form? The next example gives a negative answer to this question. Consider the following control-affine system whose control-linear part is already in the chained form $Ch_1^4$, but whose drift $f$ does not satisfy the compatibility condition $(Comp)$ and thus the system cannot be transformed into $TCh_1^4$:

$$\begin{cases} \dot{z}_0 = v_0 & \dot{z}_1 = z_3 + z_2 v_0 \\ & \dot{z}_2 = -z_4 + z_3 v_0 \\ & \dot{z}_3 = a(\bar{z}_3) + z_4 v_0 \\ & \dot{z}_4 = v_1 \end{cases}$$

where $a$ is a smooth function depending on $z_0, z_1, z_2, z_3$. The pair $(\varphi_0, \varphi_1) = (z_0, z_1)$ is an $x$-flat output. Indeed, we have $\varphi_0 = z_0$ implying $\dot{\varphi}_0 = v_0$ and $\varphi_1 = z_1$ implying

$$\begin{aligned} \dot{\varphi}_1 &= z_3 + z_2 v_0 = z_3 + z_2 \dot{\varphi}_0 \\ \ddot{\varphi}_1 &= a(\varphi_0, \varphi_1, z_2, z_3) + z_3 \dot{\varphi}_0^2 + z_2 \ddot{\varphi}_0. \end{aligned}$$

These expressions allow us to calculate $z_2$ and $z_3$ via the implicit function theorem as

$$\begin{aligned} z_2 &= \gamma_2(\bar{\varphi}_0^2, \bar{\varphi}_1^2) \\ z_3 &= \gamma_3(\bar{\varphi}_0^2, \bar{\varphi}_1^2), \end{aligned}$$

for some functions $\gamma_2$, $\gamma_3$, where $\bar{\varphi}^l$ denotes $(\varphi, \dot{\varphi}, \cdots, \varphi^{(l)})$. By differentiating $z_3$, we deduce $z_4 = \gamma_4(\bar{\varphi}_0^3, \bar{\varphi}_1^3)$ which yields $v_1 = \delta_1(\bar{\varphi}_0^4, \bar{\varphi}_1^4)$. So we have expressed all state and control variables as functions of $\varphi_0$ and $\varphi_1$ and their derivatives proving that $(\varphi_0, \varphi_1) = (z_0, z_1)$ is, indeed, an $x$-flat output.

#### 3.1.5.2 Application to mechanical systems: coin rolling without slipping on a moving table

Consider a vertical coin of radius $R$ rolling without slipping on a moving table, see Figure 3.1. Assume that the surface of the table is on the $xy$-plane and denote by $(x, y)$ the position of the contact point of the coin with the table, and by $\theta$ and $\phi$, respectively, the orientation of the vertical plane containing the coin and the rotation angle of the coin. Then the configuration space for the system is $Q = SE(2) \times S^1$ and is parameterized by the generalized coordinates $q = ((x, y, \theta), \phi)$.

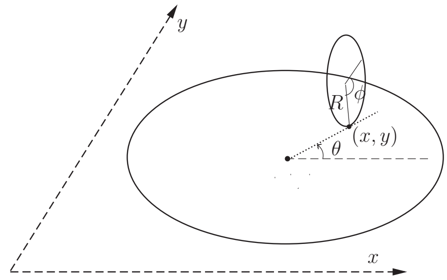


*Figure 3.1 – The coin on a moving table*

Assume that the table moves with respect to the inertial frame obeying the differential equations

$$\begin{aligned} \dot{x}_t &= \alpha(x_t, y_t) \\ \dot{y}_t &= \beta(x_t, y_t). \end{aligned} \tag{3.3}$$

for a smooth vector field $(\alpha, \beta)^\top$ on $\mathbb{R}^2$.

Therefore the nonholonomic constraints of rolling without slipping can be represented by

$$\begin{aligned} \dot{x}\sin\theta - \dot{y}\cos\theta &= 0 \\ (\dot{x} - \alpha)\cos\theta + (\dot{y} - \beta)\sin\theta &= R\dot{\phi}, \end{aligned} \tag{3.4}$$

which leads to the kinematic model of the coin on a moving table as

$$\Sigma_{coin} : \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta(\alpha\cos\theta + \beta\sin\theta) \\ \sin\theta(\alpha\cos\theta + \beta\sin\theta) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} u_2. \tag{3.5}$$

The system is control-affine because the nonholonomic constraints are affine (and not linear) as a result of the motion of the table with respect to the inertial frame.

**Remark 3.1.1.** Assume that $\alpha = -\omega y_t$, $\beta = \omega x_t$, that is, the motion equation of the table is

$$\begin{aligned} \dot{x}_t &= -\omega y_t \\ \dot{y}_t &= \omega x_t, \end{aligned}$$

meaning that the table rotates around its center point with the angular velocity $\omega$. Substituting $\alpha = -\omega y$, $\beta = \omega x$ into (3.5), we obtain the model of the coin on a rotating table as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \omega\cos\theta(x\sin\theta - y\cos\theta) \\ \omega\sin\theta(x\sin\theta - y\cos\theta) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} u_2, \tag{3.6}$$

which coincides with the model given by T. Kai [24].

**Proposition 3.1.5.** *The coin on a moving table $\Sigma_{coin}$, given by (3.5), is feedback equivalent to the triangular form $TCh_1^3$ if and only if the motion of the table is described by*

$$\begin{cases} \dot{x}_t &= & cy_t + d \\ \dot{y}_t &= & -cx_t + e \end{cases}$$

*where $c$, $d$, $e \in \mathbb{R}$ are constant.*

**Remark 3.1.2.** Notice that introducing $\tilde{x}_t = x_t - e/c$ and $\tilde{y}_t = y_t + d/c$, we obtain:

$$\begin{aligned} \dot{\tilde{x}}_t &= c\tilde{y}_t \\ \dot{\tilde{y}}_t &= -c\tilde{x}_t. \end{aligned}$$

The only motions of table that lead to the triangular form $TCh_1^3$ are thus constant speed rotations around a fixed point $(e/c, -d/c)$.

*Proof.* The system $\Sigma_{coin}$ is feedback equivalent to the triangular form $TCh_1^k$ if and only if it satisfies the conditions *(Ch1)-(Ch3)* and *(Comp)* of Theorem 3.1.2 or, equivalently, conditions *(Ch1)′-(Ch2)′* and *(Comp)*. Consider the associated distribution $\mathcal{G}$ and the drift $f$ given by:

$$\mathcal{G} = \operatorname{span}\{g_1, g_2\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } f = \begin{pmatrix} \cos\theta(\alpha\cos\theta + \beta\sin\theta) \\ \sin\theta(\alpha\cos\theta + \beta\sin\theta) \\ 0 \\ 0 \end{pmatrix}.$$

A straightforward calculation shows that

$$g_3 = [g_1, g_2] = \begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \\ 0 \end{pmatrix}, \quad g_4 = [g_1, g_3] = \begin{pmatrix} -R\cos\theta \\ -R\sin\theta \\ 0 \\ 0 \end{pmatrix}.$$

Therefore $\mathcal{G}^1 = \mathcal{G}_1 = \operatorname{span}\{g_1, g_2, g_3\}$ and $\mathcal{G}^2 = \mathcal{G}_2 = \operatorname{span}\{g_1, g_2, g_3, g_4\}$ which gives that $\operatorname{rank}\mathcal{G}^1 = \mathcal{G}_1 = 3$ and $\operatorname{rank}\mathcal{G}^2 = \mathcal{G}_2 = 4$ and thus conditions *(Ch1)′-(Ch2)′* hold. Moreover, it is easy to see that $\mathcal{C}^1 = \operatorname{span}\{c\}$ where $c = g_2$ and a direct computation gives

$$[f, c] = [f, g_2] = -\begin{pmatrix} \gamma R\cos\theta \\ \gamma R\sin\theta \\ 0 \\ 0 \end{pmatrix},$$

where

$$\gamma = \cos\theta\left(\frac{\partial\alpha}{\partial x}\cos\theta + \frac{\partial\beta}{\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial\alpha}{\partial y}\cos\theta + \frac{\partial\beta}{\partial y}\sin\theta\right).$$

The condition $(Comp)$ of Theorem 3.1.2 requires that $[f,c] \subset \mathcal{G}^1$ implying that the vector fields $[f,c]$ and $g_3$ are colinear and this is the case if and only if $\gamma \equiv 0$. We thus have to solve

$$\cos\theta\left(\frac{\partial\alpha}{\partial x}\cos\theta + \frac{\partial\beta}{\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial\alpha}{\partial y}\cos\theta + \frac{\partial\beta}{\partial y}\sin\theta\right) = 0.$$

Dividing the above equation by $\cos^2\theta$ and denoting $w = \tan\theta$, we get

$$\frac{\partial\alpha}{\partial x} + \left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right)w + \frac{\partial\beta}{\partial y}w^2 = 0,$$

which implies that

$$\frac{\partial\alpha}{\partial x} = 0, \quad \frac{\partial\beta}{\partial y} = 0, \quad \frac{\partial\alpha}{\partial y} = -\frac{\partial\beta}{\partial x}.$$

We get $\alpha = \alpha(y)$, $\beta = \beta(x)$ and then by the equality $\frac{\partial\alpha}{\partial y} = -\frac{\partial\beta}{\partial x}$, we have

$$\alpha'(y) = -\beta'(x) = c,$$

where $c \in \mathbb{R}$ is a constant. This gives

$$\begin{aligned} \alpha &= cy + d \\ \beta &= -cx + e \end{aligned}$$

where $c, e, f \in \mathbb{R}$ are constants and the motion of the table is described by

$$\begin{aligned} \dot{x}_t &= cy_t + d \\ \dot{y}_t &= -cx_t + e, \end{aligned} \tag{3.7}$$

or, equivalently,

$$\begin{aligned} \tilde{x}_t &= c\tilde{y}_t \\ \tilde{y}_t &= -c\tilde{x}_t, \end{aligned}$$

which proves the proposition. □

## 3.1.6 Proofs

### 3.1.6.1 Proof of Theorem 3.1.1

*Proof. Necessity.* Consider a two-input control-affine system $\Sigma_{aff} : \dot{x} = f(x) + u_0g_0(x) + u_1g_1(x)$ locally, around $x^*$, static feedback equivalent to $TCh_1^k$ and bring it into the form $TCh_1^k$, around $z^*$. By abuse of notation, we continue to denote by $f$, $g_0$ and $g_1$, the drift and the controlled vector fields of $TCh_1^k$. The distribution $\mathcal{G} = \text{span}\,\{g_0, g_1\}$, associated to $TCh_1^k$, is given by

$$\mathcal{G} = \text{span}\,\{\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_0} + z_2\frac{\partial}{\partial z_1} + \cdots + z_k\frac{\partial}{\partial z_{k-1}}\}.$$

By an induction argument, it is immediate to show that

$$\mathcal{G}^i = \mathcal{G}_i = \text{span}\,\{\frac{\partial}{\partial z_{k-i}}, \cdots, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_0} + z_2\frac{\partial}{\partial z_1} + \cdots + z_{k-i}\frac{\partial}{\partial z_{k-i-1}}\}.$$

Thus $\mathcal{G}^{k-1} = TX$ and the distribution $\mathcal{G}^{k-3}$ is of constant rank $k-1$. The characteristic distribution $\mathcal{C}^i$ of $\mathcal{G}^i$ is given by

$$\mathcal{C}^i = \text{span}\,\{\frac{\partial}{\partial z_{k-i+1}}, \cdots, \frac{\partial}{\partial z_k}\},\ 1 \leq i \leq k-2.$$

So it is immediate to see that $\mathcal{C}^{k-2}$ is contained in $\mathcal{G}^{k-3}$, this inclusion is of corank one and $\mathcal{G}^0(z^*) \not\subset \mathcal{C}^{k-2}(z^*)$. This shows *(Ch1)-(Ch3)*.

Moreover, we have

$$[\frac{\partial}{\partial z_k}, f] = \frac{\partial f_{k-1}}{\partial z_k}\frac{\partial}{\partial z_{k-1}} \in \mathcal{G}^1$$

and

$$[\frac{\partial}{\partial z_{k-i+1}}, f] = \frac{\partial f_{k-i}}{\partial z_{k-i+1}}\frac{\partial}{\partial z_{k-i}} \bmod \text{span}\,\{\frac{\partial}{\partial z_{k-i+1}}, \cdots, \frac{\partial}{\partial z_k}\}$$

which is clearly in $\mathcal{G}^i$, for any $2 \leq i \leq k-2$. It follows that $[f, \mathcal{C}^i] \subset \mathcal{G}^i$, for $1 \leq i \leq k-2$, which shows *(Comp)*. The conditions $(Ch1)-(Ch3)$ involve the distribution $\mathcal{G}$ only, so they are invariant under feedback of the form $g \to g\beta$. Obviously, $[g_j, \mathcal{C}^i] \in \mathcal{G}^i$ (since $\mathcal{C}^i$ is characteristic for $\mathcal{G}^i$), for $0 \leq j \leq 1$, $1 \leq i \leq k-2$, and thus $(Comp)$ is invariant under feedback of the form $f \mapsto f + \alpha_0 g_0 + \alpha_1 g_1$.

*Sufficiency.* Consider a two-input control-affine system $\Sigma_{aff} : \dot{x} = f(x) + u_0 g_0(x) + u_1 g_1(x)$ satisfying the conditions *(Ch1)-(Ch3)* and *(Comp)*. As proved in [50], the items *(Ch1)-(Ch3)* assure the existence of an invertible static feedback transformation $u = \beta\tilde{u}$ and a change of coordinates $z = \phi(x)$ bringing the distribution $\mathcal{G}^0$ into the chained form, which transform the system $\Sigma_{aff}$ into

$$\begin{cases} \dot{z}_0 = a_0(z) + \tilde{u}_0 & \dot{z}_1 = a_1(z) + z_2\tilde{u}_0 \\ & \vdots \\ & \dot{z}_{k-1} = a_{k-1}(z) + z_k\tilde{u}_0 \\ & \dot{z}_k = a_k(z) + \tilde{u}_1 \end{cases}$$

with $a_i$ smooth functions. Applying the invertible static feedback $v_0 = a_0(z) + \tilde{u}_0$ and $v_1 = a_k(z) + \tilde{u}_1$, we obtain

$$\begin{cases} \dot{z}_0 = v_0 & \dot{z}_1 = f_1(z) + z_2 v_0 \\ & \vdots \\ & \dot{z}_{k-1} = f_{k-1}(z) + z_k v_0 \\ & \dot{z}_k = v_1 \end{cases}$$

where $f_i = a_i - z_{i+1}a_0$. In these coordinates, we have

$$\mathcal{G}^i = \mathcal{G}_i = \text{span}\,\{\frac{\partial}{\partial z_{k-i}}, \cdots, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_0} + z_2\frac{\partial}{\partial z_1} + \cdots + z_{k-i}\frac{\partial}{\partial z_{k-i-1}}\},\ 0 \leq i \leq k-1,$$

and

$$\mathcal{C}^i = \text{span}\,\{\frac{\partial}{\partial z_{k-i+1}}, \cdots, \frac{\partial}{\partial z_k}\},\ 1 \leq i \leq k-2.$$

From $[f, \mathcal{C}^i] \subset \mathcal{G}^i$, for any $1 \leq i \leq k-2$, it follows immediately that

$$\frac{\partial f_i}{\partial z_j} = 0,\ \text{for } i+2 \leq j \leq k \text{ and } 1 \leq i \leq k-2,$$

which gives the triangular normal form $TCh_1^k$. □

#### 3.1.6.2 Proof of Theorem 3.1.2

*Proof. Necessity.* Consider a control-affine system $\Sigma : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$ locally, around $x^*$, static feedback equivalent to $TCh_m^k$ and bring it into the form $TCh_m^k$, around $z^*$. To simplify the notation, we continue to write $f$ and $g_i$, $0 \leq i \leq m$, for the drift and the controlled vector fields of $TCh_m^k$ and we denote

$$\text{span}\,\{\frac{\partial}{\partial z^i}\} = \text{span}\,\{\frac{\partial}{\partial z_1^i}, \cdots, \frac{\partial}{\partial z_m^i}\}.$$

The distribution $\mathcal{G}^0 = \text{span}\,\{g_i,\ 0 \leq i \leq m\}$, associated to $TCh_m^k$, is given by

$$\mathcal{G}^0 = \text{span}\,\{g_0, \frac{\partial}{\partial z^k}\}.$$

By an induction argument, it is immediate that

$$\mathcal{G}^i = \text{span}\,\{\frac{\partial}{\partial z^{k-i}}, \cdots, \frac{\partial}{\partial z^k}, g_0\},\ 0 \leq i \leq k-1.$$

It follows that $\mathcal{G}^{k-1} = TX$, the distribution $\mathcal{G}^{k-2}$ has constant rank $(k-1)m+1$ and contains an involutive subdistribution of constant corank one given by

$$\mathcal{L} = \text{span}\,\{\frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^k}\},$$

and $\mathcal{G}^0(z^*)$ is not contained in $\mathcal{L}(z^*)$. This shows *(m-Ch1)-(m-Ch3)*. The characteristic distribution of $\mathcal{G}^i$ is given by

$$\mathcal{C}^i = \text{span}\,\{\frac{\partial}{\partial z^{k-i+1}}, \cdots, \frac{\partial}{\partial z^k}\},\ 1 \leq i \leq k-2,$$

and we have, for any $k-i+1 \leq l \leq k$ and $1 \leq j \leq m$,

$$[\frac{\partial}{\partial z_j^l}, f] = \frac{\partial f_1^{l-1}}{\partial z_j^l}\frac{\partial}{\partial z_1^{l-1}} + \cdots + \frac{\partial f_m^{l-1}}{\partial z_j^l}\frac{\partial}{\partial z_m^{l-1}} \text{ mod } \mathcal{C}^i$$

which is clearly in $\mathcal{G}^i$. Thus $[f, \mathcal{C}^i] \subset \mathcal{G}^i$, for $1 \leq i \leq k-2$, which proves item $(m - Comp)$.

*Sufficiency.* Consider the control-affine system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$ satisfying the conditions *(m-Ch1)-(m-Ch3)* and *(m-Comp)*. According to Theorem 5.6 in [50], the items *(m-Ch1)-(m-Ch3)* assure the existence of an invertible static feedback transformation $u = \beta\tilde{u}$ and of a change of coordinates $z = \phi(x)$ (see Appendix 3.1.B where we explain how to construct the diffeomorphism $\phi$ and the feedback transformation) bringing the distribution $\mathcal{G}^0$ into the $m$-chained form and thus the system $\Sigma_{aff}$ into

$$\begin{cases} \dot{z}_0 = a_0(z) + \tilde{u}_0 & \dot{z}_1^1 = a_1^1(z) + z_1^2\tilde{u}_0 \quad \cdots \quad \dot{z}_m^1 = a_m^1(z) + z_m^2\tilde{u}_0 \\ & \dot{z}_1^2 = a_1^2(z) + z_1^3\tilde{u}_0 \qquad\quad \dot{z}_m^2 = a_m^2(z) + z_m^3\tilde{u}_0 \\ & \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & \dot{z}_1^{k-1} = a_1^{k-1}(z) + z_1^k\tilde{u}_0 \quad \cdots \quad \dot{z}_m^{k-1} = a_m^{k-1}(z) + z_m^k\tilde{u}_0 \\ & \dot{z}_1^k = a_1^k(z) + \tilde{u}_1 \quad \cdots \quad \dot{z}_m^k = a_m^k(z) + \tilde{u}_m \end{cases}$$

with $a_j^i$ smooth functions. Applying the invertible static feedback $v_0 = a_0(z) + \tilde{u}_0$ and $v_i = a_i^k(z) + \tilde{u}_i$, for $1 \le i \le m$, we get

$$\begin{cases} \dot{z}_0 = v_0 & \dot{z}_1^1 = f_1^1(z) + z_1^2 v_0 \quad \cdots \quad \dot{z}_m^1 = f_m^1(z) + z_m^2 v_0 \\ & \dot{z}_1^2 = f_1^2(z) + z_1^3 v_0 \qquad\quad \dot{z}_m^2 = f_m^2(z) + z_m^3 v_0 \\ & \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & \dot{z}_1^{k-1} = f_1^{k-1}(z) + z_1^k v_0 \quad \cdots \quad \dot{z}_m^{k-1} = f_m^{k-1}(z) + z_m^k v_0 \\ & \dot{z}_1^k = v_1 \quad \cdots \quad \dot{z}_m^k = v_m \end{cases}$$

with $f_j^i = a_j^i - z_j^{i+1} a_0$. In the $z$-coordinates, we have

$$\mathcal{G}^i = \operatorname{span}\{\frac{\partial}{\partial z^{k-i}}, \cdots, \frac{\partial}{\partial z^k}, g_0\},\ 0 \le i \le k-1.$$

The characteristic distribution of $\mathcal{G}^i$ is given by

$$\mathcal{C}^i = \operatorname{span}\{\frac{\partial}{\partial z^{k-i+1}}, \cdots, \frac{\partial}{\partial z^k}\},\ 1 \le i \le k-2,$$

and the corank one involutive subdistribution of $\mathcal{G}^{k-2}$ by

$$\mathcal{L} = \operatorname{span}\{\frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^k}\}.$$

We have, for $1 \le i \le k-2$,

$$[\frac{\partial}{\partial z_j^{k-i+1}}, f] = \sum_{l=1}^{m}\sum_{s=1}^{k-i-1} \frac{\partial f_l^s}{\partial z_j^{k-i+1}} \frac{\partial}{\partial z_l^s} \operatorname{mod} \operatorname{span}\{\frac{\partial}{\partial z^{k-i}}, \cdots, \frac{\partial}{\partial z^{k-1}}\}$$

and since $[\frac{\partial}{\partial z_j^{k-i+1}}, f] \in \mathcal{G}^i$, for any $1 \le j \le m$, we obtain

$$\frac{f_l^s}{z_j^{k-i+1}} = 0,\ \text{for any } 1 \le j, l \le m,\ 1 \le s \le k-i-1$$

It follows that $f$ exhibits the desired trangular form $TCh_m^k$. □

#### 3.1.6.3 Proof of Theorem 3.1.3

*Proof of (F1).* Consider the two-input control-affine system $\Sigma : \dot{x} = f(x) + u_0 g_0(x) + u_1 g_1(x)$ locally, around $x^*$, feedback equivalent to $TCh_1^k$ and bring it into the form $TCh_1^k$, around $z^*$. To simplify notation, we continue to denote by $f$, respectively by $g_0$ and $g_1$, the drift, respectively the controlled vector fields of $TCh_1^k$.

It is clear that $TCh_1^k$ is $x$-flat, with $\varphi = (z_0, z_1)$ being a flat output, at any point $(z^*, v^*)$ satisfying

$$\frac{\partial f_i}{\partial z_{i+1}}(z^*) + v_0^* \neq 0, \text{ for } 1 \leq i \leq k-1,$$

where $v^* = (v_0^*, v_1^*)$. Recall that, in coordinates $z$, we have

$$\mathcal{G}^i = \text{span}\,\{\frac{\partial}{\partial z_{k-i}}, \cdots, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + \cdots + z_{k-i}\frac{\partial}{\partial z_{k-i-1}}\}, \text{ for } 0 \leq i \leq k-1,$$

and

$$\mathcal{C}^i = \text{span}\,\{\frac{\partial}{\partial z_{k-i+1}}, \cdots, \frac{\partial}{\partial z_k}\}, \ 1 \leq i \leq k-2.$$

Notice that for each $0 \leq i \leq k-3$, the only nontrivial condition for $[f + u_0^i g_0 + u_1^i g_1, \mathcal{C}^{i+1}] \subset \mathcal{G}^i$ to be satisfied for $TCh_1^k$ is $[f + v_0^i g_0 + v_1^i g_1, \frac{\partial}{\partial z_{k-i}}] \in \mathcal{G}^i$ implying $[f, \frac{\partial}{\partial z_{k-i}}] - v_0^i \frac{\partial}{\partial z_{k-i-1}} \in \mathcal{G}^i$ and hence

$$\frac{\partial f_{k-i-1}}{\partial z_{k-i}}(z) + v_0^i = 0.$$

The latter is feedback invariant because $[f + u_0^i g_0 + u_1^i g_1, \mathcal{C}^{i+1}] \subset \mathcal{G}^i$ is feedback invariant as explained just after the definition of $U_{sing}^i$ in Section 3.1.4. Another argument proving feedback invariance is that we look for the vector field $f(x) + u_0(x)^i g_0 + u_1(x)^i g_1$ belonging to the affine distribution $f(x) + \mathcal{G}^0(x)$ which, obviously, is feedback invariant. To summarize, $v^* \in \bigcup_{i=0}^{k-3} U_{sing}^i(z^*)$ if and only if

$$\frac{\partial f_{k-i-1}}{\partial z_{k-i}}(z^*) + v_0^* = 0, \ 0 \leq i \leq k-3.$$

To analyze the condition $[f + u_0^{k-2} g_0 + u_1^{k-2} g_1, l] \in \mathcal{G}^{k-2}$, where $l \in \mathcal{L}$ and $l \notin \mathcal{C}^{k-2}$, take $l = \frac{\partial}{\partial z_2}$. Then

$$[f + v_0^{k-2} g_0 + v_1^{k-2} g_1, l] = [f, \frac{\partial}{\partial z_2}] - v_0^{k-2} \frac{\partial}{\partial z_1} \in \mathcal{G}^{k-2},$$

if and only if

$$\frac{\partial f_1}{\partial z_2}(z) + v_0^{k-2} = 0.$$

The definition of $U_{\mathcal{L}-sing}^{k-2}$ is feedback invariant (for the some reasons as those giving invariance of $U_{sing}^i$, $0 \leq i \leq k-3$) and thus $v^* \in U_{\mathcal{L}-sing}^{k-2}$ if and only if $\frac{\partial f_1}{\partial z_2}(z^*) + v_0^* = 0$, where $\mathcal{L}$ is such that $\mathcal{G}^0(x^*) \notin \mathcal{L}(x^*)$. If $\mathcal{L}$ is such that $\mathcal{G}^0(x^*) \in \mathcal{L}(x^*)$, we will show when proving the equivalence $(i) \Longleftrightarrow (ii)$, that under the assumption (which we always assume) $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$, where $\mathcal{L}^\perp = \text{span}\{d\varphi_0, d\varphi_1\}$, we have $u^* \notin U^{k-2}_{\mathcal{L}-sing}(x^*)$ and in $\mathcal{X}^* \times \mathbb{R}^2$, where $\mathcal{X}^*$ is a sufficiently small neighborhood of $x^*$, the set $U^{k-2}_{\mathcal{L}-sing}(x)$ consists of two connected components that define, for each fixed value $x \in \mathcal{X}^*$, $x \neq x^*$, an affine subspace of $U = \mathbb{R}^2$.

Now observe that the set of the singular control values $U^{k-2}_{\mathcal{L}-sing}$ (at which $(\varphi_0, \varphi_1)$ ceases to be a flat output for $TCh^k_1$) is determined by $\mathcal{L}$ which, in turn, is uniquely associated to the choice of the flat output $(\varphi_0, \varphi_1)$ by $\mathcal{L}^\perp = \text{span}\{d\varphi_0, d\varphi_1\}$. Different choices of $(\varphi_0, \varphi_1)$ lead, in general, to different distributions $\mathcal{L}$ and, consequently, to different singular control values and the system is not flat only at those that are singular for all choices of $\mathcal{L}$. Hence

$$U_{sing} = \bigcup_{i=0}^{k-3} U^i_{sing} \cup U^{k-2}_{sing}$$

where

$$U^{k-2}_{sing} = \bigcap_{\mathcal{L}} U^{k-2}_{\mathcal{L}-sing}.$$

*Proof of (F2).* It was shown in [29] that conditions $(FO2)$ and $(FO2)'$ are equivalent (for control-linear systems $\Sigma_{lin}$ but notice that $(FO2)$ and $(FO2)'$ do not involve the drift $f$). We deduce immediately that $(ii) \Leftrightarrow (iii)$. We will now prove that $(ii) \Rightarrow (i)$.

First consider the case $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0,0)$. By [29], a pair $(\varphi_0, \varphi_1)$ satisfying $(FO1) - (FO2)$ forms a flat output of the control-linear system $\Sigma_{lin}$ and, also by [29], $(\varphi_0, \varphi_1)$ is compatible with the chained form so there exists a local static feedback transformation bringing $\Sigma_{lin}$ into the chained form with $z_0 = \varphi_0$ and $z_1 = \varphi_1 + k_0\varphi_0$, $k_0 \in \mathbb{R}$, which thus transforms the control-affine system $\Sigma_{aff}$ into

$$\begin{array}{llcccc} \dot{z}_0 = f_0(z) + v_0 & \dot{z}_1 & = & f_1(z) & + & z_2 v_0 \\ & & \vdots & & & \\ & \dot{z}_{k-1} & = & f_{k-1}(z) & + & z_k v_0 \\ & \dot{z}_k & = & f_k(z) & + & v_1 \end{array}$$

Replacing $v_0$ by $v_0 - f_0$ and $v_1$ by $v_1 - f_k$ and using $[f, \mathcal{C}^i] \subset \mathcal{D}^i$, we conclude (repeating the proof of (F1)) that the system is in the triangular form and thus, flat at $(x^*, u^*)$ such that $u^* \notin U_{\mathcal{L}-sing} = \bigcup_{i=0}^{k-3} U^i_{sing} \cup U^{k-2}_{\mathcal{L}-sing}$, where $\mathcal{L} = (\text{span}\,\{d\varphi_0, d\varphi_1\})^\perp$.

Now consider the case $(L_g\varphi_0,\, L_g\varphi_1)(x^*) = (0,0)$. Since $\Sigma_{aff}\ :\ \dot{x} = f(x) + u_0 g_0(x) + u_1 g_1(x)$ is locally, around $x^*$, feedback equivalent to $TCh^k_1$, we can assume that $\Sigma_{aff}$ is in the triangular form $TCh^k_1$ around $z^* = 0$:

$$TCh^k_1 \left\{ \begin{array}{llclcl} \dot{z}_0 = v_0 & \dot{z}_1 & = & f_1(z_0, z_1, z_2) & + & z_2 v_0 \\ & \dot{z}_2 & = & f_2(z_0, z_1, z_2, z_3) & + & z_3 v_0 \\ & & \vdots & & & \\ & \dot{z}_{k-1} & = & f_{k-1}(z_0, \cdots, z_k) & + & z_k v_0 \\ & \dot{z}_k & = & v_1 & & \end{array} \right.$$

The characteristic distribution $\mathcal{C}^{k-2}$ takes the form $\mathcal{C}^{k-2} = \text{span}\{\frac{\partial}{\partial z_3}, \dots, \frac{\partial}{\partial z_k}\}$, and the condition $L_c\varphi_i = 0$, for any $c \in \mathcal{C}^{k-2}$, given by $(FO2)$ implies that $\varphi_i = \varphi_i(z_0, z_1, z_2)$, for $i = 0, 1$. Condition $(FO1)$ implies that $d\varphi_0 \wedge d\varphi_1(x^*) \neq 0$, that is equivalent to

$$\text{rk}\begin{pmatrix} \frac{\partial\varphi_0}{\partial z_0} & \frac{\partial\varphi_0}{\partial z_1} & \frac{\partial\varphi_0}{\partial z_2} \\ \frac{\partial\varphi_1}{\partial z_0} & \frac{\partial\varphi_1}{\partial z_1} & \frac{\partial\varphi_1}{\partial z_2} \end{pmatrix}(0) = 2.$$

Notice that the condition $(L_g\varphi_0, L_g\varphi_1)(x^*) = (0, 0)$ implies that $\frac{\partial\varphi_0}{\partial z_0}(0) = \frac{\partial\varphi_1}{\partial z_0}(0) = 0$ and thus we get

$$\text{rk}\begin{pmatrix} \frac{\partial\varphi_0}{\partial z_1} & \frac{\partial\varphi_0}{\partial z_2} \\ \frac{\partial\varphi_1}{\partial z_1} & \frac{\partial\varphi_1}{\partial z_2} \end{pmatrix}(0) = 2.$$

We assume $\varphi_0(0) = \varphi_1(0) = 0$ (if not, replace $\varphi_0$ by $\varphi_0 - \varphi_0(0)$ and $\varphi_1$ by $\varphi_1 - \varphi_1(0)$). We will introduce new coordinates $(\tilde{z}_1, \tilde{z}_2) = (\varphi_0, \varphi_1)$ in two steps. Assume that $\frac{\partial\varphi_1}{\partial z_2}(0) \neq 0$ (if not, permute $\varphi_0$ and $\varphi_1$) and put $\tilde{z}_2 = \varphi_1(z_0, z_1, z_2)$. Then the two first components become

$$\begin{aligned} \dot{z}_1 &= \tilde{f}_1(z_0, z_1, \tilde{z}_2) + a(z_0, z_1, \tilde{z}_2)v_0 \\ \dot{\tilde{z}}_2 &= \tilde{f}_2(z_0, z_1, \tilde{z}_2, z_3) + b(z_0, z_1, \tilde{z}_2, z_3)v_0, \end{aligned}$$

where $\tilde{f}_2 = L_f\varphi_1$, $b = L_{g_0}\varphi_1$ and $a = z_2 = \varphi_1^{-1}(z_0, z_1, \tilde{z}_2)$ is the inverse of $\varphi_1$ with respect to $z_2$. Notice that $b = L_{g_0}\varphi_1 = \frac{\partial\varphi_1}{\partial z_0} + \frac{\partial\varphi_1}{\partial z_1}z_2 + \frac{\partial\varphi_1}{\partial z_2}z_3$ is affine with respect to $z_3$ and $\frac{\partial\varphi_1}{\partial z_2}(0) \neq 0$ so $\tilde{z}_i = L_{g_0}^{i-3}b$, for $3 \leq i \leq k$, is a valid local change of coordinates in which the system, under the feedback $\tilde{v}_1 = L_f L_{g_0}^{k-3}b + v_0 L_{g_0}^{k-2}b + v_1 L_{g_1} L_{g_0}^{k-3}b$, takes the form

$$\begin{aligned} \dot{z}_0 = v_0 \quad \dot{z}_1 &= \tilde{f}_1(z_0, z_1, \tilde{z}_2) &&+ a(z_0, z_1, \tilde{z}_2)v_0 \\ \dot{\tilde{z}}_2 &= \tilde{f}_2(z_0, z_1, \tilde{z}_2, \tilde{z}_3) &&+ \tilde{z}_3 v_0 \\ &\vdots \\ \dot{\tilde{z}}_{k-1} &= \tilde{f}_{k-1}(z_0, z_1, \tilde{z}_2, \cdots, \tilde{z}_k) &&+ \tilde{z}_k v_0 \\ \dot{\tilde{z}}_k &= \tilde{v}_1. \end{aligned}$$

Now put $\tilde{z}_1 = \varphi_0(z_0, z_1, z_2)$. We get $\dot{\tilde{z}}_1 = L_f\varphi_0 + v_0 L_{g_0}\varphi_0$. Notice that $L_{g_0}\varphi_0$ is affine with respect to $z_3$ and $L_f\varphi_0$ is, in general, nonlinear with respect to $z_3$ since so is $\tilde{f}_2$. Omitting " $\sim$ " we get

$$\begin{aligned} \dot{z}_0 = v_0 \quad \dot{z}_1 &= f_1(z_0, z_1, z_2, z_3) &&+ (A + Bz_3)v_0 \\ \dot{z}_2 &= f_2(z_0, z_1, z_2, z_3) &&+ z_3 v_0 \\ &\vdots \\ \dot{z}_{k-1} &= f_{k-1}(z_0, z_1, \cdots, z_k) &&+ z_k v_0 \\ \dot{z}_k &= v_1, \end{aligned} \tag{3.8}$$

where $A$ and $B$ depend on $z_0, z_1, z_2$ only. Observe that for (3.8), we have $\varphi_0 = z_1$, $\varphi_1 = z_2$ and $\mathcal{C}^{k-2} = \text{span}\{\frac{\partial}{\partial z_3}, \dots, \frac{\partial}{\partial z_k}\}$, therefore the condition $(L_g\varphi_0)L_{[c,g]}\varphi_1 = (L_g\varphi_1)L_{[c,g]}\varphi_0$ gives $A + z_3 B = z_3 B$ and thus $A \equiv 0$ everywhere.

Notice that the function $f_2(z_0, z_1, z_2, z_3)$ can always be expressed as

$$f_2(z_0, z_1, z_2, z_3) = f_{20}(z_0, z_1, z_2) + z_3 f_{21}(z_0, z_1, z_2, z_3)$$

for some smooth functions $f_{20}$ and $f_{21}$ and thus

$$\dot{z}_2 = f_2(z_0, z_1, z_2, z_3) + z_3 v_0 = f_{20}(z_0, z_1, z_2) + z_3(f_{21}(z_0, z_1, z_2, z_3) + v_0).$$

Define the new control $\tilde{v}_0 = f_{21}(z_0, z_1, z_2, z_3) + v_0$ and denote $\eta = f_{21}$, then (3.8) becomes

$$\begin{array}{llcll} \dot{z}_0 = \tilde{v}_0 - \eta & \dot{z}_1 & = & \tilde{f}_1(z_0, z_1, z_2, z_3) & + \; z_3 B \tilde{v}_0 \\ & \dot{z}_2 & = & \tilde{f}_2(z_0, z_1, z_2) & + \; z_3 \tilde{v}_0 \\ & & \vdots & & \\ & \dot{z}_{k-1} & = & \tilde{f}_{k-1}(z_0, \cdots, z_k) & + \; z_k \tilde{v}_0 \\ & \dot{z}_k & = & v_1, & \end{array} \tag{3.9}$$

where $\tilde{f}_2 = f_{20}$ and $\tilde{f}_i = f_i - z_3 B \eta$, for $i \neq 2$.

Note that $\Sigma_{aff}$ is assumed to be locally, around $x^* \in X$, static feedback equivalent to $TCh_1^k$, hence the conditions $[f, \mathcal{C}^i] \subset \mathcal{G}^i$ hold, for $1 \leq i \leq k-2$, and are invariant under change of coordinates and feedback. Clearly, for (3.9), $\mathcal{C}^{k-2} = \text{span}\,\{\frac{\partial}{\partial z_3}, \ldots, \frac{\partial}{\partial z_k}\}$ and thus $[\tilde{f}, \mathcal{C}^{k-2}] \subset \mathcal{G}^{k-2}$ implies $[\tilde{f}, \frac{\partial}{\partial z_3}] \in \mathcal{G}^{k-2}$ and yields

$$\left[\tilde{f}, \frac{\partial}{\partial z_3}\right] = \begin{pmatrix} -\frac{\partial \eta}{\partial z_3} \\ \frac{\partial \tilde{f}_1}{\partial z_3} \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ z_3 B \\ z_3 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ B \\ 1 \end{pmatrix},$$

modulo $\mathcal{C}^{k-2}$, for some smooth functions $\alpha, \beta$ which gives $\frac{\partial \tilde{f}_1}{\partial z_3} = 0$. Therefore $\tilde{f}_1 = \tilde{f}_1(z_0, z_1, z_2)$ and thus (3.9) is, actually, in the following form

$$\begin{array}{llcll} \dot{z}_0 = \tilde{v}_0 - \eta & \dot{z}_1 & = & \tilde{f}_1(z_0, z_1, z_2) & + \; z_3 B \tilde{v}_0 \\ & \dot{z}_2 & = & \tilde{f}_2(z_0, z_1, z_2) & + \; z_3 \tilde{v}_0 \\ & & \vdots & & \\ & \dot{z}_{k-1} & = & \tilde{f}_{k-1}(z_0, \cdots, z_k) & + \; z_k \tilde{v}_0 \\ & \dot{z}_k & = & v_1, & \end{array} \tag{3.10}$$

with $(\varphi_0, \varphi_1) = (z_1, z_2)$. Define a new variable $y = z_3 \tilde{v}_0$. Notice that, although $y = z_3 \tilde{v}_0$ is not a valid control transformation (since $z_3^* = 0$), it is a system's variable under the assumption that the differentials $dy = z_3 d\tilde{v}_0 + \tilde{v}_0 dz_3$ is nonzero at $(z^*, \tilde{v}_0^*)$. Actually, $\dot{\varphi}_0$ and $\dot{\varphi}_1$ are functions of the system variables $z_0, z_1, z_2$ and $y$. Recall that $\varphi_0 = z_1$ and $\varphi_1 = z_2$. The condition $\text{rk}\frac{\partial(\varphi, \dot{\varphi})}{\partial(x, u)}(x^*, u^*) = 4$ together with

$$\frac{\partial(\varphi, \dot{\varphi})}{\partial(x, u)} = \frac{\partial(\varphi, \dot{\varphi})}{\partial(z_0, z_1, z_2, y)} \cdot \frac{\partial(z_0, z_1, z_2, y)}{\partial(x, u)}$$

implies that $\text{rk}\frac{\partial(\dot{\varphi}_0, \dot{\varphi}_1)}{\partial(z_0, y)}(z^*, v^*) = 2$. By the implicit function theorem, we can express

$$\begin{aligned} z_0 &= \zeta_0(\varphi_0, \varphi_1, \dot{\varphi}_0, \dot{\varphi}_1) \\ y &= \zeta_y(\varphi_0, \varphi_1, \dot{\varphi}_0, \dot{\varphi}_1) \end{aligned}$$

in a neighborhood of $(z^*, v^*)$, for some smooth functions $\zeta_0, \zeta_y$.

We have $\dot{z}_0 = \tilde{v}_0 - \eta = v_0$ and $\dot{z}_2 = \tilde{f}_2 + z_3\tilde{v}_0 = \tilde{f}_2 + z_3(v_0 + \eta)$. Recall that $\tilde{f}_2$ depends on $z_0, z_1, z_2$ only. So knowing $\dot{z}_0 = v_0$ and $\dot{z}_2$, we can calculate $z_3$ using the implicit functions theorem if $v_0 + \eta + z_3\frac{\partial\eta}{\partial z_3} \neq 0$. Then $\dot{z}_3$ gives $z_4$ if $v_0 + \eta + \frac{\partial f_4}{\partial z_4} \neq 0$ and so on, proving that indeed $(\varphi_0, \varphi_1)$ is an $x$-flat output at $(x^*, u^*)$.

To conclude the proof, we have to show the implication $(i) \Rightarrow (ii)$. When proving Proposition 3.1.3, we will show that any flat output $(\varphi_0, \varphi_1)$ of a system $\Sigma_{aff}$ feedback equivalent to $TCh_1^k$ satisfies $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$ and $L_c\varphi_0 = L_c\varphi_1 = (L_g\varphi_0)L_{[c,g]}\varphi_1 - (L_g\varphi_1)L_{[c,g]}\varphi_0 = 0$, for any $c \in \mathcal{C}^{k-2}$. If $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0,0)$, we conclude in the same way as for item $(F1)$ that the singular control values $v^*$ coincide with $v^* \in U_{\mathcal{L}-sing}(z^*)$.

Let us consider the case $(L_g\varphi_0, L_g\varphi_1)(x^*) = (0,0)$. Since the conditions $L_c\varphi_0 = L_c\varphi_1 = (L_g\varphi_0)L_{[c,g]}\varphi_1 - (L_g\varphi_1)L_{[c,g]}\varphi_0 = 0$ are valid everywhere on $X$, we repeat the proof of $(ii) \Rightarrow (i)$ and bring the system into the form (3.10), around $z^* = 0$, with $(\varphi_0, \varphi_1) = (z_1, z_2)$. Now we will show that the singular control values $v^*$ at which the procedures of calculating $z_0$ and $v_0$ fail, given by rk $\frac{\partial(\dot{\varphi}_0,\dot{\varphi}_1)}{\partial z_0, y}(z^*, v^*) \leq 1$ and $v_0^* = -(\eta + z_3\frac{\partial\eta}{\partial z_3})(z^*)$, coincide with $v^* \in U^{k-2}_{\mathcal{L}-sing}(z^*)$ and $v^* \in U^{k-3}_{sing}(z^*)$, respectively.

To this end, calculate $U^{k-2}_{\mathcal{L}-sing}(z) = \{v(z) = (v_0, v_1)^\top : [f + v_0g_0 + v_1g_1, l] \in \mathcal{G}^{k-2}\}$. Since $d\varphi_0 = dz_1$ and $d\varphi_1 = dz_2$, we have $\mathcal{L} = (\operatorname{span}\{d\varphi_0, d\varphi_1\})^\perp = \operatorname{span}\{\frac{\partial}{\partial z_0}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}, \ldots, \frac{\partial}{\partial z_k}\}$ and $\mathcal{G}^{k-2} = \mathcal{L} + \operatorname{span}\{B\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\}$. Thus $[f + v_0g_0 + v_1g_1, l] \in \mathcal{G}^{k-2}$, for any $l \in \mathcal{L}$, holds (taking the only nontrivial case $l = \frac{\partial}{\partial z_0}$) if and only if $[f, \frac{\partial}{\partial z_0}] + v_0[g_0, \frac{\partial}{\partial z_0}] \in \mathcal{G}^{k-2}$ which is equivalent to $[(\frac{\partial f_1}{\partial z_0} + v_0z_3\frac{\partial B}{\partial z_0})\frac{\partial}{\partial z_1} + \frac{\partial f_2}{\partial z_0}\frac{\partial}{\partial z_2}] \in \mathcal{G}^{k-2}$ and thus to $[(\frac{\partial f_1}{\partial z_0} + v_0z_3\frac{\partial B}{\partial z_0})\frac{\partial}{\partial z_1} + \frac{\partial f_2}{\partial z_0}\frac{\partial}{\partial z_2}] \wedge (B\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}) = 0$. This yields $v^* \in U^{k-2}_{\mathcal{L}-sing}(z^*)$ if and only if $\frac{\partial f_1}{\partial z_0}(z^*) - B\frac{\partial f_2}{\partial z_0}(z^*) + v_0^*z_3^*\frac{\partial B}{\partial z_0}(z^*) = 0$ which coincides with rk$\frac{\partial(\dot{\varphi}_0,\dot{\varphi}_1)}{\partial(z_0,y)}(z^*, v^*) \leq 1$.

Notice that under the assumption $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(z^*, u^*) \neq 0$, we have $\frac{\partial f_1}{\partial z_0}(z^*) - B\frac{\partial f_2}{\partial z_0}(z^*) \neq 0$ and, since $z^* = 0$, it follows that $v_0^* \notin U^{k-2}_{\mathcal{L}-sing}(z^*)$. Moreover, since $\frac{\partial B}{\partial z_0} \neq 0$ (otherwise $\mathcal{G}^{k-1} \neq TX$), for each fixed value $x \neq x^*$ in $\mathcal{X}^*$, a sufficiently small neighborhood of $x^*$, we get $(v_0, v_1) \in U^{k-2}_{\mathcal{L}-sing}(z^*)$ with $v_0 = \frac{\psi(z_0,z_1,z_2)}{z_3}$, where $\psi = (\frac{\partial f_1}{\partial z_0})(\frac{\partial B}{\partial z_0})^{-1}$, and $v_1$ any. Thus in $\mathcal{X}^* \times \mathbb{R}^2$, the set $U^{k-2}_{\mathcal{L}-sing}(x)$ consists of two connected components that define, for each fixed value $x \in \mathcal{X}^*$, $x \neq x^*$, an affine subspace of $U = \mathbb{R}^2$.

To analyze $v_0^* = -(\eta + z_3\frac{\partial\eta}{\partial z_3})(z^*)$, notice that for (3.10), $\mathcal{C}^{k-2} = \operatorname{span}\{\frac{\partial}{\partial z_3}, \cdots, \frac{\partial}{\partial z_n}\}$ and $\mathcal{G}^{n-3} = \mathcal{C}^{k-2} + \operatorname{span}\{\frac{\partial}{\partial z_0} + z_3B\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2}\}$. It follows that $[\tilde{f} + \tilde{v}_0\tilde{g}_0 + \tilde{v}_1\tilde{g}_1, \mathcal{C}^{k-2}] \in \mathcal{G}^{n-3}$ is equivalent to $[\tilde{f} + \tilde{v}_0\tilde{g}_0 + \tilde{v}_1\tilde{g}_1, \frac{\partial}{\partial z_3}] \wedge (\frac{\partial}{\partial z_0} + z_3B\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2}) = 0 \operatorname{mod} \mathcal{C}^{k-2}$, which yields $-\frac{\partial\eta}{\partial z_3} + \tilde{v}_0(\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2}) \wedge (\frac{\partial}{\partial z_0} + z_3(B\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2}))) = 0$ implying $z_3\frac{\partial\eta}{\partial z_3} + \tilde{v}_0 = z_3\frac{\partial\eta}{\partial z_3} + \eta + v_0 = 0$. Thus, indeed, $v_0^* =$

$-(z_3\frac{\partial\eta}{\partial z_3}+\eta)(z^*)$ if and only if $v^* \in U_{sing}^{n-3}(z^*)$. $\square$

#### 3.1.6.4 Proof of Proposition 3.1.1

*Proof.* In [29], the equivalence of the following conditions has been proven for any two-input system feedback equivalent to the chained form and for a pair of smooth functions $(\varphi_0, \varphi_1)$:

$(i)$ The pair $(\varphi_0, \varphi_1)$ is an $x$-flat output of $\Sigma_{lin}$ at $(x^*, u^*)$, where $u^*$ is such that $u_0^* g_0(x^*) + u_1^* g_1(x^*) \notin \mathcal{C}^1(x^*)$;

$(ii)$ The pair $(\varphi_0, \varphi_1)$ satisfies the following conditions:

$(FO1_{lin})$ $d\varphi_0 \wedge d\varphi_1(x^*) \neq 0$;

$(FO2_{lin})$ $L_c\varphi_0 = L_c\varphi_1 = L_c(\frac{L_g\varphi_1}{L_g\varphi_0}) = 0$, for any $c \in \mathcal{C}^{k-2}$, where the functions $\varphi_0$, $\varphi_1$ are ordered such that $L_g\varphi_0(x^*) \neq 0$, which is always possible due to item $(FO3_{lin})$;

$(FO3_{lin})$ $(L_g\varphi_0(x^*),\ L_g\varphi_1(x^*)) \neq (0,\ 0)$;

$(iii)$ The pair $(\varphi_0, \varphi_1)$ satisfies the following conditions:

$(FO1_{lin})'$ $d\varphi_0 \wedge d\varphi_1(x^*) \neq 0$;

$(FO2_{lin})'$ $\mathcal{L} = (\text{span}\,\{d\varphi_0, d\varphi_1\})^\perp \subset \mathcal{G}^{k-2}$;

$(FO3_{lin})'$ $\mathcal{G}^0(x^*) \not\subset \mathcal{L}(x^*)$.

In the view of the above, item $(F3)$ is obvious. So is $(F6)$ because $(FO1)'$ yields $(FO1_{lin})'$, the condition $(L_g\varphi_0(x^*),\ L_g\varphi_1(x^*)) \neq (0,\ 0)$ implies $(FO3_{lin})'$, and $(FO2)'$ and $(FO2_{lin})'$ coincide.

To show $(F5)$, notice that $(FO2)'$ and $(FO2_{lin})'$ coincide. To prove that $(\varphi_0, \varphi_1)$ satisfies $(F01)$, we can bring, see [29], the control-linear system $\Sigma_{lin}$ into the chained form compatible with the flat output $(\varphi_0, \varphi_1)$ (which is assumed to be a flat output of $\Sigma_{lin}$), that is, $Ch_1^k$ with $z_0 = \varphi_0$ and $z_1 = \varphi_1$. In the $z$-coordinates, the drift takes the triangular form for $TCh_1^k$. By a direct calculation, we can check that $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot\varphi_0 \wedge d\dot\varphi_1)(z^*, v^*) \neq 0$, where $v^* \notin U_{\mathcal{L}-sing}(z^*)$ and $\mathcal{L} = (\text{span}\,\{d\varphi_0, d\varphi_1\})^\perp$. Hence $(\varphi_0, \varphi_1)$ is an $x$-flat output of $\Sigma_{aff}$ at $(x^*, \tilde u^*)$ where $\tilde u^* \notin U_{\mathcal{L}-sing}(x^*)$.

It remains to prove $(F4)$. If $(\varphi_0, \varphi_1)$ is a flat output of $\Sigma_{lin}$, then the conditions $(FO1_{lin}) - (FO3_{lin})$ are satisfied and thus so are $(FO1) - (FO2)$ because $(FO2)$ and $(FO2_{lin})$ coincide and $(\varphi_0, \varphi_1)$ being a flat output of $\Sigma_{lin}$ satisfies $(FO1)$ with $\dot\varphi_i = L_{F_{lin}}\varphi_i$, $i = 0, 1$.

To prove the converse, we have to show that condition $(F01)$ $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot\varphi_0 \wedge d\dot\varphi_1)(x^*, u^*) \neq 0$, where $\dot\varphi_i$, for $i = 0, 1$ is understood as $\dot\varphi_i = L_{F_{lin}}\varphi_i$ and $F_{lin} = u_0 g_0 + u_1 g_1$, implies that $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0, 0)$.

Bring $\Sigma_{lin}$ into the chained form $Ch_1^k$ around $z^* = 0$ and let $(\varphi_0, \varphi_1)$ be a flat output. Since $L_c\varphi_0 = L_c\varphi_1 = 0$, for all $c \in \mathcal{C}^{k-2} = \text{span}\,\{\frac{\partial}{\partial z_3}, \cdots, \frac{\partial}{\partial z_k}\}$, it follows

$\varphi_i = \varphi_i(z_0, z_1, z_2)$, for $i = 0, 1$. Assume $(L_g\varphi_0, L_g\varphi_1)(0) = (0, 0)$, otherwise the claim holds. Thus $\frac{\partial \varphi_i}{\partial z_0}(0) = 0$, for $i = 0, 1$, and since $(d\varphi_0 \wedge d\varphi_1)(0) \neq 0$, we deduce $\mathrm{rk}\,\frac{\partial(\varphi_0,\varphi_1)}{\partial(z_1,z_2)}(0) = 2$. Assume that $\frac{\partial \varphi_1}{\partial z_2}(0) \neq 0$ (if not, permute $\varphi_0$ and $\varphi_1$) and put $\tilde{z}_2 = \varphi_1$. Notice that $b = L_{g_0}\varphi_1 = \frac{\partial \varphi_1}{\partial z_0} + \frac{\partial \varphi_1}{\partial z_1} z_2 + \frac{\partial \varphi_1}{\partial z_2} z_3$ is affine with respect to $z_3$ and $\frac{\partial \varphi_1}{\partial z_2}(0) \neq 0$ so $\tilde{z}_i = L_{g_0}^{i-3} b$, for $3 \leq i \leq k$, is a valid local change of coordinates in which the system, under the feedback $\tilde{v}_1 = v_0 L_{g_0}^{k-2} b + v_1 L_{g_1} L_{g_0}^{k-3} b$, takes the form

$$
\begin{array}{rcl}
\dot{z}_0 = v_0 \quad \dot{z}_1 & = & a(z_0, z_1, \tilde{z}_2) v_0 \\
\dot{\tilde{z}}_2 & = & \tilde{z}_3 v_0 \\
& \vdots & \\
\dot{\tilde{z}}_{k-1} & = & \tilde{z}_k v_0 \\
\dot{\tilde{z}}_k & = & \tilde{v}_1.
\end{array}
$$

where $a = z_2 = \varphi_1^{-1}(z_0, z_1, \tilde{z}_2)$. The condition $(L_g\varphi_0)L_{[c,g]}\varphi_1 = (L_g\varphi_1)L_{[c,g]}\varphi_0$ yields $\frac{\partial \varphi_0}{\partial z_0} + a\frac{\partial \varphi_0}{\partial z_1} = 0$. So omitting the tildes, we obtain $\dot{\varphi}_0 = \frac{\partial \varphi_0}{\partial z_2} z_3 v_0 = \frac{\partial \varphi_0}{\partial z_2} \dot{\varphi}_1$. Therefore the differentials satisfy $d\dot{\varphi}_0 = \dot{\varphi}_1 d\frac{\partial \varphi_0}{\partial z_2} \bmod \mathrm{span}\,\{d\dot{\varphi}_1\}$ and since $\dot{\varphi}_1(0) = 0$, we get $(d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(0) = 0$, which contradicts the independence of flat outputs and their differentials. Thus $(L_g\varphi_0, L_g\varphi_1)(0) \neq (0, 0)$. Now it is obvious that $L_c(\frac{L_g\varphi_1}{L_g\varphi_0}) = 0$ is equivalent to $(L_g\varphi_0)L_{[c,g]}\varphi_1 = (L_g\varphi_1)L_{[c,g]}\varphi_0$, where $L_g\varphi_0(x^*) \neq 0$ (after permuting $\varphi_0$ and $\varphi_1$, if necessary). □

#### 3.1.6.5 Proof of Proposition 3.1.2

*Proof.* For the proof of Proposition 3.1.2 in the case $L_g\varphi_0(x^*) \neq 0$, we refer the reader to [29]. Let us consider the case $L_g\varphi_0(x^*) = 0$. Bring the system $\Sigma_{aff}$ into the form $TCh_1^k$, around $z^* = 0$. The characteristic distribution $\mathcal{C}^{k-2}$ takes the form $\mathcal{C}^{k-2} = \mathrm{span}\,\{\frac{\partial}{\partial z_3}, \ldots, \frac{\partial}{\partial z_k}\}$, and the condition $L_c\varphi_0 = 0$, for any $c \in \mathcal{C}^{k-2}$, implies that $\varphi_0 = \varphi_0(z_0, z_1, z_2)$. From $< d\varphi_0, \mathcal{G}^{k-2} > (0) \neq 0$, we deduce $\frac{\partial \varphi_0}{\partial z_2}(0) \neq 0$. Introducing the new coordinate $\tilde{z}_2 = \varphi_0$ and following exactly the proof of item $(F2)$ of Theorem 3.1.3, we get (omitting the tildes for $\tilde{z}$)

$$
\begin{array}{rclcl}
\dot{z}_0 = \tilde{v}_0 - \eta(z_0, z_1, z_2, z_3) \quad \dot{z}_1 & = & \tilde{f}_1(z_0, z_1, z_2, z_3) & + & a(z_0, z_1, z_2)\tilde{v}_0 \\
\dot{z}_2 & = & \tilde{f}_2(z_0, z_1, z_2) & + & z_3 \tilde{v}_0 \\
& \vdots & & & \\
\dot{z}_{k-1} & = & \tilde{f}_{k-1}(z_0, \cdots, z_k) & + & z_k \tilde{v}_0 \\
\dot{z}_k & = & v_1, & &
\end{array}
\tag{3.11}
$$

with $\varphi_0 = z_2$. The condition $[f, \mathcal{C}^{k-2}] \in \mathcal{G}^{k-2}$ implies $\frac{\partial f_1}{\partial z_3} = -a\frac{\partial \eta}{\partial z_3}$. In these coordinates we have $v = (L_g\varphi_0)[c_{k-2}, g] - (L_{[c_{k-2},g]}\varphi_0)g = z_3\frac{\partial}{\partial z_2} - (\frac{\partial}{\partial z_0} + a\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2}) \bmod \mathcal{C}^{k-2}$. The distribution $\mathcal{L} = \mathcal{C}^{k-2} + \mathrm{span}\,\{\frac{\partial}{\partial z_0} + a\frac{\partial}{\partial z_1}\}$ is, indeed, involutive and of corank two in $TX$. Thus there exists a smooth function $\psi = \psi(z_0, z_1, z_2)$ such that $\frac{\partial \psi}{\partial z_1}(0) \neq 0$ and $\frac{\partial \psi}{\partial z_0} + a\frac{\partial \psi}{\partial z_1} = 0$ and we put $\tilde{z}_1 = \psi$. Then $\dot{\tilde{z}}_1 = L_f\psi + \frac{\partial \psi}{\partial z_2} z_3 \tilde{v}_0 =$

$\bar{f}_1(z_0, z_1, z_2, z_3) + z_3 B(z_0, z_1, z_2)\tilde{v}_0$. From $[f, \mathcal{C}^{k-2}] \in \mathcal{G}^{k-2}$, it follows that $\bar{f}_1 = \bar{f}_1(z_0, z_1, z_2)$. We have

$$\begin{array}{lllll} \dot{z}_0 = \tilde{v}_0 - \eta & \dot{\tilde{z}}_1 & = & \bar{f}_1(z_0, z_1, z_2) & + \quad z_3 B\tilde{v}_0 \\ & \dot{z}_2 & = & \tilde{f}_2(z_0, z_1, z_2) & + \quad z_3\tilde{v}_0 \\ & & \vdots & & \\ & \dot{z}_{k-1} & = & \tilde{f}_{k-1}(z_0, \cdots, z_k) & + \quad z_k\tilde{v}_0 \\ & \dot{z}_k & = & v_1, & \end{array}$$

with $\psi = \tilde{z}_1$ and $\varphi_0 = z_2$. The pair $(\varphi_0, \psi) = (z_2, z_1)$ is an $x$-flat output at $(z^*, v^*)$, with $v^* \notin U_{\mathcal{L}-sing}(z^*)$, if and only if $(\frac{\partial \bar{f}_1}{\partial z_0} - B\frac{\partial \tilde{f}_2}{\partial z_0})(0) \neq 0$, i.e., $(d\psi \wedge d\dot{\psi} \wedge d\varphi_0 \wedge d\dot{\varphi}_0)(0) \neq 0$.

□

### 3.1.6.6 Proof of Proposition 3.1.3

*Proof.* Consider $\Sigma_{aff}$ static feedback equivalent to $TCh_1^k$ and let $(\varphi_0, \varphi_1)$ be a flat output at $(x^*, u^*)$, such that $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0, 0)$, where $g$ is an arbitrary vector field in $\mathcal{G}$ such that $g(x^*) \notin \mathcal{C}^{k-2}(x^*)$. Form the decoupling matrix $D = (D_{ij})$, where $D_{ij} = L_{g_j}\varphi_i$, $0 \leq i, j \leq 1$. The involutive closure $\overline{\mathcal{G}}^0$ of $\mathcal{G}^0$ is $TX$, so $1 \leq \operatorname{rk} D(x) \leq 2$. If $\operatorname{rk} D(x) = 2$, then via a suitable feedback transformation $\dot{\varphi}_i = \tilde{v}$, $i = 0, 1$, which contradicts flatness. Thus $\operatorname{rk} D(x) = 1$ in a neighborhood of $x^*$, since $(L_g\varphi_0, L_g\varphi_1)(x^*) \neq (0, 0)$. We have $d\varphi_0 \wedge d\varphi_1(x) \neq 0$ so put $z_0 = \varphi_0$, $z_1 = \varphi_1$ and, after applying feedback, the first two components of the transformed system $\dot{z} = f + v_0 g_0 + v_1 g_1$ become $\dot{z}_0 = v_0$, $\dot{z}_1 = a_1(z) + b_1(z)v_0$. The successive time-derivatives $\varphi_1^{(l)}$ of $\varphi_1 = z_1$ cannot depend on $v_1$, for $0 \leq l \leq k-1$ (it would contradict flatness) and the $k$-th derivative depends explicitly on $v_1$, otherwise we would obtain a contradiction with the independence of flat outputs and their time-derivatives at $(x^*, u^*)$. Notice, however, that $\varphi_1^{(l)}$ is a polynomial of degree $l$, with respect to $v_0$, with the leading coefficient being $L_{g_0}^{l-1}b_1$. Since $\varphi_1^{(l)}$ does not depend on $v_1$, for $1 \leq l \leq k-1$, it follows that $L_{g_1}L_{g_0}^{l-1}b_1 = 0$ for $1 \leq l \leq k-2$. We claim that the functions $z_0$, $z_1$, $b_1, \ldots, L_{g_0}^{k-2}b_1$ are independent at any point of an open and dense $X' \subset X$. If not, take $x_0$ and its open neighborhood $V \subset X \backslash X'$ and let $s$ be the largest integer such that $z_0, z_1, b_1, \ldots, L_{g_0}^s b_1$ are independent in $V$. Assume $s \leq k-3$. Introduce new coordinates $z_i = L_{g_0}^{i-2}b_1$ in $V$, for $2 \leq i \leq s$. We get:

$$\begin{array}{lllll} \dot{z}_0 = v_0 & \dot{z}_1 & = & a_1(z) & + z_2 v_0 \\ & \dot{z}_2 & = & a_2(z) & + z_3 v_0 \\ & & \vdots & & \\ & \dot{z}_{s+1} & = & a_{s+1}(z) + z_{s+2}v_0 & \\ & \dot{z}_{s+2} & = & a_{s+2}(z) + b_{s+2}(z_0, \ldots, z_{s+2})v_0 & \\ & \dot{\bar{z}} & = & \bar{f} \quad + \bar{g}_0 v_0 & + \bar{g}_1 v_1 \end{array}$$

where $\bar{z} = (z_{s+3}, \ldots, z_k)$. Notice that the vector field $[g_0, g_1]$ is of the form $\sum_{i=s+3}^{k} \alpha_i \frac{\partial}{\partial z_i}$, with $\alpha_i$ smooth functions. We deduce that $\overline{\mathcal{G}}^0$, the involutive closure of $\mathcal{G}^0 = \mathrm{span}\,\{g_0, g_1\}$, satisfies $\overline{\mathcal{G}}^0 \subset \mathrm{span}\,\{g_0, \frac{\partial}{\partial z_{s+3}}, \cdots, \frac{\partial}{\partial z_k}\}$. This yields $\overline{\mathcal{G}}^0 \neq TX$, which contradicts the fact that for $\Sigma_{aff}$, static feedback equivalent to $TCh_1^k$, we have $\overline{\mathcal{G}}^0 = TX$. Thus $s = k-2$ and we put $z_2 = b_1, \dots, z_k = L_{g_0}^{k-2} b_1$, and replace $v_1$ by $L_f L_{g_0}^{k-2} b_1 + v_0(L_{g_0}^{k-1} b_1) + v_1(L_{g_1} L_{g_0}^{k-2} b_1)$. We get

$$g_0 = \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_2} + \cdots + z_{k-1} \frac{\partial}{\partial z_k} \quad \text{and} \quad g_1 = \frac{\partial}{\partial z_k}.$$

Using exactly the same arguments as in sufficiency part of the proof of Theorem 3.1.1 (the forms of $\mathcal{G}^i$ and of $\mathcal{C}^i$ and the condition $[f, \mathcal{C}^i] \in \mathcal{G}^i$) we conclude that on $X'$, open and dense in $X$, the system is locally in the triangular form

$$TCh_1^k : \begin{cases} \dot{z}_0 = v_0 \quad \dot{z}_1 & = f_1(z_0, z_1, z_2) + z_2 v_0 \\ & \vdots \\ \dot{z}_{k-1} & = f_{k-1}(z_0, \dots, z_k) + z_k v_0 \\ \dot{z}_k & = v_1 \end{cases}$$

The flat output $(\varphi_0, \varphi_1) = (z_0, z_1)$ satisfies

$$L_c \varphi_0 = L_c \varphi_1 = (L_g \varphi_0) L_{[c,g]} \varphi_1 - (L_g \varphi_1) L_{[c,g]} \varphi_0 = 0,$$

where $c \in \mathcal{C}^{k-2} = \mathrm{span}\,\{\frac{\partial}{\partial z_3}, \dots, \frac{\partial}{\partial z_k}\}$ and $g$ is any vector field such that $\mathcal{G}^0 = \mathrm{span}\,\{g, c_1\}$ where $c_1 = \frac{\partial}{\partial z_k}$ is the characteristic vector field of $\mathcal{G}^1$. In order to prove that we can bring the system into the triangular form $TCh_1^k$, around any $x^* \in X$ (and not only on $X'$), notice that the characteristic distribution $\mathcal{C}^{k-2}$ is defined everywhere (not only on $X'$) so, by continuity, the conditions $L_c \varphi_0 = L_c \varphi_1 = (L_g \varphi_0) L_{[c,g]} \varphi_1 - (L_g \varphi_1) L_{[c,g]} \varphi_0 = 0$ hold everywhere on $X$ implying that if we put the control system $\Sigma_{aff}$, around an arbitrary point $x^* \in X$, into the triangular form $TCh_1^k$, then for the flat output $(\varphi_0, \varphi_1)$, we have $\varphi_i = \varphi_i(z_0, z_1, z_2)$, $0 \leq i \leq 1$, on $X'$ and thus on $X$.

Since we have assumed that $(L_g \varphi_0, L_g \varphi_1)(x^*) \neq (0,0)$, we can apply the following change of coordinates (permute $\varphi_0$ and $\varphi_1$, if necessary) $z_0 = \varphi_0$, $z_1 = \varphi_1$ and $z_i = L_{g_0}^{i-2} \psi$, for $2 \leq i \leq k$, where $\psi = \frac{L_{g_0} \varphi_1}{L_{g_0} \varphi_0}$, in which the control vector fields are in the chained form with $(\varphi_0, \varphi_1) = (z_0, z_1)$. The system $\Sigma_{aff}$ is assumed to be feedback equivalent to the triangular form $TCh_1^k$, hence satisfies the compatibility condition *(Comp)*. Using the $z$-coordinates and applying the feedback $f \mapsto f - (L_f \varphi_0) g_0 - (L_f^{k-1} \psi) g_1$, we transform $\Sigma_{aff}$ into the triangular form $TCh_1^k$ with $(\varphi_0, \varphi_1) = (\tilde{z}_0, \tilde{z}_1)$ around any $x^* \in X$.

Notice that we have proved, in particular, that any flat output $(\varphi_0, \varphi_1)$ of a system $\Sigma_{aff}$ feedback equivalent to $TCh_1^k$ satisfies $(d\varphi_0 \wedge d\varphi_1 \wedge d\dot{\varphi}_0 \wedge d\dot{\varphi}_1)(x^*, u^*) \neq 0$ and $L_c \varphi_0 = L_c \varphi_1 = (L_g \varphi_0) L_{[c,g]} \varphi_1 - (L_g \varphi_1) L_{[c,g]} \varphi_0 = 0$, for any $c \in \mathcal{C}^{k-2}$, that is, conditions $(FO1) - (FO2)$ of Theorem 3.1.3.

□

### 3.1.6.7 Proof of Theorem 3.1.4

*Proof of (m-F1).* Consider a control-affine system $\Sigma : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$ locally, around $x^*$, static feedback equivalent to $TCh_m^k$, and bring it into the form $TCh_m^k$, around $z^*$. For simplicity of notation, we continue to denote by $f$, respectively by $g_i$, for $0 \leq i \leq m$, the drift, respectively the controlled vector fields of $TCh_m^k$.

It is clear that $TCh_m^k$ is $x$-flat, with $\varphi = (z_0, z_1^1, \cdots, z_m^1)$ being a flat output, at any point $(z^*, v^*) \in X \times \mathbb{R}^{m+1}$ satisfying

$$\operatorname{rk} F^l(z^*) = m, \text{ for } 1 \leq l \leq k-1,$$

where $F^l$, for $1 \leq l \leq k-1$, is the $m \times m$ matrix given by

$$F_{ij}^l = \frac{\partial (f_j^l + z_j^{l+1} v_0^*)}{\partial z_i^{l+1}}, \text{ for } 1 \leq i, j \leq m.$$

Moreover, the differential weight of $\varphi = (z_0, z_1^1, \cdots, z_m^1)$ is $(k+1)(m+1)$, since expressing $z$ and $v$ involves $\varphi_i^{(j)}$, for $1 \leq i \leq m$ and $0 \leq j \leq k$.

Recall that in coordinates $z$, using the notation $\operatorname{span}\{\frac{\partial}{\partial z^i}\} = \operatorname{span}\{\frac{\partial}{\partial z_1^i}, \cdots, \frac{\partial}{\partial z_m^i}\}$, we have

$$\mathcal{G}^i = \operatorname{span}\{\frac{\partial}{\partial z^{k-i}}, \cdots, \frac{\partial}{\partial z^k}, g_0\},\ 0 \leq i \leq k-1,$$

$$\mathcal{C}^i = \operatorname{span}\{\frac{\partial}{\partial z^{k-i+1}}, \cdots, \frac{\partial}{\partial z^k}\},\ 1 \leq i \leq k-2,$$

and

$$\mathcal{L} = \operatorname{span}\{\frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^k}\}.$$

We have $\mathcal{C}^1 = \operatorname{span}\{\frac{\partial}{\partial z_1^k}, \cdots, \frac{\partial}{\partial z_1^k}\}$, and thus

$$\begin{aligned}
\mathcal{G}^0 + [f + gv, \mathcal{C}^1] &= \mathcal{G}^0 + \operatorname{span}\{[f + gv, \tfrac{\partial}{\partial z_j^k}], 1 \leq j \leq m\} \\
&= \mathcal{G}^0 + \operatorname{span}\{\tfrac{\partial (f_1^{k-1} + z_1^k v_0)}{\partial z_j^k} \tfrac{\partial}{\partial z_1^{k-1}} + \cdots + \tfrac{\partial (f_m^{k-1} + z_m^k v_0)}{\partial z_j^k} \tfrac{\partial}{\partial z_m^{k-1}}, 1 \leq j \leq m\},
\end{aligned}$$

where $gv = \sum_{i=0}^{m} g_i v_i$. By induction, we obtain

$$\mathcal{G}^i + [f + gv, \mathcal{C}^{i+1}] =$$

$$\mathcal{G}^i + \operatorname{span}\{\frac{\partial (f_1^{k-i-1} + z_1^{k-i} v_0)}{\partial z_j^{k-i}} \frac{\partial}{\partial z_1^{k-i-1}} + \cdots + \frac{\partial (f_m^{k-i-1} + z_m^{k-i} v_0)}{\partial z_j^{k-i}} \frac{\partial}{\partial z_m^{k-i-1}},\ 1 \leq j \leq m\}.$$

Therefore for any $0 \leq i \leq k-2$, we have $\operatorname{rk} F^{i+1}(z^*, v^*) = m$ if and only if $\operatorname{rk}(\mathcal{G}^i + [f + gv, \mathcal{C}^{i+1}])(z^*, v^*) = (i+2)m + 1$, for $0 \leq i \leq k-3$, and $\operatorname{rk}(\mathcal{G}^{k-2} + [f + gv, \mathcal{L})(z^*, v^*) = km + 1$. It follows that the original system $\Sigma_{aff}$ is $x$-flat at $(x^*, u^*)$ such that $u^* \notin U_{m-sing}(x^*)$, of differential weight at most $(k+1)(m+1)$.

As we have noticed, $(\varphi_0, \ldots, \varphi_m) = (z_0, z_1^1, \ldots, z_m^1)$ is an $x$-flat output of $TCh_m^k$ of differential weight $(k+1)(m+1)$ since expressing $z$ and $v$ involves $\varphi_i^{(j)}$, for $0 \leq j \leq k$.

Now, we will show (which is interesting as an independent observation) that the differential weight of any $x$-flat output of $\Sigma_{aff} : \dot{x} = f + \sum_{i=0}^{m} u_i g_i$, with $m+1$ controls and $km+1$ states, is at least $(k+1)(m+1)$. Let $(\varphi_0, \ldots, \varphi_m)$ be an $x$-flat output of $\Sigma_{aff}$. Define $D = (D_{ij})$, where $D_{ij} = L_{g_i}\varphi_j$ and put $r(x) = \operatorname{rk} D(x)$. Clearly, $r(x)$ is constant on an open and dense subset $X'$ of $X$ (so denote it $r(x) = r$) and choose $x_0 \in X'$. By a suitable (local) change of coordinates and static invertible feedback, we get

$$\begin{array}{rcl} \dot{z}^0 = v^0 \quad \dot{z}^1 & = & A^1(z) \;+\; B^1(z)v^0 \\ \dot{z}^2 & = & A^2(z) \;+\; B^2(z)v \end{array}$$

where $\dim z^0 = r$, $\dim z^1 = m - r + 1$, $z_0^0 = \varphi_0, \ldots, z_{r-1}^0 = \varphi_{r-1}$ and $z_r^1 = \varphi_r$, $\ldots, z_m^1 = \varphi_m$.

Due to flatness we can express (with the help of the flat outputs $\varphi_i$ and their time-derivatives) $mk+1$ components of $z$ and $m+1$ components of $v$, i.e., $m(k+1)+2$ functions. Using $\varphi_i = z_i^0$ and $\dot{\varphi}_i = v_i^0$, $0 \leq i \leq r-1$, we express $2r$ system variables. The remaining $m(k+1)+2-2r$ system variables (that is, the components of $z^1$, $z^2$ and the remaining components of $v$) depend on derivatives of $\varphi_i$, $r \leq i \leq m$. Denote by $s_i$ the maximal order of the derivative $\varphi_i^{(s_i)}$, $r \leq i \leq m$, that is involved. Put $s = \max\{s_i : r \leq i \leq m\}$. By taking the time-derivatives of $\varphi_i$ up to order $s_i \leq s$, we can express at most $(s+1)(m-r+1)$ functions. This number cannot thus be smaller than the number of functions that remain to be expressed, that is, we need

$$(s+1)(m-r+1) \geq m(k+1) + 2 - 2r,$$

which is equivalent to

$$m(s-k) \geq (r-1)(s-1).$$

Now, three cases are possible. It is clear that if $s < k$, then the left hand side is negative, so the inequality is not satisfied. If $s = k$, then either $r = 1$ or $s = 1$. The latter is impossible since $s \geq 2$. In the case $r = 1$, we have $\dim z^0 = \dim v^0 = 1$ and in order to express all $m(k+1)+2$ variables of the system, we will use $s = k$ derivatives $v^0, \dot{v}^0, \ddot{v}^0, \ldots, (v^0)^{(s-1)}$. Thus the differential weight of $\varphi$ is at least $m(k+1)+s+1 = m(k+1)+k+1 = (m+1)(k+1)$.

Finally, if $s > k$, then there exists $\varphi_j$, for some $r+1 \leq j \leq m+1$, that we differentiate $s$ times so it involves at least $s-1$ time derivatives of $\dot{\varphi}_j = A_j^1(z) + B_j^1(z)v^0$, where $A_j^1$ is the $j$-th component of $A^1$ and $B_j^1$ is the $j$-th row of $B^1$. The involutive closure $\overline{\mathcal{G}}^0$ of the distribution $\mathcal{G}^0$ is $TX$ so $B_j^1$ is nonzero. It implies that $\varphi_j^{(s)}$ depends non-trivially on (at least one) component of $(v^0)^{(s-1)}$. To summarize, we use $mk+1$ functions to express $z$, $m+1$ functions to express $v$, and we also use the $s-1$ derivatives $\dot{v}^0, \ddot{v}^0, \ldots, (v^0)^{(s-1)}$, which gives at least $(k+1)(m+1)+1$ functions (since $s > k$). Therefore the differential weight is higher than $(k+1)(m+1)$ on $X'$ and thus on $X$.

It remains to prove that the differential weight of any flat output (not necessary an $x$-flat output) cannot be smaller than $(k+1)(m+1)$. Let $(\varphi_0, \ldots, \varphi_m)$ be an $(x, u, \dot{u}, \ldots, u^{(p)})$-flat output of $\Sigma_{aff}$. Denote by $s_i$ the highest derivative of $\varphi_i$, for

$0 \leq i \leq m$, involved in expressing the state $x$ and the control $u$, that is, by flatness, $\mathcal{X} + \mathcal{U} \subset \Phi$, where $\mathcal{X} = \mathrm{span}\,\{dx_1, \cdots, dx_n\}$, $\mathcal{U} = \mathrm{span}\,\{du_0, \cdots, du_m\}$ and $\Phi = \mathrm{span}\,\{d\varphi_i^{(j_i)}, 0 \leq i \leq m, 0 \leq j_i \leq s_i\}$. Let $s_{i^*}$ be the largest among the integers $s_i$. Either $\varphi_{i^*}$ depends on $u^{(l)}$, with $l \geq 1$ (but not on derivatives of $u$ higher than $l$) or $\varphi_{i^*}$ depends on $u$ (but not on derivatives of $u$) or $\varphi_{i^*}$ depends on $x$ only. Then the differentials $\varphi_{i^*}^{(j)}$ are independent modulo $\mathcal{X} + \mathcal{U}$, for $0 \leq j \leq s_{i^*}$ (in the first case), for $1 \leq j \leq s_{i^*}$ (in the second case) and for $2 \leq j \leq s_{i^*}$ (in the third case, since $\dot{\varphi}_{i^*}$ depends on $u$ because $\overline{\mathcal{G}}^0 = TX$). It follows that $\mathcal{X} + \mathcal{U} \subset \Psi = \mathrm{span}\,\{d\varphi_{i^*}, d\dot{\varphi}_{i^*}, d\varphi_i^{(j_i)}, 0 \leq i \leq m, i \neq i^*, 0 \leq j_i \leq s_i\}$.

We claim that $s_{i^*} \geq k$. If not, then $s_i \leq s_{i^*} \leq k - 1$, for $0 \leq i \leq m$ (recall that $s_{i^*} = \max\{s_i : 0 \leq i \leq m\}$),which implies $\mathrm{rk}\,\Psi \leq mk + 2 < m(k+1) + 2 = \mathrm{rk}\,(\mathcal{X} + \mathcal{U})$, contradicting $\mathcal{X} + \mathcal{U} \subset \Psi$. Thus $s_{i^*} \geq k$.

We have $\mathcal{X} + \mathcal{U} \subset \Phi$ (by flatness) and $d\ddot{\varphi}_{i^*}, \cdots, d\varphi_{i^*}^{(s_{i^*})}$ belong to $\Phi$ and are independent modulo $\mathcal{X} + \mathcal{U}$, so $\mathrm{rk}\,\Phi \geq \mathrm{rk}\,(\mathcal{X} + \mathcal{U}) + k - 1 = m(k+1) + 2 + k - 1 = (m+1)(k+1)$ proving that the differential weight of $\varphi$ is at least $(m+1)(k+1)$. Notice that $\mathrm{rk}\,\Phi = (m+1)(k+1)$ if and only if $s_{i^*} = s_i = k$, for any $0 \leq i \leq m$, implying that with $\varphi_i$, $i \neq i^*$, we express $mk$ system variables and the remaining two variables are expressed with $\varphi_{i^*}$. We deduce immediately that, in this case, all $\varphi_i$ depend on $x$ only.

*Proof of (m-F2).* Let $(\varphi_0, \cdots, \varphi_m)$ be a minimal $x$-flat output for $\Sigma_{aff}$. When proving *(m-F1)* we have shown that we can bring the system into the form

$$\dot{z}_0 = v_0 \quad \begin{array}{rcccl} \dot{z}^1 &=& A^1(z) &+& B^1(z)v^0 \\ \dot{z}^2 &=& A^2(z) &+& B^2(z)v \end{array}$$

where $z_0 = \varphi_0$ and $z_1^1 = \varphi_1, \ldots, z_m^1 = \varphi_m$ and $\dim z_0 = \dim v_0 = 1$, being a consequence of the minimal differential weight $(k+1)(m+1)$ of $\varphi$. For $i \leq i \leq m$, denote by $k_i$ the minimal integer such that $\varphi_i^{(k_i)}$ depends explicitly on at least one $v_j$, for $1 \leq j \leq m$. Since $\Sigma_{aff}$ is static feedback equivalent to $TCh_m^k$, it follows that $k_i \leq k$. In order to prove that $k_i = k$, for $1 \leq j \leq m$, suppose that there exists $k_i < k$ and assume, for simplicity, that $k_1 < k$. Denote $\varphi_1^{(k_1)} = v_1$ (with $v_1$ depending on $v_0, \cdots v_0^{(k_1 - 1)}$).

Like in the the proof of *(m-F1)*, notice that due to flatness we can express (with the help of the flat outputs $\varphi_i$ and their time-derivatives) $mk + 1$ components of $z$ and $m + 1$ components of $v$, i.e., $m(k+1) + 2$ functions. Using $\varphi_0 = z_0$ and $\varphi_1 = z_1^1$, we can express $2 + k_1 + 1 = k_1 + 3$ variables of the system. The remaining $m(k+1) + 2 - (k_1 + 3)$ system variables depend on derivatives of $\varphi_i$, $2 \leq i \leq m$. Denote by $s_i$ the maximal order of the derivative $\varphi_i^{(s_i)}$, $2 \leq i \leq m$, that is involved. Put $s = \max\{s_i : 2 \leq i \leq m\}$. By taking the time-derivatives of $\varphi_i$ up to order $s_i \leq s$, we can express at most $(s+1)(m-1)$ functions. This number cannot thus be smaller than the number of functions that remain to be expressed, that is, we need

$$(s+1)(m-1) \geq m(k+1) + 2 - (k_1 + 3),$$

which is equivalent to

$$m(s-k) \geq s - k_1.$$

We have $k_1 < k$ so the inequality can be satisfied only if $s > k$, but this give the differential weight of $\varphi$ at least $m(k+1) + 2 + s - 1 \geq (k+1)(m+1) + 2$, implying that $\varphi$ is not a minimal flat output. It follows that for all $1 \leq i \leq m$ we must have $k_i = k$ (and the inequality is satisfied only in this case). The distribution $\mathcal{L} = (\text{span}\{d\varphi_0, \cdots, d\varphi_m\})^\perp$ is involutive (as annihilator of exact 1-forms) and satisfies $\mathcal{L} \subset \mathcal{G}^{k-2}$ (because all $k_i = k$), as well as $\mathcal{G}^0(x^*) \not\subset \mathcal{L}(x^*)$ (since $g_0(x^*) \notin \mathcal{L}(x^*)$). It follows that $\mathcal{G}^0$ is in the $m$-chained form in $z$-coordinates, where $z_0 = \varphi_0$, $z_i^j = L_{g_0}^{j-1}\varphi_i$, for $1 \leq i \leq m$, $1 \leq j \leq k$ (see Appendix B). The compatibility condition *(m-Comp)* implies that $\Sigma_{aff}$ is in the triangular form.

*Proof of (m-F3).* We will prove the implications: $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

$(i) \Rightarrow (iii)$. Assume that the system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$ is $x$-flat at $(x^*, u^*)$, where $u^* \notin U_{m-sing}(x^*)$, and let $(\varphi_0, \cdots, \varphi_m)$ be its minimal $x$-flat output defined in a neighborhood $\mathcal{X}^*$ of $x^*$. It is well known that the differentials of flat outputs are independent at $x^*$, thus implying *(m-FO1)*. By item *(m-F2)*, that we have just proven, we can bring $\Sigma_{aff}$, around any point $x \in \mathcal{X}^*$ into the triangular form compatible with the chained form $TCh_m^k$, with $(\varphi_0, \cdots, \varphi_m) = (z_0, z_1^1, \cdots, z_m^1)$ and $x^*$ transformed into $z^* \in \mathbb{R}^{km+1}$. In coordinates $z$, the corank one involutive subdistribution $\mathcal{L}$ of $\mathcal{G}^{k-2}$ is given by

$$\mathcal{L} = \text{span}\{\frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^k}\},$$

because it is unique and we immediately have

$$\mathcal{L}^\perp = \text{span}\{d\varphi_0, \cdots, d\varphi_m\},$$

which gives *(m-FO2)* on $\mathcal{X}^*$.

$(iii) \Rightarrow (ii)$. Suppose that the $(m+1)$-tuple $(\varphi_0, \cdots, \varphi_m)$ fulfills conditions *(m-FO1)-(m-FO2)*. We apply the change of coordinates and the invertible feedback transformation presented in Appendix 3.1.B (with $\phi_i$ replaced by $\varphi_i$ and $\tilde{u}$ by $v$) that bring the control-linear system $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ into the $m$-chained form, with $z_0 = \varphi_0$ and $z_i^1 = \varphi_i$, for $1 \leq i \leq m$. Thus $(\varphi_0, \cdots, \varphi_m) = (z_0, z_1^1, \cdots, z_m^1)$ is a minimal $x$-flat output of $Ch_m^k$ at any $(z^*, v^*)$, with $v^* \neq 0$. It follows that $(\varphi_0, \cdots, \varphi_m)$ is a minimal $x$-flat output of $\Sigma_{lin}$ at any $(x^*, \tilde{u}^*)$, with $\tilde{u}^*$ such that $\sum_{i=0}^m \tilde{u}_i^* g_i(x^*) \notin \mathcal{C}^1(x^*)$.

$(ii) \Rightarrow (i)$. Assume that the system $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ is $x$-flat at $(x^*, \tilde{u}^*)$, where $\tilde{u}^*$ is such that $\sum_{i=0}^m \tilde{u}_i^* g_i(x^*) \notin \mathcal{C}^1(x^*)$, where $\mathcal{C}^1$ is the characteristic distribution of $\mathcal{G}^1$. Let $(\varphi_0, \cdots, \varphi_m)$ be its minimal $x$-flat output defined in a neighborhood $\mathcal{X}$ of $x^*$. It is known, see [28], that the minimal flat output satisfies $\mathcal{L}^\perp = \text{span}\{d\varphi_0, \cdots, d\varphi_m\}$. By the construction given in Appendix 3.1.B, bring the system into the $m$-chained form $Ch_m^k$ such that $(\varphi_0, \cdots, \varphi_m) = (z_0, z_1^1, \cdots, z_m^1)$ and $z_i^j = L_{g_0}^{j-2}\psi_i$, for $2 \leq j \leq k$ and $1 \leq i \leq m$, where $\psi_i = \frac{L_{g_0}\varphi_i}{L_{g_0}\varphi_0}$. The system $\Sigma_{aff}$ is assumed to be feedback equivalent to the triangular form $TCh_m^k$, hence satisfies the compatibility condition *(m-Comp)*. Using the $z$-coordinates and applying the feedback $f \mapsto f - \sum_{i=0}^{m} \alpha_i g_i$, where $\alpha_0 = L_f \varphi_0$ and $\alpha_i = L_f^{k-1} \psi_i$, we transform $\Sigma_{aff}$ into the triangular form $TCh_m^k$. We have proved, when showing *(m-F1)*, that $(\varphi_0, \cdots, \varphi_m) = (z_0, z_1^1, \cdots, z_m^1)$ is an $x$-flat output of $\Sigma_{aff}$ at $(x^*, u^*)$ such that $u^* \notin U_{m-sing}(x^*)$. $\square$

# Appendices

### 3.1.A. Involutive subdistribution of corank one

Consider a non involutive distribution $\mathcal{G}$ of rank $d$, defined on a manifold $X$ of dimension $n$ and define its annihilator $\mathcal{G}^\perp = \{\omega \in \Lambda^1(X) :< \omega, f >= 0, \forall f \in \mathcal{G}\}$. Let $\omega_1, \ldots, \omega_s$, where $s = n - d$, be differential 1-forms locally spanning the annihilator of $\mathcal{G}$, that is $\mathcal{G}^\perp = \mathcal{I} = \mathrm{span}\{\omega_1, \ldots, \omega_s\}$. The *Engel rank* of $\mathcal{G}$ equals 1 at $x$ if and only if $(d\omega_i \wedge d\omega_j)(x) = 0 \bmod \mathcal{I}$, for any $1 \le i, j \le s$. For any $\omega \in \mathcal{I}$, we define $\mathcal{W}(\omega) = \{f \in \mathcal{G} : f \lrcorner d\omega \in \mathcal{G}^\perp\}$, where $\lrcorner$ is the interior product. The characteristic distribution $\mathcal{C} = \{f \in \mathcal{G} : [f, \mathcal{G}] \subset \mathcal{G}\}$ of $\mathcal{G}$ is given by

$$\mathcal{C} = \bigcap\nolimits_{i=1}^{s} \mathcal{W}(\omega_i).$$

It follows directly from the Jacobi identity that the characteristic distribution is always involutive. Let $\mathrm{rk}\,[\mathcal{G}, \mathcal{G}] = d + r$. Choose the differential forms $\omega_1, \ldots, \omega_r, \ldots, \omega_s$ such that $\mathcal{I} = \mathrm{span}\{\omega_1, \ldots, \omega_s\}$ and $\mathcal{I}^1 = \mathrm{span}\{\omega_{r+1}, \ldots, \omega_s\}$, where $\mathcal{I}^1$ is the annihilator of $[\mathcal{G}, \mathcal{G}]$. Define the distribution

$$\mathcal{H} = \sum_{i=1}^{r} \mathcal{W}(\omega_i).$$

We have the following result proved by Bryant [6], see also [50].

**Proposition 3.1.6.** *Consider a distribution $\mathcal{G}$ of rank $d$ and let* $\mathrm{rk}\,[\mathcal{G}, \mathcal{G}] = d + r$.

*(i)* *Assume $r \ge 3$. The distribution $\mathcal{G}$ contains an involutive subdistribution of corank one if and only if it satisfies*

*(ISD1) The Engel rank of $\mathcal{G}$ equals one;*

*(ISD2) The characteristic distribution $\mathcal{C}$ of $\mathcal{G}$ has rank $d - r - 1$.*

*Moreover, that involutive subdistribution is unique and is given by $\mathcal{H}$.*

*(ii)* *Assume $r = 2$. The distribution $\mathcal{G}$ contains a corank one subdistribution $\mathcal{L}$ satisfying $[\mathcal{L}, \mathcal{L}] \subset \mathcal{G}$ if and only it verifies (ISD1)-(ISD2). In that case, $\mathcal{H}$ is the unique distribution with the desired properties.*

*(iii)* *Assume $r = 1$. The distribution $\mathcal{G}$ contains an involutive subdistribution of corank one if and only it satisfies the condition (ISD2). In the case $r = 1$, if an involutive subdistribution of corank one exists, it is never unique.*

### 3.1.B. Constructing coordinates for the $m$-chained form

In [59], the following characterization of the $m$-chained form was stated and proved: An $(m+1)$-input driftless control system $\Sigma_{lin} : \dot{x} = \sum_{i=0}^{m} u_i g_i(x)$, with $m \geq 2$, defined on a manifold $X$ of dimension $km+1$, is locally static feedback equivalent, in a small neighborhood of a point $x^* \in X$, to the $m$-chained form if and only if its associated distribution $\mathcal{G} = \text{span}\{g_0, \cdots, g_m\}$ satisfies conditions *(m-Ch1)-(m-Ch3)* of Theorem 3.1.2.

The prove of this result provides a method to compute the diffeomorphism bringing any control system, for which it is possible, to the $m$-chained form. Now, we will explain how to do it.

The involutive subdistribution $\mathcal{L}$ is unique and can be explicitly calculated (see Appendix 3.1.A). Choose $m+1$ independent functions $\phi_0$, $\phi_1^1, \cdots, \phi_m^1$ whose differentials annihilates $\mathcal{L}$, that is

$$\text{span}\{d\phi_0, d\phi_1^1, \cdots, d\phi_m^1\} = (\mathcal{L})^{\perp},$$

and a vector field $g \in \mathcal{G}^0$ (which always exists due to condition $(m - Ch3)$) such that $g(x^*) \notin \mathcal{L}^{k-2}(x^*)$. Without loss of generality, we can assume $g = g_0$ and $L_{g_0}\phi_0^0(x^*) \neq 0$ (otherwise permute the vector fields $g_i$ or the functions $\phi_i^1$). Define the coordinates

$$\begin{cases} z_0 &= \phi_0 \\ z_i^1 &= \phi_i^1,\ 1 \leq i \leq m, \\ z_i^j &= \phi_i^j = \frac{L_{g_0}\phi_i^{j-1}}{L_{g_0}\phi_0},\ 1 \leq i \leq m,\ 2 \leq j \leq k, \end{cases}$$

and the feedback

$$\tilde{u}_0 = u_0 L_{g_0}\phi_0 \text{ and } \tilde{u}_j = \sum_{i=0}^{m} u_i L_{g_i}\phi_j^k,\ 1 \leq j \leq m.$$

In the above coordinates, the distribution $\mathcal{G}$ takes the form

$$\mathcal{G} = \text{span}\{\frac{\partial}{\partial z_1^k}, \cdots, \frac{\partial}{\partial z_m^k}, \frac{\partial}{\partial z_0} + \sum_{j=1}^{m}\sum_{i=1}^{k-1} z_j^{i+1}\frac{\partial}{\partial z_j^i}\}$$

and, equivalently, $\Sigma_{lin}$ takes the $m$-chained form.

## 3.2 $x$-Maximal Flatness of Control-Affine Systems Compatible with the Multi-Chained Form

**Abstract**

In the second part of the paper, we introduce the concept of $x$-maximal flatness. A control system is $x$-maximally flat if the number of new states gained by each successive derivation of the flat output is the largest possible. Firstly, we show that the only control-linear systems that are $x$-maximally flat are those that are static feedback equivalent to the $m$-chained form. Secondly, we generalize that result from control-linear systems to control-affine systems whose control-linear subsystem is static feedback equivalent to the $m$-chained form. We prove that they are $x$-maximally flat if and only if the drift exhibits a triangular form compatible with the $m$-chained form (and recently characterized in [65] and [27]). We also show that if we skip the assumption of the $x$-maximal flatness, the latter condition is not necessary for $x$-flatness of control-affine system whose associated control-linear subsystem is static feedback equivalent to the $m$-chained form.

### 3.2.1 Introduction

We study flatness of nonlinear control systems of the form

$$\Xi : \ \dot{x} = \ F(x,u),$$

where $x$ is the state defined on a open subset $X$ of $\mathbb{R}^n$ and $u$ is the control taking values in an open subset $U$ of $\mathbb{R}^m$ (more generally, an $n$-dimensional manifold $X$ and an $m$-dimensional manifold $U$, respectively). The dynamics $F$ are smooth and the word smooth will always mean $\mathcal{C}^\infty$-smooth. The system $\Xi : \dot{x} = \ F(x,u)$ is *flat* if we can find $m$ functions, $\varphi_i(x,u,\ldots,u^{(r)})$, for some $r \geq 0$, called *flat outputs*, such that

$$x = \gamma(\varphi,\ldots,\varphi^{(s-1)}) \text{ and } u = \delta(\varphi,\ldots,\varphi^{(s)}), \tag{3.12}$$

for a certain integer $s$, where $\varphi = (\varphi_1,\ldots,\varphi_m)$. Therefore the evolution in time of all state and control variables can be determined from that of flat outputs without integration and all trajectories of the system can be completely parameterized.

The differential weight of a flat output $\varphi$ is, roughly speaking, the minimal number of derivatives of components of $\varphi$, needed to express $x$ and $u$ (see [45, 46, 58]. Here we propose another way of looking at that property. It is well known (see e.g. [15, 22, 54]) that for any $l \geq 0$, all time-derivatives $\varphi_i^{(j)}$, $1 \leq i \leq m$, $0 \leq j \leq l$, of flat outputs are independent. So the successive time-derivatives provide $m$ new independent functions $\varphi_i^{(l+1)}$, $1 \leq i \leq m$. The problem that we are going to study is how many new functions of the state $x$ do successive derivatives of the flat outputs provide? The system $\Xi$ will be called *x-maximal flat* if each successive time-derivative of the flat output provides the largest possible number of independent functions of the state.

Observe, first, that $x$-maximally flat systems are simply static feedback linearizable systems (see Proposition 3.2.1). Secondly, we show that, within the class of control-linear systems, the only $x$-maximally flat systems are those that are static feedback equivalent to the $m$-chained form (see Proposition 3.2.2). Thirdly, we generalize that result from control-linear systems to control-affine systems whose control-linear subsystem is static feedback equivalent to the $m$-chained form. We prove that they are $x$-maximally flat if and only if the drift is triangular in the system of coordinates in which the controlled vector fields are in the $m$-chained form (see Theorem 3.2.1). In other words, they are $x$-maximally flat if and only if they are static feedback equivalent to a triangular form compatible with the $m$-chained form. That triangular form has been recently characterized by Silveira, Pereira da Silva and Rouchon [65] (for $m = 2$) and by the authors [27] for $m \geq 2$. We also show that if we skip the assumption of the $x$-maximal flatness, the compatibility condition is not necessary for $x$-flatness of control-affine system whose associated control-linear subsystem is static feedback equivalent to the $m$-chained form.

The paper is organized as follows. In Section 3.2.2, we recall the definition of flatness, we introduce the notion of $x$-maximally flat system and we study the $x$-maximal flatness of general and then of control-linear systems. In Section 3.2.3, we give our main result: we describe $x$-maximal flatness of control-affine systems whose control-linear subsystem is static feedback equivalent to the $m$-chained form. We illustrate our results by an example in Section 3.2.4 and provide proofs in Section 3.2.5.

### 3.2.2 Preliminaries and motivation

The fundamental property of flat systems is that all their solutions can be parametrized by a finite number of functions and their time-derivatives. Fix an integer $r \geq -1$ and denote $X^r = X \times U \times \mathbb{R}^{mr}$ and $\bar{u}^r = (u, \dot{u}, \ldots, u^{(r)})$. For $r = -1$, we put $X^{-1} = X$ and $\bar{u}^{-1}$ is empty.

**Definition 3.2.1.** The system $\Xi : \dot{x} = F(x,u)$ is *flat* at $(x^*, \bar{u}^{r*}) \in X^r$, for $r \geq -1$, if there exists a neighborhood $\mathcal{O}^r$ of $(x^*, \bar{u}^{r*})$ and $m$ smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(r)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^r$, having the following property: there exist an integer $s$ and smooth functions $\gamma_i$, $1 \leq i \leq n$, and $\delta_j$, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

along any trajectory $x(t)$ given by a control $u(t)$ that satisfy $(x(t), u(t), \ldots, u^{(r)}(t)) \in \mathcal{O}^r$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ is called a *flat output*.

In the particular case $\varphi_i = \varphi_i(x)$, for $1 \leq i \leq m$, we will say that the system is $x$-flat. In our study, the flat outputs will always depend on $x$ only and $r$ is 0 or -1.

The notion of differential weight of a flat system, introduced in [58], was discussed in [45,46] in the context of system linearizable via one-fold prolongation. The differential weight of a flat output $\varphi$ is, roughly speaking, the minimal number of derivatives of components of $\varphi$ needed to express $x$ and $u$ and will be formalized as follows. By definition, for any flat output $\varphi$ of $\Xi$ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}), \end{aligned}$$

Moreover, we can choose $(s_1, \dots, s_m)$ such that (see [58]) if for any other $m$-tuple $(\tilde{s}_1, \dots, \tilde{s}_m)$ we have

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \le \tilde{s}_i$, for $1 \le i \le m$. We will call $\sum_{i=1}^m (s_i + 1) = \sum_{i=1}^m s_i + m$ the differential weight of $\varphi$. A flat output of $\Xi$ is called *minimal* if its differential weight is the lowest among all flat outputs of $\Xi$. We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output. Here we propose another way of looking at this property. Suppose that the control system $\Xi : \dot{x} = F(x, u)$ is flat at $(x^*, \bar{u}^{r*})$ and let $(\varphi_1, \dots, \varphi_m)$ be a flat output around $(x^*, \bar{u}^{r*})$. It is well known (see e.g. [15,22,54]) that for any $l \ge 0$, all time-derivatives $\varphi_i^{(j)}$, $1 \le i \le m, 0 \le j \le l$, of flat outputs are independent at $(x^*, \bar{u}^{r*})$. So successive time-derivatives provide $m$ new independent functions $\varphi_i^{(l+1)} = \varphi_i^{(l+1)}(x, u, \dot{u}, \dots, u^{(r+l+1)})$, $1 \le i \le m$. The problem that we are going to study is how many new functions of the state $x$ do successive derivatives of the flat outputs provide?

To formalize that problem, for any $j \ge 0$, we denote

$$\begin{aligned} \Phi^j &= \operatorname{span}\{d\varphi_i, \cdots, d\varphi_i^{(j)}, 1 \le i \le m\}, \\ \mathcal{A}^j &= \Phi^j \cap T^*X \\ &= \operatorname{span}\{d\varphi_i, \cdots, d\varphi_i^{(j)}, 1 \le i \le m\} \cap T^*X, \end{aligned}$$

and define $a^j(\xi) = \dim \mathcal{A}^j(\xi)$, where $\xi = (x, u, \dot{u}, \cdots)$. The vector $(a^0(\xi), a^1(\xi), \cdots, a^\rho(\xi))$ will be called the $x$-growth vector of the nested sequence of codistributions $\Phi^0 \subset \Phi^1 \subset \cdots \subset \Phi^\rho$ (equivalently, the growth vector of $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \cdots \subset \mathcal{A}^\rho$), where $\rho$ is the smallest integer such that $\mathcal{A}^\rho = T^*X$. For two codistributions $\mathcal{E}$ and $\mathcal{F}$, we define their pointwise intersection $\mathcal{E} \cap \mathcal{F}$ by $(\mathcal{E} \cap \mathcal{F})(x) = \mathcal{E}(x) \cap \mathcal{F}(x)$, for $x \in X$.

**Definition 3.2.2.** A system $\Xi$ flat at $(x^*, \bar{u}^{r*}) \in X^r$, for $r \ge -1$, is called *x-maximally flat* at $(x^*, \bar{u}^{r*})$ if there exists a flat output at $(x^*, \bar{u}^{r*})$ for which all codistributions $\mathcal{A}^j$ do not depend on the control or the control derivatives and, in a neighborhood of $x^*$, the sequence $(a^0(x), a^1(x), \cdots, a^\rho(x))$ is constant and the maximal possible among all flat systems for which $\dim U = m$ and $\dim X = n$.

Flatness is closely related to the notion of feedback linearization. The control system $\Xi : \dot{x} = F(x, u)$ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \psi(x, v)$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$. Jakubczyk and Respondek [23] and Hunt and Su [19] gave geometric necessary and sufficient conditions for a control system to be static feedback linearizable. It is well known that systems linearizable via invertible static feedback are flat. The expression of all states and controls uses the minimal possible, which is $n + m$, number of time-derivatives of the components of flat outputs $\varphi_i$. The following proposition gives an equivalent way to describe static feedback linearizable systems using the notion of $x$-maximal flatness.

Consider a control system $\Xi : \dot{x} = F(x, u)$, with $m$ inputs and defined on a state space of dimension $n = km$. Let us first introduce some notations. To $\Xi$, we associate

$\mathcal{F} = \{F_u : u \in U\}$, where $F_u = F(\cdot, u)$, i.e., $\mathcal{F}$ stands for the family of all vector fields corresponding to constant controls $u$ of $\Xi$. Define the following sequence of distributions on $X$: $\mathcal{D}^0(x,u) = \mathrm{Im}\frac{\partial F}{\partial u}(x,u)$ and $\mathcal{D}^{i+1}(x,u) = \mathcal{D}^i(x,u) + \mathrm{span}\,\{[F_u, g] : F_u \in \mathcal{F},\, g \in \mathcal{D}^i\}$, for $i \geq 0$. If $\Xi$ is a control-affine system, i.e., of the form $\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, we actually have $\mathcal{D}^0 = \mathrm{span}\,\{g_1, \cdots, g_m\}$ and $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$.

**Proposition 3.2.1.** *The following conditions are equivalent:*

*(i)* *$\Xi$ is x-maximally flat at $(x^*, \bar{u}^{r*})$, for a certain $r \geq$-1;*

*(ii)* *$\Xi$ is x-maximally x-flat at $x^*$;*

*(iii)* *There exists a flat output of $\Xi$ for which the x-growth vector is constant and equals $(m, 2m, \cdots, km)$;*

*(iv)* *$\Xi$ is static feedback equivalent to a linear system and, in particular, to the Brunovský canonical form*

$$(Br) \left\{ \begin{array}{rcl} \dot{z}_i^j &=& z_i^{j+1} \\ \dot{z}_i^k &=& v_i \end{array} \right.$$

*where $1 \leq i \leq m$ and $1 \leq j \leq k-1$.*

*(v)* *The distribution $\mathcal{D}^0$ does not depend on u and for any $0 \leq i \leq k-1$, the distributions $\mathcal{D}^i$ are involutive and of constant rank $(i+1)m$.*

According to item $(iii)$ of the above result, a control system is $x$-maximally flat if the number of new states (state functions) gained by successive derivations of the flat output is, at each step, the largest possible, which is $m$. For $x$-maximally flat systems, flatness and $x$-flatness coincide and moreover, both properties are equivalent to linearizability via an invertible static feedback transformation, and, in fact, one can bring the system into the Brunovský canonical form, see [5], with all controllability indices equal $k$. Item $(v)$ recalls the geometric necessary and sufficient conditions for a general nonlinear control system to be static feedback linearizable, see [57]. If the considered control system is affine with respect to controls it is clear that $\mathcal{D}^0$ does not depend on $u$.

In general, a flat system is not linearizable by static feedback (with the exception of the single-input case, where flatness reduces to static feedback linearization, see [9]) and therefore it is not $x$-maximally flat. We can be interested, however, in $x$-maximal flatness within a particular class of systems $\mathfrak{C}$. We will say that the system $\Xi$ is $x$-maximally flat within the class $\mathfrak{C}$ if it satisfies the conditions of Definition 3.2.2 with the sequence $(a^0, a^1, \cdots, a^\rho)$ being the maximal possible among all flat systems belonging to the class $\mathfrak{C}$ for which $\dim U = m$ and $\dim X = n$. From now on, we will denote the number of controls by $m+1$ (and not by $m$) since, as we will see below, for all classes of systems that follow one control plays a particular role. Consider a control-linear system

$$\Sigma_{lin} : \dot{x} = \sum_{i=0}^{m} u_i g_i(x),$$

where the control $u$ takes values in an open subset $U$ of $\mathbb{R}^{m+1}$, the state space $X$ is of dimension $n = km + 1$ and $g_0, \cdots, g_m$ are smooth vector fields on $X$. To $\Sigma_{lin}$ we associate the following distribution $\mathcal{G} = \text{span}\{g_0, \cdots, g_m\}$. We define inductively the derived flag of $\mathcal{G}$ by $\mathcal{G}^0 = \mathcal{G}$ and $\mathcal{G}^{i+1} = \mathcal{G}^i + [\mathcal{G}^i, \mathcal{G}^i]$, $i \geq 0$.

A *flat* control-linear system $\Sigma_{lin}$ is never static feedback linearizable (unless the number of controls, $m+1$, equals the dimension of the state space) and therefore, according to Proposition 3.2.1, cannot admit a flat output with the $x$-growth vector $(m+1, 2(m+1), 3(m+1), \cdots)$. In fact the $x$-growth vector may start with $m+1$ (if the system is $x$-flat) but, since the system is control-linear, the derivatives $\dot{\varphi}_i$, for $0 \leq i \leq m$, necessarily involve the control, hence the second component of the $x$-growth vector can be, at most, $2m+1$. So the maximal possible $x$-growth vector is $(m+1, 2m+1, 3m+1, \cdots, km+1)$ and it is, indeed, realized by control-linear systems static feedback equivalent to the $m$-chained form. An $(m+1)$-input driftless control system $\Sigma_{lin}$, defined on a manifold $X$ of dimension $km+1$, is said to be in the $m$-chained form if it is represented by

$$Ch_m^k \left\{ \begin{array}{llllll} \dot{z}_0 = v_0 & \dot{z}_1^1 = z_1^2 v_0 & \cdots & \dot{z}_m^1 & = & z_m^2 v_0 \\ & \dot{z}_1^2 = z_1^3 v_0 & & \dot{z}_m^2 & = & z_m^3 v_0 \\ & \vdots & & \vdots & & \\ & \dot{z}_1^{k-1} = z_1^k v_0 & \cdots & \dot{z}_m^{k-1} & = & z_m^k v_0 \\ & \dot{z}_1^k = v_1 & \cdots & \dot{z}_m^k & = & v_m \end{array} \right.$$

Is is clear from this representation that one control, $v_0$ in this case, is indeed "special". To simplify the notations, from now on, $z^i$ stands for $z^i = (z_1^i, \cdots, z_m^i)$, for $1 \leq i \leq k$, and $\bar{v}$ denotes the vector $(v_1, \cdots, v_m)$. The problem of characterizing systems that are locally static feedback equivalent to the $m$-chained form has been studied and solved in [50] (see also [2,11,16,31,39,49,59,63,68]). It is immediate to see that systems locally feedback equivalent to the $m$-chained form are flat with $\varphi = (z_0, z_1^1, \cdots, z_m^1)$ being a flat output, at any point $(z^*, v^*) \in X \times \mathbb{R}^{m+1}$ with $v_0^* \neq 0$, and in [58] all their minimal flat outputs have been described. Flat systems equivalent to $Ch_m^k$ exhibit singularities in the control space defined by $U_{lin}^{sing}(x) = \{u(x) \in \mathbb{R}^{m+1} : \sum_{i=0}^m u_i(x) g_i(x) \in \mathcal{C}^1(x)\}$, where $\mathcal{C}^1$ is the characteristic distribution of $\mathcal{G}^1$, see [58]. Clearly, $v_0^* = 0$ describes that singularity for $Ch_m^k$.

An invertible static feedback $u = \beta(x)\tilde{u}$ transforms the system $\Sigma_{lin}$ into the form $\tilde{\Sigma}_{lin} : \dot{x} = \sum_{i=0}^m \tilde{u}_i \tilde{g}_i(x)$, where $\tilde{g} = g\beta$, with $g = (g_0, \cdots, g_m)$ and $\tilde{g} = (\tilde{g}_0, \cdots, \tilde{g}_m)$. To $\tilde{\Sigma}_{lin}$ we associate the $(k-1)$-fold prolongation

$$\tilde{\Sigma}_{lin}^{(k-1,0,\ldots,0)} : \left\{ \begin{array}{ll} \dot{x} & = y_1 \tilde{g}_0(x) + \sum_{i=1}^m u_i^p \tilde{g}_i(x) \\ \dot{y}_1 & = y_2 \\ & \vdots \\ \dot{y}_{k-2} & = y_{k-1} \\ \dot{y}_{k-1} & = u_0^p \end{array} \right.$$

with $y_1 = \tilde{u}_0$, $u_i^p = \tilde{u}_i$, for $1 \leq i \leq m$, obtained by prolonging $k-1$ times the control $\tilde{u}_0$ as $u_0^p = \tilde{u}_0^{(k-1)}$. Denote the drift and the controlled vector fields of the prolongation $\tilde{\Sigma}_{lin}^{(k-1,0,\ldots,0)}$ by $f_p$ and $g_{pi}$, $0 \leq i \leq m$, respectively. The distributions of the prolongation will be denoted using the subindex $p$, i.e., $\mathcal{D}_p^0 = \text{span}\{g_{p0}, \cdots, g_{pm}\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [f_p, \mathcal{D}_p^i]$.

The following result characterizes control-linear systems that are locally static feedback equivalent to the $m$-chained form, from the point of view of $x$-maximal flatness.

**Proposition 3.2.2.** *The following conditions are equivalent:*

*(Lin 1)* $\Sigma_{lin}$ *is x-maximally flat at* $(x^*, \bar{u}^{r*})$*, for a certain* $r \geq -1$*, within the class of control-linear systems* $\mathfrak{C}$*;*

*(Lin 2)* $\Sigma_{lin}$ *is x-maximally x-flat at* $(x^*, u^*)$ *within the class of control-linear systems* $\mathfrak{C}$*;*

*(Lin 3) There exist a flat output of* $\Sigma_{lin}$ *at* $(x^*, u^*)$ *for which the x-growth vector is constant and equals* $(m+1, 2m+1, 3m+1, \cdots, km+1)$*;*

*(Lin 4)* $\Sigma_{lin}$ *is locally, around* $x^*$*, static feedback equivalent to the m-chained form*

$$Ch_m^k \begin{cases} \dot{z}_0 = v_0 \quad \dot{z}^1 & = \quad z^2 v_0 \\ \qquad\qquad \dot{z}^2 & = \quad z^3 v_0 \\ & \vdots \\ \qquad\qquad \dot{z}^{k-1} & = \quad z^k v_0 \\ \qquad\qquad \dot{z}^k & = \quad \bar{v} \end{cases}$$

*and* $u^* \notin U_{lin}^{sing}(x^*)$*.*

*(Lin 5)* $\Sigma_{lin}$ *satisfies, around* $(x^*, u^*)$*,* $u^* \notin U_{lin}^{sing}(x^*)$*, the conditions:*

*(m-Ch1)* $\mathcal{G}^{k-1} = TX$*;*

*(m-Ch2)* $\mathcal{G}^{k-2}$ *is of constant rank* $(k-1)m+1$ *and contains an involutive subdistribution* $\mathcal{L}$ *that has constant corank one in* $\mathcal{G}^{k-2}$*;*

*(m-Ch3)* $\mathcal{G}^0(x^*)$ *is not contained in* $\mathcal{L}(x^*)$*;*

*(Lin 6) There exists, around* $x^*$*, an invertible static feedback transformation* $u = \beta(x)\tilde{u}$*, bringing the system* $\Sigma_{lin}$ *into the form* $\tilde{\Sigma}_{lin} : \dot{x} = \sum_{i=0}^{m} \tilde{u}_i \tilde{g}_i(x)$*, such that for any* $0 \leq i \leq k-2$*, the intersections* $\mathcal{D}_p^i \cap TX$ *are involutive, of constant rank* $m(i+1)$*, and* $\mathcal{D}_p^{k-1} \cap TX = TX$*, where* $\mathcal{D}_p^i$ *are the distributions of the* $(k-1)$*-fold prolongation* $\tilde{\Sigma}_{lin}^{(k-1,0,\ldots,0)}$*.*

Proposition 3.2.2 states that the only control-linear systems that are $x$-maximally flat are those that are locally static feedback equivalent to the $m$-chained form and, as expected, $x$-maximal flatness and $x$-maximal $x$-flatness are equivalent. Conditions *(m-Ch1)-(m-Ch3)* are formally the same, independently of $m = 1$ or $m \geq 2$. Notice, however, they are checkable only if $m \geq 2$ because in that case $\mathcal{L}$, if it exists, is unique and can be calculated (see [59] and [27]). If $m = 1$, then two equivalent verifiable reformulations of the conditions *(m-Ch2)-(m-Ch3)* are:

*(m-Ch2)'* $\mathcal{G}^{k-3}$ *is of constant rank* $k-1$ *and the characteristic distribution* $\mathcal{C}^{k-2}$ *of* $\mathcal{G}^{k-2}$ *is contained in* $\mathcal{G}^{k-3}$ *and has corank one in* $\mathcal{G}^{k-3}$*;*

*(m-Ch3)′* $\mathcal{G}^0(x^*)$ *is not contained in* $\mathcal{C}^{k-2}(x^*)$*;*

or more classically (see [40]):

*(m-Ch2)″* $\dim \mathcal{G}^i(x) = \dim \mathcal{G}_i(x) = i+2$, *for* $0 \leq i \leq k-1$, *in a neighborhood of* $x^*$.

Conditions *(m-Ch1)-(m-Ch3)* characterize the $m$-chained form [59] (see also [49, 50]) and assure the existence of a change of coordinates $z = \phi(x)$ and of an invertible static feedback transformation of the form $u = \beta(x)\tilde{u}$, after which the control vector fields are in the $m$-chained form. The set of singular controls $U^{sing}_{lin}$, i.e., the controls at which the system ceases to be flat, has been described in [49], where it was also shown that all singular controls $u$ are mapped into $v = (v_0, \bar{v})$ such that $v_0 = 0$.

In item *(Lin 6)*, the system $\tilde{\Sigma}^{(k-1,0,\ldots,0)}_{lin}$ is obtained by prolonging $(k-1)$-times the control $\tilde{u}_0$ as $u_0^p = \tilde{u}_0^{(k-1)}$ and it is clear that if we bring the original system $\Sigma_{lin}$ into the $m$-chained form and we prolong the control $v_0$, the associated prolongation verifies all conditions of *(Lin 6)*. Moreover, in this case, it is easy to see that the associated $(k-1)$-prolongation is, actually, static feedback linearizable. Since for any $i \geq 0$, $\mathcal{D}^i_p \cap TX$ are involutive, it can be shown that all distributions $\mathcal{D}^i_p$ are, in fact, involutive and thus $\tilde{\Sigma}^{(k-1,0,\ldots,0)}_{lin}$ is static feedback linearizable. Notice that item *(Lin 6)* is actually the dual of *(Lin 3)*. Indeed, in the sequence of involutive distributions $\mathcal{D}^i_p \cap TX$ at each step we gain $m$ new directions, which is the maximal possible and which is also the case for the $x$-growth vector $(a^0, a^1, a^2, \cdots)$.

A natural question arises: under which conditions is $x$-maximal flatness of $\Sigma_{lin}$ conserved if we perturb the system by adding a drift $f$, thus obtaining a control-affine system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$? In other words, what are the conditions that the drift $f$ should satisfy in order that the $x$-growth vector associated to $\Sigma_{aff}$ (whose control-linear subsystem $\Sigma_{lin}$ is static feedback equivalence to the $m$-chained form) is given by $(m+1, 2m+1, 3m+1, \cdots, km+1)$? The next section of this paper answers that question and therefore generalizes Proposition 3.2.2 to the control-affine case.

### 3.2.3 Main result : $x$-maximal flatness

The purpose of this paper is to generalize Proposition 3.2.2 from control-linear systems $\Sigma_{lin}$ to control-affine systems

$$\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$$

defined on an open subset $X$ of $\mathbb{R}^{km+1}$, with $f$ and $g_0, \cdots, g_m$ smooth vector fields on $X$ and such that the associated control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ satisfies Proposition 3.2.2.

In order to describe $x$-maximal flatness of control-affine systems whose control-linear subsystem is static feedback equivalent to the $m$-chained form, consider the following triangular form generalizing the $m$-chained form:

$$TCh_m^k \begin{cases} \dot{z}_0 = v_0 & \dot{z}_i^1 = f_i^1(z_0, \bar{z}^2) + z_i^2 v_0 \\ & \dot{z}_i^2 = f_i^2(z_0, \bar{z}^3) + z_i^3 v_0 \\ & \vdots \\ & \dot{z}_i^{k-1} = f_i^{k-1}(z_0, \bar{z}^k) + z_i^k v_0 \\ & \dot{z}_i^k = v_i \end{cases}$$

where $1 \leq i \leq m$ and $\bar{z}^j$ denotes $\bar{z}^j = (z_1^1, \cdots z_m^1, z_1^2, \cdots z_m^2, \cdots, z_1^j, \cdots z_m^j)$, for $2 \leq j \leq k$. This form has been recently introduced and characterized by Silveira, Pereira da Silva and Rouchon [65] (for $m = 1$) and by the authors [27] for $m \geq 1$. It not only exhibits a formal compatibility of the triangular structure of the drift with the structure of the controlled chains but also a striking compatibility of its $x$-maximal flatness with that of the $m$-chained form. This is seen in Theorem 3.2.1 below, which is the main result of the paper, where counterparts of conditions *(Lin 1)-(Lin 6)* are given as *(Aff 1)-(Aff 6)* for the control-affine case.

It is clear, see [27], that $TCh_m^k$ is $x$-flat, with $\varphi = (z_0, z_1^1, \cdots, z_m^1)$ being a flat output, at any point $(z^*, v^*) \in X \times \mathbb{R}^{m+1}$ satisfying $\operatorname{rk} F^j(z^*) = m$, for $1 \leq j \leq k-1$, where $F^j$, for $1 \leq j \leq k-1$, is the $m \times m$ matrix given by $F_{iq}^j = \frac{\partial(f_q^j + z_q^{j+1} v_0^*)}{\partial z_i^{j+1}}$, for $1 \leq i, q \leq m$. Therefore, flat systems equivalent to $TCh_m^k$ exhibit singularities in the control space (depending on the state) defined by (see [27])

$$U_{aff}^{sing}(x) = \cup_{i=0}^{k-2} U_{sing}^i(x),$$

with $U_{sing}^i(x) = \{u(x) \in \mathbb{R}^2 : \operatorname{rk}(\mathcal{G}^i + [f + gu,\ \mathcal{L}^i])(x) < (i+2)m + 1\}$, for $0 \leq i \leq k-2$, where $gu = \sum_{i=0}^m u_i g_i$, the distribution $\mathcal{L}^i = \mathcal{C}^{i+1}$, for $0 \leq i \leq k-3$, is the characteristic distribution of $\mathcal{G}^{i+1}$ and $\mathcal{L}^{k-2} = \mathcal{L}$ is the involutive subdistribution of corank one in $\mathcal{G}^{k-2}$, if $m \geq 2$. If $m = 1$, then $U_{sing}^{k-2} = \bigcap_{\mathcal{L}} U_{\mathcal{L}-sing}^{k-2}$ where the intersection is taken over all involutive distributions $\mathcal{L}$ of corank one in $\mathcal{G}^{k-2}$ and satisfying $\mathcal{G}^0(x^*) \not\subset \mathcal{L}(x^*)$, where $x^*$ is a nominal point around which we work.

An invertible static feedback $u = \alpha(x) + \beta(x)\tilde{u}$, transforms the system $\Sigma_{aff}$ into the form $\tilde{\Sigma}_{aff} : \dot{x} = \tilde{f}(x) + \sum_{i=0}^m \tilde{u}_i \tilde{g}_i(x)$, where $\tilde{f} = f + \alpha g$ and $\tilde{g} = g\beta$, with $g = (g_0, \cdots, g_m)$ and $\tilde{g} = (\tilde{g}_0, \cdots, \tilde{g}_m)$. To $\tilde{\Sigma}_{aff}$, we associate the $(k-1)$-fold prolongation

$$\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)} \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_0(x) + \sum_{i=1}^m u_i^p \tilde{g}_i(x) \\ \dot{y}_1 &= y_2 \\ & \vdots \\ \dot{y}_{k-2} &= y_{k-1} \\ \dot{y}_{k-1} &= u_0^p \end{cases}$$

with $y_1 = \tilde{u}_0$, $u_i^p = \tilde{u}_i$, for $1 \leq i \leq m$, obtained by prolonging $(k-1)$-times the control $\tilde{u}_0$ as $u_0^p = \tilde{u}_0^{(k-1)}$. The linearizability distributions of the prolonged system $\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)}$ will be denoted using the subindex $p$, i.e., $\mathcal{D}_p^0 = \operatorname{span}\{g_{p0}, \cdots, g_{pm}\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [f_p, \mathcal{D}_p^i]$. Recall that $z^j$ stands for $z^j = (z_1^j, \cdots, z_m^j)$, for $1 \leq j \leq k$, and $\bar{v}$ denotes the vector $(v_1, \cdots, v_m)$.

**Theorem 3.2.1.** *Consider the class $\mathfrak{C}$ of control-affine system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^{m} u_i g_i(x)$ whose control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^{m} u_i g_i(x)$ is static feedback equivalent to the m-chained form, that is, satisfies the conditions (m-Ch1)-(m-Ch3) of Proposition 3.2.2. For $\Sigma_{aff} \in \mathfrak{C}$, the following conditions are equivalent:*

*(Aff 1)* *$\Sigma_{aff}$ is x-maximally flat at $(x^*, \bar{u}^{r*})$, for a certain $r \geq -1$, within the class $\mathfrak{C}$;*

*(Aff 2)* *$\Sigma_{aff}$ is x-maximally x-flat at $(x^*, u^*)$ within the class $\mathfrak{C}$;*

*(Aff 3)* *There exists a flat output of $\Sigma_{aff}$ at $(x^*, u^*)$ for which the x-growth vector is constant and equals $(m+1, 2m+1, 3m+1, \cdots, km+1)$ and all codistributions $\mathcal{A}^j(x)$, for $0 \leq j \leq k-1$, do not depend on the control or control derivatives;*

*(Aff 4)* *$\Sigma_{aff}$ is locally, around $x^*$, static feedback equivalent to the triangular form $TCh_m^k$, compatible with the m-chained form, given by*

$$TCh_m^k \left\{ \begin{array}{ll} \dot{z}_0 = v_0 & \dot{z}^1 = f^1(z_0, z^1, z^2) + z^2 v_0 \\ & \dot{z}^2 = f^2(z_0, z^1, z^2, z^3) + z^3 v_0 \\ & \vdots \\ & \dot{z}^{k-1} = f^2(z_0, z^1, \cdots, z^k) + z^k v_0 \\ & \dot{z}^k = \bar{v} \end{array} \right.$$

*and $u^* \notin U_{aff}^{sing}(x^*)$;*

*(Aff 5)* *System $\Sigma_{aff}$ satisfies, around $(x^*, u^*)$, with $u^*(x^*) \notin U_{aff}^{sing}(x^*)$, the following condition:*

*(m-Comp)* *$[f, \mathcal{C}^i] \subset \mathcal{G}^i$, for $1 \leq i \leq k-2$, where $\mathcal{C}^i$ is the characteristic distribution of $\mathcal{G}^i$.*

*(Aff 6)* *There exists, around $x^*$, an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing the system $\Sigma_{aff}$ into the form $\tilde{\Sigma}_{aff} : \dot{x} = \tilde{f}(x) + \sum_{i=0}^{m} \tilde{u}_i \tilde{g}_i(x)$, such that for any $0 \leq i \leq k-2$, the intersections $\mathcal{D}_p^i \cap TX$ do not depend on $y$, are involutive, of constants rank $m(i+1)$ and $\mathcal{D}_p^{k-1} \cap TX = TX$, where $\mathcal{D}_p^i$ are the distributions of the $(k-1)$-fold prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\ldots,0)}$.*

*Remarks:*

1) We do not claim that $\Sigma_{aff}$ satisfying one of the above conditions is $x$-maximally flat. Clearly, $x$-maximally flat control-affine systems are those that are static feedback linearizable, as assured by Proposition 3.2.1. The above theorem describes $x$-maximally flat systems within the class $\mathfrak{C}$ of control-affine ones whose control-linear subsystem is static feedback equivalent to the $m$-chained form.

2) Theorem 3.2.1 generalizes Proposition 3.2.2 and shows how $x$-maximal flatness of control-affine systems compatible with the $m$-chained form reminds, but also how it differs from, that of control-linear systems. As for control-linear systems, $x$-maximal flatness and $x$-maximal $x$-flatness are equivalent. Thus the $x$-growth vector starts with $m+1$, but since the control-linear subsystem is static feedback equivalent to the $m$-chained form, the second component can be at most $2m+1$. Condition

*(Aff 6)* of the above result is very similar to condition *(Lin 6)* of Proposition 3.2.2 but, in addition to *(Lin 6)*, it requires that the involutive distributions $\mathcal{D}_p^i \cap TX$, associated to the prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\ldots,0)}$, do not depend on $y = \tilde{u}_0$. For control-linear systems, adding that condition would be redundant, because it is a consequence of the involutivity and the proper growth vector of $\mathcal{D}_p^i \cap TX$, but for control-affine systems just involutivity and rank conditions do not give the desired triangular form. Actually, for any $x$-flat system whose prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\ldots,0)}$ possesses involutive distributions $\mathcal{D}_p^i \cap TX$ of proper growth vector, the dependence (or not) on $y = \tilde{u}_0$ distinguishes between a general $x$-flat system and the class treated here.

3) According to item *(Aff 4)*, the only $x$-maximally flat control-affine systems, compatible with the $m$-chain form, are those that are static feedback equivalent to the triangular form $TCh_m^k$. Item *(Aff 5)*, together with *(m-Ch1)-(m-Ch3)* assumed for the control-linear subsystem $\Sigma_{lin}$, provide an invariant geometric characterization of $TCh_m^k$. For two-input control systems, an equivalent description was given in [65]. In [27], the authors show that conditions *(m-Ch1)-(m-Ch3)* and *(m-Comp)* are necessary and sufficient for a control affine system to be static feedback equivalent to $TCh_m^k$, for any $m \geq 1$, and discuss flatness of that class of systems. While conditions *(m-Ch1)-(m-Ch3)* characterize the $m$-chained form, *(m-Comp)* takes into account the drift and gives the compatibility condition for the drift $f$ to have the desired triangular form in the right coordinates, i.e., in those in which the controlled vector fields are in the $m$-chained form. The involutive subdistribution $\mathcal{L}$ (which, for $m \geq 2$, is crucial for the $m$-chained form) is absent in the compatibility conditions, but plays a very important role in calculating minimal flat outputs and in describing singularities (see [27]). In order to verify the conditions *(m-Ch1)-(m-Ch3)*, we have to verify whether the distribution $\mathcal{G}^{k-2}$ contains an involutive subdistribution $\mathcal{L}$ of corank one. Checkable necessary and sufficient conditions for the existence of $\mathcal{L}$ (together with a construction), based on the work of Bryant [6], were given in [50] and is discussed in [27].

4) A natural question is whether the above theorem describes flat systems whose $x$-growth vector is $(m+1, 2m+1, 3m+1, \cdots, km+1)$ (without assuming that their control-linear subsystem is static feedback equivalent to the $m$-chained form). The answer is negative and the problem of characterizing those systems will be discussed elsewhere.

5) Now assume that $\Sigma_{aff}$ is $x$-flat with $\Sigma_{lin}$ being static feedback equivalent to the $m$-chained form. Does $\Sigma_{aff}$ satisfy the conditions of Theorem 3.2.1? In other words, are $x$-flat control-affine systems necessarily static feedback equivalent to $TCh_m^k$ if the control-linear subsystem is static feedback equivalent to $Ch_m^k$? The answer is negative as shown by the following example.

### 3.2.4 Example

Consider the following control-affine system whose associated distribution $\mathcal{G}^0$ is already in the chained form:

$$\Sigma : \begin{cases} \dot{z}_0 = v_0 & \dot{z}_1 = z_3 + z_2 v_0 \\ & \dot{z}_2 = -z_4 + z_3 v_0 \\ & \dot{z}_3 = b(z_0, z_1, z_2, z_3) + z_4 v_0 \\ & \dot{z}_4 = v_1 \end{cases}$$

where $b$ is a smooth function non involving $z_4$. Let us show that the pair $(\varphi_0, \varphi_1) = (z_0, z_1)$ is an $x$-flat output. Indeed, we have $\varphi_0 = z_0$, implying $\dot{\varphi}_0 = v_0$, and $\varphi_1 = z_1$, implying $\dot{\varphi}_1 = z_3 + z_2\dot{\varphi}_0$ and $\ddot{\varphi}_1 = b(\varphi_0, \varphi_1, z_2, z_3) + z_3\dot{\varphi}_0^2 + z_2\ddot{\varphi}_0$. From these two relations, we express $z_2$ and $z_3$, via the implicit function theorem, as: $z_2 = \gamma_2(\bar{\varphi}_0^2, \bar{\varphi}_1^2)$ and $z_3 = \gamma_3(\bar{\varphi}_0^2, \bar{\varphi}_1^2)$, where $\bar{\varphi}^j$ denotes $(\varphi, \dot{\varphi}, \cdots, \varphi^{(j)})$ and $\gamma_2$ and $\gamma_3$ are smooth functions. By differentiating $z_3$, we deduce $z_4 = \gamma_4(\bar{\varphi}_0^3, \bar{\varphi}_1^3)$ which yields $v_1 = \delta_2(\bar{\varphi}_0^4, \bar{\varphi}_1^4)$. So we have determined all state and control variables with the help of $\varphi_0$ and $\varphi_1$ and their time-derivatives and it follows that $(\varphi_0, \varphi_1) = (z_0, z_1)$ is, indeed, an $x$-flat output. However, the first derivative of $\varphi = (\varphi_0, \varphi_1)$ gives no function depending only on the state $z$ and the system is clearly not $x$-maximally flat. Moreover, the $x$-growth vector of the system is the maximal possible, i.e., equals $(m + 1, 2m + 1, \cdots, km + 1) = (2, 3, 4, 5)$, but the codistribution $\mathcal{A}^1 = \text{span}\{dz_0, dz_1, dz_3 + v_0 dz_2\}$ depends on the control. Equivalently, if we study the prolongation $\Sigma^{(3,0)}$ of the system, obtained by prolonging the control $v_0$ three times, we have $\mathcal{D}_p^1 \cap TX = \text{span}\{\frac{\partial}{\partial z_4}, y_1\frac{\partial}{\partial z_3} - \frac{\partial}{\partial z_2}\}$, where $y_1 = v_0$, which clearly depends on $y$. The above example shows that there are $x$-flat control-affine systems whose linear subsystem is static feedback equivalent to the $m$-chained form and whose drift is not compatible with the latter, i.e., the drift $f$ does not admit the desired triangular form in the system of coordinates in which the controlled vector fields exhibit the $m$-chained structure.

### 3.2.5 Proof of Theorem 3.2.1

*(Aff 1)* $\Rightarrow$ *(Aff 2)*. Assume that $\Sigma_{aff}$ is $x$-maximally flat at $(x^*, \bar{u}^{r*})$ and let $(\varphi_0, \cdots, \varphi_m)$ be a flat output such that the associated $x$-growth vector $(a_0, a_1, a_2, \cdots)$ is the maximal possible at any $x$ in a neighborhood of $x^*$. We deduce immediately that $a_0 = m + 1$ implying that all components $\varphi_i$ of the flat output are functions of $x$ only and thus the system is $x$-maximally $x$-flat.

*(Aff 2)* $\Rightarrow$ *(Aff 3)*. Assume that $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$ is $x$-maximally $x$-flat at $(x^*, u^*)$ and let $\varphi = (\varphi_0, \cdots, \varphi_m)$ be an $x$-flat output such that the associated $x$-growth vector $(a_0, a_1, a_2, \cdots)$ is the maximal possible at any $x$ in a neighborhood of $x^*$. There exists open neighborhoods $\mathcal{X}$ of $x^*$ and $\mathcal{U}$ of $u^*$ such that $\varphi$ is an $x$-flat output for any $(x, u) \in \mathcal{X} \times \mathcal{U}$.

Recall that the control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ is static feedback equivalent to the $m$-chained form. Thus $\overline{\mathcal{G}}^0$, the involutive closure of the distribution $\mathcal{G}^0 = \text{span}\{g_0, \cdots, g_m\}$, satisfies $\overline{\mathcal{G}} = \mathcal{G}^{k-1} = TX$. Therefore, on an open and dense subset $\mathcal{X}'$ of $\mathcal{X}$, for any flat output $\varphi_i$, $0 \le i \le m$, there exists at least one vector field $g_j$, $0 \leq j \leq m$, such that $L_{g_j}\varphi_i(x) \neq 0$. If not, then there exists $i$ such that $L_{g_j}\varphi_i = 0$ on $\mathcal{X}$, for $0 \leq j \leq m$, and by successive applications of Jacobi identity, it can be shown that $L_g\varphi_i = 0$ for any $g \in \mathcal{G}^{k-1} = TX$, implying that $\varphi_i$ is identically zero, which contradicts flatness of $\varphi = (\varphi_0, \cdots, \varphi_m)$. Consequently, $a_1$ can be at most $2m+1$ and the largest possible constant (see Definition 3.2.2) $x$-growth vector is $(m+1, 2m+1, 3m+1, \cdots, km+1)$.

*(Aff 3) ⇒ (Aff 4).* Let $\varphi = (\varphi_0, \cdots, \varphi_m)$ be a flat output at $(x^*, u^*)$ such that condition $(Aff\ 3)$ is satisfied. Since $a_0 = m+1$, it follows that $\varphi_i = \varphi_i(x)$, $0 \leq i \leq m$ (in other words the system is actually $x$-flat in a neighborhood of $x^*$).

There exists an open neighborhood $\mathcal{X}$ of $x^*$ and an open neighborhood $\mathcal{U}$ of $u^*$ such that $\varphi$ is an $x$-flat output at any $(x,u) \in \mathcal{X} \times \mathcal{U}$. Since the differentials of the components of flat outputs are independent at $x^*$, we can introduce new coordinates $z_0 = \varphi_0$, $z_i^1 = \varphi_i$, for $1 \leq i \leq m$, and complete them to a coordinate system $(z_0, z_1^1, \cdots, z_m^1, z_1^2, \cdots, z_1^k, \cdots, z_m^k)$. We have just seen that for any flat output $\varphi_i$, $0 \leq i \leq m$, there exists at least one vector field $g_j$, $0 \leq j \leq m$, such that $L_{g_j}\varphi_i(x) \neq 0$, on an open and dense subset $\mathcal{X}'$ of $\mathcal{X}$.

Let us now show that there exist integers $i$ and $j$ such that $L_{g_j}\varphi_i(x^*) \neq 0$. Suppose that for any flat output $\varphi_i$, we have $L_{g_j}\varphi_i(x^*) = 0$, for $0 \leq j \leq m$. We can always assume $u^* = 0$ (otherwise, apply the invertible feedback $\tilde{u} = u - u^*$ transforming $u^*$ into $\tilde{u}^* = 0$). We get $\dot{\varphi}_i = f_i(z) + \sum_{j=0}^m g_i^j u_j$, for $0 \leq i \leq m$, where $g_i^j(z^*) = 0$, for $0 \leq i,j \leq m$. This yields $d\dot{\varphi}_i = df_i(z) + \sum_{j=0}^m (u_j dg_i^j + g_i^j du_j)$, which evaluated at $(z^*, u^*) = (z^*, 0)$ gives $d\dot{\varphi}_i(z^*, 0) = df_i(z^*)$, for $0 \leq i \leq m$. Thus $\Phi^1(z^*) = \operatorname{span}\{d\varphi_i(z^*), d\dot{\varphi}_i(z^*),\ 0 \leq i \leq m\} = \operatorname{span}\{dz_0, dz_i^1, df_i(z^*),\ 0 \leq i \leq m\}$ and is clearly of dimension $2m+2$, because the differentials of flat outputs and their derivatives are independent everywhere. It follows that $\mathcal{A}^1(z^*) = \Phi^1(z^*) \cap T^*Z(z^*) = \Phi^1(z^*)$ and $a^1(z^*) = \dim \mathcal{A}^1(z^*) = 2m+2$, contradicting the fact that $a^1$ is constant and equals $2m+1$.

Without loss of generality, suppose $L_{g_0}\varphi_0(x^*) \neq 0$. After applying around $x^*$ a suitable invertible feedback, $u = \alpha(x) + \beta(x)v$, transforming $u^*$ into $v^*$, we get $\dot{\varphi}_0 = \dot{z}_0 = v_0$, $\dot{\varphi}_i = \dot{z}_i^1 = a_i^1(z) + b_i^1(z)v_0$, $1 \leq i \leq m$, where $a_i^1$ and $b_i^1$ are smooth functions. We continue to denote by $f$ and by $g_i$, $0 \leq i \leq m$, the drift and, respectively, the controlled vector fields of the feedback modified system. Since the $x$-growth vector grows always by $m$, it is immediate that $\frac{\partial \varphi_i^{(j)}}{\partial v_l} = 0$, for $1 \leq i, l \leq m$ and any $1 \leq j \leq k-1$. Now, using the fact that the control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ is static feedback equivalent to the $m$-chained form, it can be shown that $\tilde{z}_0 = \varphi_0$, $\tilde{z}_i^1 = \varphi_i$, $\tilde{z}_i^2 = L_{g_0}\varphi_i, \cdots, z_i^k = L_{g_0}^{k-1}\varphi_i$, for $1 \leq i \leq m$, is a valid local change of coordinates (to simplify notation, we continue to write $z$ instead of $\tilde{z}$) in which $\Sigma_{lin}$ is in the $m$-chained form, and after applying a suitable invertible feedback transformation, the system $\Sigma_{aff}$ takes the form

$$\begin{cases} \dot{z}_0 = v_0 & \dot{z}_i^1 \ = f_i^1(z) + z_i^2 v_0 \\ & \dot{z}_i^2 \ = f_i^2(z) + z_i^3 v_0 \\ & \quad \vdots \\ & \dot{z}_i^{k-1} = f_i^{k-1}(z) + z_i^k v_0 \\ & \dot{z}_i^k \ = v_i \end{cases}$$

where $1 \leq i \leq m$. Since $dv_0 \in \Phi^1$, the codistribution $\mathcal{A}^1 = \Phi^1 \cap T^*X$ is given by $\mathcal{A}^1 = \operatorname{span}\{dz_0, dz_i^1, \omega_i,\ 1 \leq i \leq m\}$, where $\omega_i$ is the 1-form appearing as the $i$-th line of the vector $\Omega = (\frac{\partial f^1}{\partial z^2} + v_0 \mathrm{Id})dz^2 + \sum_{j=3}^{k} \frac{\partial f^1}{\partial z^j} dz^j$, where Id denotes the identity matrix, $\frac{\partial f^1}{\partial z^j}$ is the matrix $\frac{\partial f^1}{\partial z^j} = (\frac{\partial f_i^1}{\partial z_q^j})$, for $1 \leq i, q \leq k$, and $dz^j = (dz_1^j, dz_2^j, \cdots, dz_m^j)^\top$, for $2 \leq j \leq m$. Notice that each $\omega_i$ can be written as $\omega_i = (\frac{\partial f_i^1}{\partial z_i^2} + v_0)dz_i^2 + \eta_i$, where $\eta_i$ is a 1-form not involving $v_0$. Since the codistribution $\mathcal{A}^1 = \operatorname{span}\{dz_0, dz_i^1, (\frac{\partial f_i^1}{\partial z_i^2} + v_0)dz_i^2 + \eta_i\ 1 \leq i \leq m\}$ does not depend on $v$, we have $\mathcal{A}^1(z, \tilde{v}_0) = \mathcal{A}^1(z, \bar{v}_0)$, for any fixed $\tilde{v}_0 \neq \bar{v}_0$. It follows that $((\frac{\partial f_i^1}{\partial z_i^2} + \tilde{v}_0)dz_i^2 + \eta_i) - ((\frac{\partial f_i^1}{\partial z_i^2} + \bar{v}_0)dz_i^2 + \eta_i) = (\tilde{v}_0 - \bar{v}_0)dz_i^2 \in \mathcal{A}^1$, for $1 \leq i \leq m$, thus $dz_i^2 \in \mathcal{A}^1$ and $\eta_i \in \mathcal{A}^1$, for $1 \leq i \leq m$. From this and the fact that $\mathcal{A}^1$ is of rank $2m+1$, we deduce $\mathcal{A}^1 = \operatorname{span}\{dz_0, dz_i^1, dz_i^2,\ 1 \leq i \leq m\}$, and since $\eta_i \in \mathcal{A}^1$, $1 \leq i \leq m$, it follows that $\eta_i$ cannot involve $dz_q^j$, for $j \geq 3$. Hence $\frac{\partial f_i^1}{\partial z_q^j} = 0$, for $1 \leq i, q \leq m$, $3 \leq j \leq k$, i.e., $f_i^1 = f_i^1(z_0, z^1, z^2)$, implying $\Omega = (\frac{\partial f^1}{\partial z^2} + v_0 Id)dz^2$, and the matrix $(\frac{\partial f^1}{\partial z^2} + v_0 Id)$ is of full rank at $(z^*, v_0^*)$. By induction, we show that the drift $f$ is triangular and that the regularity condition $u^* \notin U_{aff}^{sing}$ is satisfied.

*(Aff 4)* $\Rightarrow$ *(Aff 5)*. See [27].

*(Aff 5)* $\Rightarrow$ *(Aff 6)*. In [27], we have shown that conditions *(m-Ch1)-(m-Ch3)* and *(m-Comp)* of item *(Aff 5)* assure the existence of a change of coordinates $z = \phi(x)$ and of an invertible feedback transformation and $u = \alpha(x) + \beta(x)v$ that transform the system $\Sigma_{aff}$ into $TCh_m^k$. Bring the system into $TCh_m^k$ and prolong $(k-1)$-times the control $v_0$. The obtained prolongation

$$\begin{cases} \dot{z}_0 \ = y_1 & \dot{z}_i^1 \ = f_i^1(z_0, \bar{z}^2) + z_i^2 y_1 \\ \dot{y}_1 \ = y_2 & \dot{z}_i^2 \ = f_i^2(z_0, \bar{z}^3) + z_i^3 y_1 \\ \quad \vdots & \quad \vdots \\ \dot{y}_{k-2} = y_{k-1} & \dot{z}_i^{k-1} = f_i^{k-1}(z_0, \bar{z}^k) + z_i^k y_1 \\ \dot{y}_{k-1} = v_0^p & \dot{z}_i^k \ = v_i^p \end{cases}$$

where $1 \leq i \leq m$, $y_1 = v_0$ and $v_i^p = v_i$, for $1 \leq i \leq m$, clearly satisfies *(Aff 6)*.

*(Aff 6)* $\Rightarrow$ *(Aff 1)*. Assume that there exists, around $x^*$, an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, such that distributions $\mathcal{D}_p^i$ associated to the $(k-1)$-fold prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\ldots,0)}$, defined just before Theorem 3.2.1, do not depend on $y$,

are involutive and of constants rank $m(i+1)$, for any $0 \leq i \leq k-2$, and $\mathcal{D}_p^{k-1} \cap TX = TX$. For simplicity of notation, we will drop the tildes.

Recall that the linear-control sub-system associated to $\Sigma_{aff}$ is assumed to satisfy the conditions describing the $m$-chained form. An equivalent way to characterize the $m$-chained form is the following (see [59]): for $0 \leq i \leq k-1$, each element $\mathcal{G}^i$ of the derived flag has constant rank $(i+1)m+1$, contains an involutive subdistribution $\mathcal{L}^i \subset \mathcal{G}^i$ of corank one and each element $\mathcal{G}_i$ of the Lie flag, where $\mathcal{G}_{i+1} = \mathcal{G}_i + [\mathcal{G}_0, \mathcal{G}_i]$ and $\mathcal{G}_0 = \mathrm{span}\,\{g_0, \cdots, g_m\}$, has constant rank $(i+1)m+1$. Moreover, the involutive subdistribution $\mathcal{L}^i$, for $0 \leq i \leq k-3$, is the characteristic distribution $\mathcal{C}^{i+1}$ of $\mathcal{G}^{i+1}$, i.e., $\mathcal{C}^{i+1} = \mathcal{L}^i$. We will use that characterization to prove that control systems verifying item *(Aff 6)* are, in fact, feedback equivalent to $TCh_m^k$ and hence, $x$-maximally flat. To this end, we will show that the involutive distributions $\mathcal{D}_p^i \cap TX$, for $0 \leq i \leq k-2$, are, in fact, of corank one in $\mathcal{G}^i$, so let us denote $\mathcal{L}^i = \mathcal{D}_p^i \cap TX$.

For $\Sigma_{aff}^{(k-1,0,\dots,0)}$, we have $\mathcal{D}_p^0 = \mathrm{span}\,\{\frac{\partial}{\partial y_{k-1}}, g_i, 1 \leq i \leq m\}$, thus the distribution $\mathcal{L}^0 = \mathcal{D}_p^0 \cap TX = \mathrm{span}\,\{g_i, 1 \leq i \leq m\}$ is involutive and of corank one in $\mathcal{G}^0$. From this and since $\mathrm{rk}\,\mathcal{G}^1 = \mathrm{rk}\,\mathcal{G}_1 = 2m+1$, it follows that we necessarily have $\mathcal{G}^1 = \mathrm{span}\,\{g_0, g_i, [g_0, g_i], 1 \leq i \leq m\}$, where all brackets $[g_0, g_i]$ are independent modulo $\mathcal{G}^0$. We have $\mathcal{D}_p^1 = \mathrm{span}\,\{\frac{\partial}{\partial y_{k-1}}, \frac{\partial}{\partial y_{k-2}}, g_i, ad_f g_i + y_1[g_0, g_i],\ 1 \leq i \leq m\}$, thus the distribution $\mathcal{L}^1 = \mathcal{D}_p^1 \cap TX = \mathrm{span}\,\{g_i, ad_f g_i + y_1[g_0, g_i],\ 1 \leq i \leq m\}$, does not depend on $y$, is involutive and of rank $2m$. Since $\mathcal{L}^1$ does not depend on $y$, we have $\mathcal{L}^1(x, \tilde{y}_1) = \mathcal{L}^1(x, \bar{y}_1)$, for any fixed $\tilde{y}_1 \neq \bar{y}_1$. It follows that $(ad_f g_i + \tilde{y}_1[g_0, g_i]) - (ad_f g_i + \bar{y}_1[g_0, g_i]) = (\tilde{y}_1 - \bar{y}_1)[g_0, g_i] \in \mathcal{L}^1$, for $1 \leq i \leq m$, thus $\mathcal{L}^1 = \mathrm{span}\,\{g_i, ad_f g_i, [g_0, g_i],\ 1 \leq i \leq m\}$. Since $\mathrm{rk}\,\mathcal{G}^1 = \mathrm{rk}\,(\mathrm{span}\,\{g_0, g_i, [g_0, g_i], 1 \leq i \leq m\}) = 2m$, we obtain $ad_f g_i \in \mathcal{G}^1$ and we actually have $\mathcal{L}^1 = \mathrm{span}\,\{g_i, [g_0, g_i],\ 1 \leq i \leq m\}$. Thus we have just shown that $[f, \mathcal{L}^0] \subset \mathcal{G}^1$. From the fact that $\mathcal{L}^1$ is involutive, we deduce that $\mathcal{L}^1$ is the corank one involutive subdistribution of $\mathcal{G}^1$.

Repeating this argument, we prove that the involutive distributions $\mathcal{L}^i = \mathcal{D}_p^i \cap TX$, for $0 \leq i \leq k-2$, are of corank one in $\mathcal{G}^i$ and $[f, \mathcal{L}^{i-1}] \subset \mathcal{G}^i$. According to the above remark, we deduce that $\mathcal{C}^i = \mathcal{L}^{i-1} = \mathcal{D}_p^{i-1} \cap TX$ and that we actually have $[f, \mathcal{L}^{i-1}] = [f, \mathcal{C}^i] \subset \mathcal{G}^i$. It follows that the system $\Sigma_{aff}$ satisfies item *(Aff 5)* and thus, see [27], is static feedback equivalent the form $TCh_m^k$, which is clearly $x$-maximally flat.

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