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Yue Ma

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Université Pierre et Marie Curie

École Doctorale de Sciences Mathématiques de Paris Centre

THÈSE DE DOCTORAT

Discipline : Mathématiques appliquées

présentée par

Yue MA

**Équations d'onde nonlinéaires de type Klein-Gordon.
Application à la théorie $f(R)$ de la gravitation**

Directeur de thèse : Philippe LeFloch

Rapporteurs : Alain Bachelot et Anıl Zenginoğlu

Soutenue le 3 Décembre 2014 devant le jury composé de :

Alain Bachelot	Professeur
Jean-Marc Delort	Professeur
Philippe LeFloch	Directeur de recherche
Jérémie Szeftel	Directeur de recherche
Abdelghani Zeghib	Directeur de recherche

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Table des matières

Introduction	7
1. Objectifs de cette thèse	7
2. Validité mathématique de la théorie $f(R)$ de la gravitation	8
3. Méthode du feuilletage hyperboloïdal	12
Bibliographie	17
Partie I Validité mathématique de la théorie $f(R)$ de la gravitation	19
1. Introduction/ <i>Introduction</i>	21
2. Formulation du problème de Cauchy dans la métrique de Jordan/ <i>Formulation of the Cauchy problem in the Jordan metric</i>	24
3. Formulation du problème de Cauchy dans la métrique d'Einstein/ <i>Formulation of the Cauchy problem in the Einstein metric</i>	34
4. Formulation dans un repère d'onde/ <i>The conformal formulation in wave coordinates</i>	40
5. Formulation conforme augmentée/ <i>The augmented conformal formulation</i>	46
6. Existence locale, formulation et énoncé/ <i>Local existence theory - formulation and statement of the result</i>	53
7. Résultats techniques en vue de l'existence locale/ <i>Technical tools for local existence theory</i>	60
8. Démonstration de l'existence locale/ <i>Proof of the local existence theorem</i>	87
9. Comparaison entre la théorie de $f(R)$ et la relativité / <i>9. Comparing the $f(R)$ theory to general relativity</i>	99
Partie II Méthode du feuilletage hyperboloïdal	107
1. Introduction/ <i>Introduction</i>	109
2. Feuilletage hyperboloïdal et stratégie de bootstrap/ <i>The hyperboloidal foliation and the bootstrap strategy</i>	121
3. Décompositions et estimations des commutateurs/ <i>Decompositions and estimates for the commutators</i>	143
4. Structure nulle et repère semi-hyperboloïdal/ <i>The null structure in the semi-hyperboloidal frame</i>	157
5. Inégalités de type Sobolev et de type Hardy sur un hyperboloïde/ <i>Sobolev and Hardy inequalities on hyperboloids</i>	173

6. Estimations fondamentales de type L^∞ et de type L^2 / <i>Fundamental L^∞ and L^2 estimates</i>	181
7. Estimations sur les dérivées du second-ordre des composantes d'onde/ <i>Second-order derivatives of the wave components</i>	189
8. Repère nul et décroissance temporelle/ <i>Null forms and decay in time</i>	231
9. Estimations de type L^2 sur les termes de source/ <i>L^2 estimate on the source terms</i>	239
10. Retour sur les équations d'ondes scalaires / <i>Revisiting scalar wave equations</i>	249
11. Théorie d'existence locale/ <i>The local well-posedness theory</i>	257

Introduction

1 Objectifs de cette thèse

Notre objectif dans ce travail est l'étude d'une classe de systèmes nonlinéaires d'équations aux dérivées partielles (EDP) constituée par des équations d'onde et des équations de Klein-Gordon couplées, posées sur un espace-temps courbe. Un de nos résultats principaux est l'existence de solutions régulières définies globalement en temps pour le problème de Cauchy lorsque les données initiales sont suffisamment petites et que les nonlinéarités des équations satisfont à des conditions très naturelles (conditions nulles, conditions de non-explosion). Nous couvrons ainsi une classe très large de systèmes hyperboliques nonlinéaires qui apparaissent dans des nombreux contextes physiques. Notre application principale dans cette thèse est donnée par la théorie $f(R)$ de la gravitation nonlinéaire.

Cette théorie $f(R)$ est une modification de la théorie de la relativité générale introduite par Einstein en 1915. L'introduction de cette théorie plus générale est justifiée de la manière suivante : ces dernières années, de plus en plus d'observations astronomiques nous indiquent que l'expansion de l'Univers est dans une phase d'accélération, ce qui n'est pas compatible avec les modèles déduits de la relativité générale classique.

Pour résoudre cette difficulté, les physiciens ont proposé plusieurs théories modifiées, et parmi elles est l'introduction de l'énergie noire. Une autre théorie possible introduite par Buchdahl dans [3] est la théorie $f(R)$ de la gravitation nonlinéaire, que nous nous proposons d'étudier dans cette thèse. Selon le choix de la fonction f , la théorie $f(R)$ présente les aspects différents en ce qui concerne le comportement global du champ de gravité. La relativité générale est vue comme un cas limite de cette théorie, et est obtenue en prenant $f(R)$ précisément égale à la courbure scalaire R de l'espace-temps. Ce degré de liberté nous permet d'inclure plusieurs modèles physiques : l'énergie noire, la masse noire, la formation des structures, etc.

Cette théorie est très appréciée par les physiciens, mais les modèles mathématiques qui en sont issus n'ont pas été étudiés du point de vue mathématique. Les équations de champ de la théorie $f(R)$ sont algébriquement et analytiquement plus complexes que les équations d'Einstein de la relativité générale : ces équations contiennent des dérivées partielles du quatrième ordre en la métrique inconnue, alors que les équations d'Einstein ne contiennent "que" des dérivées d'ordre (au plus) deux.

Dans la première partie de la thèse, nous présentons une étude mathématique de la théorie $f(R)$ de la gravité nonlinéaire. Nos objectifs principaux sont les suivants [12] :

1. Formulation du problème de Cauchy lorsque des données initiales sont prescrites sur une hypersurface de type espace.
2. Re-formulation des équations de la théorie $f(R)$ de la gravité nonlinéaire. Nous commençons par présenter plusieurs formulations qui sont chacune pertinentes pour des techniques d'analyses mathématiques différentes. Parmi ces formulations, nous choisissons finalement la

"formulation conforme augmentée", que nous définissons et qui transforme les équations de champ de la théorie $f(R)$ sous la forme d'un système hyperbolique quasi-linéaire du second ordre, composé d'équations d'onde et d'équations de Klein-Gordon.

3. Nous démontrons l'existence de solutions (définies localement en temps) du problème de Cauchy dans le cadre de notre formulation conforme augmentée.

4. Nous montrons ensuite la propriété de convergence suivante : si la fonction f est proche de la fonction identité R (en un sens convenable), nous démontrons que la solution du problème de Cauchy donnée par la théorie $f(R)$ reste proche de la solution donnée par l'équation d'Einstein. Cette propriété est fondamentale pour affirmer que la théorie $f(R)$ est "physiquement raisonnable", en particulier parce que la relativité générale est déjà validée par de très nombreuses observations astronomiques. Une "nouvelle" théorie du champ de gravité doit admettre la relativité générale dans une certaine limite. Le passage d'un système d'ordre quatre (dans la théorie $f(R)$) à un système d'ordre deux (équations d'Einstein) rend cette analyse "limite" singulière et techniquement délicate.

Dans la deuxième partie de cette thèse, nous étudions la question de l'existence globale en temps dans le cadre des systèmes nonlinéaires composés d'équations d'onde et d'équations de Klein-Gordon [13]. Notre motivation pour ce travail est fournie par notre étude précédente des équations du champ de la théorie $f(R)$ de la gravité nonlinéaire, qui prennent la forme d'un système de ce type. Par ailleurs, de tels systèmes apparaissent fréquemment dans les contextes physiques, par exemple : le système de Dirac-Klein-Gordon, le système de Proca-Klein-Gordon, etc. Nous avons développé une nouvelle technique basée sur un feuilletage courbe de l'espace-temps, que nous qualifions de "feuilletage hyperboloidal". Avec ces techniques, nous pouvons surmonter certaines des difficultés rencontrées habituellement dans la littérature avec les techniques classiques.

2 Validité mathématique de la théorie $f(R)$ de la gravitation

2.1 Équations du champ de la théorie $f(R)$

Comme nous l'avons présenté dans la section 1, en plus des termes du deuxième ordre dus à la courbure de Ricci dans les équations d'Einstein, les équations du champ de la théorie $f(R)$ contiennent des dérivées du quatrième ordre en la métrique inconnue, qui proviennent des dérivées du second ordre de la courbure scalaire. Ces équations de type "ondes nonlinéaires" (dans une jauge bien choisie) constituent le système d'EDP à résoudre et sont considérablement plus complexes que le système d'Einstein. Cependant, nous mettons ici en évidence une structure mathématique remarquable qui est à la base de notre "formulation conforme augmentée". Cette formulation conforme augmentée est une extension du système original et dans laquelle la métrique inconnue et sa courbure scalaire sont considérées comme des inconnues indépendantes. Nous établissons l'existence locale de solutions pour ce système augmenté et nous démontrons ensuite que la solution du système augmenté coïncide avec la solution du système original de la théorie $f(R)$ de la gravitation nonlinéaire.

Rappelons que la théorie de la relativité générale est basée sur l'action d'Hilbert-Einstein :

$$\mathcal{A}_{\text{HE}}[\phi, g] := \int_M \left(\frac{R_g}{16\pi} + L[\phi, g] \right) dV_g$$

associée à l'espace-temps (M, g) inconnu de dimension $(3 + 1)$ et de signature lorentzienne $(-, +, +, +)$. La forme de volume canonique est désignée par dV_g , et nous notons par $\text{Rm} =$

Rm_g , $\text{Ric} = \text{Ric}_g$ et $R = R_g$, respectivement, les courbures de Riemann, de Ricci et la courbure scalaire de la métrique g . La fonctionnelle ci-dessus est déterminée par la courbure scalaire R_g et un lagrangien $L[\phi, g]$ décrivant la matière représentée par un champ ϕ défini sur M .

Il est bien connu que les points critiques de l'action $\mathcal{A}_{\text{HE}}[\phi, g]$ (au moins formellement) satisfont à l'équation d'Einstein :

$$G_g := \text{Ric}_g - \frac{R_g}{2} g = 8\pi T[\phi, g], \quad (1)$$

où

$$T_{\alpha\beta}[\phi, g] := -2 \frac{\delta L}{\delta g^{\alpha\beta}}[\phi, g] + g_{\alpha\beta} L[\phi, g]$$

est le tenseur énergie-impulsion déterminé par le modèle de matière. Si nous nous concentrons d'abord sur le cas du "vide", l'équation d'Einstein est équivalente à

$$\text{Ric}_g = 0.$$

La théorie $f(R)$ de la gravitation nonlinéaire peut être interprétée comme une théorie de la "gravité d'ordre élevé" et est définie comme suit. Soit f une fonction de \mathbb{R} dans \mathbb{R} fixée. Nous introduisons l'action de la théorie $f(R)$ par

$$\mathcal{A}_{\text{NG}}[\phi, g] := \int_M \left(\frac{f(R_g)}{16\pi} + L[\phi, g] \right) dV_g,$$

dont les points critiques satisfont à l'équation suivante :

$$N_g := f'(R_g) G_g - \frac{1}{2} \left(f(R_g) - R_g f'(R_g) \right) g + (g \square_g - \nabla d)(f'(R_g)) = 8\pi T[\phi, g]. \quad (2)$$

Nous observons que le tenseur N_g de la théorie $f(R)$ de la gravité nonlinéaire "remplace" le tenseur d'Einstein G_g , tandis que le tenseur énergie-impulsion reste invariant. En particulier, si nous prenons la trace de (2),

$$\text{tr } N_g = f'(R_g) R_g - 2f(R_g) + 3 \square_g f'(R_g) = 8\pi \text{tr}(T). \quad (3)$$

Cette équation peut être vue comme une équation d'évolution de la courbure scalaire R_g . Elle va jouer un rôle indispensable dans notre analyse.

Concernant le tenseur énergie-impulsion, nous observons que (voir aussi le texte plus loin) le tenseur N_g est de divergence nulle :

$$\nabla^\alpha N_{\alpha\beta} = 0.$$

Cette propriété implique que le tenseur énergie-impulsion est aussi à divergence nulle :

$$\nabla^\alpha T_{\alpha\beta} = 0.$$

Pour que la théorie $f(R)$ soit une extension de la théorie classique de la relativité générale, il est nécessaire d'imposer $f(R) \sim R$ quand $R \rightarrow 0$. De plus, nous montrons dans ce travail que le signe du coefficient $\kappa := f''(0) > 0$ est critique pour la stabilité et nous supposons donc que

$$f'(R) = 1 + \kappa(R + \kappa \mathcal{O}(R^2)).$$

Par intégration, nous avons

$$f(R) = R + \kappa \left(\frac{R^2}{2} + \kappa \mathcal{O}(R^3) \right), \quad (4)$$

où, par définition, le terme $\mathcal{O}(z^2)/z^2$ est borné quand $z \rightarrow 0$ (uniformément par rapport à κ). En particulier, la fonction f étant supposée C^∞ , elle est croissante et strictement convexe dans un voisinage de 0, et donc inversible. Un cas particulier mais intéressant en physique est obtenu en prenant le terme $\kappa \mathcal{O}(R_g^3)$ identiquement égal à zéro.

Dans un système de coordonnées locales, l'équation (2) est un système d'EDP non-linéaires du quatrième ordre, tandis que l'équation d'Einstein est un système du seconde ordre. Notre défi dans cette étude est d'analyser l'effet de ces termes du quatrième ordre et de développer un cadre mathématique qui généralise celui de la relativité générale mathématique. De plus, comme nous l'avons expliqué dans la première section, nous retrouverons la théorie d'Einstein en faisant converger le coefficient κ vers 0.

2.2 Le problème de Cauchy

Dans la littérature physique, plusieurs choix sont possibles pour le repère dans lequel les mesures physiques sont réalisées (malgré l'équivalence mathématique). Pour les physiciens, ce choix de repère est essentiel pour formuler précisément le fonctionnel d'action décrivant la matière et la couplant au champ de la gravité. Dans le "repère de Jordan", la métrique originale $g_{\alpha\beta}$ présentée dans l'équation (2) est vue comme la métrique "physique". Par ailleurs, dans le "repère d'Einstein", la métrique conforme $g^\dagger_{\alpha\beta} := f'(R_g)g_{\alpha\beta}$ est interprétée comme étant la métrique qui a une signification physique. Pour ces considérations sur les aspects physiques de la théorie, nous renvoyons à [15]. Notre objectif ici est uniquement de clarifier la terminologie du point de vue mathématique. En particulier, le terme de "repère" est utilisé en un sens différent en géométrie différentielle, et nous préférons utiliser ici les expressions "couplage de Jordan" et "couplage d'Einstein" afin d'éviter toute confusion.

Plus précisément, le "couplage de Jordan" représente le couplage minimal du champ de matière à la géométrie de l'espace-temps (représenté par le tenseur N_g) exprimé dans la "métrique de Jordan" (la métrique original g). Par ailleurs, le "couplage d'Einstein" est le couplage minimal du champ de la matière à la géométrie de l'espace-temps exprimé dans une métrique conforme g^\dagger .

Au vue des observations de [15], des couplages différents conduisent à des théories qui peuvent être non-équivalentes. Nous soulignons qu'une théorie physique peut être présentée dans les métriques différentes. Sinsi, le couplage de Jordan peut être présenté avec la métrique d'Einstein, tandis que le couplage d'Einstein peut aussi être présenté avec la métrique d'Einstein. Mais, si nous changeons la métrique de présentation avant de décider du couplage de la matière, le couplage dans une autre métrique en général ne sera plus minimal. Plus précisément, le couplage d'Einstein n'est pas minimal par rapport à la métrique de Jordan, et le couplage de Jordan n'est pas minimal par rapport à la métrique d'Einstein. Il semble que ces terminologies physiques aient conduit à de multiples confusions dans la littérature physique.

Dans ce travail, on considère le modèle de matière décrit par un champs scalaire réel de masse nulle, donné par les tenseurs énergie-impulsion suivants (selon le couplage adopté) :

$$\begin{aligned} \text{Couplage de Jordan : } T_{\alpha\beta} &:= \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\delta\lambda} \nabla_\delta \phi \nabla_\lambda \phi, \\ \text{Couplage d'Einstein : } T^\dagger_{\alpha\beta} &:= f'(R_g) \left(\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\delta\lambda} \nabla_\delta \phi \nabla_\lambda \phi \right). \end{aligned} \quad (5)$$

Nous soulignons que le couplage d'Einstein est présenté avec la métrique de Jordan. Nous remarquons que des choix de couplage différents nous conduisent à des systèmes d'EDP différents. Nous nous concentrons principalement sur le couplage de Jordan, en raison du fait que, comme nous observons dans la suite, le couplage d'Einstein fournit un problème de Cauchy *mal-posé*.

Maintenant nous énonçons le résultat principal de la première partie de la thèse sous une forme abrégée. Rappelons que le problème de Cauchy de l'équation d'Einstein est formulé comme suit (cf. Choquet-Bruhat [4]) pour la terminologie et les références historiques). Soit (\bar{M}, \bar{g}) une variété riemannienne de dimension 3 et K un champ de tenseur 2-covariant défini sur \bar{M} . Nous supposons que ces données initiales (combinées avec les données initiales qui décrivent le champ de matière) satisfont certaines contraintes de type Gauss-Coddazi que nous déterminons. Nous cherchons un développement (globalement hyperbolique) de ces données initiales, c'est-à-dire, par définition, une variété lorentzienne (M, g) qui satisfait l'équation d'Einstein et telle que la variété \bar{M} est plongée comme une sous-variété de métrique induite \bar{g} coïncidant avec la restriction de g à \bar{M} et de seconde forme fondamentale coïncidant avec K . Le développement maximal (globalement hyperbolique), par définition, est l'unique développement de ces données initiales qui contiens tous développements possibles à isométries près.

Nous généralisons cette formulation du problème de Cauchy au cas de la théorie $f(R)$. L'équation du champ (2) est du quatrième ordre, et des données initiales supplémentaires sont ainsi nécessaires. Nous ajoutons aux données initiales deux champs scalaires R_0, R_1 définis sur \bar{M} qui (essentiellement) représente la restriction de la courbure scalaire et sa dérivée temporelle. De plus, les restrictions du champ scalaire et sa dérivée temporelle sont notés, respectivement, ϕ_0, ϕ_1 .

Théorème 1 (Développements de Cauchy pour la théorie $f(R)$ de la gravité nonlinéaire). *On considère l'équation du champ gravitationnel (2) pour une fonction f de la forme (4) et pour un couplage matériel de Jordan (5). Des données initiales $(\bar{M}, \bar{g}, K, R_0, R_1, \phi_0, \phi_1)$ suffisamment régulières étant prescrites, il existe un unique développement maximal globalement hyperbolique (M, g) de ces données initiales satisfaisant l'équation (2) (en un sens que nous expliquerons dans les sections suivantes).*

Nous préciserons cet énoncé dans la partie 1 ci-dessous. Nous soulignons que notre résultat présente la première démonstration mathématique rigoureuse montrant que la théorie de $f(R)$ peut être vue comme une modification de la théorie classique introduite par Einstein.

Au-delà de ce résultat, nous présentons les équations du champ sous la forme d'un système de EDP quasi-linéaires hyperboliques. En effet, un ensemble de conditions de jauge imposées sur la métrique d'Einstein : les "conditions d'onde". Notre formulation contient une variable supplémentaire, désignée par ρ qui représente la courbure scalaire de l'espace-temps.

Definition 2. *La formulation conforme augmentée des équations de la théorie $f(R)$ (avec le couplage de Jordan et sous les conditions d'onde ajoutées sur la métrique d'Einstein) s'écrit :*

$$\begin{aligned}
g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g^{\dagger}_{\alpha\beta} &= F_{\alpha\beta}(g^{\dagger}; \partial g^{\dagger}, \partial g^{\dagger}) - 12 \partial_{\alpha} \rho \partial_{\beta} \rho + V(\rho) g^{\dagger}_{\alpha\beta} - 16\pi \partial_{\alpha} \phi \partial_{\beta} \phi, \\
g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \phi &= -2g^{\dagger\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \rho, \\
g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \rho - \frac{\rho}{3\kappa} &= W(\rho) - \frac{4\pi}{3e^{2\rho}} g^{\dagger\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi, \\
g^{\dagger\alpha\beta} \Gamma^{\lambda}_{\alpha\beta} &= 0,
\end{aligned} \tag{6}$$

où $F_{\alpha\beta}(g^\dagger; \partial g^\dagger, \partial g^\dagger)$ est une forme quadratique par rapport à la variable ∂g^\dagger (définition précise donnée dans les discussions suivantes) et la fonction $V = V(\rho)$ et $W = W(\rho)$ sont quadratiques par rapport à ρ quand $\rho = 0$.

Nous retournons la théorie d'Einstein en choisissant $f(R) = R$ et $\kappa = 0$. Nous démontrerons que si $\rho = 0$, le système ci-dessus se réduit au système standard d'Einstein.

3 Méthode du feuilletage hyperboloïdal

3.1 Objectifs de la méthode

Nous nous intéressons aux équations d'onde quasi-linéaires posées dans l'espace de Minkowski de dimension $(3+1)$ et, en particulier, aux modèles en physique mathématique et qui décrivent certains couplages non-linéaires par des équations de Klein-Gordon : par exemple, le système de Dirac-Klein-Gordon, le système de Proca-Klein-Gordon, etc. Cette étude est aussi motivée par des études réalisées sur le système (5) et le système des équations d'Einstein couplée avec un champ scalaire réel de masse positive. L'équation de Klein-Gordon est aussi essentielle pour la modélisation des ondes non-linéaires dans les fluides ou dans les matériaux élastiques.

La méthode du feuilletage hyperboloïdal est une "méthode de champ de vecteurs" à la manière de [10] et [5]. Cette approche se base sur les deux ingrédients principaux classiques : d'une part, une estimation de l'énergie (convenablement pondérée) définie à partir des champs de Killing de l'espace-temps de Minkowski et, d'autre part, une estimation du type de Sobolev qui fournit l'estimation clef dans L^∞ . Avec ces arguments, les auteurs réussissent à établir l'existence globale-en-temps des solutions pour des données initiales suffisamment petites et localisées (c'est-à-dire supportées dans un compact). De plus, cette méthode s'appuie sur un argument de bootstrap, ainsi qu'une analyse très fine de la vitesse de décroissance en temps des solutions et, de plus, l'analyse de la "structure nulle" des équations considérées. Pour un système couplé d'équations d'onde et d'équations de Klein-Gordon, le travail le plus récent [9] propose une généralisation directe de cette méthode des champs de vecteur.

Dans cette partie, nous considérons le problème de Cauchy associé à des systèmes non-linéaires couplant des équations d'onde et des équations de Klein-Gordon. Nous introduisons un nouveau cadre —le feuilletage hyperboloïdal— qui nous amène à une généralisation importante de l'état de l'art dans ce domaine. Avec des normes de type L^2 (convenablement pondérées), nous pouvons traiter ici des termes nonlinéaire qui n'étaient pas inclus dans [9] mais qui sont importants dans certains contextes de la physique mathématique. (Des applications seront ainsi traitées dans un travail en cours [14].) Ce cadre du feuilletage hyperboloïdal nous amène une nouvelle méthodologie, qui se base sur une observation d'Hörmander [8] pour l'analyse de l'équation standard de Klein-Gordon.

Nous notons par \square l'opérateur d'onde dans l'espace-temps de Minkowski. Pour mieux expliquer nos idées, nous commençons par un système simple (mais suffisamment complexe pour notre propos)

$$\begin{aligned} \square u &= P(\partial u, \partial v), \\ \square v + v &= Q(\partial u, \partial v). \end{aligned} \tag{7}$$

Il s'agit d'un système composé d'une seule équation d'onde couplée avec une seule équation de Klein-Gordon. Les termes quadratiques P et Q décrivent les interactions entre ces deux champs scalaires u et v , dont l'un est massif. Nous rappelons que, d'une part, l'existence de solutions globales-en-temps des équations d'onde quasi-linéaires (sans couplage avec Klein-Gordon) est

établie par [10] dans le cas où les termes quadratiques vérifient la "condition nulle". D'autre part, l'existence de solutions globales-en-temps (sans couplage avec une équation d'onde) est établie dans [11] et des termes quadratiques arbitraires peuvent être inclus ici. (Le même résultat est aussi établi dans [17] par une méthode différente.

Par contre, si l'on utilise ces méthodes classiques sur un système couplé d'équations d'onde et d'équations de Klein-Gordon du type (7), une difficulté essentielle apparaît : un champ de vecteur de Killing associé à l'opérateur \square , c'est-à-dire le champ $S := t\partial_t + r\partial_r$, n'est plus un champ de Killing de l'opérateur de Klein-Gordon $(\square + 1)$, de sorte que ce champ de vecteur S ne peut plus être utilisé dans l'analyse.

Dans[9], Katayama a réussi à résoudre cette difficulté et a établi un résultat d'existence nouveau pour les solutions globales-en-temps pour une classe de systèmes couplés (7). Il s'appuie sur une nouvelle estimation de type Sobolev qui remplace l'estimation classique de Klainerman-Sobolev, et il utilise aussi une estimation de type $L^\infty - L^\infty$. Notre texte fournira plus loin d'autres références sur ce sujet.

3.2 Existence globale pour les systèmes des équations d'onde et des équations de Klein-Gordon

Dans cette section, nous pouvons énoncer notre résultat principal. Par convention, tous les indices latins a, b, \dots prennent les valeurs $\{1, 2, 3\}$, tandis que les indices grecs α, β, \dots prennent leurs valeurs dans $\{0, 1, 2, 3\}$. La convention classique de "sommation d'Einstein" est aussi utilisée.

Nous nous intéressons au problème de Cauchy associé au système suivant :

$$\begin{aligned} \square w_i + G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta w_j + c_i^2 w_i &= F_i(w, \partial w), \\ w_i(B+1, x) &= w_{i0}, \\ \partial_t w_i(B+1, x) &= w_{i1}, \end{aligned} \tag{8}$$

dans lequel les inconnues sont les fonctions w_i ($1 \leq i \leq n_0$) définies sur l'espace-temps de Minkowski \mathbb{R}^{3+1} . **L'hypothèse suivante de symétrie** est imposée :

$$G_i^{j\alpha\beta} = G_j^{i\alpha\beta}, \quad G_i^{j\alpha\beta} = G_i^{j\beta\alpha} \tag{9}$$

et la structure d'**onde-Klein-Gordon** est décrite en posant :

$$c_i = \begin{cases} = 0, & 1 \leq i \leq j_0, \\ \geq \sigma, & j_0 + 1 \leq i \leq n_0, \end{cases}$$

où $\sigma > 0$ est (une constante positive et) la borne inférieure des constantes de Klein-Gordon. La partie courbe des termes principaux $G_i^{j\alpha\beta}(w, \partial w)$ et les termes quadratiques $F_i(w, \partial w)$ sont décomposés de la manière suivante :

$$G_i^{j\alpha\beta}(w, \partial w) = A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k + B_i^{j\alpha\beta k} w_k, \tag{10a}$$

$$F_i(w, \partial w) = P_i^{\alpha\beta jk} \partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha jk} w_k \partial_\alpha w_j + R_i^{jk} w_j w_k, \tag{10b}$$

où, pour simplicité dans la présentation et sans perdre en généralité, nous supposons que les coefficients $A_i^{j\alpha\beta\gamma k}$, $B_i^{j\alpha\beta k}$, $P_i^{\alpha\beta jk}$, $Q_i^{\alpha jk}$, et R_i^{jk} sont constants.

Pour simplifier la présentation, nous utilisons la convention suivante sur les indices :

tous indices i, j, k, l prennent les valeurs $1, \dots, n_0$,

tous indices $\hat{i}, \hat{j}, \hat{k}, \hat{l}$ prennent les valeurs $1, \dots, j_0$,

tous indices $\check{i}, \check{j}, \check{k}, \check{l}$ prennent les valeurs $j_0 + 1, \dots, n_0$.

Il sera utile d'introduire la notation

$$u_i := w_i$$

pour les composantes qui satisfont aux équations d'onde ou **composantes d'onde**, et

$$v_i := w_i$$

pour les composantes qui satisfont aux équations de Klein-Gordon ou **composantes de Klein-Gordon**.

Nos hypothèses principales sont la conditions nulles pour l'équations d'onde :

$$\begin{aligned} A_i^{j\alpha\beta\gamma\hat{k}} \xi_\alpha \xi_\beta \xi_\gamma &= B_i^{j\alpha\beta\hat{k}} \xi_\alpha \xi_\beta = P_i^{\alpha\beta j\hat{k}} \xi_\alpha \xi_\beta = 0 \\ \text{si } (\xi_0)^2 - \sum_a (\xi_a)^2 &= 0, \end{aligned}$$

et les conditions de **non-explosion** qui excluent les termes nonlinéaire générant une explosion en temps fini. Ce sont les termes quadratiques qui contiennent les composantess d'onde comme facteur. Plus précisément, nous supposons que

$$B_i^{j\alpha\beta\hat{k}} = R_i^{j\hat{k}} = R_i^{jk} = 0.$$

Nous soulignons que les conditions nulles sont imposées sur les termes quadratiques "purement" de type "onde" : ces conditions ne sont imposées que sur les formes quadratiques apparaissant dans les équations d'onde et ne concernent que les dérivées des composantes d'onde.

De plus, nous imposons la condition

$$Q_i^{\alpha j \hat{k}} = 0,$$

qui est la seule restriction véritable dans cette thèse. Mais c'est nécessaire à cause de l'implémentation de la méthode du feuilletage hyperboloïdal. Si cette hypothèse n'est pas réalisée, la solution ne possède plus une vitesse de décroissance "proche" de celle des solutions du système linéarisé. (Nous renvoyons à [14] pour l'analyse de tels termes.)

Nous énonçons maintenant le résultat principal de la deuxième partie de la thèse.

Théorème 3 (Existence globale du système d'onde-Klein-Gordon). *Nous considérons le problème de Cauchy (8) avec des données initiales de classe C^∞ posées sur l'hypersurface spatiale $\{t = B + 1\}$ et nous supposons que ces fonctions sont supportées dans la boule $\{|x| \leq B\}$. Alors sous les conditions (10), il existe une constante $\epsilon_0 > 0$ telle que, si*

$$\|w_{i0}\|_{\mathbf{H}^6(\mathbb{R}^3)} + \|w_{i1}\|_{\mathbf{H}^5(\mathbb{R}^3)} \leq \epsilon_0,$$

alors la solution locale associée se prolonge pour tout temps positif.

Dans le cas spécial $n_0 = j_0$, le système (8) ne contient que les composantes d'onde et le théorème 3 se réduit au résultat classique d'existence globale pour les système quasi-linéaires avec nonlinéarités quadratiques nulles. Dans ce cas, notre méthode est, dans une certaine mesure, plus simple que la méthode classique. De plus, notre méthode montre que la solution décroît plus rapidement en comparaison du resultat obtenu par la méthode classique.

3.3 Stratégie générale de la démonstration

Nous présentons dans cette section les idées cruciales de notre nouvelle méthode. Les définitions rigoureuses sur les notions utilisées dans cette discussion seront introduites dans la partie II.

– **Feuilletage hyperboloïdal**

L'idée la plus importante dans les travaux de la deuxième partie de la thèse, c'est de feuilletter l'espace-temps (la partie dans un cône de la lumière) par une famille d'*hyperboloïdes*, tandis que dans le contexte classique, l'espace-temps est feuilleté une famille de plans. En comparaison avec le feuilletage classique, ce feuilletage hyperboloïdal a l'avantage d'être invariant sous les transformations de Lorentz.

Il nous faut alors trouver une estimation d'énergie associée à ce feuilletage, ce que nous obtenons ici. (Ceci suggère qu'il faut chercher des feuilletages encore plus généraux dans une variété lorentzienne courbe [14].)

– **Repère semi-hyperboloïdal**

Nous introduisons alors le champ de repères semi-hyperboloïdals adapté à ce feuilletage hyperboloïdal. Ce champ de repère consiste en trois champs vectoriels tangents aux hyperboloïdes et un champ de type temporel. Suivant ici [8], nous préférons de ne pas utiliser le vecteur normal aux hyperboloïdes mais le champ de vecteur ∂_t . Nous rappelons que dans la littérature classique le repère nul (qui consiste en deux champs vectoriels nuls et deux champs vectoriels tangents aux cônes de lumière) est souvent utilisé.

En comparaison avec le repère nul, l'avantage du repère semi-hyperboloïdal est sa régularité dans l'intérieur du cône de la lumière, tandis que le repère nul est singulier (en $\{r = 0\}$). Cet avantage nous permet de surmonter quelques difficultés techniques.

– **Décomposition de l'opérateur d'onde adaptée au repère semi-hyperboloïdal**

Nous décomposons l'opérateur d'onde par rapport au repère semi-hyperboloïdal. Cette décomposition nous permet d'exprimer la dérivée temporelle du seconde ordre par d'autres dérivées qui possèdent des décroissances plus rapides et cette décomposition améliore donc significativement les estimations sur les composantes d'onde.

– **Énergie hyperboloïdale**

La clé de notre méthode est une analyse fine de l'énergie hyperboloïdale (introduite par [8]) : nous utilisons ici tous les termes de cette énergie afin de contrôler certaines dérivées des composantes d'onde, tandis que [8] ne s'intéresse qu'aux termes d'ordre zéro.

– **Inégalités globales de type Sobolev**

Pour établir les estimations L^∞ sur la solution (et ses dérivées), nous analysons un ensemble d'estimations de commutateurs et nous établissons une inégalité du type Sobolev globale sur les hyperboloïdes (semblable à celle établie dans [8]).

– **Inégalité de type Hardy sur un hyperboloïde**

De plus, nous avons aussi besoin d'une nouvelle inégalité, c'est-à-dire une inégalité de Hardy sur un hyperboloïde. Elle sert à contrôler la norme L^∞ des composantes de la métrique courbe.

– **Stratégie de bootstrap et hiérarchie d'énergie**

Dans l'argument de bootstrap que nous introduisons dans les sections suivantes, nous n'imposons pas une unique hypothèse de bootstrap (comme dans la littérature existante sur les équations quasi-linéaires), mais nous imposons un ensemble d'hypothèses qui, demande des comportements différents sur les énergies des dérivées d'ordres différents. Nous démontrons que les énergies d'ordre moins élevé sont bornées tandis que les énergies d'ordre plus élevé possèdent des coissances polynomiales. Cette stratégie plus complexe se trouve être indispensable pour les systèmes qui contiennent les couplages d'équations d'onde et d'équations de Klein-Gordon à cause du fait que les dérivées d'ordres différents ont des décroissances différentes.

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Partie I

Validité mathématique pour la théorie $f(R)$ de la gravitation ¹

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1 Introduction

1.1 The theory of nonlinear gravity

In recent years, new observational data have suggested that alternative theories of gravity, based on extensions of Einstein's field equations of general relativity, may be relevant in order to explain, for instance, the accelerated expansion of the Universe and certain instabilities observed in galaxies. Among these theories, the so-called $f(R)$ -theory of nonlinear gravity (where a function of the scalar curvature R is prescribed) is recognized as a physically viable alternative to Einstein's theory. Despite their important role played in physics, these field equations have not been investigated by mathematicians yet. This is due to the fact that the nonlinear gravity field equations are significantly more involved than the 'classical' Einstein equations, since they contain *fourth-order derivatives* of the unknown metric (rather than second-order derivatives).

Our purpose in the present paper is to initiate the mathematical study of these nonlinear gravity theories. Our main objective in the present paper is to define a suitable notion of 'initial data set' for this theory, formulate the initial value problem, and establish the existence of a globally hyperbolic maximal developments associated with a given initial data set. Our aim theorem will validate the physical intuition that the nonlinear gravity theory is an 'approximation' of Einstein's theory.

As already mentioned, in addition to the (second-order) Ricci curvature terms arising in Einstein equations, the field equations of the $f(R)$ -theory involve fourth-order derivatives of the metric and, specifically, second-order derivatives of its scalar curvature. The corresponding system of partial differential equations (after a choice of gauge) consists of a system of wave-like equations which is significantly more involved than the original Einstein system. Yet, a remarkable mathematical structure is uncovered in the present paper, which is based on a reformulation of the problem, referred to as the *augmented conformal formulation*: we formulate an extended system in which both the metric and its scalar curvature are regarded as *independent unknowns*; we establish the well-posedness of the initial value problem for this augmented formulation, and we then recover the solutions of field equations nonlinear gravity.

Recall first that Einstein theory is based on the Hilbert-Einstein action

$$\mathcal{A}_{\text{HE}}[\phi, g] := \int_M \left(\frac{R_g}{16\pi} + L[\phi, g] \right) dV_g$$

associated with a $(3+1)$ -dimensional spacetime (M, g) with Lorentzian signature $(-, +, +, +)$ whose canonical volume form is denoted by $dV = dV_g$. Here and thereafter, we denote by $\text{Rm} = \text{Rm}_g$, $\text{Ric} = \text{Ric}_g$, and $R = R_g$ the Riemann, Ricci, and scalar curvature of the metric g , respectively. Greek indices $\alpha, \beta = 0, \dots, 3$ represent spacetimes indices. Observe that the above functional $\mathcal{A}_{\text{EH}}[g]$ is determined from the scalar curvature R_g and a Lagrangian $L[\phi, g]$, the latter term describing the matter content represented by fields ϕ defined on M .

It is well-known that the critical metrics of the action $\mathcal{A}_{\text{EH}}[g]$ (at least formally) satisfy Einstein's equation

$$G_g := \text{Ric}_g - \frac{R_g}{2} g = 8\pi T[\phi, g], \quad (1.1)$$

in which the right-hand side

$$T_{\alpha\beta}[\phi, g] := -2 \frac{\delta L}{\delta g^{\alpha\beta}}[\phi, g] + g_{\alpha\beta} L[\phi, g] \quad (1.2)$$

is referred to as the stress-energy tensor of the matter model. For instance in the vacuum, these equations are equivalent to the Ricci-flat condition

$$\text{Ric}_g = 0. \quad (1.3)$$

The 'higher-order' gravity theory of interest in the present paper is defined as follows. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ being prescribed, we can introduce the action of the $f(R)$ -theory by

$$\mathcal{A}_{\text{NG}}[\phi, g] := \int_M \left(\frac{f(R_g)}{16\pi} + L[\phi, g] \right) dV_g, \quad (1.4)$$

whose critical points satisfy the following *field equations of nonlinear gravity*

$$N_g := f'(R_g) G_g - \frac{1}{2} \left(f(R_g) - R_g f'(R_g) \right) g + (g \square_g - \nabla d)(f'(R_g)) = 8\pi T[\phi, g]. \quad (1.5)$$

Observe that the nonlinear gravity tensor N_g now replaces Einstein's tensor G_g while the right-hand side (further discussed below) is still given by the stress-energy tensor (1.2). In particular, by taking the trace in (1.5), we find the following scalar equation

$$\text{tr } N_g = f'(R_g) R_g - 2f(R_g) + 3\square_g f'(R_g) = 8\pi \text{tr}(T), \quad (1.6)$$

which can be regarded as an evolution equation for the spacetime curvature and will play an important role in the present work.

Concerning the matter content, we observe (cf. Section 2 below for details) that the nonlinear gravity tensor N_g is *divergence free*, that is,

$$\nabla^\alpha N_{\alpha\beta} = 0, \quad (1.7)$$

so that the matter field necessarily satisfies the divergence equation

$$\nabla^\alpha T_{\alpha\beta} = 0. \quad (1.8)$$

Clearly, for the nonlinear theory to formally be an extension of the classical theory, it is necessary to assume that $f(R) \simeq R$ for small R . In addition, we will see later that the (positive) sign of the coefficient $\kappa := f''(0) > 0$ is critical for stability and, so, we set

$$f'(R) = 1 + \kappa(R + \kappa\mathcal{O}(R^2)), \quad (1.9)$$

which after integration yields

$$f(R) = R + \kappa \left(\frac{R^2}{2} + \kappa\mathcal{O}(R^3) \right), \quad (1.10)$$

where, by definition, the remainder $\mathcal{O}(z^2)/z^2$ remains bounded when $z \rightarrow 0$ (uniformly in κ). In particular, the map f (which we assume to be smooth) is increasing and strictly convex in a neighborhood of 0 and, therefore, is one-to-one. A case of particular physical interest arises if the term $\kappa\mathcal{O}(R^3)$ in (1.10) is taken to vanish identically.

In local coordinates, the field equations (1.5) consist of a nonlinear system of fourth-order partial differential equations (PDE's), while the Einstein equation (1.1) leads to only second-order equations. Our challenge in the present paper is to investigate the role of these fourth-order terms and whether they can be handled within the mathematical framework developed for Einstein equations. Furthermore, one would expect to recover Einstein's theory by letting the coefficient κ tend to zero. This formal limit is, however, *highly singular*, since the latter is a system of only second-order PDE's, while the nonlinear gravity equations are fourth-order.

1.2 The initial value problem

In the physics literature, the choice of the frame in which measurements should be made (despite being mathematically equivalent) are still under vivid discussions, and this issue is essential for physicists in order to properly formulate the coupling between the gravity equations and the matter fields. In the so-called ‘‘Jordan frame’’, the original metric $g_{\alpha\beta}$ is considered to be the ‘‘physical metric’’, while in the so-called ‘‘Einstein frame’’, the conformally-transformed metric $g^J_{\alpha\beta} := f'(R_g)g_{\alpha\beta}$ is considered to be the ‘‘physical metric’’. Some discussions in the physical aspects of the theory can be found for example in [6]. Our purpose here is simply to fix the terminology. In particular, since the term ‘‘frame’’ has a different meaning in differential geometry, we will use the expressions ‘‘Jordan coupling’’ and ‘‘Einstein coupling’’ instead.

More precisely, the ‘‘Jordan coupling’’ refers to the minimal coupling of the matter field to the geometry of the spacetime (represented by the tensor N_g) with respect to the ‘‘Jordan metric’’ (the

original metric) $g_{\alpha\beta}$. On the other hand, the “Einstein coupling” refers to the minimal coupling of the matter field to the geometry of the spacetime with respect to $g^\dagger_{\alpha\beta}$.

As discussed for instance in [6], different couplings lead to different physical theories which may not be equivalent to each other. We emphasize that a theory can be expressed in various choices of metrics, which means in our case that the “Jordan coupling” can also be expressed with the “Einstein metric” $g^\dagger_{\alpha\beta}$, while the “Einstein coupling” can also be stated in the “Jordan metric” $g_{\alpha\beta}$. Hence, if one changes the metric before defining the matter model, the coupling may no longer be the minimal coupling in the new metric. This suggests that the “Einstein metric” is not the physical metric in the “Jordan coupling” theory, while the “Jordan metric” is not the physical metric in “Einstein coupling”. This physical terminology has apparently led to great confusion in the physical literature.

Throughout this article, the matter model of interest is the real massless scalar field, defined by its stress-energy tensor and we consider the following two possible couplings:

$$\begin{aligned} T_{\alpha\beta} &:= \nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}g^{\delta\lambda}\nabla_\delta\phi\nabla_\lambda\phi, \\ T^\dagger_{\alpha\beta} &:= f'(R_g)(\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}g^{\delta\lambda}\nabla_\delta\phi\nabla_\lambda\phi). \end{aligned} \tag{1.11}$$

We emphasize that the Einstein coupling is written in the Jordan metric. (Our notion of Einstein coupling is precisely equivalent to the notion of “Einstein frame” in [6].) As we will see, different choice of couplings leads to different systems of PDE’s. Our primary interest is for the so-called Jordan coupling, since we will argue that the Einstein coupling leads to an ill-defined problem and, in the rest of this section, we thus state our conclusion for the Jordan coupling.

We are now in a position to state our main results, in a preliminary form at least. Recall that the initial value problem for the Einstein equations is formulated as follows (cf. Choquet-Bruhat [3] for the terminology and historical references). Given a Riemannian 3-manifold (\bar{M}, \bar{g}) , a 2-covariant tensor field K (together with suitable matter data) satisfying certain constraint equations, one search for a (globally hyperbolic) development of this initial data set, consisting by definition of a Lorentzian manifold (M, g) satisfying Einstein equations, such that \bar{M} is embedded in M as a spacelike hypersurface with induced metric \bar{g} and second fundamental form K . The maximal (globally hyperbolic) development, by definition, is the unique development of the initial data set in which one can isometrically embed any such development.

We generalize this standard formulation of the initial value problem to the theory of nonlinear gravity. Since the field equations (1.5) are now fourth-order, additional initial data are required, which are denoted by R_0, R_1 and are specified on the initial slice \bar{M} : they represent the spacetime scalar curvature and its time derivative of the (to-be-constructed) spacetime. In addition, initial data for the scalar field and its time-derivative are denoted by ϕ_0, ϕ_1 , respectively, and are also prescribed on the initial hypersurface.

Theorem 1.1 (Cauchy developments of nonlinear gravity). *Consider the field equations (1.5) of nonlinear gravity based on a function $f = f(R)$ satisfying (1.9) and assume that the matter is described by a scalar field with Jordan coupling (1.11). Given a “asymptotically flat” initial data set $(\bar{M}, \bar{g}, K, R_0, R_1, \phi_0, \phi_1)$ (the rigorous definition will be given in definition 6.1), there exists a unique maximal globally hyperbolic development (M, g) of these data which satisfies the nonlinear gravity equations (1.5) (in the sense of Definition 2.5, below). Furthermore: if there is a initial data set $(\bar{M}, \bar{g}', K', \phi'_0, \phi'_1)$ of the Cauchy problem associated to the classical Einstein equation which is “closed” to the initial data set $(\bar{M}, \bar{g}, K, R_0, R_1, \phi_0, \phi_1)$, then the solution generated is still “closed” to the solution generated by Einstein equation with the initial data $(\bar{M}, \bar{g}', K', \phi'_0, \phi'_1)$.*

The statements above will be made precise in the course of this paper. We emphasize that our results provide the first mathematically rigorous proof that the theory of nonlinear gravity is sound and can be regarded as an approximation of the classical theory of gravity introduced by Einstein, as expected by physicists.

Importantly, we are able here to express the field equations as a system of second-order hyperbolic equations, as now stated. In fact, we advocate here the use of wave coordinates associated

with the Einstein metric and our formulation in such coordinates is given in the following definition. Our formulation contains an *additional unknown* denoted by ρ which represents the scalar curvature of the spacetime (and specifically $\rho = \frac{1}{2} \ln f'(R_g)$).

Definition 1.2. *The augmented conformal formulation of the field equations of nonlinear gravity (with Jordan coupling and in wave coordinates associated with the Einstein metric) reads:*

$$\begin{aligned}
g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g^{\dagger}_{\alpha\beta} &= F_{\alpha\beta}(g^{\dagger}; \partial g^{\dagger}, \partial g^{\dagger}) - 12 \partial_{\alpha} \rho \partial_{\beta} \rho + V(\rho) g^{\dagger}_{\alpha\beta} - 16\pi \partial_{\alpha} \phi \partial_{\beta} \phi, \\
g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \phi &= -2g^{\dagger\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \rho, \\
g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \rho - \frac{\rho}{3\kappa} &= W(\rho) - \frac{4\pi}{3e^{2\rho}} g^{\dagger\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi, \\
g^{\dagger\alpha\beta} \Gamma^{\lambda}_{\alpha\beta} &= 0,
\end{aligned} \tag{1.12}$$

in which $F_{\alpha\beta}(g^{\dagger}; \partial g^{\dagger}, \partial g^{\dagger})$ are quadratic expressions (defined in Section 3 below) with ∂g^{\dagger} determined by the Ricci curvature, and the function $V = V(\rho)$ and $W = W(\rho)$ are of quadratic order ρ^2 as $\rho \rightarrow 0$.

We recover Einstein's theory by choosing $f(R) = R$ and $\kappa \rightarrow 0$. We will show that $\rho = 0$ and the above system reduces to the standard formulation in wave coordinates so that, in particular, we recover the expression $R = 8\pi \nabla_{\alpha} \phi \nabla^{\alpha} \phi$ of the scalar curvature in terms of the norm of the scalar field.

2 Formulation of the Cauchy problem in the Jordan metric

2.1 The 3 + 1 decomposition of spacetimes

In this section, we formulate the initial value problem for the nonlinear gravity system when initial data are prescribed on a spacelike hypersurface. Until further notice, our setup is as follows in which we follow the presentation given in [3]. We are interested in time-oriented spacetimes (M, g) endowed with a Lorentzian metric g with signature $(-, +, +, +)$, which are homeomorphic to $[0, t_{\max}) \times M_t$ and admit a global foliation by spacelike hypersurfaces $M_t \simeq \{t\} \times \overline{M}$ determined by a time function $t : M \rightarrow [0, t_{\max})$ and a fixed three-dimensional manifold \overline{M} . Throughout, we always assume that

$$\text{The spacetime } M \text{ is globally hyperbolic, and every } M_t \text{ is a Cauchy surface,} \tag{2.1}$$

so that a (system of) wave equation(s) posed on any Cauchy surface enjoys the usual uniqueness property. See the textbooks by [3, 5] for the definitions.

We take local coordinates adapted to this product structure, say $(x^{\alpha}) = (x^0 = t, x^i)$, and we choose vectors ∂_i as a natural frame of M_t . This structure defines the natural frame (∂_t, ∂_i) . We also introduce the so-called Cauchy adapted frame by setting $e_i = \partial_i$ and $e_0 = \partial_t - \beta^i \partial_i$ where $\beta = \beta^i \partial_i$ a time-dependent field tangent to M_t and called the shift vector, with the restriction that e_0 is orthogonal to M_t .

Now we introduce the dual frame θ^{α} of the Cauchy adapted frame (e_{α}) by

$$\theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt.$$

All tensors under consideration will be always expressed in a Cauchy adapted frame or in its dual frame. In particular, the spacetime metric reads

$$g = -N^2 \theta^0 \theta^0 + g_{ij} \theta^i \theta^j, \tag{2.2}$$

where the positive function N is referred to as the **lapse function**.

The structure coefficients of a Cauchy adapted frame are defined by

$$C_{0j}^i = -C_{j0}^i = \partial_j \beta^i. \quad (2.3)$$

The Riemannian connection associated to g is represented by the following differential forms ω :

$$\nabla e_\alpha = \omega_{\gamma\alpha}^\beta \theta^\gamma \otimes e_\beta,$$

and

$$\nabla \theta^\alpha = -\omega_{\gamma\beta}^\alpha \theta^\gamma \otimes \theta^\beta.$$

We denote by $\bar{g} = \bar{g}_t$ the induced Riemannian metric associated with the slices M_t and by $\bar{\nabla}$ the Levi-Civita connection of \bar{g} with the Christoffel symbols (with respect to the natural frame $\{x^i\}$) denoted by $\bar{\Gamma}_{ab}^c$. We can also introduce the second fundamental form $K = K_t$ defined, for all vectors X, Y tangent to the slices M_t , by

$$K(X, Y) := g(\nabla_X n, Y),$$

which in the Cauchy adapted frame reads

$$K_{ij} = -\frac{1}{2N} (\langle e_0, g_{ij} \rangle - g_{lj} \partial_i \beta^l - g_{il} \partial_j \beta^l). \quad (2.4)$$

Here and throughout this paper, we use the notation $\langle e_0, g_{ij} \rangle$ for the action of the vector field e_0 on the function g_{ij} . We define the operator $\bar{\partial}_0$ acting on a two-tensor defined on the slice M_t by

$$\bar{\partial}_0 T_{ij} = \langle e_0, T_{ij} \rangle - T_{lj} \partial_i \beta^l - T_{il} \partial_j \beta^l \quad (2.5)$$

which is again a two-tensor on M_t . With this notation, we thus have $K = -\frac{1}{2N} \bar{\partial}_0 \bar{g}$.

Standard calculations in section VI.3 of [3] yield the following splitting of the connection:

$$\begin{aligned} \omega_{jk}^i &= \bar{\Gamma}_{jk}^i, \\ \omega_{00}^i &= N g^{ij} \partial_j N, \quad \omega_{0i}^0 = \omega_{i0}^0 = N^{-1} \partial_i N, \quad \omega_{00}^0 = N^{-1} \langle e_0, N \rangle, \\ \omega_{ij}^0 &= \frac{1}{2} N^{-2} (\langle e_0, g_{ij} \rangle - g_{hj} \partial_i \beta^h - g_{ih} \partial_j \beta^h) = -N^{-1} K_{ij}, \\ \omega_{0j}^i &= -N K_j^i + \partial_j \beta^i, \quad \omega_{j0}^i = -N K_j^i. \end{aligned} \quad (2.6)$$

Here, $\bar{\Gamma}_{jk}^i$ denotes the Christoffel symbol of the connection of $\bar{\nabla}$ in the natural frame $\{x^i\}$. It is also a standard matter to derive the following Gauss–Codazzi equations on each slice:

$$\begin{aligned} R_{ij,kl} &= \bar{R}_{ij,kl} + K_{ik} k_{lj} - K_{il} K_{kj}, \\ R_{0i,jk} &= N (\bar{\nabla}_j K_{ki} - \bar{\nabla}_k K_{ji}), \\ R_{0i,0j} &= N (\bar{\partial}_0 K_{ij} + N K_{ik} K_j^k + \bar{\nabla}_i \partial_j N). \end{aligned} \quad (2.7)$$

Next, by suitable contractions of these identities, we arrive at

$$R_{ij} = \bar{R}_{ij} - \frac{\bar{\partial}_0 K_{ij}}{N} + K_{ij} K_l^l - 2K_{il} K_j^l - \frac{\bar{\nabla}_i \partial_j N}{N}, \quad (2.8a)$$

$$R_{0j} = N (\partial_j K_l^l - \bar{\nabla}_l K_j^l), \quad (2.8b)$$

$$G_{00} = \frac{N^2}{2} (\bar{R} - K_{ij} K^{ij} + (K_l^l)^2). \quad (2.8c)$$

These equations clarify the relations between geometric objects in M and in M_t , respectively.

The equation (2.8a) yields the evolution of the tensor K and together with the definition $\partial_0 \bar{g} = -2NK$, we thus find the evolution system

$$\begin{aligned} \bar{\partial}_0 K_{ij} &= N (\bar{R}_{ij} - R_{ij}) + N K_{ij} K_l^l - 2N K_{il} K_j^l - \bar{\nabla}_i \partial_j N, \\ \bar{\partial}_0 \bar{g}_{ij} &= -2N K_{ij}. \end{aligned} \quad (2.9)$$

On the other hand, the equations (2.8b) and (2.8c) will serve as constraint equations.

2.2 Evolution and constraint equations

Now, we are going to combine the equation (2.8) and the field equation (1.5) and obtain the fundamental equations of nonlinear gravity.

To deal with the system (1.5), we recall first some identities related to the Hessian form in a Cauchy adapted frame. Given any smooth function f defined on (M, g) , we can write

$$\nabla df = \nabla(\langle e_\gamma, f \rangle \theta^\gamma) = \langle e_\beta, \langle e_\alpha, f \rangle \rangle \theta^\beta \otimes \theta^\alpha - \langle e_\gamma, f \rangle \omega_{\beta\alpha}^\gamma \theta^\beta \otimes \theta^\alpha.$$

We need to calculate explicitly the following components:

$$\begin{aligned} \nabla_i \nabla_j f &= \partial_i \partial_j f - \partial_k f \omega_{ij}^\gamma = \partial_i \partial_j f - \bar{\Gamma}_{ij}^k \partial_k f - \omega_{ij}^0 \langle e_0, f \rangle \\ &= \bar{\nabla}_i \bar{\nabla}_j f + K_{ij} N^{-1} (\partial_t - \beta^l \partial_l) f \\ &= \bar{\nabla}_i \bar{\nabla}_j f + K_{ij} \mathcal{L}_n f, \end{aligned} \tag{2.10}$$

where \mathcal{L}_n is the Lie derivative associated to the normal unit vector of the slice M_t . Furthermore, we find

$$\begin{aligned} \nabla_j \nabla_0 f &= \nabla_0 \nabla_j f = \langle e_0, \partial_j f \rangle - \langle e_0, f \rangle \omega_{0j}^0 - \partial_i f \omega_{0j}^i \\ &= (\partial_t - \beta^i \partial_i) \partial_j f - (\partial_t - \beta^i \partial_i) f N^{-1} \partial_j N + N \partial_i f K_j^i - \partial_i f \partial_j \beta^i \\ &= (\partial_t - \beta^i \partial_i) \partial_j f - (\partial_t - \beta^i \partial_i) f \partial_j \ln N + N \partial_i f K_j^i - \partial_i f \partial_j \beta^i \\ &= \partial_j (\partial_t - \beta^i \partial_i) f - (\partial_t - \beta^i \partial_i) f \partial_j \ln N + N \partial_i f K_j^i \\ &= N \partial_j (N^{-1} (\partial_t - \beta^i \partial_i) f) + N K_j^i \partial_i f. \end{aligned}$$

The trace of the Hessian of a function, that is, the d'Alembert operator, is expressed in the Cauchy adapted frame as follows:

$$\begin{aligned} \square_g f &= g^{\alpha\beta} \nabla_\alpha \nabla_\beta f = -N^2 \nabla_0 \nabla_0 f + g^{ij} \nabla_i \nabla_j f \\ &= -N^2 \nabla_0 \nabla_0 f + \bar{g}^{ij} \bar{\nabla}_i \bar{\nabla}_j f + \bar{g}^{ij} K_{ij} N^{-1} (\partial_t - \beta^l \partial_l) f \\ &= -N^2 \nabla_0 \nabla_0 f + \Delta_{\bar{g}} f + \bar{g}^{ij} K_{ij} \mathcal{L}_n f, \end{aligned}$$

where $\Delta_{\bar{g}} f$ is the Laplace operator associated with the metric \bar{g} .

For a technical reason, we need rewrite the field equation (1.5) in a slightly different form. Define the tensor

$$E_{g_{\alpha\beta}} := N g_{\alpha\beta} - \frac{1}{2} \text{tr}(N_g) g_{\alpha\beta}, \tag{2.11}$$

where $\text{tr}(N_g)$ is the trace of the tensor N_g with respect to the metric g . Then

$$E_{g_{\alpha\beta}} = f'(R_g) R_{\alpha\beta} + \frac{1}{2} f'(R) W_1(R) g_{\alpha\beta} - \left(\frac{1}{2} g_{\alpha\beta} \square_g + \nabla_\alpha \nabla_\beta \right) f'(R_g),$$

where we have introduced the function

$$W_1(r) := \frac{f(r) - r f'(r)}{f'(r)}, \quad r \in \mathbb{R}. \tag{2.12}$$

Then by (1.5), E_g satisfies the following equation:

$$E_{g_{\alpha\beta}} = 8\pi (T_{\alpha\beta} - \frac{1}{2} \text{tr}(T) g_{\alpha\beta}). \tag{2.13}$$

By defining next by

$$H_{\alpha\beta} := (T_{\alpha\beta} - \frac{1}{2} \text{tr}(T) g_{\alpha\beta}), \tag{2.14}$$

the field equation can be written as

$$E_{g_{\alpha\beta}} = 8\pi H_{\alpha\beta}. \tag{2.15}$$

For different components, we will choose a different form of the equations, specifically:

$$\begin{aligned} E_{g_{ij}} &= 8\pi H_{ij}, \\ E_{g_{0j}} &= 8\pi H_{0j}, \\ N_{g_{00}} &= 8\pi T_{00}, \end{aligned} \tag{2.16}$$

Or equivalently:

$$f'(R_g)R_{ij} + \frac{1}{2}f'(R)W_1(R_g)g_{ij} - \left(\frac{1}{2}g_{ij}\square_g + \nabla_i\nabla_j\right)f'(R_g) = 8\pi(T_{ij} - \frac{1}{2}\text{tr}(T)g_{ij}), \tag{2.17a}$$

$$f'(R_g)R_{0j} - \nabla_0\nabla_j f'(R_g) = 8\pi T_{0j}, \tag{2.17b}$$

$$f'(R_g)G_{g_{00}} - \frac{1}{2}f'(R)W_1(R_g)g_{00} + (g_{00}\square_g - \nabla_0\nabla_0)(f'(R_g)) = 8\pi T_{00}. \tag{2.17c}$$

The following lemma shows that any solution of (1.5) is also a solution of (2.16) (and vice versa).

Lemma 2.1. *If $g_{\alpha\beta}$ and $T_{\alpha\beta}$ satisfy the equations (1.5), then they also satisfies (2.16). The converse also holds.*

Proof. By a simple calculation, (2.16) is found to be equivalent to the following three equations:

$$\begin{aligned} (N_g - 8\pi T)_{ij} &= -\frac{1}{2}g_{ij}\text{tr}(8\pi T - N_g), \\ (N_g - 8\pi T)_{ij} &= 0, \\ N_{g_{00}} - 8\pi T_{00} &= 0. \end{aligned}$$

Then by taking the trace of the tensor $(N_g - 8\pi T)$,

$$\text{tr}(N_g - 8\pi T) = -\frac{3}{2}\text{tr}(8\pi T - N_g)$$

which leads to

$$\text{tr}(8\pi T - N_g) = 0.$$

Then the desired result is proved. \square

From (2.17a) and by using (2.10), we arrive at the following form:

$$\begin{aligned} R_{ij} &= \frac{1}{f'(R_g)} \left(E_{g_{ij}} - \frac{1}{2}f'(R_g)W_1(R_g)g_{ij} + \frac{1}{2}(g_{ij}\square_g + 2\nabla_i\nabla_j)f'(R_g) \right) \\ &= \frac{E_{ij}}{f'(R_g)} - \frac{1}{2}W_1(R_g)\bar{g}_{ij} + \frac{(\bar{g}_{ij}\Delta_{\bar{g}} + 2\bar{\nabla}_i\bar{\nabla}_j)f'(R_g)}{2f'(R_g)} \\ &\quad - \frac{\bar{g}_{ij}\nabla_0\nabla_0 f'(R_g)}{2N^2 f'(R_g)} + (K_{ij} + \frac{1}{2}\bar{g}_{ij}K)\mathcal{L}_n \ln(f'(R_g)), \end{aligned} \tag{2.18a}$$

where

$$K := \bar{g}^{ij}K_{ij} \tag{2.18b}$$

is the trace of K with respect to \bar{g} . We also have

$$\begin{aligned} R_{0j} &= \frac{1}{f'(R_g)} \left(N_{g_{0j}} + \nabla_0\nabla_j f'(R_g) \right) \\ &= \frac{N_{g_{0j}}}{f'(R_g)} + \frac{N\partial_j(N^{-1}(\partial_t - \beta^i\partial_i)f'(R_g))}{f'(R_g)} + NK_j^i\partial_i(\ln(f'(R_g))) \\ &= \frac{N_{g_{0j}}}{f'(R_g)} + \frac{N\partial_j(f''(R_g)\mathcal{L}_n R_g)}{f'(R_g)} + NK_j^i\partial_i(\ln(f'(R_g))), \end{aligned} \tag{2.18c}$$

and

$$\begin{aligned}
G_{00} &= \frac{1}{f'(R_g)} \left(N_{g00} + \frac{1}{2} f'(R_g) W_1(R_g) g_{00} - (g_{00} \square_g - \nabla_0 \nabla_0) f'(R_g) \right) \\
&= \frac{N_{g00}}{f'(R_g)} + \frac{1}{2} g_{00} W_1(R_g) - \frac{g_{00}}{f'(R_g)} (\Delta_{\bar{g}} + \bar{g}^{ij} K_{ij} \mathcal{L}_n) f'(R_g).
\end{aligned} \tag{2.18d}$$

By combining (2.18a) with (2.8a), (2.18c) with (2.8b) and (2.18d) with (2.8c), the evolution equations and constraint equations for the system of nonlinear gravity are formulated as follows:

$$\begin{aligned}
\bar{\partial}_0 K_{ij} &= N \bar{R}_{ij} - N R_{ij} + N K_{ij} K_l^l - 2N K_{il} K_j^l - \bar{\nabla}_i \partial_j N \\
&= N \bar{R}_{ij} + N K_{ij} K_l^l - 2N K_{il} K_j^l - \bar{\nabla}_i \partial_j N \\
&\quad - \frac{N E_{ij}}{f'(R_g)} + \frac{N}{2} W_1(R_g) \bar{g}_{ij} - \frac{N (\bar{g}_{ij} \Delta_{\bar{g}} + 2 \bar{\nabla}_i \bar{\nabla}_j) f'(R_g)}{2 f'(R_g)} \\
&\quad + \frac{N \bar{g}_{ij} \nabla_0 \nabla_0 f'(R_g)}{2 N^2 f'(R_g)} - N \left(K_{ij} + \frac{1}{2} \bar{g}_{ij} K \right) \mathcal{L}_n \ln(f'(R_g)), \\
\bar{\partial}_0 \bar{g}_{ij} &= -2N K_{ij},
\end{aligned} \tag{2.19}$$

$$\bar{R} - K_{ij} K^{ij} + (K_j^j)^2 = \frac{2N_{g00}}{N^2 f'(R_g)} + \frac{2\Delta_{\bar{g}} f'(R_g)}{f'(R_g)} + 2\bar{g}^{ij} K_{ij} \mathcal{L}_n (\ln f'(R_g)) - W_1(R_g), \tag{2.20}$$

and

$$\partial_j K_i^i - \bar{\nabla}_i K_j^i = \frac{N_{g0j}}{N f'(R_g)} + \frac{\partial_j (f''(R_g) \mathcal{L}_n R_g)}{f'(R_g)} + K_j^i \partial_i (\ln(f'(R_g))). \tag{2.21}$$

Now, we analyze the coupling with the matter field and consider the stress energy tensor $T_{\alpha\beta}$. The equations read $N_{g\alpha\beta} = 8\pi T_{\alpha\beta}$, where $T_{\alpha\beta} = T_{\alpha\beta}$ for the Jordan coupling and $T_{\alpha\beta} = T_{\alpha\beta}^\dagger$ for the Einstein coupling. We also define the mass density σ and the momentum vector J (measured by an observer moving orthogonally to the slices) by the relations

$$\sigma := N^{-2} T_{00}, \quad J_j := -N^{-1} T_{0j}. \tag{2.22}$$

Recall first the constraint equations for the *classical* theory of general relativity, when the Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ are imposed. Therefore, the last two equations in (2.8) yield

$$\begin{aligned}
\bar{R} + K_{ij} K^{ij} - (K_i^i)^2 &= 16\pi\sigma, \\
\bar{\nabla}^i K_{ij} - \bar{\nabla}_j K_i^l &= 8\pi J_j,
\end{aligned} \quad (\text{when } f' \equiv 0). \tag{2.23}$$

In the same way, we introduce a definition suitable for nonlinear gravity.

Proposition and Definition 2.2. *The equations for nonlinear gravity in a Cauchy adapted frame $\{e_0, e_1, e_2, e_3\}$ decompose into:*

1. Evolution equations:

$$\begin{aligned}
\bar{\partial}_0 K_{ij} &= N \bar{R}_{ij} + N K_{ij} K_l^l - 2N K_{il} K_j^l - \bar{\nabla}_i \partial_j N \\
&\quad - \frac{8\pi N (T_{ij} - \frac{1}{2} g_{ij} \text{tr}(T))}{f'(R_g)} + \frac{N}{2} W_1(R_g) \bar{g}_{ij} - \frac{N (\bar{g}_{ij} \Delta_{\bar{g}} + 2 \bar{\nabla}_i \bar{\nabla}_j) f'(R_g)}{2 f'(R_g)} \\
&\quad + \frac{N \bar{g}_{ij} \nabla_0 \nabla_0 f'(R_g)}{2 N^2 f'(R_g)} - N \left(K_{ij} + \frac{1}{2} \bar{g}_{ij} K \right) \mathcal{L}_n \ln(f'(R_g)), \\
\bar{\partial}_0 \bar{g}_{ij} &= -2N K_{ij}.
\end{aligned} \tag{2.24}$$

2. Hamiltonian constraint equations:

$$\bar{R} - K_{ij} K^{ij} + (K_j^j)^2 = \frac{16\pi\sigma}{f'(R_g)} + \frac{2\Delta_{\bar{g}} f'(R_g)}{f'(R_g)} + 2\bar{g}^{ij} K_{ij} \mathcal{L}_n (\ln f'(R_g)) - W_1(R_g), \tag{2.25}$$

3. Momentum constraint equations:

$$\partial_j K_i^i - \bar{\nabla}_i K_j^i = -\frac{8\pi J_i}{f'(R_g)} + \frac{\partial_j(f''(R_g)\mathcal{L}_n R_g)}{f'(R_g)} + K_j^i \partial_i(\ln(f'(R_g))). \quad (2.26)$$

Observe that, in the classical gravity theory, the factor $f(R)$ coincides with R and $f'(R_g) \equiv 1$, so that the terms consulting $f'(R_g)$ in the right-hand sides of the constraint equations (2.25) and (2.26) vanish identically and we recover the standard equations (2.23).

We observe that these constraint equations are highly involved compared with the classical case. The main difficulty is that they consult the four-order derivatives of the metric \bar{g} in the form of the second order derivatives of the scalar curvature R_g . We also remark that these four-order derivatives destroy the elliptic structure of the classical constraint equations.

2.3 The divergence identity

As in the classical case, the evolution equation of the matter field is deduced by the relation $\nabla^\alpha T_{\alpha\beta} = 0$, which should also be guaranteed by (1.5).

Lemma 2.3 (The divergence identity in nonlinear gravity). *The contracted Bianchi identities*

$$\nabla^\alpha R_{\alpha\beta} = \frac{1}{2} \nabla_\beta R \quad (2.27)$$

imply

$$\nabla^\alpha N_{g_{\alpha\beta}} = 0. \quad (2.28)$$

Proof. The following calculation is made in a abstract (local) natural frame. Indeed, this is checked as follows by computing the three relevant terms:

$$\begin{aligned} \nabla^\alpha(\nabla_\alpha \nabla_\beta f'(R) - g_{\alpha\beta} \square_g f'(R)) &= (\nabla^\alpha \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla^\lambda \nabla_\lambda) f'(R) \\ &= (\nabla_\alpha \nabla_\beta \nabla^\alpha - \nabla_\beta \nabla_\alpha \nabla^\alpha)(f'(R)) \\ &= [\nabla_\alpha, \nabla_\beta](\nabla^\alpha(f'(R))) = R_{\alpha\beta} \nabla^\alpha(f'(R)), \end{aligned}$$

then

$$\begin{aligned} \nabla^\alpha(f'(R) R_{\alpha\beta}) &= R_{\alpha\beta} \nabla^\alpha(f'(R)) + f'(R) \nabla^\alpha R_{\alpha\beta} \\ &= R_{\alpha\beta} \nabla^\alpha(f'(R)) + \frac{1}{2} f'(R) \nabla_\beta R, \end{aligned}$$

and finally

$$\nabla^\alpha \left(\frac{1}{2} f(R) g_{\alpha\beta} \right) = \frac{1}{2} \nabla_\beta(f(R)) = \frac{1}{2} f'(R) \nabla_\beta R$$

Combine these three identities together yields the desired result. \square

As a first application of (2.27), we can now determine which coupling (in the Jordan metric under consideration in this section) is mathematically sound.

If we assume that

$$T_{\alpha\beta} = T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \phi,$$

then, by the field equation $N_{g_{\alpha\beta}} = 8\pi T_{\alpha\beta}$ and (2.27), we find

$$\nabla^{\alpha\beta} T_{\alpha\beta} = 0$$

and thus

$$\partial_\beta \phi \square_g \phi = 0.$$

Consequently, if ϕ satisfies the following wave equation on M

$$\square_g \phi = 0, \quad (2.29)$$

then the tensor $T_{\alpha\beta}$ is automatically divergence-free. We thus have a single scalar equation for the evolution of the matter field ϕ .

Next, if we assume the Einstein coupling

$$T_{\alpha\beta} = T_{\alpha\beta}^\dagger = f'(R_g)(\partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \phi),$$

then the field equation $N_{g_{\alpha\beta}} = T_{\alpha\beta}^\dagger$ together with (2.27) lead us to

$$\nabla^\alpha T_{\alpha\beta}^\dagger = 0,$$

which yields

$$f''(R_g) T_{\alpha\beta} \nabla^\alpha R_g + f'(R_g) \partial_\beta \phi \square_g \phi = 0.$$

This (vectorial) equation can be written as

$$\partial_\beta \phi \square_g \phi = \frac{f''(R_g)}{f'(R_g)} (\partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \phi) \nabla^\alpha R_g$$

or equivalently

$$\left(\frac{f''(R_g)}{f'(R_g)} \partial_\alpha \phi \nabla^\alpha R - \square_g \phi \right) \nabla_\beta \phi = \frac{f''(R_g)}{2f'(R_g)} \left(g^{\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \phi \right) \nabla_\beta R. \quad (2.30)$$

For general initial data, this always leads to an over-determined differential system (since only one scalar field should be determined) and this strongly suggests that the Einstein coupling is not compatible (unless we make the additional assumption that $\nabla\phi$ and ∇R are colinear).

From now on, we thus focus our attention on the Jordan coupling.

2.4 The initial value problem for nonlinear gravity

We now formulate the Cauchy problem for the system (1.5). To do so, we specify the expression of the stress-energy tensor $T_{\alpha\beta}$. According to our earlier discussion, we assume the Jordan coupling:

$$T_{\alpha\beta} = T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \phi \quad (2.31)$$

and we can then compute the corresponding matter fields

$$\begin{aligned} \sigma &= N^{-2} T_{00} = |\mathcal{L}_n \phi|^2 + \frac{1}{2} |\nabla \phi|_g^2 = \frac{1}{2} (|\mathcal{L}_n \phi|^2 + \bar{g}^{ij} \partial_i \phi \partial_j \phi), \\ J_i &= -\mathcal{L}_n \phi \partial_i \phi. \end{aligned} \quad (2.32)$$

Definition 2.4. *An initial data set for the nonlinear gravity theory $(\bar{M}, \bar{g}, K, R_0, R_1, \phi_0, \phi_1)$ consists of the following data:*

- a 3-dimensional manifold \bar{M} endowed with a Riemannian metric \bar{g} and a symmetric $(0, 2)$ -tensor field K ,
- two scalar fields denoted by R_0 and R_1 defined on \bar{M} and representing the (to-be-constructed) spacetime curvature and its time derivative,
- two scalar fields ϕ_0 and ϕ_1 defined on \bar{M} ,

- which are assumed to satisfy the **Hamiltonian constraint of nonlinear gravity**

$$\begin{aligned} \bar{R} - K_{ij}K^{ij} + (K_j^j)^2 &= \frac{8(\phi_1^2 + \bar{g}^{ij}\partial_i\phi_0\partial_j\phi_0)}{f'(R_0)} + \frac{2\Delta_{\bar{g}}f'(R_0)}{f'(R_0)} \\ &+ 2\bar{g}^{ij}K_{ij}\mathcal{L}_n\frac{f''(R_0)R_1}{f'(R_0)} - W_1(R_0), \end{aligned} \quad (2.33)$$

and the **momentum constraint of nonlinear gravity**

$$\partial_j K_i^i - \bar{\nabla}_i K_j^i = \frac{8\pi\phi_1\partial_i\phi_0}{f'(R_0)} + \frac{\partial_j(f''(R_0)R_1)}{f'(R_0)} + K_j^i\partial_i(\ln(f'(R_0))). \quad (2.34)$$

Definition 2.5 (Formulation of the initial value problem for nonlinear gravity). *Given an initial data set $(\bar{M}, \bar{g}, K, R_0, R_1, \phi_0, \phi_1)$ as in Definition 2.4, the initial value problem associated with the nonlinear gravity theory consists of finding a Lorentzian manifold (M, g) and a matter field ϕ defined on M*

1. The field equations of nonlinear gravity (1.5) are satisfied,
2. There exists an embedding $i: \bar{M} \rightarrow M$ with pull back metric $\bar{g} = i^*g$ and second fundamental form K .
3. The field R_0 coincides with the restriction of the spacetime scalar curvature R on \bar{M} , while R_1 coincides with the Lie derivative $\mathcal{L}_n R$ restricted to \bar{M} , where n denotes the normal to \bar{M} .
4. The scalar fields ϕ_0, ϕ_1 coincides with the restriction of $\phi, \mathcal{L}_n\phi$ on \bar{M} .

Such a solution to (1.5) is referred to as a **nonlinear gravity development of the initial data set** $(\bar{M}, \bar{g}, K, R_0, R_1, \phi_0, \phi_1)$.

Similarly as in classical gravity, we can define the notion of *maximally hyperbolic nonlinear gravity development of the initial data set*.

We first observe that the initial value problem for nonlinear gravity reduces to the classical formulation in the special case of vanishing geometric data $\phi_0 = \phi_1 = R_0 = R_1 \equiv 0$. Similarly as in classical gravity, these fields can not be fully arbitrary prescribed but certain constraints (given above) must be assumed.

2.5 Preservation of the constraints

We now address the following issue: Suppose that the evolution equations are satisfied by a pair of symmetric two-tensors (\bar{g}, K) in the spacetime and the stress-energy tensor T is divergence-free, and suppose that if the constraint equations are satisfied on the initial slice M_0 , then we will prove that the field equations hold on the spacetime – that is the preservation of the constraint equations.

This is the first example of the “preservation-type” results. We will see other examples in the following discussion. The common character of these results are: in a differential system the equations are classified into two categories: one is easier to handle (the evolution equations in this example) while the other is difficult (the constraint equations). Our strategy is to replace the difficult equations by some equations which can be deduced from the original system (in this example, the trace-free equation of T), and which are also easier to handle. This forms a new system and our task is to prove the equivalence between the original system and this newly formed system.

Before the statement of the main result, we remark the following observations. The evolution equations (2.24) are equivalent to (2.17a) and the constraint equations (2.25) and (2.26) are equivalent to (2.17b) and (2.17c). So we suppose that (2.17a) together with the divergence condition

$\nabla^\alpha T_{\alpha\beta} = 0$ are satisfied in the spacetime $M = \cup_{t \in [0, t_{\max})} M_t$, and the constraint equations (2.17b) and (2.17c) are satisfied on the initial slice. Then we will prove that the equations (2.17b) and (2.17c) are satisfied in the whole spacetime. More precisely, the following proposition holds:

Proposition 2.6. *With the previous notation, suppose that equations*

$$E_{g_{ij}} - 8\pi H_{ij} = 0 \quad (2.35)$$

and

$$\nabla^\alpha T_{\alpha\beta} = 0 \quad (2.36)$$

hold in $\cup_{t \in [0, t_{\max})} M_t$ and

$$E_{g_{0j}} = 8\pi H_{0j}, \quad N_{g_{00}} = 8\pi T_{00} \quad (2.37)$$

hold on the initial slice $M_0 = \{t = 0\}$. Then

$$E_{g_{0j}} = 8\pi H_{0j}, \quad N_{g_{00}} = 8\pi T_{00} \quad \text{in the spacetime } \bigcup_{t \in [0, t_{\max})} M_t. \quad (2.38)$$

Proof. The calculations are made in a Cauchy adapted frame. For convenience in the proof, we introduce the notation

$$\Sigma_{\alpha\beta} := N_{g_{\alpha\beta}} - 8\pi T_{\alpha\beta},$$

and we will prove that $\sigma_{\alpha\beta} = 0$ which leads to the desired result.

By the hypothesis $\nabla_\alpha T_{\alpha\beta} = 0$ and the identity (2.27),

$$\nabla^\alpha \Sigma_{\alpha\beta} = 0. \quad (2.39)$$

Recall the definition of $E_{g_{\alpha\beta}}$ and $H_{\alpha\beta}$, the following relation holds:

$$\Sigma_{\alpha\beta} = E_{g_{\alpha\beta}} - H_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \text{tr}(E_g - H). \quad (2.40)$$

In particular

$$\Sigma_{00} = E_{g_{00}} - H_{00} - \frac{1}{2} g_{00} \text{tr}(E_g - H) \quad (2.41)$$

Now by considering the fact that $g^{0i} = g_{0i} = 0$ and (2.35),

$$\text{tr}(E_g - H) = g^{00}(E_{g_{00}} - H_{00}). \quad (2.42)$$

Combine (2.41) and (2.42),

$$\text{tr}(E_g - H) = 2g^{00}\Sigma_{00}. \quad (2.43)$$

Substitute this equation into (2.40),

$$\Sigma_{\alpha\beta} = E_{g_{\alpha\beta}} - H_{\alpha\beta} - g_{\alpha\beta} g^{00} \Sigma_{00}. \quad (2.44)$$

where we can calculate more precisely the spacial components with the aid of (2.35)

$$\Sigma_{ij} = -g_{ij} g^{00} \Sigma_{00}. \quad (2.45)$$

Now with these preparations we calculate

$$\Sigma_\beta^\alpha = g^{\alpha\alpha'} \Sigma_{\alpha'\beta}.$$

When $\alpha = \beta = 0$,

$$\Sigma_0^0 = \Sigma_{0\alpha} g^{\alpha 0} = g^{00} \Sigma_{00}. \quad (2.46)$$

When $\beta = 0, 1 \leq \alpha \leq 3$. We denote by $a = \alpha$. Then recall that $g^{a0} = 0$,

$$\Sigma_0^a = g^{a\alpha'} \Sigma_{\alpha'0} = g^{a\alpha'} \Sigma_{\alpha'0}$$

For $1 \leq b \leq 3$

$$\Sigma_b^0 = g_{b\beta} g^{0\alpha} \Sigma_\alpha^\beta = g^{00} g_{bc} \Sigma_0^c.$$

For $1 \leq b \leq 3$ and $1 \leq a \leq 3$, by applying (2.45) and (2.46), then

$$\Sigma_b^a = g^{a\alpha} \Sigma_{b\alpha} = g^{aa'} \Sigma_{ba'} = -g^{aa'} g_{ba'} g^{00} \Sigma_{00} = -\delta_b^a \Sigma_0^0.$$

So we conclude by

$$\Sigma_b^0 = g^{00} g_{bc} \Sigma_0^c, \quad \Sigma_b^a = -\delta_b^a \Sigma_0^0. \quad (2.47)$$

Now recall that the identity (2.39) can be written as

$$\nabla_\alpha \Sigma_\beta^\alpha = 0$$

which leads to

$$\langle e_\alpha, \Sigma_\beta^\alpha \rangle - \omega_{\alpha\beta}^\delta \Sigma_\delta^\alpha + \omega_{\alpha\delta}^\alpha \Sigma_\beta^\delta = 0.$$

When $\beta = 0$,

$$\langle e_0, \Sigma_0^0 \rangle + \partial_a \Sigma_0^a - \omega_{\alpha\beta}^\delta \Sigma_\delta^\alpha + \omega_{\alpha\delta}^\alpha \Sigma_\beta^\delta = 0. \quad (2.48)$$

For $1 \leq b \leq 3$, taking the equation (2.47),

$$\langle e_0, g^{00} g_{bc} \Sigma_0^c \rangle + \partial_a (-\delta_b^a \Sigma_0^0) - \omega_{\alpha b}^\gamma \Sigma_\gamma^\alpha + \omega_{\alpha\gamma}^\alpha \Sigma_b^\gamma = 0$$

which leads to

$$\langle e_0, \Sigma_0^c \rangle - g_{00} \bar{g}^{bc} \partial_b \Sigma_0^0 - g_{00} \bar{g}^{bc} (\omega_{\alpha b}^\gamma \Sigma_\gamma^\alpha - \omega_{\alpha\gamma}^\alpha \Sigma_b^\gamma) + g_{00} \bar{g}^{bc} \langle e_0, g^{00} g_{bc} \rangle \Sigma_0^c = 0. \quad (2.49)$$

Consider (2.48) and (2.49) together, we remark that by (2.47), the lower order terms are linear combinations Σ_0^α with $0 \leq \alpha \leq 3$. So these equations forms a first-order differential system.

Furthermore we remark that this system is symmetrizable. Recalling that we denote by $\bar{g} = \bar{g}_t$ the induced Riemannian metric on the slices M_t , we introduce the following notation:

$$V := (\Sigma_0^0, \Sigma_0^a)^T, \quad \rho^a := g_{00} (\bar{g}^{a1}, \bar{g}^{a2}, \bar{g}^{a3})^T,$$

$$\sigma_1 = (1, 0, 0), \quad \sigma_2 = (0, 1, 0), \quad \sigma_3 = (0, 0, 1).$$

Then the principal part of the system formed by (2.48) and (2.49) can be written as

$$\langle e_0, V \rangle + \sum_a A^a \partial_a V = F \quad (2.50)$$

where

$$A^a = \begin{pmatrix} 0 & \sigma_a \\ -g_{00} \rho^a & 0 \end{pmatrix}$$

and F a linear form on V . Then by multiply this equation by the matrix A_0 defined as following:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -g^{00} g_{ab} \end{pmatrix} = -g_{00} \begin{pmatrix} -g_{00} & 0 \\ 0 & g_{ab} \end{pmatrix}.$$

Then the system (2.50) becomes

$$A_0 \langle e_0, V \rangle + \sum_a A_0 A^a \partial_a V = A_0 F$$

and remark that $A_0 A^a$ are symmetric:

$$A_0 A^a = \begin{pmatrix} 0 & \sigma^a \\ (\sigma^a)^T & 0 \end{pmatrix}.$$

Then the system (2.50) is symmetrizable. We also remark that by (2.37) implies that $V = 0$ on the initial slice $\{t = 0\}$. Then, thanks to our global hyperbolicity assumption (2.1) and by a standard uniqueness theorem, we conclude that $V = 0$ in the whole spacetime. \square

3 Formulation of the Cauchy problem in the Einstein metric

3.1 Conformal transformation

From the last section we observe that both evolution equations and constraint equations for nonlinear gravity are highly involved and difficult to be treated with standard PDE's techniques. Then main difficulty comes from the forth-order term $\nabla_\alpha \nabla_\beta f'(R_g)$ in (1.5). To overcome this difficulty, we adopt the following conformal transformation:

$$g^\dagger_{\alpha\beta} := e^{2\rho} g_{\alpha\beta}, \quad g^{\dagger\alpha\beta} = e^{-2\rho} g^{\alpha\beta}, \quad (3.1)$$

where the conformal factor is defined by

$$\rho := \frac{1}{2} \ln f'(R_g) \quad \text{or equivalently} \quad f'(R_g) = e^{2\rho}. \quad (3.2)$$

We have the following expression of the tensor N_g in the Einstein metric.

Lemma 3.1. *With the notation above, the following identities hold:*

$$e^{2\rho} R^\dagger_{\alpha\beta} - 6e^{2\rho} \partial_\alpha \rho \partial_\beta \rho + \frac{g^\dagger_{\alpha\beta}}{2} W_2(\rho) = N_{g_{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} \text{tr}(N_g), \quad (3.3)$$

where the function $W_2 = W_2(\rho)$ is defined implicitly by the relation

$$W_2(s) = \frac{f(r) - f'(r)r}{f'(r)}, \quad e^{2s} = f'(r). \quad (3.4)$$

where W_1 is defined by (2.12). We also define the function

$$W_3(s) := f(r), \quad e^{2s} = f'(r)$$

which will be used in the proof.

Proof. Recall the tensor

$$N_{g_{\alpha\beta}} = f'(R_g) R_{\alpha\beta} - \frac{1}{2} f(R_g) g_{\alpha\beta} + (g_{\alpha\beta} \square_g - \nabla_\alpha \nabla_\beta) f'(R_g)$$

and the trace

$$\text{tr}(N_g) = f'(R_g) R_g - 2f(R_g) + 3\square_g f'(R_g).$$

We remark the following identities:

$$\begin{aligned} \nabla_\alpha \nabla_\beta e^{2\rho} &= 2e^{2\rho} \nabla_\alpha \nabla_\beta \rho + 4e^{2\rho} \nabla_\alpha \rho \nabla_\beta \rho, \\ \square_g e^{2\rho} &= 2e^{2\rho} \square_g \rho + 4e^{2\rho} g(\nabla \rho, \nabla \rho). \end{aligned}$$

With these identities, we get

$$N_{g_{\alpha\beta}} = e^{2\rho} R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} W_3(\rho) + 2e^{2\rho} (g_{\alpha\beta} \square_g - \nabla_\alpha \nabla_\beta) \rho + 4e^{2\rho} (g_{\alpha\beta} g(\partial \rho, \partial \rho) - \partial_\alpha \rho \partial_\beta \rho)$$

and

$$\square_g \rho - \frac{W_3(\rho)}{6e^{2\rho}} + 2g(\partial \rho, \partial \rho) - \frac{1}{6} W_2(\rho) = \frac{\text{tr}(N_g)}{6e^{2\rho}}.$$

In general, the following relation hold between the Ricci curvature tensor of g and g^\dagger :

$$R^\dagger_{\alpha\beta} = R_{\alpha\beta} - 2(\nabla_\alpha \nabla_\beta \rho - \nabla_\alpha \rho \nabla_\beta \rho) - (\square_g \rho + 2g(\nabla \rho, \nabla \rho)) g_{\alpha\beta}. \quad (3.5)$$

By (3.5), N_g can be expressed as following:

$$N_{g_{\alpha\beta}} = e^{2\rho} R^\dagger_{\alpha\beta} - 6e^{2\rho} \nabla_\alpha \rho \nabla_\beta \rho + 6e^{2\rho} g_{\alpha\beta} g(\nabla \rho, \nabla \rho) + 3e^{2\rho} g_{\alpha\beta} \square_g \rho - \frac{1}{2} W_3(\rho) g_{\alpha\beta}.$$

Combined with the trace equation, the desired result is proved. \square

Now the field equation in conformal metric is formulated as following.

Proposition and Definition 3.2. *The field equations of nonlinear gravity in the Einstein metric $g^\dagger_{\alpha\beta} = e^{2\rho}g_{\alpha\beta}$ with $\rho = \frac{1}{2}\ln(f'(R_g))$ read as follows:*

$$e^{2\rho}R^\dagger_{\alpha\beta} - 6e^{2\rho}\partial_\alpha\rho\partial_\beta\rho + \frac{g^\dagger_{\alpha\beta}}{2}W_2(\rho) = 8\pi(T_{\alpha\beta} - \frac{1}{2}g^\dagger_{\alpha\beta}g^{\dagger\alpha'\beta'}T_{\alpha'\beta'}). \quad (3.6)$$

At this juncture, it is unclear how the scalar field ρ should be recovered in term of the Einstein metric. We will derive below an evolution equation on ρ .

Remark 3.3. *For later use, we also observe here that for an arbitrary C^2 function w ,*

$$\square_{g^\dagger}w = e^{-2\rho}(\square_g w + 2g^{\alpha\beta}\partial_\alpha\rho\partial_\beta w) = e^{-2\rho}\square_g w + 2g^{\dagger\alpha\beta}\partial_\alpha\rho\partial_\beta w. \quad (3.7)$$

Then the trace equation is transformed into

$$\square_{g^\dagger}\rho = \frac{W_2(\rho)}{6e^{2\rho}} + \frac{W_3(\rho)}{6e^{4\rho}} + \frac{1}{6e^{4\rho}}\text{tr}(N_g). \quad (3.8)$$

3.2 Evolution and constraint equations in the conformal metric

As in the last section, we can formulate the evolution equations and constraint equations with respect to the transformed field equation (3.3). To do so, we introduce as before a foliation of spacetime $M = [0, +\infty) \times M_t$ and a Cauchy adapted frame $\{e_0, e_1, e_2, e_3\}$ with respect to the transformed metric g^\dagger . Then by exactly the same calculation, we find

$$R^\dagger_{ij} = \bar{R}_{ij} - \frac{\bar{\partial}_0 K^\dagger_{ij}}{N} + K^\dagger_{ij}K^{\dagger l}_l - 2K^\dagger_{il}K^{\dagger l}_j - \frac{\bar{\nabla}^\dagger_i \partial_j N^\dagger}{N^\dagger}, \quad (3.9a)$$

$$R^\dagger_{0j} = N^\dagger(\partial_j K^{\dagger l}_l - \bar{\nabla}^\dagger_l K^{\dagger l}_j), \quad (3.9b)$$

$$G^\dagger_{00} = \frac{N^{\dagger 2}}{2}(\bar{R}^\dagger - K^\dagger_{ij}K^{\dagger ij} + (K^{\dagger l}_l)^2). \quad (3.9c)$$

Here $\bar{\nabla}^\dagger$ refers to the covariant derivative on the slice M_t with respect to g^\dagger . Then (3.9a) represents the evolution equations

$$\begin{aligned} \bar{\partial}_0 K^\dagger_{ij} &= N^\dagger(\bar{R}^\dagger_{ij} - R^\dagger_{ij}) + N^\dagger K^\dagger_{ij}K^{\dagger l}_l - 2N^\dagger K^\dagger_{il}K^{\dagger l}_j - \bar{\nabla}^\dagger_i \partial_j N^\dagger, \\ \bar{\partial}_0 \bar{g}^\dagger_{ij} &= -2N^\dagger K^\dagger_{ij}. \end{aligned} \quad (3.10)$$

The transformed field equations (3.3) can be written as following:

$$R^\dagger_{\alpha\beta} = e^{-2\rho}(N_{g_{\alpha\beta}} - \frac{1}{2}g_{\alpha\beta}\text{tr}(N_g)) + 6\partial_\alpha\rho\partial_\beta\rho - \frac{1}{2e^{2\rho}}g^\dagger_{\alpha\beta}W_2(\rho).$$

By taking the trace of this equation **with respect to the metric g^\dagger** ,

$$R^\dagger = -e^{-4\rho}\text{tr}(N_g) + 6g^\dagger(\partial\rho, \partial\rho)g_{\alpha\beta} + e^{-2\rho}g^\dagger_{\alpha\beta}W_2(\rho)$$

which leads to

$$G^\dagger_{\alpha\beta} = N_{g_{\alpha\beta}} + 6\nabla_\alpha\varrho\nabla_\beta\varrho - 3g^\dagger(\partial\rho, \partial\rho)g^\dagger_{\alpha\beta} + e^{-2\rho}g^\dagger_{\alpha\beta}W_2(\rho),$$

So the evolution equations and constraint equations are found: The evolution equations are

$$\begin{aligned} \bar{\partial}_0 K^\dagger_{ij} &= N^\dagger \bar{R}^\dagger_{ij} + N^\dagger K^\dagger_{ij}K^{\dagger l}_l - 2N^\dagger K^\dagger_{il}K^{\dagger l}_j - \bar{\nabla}^\dagger_i \partial_j N^\dagger \\ &\quad - N^\dagger e^{-2\rho} \left(N_{g_{ij}} - \frac{1}{2e^{2\rho}}g^\dagger_{ij}\text{tr}(N_g) + 6e^{2\rho}\partial_i\rho\partial_j\rho + \frac{1}{2}g^\dagger_{ij}W_2(R_g) \right) \\ \bar{\partial}_0 \bar{g}^\dagger_{ij} &= -2N^\dagger K^\dagger_{ij}, \end{aligned}$$

while the Hamilton constraint equation is

$$\bar{R}^\dagger - K^\dagger_{ij}K^{\dagger ij} + (K^\dagger_l)^2 = \frac{2N_{g00}}{e^{2\rho}N^\dagger} + 12|\mathcal{L}_{n^\dagger}\rho|^2 + 6g^\dagger(\nabla^\dagger\rho, \nabla^\dagger\rho) - e^{-2\rho}W_2(\rho)$$

and the momentum constraint equations read

$$\partial_j K^\dagger_l{}^l - \bar{\nabla}^\dagger K^\dagger_j{}^l = \frac{N_{g0j}}{e^{2\rho}N^\dagger} + 6\mathcal{L}_{n^\dagger}\rho \partial_j\rho.$$

Here n^\dagger denotes the normal unit vector of the slice M_t .

We consider again the Jordan coupling with matter field (the reason of this choice of coupling is explained in the next subsection):

$$N_{g_{\alpha\beta}} = 8\pi T_{\alpha\beta}.$$

Furthermore, define

$$J^\dagger_j = -\frac{T_{0j}}{N^\dagger}, \quad \sigma^\dagger = \frac{T_{00}}{N^\dagger{}^2}.$$

Then the evolution and constraint equations with conformal metric are formulated as following.

Definition 3.4. *In the Einstein metric, the equations of nonlinear gravity in a Cauchy adapted frame $\{e_0, e_1, e_2, e_3\}$ can be decomposed as follows:*

1. Evolution equations:

$$\begin{aligned} \bar{\partial}_0 K^\dagger_{ij} &= N^\dagger \bar{R}^\dagger_{ij} + N^\dagger K^\dagger_{ij} K^\dagger_l{}^l - 2N^\dagger K^\dagger_{il} K^\dagger_j{}^l - \bar{\nabla}^\dagger_i \partial_j N^\dagger \\ &\quad - N^\dagger e^{-2\rho} \left(8\pi T_{ij} - 4\pi e^{-2\rho} g^\dagger_{ij} \text{tr}(T) + 6e^{2\rho} \partial_i \rho \partial_j \rho + \frac{1}{2} g^\dagger_{ij} W_1(R_g) \right), \\ \bar{\partial}_0 \bar{g}^\dagger_{ij} &= -2N^\dagger K^\dagger_{ij}. \end{aligned}$$

2. Hamiltonian constraint equation:

$$\bar{R}^\dagger - K^\dagger_{ij}K^{\dagger ij} + (K^\dagger_l)^2 = \frac{16\sigma^\dagger}{e^{2\rho}} + 12|\mathcal{L}_{n^\dagger}\rho|^2 + 6g^\dagger(\nabla^\dagger\rho, \nabla^\dagger\rho) - e^{-2\rho}W_2(\rho) \quad (3.11)$$

3. Momentum constraint equations:

$$\partial_j K^\dagger_l{}^l - \bar{\nabla}^\dagger K^\dagger_j{}^l = -\frac{J^\dagger_j}{e^{2\rho}} + 6\mathcal{L}_{n^\dagger}\rho \partial_j\rho. \quad (3.12)$$

Remark 3.5. *We emphasize again that the constraint equations are equivalent to*

$$\begin{aligned} N_{g_{00}} &= 8\pi T_{00}, \\ N_{g_{0a}} &= 8\pi T_{0a}. \end{aligned}$$

The evolution equations are equivalent to

$$N_{g_{ab}} - \frac{1}{2}g_{ab}\text{tr}(N_g) = 8\pi(T_{ab} - \frac{1}{2}g_{ab}\text{tr}(T)).$$

3.3 The divergence identity

As before, in order to get the evolution equation of the matter field, we need to calculate the divergence of the tensor N_g with respect to the conformal metric g^\dagger . This is concluded by the following lemma:

Lemma 3.6.

$$\nabla^{\dagger\alpha} N_{g_{\alpha\beta}} = e^{-2\rho} (2g^{\gamma\delta} \partial_\gamma \rho N_{g_{\delta\beta}} - \text{tr}(N_g) \partial_\beta \rho). \quad (3.13)$$

Proof. The following calculations are made in a abstract (local) natural frame. This identity comes from the relation

$$\Gamma^{\dagger\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \delta^{\gamma}_{\alpha}\partial_{\beta}\rho + \delta^{\gamma}_{\beta}\partial_{\alpha}\rho - g_{\alpha\beta}\nabla^{\gamma}\rho.$$

Then

$$\begin{aligned} & \nabla^{\dagger\alpha}N_{g_{\alpha\beta}} \\ &= e^{-2\rho}g^{\alpha\gamma}\nabla^{\dagger\gamma}N_{g_{\alpha\beta}} \\ &= e^{-2\rho}g^{\gamma\alpha}(\partial_{\gamma}N_{g_{\alpha\beta}} - \Gamma^{\delta}_{\gamma\alpha}N_{g_{\beta\delta}} - \Gamma^{\delta}_{\gamma\beta}N_{g_{\alpha\delta}}) \\ &\quad - e^{-2\rho}g^{\alpha\gamma}(\delta^{\delta}_{\gamma}\partial_{\alpha}\rho + \delta^{\delta}_{\alpha}\partial_{\gamma}\rho - g_{\gamma\alpha}\nabla^{\delta}\rho)N_{g_{\beta\delta}} - e^{-2\rho}g^{\alpha\gamma}(\delta^{\delta}_{\gamma}\partial_{\beta}\rho + \delta^{\delta}_{\beta}\partial_{\gamma}\rho - g_{\gamma\beta}\nabla^{\delta}\rho)N_{g_{\alpha\delta}} \\ &= e^{-2\rho}\nabla^{\alpha}N_{g_{\alpha\beta}} \\ &\quad - e^{-2\rho}(\nabla^{\delta}\rho + \nabla^{\delta}\rho - 4\nabla^{\delta}\rho)N_{g_{\beta\delta}} - e^{-2\rho}(\partial_{\beta}\rho\text{tr}(N_g) + \nabla^{\alpha}\rho N_{g_{\alpha\beta}} - \nabla^{\alpha}\rho N_{g_{\alpha\beta}}) \end{aligned}$$

Remark that $\nabla^{\alpha}N_{g_{\alpha\beta}} = 0$ by lemma 2.3, then

$$\nabla^{\dagger\alpha}N_{g_{\alpha\beta}} = e^{-2\rho}(2g^{\gamma\delta}\partial_{\gamma}\rho N_{g_{\delta\beta}} - \text{tr}(N_g)\partial_{\beta}\rho).$$

□

With the Jordan coupling, the divergence of the stress-energy tensor is expressed by

$$\nabla^{\dagger\alpha}T_{\alpha\beta} = (2g^{\delta\gamma}\partial_{\delta}\rho T_{\gamma\beta} - \text{tr}(T)\partial_{\beta}\rho)e^{-2\rho}$$

which (together with the state equations of the matter field coupled) will supply the evolution equation of the matter field.

Here parallel to the case of Jordan metric, we will determine the choice of coupling in the Einstein metric with the aid of (3.13). Surprisingly, we will see that in Einstein metric, the reasonable choice is again the Jordan coupling.

We remark the following calculation. When

$$T_{\alpha\beta} = T_{\alpha\beta} = \partial_{\alpha}\phi\partial_{\beta}\phi - \frac{1}{2}g_{\alpha\beta}|\nabla\phi|_g^2,$$

which is the case of Jordaon coupling,

$$\begin{aligned} \partial_{\beta}\rho\text{tr}(T) - 2g^{\delta\gamma}\partial_{\delta}\rho T_{\gamma\beta} &= -|\nabla\phi|_g^2\partial_{\beta}\rho - 2g(\partial\rho, \partial\phi)\partial_{\beta}\phi + |\nabla\phi|_g^2\partial_{\beta}\rho \\ &= -2g(\partial\rho, \partial\phi)\partial_{\beta}\phi \end{aligned}$$

which leads to

$$(2g^{\delta\gamma}\partial_{\delta}\rho T_{\gamma\beta} - \text{tr}(T)\partial_{\beta}\rho)e^{-2\rho} = 2g^{\dagger}(\partial\rho, \partial\rho)\partial_{\beta}\phi. \quad (3.14)$$

Then we remark at first the following identities:

$$\nabla^{\dagger\alpha}T_{\alpha\beta} = \partial_{\beta}\phi\Box_{g^{\dagger}}\phi.$$

Combined with (3.13),

$$\partial_{\beta}\phi\Box_{g^{\dagger}}\phi = 2e^{-2\rho}g^{\alpha'\beta'}\partial_{\alpha'}\phi\partial_{\beta'}\rho\partial_{\beta}\phi.$$

This leads to the following wave equation:

$$\Box_{g^{\dagger}}\phi = 2g^{\dagger\alpha'\beta'}\partial_{\alpha'}\phi\partial_{\beta'}\rho. \quad (3.15)$$

Then we consider the Einstein coupling:

$$T_{\alpha\beta} = T^{\dagger}_{\alpha\beta} = e^{2\rho}(\partial_{\alpha}\phi\partial_{\beta}\phi - \frac{1}{2}g_{\alpha\beta}|\nabla\phi|_g^2),$$

then

$$\begin{aligned}\nabla^{\dagger\alpha}T_{\alpha\beta} &= e^{2\rho}\partial_\beta\phi\Box_{g^\dagger}\phi + 2e^{2\rho}g^{\dagger\alpha\alpha'}\partial_{\alpha'}\rho(\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}|\nabla\phi|_g^2) \\ &= e^{2\rho}\partial_\beta\phi(\Box_{g^\dagger}\phi + 2g^\dagger(\partial\rho, \partial\phi)) - \partial_\beta\rho|\nabla\phi|_g^2.\end{aligned}$$

Combined with (3.18),

$$e^{2\rho}\partial_\beta\phi(\Box_{g^\dagger}\phi + 2g^\dagger(\partial\rho, \partial\phi)) - \partial_\beta\rho|\nabla\phi|_g^2 = 2g^\dagger(\partial\rho, \partial\phi)\partial_\beta\phi.$$

This leads to

$$e^{2\rho}\partial_\beta\phi(\Box_{g^\dagger}\phi + 2g^\dagger(\partial\rho, \partial\phi) - 2e^{-2\rho}g^\dagger(\nabla^\dagger\rho, \nabla^\dagger\phi)) = \partial_\beta\rho|\nabla\phi|_g^2,$$

which, as in the case of Jordan metric, leads to an over-determined differential system. This suggests again that the Einstein coupling is not compatible.

3.4 The initial value problem in nonlinear gravity (conformal version)

As in the case of Jordan metric, we formulate the initial value problem and the constraint equations on initial data in conformal metric. To do so, we take again the Jordan coupling of a real massless scalar field:

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}|\nabla\phi|_g^2.$$

Then taking the expression of σ^\dagger and J^\dagger ,

$$\begin{aligned}\sigma^\dagger &= \frac{1}{2}(|\mathcal{L}_{n^\dagger}\phi|^2 + \bar{g}^{ij}\partial_i\phi\partial_j\phi), \\ J^\dagger_j &= -\mathcal{L}_{n^\dagger}\phi\partial_j\phi.\end{aligned}$$

As in the classical case, define the following initial data set for this conformal formulation.

Definition 3.7. *An initial data set for the nonlinear gravity theory in Einstein metric $(\bar{M}, \bar{g}^\dagger, K^\dagger, \rho_0^\dagger, \rho_1^\dagger, \phi_0^\dagger, \phi_1^\dagger)$ consists the following data:*

- a 3-dimensional manifold \bar{M} endowed with a Riemannian metric \bar{g}^\dagger and a symmetric $(0, 2)$ -tensor field K^\dagger ,
- two scalar fields denoted by ρ_0^\dagger and ρ_1^\dagger on \bar{M} and representing the (to-be-constructed) conformal factor and its time derivative,
- two scalar field ϕ_0^\dagger and ϕ_1^\dagger defined on \bar{M} ,
- which are assumed to satisfy the **Hamiltonian constraint of nonlinear gravity in Einstein metric**

$$\bar{R}^\dagger - K^\dagger_{ij}K^{\dagger ij} + (K^{\dagger j}_j)^2 = 8e^{-2\rho} \left((\phi_1^\dagger)^2 + \bar{g}^{\dagger ij}\partial_i\phi_0^\dagger\partial_j\phi_0^\dagger \right) + 6(\rho_1^\dagger)^2 + 6\bar{g}^{\dagger ij}\partial_i\rho_0^\dagger\partial_j\rho_0^\dagger - e^{-2\rho_0^\dagger}W_2(\rho_0^\dagger), \quad (3.16)$$

and the **momentum constraint of nonlinear gravity in Einstein metric**

$$\partial_j K^{\dagger i}_i - \bar{\nabla}^\dagger_i K^{\dagger i}_j = \frac{\phi_1^\dagger\partial_j\phi_0^\dagger}{e^{2\rho}} + 6\rho_1^\dagger\partial_j\rho_0^\dagger. \quad (3.17)$$

Then similarly, we formulate the **initial value problem of nonlinear gravity in Einstein metric**.

Definition 3.8 (Formulation of the initial value problem in Einstein metric). *Given an initial data set $(\bar{M}, \bar{g}^\dagger, K^\dagger, \rho_0^\dagger, \rho_1^\dagger, \phi_0^\dagger, \phi_1^\dagger)$ as in Definition 3.7, the initial value problem associated with the nonlinear gravity theory consists of finding a Lorentzian manifold (M, g) and a two-tensor field $T_{\alpha\beta}$ on M*

1. The conformal metric g^\dagger is defined with the relation $g^\dagger_{\alpha\beta} = e^{2\rho} g_{\alpha\beta}$ with the conformal factor $\rho = \frac{1}{2} \ln(f'(R_g))$ where R_g is the scalar curvature of g .
2. The field equations of nonlinear gravity (3.6) are satisfied with $\rho = \frac{1}{2} \ln f'(R_g)$.
3. There exists an embedding $i : \bar{M} \rightarrow M$ with pull back metric $\bar{g}^\dagger = i^* g^\dagger$ and second fundamental form K^\dagger .
4. The field ρ_0^\dagger coincides with the restriction of the conformal factor ρ on \bar{M} , while ρ_1^\dagger coincides with the Lie derivative $\mathcal{L}_{n^\dagger} \rho$ restricted to \bar{M} , where n^\dagger denotes the normal unit vector of \bar{M} .
5. The scalar fields $\phi_0^\dagger, \phi_1^\dagger$ coincides with the restriction of $\phi, \mathcal{L}_{n^\dagger} \phi$ on \bar{M} .

Such a solution to (3.6) is referred to as a **nonlinear gravity development of the initial data set** $(\bar{M}, \bar{g}^\dagger, K^\dagger, \rho_0^\dagger, \rho_1^\dagger, \phi_0^\dagger, \phi_1^\dagger)$.

Similarly as in classical gravity, we can define the notion of *maximally hyperbolic nonlinear gravity development of the initial data set*.

We first observe that the initial value problem for nonlinear gravity reduces to the classical formulation in the special case of vanishing geometric data $\phi_0^\dagger = \phi_1^\dagger = R_0 = R_1 \equiv 0$. On the other hand, without matter fields and for non-vanishing geometric data R_0 and R_1 , the spacetimes under consideration do not satisfy Einstein vacuum equations. Similarly as in classical gravity, these fields can not be fully arbitrary prescribed but certain constraints (given above) must be assumed.

3.5 Preservation of the constraints

As in the case of original (Jordan) metric, we discuss the preservation of the constraint equations. This result is concluded in the following proposition.

Proposition 3.9. *Let $(\bar{g}^\dagger, K^\dagger)$ be a pair of symmetric two-tensor of class C^2 defined in the spacetime $M = \cup_{t \in [0, t_{\max})} M_t$. If the following equations hold in M*

$$N_{g_{ij}} - \frac{1}{2} \text{tr} (N_g) g_{ij} = 8\pi (T_{ij} - \frac{1}{2} \text{tr} (N_g) g_{ij}), \quad (3.18)$$

$$\nabla^{\dagger\alpha} T_{\alpha\beta} = e^{-2\rho} (g^{\gamma\delta} \partial_\gamma \rho T_{\gamma\beta} - \partial_\beta \rho \text{tr} (T)), \quad (3.19)$$

and on the initial slice M_0

$$N_{g_{0\beta}} = 8\pi T_{0\beta}. \quad (3.20)$$

Then (3.20) hold in M .

Proof. Recall the definition

$$\Sigma_{\alpha\beta} = N_{g_{\alpha\beta}} - 8\pi T_{\alpha\beta}.$$

We will prove that $\sigma_{00} = 0$.

Then (3.18) can be written as

$$\Sigma_{ij} - \frac{1}{2} \text{tr}^\dagger(\Sigma) g^\dagger_{ij} = 0. \quad (3.21)$$

Taking the trace of the tensor $\Sigma_{\alpha\beta} - \frac{1}{2} \text{tr} \Sigma g_{\alpha\beta}$ **with respect to g^\dagger** :

$$\text{tr}^\dagger \Sigma - 2 \text{tr}^\dagger \Sigma = -N^{\dagger-2} \Sigma_{00} + g^{\dagger ij} (\Sigma_{ij} - \frac{1}{2} \text{tr} (\Sigma) g_{ij}),$$

combined with (3.21),

$$\text{tr } \dagger \Sigma = -N^{\dagger -2} \Sigma_{00}. \quad (3.22)$$

Combine (3.22) with (3.21), then

$$\Sigma_{ij} = -\frac{\Sigma_{00}}{2N^{\dagger 2}} g^{\dagger ij}. \quad (3.23)$$

By the same procedure in the proof of proposition 2.6, the following relations hold:

$$\Sigma_b^0 = g^{\dagger 00} g^{\dagger bc} \Sigma_{0c}, \quad \Sigma_b^a = -\delta_b^a \Sigma_0^0. \quad (3.24)$$

Now consider the identity (3.13) combined with (3.19), the following identity holds:

$$\nabla^{\dagger \alpha} \Sigma_{\alpha\beta} = e^{-2\rho} (2\partial_\gamma \rho \Sigma_\beta^\gamma - \partial_\beta \rho \text{tr}(\Sigma)) \quad (3.25)$$

We remark that by (3.24), the right-hand-side is a linear form of the function Σ_0^β . Recall that by definition,

$$\nabla^{\dagger \alpha} \Sigma_\beta^\alpha = \langle e_\alpha, \Sigma_\beta^\alpha \rangle - \omega^{\dagger \delta}_{\alpha\beta} \Sigma_\delta^\alpha + \omega^{\dagger \alpha}_{\alpha\delta} \Sigma_\beta^\delta. \quad (3.26)$$

Combine (3.25) and (3.26), a first-order linear differential system is formed:

$$\langle e_\alpha, \Sigma_\beta^\alpha \rangle - \omega^{\dagger \delta}_{\alpha\beta} \Sigma_\delta^\alpha + \omega^{\dagger \alpha}_{\alpha\delta} \Sigma_\beta^\delta = e^{-2\rho} (2\partial_\gamma \rho \Sigma_\beta^\gamma - \partial_\beta \rho \text{tr}(\Sigma))$$

Remark that the principle part of this system

$$\begin{aligned} \langle e_0, \Sigma_0^0 \rangle + \partial_a \Sigma_0^a &= \text{lower order terms,} \\ \langle e_0, g^{\dagger 00} g^{\dagger bc} \Sigma_{0c} \rangle - \partial_b \Sigma_0^0 &= \text{lower order terms.} \end{aligned} \quad (3.27)$$

can be symmetrized by the same procedure of the system formed by (2.48) and (2.49). Recall also that by (3.20), this system has vanishing initial data. Then, recalling our global hyperbolicity assumption (2.1), the desired result is proved. \square

4 The conformal formulation in wave coordinates

4.1 Gravity tensors in wave coordinates

From this section we will begin to solve the system (3.6) with the techniques of PDE. To do so, we need firstly write (3.6) in a coordinate system well chosen. If we observe the left-hand-side of (3.6), we remark that if we neglect the terms on ρ , the principle part (the second-order derivatives on g^\dagger) is contained in $R^\dagger_{\alpha\beta}$. To investigate the structure, we perform the following calculation which is valid for a general Lorentzian manifold with arbitrary local coordinate system.

Let (M, g) be a Lorentzian manifold with metric g of sign $(-, +, +, +)$ in a local coordinate system $\{x^0, x^1, x^2, x^3\}$. We denote by $\Gamma_{\alpha\beta}^\gamma$ the associated Christoffel symbols. $\square_g := \nabla^\alpha \nabla_\alpha$ is called the d'Alembert operator (associated to g). The following lemma is the result of a direct but tedious calculation:

Lemma 4.1 (Ricci tensor in general coordinates). *With the notation¹ $\Gamma^\lambda := g^{\alpha\beta} \Gamma_{\alpha\beta}^\lambda$, and $\Gamma_\lambda := g_{\lambda\beta} \Gamma^\beta$, we have*

$$R_{\alpha\beta} = -\frac{1}{2} g^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g_{\alpha\beta} + \frac{1}{2} (\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha) + \frac{1}{2} F_{\alpha\beta}(g; \partial g, \partial g), \quad (4.1)$$

where $F_{\alpha\beta}(g; \partial g, \partial g)$ are nonlinear functions in the metric coefficients and are quadratic in their first-order derivatives and the second-order operator $g^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'}$ is called the reduced d'Alembert operator. Furthermore

$$\square_g u = g^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} u + \Gamma^\delta \partial_\delta u.$$

¹We emphasize that here Γ_γ are regarded as functions but **not** the components of a vector.

Proof. These are direct but tedious calculations:

$$R_{\alpha\beta} = \partial_\lambda \Gamma_{\alpha\beta}^\lambda - \partial_\alpha \Gamma_{\beta\lambda}^\lambda + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda\delta}^\delta - \Gamma_{\alpha\delta}^\lambda \Gamma_{\beta\lambda}^\delta$$

combined with

$$\Gamma_{\alpha\beta}^\lambda = \frac{1}{2} g^{\lambda\lambda'} (\partial_\alpha g_{\beta\lambda'} + \partial_\beta g_{\alpha\lambda'} - \partial_{\lambda'} g_{\alpha\beta}).$$

Then we calculate the expression of Ricci tensor,

$$R_{\alpha\beta} = \partial_\lambda \Gamma_{\alpha\beta}^\lambda - \partial_\alpha \Gamma_{\beta\lambda}^\lambda + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda\delta}^\delta - \Gamma_{\alpha\delta}^\lambda \Gamma_{\beta\lambda}^\delta,$$

where the second order derivatives come from the first two terms. So we focus on the first two terms,

$$\begin{aligned} \partial_\lambda \Gamma_{\alpha\beta}^\lambda - \partial_\alpha \Gamma_{\beta\lambda}^\lambda &= \frac{1}{2} \partial_\lambda (g^{\lambda\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta})) - \frac{1}{2} (\partial_\alpha (g^{\lambda\delta} (\partial_\beta g_{\lambda\delta} + \partial_\lambda g_{\beta\delta} - \partial_\delta g_{\beta\lambda}))) \\ &= -\frac{1}{2} \partial_\lambda (g^{\lambda\delta} \partial_\delta g_{\alpha\beta}) + \frac{1}{2} \partial_\lambda (g^{\lambda\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta})) - \frac{1}{2} \partial_\alpha (g^{\lambda\delta} \partial_\beta g_{\lambda\delta}), \end{aligned}$$

then

$$\partial_\lambda \Gamma_{\alpha\beta}^\lambda - \partial_\alpha \Gamma_{\beta\lambda}^\lambda = -\frac{1}{2} g^{\lambda\delta} \partial_\lambda \partial_\delta g_{\alpha\beta} + \frac{1}{2} g^{\lambda\delta} \partial_\alpha \partial_\lambda g_{\delta\beta} + \frac{1}{2} g^{\lambda\delta} \partial_\beta \partial_\lambda g_{\delta\alpha} - \frac{1}{2} g^{\lambda\delta} \partial_\alpha \partial_\beta g_{\lambda\delta} + \text{l.o.t.} \quad (4.2)$$

where l.o.t. refers to the quadratic terms of first-order or zero-order derivatives.

Then we focus on the term $\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha$.

$$\begin{aligned} \Gamma^\gamma &= \Gamma_{\alpha\beta}^\gamma g^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) \\ &= g^{\gamma\delta} g^{\alpha\beta} \partial_\alpha g_{\beta\delta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \partial_\delta g_{\alpha\beta} \end{aligned}$$

and

$$\Gamma_\lambda = g_{\lambda\gamma} \Gamma^\gamma = g^{\alpha\beta} \partial_\alpha g_{\beta\lambda} - \frac{1}{2} g^{\alpha\beta} \partial_\lambda g_{\alpha\beta}.$$

So

$$\partial_\alpha \Gamma_\beta = \partial_\alpha (g^{\delta\lambda} \partial_\delta g_{\lambda\beta}) - \frac{1}{2} \partial_\alpha (g^{\lambda\delta} \partial_\beta g_{\lambda\delta}),$$

then

$$\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha = g^{\gamma\delta} \partial_\alpha \partial_\lambda g_{\delta\beta} + g^{\lambda\delta} \partial_\beta \partial_\lambda g_{\delta\alpha} - g^{\lambda\delta} \partial_{\alpha\beta} g_{\lambda\delta} + \text{l.o.t.} \quad (4.3)$$

Then compare (4.2) and (4.3), the desired result is proved. \square

Observe the field equation (1.5) or the transformed field equation in conformal metric (3.3), there are linear terms of Ricci curvature tensor. To guarantee the hyperbolicity of the linear part (at least for the second order terms in (1.5)), we will use the wave coordinate conditions: a local coordinate system is called a **wave coordinate system** if its Christoffel symbols satisfies the following conditions:

$$\Gamma^\lambda = g^{\alpha\beta} \Gamma_{\alpha\beta}^\lambda = 0. \quad (4.4)$$

Now, by (3.3), the principle part (neglige the terms on ρ) of the expression

$$N_{g_{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} \text{tr} (N_g) - \frac{1}{2} e^{2\rho} (\partial_\alpha \Gamma^\dagger + \partial_\beta \Gamma^\dagger)$$

is $\frac{1}{2} e^{2\rho} g^{\dagger\alpha\beta} \partial_{\alpha'} \partial_{\beta'} g^\dagger_{\alpha\beta}$ which is a (quasi-linear) wave operator. Motivate by this expression, we introduce the following formulation of (3.6) in a wave coordinate system:

The equation (3.6) written in a wave coordinate system takes the following form

$$N_{g_{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} \text{tr} (N_g) - \frac{1}{2} e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha) = 8\pi (T_{\alpha\beta} - \frac{1}{2} \text{tr} (T) g_{\alpha\beta}), \quad (4.5)$$

The trace equation (3.8) becomes

$$\square_{g^\dagger} \rho + \Gamma^{\dagger\delta} \partial_\delta \rho = \frac{W_2(\rho)}{6e^{2\rho}} + \frac{W_3(\rho)}{6e^{4\rho}} + \frac{4\pi \text{tr}(T)}{3e^{4\rho}} \quad (4.6)$$

with

$$\Gamma^\dagger_\lambda = 0.$$

Then by lemma 4.1, this system, written with the metric g^\dagger and its derivatives, is as following

Lemma 4.2. *The field equation (3.6) written in a wave coordinate system is expressed as following:*

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g^\dagger_{\alpha\beta} = F_{\alpha\beta}(g^\dagger; \partial g^\dagger, \partial g^\dagger) - 12\partial_\alpha \rho \partial_\beta \rho + \frac{W_2(\rho)}{e^{2\rho}} g^\dagger_{\alpha\beta} - \frac{8\pi}{e^{2\rho}} (2T_{\alpha\beta} - \text{tr}(T)g_{\alpha\beta}) \quad (4.7a)$$

with the constraint equations given by wave coordinate conditions

$$g^{\dagger\alpha\beta} \Gamma^{\dagger\gamma}_{\alpha\beta} = 0. \quad (4.7b)$$

Together with the trace equation

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \rho = \frac{W_2(\rho)}{6e^{2\rho}} + \frac{W_3(\rho)}{6e^{4\rho}} + \frac{4\pi \text{tr}(T)}{3e^{4\rho}} \quad (4.7c)$$

where $\rho = \frac{1}{2} f'(R_g)$ and the evolution equation of matter field

$$\nabla^{\dagger\alpha} T_{\alpha\beta} = (\text{tr}(T) \partial_\beta \rho - 2g^{\delta\gamma} \partial_\gamma \rho T_{\gamma\beta}) e^{-2\rho}. \quad (4.7d)$$

We emphasize here in in this formulation, the trace equation (4.7c) and the evolution equation of matter field (4.7d) are not independent of the field equation. The **essential** equations are (4.7a) and (4.7b).

4.2 The wave-like system of nonlinear gravity theory

The aim of this subsection is to study the system formed by (4.7a) and (4.7b). The main difficulty is that this is a (neglige the terms on ρ) quasi-linear wave system but with constraints (4.7b) which is difficult to manage. The strategy is try to replace these constraints by another differential equation which is simpler to manage. This forms a new system and (a priori) not equivalent to the original (3.6). This system is call the **wave-reduced system** which is defined as following:

Definition 4.3 (Wave-reduced system, geometric form). *The wave-reduced system associated to (3.6) is formulated as following*

$$N_{g_{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} \text{tr}(N_g) - \frac{1}{2} e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha) = 8\pi (T_{\alpha\beta} - \frac{1}{2} \text{tr}(T) g_{\alpha\beta}), \quad (4.8a)$$

$$\nabla^{\dagger\alpha} T_{\alpha\beta} = (\text{tr}(T) \partial_\beta \rho - 2g^{\delta\gamma} \partial_\gamma \rho T_{\gamma\beta}) e^{-2\rho} \quad (4.8b)$$

where $N_{g_{\alpha\beta}}$ is defined by (3.3) and we recall the relation $e^{2\rho} = f'(R_g)$.

Then by (3.3) and Lemma 4.1, this system is written as following:

$$\begin{aligned} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g^\dagger_{\alpha\beta} &= F_{\alpha\beta}(g^\dagger; \partial g^\dagger, \partial g^\dagger) - 12\partial_\alpha \rho \partial_\beta \rho + \frac{W_2(\rho)}{e^{2\rho}} g^\dagger_{\alpha\beta} - \frac{8\pi}{e^{2\rho}} (2T_{\alpha\beta} - \text{tr}(T)g_{\alpha\beta}) \\ \nabla^{\dagger\alpha} T_{\alpha\beta} &= (\text{tr}(T) \partial_\beta \rho - 2g^{\delta\gamma} \partial_\gamma \rho T_{\gamma\beta}) e^{-2\rho}. \\ \rho &= \frac{1}{2} \ln f'(R_g). \end{aligned} \quad (4.9)$$

which is (4.7) without the constraint equations (4.7b) but combined with the evolution equation (4.7d). We will establish the following result.

Proposition 4.4. Consider a spacetime $M = \cup_{t \in [0, t_{\max})} M_t$ with metric g^\dagger which is supposed to be globally hyperbolic, together with a matter field T . Suppose that the following wave coordinate conditions hold

$$\Gamma^{\dagger\gamma} := g^{\dagger\alpha\beta} \Gamma_{\alpha\beta}^{\dagger\gamma} = 0 \quad (4.10)$$

on the initial slice M_0 , together with the constraint equations (3.11) and (3.12). Then, the conditions (4.10) are satisfied within the domain of determinacy of M_0 .

This proposition means that if we want to find a solution of (3.6), what we need to do is to find a solution of (4.9) with the constraint equations (4.10) satisfied on the initial slice, which will turn to be four additional constraint equations demanded on initial data set. Recall again that the interest of seeing (4.8) rather than (3.6) is that the former one has a hyperbolic ‘‘principle part’’ (neglige the terms on ρ).

The key to proving this proposition is using contracted Bianchi identity (in the sense of (3.13)) and, amazingly, the precise form of the nonlinear gravity tensor N_g is not used for this step !

4.3 Preservation of the wave coordinate condition

Before prove this proposition, we need some preparations. The following lemmas gives the key identities to be used.

Lemma 4.5. Let (M, g) be a regular pseudo-riemannian manifold. Then the following identity holds:

$$\begin{aligned} \nabla^\alpha (\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha - g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} \Gamma_{\beta'}) &= g^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma_\beta - g^{\alpha\alpha'} \Gamma_{\alpha\alpha'}^\gamma \partial_\gamma \Gamma_\beta - g^{\alpha\alpha'} \Gamma_{\alpha'\beta}^\delta \partial_\delta \Gamma_\alpha \\ &\quad - \partial_\beta g^{\alpha'\beta'} (\partial_{\alpha'} \Gamma_{\beta'}) \end{aligned} \quad (4.11)$$

where $\Gamma_{\alpha\beta}^\gamma$ are the Christoffel symbols, $\Gamma^\gamma := g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma$ and $\Gamma_\gamma = g_{\gamma'\gamma} \Gamma^{\gamma'}$.

Proof. This is by direct calculation. First we remark the following identities:

$$\begin{aligned} \nabla^\alpha \partial_\beta u &= g^{\alpha\alpha'} \partial_{\alpha'} \partial_\beta u - \Gamma_{\alpha'\beta}^\gamma \partial_\gamma u, \\ \square_g u &= g^{\alpha\beta} \partial_\alpha \partial_\beta u - g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma \partial_\gamma u. \end{aligned}$$

Then we find

$$\begin{aligned} &\nabla^\alpha (\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha - g_{\alpha\beta} g^{\alpha'\beta'} \partial_{\alpha'} \Gamma_{\beta'}) \\ &= \nabla^\alpha \partial_\alpha \Gamma_\beta + \nabla^\alpha \partial_\beta \Gamma_\alpha - \partial_\beta (g^{\alpha'\beta'} \partial_{\alpha'} \Gamma_{\beta'}) \\ &= \square_g \Gamma_\beta + g^{\alpha\alpha'} \partial_{\alpha'} \partial_\beta \Gamma_\alpha - g^{\alpha\alpha'} \Gamma_{\alpha'\beta}^\delta \partial_\delta \Gamma_\alpha - \partial_\beta g^{\alpha'\beta'} (\partial_{\alpha'} \Gamma_{\beta'}) - g^{\alpha'\beta'} \partial_\beta \partial_{\alpha'} \Gamma_{\beta'} \\ &= \square_g \Gamma_\beta - g^{\alpha\alpha'} \Gamma_{\alpha'\beta}^\delta \partial_\delta \Gamma_\alpha - \partial_\beta g^{\alpha'\beta'} (\partial_{\alpha'} \Gamma_{\beta'}) \\ &= g^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma_\beta - g^{\alpha\alpha'} \Gamma_{\alpha\alpha'}^\gamma \partial_\gamma \Gamma_\beta - g^{\alpha\alpha'} \Gamma_{\alpha'\beta}^\delta \partial_\delta \Gamma_\alpha - \partial_\beta g^{\alpha'\beta'} (\partial_{\alpha'} \Gamma_{\beta'}). \end{aligned}$$

□

The following lemma gives the relation between the wave condition and the evolution equation of the wave-reduced system (4.9).

Lemma 4.6. Let (M, g^\dagger) be a Lorentzian manifold. Suppose that g^\dagger in class C^4 and $T_{\alpha\beta}$ a tensor of class C^2 . If in a local coordinate system $\{x^0, x^1, x^2, x^3\}$, $(g^\dagger, T_{\alpha\beta})$ satisfy the system (4.8), then the following equation holds in the domain of the coordinate system:

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma_\beta^\dagger = F_\beta(\rho, g^\dagger; \Gamma_\gamma^\dagger, \partial \Gamma_\gamma^\dagger) \quad (4.12)$$

where $F_\beta(\rho, g^\dagger; \cdot, \cdot)$ is a combination of linear and bilinear forms and we recall $\Gamma_{\alpha\beta}^{\dagger\gamma}$ the Christoffel symbol and

$$\Gamma_\beta^\dagger = g_{\beta\beta'} \Gamma^{\dagger\beta'} = g_{\beta\beta'} g^{\alpha\gamma} \Gamma_{\alpha\gamma}^{\dagger\beta'}$$

Proof. Taking the trace of (4.8a) with respect to the metric g ,

$$\text{tr} (N_g - 8\pi T) = -e^{2\rho} g^{\alpha\beta} \partial_\alpha \Gamma^\dagger_\beta. \quad (4.13)$$

Combined with (4.8a),

$$\begin{aligned} N_{g_{\alpha\beta}} - 8\pi T_{\alpha\beta} &= \frac{1}{2} g_{\alpha\beta} \text{tr} (N_g - 8\pi T) + \frac{1}{2} e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha) \\ &= \frac{1}{2} e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'}) \end{aligned} \quad (4.14)$$

Taking the trace of (4.14) **with respect to** g^\dagger ,

$$\frac{1}{2} \nabla^{\dagger\alpha} (e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'})) = \nabla^{\dagger\alpha} (N_{g_{\alpha\beta}} - 8\pi T_{\alpha\beta}). \quad (4.15)$$

The left-hand-side is calculated by using (4.11)

$$\begin{aligned} &\frac{1}{2} \nabla^{\dagger\alpha} (e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'})) \\ &= \frac{1}{2} e^{2\rho} \nabla^{\dagger\alpha} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'}) \\ &\quad + e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'}) \nabla^{\dagger\alpha} \rho \\ &= \frac{1}{2} e^{2\rho} (g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma^\dagger_\beta - g^{\dagger\alpha\alpha'} \Gamma^{\dagger\gamma}_{\alpha\alpha'} \partial_\gamma \Gamma^\dagger_\beta - g^{\dagger\alpha\alpha'} \Gamma^{\dagger\delta}_{\alpha'\beta} \partial_\delta \Gamma^\dagger_\alpha - \partial_\beta g^{\dagger\alpha'\beta'} (\partial_{\alpha'} \Gamma^\dagger_{\beta'})) \\ &\quad + e^{2\rho} (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'}) \nabla^{\dagger\alpha} \rho \\ &=: \frac{1}{2} e^{2\rho} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma^\dagger_\beta + \frac{1}{2} e^{2\rho} \tilde{F}_\beta(\rho, g^\dagger; \Gamma^\dagger_\gamma, \partial \Gamma^\dagger_\gamma) \end{aligned}$$

where $F_\beta(\rho, g^\dagger; \cdot)$ is a combination of linear and bilinear forms of the functions Γ^\dagger_γ and $\partial \Gamma^\dagger_\gamma$.

The right-hand-side of (4.15) is calculated by using the identity (3.13) and (4.8b).

$$\nabla^{\dagger\alpha} (N_{g_{\alpha\beta}} - 8\pi T_{\alpha\beta}) = e^{-2\rho} (\text{tr} (N_g - 8\pi T) \partial_\beta \rho - 2g^{\gamma\delta} \partial_\gamma \rho (N_{g_{\delta\beta}} - 8\pi T_{\delta\beta})).$$

Then by (4.13),

$$\begin{aligned} &\nabla^{\dagger\alpha} (N_{g_{\alpha\beta}} - 8\pi T_{\alpha\beta}) \\ &= e^{-2\rho} \left(-e^{2\rho} g^{\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'} \partial_\beta \rho - e^{2\rho} g^{\gamma\delta} \partial_\gamma \rho (\partial_\alpha \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\alpha - g^\dagger_{\alpha\beta} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \Gamma^\dagger_{\beta'}) \right) \\ &= -g^{\gamma\delta} \partial_\gamma \rho (\partial_\beta \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\delta). \end{aligned}$$

Then by (4.15),

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma^\dagger_\beta = F_\beta(\rho, g^\dagger; \Gamma^\dagger_\gamma, \partial \Gamma^\dagger_\gamma), \quad (4.16)$$

where

$$F_\beta(\rho, g^\dagger; \Gamma^\dagger_\gamma, \partial \Gamma^\dagger_\gamma) = -\tilde{F}_\beta(\rho, g^\dagger; \Gamma^\dagger_\gamma, \partial \Gamma^\dagger_\gamma) - 2e^{-2\rho} g^{\gamma\delta} \partial_\gamma \rho (\partial_\delta \Gamma^\dagger_\beta + \partial_\beta \Gamma^\dagger_\delta)$$

is a combination of linear and bilinear forms of the functions Γ^\dagger_γ and $\partial \Gamma^\dagger_\gamma$. \square

Lemma 4.7. *Let (M, g^\dagger) be a Lorentzian manifold with space-like foliation $M = [0, t_{max}) \times M_t$ and the sign of g^\dagger is $(-, +, +, +)$. Suppose that the metric is in class C^4 and the tensor field $T_{\alpha\beta}$ is in class C^2 and the equation (4.8) holds on the initial slice M_0 . Suppose furthermore on the slice M_0 the wave coordinate conditions and the constraint equations hold, which are*

$$\Gamma^{\dagger\gamma} := g^{\dagger\alpha\beta} \Gamma^{\dagger\gamma}_{\alpha\beta} = 0$$

and (as stated in (3.16) and (3.17))

$$\begin{aligned}\bar{R}^\dagger - K^\dagger_{ij}K^{\dagger ij} + (K^\dagger_l)^2 &= \frac{16\sigma^\dagger}{e^{2\rho}} + \frac{12|\langle e_0, \rho \rangle|^2}{N^{\dagger 2}} + 6g^\dagger(\nabla\rho, \nabla\rho) - e^{-2\rho}W_2(\rho), \\ \partial_j K^\dagger_l - \bar{\nabla}^\dagger K^\dagger_j &= -\frac{J^\dagger_j}{e^{2\rho}} + \frac{6\langle e_0, \rho \rangle \partial_j \rho}{N^\dagger}.\end{aligned}$$

Then on M_0 ,

$$\partial_0 \Gamma^{\dagger \lambda} = 0.$$

Proof. We recall the Cauchy adapted frame (e_0, e_1, e_2, e_3) defined as following:

$$\begin{aligned}e_0 &:= \partial_0 - \beta^i \partial_i, \quad \beta^i := \frac{g^\dagger_{0i}}{g^\dagger_{ii}}, \\ e_i &:= \partial_i.\end{aligned}$$

So $g^\dagger(e_0, e_i) = 0$.

A tensor can be written in both natural frame and a Cauchy adapted frame. We denote by a underlined letter with index the components of a tensor written in a Cauchy adapted frame. For example, $\underline{T}_{\alpha\beta}$ is the (α, β) component of the tensor T in a Cauchy adapted frame.

Recall that the momentum constraint equations are equivalent to

$$\underline{N}_{g_{0j}} = 8\pi \underline{T}_{0j}, \quad (4.17)$$

the Hamiltonian constraint equation is equivalent to

$$\underline{N}_{g_{00}} = 8\pi \underline{T}_{00}. \quad (4.18)$$

Recall that the Cauchy adapted frame is expressed by the natural frame via

$$e_\alpha = \Phi_\alpha^\beta \partial_\beta,$$

where

$$(\Phi_\alpha^\beta)_{\alpha\beta} = \begin{pmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\underline{N}_{g_{\alpha\beta}} = N_{g_{\alpha'\beta'}} \Phi_\alpha^{\alpha'} \Phi_\beta^{\beta'}, \quad \underline{T}_{\alpha\beta} = T_{\alpha'\beta'} \Phi_\alpha^{\alpha'} \Phi_\beta^{\beta'}.$$

Now remark that the wave-reduced field equation (4.8a) can be rewritten with the components in Cauchy adapted frame:

$$\underline{N}_{g_{\alpha\beta}} - \frac{1}{2} \text{tr} N_g \underline{g}_{\alpha\beta}^\dagger - \frac{1}{2} e^{2\rho} \Phi_\alpha^{\alpha'} \Phi_\beta^{\beta'} (\partial_{\alpha'} \Gamma_{\beta'}^\dagger + \partial_{\beta'} \Gamma_{\alpha'}^\dagger) = 8\pi (\underline{T}_{\alpha\beta} - \frac{1}{2} \text{tr} (T) \underline{g}_{\alpha\beta}^\dagger)$$

which is

$$\underline{N}_{g_{\alpha\beta}} - \frac{1}{2} \text{tr} N_g \underline{g}_{\alpha\beta}^\dagger - \frac{1}{2} e^{2\rho} (\Phi_\beta^{\beta'} \langle e_\alpha, \Gamma_{\beta'}^\dagger \rangle + \Phi_\alpha^{\alpha'} \langle e_\beta, \Gamma_{\alpha'}^\dagger \rangle) = 8\pi (\underline{T}_{\alpha\beta} - \frac{1}{2} \text{tr} (T) \underline{g}_{\alpha\beta}^\dagger). \quad (4.19)$$

Now combine (4.19) with (4.17) by fixing $\alpha = 0, 1 \leq \beta = b \leq 3$ in (4.19) and remark that $\underline{g}_{0j}^\dagger = 0$:

$$\Phi_b^{\beta'} \langle e_0, \Gamma_{\beta'}^\dagger \rangle + \Phi_0^{\alpha'} \langle e_b, \Gamma_{\alpha'}^\dagger \rangle = 0.$$

Consider this equation on the initial slice M_0 . Recall that $\Gamma^{\dagger \lambda} = 0$ so that $\partial_b \Gamma_\beta = 0$ for any $1 \leq b \leq 3$ and $0 \leq \beta \leq 3$. Then remark that $\Phi_b^{\beta'} = \delta_b^{\beta'}$ and $\langle e_b, \Gamma_{\alpha'}^\dagger \rangle = \partial_b \Gamma_{\alpha'}^\dagger = 0$. So that

$$\langle e_0, \Gamma_b^\dagger \rangle = 0, \quad (4.20)$$

which leads to $\partial_t \Gamma^\dagger_b = 0$.

Now combine (4.19) with (4.17) and (4.18) by fixing $\alpha = \beta = 0$ in (4.19):

$$\Phi_0^{\beta'} \langle e_0, \Gamma^\dagger_{\beta'} \rangle + \Phi_0^{\alpha'} \langle e_0, \Gamma^\dagger_{\alpha'} \rangle = -\text{tr} (N_g - 8\pi T) g^\dagger_{00}.$$

Now recall (4.17) and (4.18) and the fact that $g^\dagger_{0j} = 0$, then

$$\text{tr} (N_g - 8\pi T) = \underline{g}^{\alpha\beta} (\underline{N}_{g_{\alpha\beta}} - 8\pi \underline{T}_{\alpha\beta}) = \underline{g}^{ij} (\underline{N}_{g_{ij}} - 8\pi \underline{T}_{ij}). \quad (4.21)$$

Recall (4.19) with $1 \leq \alpha = i \leq 3$ and $1 \leq \beta = j \leq 3$, and remark that $\langle e_i, \Gamma^\dagger_\alpha \rangle = 0$, then

$$\underline{N}_{g_{ij}} - 8\pi \underline{T}_{ij} = \frac{1}{2} \text{tr} (N_g - 8\pi T) \underline{g}_{ij}$$

which leads to

$$\underline{g}^{ij} (\underline{N}_{g_{ij}} - 8\pi \underline{T}_{ij}) = \frac{1}{2} \text{tr} (N_g - 8\pi T) \underline{g}^{ij} \underline{g}_{ij}$$

so that (by (4.21))

$$\text{tr} (N_g - 8\pi T) = \frac{3}{2} \text{tr} (N_g - 8\pi T)$$

then

$$\text{tr} (N_g - 8\pi T) = 0.$$

Substitute this into (4.20),

$$\Phi_0^{\beta'} \langle e_0, \Gamma^\dagger_{\beta'} \rangle + \Phi_0^{\alpha'} \langle e_0, \Gamma^\dagger_{\alpha'} \rangle = 0.$$

Recall (4.20),

$$\Phi_0^0 \langle e_0, \Gamma^\dagger_0 \rangle + \Phi_0^0 \langle e_0, \Gamma^\dagger_0 \rangle = 0.$$

So the desired result is proved. \square

Proof of Proposition 4.4. Combine lemma 4.6 and lemma 4.7, we see that Γ^\dagger_β satisfies the following initial value problem

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \Gamma^\dagger_\beta = F_\beta(\rho, g^\dagger; \Gamma^\dagger_\gamma, \partial \Gamma^\dagger_\gamma)$$

with initial data

$$\Gamma^\dagger_\beta|_{x^0=0} = 0, \quad \partial_0 \Gamma^\dagger_\beta|_{x^0=0} = 0.$$

Recall that x^0 is the time-time like direction and the symmetry of g^\dagger guarantees the hyperbolicity of $g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'}$.

We also observe that $\Gamma^\dagger = 0$ is a solution of this initial value problem. Then, by recalling the global hyperbolicity of g^\dagger , the desired result of uniqueness within the domain of determinacy is proved. \square

5 The augmented conformal formulation

5.1 Definition

In this section we will re-formulate again our system in order to get the local-in-time existence of the system (3.6). The main difficulty is that if we take into consideration of the fact that ρ is function of the scalar curvature, this is a third-order system without any special structure on its principle part (remark that the third order terms are $\partial\rho$ where ρ is a function of the second-order derivatives of $g^\dagger_{\alpha\beta}$.) To over pass this difficulty, we make the following trick. We replace in system (4.8) the constraint $e^{2\rho} = f'(R_g)$ by the trace equation which gives the evolution law of ρ . This leads to the formulation to be presented. Here we replace ρ by ϱ in order to emphasize that the relation $e^{2\rho} = f'(R_g)$ is no longer **a priori** valid.

For the convenience of discussion, we introduce the following notation. We define the tensor $N_{g\alpha\beta}^\ddagger$ by the following equation:

$$N_{g\alpha\beta}^\ddagger - \frac{1}{2}g_{\alpha\beta}\text{tr}(N_g^\ddagger) = e^{2\varrho}R_{\alpha\beta}^\ddagger - 6e^{2\varrho}\partial_\alpha\varrho\partial_\beta\varrho + \frac{1}{2}g_{\alpha\beta}^\ddagger W_2(\varrho). \quad (5.1)$$

Here we use ϱ to take the role of $\rho = \frac{1}{2}\ln f'(R_g)$ and we define $g_{\alpha\beta}^\ddagger = e^{2\varrho}g_{\alpha\beta}$. We also use the notation ∇^\ddagger , R_α^\ddagger and R^\ddagger for the connection, the Ricci curvature tensor and the scalar curvature associated with the metric g^\ddagger . Also, $\Gamma_{\alpha\beta}^{\ddagger\gamma}$ refers to the Christoffel symbols of g^\ddagger under a natural frame, and $\Gamma^{\ddagger\alpha\beta} := g^{\ddagger\alpha\beta}\Gamma_{\alpha\beta}^{\ddagger\gamma}$ and $\Gamma^{\ddagger\gamma} := g^{\ddagger\gamma\alpha}\Gamma_{\alpha\beta}^{\ddagger\gamma}$.

Definition 5.1. *The conformal augmented formulation for the nonlinear gravity theory is the following differential system:*

$$N_{g\alpha\beta}^\ddagger - \frac{1}{2}e^{2\varrho}(\partial_\alpha\Gamma_{\beta}^{\ddagger\alpha} + \partial_\beta\Gamma_{\alpha}^{\ddagger\beta}) = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\text{tr}(T)). \quad (5.2a)$$

$$\square_{g^\ddagger}\varrho = \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{f(\theta(\varrho))}{6e^{4\varrho}} + \frac{4\pi\text{tr}(T)}{3e^{4\varrho}}g_{\alpha\beta}^\ddagger. \quad (5.2b)$$

A priori, the variable ϱ becomes an independent unknown that is, it is no longer regarded as a function of the scalar curvature R_g . Then this system becomes a second-order system.

First we calculate the divergence of the tensor $N_{g\alpha\beta}^\ddagger$. This is conclude in the following lemma:

Lemma 5.2. *When (5.2b) holds, the following identity holds:*

$$\nabla^{\ddagger\alpha}N_{g\alpha\beta}^\ddagger = e^{-2\varrho}(2g^{\alpha\alpha'}\partial_{\alpha'}\varrho N_{g\alpha\beta}^\ddagger - \text{tr}(8\pi T)\partial_\beta\varrho). \quad (5.3)$$

Proof. This is by direct calculation. First, by (5.2),

$$N_{g\alpha\beta}^\ddagger = e^{2\varrho}G_{\alpha\beta}^\ddagger - 6e^{2\varrho}(\partial_\alpha\varrho\partial_\beta\varrho - \frac{1}{2}g_{\alpha\beta}|\nabla\varrho|_{g^\ddagger}^2) - \frac{1}{2}g_{\alpha\beta}^\ddagger W_2(\varrho)$$

where

$$G_{\alpha\beta}^\ddagger := R_{\alpha\beta}^\ddagger - \frac{1}{2}g_{\alpha\beta}^\ddagger R^\ddagger$$

is the Einstein tensor associated to the metric g^\ddagger .

Now we remark the following identities:

$$\nabla^{\ddagger\alpha\beta}G_{\alpha\beta}^{(\varrho)} = 0,$$

$$\nabla^{\ddagger\alpha\beta}(\partial_\alpha\varrho\partial_\beta\varrho - \frac{1}{2}g_{\alpha\beta}|\nabla\varrho|_{g^\ddagger}^2) = \partial_\beta\varrho\square_{g^\ddagger}\varrho.$$

For the convenience of expression, we introduce the following notation: Let the application $s(\cdot)$

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R} \\ r &\longrightarrow s(r) = \frac{1}{2}\ln f'(r) \end{aligned}$$

and we denote by $\theta(\cdot)$ the (local) inverse of $s(\cdot)$ near 0. Then

$$\begin{aligned} \nabla^{\ddagger\alpha\beta}(g_{\alpha\beta}^\ddagger W_2(\varrho)) &= \partial_\beta(W_2(\varrho)) \\ &= \partial_\beta(e^{-2\varrho}f(\theta(\varrho)) - \theta(\varrho)) \\ &= -2e^{-2\varrho}\partial_\beta\varrho f(\theta(\varrho)) + e^{-2\varrho}f'(\theta(\varrho))\theta'(\varrho)\partial_\beta\varrho - \theta'(\varrho)\partial_\beta\varrho \\ &= -2e^{-2\varrho}\partial_\beta\varrho f(\theta(\varrho)) + e^{-2\varrho}e^{2\varrho}\theta'(\varrho)\partial_\beta\varrho - \theta'(\varrho)\partial_\beta\varrho \\ &= -2e^{-2\varrho}\partial_\beta\varrho f(\theta(\varrho)). \end{aligned}$$

With these identities we can calculate the divergence of $N_{g_{\alpha\beta}}^\ddagger$:

$$\begin{aligned}
\nabla^{\ddagger\alpha} N_{g_{\alpha\beta}}^\ddagger &= 2e^{2\varrho} G_{\alpha\beta}^\varrho \nabla^{\ddagger\alpha} \varrho - 12e^{2\varrho} (\partial_\alpha \varrho \partial_\beta \varrho - \frac{1}{2} g_{\alpha\beta} |\nabla \varrho|_{g^\ddagger}^2) \nabla^{\ddagger\alpha} \varrho \\
&\quad - 6e^{2\varrho} \partial_\beta \varrho \square_{g^\ddagger} \varrho + e^{2\varrho} \partial_\beta f(\theta(\varrho)) \\
&= 2e^{2\varrho} G_{\alpha\beta}^\varrho \nabla^{\ddagger\alpha} \varrho - 12e^{2\varrho} (\partial_\alpha \varrho \partial_\beta \varrho - \frac{1}{2} g_{\alpha\beta} |\nabla \varrho|_{g^\ddagger}^2) \nabla^{\ddagger\alpha} \varrho - g_{\alpha\beta}^\ddagger W_2(\varrho) \nabla^{\ddagger\alpha} \varrho \\
&\quad - 6e^{2\varrho} \partial_\beta \varrho \square_{g^\ddagger} \varrho + e^{-2\varrho} \partial_\beta f(\theta(\varrho)) + g_{\alpha\beta}^\ddagger W_2(\varrho) \nabla^{\ddagger\alpha} \varrho \\
&= 2N_{g_{\alpha\beta}}^\ddagger \nabla^{\ddagger\alpha} \varrho - 6e^{2\varrho} \partial_\beta \varrho \left(\square_{g^\ddagger} - \frac{f(\theta(\varrho))}{6e^{4\varrho}} - \frac{W_2(\varrho)}{6e^{2\varrho}} \right).
\end{aligned}$$

Then by (5.7c),

$$\square_{g^\ddagger} \varrho = \frac{f(\theta(\varrho))}{6e^{4\varrho}} + \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{4\pi \text{tr}(T)}{3e^{4\varrho}}.$$

Then the desired result is proved by substitution. \square

Then by this lemma, we see that the evolution law of T is given by

$$\nabla^{\ddagger\alpha} T_{\alpha\beta} = (2g^{\delta\gamma} \partial_\gamma \varrho T_{\gamma\beta} - \text{tr}(T) \partial_\beta \varrho) e^{-2\varrho}, \quad (5.4)$$

Then there arise the following questions: will the relation $e^{2\varrho} = f'(R_g)$ (with $g_{\alpha\beta}^\ddagger = e^{2\varrho} g_{\alpha\beta}$) be guaranteed by (5.2)? The following subsections of this section are devoted to answer this question.

First, in the second subsection, we will re-formulate the initial value problem for the augmented conformal formulation, which we have formulated for the nonlinear field equation in both Jordan and Einstein metric.

Then in the third subsection, to make the principle part of (5.2) hyperbolic, we will reduce this system in a wave coordinate. We will prove in the fourth subsection that once the wave constraint equations $\Gamma^{\ddagger\lambda} = 0$ hold on the initial slice, then the augmentation relation $e^{2\varrho} = f'(R_g)$ is guaranteed by (5.8).

5.2 Formulation of the initial value problem for the augmented conformal formulation

In this section we formulate the initial value problem for the augmented conformal formulation. This is rather a repeat of the definition 3.7 and 3.8. But we would rather like to do this for the clarity of the following discussion.

As in the classical case, define the following initial data set for this conformal formulation.

Definition 5.3. *An initial data set for the augmented conformal formulation of nonlinear gravity theory $(\overline{M}, \overline{g}^\ddagger, K^\ddagger, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$ consists the following data:*

- a 3-dimensional manifold \overline{M} endowed with a Riemannian metric \overline{g}^\ddagger and a symmetric $(0, 2)$ -tensor field K^\ddagger ,
- two scalar fields denoted by ϱ_0 and ϱ_1 on \overline{M} and representing the (to-be-constructed) conformal factor and its time derivative,
- two scalar field ϕ_0^\ddagger and ϕ_1^\ddagger defined on \overline{M} ,

- which are assumed to satisfy the **Hamiltonian constraint of nonlinear gravity of augmented conformal formulation**

$$\overline{R}^\ddagger - K^\ddagger_{ij} K^\ddagger{}^{ij} + (K^\ddagger_j{}^j)^2 = 8e^{-2\rho} \left((\phi_1^\ddagger)^2 + \overline{g}^\ddagger{}^{ij} \partial_i \phi_0^\ddagger \partial_j \phi_0^\ddagger \right) + 6(\varrho_1)^2 + 6\overline{g}^\ddagger{}^{ij} \partial_i \varrho_0 \partial_j \varrho_0 - e^{-2\varrho_0} W_2(\varrho_0), \quad (5.5)$$

and the **momentum constraint of augmented conformal formulation**

$$\partial_j K^\ddagger{}^i{}_i - \overline{\nabla}^\ddagger{}_i K^\ddagger{}^i{}_j = \frac{\phi_1^\ddagger \partial_j \phi_0^\ddagger}{e^{2\varrho_0}} + 6\varrho_1 \partial_j \varrho_0. \quad (5.6)$$

Here \overline{R}^\ddagger and $\overline{\nabla}^\ddagger$ refers to the scalar curvature and connection with respect to the metric \overline{g}^\ddagger .

Then similarly, we formulate the **initial value problem of augmented conformal formulation**.

Definition 5.4 (Formulation of the initial value problem of augmented conformal formulation). *Given an initial data set $(\overline{M}, \overline{g}^\ddagger, K^\ddagger, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$ as in Definition 5.3, the initial value problem associated with the nonlinear gravity theory consists of finding a Lorentzian manifold (M, g) and a two-tensor field $T_{\alpha\beta}$ on M*

1. The augmented conformal metric g^\ddagger is defined with the relation $g^\ddagger{}_{\alpha\beta} = e^{2\varrho} g_{\alpha\beta}$ with the conformal factor ϱ satisfying the evolution equation (5.7c).
2. The augmented conformal field equations (5.7) are satisfied.
3. There exists an embedding $i : \overline{M} \rightarrow M$ with pull back metric $\overline{g}^\ddagger = i^* g^\ddagger$ and second fundamental form K^\ddagger .
4. The field ϱ_0 coincides with the restriction of the conformal factor ϱ on \overline{M} , while ϱ_1 coincides with the Lie derivative $\mathcal{L}_{n^\ddagger} \varrho$ restricted to \overline{M} , where n^\ddagger denotes the normal unit vector of \overline{M} (with respect to g^\ddagger).
5. The scalar fields $\phi_0^\ddagger, \phi_1^\ddagger$ coincides with the restriction of $\phi, \mathcal{L}_{n^\ddagger} \phi$ on \overline{M} .

Such a solution to (3.6) is referred to as a **nonlinear gravity development of the initial data set** $(\overline{M}, \overline{g}^\ddagger, K^\ddagger, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$.

Exactly as formulated in Section 3, the constraint equations can be written in the following geometric form: The **Hamiltonian constraint**:

$$N_{g_{00}} = 8\pi T_{00}.$$

The **momentum constraint**:

$$N_{g_{0a}} = 8\pi T_{0a}.$$

5.3 Wave-reduced augmented conformal formulation

As explained above, in this subsection we are going to reduce the conformal augmented system (5.2) in a wave coordinate system in order to form a hyperbolic principle part. This leads to the following formulation (where we replaced the wave constraints by the evolution law of T given by (5.3)):

Definition 5.5. *The conformal augmented formulation for the nonlinear gravity theory is the following differential system:*

$$N_{g_{\alpha\beta}}^\ddagger - \frac{1}{2} g_{\alpha\beta} \text{tr} (N_g^\ddagger) - \frac{1}{2} e^{2\varrho} (\partial_\alpha \Gamma^\ddagger{}_\beta + \partial_\beta \Gamma^\ddagger{}_\alpha) = 8\pi (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \text{tr} (T)). \quad (5.7a)$$

$$\nabla^\ddagger{}^\alpha T_{\alpha\beta} = (2g^{\delta\gamma} \partial_\gamma \varrho T_{\gamma\beta} - \text{tr} (T) \partial_\beta \varrho) e^{-2\varrho}, \quad (5.7b)$$

$$\square_{g^\ddagger} \varrho + \Gamma^\ddagger{}^\delta \partial_\delta \varrho = \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{f(\theta(\varrho))}{6e^{4\varrho}} + \frac{4\pi \text{tr} (T)}{3e^{4\varrho}} g^\ddagger{}_{\alpha\beta}. \quad (5.7c)$$

By lemma 4.1, the following PDE form of the system (5.7) is direct:

Lemma 5.6. *The conformal augmented formulation for the nonlinear gravity theory is the following differential system*

$$g^{\ddagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g^{\ddagger}_{\alpha\beta} = F_{\alpha\beta}(g^{\ddagger}; \partial g^{\ddagger}, \partial g^{\ddagger}) - 12\partial_{\alpha}\varrho \partial_{\beta}\varrho + \frac{W_2(\varrho)}{e^{2\varrho}} g^{\ddagger}_{\alpha\beta} - 16\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta} \text{tr}(T)), \quad (5.8a)$$

$$\nabla^{\ddagger\alpha} T_{\alpha\beta} = (2g^{\delta\gamma} \partial_{\gamma} \varrho T_{\gamma\beta} - \text{tr}(T) \partial_{\beta} \varrho) e^{-2\varrho}, \quad (5.8b)$$

$$g^{\ddagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \varrho = \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{f(\theta(\varrho))}{6e^{4\varrho}} + \frac{4\pi \text{tr}(T)}{3e^{4\varrho}}. \quad (5.8c)$$

where

$$g^{\ddagger}_{\alpha\beta} = e^{2\varrho} g_{\alpha\beta}$$

which is **not** a priori coincide with g^{\ddagger} .

5.4 Preservation of the constraints

In this subsection we will solve the first problem of preservation of the constraints. The answer is concluded by the following proposition.

Proposition 5.7. *Let (M, g^{\ddagger}) be a Lorentzian manifold with space-like foliation $M = [0, t_{max})$ with the sign of g^{\ddagger} $(-, +, +, +)$. Let T be a C^1 symmetric two tensor and ϱ be a C^4 scalar fields defined in $(-\epsilon, \epsilon) \times \mathbb{R}^3$. Furthermore, assume that g^{\ddagger} is in class C^4 and the triple $(g^{\ddagger}_{\alpha\beta}, \phi, \varrho)$ is a classical solution of (5.8). Denote by $g_{\alpha\beta} := e^{-2\varrho} g^{\ddagger}_{\alpha\beta}$ a metric conformal to g^{\ddagger} . Then if on the slice $\{x^0 = 0\}$, the constraint equations (5.5), (5.6) together with the following constraint equations hold:*

$$g^{\ddagger\alpha\beta} \Gamma^{\ddagger\gamma}_{\alpha\beta} = 0, \quad (5.9)$$

where R_g is the scalar curvature associated with $g_{\alpha\beta}$ and $\Gamma^{\ddagger\gamma}_{\alpha\beta}$ are the Christoffel symbols of g^{\ddagger} . Then (5.9) hold in $(-\epsilon, \epsilon) \times \mathbb{R}^3$. Furthermore,

$$e^{2\varrho} = e^{2\rho} = \frac{1}{2} \ln(f'(R_g)) \quad (5.10)$$

holds in $(-\epsilon, \epsilon) \times \mathbb{R}^3$. So that $(g^{\ddagger}_{\alpha\beta}, \phi, \varrho)$ is also a solution of (3.6).

The rest of this section is devoted to the proof of this proposition. From now on in this section we assume that the function $g^{\ddagger}_{\alpha\beta}$, ϱ and $T_{\alpha\beta}$ are regular enough such that all the derivatives appear in the equations are continuous.

To prove this proposition, we make the following preparations.

First we calculate the divergence of the tensor $N^{\ddagger}_{g_{\alpha\beta}}$. This is conclude in the following lemma:

Lemma 5.8. *When (5.7c) holds, the following identity holds:*

$$\nabla^{\ddagger\alpha} N^{\ddagger}_{g_{\alpha\beta}} = e^{-2\varrho} (2g^{\alpha\alpha'} \partial_{\alpha'} \varrho N^{\ddagger}_{g_{\alpha\beta}} - \text{tr}(8\pi T) \partial_{\beta} \varrho) + 6e^{2\varrho} \partial_{\beta} \varrho \Gamma^{\ddagger\delta}_{\alpha\beta} \partial_{\delta} \varrho. \quad (5.11)$$

Proof. This is by direct calculation. First, by (5.2),

$$N^{\ddagger}_{g_{\alpha\beta}} = e^{2\varrho} G^{\ddagger}_{\alpha\beta} - 6e^{2\varrho} (\partial_{\alpha} \varrho \partial_{\beta} \varrho - \frac{1}{2} g_{\alpha\beta} |\nabla \varrho|_{g^{\ddagger}}^2) - \frac{1}{2} g^{\ddagger}_{\alpha\beta} W_2(\varrho)$$

where

$$G^{\ddagger}_{\alpha\beta} := R^{\ddagger}_{\alpha\beta} - \frac{1}{2} g^{\ddagger}_{\alpha\beta} R^{\ddagger}$$

is the Einstein tensor associated to the metric g^{\ddagger} .

Now we remark the following identities:

$$\begin{aligned}\nabla^{\dagger\alpha\beta} G_{\alpha\beta}^{(\varrho)} &= 0, \\ \nabla^{\dagger\alpha\beta} \left(\partial_\alpha \varrho \partial_\beta \varrho - \frac{1}{2} g_{\alpha\beta} |\nabla \varrho|_{g^\dagger}^2 \right) &= \partial_\beta \varrho \square_{g^\dagger} \varrho.\end{aligned}$$

For the convenience of expression, we introduce the following notation: Let the application $s(\cdot)$

$$\begin{aligned}\mathbb{R} &\longrightarrow \mathbb{R} \\ r &\longrightarrow s(r) = \frac{1}{2} \ln f'(r)\end{aligned}$$

and we denote by $\theta(\cdot)$ the (local) inverse of $s(\cdot)$ near 0. Then

$$\begin{aligned}\nabla^{\dagger\alpha\beta} (g_{\alpha\beta}^\dagger W_2(\varrho)) &= \partial_\beta (W_2(\varrho)) \\ &= \partial_\beta (e^{-2\varrho} f(\theta(\varrho)) - \theta(\varrho)) \\ &= -2e^{-2\varrho} \partial_\beta \varrho f(\theta(\varrho)) + e^{-2\varrho} f'(\theta(\varrho)) \theta'(\varrho) \partial_\beta \varrho - \theta'(\varrho) \partial_\beta \varrho \\ &= -2e^{-2\varrho} \partial_\beta \varrho f(\theta(\varrho)) + e^{-2\varrho} e^{2\varrho} \theta'(\varrho) \partial_\beta \varrho - \theta'(\varrho) \partial_\beta \varrho \\ &= -2e^{-2\varrho} \partial_\beta \varrho f(\theta(\varrho)).\end{aligned}$$

With these identities we can calculate the divergence of $N_{g_{\alpha\beta}}^\dagger$:

$$\begin{aligned}\nabla^{\dagger\alpha} N_{g_{\alpha\beta}}^\dagger &= 2e^{2\varrho} G_{\alpha\beta}^e \nabla^{\dagger\alpha} \varrho - 12e^{2\varrho} \left(\partial_\alpha \varrho \partial_\beta \varrho - \frac{1}{2} g_{\alpha\beta} |\nabla \varrho|_{g^\dagger}^2 \right) \nabla^{\dagger\alpha} \varrho \\ &\quad - 6e^{2\varrho} \partial_\beta \varrho \square_{g^\dagger} \varrho + e^{2\varrho} \partial_\beta f(\theta(\varrho)) \\ &= 2e^{2\varrho} G_{\alpha\beta}^e \nabla^{\dagger\alpha} \varrho - 12e^{2\varrho} \left(\partial_\alpha \varrho \partial_\beta \varrho - \frac{1}{2} g_{\alpha\beta} |\nabla \varrho|_{g^\dagger}^2 \right) \nabla^{\dagger\alpha} \varrho - g_{\alpha\beta}^\dagger W_2(\varrho) \nabla^{\dagger\alpha} \varrho \\ &\quad - 6e^{2\varrho} \partial_\beta \varrho \square_{g^\dagger} \varrho + e^{-2\varrho} \partial_\beta f(\theta(\varrho)) + g_{\alpha\beta}^\dagger W_2(\varrho) \nabla^{\dagger\alpha} \varrho \\ &= 2N_{g_{\alpha\beta}}^\dagger \nabla^{\dagger\alpha} \varrho - 6e^{2\varrho} \partial_\beta \varrho \left(\square_{g^\dagger} - \frac{f(\theta(\varrho))}{6e^{4\varrho}} - \frac{W_2(\varrho)}{6e^{2\varrho}} \right).\end{aligned}$$

Then by (5.7c),

$$\square_{g^\dagger} \varrho = -\Gamma^{\dagger\delta} \partial_\delta \varrho + \frac{f(\theta(\varrho))}{6e^{4\varrho}} + \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{4\pi \text{tr}(T)}{3e^{4\varrho}}$$

Then the desired result is proved by substitution. \square

Lemma 5.9. (5.8a) leads to the following identity:

$$\square_{g^\dagger} \varrho = \frac{R_g}{6e^{2\varrho}} + \frac{W_2(\varrho)}{3e^{2\varrho}} + \frac{4\pi \text{tr}(T)}{3e^{4\varrho}} - \frac{g^{\alpha\beta} \partial_\alpha \Gamma_{\beta}^\dagger}{6e^{2\varrho}}. \quad (5.12)$$

Proof. This is also a direct calculation. Recall that (5.8a) is written as

$$e^{2\varrho} R_{\alpha\beta}^\dagger - 6e^{2\varrho} \partial_\alpha \varrho \partial_\beta \varrho + \frac{g_{\alpha\beta}^\dagger}{2} W_2(\varrho) - \frac{e^{2\varrho}}{2} (\partial_\alpha \Gamma_{\beta}^\dagger + \partial_\beta \Gamma_{\alpha}^\dagger) = 8\pi (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \text{tr}(T)). \quad (5.13)$$

Recall that (3.5) leads to

$$R_{\alpha\beta}^\dagger = R_{\alpha\beta} - 2(\nabla_\alpha \nabla_\beta \varrho - \partial_\alpha \varrho \partial_\beta \varrho) - (\square_g \varrho + 2g(\nabla \varrho, \nabla \varrho)) g_{\alpha\beta}$$

Substitute this into (5.13),

$$\begin{aligned}& e^{2\varrho} R_{\alpha\beta} - 2e^{2\varrho} \nabla_\alpha \nabla_\beta \varrho - 4e^{2\varrho} \partial_\alpha \varrho \partial_\beta \varrho - g_{\alpha\beta} e^{2\varrho} \square_g \varrho - 2e^{2\varrho} g_{\alpha\beta} g(\nabla \varrho, \nabla \varrho) + \frac{g_{\alpha\beta}^\dagger}{2} W_2(\varrho) \\ &= 8\pi (T_{\alpha\beta} - \frac{1}{2} \text{tr}(T) g_{\alpha\beta}) + \frac{1}{2} e^{2\varrho} (\partial_\alpha \Gamma_{\beta}^\dagger + \partial_\beta \Gamma_{\alpha}^\dagger).\end{aligned}$$

Taking the trace of this equation **with respect to g** ,

$$e^{2\varrho}R_g - 6e^{2\varrho}\square_g\varrho - 12e^{2\varrho}g(\nabla\varrho, \nabla\varrho) + 2e^{2\varrho}W_2(\varrho) = -8\pi\text{tr}(T) + e^{2\varrho}g^{\alpha\beta}\partial_\alpha\Gamma^\dagger_\beta$$

which can also be written as

$$\square_g\varrho + 2g(\nabla\varrho, \nabla\varrho) = \frac{R_g}{6} + \frac{W_2(\varrho)}{3} + \frac{4\pi\text{tr}(T)}{3e^{2\varrho}} - \frac{g^{\alpha\beta}}{6}\partial_\alpha\Gamma^\dagger_\beta.$$

Recall (3.7), the desired result is proved. \square

The following lemma is to clarify the geometric form of the system (5.8)

Lemma 5.10. *Let (M, g^\dagger) be a Lorentzian manifold. Suppose that g^\dagger in class C^4 and $T_{\alpha\beta}$ a tensor of class C^2 . If in a local coordinate system $\{x^0, x^1, x^2, x^3\}$, $(g^\dagger, T_{\alpha\beta})$ satisfy (5.8) Then the following equation holds in the domain of the coordinate system:*

$$g^{\dagger\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}\Gamma^\dagger_\beta = F_\beta(\varrho, g^\dagger; \Gamma^\dagger_\gamma, \partial\Gamma^\dagger_\gamma) \quad (5.14)$$

where $F_\beta(\varrho, g^\dagger; \cdot, \cdot)$ is a combination of linear and bilinear forms and we recall $\Gamma^{\dagger\gamma}_{\alpha\beta}$ the Christoffel symbol and

$$\Gamma^\dagger_\beta = g^\dagger_{\beta\beta'}\Gamma^{\dagger\beta'} = g^\dagger_{\beta\beta'}g^{\dagger\alpha\gamma}\Gamma^{\dagger\beta'}_{\alpha\gamma}.$$

Proof. Taking the trace of (5.7a) leads to the following equation

$$e^{2\varrho}g^{\alpha'\beta'}\partial_{\alpha'}\Gamma^\dagger_{\beta'} = -\text{tr}(N_g^\dagger - 8\pi T). \quad (5.15)$$

Combine this with (5.7a),

$$\frac{e^{2\varrho}}{2}(\partial_\alpha\Gamma^\dagger_\beta + \partial_\beta\Gamma^\dagger_\alpha - g_{\alpha\beta}g^{\alpha'\beta'}\partial_{\alpha'}\Gamma^\dagger_{\beta'}) = N_{g_{\alpha\beta}}^\dagger - 8\pi T_{\alpha\beta}. \quad (5.16)$$

Taking the divergence of this equation with respect to the metric g^\dagger , the left-hand-side is calculated by using (4.11):

$$\begin{aligned} & \frac{1}{2}\nabla^{\dagger\alpha}\left(e^{2\varrho}(\partial_\alpha\Gamma^\dagger_\beta + \partial_\beta\Gamma^\dagger_\alpha - g_{\alpha\beta}g^{\alpha'\beta'}\partial_{\alpha'}\Gamma^\dagger_{\beta'})\right) \\ &= \frac{1}{2}e^{2\varrho}\left(g^{\dagger\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}\Gamma^\dagger_\beta - g^{\dagger\alpha\alpha'}\Gamma^{\dagger\gamma}_{\alpha\alpha'}\partial_\gamma\Gamma^\dagger_\beta - g^{\dagger\alpha\alpha'}\Gamma^{\dagger\delta}_{\alpha'\beta}\partial_\delta\Gamma^\dagger_\alpha - \partial_\beta g^{\dagger\alpha'\beta'}(\partial_{\alpha'}\Gamma^\dagger_{\beta'})\right) \\ & \quad + e^{2\varrho}\left(e^{2\varrho}(\partial_\alpha\Gamma^\dagger_\beta + \partial_\beta\Gamma^\dagger_\alpha - g_{\alpha\beta}g^{\alpha'\beta'}\partial_{\alpha'}\Gamma^\dagger_{\beta'})\right)\nabla^{\dagger\alpha}\varrho \\ &=: \frac{1}{2}e^{2\varrho}g^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}\Gamma^\dagger_\beta + \tilde{F}_\beta(\varrho, g^\dagger; \Gamma^\dagger_\gamma, \partial\Gamma^\dagger_\gamma) \end{aligned} \quad (5.17)$$

where \tilde{F} is a combination of linear and bilinear forms on Γ^\dagger_γ and $\partial\Gamma^\dagger_\gamma$ depending on ϱ , g^\dagger and their derivatives.

The right-hand-side is calculated by using (5.7b) and (5.11):

$$\begin{aligned} & \nabla^{\dagger\alpha}(N_{g_{\alpha\beta}}^\dagger - 8\pi T_{\alpha\beta}) \\ &= e^{-2\varrho}(2g^{\alpha\alpha'}\partial_{\alpha'}\varrho N_{g_{\alpha\beta}}^\dagger - \text{tr}(8\pi T)\partial_\beta\varrho) + 6e^{2\varrho}\partial_\beta\varrho\Gamma^{\dagger\delta}\partial_\delta\varrho - 8\pi e^{-2\varrho}(2g^{\delta\gamma}\partial_\gamma\varrho T_{\gamma\beta} - \text{tr}(T)\partial_\beta\varrho) \\ &= e^{-2\varrho}\left(2g^{\gamma\alpha}\partial_\gamma\varrho(N_{g_{\alpha\beta}}^\dagger - 8\pi T_{\alpha\beta}) - \text{tr}(N_g^\dagger - 8\pi T)\partial_\beta\varrho\right) + 6\partial_\beta\varrho\Gamma^{\dagger\delta}\partial_\delta\varrho. \end{aligned}$$

Then by (5.15) and (5.16),

$$\nabla^{\dagger\alpha}(N_{g_{\alpha\beta}}^\dagger - 8\pi T_{\alpha\beta}) = g^{\gamma\alpha}\partial_\gamma\varrho(\partial_\alpha\Gamma^\dagger_\beta + \partial_\beta\Gamma^\dagger_\alpha) + 6\partial_\beta\varrho\Gamma^{\dagger\delta}\partial_\delta\varrho \quad (5.18)$$

which is a linear form of the functions Γ^\dagger_α and $\partial_\gamma\Gamma^\dagger_\alpha$.

Combine (5.17) and (5.18) with (5.16), the desired result is established. \square

Lemma 5.11. *Assume that on the initial slice $\{t = 0\}$ the equations (5.8) holds. Furthermore we assume that*

$$\Gamma^\ddagger_\lambda = 0$$

and the constraint equations (5.5) and (5.6) hold on the initial slice $\{t = 0\}$.

Then on the initial slice

$$\partial_t \Gamma^\ddagger_\lambda = 0.$$

Proof. We remark that the constraint equations (5.5) and (5.6) are equivalent to

$$N^\ddagger_{g_{0j}} = 8\pi T_{00},$$

and

$$N^\ddagger_{g_{00}} = 8\pi T_{00}.$$

Then the proof is exactly the same calculation which is performed in the proof of lemma 4.7 that we omit the detail. \square

Proof of proposition 5.7. By lemma 5.10 and lemma 5.11 and by the global hyperbolicity of the metric g^\ddagger , we see that in the spacetime M ,

$$\Gamma^\ddagger_\beta = 0. \tag{5.19}$$

Now recall the lemma 5.9. Combine (5.12) with (5.19),

$$\square_{g^\ddagger} \varrho = \frac{R_g}{6e^{2\varrho}} + \frac{W_2(\varrho)}{3e^{2\varrho}} + \frac{4\pi \text{tr}(T)}{3e^{4\varrho}}. \tag{5.20}$$

In the same way (5.7c) becomes

$$\square_{g^\ddagger} \varrho = \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{f(\theta(\varrho))}{6e^{4\varrho}} + \frac{4\pi \text{tr}(T)}{3e^{4\varrho}}. \tag{5.21}$$

Compare these two equations we get

$$R_g + W_2(\varrho) - f(\theta(\varrho)) = 0$$

which leads to (by the definition of W_2 given in)

$$\theta(\varrho) = R_g.$$

Then by the definition of θ , $e^{2\varrho} = R_g$. \square

6 Local existence theory – formulation and statement of the result

6.1 Overview

In this section we will establish the local existence of the Cauchy problem associated to system (5.8) with a type of initial data called asymptotic flat initial data and a special (but important and typical) nonlinear function f . In the following subsections we will make a sequence of notation and definition for the statement of the main result of local existence announced in theorem 6.2.

6.2 Construction of the initial data set for PDE system

Recall the mater model we are considering is the real massless scalar field with Jordan coupling. Recall the discussion made on section 3.3 (the equation (3.15)), the system (5.8) is written as

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} g^{\dagger}_{\alpha\beta} = F_{\alpha\beta}(g^{\dagger}; \partial g^{\dagger}, \partial g^{\dagger}) - 12\partial_{\alpha} \varrho \partial_{\beta} \varrho + \frac{W_2(\varrho)}{e^{2\varrho}} g^{\dagger}_{\alpha\beta} - 16\pi \partial_{\alpha} \phi \partial_{\beta} \phi, \quad (6.1a)$$

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \phi = 2g^{\dagger\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \varrho, \quad (6.1b)$$

$$g^{\dagger\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \varrho = \frac{W_2(\varrho)}{6e^{2\varrho}} + \frac{f(\theta(\varrho))}{6e^{4\varrho}} - \frac{4\pi}{3e^{2\varrho}} g^{\dagger\alpha'\beta'} \partial_{\alpha'} \phi \partial_{\beta'} \phi. \quad (6.1c)$$

Remark that this is a diagonalized quasi-linear wave system. To make the associated initial value problem well posed, the initial data should be composed by the following functions:

$$\begin{aligned} g^{\dagger}_{\alpha\beta}(0, x) &= g^{\dagger}_{0\alpha\beta}, & \partial_t g^{\dagger}_{\alpha\beta}(0, x) &= g^{\dagger}_{1\alpha\beta}, \\ \varrho(0, x) &= \varrho_0, & \partial_t \varrho(0, x) &= \varrho_1, \\ \phi(0, x) &= \phi_0 = \phi_0^{\dagger}, & \partial_t \phi(0, x) &= \phi_1 = \phi_1^{\dagger}. \end{aligned}$$

There are 24 functions to be supplied, but the geometrical initial data set $(\overline{M}, \overline{g}^{\dagger}, K^{\dagger}, \varrho_0, \varrho_1, \phi_0^{\dagger}, \phi_1^{\dagger})$ supplies only 16 functions.

To construct a solution of (6.1) which solves also (3.6), by proposition 5.7, the condition (5.9), (5.5) and (5.6) should be satisfied on the initial hypersurface. This leads to a nonlinear PDE system composed by eight equations. It can be expected that the 8 supplementary initial data can be determined by these 8 constraint equations. However it is not a trivial task. We will not investigate this system in detail but assume that such a system of 8 constraint equations plus 16 functions supplied by the geometric initial data set will determine a PDE initial data set.

We denote by $(\overline{M}, g^{\dagger}_{0\alpha\beta}, g^{\dagger}_{1\alpha\beta}, \varrho_0, \varrho_1, \phi_0, \phi_1)$ the PDE initial data determined by $(\overline{M}, \overline{g}^{\dagger}, K^{\dagger}, \varrho_0, \varrho_1, \phi_0^{\dagger}, \phi_1^{\dagger})$ and the constraint equations (5.5), (5.6) and (5.9)

6.3 Simplification of the $f(R)$ field equation

To clarify the nonlinear effect given by f , we focus on the second order term of f and for simplicity, we assume that

$$f(r) := r + \frac{\kappa}{2} r^2 \quad (6.2)$$

with $\kappa \geq 0$. Recall $e^{2\rho} = f'(R_g)$, then

$$e^{2\rho} = 1 + \kappa R_g, \quad R_g = \frac{e^{2\rho} - 1}{\kappa},$$

then

$$W_2(s) = \frac{f(r) - r f'(r)}{f'(r)} = \frac{r + \frac{\kappa}{2} r^2 - r(1 + \kappa r)}{1 + \kappa r} = -\frac{\kappa r^2}{2(1 + \kappa r)}$$

where

$$e^{2s} = f'(r) = 1 + \kappa r.$$

So

$$W_2(s) = -\frac{(e^{2s} - 1)^2}{2\kappa e^{2s}}.$$

The spacetime we will consider in the rest of this section, denoted by M , is equipped with a 1-parameterized space-like foliation

$$M = [0, T] \times M_t, \quad t \in [0, T]$$

and we assume that for each t , M_t is diffeomorphic to \mathbb{R}^3 . That is, M is diffeomorphic to $[0, T] \times \mathbb{R}^3$. We assume that in its canonical coordinate system (x^0, x^1, x^2, x^3) the metric g^\ddagger is of signature $(-, +, +, +)$.

We will work in the case where the space-time metric is “near” the Minkowski metric. So the following notation is found to be convenient: Denote by

$$h_{\alpha\beta} := g^\ddagger_{\alpha\beta} - m_{\alpha\beta}.$$

Then a triple $(g^\ddagger_{\alpha\beta}, \varrho, \phi)$ is also represented by $(h_{\alpha\beta}, \varrho, \phi)$.

When κ is given, we will write the triple $(h_{\alpha\beta}, \varrho, \phi)$ as $(h_{\alpha\beta}^\kappa, \varrho^\kappa, \phi^\kappa)$ to emphasize that the solution of (5.8) depends on the coefficient κ .

With these notation the system (5.8) is reduced in the following form:

$$\begin{aligned} (m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^\kappa))\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^\kappa &= F_{\alpha\beta}(h^\kappa; \partial h^\kappa, \partial h^\kappa) - 16\pi\partial_\alpha\phi^\kappa\partial_\beta\phi^\kappa \\ &\quad - 12\partial_\alpha\varrho^\kappa\partial_\beta\varrho^\kappa - \frac{(e^{2\varrho^\kappa} - 1)^2}{2\kappa e^{4\varrho^\kappa}}(m_{\alpha\beta} + h_{\alpha\beta}^\kappa), \end{aligned} \quad (6.3a)$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^\kappa))\partial_{\alpha'}\partial_{\beta'}\phi^\kappa = 2(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^\kappa))\partial_{\alpha'}\phi^\kappa\partial_{\beta'}\varrho^\kappa, \quad (6.3b)$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^\kappa))\partial_{\alpha'}\partial_{\beta'}\varrho^\kappa - \frac{e^{2\varrho^\kappa} - 1}{6\kappa e^{4\varrho^\kappa}} = -\frac{4\pi}{3e^{2\varrho^\kappa}}(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^\kappa))\partial_{\alpha'}\phi^\kappa\partial_{\beta'}\phi^\kappa, \quad (6.3c)$$

where

$$(H^{\alpha\beta}(h^\kappa) + m_{\alpha\beta})_{\alpha\beta} = ((m_{\alpha\beta} + h_{\alpha\beta}^\kappa)_{\alpha\beta})^{-1}, \quad (6.4)$$

h^κ denotes the matrix $h_{\alpha\beta}^\kappa$. We remark that $H^{\alpha\beta}(\cdot)$ are C^∞ functions defined in a compact neighborhood of 0 in \mathbb{R}^{10} , which means there exists a positive constant ε_0 such that if

$$\max_{1 \leq i \leq 10} |x^i| \leq \varepsilon_0,$$

then for all $k \in \mathbb{N}$, the k -th order derivatives of $H^{\alpha\beta}$, denoted by $D^k H^{\alpha\beta}$, is well defined at $x = (x^1, \dots, x^{10})$ and

$$\sup_{|x^i| \leq \varepsilon'} |D^k H^{\alpha\beta}(x)| \leq C(k, \varepsilon_0)$$

A classical result of linear algebra shows that when $\max_{\alpha\beta} h_{\alpha\beta} \leq \varepsilon_0$ with ε_0 sufficiently small, then

$$H^{\alpha\beta}(h) = -h_{\alpha\beta} + Q^{\alpha\beta}(h_{\alpha'\beta'}, h_{\alpha'\beta'}) (1 + R^{\alpha\beta}(h_{\alpha'\beta'})) \quad (6.5)$$

where $Q^{\alpha\beta}(\cdot, \cdot)$ is a quadratic form and

$$|R^{\alpha\beta}(h_{\alpha'\beta'})| \leq C(\varepsilon_0) \max_{\alpha'\beta'} |h_{\alpha'\beta'}|.$$

Remark that with this notation, the PDE initial data can be rewritten with $h_{\alpha\beta}$ as $(\overline{M}, h_0, h_1, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$ with

$$h_{\alpha\beta}|_{t=0} = h_{0\alpha\beta} := g^\ddagger_{0\alpha\beta} - m_{\alpha\beta}, \quad \partial_t h_{\alpha\beta} = h_{1\alpha\beta} := g^\ddagger_{1\alpha\beta}.$$

Some observations can be already made. This system is composed by 12 quasi-linear wave-type equations where 11 of them (those on $h_{\alpha\beta}^\kappa$ and ϕ^κ) are quasi-linear wave equations while the equation on ϱ^κ is a quasi-linear Klein-Gordon equation with a potential containing a linear defocusing part.

6.4 Vector fields and index

For the convenience expression, we introduce in this section a series of notation:

We introduce the vector fields which are the generators of the spacial rotation. They are commute to the linear wave and linear Klein-Gordon operator:

$$\Omega_{ij} := x_i \partial_j - x_j \partial_i = x^i \partial_j - x^j \partial_i,$$

where the index of coordinates are rising and lowering by the Minkowski metric. There are totally 3 such rotation vector fields. We denote by

$$\Omega_1 := \Omega_{12}, \quad \Omega_2 := \Omega_{23}, \quad \Omega_3 := \Omega_{13}.$$

Remark that

$$[\Omega_a, \square] = 0, \quad [\Omega_a, \square + 1] = 0,$$

that is, the Ω_a commute with the linear wave and Klein-Gordon operator.

For the convenience of proof, we introduce the following notion on multi-index. Let \mathcal{I} be a finite set composed by n letters $\mathcal{I} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. We call n the order of \mathcal{I} , denote by $|\mathcal{I}| = n$. We introduce a order " \preceq " on \mathcal{I} defined by

$$\alpha_i \preceq \alpha_j \quad \text{if and only if} \quad i \leq j.$$

The pair (\mathcal{I}, \preceq) is called a abstract multi-index. Obviously a subset of \mathcal{I} can also be regard as a multi-index with the restricted order. The order \preceq describe the location of each differential operator in a product.

Now we introduce the partition of a abstract index \mathcal{I} . Let \mathcal{J}_k be family of subset of \mathcal{I} with

$$\bigcup_{k=1}^m \mathcal{J}_k = \mathcal{I}, \quad \mathcal{J}_k \cap \mathcal{J}_{k'} = \emptyset.$$

we say $\{\mathcal{J}_k\}$ is a m -partition of \mathcal{I} and denote by $\sum_{k=1}^m \mathcal{J}_k = \mathcal{I}$. We observe that each \mathcal{J}_k can be regard as a multi-index and $\sum_{k=1}^m |\mathcal{J}_k| = |\mathcal{I}|$.

If for all $k = 1, 2, \dots, m$, $\mathcal{J}_k \neq \emptyset$, we say $\{\mathcal{J}_k\}$ is a proper m -partition of \mathcal{I} and denote by $\sum_{k=1}^m \mathcal{J}_k = \mathcal{I}, |\mathcal{J}_k| \geq 1$.

Now return to the case of multi-index in the context of differential operators. Let Z be a family of one-order differential operator, $Z = \{Z_1, Z_2, \dots, Z_p\}$. An n order multi-index on family Z is a map

$$\begin{aligned} I : \mathcal{I} &\longrightarrow \{1, 2, \dots, n\} \\ \alpha_i &\longrightarrow I(\alpha_i) \in \{1, 2, \dots, n\} \end{aligned}$$

We write $Z^I := Z_{I(\alpha_1)} \circ Z_{I(\alpha_2)} \circ \dots \circ Z_{I(\alpha_n)}$. We often abuse a bit the notation by writing $I = (\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ with $\alpha_i \in \{1, 2, \dots, n\}$ where we have valuate each letter α_i by $I(\alpha_i)$.

A m -partition of index I is defined as following: Let $\{\mathcal{J}_k\}$ be a (proper) m -partition of the abstract index \mathcal{I} . Then we restrict the map I on each ordered set \mathcal{J}_k . This gives m multi-index, denoted by $\{J_k\} = I(\{\mathcal{J}_k\})$. Then we call J_k a (proper) m -partition of I , denoted by $I = \sum_{k=1}^m J_k, (|J_k| \geq 1)$.

We often consider the set $\mathcal{P}(\mathcal{I}, m)$ composed by all the possible m -partition of \mathcal{I} . Then each partition in $\mathcal{P}(\mathcal{I}, m)$ can associated to a partition of I . We remark that if $\prod_{k=1}^m u_k$ is a product of m C^∞ functions, then

$$Z^I \left(\prod_{k=1}^m u_k \right) = \sum_{\{\mathcal{J}_k\} \in \mathcal{P}(\mathcal{I}, m)} \prod_{i=1}^m Z^{J_k} u_k$$

with $\{J_k\} = I(\{\mathcal{J}_k\})$. But for simplicity we often take the notation

$$Z^I \left(\prod_{k=1}^m h_k \right) = \sum_{\sum_{k=1}^m J_k = I} \prod_{k=1}^m Z^{J_k} u_k$$

when there no risk of misunderstanding.

6.5 Functional spaces

We recall that in the classical case, a initial data set of a Cauchy problem of the general relativity satisfies the constraint equations (2.23). This system leads to a nonlinear elliptic system and by the theorem of positive mass, the non-trivial part of the metric $\bar{g} - \bar{m}$ always has (at most) a decreasing rate as r^{-1} at spacial infinity which gives the definition of the ADM mass. If $\bar{g} - \bar{m}$ decreases faster than r^{-1} , then $\bar{g} = \bar{m}$.

In our case, the constraint equations (5.5) and (5.6) are much more complicated than the classical system, the parallel result to the positive mass theorem is totally known. But as we will concentrated on the convergence result where we take a ‘‘Einstein’’ initial data set (which is $\varrho_0 = \varrho_1 \equiv 0$) which satisfies the classical constraint equations. So we have to handle the quasi-linear wave system with initial data decreasing as r^{-1} at spacial infinity. These functions as (in general) not in $L^2(\mathbb{R}^3)$. So we need to construct our local solution with the aid of the following functional space.

For the convenience of discussion we first define the following norms for the $C_c^\infty(\mathbb{R}^3)$ functions which are the smooth functions with compact support set. A first norm to be introduced is

$$\|\varrho\|_{X^d} := \sum_{|I_1|+|I_2|\leq d} \|\partial_x^{I_1} \Omega^{I_2} \varrho\|_{L^2(\mathbb{R}^3)}.$$

The L^2 norm is with respect to the standard volume form, i.e. the Lebesgue measure.

The second norm defined for the $C_c^\infty(\mathbb{R}^3)$ functions is

$$\|\varrho\|_{X_P^{d+1}} := \sum_{|I_1|+|I_2|\leq d} \|\partial_x \partial_x^{I_1} \Omega^{I_2} \varrho\|_{L^2(\mathbb{R}^3)},$$

where $\partial_x \phi$ refers to the spacial gradient of ϕ .

The first functional space to be used in the following discussion, the space X_P^{d+1} , is defined as the completion of the norm $\|\cdot\|_{X_P^{d+1}}$ on the C_c^∞ functions.

We denote by $\|\cdot\|_{X_R^{d+1}} := \|\cdot\|_{X^d} + \|\cdot\|_{X_P^{d+1}}$. The second functional space, X_R^{d+1} , to be used in the following discussion, is defined as the completion of the norm $\|\cdot\|_{X_R^{d+1}}$ on the C_c^∞ functions.

The last functional space to be used, denoted by X_H^{d+1} , is defined as following: we define the following norms,

$$\begin{aligned} \|f\|_{\mathcal{B}^{-1}} &:= \sup_{r \geq 0} \{(1+r)|f|\} \\ \|f\|_{X_H^{d+1}} &:= \|f\|_{\mathcal{B}^{-1}} + \|f\|_{X_P^{d+1}}, \end{aligned}$$

We define the functional space X_H^{d+1} as the completion of the $C_c(\mathbb{R}^3)$ functions with respect of this norm.

The relations among these functional spaces are:

$$X^{d+1} \subset X_R^{d+1} \subset X^d, \quad X_R^d \subset X_P^d, \quad X_H^d \subset X_P^d.$$

And, as we will see in the next section, when $d \geq 2$, by (7.8),

$$X^d \subset \mathcal{B}^{-1}, \quad X_R^d \subset X_H^d \subset X_P^d.$$

Finally, for the triple $S_0 := (h_{0\alpha\beta}, \varrho_0, \phi_0)$, we define the norm:

$$\|S\|_{X_0^{d+1}} := \sum_{\alpha, \beta} \|h_{0\alpha\beta}\|_{X_H^{d+1}} + \|\varrho_0\|_{X_R^{d+1}} + \|\phi_0\|_{X_P^{d+1}}$$

and denote by $X_0^{d+1} = X_H^{d+1} \times X_R^{d+1} \times X_P^{d+1}$. For the triple $S_1 = (h_{1\alpha\beta}, \phi_1, \varrho_1)$, we define

$$\|S_1\|_{X_1^d} := \sum_{\alpha, \beta} \|h_{1\alpha\beta}\|_{X^d} + \|\varrho_1\|_{X^d} + \|\phi_1\|_{X^d}$$

Now we define the notion of asymptotically flat PDE initial data.

Definition 6.1 (Definition of the asymptotically flat initial data). *A PDE initial data set in wave coordinate system $(\overline{M}, h_{0\alpha\beta}, h_{1\alpha\beta}, \varrho_0, \varrho_1, \phi_0, \phi_1)$ of initial value problem defined in definition 3.7 is called asymptotically flat if*

- the initial slice M_0 is diffeomorphic to \mathbb{R}^3 and in its canonical coordinate system, the initial data set satisfies the wave constraint equations.
- in the canonical coordinate system (x^1, x^2, x^3) ,

$$\|h_{0\alpha\beta}\|_{\mathcal{B}^{-1}} \leq \varepsilon_0.$$

where ε_0 represents the ADM mass.

- the $\mathcal{B}^{-1}(\mathbb{R}^3)$ norm of $\varrho_0, \varrho_1, \phi_1$ are finite.

This means that the initial data decreasing as r^{-1} at spacial infinity.

A geometrical initial data $(\overline{M}, \overline{g}^\ddagger, K^\ddagger, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$ is called asymptotically flat, if it gives a asymptotically flat PDE initial data.

We recall that the components of the solution to the system (6.1) are functions defined in \mathbb{R}^4 with three spacial variables and one time variable. To describe these functions, we need to introduce the following functions space.

Let C_c^∞ denotes the smooth functions with compact support defined in \mathbb{R}^4 . Let $u \in C_c^\infty$. Then for all $t \in \mathbb{R}$, the function $u(t, \cdot)$ defined in \mathbb{R}^3 is a C_c^∞ function. This observation leads to the following definitions:

We define

$$\begin{aligned} \|u(t, \cdot)\|_{E^d} &:= \sum_{|I_1|+|I_2|\leq d} \|\partial^{I_1}\Omega^{I_2}u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ &= \sum_{k+|J_1|+|J_2|\leq d} \|\partial_t^k \partial_x^{J_1} \Omega^{J_2} u(t, \cdot)\| = \sum_{k+l\leq d} \|\partial_t^k u(t, \cdot)\|_{X^l}, \\ \|u(t, \cdot)\|_{E_P^{d+1}} &:= \sum_{\substack{\alpha \\ |I_1|+|I_2|\leq d}} \|\partial_\alpha \partial^{I_1} \Omega^{I_2} u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ &= \sum_{\alpha} \sum_{k+|J_1|+|J_2|\leq d} \|\partial_\alpha \partial_t^k \partial_x^{J_1} \Omega^{J_2} u(t, \cdot)\|_{L^2(\mathbb{R}^3)} = \sum_{k+l\leq d} \|\partial_t^k u(t, \cdot)\|_{X_P^l}, \\ \|u(t, \cdot)\|_{E_R^{d+1}} &:= \|u(t, \cdot)\|_{E^d} + \|u(t, \cdot)\|_{E_P^{d+1}}, \\ \|u(t, \cdot)\|_{E_H^{d+1}} &:= \|u(t, \cdot)\|_{\mathcal{B}^{-1}} + \|u(t, \cdot)\|_{E_P^{d+1}}. \end{aligned}$$

We also define the following norms on time interval $[0, T]$ for $p \in [1, \infty]$:

$$\begin{aligned} \|u\|_{L^p([0, T]; E^d)} &:= \left\| \|u(t, \cdot)\|_{E^d} \right\|_{L^p([0, T])}, & \|u\|_{L([0, T]; E_P^d)} &:= \left\| \|u(t, \cdot)\|_{E_P^d} \right\|_{L^p([0, T])}, \\ \|u\|_{L^p([0, T]; E_R^d)} &:= \left\| \|u(t, \cdot)\|_{E_R^d} \right\|_{L^p([0, T])}, & \|u\|_{L^p([0, T]; E_H^d)} &:= \left\| \|u(t, \cdot)\|_{E_H^d} \right\|_{L^p([0, T])}. \end{aligned}$$

Finally we define, for $p < \infty$, the functional space $L^p([0, T]; E^d)$, $L^p([0, T]; E_P^d)$, $L^p([0, T]; E_R^d)$ and $L^p([0, T]; E_H^d)$ the completion with respect to the corresponding norm on the $C_c^\infty(\mathbb{R}^4)$ function and denoted by $C([0, T]; E^d)$, $C([0, T]; E_P^d)$ and $C([0, T]; E_H^d)$ the completion of C_c^∞ functions with respect to the corresponding norms.

6.6 Statement of the theorem of local existence

For the convenience of discussion, we introduce the following norm on initial data $S_0 := (h_0, h_1, \phi_0, \phi_1, \varrho_0, \varrho_1)$:

$$\|S_0\|_{X_\kappa^{d+1}} := \max\{\|h_{0\alpha\beta}\|_{X_H^{d+1}}, \|h_{1\alpha\beta}\|_{X^d}, \kappa^{-(1/2)[d/2]+1/4}\|\phi_0\|_{X_P^{d+1}}, \\ \kappa^{-(1/2)[d/2]+1/4}\|\phi_1\|_{X^d}, \kappa^{-[d/2]-1/2}\|\varrho_0\|_{X_P^{d+1}}, \kappa^{-[d/2]-1/2}\|\varrho_1\|_{X^d}\}.$$

Now we are ready to state the local theorem of existence.

Theorem 6.2 (Local existence and uniform bound). *Let $d \geq 4$ be an integer. And assume that*

$$(h_{0\alpha\beta}, h_{1\alpha\beta}) \in X_H^{d+1} \times X^d, \quad (\phi_0, \phi_1) \in X_P^{d+1} \times X^d, \quad (\varrho_0, \varrho_1) \in X_R^{d+1} \times X^d.$$

Denote by $S_0 := (h_0, h_1, \phi_0, \phi_1, \varrho_0, \varrho_1)$ and suppose that $\|S_0\|_{X_\kappa^{d+1}} \leq \varepsilon \leq \varepsilon_0$ with $\varepsilon_0 > 0$ sufficiently small. Then there exist two constants $A > 0$ and $T^* > 0$ **independent** of κ such that on the time interval $[0, T^*]$, the Cauchy problem (6.3) (with $0 < \kappa \leq 1$) has a unique (distributional) solution $(h_{\alpha\beta}^\kappa, \phi^\kappa, \varrho^\kappa)$ in the following functional space:

$$\partial_t^k h_{\alpha\beta}^\kappa \in C([0, T], X_H^{d-k}) \cap C^1([0, T], X_H^{d-k-1}), \quad \partial_t^k \phi^\kappa \in C([0, T], X_P^{d-k}) \cap C^1([0, T], X_P^{d-k-1}), \\ \partial_t^k \varrho^\kappa \in C([0, T], X_R^{d-k}) \cap C^1([0, T], X_R^{d-k-1})$$

for $0 \leq k \leq d-1$. Furthermore the following estimates hold with constant **independent** of κ :

$$\|h_{\alpha\beta}^\kappa(t, \cdot)\|_{E_H^d} \leq A\varepsilon, \tag{6.6a}$$

$$\|\phi^\kappa\|_{E_P^d} \leq A\varepsilon, \tag{6.6b}$$

$$\|\varrho_n^\kappa\|_{E_P^d} + \kappa^{-1/2}\|\varrho_n^\kappa\|_{E^{d-1}} \leq A\varepsilon. \tag{6.6c}$$

where $C(\varepsilon_0)$ is a constant determined by ε_0 .

6.7 Existence result for nonlinear field equation

Equipped with the local existence theorem 6.2, we are able to build the local solution of the original Cauchy problem defined in definition 5.4. This is concluded in the following theorem.

Theorem 6.3 (Existence of nonlinear gravity development). *Let $(\bar{M}, \bar{g}^\ddagger, K^\ddagger, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$ be a initial data set of the Cauchy problem defined in definition 5.3 and assume its associated PDE initial data $S_0 = (h_0, h_1, \phi_0, \phi_1, \varrho_0, \varrho_1)$ is asymptotically flat and satisfies the conditions of theorem 6.2. Let $(h_{\alpha\beta}^\kappa, \phi^\kappa, \varrho^\kappa)$ the solution of (5.8) associated to S_0 . Then the space-time $([0, T] \times \mathbb{R}^3, g^\ddagger)$ is a nonlinear gravity development of the initial data $(\bar{M}, \bar{g}^\ddagger, K^\ddagger, \varrho_0, \varrho_1, \phi_0^\ddagger, \phi_1^\ddagger)$.*

Proof. We just inverse the procedure of formulation. Remark that the local solution $(h_{\alpha\beta}^\kappa, \phi^\kappa, \varrho^\kappa)$ constructed is sufficiently regular with $h = g^\ddagger - m$ sufficiently small. This guarantees that the metric g^\ddagger is globally hyperbolic on $[0, T] \times \mathbb{R}^3$. So we apply the result on preservation of constraints with is proportion (5.7). Once the constraints

$$e^{2\varrho} = \frac{1}{2} \ln f'(R_g), \\ \Gamma^{\ddagger\lambda} = 0$$

are verified, we see that the pair (g^\ddagger, ϕ) satisfies the conformal field equation (3.6). \square

7 Technical tools for local existence theory

7.1 Estimates on commutators

From this subsection we will make some preparations in order to prove the theorem 6.2. In this subsection we will concentrate on the estimates on commutators. First we remark the following basic commutative relation:

$$[\partial_a, \Omega_{bc}] = \delta_{ab}\partial_c - \delta_{ac}\partial_b, \quad [\partial_t, \Omega_{ab}] = 0. \quad (7.1)$$

Lemma 7.1. *Let u be a C^∞ function defined in $[0, T] \times \mathbb{R}^3$. Then for any multi-index I_1, I_2 with $|I_1| + |I_2| = n$, the following estimates hold:*

$$[\Omega^I, \partial_\alpha]u = \sum_{|J| < |I|} \Theta_{\alpha J}^{Ic} \partial_c \Omega^J u \quad (7.2)$$

where $\Theta_{\alpha J}^{Ic}$ are constants. When $\alpha = 0$, $\Theta_{\alpha J}^{Ic} = 0$.

Proof. First we remark that when $\alpha = 0$, $\partial_0 = \partial_t$ commutes with Ω^I .

When $\alpha > 0$, this is proved by induction on the order of $|J|$. When $|J| = 1$, the result is proved by (7.1). We denote by

$$[\Omega_a, \partial_\beta] = \theta_{a\beta}^c \partial_c.$$

Now assume that for $|J| \leq k$, (7.2) is valid. Then for $|J| = k$,

$$\Omega^J = \Omega_{a_1} \Omega^{J_1}$$

where $|J_1| = k - 1$. Then

$$\begin{aligned} [\Omega^J, \partial_a]u &= \Omega_{a_1} \Omega^{J_1} \partial_a u - \partial_a \Omega_{a_1} \Omega^{J_1} u = \Omega_{a_1} \Omega^{J_1} \partial_a u - \Omega_{a_1} \partial_a \Omega^{J_1} u + \Omega_{a_1} \partial_a \Omega^{J_1} u - \partial_a \Omega_{a_1} \Omega^{J_1} u \\ &= \Omega_{a_1} ([\Omega^{J_1}, \partial_a]u) + [\Omega_{a_1}, \partial_a] \Omega^{J_1} u \\ &= \Omega_{a_1} \left(\sum_{|K_1| < |J_1|} \Theta_{\alpha K_1}^{J_1 b} \partial_b \Omega^{K_1} u \right) + \theta_{a_1 a}^b \partial_b \Omega^{J_1} u \\ &= \sum_{|K_1| < |J_1|} \Theta_{\alpha K_1}^{J_1 b} \Omega_{a_1} \partial_b \Omega^{K_1} u + \theta_{a_1 a}^b \partial_b \Omega^{J_1} u \\ &= \sum_{|K_1| < |J_1|} \Theta_{\alpha K_1}^{J_1 b} \partial_b \Omega_{a_1} \Omega^{K_1} u + \theta_{a_1 a}^b \partial_b \Omega^{J_1} u + \sum_{|K_1| < |J_1|} \Theta_{\alpha K_1}^{J_1 b} [\Omega_{a_1}, \partial_b] \Omega^{K_1} u \\ &= \sum_{|K_1| < |J_1|} \Theta_{\alpha K_1}^{J_1 b} \partial_b \Omega_{a_1} \Omega^{K_1} u + \theta_{a_1 a}^b \partial_b \Omega^{J_1} u + \sum_{|K_1| < |J_1|} \Theta_{\alpha K_1}^{J_1 b} \theta_{a_1 b}^c \partial_c \Omega^{K_1} u. \end{aligned}$$

This concludes the desired result. \square

Lemma 7.2. *Let u be a C_c^∞ function defined in $[0, T] \times \mathbb{R}^3$. Then the following estimates hold for $|I_1| + |I_2| = d$:*

$$|\partial_x^{I_1} \Omega^{I_2} \partial_x u| \leq |\partial_x \partial_x^{I_1} \Omega^{I_2} u| + C(d) \sum_{|I_2| < |I_2|} |\partial_x \partial_x^{I_1} \Omega^{I_2} u|, \quad (7.3a)$$

$$|\partial_x \partial_x^{I_1} \Omega^{I_2} u| \leq |\partial_x^{I_1} \Omega^{I_2} \partial_x u| + C(d) \sum_{|I_1| + |I_2| < d} |\partial_x^{I_1} \Omega^{I_2} \partial_x u|, \quad (7.3b)$$

$$|[\partial_x^{I_1} \Omega^{I_2}, \partial_\alpha \partial_\beta]u| \leq C(d) \sum_{\substack{a, \alpha \\ |J_2| < |I_2|}} |\partial_a \partial_\alpha \partial_x^{I_1} \Omega^{J_2} u|, \quad (7.3c)$$

$$|\partial_r^k \Omega^{I_1} u(t, x)| \leq C(k) \sum_{|J_1| \leq k} |\partial_x^{J_1} \Omega^{I_1} u(t, x)|, \quad \text{for } |x| \geq 1, \quad (7.3d)$$

$$\begin{aligned}
|[\partial_x^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta]u| &\leq \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2 \\ |J'_1|+|J'_2|>0}} |\partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta}| |\partial_\alpha \partial_\beta \partial_x^{J_1} \Omega^{J_2} u| \\
&+ C(d) \sum_{\substack{J_1+J'_1=I_1 \\ |J_2|+|J'_2|<|I_2| \\ \alpha, \beta, \alpha', a}} |\partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta}| |\partial_{\alpha'} \partial_a \partial_x^{J_1} \Omega^{J_2} u|
\end{aligned} \tag{7.3e}$$

where $H^{\alpha\beta}$ are C^∞ functions.

Proof. We remark that from (7.2), the following identity is direct:

$$[\partial_x^{I_1} \Omega^{I_2}, \partial_\alpha] = \sum_{|K|<|I_2|} \Theta_{\alpha K}^{I_2 b} \partial_b \partial_x^{I_1} \Omega^K u.$$

- To prove (7.3a), we remark that

$$[\partial_x^{I_1} \Omega^{I_2}, \partial_\alpha]u = \partial_x^{I_1} ([\Omega^{I_1}, \partial_\alpha]u) = \sum_{c, |J_2| \leq |I_2|} \partial_x^{I_1} (\Theta_{\alpha J_2}^{I_1 c} \partial_c \Omega^{J_2} u) = \sum_{c, |J_2| \leq |I_2|} \Theta_{\alpha J_2}^{I_1 c} \partial_c \partial_x^{I_1} \Omega^{J_2} u$$

and

$$\sum_{c, |J_2| \leq |I_2|} |\Theta_{\alpha J_2}^{I_1 c} \partial_c \partial_x^{I_1} \Omega^{J_2} u| \leq C(d) \sum_{c, |I_2'| < |I_2|} |\partial_c \partial^{I_1} \Omega^{I_2'} u|. \tag{7.4}$$

This proves (7.3a).

- (7.3b) is proved by induction on d . We see clearly that (7.3b) holds for $d = 0$. If it holds for $d \leq k$, let us prove the case $d = k + 1$:

$$|\partial_\alpha \partial^{I_1} \Omega^{I_2} u| \leq |\partial^{I_1} \Omega^{I_2} \partial_\alpha u| + |[\partial_\alpha, \partial^{I_1} \Omega^{I_2}]u|.$$

Then by (7.4),

$$|\partial_\alpha \partial^{I_1} \Omega^{I_2} u| \leq |\partial^{I_1} \Omega^{I_2} \partial_\alpha u| + C(d) \sum_{c, |I_2'| < |I_2|} |\partial_c \partial^{I_1} \Omega^{I_2'} u|$$

Remark here, $|I_2'| \leq |I_2| - 1 \leq k$. Then we apply the assumption of induction on the second term in right-hand-side:

$$|\partial_c \partial^{I_1} \Omega^{I_2'} u| \leq |\partial^{I_1} \Omega^{I_2'} \partial_c u| + C(d-1) \sum_{c', |I_2''| > |I_2'|} |\partial^{I_1} \Omega^{I_2''} \partial_{c'} u|$$

which proves (7.3b).

- The proof of (7.3c) is a direct application of (7.2).

$$\begin{aligned}
[\partial_x^{I_1} \Omega^{I_2}, \partial_\alpha \partial_\beta]u &= [\partial_x^{I_1} \Omega^{I_2}, \partial_\alpha] \partial_\beta u + \partial_\alpha ([\partial_x^{I_1} \Omega^{I_2}, \partial_\beta]u) \\
&= \sum_{|J_2| < |I_2|} \Theta_{\alpha J_2}^{I_2 c} \partial_c \partial_x^{I_1} \Omega^{J_2} \partial_\beta u + \sum_{|J_2| < |I_2|} \Theta_{\beta J_2}^{I_2 c} \partial_\alpha \partial_c \partial_x^{I_1} \Omega^{J_2} u \\
&= \sum_{|J_2| < |I_2|} \Theta_{\alpha J_2}^{I_2 c} \partial_c \partial_x^{I_1} \partial_\beta \Omega^{J_2} u + \sum_{|J_2| < |I_2|} \Theta_{\alpha J_2}^{I_2 c} \partial_c \partial_x^{I_1} [\Omega^{J_2}, \partial_\beta]u \\
&\quad + \sum_{|J_2| < |I_2|} \Theta_{\beta J_2}^{I_2 c} \partial_\alpha \partial_c \partial_x^{I_1} \Omega^{J_2} u \\
&= \sum_{\substack{|J_2| < |I_2| \\ |K_2| < |I_2|}} \Theta_{\alpha J_2}^{I_2 c} \Theta_{\beta K_2}^{J_2 c'} \partial_c \partial_{c'} \partial_x^{I_1} \Omega^{K_2} u \\
&\quad + \sum_{|J_2| < |I_2|} \Theta_{\beta J_2}^{I_2 c} \partial_\alpha \partial_c \partial_x^{I_1} \Omega^{J_2} u + \sum_{|J_2| < |I_2|} \Theta_{\alpha J_2}^{I_2 c} \partial_c \partial_x^{I_1} \partial_\beta \Omega^{J_2} u
\end{aligned} \tag{7.5}$$

which leads to (7.3c).

• The proof of (7.3d) consults the notion of homogeneous functions. A function f of class $C^\infty(\mathbb{R}^3 \setminus \{0\})$ is called homogeneous of degree i if

$$f(rx) = r^i f(x), \text{ for any } r > 0 \text{ and } x \in \mathbb{R}^3 \setminus \{0\}.$$

Recall the classical result that a partial derivative of a homogeneous function of degree i is again homogeneous and of degree $i - 1$.

We denote by $\omega^a := \frac{x^a}{r}$. And we notice that they are homogeneous functions of degree 0. And recall the definition of radical derivative $\partial_r = \omega^a \partial_a$. We will prove that

$$\partial_r^k = \sum_{\substack{a, \\ |I| \leq k}} A_I^k \partial_x^I \quad (7.6)$$

where A_I^k is a homogeneous function of degree $-k + |I|$.

For $k = 1$, this is guaranteed by the expression of ∂_r . Assume that this holds for the integer less than or equal to k , we will prove the case of $k + 1$:

$$\begin{aligned} \partial_r^{k+1} u &= \partial_r \partial_r^k u = \partial_r \left(\sum_{I,a} A_I^k \partial_x^I \right) \\ &= \sum_{I,a} \omega^b \partial_b (A_I^k \partial_x^I) = \sum_{I,a} \omega^b \partial_b A_I^k \partial_x^I + \sum_{I,a} \omega^b A_I^k \partial_b \partial_x^I \end{aligned}$$

We remark that $\omega^b \partial_b A_I^k$ is homogeneous of degree $-k - 1 + |I|$ and $\omega^b A_I^k$ is homogeneous of degree $-(k + 1) + |I| + 1$. This concludes (7.6).

Then we sees that (7.3d) is a direct result of (7.6).

• To prove (7.3e), we preform the following calculation:

$$\begin{aligned} &[\partial_x^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta u] \\ &= \sum_{\substack{J_1+J'_1=I_1, J_2+J'_2=I_2 \\ |J'_1|+|J'_2|>0}} \partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_x^{J_1} \Omega^{J_2} \partial_\alpha \partial_\beta u + H^{\alpha\beta} [\partial_x^{I_1} \Omega^{I_2}, \partial_\alpha \partial_\beta] u \\ &= \sum_{\substack{J_1+J'_1=I_1, J_2+J'_2=I_2 \\ |J'_1|+|J'_2|>0}} \partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_\alpha \partial_\beta \partial_x^{J_1} \Omega^{J_2} u + \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta} [\partial_x^{J_1} \Omega^{J_2}, \partial_\alpha \partial_\beta] u \\ &= \sum_{\substack{J_1+J'_1=I_1, J_2+J'_2=I_2 \\ |J'_1|+|J'_2|>0}} \partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_\alpha \partial_\beta \partial_x^{J_1} \Omega^{J_2} u \\ &+ \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \partial_x^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \left(\sum_{\substack{|K_2|<|J_2| \\ |K'_2|<|J'_2|}} \Theta_{\alpha K_2}^{J_2 c} \Theta_{\beta K'_2}^{c} \partial_c \partial_c \partial_x^{J_1} \Omega^{K_2} u + \sum_{|K_2|<|J_2|} \Theta_{\beta K_2}^{J_2 c} \partial_\alpha \partial_c \partial_x^{J_1} \Omega^{K_2} u \right) \end{aligned}$$

where (7.5) is applied. Then (7.3e) comes directly from this identity. \square

We also need the commutator estimates on the product in the form $\partial^{I_1} \Omega^{I_2}$. These estimates are included in the following lemma

Lemma 7.3. *Let u be a C_c^∞ function defined in $[0, T] \times \mathbb{R}^3$. Then the following estimates hold for $|I_1| + |I_2| = d$:*

$$|\partial^{I_1} \Omega^{I_2} \partial_\alpha u| \leq |\partial_\alpha \partial^{I_1} \Omega^{I_2} u| + C(d) \sum_{a, |I'_2| < |I_2|} |\partial_a \partial^{I_1} \Omega^{I'_2} u|, \quad (7.7a)$$

$$|\partial_\alpha \partial^{I_1} \Omega^{I_2} u| \leq |\partial^{I_1} \Omega^{I_2} \partial_\alpha u| + C(d) \sum_{\substack{\beta \\ |I'_1|+|I'_2|<d}} |\partial^{I_1} \Omega^{I'_2} \partial_\beta u|, \quad (7.7b)$$

$$|[\partial^{I_1} \Omega^{I_2}, \partial_\alpha \partial_\beta] u| \leq C(d) \sum_{\substack{a, \alpha \\ |J_2| < |I_2|}} |\partial_a \partial_\alpha \partial^{I_1} \Omega^{J_2} u|, \quad (7.7c)$$

$$\begin{aligned} |[\partial^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta] u| &\leq \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2 \\ |J'_1|+|J'_2|>0}} |\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta}| |\partial_\alpha \partial_\beta \partial^{J_1} \Omega^{J_2} u| \\ &+ C(d) \sum_{\substack{J_1+J'_1=I_1 \\ |J_2|+|J'_2|<|I_2| \\ \alpha, \beta, \alpha', a}} |\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta}| |\partial_{\alpha'} \partial_a \partial^{J_1} \Omega^{J_2} u| \end{aligned} \quad (7.7d)$$

Proof. We remark that from (7.2), the following identity is direct:

$$[\partial^{I_1} \Omega^{I_2}, \partial_\alpha] = \sum_{|K| < |I_2|} \Theta_{\alpha K}^{I_2 b} \partial_b \partial_x^{I_1} \Omega^K u.$$

Then we perform exactly the same calculation to the proof of lemma 7.2 with $\partial_x^{I_1}$ replaced by ∂^{I_1} . \square

7.2 Global type Sobolev inequalities and embedding results

In this subsection we will reprove the following estimate due to [cite Klainerman]

Lemma 7.4. *Let $u \in X^d$ with $d \geq 2$. Then for*

$$\|u\|_{\mathbb{B}^{-1}} \leq C \|u\|_{X^d}. \quad (7.8)$$

Proof. We will only prove this inequality for the C_c^∞ functions. Then by the procedure of regularization, it will extend to the whole X^d .

We consider \mathbb{R}^3 equipped with the polar coordinates, i.e.:

$$x^1 = r \sin \theta \cos \varphi, \quad x^2 = r \sin \theta \sin \varphi, \quad x^3 = r \cos \theta.$$

Remark that (7.8) is equivalent to the following inequality for all $x_0 \in \mathbb{R}^3$:

$$|u(x_0)|(1+r_0) \leq C \|u\|_{X^d}$$

with $x_0 = (x_0^1, x_0^2, x_0^3) = (r_0 \sin \theta_0 \cos \varphi_0, r_0 \sin \theta_0 \sin \varphi_0, r_0 \cos \theta_0)$.

The case where $r_0 \leq 1$ is direct by classical Sobolev inequality. We will concentrate on the case where $r_0 > 1$.

We will consider the estimate on the following open subset of \mathbb{R}^3 defined by

$$V = (r_0 - 1/2, r_0 + 1/2) \times (\theta_0 - 1/2, \theta_0 + 1/2) \times (\varphi_0 - 1/2, \varphi_0 + 1/2)$$

Now let u be a C_c^∞ function, we denote also by u its restriction on V and define:

$$v(r, \theta, \varphi) := u(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$$

with $r_0 > 1$. Then by classical Sobolev inequality,

$$|u(x_0)|^2 = |v(r_0, \theta_0, \varphi_0)|^2 \leq C \sum_{k_0+k_1+k_2 \geq 2} \int_V |\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v|^2 dr d\theta d\varphi.$$

Remark that in V , $r_0 - 1/2 < r < r_0 + 1/2$, which leads to $1 - 1/(2r_0) < r/r_0 < 1 + 1/(2r_0)$. Recall that $r_0 > 1$, then

$$\frac{1}{2} < \frac{r}{r_0} < \frac{3}{2}.$$

Then

$$\begin{aligned}
\frac{1}{2} \int_V |\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v|^2 dr d\theta d\varphi &\leq \int_V |\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v|^2 \frac{r^2}{r_0^2} dr d\theta d\varphi \\
&= r_0^{-2} \int_V |\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v|^2 r^2 dr d\theta d\varphi \\
&= r_0^{-2} \int_V |\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v|^2 dx \\
&\leq r_0^{-2} \int_{\mathbb{R}^3} |\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v|^2 dx
\end{aligned}$$

where dx is the standard volume form of \mathbb{R}^3 .

Here we remark that

$$\partial_\theta v = \cos \varphi \Omega_{31} u + \sin \varphi \Omega_{32} u = \frac{x^1}{((x^1)^2 + (x^2)^2)^{1/2}} \Omega_{31} u + \frac{x^2}{((x^1)^2 + (x^2)^2)^{1/2}} \Omega_{32} u$$

and

$$\partial_\varphi v = \Omega_{12} u, \quad \partial_r v = \partial_r u.$$

Remark that $\Omega^J \cos \varphi$, $\Omega^J \sin \varphi$ are homogeneous of degree 0. So we see, by homogeneity, that for $r \geq 1/2$,

$$|\partial_r^{k_0} \partial_\theta^{k_1} \partial_\varphi^{k_2} v| = |\partial_r^{k_0} (\cos \varphi \Omega_{31} + \sin \varphi \Omega_{33})^{k_1} \Omega_{12}^{k_2} u| \leq C \sum_{|I| \leq k_1 + k_2} |\partial_r^{k_0} \Omega^I u|$$

which leads to

$$|u(x_0)|^2 \leq C r_0^{-2} \sum_{|I| + k_0 \leq 2} \int_{\mathbb{R}^3} |\partial_r^{k_0} \Omega^I u|^2 dx \tag{7.9}$$

Then by (7.3d), the desired result is proved. \square

For the convenience of application, we will also need the following embedding results.

Lemma 7.5. *Let u be a function in X_H^{d+2} and $v \in L^\infty([0, T]; E_H^{d+2})$. Then for all pair of multi-index (I_1, I_2) with $|I_1| + |I_2| \leq d$, the following estimate holds:*

$$\|\partial_x^{I_1} \Omega^{I_2} u\|_{L^\infty(\mathbb{R}^3)} \leq C \|u\|_{X_H^{d+2}}. \tag{7.10a}$$

$$\|\partial^{I_1} \Omega^{I_2} v(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \|v(t, \cdot)\|_{E_H^{d+2}} \quad \text{for } t \in [0, T]. \tag{7.10b}$$

Proof. We will only prove this inequality when $u \in C_c^\infty(\mathbb{R}^3)$ and $v \in C_c^\infty(\mathbb{R}^4)$. Then by the regularization procedure, it will extends on X_H^{d+2} and $L^\infty([0, T]; E_H^{d+2})$.

• Let us begin with (7.10a). The proof is composed by estimates in different cases.

CASE 1. $I_1 = I_2 = 0$. Then the left-hand-side of (7.10a) is controlled by its \mathcal{B}^{-1} norm so (7.10a) holds.

CASE 2. $|I_1| > 0$. In this case we suppose that $I_1 = (a_1, a_2, \dots, a_n)$ and denote by $I_1' = (a_2, \dots, a_n)$. Then I_1' is of order $n - 1 \geq 0$. Then by classical Sobolev's inequality,

$$\|\partial_x^{I_1} \Omega^{I_2} u\|_{L^\infty(\mathbb{R}^3)} = \|\partial_{a_1} \partial_x^{I_1'} \Omega^{I_2} u\|_{L^\infty(\mathbb{R}^3)} \leq C \sum_{|J_1| \leq 2} \|\partial_x^{J_1} \partial_{a_1} \partial_x^{I_1'} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{X_P^{d+2}}.$$

CASE 3. $|I_2| = 0, |I_2| > 0$. In this case we suppose that $I_2 = (b_1, b_2, \dots, b_n)$ and denote by $I_2' = (b_2, \dots, b_n)$. Then I_2' is of order $n - 1 \geq 0$. Then

$$\|\partial_x^{I_1} \Omega^{I_2} u\|_{L^\infty(\mathbb{R}^3)} = \|\Omega_{b_1} \Omega^{I_2'} u\|_{L^\infty(\mathbb{R}^3)} \leq C \sum_b \|(1+r) \partial_b \Omega^{I_2'} u\|_{L^\infty(\mathbb{R}^3)} = C \sum_b \|\partial_b \Omega^{I_2'} u\|_{\mathcal{B}^{-1}}$$

Then by (7.8),

$$\begin{aligned} \|\partial_x^{I_1} \Omega^{I_2} u\|_{L^\infty(\mathbb{R}^3)} &\leq C \sum_b \|\partial_b \partial_x^{I_1} \Omega^{I_2} u\|_{X^2} = C \sum_{|J_1|+|J_2|\leq 2} \sum_b \|\partial_x^{J_1} \Omega^{J_2} \partial_b \partial_x^{I_1} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} \\ &\leq C \sum_{\substack{|J_1|+|J_2|\leq 2 \\ b'}} \|\partial_{b'} \partial_x^{J_1} \partial_x^{J_2} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{X_P^{d+2}} \end{aligned}$$

where the estimate of commutator (7.3a) is applied.

Combine these three cases, (7.10a) is proved.

• Then we will prove (7.10b). The proof is similar: we also discuss in three different cases:

CASE 1. $I_1 = I_2 = 0$. Then the left-hand-side of (7.10b) is controlled by its \mathcal{B}^{-1} norm so (7.10b) holds.

CASE 2. $I_1 > 0$. In this case we also suppose that $I_1 = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and denote by $I_1' = (\alpha_2, \dots, \alpha_n)$. Then $|I_1'| \geq 0$. Then also by classical Sobolev's inequality:

$$\begin{aligned} \|\partial^{I_1} \Omega^{I_2} v(t, \cdot)\|_{L^\infty} &\leq C \sum_{|J_1|\leq 2} \|\partial_x^{J_1} \partial_{\alpha_1} \partial^{I_1'} \Omega^{I_2} v(t, \cdot)\|_{L^2(\mathbb{R}^3)} = C \sum_{|J_1|\leq 2} \|\partial_{\alpha_1} \partial_x^{J_1} \partial^{I_1'} \Omega^{I_2} v(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|v(t, \cdot)\|_{E_P^{d+2}}. \end{aligned}$$

CASE 3. $|I_1| = 0, |I_2| > 0$. Then we suppose that $I_2 = (b_1, b_2, \dots, a_n)$, and denote by $I_2' = (b_2, \dots, b_n)$. Then $|I_2'| \geq 0$. Then

$$\begin{aligned} \|\partial^{I_1} \Omega^{I_2} v(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} &= \|\Omega_{b_1} \Omega^{I_2'} v(t, \cdot)\|_{L^\infty} \\ &\leq C \sum_b \|(1+r) \partial_b \Omega^{I_2'} v(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} = C \sum_b \|\Omega^{I_2'} v(t, \cdot)\|_{\mathcal{B}^{-1}} \\ &\leq C \sum_b \|\partial_b \Omega^{I_2'} v(t, \cdot)\|_{X^2} = C \sum_{\substack{|J_1|+|J_2|\leq 2 \\ b}} \|\partial_x^{J_1} \Omega^{J_2} \partial_b \Omega^{I_2'} v(t, \cdot)\|_{X^2} \\ &\leq C \sum_{\substack{b' \\ |J_2|+|J_2'|\leq 2}} \|\partial_x^{J_1} \partial_{b'} \Omega^{J_2'} \Omega^{I_2'} v(t, \cdot)\|_{X^2} \leq C \|v(t, \cdot)\|_{E_P^{d+2}}. \end{aligned}$$

Combine these three cases, (7.10b) is established. \square

Lemma 7.6. *Let $u \in X_H^d$ and $v \in L^\infty([0, T]; E_P^d)$, then the following estimate holds for all pair of multi-index (I_1, I_2) with $1 \leq |I_1| + |I_2| \leq d$:*

$$\|(1+r)^{-1} \partial_x^{I_1} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{X_P^d}. \quad (7.11a)$$

$$\|(1+r)^{-1} \partial^{I_1} \Omega^{I_2} v(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C \|v(t, \cdot)\|_{E_P^d}, \quad \text{for } t \in [0, T] \quad (7.11b)$$

Proof. • The proof of (7.11a) is composed by estimates in different cases:

CASE 1. $|I_1| > 0$. In this case we suppose that $I_1 = (a_1, a_2, \dots, a_n)$ and denote by $I_1' = (a_2, \dots, a_n)$. Then I_1' is of order $n-1 \geq 0$. Then

$$\|\partial_x^{I_1} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} = \|\partial_{a_1} \partial_x^{I_1'} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{X_P^d}.$$

CASE 2. $|I_1| = 0, |I_2| > 0$. In this case we suppose that $I_2 = (b_1, b_2, \dots, b_n)$ and denote by $I_2' = (b_2, \dots, b_n)$. Then I_2' is of order $n-1 \geq 0$. Then

$$\|(1+r)^{-1} \partial_x^{I_1} \Omega^{I_2} u\|_{L^2(\mathbb{R}^3)} = \|(1+r)^{-1} \Omega_{b_1} \Omega^{I_2'} u\|_{L^2(\mathbb{R}^3)} \leq C \sum_b \|\partial_b \Omega^{I_2'} u\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{X_P^d}.$$

Combine these two cases, (7.11a) is proved.

• The proof of (7.11b) is exactly the same if we replace ∂_x by ∂ in the above proof. \square

Lemma 7.7. *Let u be a in class X^2 . Then*

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C \sum_{|I_1|+|I_2|\leq 2} \|(1+|x|)^{-1} \partial_x^{I_1} \Omega^{I_2} u\|_{L^2}. \quad (7.12)$$

Proof. This inequality is equivalent to

$$|u(x)| \leq C \sum_{|I_1|+|I_2|\leq 2} \|(1+|x|)^{-1} \partial_x^{I_1} \Omega^{I_2} u\|_{L^2}, \quad x \in \mathbb{R}^3 \quad (7.13)$$

for all $u \in C_c^\infty(\mathbb{R}^3)$. Then by regularization, this inequality is hold for all $u \in X^2$. This is proved by distinguish different x .

Let $\chi(\cdot)$ a C^∞ function defined on $[0, \infty)$ with

$$\chi(s) = \begin{cases} 1, & s \leq 1/2, \\ 0, & 1 \leq s < \infty. \end{cases}$$

Then $\chi(|\cdot|)$ is a $C_c^\infty(\mathbb{R}^3)$ function.

Now we consider the case $|x| \leq 1/2$. We consider the function

$$f(x) := \chi(|x|)u(x)$$

which is in class $C_c^\infty(\{|x| < 1\})$. Then we apply the classical Sobolev's inequality:

$$\begin{aligned} |u(x)| = |f(x)| &\leq C \sum_{|I|\leq 2} \|\partial_x^I f\|_{L^2(\{|x|<1\})} = C \sum_{|I|\leq 2} \|\partial_x^I (\chi(|x|)u(x))\|_{L^2(\{|x|<1\})} \\ &= C \sum_{\substack{|I_1|+|I_2|\leq |I|, \\ |I|\leq 2}} \|\partial_x^{I_1} \chi(|x|) \partial_x^{I_2} u(x)\|_{L^2(\{|x|<1\})} \leq C \sum_{|I_2|\leq 2} \|\partial_x^{I_2} u(x)\|_{L^2(\{|x|<1\})} \\ &\leq C \sum_{|I_2|\leq 2} \|(1+|x|)^{-1} \partial_x^{I_2} u(x)\|_{L^2(\{|x|<1\})} \leq C \sum_{|I_2|\leq 2} \|(1+|x|)^{-1} \partial_x^{I_2} u(x)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

where we remarked that $\partial_x^{I_1} \chi(|x|)$ are bounded.

When $|x| \geq 1/2$, we consider the following calculation:

$$\begin{aligned} |u(x)| &= |(1 - \chi(2|x|))u(x)| = \|(1+|x|) \cdot (1+|x|)^{-1} (1 - \chi(2|x|))u(x)\|_{L^\infty(\mathbb{R}^3)} \\ &\leq \|(1+|x|)^{-1} (1 - \chi(2|x|))u(x)\|_{\mathcal{B}^{-1}} \end{aligned}$$

and the by (7.8),

$$\begin{aligned} |u(x)| &\leq \|(1+|x|)^{-1} (1 - \chi(2|x|))u(x)\|_{\mathcal{B}^{-1}} \\ &\leq C \sum_{|I_1|+|I_2|\leq 2} \|\partial_x^{I_1} \Omega^{I_2} ((1+|x|)^{-1} (1 - \chi(2|x|))u(x))\|_{L^2(\mathbb{R}^3)} \\ &= C \sum_{\substack{|J_1|+|J_2|=|I_1| \\ |I_1|+|I_2|\leq 2}} \|\partial_x^{J_1} ((1+|x|)^{-1} (1 - \chi(2|x|))) \partial_x^{J_2} \Omega^{I_2} ((1+|x|)^{-1} (1 - \chi(2|x|))u(x))\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

where for the last equality we have used the fact that

$$\Omega_a((1+|x|)^{-1} (1 - \chi(2|x|))) = 0$$

because both factors are radial symmetric. Then we will prove that

$$\partial_x^{J_1} ((1+|x|)^{-1} (1 - \chi(2|x|))) \leq C(1+|x|)^{-1}$$

for $|J_1| \leq 2$. This is check directly by calculating and the fact that $(1 - \chi(2|x|))$ and its derivatives are supported out of the ball $\{|x| < 1/4\}$. Then the desired result is established. \square

7.3 Linear estimates

We begin with the linear theory of wave equation with the initial data given in $X_H^{d+1}(\mathbb{R}^3) \times X^d(\mathbb{R}^3)$. For the simplicity of proof, we introduce the energy functional with respect to a metric $g^{\alpha\beta}$ defined in \mathbb{R}^4 as following:

$$E_g(t, u) := \left(\int_{\mathbb{R}^3} (-g^{00}|\partial_t u|^2 + g^{ab}\partial_a u \partial_b u)(t, \cdot) dx \right)^{1/2} \quad (7.14)$$

and

$$E_{g,c}(t, u) := \left(\int_{\mathbb{R}^3} (-g^{00}|\partial_t u|^2 + g^{ab}\partial_a u \partial_b u + c^2 v^2)(t, \cdot) dx \right)^{1/2} \quad (7.15)$$

A metric $g^{\alpha\beta}$ is said to be coercive with constant $C > 0$, means that

$$C^{-1}\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq E_g(t, u) \leq C\|\nabla u\|_{L^2(\mathbb{R}^3)}, \quad (7.16)$$

where ∇u refers to the space-time gradient of u . For the convenience of discussion, we introduce the notation for the C_c^∞ functions defined in the region $[0, T] \times \mathbb{R}^3$:

$$\|u(t, \cdot)\|_{X_E^{d+1}} := \sum_{|I_1|+|I_2|\leq d} \|\nabla \partial_x^{I_1} \Omega^{I_2} u(t, \cdot)\|_{L^2(\mathbb{R}^3)}.$$

By (7.3a) and (7.3b), the norm $\|\cdot\|_{X_E^{d+1}}$ and $\|\nabla(\cdot)\|_{X^d}$ are equivalent.

We also introduce the following notation:

$$E_g^d(s, u) := \sum_{|I_1|+|I_2|\leq d} E_g(s, \partial^{I_1} \Omega^{I_2} u), \quad E_{g,c}^d(s, u) := \sum_{|I_1|+|I_2|\leq d} E_{g,c}(s, \partial^{I_1} \Omega^{I_2} u).$$

Then when the coercive condition (7.16) is assumed,

$$C^{-1}\|u(t, \cdot)\|_{E_P^{d+1}} \leq E_g^d(s, u) \leq C\|u(t, \cdot)\|_{E_P^{d+1}}. \quad (7.17)$$

The existence results to be established in the next subsection is essentially based on the following linear estimates.

Lemma 7.8 (L^∞ type estimate for wave equation). *Let u be a C_c^∞ function defined in the region $[0, T]$ and let $F = -\square u$, then for any $0 \leq t \leq T$*

$$\|u(t, \cdot)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)} \leq Ct \int_0^t \|F(s, \cdot)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)} ds + C(\|u(0, x)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)} + t\|\nabla u(0, x)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)}) \quad (7.18)$$

where ∇u refers to the space-time gradient of u .

Proof. This estimate based on the explicit expression of the linear wave equation. We consider the Cauchy problem:

$$\begin{aligned} \square u &= -F \\ u(0, x) &= -f(x), \quad \partial_t u(0, x) = -g(x) \end{aligned} \quad (7.19)$$

The u can be expressed by

$$\begin{aligned} u(t, x) &= \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_{|y|=t-s} F(s, x-y) d\sigma(y) ds \\ &\quad + \frac{1}{4\pi t} \int_{|y|=t} g(x-y) d\sigma(y) + \frac{1}{4\pi t^2} \int_{|y|=t} (f(x-y) - \langle \partial_x f(x-y), y \rangle) d\sigma(y) \\ &:= u_1(t, x) + u_2(t, x) + u_3(t, x). \end{aligned}$$

Here $d\sigma(y)$ refers to the standard Lebesgue measure on the sphere $|y| = t - s$ or $|y| = t$, and $\langle \cdot, \cdot \rangle$ refers to the standard scalar product on \mathbb{R}^3 . The notation $\partial_x f$ stands for the (spacial) gradient of f . The remained work is to estimate u_i with $i = 1, 2, 3$.

When $|x| \leq 1$, we make use of the fact that $\|\cdot\|_{L^\infty(\mathbb{R}^3)} \leq \|\cdot\|_{\mathbb{B}^{-1}}$. Then

$$|u_1(t, x)| \leq \frac{1}{4\pi} \int_0^t \frac{\|F(s, \cdot)\|_{\mathbb{B}^{-1}}}{t-s} \int_{|y|=t-s} d\sigma(y) ds \leq \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}}(t-s) ds \leq t \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}}.$$

The other terms u_2 and u_3 are estimated similarly. Then (7.18) is proved in this case.

When $|x| > 1$, we need to establish the decay estimate of the solution at spacial infinity. We begin with u_1 .

$$\begin{aligned} |u_1(t, x)| &\leq \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_{|y|=t-s} \frac{\|F(s, \cdot)\|_{\mathbb{B}^{-1}}}{1+|x-y|} d\sigma(y) ds \\ &= \frac{1}{4\pi} \int_0^t \frac{\|F(s, \cdot)\|_{\mathbb{B}^{-1}}}{t-s} \int_{|y|=t-s} (1+|x-y|)^{-1} d\sigma(y) ds \end{aligned}$$

We focus on the expression $\int_{|y|=t-s} (1+|x-y|)^{-1} d\sigma(y)$. We make the following parametrization of the sphere $\{|y| = t-s\}$. Let θ be the angle from the vector $-x$ to the vector $-y$, and φ refers the angle from the plan determined by pair of vector (x, y) to a fixed plan containing x (for example the plan determined by x and $(0, 0, 1)$). With this parametrization, the volume form $\sigma(y)$ has the following expression:

$$d\sigma(y) = (t-s)^2 \sin \theta d\theta d\varphi$$

Also, by the classical trigonometrical theorem ‘‘Law of cosines’’,

$$|x-y|^2 = |x|^2 + |t-s|^2 - 2|x|(t-s) \cos \theta.$$

Then

$$\begin{aligned} \int_{|y|=t-s} (1+|x-y|)^{-1} d\sigma(y) &= \int_0^{2\pi} \int_0^\pi \frac{(t-s)^2 \sin \theta d\theta d\varphi}{1 + (|x|^2 + |t-s|^2 - 2|x|(t-s) \cos \theta)^{1/2}} \\ &= 2\pi \int_{-1}^1 \frac{(t-s)^2 d\gamma}{1 + (|x|^2 + |t-s|^2 - 2|x|(t-s)\gamma)^{1/2}} \end{aligned}$$

where $\gamma := \cos \theta$. Then

$$\begin{aligned} \int_{|y|=t-s} (1+|x-y|)^{-1} d\sigma(y) &= \frac{2\pi(t-s)}{|x|} \int_{-1}^1 \frac{|x|(t-s) d\gamma}{1 + (|x|^2 + |t-s|^2 - 2|x|(t-s)\gamma)^{1/2}} \\ &= \frac{2\pi(t-s)}{|x|} \int_{||x|-(t-s)|}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} \end{aligned}$$

where $\tau := (|x|^2 + |t-s|^2 - 2|x|(t-s)\gamma)^{1/2}$. Then,

$$\begin{aligned} |u_1(t, x)| &\leq \frac{1}{4\pi} \int_0^t \frac{\|F(s, \cdot)\|_{\mathbb{B}^{-1}}}{t-s} \int_{|y|=t-s} (1+|x-y|)^{-1} d\sigma(y) ds \\ &\leq \frac{1}{2|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{||x|-(t-s)|}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} ds. \end{aligned}$$

Now the discussion should be made in the following two cases:

1. $|x| > t$. In this case $|x| > t - s$ always holds. then

$$\begin{aligned}
|u_1(t, x)| &\leq \frac{1}{2|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{||x|-(t-s)|}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} ds \\
&= \frac{1}{2|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{|x|-(t-s)}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} ds \\
&\leq \frac{1}{2|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{|x|-(t-s)}^{|x|+(t-s)} d\tau ds \\
&= \frac{1}{|x|} \int_0^t (t-s) \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds \leq \frac{t}{|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds
\end{aligned}$$

which leads to

$$|x| |u(t, x)| \leq t \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds.$$

2. $|x| \leq t$. In this case we need a more precise calculation.

$$\begin{aligned}
|u_1(t, x)| &\leq \frac{1}{2|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{||x|-(t-s)|}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} ds \\
&= \frac{1}{2|x|} \int_0^{t-|x|} \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{(t-s)-|x|}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} ds + \frac{1}{2|x|} \int_{t-|x|}^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} \int_{|x|-(t-s)}^{|x|+(t-s)} \frac{\tau d\tau}{1+\tau} ds \\
&\leq \frac{1}{2|x|} \int_0^{t-|x|} 2|x| \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds + \frac{1}{2|x|} \int_{t-|x|}^t 2(t-s) \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds \\
&\leq \int_0^{t-|x|} \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds + \int_{t-|x|}^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds = \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds
\end{aligned}$$

Recall that $|x| \leq t \leq T$, then

$$|u_1(t, x)| \leq \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds \leq \frac{t}{|x|} \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds$$

so that

$$|x| |u_1(t, x)| \leq t \int_0^t \|F(s, \cdot)\|_{\mathbb{B}^{-1}} ds.$$

The estimate of u_2 and u_3 are similar.

$$\begin{aligned}
|u_2(t, x)| &\leq \frac{1}{4\pi t} \int_{|y|=t} |g(x-y)| d\sigma(y) \\
&\leq \frac{\|g\|_{\mathbb{B}^{-1}}}{4\pi t} \int_{|y|=t} (|x-y|+1)^{-1} d\sigma(y)
\end{aligned}$$

By the same parametrization made in the estimate of u_1 and similar calculation, we can conclude by

$$|x| |u_2(t, x)| \leq t \|g\|_{\mathbb{B}^{-1}}.$$

In the same way:

$$\begin{aligned}
|u_3(t, x)| &\leq \frac{1}{4\pi t^2} \int_{|y|=t} \frac{\|f\|_{\mathbb{B}^{-1}} d\sigma(y)}{|x-y|+1} + \frac{1}{4\pi t} \int_{|y|=t} \frac{\|\partial_x f\|_{\mathbb{B}^{-1}} d\sigma(y)}{|x-y|+1} \\
&\leq \frac{1}{|x|} \|f\|_{\mathbb{B}^{-1}} + \frac{t}{|x|} \|\partial_x f\|_{\mathbb{B}^{-1}}.
\end{aligned}$$

Combine the estimates made in $|x| \leq 1$ and $|x| > 1$, the desired result is established. \square

Lemma 7.9 (L^2 type estimate for wave equation). *Let u be a C_c^∞ function defined in the region $[0, T] \times \mathbb{R}^3$ and $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$ be a C^∞ metric where $m^{\alpha\beta}$ is the Minkowski metric with signature $(-, +, +, +)$. Assume that $g^{\alpha\beta}$ is coercive with constant C . Let $f = -g^{\alpha\beta} \partial_\alpha \partial_\beta u$, then for any $0 \leq t \leq T$,*

$$\frac{d}{dt} E_g(t, u) \leq C \|f(t, \cdot)\|_{L^2(\mathbb{R}^3)} + C \sum_{\alpha, \beta} \|\nabla g^{\alpha\beta}(t, \cdot)\|_{L^\infty} E_g(t, u). \quad (7.20)$$

where ∇u refers to the space-time gradient of u .

Proof. This is a standard calculation.

$$\partial_t u g^{\alpha\beta} \partial_\alpha \partial_\beta u = \frac{1}{2} \partial_0 (g^{00} (\partial_0 u)^2 - g^{ab} \partial_a u \partial_b u) + \partial_a (g^{\alpha\beta} \partial_t u \partial_\beta u) + \frac{1}{2} \partial_t g^{\alpha\beta} \partial_\alpha u \partial_\beta u - \partial_\alpha g^{\alpha\beta} \partial_t u \partial_\beta u$$

Integrate on the slice $\{t = \tau\}$ and apply the Stocks' formula,

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} \partial_0 (-g^{00} (\partial_t u)^2 + g^{ab} \partial_a u \partial_b u) dx &= \int_{\mathbb{R}^3} \partial_t u f dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t g^{\alpha\beta} \partial_\alpha u \partial_\beta u - 2 \partial_\alpha g^{\alpha\beta} \partial_t u \partial_\beta u) dx \end{aligned}$$

which leads to (by coercive condition,)

$$\frac{1}{2} \frac{d}{dt} (E_g(t, u))^2 = \int_{\mathbb{R}^3} \partial_t u f dx + \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t g^{\alpha\beta} \partial_\alpha u \partial_\beta u - 2 \partial_\alpha g^{\alpha\beta} \partial_t u \partial_\beta u) dx$$

which is

$$E_g(t, u) \frac{d}{dt} E_g(t, u) = \int_{\mathbb{R}^3} \partial_t u f dx + \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t g^{\alpha\beta} \partial_\alpha u \partial_\beta u - 2 \partial_\alpha g^{\alpha\beta} \partial_t u \partial_\beta u) dx.$$

So that

$$E_g(t, u) \frac{d}{dt} E_g(t, u) \leq \|\partial_t u\|_{L^2(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)} + C \|\nabla g^{\alpha\beta}\|_{L^\infty(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2.$$

Then recall (7.16)

$$\frac{d}{dt} E_g(t, u) \leq C \|f\|_{L^2(\mathbb{R}^3)} + C \|\nabla g^{\alpha\beta}\|_{L^\infty(\mathbb{R}^3)} E_g(t, u).$$

□

Now we will combine lemma 7.8, 7.9 with the Sobolev estimate (7.8) in order to get a estimate on the E_H^{d+1} norm. For the convenience of the discussion, With this notation, we can state the following estimate:

Lemma 7.10 (E_H^d norm estimate on wave equation). *Let u be a C_c^∞ function defined in the region $[0, T] \times \mathbb{R}^3$. Let $g^{\alpha\beta}$ be a C^∞ metric, $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$ with $m^{\alpha\beta}$ the standard Minkowski metric. Assume that g satisfies the following coercive condition with a constant $C > 0$ and $|H^{00}| \leq 1/2$. Let $g^{\alpha\beta} \partial_\alpha \partial_\beta u = -F$. Then the following estimates hold for $d \geq 3$:*

$$\|u(t, \cdot)\|_{E_P^{d+1}} \leq C \|u(0, \cdot)\|_{E_P^{d+1}} e^{C \int_0^t D_d(\tau) d\tau} + C \int_0^t \|F(\tau, \cdot)\|_{E^d} e^{C \int_\tau^t D_d(s) ds} d\tau, \quad (7.21a)$$

$$\begin{aligned} \|u(t, \cdot)\|_{\mathcal{B}^{-1}} &\leq C t \int_0^t \|F(s, \cdot)\|_{\mathcal{B}^{-1}} ds + C t E_g^3(0, u) \sum_{\alpha, \beta} \int_0^t \|H^{\alpha\beta}(s, \cdot)\|_{L^\infty} e^{C \int_0^s D_3(\tau) d\tau} ds \\ &+ C t \sum_{\alpha, \beta} \int_0^t \|H^{\alpha\beta}(s, \cdot)\|_{L^\infty} \int_0^s \|F(\tau, \cdot)\|_{E^3} e^{C \int_\tau^s D_3(\lambda) d\lambda} ds + C t E_g^3(0, u) \end{aligned} \quad (7.22)$$

where

$$D_k(t) := \max_{\alpha, \beta} \|H^{\alpha\beta}(t, \cdot)\|_{E_H^{k+1}}.$$

Proof. We begin with the estimate on E^d norm of ∇u . We derive the equation with respect to $\partial^{I_1} \Omega^{I_2}$ where $|I_1| + |I_2| = l$. Remember that this product of operator commute with the linear wave operator, Then

$$-\partial^{I_1} \Omega^{I_2} F = \square \partial^{I_1} \Omega^{I_2} u + H^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{I_1} \Omega^{I_2} u + [\partial^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta] u$$

which is

$$g^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{I_1} \Omega^{I_2} u = -F - [\partial^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta] u.$$

Apply (7.20),

$$\begin{aligned} \frac{d}{dt} E_g(t, \partial^{I_1} \Omega^{I_2} u) &\leq C \|\partial^{I_1} \Omega^{I_2} F(t, \cdot)\|_{L^2(\mathbb{R}^3)} + C \|[\partial^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta] u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ &\quad + C \sum_{\alpha, \beta} \|\nabla H^{\alpha\beta}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} E_g(t, \partial^{I_1} \Omega^{I_2} u). \end{aligned} \quad (7.23)$$

We should estimate the second term in right-hand-side. By (7.3e),

$$\begin{aligned} \|[\partial^{I_1} \Omega^{I_2}, H^{\alpha\beta} \partial_\alpha \partial_\beta] u\|_{L^2(\mathbb{R}^3)} &\leq \sum_{\substack{J_1 + J'_1 = I_1 \\ J_2 + J'_2 = I_2 \\ |J'_1| + |J'_2| > 0}} \|\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{J_1} \Omega^{J_2} u\|_{L^2(\mathbb{R}^3)} \\ &\quad + \sum_{\substack{J_1 + J'_1 = I_1 \\ |J_2| + |J'_2| < |I_2| \\ \alpha, \beta, \alpha', a}} \|\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_{\alpha'} \partial_a \partial^{J_1} \Omega^{J_2} u\|_{L^2(\mathbb{R}^3)} \\ &=: T_1 + T_2. \end{aligned} \quad (7.24)$$

Here we make the convention that when $l \leq 0$, $[\partial^{I_1} \Omega^{I_1}, \partial_\alpha \partial_\beta] = 0$. We see that both terms can be bounded by $CD_d E^d(s, u)$: For T_1 , when $|J'_1| + |J'_2| \leq d - 1$, Then by (7.10b),

$$\begin{aligned} \|\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{J_1} \Omega^{J_2} u\|_{L^2(\mathbb{R}^3)} &\leq \|\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta}\|_{L^\infty(\mathbb{R}^3)} \|\partial_\alpha \partial_\beta \partial^{J_1} \Omega^{J_2} u\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|H^{\alpha\beta}\|_{E_H^{d+1}} \|u\|_{E_P^{d+1}}. \end{aligned}$$

When $|J'_1| + |J'_2| \geq d$, then $J_1 = J_2 = 0$. Then by (7.11b) and (7.8),

$$\begin{aligned} \|\partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{J_1} \Omega^{J_2} u\|_{L^2(\mathbb{R}^3)} &\leq \|(1+r)^{-1} \partial^{J'_1} \Omega^{J'_2} H^{\alpha\beta}\|_{L^2(\mathbb{R}^3)} \|(1+r) \partial_\alpha \partial_\beta u\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C \|H^{\alpha\beta}\|_{E_H^{d+1}} \|\partial_\alpha \partial_\beta u\|_{E^2} \leq C \|H^{\alpha\beta}\|_{E_H^{d+1}} \|u\|_{E_P^4}. \end{aligned}$$

The term T_2 is bounded in the same manner and we omit the detail.

Combine this with (7.23),

$$\begin{aligned} \frac{d}{dt} E_g(t, \partial^{I_1} \Omega^{I_2} u) &\leq C \|\partial^{I_1} \Omega^{I_2} F(t, \cdot)\|_{L^2(\mathbb{R}^3)} + CD_d E_g^d(t, u) \\ &\quad + C \sum_{\alpha, \beta} \|\nabla H^{\alpha\beta}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} E_g(t, \partial^{I_1} \Omega^{I_2} u) \end{aligned} \quad (7.25)$$

Taking the sum with respect to (I_1, I_2) for $|I_1| + |I_2| \leq d$, then

$$\frac{d}{dt} E_g^d(t, u) \leq C \|F(t, \cdot)\|_{E^d} + CD_d E_g^d(t, u). \quad (7.26)$$

Integrate (7.26):

$$E_g^d(t, u) \leq E_g^d(0, u) e^{C \int_0^t D_d(\tau) d\tau} + C \int_0^t \|F(\tau, \cdot)\|_{E^d} e^{C \int_\tau^t D_d(s) ds} d\tau, \quad (7.27)$$

which leads to (by (7.17)) (7.21a).

We now turns to estimate the \mathcal{B}^{-1} norm. We can easily deduce that

$$\square u = -(1 - H^{00})^{-1}(H^{ab} + H^{00}m^{ab})\partial_a\partial_b u - 2(1 - H^{00})^{-1}H^{a0}\partial_t\partial_a u - (1 - H^{00})^{-1}F \quad (7.28)$$

here we use the assumption that $H^{00} \leq 1/2$ to make sure that $(1 - H^{00})^{-1}$ is well defined. Then and then by lemma 7.8,

$$\begin{aligned} \|u\|_{\mathcal{B}^{-1}} &\leq Ct \int_0^t \|(1 - H^{00})^{-1}F(s, \cdot)\|_{\mathcal{B}^{-1}} ds \\ &\quad + Ct \int_0^t \|(1 - H^{00})^{-1}H^{ab}\partial_a\partial_b u\|_{\mathcal{B}^{-1}} ds + Ct \int_0^t \|(1 - H^{00})^{-1}H^{a0}\partial_a\partial_t u\|_{\mathcal{B}^{-1}} ds \\ &\quad + C(\|u(0, x)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)} + t\|\nabla u(0, x)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)}) \\ &\leq Ct \int_0^t \|F(s, \cdot)\|_{\mathcal{B}^{-1}} ds + Ct \int_0^t \|H^{ab}\partial_a\partial_b u\|_{\mathcal{B}^{-1}} ds + Ct \int_0^t \|H^{a0}\partial_a\partial_t u\|_{\mathcal{B}^{-1}} ds \\ &\quad + C(\|u(0, x)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)} + t\|\nabla u(0, x)\|_{\mathcal{B}^{-1}(\mathbb{R}^3)}) \end{aligned}$$

We remark that the following estimates are guaranteed by (7.8),

$$\|H^{a\beta}\partial_a\partial_\beta u\|_{\mathcal{B}^{-1}} \leq \|H^{a\beta}\|_{L^\infty} \|\partial_a\partial_\beta u\|_{\mathcal{B}^{-1}} \leq \|H^{a\beta}\|_{L^\infty} \|\partial_a\partial_\beta u\|_{E^2} \leq C \sum_{a,\beta} \|H^{a\beta}\|_{L^\infty} E_g^3(s, u)$$

and

$$\|\nabla u(0, \cdot)\|_{\mathcal{B}^{-1}} \leq C\|\nabla u(0, \cdot)\|_{E^2},$$

Then combined these two estimate:

$$\|u(t, \cdot)\|_{\mathcal{B}^{-1}} \leq Ct \int_0^t \|F(s, \cdot)\|_{\mathcal{B}^{-1}} ds + Ct \sum_{\alpha,\beta} \int_0^t \|H^{\alpha\beta}(s, \cdot)\|_{L^\infty} E_g^3(s, u) ds + Ct E_g^2(0, u). \quad (7.29)$$

Now combine (7.27) and (7.29),

$$\begin{aligned} \|u(t, \cdot)\|_{\mathcal{B}^{-1}} &\leq Ct \int_0^t \|F(s, \cdot)\|_{\mathcal{B}^{-1}} ds + Ct E_g^3(0, u) \sum_{a,\beta} \int_0^t \|H^{a\beta}(s, \cdot)\|_{L^\infty} e^{C \int_0^s D_3(\tau) d\tau} ds \\ &\quad + Ct \sum_{a,\beta} \int_0^t \|H^{a\beta}(s, \cdot)\|_{L^\infty} \int_0^s \|F(\tau, \cdot)\|_{E_R^3} e^{C \int_\tau^s D_3(\lambda) d\lambda} ds + Ct E_g^3(0, u). \end{aligned} \quad (7.30)$$

□

Furthermore, the following L^2 type estimate on Klein-Gordon equation is important:

Lemma 7.11 (L^2 type estimate for KG equation). *Let v be in class C_c^∞ defined $[0, T] \times \mathbb{R}^3$ and let $F = c^2 v - g^{\alpha\beta}\partial_\alpha\partial_\beta v$, $c > 0$. Let $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$ be a C^∞ metric and satisfies the coercive condition with a constant C , i.e. (7.16), then the following estimate holds for $0 \leq t < T$:*

$$\begin{aligned} c\|v(t, \cdot)\|_{E^d} + \|v(t, \cdot)\|_{E_P^{d+1}} &\leq C(\|v(0, \cdot)\|_{E_P^{d+1}} + c\|v(0, \cdot)\|_{E^d}) e^{C \int_0^t D_a(\tau) d\tau} \\ &\quad + C \int_0^t \|F(s, \cdot)\|_{E^d} e^{C \int_s^t D_d(\tau) d\tau} ds. \end{aligned} \quad (7.31a)$$

Proof. The proof is essentially the same to that of lemma 7.8. The only difference comes from the potential term:

$$\begin{aligned} &\partial_t v (g^{\alpha\beta}\partial_\alpha\partial_\beta v - c^2 v) \\ &= \frac{1}{2}\partial_0 (g^{00}(\partial_0 v)^2 - g^{ab}\partial_a v \partial_b v) - \frac{1}{2}(c\partial_0 v)^2 + \partial_a (g^{a\beta}\partial_t v \partial_\beta v) + \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha v \partial_\beta v - \partial_\alpha g^{\alpha\beta}\partial_t v \partial_\beta v. \end{aligned}$$

Then the same calculation of the proof in (7.8) leads to

$$\frac{d}{dt}E_{g,c}(t, v) \leq C\|F(t, \cdot)\|_{L^2(\mathbb{R}^3)} + C \sum_{\alpha, \beta} \|\nabla H^{\alpha\beta}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} E_{g,c}(t, v). \quad (7.32)$$

Now we derive the equation with respect to $\partial^{I_1}\Omega^{I_2}$, and preform the same calculation as we done in the proof of lemma 7.10, the we arrive at:

$$E_{g,c}^d(t, u) \leq E_{g,c}^d(0, u)e^{C \int_0^t D_d(\tau)d\tau} + C \int_0^t \|F(\tau, \cdot)\|_{E^d} e^{C \int_\tau^t D_d(s)ds} d\tau. \quad (7.33)$$

Then combined with the expression of $E_{g,c}$ and the coercive condition (7.16), the desired result is proved. \square

At the end of this subsection, we establish the following estimate on second order time derivative of the solution.

Lemma 7.12. *Let u be a C_c^∞ function defined in \mathbb{R}^4 . Suppose that u satisfies the following wave/Klein-Gordon equation:*

$$g^{\alpha\beta}\partial_\alpha\partial_\beta u - c^2u = -F$$

where $c \geq 0$. Suppose that $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$ with $|H^{00}| \leq 1/2$. Then the following estimate hold for all pair of multi-index (I_1, I_2)

$$\begin{aligned} & \left\| \partial_x^{I_1}\Omega^{I_2}\partial_t\partial_t u \right\|_{L^2(\mathbb{R}^3)} \\ & \leq \left\| \partial_x^{I_1}\Omega^{I_2}((1 - H^{00})^{-1}(m^{ab} + H^{ab})\partial_a\partial_b u) \right\|_{L^2(\mathbb{R}^3)} + c^2 \left\| \partial_x^{I_1}\Omega^{I_2}((1 - H^{00})^{-1}u) \right\|_{L^2(\mathbb{R}^3)} \\ & \quad + 2 \left\| \partial_x^{I_1}\Omega^{I_2}((1 - H^{00})^{-1}H^{0a}\partial_t\partial_a u) \right\|_{L^2(\mathbb{R}^3)} + \left\| \partial_x^{I_1}\Omega^{I_2}((1 - H^{00})^{-1}F) \right\|_{L^2(\mathbb{R}^3)}. \end{aligned} \quad (7.34)$$

Proof. This is by decomposition of the wave operator:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta u = (-1 + H^{00})\partial_t\partial_t u + 2H^{0a}\partial_t\partial_a u + (m^{ab} + H^{ab})\partial_a\partial_b u$$

Then by the equation

$$(-1 + H^{00})\partial_t\partial_t u + 2H^{0a}\partial_t\partial_a u + (m^{ab} + H^{ab})\partial_a\partial_b u - c^2u = -F,$$

we have

$$\partial_t\partial_t u = -\frac{c^2u}{1 - H^{00}} + \frac{2H^{0a}\partial_t\partial_a u}{1 - H^{00}} + \frac{(m^{ab} + H^{ab})\partial_a\partial_b u}{1 - H^{00}} + \frac{F}{1 - H^{00}}$$

Then the desired result is proved. \square

7.4 Existence results for linear equations

In this subsection we will establish the existence theory for linear wave and Klein-Gordon equations with initial data in the corresponding functional spaces defined in subsection 6.5. We begin with the wave equation.

Proposition 7.13 (Existence of linear wave equation in E_H^{d+1}). *Let $d \geq 3$ be an integer. Assume that $F \in L^1([0, T]; E^d)$, $(u_0, u_1) \in E_H^{d+1} \times E^d$. Assume that $g^{\alpha\beta}$ is a C^∞ metric defined on $[0, T] \times \mathbb{R}^3$ and coercive with constant $C > 0$, and $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ is in the class $C([0, T]; E_H^{d+1})$ with $|H^{00}| < 1/2$. Then the following Cauchy problem*

$$\begin{aligned} g^{\alpha\beta}\partial_\alpha\partial_\beta u &= -F, \\ u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x) \end{aligned} \quad (7.35)$$

has a unique solution in class $C([0, T]; E_H^{d+1})$ with $\partial_t^k u \in C([0, T]; E^{d+1-k})$, $1 \leq k \leq d$. Furthermore, this solution satisfies the following estimate:

$$\|u(t, \cdot)\|_{E_P^{d+1}} \leq C \|\nabla u(0, \cdot)\|_{E^d} e^{C \int_0^t D_d(\tau) d\tau} + C \int_0^t \|F(\tau, \cdot)\|_{E^d} e^{C \int_\tau^t D_d(s) ds} d\tau, \quad (7.36a)$$

$$\begin{aligned} \|u(t, \cdot)\|_{\mathcal{B}^{-1}} &\leq Ct \int_0^t \|F(s, \cdot)\|_{\mathcal{B}^{-1}} ds + Ct E_g^3(0, u) \sum_{\alpha, \beta} \int_0^t \|H^{\alpha\beta}(s, \cdot)\|_{L^\infty} e^{\int_0^s D_3(\tau) d\tau} ds \\ &+ Ct \sum_{\alpha, \beta} \int_0^t \|H^{\alpha\beta}(s, \cdot)\|_{L^\infty} \int_0^s \|F(\tau, \cdot)\|_{E^3} e^{\int_\tau^s D_3(\lambda) d\lambda} ds + Ct E_g^3(0, u) \end{aligned} \quad (7.37)$$

where

$$D_k(t) := \max_{\alpha, \beta} \|H^{\alpha\beta}(t, \cdot)\|_{E_H^{k+1}}.$$

Proof of proposition 7.13. The uniqueness is direct by applying the lemma 7.9.

The existence is based on the regularization and the estimate proved in lemma 7.10. We proceed by make a series of triple (u_0^n, u_1^n, F^n) which converges to (u_0, u_1, F) in the following sense:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_0^n - u_0\|_{E_H^{d+1}} &= 0, \quad \lim_{n \rightarrow \infty} \|u_1^n - u_0\|_{E^d} = 0 \\ \lim_{n \rightarrow \infty} \|F^n - F\|_{L^1([0, T]; E^d)} &= 0, \quad \lim_{n \rightarrow \infty} \|H_n^{\alpha\beta} - H^{\alpha\beta}\|_{L^\infty([0, T]; E^d)} = 0 \end{aligned}$$

where u_0^n, u_1^n are $C_c^\infty(\mathbb{R}^3)$ functions and for all $t \in [0, T]$, $F^n(t, \cdot) \in C_c^\infty(\mathbb{R}^3)$.

By classical existence theorem of linear wave equation (see for example [8]), fix the time interval $[0, T]$, each triple (u_0^n, u_1^n, F^n) determines a unique C_c^∞ solution by (7.35). These solution, denoted by u^n , formes a series.

By estimate (7.21a), $\{u^n\}$ is a Cauchy sequence in $L^\infty([0, T], X_P^{d+1})$. When $d \geq 3$, (7.22) shows that $\{u^n\}$ is a Cauchy sequence in $L^\infty([0, T], \mathcal{B}^{-1})$. So that $\{u^n\}$ converges in $C([0, T], E_H^{d+1})$ (u^n are C_c^∞ functions so $u^n \in C([0, T]; \mathcal{B}^{-1})$). Furthermore, because u^n are C_c^∞ functions so they are in $C([0, T], E_H^{d+1})$ which is a closed subspace of $L^\infty([0, T], E_H^{d+1})$. Then $\{u^n\}$ converges in $C([0, T], E_H^{d+1})$. We denote by u the $C([0, T], E_H^{d+1})$ -limit of $\{u^n\}$. Then we see that $u \in C([0, T]; E_H^d)$.

We call apply the same argument on $\{\partial_t^k u\}$ and get the desired regularity.

The estimate on u is gained by taking limit of the estimate on u^n . \square

If we analyse carefully the proof of proposition 7.13, we can conclude that if the triple (F, u_0, u_1) is only supposed to be in $L^1([0, T], E^d) \times E_P^{d+1} \times E^d$, the Cauchy problem (7.35) determines also a unique solution in $C([0, T], E_P^{d+1})$. We prefer to state this result separately in the following proposition:

Proposition 7.14. *Let $d \geq 3$ and assume that the triple (F, u_0, u_1) is only supposed to be in $L^1([0, T], E^d) \times E_P^{d+1} \times E^d$. And assume that $g^{\alpha\beta}$ is a C^∞ metric defined on $[0, T] \times \mathbb{R}^3$ and coercive with constant $C > 0$, and $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ is in the class $C([0, T]; E_H^{d+1})$. Then the Cauchy problem (7.35) has a unique solution u in $C([0, T], E_P^{d+1})$ and $\partial_t^k u \in C([0, T]; E^{d+1-k})$ for $0 \leq k \leq d$. Furthermore it satisfies the following estimate*

$$\|u(t, \cdot)\|_{E_P^{d+1}} \leq C \|\nabla u(0, \cdot)\|_{E^d} e^{\int_0^t C D_d(\tau) d\tau} + C \int_0^t \|F(\tau, \cdot)\|_{E^d} e^{\int_\tau^t D_d(s) ds} d\tau, \quad (7.38a)$$

where

$$D_k(t) := \max_{\alpha, \beta} \|H^{\alpha\beta}(t, \cdot)\|_{E_H^{k+1}}.$$

Apply lemma 7.11 and taking the same procedure of regularization used in the proof of proposition 7.13, the following existence result for linear Klein-Gordon equation holds:

Proposition 7.15 (Existence for KG equation). *Let $d \geq 3$ and the triple (v_0, v_1, F) be in class $E_R^{d+1} \times E_R^d \times L^1([0, T], E_R^d)$. Assume that $g^{\alpha\beta}$ is a C^∞ metric defined on $[0, T] \times \mathbb{R}^3$ and coercive with constant $C > 0$, and $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ is in the class $C[0, T]; E_H^{d+1}$. Then the following Cauchy problem*

$$\begin{aligned} g^{\alpha\beta} \partial_\alpha \partial_\beta v - c^2 v &= F, \quad c > 0, \\ v(0, x) &= v_0(x), \quad \partial_t v(0, x) = v_1(x) \end{aligned} \quad (7.39)$$

has a unique solution in class $C([0, T]; E_R^{d+1}) \cap C^1([0, T]; E_R^d)$. Furthermore, it satisfies the following estimate

$$\begin{aligned} \|v(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t v(t, \cdot)\|_{E^d} + c\|v(t, \cdot)\|_{E^d} &\leq C(\|\nabla u(0, \cdot)\|_{E^d} + c\|v(0, \cdot)\|_{E^d}) e^{C \int_0^t D_d(\tau, \cdot) d\tau} \\ &\quad + C \int_0^t \|F(s, \cdot)\|_{E^d} e^{C \int_s^t D_d(\tau, \cdot) d\tau} ds, \end{aligned} \quad (7.40a)$$

$$\begin{aligned} \|v(t, \cdot)\|_{E_P^{d+1}} + c\|v(t, \cdot)\|_{E^d} &\leq C(\|\nabla u(0, \cdot)\|_{E^d} + c\|v(0, \cdot)\|_{E^d}) e^{C \int_0^t D_d(\tau, \cdot) d\tau} \\ &\quad + C \int_0^t \|F(s, \cdot)\|_{E^d} e^{C \int_s^t D_d(\tau, \cdot) d\tau} ds, \end{aligned} \quad (7.40b)$$

where

$$D_k(t) := \max_{\substack{|I_1|+|I_2| \leq k \\ \alpha, \beta}} \|\partial^{I_1} \Omega^{I_2} H^{\alpha\beta}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}.$$

7.5 Nonlinear estimates

To estimate the solution of quasi-linear system, we will need the following estimates on nonlinear terms.

Lemma 7.16. *Let F be a C^∞ function from \mathbb{R}^m to \mathbb{R} and u a C^∞ application form \mathbb{R}^4 to \mathbb{R}^m with components denoted by $u = (u_1, u_2, \dots, u_m)$. Let Z be a family of one order linear differential operate $Z = \{Z_\alpha\}$ with $\alpha \in \Lambda$ where Λ is a subset of \mathbb{N}^* . Then the following identity holds for all multi-index $I = (\alpha_1, \alpha_2, \dots, \alpha_{|I|})$ with $|I| \geq 1$:*

$$Z^I(F(u)) = \sum_{1 \leq |L| \leq |I|} P^L F(u) \sum_{\sum_j K_{ji} = I} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j. \quad (7.41)$$

Here

$$P^L = \prod_{i=1}^m \partial_i^{l_i}$$

is a product of partial differential operator with $L = (l_1, \dots, l_m)$ and the convention:

$$Z^I u = 1, \quad \text{if } |I| = 0$$

is applied. Furthermore, in a product if the set of index is empty, this product is regarded as 1. For example,

$$\prod_{i=1}^{l_j} Z^{K_{ji}} u_j = 1, \quad \text{if } l_j = 0.$$

Proof. We remark that in the right-hand-side, for a fixed L , the sum is taken over all the proper $|L|$ -partition of index I . That is, over all the proper L -partition of abstract index \mathcal{I} with $|\mathcal{I}| = I$. We denote by

$$\mathcal{K}_{ji} = \mathcal{I} \quad \text{with} \quad \{K_{ji}\} = I(\{\mathcal{K}_{ji}\})$$

and we denote by $\mathcal{P}_p(\mathcal{I}, |L|)$ the set of all proper $|L|$ -partition of \mathcal{I} . Then (7.41) can be written as

$$Z^I(F(u)) = \sum_{1 \leq |L| \leq |I|} P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j, \quad \{K_{ji}\} = I(\{\mathcal{K}_{ji}\}).$$

Now we associate each term in right-hand-side to a pair $(L, \{\mathcal{K}_{ji}\})$ by the following manner: in the sum, each term $P^L F(u) \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j$ corresponds to an operator P^L which supply L . The $\{K_{ji}\}$ is a partition of I which is a restriction of I on a abstract partition $\{\mathcal{K}_{ji}\}$. Remark that for fixed L the sum is taken on $\mathcal{P}_p(\mathcal{I}, |L|)$, so we have constructed a bijection from the terms in right hand side to the following set

$$\mathcal{Q}(\mathcal{I}) = \{(L, \{\mathcal{K}_{ji}\}) | \mathcal{K}_{ji} \in \mathcal{P}_p(\mathcal{I}, |L|), 1 \leq |L| \leq |\mathcal{I}|\}$$

We will prove (7.16) by induction on the order of $|I|$ with associated abstract multi-index $\mathcal{I} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. We check by direct calculation that this identity is valid for $|I| = 1$. Suppose that it holds for $|I| \leq n$, we consider $|I'| = n + 1$. Let \mathcal{I}' be the associated $n + 1$ order abstract multi-index composed by $\{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}\}$ and the restriction of I' on \mathcal{I} is coincide with I .

Then

$$\begin{aligned} & Z^{I'}(F(u)) \\ &= Z^{I'(\alpha_{n+1})} \left(\sum_{1 \leq |L| \leq |I|} P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \right) \\ &= \sum_{1 \leq |L| \leq |I|} Z_{I'(\alpha_{n+1})}(P^L F(u)) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \\ &\quad + \sum_{1 \leq |L| \leq |I|} P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} Z_{I'(\alpha_{n+1})} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \\ &=: T_1 + T_2. \end{aligned}$$

For T_1 , we remark that

$$\begin{aligned} T_1 &= \sum_{1 \leq |L| \leq |I|} \sum_{k=1}^m P^{L'_k} F(u) Z_{I'(\alpha_{n+1})} u_k \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \\ &= \sum_{1 \leq |L| \leq |I|} \sum_{k=1}^m P^{L'_k} F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l'_j} Z^{K'_{ji}} u_j \end{aligned}$$

with $L'_k = (l'_1, l'_2, \dots, l'_k, \dots, l'_m)$ with $l'_j = l_j$ for $j \neq k$ and $l'_k = l_k + 1$ and $\mathcal{K}'_{ji} = \mathcal{K}_{ji}$ with $(j, i) \neq (k, l'_k)$ and $\mathcal{K}'_{kl'_k} = \alpha_{n+1}$. Here K_{ij} is the restriction of I on \mathcal{K}_{ji} while K'_{ji} is the restriction of I' on \mathcal{K}'_{ji} .

For T_2 ,

$$\begin{aligned} T_2 &= \sum_{1 \leq |L| \leq |I|} P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} Z_{I'(\alpha_{n+1})} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \\ &= \sum_{1 \leq |L| \leq |I|} P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{I}, |L|)} \sum_{\substack{1 \leq j_0 \leq m \\ 1 \leq i_0 \leq l_{j_0}}} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K'_{ji}} u_j \end{aligned}$$

where $\mathcal{K}'_{ji} = \mathcal{K}_{ji}$ when $(j, i) \neq (j_0, i_0)$ and $\mathcal{K}'_{j_0 i_0} = \mathcal{K}_{j_0 i_0} \cup \{\alpha_{n+1}\}$.

Now we associate each term in T_1 and T_2 a pair of form $(L', \{\mathcal{K}_{ji}\})$ by the same manner. We remark that this defines an injection from the terms contained in T_1 and T_2 to the set

$$\mathcal{Q}(\mathcal{S}') = \{(L', \{\mathcal{K}_{ji}\}) | 1 \leq |L'| \leq |\mathcal{S}'|, \{\mathcal{K}_{ji}\} \in \mathcal{P}_P(|L|, \mathcal{S}')\}.$$

The injectivity is by checking the fact that for two terms if $L' = \tilde{L}'$, the different terms correspond to different partition by the definition of “sum over partitions”.

Denote by \mathcal{A} the image of the terms in T_1 and T_2 under this injection. This is a subset of $\mathcal{Q}(\mathcal{S}')$. We will prove that $\mathcal{A} = \mathcal{Q}(\mathcal{S}')$ which leads to the equality in the case where $|I'| = n + 1$. To do so, let

$$(L', \{\mathcal{K}'_{ji}\}) \in \mathcal{Q}(\mathcal{S}'), \quad L' = (l'_1, l'_2, \dots, l'_m)$$

Then we see that as $\alpha_{n+1} \in \mathcal{S}'$ there is one and only one (j_0, i_0) such that $\alpha_{n+1} \in \mathcal{K}'_{j_0 i_0}$ because $\bigcup_{(j,i)} \mathcal{K}'_{ji} = \mathcal{S}'$ and they are disjoint to each other. We will prove that $(L', \{\mathcal{K}'_{ji}\}) \in \mathcal{A}$.

We define $L = (l_1, l_2, \dots, l_m)$ with $l_j = l'_j$ for $j \neq j_0$ and $l_{j_0} = l'_{j_0} - 1$. We construct $\{\mathcal{K}_{ji}\}$ as following: $\mathcal{K}_{ji} = \mathcal{K}'_{ji}$ if $(j, i) \neq (j_0, i_0)$ and $\mathcal{K}_{j_0 i_0} = \mathcal{K}'_{j_0 i_0} \cap \{\alpha_{n+1}\}^c$. Such constructed pair (L, \mathcal{K}_{ji}) to be in $\mathcal{Q}(\mathcal{S})$.

When $\mathcal{K}_{j_0 i_0} = \emptyset$, we see that $(L', \{\mathcal{K}'_{ji}\})$ corresponds to a term in T_1 . More precisely in the following term:

$$\begin{aligned} & Z_{I'(\alpha_{n+1})}(P^L F(u)) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{S}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \\ &= P^{L'} F(u) \sum_{k=1}^m Z_{\alpha_{n+1}} u_k \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{S}, |L|)} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \end{aligned}$$

where we can see it by fixing $k = j_0$ and $\{\mathcal{K}_{ji}\}$ in the sum.

When $\mathcal{K}_{j_0 i_0} \neq \emptyset$, $(L', \{\mathcal{K}'_{ji}\})$ corresponds to a term in T_2 :

$$\begin{aligned} & P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{S}, |L|)} Z_{I'(\alpha_{n+1})} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{ji}} u_j \\ &= P^L F(u) \sum_{\{\mathcal{K}_{ji}\} \in \mathcal{P}_p(\mathcal{S}, |L|)} \sum_{\substack{1 \leq j_0 \leq m \\ 1 \leq i_0 \leq l_{j_0}}} \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K'_{ji}} u_j \end{aligned}$$

we see it by fixing (i_i, j_0) and $\{\mathcal{K}_{ji}\}$ in the sum. \square

Lemma 7.17. *Let F be a C^∞ function from \mathbb{R}^m to \mathbb{R} and u a C^∞ application form \mathbb{R}^4 to \mathbb{R}^m . Then the following identity holds for any multi-index I_1, I_2 with $|I_1| + |I_2| \geq 1$:*

$$\partial^{I_1} \Omega^{I_2} (F(u)) = \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_{j=1}^m l_j = l} P^L F(u) \sum_{\substack{\sum_{j,i} K_{1ji} = I_1 \\ \sum_{j,i} K_{2ji} = I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1ji}} \Omega^{K_{2ji}} u_j \quad (7.42)$$

where $L = (l_1, l_2, \dots, l_m)$ is a m -dimensional vector with its components taking value in \mathbb{N} and P^L the partial differential operator

$$P^L = \prod_{i=1}^m \partial_i^{l_i}$$

and the convention:

$$\partial^{I_1} \Omega^{I_2} u = 1, \quad \text{if } |I_1| = |I_2| = 0$$

is applied.

Proof. The proof is an application of (7.41). Let $D^I = \partial^{I_1} \Omega^{I_2}$ with $D = \{\partial_\alpha, \Omega_a\}$. We denote by

$$D_\alpha = \partial_\alpha \text{ for } \alpha = 0, 1, 2, 3, \quad D_\alpha = \Omega_{\alpha-3}, \text{ for } \alpha = 4, 5, 6.$$

We denote the components of I_1 and I_2 by

$$I_1 = (\beta_1, \beta_2, \dots, \beta_{n_1}), \quad I_2 = (\gamma_1, \gamma_2, \dots, \gamma_{n_2})$$

Then I is determined by

$$I = (\alpha_1, \alpha_2, \dots, \alpha_{n_1}, \alpha_{n_1+1}, \dots, \alpha_{n_1+n_2})$$

with $\alpha_i = \beta_i$ for $i = 1, 2, n_1$ and $\alpha_{i+n_1} = \gamma_i + 3$ for $i = 1, 2, n_2$.

Remark that D is a family of first-order linear differential operator. Then by (7.41)

$$\partial^{I_1} \Omega^{I_2} (F(u)) = \sum_{1 \leq |L| \leq |I|} \sum_{\sum_{j_i} K_{j_i} = I} P^L F(u) \prod_{j=1}^m \prod_{i=1}^{l_j} Z^{K_{j_i}} u_j. \quad (7.43)$$

Then, because $\sum_{j_i} K_{j_i} = I$ is a partition of I . Then

$$D^{K_{j_i}} = \partial^{K_{1j_i}} \Omega^{K'_{2j_i}}$$

with $\sum_{j_i} K_{1j_i} = I_1$ a partition of I_1 and $\sum_{j_i} K_{2j_i} = I_2$ a partition of I_2 . This gives

$$\partial^{I_1} \Omega^{I_2} (F(u)) = \sum_{1 \leq |L| \leq |I|} \sum_{\substack{\sum_{j_i} K_{1j_i} = I_1 \\ \sum_{j_i} K_{2j_i} = I_2}} P^L F(u) \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j_i}} \Omega^{K_{2j_i}} u_j.$$

Then the desired result is proved. \square

The result of lemma 7.17 will be applied in the following case where $F(\cdot)$ is supposed to vanish at 0 in second order.

Lemma 7.18. *Let F be a C^∞ function defined in a compact neighborhood V of 0 in \mathbb{R}^m and $F(0) = \nabla F(0) = 0$. Let $d \geq 3$ and suppose that u map from \mathbb{R}^4 to V with its components u_j in $L^\infty([0, T]; E^d)$. Then the following estimates hold for any couple of index (I_1, I_2) with $|I_1| + |I_2| \leq d$:*

$$\|\partial^{I_1} \Omega^{I_2} (F(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C(F, V, d) \sum_{k=2}^{|I_1|+|I_2|} \|u(t, \cdot)\|_{E^d}^k, \quad \text{for } t \in [0, T] \quad (7.44)$$

where $C(F, V, d)$ is a constant determined by F , V and d , and $\|u(t, \cdot)\|_{E^d} := \max_j \|u_j(t, \cdot)\|_{E^d}$.

Proof. When $|I_1| + |I_2| = 0$, by the condition $F(0) = \nabla F(0) = 0$,

$$|F(u)| \leq C(F, V) \sum_{j=1}^m |u_j|^2$$

which leads to the desired result.

For $|I_1| + |I_2| \geq 1$, proof is based on the lemma 7.17. We take the expression:

$$\partial^{I_1} \Omega^{I_2} (F(u)) = \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_{j=1}^m l_j = l} P^L F(u) \sum_{\substack{\sum_{j,i} K_{1j_i} = I_1 \\ \sum_{j,i} K_{2j_i} = I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j_i}} \Omega^{K_{2j_i}} u_j$$

and remark that for $|L| = \sum_{j=1}^m l_j = 1$, $P^L F(0) = 0$. Then we have, in the compact neighborhood V of 0,

$$|\partial_j F(u)| \leq C(F, V) |u|$$

where $C(F, V)$ is determined by V and F and $|u| := \max_j |u_j|$. Then,

$$\begin{aligned}
|\partial^{I_1} \Omega^{I_2} (F(u))| &\leq \sum_{j=1}^m |\partial_j F(u)| |\partial^{I_1} \Omega^{I_2} u_j| \\
&+ \sum_{l=2}^{|I_1|+|I_2|} \sum_{\sum_{j=1}^m l_j=l} |P^L F(u)| \sum_{\substack{\sum_{j,i} K_{1ji}=I_1 \\ \sum_{j,i} K_{2ji}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} |\partial^{K_{1ji}} \Omega^{K_{2ji}} u_j| \\
&\leq C(F, V) |u| |\partial^{I_1} \Omega^{I_2} u| + \sum_{l=2}^{|I_1|+|I_2|} \sum_{\sum_{j=1}^m l_j=l} |P^L F(u)| \sum_{\substack{\sum_{j,i} K_{1ji}=I_1 \\ \sum_{j,i} K_{2ji}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} |\partial^{K_{1ji}} \Omega^{K_{2ji}} u_j|.
\end{aligned} \tag{7.45}$$

The first term in right-hand-side is estimated as following:

$$\begin{aligned}
\|u(t, \cdot) \partial^{I_1} \Omega^{I_2} u(t, \cdot)\|_{L^2(\mathbb{R}^3)} &\leq \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \|\partial^{I_1} \Omega^{I_2} u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\
&\leq C \|u(t, \cdot)\|_{E^2} \|u(t, \cdot)\|_{X^d} \leq C \|u(t, \cdot)\|_{E^d}^2
\end{aligned}$$

where the Sobolev's inequality is applied.

The second term in right-hand-side of (7.45) is estimated as following: we observe the term :

$$\sum_{\substack{\sum_{j,i} K_{1ji}=I_1 \\ \sum_{j,i} K_{2ji}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} |\partial^{K_{1ji}} \Omega^{K_{2ji}} u_j|$$

Recall that $|L| = \sum_j l_j \geq 2$, then we see that in the product there are at least two factors and

$$\sum_{ji} K_{1ji} = I_1, \quad \sum_{ji} K_{2ji} = I_2$$

is a partition of (I_1, I_2) in $|L| = \sum_j l_j$ pieces with $|L| \geq 2$. Remark that

$$\sum_{ji} |K_{1ji}| = |I_1|, \quad \sum_{ji} |K_{2ji}| = |I_2|.$$

We observe that there among the index K_{1ji} there is at most one, denoted by $K_{1i_0j_0}$, is of order higher than $|I_1|/2$. In an other world,

$$|K_{1ji}| \leq \lceil |I_2|/2 \rceil, \quad \text{if } i \neq i_0, j \neq j_0,$$

where $[x]$ denotes the biggest integer less than or equal to x . The same result holds for the index K_{2ji} . Then we conclude that in the decomposition of (I_1, I_2) there is at most one pair of index, denoted by $(K_{1i_0j_0}, K_{2i_0j_0})$, is of order higher than $d/2$. In an other world,

$$|K_{1ji}| + |K_{2ji}| \leq \lfloor d/2 \rfloor, \quad \text{if } i \neq i_0, j \neq j_0.$$

Then if we take the L^2 norm, we will find that

$$\begin{aligned}
\left\| \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1ji}} \Omega^{K_{2ji}} u_j(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} &\leq \prod_{\substack{j=1 \\ j \neq j_0}}^m \prod_{\substack{i=1 \\ i \neq i_0}}^{l_j} \|\partial^{K_{1ji}} \Omega^{K_{2ji}} u_j(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \cdot \|\partial^{K_{1i_0j_0}} \Omega^{K_{2i_0j_0}} u_{j_0}(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\
&\leq \prod_{\substack{j=1 \\ j \neq j_0}}^m \prod_{\substack{i=1 \\ i \neq i_0}}^{l_j} \|\partial^{K_{1ji}} \Omega^{K_{2ji}} u_j(t, \cdot)\|_{E^2} \cdot \|u_{j_0}(t, \cdot)\|_{E^d} \\
&\leq \prod_{\substack{j=1 \\ j \neq j_0}}^m \prod_{\substack{i=1 \\ i \neq i_0}}^{l_j} \|u(t, \cdot)\|_{E^{2+\lfloor d/2 \rfloor}} \|u(t, \cdot)\|_{E^d}.
\end{aligned}$$

Here we have applied (7.3a). Now recall $d \geq 3$, then $2 + [d/2] \leq d$. Then

$$\left\| \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1ji}} \Omega^{K_{2ji}} u(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{E^d}^{|L|}.$$

Also, we remark that in the compact neighborhood V , $\sup_{x \in V} |P^L F(u)| \leq C(F, V)$ with $C(F, V, |L|)$ a constant determined by F and V and $|L|$. Then the desired result is proved. \square

Now combine lemma 7.18 with the global Sobolev inequality (7.8), the following estimate holds:

Lemma 7.19. *Taking the assumptions on F and u as in lemma 7.18. Then for $|I_1| + |I_2| \leq d - 2$, the following estimate holds:*

$$\|\partial^{I_1} \Omega^{I_2} (F(u))(t, \cdot)\|_{\mathcal{B}^{-1}} \leq C(F, V, d) \sum_{k=2}^{|I_1|+|I_2|+2} \|u(t, \cdot)\|_{E^d}^k. \quad (7.46)$$

We also need the following type of estimate in the following discussion:

Lemma 7.20. *Let $d \geq 3$ and assume that $H(\cdot)$ be a C^∞ function defined in a compact neighbourhood V of 0 in \mathbb{R}^m and assume that u is a map from \mathbb{R}^3 to V with its components u_j in class $L^\infty([0, T]; E_H^d)$. Then the following estimate holds for $d \geq |I_1| + |I_2| \geq 1$:*

$$\|(1+r)^{-1} \partial^{I_1} \Omega^{I_2} (H(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C(H, V, d) \sum_{k=1}^{|I_1|+|I_2|} \|u(t, \cdot)\|_{E_H^d}^k, \quad t \in [0, T]. \quad (7.47)$$

Proof. We apply the expression (7.42):

$$\partial^{I_1} \Omega^{I_2} (H(u)) = \sum_{l=1}^{|I_1|+|I_2|} \sum_{\substack{m \\ \sum_{j=1}^m l_j=l}} P^L H(u) \sum_{\substack{\sum_{j_i} K_{1ji}=I_1 \\ \sum_{j_i} K_{2ji}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1ji}} \Omega^{K_{2ji}} u_j$$

Remark that $P^L H(u)$ is bounded by a constant $C(H, V, |L|)$ determined by V . The estimate of $\|\partial^{I_1} \Omega^{I_2} (H(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)}$ reduced into the estimate of

$$\left\| (1+r)^{-1} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1ji}} \Omega^{K_{2ji}} u_j(t, \cdot) \right\|_{L^2(\mathbb{R}^3)}$$

where K_{1ji} and K_{2ji} is a possible partition of (I_1, I_2) . We take the same argument to that the proof of lemma 7.18. Suppose that $|K_{1j_0 i_0}| + |K_{2j_0 i_0}| \geq |K_{1ji}| + |K_{2ji}|$ for all pares (j, i) , i.e. $(K_{1j_0 i_0}, K_{2j_0 i_0})$ is the pair of index with highest order. Then

$$|K_{1ji}| + |K_{2ji}| \leq [d/2], \quad \text{if } i \neq i_0, j \neq j_0$$

and

$$|K_{1j_0 i_0}| + |K_{2j_0 i_0}| \geq 1.$$

Then

$$\begin{aligned} & \left\| (1+r)^{-1} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1ji}} \Omega^{K_{2ji}} u_j(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \\ & \leq \prod_{\substack{j=1, \\ j \neq j_0}}^m \prod_{\substack{i=1, \\ i \neq i_0}}^{l_j} \|\partial^{K_{1ji}} \Omega^{K_{2ji}} u_j(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \|(1+r)^{-1} \partial^{K_{1j_0 i_0}} \Omega^{K_{2j_0 i_0}} u_j(t, \cdot)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

Then by (7.10b) and (7.11b), the desired result is proved. \square

Lemma 7.21. *Taking the same assumption on H and u as in lemma 7.20, and assume furthermore that $H(0) = 0$. Then for all $|I_1| + |I_2| \leq d - 2$, the following estimate holds:*

$$\|\partial^{I_1} \Omega^{I_2} (H(u))(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(H, V, d) \sum_{k=1}^{|I_1|+|I_2|+2} \|u(t, \cdot)\|_{E_H^d}^k. \quad (7.48)$$

Proof. When $I_1 = I_2 = 0$, we recall that the condition $H(0) = 0$ implies

$$\|H(u)\|_{L^\infty(\mathbb{R}^3)} \leq C(H, V) \|u\|_{L^\infty(\mathbb{R}^3)} \leq C(H, V) \|u\|_{\mathcal{B}^{-1}}.$$

When $|I_1| + |I_2| \geq 1$, we apply (7.47) combined with lemma 7.7:

$$\begin{aligned} \|\|\partial^{I_1} \Omega^{I_2} (H(u))(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}\| &\leq C \sum_{|J_1|+|J_2| \leq 2} \|\partial_x^{J_1} \Omega^{J_2} \partial^{I_1} \Omega^{I_2} (H(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ &\leq C \sum_{\substack{|J_1|+|J_2| \leq 2 \\ |I'_1| \leq |I_1|}} \|\partial_x^{J_1} \partial^{I'_1} \Omega^{J_2} \Omega^{I_2} (H(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

Here we remark that when $|I_1| \geq 1$, by applying (7.7a) successively ($|I_1|$ times), we see that $|I'_1| \geq 1$, this leads to the fact that $|J_1| + |I'_1| + |J_2| + |I'_2| \geq 1$. Then we apply (7.47).

When $|I_1| = 0$, then $|I_2| \geq 1$ then we can also apply (7.47). □

Lemma 7.22. *Let F be a C^∞ function defined in a compact neighbourhood V_1 of 0 in \mathbb{R}^m and H be a C^∞ function defined in a compact neighbourhood V_2 of 0 in \mathbb{R}^n . Assume that $F(0) = 0$ and $\nabla(F) = 0$. Let u be a map from \mathbb{R}^4 to V_1 with its components u_j in class $L^\infty([0, T]; E_H^d)$ and v be a map from \mathbb{R}^4 to V_2 with its components v_i in class $L^\infty([0, T]; E^d)$. Then the following estimate holds for $|I_1| + |I_2| \leq d$ with $d \geq 3$:*

$$\|\partial^{I_1} \Omega^{I_2} (H(u)F(v))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C(H, F, V, d) \left(\sum_{k=0}^d \|u(t, \cdot)\|_{E_H^d}^k \right) \left(\sum_{k=2}^d \|v(t, \cdot)\|_{E^d}^k \right), \quad t \in [0, T]. \quad (7.49)$$

Proof. The proof is based on the lemma 7.18, lemma 7.19, lemma 7.20 and lemma 7.21.

$$\partial^{I_1} \Omega^{I_2} (H(u)F(v)) = \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \partial^{J_1} \Omega^{J_2} (H(u) - H(0)) \partial^{J'_1} \Omega^{J'_2} (F(v)) + H(0) \partial^{I_1} \Omega^{I_2} (F(v)).$$

When $d \geq |J'_1| + |J'_2| \geq d/2$, $|J_1| + |J_2| \leq [d/2] \leq d - 2$:

$$\begin{aligned} &\|\|\partial^{J_1} \Omega^{J_2} (H(u) - H(0))(t, \cdot) \partial^{J'_1} \Omega^{J'_2} (F(v))(t, \cdot)\|_{L^2(\mathbb{R}^3)}\| \\ &\leq \|\|\partial^{J_1} \Omega^{J_2} (H(u) - H(0))(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}\| \|\|\partial^{J'_1} \Omega^{J'_2} (F(v))(t, \cdot)\|_{L^2(\mathbb{R}^3)}\| \\ &\leq C(H, F, V, d) \sum_{k=1}^{|J_1|+|J_2|+2} \|u(t, \cdot)\|_{E_H^d}^k \sum_{k=2}^{|J'_1|+|J'_2|} \|v(t, \cdot)\|_{E^d}^k, \end{aligned}$$

where lemma 7.21 and lemma 7.19 are applied (on function $H(u) - H(0)$ and $F(v)$). The term $H(0) \partial^{I_1} \Omega^{I_2} (F(v))$ is estimated by (7.44).

When $|J_1| + |J_2| \geq d/2 > 1$, $|J'_1| + |J'_2| \leq [d/2] \leq d - 2$:

$$\begin{aligned} &\|\|\partial^{J_1} \Omega^{J_2} (H(u))(t, \cdot) \partial^{J'_1} \Omega^{J'_2} (F(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)}\| \\ &\leq \|(1+r)^{-1} \partial^{J_1} \Omega^{J_2} (H(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \|(1+r) \partial^{J'_1} \Omega^{J'_2} (F(u))(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \\ &= \|(1+r)^{-1} \partial^{J_1} \Omega^{J_2} (H(u))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \|\|\partial^{J'_1} \Omega^{J'_2} (F(u))(t, \cdot)\|_{\mathcal{B}^{-1}}\| \\ &\leq C(H, F, V, d) \left(\sum_{k=1}^{|J_1|+|J_2|} \|u(t, \cdot)\|_{E_H^d}^k \right) \left(\sum_{k=2}^{|J'_1|+|J'_2|+2} \|v(t, \cdot)\|_{E^d}^k \right). \end{aligned}$$

where lemma 7.20 and lemma 7.19 are applied. \square

We will need the following estimate on multi-linear functions.

Lemma 7.23. *Let $d \geq 3$ be an integer and $u_i, v_i, i = 1, 2, \dots, m$ be functions in class $L^\infty([0, T]; E^d)$. Then the following estimates hold for $m \geq 2$ and $|I_1| + |I_2| \leq d$:*

$$\left\| \partial^{I_1} \Omega^{I_2} \left(\prod_{i=1}^m u_i - \prod_{i=1}^m v_i \right) (t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \leq C(V, d) \|u(t, \cdot) - v(t, \cdot)\|_{E^d} \sum_{k=1}^{m-1} \sum_{k_1+k_2=k} \|u(t, \cdot)\|_{E^d}^{k_1} \|v(t, \cdot)\|_{E^d}^{k_2} \quad (7.50)$$

where $\|u\|_{X^d} := \max_i \|u_i\|_{X^d}$ and $\|v\|_{X^d} := \max_i \|v_i\|_{X^d}$.

Proof. We remark the following identity

$$\prod_{i=1}^m u_i - \prod_{i=1}^m v_i = \sum_{k=1}^m (u_k - v_k) \prod_{j < k} u_j \prod_{j > k} v_j.$$

The for each term,

$$\partial^{I_1} \Omega^{I_2} \left((u_k - v_k) \prod_{j < k} u_j \prod_{j > k} v_j \right) = \sum_{\substack{\sum_i K_{1i} = I_1 \\ \sum_i K_{2i} = I_2}} \partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \prod_{j > k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j < k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j.$$

To estimate the product

$$\partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \prod_{j > k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j < k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j,$$

we apply the same reasoning in the proof of lemma 7.18, there is at most one pair of multi index of order bigger than $d/2$. Then (7.50) is proved by applying the classical Sobolev's inequality. \square

Lemma 7.24. *Let $d \geq 3$ be an integer and $u_i, v_i, i = 1, 2, \dots, m$ be functions in class $L^\infty([0, T]; E_H^d)$. Then the following estimates hold for $m \geq 2$ for $1 \leq |I_1| + |I_2| \leq d$:*

$$\begin{aligned} & \left\| (1+r)^{-1} \partial^{I_1} \Omega^{I_2} \left(\prod_{i=1}^m u_i - \prod_{i=1}^m v_i \right) (t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \\ & \leq C(V, d) \|u(t, \cdot) - v(t, \cdot)\|_{E_H^d} \sum_{k=1}^{m-1} \sum_{k_1+k_2=k} \|u(t, \cdot)\|_{E_H^d}^{k_1} \|v(t, \cdot)\|_{E_H^d}^{k_2}, \quad t \in [0, T], \end{aligned} \quad (7.51)$$

where $\|u\|_{E_H^d} := \max_i \|u_i\|_{E_H^d}$ and $\|v\|_{E_H^d} := \max_i \|v_i\|_{E_H^d}$.

Proof. We remark the following identity

$$\prod_{i=1}^m u_i - \prod_{i=1}^m v_i = \sum_{k=1}^m (u_k - v_k) \prod_{j < k} u_j \prod_{j > k} v_j.$$

The for each term,

$$\partial^{I_1} \Omega^{I_2} \left((u_k - v_k) \prod_{j < k} u_j \prod_{j > k} v_j \right) = \sum_{\substack{\sum_i K_{1i} = I_1 \\ \sum_i K_{2i} = I_2}} \partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \prod_{j > k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j < k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j.$$

To estimate the product

$$\partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \prod_{j > k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j < k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j,$$

we also apply the same reasoning in the proof of lemma 7.18, there is at most one pair of multi index of order bigger than $d/2$. We discuss in the following three different cases.

CASE 1. $|K_{1k}| + |K_{2k}| \geq |K_{1j}| + |K_{2j}|$ for $j = 1, 2, \dots, m$, i.e. (K_{1k}, K_{2k}) takes the highest order. Then for $j \neq k$, $|K_{1j}| + |K_{2j}| \leq [d/2] \leq d - 2$ and $|K_{1k}| + |K_{2k}| \geq 1$. So

$$\begin{aligned} & \left\| (1+r)^{-1} \partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \prod_{j>k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j<k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j \right\|_{L^2(\mathbb{R}^3)} \\ & \leq \left\| (1+r)^{-1} \partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \right\|_{L^2(\mathbb{R}^3)} \left\| \prod_{j>k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j<k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j \right\|_{L^\infty(\mathbb{R}^3)} \\ & \leq C \|u_k - v_k\|_{E_P^d} \left\| \prod_{j>k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j<k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j \right\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

Here we applied (7.10a) on the first factor. Remark that $m \geq 2$ then there exists a $j_0 \neq k$. So the second factor is bounded by applying (7.48) and the fact that $|K_{1j}| + |K_{2j}| \leq d - 2$.

CASE 2. $|K_{1k}| + |K_{2k}| \leq [d/2] \leq d - 2$ and there exists $j_0 < k$ such that $|K_{1j_0}| + |K_{2j_0}| \geq |K_{1j}| + |K_{2j}|$. Then

$$\begin{aligned} & \left\| (1+r)^{-1} \partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \prod_{j>k} \partial^{K_{1j}} \Omega^{K_{2j}} u_j \prod_{j<k} \partial^{K_{1j}} \Omega^{K_{2j}} v_j \right\|_{L^2(\mathbb{R}^3)} \\ & \leq \left\| \partial^{K_{1k}} \Omega^{K_{2k}} (u_k - v_k) \right\|_{L^\infty(\mathbb{R}^3)} \prod_{j>k} \left\| \partial^{K_{1j}} \Omega^{K_{2j}} u_j \right\|_{L^\infty(\mathbb{R}^3)} \prod_{\substack{j<k \\ j \neq j_0}} \left\| \partial^{K_{1j}} \Omega^{K_{2j}} v_j \right\|_{L^\infty(\mathbb{R}^3)} \\ & \quad \times \left\| (1+r)^{-1} \partial^{K_{1j_0}} \Omega^{K_{2j_0}} u^{j_0} \right\|_{L^2(\mathbb{R}^3)} \\ & \leq C \|u - v\|_{E_H^d} \prod_{j>k} \|u_j\|_{E_H^d} \prod_{\substack{j<k \\ j \neq j_0}} \|v_j\|_{E_H^d} \|u^{j_0}\|_{E_H^d} \end{aligned}$$

where (7.10b) and (7.11b) are applied.

CASE 3. $|K_{1k}| + |K_{2k}| \leq [d/2] \leq d - 2$ and there exists $j_0 > k$ such that $|K_{1j_0}| + |K_{2j_0}| \geq |K_{1j}| + |K_{2j}|$. The prove is exactly the same to the last case if we exchange the roles of u_j and v_j . \square

Finally we should be able to estimate the difference of two quadratic functions. This is concluded in the following lemma:

Lemma 7.25. *Let $d \geq 3$ be an integer. Let F be a C^∞ function defined in a compact neighbourhood V of 0 in \mathbb{R}^m with $F(0) = \nabla F(0) = 0$. Assume that u and v are maps from \mathbb{R}^d to V with their components in $L^\infty([0, T]; E^d)$. Then the following estimate hold for all pair of index (I_1, I_2) with $|I_1| + |I_2| \leq d$:*

$$\left\| \partial_x^{I_1} \Omega^{I_2} (F(u) - F(v)) \right\|_{L^2(\mathbb{R}^3)} \leq C(F, V, d) \|u - v\|_{X^d} \sum_{k=1}^{|I_1|+|I_2|-1} \sum_{k_1+k_2=k} \|u\|_{X^d}^{k_1} \|v\|_{X^d}^{k_2}. \quad (7.52)$$

Proof. When $I_1 = I_2 = 0$, we apply the mean value theorem: there is a $\theta \in [0, 1]$ such that

$$|F(u) - F(v)| \leq |\nabla F(\theta u + (1 - \theta)v)| |u - v|.$$

Then by the condition $\nabla F(0) = 0$,

$$|F(u) - F(v)| \leq |\nabla F(\theta u + (1 - \theta)v)| |u - v| \leq C(F, V) |\theta u + (1 - \theta)v| |u - v|.$$

Then

$$\begin{aligned} \|F(u)(t, \cdot) - F(v)(t, \cdot)\|_{L^2(\mathbb{R}^3)} & \leq C(F, V) \|u - v\|_{L^2(\mathbb{R}^3)} \|\theta u + (1 - \theta)v\|_{L^\infty(\mathbb{R}^3)} \\ & \leq C(V) \|u(t, \cdot) - v(t, \cdot)\|_{E^d} (\|u(t, \cdot)\|_{E^d} + \|v(t, \cdot)\|_{E^d}). \end{aligned}$$

For the case for $|I_1| + |I_2| \geq 1$, we recall the expression (7.42) :

$$\begin{aligned}
& \partial^{I_1} \Omega^{I_2} (F(u) - F(v)) \\
&= \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} \left(P^L F(u) \sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} u_j - P^L F(v) \sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} v_j \right) \\
&= \sum_j \partial_j F(u) (\partial^{I_1} \Omega^{I_2} u_j - \partial^{I_1} \Omega^{I_2} v_j) \\
&+ \sum_{l=2}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} P^L F(u) \sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \left(\prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} u_j - \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} v_j \right) \\
&+ \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} (P^L F(u) - P^L F(v)) \sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} v_j \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

To estimate T_1 , we take into consideration of the condition $\nabla F(0) = 0$ with leads to

$$|\partial_j F(u)(t, \cdot)| \leq C(V) |u|.$$

So $\|\partial_j F(u)(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(V) \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(F, V) \|u(t, \cdot)\|_{X^d}$. Then

$$\|T_1(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C(F, V) \|u(t, \cdot) - v(t, \cdot)\|_{X^d} \|u(t, \cdot)\|_{X^d}.$$

To estimate T_2 , we need to apply lemma 7.23. To do so, we remark the following relation:

$$\partial^{I_1} \Omega^{I_2} \left(\prod_{i=1}^m u_j^{l_j} - \prod_{i=1}^m v_j^{l_j} \right) = \sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \left(\prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} u_j - \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} v_j \right) \quad (7.53)$$

Then

$$T_2 = \sum_{l=2}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} P^L F(u) \partial^{I_1} \Omega^{I_2} \left(\prod_{j=1}^m u_j^{l_j} - \prod_{j=1}^m v_j^{l_j} \right)$$

Recall that $|P^L F(u)|$ is bounded by a constant $C(F, V, |L|)$ determined by the neighbourhood V , the function F and the order $|L|$. Then we apply (7.50).

The estimate of T_3 is as following:

$$\|T_3\|_{L^2(\mathbb{R}^3)} \leq \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} \|P^L F(u) - P^L F(v)\|_{L^\infty(\mathbb{R}^3)} \sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \left\| \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} v_j \right\|_{L^2(\mathbb{R}^3)} \quad (7.54)$$

As in the estimate on T_2 , we see that

$$\sum_{\substack{\sum_{j,i} K_{1j^i}=I_1 \\ \sum_{j,i} K_{2j^i}=I_2}} \left\| \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j^i}} \Omega^{K_{2j^i}} v(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \leq \sum_{l=1}^{|I_1|+|I_2|} \|v(t, \cdot)\|_{X^d}^l.$$

and

$$\|P^L F(u)(t, \cdot) - P^L F(v)(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(F, V, |L|) \|u(t, \cdot) - v(t, \cdot)\|_{L^\infty} \leq C(V, |L|) \|u(t, \cdot) - v(t, \cdot)\|_{E^d}.$$

Then

$$\|T_3(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C(V) \|u(t, \cdot) - v(t, \cdot)\|_{E^d} \sum_{k=1}^{d-1} \|v\|_{E^d}^k$$

which concludes (7.52). \square

The following L^∞ type estimate is a direct application of lemma 7.25 and the global Sobolev inequality (7.8):

Lemma 7.26. *Let F, G, u and v take the assumption as in lemma 7.25, then for any $|I_1| + |I_2| \leq d - 2$, the following estimate holds:*

$$\begin{aligned} & \left\| \partial^{I_1} \Omega^{I_2} (F(u)(t, \cdot) - F(v)(t, \cdot))(t, \cdot) \right\|_{\mathcal{B}^{-1}(\mathbb{R}^3)} \\ & \leq C(F, V, d) \|u(t, \cdot) - v(t, \cdot)\|_{E^d} \sum_{k=1}^{d-1} \sum_{k_1+k_2=k} \|u(t, \cdot)\|_{E^d}^{k_1} \|v(t, \cdot)\|_{E^d}^{k_2}. \end{aligned} \quad (7.55)$$

Lemma 7.27. *Let $d \geq 3$ be an integer. Let H be a C^∞ function which is defined in a compact neighbourhood V of 0 in \mathbb{R}^m and let u, v be maps from \mathbb{R}^4 to V with their components in class $L^\infty([0, T], E_H^d)$. Then the following estimate holds for $1 \leq |I_1| + |I_2| \leq d$:*

$$\begin{aligned} & \left\| (1+r)^{-1} \partial^{I_1} \Omega^{I_2} (H(u) - H(v))(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \\ & \leq C(V) \|u(t, \cdot) - v(t, \cdot)\|_{E_H^d} \sum_{k=0}^{|I_1|+|I_2|} \sum_{k_1+k_2=k} \|u(t, \cdot)\|_{E_H^d}^{k_1} \|v(t, \cdot)\|_{E_H^d}^{k_2}. \end{aligned} \quad (7.56)$$

Proof. The proof is quite similar to that of lemma 7.25.

When $|I_1| + |I_2| \geq 1$, we apply the same calculation:

$$\begin{aligned} & \partial^{I_1} \Omega^{I_2} (H(u) - H(v)) \\ & = \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} \left(P^L H(u) \sum_{\substack{\sum_{j,i} K_{1j} i = I_1 \\ \sum_{j,i} K_{2j} i = I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j} i} \Omega^{K_{2j} i} u_j - P^L H(v) \sum_{\substack{\sum_{j,i} K_{1j} i = I_1 \\ \sum_{j,i} K_{2j} i = I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j} i} \Omega^{K_{2j} i} v_j \right) \\ & = \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} P^L H(u) \sum_{\substack{\sum_{j,i} K_{1j} i = I_1 \\ \sum_{j,i} K_{2j} i = I_2}} \left(\prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j} i} \Omega^{K_{2j} i} u_j - \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j} i} \Omega^{K_{2j} i} v_j \right) \\ & \quad + \sum_{l=1}^{|I_1|+|I_2|} \sum_{\sum_j l_j=l} (P^L H(u) - P^L H(v)) \sum_{\substack{\sum_{j,i} K_{1j} i = I_1 \\ \sum_{j,i} K_{2j} i = I_2}} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial^{K_{1j} i} \Omega^{K_{2j} i} v_j \\ & = : T_1 + T_2. \end{aligned}$$

The estimate of T_1 and T_2 are similar to the estimate made in the proof of lemma 7.25. T_1 is estimated by apply lemma 7.24 and T_2 is by applying the following estimate

$$\|P^L H(u) - P^L H(v)\|_{L^\infty(\mathbb{R}^3)} \leq C(H, V, |L|) \|u - v\|_{L^\infty} \leq C(V, |L|) \|u - v\|_{X^d}$$

and the fact that (by applying (7.47))

$$\left\| (1+r)^{-1} \prod_{j=1}^m \prod_{i=1}^{l_j} \partial_x^{K_{1j} i} \Omega^{K_{2j} i} v_j \right\|_{L^2(\mathbb{R}^3)} \leq C(V) \|v\|_{X_H^d}^{|L|}.$$

\square

We also need a L^∞ estimate on $H(u) - H(v)$:

Lemma 7.28. *Let $d \geq 3$. Assume that H be a C^∞ function defined in a compact neighbourhood V of 0 in \mathbb{R}^m and assume that u, v are maps from \mathbb{R}^4 to V with their components in the class $L^\infty([0, T]; E_H^d)$. Then the following estimate holds for $|I_1| + |I_2| \leq d - 2$:*

$$\begin{aligned} & \|\partial^{I_1} \Omega^{I_2} (H(u) - H(v))(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \\ & \leq C(H, V, d) \|u(t, \cdot) - v(t, \cdot)\|_{E_H^d} \sum_{k=0}^{|I_1|+|I_2|+1} \sum_{k_1+k_2=k} \|u(t, \cdot)\|_{E_H^d}^{k_1} \|v(t, \cdot)\|_{E_H^d}^{k_2} \end{aligned} \quad (7.57)$$

Proof. When $I_1 = I_2 = 0$, then there exists $\theta \in [0, 1]$ such that

$$|H(u) - H(v)| \leq |\nabla H(\theta u + (1 - \theta)v)| |u - v| \leq C(H, V) |u - v|.$$

which proves (7.57).

For $|I_1| + |I_2| \geq 1$, we apply lemma 7.7 combined with (7.56). \square

Lemma 7.29. *Let F be a C^∞ function defined in a compact neighbourhood V_1 of 0 in \mathbb{R}^m and H be a C^∞ function defined in a neighbourhood V_2 of 0 in \mathbb{R}^n . Let d be an integer and $d \geq 3$. Assume that $F(0) = 0, \nabla F(0) = 0$. Let u_1 and u_2 be maps from \mathbb{R}^4 to V_1 with their components in $L^\infty([0, T]; E_H^d)$ and v_1, v_2 be maps from \mathbb{R}^4 to V_2 with their components in class $L^\infty([0, T]; E^d)$. Then the following estimate holds for $|I_1| + |I_2| \leq d$ and $d \geq 3$. Then*

$$\begin{aligned} & \|\partial^{I_1} \Omega^{I_2} (H(u_1)F(v_1) - H(u_2)F(v_2))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ & \leq C(V) (\|u_1(t, \cdot) - u_2(t, \cdot)\|_{E_H^d} + \|v_1(t, \cdot) - v_2(t, \cdot)\|_{E^d}) \sum_{k=1}^{d-1} M^k(t) \end{aligned} \quad (7.58)$$

where

$$M(t) := \max\{\|u_{1j}(t, \cdot)\|_{E_H^d}, \|u_{2j}(t, \cdot)\|_{E_H^d}, \|v_{1j}(t, \cdot)\|_{E^d}, \|v_{2j}(t, \cdot)\|_{E^d}\}.$$

Proof. We remark the following calculation:

$$\begin{aligned} & \partial^{I_1} \Omega^{I_2} (H(u_1)F(v_1) - H(u_2)F(v_2)) \\ & = \partial^{I_1} \Omega^{I_2} (H(u_1)F(v_1) - H(u_1)F(v_2) + H(u_1)F(v_2) - H(u_2)F(v_2)) \\ & = \partial^{I_1} \Omega^{I_2} (H(u_1)(F(v_1) - F(v_2))) + \partial^{I_1} \Omega^{I_2} (F(v_2)(H(u_1) - H(u_2))) \\ & = : T_1 + T_2 \end{aligned}$$

The term T_1 is estimated as following:

$$T_1 = \partial^{I_1} \Omega^{I_2} (H(u_1)(F(v_1) - F(v_2))) = \sum_{\substack{J_1+J_2=I_1 \\ J_1'+J_2'=I_2}} \partial^{J_1} \Omega^{J_2} (H(u_1)) \partial^{J_1'} \Omega^{J_2'} (F(v_1) - F(v_2)).$$

When $|J_1| + |J_2| \leq [d/2] \leq d - 2$, by applying (7.48) and (7.52)

$$\begin{aligned} & \|\partial^{J_1} \Omega^{J_2} (H(u_1)) \partial^{J_1'} \Omega^{J_2'} (F(v_1) - F(v_2))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ & = \|\partial^{J_1} \Omega^{J_2} (H(u_1))(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \|\partial^{J_1'} \Omega^{J_2'} (F(v_1) - F(v_2))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\ & \leq C(H, F, V, d) \left(\sum_{k=0}^{|J_1|+|J_2|+2} \|u_1(t, \cdot)\|_{E_H^d} \right) \left(\|v_1(t, \cdot) - v_2(t, \cdot)\|_{E^d} \sum_{k=1}^{|J_1'|+|J_2'|-1} \sum_{k_1+k_2=k} \|v_1(t, \cdot)\|_{E^d}^{k_1} \|v_2(t, \cdot)\|_{E^d}^{k_2} \right) \end{aligned}$$

When $|J_1| + |J_2| \geq d/2 > 1$, $|J'_1| + |J'_2| \leq [d/2]$, we apply (7.47):

$$\begin{aligned}
& \left\| \partial^{J_1} \Omega^{J_2} (H(u_1)) \partial^{J'_1} \Omega^{J'_2} (F(v_1) - F(v_2))(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \\
& \leq \|(1+r)^{-1} \partial^{J_1} \Omega^{J_2} (H(u_1))(t, \cdot)\|_{L^2(\mathbb{R}^3)} \|\partial^{J'_1} \Omega^{J'_2} (F(v_1) - F(v_2))(t, \cdot)\|_{\mathcal{B}^{-1}} \\
& \leq C(V) \sum_{k=1}^{|J_1|+|J_2|} \|u\|_{E_H^d}^k \cdot C(H, F, V, d) \|v_1(t, \cdot) - v_2(t, \cdot)\|_{E^d} \sum_{k=1}^{|J'_1|+|J'_2|+1} \sum_{k_1+k_2=k} \|v_1(t, \cdot)\|_{E^d}^{k_1} \|v_2(t, \cdot)\|_{E^d}^{k_2} \\
& \leq C(H, F, V, d) \|v_1(t, \cdot) - v_2(t, \cdot)\|_{E^d} \sum_{k=1}^{|I_1|+|I_2|+2} \sum_{k_1+k_2+k_3=k} \|u_1(t, \cdot)\|_{E_H^d}^{k_1} \|v_1(t, \cdot)\|_{E^d}^{k_2} \|v_2(t, \cdot)\|_{E^d}^{k_3}
\end{aligned}$$

The term T_2 is estimated similarly by applying (7.44), (7.56), (7.46) and (7.57). We omit the detail but write out the estimate

$$\|T_2(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq \|u_1(t, \cdot) - u_2(t, \cdot)\|_{E_H^d} \sum_{k=2}^{|I_1|+|I_2|+1} \sum_{k_1+k_2+k_3=k} \|u_1\|_{E_H^d}^{k_1} \|u_2\|_{E_H^d}^{k_2} \|v_2\|_{E^d}^{k_3}.$$

□

8 Proof of the local existence theorem

8.1 Iteration and uniform bound

In this section we will begin the proof of theorem 6.2. The proof of this theorem will occupy the following two subsections. The prove is a classical iteration procedure constructed below:

$$\begin{aligned}
(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n+1} &= F_{\alpha\beta}(h_n^\kappa; \partial h_n^\kappa, \partial h_n^\kappa) - 16\pi \partial_\alpha \phi_n^\kappa \partial_\beta \phi_n^\kappa \\
&\quad - 12 \partial_\alpha \varrho_n^\kappa \partial_\beta \varrho_n^\kappa - \kappa^{-1} V_h(\varrho_n^\kappa) (m_{\alpha\beta} + h_{\alpha\beta}^{\kappa, n}), \tag{8.1a}
\end{aligned}$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} \phi_{n+1}^\kappa = 2(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \phi_n^\kappa \partial_{\beta'} \varrho_n^\kappa, \tag{8.1b}$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} \varrho_{n+1}^\kappa - \frac{\varrho_{n+1}^\kappa}{3\kappa} = \kappa^{-1} V_\rho(\varrho_n^\kappa) - \frac{4\pi}{3e^{2\varrho_n^\kappa}} (m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \phi_n^\kappa \partial_{\beta'} \phi_n^\kappa, \tag{8.1c}$$

where

$$\kappa^{-1} V_h(s) := \frac{(e^{2s} - 1)^2}{2\kappa e^{4s}}, \quad \kappa^{-1} V_\rho(s) := \frac{e^{2s} - 1}{6\kappa e^{4s}} - \frac{s}{3\kappa}$$

and with initial data

$$\begin{aligned}
h_{\alpha\beta}^{\kappa, n}(0, x) &= h_{0\alpha\beta}, \quad \phi_n^\kappa(0, x) = \phi_0(x), \quad \varrho_n^\kappa(0, x) = \varrho_0(x), \\
\partial_t h_{\alpha\beta}^{\kappa, n}(0, x) &= h_{1\alpha\beta}, \quad \partial_t \phi_n^\kappa(0, x) = \phi_1(x) \quad \partial_t \varrho_n^\kappa(0, x) = \varrho_1(x).
\end{aligned}$$

Recall the function $H^{\alpha\beta}(h)$ are defined in (6.4) and the associated estimates are in (6.5). We take $(h_{\alpha\beta}^{\kappa, 0})$. For the convenience of discussion, we denote by

$$S_n^\kappa := (h_{\alpha\beta}^{\kappa, n}, \phi_n^\kappa, \varrho_n^\kappa).$$

We also denote by $F_H(S_n^\kappa)$, $F_P(S_n^\kappa)$ and $F_R(S_n^\kappa)$ the terms in right-hand-side of (8.1a), (8.1b) and (8.1c).

We take $S_0^\kappa = (h_{\alpha\beta}^{\kappa, 0}, \phi_0^\kappa, \varrho_0^\kappa)$ as the solution of the following homogeneous Cauchy problem:

$$\begin{aligned}
\Box h_{\alpha\beta} &= 0, \quad \Box \phi = 0, \quad \Box \varrho - \frac{\varrho}{3\kappa} = 0, \\
h_{\alpha\beta}^{\kappa, 0}(0, x) &= h_{0\alpha\beta}, \quad \phi_0^\kappa(0, x) = \phi_0(x), \quad \varrho_0^\kappa(0, x) = \varrho_0(x), \\
\partial_t h_{\alpha\beta}^{\kappa, 0}(0, x) &= h_{1\alpha\beta}, \quad \partial_t \phi_0^\kappa(0, x) = \phi_1(x) \quad \partial_t \varrho_0^\kappa(0, x) = \varrho_1(x).
\end{aligned}$$

we see that the source terms and metric coefficients in (8.1) are sufficiently regular and the initial data are in the corresponding class demanded in proposition 7.13, 7.14 and 7.15. Then we conclude that by the theory of local existence for linear equations, this iteration procedure is well defined in a fixed time interval $[0, T]$ where the metric coefficients and source terms are in the corresponding class and $|H(h_n^\kappa)^{00}| \leq 1/2$. We see that this iteration defines a sequence of triple $S_n^\kappa := (h_{\alpha\beta}^{\kappa,n}, \phi_n^\kappa, \varrho_n^\kappa)$. In order to get the local existence, we will prove that S_n^κ converges with respect to the following norm:

$$\begin{aligned} & \|S(t, \cdot)_n^\kappa\|_{E_\kappa^{d+1}} \\ & := \max\{\|h_{\alpha\beta}^{\kappa,n}(t, \cdot)\|_{E_H^{d+1}}, \|\phi_n^\kappa(t, \cdot)\|_{E_P^{d+1}}, \|\varrho_n^\kappa(t, \cdot)\|_{E_P^{d+1}}, \kappa^{-1/2}\|\varrho_n^\kappa(t, \cdot)\|_{E^d}\}, \end{aligned}$$

For all $t \in [0, T^*]$ where $T^* > 0$ will be constructed.

We suppose that for certain $d \geq 3$, for all $k \leq n$ and all $t \in [0, T]$,

$$\|S_k^\kappa(t, \cdot)\|_{E_\kappa^{d+1}} \leq A\varepsilon, \quad (8.2)$$

we will show that $\|S_{n+1}^\kappa(t, \cdot)\|_{X_\kappa^{d+1}} \leq A\varepsilon$ with T and A well chosen.

First, we remark that when (8.2) holds with $A\varepsilon \leq \varepsilon_0$ with ε_0 small enough, the metric $g_{\alpha\beta}^{\kappa,k} := m^{\alpha\beta} + h_{\alpha\beta}^{\kappa,k}$ are coercive with constant $C(\varepsilon_0)$ where $C(\varepsilon_0)$ is determined by ε_0 .

Now combined (8.2) with lemma 7.18, the following estimates on source terms are direct:

Lemma 8.1. *Assume that (8.2) holds with $d \geq 3$. Suppose that $0 \leq A\varepsilon \leq \varepsilon_0$ where ε_0 is a constant sufficiently small. Then the following estimates hold for $k \leq n$:*

$$\|F_{\alpha\beta}(h_k^\kappa, \partial h_k^\kappa, \partial h_k^\kappa)(t, \cdot)\|_{E^d} + \|\partial_\alpha \phi_k^\kappa \partial_\beta \phi_k^\kappa(t, \cdot)\|_{E^d} + \|\partial_\alpha \varrho_k^\kappa \partial_\beta \varrho_k^\kappa(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(A\varepsilon)^2, \quad (8.3a)$$

$$\|(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_k^\kappa))\partial_{\alpha'} \phi_k^\kappa \partial_{\beta'} \phi_k^\kappa(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(A\varepsilon)^2, \quad (8.3b)$$

$$\|e^{-2\varrho_k^\kappa}(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_k^\kappa))\partial_{\alpha'} \phi_k^\kappa \partial_{\beta'} \phi_k^\kappa(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(A\varepsilon)^2, \quad (8.3c)$$

$$\|\kappa^{-1}V_\rho(\varrho_k^\kappa)(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(A\varepsilon)^2, \quad \|\kappa^{-1}V_h(\varrho_k^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^{\kappa,k})(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(A\varepsilon)^2. \quad (8.3d)$$

These are classical estimates when we establish the local theory of existence for quasi-linear wave equations with quadratic nonlinearity. The only thing important is (8.3d) where the κ appears in the left-hand-side in order to get estimates independent of κ .

Proof. These estimates are applications of (7.44) and (7.49) combined with (8.2). The only thing we need to pay attention is that to guarantee the C^∞ regularity of the function $H^{\alpha\beta}(\cdot)$, we need to restrict its defined in a compact neighborhood V of 0 in \mathbb{R}^{10} with

$$V = \left\{ \max_{1 \leq i \leq 10} |x^i| \leq \varepsilon_0 \right\}$$

with ε_0 sufficiently small. This can be guaranteed by taking $\varepsilon \leq \varepsilon_0$ in (8.2).

We remark that $F_{\alpha\beta}(h_n^\kappa, \partial h_n^\kappa, \partial h_n^\kappa)$ is quadratic with respect to ∂h_n^κ and C^∞ with respect to h_n^κ . Then we apply (7.49). The estimate on the term

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_k^\kappa))\partial_{\alpha'} \phi_k^\kappa \partial_{\beta'} \phi_k^\kappa$$

is established in the same manner.

The estimate of (8.3d) is proved by making use of the estimate

$$\kappa^{-1/2}\|\varrho_k^\kappa\|_{E^d} \leq A\varepsilon,$$

which is a direct result of (8.2). □

Now we begin the estimates on the commutators such as $[\partial_x^{I_1} \Omega^{I_2}, H^{\alpha' \beta'} (h_n^\kappa) \partial_{\alpha'} \partial_{\beta'}] h_{\alpha\beta}^{\kappa, n}$ who appears in the estimates of $\|h_{\alpha\beta}^{\kappa, n+1}\|_{X_E^d}$.

Lemma 8.2. *There exists a positive constant ε_0 such that if (8.2) holds with $d \geq 3$, $A\varepsilon \leq \varepsilon_0 \leq 1$, then the following estimates hold for all couple of index (I_1, I_2) with $1 \leq |I_1| + |I_2| \leq d$:*

$$\|[\partial^{I_1} \Omega^{I_2}, H^{\alpha' \beta'} (h_n^\kappa) \partial_{\alpha'} \partial_{\beta'}] h_{\alpha\beta}^{\kappa, n+1}(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d) A\varepsilon \|h_{\alpha\beta}^{\kappa, n+1}(t, \cdot)\|_{E_P^{d+1}}, \quad (8.4a)$$

$$\|[\partial^{I_1} \Omega^{I_2}, H^{\alpha' \beta'} (h_n^\kappa) \partial_{\alpha'} \partial_{\beta'}] \phi_{n+1}^\kappa(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d) A\varepsilon \|\phi_{n+1}^\kappa(t, \cdot)\|_{E_P^{d+1}}, \quad (8.4b)$$

$$\|[\partial^{I_1} \Omega^{I_2}, H^{\alpha' \beta'} (h_n^\kappa) \partial_{\alpha'} \partial_{\beta'}] \varrho_{n+1}^\kappa(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d) A\varepsilon \|\varrho_{n+1}^\kappa(t, \cdot)\|_{E_P^{d+1}}. \quad (8.4c)$$

Proof. The estimate of these three commutators are similar. We will only prove the first in detail. Let (I_1, I_2) be a pair of multi-index, $|I_1| + |I_2| \leq d$. Recall the estimate of commutator (7.3e):

$$\begin{aligned} |[\partial^{I_1} \Omega^{I_2}, H^{\alpha' \beta'} \partial_{\alpha'} \partial_{\beta'}] h_{\alpha' \beta'}^{\kappa, n+1}| &\leq \sum_{\substack{J_1 + J'_1 = I_1 \\ |J_2| + |J'_2| \leq |I_2| \\ \alpha', \beta', \alpha'', a}} |\partial^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'}| |\partial_{\alpha''} \partial_a \partial^{J_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}| \\ &+ \sum_{\substack{J_1 + J'_1 = I_1 \\ J_2 + J'_2 = I_2 \\ |J'_1| + |J'_2| > 0}} |\partial^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'}| |\partial_{\alpha'} \partial_{\beta'} \partial^{J_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}| \\ &=: T_1 + T_2. \end{aligned}$$

• We begin with T_2 . The estimate is established in the following two cases.

CASE 1. $1 \leq |J'_1| + |J'_2| \leq d - 1$, $1 \leq |J_1| + |J_2| \leq d - 1$.

In this case,

$$\begin{aligned} &\|\partial^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'} (h_n^\kappa) \partial_{\alpha'} \partial_{\beta'} \partial^{J_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}\|_{L^2(\mathbb{R}^3)} \\ &\leq \|\partial^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'} (h_n^\kappa)\|_{L^\infty(\mathbb{R}^3)} \|\partial_{\alpha'} \partial_{\beta'} \partial^{J_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}\|_{L^2(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0, d) A\varepsilon \|h_n^\kappa(t, \cdot)\|_{E_P^{d+1}}, \end{aligned}$$

where we applied $A\varepsilon \leq 1$ and (7.48).

CASE 2. $J'_1 = I_1$, $J'_2 = I_2$, $J_1 = J_2 = 0$. Recall that $d \geq 3$ then $|J_1| + |J_2| \leq d - 3$.

$$\begin{aligned} &\|\partial_x^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'} (h_n^\kappa) \partial_{\alpha'} \partial_{\beta'} \partial_x^{J_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}\|_{L^2(\mathbb{R}^3)} \\ &\leq \|(1+r)^{-1} \partial_x^{I_1} \Omega^{I_2} H^{\alpha' \beta'} (h_n^\kappa)\|_{L^2(\mathbb{R}^3)} \|\partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n+1}\|_{\mathfrak{B}^{-1}} \\ &\leq C(\varepsilon_0, d) A\varepsilon \|h_{\alpha\beta}^{\kappa, n+1}\|_{E_P^{d+1}}, \end{aligned}$$

where we applied $A\varepsilon \leq 1$ and (7.47).

The estimate of term T_1 is quite simpler. Recall that in the expression of T_1 , the sum is taken over the index satisfying the following conditions:

$$J_1 + J'_1 = I_1, \quad |J_2| + |J'_2| \leq |I_2| - 1$$

So

$$(|J_1| + |J'_1|) + (|J_2| + |J'_2|) \leq d - 1$$

which leads to

$$|J'_1| + |J'_2| \leq d - 1.$$

So

$$\|\partial_x^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'} \partial_{\alpha''} \partial_a \partial_x^{I_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}\|_{L^2(\mathbb{R}^3)} \leq \|\partial_x^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'}\|_{L^\infty} \|\partial_{\alpha''} \partial_a \partial_x^{I_1} \Omega^{J_2} h_{\alpha\beta}^{\kappa, n+1}\|_{L^2(\mathbb{R}^3)}$$

As in the estimate of T_2 , we see that with $|J'_1| + |J'_2| \leq d - 1$,

$$\|\partial_x^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'}\|_{L^\infty(\mathbb{R}^3)} \leq C(\varepsilon_0, d) A \varepsilon.$$

The second factor on $h_{\alpha\beta}^{\kappa, n+1}$ is bounded directly by $\|h_{\alpha\beta}^{\kappa, n+1}\|_{X_E^{d+1}}$. So we conclude by (8.4a). \square

Now we need to discuss the bound of the initial data $E_{g_n}^d(0, h_{\alpha\beta}^{\kappa, n+1})$, $E_{g_n}^d(0, \phi_{n+1}^\kappa)$ and $E_{g_n, c}^d(0, \varrho_{n+1}^\kappa)$. We will see that these norms are controlled by $\|S_0\|_{X_\kappa^{d+1}}$:

Lemma 8.3. *When $\|S_0\|_{X_\kappa^{d+1}}$ is supposed to be bounded by $\varepsilon \leq \varepsilon_0$ for ε_0 sufficiently small, there exists a positive constant determined by ε_0 and d such that*

$$E_{g_n}^d(0, h_{n+1}^\kappa) + E_{g_{n+1}}^d(0, \phi^\kappa) + E_{g_{n+1}, \kappa^{-1/2}}^d(0, \varrho^\kappa) \leq C(\varepsilon_0, d) \varepsilon.$$

Proof. We recall that

$$\begin{aligned} \|S_0\|_{X_\kappa^{d+1}} := \max\{ & \|h_{0\alpha\beta}\|_{X_H^{d+1}}, \|h_{1\alpha\beta}\|_{X^d}, \kappa^{-(1/2)[d/2]+1/4} \|\phi_0\|_{X_P^{d+1}}, \\ & \kappa^{-(1/2)[d/2]+1/4} \|\phi_1\|_{X^d}, \kappa^{-[d/2]-1/2} \|\varrho_0\|_{X_P^{d+1}}, \kappa^{-[d/2]-1/2} \|\varrho_1\|_{X^d}\}. \end{aligned}$$

We remark that when $0 \leq k \leq 1$ the norm $\|\partial_t^k \partial_x^{I_1} \Omega^{I_2} \varrho^\kappa(0, \cdot)\|_{L^2(\mathbb{R}^3)}$ is determined directly by ϱ_0 and ϱ_1 thus, bounded by $\kappa^{-[d/2]-1/2} \leq \kappa^{-1/2}$.

When $2 \leq k \leq d$, we need to use the equation.

We will prove that for $0 \leq |k| \leq d - 2$ and $|I_1| + |I_2| \leq d - 1 - k$

$$\begin{aligned} \|\partial_t \partial_t \partial_t^k \partial_x^{I_1} \Omega^{I_2} h_{\alpha\beta}^\kappa(0, \cdot)\|_{L^2(\mathbb{R}^3)} &\leq C(\varepsilon_0, d) \varepsilon, \\ \|\partial_t \partial_t \partial_t^k \partial_x^{I_1} \Omega^{I_2} \varrho^\kappa(0, \cdot)\|_{L^2(\mathbb{R}^3)} &\leq C(\varepsilon_0, d) \kappa^{[d/2]+1/2-[k/2]} \varepsilon, \\ \|\partial_t \partial_t \partial_t^k \partial_x^{I_1} \Omega^{I_2} \phi^\kappa(0, \cdot)\|_{L^2(\mathbb{R}^3)} &\leq C(\varepsilon_0, d) \kappa^{1/2[d/2]-1/4} \varepsilon \end{aligned} \tag{8.5}$$

This is proved by induction on k . We see that for $k = 0, 1$, the estimates hold by direct verification. Suppose that (8.5) holds for $(k - 1, k)$ we will prove the case $k + 1$.

The estimate of $\|\partial_t \partial_t \partial_t^{k+1} \partial_x^{I_1} \Omega^{I_2} \phi^\kappa\|_{L^2(\mathbb{R}^3)}$, is a bit complicated. We see that by lemma 7.12,

$$\begin{aligned} \|\partial_t \partial_t \partial_t^{k+1} \partial_x^{I_1} \Omega^{I_2} \varrho^\kappa\|_{L^2(\mathbb{R}^3)} &\leq \|\partial_t^{k+1} \partial_x^{I_1} \Omega^{I_2} ((1 - H^{00})^{-1} (m^{ab} + H^{ab}) \partial_a \partial_b \varrho^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\quad + 2 \|\partial_t^{k+1} \partial_x^{I_1} \Omega^{I_2} ((1 - H^{00})^{-1} H^{0a} \partial_t \partial_a \varrho^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\quad + \frac{3}{\kappa} \|\partial_t^{k+1} \partial_x^{I_1} \Omega^{I_2} \varrho^\kappa\|_{L^2(\mathbb{R}^3)} \\ &\quad + \|\partial_t^{k+1} \partial_x^{I_1} \Omega^{I_2} ((1 - H^{00}) F_R)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

Then we see that

$$\|\partial_t \partial_t \partial_t^k \partial_x^{I_1} \Omega^{I_2} \varrho^\kappa\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon_0, d) \kappa^{[d/2]+1/2-[d/2]} \varepsilon.$$

We remark that by lemma 7.12,

$$\begin{aligned} \|\partial_t \partial_t \partial_t^k \partial_x^{I_1} \Omega^{I_2} h_{\alpha\beta}^\kappa\|_{L^2(\mathbb{R}^3)} &\leq \|\partial_t^k \partial_x^{I_1} \Omega^{I_2} ((1 - H^{00})^{-1} (m^{ab} + H^{ab}) \partial_a \partial_b h_{\alpha\beta}^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\quad + 2 \|\partial_t^k \partial_x^{I_1} \Omega^{I_2} ((1 - H^{00})^{-1} H^{0a} \partial_t \partial_a h_{\alpha\beta}^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\quad + \|\partial_t^k \partial_x^{I_1} \Omega^{I_2} ((1 - H^{00}) F_H)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

Then by the bounds supplied by $\|S_0\|_{X_\kappa^{d+1}}$ and (8.5), we see that $\|\partial_t \partial_t \partial_t^{I_1} \Omega^{I_2} h_{\alpha\beta}^\kappa\|_{L^2(\mathbb{R}^3)}$ is bounded by $C(\varepsilon_0, d) \varepsilon$.

In the same manner, we see that for $0 \leq k \leq d - 2$

$$\|\partial_t \partial_t \partial_t^k \partial_x^{I_1} \Omega^{I_2} \phi^\kappa\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon_0, d) \kappa^{-(1/2)[d/2]+1/4} \varepsilon.$$

\square

Now we are ready to estimate the L^2 type norm of S_n^κ :

Lemma 8.4. *There exists a positive constant ε_0 such that if (8.2) holds for $A\varepsilon \leq \varepsilon_0 \leq 1$ and $d \geq 3$, then*

$$\|h_{\alpha\beta}^{\kappa,n+1}(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t h_{\alpha\beta}^{\kappa,n+1}(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(\varepsilon e^{C(\varepsilon_0, d)A\varepsilon t} + A\varepsilon(e^{C(\varepsilon_0, d)A\varepsilon t} - 1)) \quad (8.6a)$$

$$\|\phi_{n+1}^\kappa(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t \phi_{n+1}^\kappa(t, \cdot)\|_{E^d} \leq C(\varepsilon_0, d)(\varepsilon e^{C(\varepsilon_0, d)A\varepsilon t} + A\varepsilon(e^{C(\varepsilon_0, d)A\varepsilon t} - 1)) \quad (8.6b)$$

$$\begin{aligned} \|\varrho_{n+1}^\kappa(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t \varrho_{n+1}^\kappa(t, \cdot)\|_{E^d} + \kappa^{-1/2} \|\varrho_{n+1}^\kappa(t, \cdot)\|_{E^d} \\ \leq C(\varepsilon_0, d)(\varepsilon e^{C(\varepsilon_0, d)A\varepsilon t} + A\varepsilon(e^{C(\varepsilon_0, d)A\varepsilon t} - 1)) \end{aligned} \quad (8.6c)$$

where $C(\varepsilon_0, d)$ is a positive constant determined by ε_0 and d .

Proof. To prove this an application of the L^2 estimate (7.20). We first concentrate on the estimate of $h_{\alpha\beta}^{\kappa,n+1}$. To do so, we derive the equation (8.1a) with respect to a product $\partial^{I_1}\Omega^{I_2}$ with $|I_1| + |I_2| \leq d$:

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa))\partial_{\alpha'}\partial_{\beta'}\partial^{I_1}\Omega^{I_2}h_{\alpha\beta}^{\kappa,n+1} = \partial^{I_1}\Omega^{I_2}F_H(h_n^\kappa, \phi_n^\kappa, \varrho_n^\kappa) - [\partial^{I_1}\Omega^{I_2}, H^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}]h_{\alpha\beta}^{\kappa,n+1},$$

where F_H denotes the terms in right-hand-side of (8.1a), which is

$$F_H(h_n^\kappa, \phi_n^\kappa, \varrho_n^\kappa) = F_{\alpha\beta}(h_n^\kappa; \partial h_n^\kappa, \partial h_n^\kappa) - 16\pi\partial_\alpha\phi_n^\kappa\partial_\beta\phi_n^\kappa - 12\partial_\alpha\varrho_n^\kappa\partial_\beta\varrho_n^\kappa - \kappa^{-1}V_h(\varrho_n^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^{\kappa,n}).$$

Then by (7.20),

$$\begin{aligned} \frac{d}{dt}E_{g_n}(t, \partial^{I_1}\Omega^{I_2}h_{\alpha\beta}^{\kappa,n+1}) \leq C\|\partial^{I_1}\Omega^{I_2}F_H\|_{L^3(\mathbb{R}^3)} + C\|[\partial^{I_1}\Omega^{I_2}, H^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}]h_{\alpha\beta}^{\kappa,n+1}\|_{L^2(\mathbb{R}^3)} \\ + C\sum_{\alpha\beta}\|\nabla H^{\alpha'\beta'}(h_n^\kappa)\|_{L^\infty(\mathbb{R}^3)}E_{g_n}(t, \partial^{I_1}\Omega^{I_2}h_{\alpha\beta}^{\kappa,n+1}). \end{aligned}$$

By lemma 8.1 and (8.2), and the (equi-)coercivity of g_n guaranteed by $A\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \frac{d}{dt}E_{g_n}(t, \partial^{I_1}\Omega^{I_2}h_{\alpha\beta}^{\kappa,n+1}) \leq C(\varepsilon_0, d)(A\varepsilon)^2 + C(\varepsilon_0, d)A\varepsilon\sum_{\alpha\beta}E_g^d(t, h_{\alpha\beta}^{\kappa,n+1}) \\ + C(\varepsilon_0, d)A\varepsilon E_{g_n}(t, \partial^{I_1}\Omega^{I_2}h_{\alpha\beta}^{\kappa,n+1}). \end{aligned}$$

Taking the sum over the index (I_1, I_2) with $|I_1| + |I_2| \leq d$ and α, β :

$$\frac{d}{dt}\sum_{\alpha\beta}E_{g_n}^d(t, h_{\alpha\beta}^{\kappa,n+1}) \leq C(\varepsilon_0, d)(A\varepsilon)^2 + C(\varepsilon_0, d)A\varepsilon\sum_{\alpha\beta}E_{g_n}^d(t, h_{\alpha\beta}^{\kappa,n+1})$$

which leads to

$$\sum_{\alpha\beta}E_{g_n}^d(t, h_{\alpha\beta}^{\kappa,n+1}) \leq \sum_{\alpha\beta}E_{g_n}^d(0, h_{\alpha\beta}^{\kappa,n+1})e^{C(\varepsilon_0, d)A\varepsilon t} + A\varepsilon(e^{C(\varepsilon_0, d)A\varepsilon t} - 1) \quad (8.7)$$

Remark that by lemma 8.3, $E_{g_n}^d(0, h_{\alpha\beta}^{\kappa,n+1})$ is controlled by $\|h_0\|_{X_P^{d+1}}$ and $\|h_1\|_{X^d}$, so it can be controlled by $C(\varepsilon_0, d)\varepsilon$ where $C(\varepsilon_0, d)$ is a constant depending only on d and ε_0 .

The estimate of $E_{g_n}^d(t, \phi_{n+1}^\kappa)$ is exactly the same, we omit the detail.

The estimate of on ϱ_{n+1}^κ is similar. By the same procedure of energy estimate, we arrive at the following estimate:

$$\begin{aligned} \frac{d}{dt}E_{g_n, \kappa^{-1/2}}(t, \partial^{I_1}\Omega^{I_2}\varrho_{n+1}^\kappa) \leq C\|\partial^{I_1}\Omega^{I_2}F_R\|_{L^3(\mathbb{R}^3)} + C\|[\partial^{I_1}\Omega^{I_2}, H^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}]\varrho_{n+1}^\kappa\|_{L^2(\mathbb{R}^3)} \\ + C\sum_{\alpha\beta}\|\nabla H^{\alpha'\beta'}(h_n^\kappa)\|_{L^\infty(\mathbb{R}^3)}E_{g_n, \kappa^{-1/2}}(t, \partial^{I_1}\Omega^{I_2}\varrho_{n+1}^\kappa). \end{aligned} \quad (8.8)$$

where F_R denotes the right-hand-side of (8.1c). Then, also by lemma 8.1 and (8.4c) and the same calculation in the estimate of $h_{\alpha\beta}^{\kappa,n+1}$,

$$E_{g_n, \kappa^{-1/2}}^d(t, \varrho_{n+1}^\kappa) \leq E_{g_n, \kappa^{-1/2}}^d(0, \varrho_{n+1}^\kappa) e^{C(\varepsilon_0, d)A\varepsilon t} + A\varepsilon(e^{C(\varepsilon_0, d)A\varepsilon t} - 1). \quad \square$$

Now we begin to make the choice of the couple (A_0, T_0) such that when $A \leq A_0$, $T \leq T_0$, (8.6) implies

$$\begin{aligned} \|h_{\alpha\beta}^{\kappa,n+1}(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t h_{\alpha\beta}^{\kappa,n+1}(t, \cdot)\|_{E^d} &\leq A\varepsilon, \\ \|\phi_{n+1}^\kappa(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t \phi_{n+1}^\kappa(t, \cdot)\|_{E^d} &\leq A\varepsilon, \\ \|\varrho_{n+1}^\kappa(t, \cdot)\|_{E_P^{d+1}} + \|\partial_t \varrho_{n+1}^\kappa(t, \cdot)\|_{E^d} + \kappa^{-1/2} \|\varrho_{n+1}^\kappa(t, \cdot)\|_{E^d} &\leq A\varepsilon \end{aligned} \quad (8.9)$$

on the time interval $[0, T]$. This is concluded in the following lemma.

Lemma 8.5. *There exists a couple of positive constants $(\varepsilon_0, A_0(\varepsilon_0, d))$ where ε_0 is a universal constant and $A_0(\varepsilon_0, d)$ is determined by d and ε_0 such that when (8.2) is valid with $A \geq A_0$ and $A\varepsilon \leq \varepsilon_0 \leq 1$ on the time interval $[0, T]$ with*

$$T \leq T_0 := \frac{\ln(1 + (2C(\varepsilon_0, d))^{-1})}{C(\varepsilon_0, d)A\varepsilon}.$$

where $C(\varepsilon_0, d)$ is a constant determined by ε_0 and d . Then (8.9) hold.

Proof. By lemma 8.4, we chose $A_0(\varepsilon_0, d)$ and T such that when $A \geq A_0(\varepsilon_0, d)$ and $t \leq T$

$$C(\varepsilon_0, d)(\varepsilon e^{C(\varepsilon_0, d)A\varepsilon t} + A\varepsilon(e^{C(\varepsilon_0, d)A\varepsilon t} - 1)) \leq A\varepsilon.$$

This can be guaranteed by

$$\begin{cases} e^{C(\varepsilon_0, d)A\varepsilon T_0} - 1 \leq \frac{1}{2C(\varepsilon_0, d)}, \\ e^{C(\varepsilon_0, d)A\varepsilon T_0} \leq \frac{A}{2C(\varepsilon_0, d)} \end{cases}$$

which is equivalent to

$$T \leq \frac{\ln(1 + (2C(\varepsilon_0, d))^{-1})}{C(\varepsilon_0, d)A\varepsilon} \quad A \geq 2C(\varepsilon_0, d) + 1.$$

Then we can take $A_0 = 2C(\varepsilon_0, d) + 1$ and $T_0 = \frac{\ln(1 + (2C(\varepsilon_0, d))^{-1})}{C(\varepsilon_0, d)A\varepsilon}$. □

Then we are about to estimate the \mathcal{B}^{-1} norm of $h_{\alpha\beta}^{\kappa,n+1}$.

Lemma 8.6. *There exists a positive constant ε_0 such that if (8.2) holds with $d \geq 3$, $A \geq A_0$, $A\varepsilon \leq \varepsilon_0$ on the time interval $[0, T]$, $T \leq T_0$ where (A_0, T_0) are constants determined in lemma 8.5. Then the following estimate hold*

$$\|h_{\alpha\beta}^{\kappa,n+1}(t, \cdot)\|_{\mathcal{B}^{-1}} \leq C(\varepsilon_0)T^2(A\varepsilon)^2 + C(T+1)\varepsilon. \quad (8.10)$$

Proof. We will apply (7.18). Remark that

$$\square h_{\alpha\beta}^{\kappa,n+1} = F_H - H^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa,n+1}.$$

Then by (7.18),

$$\begin{aligned} \|h_{\alpha\beta}^{\kappa,n+1}(t, \cdot)\|_{\mathcal{B}^{-1}} &\leq Ct \int_0^t \|F_H(\tau, \cdot)\|_{\mathcal{B}^{-1}} d\tau + Ct \sum_{\alpha, \beta} \int_0^t \|H^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa,n+1}(\tau, \cdot)\|_{\mathcal{B}^{-1}} d\tau \\ &\quad + C(\|h_{\alpha\beta}^{\kappa,n+1}(0, \cdot)\|_{\mathcal{B}^{-1}} + t \|\nabla h_{\alpha\beta}^{\kappa,n+1}(0, \cdot)\|_{\mathcal{B}^{-1}}) \end{aligned}$$

Then we can apply on each term the global Sobolev inequality (7.8) to get estimates on the \mathcal{B}^{-1} norms:

$$\|F_H\|_{\mathcal{B}^{-1}} \leq C\|F_H\|_{X^2} \leq C\|F_H\|_{E^2} \leq C(\varepsilon_0)(A\varepsilon)^2,$$

where we applied (8.3a) and (7.8).

$$\begin{aligned} \|H^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n+1}(t, \cdot)\|_{\mathcal{B}^{-1}} &\leq \|H^{\alpha'\beta'}\|_{L^\infty(\mathbb{R}^3)} \|\partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n+1}\|_{X^2} \\ &\leq C \|H^{\alpha\beta}\|_{E_H^2} \|h_{\alpha\beta}^{\kappa, n+1}\|_{E_P^4} \\ &\leq C(\varepsilon_0)(A\varepsilon)^2. \end{aligned}$$

Here we applied lemma 8.5.

And we remark that the initial terms $\|h_{\alpha\beta}^{\kappa, n+1}(0, x)\|_{\mathcal{B}^{-1}}$ and $\|\nabla h_{\alpha\beta}^{\kappa, n+1}\|_{\mathcal{B}^{-1}}$ are determined by the initial data h_0 and h_1 , hence, can be controlled by $C\varepsilon$ where C is a universal constant. So we conclude by the desired result. \square

Now we can conclude that, with suitable choice of A and T and sufficient small ε , the sequence $\{S_n\}$ is bounded with respect to the norm $\|\cdot\|_{X_S^{d+1}}$. More rigorously, the following proposition.

Proposition 8.7. *There exists a couple of positive constant (A, T) depends only on ε_0 , ε and d such that if (8.2) holds on $[0, T]$, then*

$$\|S_n^\kappa(t, \cdot)\|_{E_\kappa^{d+1}} \leq A\varepsilon \leq \varepsilon_0 \leq 1, \quad (8.11)$$

which means that the sequence of triple $\{S_n^\kappa\}$ is bounded in the Banach space E_κ^{d+1} . Furthermore, if $\varepsilon \rightarrow 0^+$,

$$T \rightarrow \infty.$$

Remark that the choice of (A, T) are **independent** of κ .

Proof. By lemma 8.5, we take already $A \geq A_0(\varepsilon_0, d)$ and $T \leq T_0$ such that (8.9) holds. In order to prove (8.11), we need only to guarantee, by (8.10), the following inequality:

$$C(\varepsilon_0)T^2(A\varepsilon)^2 + C(T+1)\varepsilon \leq A\varepsilon.$$

This can be guaranteed by

$$\begin{cases} T \leq \frac{A}{2C} - 1, \\ T^2 \leq \frac{1}{2C(\varepsilon_0)A\varepsilon}. \end{cases}$$

So we demand $\frac{A}{2C} - 1 > 0$. Taking into consideration of the conditions demanded in lemma 8.5:

$$\begin{cases} A \geq A_0(\varepsilon_0, d) = 2C(\varepsilon_0, d) + 1 \\ T \leq T_0 = \frac{\ln(1 + (2C(\varepsilon_0, d))^{-1})}{C(\varepsilon_0, d)A\varepsilon} \end{cases}$$

together with the condition $A\varepsilon \leq \varepsilon_0$. So we see that when ε sufficiently small such that

$$A_0(\varepsilon_0, d) \leq \varepsilon_0 \varepsilon^{-1/3},$$

we can take $A = \varepsilon_0 \varepsilon^{-1/3}$ and $T = \min\{A(2C)^{-1} - 1, (2C(\varepsilon_0)A\varepsilon)^{-1/2}, T_0\}$. Then we see that there exist a constant $C'(\varepsilon_0, d)$ such that $T \geq C'(\varepsilon_0, d)\varepsilon^{-1/3}$. This proves the desired result. \square

8.2 Contraction and local existence

To establish theorem 6.2, we need to prove that the sequence $\{S_n^\kappa\}$ is contract. This is conclude in the following proposition:

Proposition 8.8. *Let (8.11) holds with (A, T) determined in by proposition 8.7. Assume that $d \geq 4$. Then there exist a time interval $[0, T^*]$ determined by $\varepsilon_0, \varepsilon$ and d such that the sequence $\{S_n\}$ is contract in the following sense:*

$$\|S_{n+1}^\kappa - S_n^\kappa\|_{L^\infty([0, T^*]; X_\kappa^d)} \leq \lambda \|S_n^\kappa(t, \cdot) - S_{n-1}^\kappa(t, \cdot)\|_{L^\infty([0, T^*]; X_\kappa^d)}. \quad (8.12)$$

with a fixed $0 < \lambda < 1$. Furthermore, when $\varepsilon \rightarrow 0^+$, we can take T^* such that $T^*(\varepsilon) \rightarrow +\infty$.

We emphasize that here the lower bound of the life-span-time T^* given here **does not** depend on the coefficient κ .

The rest of this section is mainly devoted to the proof of this proposition. To do so, we start by taking the difference of between the iteration relation for the pair $(S_{n+1}^\kappa, S_n^\kappa)$ and that of $(S_n^\kappa, S_{n-1}^\kappa)$. This leads to the following differential system

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}) = (H^{\alpha'\beta'}(h_{n-1}^\kappa) - H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n} + F_H(S_n^\kappa) - F_H(S_{n-1}^\kappa), \quad (8.13a)$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} (\phi_{n+1}^\kappa - \phi_n^\kappa) = (H^{\alpha'\beta'}(h_{n-1}^\kappa) - H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} \phi_n^\kappa + F_P(S_n^\kappa) - F_P(S_{n-1}^\kappa), \quad (8.13b)$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} (\varrho_{n+1}^\kappa - \varrho_n^\kappa) = (H^{\alpha'\beta'}(h_{n-1}^\kappa) - H^{\alpha'\beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} \varrho_n^\kappa + F_R(S_n^\kappa) - F_R(S_{n-1}^\kappa) \quad (8.13c)$$

with zero initial data

$$\begin{aligned} h_{\alpha\beta}^{\kappa, n+1}(0, x) - h_{\alpha\beta}^{\kappa, n}(0, x) &= 0, & \partial_t (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})(0, x) &= 0, \\ \phi_{n+1}^\kappa(0, x) - \phi_n^\kappa(0, x) &= 0, & \partial_t (\phi_{n+1}^\kappa - \phi_n^\kappa) &= 0, \\ \varrho_{n+1}^\kappa(0, x) - \varrho_n^\kappa(0, x) &= 0, & \partial_t (\varrho_{n+1}^\kappa - \varrho_n^\kappa) &= 0 \end{aligned}$$

For simplicity of expression, we denote by $D_H(S_n, S_{n-1})$ the right-hand-side of (8.13a), by $D_P(S_n, S_{n-1})$ the right-hand-side of (8.13b) and by $D_R(S_n, S_{n-1})$ the right-hand-side of (8.13c). Then we need to estimate

$$\begin{aligned} \|h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}\|_{X_E^d}, \quad \|\phi_{n+1}^\kappa - \phi_n^\kappa\|_{X_E^d}, \quad \|\varrho_{n+1}^\kappa - \varrho_n^\kappa\|_{X_E^d}, \\ \|h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}\|_{\mathbb{B}^{-1}}. \end{aligned} \quad (8.14)$$

First we recall the uniform bound of the sequence constructed in the last subsection:

$$\|S_n^\kappa(t, \cdot)\|_{E_\kappa^{d+1}} \leq A\varepsilon \leq \varepsilon_0 \leq 1 \quad (8.15)$$

with $d \geq 3$ for $0 \leq t \leq T$. We remark that this condition is equivalent to (8.9) for all $n \in \mathbb{N}^*$.

Now we will make a series of estimates to bound the norm listed in (8.14).

Lemma 8.9. *Let $\{S_n^\kappa\}$ be the sequence constructed by (8.1) who satisfies the uniform bound condition (8.15) with $d \geq 4$. Then the following estimate holds for $|I_1| + |I_2| \leq d - 1$:*

$$\|\partial^{I_1} \Omega^{I_2} D_H(S_{n+1}^\kappa, S_n^\kappa)(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon_0, d) A\varepsilon \|S_n^\kappa(t, \cdot) - S_{n-1}^\kappa(t, \cdot)\|_{E_\kappa^d} \quad (8.16)$$

Proof. This is guaranteed by (7.58) and (8.2). Recall that

$$\begin{aligned} D_H(S_n, S_{n-1}) &= - (H^{\alpha'\beta'}(h_n^\kappa) - H^{\alpha'\beta'}(h_{n-1}^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n} \\ &\quad + (F_{\alpha\beta}(h_n^\kappa, \partial h_n^\kappa, \partial h_n^\kappa) - F_{\alpha\beta}(h_{n-1}^\kappa, \partial h_{n-1}^\kappa, \partial h_{n-1}^\kappa)) \\ &\quad - 16\pi (\partial_\alpha \phi_n^\kappa \partial_\beta \phi_n^\kappa - \partial_\alpha \phi_{n-1}^\kappa \partial_\beta \phi_{n-1}^\kappa) - 12 (\partial_\alpha \varrho_n^\kappa \partial_\beta \varrho_n^\kappa - \partial_\alpha \varrho_{n-1}^\kappa \partial_\beta \varrho_{n-1}^\kappa) \\ &\quad - \kappa^{-1} (V_h(\varrho_n^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^{\kappa, n}) - V_h(\varrho_{n-1}^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^{\kappa, n-1})) \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

- We observe that $\|T_1\|_{E^d}$ is bounded by $C(\varepsilon_0, d)A\varepsilon\|h_n^\kappa - h_{n-1}^\kappa\|_{X_H^d}$.

$$\|\partial^{J_1}\Omega^{J_2}T_1\|_{L^2(\mathbb{R}^3)} \leq \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \|\partial^{J_1}\Omega^{J_2}(H^{\alpha'\beta'}(h_n^\kappa) - H^{\alpha'\beta'}(h_{n-1}^\kappa))\partial^{J'_1}\Omega^{J'_2}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}\|_{L^2(\mathbb{R}^3)}.$$

When $|J_1| + |J_2| \leq d - 3$, $d - 1 \geq |J'_1| + |J'_2| \geq 2$,

$$\begin{aligned} & \|\partial^{J_1}\Omega^{J_2}(H^{\alpha'\beta'}(h_n^\kappa) - H^{\alpha'\beta'}(h_{n-1}^\kappa))\partial^{J'_1}\Omega^{J'_2}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}\|_{L^2(\mathbb{R}^3)} \\ & \leq \|\partial^{J_1}\Omega^{J_2}(H^{\alpha'\beta'}(h_n^\kappa) - H^{\alpha'\beta'}(h_{n-1}^\kappa))\|_{L^\infty(\mathbb{R}^3)} \|\partial^{J'_1}\Omega^{J'_2}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}\|_{L^2(\mathbb{R}^3)} \\ & \leq C(\varepsilon_0, d)\|h_n^\kappa - h_{n-1}^\kappa\|_{E_H^d} \|h_{\alpha\beta}^{\kappa,n}\|_{E_P^{d+1}} \\ & \leq C(\varepsilon_0, d)A\varepsilon\|h_n^\kappa - h_{n-1}^\kappa\|_{E_H^d} \\ & \leq C(\varepsilon_0, d)A\varepsilon\|S_n^\kappa - S_{n-1}^\kappa\|_{E_\kappa^d}, \end{aligned}$$

Where (7.57) is applied.

When $d - 1 \geq |J_1| + |J_2| \geq d - 2$, we have $0 \leq |J'_1| + |J'_2| \leq 1$. Then recall that $d \geq 4$, $|J'_1| + |J'_2| + 1 \leq d - 2$. Then

$$\begin{aligned} & \|\partial^{J_1}\Omega^{J_2}(H^{\alpha'\beta'}(h_n^\kappa) - H^{\alpha'\beta'}(h_{n-1}^\kappa))\partial^{J'_1}\Omega^{J'_2}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}\|_{L^2(\mathbb{R}^3)} \\ & \leq \|(1+r)^{-1}\partial^{J_1}\Omega^{J_2}(H^{\alpha'\beta'}(h_n^\kappa) - H^{\alpha'\beta'}(h_{n-1}^\kappa))\|_{L^2(\mathbb{R}^3)} \|\partial^{J'_1}\Omega^{J'_2}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}\|_{\mathcal{B}^{-1}} \\ & \leq C(\varepsilon_0, d)\|h_n^\kappa - h_{n-1}^\kappa\|_{E_H^d} \|\partial^{J'_1}\Omega^{J'_2}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}\|_{X^2} \\ & \leq C(\varepsilon_0, d)\|S_n^\kappa - S_{n-1}^\kappa\|_{E_\kappa^d} \|h_{\alpha\beta}^{\kappa,n}\|_{E_P^d} \\ & \leq C(\varepsilon_0, d)A\varepsilon\|S_n^\kappa - S_{n-1}^\kappa\|_{E_\kappa^d} \end{aligned}$$

where (7.8), (7.56) and (7.7a) are applied. Remark that because of the term with second order derivative $\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}^{\kappa,n}$, we can only bound the E^d norm, i.e. one order of regularity is lost.

- The E^d term T_2 and T_3 are bounded by (7.52) and (8.15).
- We should pay additional attention to the term T_4 :

$$\kappa^{-1}\left(V_h(\varrho_n^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^{\kappa,n}) - V_h(\varrho_{n-1}^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^{\kappa,n-1})\right)$$

The E^d norm of this term can be bounded by $C(\varepsilon_0, d)(A\varepsilon)^2$. This is guaranteed by (7.44) and the assumption

$$\kappa^{-1/2}\|\varrho_n^\kappa\|_{X^d} \leq C(\varepsilon_0, d)A\varepsilon$$

deduced from (8.15). □

Lemma 8.10. *Let $\{S_n^\kappa\}$ be the sequence constructed by (8.1) who satisfies the uniform bound condition (8.15). Then the following estimate holds for $|I_1| + |I_2| \leq d - 1$:*

$$\|[\partial^{I_1}\Omega^{I_2}, H^{\alpha'\beta'}(h_n^\kappa)\partial_{\alpha'}\partial_{\beta'}](h_{\alpha\beta}^{\kappa,n+1} - h_{\alpha\beta}^{\kappa,n})\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon_0)A\varepsilon\|h_{\alpha\beta}^{\kappa,n+1} - h_{\alpha\beta}^{\kappa,n}\|_{E_P^d} \quad (8.17)$$

Proof. we preform the same calculation as in the proof of lemma 8.2:

$$\begin{aligned} & |[\partial^{I_1}\Omega^{I_2}, H^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}](h_{\alpha\beta}^{\kappa,n+1} - h_{\alpha\beta}^{\kappa,n})| \\ & \leq \sum_{\substack{J_1+J'_1=I_1 \\ |J_2|+|J'_2| \leq |I_2| \\ \alpha', \beta', \alpha'', a}} |\partial^{J'_1}\Omega^{J'_2}H^{\alpha'\beta'}| |\partial_{\alpha''}\partial_a\partial^{J_1}\Omega^{J_2}(h_{\alpha\beta}^{\kappa,n+1} - h_{\alpha\beta}^{\kappa,n})| \\ & \quad + \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2 \\ |J'_1|+|J'_2| > 0}} |\partial^{J'_1}\Omega^{J'_2}H^{\alpha'\beta'}| |\partial_{\alpha'}\partial_{\beta'}\partial^{J_1}\Omega^{J_2}(h_{\alpha\beta}^{\kappa,n+1} - h_{\alpha\beta}^{\kappa,n})| \\ & =: T_1 + T_2. \end{aligned}$$

To estimate T_1 , we remark that because $|J'_1| + |J'_2| + |J_2| + |J'_2| \leq d - 2$, by (7.48) and (8.15),

$$\|\partial^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'} (h_{\alpha\beta}^{\kappa, n})\|_{L^\infty(\mathbb{R}^3)} \leq C(\varepsilon_0, d) A \varepsilon$$

and also because $|J_1| + |J_2| \leq d - 2$:

$$\|\partial_{\alpha'} \partial_a \partial^{I_1} \Omega^{J_2} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})\|_{L^2(\mathbb{R}^3)} \leq C \|h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}\|_{E_P^d}$$

Then we see that

$$\|T_1\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon_0) A \varepsilon \|h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}\|_{E_P^d}.$$

The estimate on T_2 is established in a bit complicated. We see that in T_2 , $|J_1| + |J_2| \leq d - 2$ so

$$\|\partial_{\alpha'} \partial_a \partial^{I_1} \Omega^{J_2} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})\|_{L^2(\mathbb{R}^3)} \leq C \|h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}\|_{E_P^d}$$

When $1 \leq |J'_1| + |J'_2| \leq d - 1$

$$\|\partial^{J'_1} \Omega^{J'_2} H^{\alpha' \beta'} (h_{\alpha\beta}^{\kappa, n})\|_{L^\infty(\mathbb{R}^3)} \leq C(\varepsilon_0, d) A \varepsilon$$

where we applied (7.48) combined with (8.15). □

Now we are ready to estimate the term $\|h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}\|_{E_P^d}$

Lemma 8.11. *Let $\{S_n^\kappa\}$ be the sequence constructed by (8.1) who satisfies the uniform bound condition (8.15) with $d \geq 4$. Then the following estimate holds:*

$$\begin{aligned} \frac{d}{dt} E_{g_n}^{d-1}(t, (h_{\alpha, \beta}^{\kappa, n+1} - h_{\alpha, \beta}^{\kappa, n})) &\leq C(\varepsilon_0, d) A \varepsilon \|S_n(t, \cdot) - S_{n-1}(t, \cdot)\|_{E_\kappa^d} \\ &+ C(\varepsilon_0, d) A \varepsilon E_{g_n}^{d-1}(t, (h_{\alpha, \beta}^{\kappa, n+1} - h_{\alpha, \beta}^{\kappa, n})). \end{aligned} \quad (8.18)$$

Proof. We derive the equation (8.13a) with respect to a product $\partial^{I_1} \Omega^{I_2}$ with $|I_1| + |I_2| \leq d - 1$. Recall the relation of commutation, we get

$$\begin{aligned} (m^{\alpha' \beta'} + H^{\alpha' \beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} \partial^{I_1} \Omega^{I_2} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}) &= \partial^{I_2} \Omega^{I_2} \left((H^{\alpha' \beta'}(h_{n-1}^\kappa) - H^{\alpha' \beta'}(h_n^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^{\kappa, n} \right) \\ &+ \partial^{I_1} \Omega^{I_2} (F_H(S_n^\kappa) - F_H(S_{n-1}^\kappa)) \\ &- [\partial^{I_1} \Omega^{I_2}, H^{\alpha' \beta'} \partial_{\alpha'} \partial_{\beta'}] (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n}). \end{aligned}$$

Then we apply (7.20),

$$\begin{aligned} &\frac{d}{dt} E_{g_n}(t, \partial^{I_1} \Omega^{I_2} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})) \\ &\leq C \|\partial^{I_1} \Omega^{I_2} D_H(S_n^\kappa, S_{n-1}^\kappa)\|_{L^2(\mathbb{R}^3)} + \|[\partial^{I_1} \Omega^{I_2}, H^{\alpha' \beta'}(h_n^\kappa) \partial_{\alpha'} \partial_{\beta'}] (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})\|_{L^2(\mathbb{R}^3)} \\ &+ \sum_{\alpha', \beta'} \|\nabla H^{\alpha' \beta'}(h_n^\kappa)\|_{L^\infty(\mathbb{R}^3)} E_{g_n}(t, \partial^{I_1} \Omega^{I_2} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})). \end{aligned}$$

Remark that by (7.48) and (8.15),

$$\|\nabla H^{\alpha' \beta'}(h_n^\kappa)\|_{L^\infty(\mathbb{R}^3)} \leq C(\varepsilon_0, d) A \varepsilon.$$

Then by lemma 8.9 and 8.10,

$$\begin{aligned} \frac{d}{dt} E_{g_n}(t, \partial^{I_1} \Omega^{I_2} (h_{\alpha\beta}^{\kappa, n+1} - h_{\alpha\beta}^{\kappa, n})) &\leq C(\varepsilon_0, d) A \varepsilon \|S_n(t, \cdot) - S_{n-1}(t, \cdot)\|_{E_\kappa^d} \\ &+ C(\varepsilon_0, d) A \varepsilon \|h_{\alpha, \beta}^{\kappa, n+1} - h_{\alpha, \beta}^{\kappa, n}\|_{E_P^d} \\ &+ C(\varepsilon_0) A \varepsilon E_{g_n}(t, \partial_x^{I_1} \Omega^{I_2} (h_{\alpha, \beta}^{\kappa, n+1} - h_{\alpha, \beta}^{\kappa, n})). \end{aligned}$$

Then by taking the sum over all the pair of multi-index (I_1, I_2) with $|I_1| + |I_2| \leq d-1$, and remark that (by (8.15)):

$$\|h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n}\|_{E_P^d} \leq C(\varepsilon_0, d)E_{g_n}^{d-1}(t, (h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})),$$

Then the desired result is proved. \square

The estimates on $(\phi_{n+1}^\kappa - \phi_n^\kappa)$ and $(\varrho_{n+1}^\kappa - \varrho_n^\kappa)$ are established in the same manner. We will only write the result:

Lemma 8.12. *Let $\{S_n^\kappa\}$ be the sequence constructed by (8.1) who satisfies the uniform bound condition (8.15) with $d \geq 4$. Then the following estimates hold:*

$$\frac{d}{dt}E_{g_n}^{d-1}(t, (\phi_{n+1}^\kappa - \phi_n^\kappa)) \leq C(\varepsilon_0, d)A\varepsilon\|S_n(t, \cdot) - S_{n-1}(t, \cdot)\|_{E_\kappa^d} + C(\varepsilon_0, d)A\varepsilon E_{g_n}^{d-1}(t, (\phi_{n+1}^\kappa - \phi_n^\kappa)), \quad (8.19a)$$

$$\frac{d}{dt}E_{g_n, \kappa^{-1/2}}^{d-1}(t, (\varrho_{n+1}^\kappa - \varrho_n^\kappa)) \leq C(\varepsilon_0, d)A\varepsilon\|S_n(t, \cdot) - S_{n-1}(t, \cdot)\|_{E_\kappa^d} + C(\varepsilon_0, d)A\varepsilon E_{g_n, \kappa^{-1/2}}^{d-1}(t, (\varrho_{n+1}^\kappa - \varrho_n^\kappa)). \quad (8.19b)$$

Then we are at the point to estimate the \mathcal{B}^{-1} norm of $(h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})$.

Lemma 8.13. *Let $\{S_n^\kappa\}$ be the sequence constructed by (8.1) who satisfies the uniform bound condition (8.15) with $d \geq 4$. Then the following estimate holds:*

$$\|h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n}\|_{\mathcal{B}^{-1}} \leq C(\varepsilon_0)t^2 A\varepsilon\|S_n^\kappa - S_{n-1}^\kappa\|_{E_\kappa^d}. \quad (8.20)$$

Proof. This is by applying (7.18). To do so we need to establish the following estimates:

$$\|H^{\alpha'\beta'}(h_n^\kappa)\partial_{\alpha'}\partial_{\beta'}(h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})\|_{\mathcal{B}^{-1}} \leq C(\varepsilon_0)A\varepsilon\|h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n}\|_{E_H^d}, \quad (8.21)$$

$$\|D_H(S_n^\kappa, S_{n-1}^\kappa)\|_{\mathcal{B}^{-1}} \leq C(\varepsilon_0)A\varepsilon\|S_n^\kappa - S_{n-1}^\kappa\|_{E_\kappa^d}. \quad (8.22)$$

We see that (8.22) is direct by (8.16) and (7.8).

To establish (8.21), we see that

$$\begin{aligned} \|H^{\alpha'\beta'}(h_n^\kappa)\partial_{\alpha'}\partial_{\beta'}(h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})\|_{\mathcal{B}^{-1}} &\leq \|H^{\alpha'\beta'}(h_n^\kappa)\|_{\mathcal{B}^{-1}}\|\partial_{\alpha'}\partial_{\beta'}(h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0)A\varepsilon\|h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n}\|_{E_P^4} \end{aligned}$$

where (7.7) is applied. \square

Now we are ready to prove proposition 8.8.

Proof of proposition 8.8. We integrate (8.18), (8.19a) and (8.19b) and get the following estimates:

$$E_{g_n}^{d-1}(t, (h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})) \leq (e^{C(\varepsilon_0, d)A\varepsilon t} - 1)\|S_n^\kappa - S_{n-1}^\kappa\|_{L^\infty([0, T^*]; E_\kappa^d)} \quad (8.23a)$$

$$E_{g_n}^{d-1}(t, (\phi_{n+1}^\kappa - \phi_n^\kappa)) \leq (e^{C(\varepsilon_0, d)A\varepsilon t} - 1)\|S_n^\kappa - S_{n-1}^\kappa\|_{L^\infty([0, T^*]; E_\kappa^d)} \quad (8.23b)$$

$$E_{g_n, \kappa^{-1/2}}^{d-1}(t, (\varrho_{n+1}^\kappa - \varrho_n^\kappa)) \leq (e^{C(\varepsilon_0, d)A\varepsilon t} - 1)\|S_n^\kappa - S_{n-1}^\kappa\|_{L^\infty([0, T^*]; E_\kappa^d)} \quad (8.23c)$$

Recall that the metric g_n is coercive with constant $C(\varepsilon_0)$. Then

$$\begin{aligned} &\|S_{n+1}^\kappa(t, \cdot) - S_n^\kappa(t, \cdot)\|_{E_\kappa^d} \\ &\leq C(\varepsilon_0, d)\max\{E_{g_n}^{d-1}(t, (h_{\alpha,\beta}^{\kappa,n+1} - h_{\alpha,\beta}^{\kappa,n})), E_{g_n}^{d-1}(t, (\phi_{n+1}^\kappa - \phi_n^\kappa)), E_{g_n, \kappa^{-1/2}}^{d-1}(t, (\varrho_{n+1}^\kappa - \varrho_n^\kappa))\}. \end{aligned}$$

Then we conclude by

$$\|S_{n+1}^\kappa - S_n^\kappa\|_{L^\infty([0, T^*]; E_\kappa^d)} \leq C(\varepsilon_0, d)(e^{C(\varepsilon_0, d)A\varepsilon T^*} - 1)\|S_n^\kappa - S_{n-1}^\kappa\|_{L^\infty([0, T^*]; E_\kappa^d)}. \quad (8.24)$$

Then if we chose

$$T^* = \frac{\ln(1 + (2C(\varepsilon_0))^{-1})}{C(\varepsilon_0)A\varepsilon},$$

then

$$\lambda := e^{C(\varepsilon_0)A\varepsilon t} - 1 = 1/2 < 1$$

which satisfies the condition of contraction. Furthermore, recall that in proposition 8.7 we can take $A = \varepsilon_0\varepsilon^{-1/3}$ and $T \geq C'(\varepsilon_0, d)\varepsilon^{-1/3}$ when ε sufficiently small. So here we can also take $T^* = C''(\varepsilon_0, d)\varepsilon^{-1/3}$ for ε sufficiently small. This leads to the limit of $T^*(\varepsilon)$ when $\varepsilon \rightarrow 0^+$. \square

Now we apply the fixed point theorem of Banach and see that $\{S_n^\kappa\}$ converges to a triple $S^\kappa := (h^\kappa, \phi^\kappa, \varrho^\kappa)$ in the sense of $L^\infty([0, T^*], E_\kappa^d)$. Then we will prove that S^κ is a solution of (8.1). This is concluded by the following proposition.

Proposition 8.14. *When $d \geq 4$, the S^κ constructed above is a solution of (8.1) in the sense of distribution. Furthermore,*

$$\begin{aligned} h_{\alpha\beta}^\kappa &\in C([0, T_0], E_H^d), & \partial_t^k h_{\alpha\beta}^\kappa &\in C([0, T_0], E_P^{d-k}), k \leq d \\ \phi^\kappa &\in C([0, T_0], E_P^d), & \partial_t^k h_{\alpha\beta}^\kappa &\in C([0, T_0], E_P^{d-k}), k \leq d \\ \varrho^\kappa &\in C([0, T_0], E_R^d), & \partial_t^k h_{\alpha\beta}^\kappa &\in C([0, T_0], E_R^{d-k}), k \leq d. \end{aligned}$$

Furthermore, $\|S^\kappa(t, \cdot)\|_{E_\kappa^d} \leq A\varepsilon$ with $0 \leq t \leq T^*$.

Proof. The prove is by taking the limit in both side of (8.1). We will prove that the convergence of $\{S_n^\kappa\}$ in sense of $L^\infty([0, T^*]; E_\kappa^d)$ can guarantee the convergence of both side of (8.1).

Recall that the sequence $\{\partial_t \partial_t h_{\alpha\beta}^{\kappa, n}\}$ also converges in the sense $L^\infty([0, T^*]; E^{d-2})$ and so does $\{\partial_t \partial_t \phi_n^\kappa\}$ and $\{\partial_t \partial_t \varrho_n^\kappa\}$.

The convergence of $\{S_n^\kappa\}$ in E_κ^d guarantees the following convergence (remark that $\kappa \leq 1$):

$$\begin{aligned} h_{\alpha\beta}^{\kappa, n} &\rightarrow h_{\alpha\beta}^\kappa \text{ in } L^\infty([0, T^*]; E_H^d), & \nabla h_{\alpha\beta}^{\kappa, n} &\rightarrow \nabla h_{\alpha\beta}^\kappa \text{ in } L^\infty([0, T^*]; E^{d-1}), \\ \phi_n^\kappa &\rightarrow \phi^\kappa \text{ in } L^\infty([0, T^*]; E_P^d), & \nabla \phi_n^\kappa &\rightarrow \nabla \phi^\kappa \text{ in } L^\infty([0, T^*]; E^{d-1}), \\ \varrho_n^\kappa &\rightarrow \varrho^\kappa \text{ in } L^\infty([0, T^*]; E_P^d) \cap L^\infty([0, T^*]; E^{d-1}), & \nabla \varrho_n^\kappa &\rightarrow \nabla \varrho^\kappa \text{ in } L^\infty([0, T^*]; E^{d-1}). \end{aligned} \quad (8.25)$$

here ∇ denotes the space-time divergence. So by Sobolev embedding ($d - 1 \geq 2$), $\{h_n^\kappa\}$, $\{\nabla \phi_n^\kappa\}$ and $\{\varrho_n^\kappa\}$ converges in $L^\infty([0, T^*] \times \mathbb{R}^3)$. Furthermore, we have

$$\begin{aligned} \partial_t \partial_x h_{\alpha\beta}^{\kappa, n} &\rightarrow \partial_t \partial_x h_{\alpha\beta}^\kappa \text{ in } L^\infty([0, T^*]; E^{d-2}), & \partial_x \partial_x h_{\alpha\beta}^{\kappa, n} &\rightarrow \partial_x \partial_x h_{\alpha\beta}^\kappa \text{ in } L^\infty([0, T^*]; E^{d-2}) \\ \partial_t \partial_x \phi_n^\kappa &\rightarrow \partial_t \partial_x \phi^\kappa \text{ in } L^\infty([0, T^*]; E^{d-2}), & \partial_x \partial_x \phi_n^\kappa &\rightarrow \partial_x \partial_x \phi^\kappa \text{ in } L^\infty([0, T^*]; E^{d-2}) \\ \partial_t \partial_x h_{\alpha\beta}^{\kappa, n} &\rightarrow \partial_t \partial_x h_{\alpha\beta}^\kappa \text{ in } L^\infty([0, T^*]; E^{d-2}), & \partial_x \partial_x h_{\alpha\beta}^{\kappa, n} &\rightarrow \partial_x \partial_x h_{\alpha\beta}^\kappa \text{ in } L^\infty([0, T^*]; E^{d-2}) \end{aligned} \quad (8.26)$$

These convergence are sufficient to guarantee the convergence of both side of (8.1) because both side depend linearly the terms with second order derivatives. And the lower order terms converge in L^∞ sense. \square

Now we are ready to prove the theorem 6.2:

Proof of theorem 6.2. We have verified that the triple S^κ is a local solution of (6.3). Furthermore, we notice that the lower bound of life-span-time T^* constructed in proposition 8.14 does not depend on κ . The estimates are established by taking the limit of the (8.15). \square

9 Comparing the $f(R)$ theory to general relativity

9.1 Overview

In this section we will compare the solution given by $f(R)$ theory with the solution of classical Einstein equation. For the convenience of discussion, we introduce the following notation: we denote by $S^0 := (h_{\alpha\beta}, \phi)$ the triple composed by $h_{\alpha\beta}$ and ϕ which are determined by the following Cauchy problem:

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h))\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta} = F_{\alpha\beta}(h, \partial h, \partial h) - 16\pi\partial_{\alpha}\phi\partial_{\beta}\phi, \quad (9.1a)$$

$$(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h))\partial_{\alpha'}\partial_{\beta'}\phi = 0 \quad (9.1b)$$

with the initial data

$$\begin{aligned} h_{\alpha\beta}(0, x) &= h_{0\alpha\beta}, & \partial_t h_{\alpha\beta}(0, x) &= h_{1\alpha\beta}, \\ \phi(0, x) &= \phi_0, & \partial_t \phi(0, x) &= \phi_1. \end{aligned}$$

We claim that if this initial data satisfies the following constraint conditions,

$$(9.2)$$

then $g_{\alpha\beta} = m_{\alpha\beta} + h_{\alpha\beta}$ and ϕ satisfies the classical Einstein's field equation coupled with the real masse less scalar field ϕ . For the convenience of discussion we introduce the following norm of S_0 :

$$\|S_0\|_{X_0^{d+1}} := \max\{\|h_{0\alpha\beta}\|_{X_H^{d+1}}, \|h_{1\alpha\beta}\|_{X^d}, \|\phi_0\|_{X_P^{d+1}}, \|\phi_1\|_{X^d}\}.$$

We will state the following local existence result on this system:

Theorem 9.1. *Suppose that $(h_{0\alpha\beta}, h_{1\alpha\beta}) \in X_H^{d+1} \times X^d$ and $(\phi_0, \phi_1) \in X_P^{d+1} \times X^d$ and with $d \geq 4$. Denote by $S_0 = (h_0, h_1, \phi_0, \phi_1)$ and assume that*

$$\|S_0\|_{X_0^{d+1}} \leq \varepsilon \leq \varepsilon_0 \leq 1$$

with ε_0 a sufficiently small constant. Then there exists a pair of positive constant (A, T^*) determined by $\varepsilon_0, \varepsilon$ and d such that the Cauchy problem (9.1) with initial data S_0 has a unique solution (in sense of distribution) $(h_{\alpha\beta}, \phi)$ in the time interval $[0, T^*]$. Here

$$h_{\alpha\beta} \in C([0, T^*]; E_H^d), \quad \phi \in C([0, T^*]; E_P^d).$$

When $\varepsilon \rightarrow 0^+$, we can take

$$\lim_{\varepsilon \rightarrow 0^+} T^* = +\infty.$$

Furthermore, the local solution satisfies the following estimates in the time interval $[0, T^*]$:

$$\|h_{\alpha\beta}\|_{E_H^d} + \|\phi\|_{E_P^d} \leq A\varepsilon. \quad (9.3)$$

The proof is similar to that of the theorem 6.2: we make iteration and estimate the sequence constructed by this iteration and prove that with suitable choice of (A, T^*) , this sequence is contract. The detailed prove is omit.

For the convenience of discussion, we introduce the following notation: Let $S_0 = (h_{0\alpha\beta}^0, h_{1\alpha\beta}^0, \phi_0^0, \phi_1^0)$ be a initial data who satisfies the Einstein's constraint equation (2.23) and $S_1 = (h_{0\alpha\beta}, h_{1\alpha\beta}, \phi_0, \phi_1, \varrho_0, \varrho_1)$ be a initial data who satisfies the nonlinear constraint equations (5.5) and (5.6). Define the following function $\mathcal{D}_\kappa(S_0, S_1)$:

$$\begin{aligned} \mathcal{D}_\kappa^d(S_0, S_1) := \max\{ & \|h_{0\alpha\beta}^0 - h_{0\alpha\beta}\|_{X_H^{d+1}}, \|h_{1\alpha\beta}^0 - h_{1\alpha\beta}\|_{X^d}, \|\phi_0^0 - \phi_0\|_{X_P^{d+1}}, \\ & \|\phi_1^0 - \phi_1\|_{X_P^{d+1}}, \|\varrho_0\|_{X_P^{d+1}}, \|\varrho_1\|_{X^d}, \kappa^{-1/2}\|\varrho_0\|_{X^d}\}. \end{aligned}$$

Denote by

$$S^0(t) = (h_{\alpha\beta}^0(t), \phi^0(t)) \in C([0, T^*]; X_H^d) \cap C^1([0, T]; X_H^{d-1}) \times C([0, T^*]; X_P^d) \cap C^1([0, T]; X_P^{d-1}).$$

the local solution of Cauchy problem (9.1) with initial data $S^0(0) = S_0$, and

$$S^\kappa(t) = (h_{\alpha\beta}^\kappa, \phi^\kappa, \varrho^\kappa) \in C([0, T^*]; X_H^d) \cap C^1([0, T]; X_H^{d-1}) \times C([0, T^*]; X_P^d) \cap C^1([0, T]; X_P^{d-1}) \\ \times C([0, T^*]; X_R^d) \cap C^1([0, T]; X_R^{d-1}) \cap C([0, T^*]; X_R^{d-1}).$$

We introduce the “distance” form S^0 to S^κ :

$$\mathcal{D}^d(S^0, S^\kappa)(t) := \sum_{\alpha\beta} \|h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa\|_{E_H^d} + \|\phi^0 - \phi^\kappa\|_{E_P^d}.$$

Now we are ready to state key estimate.

Theorem 9.2 (Theorem of comparison). *There exists a positive constant ε_0 such that if*

$$\max\{\|S_1\|_{X_\kappa^{d+1}}, \|S_0\|_{X_\kappa^{d+1}}\} \leq \varepsilon \leq \varepsilon_0 \leq 1,$$

with $d \geq 4$. Then in the common interval of existence $[0, T^*]$ (which depends only on $\varepsilon, \varepsilon_0$ and d), the following estimates hold:

$$\mathcal{D}^{d-1}(S^0, S^\kappa)(t) \leq C(\varepsilon_0, d)(\mathcal{D}^{d-1}(S^0, S^\kappa)(0) + (E_{g, \kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2 + \kappa^{1/2}(A\varepsilon)^3). \quad (9.4)$$

9.2 Proof of the result of comparison

The proof of theorem 9.2 demands better estimate on $\|\varrho_{n+1}^\kappa\|_{X^{d-1}}$. We will improve it by the following estimates:

First we establish a better bound of the L^2 norm of $\partial_t \partial^{I_1} \Omega^{I_2} \varrho^\kappa$. This is by the following lemmas:

Lemma 9.3. *Let $S^\kappa = (h_{\alpha\beta}^\kappa, \phi^\kappa, \varrho^\kappa)$ be the solution of Cauchy problem (6.3) with $d \geq 4$. Then the following estimate holds for all $|I_1| + |I_2| \leq d - 2$:*

$$\kappa^{-1/2} \|\partial_t \partial^{I_1} \Omega^{I_2} \varrho^\kappa\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon_0, d) A\varepsilon. \quad (9.5)$$

Proof. This is a direct result of (6.6c). \square

Lemma 9.4. *There exists a positive constant ε_0 such that if (8.2) holds for $d \geq 4$ and $A\varepsilon \leq \varepsilon_0$, then*

$$\|\varrho^\kappa(t, \cdot)\|_{E^{d-2}}^2 \leq \kappa^{3/2} C(\varepsilon_0, d) t (A\varepsilon)^3 + C(\varepsilon_0, d) \kappa (E_{g, \kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2. \quad (9.6)$$

Proof. This is a modified energy estimate. We derive the equation (6.3c) with respect to $\partial^{I_1} \Omega^{I_2}$ with $|I_1| + |I_2| \leq d - 2$: As in the proof of lemma 7.11:

$$\begin{aligned} & \partial_t \partial^{I_1} \Omega^{I_2} \varrho^\kappa (g^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{I_1} \Omega^{I_2} \varrho^\kappa - 3\kappa^{-1} \partial^{I_1} \Omega^{I_2} \varrho^\kappa) \\ &= \frac{1}{2} \partial_0 (g^{00} (\partial_0 \partial^{I_1} \Omega^{I_2} \varrho^\kappa)^2 - g^{ab} \partial_a \partial^{I_1} \Omega^{I_2} \varrho^\kappa \partial_b \partial^{I_1} \Omega^{I_2} \varrho^\kappa) - \frac{1}{2} \partial_0 ((3\kappa)^{-1/2} \partial^{I_1} \Omega^{I_2} \varrho^\kappa)^2 \\ & \quad + \partial_a (g^{a\beta} \partial^{I_1} \Omega^{I_2} \varrho^\kappa \partial_\beta \partial^{I_1} \Omega^{I_2} \varrho^\kappa) + \frac{1}{2} \partial_t g^{\alpha\beta} \partial_\alpha \partial^{I_1} \Omega^{I_2} \varrho^\kappa \partial_\beta \partial^{I_1} \Omega^{I_2} \varrho^\kappa \\ & \quad - \partial_\alpha g^{\alpha\beta} \partial_0 \partial^{I_1} \Omega^{I_2} \varrho^\kappa \partial_\beta \partial^{I_1} \Omega^{I_2} \varrho^\kappa. \end{aligned}$$

For simplicity, denote by $v = \partial^{I_1} \Omega^{I_2} \varrho^\kappa$. Then

$$\begin{aligned} & \partial_t v (\partial^{I_1} \Omega^{I_2} F_R(h^\kappa, \phi^\kappa, \varrho^\kappa)) \\ &= \frac{1}{2} \partial_t (g^{00} (\partial_t v)^2 - g^{ab} \partial_a v \partial_b v) - \frac{1}{2} \partial_t ((3\kappa)^{-1/2} v)^2 + \partial_a (g^{a\beta} \partial_t v \partial_\beta v) + \frac{1}{2} \partial_t g^{\alpha\beta} \partial_\alpha v \partial_\beta v - \partial_\alpha g^{\alpha\beta} \partial_t v \partial_\beta v \end{aligned}$$

Integrate this equation in the region $[0, t] \times \mathbb{R}^3$ and use the Stokes' formula,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \partial_t v (\partial^{I_1} \Omega^{I_2} F_R(h^\kappa, \phi^\kappa, \varrho^\kappa)) dx dt \\ &= \frac{1}{2} (E_{g, \kappa^{-1/2}}(t, \partial^{I_1} \Omega^{I_2} \varrho^\kappa))^2 - \frac{1}{2} (E_{g, \kappa^{-1/2}}(0, \partial^{I_1} \Omega^{I_2} \varrho^\kappa))^2 \\ & \quad + \int_0^t \int_{\mathbb{R}^3} \left(\frac{1}{2} \partial_t g^{\alpha\beta} \partial_\alpha v \partial_\beta v - \partial_\alpha g^{\alpha\beta} \partial_t v \partial_\beta v \right) dx dt \end{aligned} \quad (9.7)$$

Then by (7.44), (7.49), (9.5) and (6.6), we see that

$$\left| \int_0^t \int_{\mathbb{R}^3} \partial_t v (\partial^{I_1} \Omega^{I_2} F_R(h^\kappa, \phi^\kappa, \varrho^\kappa)) dx dt \right| \leq C(\varepsilon_0, d) t (A\varepsilon)^3 \kappa^{1/2}.$$

Also by (6.6) and the fact that $d \geq 4$

$$|\partial_\alpha g^{\alpha\beta}|_{L^\infty(\mathbb{R}^3)} \leq C(\varepsilon_0) \|h_{\alpha\beta}\|_{E_P^{d-1}} \leq C(\varepsilon_0) A\varepsilon,$$

then

$$\left| \int_0^t \int_{\mathbb{R}^3} \left(\frac{1}{2} \partial_t g^{\alpha\beta} \partial_\alpha v \partial_\beta v - \partial_\alpha g^{\alpha\beta} \partial_t v \partial_\beta v \right) dx dt \right| \leq C(\varepsilon_0, d) t (A\varepsilon)^3 \kappa.$$

Combine these two estimate with (9.7), we see that (remark $0 < \kappa \leq 1$)

$$(E_{g, \kappa^{-1/2}}(t, \partial^{I_1} \Omega^{I_2} \varrho^\kappa))^2 \leq (E_{g, \kappa^{-1/2}}(0, \partial^{I_1} \Omega^{I_2} \varrho^\kappa))^2 + C(\varepsilon_0, d) t (A\varepsilon)^3 \kappa^{1/2}.$$

This leads to (recall the definition of $(E_{g, \kappa^{-1/2}}(t, u))$)

$$\kappa^{-1} \|\partial^{I_1} \Omega^{I_2} \varrho^\kappa(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \leq C(\varepsilon_0, d) (E_{g, \kappa^{-1/2}}(0, \partial^{I_1} \Omega^{I_2} \varrho^\kappa))^2 + C(\varepsilon_0, d) t (A\varepsilon)^3 \kappa^{1/2}.$$

□

Now we are ready to prove the theorem 9.2.

Proof of theorem 9.2. By taking the difference of (6.3a) and (9.1a), and the difference of (6.3b) and (9.1b), we obtain the following two equations:

$$\begin{aligned} (m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^0)) \partial_{\alpha'} \partial_{\beta'} ((h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)) &= - (H^{\alpha'\beta'}(h^0) - H^{\alpha'\beta'}(h^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^\kappa \\ & \quad + (F_{\alpha\beta}(h^0, \partial h^0, \partial h^0) - F_{\alpha\beta}(h^\kappa, \partial h^\kappa, \partial h^\kappa)) \\ & \quad - 16\pi (\partial_\alpha \phi^0 \partial_\beta \phi^0 - \partial_\alpha \phi^\kappa \partial_\beta \phi^\kappa) \\ & \quad + 12 \partial_\alpha \varrho^\kappa \partial_\beta \varrho^\kappa + \kappa^{-1} V_h(\varrho^\kappa) (m_{\alpha\beta} + h_{\alpha\beta}^\kappa), \end{aligned} \quad (9.8a)$$

$$\begin{aligned} (m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^0)) \partial_{\alpha'} \partial_{\beta'} (\phi^0 - \phi^\kappa) &= - (H^{\alpha'\beta'}(h^0) - H^{\alpha'\beta'}(h^\kappa)) \partial_{\alpha'} \partial_{\beta'} \phi^\kappa \\ & \quad - 2(m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^\kappa)) \partial_{\alpha'} \phi^\kappa \partial_\beta \varrho^\kappa. \end{aligned} \quad (9.8b)$$

Then we will establish the estimates (9.4) from these two equations. The proof is quite similar to that of lemma 8.11 and lemma 8.12.

Step I. The estimates on L^2 type norms. Let us begin with the E_P^{d-2} norm of $(h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)$. Let (I_1, I_2) be a pair of multi-index with $|I_1| + |I_2| \leq d-2$. We derive (9.8) with respect to $\partial^{I_1} \Omega^{I_2}$:

$$\begin{aligned} & (m^{\alpha'\beta'} + H^{\alpha'\beta'}(h^0)) \partial_{\alpha'} \partial_{\beta'} \partial^{I_1} \Omega^{I_2} (h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa) \\ &= - \partial^{I_1} \Omega^{I_2} \left((H^{\alpha'\beta'}(h^0) - H^{\alpha'\beta'}(h^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^\kappa \right) + \partial^{I_1} \Omega^{I_2} (F_{\alpha\beta}(h^0, \partial h^0, \partial h^0) - F_{\alpha\beta}(h^\kappa, \partial h^\kappa, \partial h^\kappa)) \\ & \quad - 16\pi \partial^{I_1} \Omega^{I_2} (\partial_\alpha \phi^0 \partial_\beta \phi^0 - \partial_\alpha \phi^\kappa \partial_\beta \phi^\kappa) - [\partial^{I_1} \Omega^{I_2}, H^{\alpha'\beta'}(h^0) \partial_{\alpha'} \partial_{\beta'}] (h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa) \\ & \quad + 12 \partial^{I_2} \Omega^{I_2} (\partial_\alpha \varrho^\kappa \partial_\beta \varrho^\kappa) + \kappa^{-1} \partial_x^{I_1} \Omega^{I_2} \left(V_h(\varrho^\kappa) (m_{\alpha\beta} + h_{\alpha\beta}^\kappa) \right) \\ &=: T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \end{aligned}$$

Then combined with (7.20),

$$\frac{d}{dt}E_g(t, \partial^{I_1}\Omega^{I_2}(h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)) \leq C \sum_{i=1}^6 \|T_i\|_{L^2} + C \sum_{\alpha,\beta} \|H^{\alpha'\beta'}(h^0)\|_{L^\infty(\mathbb{R}^3)} E_g(t, \partial^{I_1}\Omega^{I_2}(h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)). \quad (9.9)$$

We will need to control the L^2 norm of these T_i for $i = 1, \dots, 6$. The term T_i for $i = 1, 2, 3, 4$ can be bounded as following:

$$\begin{aligned} \|T_i\|_{L^2(\mathbb{R}^3)} &\leq C(\varepsilon_0, d)A\varepsilon(\|h^0 - h^\kappa\|_{E_H^{d-1}} + \|h^0 - h^\kappa\|_{E_P^{d-1}}\|\phi^0 - \phi^\kappa\|_{E_P^{d-1}}) \\ &\leq C(\varepsilon_0, d)A\varepsilon\mathcal{D}^{d-1}(S^0, S^\kappa)(t). \end{aligned} \quad (9.10)$$

The proof is exactly the same to the proof of (8.16) and (8.17) and we omit the detail.

The **crucial** terms T_5 and T_6 are bounded as following:

$$\begin{aligned} \|T_5\|_{L^2(\mathbb{R}^3)} &\leq C(\varepsilon_0, d)\kappa(A\varepsilon)^2 \\ \|T_6\|_{L^2(\mathbb{R}^3)} &\leq C(\varepsilon_0, d)\kappa^{1/2}(A\varepsilon)^3 + C(\varepsilon_0, d)(E_{g,\kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2 \end{aligned} \quad (9.11)$$

The estimates on T_5 and T_6 consult the refined estimates (9.5) and (9.6). More precisely, T_5 is estimated by (7.44), (6.6b) and (9.5). T_6 is estimated by (7.44) and (9.6):

Now combine these estimates on T_i and remark that

$$\|H^{\alpha'\beta'}(h^0)\|_{L^\infty} \leq C(\varepsilon_0)A\varepsilon$$

and

$$E_g(t, \partial^{I_1}\Omega^{I_2}(h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)) \leq C(\varepsilon_0, d)\mathcal{D}^{d-1}(S^0, S^\kappa)(t),$$

we can deduce from (9.9), that

$$\begin{aligned} \frac{d}{dt}E_g^{d-2}(t, (h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)) &\leq C(\varepsilon_0, d)A\varepsilon\mathcal{D}^{d-1}(S^0, S^\kappa)(t) + C(\varepsilon_0, d)\kappa^{1/2}(A\varepsilon)^3 \\ &\quad + C(\varepsilon_0, d)(E_{g,\kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2. \end{aligned} \quad (9.12)$$

where we recall the definition of \mathcal{D} :

$$\mathcal{D}^d(S^0, S^\kappa)(t) := \sum_{\alpha\beta} \|h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa\|_{E_H^d} + \|\phi^0 - \phi^\kappa\|_{E_P^d}.$$

The estimate on the norm $\|\phi^0 - \phi^1\|_{X_E^d}$ is similar to that of $h^0 - h^\kappa$ (even simpler). We claim that the following estimate on the right-hand-side of (9.8b):

$$\|(H^{\alpha'\beta'}(h^0) - H^{\alpha'\beta'}(h^\kappa))\partial_{\alpha'}\partial_{\beta'}\phi^\kappa\|_{E^{d-2}} \leq C(\varepsilon_0)A\varepsilon\|h^0 - h^\kappa\|_{E_H^{d-1}}, \quad (9.13a)$$

$$\|(m^{\alpha\beta} + H^{\alpha\beta}(h^\kappa))\partial_\alpha\phi^\kappa\partial_\beta\varrho^\kappa\|_{E^{d-2}} \leq C(\varepsilon_0)\kappa^{1/2}(A\varepsilon)^2. \quad (9.13b)$$

The first can be proved exactly as in the proof of (9.12). The second one is proved as following: for any (I_1, I_2) with $|I_1| + |I_2| \leq d - 2$,

$$\begin{aligned} &\|\partial^{I_2}\Omega^{I_2}((m^{\alpha\beta} + H^{\alpha\beta}(h^\kappa))\partial_\alpha\phi^\kappa\partial_\beta\varrho^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\leq \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \|\partial^{J_1}\Omega^{J_2}(m^{\alpha\beta} + H^{\alpha\beta}(h^\kappa))\|_{L^\infty(\mathbb{R}^3)} \|\partial^{J'_1}\Omega^{J'_2}(\partial_\alpha\phi^\kappa\partial_\beta\varrho^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0, d) \sum_{\substack{\alpha,\beta \\ |J'_1|+|J'_2|\leq d-2}} \|\partial^{J'_1}\Omega^{J'_2}(\partial_\alpha\phi^\kappa\partial_\beta\varrho^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0, d) \sum_{\substack{\alpha,\beta \\ |J'_1|+|J'_2|\leq d-2}} \sum_{\substack{K_1+K'_1=J'_1 \\ K_2+K'_2=J'_2}} \|\partial^{K_1}\Omega^{K_2}\partial_\alpha\phi^\kappa\partial_x^{K'_1}\Omega^{K'_2}\partial_\beta\varrho^\kappa\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

Then, when $|K_1| + |K_2| \leq d - 3$,

$$\begin{aligned} \|\partial^{K_1} \Omega^{K_2} \partial_\alpha \phi^\kappa \partial^{K'_1} \Omega^{K'_2} \partial_\beta \varrho^\kappa\|_{L^2(\mathbb{R}^3)} &\leq \|\partial^{K_1} \Omega^{K_2} \partial_\alpha \phi^\kappa\|_{L^\infty(\mathbb{R}^3)} \|\partial^{K'_1} \Omega^{K'_2} \partial_\beta \varrho^\kappa\|_{L^2(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0, d) A\varepsilon \|\varrho^\kappa\|_{E^{d-1}} \\ &\leq C(\varepsilon_0, d) \kappa^{1/2} (A\varepsilon)^2. \end{aligned}$$

When $|K_1| + |K_2| = d - 2$ and $K'_1 = K'_2 = 0$, recall that $d \geq 4$:

$$\begin{aligned} \|\partial^{K_1} \Omega^{K_2} \partial_\alpha \phi^\kappa \partial^{K'_1} \Omega^{K'_2} \partial_\beta \varrho^\kappa\|_{L^2(\mathbb{R}^3)} &\leq \|\partial^{K_1} \Omega^{K_2} \partial_\alpha \phi^\kappa\|_{L^2(\mathbb{R}^3)} \|\partial_\beta \varrho^\kappa\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0, d) A\varepsilon \|\varrho^\kappa\|_{E^{d-1}} \leq C(\varepsilon_0, d) \kappa^{1/2} (A\varepsilon)^2 \end{aligned}$$

So we conclude by (9.13b), and combined with (7.20),

$$\frac{d}{dt} E_g^{d-2}(t, (\phi^0 - \phi^\kappa)) \leq C(\varepsilon_0, d) A\varepsilon \mathcal{D}^{d-1}(S^0, S^\kappa)(t) + C(\varepsilon_0, d) \kappa^{1/2} (A\varepsilon)^2. \quad (9.14)$$

Step II. The estimate on \mathcal{B}^{-1} norm. To do so we rewrite the equation (9.8a) into the following form:

$$\begin{aligned} \square((h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)) &= -H^{\alpha'\beta'}(h^0) \partial_{\alpha'} \partial_{\beta'} ((h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)) - (H^{\alpha'\beta'}(h^0) - H^{\alpha'\beta'}(h^\kappa)) \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}^\kappa \\ &\quad + (F_{\alpha\beta}(h^0, \partial h^0, \partial h^0) - F_{\alpha\beta}(h^\kappa, \partial h^\kappa, \partial h^\kappa)) - 16\pi(\partial_\alpha \phi^0 \partial_\beta \phi^0 - \partial_\alpha \phi^\kappa \partial_\beta \phi^\kappa) \\ &\quad + 12\partial_\alpha \varrho^\kappa \partial_\beta \varrho^\kappa + \kappa^{-1} V_h(\varrho^\kappa)(m_{\alpha\beta} + h_{\alpha\beta}^\kappa) \\ &=: T_0 + T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \end{aligned} \quad (9.15)$$

By (7.18) we need to control the \mathcal{B}^{-1} norm of the terms T_i for $i = 1, \dots, 6$. By (7.8), we need only to control the X^2 norm of these terms. Recall the condition $d \geq 4$, then $d - 2 \geq 2$. So we only need to control the X^{d-2} norm of these terms. Remark that in the **Step I** we have already controlled this norm for the terms T_i with $i \geq 1$. Now we only need to control the X^2 norm of T_0 . Let (I_1, I_2) be a pair of multi-index with $|I_1| + |I_2| \leq 2$. Then

$$\begin{aligned} &\|\partial_x^{I_1} \Omega^{I_2} (H^{\alpha'\beta'}(h^0) \partial_{\alpha'} \partial_{\beta'} (h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa))\|_{L^2(\mathbb{R}^3)} \\ &\leq \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \|\partial_x^{J_1} \Omega^{J_2} (H^{\alpha'\beta'}(h^0)) \partial_x^{J'_1} \Omega^{J'_2} \partial_{\alpha'} \partial_{\beta'} (h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\leq \sum_{\substack{J_1+J'_1=I_1 \\ J_2+J'_2=I_2}} \|\partial_x^{J_1} \Omega^{J_2} (H^{\alpha'\beta'}(h^0))\|_{L^\infty(\mathbb{R}^3)} \|\partial_x^{J'_1} \Omega^{J'_2} \partial_{\alpha'} \partial_{\beta'} (h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa)\|_{L^2(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0) \|h^0\|_{X^4} \|h^0 - h^\kappa\|_{E_P^3} \\ &\leq C(\varepsilon_0) A\varepsilon \|h^0 - h^\kappa\|_{E_P^{d-1}}. \end{aligned}$$

Then by (7.18), the following estimate on \mathcal{B}^{-1} norm holds:

$$\|h_{\alpha\beta}^0(t, \cdot) - h_{\alpha\beta}^\kappa(t, \cdot)\|_{\mathcal{B}^{-1}} \leq C(\varepsilon_0) t A\varepsilon \int_0^t \mathcal{D}^{d-1}(S^0, S^\kappa)(\tau) d\tau + C(1+t) (\|h_{0\alpha\beta}^0 - h_{0\alpha\beta}^\kappa\|_{X_H^{d-1}}) \quad (9.16)$$

Step III. Conclusion. Now by integrating (9.12) and (9.15), we get the following estimate:

$$\begin{aligned} E_g^{d-2}(t, h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa) &\leq E_g^{d-2}(0, h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa) + C(\varepsilon_0, d) A\varepsilon \int_0^t \mathcal{D}^{d-1}(S^0, S^\kappa)(\tau) d\tau \\ &\quad + C(\varepsilon_0, d) \kappa^{1/2} (A\varepsilon)^3 t + C(\varepsilon_0, d) t (E_{g, \kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2 \end{aligned} \quad (9.17a)$$

$$E_g^{d-2}(t, \phi^0 - \phi^\kappa) \leq E_g^{d-2}(0, (\phi^0 - \phi^\kappa)) + C(\varepsilon_0, d)A\varepsilon \int_0^t \mathcal{D}^{d-1}(S^0, S^\kappa)(\tau)d\tau + C(\varepsilon_0, d)\kappa^{1/2}(A\varepsilon)^2t. \quad (9.17b)$$

Recall that g is coercive with constant $C(\varepsilon_0)$ when ε_0 is sufficiently small. Then

$$\begin{aligned} \mathcal{D}^{d-1}(S^0 - S^\kappa)(t) &\leq C(\varepsilon_0, d) \sum_{\alpha, \beta} E_g^{d-2}(t, h_{\alpha\beta}^0 - h_{\alpha\beta}^\kappa) + C(\varepsilon_0, d) \sum_{\alpha, \beta} \|h_{\alpha\beta}^0(t, \cdot) - h_{\alpha\beta}^\kappa(t, \cdot)\|_{\mathcal{B}^{-1}} \\ &\quad + C(\varepsilon_0, d)E_g^{d-2}(t, \phi^0 - \phi^\kappa) \leq C(\varepsilon_0, d)^2 \mathcal{D}^{d-1}(S^0 - S^\kappa)(t). \end{aligned} \quad (9.18)$$

Then combine (9.16), (9.17a), (9.17b) and (9.18), the following estimate holds:

$$\begin{aligned} \mathcal{D}^{d-1}(S^0, S^\kappa)(t) &\leq C(\varepsilon_0, d)(1 + T^*)\mathcal{D}^{d-1}(S^0, S^\kappa)(0) + C(\varepsilon_0, d)T^*(E_{g, \kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2 \\ &\quad + C(\varepsilon_0, d)T^*\kappa^{1/2}(A\varepsilon)^3 \\ &\quad + C(\varepsilon_0, d)(1 + T^*)A\varepsilon \int_0^t \mathcal{D}^{d-1}(S^0, S^\kappa)(\tau)d\tau \end{aligned} \quad (9.19)$$

which leads to

$$\begin{aligned} &\mathcal{D}^{d-1}(S^0, S^\kappa)(t) \\ &\leq C(\varepsilon_0, d)(1 + T^*)\left(\mathcal{D}^{d-1}(S^0, S^\kappa)(0) + (E_{g, \kappa^{-1/2}}^{d-2}(0, \varrho^\kappa))^2 + \kappa^{1/2}(A\varepsilon)^3\right)e^{C(\varepsilon_0, d)A\varepsilon(1+T^*)t} \end{aligned}$$

which leads to the desired result. \square

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Partie II

Méthode du feuilletage hyperboloïdal ²

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Chapter 1

Introduction

1.1 Main objective

We are interested in nonlinear wave equations posed on the $(3 + 1)$ -dimensional Minkowski spacetime and, especially, in models arising in mathematical physics and involving a nonlinear coupling between with the Klein-Gordon equation, such as the Dirac-Klein-Gordon equation, the Proca-Klein-Gordon equation, etc. (See Section 1.4, below.) Our study is also motivated by the Einstein equations of general relativity when the matter model is a massive scalar field. The Klein-Gordon equation is also important to describe nonlinear waves propagating in fluid or elastic materials.

The so-called ‘vector field method’ was introduced by [Klainerman (1986, 1987)] and [Christodoulou (1986)]. It is based on weighted norms defined from the conformal Killing fields of Minkowski spacetime and Sobolev-type arguments adapted to the wave operator, and yields a global-in-time, well-posedness theory for the initial value problem for the nonlinear wave equation, when the initial data have small amplitude and are “localized”, that is, are compactly supported. This method relies on bootstrap strategies and an analysis of the decay of solutions in time, and applies to quadratic nonlinearities satisfying the so-called ‘null condition’. (See Section 1.3.) Furthermore, as far as coupled systems of wave and Klein-Gordon equations are concerned, the current state-of-the-art theory is given by the recent work by [Katayama (2012a)] who suitably extended the vector field method. (See the presentation below.)

In this monograph, building upon these earlier works, we consider the Cauchy problem for a broad class of nonlinear coupled systems consisting of wave equations and Klein-Gordon equations, and we introduce a novel

approach, which we refer to as the ‘hyperboloidal foliation method’ and leads us to a vast generalization of the state-of-the-art. By working in suitably (spacetime) weighted norms, we are able to include nonlinearities that were not covered by [Katayama (2012a)], but yet are important in certain applications (to be presented in [LeFloch and Ma (2015)]). The hyperboloidal foliation method introduces a novel methodology, which takes its roots in an observation (recalled in Section 2.1) made by [Hörmander (1997)] for the Klein-Gordon equation.

Let us denote by \square the wave operator in Minkowski space. For our purpose in this monograph, a simpler (yet challenging) model of interest is provided by the system (see notation below)

$$\begin{aligned}\square u &= P(\partial u, \partial v), \\ \square v + v &= Q(\partial u, \partial v),\end{aligned}\tag{1.1.1}$$

which couples the wave equation and the Klein Gordon equation and describe the nonlinear interactions of, say, two scalar fields, one of them being massive. Here, the nonlinear terms $P(\partial u, \partial v)$ and $Q(\partial u, \partial v)$ are quadratic forms in the first-order spacetime derivatives $\partial u, \partial v$ and account for both self-interactions and interactions between the two unknown scalar fields u and v . On one hand, global existence of solutions to nonlinear wave equations without Klein-Gordon terms is established when the nonlinearities satisfy the ‘null condition’ (cf. 1.1.5c, below). On the other hand, the vector field method also applies to nonlinear Klein-Gordon equations, as shown by¹ [Klainerman (1987)]. However, when one attempts to tackle *coupled* systems of wave and Klein-Gordon equations like (1.1.1), one faces a major challenge due to the fact that one of the conformal Killing fields associated with the wave equation (the scaling vector field $t\partial_t + r\partial_r$) is *not conformal Killing* for the Klein-Gordon equation and, therefore, can no longer be used in the analysis.

[Katayama (2012a,b)] precisely succeeded to circumvent this difficulty and established a global existence theory for a class of coupled systems like (1.1.1) satisfying certain structural conditions. He relied on an alternative Sobolev-type estimate in place of the classical Klainerman-Sobolev inequality and an L^∞ - L^∞ estimate. We refer to Section 2.1, below, for further historical background.

The rest of this introduction is devoted to the main result we will establish by our method. Throughout, Latin indices a, b, c will take values within

¹The global existence problem for the nonlinear Klein-Gordon equation in dimension three was solved independently by [Klainerman (1985)] and [Shatah (1985)] by different methods.

1, 2, 3, while Greek indices α, β, γ denote 0, 1, 2, 3. Einstein's convention on repeated indices is in order.

We are interested in the Cauchy problem for the following broad class of **nonlinear systems of wave–Klein-Gordon equations**:

$$\begin{aligned} \square w_i + G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta w_j + c_i^2 w_i &= F_i(w, \partial w), \\ w_i(B+1, x) &= w_{i0}, \\ \partial_t w_i(B+1, x) &= w_{i1}, \end{aligned} \quad (1.1.2)$$

in which the unknowns are the functions w_i ($1 \leq i \leq n_0$) defined on Minkowski space \mathbb{R}^{3+1} . We assume the **symmetry conditions**

$$G_i^{j\alpha\beta} = G_j^{i\alpha\beta}, \quad G_i^{j\alpha\beta} = G_i^{j\beta\alpha} \quad (1.1.3)$$

and, for definiteness, the **wave-Klein-Gordon structure condition**

$$c_i \begin{cases} = 0, & 1 \leq i \leq j_0, \\ \geq \sigma, & j_0 + 1 \leq i \leq n_0, \end{cases} \quad (1.1.4)$$

where $\sigma > 0$ is a (constant, positive) lower bound for the mass coefficients of Klein-Gordon equations. We decompose the **curved metric**² $G_i^{j\alpha\beta}(w, \partial w)$ and the quadratic **interaction terms** $F_i(w, \partial w)$ in the form

$$G_i^{j\alpha\beta}(w, \partial w) = A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k + B_i^{j\alpha\beta k} w_k, \quad (1.1.5a)$$

$$F_i(w, \partial w) = P_i^{\alpha\beta j k} \partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha j k} w_k \partial_\alpha w_j + R_i^{j k} w_j w_k, \quad (1.1.5b)$$

in which, for simplicity in the presentation of the method and without genuine loss of generality, we assume that the coefficients $A_i^{j\alpha\beta\gamma k}$, $B_i^{j\alpha\beta k}$, $P_i^{\alpha\beta j k}$, $Q_i^{\alpha j k}$, and $R_i^{j k}$ are constants.

Then, in order to simplify the notation, we adopt the index convention that

- all indices i, j, k, l take the values $1, \dots, n_0$,
- all indices $\hat{i}, \hat{j}, \hat{k}, \hat{l}$ take the values $1, \dots, j_0$,
- and all indices $\check{i}, \check{j}, \check{k}, \check{l}$ take the values $j_0 + 1, \dots, n_0$.

It is convenient also to write

$$u_{\hat{i}} := w_{\hat{i}}$$

for the components satisfying wave equations or **wave components** for short, and to write

$$v_{\check{i}} := w_{\check{i}}$$

²More precisely, the underlying curved metric of the problem is the Minkowski metric plus this term.

for the components satisfying Klein-Gordon equations, also referred to as **Klein-Gordon components** for short.

Our main assumptions are the **null conditions for the wave components**

$$\begin{aligned} A_i^{\hat{j}\alpha\beta\gamma\hat{k}} \xi_\alpha \xi_\beta \xi_\gamma &= B_i^{\hat{j}\alpha\beta\hat{k}} \xi_\alpha \xi_\beta = P_i^{\alpha\beta\hat{j}\hat{k}} \xi_\alpha \xi_\beta = 0 \\ \text{whenever } (\xi_0)^2 - \sum_a (\xi_a)^2 &= 0, \end{aligned} \tag{1.1.5c}$$

and the **non-blow-up conditions** (imposing that only Klein-Gordon components have zero-order quadratic interactions and suppressing also other products which are known to lead to finite time blow-up)

$$B_i^{\check{j}\alpha\beta\hat{k}} = R_i^{j\hat{k}} = R_i^{\hat{j}k} = 0. \tag{1.1.5d}$$

Moreover, we impose that

$$Q_i^{\alpha j\hat{k}} = 0, \tag{1.1.5e}$$

which is our only genuine restriction in this monograph, but is necessary for the present implementation of the hyperboloidal foliation method. Namely, when this condition is violated, solutions do not have the decay and asymptotics of solutions to the linear wave or Klein-Gordon equations in Minkowski space; this issue is postponed to [LeFloch and Ma (2015)]. We emphasize that the null condition in (1.1.5c) is imposed for the quadratic forms associated with the *wave components*, only, and that no such restriction is required for the Klein-Gordon components.

We are now in a position to state the main result of the proposed theory.

Theorem 1.1 (Global well-posedness for nonlinear wave systems).

Consider the initial value problem (1.1.2) with smooth initial data posed on the spacelike hypersurface $\{t = B + 1\}$ of constant time and suppose these data are compactly supported in the ball $\{|x| \leq B\}$. Then, under the conditions (1.1.5), there exists a real $\epsilon_0 > 0$ such that, provided the initial data $w_{i0}, w_{i1} : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfy the smallness condition

$$\|w_{i0}\|_{\mathbf{H}^6(\mathbb{R}^3)} + \|w_{i1}\|_{\mathbf{H}^5(\mathbb{R}^3)} < \epsilon_0, \tag{1.1.6}$$

the Cauchy problem (1.1.2) admits a unique, smooth global-in-time solution (w_i) .

In the special case $n_0 = j_0$, the system (1.1.2) contains only wave equations and the statement in Theorem 1.1 reduces to the classical existence

result for quasilinear wave equations satisfying the null condition. Our method is somewhat simpler than the classical proof in this case.

An outline of this monograph is as follows. In Chapter 2, we introduce some basic notations on the hyperboloidal foliation and the associated energy, and we formulate our bootstrap assumptions. Chapter 3 is devoted to the properties of vector fields and commutators and their decompositions, which we will use throughout this book. In Chapter 4, we discuss the null condition in the proposed semi-hyperboloidal frame and we derive preliminary estimates on first- and second-order derivatives of the solutions. In Chapter 5, we present some technical tools and, especially, a global Sobolev inequality on hyperboloids and a Hardy-type inequality on hyperboloids. We continue in Chapter 6 with fundamental estimates in L^∞ and L^2 norms, which follows from our bootstrap assumptions. Chapter 7 is devoted to controlling the second-order derivatives of the wave components, while Chapter 8 deals with quadratic terms satisfying the null condition and also discusses the time decay of solutions. Next, in Chapter 9, we derive L^2 estimates on interaction terms, which completes our bootstrap argument. Furthermore, to illustrate the advantage of our new theory, in Chapter 10 we include a simple (but typical) example, that is, the application of the hyperboloidal foliation method to scalar quasilinear wave equations. In fact, the reader could start by reading this chapter, although some material from earlier chapters is used. Finally, the local existence theory is sketched in Chapter 11. Furthermore, the bibliography at the end of this book provides the reader with further material of interest.

1.2 General strategy of proof and scope of the theory

We present here several key features of our method of proof of Theorem 1.1, while referring to Chapter 2 below for the relevant notions (hyperboloidal foliation, bootstrap estimates, etc.).

- **Hyperboloidal foliation.**

Most importantly, in this book we propose to work with the *family of hyperboloids* which generate a foliation of the interior of the light cone in Minkowski spacetime. (Cf. (2.1.4), below). In contrast with other foliations of Minkowski space which are adopted in the literature, the hyperboloidal foliation has the advantage of being fully *geometric in nature* and is *invariant* under Lorentz transformations.

It is therefore quite natural to search for an estimate of the energy defined on these hypersurfaces, rather than the energy defined on flat hypersurfaces of constant time, as is classically done. This also suggests that our method should generalize to more general curved Lorentzian manifolds, as we will discuss in [LeFloch and Ma (2015)].

- **The semi-hyperboloidal frame.**

Furthermore, to the hyperboloidal foliation we attach a *semi-hyperboloidal frame* (as we call it), which consists of three vectors tangent to the hyperboloids and a timelike vector. (Following here [Hörmander (1997)], we do not use the normal to the hyperboloids.) This frame has several advantages in the analysis in comparison with, for instance, the ‘null frame’ which is often used in the literature and, instead, is defined from vectors tangent to the light cone. Importantly, the semi-hyperboloidal frame is *regular* (in the interior of the light cone, which is the region of interest), while the null frame is singular at the center ($\{r = 0\}$, say). This regularity allows us to bypass certain technical difficulties.

- **Adapted decomposition of the wave operator.**

We also introduce a decomposition of the wave operator \square with respect to the semi-hyperboloidal frame, which, the wave components, provides us with an expression of the second-order time derivatives $\partial_t \partial_t$ in terms of better-behaved derivatives. Cf. in Lemma 2.4, below.

- **The hyperboloidal energy.**

Our method takes advantage of the *full* expression of the energy flux induced on the hyperboloids, in order to estimate certain *weighted derivatives on the hyperboloids*. This is one of the keys which allows us to encompass the wave and the Klein-Gordon equations within a single framework. In contrast, [Hörmander (1997)] was only interested in controlling the zero-order term of the energy.

- **Global Sobolev inequality on hyperboloids.**

In order to establish uniform bounds on the solutions, we analyze various commutators of fields and operators arising in the problem and, then, apply suitable embedding theorems. In this direction, we rely here on a Sobolev inequality on hyperboloids, which was discovered by [Hörmander (1997)].

- **Global Hardy inequality on hyperboloids.**

Furthermore, we also need a new embedding estimate, that is, a

global Hardy inequality on hyperboloids, which we establish in this book and is essential in eventually deriving an L^∞ estimate on the metric coefficients. Section 5.2 for a precise statement.

- **Bootstrap strategy and hierarchy of energy bounds**

In the bootstrap formulation we introduce below, we do not impose a single bootstrap assumption as done in most of the literature on nonlinear wave equations, but we impose a hierarchy of assumptions. This rather involved bootstrap argument is found to be necessary in order to handle systems of coupled wave and Klein-Gordon equations: this is due to the fact that, for Klein-Gordon equations, the derivatives of different order admit different decay behavior and, therefore, different energy bounds.

We refer to Chapter 2 for further details and continue with several observations concerning the scope of Theorem 1.1.

1. First of all, in the course of the proof, we will show that the energy on the hyperboloids for the so-called wave components (that is, the quantity $E_{m,c_i}(s, Z^I u_{\hat{\tau}})$ defined in Section 2.1 below) remains *globally bounded* for all $|I| \leq 3$, that is, up to fourth-order derivatives. Hence, the wave components have not only “small amplitude” but also “small energy”. Other methods of proof for such problems often lead to *unbounded* energy. In contrast, in Chapter 2, we will actually establish that the “standard” flat energy (that is, the quantity $\|\partial_\alpha Z^I u_{\hat{\tau}}(t, \cdot)\|_{L^2(\mathbb{R}^3)}$ defined at the end of Chapter 2, below) is uniformly bounded for all times. Hence, in comparison with the standard method based on the “flat energy”, a much better bound is achieved with the help of the hyperboloidal energy.

2. We also emphasize that, in Theorem 1.1, the initial data belong to H^6 , while [Katayama (2012a)] assumes a very high regularity on the initial data (that is, 19 derivatives). In principle, H^4 would be the optimal regularity in order to work with a uniformly bounded metric and apply the vector field technique: namely, the metric term $G_i^{j\alpha\beta}(w, \partial w)$ in (1.1.2) involves second-order derivatives and, in the spatial dimension 3 under consideration, Sobolev’s embedding theorem $H^m \subset L^\infty$ holds provided $m > 3/2$. Allowing only integer exponents, we see that H^4 would be optimal.

3. Certain nonlinear interaction terms may lead to a *finite time blow-up* of the solutions, especially

$$u_{\hat{\tau}} u_{\hat{\tau}}, \quad u_{\hat{\tau}} v_{\hat{\tau}}, \quad u_{\hat{\tau}} \partial v_{\hat{\tau}}$$

and are, therefore, naturally excluded in Theorem 1.1.

4. We only impose the standard null condition. In contrast, a more restrictive null condition was imposed in [Katayama (2012a)] as certain interactions were excluded. Our method appears to be more robust and allow still further generalizations; cf. [LeFloch and Ma (2015)].

5. In short, by denoting by Q an arbitrary quadratic nonlinearity and by N an arbitrary null form and by using the notation u, v for arbitrary wave/Klein-Gordon components, the terms allowed in Theorem 1.1 for, both, wave and Klein-Gordon equations, are

$$Q(v, v), \quad Q(v, \partial\partial v), \quad Q(v, \partial\partial u) \quad Q(\partial v, \partial v), \\ Q(\partial v, \partial u), \quad Q(\partial v, \partial\partial v), \quad Q(\partial v, \partial\partial u),$$

while, in wave equations only, we impose the null condition on quadratic forms in ∂u and $\partial\partial u$, so that the following terms can be included

$$N(u, \partial\partial u), \quad N(\partial u, \partial u), \quad N(\partial u, \partial\partial u),$$

while, in Klein-Gordon equations, we can include the terms

$$Q(\partial u, \partial u), \quad Q(\partial u, \partial\partial u), \quad Q(u, \partial\partial u).$$

1.3 Earlier works

For a background on the vector field method and the global well-posedness for nonlinear wave equations, we refer to the textbooks [Hörmander (1997)] and [Sogge (2008)]. Additional background on nonlinear wave equations is found in [Strauss (1989)]. We do not attempt to review the large literature and only mention some selected results, while refereing the reader to the bibliography at the end of this monograph.

As already mentioned, the first results of global existence for nonlinear wave equations in three spatial dimensions were established by [Klainerman (1986, 1987)] and [Christodoulou (1986)] under the assumption that the nonlinearities satisfy the null condition and when the equation is posed in the flat Minkowski space. A (very) large literature is available for equations posed on curved spaces and, once more, we do not try to be exhaustive. We refer to [Lindblad (1990)], [Klainerman and Sideris (1996)], [Klainerman and Selberg (1997)], [Klainerman and Machedon (1997)], [Bahouri and Chemin (1999)], [Tataru (2000, 2001, 2002)], [Alinhac (2004, 2006)], [Lindblad and Rodnianski (2005)], and [Lindblad, Nakamura, and Sogge (2012)].

The Klein-Gordon equation on curved spaces was also studied in [Bachelot (1994, 2011)].

Since the decay of solutions to the (linear) Klein-Gordon equation is $t^{-d/2}$ in dimension $d \geq 1$, the decay function $t^{-d/2}$ is not integrable in dimension two and specific arguments are required in two dimensions: [Delort, Fang, and Xue (2004)] have treated quadratic quasilinear Klein-Gordon systems in two space dimensions and, more precisely, coupled systems of two equations with masses satisfying $m_1 \neq 2m_2$ and $m_2 \neq 2m_1$ with general nonlinearities. Furthermore, [Delort, Fang, and Xue (2004)] could treat the case of equality when the null condition is assumed. This work simplified and generalized (by including resonant cases) the earlier works by [Ozawa, Tsutaya and Tsutsumi (1995, 1996)], [Tsutsumi (2003a,b)] and [Sunagawa (2004)].

More recently, [Germain (2010)] revisited the global existence theory in dimension three for coupled Klein-Gordon equations with different speeds and systematically analyzed resonance effects. This problem was also studied by [Sideris and Tu (2001)] under the null condition.

Wave equations in one space dimension are treated by quite different methods; see [Delort (2001); Sunagawa (2003); Candy (2013)].

The problem of global existence for coupled systems of wave and Klein-Gordon equations have attracted much less attention so far in the literature. In addition to the references already quoted, let us mention [Bachelot (1988)] who first treated the Dirac-Klein-Gordon system and [Georgiev (1990, 1992)] who treated general systems but imposed a strong version of the null condition.

Results on the blow-up of solutions were established by [John (1979, 1981)] and, more recently, [Alinhac (2003)].

Hyperboloidal foliations were used first by [Friedrich (1981, 1983, 2002)] in order to establish a global existence result for the Einstein equations. His proof was based on a conformal transformation of the Einstein equations and an analysis of the regularity of its solutions at infinity. This research was motivated by earlier work by [Penrose (1963)] on the compactification of spacetimes. This idea was later developed by [Frauendiener (1998, 2002, 2004)] and [Rinne and Moncrief (2013)]. The importance of hyperboloidal foliations for general hyperbolic systems was emphasized in [Zenginoğlu (2008, 2011)] in order to numerically compute solutions within an unbounded domain. The general standpoint in these works is that, by compactification of the spacetime, one can conveniently formulate an ‘artificial’ outer boundary and numerically compute asymptotic properties of interest.

1.4 Examples and applications

The Maxwell-Klein-Gordon system

The theory presented in this book applies to many systems arising in mathematical physics; we present here a few of them. For instance, the Maxwell-Klein-Gordon system in Coulomb gauge takes the form of a system of non-linear wave equations for real-valued unknown A_j and a complex-valued field ϕ

$$\begin{aligned}\square A_j &= -\Im(\phi\overline{\partial_j\phi}) + |\phi|^2 A_j - \partial_j\partial_t A_0, \\ \square\phi &= 2\sqrt{-1}(-A^j\partial_j\phi + A_0\partial_t\phi) + \sqrt{-1}\partial_t A_0\phi + (A_\alpha A^\alpha + m^2)\phi,\end{aligned}\tag{1.4.1}$$

with auxiliary unknown A_0 given by

$$\Delta A_0 = -\Im(\phi\overline{\partial_t\phi}) + |\phi|^2 A_0,\tag{1.4.2}$$

supplemented with an elliptic constraint equation imposed on the initial data

$$\partial^j A_j = 0.\tag{1.4.3}$$

The Dirac-Klein-Gordon system

Consider next the following coupling between the Dirac equation and the wave or Klein-Gordon equation (with $m, \sigma \geq 0$):

$$\begin{aligned}-\sqrt{-1}\Gamma^\alpha\partial_\alpha\psi + m\psi &= \beta v\Gamma_0\Gamma_1\Gamma_2\Gamma_3\psi, \\ -\square v + \sigma v &= \psi^\dagger\Gamma_6\psi,\end{aligned}\tag{1.4.4}$$

in which the unknown are the (\mathbb{C}^4 -valued) spinor field ψ and the (real-valued) scalar field v . We have denoted by ψ^\dagger the complex conjugate transpose of ψ . Here, $\beta > 0$ is a coupling constant, while the 4×4 matrices Γ^α are the so-called Dirac matrices which are essentially characterized by the commutation conditions

$$\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha = 2I m_{\alpha\beta},\tag{1.4.5}$$

where $m_{\alpha,\beta}$ is the Minkowski metric $\text{diag}(-, 1, 1, 1)$ and I denotes the 4×4 identity matrix. It is a standard matter to deduce from the Dirac equation second-order wave/ Klein-Gordon equations for real-valued unknowns. This is obtained by composing the Dirac operator with itself (since, roughly speaking, the Dirac operator is the “square-root” of the wave operator) and considering real and imaginary parts of ψ .

The Einstein equations

Although our theory in its present form does not directly apply to the **Einstein equations** of general relativity

$$G_{\alpha\beta} = T_{\alpha\beta}, \tag{1.4.6}$$

it however is directly motivated by this system. The left-hand side $G_{\alpha\beta}$ of (1.4.6) is the Einstein tensor of a spacetime (M, g) , that a Lorentzian $(3 + 1)$ -dimensional manifold, while the right-hand side $T_{\alpha\beta}$ denotes the energy momentum tensor of a matter field, which in our context can be assumed to be a set of massless and massive scalar fields. We refer the reader to the follow-up work [LeFloch and Ma (2015)].

Chapter 2

The hyperboloidal foliation and the bootstrap strategy

2.1 The hyperboloidal foliation and the Lorentz boosts

We will work with the foliation of the interior of the light cone in Minkowski spacetime \mathbb{R}^{3+1} , defined as below.

We introduce the **hyperboloidal hypersurfaces**

$$\mathcal{H}_s := \{(t, x) / t > 0; t^2 - |x|^2 = s^2\} \quad (2.1.1)$$

with hyperbolic radius $s > 0$, where $(t, x) = (t, x^j) = (t, x^1, x^2, x^3)$ denote Cartesian coordinates. We will also write $r^2 := |x|^2 = \sum_i (x^i)^2$. We also introduce the (interior of the half) **cone**

$$\mathcal{K} := \{(t, x) / |x| < t - 1\} \quad (2.1.2)$$

and, with $s_0 = s_0 > 1$, the **truncated cone** region limited by two hyperboloids

$$\begin{aligned} \mathcal{K}_{[s_0, s_1]} &:= \{(t, x) / |x| < t - 1, \quad (s_0)^2 \leq t^2 - |x|^2 \leq (s_1)^2, t > 0\} \\ &= \bigcup_{s_0 \leq s \leq s_1} (\mathcal{H}_s \cap \mathcal{K}). \end{aligned} \quad (2.1.3)$$

This set is thus limited by two hyperboloids and can be naturally foliated by hyperboloids. See Figure 2.1 for a display of the set $\mathcal{K}_{[s_0, s_1]}$.

By taking $s_1 = +\infty$, we can also use the notation

$$\begin{aligned} \mathcal{K}_{[s_0, +\infty)} &:= \{(t, x) / |x| < t - 1, \quad (s_0)^2 \leq t^2 - |x|^2\} \\ &= \bigcup_{s \geq s_0} (\mathcal{H}_s \cap \mathcal{K}). \end{aligned} \quad (2.1.4)$$

We refer to Figure 2.2 for an illustration of the set $\mathcal{K}_{[2, +\infty)}$.

In the following, we will be interested in functions supported in the cone $\mathcal{K}_{[s_0, +\infty)}$ (for instance in solutions to the Klein-Gordon equation (2.1.5),

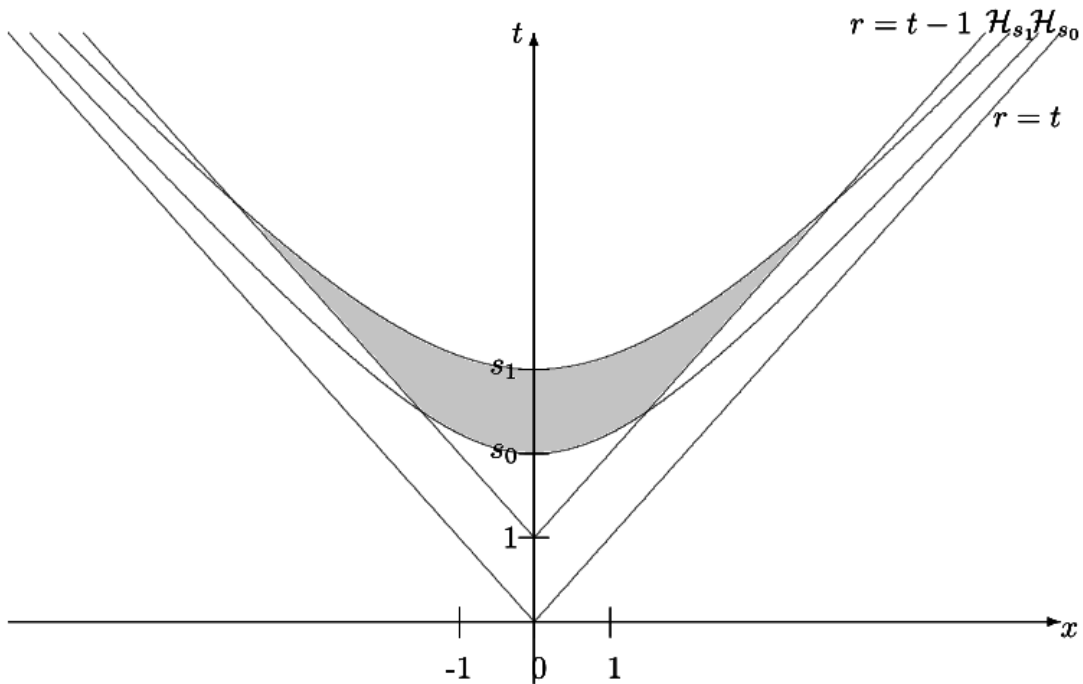


Fig. 2.1 $\mathcal{K}_{[s_0, s_1]}$

below) which is defined in the cone $\mathcal{K}_{[s_0, +\infty)}$ and vanishing on the light cone $\{t > 0, |x| = t - 1\}$.

Remark 2.1. The estimate $t \leq \frac{2\sqrt{3}}{3}s$ holds in the region $\mathcal{K} \cap \{|x| \leq t/2\}$ —a region of interest in our estimates below.

It is convenient to consider first the **Klein-Gordon equation**

$$\begin{aligned} \square u + \sigma^2 u &= f, \\ u(t, x)|_{\mathcal{H}_{s_0}} &= u_0(t, x), \quad u_t(t, x)|_{\mathcal{H}_{s_0}} = u_1(t, x), \end{aligned} \tag{2.1.5}$$

with given $\sigma, s_0 = B + 1 > 0$. Recall that the symbol \square denotes the wave operator in Minkowski spacetime. In (2.1.5), the initial data u_0, u_1 are prescribed and compactly supported in the ball $\{|x| \leq B\}$ of radius $B = s_0 - 1$. The source-term function f is defined in $\mathcal{K}_{[s_0, +\infty)}$, so that, by the principle of propagation at finite speed, the solution $u = u(t, x)$ to (2.1.5) is also defined in $\mathcal{K}_{[s_0, +\infty)}$ and vanishes on the light cone $\{t > 0, |x| = t - 1\}$. In the same manner, the solution of our main system (1.1.2) studied in this monograph is also defined in $\mathcal{K}_{[s_0, +\infty)}$ and vanishes on the light cone.

Furthermore, we introduce the hyperbolic rotations or **Lorentz boosts** (by rising and lowering the indices with the Minkowski metric)

$$L_a := -x_a \partial_0 + x_0 \partial_a = x^a \partial_t + t \partial_a, \tag{2.1.6}$$

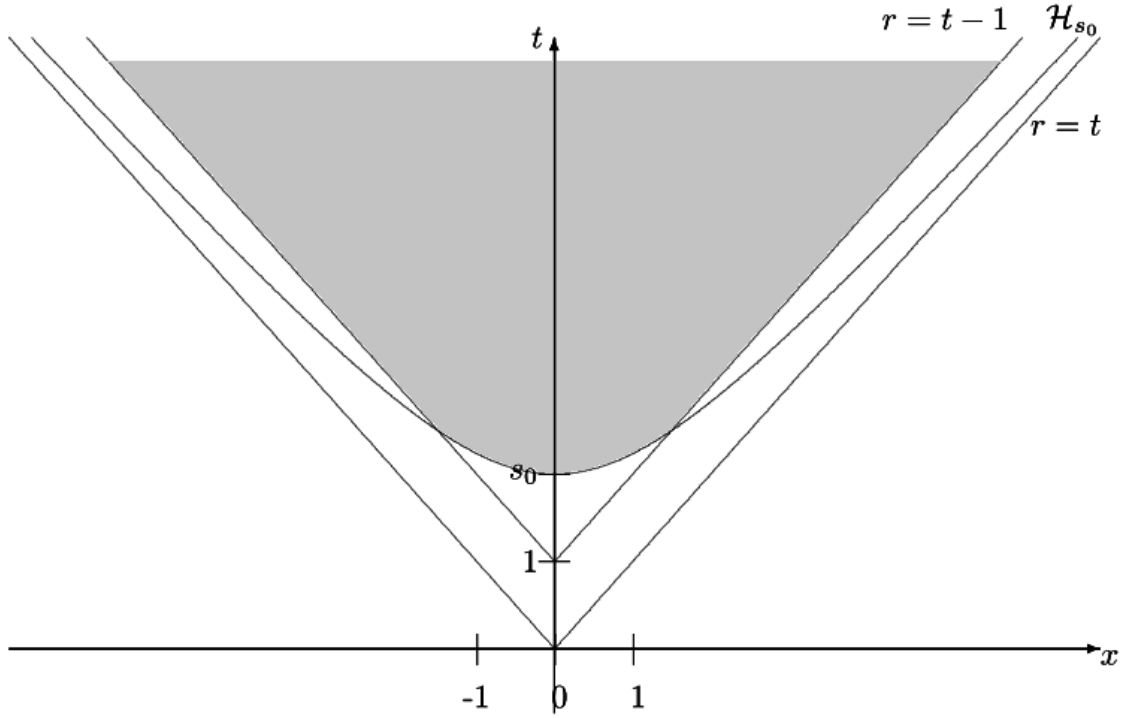


Fig. 2.2 $\mathcal{K}_{[s_0, +\infty)}$

and we denote by \mathcal{L} the family of **admissible vector fields** consisting, by definition, of all vectors

$$Z_\alpha := \partial_\alpha, \quad Z_{3+a} := L_a. \quad (2.1.7)$$

Observe that, for any $Z, Z' \in \mathcal{L}$, the Lie bracket $[Z, Z']$ also belongs to \mathcal{L} , so that this set is a Lie algebra. For any multi-index $I = (\alpha_1, \alpha_2, \dots, \alpha_m)$ of length $|I| := m$, we denote by Z^I the m -th order differential operator $Z^I := Z_{\alpha_1} \dots Z_{\alpha_m}$. We also denote by ∂^I the m -th order derivative operator $\partial^I := \partial_{\alpha_1} \partial_{\alpha_2} \dots \partial_{\alpha_m}$ (here $0 \leq \alpha_i \leq 3$) and L^I the m -th order derivative operator $L^I := L_{\alpha_1} L_{\alpha_2} \dots L_{\alpha_m}$ (here $4 \leq \alpha_j \leq 7$).

Since we will be working within the cone \mathcal{K} , we have $|x^a/t| \leq 1$ in \mathcal{K} and therefore the **spatial rotations**

$$\Omega_{ab} := x^a \partial_b - x^b \partial_a \quad (2.1.8)$$

need not be included here since they can be recovered from \mathcal{L} , namely

$$\Omega_{ab} = \frac{x^a}{t} L_b - \frac{x^b}{t} L_a. \quad (2.1.9)$$

Observe that, within the cone \mathcal{K} , the coefficients x^a/t are smooth, bounded, and homogeneous of degree zero (see the definition below in lemma 2.2).

Now; we study the energy associated with the hyperboloidal foliation. Using $\partial_t u$ as multiplier for the equation (2.1.5), it is easy to derive the following **energy inequality** for all $s_1 \geq s_0$:

$$(E_{m,\sigma}(s_1, u))^{1/2} \leq (E_{m,\sigma}(s_0, u))^{1/2} + \int_{s_0}^{s_1} \left(\int_{\mathcal{H}_s} f^2 dx \right)^{1/2} ds, \quad (2.1.10)$$

where $dx = dx^1 dx^2 dx^3$ and¹ the **energy on the hyperboloids** is defined as

$$E_{m,\sigma}(s_1, u) := \int_{\mathcal{H}_{s_1}} \left(\sum_{i=1}^3 ((x^i/t)\partial_t u + \partial_i u)^2 + ((s_1/t)\partial_t u)^2 + au^2 \right) dx \quad (2.1.11)$$

with $a > 0$. Then, [Hörmander (1997)], established a Sobolev-type estimate adapted to this inequality (cf. Lemma 7.6.1 therein or (5.1.1) in chapter 5) and arrived at the L^∞ estimate

$$\begin{aligned} \sup_{\mathcal{H}_{s_1}} t^{3/2} |u| &\leq C \sum_{|I| \leq 2} E_{m,\sigma}(s_1, Z^I u)^{1/2} \\ &\leq C E_{m,\sigma}(s_0, Z^I u)^{1/2} + C \sum_{|I| \leq 2} \int_{s_0}^{s_1} \left(\int_{\mathcal{H}_s} Z^I f^2 dx \right)^{1/2} ds. \end{aligned} \quad (2.1.12)$$

Most importantly, this calculation yields the optimal rate of decay $t^{-3/2}$ for solutions to the Klein-Gordon equation.

2.2 Semi-hyperboloidal frame

Our analysis in the present work is based on the **semi-hyperboloidal frame** as we call it, which is defined by

$$\underline{\partial}_0 := \partial_t, \quad \underline{\partial}_a := t^{-1} L_a = (x^a/t)\partial_t + \partial_a, \quad (2.2.1)$$

while standard methods rely on the so-called null frame (containing two null vectors tangent to the light cone). Note that the transition matrix between the semi-hyperboloidal frame and the natural frame ∂_β is given by $\underline{\partial}_\alpha = \Phi_\alpha^\beta \partial_\beta$ and $\partial_\alpha = \Psi_\alpha^\beta \underline{\partial}_\beta$ with

$$\Phi := \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix}, \quad \Psi := \Phi^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \quad (2.2.2)$$

¹The subscript refers to the Minkowski metric.

Let us emphasize here an advantage of the semi-hyperboloidal frame: the “bad” direction $\underline{\partial}_0$ (along which the derivative of solutions to the wave equation decrease slower than the tangent directions of \mathcal{H}_s) is a unit time-like vector (while in the null frame the “bad” direction is a null vector) and, with our choice of frame, the transition matrices Φ and Ψ are *smooth within the cone* \mathcal{K} .

In the following, we adopt the following notation and convention. In order to express the components of a tensor in a frame, we always use Roman font with upper and lower indices for its components in the natural frame, while we use underlined Roman font for its components in the semi-hyperboloidal frame. In other words, the coordinates of a tensor T (defined in \mathcal{K}) are denoted by $T^{\alpha\beta}$ in the frame ∂_α and by $\underline{T}^{\alpha\beta}$ in the frame $\underline{\partial}_\alpha$. By observing that $|\partial_\alpha \Phi_\beta^\gamma| \leq 1$ in \mathcal{K} , we obtain the following estimate.

Lemma 2.1. *For any two-tensor T defined in the cone \mathcal{K} and for all indices α, β, I and admissible field Z , one has*

$$|Z^I \underline{T}^{\alpha\beta}| \lesssim \sum_{\substack{\alpha', \beta' \\ |I'| \leq |I|}} |Z^{I'} T^{\alpha' \beta'}| \quad \text{in } \mathcal{K}.$$

The proof of this estimate relies on the following lemma:

Lemma 2.2 (Homogeneity lemma). *Let $F = F(t, x)$ be a C^∞ function defined in the closed region $\{t \geq 1, |x| \leq t\}$ and assumed to be homogeneous of degree η in the first variable, in the sense that:*

$$f(t, x) = t^\eta f(1, x/t) \quad \text{in the region } \{t \geq 1, |x| \leq t\}. \quad (2.2.3)$$

Then, for all multi-indices I_1, I_2 , the following estimate holds for some positive constant $C(n, |I_1|, |I_2|, f)$:

$$|\partial^{I_1} Z^{I_2} f(t, x)| \leq C(n, |I_1|, |I_2|, f) t^{-|I_1|+\eta} \quad \text{in } \mathcal{K} = \{|x| < t - 1\}. \quad (2.2.4)$$

Proof. By differentiating the relation $f(t, x) = t^\eta f(1, x/t)$ with respect to x^a we find

$$\partial_a f(t, x) = t^{\eta-1} \partial_a f(1, x/t)$$

and, with respect to t ,

$$\partial_t f(t, x) = -t^{\eta-1} (x^a/t) \partial_a f(1, x/t) + \eta t^{\eta-1} f(1, x/t).$$

Hence, $\partial_a f$ and $\partial_t f$ are homogeneous of degree $(\eta - 1)$. Observe next that

$$L_a f(t, x) = -t^\eta (x^a/t) (x^b/t) \partial_b f(1, x/t) + \eta t^\eta f(1, x/t) + t^\eta \partial_a f(1, x/t)$$

is also homogeneous of degree η .

We observe that if some functions f_i are homogeneous of degree η_i (in their first variable), then the product $\prod_i f_i$ is homogeneous of degree $\sum_i \eta_i$ and then any $Z^I f$ is homogeneous.

We claim that the degree of homogeneity of $Z^I f$ is not higher than η . This follows easily by induction. Namely, this is clear when $|I| = 1$. Moreover, assume that for all $|I| \leq m$, $Z^I f$ is homogeneous of degree $\eta' \leq \eta$, then we now check the same property for all $|I| = m + 1$. Namely, assume that $Z^I = Z_1 Z^{I'}$ with $|I'| = m$. When $Z_1 = \partial_\alpha$, then $Z^I f = \partial_\alpha(Z^{I'} f)$ and we observe that $Z^{I'} f$ is a homogeneous of degree $\eta' \leq \eta$. Then we see that $\partial_\alpha(Z^{I'} f)$ is again homogeneous of degree $\eta' - 1 < \eta' \leq \eta$. When $Z_1 = L_a$, then $L_a(Z^{I'} f)$ is homogeneous of degree $\eta' \leq \eta$. This completes the induction argument that $Z^I f$ is homogeneous of degree at most η .

Furthermore, observe that if f is homogeneous of degree η , then $\partial^{I_1} f$ is homogeneous of degree $\eta - |I_1|$. This is so since $\partial_\alpha f$ is homogeneous of degree $\eta - 1$. So $\partial^{I_1} Z^{I_2} f$ is homogeneous of degree $\eta - |I_1|$ and is C^∞ within the cone \mathcal{K} .

Now, in order to establish the estimate (2.2.4), we perform the following calculation. Given $(t, x) \in \mathcal{K}$, we have

$$\partial^{I_1} Z^{I_2} f(t, x) = t^{-|I_1| + \eta} \partial^{I_1} Z^{I_2} f(1, x/t)$$

and thus

$$|\partial^{I_1} Z^{I_2} f(t, x)| = t^{-|I_1| + \eta} |\partial^{I_1} Z^{I_2} f(1, x/t)|.$$

We observe that, by a continuity argument in the compact set $\{t = 1, |x| \leq 1\}$, there exists a positive constant $C(n, |I_1|, |I_2|)$ such that

$$\sup_{|x| \leq 1} |\partial^{I_1} Z^{I_2} f(1, x)| \leq C(n, |I_1|, |I_2|)$$

and the desired result is proven. \square

We now estimate the coefficients of the matrices Φ and Ψ . First, we recall the following property of homogeneous functions. We observe that x^a/t is obviously C^∞ in the closed region $\{t \geq 1, |x| \leq t\}$.

Lemma 2.3 (Changes of frame). *With the notation above, the following two estimates hold for all multi-indices I_1, I_2 :*

$$|\partial^{I_1} Z^{I_2} \Phi_\alpha^\beta| \leq C(n, |I_1|, |I_2|) t^{-|I_1|} \quad \text{in } \mathcal{K}. \quad (2.2.5)$$

$$|\partial^{I_1} Z^{I_2} \Psi_\alpha^\beta| \leq C(n, |I_1|, |I_2|) t^{-|I_1|} \quad \text{in } \mathcal{K}. \quad (2.2.6)$$

Proof of lemma 2.1. Recall that

$$\underline{T}^{\alpha\beta} = T^{\alpha'\beta'} \Psi_{\alpha'}^{\alpha} \Psi_{\beta'}^{\alpha}.$$

Then

$$Z^I \underline{T}^{\alpha\beta} = Z^I (T^{\alpha'\beta'} \Psi_{\alpha'}^{\alpha} \Psi_{\beta'}^{\alpha}) = \sum_{I_1+I_2=I} Z^{I_1} (\Psi_{\alpha'}^{\alpha} \Psi_{\beta'}^{\alpha}) Z^{I_2} T^{\alpha'\beta'}.$$

Then

$$\begin{aligned} |Z^I \underline{T}^{\alpha\beta}| &\leq \sum_{I_1+I_2=I} |Z^{I_1} (\Psi_{\alpha'}^{\alpha} \Psi_{\beta'}^{\alpha})| |Z^{I_2} T^{\alpha'\beta'}| \\ &\leq \sum_{|I_2| \leq |I|} |Z^{I_2} T^{\alpha'\beta'}| \end{aligned}$$

where the fact

$$|Z^{I_1} (\Psi_{\alpha'}^{\alpha} \Psi_{\beta'}^{\alpha})| \leq C(|I_1|)$$

is applied, which is a direct result of (2.2.6). \square

More generally, any second-order differential operator $T^{\alpha\beta} \partial_{\alpha} \partial_{\beta}$ can be written in the semi-hyperboloidal frame so that, for arbitrary functions u , we can write $T^{\alpha\beta} = \underline{T}^{\alpha'\beta'} \Phi_{\beta'}^{\beta} \Phi_{\alpha'}^{\alpha}$ (with obvious notation) and

$$T^{\alpha\beta} \partial_{\alpha} \partial_{\beta} u = \underline{T}^{\alpha\beta} \underline{\partial}_{\alpha} \underline{\partial}_{\beta} u - \underline{T}^{\alpha\beta} (\underline{\partial}_{\alpha} \Psi_{\beta}^{\beta'}) \partial_{\beta'} u. \quad (2.2.7)$$

In particular, for the wave operator \square , we obtain

$$\square u = \underline{m}^{\alpha\beta} \underline{\partial}_{\alpha} \underline{\partial}_{\beta} u - \underline{m}^{\alpha\beta} (\underline{\partial}_{\alpha} \Psi_{\beta}^{\beta'}) \partial_{\beta'} u,$$

where $\underline{m}^{\alpha\beta}$ are the components of Minkowski metric tensor in the semi-hyperboloidal frame. Then, a simple calculation gives $\underline{m}^{00} = (t^2 - r^2)/t^2 = s^2/t^2$ and leads us to the following important identity.

Lemma 2.4 (Decomposition of the wave operator). *In the semi-hyperboloidal frame, the wave operator admits the decomposition*

$$\begin{aligned} (s/t)^2 \underline{\partial}_0 \underline{\partial}_0 u &= \square u - \underline{m}^{0a} \underline{\partial}_0 \underline{\partial}_a u - \underline{m}^{a0} \underline{\partial}_a \underline{\partial}_0 u - \underline{m}^{ab} \underline{\partial}_{ab} u \\ &\quad + \underline{m}^{\alpha\beta} (\underline{\partial}_{\alpha} \Psi_{\beta}^{\beta'}) \partial_{\beta'} u, \end{aligned} \quad (2.2.8)$$

where $\underline{m}^{0a} = x^a/t$ and $\underline{m}^{ab} = \delta_a^b$.

2.3 Energy estimate for the hyperboloidal foliation

As presented in Chapter 1, we are interested in the following class of wave-Klein-Gordon type systems

$$\begin{aligned} \square w_i + G_i^{j\alpha\beta} \partial_{\alpha\beta} w_j + c_i^2 w_i &= F_i, \\ w_i|_{\mathcal{H}_{s_0}} &= w_{i0}, \quad \partial_t w_i|_{\mathcal{H}_{s_0}} = w_{i1}, \end{aligned} \quad (2.3.1)$$

with unknowns w_i ($1 \leq i \leq n_0$), where the **metric** (with some abuse of notation) $G_i^{j\alpha\beta}$ and the source-terms F_i are defined in the cone \mathcal{K} , and w_{i0} , w_{i1} are supported on the initial hypersurface $\mathcal{H}_{s_0} \cap \mathcal{K}$. To guarantee the hyperbolicity, we assume the symmetry property (1.1.3) and we also assume that the constants c_i satisfy (1.1.4).

We introduce the following energy associated with the Minkowski metric on each hyperboloid \mathcal{H}_s :

$$\begin{aligned} &E_{m,c_i}(s, w_i) \\ &:= \int_{\mathcal{H}_s} \left(|\partial_t w_i|^2 + \sum_a |\partial_a w_i|^2 + (2x^a/t) \partial_t w_i \partial_a w_i + c_i^2 w_i^2 \right) dx \\ &= \int_{\mathcal{H}_s} \left(c_i^2 w_i^2 + \sum_a |\underline{\partial}_a w_i|^2 + ((s/t) \partial_t w_i)^2 \right) dx \\ &= \int_{\mathcal{H}_s} \left(c_i^2 w_i^2 + \sum_a ((s/t) \partial_a w_i)^2 + t^{-2} (S w_i)^2 + t^{-2} \sum_{a \neq b} |\Omega_{ab} w_i|^2 \right) dx, \end{aligned} \quad (2.3.2)$$

where we use the notation (2.1.8) and

$$t^{-1} S := \partial_t + \sum_a \partial_a (x^a/t). \quad (2.3.3)$$

When $c_i = 0$, we write $E_m(s, w_i) := E_{m,0}(s, w_i)$ for short.

On the other hand, the **curved energy** associated with the principal part of (2.3.1) is defined as

$$\begin{aligned} E_{G,c_i}(s, w_i) &:= E_{m,c_i}(s, w_i) + 2 \int_{\mathcal{H}_s} (\partial_t w_i \partial_\beta w_j G_i^{j\alpha\beta}) \cdot (1, -x^a/t) dx \\ &\quad - \int_{\mathcal{H}_s} (\partial_\alpha w_i \partial_\beta w_j G_i^{j\alpha\beta}) dx, \end{aligned} \quad (2.3.4)$$

where the second term in the right-hand-side involves the Euclidian inner product of the vectors $(\partial_t w_i \partial_\beta w_j G_i^{j\alpha\beta})$ and $(1, -x^a/t)$.

Proposition 2.1 (Energy estimate for the hyperboloidal foliation).

Let $(w_i)_{1 \leq i \leq n_0}$ be the (local-in-time) solution to (2.3.1) and suppose that,

on some interval $[s_0, s_1]$, the following coercivity conditions are satisfied for some constant $\kappa_1 > 1$ by the "metric" G

$$\kappa_1^{-1} \sum_i E_{m,c_i}(s, w_i) \leq \sum_i E_{G,c_i}(s, w_i) \leq \kappa_1 \sum_i E_{m,c_i}(s, w_i), \quad (2.3.5)$$

$$\left| \int_{\mathcal{H}_s} \frac{s}{t} \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j \right) dx \right| \leq M(s) E_{m,c_i}(s, w_i)^{1/2}, \quad (2.3.6)$$

and the sources satisfy the bound

$$\sum_i \|F_i\|_{L^2(\mathcal{H}_s)} dx \leq L(s) \quad (2.3.7)$$

for some locally integrable functions $L, M \geq 0$. Then the following energy estimate holds (for all $s \in [s_0, s_1]$):

$$\begin{aligned} & \left(\sum_i E_{m,c_i}(s, w_i) \right)^{1/2} \\ & \leq \kappa_1^2 \left(\sum_i E_{m,c_i}(s_0, w_i) \right)^{1/2} + C \kappa_1^2 \int_{s_0}^s (L(\tau) + M(\tau)) d\tau, \end{aligned} \quad (2.3.8)$$

where the constant $C > 0$ depends on the the structure of the system (2.3.1) and is independent on the initial data.

The following remarks are in order:

- In view of (2.3.2) and (2.3.8), the L^2 norm of $\underline{\partial}_a w$ and $(s/t)\partial_\alpha w$ are uniformly controlled on each hypersurface \mathcal{H}_s . It is expected that these derivatives enjoy better decay than $\partial_\alpha w$ itself.
- Note also that the term $\sum_a (x^a/r)((r/t)\partial_a + (x^a/r)\partial_t)w = t^{-1}Sw$ so that $t^{-1}Sw$ is also a "good" term (although we will not use this term directly in the following analysis).
- In Chapter 10, we will see that the energy on the initial hyperboloid $E_m(s_0, w_i)$ is controlled by the \mathbf{H}^1 norm of the initial data on the initial slice $t = s_0$.

Proof. In view of the symmetry (1.1.3) and by using $\partial_t w_i$ as a multiplier, we easily derive the energy identity

$$\begin{aligned} & \sum_i \left(\frac{1}{2} \partial_t \sum_\alpha (\partial_\alpha w_i + c_i^2 w_i^2)^2 + \sum_a \partial_a (\partial_a w_i \partial_t w_i) \right. \\ & \quad \left. + \partial_\alpha (G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j) - \frac{1}{2} \partial_t (G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j) \right) \\ & = \sum_i \partial_t w_i F_i + \sum_i \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j \right). \end{aligned}$$

We integrate this identity over the region $\mathcal{K}_{[s_0, s]}$ and use Stokes' formula. Note that by the property of propagation at finite speed, the solution (w_i) is defined in $\mathcal{K}_{[s_0, +\infty)}$ and vanishing on the light cone. We obtain

$$\begin{aligned}
& \frac{1}{2} \sum_i (E_{G, c_i}(s, w_i) - E_{G, c_i}(s_0, w_i)) \\
&= \sum_i \int_{\mathcal{K}_{[s_0, s]}} \left(\partial_t w_i F_i + \partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j \right) dt dx \\
&= \sum_i \int_{s_0}^s d\tau \int_{\mathcal{H}_\tau} (\tau/t) \partial_t w_i F_i dx \\
&\quad + \sum_i \int_{s_0}^s d\tau \int_{\mathcal{H}_\tau} \left((\tau/t) \partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j \right) dx,
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{d}{ds} \sum_i E_{G, c_i}(s, w_i) &= 2 \sum_i \int_{\mathcal{H}_s} \left((s/t) \partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j \right. \\
&\quad \left. - (s/2t) \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j + (s/t) \partial_t w_i F_i \right) dx
\end{aligned}$$

So, with the assumptions (2.3.6) and (2.3.7), we get

$$\begin{aligned}
& \left(\sum_i E_{G, c_i}(s, w_i) \right)^{1/2} \frac{d}{ds} \left(\sum_i E_{G, c_i}(s, w_i) \right)^{1/2} \\
&\leq \sum_i \left(\int_{\mathcal{H}_s} |F_i|^2 dx \right)^{1/2} E_{m, c_i}(s, w_i)^{1/2} + M(s) \sum_i E_{m, c_i}(s, w_i)^{1/2} \\
&\leq C\kappa_1 \left(\sum_i \int_{\mathcal{H}_s} |F_i|^2 dx \right)^{1/2} \left(\sum_i E_{G, c_i}(s, w_i) \right)^{1/2} + C\kappa_1 M(s) \sum_i E_{G, c_i}(s, w_i)^{1/2} \\
&\leq C\kappa_1 L(s) \left(\sum_i E_{G, c_i}(s, w_i) \right)^{1/2} + C\kappa_1 M(s) \left(\sum_i E_{G, c_i}(s, w_i) \right)^{1/2},
\end{aligned}$$

which yields

$$\frac{d}{ds} \left(\sum_i E_{G, c_i}(s, w_i) \right)^{1/2} \leq C\kappa_1 (L(s) + M(s)).$$

We integrate this inequality over the interval $[s_0, s]$ and obtain

$$\begin{aligned}
& \left(\sum_{i=1}^{n_0} E_{G, c_i}(s, w_i) \right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n_0} E_{G, c_i}(s_0, w_i) \right)^{1/2} + C\kappa_1 \int_{s_0}^s L(\tau) + C\kappa_1 \int_{s_0}^s M(\tau) d\tau.
\end{aligned}$$

By using the condition (2.3.5), we find

$$\sum_i E_{m,c_i}(s, w_i) \leq \kappa_1 \sum_i E_{G,c_i}(s, w_i) \leq \kappa_1^2 \sum_i E_{m,c_i}(s, w_i),$$

which completes the proof. \square

2.4 The bootstrap strategy

We are now in a position to outline our method of proof. It will be necessary to distinguish between three levels of regularity and, in order to describe this *scale of regularity*, we will use the following convention on the indices in use:

$$\begin{aligned} I^\sharp &: \text{multi-index of order } \leq 5, \\ I^\dagger &: \text{multi-index of order } \leq 4, \\ I &: \text{multi-index of order } \leq 3, \end{aligned} \tag{2.4.1}$$

which we call **admissible indices**. Throughout, we denote by C a constant depending only on the structural constants of the system (1.1.2), which are j_0 , k_0 , B , and σ . We emphasize that these constants do not depend on $\epsilon \in (0, 1]$.

It is convenient to introduce the norm of a function on the hyperboloids, so if u is a function supported in the half-cone \mathcal{K} , we set

$$\|u\|_{L^p(\mathcal{H}_s)} := \left(\int_{H_s} |u(t, x)|^p dx \right)^{1/p} = \left(\int_{\mathbb{R}^3} |u(\sqrt{s^2 + |x|^2}, x)|^p dx \right)^{1/p}. \tag{2.4.2}$$

The proof of Theorem 1.1 relies on Propositions 2.2 and 2.4, stated now. The first proposition below will be established in Chapter 11.

Proposition 2.2 (Initialization of the argument). *For any sufficiently large constant $C_0 > 0$, there exists a positive constant $\epsilon'_0 \in (0, 1)$ depending only on B and C_0 such that, for every initial data satisfying*

$$\sum_i \|w_{i0}\|_{H^6(\mathbb{R}^3)} + \|w_{i1}\|_{H^5(\mathbb{R}^3)} \leq \epsilon'_0, \tag{2.4.3}$$

a corresponding solution exists in the region limited by the constant time hypersurface $t = B + 1$ and to hyperboloid at the time $s_0 = B + 1$, which also satisfies the uniform bound

$$\sum_j E_m(s_0, Z^{I^\sharp} w_j)^{1/2} \leq C_0 \left(\sum_i \|w_{i0}\|_{H^6(\mathbb{R}^3)} + \|w_{i1}\|_{H^5(\mathbb{R}^3)} \right) \tag{2.4.4}$$

for all admissible vector fields Z and indices I^\sharp .

Given some constants, $C_1, \epsilon > 0$, $1/6 > \delta > 0$ and a hyperbolic time interval $[s_0, s_1]$, we call **hierarchy of energy bounds** the following conditions (for all $s \in [s_0, s_1]$ and all admissible fields and indices):

$$E_{m,\sigma}(s, Z^{I^\sharp} v_{\check{j}})^{1/2} \leq C_1 \epsilon s^\delta \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \quad (2.4.5a)$$

$$E_m(s, Z^{I^\sharp} u_{\hat{i}})^{1/2} \leq C_1 \epsilon s^\delta \quad \text{for } 1 \leq \hat{i} \leq j_0, \quad (2.4.5b)$$

$$E_{m,\sigma}(s, Z^{I^\dagger} v_{\check{j}})^{1/2} \leq C_1 \epsilon s^{\delta/2} \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \quad (2.4.5c)$$

$$E_m(s, Z^{I^\dagger} u_{\hat{i}})^{1/2} \leq C_1 \epsilon s^{\delta/2} \quad \text{for } 1 \leq \hat{i} \leq j_0, \quad (2.4.5d)$$

$$E_m(s, Z^I u_{\hat{i}})^{1/2} \leq C_1 \epsilon \quad \text{for } 1 \leq \hat{i} \leq j_0. \quad (2.4.5e)$$

Observe that (2.4.5e) is specific to the wave components. We also remark that, because $c_{\check{i}} \geq \sigma > 0$ for $j_0 + 1 \leq \check{i} \leq n + 0$,

$$E_{m,\sigma}(s, Z^{I^\sharp} v_{\check{i}}) \leq E_{m,c_{\check{i}}}(s, Z^{I^\sharp} v_{\check{i}}) \leq (c_{\check{i}}/\sigma) E_{m,c_{\check{i}}}(s, Z^{I^\sharp} v_{\check{i}}), \quad (2.4.6)$$

which means these two energy are equivalent.

Given some constants $\kappa_1 > 1$, $C_1, \epsilon > 0$ and a hyperbolic time interval $[s_0, s_1]$, we call **hierarchy of metric and source bounds**² the following three sets of estimates:

- For all $|I^\sharp| \leq 5$,

$$\kappa_1^{-1} E_{G,c_i}(s, Z^{I^\sharp} w_i) \leq \sum_i E_m(s, Z^{I^\sharp} w_i) \leq \kappa_1 \sum_i E_{G,c_i}(s, Z^{I^\sharp} w_i), \quad (2.4.7a)$$

$$\left| \int_{\mathcal{H}_s} \frac{s}{t} \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t Z^{I^\sharp} w_i \partial_\beta Z^{I^\sharp} w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha Z^{I^\sharp} w_i \partial_\beta Z^{I^\sharp} w_j \right) dx \right| \leq M(I^\sharp, s) E_{m,c_i}(s, Z^{I^\sharp} w_i) := C(C_1 \epsilon)^2 s^{-1+\delta} E_{m,c_i}(s, Z^{I^\sharp} w_i) \quad (2.4.7b)$$

and (with the notation (2.4.2))

$$\sum_{i=1}^{n_0} \left\| [G_i^{j\alpha\beta} \partial_\alpha \partial_\beta, Z^{I^\sharp}] w_j \right\|_{L^2(\mathcal{H}_s)} + \sum_{i=1}^{n_0} \left\| Z^{I^\sharp} F_i \right\|_{L^2(\mathcal{H}_s)} \leq L(I^\sharp, s) := C(C_1 \epsilon)^2 s^{-1+\delta}. \quad (2.4.7c)$$

²Throughout, $C > 0$ denotes a constant depending only on the structure of (2.3.1) and independent on the initial data.

- For all $|I^\dagger| \leq 4$,

$$\begin{aligned} & \left| \int_{\mathcal{H}_s} \frac{s}{t} \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t Z^{I^\dagger} w_i \partial_\beta Z^{I^\dagger} w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha Z^{I^\dagger} w_i \partial_\beta Z^{I^\dagger} w_j \right) dx \right| \\ & \leq M(I^\dagger, s) E_{m, c_i}(s, Z^{I^\dagger} w_i) := C(C_1 \epsilon)^2 s^{-1+\delta/2} E_{m, c_i}(s, Z^{I^\dagger} w_i) \end{aligned} \quad (2.4.8a)$$

and

$$\begin{aligned} & \sum_{i=1}^{n_0} \|[G_i^{j\alpha\beta} \partial_\alpha \partial_\beta, Z^{I^\dagger}] w_j\|_{L^2(\mathcal{H}_s)} + \sum_{i=1}^{n_0} \|Z^{I^\dagger} F_i\|_{L^2(\mathcal{H}_s)} \\ & \leq L(I^\dagger, s) := C(C_1 \epsilon)^2 s^{-1+\delta/2}. \end{aligned} \quad (2.4.8b)$$

- For all $|I| \leq 3$,

$$\begin{aligned} & \left| \int_{\mathcal{H}_s} \frac{s}{t} \left(\partial_\alpha G_{\hat{i}}^{j\alpha\beta} \partial_t Z^I u_{\hat{i}} \partial_\beta Z^I u_{\hat{j}} - \frac{1}{2} \partial_t G_{\hat{i}}^{j\alpha\beta} \partial_\alpha Z^I u_{\hat{i}} \partial_\beta Z^I u_{\hat{j}} \right) dx \right| \\ & \leq M(I, s) E_m(s, Z^I u_{\hat{i}}) := C(C_1 \epsilon)^2 s^{-3/2+2\delta} E_m(s, Z^I u_{\hat{i}}) \end{aligned} \quad (2.4.9a)$$

and

$$\begin{aligned} & \sum_{i=1}^{n_0} \|[G_i^{j\alpha\beta} \partial_\alpha \partial_\beta, Z^I] w_j\|_{L^2(\mathcal{H}_s)}^2 \\ & + \sum_{i=1}^{n_0} \|Z^I F_i\|_{L^2(\mathcal{H}_s)} + \|Z^I (G_{\hat{i}}^{\check{j}\alpha\beta} v_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \\ & \leq L(I, s) := C(C_1 \epsilon)^2 s^{-3/2+2\delta}. \end{aligned} \quad (2.4.9b)$$

Observe that (2.4.9a) concerns the wave components, only.

Proposition 2.3. *There exist positive constants ϵ_0'', C_1 such that if the hierarchy of energy bounds (2.4.5) holds for $\epsilon \leq \epsilon_0''$ and $C_1 \epsilon < 1$, then there exists a constant $\kappa_1 > 1$ (determined by ϵ_0'') and a constant $C > 0$ (determined by the structure of the system (1.1.2)) such that hierarchy of metric and source bounds (2.4.7a) also hold.*

The proof of this proposition will occupy a major part of this monograph, especially Chapter 6 to 9.

Proposition 2.4 (Enhancing the hierarchy of energy bounds).

Fix some parameter $\delta \in (0, 1/6)$. Suppose that Proposition 2.2 holds for some positive constant C_0 and ϵ_0' . Then, there exists a sufficiently large constant $C_1 \geq C_0$ such that, if the solution to (1.1.2) with data (2.4.3) satisfies the hierarchy of energy bounds (2.4.5), then there exists some $\epsilon_0 \in (0, \epsilon_0')$

such that for all $\epsilon \in (0, \epsilon_0)$, the following **hierarchy of enhanced energy bounds** also hold (for all $s \in [s_0, s_1]$ and all admissible indices):

$$E_{m,\sigma}(s, Z^{I^\sharp} v_{\check{j}})^{1/2} \leq \frac{1}{2} C_1 \epsilon s^\delta \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \quad (2.4.10a)$$

$$E_m(s, Z^{I^\sharp} u_{\hat{i}})^{1/2} \leq \frac{1}{2} C_1 \epsilon s^\delta \quad \text{for } 1 \leq \hat{i} \leq j_0, \quad (2.4.10b)$$

$$E_{m,\sigma}(s, Z^{I^\dagger} v_{\check{j}})^{1/2} \leq \frac{1}{2} C_1 \epsilon s^{\delta/2} \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \quad (2.4.10c)$$

$$E_m(s, Z^{I^\dagger} u_{\hat{i}})^{1/2} \leq \frac{1}{2} C_1 \epsilon s^{\delta/2} \quad \text{for } 1 \leq \hat{i} \leq j_0, \quad (2.4.10d)$$

$$E_m(s, Z^I u_{\hat{i}})^{1/2} \leq \frac{1}{2} C_1 \epsilon \quad \text{for } 1 \leq \hat{i} \leq j_0. \quad (2.4.10e)$$

We now assume that (2.4.5) hold and we provide a proof of Proposition 2.4 which is given *under the assumption* that Proposition 2.3 holds.

Proof. Step I. High-order energy estimates.

Suppose the hierarchy of metric and source bounds (2.4.7a)-(2.4.7c) hold (for all $|I^\sharp| \leq 5$). Let us apply to the equations (1.1.2) the operator Z^{I^\sharp} (with $|I^\sharp| \leq 5$):

$$\square Z^{I^\sharp} w_i + G_i^{j\alpha\beta} \partial_\alpha \partial_\beta Z^{I^\sharp} w_j + c_i^2 w_i = [G_i^{j\alpha\beta} \partial_\alpha \partial_\beta, Z^{I^\sharp}] w_j + Z^{I^\sharp} F_i.$$

Then, by using Lemma 2.1 (under the condition (2.4.7a)), for $|I^\sharp| \leq 5$ we find

$$\begin{aligned} & \left(\sum_i E_{m,c_i}(s, Z^{I^\sharp} w_i) \right)^{1/2} \\ & \leq \kappa_1^2 \left(\sum_i E_{m,c_i}(s_0, Z^{I^\sharp} w_i) \right)^{1/2} + C \kappa_1^2 \int_{s_0}^s (L(I^\sharp, \tau) + M(I^\sharp, \tau)) d\tau. \end{aligned}$$

Then we have

$$\left(\sum_i E_{m,c_i}(s, Z^{I^\sharp} w_i) \right)^{1/2} \leq \kappa_1^2 C_0 \epsilon + C \kappa_1^2 (C_1 \epsilon)^2 \int_{s_0}^s \tau^{-1+\delta} d\tau.$$

Observing that the integral of the function is bounded independently of s and provided $\epsilon \leq \frac{\delta(C_1 - 2\kappa_1^2 C_0)}{2C\kappa_1^2 C_1^2}$, we obtain

$$\left(\sum_i E_{m,c_i}(s, Z^{I^\sharp} w_i) \right)^{1/2} \leq (1/2) C_1 \epsilon s^\delta,$$

which leads to the enhanced energy bound

$$E_{m,\sigma}(s, Z^{I^\sharp} w_i) \leq E_{m,c_i}(s, Z^{I^\sharp} w_i) \leq \frac{1}{2} C_1 \epsilon s^\delta. \quad (2.4.11)$$

This establishes (2.4.10a)-(2.4.10b).

Step II. Intermediate energy estimates.

Suppose here the hierarchy of metric and source bounds (2.4.8a)-(2.4.8b) hold for all $|I^\dagger| \leq 4$. Let us apply to the equation (1.1.2) the operator Z^{I^\dagger} (with $|I^\dagger| \leq 4$):

$$\square Z^{I^\dagger} w_i + G_i^{j\alpha\beta} \partial_\alpha \partial_\beta Z^{I^\dagger} w_j + c_i^2 w_i = [G_i^{j\alpha\beta} \partial_\alpha \partial_\beta, Z^{I^\dagger}] w_j + Z^{I^\dagger} F_i.$$

Then, by using Lemma 2.1 (under the condition (2.4.7a)), for $|I^\dagger| \leq 5$ we find

$$\begin{aligned} & \left(\sum_i E_{m,c_i}(s, Z^{I^\dagger} w_i) \right)^{1/2} \\ & \leq \kappa_1^2 \left(\sum_i E_{m,c_i}(s_0, Z^{I^\dagger} w_i) \right)^{1/2} + C \kappa_1^2 \int_{s_0}^s (L(I^\dagger, \tau) + M(I^\dagger, \tau)) d\tau. \end{aligned}$$

Then we have

$$\left(\sum_i E_{m,c_i}(s, Z^{I^\dagger} w_i) \right)^{1/2} \leq \kappa_1^2 C_0 \epsilon + C \kappa_1^2 (C_1 \epsilon)^2 \int_{s_0}^s \tau^{-1+\delta/2} d\tau.$$

Observing that the integral of the function is bounded independently of s and provided ϵ is sufficiently small in the sense that

$$\epsilon \leq \frac{\delta(C_1 - 2\kappa_1^2 C_0)}{4C \kappa_1^2 C_1^2},$$

we find

$$\left(\sum_i E_{m,c_i}(s, Z^{I^\dagger} w_i) \right)^{1/2} \leq (1/2) C_1 \epsilon s^\delta,$$

which leads to the enhanced energy bound

$$E_{m,\sigma}(s, Z^{I^\dagger} w_i) \leq E_{m,c_i}(s, Z^{I^\dagger} w_i)^{1/2} \leq \frac{1}{2} C_1 \epsilon s^{\delta/2}. \quad (2.4.12)$$

This establishes (2.4.10c)-(2.4.10d).

Step III. Low-order energy estimates.

Assume now that the hierarchy of metric and source bounds (2.4.9a)-(2.4.9b) holds for all $|I| \leq 3$.

$$\square Z^I u_{\hat{i}} + G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_{\alpha\beta} Z^I u_{\hat{j}} = [G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_{\alpha\beta}, Z^I] u_{\hat{j}} - Z^I (G_{\hat{i}}^{\check{j}\alpha\beta} \partial_{\alpha\beta} v_{\check{j}}) + Z^I F_{\hat{i}}.$$

By Lemma 2.1 (under the condition (2.4.7a)), we have

$$\begin{aligned} & \left(\sum_i E_m(s, Z^I u_{\hat{i}}) \right)^{1/2} \\ & \leq \kappa_1^2 \left(\sum_i E_m(s_0, Z^I u_{\hat{i}}) \right)^{1/2} + C \kappa_1^2 \int_{s_0}^s L(I, \tau) + M(I, \tau) d\tau \end{aligned}$$

and thus

$$\left(\sum_i E_m(s, Z^I u_{\hat{i}}) \right)^{1/2} \leq \kappa_1^2 C_0 \epsilon + C \kappa_1^2 (C_1 \epsilon)^2 \int_{s_0}^s \tau^{-3/2+2\delta} d\tau.$$

By recalling $\delta < 1/6$, it follows that

$$\left(\sum_i E_m(s, Z^I u_{\hat{i}}) \right)^{1/2} \leq \kappa_1^2 C_0 \epsilon + \frac{C \kappa_1^2 C_1^2 \epsilon^2 B^{2\delta-1/2}}{(1/2) - 2\delta}.$$

Observing again that the integral of the function is bounded independently of s and provided $\epsilon \leq \frac{(1-4\delta)(C_1-2\kappa_1^2 C_0)}{4C\kappa_1^2 C_1^2 B^{2\delta-1/2}}$, we obtain the enhanced energy bound for the wave components

$$E_m(s, Z^I u_{\hat{i}})^{1/2} \leq \frac{1}{2} C_1 \epsilon. \quad (2.4.13)$$

This establishes (2.4.10e).

Step IV. Conclusion. Assuming now that

$$\epsilon \leq \min \left(\frac{\delta(C_1 - 2\kappa_1^2 C_0)}{4C\kappa_1^2 C_1^2}, \frac{(1-4\delta)(C_1 - 2\kappa_1^2 C_0)}{4C\kappa_1^2 C_1^2 B^{2\delta-1/2}} \right),$$

then we see that (2.4.10) is established. \square

Observe that (2.4.7a), (2.4.7b), and (2.4.9a) concern only the metric and depend mainly on L^∞ bounds on the solution and its derivatives: these bounds will be established in Chapter 6. On the other hand, the inequalities (2.4.7c) and (2.4.9b) are L^2 type estimates and will be eventually derived in Chapter 9.

We complete this chapter with a proof of our main result (by assuming Propositions 2.2 and 2.4).

Proof of Theorem 1.1. Let (w_i) be the unique local-in-time solution to (1.1.2), and let $[s_0, s^*]$ be the largest interval (containing s_0) on which the inequalities (2.4.5) hold for all $s \in [s_0, s^*]$. As guaranteed by Proposition 2.2, there exists some ϵ'_0 such that, provided $\|w_{i0}\|_{H^6(\mathbb{R}^3)} +$

$\|w_{i1}\|_{H^5(\mathbb{R}^3)} < \epsilon < \epsilon'_0$, we have $E_G(s_0, Z^{I^\sharp} w_j)^{1/2} \leq C_0 \epsilon$ with $C_0 < C_1$). By continuity, we certainly have $s^* > s_0$.

Proceeding by contradiction, we assume that $s^* < +\infty$, so that, at the time $s = s^*$, at least one of the inequalities (2.4.5) must be an equality. That is, at least one of the following conditions holds:

$$\begin{aligned}
E_{m,\sigma}(s^*, Z^{I^\sharp} v_{\check{j}})^{1/2} &= C_1 \epsilon s^{*\delta} \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \\
E_m(s^*, Z^{I^\sharp} u_{\hat{i}})^{1/2} &= C_1 \epsilon s^{*\delta} \quad \text{for } 1 \leq \hat{i} \leq j_0, \\
E_{m,\sigma}(s^*, Z^{I^\dagger} v_{\check{j}})^{1/2} &= C_1 \epsilon s^{*\delta/2} \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \\
E_m(s^*, Z^{I^\dagger} u_{\hat{i}})^{1/2} &= C_1 \epsilon s^{*\delta/2} \quad \text{for } 1 \leq \hat{i} \leq j_0, \\
E_m(s^*, Z^I u_{\hat{i}})^{1/2} &= C_1 \epsilon, \quad \text{for } 1 \leq \hat{i} \leq j_0.
\end{aligned} \tag{2.4.14}$$

Yet, for $\epsilon \leq \epsilon_0$ and according to Proposition 2.4, the enhanced energy bounds also hold:

$$\begin{aligned}
E_{m,\sigma}(s^*, Z^{I^\sharp} v_{\check{j}})^{1/2} &\leq \frac{1}{2} C_1 \epsilon s^{*\delta} \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \\
E_m(s^*, Z^{I^\sharp} u_{\hat{i}})^{1/2} &\leq \frac{1}{2} C_1 \epsilon s^{*\delta} \quad \text{for } 1 \leq \hat{i} \leq j_0, \\
E_{m,\sigma}(s^*, Z^{I^\dagger} v_{\check{j}})^{1/2} &\leq \frac{1}{2} C_1 \epsilon s^{*\delta/2} \quad \text{for } j_0 + 1 \leq \check{j} \leq n_0, \\
E_m(s^*, Z^{I^\dagger} u_{\hat{i}})^{1/2} &\leq \frac{1}{2} C_1 \epsilon s^{*\delta/2} \quad \text{for } 1 \leq \hat{i} \leq j_0, \\
E_m(s^*, Z^I u_{\hat{i}})^{1/2} &\leq \frac{1}{2} C_1 \epsilon \quad \text{for } 1 \leq \hat{i} \leq j_0.
\end{aligned}$$

This is a contradiction and therefore it must be that $s^* = +\infty$. Hence, the inequalities (2.4.5) hold for all $s \in [s_0, +\infty)$ and, by the local existence criteria in Theorem 11.2 (combined with Sobolev's inequalities), this local solution extends for all times. This completes the proof of Theorem 1.1. \square

2.5 Energy on the hypersurfaces of constant time

To end this chapter, we prove that the global-in-time solutions we have constructed for (1.1.2) enjoy a uniform energy estimate along the standard hypersurface of constant time t .

Proposition 2.5. *Suppose that the wave components u_i of the global-in-time solution w_i satisfying (2.4.5) and (2.4.9) (with ϵ_0 sufficiently small and $C_1 \epsilon < 1$). Then, the following estimate also holds for all $t \geq B + 1$:*

$$\|\partial_t Z^I u(t, \cdot)\|_{L^2(\mathbb{R}^3)} + \sum_a \|\partial_a Z^I u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C_3(\delta, \epsilon_0) C_1 \epsilon, \tag{2.5.1}$$

where C_3 is depends upon δ, ϵ_0 and the structure of the system.

The proof of this result relies on a modified version of the basic energy estimate, as follows.

Lemma 2.5. *Let (u_i) be the solution to the following Cauchy problem*

$$\begin{aligned} \square u_i + G_i^{j\alpha\beta} \partial_{\alpha\beta} u_i &= F_i, \\ u_i|_{\mathcal{H}_{s_0}} &= u_{i0}, \quad \partial_t u_i|_{\mathcal{H}_{s_0}} = u_{i1}, \end{aligned} \tag{2.5.2}$$

where F_i and $G_i^{j\alpha\beta}$ are defined in $\mathcal{K}_{[s_0, +\infty)}$. Then, there exists a positive constant ϵ_0 such that provided

$$\max_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| \leq \epsilon_0,$$

there exists a positive constant $C(\epsilon_0)$ (depending upon ϵ_0) such that

$$\begin{aligned} & \left(\|\partial_t u(t_0, \cdot)\|_{L^2(\mathbb{R}^3)} + \sum_a \|\partial_a u(t_0, \cdot)\|_{L^2(\mathbb{R}^3)} \right)^2 \\ & \leq C \sum_i E_G(t, u_i) + C \sum_i \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t u_i F_i| dx \\ & \quad + C \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_\alpha G_i^{j\alpha\beta} \partial_t u_i \partial_\beta u_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha u_i \partial_\beta u_j| dx ds. \end{aligned} \tag{2.5.3}$$

Proof. The proof relies on an energy estimate within the region $\mathcal{K}_{t_0} := \{(t, x) \mid |x| \leq t - 1, t_0 \leq t \leq \sqrt{|x|^2 + t_0^2}\}$. The corresponding figure 2.3 displays the set \mathcal{K}_{t_0} . We start from the identity

$$\begin{aligned} & \sum_i \left(\frac{1}{2} \partial_t \sum_\alpha (\partial_\alpha u_i)^2 + \sum_a \partial_a (\partial_a u_i \partial_t u_i) \right. \\ & \quad \left. + \partial_\alpha (G_i^{j\alpha\beta} \partial_t u_i \partial_\beta u_j) - \frac{1}{2} \partial_t (G_i^{j\alpha\beta} \partial_\alpha u_i \partial_\beta u_j) \right) \\ & = \sum_i \partial_t u_i F_i + \sum_i \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t u_i \partial_\beta u_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha u_i \partial_\beta u_j \right) \end{aligned}$$

and integrate it within \mathcal{K}_{t_0} with respect to the volume form $dt dx$. Applying

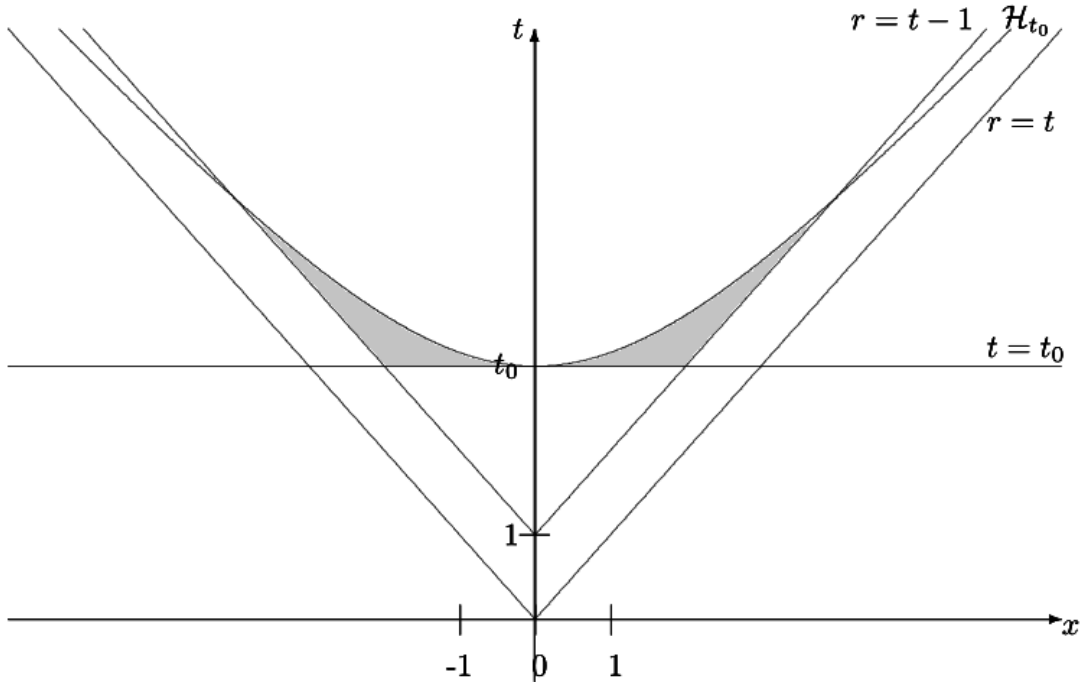


Fig. 2.3 \mathcal{K}_{t_0}

Stokes' formula, we get

$$\begin{aligned}
& \frac{1}{2} \sum_i E_G(t, u_i) \\
& - \frac{1}{2} \sum_i \int_{\mathbb{R}^3} \left(\sum_{\alpha} |\partial_{\alpha} u_i|^2 + 2G_i^{j_0\beta} \partial_t u_i \partial_{\beta} u_j - G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j \right) (t, \cdot) dx \\
& = \int_{\mathcal{K}_{t_0}} \sum_i \left(\partial_t u_i F_i + \partial_{\alpha} G_i^{j\alpha\beta} \partial_t u_i \partial_{\beta} u_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j \right) dx dt \\
& = \int_{\mathcal{K}_{t_0}} (s/t) \sum_i \left(\partial_t u_i F_i + \partial_{\alpha} G_i^{j\alpha\beta} \partial_t u_i \partial_{\beta} u_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j \right) dx ds,
\end{aligned}$$

where we recall that $s = \sqrt{t^2 - |x|^2}$. This implies

$$\begin{aligned}
& \sum_i \int_{\mathbb{R}^3} \left(\sum_{\alpha} |\partial_{\alpha} u_i|^2 + 2G_i^{j0\beta} \partial_t u_i \partial_{\beta} u_j - G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j \right) (t, \cdot) dx \\
& \leq \sum_i E_G(t, u_i) \\
& \quad + \sum_i \int_{\mathcal{H}_{t_0}} (s/t) |\partial_t u_i F_i + \partial_{\alpha} G_i^{j\alpha\beta} \partial_t u_i \partial_{\beta} u_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j| dx ds \\
& \leq \sum_i E_G(t, u_i) \\
& \quad + \sum_i \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t u_i F_i| dx \\
& \quad + \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_{\alpha} G_i^{j\alpha\beta} \partial_t u_i \partial_{\beta} u_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j| dx ds.
\end{aligned}$$

Then we observe that

$$\sum_i \int_{\mathbb{R}^3} |2G_i^{j0\beta} \partial_t u_i \partial_{\beta} u_j - G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j| dx \leq C \max_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| \sum_{i,\alpha} \|\partial_{\alpha} u_i(t, \cdot)\|_{L^2(\mathbb{R}^3)}$$

and thus, if $\max_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}|$ is sufficiently small,

$$\sum_i \int_{\mathbb{R}^3} |2G_i^{j0\beta} \partial_t u_i \partial_{\beta} u_j - G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j| dx \leq \frac{1}{2} \sum_{i,\alpha} \|\partial_{\alpha} u_i(t, \cdot)\|_{L^2(\mathbb{R}^3)}.$$

Then, we have

$$\frac{1}{2} \sum_{i,\alpha} \|\partial_{\alpha} u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq \sum_i \int_{\mathbb{R}^3} \left(\sum_{\alpha} |\partial_{\alpha} u_i|^2 + 2G_i^{j0\beta} \partial_t u_i \partial_{\beta} u_j - G_i^{j\alpha\beta} \partial_{\alpha} u_i \partial_{\beta} u_j \right) dx$$

and the desired result is proven. \square

Proof of Proposition 2.5. We recall the equations satisfied by the wave components:

$$\square Z^I u_{\hat{i}} + G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_{\alpha\beta} Z^I u_{\hat{j}} = [G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_{\alpha} \partial_{\beta}, Z^I] u_{\hat{j}} - Z^I (G_{\hat{i}}^{\check{j}\alpha\beta} \partial_{\alpha} \partial_{\beta} v_{\hat{j}}) + Z^I F_{\hat{i}}$$

and apply (2.5.3), so that:

$$\begin{aligned}
& \sum_{\hat{i}, \alpha} \|\partial_\alpha Z^I u_{\hat{i}}(t, \cdot)\|_{L^2(\mathbb{R}^3)} \\
& \leq C \sum_{\hat{i}} E_G(t, Z^I u_{\hat{i}}) + C \sum_{\hat{i}} \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t Z^I u_{\hat{i}} Z^I F_{\hat{i}}| dx \\
& \quad + \sum_{\hat{i}} \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t Z^I u_{\hat{i}} [G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_\alpha \partial_\beta, Z^I] u_{\hat{j}}| dx \\
& \quad + \sum_{\hat{i}} \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t Z^I u_{\hat{i}} Z^I (G_{\hat{i}}^{\check{j}\alpha\beta} \partial_\alpha \partial_\beta v_{\check{j}})| dx \\
& \quad + C \sum_{\hat{i}} \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_\alpha G_{\hat{i}}^{j\alpha\beta} \partial_t Z^I u_{\hat{i}} \partial_\beta Z^I u_j - \frac{1}{2} \partial_t G_{\hat{i}}^{j\alpha\beta} \partial_\alpha Z^I u_{\hat{i}} \partial_\beta Z^I u_j| dx ds.
\end{aligned}$$

By (2.4.5e) (which is already established thanks to our bootstrap argument) which holds on the time interval $[B+1, +\infty)$:

$$\sum_{\hat{i}} E_G(t, Z^I u_{\hat{i}}) \leq (C_1 \epsilon)^2,$$

and the second term in the right-hand side is uniformly bounded as follows:

$$\begin{aligned}
& \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t Z^I u_{\hat{i}} Z^I F_{\hat{i}}| dx \\
& \leq \int_{t_0}^{(t_0^2+1)/2} \|(s/t) \partial_t Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \|Z^I F_{\hat{i}}\|_{L^2(\mathcal{H}_s)} ds \\
& \leq C_1 \epsilon \int_{t_0}^{(t_0^2+1)/2} \|Z^I F_{\hat{i}}\|_{L^2(\mathcal{H}_s)} ds \leq C_1 \epsilon \int_{t_0}^{(t_0^2+1)/2} C(C_1 \epsilon)^2 s^{-3/2+2\delta} ds \\
& \leq C(C_1 \epsilon)^3 (1/2 - 2\delta)^{-1}.
\end{aligned}$$

The third and fourth terms are bounded in the same manner:

$$\begin{aligned}
& \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t Z^I u_{\hat{i}} [G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_\alpha \partial_\beta, Z^I] u_{\hat{j}}| dx \leq C(C_1 \epsilon)^3 (1/2 - 2\delta)^{-1}, \\
& \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_t Z^I u_{\hat{i}} Z^I (G_{\hat{i}}^{\check{j}\alpha\beta} \partial_\alpha \partial_\beta v_{\check{j}})| dx \leq C(C_1 \epsilon)^3 (1/2 - 2\delta)^{-1}.
\end{aligned}$$

The last term is bounded by applying (2.4.9a):

$$\begin{aligned}
& \int_{t_0}^{(t_0^2+1)/2} ds \int_{\mathcal{H}_s} (s/t) |\partial_\alpha G_{\hat{i}}^{j\alpha\beta} \partial_t Z^I u_{\hat{i}} \partial_\beta Z^I u_j - \frac{1}{2} \partial_t G_{\hat{i}}^{j\alpha\beta} \partial_\alpha Z^I u_{\hat{i}} \partial_\beta Z^I u_j| dx ds \\
& \leq C(C_1 \epsilon)^3 (1/2 - 2\delta)^{-1}.
\end{aligned}$$

□

Chapter 3

Decompositions and estimates for the commutators

3.1 Algebraic decomposition of commutators

In this chapter, we are going to estimate the commutators $[X, Y]u := X(Yu) - Y(Xu)$ of two operators X, Y applied to functions u defined in the cone $\mathcal{K} = \{|x| < t - 1\}$.

First of all, all admissible vector fields Z_α (cf. Chapter 2) are Killing fields for the flat wave operator \square , so that the commutation relations

$$[\partial_\alpha, \square] = 0, \quad [L_a, \square] = 0 \quad (3.1.1)$$

hold. Next, we introduce the notation

$$[L_a, \partial_\beta]u =: \Theta_{a\beta}^\gamma \partial_\gamma u, \quad (3.1.2a)$$

$$[\partial_\alpha, \underline{\partial}_\beta]u =: t^{-1} \underline{\Gamma}_{\alpha\beta}^\gamma \partial_\gamma u, \quad (3.1.2b)$$

$$[L_a, \underline{\partial}_\beta]u =: \underline{\Theta}_{a\beta}^\gamma \underline{\partial}_\gamma u, \quad (3.1.2c)$$

in which

$$\begin{aligned} \Theta_{ab}^\gamma &= -\delta_{ab} \delta_0^\gamma, & \Theta_{a0}^\gamma &= -\delta_a^\gamma, \\ \underline{\Gamma}_{ab}^\gamma &= \delta_{ab} \delta_0^\gamma, & \underline{\Gamma}_{0b}^\gamma &= -\frac{x^b}{t} \delta_0^\gamma = \Psi_b^0 \delta_0^\gamma, & \underline{\Gamma}_{\alpha 0}^\gamma &= 0, \\ \underline{\Theta}_{ab}^\gamma &= -\frac{x^b}{t} \delta_a^\gamma = \Psi_b^0 \delta_a^\gamma, & \underline{\Theta}_{a0}^\gamma &= -\delta_a^\gamma + \frac{x^a}{t} \delta_0^\gamma = -\delta_a^\gamma + \Phi_0^a \delta_0^\gamma. \end{aligned} \quad (3.1.3)$$

All of these coefficients are C^∞ in the cone \mathcal{K} and for each multi-index $|I|$, the function $|Z^I \Pi|$ is bounded for all $\Pi \in \{\Theta, \underline{\Theta}, \underline{\Gamma}\}$ and, furthermore,

$$\underline{\Theta}_{ab}^0 = 0, \quad [L_a, \underline{\partial}_b] = \underline{\Theta}_{ab}^\beta \underline{\partial}_\beta = \Theta_{ab}^c \underline{\partial}_c. \quad (3.1.4)$$

Lemma 3.1 (Algebraic decomposition of commutators. I).

There exist some constants $\theta_{\alpha J}^{I\beta}$ such that, for any function u defined in the cone \mathcal{K} , the following identities hold (for all α and I):

$$[Z^I, \partial_\alpha]u = \sum_{|J| < |I|} \theta_{\alpha J}^{I\beta} \partial_\beta Z^J u. \quad (3.1.5)$$

Proof. The proof is done by induction on $|I|$. Observe that when $|I| = 1$, (3.1.5) is covered by (3.1.2). Assume next that (3.1.5) is valid for $|I| \leq k$ and let us derive this property for all $|I| \leq k + 1$. Let Z^I be a product with multi-index satisfying $|I| = k + 1$, where $Z^I = Z_1 Z^{I'}$ with $|I'| = k$ and Z_1 is one of the fields ∂_γ, L_a . Recall that

$$[Z^I, \partial_\alpha]u = [Z_1 Z^{I'}, \partial_\alpha]u = Z_1([Z^{I'}, \partial_\alpha]u) + [Z_1, \partial_\alpha]Z^{I'}u$$

and

$$Z_1([Z^{I'}, \partial_\alpha]u) = Z_1\left(\sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} \partial_\gamma Z^J u\right) = \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} Z_1 \partial_\gamma Z^J u.$$

Then, we have

$$[Z^I, \partial_\alpha]u = \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} Z_1 \partial_\gamma Z^J u + [Z_1, \partial_\alpha]Z^{I'}u.$$

First of all, when $Z_1 = \partial_\beta$, we can commute and obtain $[Z_1, \partial_\alpha]Z^{I'}u = 0$ and

$$\sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} Z_1 \partial_\gamma Z^J u = \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} \partial_\gamma Z_1 Z^J u,$$

so that (3.1.5) is established in this case.

Second, when $Z_1 = L_a$, we apply (3.1.2a) and write

$$\begin{aligned} \sum_{|J| \leq k-1} \theta_{\alpha J}^{I_1\gamma} L_a \partial_\gamma Z^J u &= \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} \partial_\gamma L_a Z^J u + \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} [L_a, \partial_\gamma] Z^J u \\ &= \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\gamma} \partial_\gamma L_a Z^J u + \sum_{|J| \leq k-1} \theta_{\alpha J}^{I'\beta} \Theta_{\alpha\beta}^\gamma \partial_\gamma Z^J u \end{aligned}$$

and $[L_a, \partial_\alpha]Z^{I'}u = \Theta_{\alpha\beta}^\gamma \partial_\beta Z^{I'}u$. So, (3.1.5) holds for $|I| = k + 1$, which completes the induction argument. \square

3.2 Derivation of the decomposition

Lemma 3.2 (Algebraic decomposition of commutators. II). *For any function u defined in the cone \mathcal{K} , the following identity holds:*

$$[Z^I, \underline{\partial}_\beta]u = \sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \partial_\gamma Z^J u \quad (3.2.1)$$

where $\underline{\theta}_{\beta J}^{I\gamma}$ are C^∞ functions satisfying the following estimates in \mathcal{K} :

$$|\partial^{I_1} Z^{I_2} \underline{\theta}_{\beta J}^{I\gamma}| \leq C(n, |I|, |I_1|, |I_2|) t^{-|I_1|}. \quad (3.2.2)$$

Proof. To establish (3.2.1) and (3.2.2), we perform the following calculation:

$$[Z^I, \underline{\partial}_\beta]u = [Z^I, \Phi_\beta^\gamma \partial_\gamma]u = \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} \Phi_\beta^\gamma Z^{I_2} \partial_\gamma u + \Phi_\beta^\gamma [Z^I, \partial_\gamma]u.$$

In the first sum, we commute Z^{I_2} and ∂_γ and obtain

$$\begin{aligned} [Z^I, \underline{\partial}_\beta]u &= \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} \Phi_\beta^\gamma \partial_\gamma Z^{I_2} u + \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} \Phi_\beta^\gamma [Z^{I_2}, \partial_\gamma]u + \Phi_\beta^\gamma [Z^I, \partial_\gamma]u \\ &= \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} \Phi_\beta^\gamma \partial_\gamma Z^{I_2} u + \sum_{I_1+I_2=I} Z^{I_1} \Phi_\beta^\gamma [Z^{I_2}, \partial_\gamma]u \\ &= \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} \Phi_\beta^\gamma \partial_\gamma Z^{I_2} u + \sum_{\substack{I_1+I_2=I \\ |J| < |I_2|}} (Z^{I_1} \Phi_\beta^\gamma) \theta_{\gamma J}^{I_2\alpha} \partial_\alpha Z^J u. \end{aligned}$$

Hence, $\underline{\theta}_{\gamma J}^{I\alpha}$ are linear combinations of $Z^{I_1} \Phi_\beta^\gamma$ and $\theta_{\gamma J}^{I_2\alpha} Z^{I_1} \Phi_\beta^\gamma$ with $|I_1| \leq |I|$ and $|I_2| \leq |I|$, which yields (3.2.1). Note that $\theta_{\gamma J}^{I_2\alpha}$ are constants, so that

$$\partial^{I_3} Z^{I_4} (\theta_{\gamma J}^{I_2\alpha} Z^{I_1} \Phi_\beta^\gamma) = \theta_{\gamma J}^{I_2\alpha} \partial^{I_3} Z^{I_4} Z^{I_1} \Phi_\beta^\gamma$$

and, by (2.2.5), we arrive at (3.2.2). \square

Lemma 3.3 (Algebraic decomposition of commutators. III). *For any function u defined in the cone \mathcal{K} , the following identities hold:*

$$[Z^I, \underline{\partial}_c]u = \sum_{|J| < |I|} \sigma_{cJ}^{Ib} \underline{\partial}_b Z^J u + t^{-1} \sum_{|J'| < |I|} \rho_{cJ'}^{I\gamma} \partial_\gamma Z^{J'} u, \quad (3.2.3)$$

where $\sigma_{cJ}^{I\gamma}$ and $\rho_{cJ}^{I\gamma}$ are C^∞ functions satisfying the following estimates in \mathcal{K} :

$$|\partial^{I_1} Z^{I_2} \sigma_{\beta J}^{I\gamma}| \leq C(n, |I|, |I_1|, |I_2|) t^{-|I_1|}, \quad (3.2.4a)$$

$$|\partial^{I_1} Z^{I_2} \rho_{\beta J}^{I\gamma}| \leq C(n, |I|, |I_1|, |I_2|) t^{-|I_1|}. \quad (3.2.4b)$$

Proof. We proceed by induction. The case $|I| = 1$ is easily checked from (3.1.2b) and (3.1.2c). By assuming that (3.2.3) with (3.2.4a) and (3.2.4b) hold for $|I| \leq k$, we want to treat indices $|I| = k + 1$. To do this, let Z^I be a product of operators with multi-index $|I| = k + 1$. Then, $Z^I = Z_1 Z^{I'}$ with $|I'| = k$ and the following holds:

$$[Z^I, \underline{\partial}_c]u = [Z_1 Z^{I'}, \underline{\partial}_c]u = Z_1 ([Z^{I'}, \underline{\partial}_c]u) + [Z_1, \underline{\partial}_c]Z^{I'}u.$$

Case I: $Z_1 = \partial_\alpha$. We consider first the case $Z_1 = \partial_\alpha$ and write

$$\begin{aligned} & Z_1([Z^{I'}, \underline{\partial}_c]u) \\ &= \partial_\alpha \left(\sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\partial}_b Z^J u + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma Z^{J'} u \right) \\ &= \sum_{|J| < |I'|} \partial_\alpha(\sigma_{cJ}^{I'b}) \underline{\partial}_b Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\partial}_b \partial_\alpha Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} [\partial_\alpha, \underline{\partial}_b] Z^J u \\ &\quad + \partial_\alpha t^{-1} \sum_{|J'| < |I'|} (\rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u + t^{-1} \sum_{|J'| < |I'|} \partial_\alpha(\rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u \\ &\quad + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma \partial_\alpha Z^{J'} u + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} [\partial_\alpha, \partial_\gamma] Z^{J'} u. \end{aligned}$$

For the second term in the right-hand-side, we make a change of frame, i.e.

$$\sigma_{cJ}^{I'b} \underline{\partial}_b \partial_\alpha Z^J u = \sigma_{cJ}^{I'b} \underline{\partial}_b (\Psi_\alpha^\gamma \underline{\partial}_\gamma Z^J u).$$

For the third term, we recall (3.1.2b) and write

$$\sigma_{cJ}^{I'b} [\partial_\alpha, \underline{\partial}_b] Z^J u = t^{-1} \sigma_{cJ}^{I'b} \Gamma_{\alpha b}^\gamma \partial_\gamma Z^J u.$$

For the last term, we remark that $[\partial_\alpha, \partial_\gamma] = 0$. Hence, we conclude that

$$\begin{aligned} & Z_1([Z^{I'}, \underline{\partial}_c]u) \\ &= \sum_{|J| < |I'|} \partial_\alpha(\sigma_{cJ}^{I'b}) \underline{\partial}_b Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\partial}_b (\Psi_\alpha^\gamma \underline{\partial}_\gamma Z^J u) \\ &\quad + t^{-1} \sum_{|J| < |I'|} (\sigma_{cJ}^{I'b} \underline{\partial}_b \Gamma_{\alpha b}^\gamma) \partial_\gamma Z^J u + \partial_\alpha t^{-1} \sum_{|J'| < |I'|} (\rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u \\ &\quad + t^{-1} \sum_{|J'| < |I'|} \partial_\alpha(\rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma \partial_\alpha Z^{J'} u. \end{aligned}$$

So, we have

$$\begin{aligned}
& Z_1([Z^{I'}, \underline{\partial}_c]u) \\
&= \sum_{|J| < |I'|} (\sigma_{cJ}^{I'b} \Psi_\alpha^\gamma) \underline{\partial}_b \underline{\partial}_\gamma Z^J u \\
&\quad + \sum_{|J| < |I'|} \partial_\alpha (\sigma_{cJ}^{I'b}) \Psi_b^\gamma \partial_\gamma Z^J u + \sum_{|J| < |I'|} (\sigma_{cJ}^{I'b} \underline{\partial}_b \Psi_\alpha^\gamma) \Phi_\gamma^{\gamma'} \partial_{\gamma'} Z^J u \\
&\quad + t^{-1} \sum_{|J| < |I'|} (\sigma_{cJ}^{I'b} \underline{\partial}_b \underline{\Gamma}_{\alpha b}^\gamma) \partial_\gamma Z^J u + t^{-1} \sum_{|J'| < |I'|} \partial_\alpha (\rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u \\
&\quad + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma \partial_\alpha Z^{J'} u + \partial_\alpha t^{-1} \sum_{|J'| < |I'|} (\rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u.
\end{aligned}$$

We also recall that

$$[Z_1, \underline{\partial}_c] Z^{I'} u = [\partial_\alpha, \underline{\partial}_c] Z^{I'} u = t^{-1} \underline{\Gamma}_{\alpha c}^\gamma \partial_\gamma Z^{I'} u.$$

Then we conclude that, when $Z_1 = \partial_\alpha$, $\sigma_{\alpha J}^{I'\gamma}$ are linear combinations of $\sigma_{cJ}^{I'b} \Psi_\alpha^\gamma$. For all $\partial^{I_1} Z^{I_2}$, we have

$$\begin{aligned}
\partial^{I_1} Z^{I_2} (\sigma_{cJ}^{I'b} \Psi_\alpha^\gamma) &= \sum_{I_3+I_4=I_2} \partial^{I_1} (Z^{I_3} \sigma_{cJ}^{I'b} Z^{I_4} \Psi_\alpha^\gamma) \\
&= \sum_{\substack{I_3+I_4=I_2 \\ I_5+I_6=I_1}} \partial^{I_5} Z^{I_3} \sigma_{cJ}^{I'b} \partial^{I_6} Z^{I_4} \Psi_\alpha^\gamma.
\end{aligned}$$

Now, we apply (2.2.6) on $\partial^{I_6} Z^{I_4} \Psi_\alpha^\gamma$ and our induction assumption (3.2.4a) on $\partial^{I_5} Z^{I_3} \sigma_{cJ}^{I'b}$ for $|I'| \leq k$ and obtain

$$|\partial^{I_1} Z^{I_2} (\sigma_{cJ}^{I'b} \Psi_\alpha^\gamma)| \leq C(n, |I_1|, |I_2|) t^{-|I_1|}.$$

Note that $\rho_{\alpha J}^{I'\gamma}$ are linear combinations of the following terms:

$$\begin{aligned}
& t \partial_\alpha (\sigma_{cJ}^{I'b}) \Psi_b^\gamma, \quad t (\sigma_{cJ}^{I'b} \underline{\partial}_b \Psi_\alpha^\gamma) \Phi_\gamma^{\gamma'}, \quad \sigma_{cJ}^{I'b} \underline{\partial}_b \underline{\Gamma}_{\alpha b}^\gamma, \\
& \rho_{cJ'}^{I'\gamma}, \quad \partial_\alpha (\rho_{cJ'}^{I'\gamma}), \quad t \partial_\alpha t^{-1} (\rho_{cJ'}^{I'\gamma}), \quad \underline{\Gamma}_{\alpha c}^\gamma.
\end{aligned}$$

For the term $t \partial_\alpha (\sigma_{cJ}^{I'b}) \Psi_b^\gamma$, we observe that

$$\partial^{I_1} Z^{I_2} (t \partial_\alpha (\sigma_{cJ}^{I'b}) \Psi_b^\gamma) = \sum_{\substack{I_3+I_4=I_1 \\ I_5+I_6=I_2}} \partial^{I_3} Z^{I_5} (t \Psi_b^\gamma) \partial^{I_4} Z^{I_6} (\partial_\alpha \sigma_{cJ}^{I'b})$$

Note that $t \Psi_b^\gamma$ is homogeneous of degree 1 and continuous in the region $\{t \geq 1, |x| \leq t\}$. Then by Lemma 2.2,

$$|\partial^{I_3} Z^{I_5} (t \Psi_b^\gamma)| \leq C(n, |I_3|, |I_6|) t^{-|I_3|+1}. \quad (3.2.5)$$

By Lemma 3.1, we have

$$\begin{aligned}\partial^{I_4} Z^{I_6} (\partial_\alpha \sigma_{cJ}^{I'b}) &= \partial^{I_4} \partial_\alpha Z^{I_6} \sigma_{cJ}^{I'b} + \partial^{I_4} [Z^{I_6}, \partial_\alpha] \sigma_{cJ}^{I'b} \\ &= \partial^{I_4} \partial_\alpha Z^{I_6} \sigma_{cJ}^{I'b} + \sum_{\substack{\beta \\ |I'_6| < |I_6|}} \theta_{\alpha I'_6}^{I_6 \beta} \partial^{I_4} \partial_\beta Z^{I'_6} \sigma_{cJ}^{I'b}.\end{aligned}$$

Then, by the induction assumption (3.2.4b) and (2.2.6)

$$\begin{aligned}|\partial^{I_4} \partial_\alpha Z^{I_6} \sigma_{cJ}^{I'b}| &\leq C(n, |I_4|, |I_6|) t^{-|I_4|-1}, \\ |\partial^{I_4} \partial_\beta Z^{I'_6} \sigma_{cJ}^{I'b}| &\leq C(n, |I_4|, |I_6|) t^{-|I_4|-1}.\end{aligned}$$

Then, we obtain

$$|\partial^{I_4} Z^{I_6} (\partial_\alpha \sigma_{cJ}^{I'b})| \leq C(n, |I_4|, |I_6|) t^{-|I_4|-1},$$

combined with (3.2.5), we get

$$|\partial^{I_1} Z^{I_2} (t \partial_\alpha (\sigma_{cJ}^{I'b}) \Psi_b^\gamma)| \leq C(n, |I_1|, |I_2|) t^{-|I_1|}$$

which is the right-hand-side of (3.2.4b).

Next, for the term $t(\sigma_{cJ}^{I'b} \underline{\partial}_b \Psi_\alpha^\gamma) \Phi_\gamma^{\gamma'}$, we observe that

$$t(\sigma_{cJ}^{I'b} \underline{\partial}_b \Psi_\alpha^\gamma) \Phi_\gamma^{\gamma'} = t \Phi_b^\beta \Phi_\gamma^{\gamma'} \partial_\beta \Psi_\alpha^\gamma \sigma_{cJ}^{I'b},$$

and, similarly to the cases above,

$$\partial^{I_1} Z^{I_2} (t \Phi_b^\beta \Phi_\gamma^{\gamma'} \partial_\beta \Psi_\alpha^\gamma \sigma_{cJ}^{I'b}) = \sum_{\substack{I_3+I_5=I_1 \\ I_4+I_6=I_2}} \partial^{I_3} Z^{I_4} (t \Phi_b^\beta \Phi_\gamma^{\gamma'} \partial_\beta \Psi_\alpha^\gamma) \partial^{I_5} Z^{I_6} \sigma_{cJ}^{I'b}$$

where in each term of the sum the first factor is homogeneous of degree 0 and can be controlled by lemma 2.2

$$|\partial^{I_3} Z^{I_4} (t \Phi_b^\beta \Phi_\gamma^{\gamma'} \partial_\beta \Psi_\alpha^\gamma)| \leq C(n, |I_3|, |I_5|) t^{-|I_3|}$$

and the second factor is controlled by the assumption of induction (3.2.4b):

$$|\partial^{I_5} Z^{I_6} \sigma_{cJ}^{I'b}| \leq C(n, |I_5|, |I_6|) t^{-|I_5|}.$$

Then we conclude that

$$|\partial^{I_1} Z^{I_2} (t \Phi_b^\beta \Phi_\gamma^{\gamma'} \partial_\beta \Psi_\alpha^\gamma \sigma_{cJ}^{I'b})| \leq C(n, |I_1|, |I_2|) t^{-|I_1|}.$$

The term $\sigma_{cJ}^{I'b} \underline{\partial}_b \Gamma_{\alpha b}^\gamma$ is estimated in the same manner by noting that $\Gamma_{\alpha b}^\gamma$ is nothing but a linear combination of Ψ_b^γ with constant coefficients. So they are homogeneous of degree 0.

The term $\rho_{cJ}^{I'\gamma}$ is bounded by our induction assumption (3.2.4b).

For the term $\partial_\alpha(\rho_{cJ'}^{I'\gamma})$, we apply (3.1.5):

$$\begin{aligned}\partial^{I_1} Z^{I_2} \partial_\alpha(\rho_{cJ'}^{I'\gamma}) &= \partial^{I_1} \partial_\alpha Z^{I_2} \rho_{cJ'}^{I'\gamma} + \partial^{I_1} ([Z^{I_2}, \partial_\alpha] \rho_{cJ'}^{I'\gamma}) \\ &= \partial^{I_1} \partial_\alpha Z^{I_2} \rho_{cJ'}^{I'\gamma} + \partial^{I_1} \left(\sum_{|J_2| < |I_2|} \theta_{\alpha J_2}^{I_2 \delta} \partial_\delta Z^{J_2} \rho_{cJ'}^{I'\gamma} \right) \\ &= \partial^{I_1} \partial_\alpha Z^{I_2} \rho_{cJ'}^{I'\gamma} + \sum_{|J_2| < |I_2|} \theta_{\alpha J_2}^{I_2 \delta} \partial^{I_1} \partial_\delta Z^{J_2} \rho_{cJ'}^{I'\gamma}.\end{aligned}$$

Then, by applying the assumption if induction, we see that the term $\partial^{I_1} Z^{I_2} \partial_\alpha(\rho_{cJ'}^{I'\gamma})$ can be bounded by the right-hand-side of (3.2.4b).

Then, we consider the term $t\partial_\alpha t^{-1}(\rho_{cJ'}^{I'\gamma})$ and observe that it vanishes when $\alpha \neq 0$, while for $\alpha = 0$ we obtain $t\partial_\alpha t^{-1}(\rho_{cJ'}^{I'\gamma}) = -t^{-1}(\rho_{cJ'}^{I'\gamma})$ and

$$\partial^{I_1} Z^{I_2} (-t^{-1} \rho_{cJ'}^{I'\gamma}) = - \sum_{\substack{I_3+I_4=I_1 \\ I_5+I_6=I_2}} \partial^{I_3} Z^{I_5} t^{-1} \partial^{I_4} Z^{I_6} \rho_{cJ'}^{I'\gamma}.$$

By a direct calculation, we find

$$|\partial^{I_3} Z^{I_5} t^{-1}| \leq C(n, |I_3|, |I_5|) t^{-|I_3|}$$

and, by our induction assumption (3.2.4b),

$$|\partial^{I_1} Z^{I_2} (-t^{-1} \rho_{cJ'}^{I'\gamma})| \leq C(n, |I_1|, |I_2|) t^{-|I_1|}.$$

At last, we observe that the term $\underline{\Gamma}_{\alpha c}^\gamma$ is directly handled with (3.1.3).

Finally, we combine together our estimates on the components of $\rho_{cJ'}^{I'\gamma}$ and thus arrive at (3.2.4b) when $Z_1 = \partial_\alpha$.

Case 2, $Z_1 = L_a$.

We now treat the case $Z_1 = L_a$ and write

$$\begin{aligned}& Z_1([Z^{I'}, \underline{\partial}_c]u) \\ &= L_a \left(\sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\partial}_b Z^J u + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma Z^{J'} u \right) \\ &= \sum_{|J| < |I'|} (L_a \sigma_{cJ}^{I'b}) \underline{\partial}_b Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\partial}_b L_a Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} [L_a, \underline{\partial}_b] Z^J u \\ &\quad + t^{-1} (t L_a t^{-1}) \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma Z^{J'} u \\ &\quad + t^{-1} \sum_{|J'| < |I'|} (L_a \rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma L_a Z^{J'} u \\ &\quad + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} [L_a, \partial_\gamma] Z^{J'} u.\end{aligned}$$

In right-hand-side, we apply (3.1.2c) on the third term and (3.1.2a) on the last term, and remark that the coefficient of the fourth term $tL_a(t^{-1}) = -\frac{x^a}{t} = \Psi_a^0$. Then,

$$\begin{aligned}
& Z_1([Z^{I'}, \underline{\partial}_c]u) \\
&= \sum_{|J| < |I'|} (L_a \sigma_{cJ}^{I'b}) \underline{\partial}_b Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\partial}_b L_a Z^J u + \sum_{|J| < |I'|} \sigma_{cJ}^{I'b} \underline{\Theta}_{ab}^c \underline{\partial}_c Z^J u \\
&\quad + t^{-1} \Psi_a^0 \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma Z^{J'} u \\
&\quad + t^{-1} \sum_{|J'| < |I'|} (L_a \rho_{cJ'}^{I'\gamma}) \partial_\gamma Z^{J'} u + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \partial_\gamma L_a Z^{J'} u \\
&\quad + t^{-1} \sum_{|J'| < |I'|} \rho_{cJ'}^{I'\gamma} \Theta_{a\gamma}^\beta \partial_\beta Z^{J'} u.
\end{aligned}$$

We also recall that

$$[Z_1, \underline{\partial}_c] Z^{I'} u = [L_a, \partial_c] Z^{I'} u = \underline{\Theta}_{ac}^b \underline{\partial}_b Z^{I'} u.$$

We observe that $\sigma_{cJ}^{I'b}$ are linear combinations of $L_a \sigma_{cJ}^{I'b}$, $\sigma_{cJ}^{I'b}$, $\sigma_{cJ}^{I'b} \underline{\Theta}_{ab}^c$ and $\underline{\Theta}_{ac}^b$. The estimate on the first two terms follows directly from the induction assumption (3.2.4a). On the other hand, the third term is estimated as follows:

$$\partial^{I_1} Z^{I_2} \sigma_{cJ}^{I'b} \underline{\Theta}_{ab}^c = \sum_{\substack{I_3 + I_4 = I_1 \\ I_5 + I_6 = I_2}} \partial^{I_3} Z^{I_5} \sigma_{cJ}^{I'b} \partial^{I_4} Z^{I_6} \underline{\Theta}_{ab}^c.$$

The first factor is bounded by the induction assumption (3.2.4a) and for the second factor, remarking that $\underline{\Theta}_{ab}^c$ is a linear combination of Ψ_γ^δ , can be bounded by (2.2.6). In the same manner we can get the desired estimate on $\underline{\Theta}_{ac}^b$. So, (3.2.4a) is established in the case $Z_1 = L_a$, $|I| = k + 1$.

In the same way, $\rho_{cJ}^{I'\gamma}$ are linear combinations of $\Psi_a^0 \rho_{cJ'}^{I'\gamma}$, $L_a \rho_{cJ'}^{I'\gamma}$, $\rho_{cJ'}^{I'\gamma}$, $\rho_{cJ'}^{I'\gamma} \Theta_{a\gamma}^\beta$. The second and the third terms are bounded by the induction assumption (3.2.4b). The first term is estimated by applying (2.2.6) and the induction assumption (3.2.4b). For the last term, note that $\Theta_{a\gamma}^\beta$ are linear combinations of $\Psi_\delta^{\delta'}$ and constants. Hence, by applying (2.2.5) and the induction assumption (3.2.4b), the desired bound is reached. Finally, by combining these estimates together, (3.2.4b) is established in the case $|I| = k + 1$, which completes the argument and, consequently, the proof of Lemma 3.2. \square

3.3 Estimates of the commutators

The following statement is now immediate in view of (3.1.5), (3.2.1), and (3.2.3).

Lemma 3.4. *For any function u defined in the cone \mathcal{K} , the following estimates hold:*

$$|[Z^I, \underline{\partial}_\alpha]u| + |[Z^I, \partial_\alpha]u| \leq C(n, |I|) \sum_{\beta, |J| < I} |\partial_\beta Z^J u|, \quad (3.3.1)$$

$$|[Z^I, \underline{\partial}_a]u| \leq C(n, |I|) \sum_{b, |J_1| < |I|} |\underline{\partial}_b Z^{J_1} u| + C(n, |I|) \sum_{|J_2| \leq |I|-1} |t^{-1} \partial_\gamma Z^{J_2} u|. \quad (3.3.2)$$

Furthermore, from (3.1.5), (3.2.1), and (3.2.3), we deduce the following estimates. Recall the convention $Z^I = 0$ when $|I| < 0$.

Lemma 3.5. *For any function u defined in the cone \mathcal{K} , the following estimates hold:*

$$|[Z^I, \partial_\alpha \partial_\beta]u| \leq C(n, |I|) \sum_{\substack{\gamma, \gamma' \\ |J| < |I|}} |\partial_\gamma \partial_{\gamma'} Z^J u|, \quad (3.3.3)$$

$$\begin{aligned} |[Z^I, \underline{\partial}_a \underline{\partial}_\beta]u| + |[Z^I, \underline{\partial}_\alpha \underline{\partial}_b]u| &\leq C(n, |I|) \sum_{\substack{a, \beta \\ |J_1| \leq |I|}} |\underline{\partial}_a \underline{\partial}_\beta Z^{J_1} u| \\ &+ C(n, |I|) t^{-1} \sum_{\substack{\gamma \\ |J_2| \leq |I|}} |\partial_\gamma Z^{J_2} u|. \end{aligned} \quad (3.3.4)$$

Proof. 1. To derive (3.3.3), we write

$$[Z^I, \partial_\alpha \partial_\beta]u = \partial_\alpha([Z^I, \partial_\beta]u) + [Z^I, \partial_\alpha] \partial_\beta u$$

and, by applying (3.1.5), the first term in right-hand-side can be written as

$$\partial_\alpha([Z^I, \partial_\beta]u) = \partial_\alpha \left(\sum_{|J| < |I|} \theta_{\beta J}^{I\gamma} \partial_\gamma Z^J u \right) = \sum_{|J| < |I|} \theta_{\beta J}^{I\gamma} \partial_\alpha \partial_\gamma Z^J u.$$

This expression is bounded by $C(n, |I|) \sum_{|J| < |I|} \sum_{\alpha, \beta} |\partial_\alpha \partial_\beta Z^J u|$. The second term

is estimated as follows:

$$\begin{aligned}
[Z^I, \partial_\alpha] \partial_\beta u &= \sum_{|J| < |I|} \theta_{\alpha J}^{I\gamma} \partial_\gamma Z^J \partial_\beta u \\
&= \sum_{|J| < |I|} \theta_{\alpha J}^{I\gamma} \partial_\gamma \partial_\beta Z^J u + \sum_{|J| < |I|} \theta_{\alpha J}^{I\gamma} \partial_\gamma [Z^J, \partial_\beta] u \\
&= \sum_{|J| < |I|} \theta_{\alpha J}^{I\gamma} \partial_\gamma \partial_\beta Z^J u + \sum_{|J| < |I|} \theta_{\alpha J}^{I\gamma} \partial_\gamma \left(\sum_{|J'| < |J|} \theta_{\beta J'}^{J\delta} \partial_\delta Z^{J'} u \right) \\
&= \sum_{|J| < |I|} \theta_{\alpha J}^{I\gamma} \partial_\gamma \partial_\beta Z^J u + \sum_{\substack{|J| < |I| \\ |J'| < |J|}} \theta_{\alpha J}^{I\gamma} \theta_{\beta J'}^{J\delta} \partial_\gamma \partial_\delta Z^{J'} u.
\end{aligned}$$

This latter expression is bounded by $C(n, |I|) \sum_{\substack{\alpha, \beta \\ |J| < |I|}} |\partial_\alpha \partial_\beta Z^J u|$. Therefore, (3.3.3) is established.

2. We now derive (3.3.4) and we will begin with $[Z^I, \underline{\partial}_a \underline{\partial}_\beta] u$. By (3.2.1) and (3.2.3),

$$\begin{aligned}
u &= \underline{\partial}_a [Z^I, \underline{\partial}_\beta] u + [Z^I, \underline{\partial}_a] \underline{\partial}_\beta u \\
&= \underline{\partial}_a \left(\sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \partial_\gamma Z^J u \right) + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c Z^J \underline{\partial}_\beta u + t^{-1} \sum_{|J| < |I|} \rho_{aJ}^{I\gamma} \partial_\gamma Z^J \underline{\partial}_\beta u \\
&= \sum_{|J| < |I|} \underline{\partial}_a \underline{\theta}_{\beta J}^{I\gamma} \partial_\gamma Z^J u + \sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \underline{\partial}_a \partial_\gamma Z^J u \\
&\quad + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c Z^J \underline{\partial}_\beta u + t^{-1} \sum_{|J| < |I|} \rho_{aJ}^{I\gamma} \partial_\gamma Z^J \underline{\partial}_\beta u,
\end{aligned}$$

thus

$$\begin{aligned}
u &= \sum_{|J| < |I|} \underline{\partial}_a \underline{\theta}_{\beta J}^{I\gamma} \partial_\gamma Z^J u + \sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \underline{\partial}_a \partial_\gamma Z^J u \\
&\quad + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c \underline{\partial}_\beta Z^J u + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c ([Z^J, \underline{\partial}_\beta] u) + t^{-1} \sum_{|J| < |I|} \rho_{aJ}^{I\gamma} \partial_\gamma Z^J \underline{\partial}_\beta u,
\end{aligned}$$

and, therefore,

$$\begin{aligned}
u &= \sum_{|J| < |I|} \underline{\partial}_a \underline{\theta}_{\beta J}^{I\gamma} \partial_\gamma Z^J u + \sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \underline{\partial}_a (\Psi_\gamma^{\gamma'} \underline{\partial}_{\gamma'} Z^J u) \\
&\quad + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c \underline{\partial}_\beta Z^J u + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c ([Z^J, \underline{\partial}_\beta] u) + t^{-1} \sum_{|J| < |I|} \rho_{aJ}^{I\gamma} \partial_\gamma Z^J \underline{\partial}_\beta u \\
&= \sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \Psi_\gamma^{\gamma'} \underline{\partial}_a \underline{\partial}_{\gamma'} Z^J u + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c \underline{\partial}_\beta Z^J u \\
&\quad + \sum_{|J| < |I|} \underline{\theta}_{\beta J}^{I\gamma} \underline{\partial}_a (\Psi_\gamma^{\gamma'}) \underline{\partial}_{\gamma'} Z^J u + \sum_{|J| < |I|} \underline{\partial}_a \underline{\theta}_{\beta J}^{I\gamma} \partial_\gamma Z^J u \\
&\quad + \sum_{|J| < |I|} \sigma_{aJ}^{Ic} \underline{\partial}_c ([Z^J, \underline{\partial}_\beta] u) + t^{-1} \sum_{|J| < |I|} \rho_{aJ}^{I\gamma} \partial_\gamma Z^J \underline{\partial}_\beta u.
\end{aligned}$$

Here, the first and second terms can be bounded by

$$\sum_{a,\beta,|J|<|I|} |\underline{\partial}_a \underline{\partial}_\beta Z^J u|.$$

Recall that $|\underline{\partial}_a \Psi_\gamma^{\gamma'}| \leq C(n)t^{-1}$ and $|\underline{\partial}_a \underline{\theta}_{\beta J}^{J\gamma}| \leq C(n, |I|)t^{-1}$, the third, fourth and last terms can be bounded by $C(n, |I|)t^{-1} \sum_{\gamma, |J_2| \leq |I|} |\partial_\gamma Z^{J_2} u|$.

So we focus on the fifth term:

$$\begin{aligned} & \sum_{|J|<|I|} \sigma_{aJ}^{Ic} \underline{\partial}_c ([Z^J, \underline{\partial}_\beta] u) \\ &= \sum_{|J|<|I|} \sigma_{aJ}^{Ic} \underline{\partial}_c \left(\sum_{|J'|<|J|} \underline{\theta}_{\beta J'}^{J\gamma} \partial_\gamma u \right) \\ &= \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\partial}_c (\underline{\theta}_{\beta J'}^{J\gamma}) \partial_\gamma u + \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\theta}_{\beta J'}^{J\gamma} \underline{\partial}_c \partial_\gamma u \\ &= \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\partial}_c (\underline{\theta}_{\beta J'}^{J\gamma}) \partial_\gamma u + \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\theta}_{\beta J'}^{J\gamma} \underline{\partial}_c (\Psi_\gamma^{\gamma'} \partial_{\gamma'} u) \end{aligned}$$

so

$$\begin{aligned} & \sum_{|J|<|I|} \sigma_{\beta J}^{Ic} \underline{\partial}_c ([Z^J, \underline{\partial}_\beta] u) \\ &= \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\partial}_c (\underline{\theta}_{\beta J'}^{J\gamma}) \partial_\gamma u + \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\theta}_{\beta J'}^{J\gamma} \underline{\partial}_c (\Psi_\gamma^{\gamma'}) \underline{\partial}_{\gamma'} u \\ & \quad + \sum_{\substack{|J|<|I| \\ |J'|<|J|}} \sigma_{aJ}^{Ic} \underline{\theta}_{\beta J'}^{J\gamma} \Psi_\gamma^{\gamma'} \underline{\partial}_c \underline{\partial}_{\gamma'} u. \end{aligned}$$

Similarly, the first two terms can be bounded by $C(n, |I|)t^{-1} \sum_{\gamma, |J_2| \leq |I|} |\partial_\gamma Z^{J_2} u|$ and the last term can be bounded by $\sum_{a,\beta,|J|<|I|} |\underline{\partial}_a \underline{\partial}_\beta Z^J u|$. This completes the proof of (3.3.4).

Now we regard the estimate on $[Z^I, \underline{\partial}_\alpha \underline{\partial}_b] u$. First we perform the following calculation:

$$[Z^I, \underline{\partial}_\alpha \underline{\partial}_b] u = [Z^I, \underline{\partial}_b \underline{\partial}_\alpha] u + [Z^I, [\underline{\partial}_\alpha, \underline{\partial}_b]] u.$$

The estimate on the first term in right-hand-side is already done. We

concentrate on the second term. This leads to the following calculation:

$$\begin{aligned}
[Z^I, [\underline{\partial}_\alpha, \underline{\partial}_b]]u &= [Z^I, \Phi_\alpha^\beta[\partial_\beta, \underline{\partial}_b]]u - [Z^I, \underline{\partial}_b \Phi_\alpha^\beta \partial_\beta]u \\
&= [Z^I, t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma \partial_\gamma]u - [Z^I, \underline{\partial}_b \Phi_\alpha^\beta \partial_\beta]u \\
&= \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma) Z^{I_2} \partial_\gamma u + t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma [Z^I, \partial_\gamma]u \\
&\quad - \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (\underline{\partial}_b \Phi_\alpha^\beta) Z^{I_2} \partial_\beta u - \underline{\partial}_b \phi_\alpha^\beta [Z^I, \partial_\beta]u \\
&= \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma) \partial_\gamma Z^{I_2} u \\
&\quad + \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma) [Z^{I_2}, \partial_\gamma]u + t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma [Z^I, \partial_\gamma]u \\
&\quad - \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (\underline{\partial}_b \Phi_\alpha^\beta) \partial_\beta Z^{I_2} u \\
&\quad - \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (\underline{\partial}_b \Phi_\alpha^\beta) [Z^{I_2}, \partial_\beta]u - \underline{\partial}_b \phi_\alpha^\beta [Z^I, \partial_\beta]u.
\end{aligned}$$

So, we have

$$\begin{aligned}
&[Z^I, [\underline{\partial}_\alpha, \underline{\partial}_b]]u \\
&= \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma) \partial_\gamma Z^{I_2} u + \sum_{I_1+I_2=I} Z^{I_1} (t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma) [Z^{I_2}, \partial_\gamma]u \\
&\quad - \sum_{\substack{I_1+I_2=I \\ |I_2| < |I|}} Z^{I_1} (\underline{\partial}_b \Phi_\alpha^\beta) \partial_\beta Z^{I_2} u - \sum_{I_1+I_2=I} Z^{I_1} (\underline{\partial}_b \Phi_\alpha^\beta) [Z^{I_2}, \partial_\beta]u.
\end{aligned}$$

We remark that the function $t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma$ and $\underline{\partial}_b \Phi_\alpha^\beta$ are homogeneous of degree -1 and C^∞ in the closed region $\{t \geq 1, |x| \leq t\}$. Then by Lemma 2.2, we have

$$\begin{aligned}
|Z^{I_1} (t^{-1} \Phi_{\alpha \underline{\beta} b}^\beta \Gamma_\gamma^\gamma)| &\leq C(n, |I|) t^{-1}, \\
|Z^{I_1} (\underline{\partial}_b \Phi_\alpha^\beta)| &\leq C(n, |I|) t^{-1}.
\end{aligned}$$

So we conclude by

$$|[Z^I, [\underline{\partial}_\alpha, \underline{\partial}_b]]u| \leq C(n, |I|) t^{-1} \sum_{|J| < |I|} |\partial_\gamma Z^J u|.$$

This complete the proof of (3.3.4). \square

Lemma 3.6. *For any function u defined in the cone \mathcal{K} , the following estimates hold:*

$$|Z^I((s/t)\partial_\alpha u)| \leq |(s/t)\partial_\alpha Z^I u| + C(n, |I|) \sum_{\beta, |J| < |I|} |(s/t)\partial_\beta Z^J u|. \quad (3.3.5)$$

To prove this lemma, we need the following result:

Lemma 3.7. *Let I be a multi-index. Then*

$$\Xi^I := (t/s)Z^I(s/t) \quad (3.3.6)$$

is a C^∞ function defined in the closed cone $\overline{\mathcal{K}} = \{|x| \leq t - 1\}$ and all of its derivatives (of any order) are bounded in $\overline{\mathcal{K}}$. Furthermore, it is homogeneous of degree η with $\eta \leq 0$.

We admit here this result and give the proof of (3.3.5). The proof of (3.3.6) will be given at the end of this chapter.

Proof of (3.3.5). We observe that

$$\begin{aligned} [Z^I, (s/t)\partial_\alpha]u &= \sum_{\substack{I_1+I_2=I \\ |I_1| \leq |I|-1}} Z^{I_2}(s/t) Z^{I_1} \partial_\alpha u + (s/t)[Z^I, \partial_\alpha]u \\ &= \sum_{\substack{I_1+I_2=I \\ |I_1| \leq |I|-1}} Z^{I_2}(s/t) [Z^{I_1}, \partial_\alpha]u + \sum_{\substack{I_1+I_2=I \\ |I_1| \leq |I|-1}} Z^{I_2}(s/t) \partial_\alpha Z^{I_1}u \\ &\quad + (s/t)[Z^I, \partial_\alpha]u \\ &= \sum_{I_1+I_2=I} Z^{I_2}(s/t) [Z^{I_1}, \partial_\alpha]u + \sum_{\substack{I_1+I_2=I \\ |I_1| \leq |I|-1}} Z^{I_2}(s/t) \partial_\alpha Z^{I_1}. \end{aligned}$$

By applying (3.3.6), we find

$$\begin{aligned} \left| \sum_{I_1+I_2=I} Z^{I_2}(s/t) [Z^{I_1}, \partial_\alpha]u \right| &\leq \sum_{I_1+I_2=I} |Z^{I_2}(s/t)| |[Z^{I_1}, \partial_\alpha]u| \\ &\leq C(n, |I|) \sum_{|J| \leq |I|-1} |(s/t)\partial_\alpha Z^J u| \end{aligned}$$

and

$$\left| \sum_{\substack{I_1+I_2=I \\ |I_1| \leq |I|-1}} Z^{I_2}(s/t) \partial_\alpha Z^{I_1}u \right| \leq C(n, |I|) \sum_{|J| \leq |I|-1} |(s/t)\partial_\alpha Z^J u|.$$

Consequently, (3.3.5) is proven. \square

Proof of (3.3.6). First we remark the following identities:

$$\begin{aligned} (t/s)L_a(s/t) &= -x^a/t, \\ (t/s)\partial_a(s/t) &= -x^a/s^2, \quad (t/s)\partial_t(s/t) = |x|^2s^{-2}t^{-1}. \end{aligned} \tag{3.3.7}$$

Remark that in the cone $\mathcal{K} = \{|x| < t - 1\}$, the function x^a/s^2 , $|x|^2s^{-2}t^{-1}$, x^a/t are bounded and C^∞ . x^a/s^2 and $|x|^2s^{-2}t^{-1}$ are homogeneous of degree -1 while x^a/t is homogeneous of degree 0 , we also remark that all of their derivatives (of any order) are bounded in $\overline{\mathcal{K}}$. Then we have proved (3.3.6) in the case where $|I| = 1$.

For the case $|I| > 1$, we perform the induction on $|I|$. Assume that for $|I| \leq k$, (3.3.6) holds. Then for an operator Z^I with $|I| = k + 1$, we suppose that $Z^I = Z_1 Z^{I'}$ where $|I'| = k$. Then

$$(t/s)Z^I(s/t) = (t/s)Z_1 Z^{I'}(s/t) = Z_1((t/s)Z^{I'}(s/t)) - Z_1(s/t)Z^{I'}(s/t),$$

where Z_1 can be ∂_α or L_a . Remark that by the assumption of induction and (3.3.7), the second term is a C^∞ function homogeneous of degree non-positive and all of its derivatives are bounded in $\overline{\mathcal{K}}$. Now we focus on the first term. By the assumption of induction, $(t/s)Z^{I'}(s/t)$ is C^∞ , homogeneous of degree non-positive and bounded in $\overline{\mathcal{K}}$. We need to distinguish two different cases.

CASE 1, $Z_1 = \partial_\alpha$. In this case, $Z_1((t/s)Z^{I'}(s/t))$ is homogeneous of degree less than or equal to -1 , and by the assumption of induction, all of its derivatives are bounded in $\overline{\mathcal{K}}$.

CASE 2, $Z_1 = L_a$. In this case, $Z_1((t/s)Z^{I'}(s/t))$ is homogeneous of degree less than or equal to 0 .

$$Z_1((t/s)Z^{I'}(s/t)) = x^a\partial_t((t/s)Z^{I'}(s/t)) + t\partial_a((t/s)Z^{I'}(s/t)).$$

Denote by $f(t, x) = \partial_\alpha((t/s)Z^{I'}(s/t))$ and recall that $f(t, x)$ is homogeneous of degree η with $\eta \leq -1$ so

$$f(t, x) = (t/2)^\eta f(2, 2x/t).$$

Recall that $f(2, x)$ is bounded when $|x| \leq 2$. Then (recall that $t \geq 1$ in $t \geq 1$)

$$|\partial_\alpha((t/s)Z^{I'}(s/t))| = |f(t, x)| \leq C(n, I')t^\eta \leq C(n, I')t^{-1}.$$

Then

$$|Z_1((t/s)Z^{I'}(s/t))| \leq C(n, I')(1 + |x^a|/t).$$

Taking into consideration the fact that in $\overline{\mathcal{K}}$, $|x^a| \leq t$, the desired result is proven. \square

Chapter 4

The null structure in the semi-hyperboidal frame

4.1 Estimating first-order derivatives

In this chapter, we discuss the null condition and derive estimates on several null quadratic forms. A quadratic form $T^{\alpha\beta}\partial_\alpha u\partial_\beta v$ acting on the gradient of two functions u, v is said to satisfy the **null condition** if, for all null vectors $\xi \in \mathbb{R}^4$, i.e. satisfying $(\xi_0)^2 - \sum_a (\xi_a)^2 = 0$, one has

$$T^{\alpha\beta}\xi_\alpha\xi_\beta = 0. \quad (4.1.1)$$

Similarly, a cubic form $A^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma$ is said to satisfy the null condition if, for all null vectors, one has

$$A^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0. \quad (4.1.2)$$

All of the terms of interest here are linear combinations of factors $\partial_\alpha u\partial_\beta v$ and $\partial_\gamma u\partial_\alpha\partial_\beta v$ and $u\partial_\alpha\partial_\beta v$. Throughout, we constantly use the notation $s^2 = t^2 - r^2$.

Proposition 4.1 (Bounds for null forms). *If $T^{\alpha\beta}\xi_\alpha\xi_\beta$ is a quadratic form satisfying the null condition, then for every multi-index I there exists a constant $C(I) > 0$ such that*

$$|Z^I \underline{T}^{00}| \leq C(I) \frac{(t^2 - r^2)}{t^2} = C(I) (s/t)^2 \quad \text{in the cone } \mathcal{K}. \quad (4.1.3)$$

Similarly, if a cubic form $A^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma$ satisfies the null condition, then for every multi-index I there exists a constant $C(I) > 0$ such that

$$|Z^I \underline{A}^{000}| \leq C(I) \frac{(t^2 - r^2)}{t^2} = C(I) (s/t)^2 \quad \text{in the cone } \mathcal{K}. \quad (4.1.4)$$

We first state an elementary estimate based on homogeneity, which is immediate from the observation that, with the notation in the statement below, $Z^I f$ is homogeneous of degree $\eta' \leq \eta$; namely, the proof is similar to the one of Lemma 2.2.

Lemma 4.1 (Homogeneity lemma). *Let f be a C^∞ function defined in the closed set $\{t \geq 1\} \cap \{t/2 \leq r \leq t\}$. Assume that f is homogeneous (in first variable) of degree η in the sense that $f(t, x) = t^\eta f(1, x/t)$. Then, the following estimate holds:*

$$|Z^I f(t, x)| \leq C(n, |I|) t^\eta \quad \text{in the region } \{r \geq t/2\} \cap \mathcal{K}. \quad (4.1.5)$$

Proof of Proposition 4.1. We use the notation $\omega_a = \omega^a := x^a/|x|$ and $\omega_0 = -\omega^0 := -1$, so that $(\omega_0)^2 - \sum_a (\omega_a)^2 = 0$. We consider the component \underline{T}^{00} and write

$$\begin{aligned} \underline{T}^{00} &= T^{\alpha\beta} \Psi_\alpha^0 \Psi_\beta^0 = T^{\alpha\beta} \Psi_\alpha^0 \Psi_\beta^0 - T^{\alpha\beta} \omega_\alpha \omega_\beta \\ &= T^{\alpha\beta} (\Psi_\alpha^0 \Psi_\beta^0 - \omega_\alpha \omega_\beta). \end{aligned}$$

First, when $r \leq t/2$, we have

$$Z^I (T^{\alpha\beta} \Psi_\alpha^0 \Psi_\beta^0) = \sum_{I_1+I_2=I} T^{\alpha\beta} Z^{I_1} \Psi_\alpha^0 Z^{I_2} \Psi_\beta^0.$$

Applying Lemma 2.2, we obtain

$$|Z^I (T^{\alpha\beta} \Psi_\alpha^0 \Psi_\beta^0)| \leq \sum_{I_1+I_2=I} K |Z^{I_1} \Psi_\alpha^0| |Z^{I_2} \Psi_\beta^0| \leq C(n, |I|) K (s/t)^2 (t/s)^2.$$

Recall that when $0 \leq r \leq t/2$, we have $(t/s)^2 \leq 4/3$ so in the region $\mathcal{K} \cap \{r \leq t/2\}$,

$$|Z^I (T^{\alpha\beta} \Psi_\alpha^0 \Psi_\beta^0)| \leq C(n, |I|) K (s/t)^2.$$

Second, in the region $\{r \geq t/2\} \cap \mathcal{K}$, we have the expression

$$\underline{T}^{00} = T^{\alpha\beta} (\Psi_\alpha^0 \Psi_\beta^0 - \omega_\alpha \omega_\beta)$$

and thus

$$Z^I \underline{T}^{00} = T^{\alpha\beta} Z^I (\Psi_\alpha^0 \Psi_\beta^0 - \omega_\alpha \omega_\beta).$$

When $\alpha = \beta = 0$, $(\Psi_\alpha^0 \Psi_\beta^0 - \omega_\alpha \omega_\beta) = 0$. When $\alpha = a > 0, \beta = 0$, we have

$$\begin{aligned} Z^I (\Psi_\alpha^0 \Psi_\beta^0 - \omega_\alpha \omega_\beta) &= Z^I (\omega_a (1 - (r/t))) \\ &= \sum_{I_1+I_2=I} Z^{I_1} \omega_a Z^{I_2} \left(\frac{t-r}{t} \right). \end{aligned}$$

When $\alpha = a > 0, \beta = 0 > 0$, we obtain

$$\begin{aligned} Z^I (\Psi_\alpha^0 \Psi_\beta^0 - \omega_\alpha \omega_\beta) &= Z^I \left(\frac{x^a x^b}{t^2} - \frac{x^a x^b}{r^2} \right) \\ &= \sum_{I_1+I_2+I_3+I_4=I} Z^{I_1} \omega_a Z^{I_2} \omega_b Z^{I_3} \left(1 + \frac{r}{t} \right) Z^{I_4} \left(\frac{r-t}{t} \right). \end{aligned}$$

We focus on the estimates of $Z^I \omega_a$, $Z^I(1 + \frac{r}{t})$ and $Z^I(\frac{t-r}{t})$. By Lemma 4.1, $Z^I \omega_a$ and $Z^I(1 + \frac{r}{t})$ are bounded by $C(n, |I|)$. For the estimate of $Z^I(\frac{t-r}{t})$, we perform the following calculation:

$$\frac{t-r}{t} = \frac{s^2}{t^2} \frac{t}{t+r},$$

and then

$$Z^I((t-r)/t) = \sum_{I_1+I_2+I_3=I} Z^{I_1}(t/(t+r)) Z^{I_2}(s/t) Z^{I_3}(s/t).$$

We observe that $t/(t+r)$ is C^∞ regular, defined in $\{t \geq 1\} \cap \{t/2 \leq r \leq t\}$, and homogeneous of degree 0. Then, we have

$$|Z^{I_1}(t/(t+r))| \leq C(|I|).$$

The term $Z^{I_2}(s/t)$ is bounded by $C(|I|)(s/t)$ by applying Lemma 3.7. Then we conclude with

$$|Z^I((t-r)/t)| \leq C(|I|)(s/t)^2,$$

which proves the desired result. \square

Then we have the following estimates on the null quadratic forms.

Proposition 4.2. *For any null quadratic form $T^{\alpha\beta} \partial_\alpha u \partial_\beta v$, where $T^{\alpha\beta}$ are constants, and for any multi-index I , one has*

$$\begin{aligned} & |Z^I(T^{\alpha\beta} \partial_\alpha u \partial_\beta v)| \\ & \leq CK(s/t)^2 \sum_{|I_1|+|I_2| \leq |I|} |Z^{I_1} \partial_t u Z^{I_2} \partial_t v| \\ & \quad + CK \sum_{\substack{\alpha, \beta, \\ |I_1|+|I_2| \leq |I|}} \left(|Z^{I_1} \underline{\partial}_\alpha u Z^{I_2} \underline{\partial}_\beta v| + |Z^{I_1} \underline{\partial}_\beta u Z^{I_2} \underline{\partial}_\alpha v| \right), \end{aligned}$$

where $K := \max_{\alpha, \beta} |T^{\alpha\beta}|$.

The importance of this estimate relies in the factor $(s/t)^2$ before the component of $\partial_t u \partial_t v$. As we will see later, the other derivative term $\underline{\partial}_a$ enjoys better L^∞ and L^2 estimates in our framework. The derivatives of direction ∂_t do not always have enough decay, and the factor $(s/t)^2$ precisely allows us to overcome this lack of sufficient L^2 and L^∞ decay.

Proof. The proof relies on a change of frame. In the semi-hyperboloidal frame we have

$$\begin{aligned} T^{\alpha\beta} \partial_\alpha u \partial_\beta v &= \underline{T}^{\alpha\beta} \underline{\partial}_\alpha u \underline{\partial}_\beta v \\ &= \underline{T}^{00} \partial_t u \partial_t v + \underline{T}^{0b} \partial_t u \underline{\partial}_b v + \underline{T}^{a0} \underline{\partial}_a u \partial_t v + \underline{T}^{ab} \underline{\partial}_a u \underline{\partial}_b v. \end{aligned}$$

Then, we have

$$\begin{aligned}
& Z^I(T^{\alpha\beta}\partial_\alpha u\partial_\beta v) \\
&= Z^I(\underline{T}^{\alpha\beta}\underline{\partial}_\alpha u\underline{\partial}_\beta v) \\
&= Z^I(\underline{T}^{00}\partial_t u\partial_t v) + Z^I(\underline{T}^{0b}\partial_t u\underline{\partial}_b v) + Z^I(\underline{T}^{a0}\underline{\partial}_a u\partial_t v) + Z^I(\underline{T}^{ab}\underline{\partial}_a u\underline{\partial}_b v) \\
&:= R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

Recalling the null condition satisfied by the null form under consideration and by Proposition 4.1, we get

$$\begin{aligned}
|R_1| &\leq \sum_{I_1+I_2+I_3=I} |Z^{I_3}(\underline{T}^{00}) Z^{I_1}(\partial_t u) \cdot Z^{I_2}(\partial_t v)| \\
&\leq CK(s/t)^2 \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1}\partial_t u| \cdot |Z^{I_2}\partial_t v|.
\end{aligned}$$

The term R_2 are estimated directly. Recall that by Lemma 2.1, $|Z^I \underline{T}^{\alpha\beta}| \leq C(I)K$. Then

$$\begin{aligned}
|R_2| &\leq \sum_{\substack{b, \\ I_1+I_2+I_3=I}} |Z^{I_3}(\underline{T}^{0b}) Z^{I_1}(\partial_t u) \cdot Z^{I_2}(\underline{\partial}_b v)| \\
&\leq CK \sum_{\substack{b, \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}(\partial_t u) \cdot Z^{I_2}(\underline{\partial}_b v)|.
\end{aligned}$$

The terms R_3 and R_4 are estimated similarly. Then the desired result is proved. \square

4.2 Estimating second-order derivatives

In the following sections, we will need to treat the second-order derivatives $B^{\alpha\beta}\partial_\alpha\partial_\beta u$, where $B^{\alpha\beta}$ are constants. A second-order operator of this form is said to satisfy the null condition if

$$B^{\alpha\beta}\xi_\alpha\xi_\beta = 0 \quad \text{for } \xi_0\xi_0 - \sum_{i=1}^3 \xi_i\xi_i = 0.$$

Proposition 4.3. *Let $B^{\alpha\beta}\partial_\alpha\partial_\beta$ be a second-order operator satisfying the null condition with constants $B^{\alpha\beta}$ bounded by K . Then, one has*

$$\begin{aligned}
& |Z^I(B^{\alpha\beta}\partial_\alpha\partial_\beta u)| \\
&\leq C(n, |I|)K(s/t)^2 \sum_{|I_1|\leq|I|} Z^I(\partial_t\partial_t u) + C(n, |I|)K \sum_{\substack{a, \alpha \\ |I_1|\leq|I|}} Z^I(\underline{\partial}_a\partial_\alpha u) \\
&\quad + C(n, |I|)\frac{K}{t} \sum_{\alpha, |I_1|\leq|I|} |Z^{I_1}(\partial_\alpha u)|.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
B^{\alpha\beta} \partial_\alpha \partial_\beta u &= \underline{B}^{\alpha\beta} \underline{\partial}_\alpha \underline{\partial}_\beta u + B^{\alpha\beta} \partial_\alpha (\Psi_\beta^{\beta'}) \underline{\partial}_{\beta'} u \\
&= \underline{B}^{00} \partial_t \partial_t u + \underline{B}^{a0} \underline{\partial}_a \partial_t u + \underline{B}^{0b} \partial_t \underline{\partial}_b u + \underline{B}^{ab} \underline{\partial}_a \underline{\partial}_b u \\
&\quad + B^{\alpha\beta} \partial_\alpha (\Psi_\beta^{\beta'}) \underline{\partial}_{\beta'} u.
\end{aligned}$$

Recall that \mathcal{B} is a null quadratic form, then $|Z^J \underline{B}^{00}| \leq C(n, |J|)K(s/t)^2$ and

$$\begin{aligned}
|Z^I \underline{B}^{00} \partial_t \partial_t u| &\leq \sum_{I_1+I_2=I} |Z^{I_2} \underline{B}^{00}| |Z^{I_1} \partial_t \partial_t u| \\
&\leq C(n, |I|)K(s/t)^2 \sum_{|I_1| \leq |I|} |Z^{I_1} \partial_t \partial_t u|.
\end{aligned}$$

Also, from Lemma 2.1 $|Z^I \underline{B}| \leq C(n, |I|)|B|$. Then we have

$$\begin{aligned}
|Z^I (\underline{B}^{a0} \underline{\partial}_a \partial_t u)| &\leq \sum_{I_1+I_2=I} |Z^{I_2} \underline{B}^{a0}| |Z^{I_1} \underline{\partial}_a \partial_t u| \\
&\leq C(n, |I|)K \sum_{|I_1| \leq |I|} |Z^{I_1} \underline{\partial}_a \partial_t u|.
\end{aligned}$$

For the term $Z^I (\underline{B}^{0b} \partial_t \underline{\partial}_b u)$ by applying (3.1.2) we have

$$\begin{aligned}
|Z^I (\underline{B}^{0b} \partial_t \underline{\partial}_b u)| &\leq \sum_{I_1+I_2=I} |Z^{I_2} \underline{B}^{0b}| |Z^{I_1} \partial_t \underline{\partial}_b u| \\
&\leq C(n, |I|)K \sum_{|I_1| \leq |I|} |Z^{I_1} \partial_t \underline{\partial}_b u|
\end{aligned}$$

and

$$\begin{aligned}
|Z^{I_1} \partial_t \underline{\partial}_b u| &\leq |Z^{I_1} \underline{\partial}_b \partial_t u| + |Z^{I_1} [\underline{\partial}_b, \partial_t] u| \\
&\leq |Z^{I_1} \underline{\partial}_b \partial_t u| + |Z^{I_1} (t^{-1} \underline{\Gamma}_{0b}^\gamma \partial_\gamma u)| \\
&\leq |Z^{I_1} \underline{\partial}_b \partial_t u| + \sum_{I_3+I_4=I_2} |Z^{I_4} (t^{-1} \underline{\Gamma}_{0b}^\gamma) Z^{I_3} \partial_\gamma u| \\
&\leq |Z^{I_1} \underline{\partial}_b \partial_t u| + Ct^{-1} \sum_{|I_3| \leq |I_1|} |Z^{I_3} \partial_\gamma u|.
\end{aligned}$$

Here, we have used the estimate $|Z^J (t^{-1} \underline{\Gamma}_{0b})| \leq C(n, |J|)$. So we have proven that

$$|Z^{I_1} \underline{B}^{0b} \partial_t \underline{\partial}_b u| \leq C(n, |I|)K \sum_{|I_1| \leq |I|} |Z^{I_1} \underline{\partial}_b \partial_t u| + C(n, |I|)Kt^{-1} \sum_{|I_1| \leq |I|} |Z^{I_1} \partial_\gamma u|.$$

The term $Z^I(\underline{B}^{ab}\underline{\partial}_a\underline{\partial}_b u)$ is estimated as follows:

$$\begin{aligned} |Z^I(\underline{B}^{ab}\underline{\partial}_a\underline{\partial}_b u)| &\leq C(n, |I|) \sum_{\substack{a,b \\ I_1+I_2=I}} |Z^{I_2}\underline{B}^{ab}| |Z^{I_1}\underline{\partial}_a\underline{\partial}_b u| \\ &\leq C(n, |I|)K \sum_{\substack{\beta,a \\ I_1+I_2=I}} |Z^{I_1}\underline{\partial}_a\underline{\partial}_\beta u|. \end{aligned}$$

The term $Z^I(B^{\alpha\beta}\partial_\alpha(\Psi_\beta^{\beta'})\underline{\partial}_{\beta'}u)$ is estimated by applying the decay rate supplied by $Z^J|\partial_\alpha(\Psi_\beta^{\beta'})| \leq C(n, |J|)t^{-1}$. This completes the proof. \square

As a direct corollary, the following estimates hold.

Proposition 4.4. *Let $\mathcal{B}(u, \partial\partial v)$ be a bilinear form acting on the function u and the Hessian form of v by*

$$\mathcal{B}(u, \partial\partial v) = B^{\alpha\beta}u\partial_\alpha\partial_\beta v.$$

We assume that the quadratic form $B^{\alpha\beta}\partial_\alpha\partial_\beta$ satisfies the null condition. Then the following estimates hold:

$$\begin{aligned} |Z^I(B^{\alpha\beta}u\partial_\alpha\partial_\beta v)| &\leq C(n, |I|)K(s/t)^2 \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1}u| |Z^{I_2}\partial_t\partial_t v| \\ &\quad + C(n, |I|)K \sum_{\substack{\alpha,b \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}u| |Z^{I_2}\underline{\partial}_\alpha\underline{\partial}_b v| \\ &\quad + C(n, |I|)K \sum_{\substack{\alpha,\beta \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}u| |Z^{I_2}\underline{\partial}_\alpha\underline{\partial}_\beta v| \\ &\quad + C(n, |I|)Kt^{-1} \sum_{\substack{\alpha \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}u| |Z^{I_2}\underline{\partial}_\alpha v|, \end{aligned} \tag{4.2.1a}$$

$$\begin{aligned} |[Z^I, B^{\alpha\beta}u\partial_\alpha\partial_\beta]v| &\leq C(n, |I|)K(s/t)^2 \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|}} |Z^{I_1}u| |Z^{I_2}\partial_t\partial_t v| \\ &\quad + C(n, |I|)K \sum_{a,\beta} \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|}} |Z^{I_1}u| |Z^{I_2}\underline{\partial}_a\underline{\partial}_\beta v| \\ &\quad + C(n, |I|)K \sum_{\alpha,b} \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|}} |Z^{I_1}u| |Z^{I_2}\underline{\partial}_\alpha\underline{\partial}_b v| \\ &\quad + C(n, |I|)Kt^{-1} \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|,\beta}} |Z^{I_1}u| |Z^{I_2}\partial_{\beta'} v|. \end{aligned} \tag{4.2.1b}$$

Proof. We first establish (4.2.1a):

$$\begin{aligned}
Z^I (B^{\alpha\beta} u \partial_\alpha \partial_\beta v) &= Z^I (\underline{B}^{\alpha\beta} u \underline{\partial}_\alpha \underline{\partial}_\beta v + B^{\alpha\beta} u \partial_\alpha \Psi_\beta^{\beta'} \underline{\partial}_{\beta'} v) \\
&= \sum_{I_1+I_2+I_3=I} Z^{I_3} \underline{B}^{\alpha\beta} Z^{I_1} u Z^{I_2} \underline{\partial}_\alpha \underline{\partial}_\beta v \\
&\quad + \sum_{I_1+I_2+I_3=I} B^{\alpha\beta} Z^{I_1} u Z^{I_3} \partial_\alpha \Psi_\beta^{\beta'} Z^{I_2} \underline{\partial}_{\beta'} v =: R_1 + R_2,
\end{aligned}$$

in which R_1 can be estimated as follows:

$$\begin{aligned}
R_1 &= \sum_{I_1+I_2+I_3=I} Z^{I_3} \underline{B}^{\alpha\beta} Z^{I_1} u Z^{I_2} \underline{\partial}_\alpha \underline{\partial}_\beta v \\
&= \sum_{I_1+I_2+I_3=I} Z^{I_3} \underline{B}^{00} Z^{I_1} u Z^{I_2} \partial_t \partial_t v \\
&\quad + \sum_{I_1+I_2+I_3=I} Z^{I_3} \underline{B}^{a0} Z^{I_1} u Z^{I_2} \underline{\partial}_a \partial_t v + \sum_{I_1+I_2+I_3=I} Z^{I_3} \underline{B}^{0b} Z^{I_1} u Z^{I_2} \partial_t \underline{\partial}_b v \\
&\quad + \sum_{I_1+I_2+I_3=I} Z^{I_3} \underline{B}^{ab} Z^{I_1} u Z^{I_2} \underline{\partial}_a \underline{\partial}_b v.
\end{aligned}$$

Recall that $|Z^I \underline{B}^{\alpha\beta}| \leq C(n, |I|)K$ and $|Z^I \underline{B}^{00}| \leq C(n, |I|)(s/t)^2$, then

$$\begin{aligned}
|R_1| &\leq C(n, |I|)K(s/t)^2 \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1} u| |Z^{I_2} \partial_t \partial_t v| \\
&\quad + C(n, |I|)K \sum_{\substack{\alpha, b \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1} u| |Z^{I_2} \underline{\partial}_\alpha \underline{\partial}_b v| \\
&\quad + C(n, |I|)K \sum_{\substack{a, \beta \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1} u| |Z^{I_2} \underline{\partial}_a \underline{\partial}_\beta v|.
\end{aligned}$$

Observe now that $|Z^I \partial_\alpha \Psi_\beta^{\beta'}| \leq C(n, |I|)t^{-1}$, then R_2 is bounded by

$$\begin{aligned}
|R_2| &\leq \sum_{I_1+I_2+I_3=I} |B^{\alpha\beta}| |Z^{I_1} u| |Z^{I_3} \partial_\alpha \Psi_\beta^{\beta'}| |Z^{I_2} \underline{\partial}_{\beta'} v| \\
&\leq C(n, |I|)Kt^{-1} \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1} u| |Z^{I_2} \underline{\partial}_\alpha v|.
\end{aligned}$$

Thus the estimate (4.2.1a) is proven.

Next, in order to derive (4.2.1b), we observe that

$$[Z^I, B^{\alpha\beta} u \partial_\alpha \partial_\beta]v = [Z^I, \underline{B}^{\alpha\beta} u \underline{\partial}_\alpha \underline{\partial}_\beta]v + [Z^I, B^{\alpha\beta} u \partial_\alpha \Psi_\beta^{\beta'} \underline{\partial}_{\beta'}]v =: R_3 + R_4.$$

The term R_3 is decomposed as follows:

$$\begin{aligned}
&[Z^I, \underline{B}^{\alpha\beta} u \underline{\partial}_\alpha \underline{\partial}_\beta]v \\
&= \sum_{\substack{I_1+I_2+I_3 \\ |I_2|<|I|}} Z^{I_3} \underline{B}^{\alpha\beta} Z^{I_1} u Z^{I_2} \underline{\partial}_\alpha \underline{\partial}_\beta v + \underline{B}^{\alpha\beta} u [Z^I, \underline{\partial}_\alpha \underline{\partial}_\beta]v \\
&=: R_5 + R_6.
\end{aligned}$$

The term R_5 is estimated as follows:

$$\begin{aligned}
R_5 &= \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} Z^{I_3} \underline{B}^{\alpha\beta} Z^{I_1} u Z^{I_2} \underline{\partial}_\alpha \underline{\partial}_\beta v \\
&= \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} Z^{I_3} \underline{B}^{00} Z^{I_1} u Z^{I_2} \partial_t \partial_t v \\
&\quad + \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} Z^{I_3} \underline{B}^{a0} Z^{I_1} u Z^{I_2} \underline{\partial}_a \partial_t v + \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} Z^{I_3} \underline{B}^{0b} Z^{I_1} u Z^{I_2} \partial_t \underline{\partial}_b v \\
&\quad + \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} Z^{I_3} \underline{B}^{ab} Z^{I_1} u Z^{I_2} \underline{\partial}_a \underline{\partial}_b v
\end{aligned}$$

and thus

$$\begin{aligned}
|R_5| &\leq \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} |Z^{I_3} \underline{B}^{00}| |Z^{I_1} u| |Z^{I_2} \partial_t \partial_t v| \\
&\quad + \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} |Z^{I_3} \underline{B}^{a0}| |Z^{I_1} u| |Z^{I_2} \underline{\partial}_a \partial_t v| \\
&\quad + \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} |Z^{I_3} \underline{B}^{0b}| |Z^{I_1} u| |Z^{I_2} \partial_t \underline{\partial}_b v| \\
&\quad + \sum_{\substack{I_1+I_2+I_3=I \\ |I_2|<|I|}} |Z^{I_3} \underline{B}^{ab}| |Z^{I_1} u| |Z^{I_2} \underline{\partial}_a \underline{\partial}_b v|,
\end{aligned}$$

hence

$$\begin{aligned}
|R_5| &\leq C(n, |I|) K(s/t)^2 \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|}} |Z^{I_1} u| |Z^{I_2} \partial_t \partial_t v| \\
&\quad + C(n, |I|) K \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|}} |Z^{I_1} u| |Z^{I_2} \underline{\partial}_a \underline{\partial}_\beta v| \\
&\quad + C(n, |I|) K \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_2|<|I|}} |Z^{I_1} u| |Z^{I_2} \underline{\partial}_\alpha \underline{\partial}_b v|.
\end{aligned}$$

The term R_6 is decomposed as follows:

$$\begin{aligned}
\underline{B}^{\alpha\beta} u[Z^I, \underline{\partial}_\alpha \underline{\partial}_\beta] v &= \underline{B}^{00} u[Z^I, \partial_t \partial_t] v + \underline{B}^{a0} u[Z^I, \underline{\partial}_a \partial_t] v \\
&\quad + \underline{B}^{0b} u[Z^I, \partial_t \underline{\partial}_b] v + \underline{B}^{ab} u[Z^I, \underline{\partial}_a \underline{\partial}_b] v.
\end{aligned}$$

Then we apply (3.3.3) and (3.3.4) and the fact that $|\underline{B}^{00}| \leq C(s/t)^2$:

$$\begin{aligned}
& |\underline{B}^{\alpha\beta} u [Z^I, \underline{\partial}_\alpha \underline{\partial}_\beta] v| \\
&= |\underline{B}^{00} u [Z^I, \partial_t \partial_t] v| + |\underline{B}^{a0} u [Z^I, \underline{\partial}_a \partial_t] v| \\
&\quad + |\underline{B}^{0b} u [Z^I, \partial_t \underline{\partial}_b] v| + |\underline{B}^{ab} u [Z^I, \underline{\partial}_a \underline{\partial}_b] v| \\
&\leq C(n, |I|) K (s/t)^2 |u| \sum_{\substack{\gamma, \gamma' \\ |I'| < |I|}} |Z^{I_1} \partial_\gamma \partial_{\gamma'} v| \\
&\quad + C(n, |I|) K |u| \sum_{\substack{\alpha, \beta \\ |I_1| < |I|}} |\underline{\partial}_\alpha \underline{\partial}_\beta Z^{I_1} v| + C(n, |I|) K t^{-1} |u| \sum_{\substack{\alpha, \beta \\ |I_1| < |I|}} |\partial_\gamma Z^{I_1} v|.
\end{aligned}$$

The term R_4 is estimated as follows:

$$\begin{aligned}
& [Z^I, B^{\alpha\beta} u \partial_\alpha \Psi_\beta^{\beta'} \underline{\partial}_{\beta'}] v \\
&= \sum_{\substack{I_1 + I_2 + I_3 = I \\ |I_2| < |I|}} B^{\alpha\beta} Z^{I_1} u Z^{I_3} \partial_\alpha \Psi_\beta^{\beta'} Z^{I_2} \underline{\partial}_{\beta'} v + B^{\alpha\beta} u \partial_\alpha \Psi_\beta^{\beta'} [Z^I, \underline{\partial}_{\beta'}] v.
\end{aligned}$$

Recall the additional decreasing rate supplied by $|Z^I \partial_\alpha \Psi_\beta^{\beta'}| \leq C(n, |I|) t^{-1}$, the first term is bounded by

$$C(n, |I|) K t^{-1} \sum_{\substack{|I_1| + |I_2| \leq |I| \\ |I_2| < |I|, \beta'}} |Z^{I_1} u| |Z^{I_2} \partial_{\beta'} v|.$$

The second term is estimated by (3.3.2) and the additional decreasing rate supplied by $|Z^I \partial_\alpha \Psi_\beta^{\beta'}| \leq C(n, |I|) t^{-1}$. It can also be bounded by

$$C(n, |I|) K t^{-1} \sum_{\substack{|I_1| + |I_2| \leq |I| \\ |I_2| < |I|, \beta}} |Z^{I_1} u| |Z^{I_2} \partial_{\beta'} v|.$$

This complete the proof of (4.2.1b). \square

4.3 Products of first-order and second-order derivatives

The third type of null form we treat is a null quadratic form acting on the gradient and the Hessian, as now stated.

Proposition 4.5. *Let \mathcal{A} be a cubic form acting on the gradient of u and the Hessian matrix of v by*

$$\mathcal{A}(\partial u, \partial \partial v) = A^{\alpha\beta\gamma} \partial_\gamma u \partial_\alpha \partial_\beta v.$$

where u, v are C^∞ function supported in the cone \mathcal{K} . If $A^{\alpha\beta\gamma}$ satisfies the null condition, then the following estimate holds for any multi-index I :

$$\begin{aligned} & |Z^I(A^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v)| \\ & \leq CK(s/t)^2 \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1}\partial_t u Z^{I_2}\partial_t\partial_t v| + CK\Omega_1(I, u, v) + CK\Omega_2(I, u, v), \end{aligned}$$

where

$$\begin{aligned} \Omega_1(I, u, v) &= \sum_{\substack{\alpha, \beta, \gamma \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}\partial_\alpha u| |Z^{I_2}\partial_\beta\partial_\gamma v| + \sum_{\substack{\alpha, b, \gamma \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}\partial_\alpha u| |Z^{I_2}\partial_b\partial_\gamma v| \\ &+ \sum_{\substack{\alpha, \beta, c \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}\partial_\alpha u| |Z^{I_2}\partial_\beta\partial_c v| \end{aligned}$$

and

$$\Omega_2(I, u, v) \leq t^{-1} \sum_{\substack{\alpha, \beta \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}\partial_\alpha u \cdot Z^{I_2}\partial_\beta v|.$$

Proof. We observe the following relation of change of frame:

$$\begin{aligned} A^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v &= \underline{A}^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v + A^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha(\Psi_\beta^{\beta'})\partial_{\beta'} v \\ &:= R_1(u, v) + R_2(u, v). \end{aligned}$$

The term $Z^I R_2$ can be estimated as follows. Recall that $|\partial_\alpha(\Psi_\beta^{\beta'})| \leq \frac{C}{t}$, then

$$Z^I(A^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha(\Psi_\beta^{\beta'})\partial_{\beta'} v) = \sum_{I_1+I_2+I_3=I} A^{\alpha\beta\gamma} Z^{I_1}\partial_\gamma u Z^{I_2}\partial_{\beta'} v \cdot Z^{I_3}\partial_\alpha(\Psi_\beta^{\beta'}).$$

Then it can be estimated as

$$|Z^I(A^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha(\Psi_\beta^{\beta'})\partial_{\beta'} v)| \leq CKt^{-1} \sum_{\substack{\alpha, \beta \\ |I_1|+|I_2|\leq|I|}} |Z^{I_1}\partial_\alpha u Z^{I_2}\partial_\beta v| = \Omega_2.$$

The term R_1 can be estimated as follows:

$$\begin{aligned} R_1 &= \underline{A}^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v \\ &= \underline{A}^{000}\partial_t u\partial_t\partial_t v \\ &\quad + \underline{A}^{00c}\partial_c u\partial_t\partial_t v + \underline{A}^{0b0}\partial_t u\partial_t\partial_b v + \underline{A}^{a00}\partial_t u\partial_a\partial_t v \\ &\quad + \underline{A}^{ab0}\partial_t u\partial_a\partial_b v + \underline{A}^{0bc}\partial_c u\partial_t\partial_b v + \underline{A}^{a0c}\partial_c u\partial_a\partial_t v + \underline{A}^{abc}\partial_a u\partial_b\partial_c v \\ &:= R_3(u, v) + R_4(u, v) \end{aligned}$$

where

$$R_3(u, v) := \underline{A}^{000}\partial_t u\partial_t\partial_t v$$

and $R_4(u, v)$ denotes the remaining terms. Then $Z^I R_1 = Z^I R_3 + Z^I R_4$. To estimate $|Z^I R_3|$, we observe that $A^{\alpha\beta\gamma}$ is a null cubic form so by Proposition 4.1, $|Z^I A^{000}| \leq C(s/t)^2$. Then

$$Z^I (\underline{A}^{000} \partial_t u \partial_t \partial_t v) = \sum_{I_1+I_2+I_3=I} Z^{I_3} A^{000} \cdot Z^{I_1} \partial_t u \cdot Z^{I_2} \partial_t \partial_t v.$$

So that

$$|Z^I (\underline{A}^{000} \partial_t u \partial_t \partial_t v)| \leq CK(s/t)^2 \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1} \partial_t u \cdot Z^{I_2} \partial_t \partial_t v|$$

To see the estimates on $Z^I R_4$ terms, one just remark that by Lemma 2.1, $|Z^I \underline{A}^{\alpha\beta\gamma}| \leq C(I)K$. Then one sees directly the controlling of $Z^I R_3$ by Ω_1 .

Finally, by combining the estimate on R_2, R_3 and R_4 , the desired result is established. \square

The fourth type of null terms we need to control is the commutator between Z^I and A .

Proposition 4.6. *Let A be a null quadratic form acting on the gradient of function u and the Hessian matrix of v by*

$$A(\partial u, \partial \partial v) := A^{\alpha\beta\gamma} \partial_\gamma u \partial_\alpha \partial_\beta v$$

where u and v are C^∞ functions supported in the cone \mathcal{K} . Then the following estimate holds:

$$\begin{aligned} |[Z^I, A^{\alpha\beta\gamma} \partial_\gamma u \partial_\alpha \partial_\beta v]v| &\leq CK(s/t)^2 \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \partial_t u Z^{I_3} \partial_t \partial_t v| \\ &+ CK \sum_{c,\alpha,\beta} \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_\alpha \underline{\partial}_\beta v| \\ &+ CK \sum_{c,\alpha,\beta} \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_\alpha u Z^{I_3} \underline{\partial}_c \underline{\partial}_\beta v| \\ &+ CK \sum_{c,\alpha,\beta} \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_\alpha u Z^{I_3} \underline{\partial}_\beta \underline{\partial}_c v| \\ &+ t^{-1} CK \sum_{\alpha,\gamma} \sum_{\substack{|I_2|\leq|I|-1 \\ |I_1|+|I_2|\leq|I|}} |\partial_\gamma Z^{I_1} u \partial_\alpha Z^{I_2} v|. \end{aligned}$$

Proof. We use the change of frame formula

$$A^{\alpha\beta\gamma} \partial_\gamma u \partial_\alpha \partial_\beta v = \underline{A}^{\alpha\beta\gamma} \underline{\partial}_\gamma u \cdot \underline{\partial}_\alpha \underline{\partial}_\beta v + A^{\alpha\beta\gamma} \partial_\gamma u \cdot \partial_\alpha (\Psi_\beta^{\beta'}) \underline{\partial}_{\beta'} v$$

and find

$$\begin{aligned}
& [Z^I, A^{\alpha\beta\gamma} \partial_\gamma u \partial_\alpha \partial_\beta] v \\
&= [Z^I, \underline{A}^{\alpha\beta\gamma} \underline{\partial}_\gamma u \underline{\partial}_\alpha \underline{\partial}_\beta] v + [Z^I, A^{\alpha\beta\gamma} \partial_\gamma u \partial_\alpha (\Psi_\beta^{\beta'}) \underline{\partial}_{\beta'}] v \\
&:= R_1(I, u, v) + R_2(I, u, v).
\end{aligned}$$

We first decompose $R_1(I, u, v)$ as

$$\begin{aligned}
& [Z^I, \underline{A}^{\alpha\beta\gamma} \underline{\partial}_\gamma u \underline{\partial}_\alpha \underline{\partial}_\beta] v \\
&= \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{\alpha\beta\gamma} Z^{I_2} (\underline{\partial}_\gamma u) Z^{I_3} (\underline{\partial}_\alpha \underline{\partial}_\beta v) + \underline{A}^{\alpha\beta\gamma} \underline{\partial}_\gamma u [Z^I, \underline{\partial}_\alpha \underline{\partial}_\beta] v \\
&:= R_3(I, u, v) + R_4(I, u, v),
\end{aligned}$$

while $R_3(I, u, v)$ is decomposed as

$$\begin{aligned}
& R_3(I, u, v) \\
&= \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{000} Z^{I_2} \partial_t u Z^{I_3} \partial_t \partial_t v \\
&+ \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{00c} Z^{I_2} \underline{\partial}_c u Z^{I_3} \partial_t \partial_t v + \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{0b0} Z^{I_2} \partial_t u Z^{I_3} \partial_t \underline{\partial}_c v \\
&+ \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{a00} Z^{I_2} \partial_t u Z^{I_3} \underline{\partial}_a \partial_t v + \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{0bc} Z^{I_2} \underline{\partial}_c u Z^{I_3} \partial_t \underline{\partial}_b v \\
&+ \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{a0c} Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_a \partial_t v + \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{ab0} Z^{I_2} \partial_t u Z^{I_3} \underline{\partial}_a \underline{\partial}_b v \\
&+ \sum_{\substack{I_2+I_3+I_4=I, \\ |I_3|<|I|}} Z^{I_4} \underline{A}^{abc} Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_a \underline{\partial}_b v.
\end{aligned}$$

Then, we observe that \mathcal{A} is a null cubic form and, by Proposition 4.1,

$$|Z^{I_4} \underline{A}^{000}| \leq C(|I|) K(s/t)^2.$$

Then, by Lemma 2.1, we have

$$|Z^{I_4} \underline{A}^{\alpha\beta\gamma}| \leq K.$$

The term $R_3(I, u, v)$ is estimated as follows:

$$\begin{aligned}
& |R_3(I, u, v)| \\
& \leq CK(s/t)^2 \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \partial_t u Z^{I_3} \partial_t \partial_t v| \\
& + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \partial_t \partial_t v| + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \partial_t u Z^{I_3} \partial_t \underline{\partial}_c v| \\
& + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \partial_t u Z^{I_3} \underline{\partial}_a \partial_t v| + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \partial_t \underline{\partial}_b v| \\
& + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_a \partial_t v| + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \partial_t u Z^{I_3} \underline{\partial}_a \underline{\partial}_b v| \\
& + CK \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_a \underline{\partial}_b v|,
\end{aligned}$$

so that

$$\begin{aligned}
|R_3(I, u, v)| & \leq CK(s/t)^2 \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \partial_t u Z^{I_3} \partial_t \partial_t v| \\
& + CK \sum_{c, \alpha, \beta} \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_\alpha \underline{\partial}_\beta v| \\
& + CK \sum_{c, \alpha, \beta} \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_\alpha u Z^{I_3} \underline{\partial}_c \underline{\partial}_\beta v| \\
& + CK \sum_{c, \alpha, \beta} \sum_{\substack{|I_2|+|I_3|\leq|I|, \\ |I_3|<|I|}} |Z^{I_2} \underline{\partial}_\alpha u Z^{I_3} \underline{\partial}_\beta \underline{\partial}_c v|.
\end{aligned}$$

The term $R_4(I, u, v)$ is also decomposed as:

$$\begin{aligned}
R_4(I, u, v) & = \underline{A}^{000} \partial_t u [Z^I, \partial_t \partial_t] v + \underline{A}^{00c} \underline{\partial}_c u [Z^I, \partial_t \partial_t] v \\
& + \underline{A}^{0b0} \partial_t u [Z^I, \partial_t \underline{\partial}_c] v + \underline{A}^{a00} \partial_t u [Z^I, \underline{\partial}_a \partial_t] v \\
& + \underline{A}^{0bc} \underline{\partial}_c u [Z^I, \partial_t \underline{\partial}_b] v + \underline{A}^{a0c} \underline{\partial}_c u [Z^I, \underline{\partial}_a \partial_t] v \\
& + \underline{A}^{ab0} \partial_t u [Z^I, \underline{\partial}_a \underline{\partial}_b] v + \underline{A}^{abc} \underline{\partial}_c u [Z^I, \underline{\partial}_a \underline{\partial}_b] v.
\end{aligned}$$

Thanks to the null condition, we have

$$\begin{aligned}
|R_4(I, u, v)| & \leq CK(s/t)^2 \partial_t u [Z^I, \partial_t \partial_t] v + CK |\underline{\partial}_c u [Z^I, \partial_t \partial_t] v| \\
& + CK |\partial_t u [Z^I, \partial_t \underline{\partial}_c] v| + CK |\partial_t u [Z^I, \underline{\partial}_a \partial_t] v| \\
& + CK |\underline{\partial}_c u [Z^I, \partial_t \underline{\partial}_b] v| + CK |\underline{\partial}_c u [Z^I, \underline{\partial}_a \partial_t] v| \\
& + CK |\partial_t u [Z^I, \underline{\partial}_a \underline{\partial}_b] v| + CK |\underline{\partial}_c u [Z^I, \underline{\partial}_a \underline{\partial}_b] v|.
\end{aligned}$$

Observe now that thanks to the estimates on commutators (3.3.3) and (3.3.4), we have

$$|\underline{\partial}_\alpha u[Z^I, \partial_t \partial_t]v| \leq C \sum_{\substack{\alpha, \beta \\ |I'| \leq |I|-1}} |\underline{\partial}_\alpha u \partial_\alpha \partial_\beta Z^{I'} v|,$$

$$|\underline{\partial}_\alpha u[Z^I, \underline{\partial}_a \underline{\partial}_\beta]v| \leq C \sum_{\substack{\alpha, \beta \\ |I'| \leq |I|-1}} |\underline{\partial}_\alpha u \underline{\partial}_a \underline{\partial}_\beta Z^{I'} v| + Ct^{-1} \sum_{\substack{\gamma \\ |I'| < |I|}} |\underline{\partial}_\alpha u \partial_\gamma Z^{I'} v|,$$

and

$$|\underline{\partial}_\alpha u[Z^I, \underline{\partial}_\beta \underline{\partial}_c]v| \leq C \sum_{\substack{\beta, c \\ |I'| \leq |I|-1}} |\underline{\partial}_\alpha u \underline{\partial}_c \underline{\partial}_\beta Z^{I'} v| + Ct^{-1} \sum_{\substack{\gamma \\ |I'| < |I|}} |\underline{\partial}_\alpha u \partial_\gamma Z^{I'} v|.$$

We also note that $\sum_\alpha |\underline{\partial}_\alpha u| \leq C \sum_\alpha |\partial_\alpha u|$, so that $|R_4(I, u, v)|$ is bounded by

$$\begin{aligned} & |R_4(I, u, v)| \\ & \leq CK(s/t)^2 \sum_{\substack{\alpha, \beta \\ |I'| \leq |I|-1}} |\partial_t u \partial_\alpha \partial_\beta Z^{I'} v| + CK \sum_{\substack{\alpha, \beta, c \\ |I'| \leq |I|-1}} |\underline{\partial}_c u \partial_\alpha \partial_\beta Z^{I'} v| \\ & + CK \sum_{\substack{\alpha, \beta, c \\ |I'| \leq |I|-1}} |\partial_\alpha u \underline{\partial}_c \underline{\partial}_\beta v| + CKt^{-1} \sum_{\substack{\alpha, \gamma \\ |I'| \leq |I|-1}} |\partial_\gamma u \partial_\alpha Z^{I'} v|. \end{aligned}$$

In conclusion, $R_1(I, u, v)$ is bounded by

$$\begin{aligned} |R_1(I, u, v)| & \leq CK(s/t)^2 \sum_{\substack{|I_2|+|I_3| \leq |I|, \\ |I_3| < |I|}} |Z^{I_2} \partial_t u Z^{I_3} \partial_t \partial_t v| \\ & + CK \sum_{c, \alpha, \beta} \sum_{\substack{|I_2|+|I_3| \leq |I|, \\ |I_3| < |I|}} |Z^{I_2} \underline{\partial}_c u Z^{I_3} \underline{\partial}_\alpha \underline{\partial}_\beta v| \\ & + CK \sum_{c, \alpha, \beta} \sum_{\substack{|I_2|+|I_3| \leq |I|, \\ |I_3| < |I|}} |Z^{I_2} \underline{\partial}_\alpha u Z^{I_3} \underline{\partial}_c \underline{\partial}_\beta v| \\ & + CK \sum_{c, \alpha, \beta} \sum_{\substack{|I_2|+|I_3| \leq |I|, \\ |I_3| < |I|}} |Z^{I_2} \underline{\partial}_\alpha u Z^{I_3} \underline{\partial}_\beta \underline{\partial}_c v| \\ & + t^{-1} CK \sum_{\substack{\alpha, \gamma \\ |I'| \leq |I|-1}} |\partial_\gamma u \partial_\alpha Z^{I'} v|. \end{aligned}$$

Next, we turn our attention to the estimate of $R_2(I, u, v)$:

$$\begin{aligned} R_2(I, u, v) & = A^{\alpha\beta\gamma} \partial_\gamma u \cdot \partial_\alpha \Psi_\beta^{\beta'} [Z^I, \underline{\partial}_{\beta'}]v \\ & + \sum_{\substack{I_1+I_2+I_3=I, \\ |I_2| < |I|}} A^{\alpha\beta\gamma} Z^{I_1} \partial_\gamma u \cdot Z^{I_2} \underline{\partial}_{\beta'} v \cdot Z^{I_3} \partial_\alpha \Psi_\beta^{\beta'} \\ & := R_5(I, u, v) + R_6(I, u, v). \end{aligned}$$

To estimate the term $R_5(I, u, v)$, we recall (3.2.1) and the fact that $|\partial_\alpha \Psi_\beta^{\beta'}| \leq Ct^{-1}$, so that

$$|A^{\alpha\beta\gamma} \partial_\gamma u \cdot \partial_\alpha \Psi_\beta^{\beta'} [Z^I, \underline{\partial}_{\beta'}] v| \leq CKt^{-1} \sum_{\substack{\beta, \gamma \\ |I'| \leq |I| - 1}} |\partial_\gamma u \partial_\beta Z^{I'} v|.$$

In the same way, for $R_6(I, u, v)$ we have

$$R_6(I, u, v) \leq CKt^{-1} \sum_{\beta, \gamma} \sum_{\substack{|I_1| + |I_2| \leq |I| \\ |I_2| < |I|}} |\partial_\beta Z^{I_1} u \partial_\gamma Z^{I_2} v|.$$

This establishes the desired conclusion. □

Chapter 5

Sobolev and Hardy inequalities on hyperboloids

5.1 A global Sobolev inequality on hyperboloids

To turn the L^2 energy estimates into L^∞ estimates, we will rely on the following Sobolev inequality. Here, we consider this issue in general spatial dimension $n \geq 1$.

Lemma 5.1 (Sobolev-type estimate on hyperboloid). *Let $p(n)$ be the smallest integer greater than $n/2$. For any function u defined on a hyperboloid \mathcal{H}_T in Minkowski spacetime \mathbb{R}^{n+1} , one has*

$$\sup_{\mathcal{H}_T} t^{n/2} |u(t, x)| \leq C(n) \sum_{|I| \leq p(n)} \|L^I u\|_{L^2(\mathcal{H}_T)}, \quad (5.1.1)$$

where the constant $C(n) > 0$ depends only on the dimension.

This result slightly improves upon Lemma 7.6.1 in [Hörmander (1997)], since the right-hand side of (5.1.1) does not contain the rotation fields $\Omega_{ab} := x^a \partial_b - x^b \partial_a$.

Proof. Recall the relation $t = \sqrt{s^2 + |x|^2}$ in \mathcal{H}_s and consider any function u be a defined in some hyperboloid \mathcal{H}_s . The restriction of the function u to the hyperboloid \mathcal{H}_s is

$$w_s(x) := u(\sqrt{s^2 + |x|^2}, x).$$

Fix a point (t_0, x_0) on the hyperboloid \mathcal{H}_{s_0} where $t_0 = \sqrt{s_0^2 + x_0^2}$. Observe that

$$\partial_a w_{s_0}(x) = \underline{\partial}_a u((s_0^2 + |x|^2)^{1/2}, x) = \underline{\partial}_a u(t, x),$$

where $t = \sqrt{s_0^2 + |x|^2}$. Then, we have

$$t \partial_a w_{s_0}(x) = t \underline{\partial}_a u((s_0^2 + |x|^2)^{1/2}, t) = L_a u(t, x). \quad (5.1.2)$$

We define the following function

$$g_{s_0, t_0}(y) := w_{s_0}(t_0 y)$$

and note that

$$g_{s_0, t_0}(y_0) = w_{s_0}(x_0) = u((s_0^2 + |x_0|^2)^{1/2}, x_0) = u(t_0, x_0).$$

Applying the standard Sobolev inequality to g_{s_0, t_0} , we obtain

$$|g_{s_0, t_0}(y_0)|^2 \leq C(n) \int_{B(y_0, 1/3)} |\partial^I g_{s_0, t_0}(y)|^2 dy.$$

Taking into account the identity

$$\begin{aligned} \partial_a g_{s_0, t_0}(y) &= t_0 \partial_a w_{s_0}(t_0 y) = t_0 \partial_a w_{s_0}(x) \\ &= t_0 \underline{\partial}_a u((t_0 + |x^2|)^{1/2}, x) = t_0 \underline{\partial}_a u(t, x), \end{aligned}$$

we see that

$$\partial^I g_{s_0, t_0}(y) = (t_0 \underline{\partial})^I u(t, x)$$

and, therefore,

$$\begin{aligned} |g_{s_0, t_0}(y_0)|^2 &\leq C(n) \int_{B(y_0, 1/3)} |(t_0 \underline{\partial})^I u(t, x)|^2 dy \\ &= C(n) t_0^{-n} \int_{B((t_0, x_0), t_0/3) \cap \mathcal{H}_{s_0}} |(t_0 \underline{\partial})^I u(t, x)|^2 dx. \end{aligned}$$

Observe also the relation

$$\begin{aligned} (t_0 \partial)^I w_{s_0}(x) &= (t_0/t)^I (t \partial)^I w_{s_0}(x) \\ &= (t_0/t)^I L^I u((s_0^2 + |x|^2)^{1/2}, x) = (t_0/t)^I L^I u(t, x). \end{aligned}$$

So, when $|x_0| \leq t_0/2$ then $t_0 \leq \frac{2}{\sqrt{3}} s_0$ and thus

$$t_0 \leq C s_0 \leq C \sqrt{|x|^2 + s_0^2} = Ct.$$

When $|x_0| \geq t_0/2$ then in the region $B((t_0, x_0), t_0/3) \cap \mathcal{H}_{s_0}$ we have

$$t_0 \leq C|x| \leq C \sqrt{|x|^2 + s_0^2} = Ct.$$

Here C is a universal constant. Then, it follows that

$$\begin{aligned} |g_{s_0, t_0}(y_0)|^2 &= |g_{s_0, t_0}(y_0)|^2 \\ &\leq C(n) t_0^{-n} \int_{B(x_0, t_0/3) \cap \mathcal{H}_{s_0}} |(t \underline{\partial})^I u(t, x)|^2 dx \\ &\leq C(n) t_0^{-n} \int_{\mathcal{H}_{s_0}} |L^I u(t, x)|^2 dx, \end{aligned}$$

which completes the proof. □

Combining this result with the estimate (3.3.5) and recalling the equation (2.3.2), we arrive at the following conclusion. For clarity in the following we use now the notation $E_{m,c}$ (defined in Chapter 2) in order to emphasize the dependency of the hyperboloidal energy upon the “mass” coefficient c in the Klein-Gordon equation.

Lemma 5.2 (L^∞ estimates on zero and first-order derivatives).

If u is a function supported in the cone \mathcal{K} , then the following estimates hold:

$$\begin{aligned}
(a) \sup_{\mathcal{H}_s} |t^{(n-2)/2} s \partial_\alpha u| &\leq C(n) \sum_{|I| \leq p(n)} E_{m,\sigma}(s, Z^I u)^{1/2}, \\
(b) \sup_{\mathcal{H}_s} |t^{n/2} \underline{\partial}_a u| &\leq C(n) \sum_{|I| \leq p(n)} E_{m,\sigma}(s, Z^I u)^{1/2}, \\
(c) \sup_{\mathcal{H}_s} |\sigma t^{n/2} u| &\leq C(n) \sum_{|I| \leq p(n)} E_{m,\sigma}(s, Z^I u)^{1/2},
\end{aligned} \tag{5.1.3}$$

where $C(n)$ depends only on the dimension n .

We are now illustrate our result with the case of the homogeneous linear wave equation

$$\square w = 0, \quad w|_{\mathcal{H}_{B+1}} = w_0, \quad \partial_t w|_{\mathcal{H}_{B+1}} = w_1, \tag{5.1.4}$$

where the solution w_i is defined in $\mathcal{H}_{B+1} \cap K$. By the energy estimate in Lemma 2.1, the associated energy $E_m(s, Z^I w)$ is essentially conserved. Then, by the commutator estimates 3.5 and Sobolev inequality in Lemma 5.1, we find

$$|\underline{\partial}_a w| \leq C(n) t^{-n/2}, \quad |\partial_\alpha w| \leq C(n) t^{-n/2+1} (t^2 - r^2)^{-1/2}. \tag{5.1.5}$$

This is exactly the classical result but we emphasize that our proof uses neither the explicit expression of the solution operator nor the scaling vector field $S = r\partial_r + t\partial_t$.

Now, we turn our attention to the energy and decay estimates for the “good” second-order derivatives, that is, derivatives such as $\underline{\partial}_a \partial_\alpha u$. As we will see, these derivatives have better decay than that of $\partial_\alpha u$ or even ∂u_a .

Lemma 5.3 (Bounds on second-order derivatives). For every function u supported in the cone \mathcal{K} , the following estimates hold:

$$\sup_{\mathcal{H}_s} |t^{n/2} s \underline{\partial}_a \underline{\partial}_\alpha u|^2 + \sup_{\mathcal{H}_s} |t^{n/2} s \underline{\partial}_\alpha \underline{\partial}_a u|^2 \leq C(n) \sum_{|I| \leq p(n)+1} E_m(s, Z^I u), \tag{5.1.6}$$

$$\int_{\mathcal{H}_s} |s \underline{\partial}_a \underline{\partial}_\alpha u|^2 dx + \int_{\mathcal{H}_s} |s \underline{\partial}_\alpha \underline{\partial}_a u|^2 dx \leq C \sum_{|I| \leq 1} E_m(s, Z^I u). \tag{5.1.7}$$

Proof. Noting that

$$\underline{\partial}_a = t^{-1}L_a,$$

we obtain

$$|\underline{\partial}_a \partial_\alpha u| \leq t^{-1}|L_a \partial u|.$$

Then by Lemma (5.2) we obtain the first estimate. The second is a trivial result of the expression (2.3.2). \square

Remark 5.1. The energy estimates and L^∞ estimates of $\underline{\partial}_0 \underline{\partial}_0 u$ will consult the wave equation it-self. Roughly saying, by Lemma 2.4. These estimates will be established later.

At the end of this section, we state the L^∞ estimates of the solution of wave equation (i.e. $c_i = 0$).

Lemma 5.4. *If u is a function supported in the cone \mathcal{K} , then for any multi-index J , if $\sum_{|I| \leq |J| + p(n)} E_m(s, Z^I u)^{1/2} \leq C' s^\epsilon$ for any $\epsilon \geq 0$, then one has*

$$|Z^J u| \leq CC' t^{(-n+1+\epsilon)/2} (t-r)^{(1+\epsilon)/2} = CC' t^{-n/2} s^{1+\epsilon}. \quad (5.1.8)$$

Proof. When $\sum_{|I| \leq |J| + p(n)} E_m(s, Z^I u)^{1/2}$ is bounded by $C' s^\epsilon$, by Lemma 5.2, in the cone \mathcal{K} ,

$$|\partial_r u| \leq C(n) t^{-(n-1-\epsilon)/2} (t-r)^{-(1-\epsilon)/2}.$$

Then the proof of (5.1.8) follows by integration along radial directions. \square

5.2 Hardy inequality on hyperboloids

In this section we will establish an analogue of the classical Hardy's inequality on hyperboloid. This inequality are used to control the L^2 norm of the wave components in the following chapter.

Proposition 5.1. *Let u be a function supported in the cone \mathcal{K} . Then the following estimate holds:*

$$\begin{aligned} \|s^{-1}u\|_{L^2(\mathcal{H}_s)} &\leq C \|s^{-1}u\|_{L^2(\mathcal{H}_{B+1})} + C \sum_a \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_s)} \\ &\quad + C \sum_a \int_{B+1}^s \tau^{-1} \left(\|(s/t)\partial_a u\|_{L^2(\mathcal{H}_\tau)} + \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_s)} \right) d\tau. \end{aligned} \quad (5.2.1)$$

We will rely on the following technical observation.

Lemma 5.5. *Let u be a function supported in the cone \mathcal{K} then the following estimate holds:*

$$\left\| \frac{u}{|x|} \right\|_{L^2(\mathcal{H}_s)} \leq C \sum_a \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_s)}.$$

Proof. This is a modified version of the classical Hardy's inequality. We introduce the following function as in the proof of Lemma 5.1:

$$w_s(x) := u(\sqrt{s^2 + |x|^2}, x).$$

Then we note that

$$\partial_a w_s(x) = \underline{\partial}_a u(\sqrt{s^2 + |x|^2}, x),$$

and we apply the classical Hardy inequality to w_s :

$$\begin{aligned} \int_{\mathbb{R}^3} |r^{-1} w_s(x)|^2 dx &\leq C \int_{\mathbb{R}^3} |\nabla w_s(x)|^2 dx = C \sum_a \int_{\mathbb{R}^3} |\underline{\partial}_a u(\sqrt{s^2 + r^2}, x)|^2 dx \\ &\leq C \sum_a \int_{\mathcal{H}_s} |\underline{\partial}_a u(t, x)|^2 dx. \end{aligned}$$

This completes the proof. □

Proof of Proposition 5.1. Recall that $s^2 = t^2 - r^2$ and introduce a smooth cut-off function χ satisfying

$$\chi(r) = \begin{cases} 1, & 2/3 \leq r, \\ 0, & 0 \leq r \leq 1/3. \end{cases}$$

We also consider the vector field (defined in \mathcal{K})

$$\mathbf{W} = \left(0, \frac{-x^a t (u \chi(r/t))^2}{(1+r^2)s^2} \right)$$

and compute its divergence

$$\begin{aligned} &\text{Div } \mathbf{W} \\ &= s^{-1} \cdot \partial_a u \cdot \frac{r \chi(r/t) u}{(1+r^2)^{1/2} s} \cdot \frac{-2x^a t \chi(r/t)}{r(1+r^2)^{1/2}} - s^{-1} \frac{u}{r} \cdot \frac{r \chi(r/t) u}{s(1+r^2)^{1/2}} \cdot \frac{2\chi'(r/t)r}{(1+r^2)^{1/2}} \\ &\quad - \left(\frac{r^2 t + 3t}{(1+r^2)^2 s^2} + \frac{2r^2 t}{(1+r^2)s^4} \right) (u \chi(r/t))^2. \end{aligned}$$

Next, we integrate the above inequality in the region $\mathcal{K}_{[B+1, s_0]} \subset \mathcal{K} \cap \{B+1 \leq \sqrt{t^2 - r^2} \leq s_0\}$ with respect to the Lebesgue measure of \mathbb{R}^4 :

$$\begin{aligned}
& \int_{\mathcal{K}_{[B+1, s_0]}} \text{Div} \mathbf{W} dx dt \\
&= -2 \int_{\mathcal{K}_{[B+1, s_0]}} s^{-1} \left(\partial_a u \cdot \frac{r\chi(r/t)u}{(1+r^2)^{1/2}s} \cdot \frac{x^a t \chi(r/t)}{r(1+r^2)^{1/2}} \right) dx dt \\
&\quad - 2 \int_{\mathcal{K}_{[B+1, s_0]}} s^{-1} \frac{u}{r} \cdot \frac{r\chi(r/t)u}{s(1+r^2)^{1/2}} \cdot \frac{\chi'(r/t)r}{(1+r^2)^{1/2}} dx dt \\
&\quad - \int_{\mathcal{K}_{[B+1, s_0]}} \left(\frac{r^2 t + 3t}{(1+r^2)^2 s^2} + \frac{2r^2 t}{(1+r^2)s^4} \right) (u\chi(r/t))^2 dx dt \\
&= -2 \int_{B+1}^{s_0} \int_{\mathcal{H}_s} (s/t) s^{-1} \left(\partial_a u \cdot \frac{r\chi(r/t)u}{(1+r^2)^{1/2}s} \cdot \frac{x^a t \chi(r/t)}{r(1+r^2)^{1/2}} \right) dx ds \\
&\quad - 2 \int_{B+1}^{s_0} \int_{\mathcal{H}_s} (s/t) s^{-1} \frac{u}{r} \cdot \frac{r\chi(r/t)u}{s(1+r^2)^{1/2}} \cdot \frac{\chi'(r/t)r}{(1+r^2)^{1/2}} dx ds \\
&\quad - \int_{B+1}^{s_0} \int_{\mathcal{H}_s} (s/t) \left(\frac{r^2 t + 3t}{(1+r^2)^2 s^2} + \frac{2r^2 t}{(1+r^2)s^4} \right) (u\chi(r/t))^2 dx ds \\
&=: \int_{B+1}^{s_0} T_1(s) ds + \int_{B+1}^{s_0} T_2(s) ds + \int_{B+1}^{s_0} T_3(s) ds,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= -2s^{-1} \int_{\mathcal{H}_s} (s/t) \left(\partial_a u \cdot \frac{r\chi(r/t)u}{(1+r^2)^{1/2}s} \cdot \frac{x^a t \chi(r/t)}{r(1+r^2)^{1/2}} \right) dx \\
&\leq 2s^{-1} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{\mathcal{H}_s} \sum_a \left\| (s/t) \partial_a u \right\|_{L^2(\mathcal{H}_s)} \left\| \chi(r/t) x^a t r^{-1} (1+r^2)^{-1/2} \right\|_{L^\infty(\mathcal{H}_s)} \\
&\leq Cs^{-1} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{\mathcal{H}_s} \sum_a \left\| (s/t) \partial_a u \right\|_{L^2(\mathcal{H}_s)},
\end{aligned}$$

$$\begin{aligned}
T_2 &= -2s^{-1} \int_{\mathcal{H}_s} (s/t) \frac{u}{r} \cdot \frac{r\chi(r/t)u}{s(1+r^2)^{1/2}} \cdot \frac{\chi'(r/t)r}{(1+r^2)^{1/2}} dx \\
&\leq Cs^{-1} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{L^2(\mathcal{H}_s)} \left\| ur^{-1} \right\|_{L^2(\mathcal{H}_s)} \left\| r\chi'(r/t)(1+r^2)^{-1/2} \right\|_{L^\infty(\mathcal{H}_s)} \\
&\leq Cs^{-1} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{L^2(\mathcal{H}_s)} \sum_a \left\| \partial_a u \right\|_{L^2(\mathcal{H}_s)},
\end{aligned}$$

$$T_3 \leq 0.$$

We write our identity in the form

$$\frac{d}{ds_0} \left(\int_{K_{[B+1, s_0]}} \operatorname{Div} \mathbf{W} dx dt \right) = T_1 + T_2 + T_3 \quad (5.2.2)$$

and obtain

$$\begin{aligned} & \frac{d}{ds_0} \left(\int_{K_{[B+1, s_0]}} \operatorname{Div} \mathbf{W} dx dt \right) \\ & \leq C s^{-1} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{L^2(\mathcal{H}_s)} \sum_a \left(\|(s/t)\partial_a\|_{L^2(\mathcal{H}_s)} + \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_s)} \right). \end{aligned} \quad (5.2.3)$$

On the other hand, we apply Stokes' formula in the region $\mathcal{K}_{[B+1, s_0]}$, and obtain

$$\begin{aligned} & \int_{\mathcal{K}_{[B+1, s_0]}} \operatorname{Div} \mathbf{W} dx dt \\ & = \int_{\mathcal{H}_{s_0}} W \cdot n d\sigma + \int_{\mathcal{H}_{B+1}} W \cdot n d\sigma \\ & = \int_{\mathcal{H}_{s_0}} \frac{r^2}{1+r^2} |u\chi(r/t)s^{-1}|^2 dx - \int_{\mathcal{H}_{B+1}} \frac{r^2}{1+r^2} |u\chi(r/t)s^{-1}|^2 dx. \end{aligned}$$

Differentiating this identity with respect to s_0 , we find

$$\begin{aligned} & \frac{d}{ds_0} \left(\int_{\mathcal{K}_{[B+1, s_0]}} \operatorname{Div} \mathbf{W} dx dt \right) = \frac{d}{ds_0} \left(\int_{\mathcal{H}_{s_0}} \frac{r^2}{1+r^2} |u\chi(r/t)s^{-1}|^2 dx \right) \\ & = 2 \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{L^2(\mathcal{H}_{s_0})} \frac{d}{ds_0} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{L^2(\mathcal{H}_{s_0})}. \end{aligned} \quad (5.2.4)$$

Combining (5.2.3) and (5.2.4) yields us

$$\frac{d}{ds_0} \left\| \frac{ru\chi(r/t)}{s(1+r^2)^{1/2}} \right\|_{L^2(\mathcal{H}_{s_0})} \leq C s_0^{-1} \sum_a \left(\|\underline{\partial}_a u\|_{L^2(\mathcal{H}_{s_0})} + \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_{s_0})} \right).$$

Then, by integrating this identity on the interval $[B+1, s]$, we obtain

$$\begin{aligned} & \|r(1+r^2)^{-1/2}\chi(r/t)s^{-1}u\|_{L^2(\mathcal{H}_s)} \leq \|r(1+r^2)^{-1/2}\chi(r/t)s^{-1}u\|_{L^2(\mathcal{H}_{B+1})} \\ & \quad + C \sum_a \int_{B+1}^s \tau^{-1} \left(\|\underline{\partial}_a u\|_{L^2(\mathcal{H}_\tau)} + \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_\tau)} \right). \end{aligned} \quad (5.2.5)$$

Furthermore, we observe that, in the region $\mathcal{K}_{[B+1, s_0]}$, there is a positive constant C

$$\chi(r/t) \leq C \frac{\chi(r/t)r}{(1+r)^{1/2}}.$$

Then by Lemma 5.5, we find

$$\begin{aligned}
\|\chi(r/t)s^{-1}u\|_{L^2(\mathcal{H}_s)} &\leq C\|r(1+r)^{-1/2}\chi(r/t)s^{-1}u\|_{L^2(\mathcal{H}_s)} \\
&\leq C\|s^{-1}u\|_{L^2(\mathcal{H}_{B+1})} \\
&\quad + C\sum_a\int_{B+1}^s\tau^{-1}(\|\underline{\partial}_a u\|_{L^2(\mathcal{H}_\tau)} + \|\underline{\partial}_a u\|_{L^2(\mathcal{H}_\tau)}).
\end{aligned} \tag{5.2.6}$$

Furthermore, we observe that in the cone \mathcal{K} ,

$$(1 - \chi(r/t))s^{-1} \leq Ct^{-1}.$$

Hence, by Lemma 5.5, we have

$$\|(1 - \chi(r/t))us^{-1}\|_{L^2(\mathcal{H}_s)} \leq \|t^{-1}u\|_{L^2(\mathcal{H}_s)} \leq C\sum_a\|\underline{\partial}_a u\|_{L^2(\mathcal{H}_s)}.$$

Combining this with (5.2.6), we obtain the desired result. \square

Chapter 6

Fundamental L^∞ and L^2 estimates

6.1 Objective of this chapter

We begin the discussion of the bootstrap arguments and assume that (2.4.5) hold on some time interval. Our aim is to derive additional estimates which follows from these assumptions. In the present chapter, we are able to deduce several L^2 and L^∞ estimates on the solution and its derivatives. These rather direct estimates will serve as a basis for the following Chapters. The estimates we are going to now establish are classified into two groups: L^2 estimates or L^∞ estimates:

- The L^2 -type estimates are classified into two generations:
 - The estimates of the first generation are immediate consequences of (2.4.5).
 - The estimates of the second generation are deduced from those of the first generation, by recalling the commutator estimates (cf. Lemmas 3.4 and 3.5). These are the L^2 bounds that will be more often used in the following discussion.
- The L^∞ -type estimates are also classified into two generations:
 - The estimates of the first generation follow immediately from (2.4.5) and the Sobolev inequalities (5.1).
 - The estimates of the second generation are deduced from the ones of the first generation and the commutator estimates in Lemmas 3.4 and 3.5.

In the following, the letter C will be used to represent a constant who depends only on the structure of the system (1.1.2), that is, the number

of equations n_0 , the number of wave components j_0 , the Klein-Gordon constant c_i , and the structural constant K .

6.2 L^2 estimates of the first generation

From the structure of the energy given in (2.3.2) and the energy assumption (2.4.5), the following estimates hold of $|I^\sharp| \leq 5$ and $|I^\dagger| \leq 4$:

$$\sum_{\alpha, i} \|(s/t)\partial_\alpha Z^{I^\sharp} w_i\|_{L^2(\mathcal{H}_s)} + \sum_{\alpha, i} \|(s/t)\underline{\partial}_\alpha Z^{I^\sharp} w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.2.1a)$$

$$\sum_{\alpha, i} \|(s/t)\partial_\alpha Z^{I^\dagger} w_i\|_{L^2(\mathcal{H}_s)} + \sum_{\alpha, i} \|(s/t)\underline{\partial}_\alpha Z^{I^\dagger} w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.2.1b)$$

$$\sum_{a, i} \|\underline{\partial}_a Z^{I^\sharp} w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.2.1c)$$

$$\sum_{a, i} \|\underline{\partial}_a Z^{I^\dagger} w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.2.1d)$$

$$\sum_{\check{i}} \|Z^{I^\sharp} v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.2.1e)$$

$$\sum_{\check{i}} \|Z^{I^\dagger} v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}. \quad (6.2.1f)$$

By taking $Z^{I^\sharp} = \partial_\alpha Z^{I^\dagger}$, $Z^{I^\dagger} = \partial_\alpha Z^I$, $Z^{I^\sharp} = t\underline{\partial}_a Z^{I^\dagger} = H_a Z^{I^\dagger}$ and $Z^{I^\dagger} = t\underline{\partial}_a Z^I = H_a Z^I$ in (6.2.1e) and (6.2.1f), we have especially the following estimates on Klein-Gordon components for $|I^\dagger| \leq 4$ and $|I| \leq 3$:

$$\sum_{\check{i}, \alpha} \|\partial_\alpha Z^{I^\dagger} v_{\check{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{\check{i}, a} \|\underline{\partial}_\alpha Z^{I^\dagger} v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.2.2a)$$

$$\sum_{\check{i}, \alpha} \|\partial_\alpha Z^I v_{\check{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{\check{i}, a} \|\underline{\partial}_\alpha Z^I v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.2.2b)$$

$$\sum_{\check{i}, a} \|t\underline{\partial}_a Z^{I^\dagger} v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.2.2c)$$

$$\sum_{\check{i}, a} \|t\underline{\partial}_a Z^I v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}. \quad (6.2.2d)$$

The estimate on $\underline{\partial}_\alpha Z^{I^\dagger} v_{\check{i}}$ in (6.2.2a) is covered by the estimate on $\partial_t Z^{I^\dagger} v_{\check{i}}$ and the estimates on $\underline{\partial}_a Z^{I^\dagger} v_{\check{i}}$.

For any $|I| \leq 3$, we deduce from (2.3.2) and (2.4.5e) the following estimates for the wave components:

$$\sum_{\alpha, \hat{i}} \|(s/t)\partial_\alpha Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{\alpha, \hat{i}} \|(s/t)\underline{\partial}_\alpha Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (6.2.3a)$$

$$\sum_{a, \hat{i}} \|\underline{\partial}_a Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon. \quad (6.2.3b)$$

6.3 L^2 estimates of the second generation

By using Lemma 3.5, we can commute the vector fields under consideration and, by relying on the estimates established in the previous section, we obtain the following result.

The first group of estimates is obtained by (3.3.1), (3.3.2) and (6.2.1):

$$\sum_{\hat{i}, \alpha} \|(s/t)Z^{I^\sharp} \partial_\alpha w_i\|_{L^2(\mathcal{H}_s)} + \sum_{\hat{i}, \alpha} \|(s/t)Z^{I^\sharp} \underline{\partial}_\alpha w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.3.1a)$$

$$\sum_{\hat{i}, \alpha} \|(s/t)Z^{I^\dagger} \partial_\alpha w_i\|_{L^2(\mathcal{H}_s)} + \sum_{\hat{i}, \alpha} \|(s/t)Z^{I^\dagger} \underline{\partial}_\alpha w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.1b)$$

$$\sum_{\hat{i}, a} \|Z^{I^\sharp} \underline{\partial}_a w_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.3.1c)$$

$$\sum_{\hat{i}, a} \|Z^{I^\dagger} \underline{\partial}_a w_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.1d)$$

$$\sum_{\check{j}} \|Z^{I^\sharp} v_{\check{j}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.1e)$$

$$\sum_{\check{j}} \|Z^{I^\dagger} v_{\check{j}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta. \quad (6.3.1f)$$

The second group of estimates follows from (3.3.1), (3.3.2)s and (6.2.2):

$$\sum_{\check{i}, \alpha} \|Z^{I^\dagger} \partial_\alpha v_{\check{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{\check{i}, \alpha} \|Z^{I^\dagger} \underline{\partial}_\alpha v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.3.2a)$$

$$\sum_{\check{i}, \alpha} \|Z^I \partial_\alpha v_{\check{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{\check{i}, \alpha} \|Z^I \underline{\partial}_\alpha v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.2b)$$

$$\sum_{\check{i}, a} \|tZ^{I^\dagger} \underline{\partial}_a v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.3.2c)$$

$$\sum_{\check{i}, a} \|tZ^I \underline{\partial}_a v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}. \quad (6.3.2d)$$

The third group of estimates follows from (3.3.1), (3.3.2), and (6.2.3):

$$\sum_{\hat{i}, \alpha} \|(s/t)Z^I \partial_\alpha u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{\hat{i}, \alpha} \|(s/t)Z^I \underline{\partial}_\alpha u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (6.3.3a)$$

$$\sum_{\hat{i}, a} \|Z^I \underline{\partial}_a u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon. \quad (6.3.3b)$$

The fourth group of estimates concerns the second order derivatives. The first is deduced from (5.1.7), (2.4.5a), and (2.4.5b), while the second is deduced from (5.1.7) and (2.4.5e):

$$\sum_{a, \beta, i} \|s \underline{\partial}_a \underline{\partial}_\beta Z^{I^\dagger} w_i\|_{L^2(\mathcal{H}_s)} + \sum_{a, \beta, i} \|s \underline{\partial}_\beta \underline{\partial}_a Z^{I^\dagger} w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.3.4a)$$

$$\sum_{a, \beta, i} \|s \underline{\partial}_a \underline{\partial}_\beta Z^I w_i\|_{L^2(\mathcal{H}_s)} + \sum_{a, \beta, i} \|s \underline{\partial}_\beta \underline{\partial}_a Z^I w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.4b)$$

$$\sum_{a, \beta, \hat{i}} \|s \underline{\partial}_a \underline{\partial}_\beta Z^{I^b} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + \sum_{a, \beta, \hat{i}} \|s \underline{\partial}_\beta \underline{\partial}_a Z^{I^b} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (6.3.4c)$$

where I^b denotes the multi-index of order not bigger than 2. We can also use estimates on commutators (3.3.4) and get:

$$\sum_{a, \beta, i} \|s Z^{I^\dagger} \underline{\partial}_a \underline{\partial}_\beta w_i\|_{L^2(\mathcal{H}_s)} + \sum_{a, \beta, i} \|s Z^{I^\dagger} \underline{\partial}_\beta \underline{\partial}_a w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (6.3.5a)$$

$$\sum_{a, \beta, i} \|s Z^I \underline{\partial}_a \underline{\partial}_\beta w_i\|_{L^2(\mathcal{H}_s)} + \sum_{a, \beta, i} \|s Z^I \underline{\partial}_\beta \underline{\partial}_a w_i\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.5b)$$

$$\sum_{a, \beta, \hat{i}} \|s Z^{I^b} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{i}}\|_{\mathcal{H}_s} + \sum_{a, \beta, \hat{i}} \|s Z^{I^b} \underline{\partial}_\beta \underline{\partial}_a u_{\hat{i}}\|_{\mathcal{H}_s} \leq CC_1 \epsilon, \quad (6.3.5c)$$

Finally, we have the L^2 estimates for the wave components themselves: for all $I^\sharp \leq 5$ and all $|I^\dagger| \leq 4$,

$$\|s^{-1} Z^{I^\sharp} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta, 1} \quad (6.3.6a)$$

$$\|s^{-1} Z^{I^\dagger} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (6.3.6b)$$

$$\|t^{-1} Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon. \quad (6.3.6c)$$

The first two inequalities are direct results of (6.2.1a), (6.2.1b), (6.2.1c), (6.2.1d) combined with lemma 5.1. The last inequality is a result of Lemma 5.5 combined with (6.2.3b).

¹Here if we want to be more precise, $C = C'(C_0/C_1 + 1 + \delta^{-1})$ where C' is a constant depending only on the structure of the system. We see that it also depends on C_0 and δ . But as we can assume that $1/12 \leq \delta < 1/6$ and $C_1 \geq C_0$, we denote again by C here and regard it as a constant determined by the structure of the system.

6.4 L^∞ estimates of the first generation

For convenience in the presentation, we introduce here a new convention (which parallels the index convention already made in (2.4.1)):

$$\begin{aligned} J^\sharp &: \text{multi-index of order } \leq 3, \\ J^\dagger &: \text{multi-index of order } \leq 2, \\ J &: \text{multi-index of order } \leq 1. \end{aligned} \tag{6.4.1}$$

Now we give the decay estimates based on the energy assumption (2.4.5) and the Sobolev-type inequality given in Lemma 5.1.

The first group of inequalities follows directly from (5.1.3), (2.4.5a), and (2.4.5b):

$$\sup_{\mathcal{H}_s} \left(st^{1/2} |\partial_\alpha Z^{J^\sharp} w_j| \right) + \sup_{\mathcal{H}_s} \left(st^{1/2} |\underline{\partial}_\alpha Z^{J^\sharp} w_j| \right) \leq CC_1 \epsilon s^\delta \tag{6.4.2a}$$

$$\sup_{\mathcal{H}_s} \left(st^{1/2} |\partial_\alpha Z^{J^\dagger} w_j| \right) + \sup_{\mathcal{H}_s} \left(st^{1/2} |\underline{\partial}_\alpha Z^{J^\dagger} w_j| \right) \leq CC_1 \epsilon s^{\delta/2} \tag{6.4.2b}$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |\underline{\partial}_a Z^{J^\sharp} w_j| \right) \leq CC_1 \epsilon s^\delta \tag{6.4.2c}$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |\underline{\partial}_a Z^{J^\dagger} w_j| \right) \leq CC_1 \epsilon s^{\delta/2} \tag{6.4.2d}$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |Z^{J^\sharp} v_j| \right) \leq CC_1 \epsilon s^\delta. \tag{6.4.2e}$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |Z^{J^\dagger} v_j| \right) \leq CC_1 \epsilon s^{\delta/2}. \tag{6.4.2f}$$

The second group of estimates is a special case of the second one by taking $Z^{J^\sharp} = \partial_\alpha Z^{J^\dagger}$, $Z^{J^\sharp} = H_a Z^{J^\dagger} = t \underline{\partial}_a Z^{J^\dagger}$, $Z^{J^\dagger} = \partial_\alpha Z^J$ and $Z^{J^\dagger} = H_a Z^J = t \underline{\partial}_a Z^J$ in (6.4.2e) and (6.4.2f):

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |\partial_\alpha Z^{J^\dagger} v_j| \right) + \sup_{\mathcal{H}_s} \left(t^{3/2} |\underline{\partial}_\alpha Z^{J^\dagger} v_j| \right) \leq CC_1 \epsilon s^\delta, \tag{6.4.3a}$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |\partial_\alpha Z^J v_j| \right) + \sup_{\mathcal{H}_s} \left(t^{3/2} |\underline{\partial}_\alpha Z^J v_j| \right) \leq CC_1 \epsilon s^{\delta/2}, \tag{6.4.3b}$$

$$\sup_{\mathcal{H}_s} \left(t^{5/2} |\underline{\partial}_a Z^{J^\dagger} v_j| \right) \leq CC_1 \epsilon s^\delta, \tag{6.4.3c}$$

$$\sup_{\mathcal{H}_s} \left(t^{5/2} |\underline{\partial}_a Z^J v_j| \right) \leq CC_1 \epsilon s^{\delta/2}. \quad (6.4.3d)$$

The estimates in the third group follow from (5.1.3) (a) and (b) combined with (2.4.5e):

$$\sup_{\mathcal{H}_s} \left(st^{1/2} |\partial_\alpha Z^J u_{\widehat{k}}| \right) + \sup_{\mathcal{H}_s} \left(st^{1/2} |\underline{\partial}_\alpha Z^J u_{\widehat{k}}| \right) \leq CC_1 \epsilon, \quad (6.4.4a)$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} |\underline{\partial}_a Z^J u_{\widehat{k}}| \right) \leq CC_1 \epsilon. \quad (6.4.4b)$$

The fourth group concerns the “second order derivative” of the solution. They are deduced from (5.1.6) and (2.4.5):

$$\sup_{\mathcal{H}_s} \left(t^{3/2} s |\underline{\partial}_\alpha \underline{\partial}_a Z^{J^\dagger} w_j| \right) + \sup_{\mathcal{H}_s} \left(t^{3/2} s |\underline{\partial}_a \underline{\partial}_\alpha Z^{J^\dagger} w_j| \right) \leq CC_1 \epsilon s^\delta, \quad (6.4.5a)$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} s |\underline{\partial}_\alpha \underline{\partial}_a Z^J w_j| \right) + \sup_{\mathcal{H}_s} \left(t^{3/2} s |\underline{\partial}_a \underline{\partial}_\alpha Z^J w_j| \right) \leq CC_1 \epsilon s^{\delta/2}, \quad (6.4.5b)$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} s |\underline{\partial}_\alpha \underline{\partial}_a u_{\widehat{j}}| \right) + \sup_{\mathcal{H}_s} \left(t^{3/2} s |\underline{\partial}_a \underline{\partial}_\alpha u_{\widehat{j}}| \right) \leq CC_1 \epsilon. \quad (6.4.5c)$$

6.5 L^∞ estimates of the second generation

The estimates in this section follow from the first generation L^∞ estimates established in the previous section, combined with commutator estimates in Lemma 3.5.

The first group of estimates are results of (3.3.1), (3.3.2) combined with (6.4.2):

$$\sup_{\mathcal{H}_s} |st^{1/2} Z^{J^\sharp} \partial_\alpha w_j| + \sup_{\mathcal{H}_s} |st^{1/2} Z^{J^\sharp} \underline{\partial}_\alpha w_j| \leq CC_1 \epsilon s^\delta, \quad (6.5.1a)$$

$$\sup_{\mathcal{H}_s} |st^{1/2} Z^{J^\dagger} \partial_\alpha w_j| + \sup_{\mathcal{H}_s} |st^{1/2} Z^{J^\dagger} \underline{\partial}_\alpha w_j| \leq CC_1 \epsilon s^{\delta/2}, \quad (6.5.1b)$$

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^{J^\sharp} \underline{\partial}_a w_j| \leq CC_1 \epsilon s^\delta, \quad (6.5.1c)$$

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^{J^\dagger} \underline{\partial}_a w_j| \leq CC_1 \epsilon s^{\delta/2}, \quad (6.5.1d)$$

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^{J^\sharp} v_{\check{k}}| \leq CC_1 \epsilon s^\delta, \quad (6.5.1e)$$

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^{J^\dagger} v_k^\dagger| \leq CC_1 \epsilon s^{\delta/2}. \quad (6.5.1f)$$

The second group consists of the following estimates. They are results of (3.3.1), (3.3.2) combined with (6.4.3)

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^{J^\dagger} \partial_a v_j^\dagger| + \sup_{\mathcal{H}_s} |t^{3/2} Z^{J^\dagger} \underline{\partial}_\alpha v_j^\dagger| \leq CC_1 \epsilon s^\delta, \quad (6.5.2a)$$

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^J \partial_a v_j| + \sup_{\mathcal{H}_s} |t^{3/2} Z^J \underline{\partial}_\alpha v_j| \leq CC_1 \epsilon s^{\delta/2}, \quad (6.5.2b)$$

$$\sup_{\mathcal{H}_s} |t^{5/2} Z^{J^\dagger} \underline{\partial}_a v_j^\dagger| \leq CC_1 \epsilon s^\delta, \quad (6.5.2c)$$

$$\sup_{\mathcal{H}_s} |t^{5/2} Z^J \underline{\partial}_a v_j| \leq CC_1 \epsilon s^{\delta/2}. \quad (6.5.2d)$$

The third group consists of the following estimates which are direct results of (3.3.1), (3.3.2) combined with (6.4.4).

$$\sup_{\mathcal{H}_s} |st^{1/2} Z^J \partial_\alpha u_j| + \sup_{\mathcal{H}_s} |st^{1/2} Z^J \underline{\partial}_\alpha u_j| \leq CC_1 \epsilon, \quad (6.5.3a)$$

$$\sup_{\mathcal{H}_s} |t^{3/2} Z^J \underline{\partial}_a u_j| \leq CC_1 \epsilon, \quad (6.5.3b)$$

The estimates of fourth group are deduced from (3.3.4) and (6.4.5a) and (6.4.5c).

$$\sup_{\mathcal{H}_s} (t^{3/2} s Z^{J^\dagger} \underline{\partial}_\alpha \underline{\partial}_a w_j) + \sup_{\mathcal{H}_s} (t^{3/2} s Z^{J^\dagger} \underline{\partial}_a \underline{\partial}_\alpha w_j) \leq CC_1 \epsilon s^\delta, \quad (6.5.4a)$$

$$\sup_{\mathcal{H}_s} (t^{3/2} s Z^J \underline{\partial}_\alpha \underline{\partial}_a w_j) + \sup_{\mathcal{H}_s} (t^{3/2} s Z^J \underline{\partial}_a \underline{\partial}_\alpha w_j) \leq CC_1 \epsilon s^{\delta/2}, \quad (6.5.4b)$$

$$\sup_{\mathcal{H}_s} |st^{3/2} \underline{\partial}_a \underline{\partial}_\beta u_j| + \sup_{\mathcal{H}_s} |st^{3/2} \underline{\partial}_\alpha \underline{\partial}_b u_j| \leq CC_1 \epsilon s. \quad (6.5.4c)$$

We finally can write down the decay estimates for the wave components, which follow from Lemma 5.4 combined with (2.4.5):

$$\sup_{\mathcal{H}_s} \left(t^{3/2} s^{-1} |Z^{J^\dagger} u_{\hat{i}}| \right) \leq CC_1 \epsilon s^\delta, \quad (6.5.5a)$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} s^{-1} |Z^{J^\dagger} u_{\hat{i}}| \right) \leq CC_1 \epsilon s^{\delta/2}, \quad (6.5.5b)$$

$$\sup_{\mathcal{H}_s} \left(t^{3/2} s^{-1} |Z^J u_{\hat{i}}| \right) \leq CC_1 \epsilon. \quad (6.5.5c)$$

Chapter 7

Second-order derivatives of the wave components

7.1 Preliminaries

The estimates derived in this chapter concern the terms $u_{\hat{\gamma}}\partial_{\alpha}\partial_{\beta}u_{\hat{\gamma}}$, while the phrase “second-order derivatives of wave components” refers here to the terms

$$\partial_{\alpha}\partial_{\beta}Z^I u_{\hat{\gamma}}, \quad Z^I\partial_{\alpha}\partial_{\beta}u_{\hat{\gamma}}.$$

Controlling these terms is particularly delicate, in comparison with the estimate already derived in previous chapter. Our strategy here is as follows. We will analyse the structure of the second-order derivatives of the wave components and we will see in the following section that the only “bad” component of the second order-derivative is $\underline{\partial}_0\underline{\partial}_0 u_{\hat{\gamma}}Z^I u$ or equivalently, $\partial_t\partial_t Z^I u_{\hat{\gamma}}$. Then in the third section we will analyse the estimate on $\partial_t\partial_t Z^I u_{\hat{\gamma}}$ and give a general description. The proofs are presented in the last two sections.

The aim of this section is to prove that in order to get the estimate on $\partial_{\alpha}\partial_{\beta}Z^I u_{\hat{\gamma}}$, we need only estimate $\partial_t\partial_t Z^I u_{\hat{\gamma}}$ since of the better estimates enjoyed by the other components of the Hessian matrix of $Z^I u_{\hat{\gamma}}$. More precisely, the following estimate will be established:

Lemma 7.1. *Let u be a C^{∞} function compactly supported in the half-cone $\mathcal{K} = \{|x| \leq t - 1\}$. Then for any multi-index I the following estimate holds:*

$$|\partial_{\alpha}\partial_{\beta}Z^I u| \leq |\partial_t\partial_t Z^I u| + C \sum_{\alpha,\beta} |\underline{\partial}_{\alpha}\partial_{\beta}Z^I u| + \frac{C}{t} \sum_{\gamma} |\partial_{\gamma}Z^I u|,$$

where C is a universal constant.

Proof. We recall the following relation based on the change of frame (between natural frame and semi-hyperboloidal frame):

$$\begin{aligned}\partial_\alpha \partial_\beta Z^I u &= \Psi_\alpha^{\alpha'} \Psi_\beta^{\beta'} \underline{\partial}_{\alpha'} \underline{\partial}_{\beta'} Z^I u + \partial_\alpha (\Psi_\beta^{\beta'}) \underline{\partial}_\beta Z^I u \\ &= \Psi_\alpha^0 \Psi_\beta^0 \underline{\partial}_0 \underline{\partial}_0 Z^I u + \sum_b \Psi_\alpha^0 \Psi_\beta^b \underline{\partial}_0 \underline{\partial}_b Z^I u \\ &\quad + \sum_a \Psi_\alpha^a \Psi_\beta^0 \underline{\partial}_a \underline{\partial}_0 Z^I u + \sum_{a,b} \Psi_\alpha^a \Psi_\beta^b \underline{\partial}_a \underline{\partial}_b Z^I u + \partial_\alpha (\Psi_\beta^{\beta'}) \underline{\partial}_\beta Z^I u.\end{aligned}$$

Observing that

$$|\partial_\alpha \Psi_\beta^{\beta'}| \leq \frac{C}{t}$$

and

$$|[\underline{\partial}_b, \partial_t](Z^I u)| \leq Ct^{-1} \sum_\alpha |\partial_\alpha Z^I u|,$$

we can estimate the term

$$\sum_b |\partial_t \underline{\partial}_b Z^I u| + \sum_{a,b} |\underline{\partial}_a \underline{\partial}_b Z^I u| \leq \sum_{a,\beta} |\underline{\partial}_a \partial_\beta Z^I u| + Ct^{-1} \sum_\gamma |\partial_\gamma Z^I u|.$$

then the desired result is proved. \square

Now we consider the case under energy assumption (2.4.5). Then the following estimates hold.

Lemma 7.2. *Under the assumption (2.4.5), the following L^2 estimates hold for $|I^\dagger| \leq 4, |I^b| \leq 2$*

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^{I^\dagger} u_{\hat{v}}\|_{L^2(\mathcal{H}_s)} \leq \|s^3 t^{-2} \partial_t \partial_t Z^{I^\dagger} u_{\hat{v}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon s^\delta, \quad (7.1.1a)$$

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^I u_{\hat{v}}\|_{L^2(\mathcal{H}_s)} \leq \|s^3 t^{-2} \partial_t \partial_t Z^I u_{\hat{v}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon s^{\delta/2}, \quad (7.1.1b)$$

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^{I^b} u_{\hat{v}}\|_{L^2(\mathcal{H}_s)} \leq \|s^3 t^{-2} \partial_t \partial_t Z^{I^b} u_{\hat{v}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon. \quad (7.1.1c)$$

And the following L^∞ estimates hold for $|J^\dagger| \leq 2$:

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta Z^{J^\dagger} u| \leq \sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_t \partial_t Z^{J^\dagger} u| + CC_1 \epsilon s^\delta, \quad (7.1.2a)$$

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta Z^J u| \leq \sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_t \partial_t Z^J u| + CC_1 \epsilon s^{\delta/2}, \quad (7.1.2b)$$

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta u| \leq \sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_t \partial_t u| + CC_1 \epsilon. \quad (7.1.2c)$$

Proof. The estimate (7.1.1a) is a combination of Lemma 7.1 and the estimate (6.3.4a) while (7.1.1c) is a combination of Lemma 7.1 and the estimate (6.3.4c). For the L^∞ estimates, we combine Lemma 7.1 with (6.4.5). \square

7.2 Analysis of the algebraic structure

The aim of this section is to establish a general estimate on the term $\partial_t \partial_t Z^I u_{\hat{i}}$ by making use of the identity (2.2.8). The strategy is to express the term $\partial_t \partial_t Z^I u_{\hat{i}}$ by other terms in (2.2.8).

In this first lemma, we establish the algebraic relation between $\partial_t \partial_t Z^I u_{\hat{i}}$ and the remaining terms which decrease faster.

Lemma 7.3. *Let $U_{tt}(I) := (\partial_t \partial_t Z^I u_1, \partial_t \partial_t Z^I u_2, \dots, \partial_t \partial_t Z^I u_{j_0})^T$ which is a j_0 -dimension vector and \mathbf{I}_{j_0} be the j_0 -dimensional matrix of identity. Then the following relation holds:*

$$\begin{aligned} & ((s/t)^2 \mathbf{I}_{j_0} + \mathbf{G}(w, \partial w)) U_{tt}(I) \\ &= ([Z^I, G_{\hat{i}}^{\hat{j}00} \partial_t \partial_t] u_{\hat{j}} - Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w) + Z^I F_{\hat{i}} + R(Z^I u_{\hat{i}}))_{\hat{i}}, \end{aligned} \quad (7.2.1)$$

where

$$(\mathbf{G}(w, \partial w))_{\hat{i}, \hat{j}} = (\underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w))_{\hat{i}, \hat{j}},$$

is a $j_0 \times j_0$ order matrix and

$$\begin{aligned} & Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w) \\ &= Z^I (\underline{G}_{\hat{i}}^{\hat{j}a0}(w, \partial w) \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}0b}(w, \partial w) \partial_t \underline{\partial}_b u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}ab}(w, \partial w) \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) \\ &+ Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) + Z^I (G_{\hat{i}}^{\check{\alpha}\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} v_{\hat{j}}) \end{aligned}$$

and

$$\begin{aligned} R(Z^I u_{\hat{i}}) &:= -\underline{m}^{0a} \underline{\partial}_0 \underline{\partial}_a Z^I u_{\hat{i}} - \underline{m}^{a0} \underline{\partial}_a \underline{\partial}_0 Z^I u_{\hat{i}} - \underline{m}^{ab} \underline{\partial}_a \underline{\partial}_b Z^I u_{\hat{i}} \\ &+ \underline{m}^{\alpha\beta} (\underline{\partial}_{\alpha} \Phi_{\beta}^{\beta'}) \underline{\partial}_{\beta'} Z^I u_{\hat{i}}. \end{aligned}$$

We note that the term $Z^I F_{\hat{i}}$ and $Q_{G_{\hat{i}}}$ are bilinear terms which can be expected to enjoy better estimates. The term $R(Z^I u_{\hat{i}})$ contains only the “good” second order derivatives and can also be expected to enjoy better estimates.

Proof. Recall the identity (2.2.8) with the function $Z^I u_{\hat{i}}$

$$\begin{aligned} (s/t)^2 \underline{\partial}_0 \underline{\partial}_0 Z^I u_{\hat{i}} &= \square Z^I u_{\hat{i}} - \underline{m}^{0a} \underline{\partial}_0 \underline{\partial}_a Z^I u_{\hat{i}} - \underline{m}^{a0} \underline{\partial}_a \underline{\partial}_0 Z^I u_{\hat{i}} \\ &\quad - \underline{m}^{ab} \underline{\partial}_a \underline{\partial}_b Z^I u_{\hat{i}} + \underline{m}^{\alpha\beta} (\underline{\partial}_{\alpha} \Phi_{\beta}^{\beta'}) \underline{\partial}_{\beta'} Z^I u_{\hat{i}} \quad (7.2.2) \\ &:= \square Z^I u_{\hat{i}} + R(Z^I u_{\hat{i}}). \end{aligned}$$

By equation (1.1.2), the first term in right-hand side can be written as

$$\square Z^I u_{\hat{i}} = Z^I \square u_{\hat{i}} = -Z^I (G_{\hat{i}}^{j\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} w_j) + Z^I (F_{\hat{i}}(w, \partial w)).$$

where

$$G_{\hat{i}}^{j\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} w_j = G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} u_{\hat{j}} + G_{\hat{i}}^{\check{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} v_{\check{j}}$$

and

$$G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} u_{\hat{j}} = \underline{G}_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\alpha} \underline{\partial}_{\beta} u_{\hat{j}} + G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}.$$

Then, we have

$$\begin{aligned} & Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} u_{\hat{j}}) \\ &= Z^I (\underline{G}_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\alpha} \underline{\partial}_{\beta} u_{\hat{j}}) + Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) \\ &= Z^I (\underline{G}_{\hat{i}}^{\hat{j}00} \partial_t \partial_t u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}a0} \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}0b} \partial_t \underline{\partial}_b u_{\hat{j}} \\ &\quad + \underline{G}_{\hat{i}}^{\hat{j}ab} \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) + Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) \\ &= \underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w) \partial_t \partial_t Z^I u_{\hat{j}} + [Z^I, \underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w) \partial_t \partial_t] u_{\hat{j}} \\ &\quad + Z^I (\underline{G}_{\hat{i}}^{\hat{j}0b}(w, \partial w) \partial_t \underline{\partial}_b u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}a0}(w, \partial w) \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}ab}(w, \partial w) \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) \\ &\quad + Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}). \end{aligned}$$

So we conclude with

$$\begin{aligned} \square Z^I u_{\hat{i}} &= - \underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w) \partial_t \partial_t Z^I u_{\hat{j}} - [Z^I, \underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w) \partial_t \partial_t] u_{\hat{j}} \\ &\quad - Z^I (\underline{G}_{\hat{i}}^{\hat{j}a0}(w, \partial w) \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}0b}(w, \partial w) \partial_t \underline{\partial}_b u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}ab}(w, \partial w) \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) \\ &\quad - Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) - Z^I (G_{\hat{i}}^{\check{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} v_{\check{j}}) \\ &\quad + Z^I F_{\hat{i}}(w, \partial w). \end{aligned}$$

Substituting this into the equation (7.2.2), we obtain

$$\begin{aligned} & (s/t)^2 \partial_t \partial_t Z^I u_{\hat{i}} + \underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w) \partial_t \partial_t Z^I u_{\hat{j}} \\ &= - [Z^I, \underline{G}_{\hat{i}}^{\hat{j}00}(w, \partial w) \partial_t \partial_t] u_{\hat{j}} \\ &\quad - Z^I (\underline{G}_{\hat{i}}^{\hat{j}a0}(w, \partial w) \underline{\partial}_a \partial_t u_{\hat{j}} - \underline{G}_{\hat{i}}^{\hat{j}0b}(w, \partial w) \partial_t \underline{\partial}_b u_{\hat{j}} - \underline{G}_{\hat{i}}^{\hat{j}ab}(w, \partial w) \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) \\ &\quad - Z^I (G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) - Z^I (G_{\hat{i}}^{\check{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} v_{\check{j}}) \\ &\quad + Z^I F_{\hat{i}}(w, \partial w) + R_1(Z^I, u_{\hat{i}}) \end{aligned}$$

and the desired result is proven. \square

Now we derive estimates from the algebraic relation (7.2.1). A first step is to get the estimate of the inverse of the linear operator $(\mathbf{I}_{j_0} + (t/s)^2 \mathbf{G})$. We can expect that when $|(t/s)^2 \mathbf{G}|$ is small, $(\mathbf{I}_{j_0} + (t/s)^2 \mathbf{G})$ is invertible and can get estimate on $U_{tt}(I)$ form (7.2.1).

Lemma 7.4. *If the following sup-norm estimates*

$$\begin{aligned} |\partial u_{\hat{i}}| &\leq CC_1 \epsilon t^{-1/2} s^{-1}, & |\partial v_{\hat{i}}| &\leq CC_1 \epsilon t^{-3/2} s^\delta, \\ |v_{\hat{i}}| &\leq CC_1 \epsilon t^{-3/2} s^\delta, & |u_{\hat{i}}| &\leq CC_1 \epsilon t^{-3/2} s \end{aligned} \quad (7.2.3)$$

hold for some $CC_1 \epsilon$ sufficiently small, then the following estimate holds:

$$\begin{aligned} |\partial_t \partial_t Z^I u_{\hat{i}}| &\leq C(t/s)^2 (|[Z^I, G_{\hat{i}}^{\hat{j}00} \partial_t \partial_t] u_{\hat{j}}| + |Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w)| \\ &\quad + |Z^I F_{\hat{i}}| + |R(Z^I u_{\hat{i}})|). \end{aligned} \quad (7.2.4)$$

Remark 7.1. (7.2.3) can be guaranteed by the energy assumption (2.4.5) via the L^∞ estimates (6.5.1), (6.5.2), (6.5.3) and (6.5.5).

Proof. By the structure of $\mathbf{G}(w, \partial w)$

$$\underline{G}_{\hat{i}}^{\hat{j}00} = \underline{A}_{\hat{i}}^{\hat{j}00\gamma\hat{k}} \underline{\partial}_\gamma u_{\hat{k}} + \underline{A}_{\hat{i}}^{\hat{j}00\gamma\hat{k}} \underline{\partial}_\gamma v_{\hat{k}} + \underline{B}_{\hat{i}}^{\hat{j}00\hat{k}} u_{\hat{k}} + \underline{B}_{\hat{i}}^{\hat{j}00\hat{k}} v_{\hat{k}}.$$

Taking into account the assumption of (7.2.3),

$$|\underline{G}_{\hat{i}}^{\hat{j}00}| \leq CC_1 \epsilon (s/t)^2.$$

When $CC_1 \epsilon$ sufficiently small, the linear operator $\mathbf{I}_{j_0} + (t/s)^2 \mathbf{G}(w, \partial w)$ is invertible (viewed as a linear mapping from $(\mathbb{R}^{j_0}, \|\cdot\|_\infty)$ to itself) and $\|(\mathbf{I}_{j_0} + (t/s)^2 \mathbf{G}(w, \partial w))^{-1}\|_{\infty, \infty}$ is bounded by a fixed constant. Then by Lemma 7.3, we have

$$\begin{aligned} (s/t)^2 \partial_t \partial_t Z^I u_{\hat{i}} &= (\mathbf{I}_{j_0} + (t/s)^2 \mathbf{G}(w, \partial w))^{-1} \\ &\quad \left([Z^I, G_{\hat{i}}^{\hat{j}00} \partial_t \partial_t] u_{\hat{j}} - Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w) + Z^I F_{\hat{i}} + R(Z^I u_{\hat{i}}) \right)_{\hat{i}} \end{aligned}$$

and so

$$\begin{aligned} |(s/t)^2 \partial_t \partial_t Z^I u_{\hat{i}}| &\leq C (|[Z^I, G_{\hat{i}}^{\hat{j}00} \partial_t \partial_t] u_{\hat{j}}| + |Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w)| \\ &\quad + C(|Z^I F_{\hat{i}}| + |R(Z^I u_{\hat{i}})|)). \end{aligned}$$

□

Then we need to estimate the terms appearing in the right-hand side of (7.2.4). We observe that the term $[Z^I, G_{\hat{i}}^{\hat{j}00} \partial_t \partial_t] u_{\hat{j}}$ contains the factor $\partial_t \partial_t Z^J u_{\hat{j}}$ which is also a “second-order” derivative but with $|J| < |I|$. This structure leads us to the following estimates of induction type. Recall again our convention: $Z^I = 0$ when $|I| < 0$.

Lemma 7.5. *Followed by the notation of Lemma 7.3 and 7.4, the following induction estimate holds*

$$\begin{aligned} |\partial_t \partial_t Z^I u_{\hat{i}}| &\leq CK(t/s)^2 \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_1|<|I|, \gamma, \hat{j}, i}} (|Z^{I_2} \partial_\gamma w_i| + |Z^{I_2} w_i|) |\partial_t \partial_t Z^{I_1} u_{\hat{j}}| \\ &\quad + C(t/s)^2 Q_{T_{\hat{i}}}(I, w, \partial w, \partial \partial w) + C(t/s)^2 |Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w)| \\ &\quad + C(t/s)^2 |Z^I F_{\hat{i}}| + C(t/s)^2 |R(Z^I u_{\hat{i}})|, \end{aligned} \quad (7.2.5)$$

where

$$Q_{T\hat{\gamma}} = \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_1|<|I|, \gamma, j, \hat{\gamma}}} \sum_{a, \beta, \gamma'} (|Z^{I_2} \partial_\gamma w_j| + |Z^{I_2} w_j|) (|\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{\gamma}}| + t^{-1} |\partial_{\gamma'} Z^{I_1} u_{\hat{\gamma}}|)$$

Proof. This is purely an estimate on the term $[[Z^I, \underline{G}_{\hat{\gamma}}^{\hat{J}00} \partial_t \partial_t] u_{\hat{\gamma}}]$. The estimate on this term is as follows:

$$[Z^I, \underline{G}_{\hat{\gamma}}^{\hat{J}00} \partial_t \partial_t] u_{\hat{\gamma}} = \sum_{\substack{I_1+I_2=I \\ |I_1|<|I|}} Z^{I_2} (\underline{G}_{\hat{\gamma}}^{\hat{J}00}) \cdot Z^{I_1} (\partial_t \partial_t u_{\hat{\gamma}}) + \underline{G}_{\hat{\gamma}}^{\hat{J}00} [Z^I, \partial_t \partial_t] u_{\hat{\gamma}}. \quad (7.2.6)$$

Remark that

$$\begin{aligned} |Z^{I_2} (\underline{G}_{\hat{\gamma}}^{\hat{J}00})| &\leq |Z^{I_2} (\underline{A}_{\hat{\gamma}}^{\hat{J}00\gamma k} \partial_\gamma w_k)| + |Z^{I_2} (\underline{B}_{\hat{\gamma}}^{\hat{J}00k} w_k)| \\ &\leq CK \sum_{\substack{j, \gamma, \\ |I_3| \leq |I_2|}} (|Z^{I_3} \partial_\gamma w_j| + |Z^{I_3} w_j|). \end{aligned}$$

Then the first term in right-hand side of (7.2.6) is bounded by

$$CK \sum_{\substack{\gamma, j \\ |I_1|+|I_2|\leq|I|}} (|Z^{I_2} \partial_\gamma w_j| + |Z^{I_2} w_j|) |Z^{I_1} \partial_t \partial_t u_{\hat{\gamma}}|.$$

Recall that

$$|Z^{I_1} \partial_t \partial_t u_{\hat{\gamma}}| \leq |\partial_t \partial_t Z^{I_1} u_{\hat{\gamma}}| + |[Z^{I_1}, \partial_t \partial_t] u_{\hat{\gamma}}|,$$

and, again by (3.3.3) and Lemma 7.1,

$$\begin{aligned} |[Z^{I_1}, \partial_t \partial_t] u_{\hat{\gamma}}| &\leq \sum_{|I_3|<|I_1|} |\partial_t \partial_t Z^{I_3} u_{\hat{\gamma}}| + C \sum_{\substack{a, \beta \\ |I_3|<|I_1|}} |\underline{\partial}_a \underline{\partial}_\beta Z^{I_3} u_{\hat{\gamma}}| \\ &\quad + Ct^{-1} \sum_{\substack{\gamma \\ |I_3|<|I_1|}} |\partial_\gamma Z^{I_3} u_{\hat{\gamma}}|. \end{aligned}$$

Then the first term of the right-hand side of (7.2.6) is bounded by

$$\begin{aligned} &CK \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_1|<|I|, \gamma, j, \hat{\gamma}}} (|Z^{I_2} \partial_\gamma w_j| + |Z^{I_2} w_j|) |\partial_t \partial_t Z^{I_1} u_{\hat{\gamma}}| \\ &+ CK \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_1|<|I|, \gamma, j, \hat{\gamma}}} \sum_{a, \beta, \gamma'} (|Z^{I_2} \partial_\gamma w_j| + |Z^{I_2} w_j|) (|\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{\gamma}}| + t^{-1} |\partial_{\gamma'} Z^{I_1} u_{\hat{\gamma}}|) \end{aligned}$$

Similarly, the second term of the right-hand side of (7.2.6) is estimated as follows: by (3.3.3) and Lemma 7.1,

$$\begin{aligned} |[Z^I, \partial_t \partial_t] u_{\hat{\gamma}}| &\leq C \sum_{\substack{\alpha, \beta \\ |J|<|I|}} |\partial_\alpha \partial_\beta Z^J u_{\hat{\gamma}}| \\ &\leq C \sum_{|J|<|I|} |\partial_t \partial_t Z^J u_{\hat{\gamma}}| + C \sum_{\substack{a, \beta \\ |J|<|I|}} |\underline{\partial}_a \underline{\partial}_\beta Z^J u_{\hat{\gamma}}| \\ &\quad + Ct^{-1} \sum_{\substack{\gamma \\ |J|<|I|}} |\partial_\gamma Z^J u_{\hat{\gamma}}|. \end{aligned}$$

Combined with the following estimate

$$|\underline{G}_{\hat{i}}^{\hat{j}00}| \leq CK \sum_{\hat{j}, \gamma} (|\partial_\gamma w_j| + |w_j|).$$

Then the second term of right-hand side of (7.2.6) bounded by

$$\begin{aligned} & CK \sum_{\substack{\hat{j}, \hat{j}, \gamma \\ |\hat{j}| < |I|}} (|\partial_\gamma w_j| + |w_j|) |\partial_t \partial_t Z^J u_{\hat{j}}| \\ & + CK \sum_{\substack{\hat{j}, \hat{j}, \gamma, \gamma' \\ |\hat{j}| < |I|}} (|\partial_\gamma w_j| + |w_j|) (|\underline{\partial}_a \underline{\partial}_\beta Z^J u_{\hat{j}} + t^{-1} |\partial_{\gamma'} Z^J u_{\hat{j}}|) \end{aligned}$$

and the desired result is proved. \square

7.3 Structure of the quadratic terms

The aim of this section is to analyse the structure of the quadratic terms $Q_{T_{\hat{i}}}$, $Q_{G_{\hat{i}}}$ and $Z^I F_{\hat{i}}$. We emphasize that some components of these terms satisfy the null condition and we can use it to make the estimates a bit simpler, but we prefer to *avoid the use of the null structure* here in order to show the independence of these analysis on this structure. First, we study the term $Q_{T_{\hat{i}}}$ and note that the $Q_{T_{\hat{i}}}$'s are a linear combination of the following terms with constant coefficients with $|I_1| + |I_2| \leq |I|$:

$$\begin{aligned} & |Z^{I_2} \partial_\gamma u_{\hat{k}}| |\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}|, \quad |Z^{I_2} \partial_\gamma v_{\check{k}}| |\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}|, \\ & |Z^{I_2} u_{\hat{k}}| |\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}|, \quad |Z^{I_2} v_{\check{k}}| |\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}|, \\ & t^{-1} |Z^{I_2} \partial_\gamma u_{\hat{k}}| |\partial_{\gamma'} Z^{I_1} u_{\hat{j}}|, \quad t^{-1} |Z^{I_2} \partial_\gamma v_{\check{k}}| |\partial_{\gamma'} Z^{I_1} u_{\hat{j}}|, \\ & t^{-1} |Z^{I_2} u_{\hat{k}}| |\partial_{\gamma'} Z^{I_1} u_{\hat{j}}|, \quad t^{-1} |Z^{I_2} v_{\check{k}}| |\partial_{\gamma'} Z^{I_1} u_{\hat{j}}|, \end{aligned} \tag{7.3.1}$$

Then, we consider $Q_{G_{\hat{i}}}$:

$$\begin{aligned}
& Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w) \\
&= Z^I (\underline{G}_{\hat{i}}^{\hat{j}a0}(w, \partial w) \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}0b}(w, \partial w) \partial_t \underline{\partial}_b u_{\hat{j}} + \underline{G}_{\hat{i}}^{\hat{j}ab}(w, \partial w) \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) \\
&\quad + Z^I (\underline{G}_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) + Z^I (\underline{G}_{\hat{i}}^{\check{j}\alpha\beta}(w, \partial w) \partial_{\alpha} \partial_{\beta} v_{\check{j}}) \\
&= Z^I (\underline{A}_{\hat{i}}^{\hat{j}a0\gamma\hat{k}} \underline{\partial}_{\gamma} u_{\hat{k}} \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{B}_{\hat{i}}^{\hat{j}a0\hat{k}} u_{\hat{k}} \underline{\partial}_a \partial_t u_{\hat{j}} + \underline{A}_{\hat{i}}^{\check{j}a0\gamma\check{k}} \underline{\partial}_{\gamma} v_{\check{k}} \underline{\partial}_a \partial_t u_{\hat{j}} \\
&\quad + \underline{B}_{\hat{i}}^{\check{j}a0\check{k}} v_{\check{k}} \underline{\partial}_a \partial_t u_{\hat{j}}) \\
&\quad + Z^I (\underline{A}_{\hat{i}}^{\hat{j}0b\gamma\hat{k}} \underline{\partial}_{\gamma} u_{\hat{k}} \partial_t \underline{\partial}_b u_{\hat{j}} + \underline{B}_{\hat{i}}^{\hat{j}0b\hat{k}} u_{\hat{k}} \partial_t \underline{\partial}_b u_{\hat{j}} + \underline{A}_{\hat{i}}^{\check{j}0b\gamma\check{k}} \underline{\partial}_{\gamma} v_{\check{k}} \partial_t \underline{\partial}_b u_{\hat{j}} \\
&\quad + \underline{B}_{\hat{i}}^{\check{j}0b\check{k}} v_{\check{k}} \partial_t \underline{\partial}_b u_{\hat{j}}) \\
&\quad + Z^I (\underline{A}_{\hat{i}}^{\hat{j}ab\gamma\hat{k}} \underline{\partial}_{\gamma} u_{\hat{k}} \underline{\partial}_a \underline{\partial}_b u_{\hat{j}} + \underline{B}_{\hat{i}}^{\hat{j}ab\hat{k}} u_{\hat{k}} \underline{\partial}_a \underline{\partial}_b u_{\hat{j}} + \underline{A}_{\hat{i}}^{\check{j}ab\gamma\check{k}} \underline{\partial}_{\gamma} v_{\check{k}} \underline{\partial}_a \underline{\partial}_b u_{\hat{j}} \\
&\quad + \underline{B}_{\hat{i}}^{\check{j}ab\check{k}} v_{\check{k}} \underline{\partial}_a \underline{\partial}_b u_{\hat{j}}) \\
&\quad + Z^I (A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\hat{k}} \partial_{\gamma} u_{\hat{k}} \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'} + B_{\hat{i}}^{\hat{j}\alpha\beta\hat{k}} u_{\hat{k}} \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'} \\
&\quad\quad + A_{\hat{i}}^{\check{j}\alpha\beta\gamma\check{k}} \partial_{\gamma} v_{\check{k}} \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'} + B_{\hat{i}}^{\check{j}\alpha\beta\check{k}} v_{\check{k}} \underline{\partial}_{\beta'} u_{\hat{j}} \partial_{\alpha} \Psi_{\beta}^{\beta'}) \\
&\quad + Z^I (A_{\hat{i}}^{\check{j}\alpha\beta\gamma\check{k}} \partial_{\gamma} u_{\hat{k}} \partial_{\alpha} \partial_{\beta} v_{\check{j}} + A_{\hat{i}}^{\check{j}\alpha\beta\gamma\check{k}} \partial_{\gamma} v_{\check{k}} \partial_{\alpha} \partial_{\beta} v_{\check{j}} + B_{\hat{i}}^{\check{j}\alpha\beta\check{k}} v_{\check{k}} \partial_{\alpha} \partial_{\beta} v_{\check{j}}).
\end{aligned}$$

Let $K := \max_{\alpha, \beta, \gamma, i, j, k} \{A_{\hat{i}}^{j\alpha\beta\gamma k}, B_{\hat{i}}^{j\alpha\beta}\}$ and remark that by (2.1),

$$|Z^I \underline{A}_{\hat{i}}^{\hat{j}\alpha\beta\gamma k}| + |Z^I \underline{B}_{\hat{i}}^{\hat{j}\alpha\beta k}| \leq C(I)K.$$

Then we observe that $Q_{G_{\hat{i}}}(I, w, \partial w, \partial \partial w)$ is a linear combination with bounded coefficients of the following terms:

$$\begin{aligned}
& Z^I (u_{\hat{i}} \underline{\partial}_a \underline{\partial}_{\beta} u_{\hat{j}}), \quad Z^I (v_{\check{i}} \underline{\partial}_a \underline{\partial}_{\beta} u_{\hat{j}}), \quad Z^I (u_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}), \quad Z^I (v_{\check{i}} \partial_t \underline{\partial}_a u_{\hat{j}}), \\
& Z^I (\underline{\partial}_{\gamma} u_{\hat{i}} \underline{\partial}_a \underline{\partial}_{\beta} u_{\hat{j}}), \quad Z^I (\underline{\partial}_{\gamma} v_{\check{i}} \underline{\partial}_a \underline{\partial}_{\beta} u_{\hat{j}}), \quad Z^I (\underline{\partial}_{\gamma} u_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}), \quad Z^I (\underline{\partial}_{\gamma} v_{\check{i}} \partial_t \underline{\partial}_a u_{\hat{j}}), \\
& Z^I (\partial_{\alpha} \Psi_{\beta}^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \quad Z^I (\partial_{\alpha} \Psi_{\beta}^{\beta'} v_{\check{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \\
& Z^I (\partial_{\alpha} \Psi_{\beta}^{\beta'} \underline{\partial}_{\gamma} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \quad Z^I (\partial_{\alpha} \Psi_{\beta}^{\beta'} \underline{\partial}_{\gamma} v_{\check{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \\
& Z^I (v_{\check{i}} \partial_{\alpha} \partial_{\beta} v_{\check{j}}), \quad Z^I (\partial_{\gamma} u_{\hat{i}} \partial_{\alpha} \partial_{\beta} v_{\check{j}}), \quad Z^I (\partial_{\gamma} v_{\check{i}} \partial_{\alpha} \partial_{\beta} v_{\check{j}})
\end{aligned} \tag{7.3.2}$$

Then for the term $Z^I F_{\hat{i}}$, recall its definition

$$Z^I F_{\hat{i}} = Z^I (P_{\hat{i}}^{\alpha\beta j k} \partial_{\alpha} w_j \partial_{\beta} w_k + Q_{\hat{i}}^{\alpha j \check{k}} v_{\check{k}} \partial_{\alpha} w_j + R_{\hat{i}}^{\check{j} k} v_{\check{j}} v_{\check{k}})$$

We classify its components into the following groups:

$$\partial_{\alpha} u_{\hat{j}} \partial_{\beta} u_{\hat{k}}, \quad \partial_{\alpha} v_{\check{j}} \partial_{\beta} w_k, \quad v_{\check{k}} \partial_{\alpha} w_j, \quad v_{\check{j}} v_{\check{k}}.$$

We regard $Z^I F_{\hat{i}}$ as a linear combination with constant coefficients bounded by K of the following terms:

$$\begin{aligned} & Z^I (\partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}}), \quad Z^I (\partial_\alpha v_{\hat{j}} \partial_\beta w_{\hat{k}}), \\ & Z^I (v_{\hat{k}} \partial_\alpha u_{\hat{j}}), \quad Z^I (v_{\hat{j}} v_{\hat{k}}). \end{aligned} \quad (7.3.3)$$

Then the estimates on $\partial_t \partial_t Z^I u_{\hat{i}}$ turn out to be the estimates on these terms listed in (7.3.2) and (7.3.3).

7.4 L^∞ estimates

The purpose of the section is to establish the L^∞ estimates on $\partial_t \partial_t Z^J u_{\hat{i}}$ (and then $\partial_\alpha \partial_\beta Z^J u_{\hat{i}}$) under the energy assumption (2.4.5). It is a combination of the estimates on the terms $Q_{T_{\hat{i}}}$, $Q_{G_{\hat{i}}}$, $Z^{J^\dagger} F_{\hat{i}}$ and $R(Z^{J^\dagger} u_{\hat{i}})$ with Lemma 7.5. So first we estimate these terms in the following two lemmas.

Lemma 7.6. *Under the energy assumption (2.4.5), the following estimates hold for any $|J^\dagger| \leq 2$*

$$|Z^{J^\dagger} F_{\hat{i}}| \leq CKt^{-3/2} s^{-1+\delta}, \quad (7.4.1a)$$

$$|Q_{G_{\hat{i}}}(J^\dagger, w, \partial w, \partial \partial w)| \leq CKt^{-3/2} s^{-1+\delta}, \quad (7.4.1b)$$

$$Q_{T_{\hat{i}}}(J^\dagger, w, \partial w, \partial \partial w) \leq CKt^{-3/2} s^{-1+\delta}, \quad (7.4.1c)$$

$$|R(Z^{J^\dagger} u_{\hat{i}})| \leq Ct^{-3/2} s^{-1+\delta}. \quad (7.4.1d)$$

Lemma 7.7. *Under the energy assumption (2.4.5), the following estimates hold for any $|J| \leq 1$*

$$|Z^J F_{\hat{i}}| \leq CKt^{-3/2} s^{-1+\delta/2}, \quad (7.4.2a)$$

$$|Q_{G_{\hat{i}}}(J, w, \partial w, \partial \partial w)| \leq CKt^{-3/2} s^{-1+\delta/2}, \quad (7.4.2b)$$

$$Q_{T_{\hat{i}}}(J, w, \partial w, \partial \partial w) \leq CKt^{-3/2} s^{-1+\delta/2}, \quad (7.4.2c)$$

$$|R(Z^J u_{\hat{i}})| \leq Ct^{-3/2} s^{-1+\delta/2}. \quad (7.4.2d)$$

Lemma 7.8. *Under the energy assumption (2.4.5), the following estimates hold:*

$$|F_{\hat{i}}| \leq CKt^{-3/2} s^{-1}, \quad (7.4.3a)$$

$$|Q_{G_{\hat{i}}}(0, w, \partial w, \partial \partial w)| \leq CKt^{-3/2} s^{-1}, \quad (7.4.3b)$$

$$Q_{T_{\hat{i}}}(0, w, \partial w, \partial \partial w) \leq CKt^{-3/2} s, \quad (7.4.3c)$$

$$|R(u_{\hat{i}})| \leq Ct^{-3/2} s^{-1}. \quad (7.4.3d)$$

The proofs of these two lemmas are essentially the same. We compute the relevant terms and express them as linear combinations of some bilinear terms, then suitably estimate each of them. The difference of the decreasing rate between these two lemmas is the due to the difference of regularity.

Proof of Lemma 7.6. The estimates of $|Q_{G\hat{\imath}}(J^\dagger, w, \partial w, \partial\partial w)|$ are as follows. Recall that $Q_{G\hat{\imath}}(J^\dagger, w, \partial w, \partial\partial w)$ is a linear combination of the terms listed in (7.3.2) with the index I replaced by J^\dagger . We estimate of $Z^{J^\dagger}(u_{\hat{\imath}}\underline{\partial}_a\partial_t u_{\hat{\jmath}})$ in detail:

$$\begin{aligned} |Z^{J^\dagger}(u_{\hat{\imath}}\underline{\partial}_a\partial_t u_{\hat{\jmath}})| &\leq \sum_{J_2+J_3=J^\dagger} |Z^{J_2}u_{\hat{\imath}}||Z^{J_3}\underline{\partial}_a\partial_t u_{\hat{\jmath}}| \\ &= |u_{\hat{\imath}}||Z^{J^\dagger}\underline{\partial}_a\partial_t u_{\hat{\jmath}}| + |Z^{J_2}u_{\hat{\imath}}||Z^{J_3}\underline{\partial}_a\partial_t u_{\hat{\jmath}}| + |Z^{J^\dagger}u_{\hat{\imath}}||\underline{\partial}_a\partial_t u_{\hat{\jmath}}|. \end{aligned}$$

Then when $J_2 = 0$ and $J_3 = J^\dagger$, we apply (6.5.5c) and (6.5.4a); when $J_3 = 0$ and $J_2 = J^\dagger$, we apply (6.5.5a) and (6.5.4c); and when $|J_2| = 1$ and $|J_3| = 1$, we apply (6.5.5c) and (6.5.4a);

$$|Z^{J^\dagger}(w_i\underline{\partial}_a\partial_t u_{\hat{\jmath}})| \leq C(C_1\epsilon)^2 t^{-3+\delta}.$$

For the other terms, we will specify the L^∞ estimates to be used but omits the details. This makes the following list:

Here the in the different colones represent the different partition of $J^\dagger = J_2 + J_3$. $(a, \leq b)$ means $|J_2| = a$, $|J_3| \leq b$. In the last colon, we specify the decreasing rate obtained directly by applying the given inequalities (modulo the constant $C(C_1\epsilon)^2$). The first number of inequality is that used on the first factor and the second number of inequality is that used on the second factor.

Product	(2, \leq 0)	(1, \leq 1)	(0, \leq 2)	Decreasing rate
$u_{\hat{\imath}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\jmath}}$	(6.5.5a), (6.5.4c)	(6.5.5c), (6.5.4a)	(6.5.5c), (6.5.4a)	$t^{-3}s^\delta$
$v_{\hat{\imath}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\jmath}}$	(6.5.1e), (6.5.4c)	(6.5.1e), (6.5.4a)	(6.5.1e), (6.5.4a)	$t^{-3}s^{-1+2\delta}$
$u_{\hat{\imath}}\partial_t\underline{\partial}_a u_{\hat{\jmath}}$	(6.5.5a), (6.5.4c)	(6.5.5c), (6.5.4a)	(6.5.5c), (6.5.4a)	$t^{-3}s^\delta$
$v_{\hat{\imath}}\partial_t\underline{\partial}_a u_{\hat{\jmath}}$	(6.5.1e), (6.5.4c)	(6.5.1e), (6.5.4a)	(6.5.1e), (6.5.4a)	$t^{-3}s^{-1+2\delta}$
$\underline{\partial}_\gamma v_{\hat{\imath}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\jmath}}$	(6.5.2a), (6.5.4c)	(6.5.2a), (6.5.4a)	(6.5.2a), (6.5.4a)	$t^{-3}s^{-1+2\delta}$
$\underline{\partial}_\gamma u_{\hat{\imath}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\jmath}}$	(6.5.1a), (6.5.4c)	(6.5.3a), (6.5.4a)	(6.5.3a), (6.5.4a)	$t^{-2}s^{-2+\delta}$

Product	(2, \leq 0)	(1, \leq 1)	(0, \leq 2)	Decreasing rate
$\underline{\partial}_\gamma u_{\hat{\imath}}\partial_t\underline{\partial}_a u_{\hat{\jmath}}$	(6.5.1a), (6.5.4c)	(6.5.3a), (6.5.4a)	(6.5.3a), (6.5.4a)	$t^{-2}s^{-2+\delta}$
$\underline{\partial}_\gamma v_{\hat{\imath}}\partial_t\underline{\partial}_a u_{\hat{\jmath}}$	(6.5.2a), (6.5.4c)	(6.5.2a), (6.5.4a)	(6.5.2a), (6.5.4a)	$t^{-3}s^{-1+2\delta}$
$v_{\hat{\imath}}\partial_\alpha\partial_\beta v_{\hat{\jmath}}$	(6.5.1e), (6.5.2a)	(6.5.1e), (6.5.2a)	(6.5.1e), (6.5.2a)	$t^{-3}s^{2\delta}$
$\partial_\gamma u_{\hat{\imath}}\partial_\alpha\partial_\beta v_{\hat{\jmath}}$	(6.5.1a), (6.5.2a)	(6.5.3a), (6.5.2a)	(6.5.3a), (6.5.2a)	$t^{-2}s^{-1+2\delta}$
$\partial_\gamma v_{\hat{\imath}}\partial_\alpha\partial_\beta v_{\hat{\jmath}}$	(6.5.2a), (6.5.2a)	(6.5.2a), (6.5.2a)	(6.5.2a), (6.5.2a)	$t^{-3}s^{2\delta}$

Taking into account the fact that $s \leq Ct \leq Cs^2$ and $\delta < 1/6$, we conclude that these terms are bounded by $C(C_1\epsilon)^2 t^{-3/2} s^{-1+\delta}$.

There are again the following four terms of $Q_{G_{\hat{i}}}$, which are estimated differently:

$$\begin{aligned} & Z^{J^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \quad Z^{J^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} v_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \\ & Z^{J^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \quad Z^{J^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma v_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}). \end{aligned}$$

By observing that $|Z^I \partial_\alpha \Psi_\beta^{\beta'}| \leq C(I)t^{-1}$, these terms can be estimated by $C(C_1\epsilon)^2 t^{-3} s^{2\delta}$. We give the proof for $Z^{J_1}(\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}})$:

$$\begin{aligned} & |Z^{J^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}})| \leq \sum_{J_2+J_3+J_4=J_1} |Z^{J_4} \partial_\alpha \Psi_\beta^{\beta'}| \cdot |Z^{J_2} u_{\hat{i}}| \cdot |Z^{J_3} \underline{\partial}_{\beta'} u_{\hat{j}}| \\ & \leq Ct^{-1} \sum_{|J_2|+|J_3| \leq |J^\dagger|} |Z^{J_2} u_{\hat{i}}| \cdot |Z^{J_3} \underline{\partial}_{\beta'} u_{\hat{j}}| \leq Ct^{-1} C_1 \epsilon t^{-3/2} s^{1+\delta} C_1 \epsilon t^{-1/2} s^{-1+\delta} \\ & = C(C_1\epsilon)^2 t^{-3} s^{2\delta} \leq C(C_1\epsilon)^2 t^{-3/2} s^{-1+\delta} \end{aligned}$$

where we recall that $\delta < 1/6$ and (6.5.5a) and (6.5.1a) are used. The other terms are estimated similarly, we omit the details and list out the inequalities used on each term and each partition of indices:

terms	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)	Decreasing rate
$t^{-1}Z^{J_2}u_{\hat{\gamma}}Z^{J_3}\underline{\partial}_{\beta'}u_{\hat{\gamma}}$	(6.5.5a), (6.5.3a)	(6.5.5c), (6.5.3a)	(6.5.5c), (6.5.1a)	$t^{-3}s^{\delta}$
$t^{-1}Z^{J_2}v_{\hat{\gamma}}Z^{J_3}\underline{\partial}_{\beta'}u_{\hat{\gamma}}$	(6.5.1e), (6.5.3a)	(6.5.1e), (6.5.3a)	(6.5.1e), (6.5.1a)	$t^{-2}s^{-1+2\delta}$
$t^{-1}Z^{J_2}\partial_{\gamma}u_{\hat{\gamma}}Z^{J_3}\underline{\partial}_{\beta'}u_{\hat{\gamma}}$	(6.5.1a), (6.5.3a)	(6.5.3a), (6.5.3a)	(6.5.3a), (6.5.1a)	$t^{-2}s^{-2+\delta}$
$t^{-1}Z^{J_2}\partial_{\gamma}v_{\hat{\gamma}}Z^{J_3}\underline{\partial}_{\beta'}u_{\hat{\gamma}}$	(6.5.1a), (6.5.3a)	(6.5.1a), (6.5.3a)	(6.5.1a), (6.5.1a)	$t^{-3}s^{-1+2\delta}$

Then we focus on the term $Q_{T_{\hat{\gamma}}}$. Recall that it is a linear combinations of the terms presented in (7.3.1) with I_i replace by J_i where $i = 1, 2$. Then each term is estimated as in the estimates of Q_G . We give the list of inequalities used for every partition $|J_1| + |J_2| \leq |J^{\dagger}|$. Recall that in the following list the symbol $(a, \leq b)$ means $|J_1| = a, |J_2| \leq b$.

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)	Decreasing rate
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1}u_{\hat{\gamma}}Z^{J_2}\partial_{\gamma}u_{\hat{k}}$	(6.4.5a), (6.5.3a)	(6.4.5a), (6.5.3a)	(6.4.5c), (6.5.1a)	$t^{-2}s^{-2+\delta}$
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1}u_{\hat{\gamma}}Z^{J_2}\partial_{\gamma}v_{\hat{k}}$	(6.4.5a), (6.4.3a)	(6.4.5a), (6.4.3a)	(6.4.5c), (6.4.3a)	$t^{-3}s^{-1+2\delta}$
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1}u_{\hat{\gamma}}Z^{J_2}u_{\hat{k}}$	(6.4.5a), (6.5.5c)	(6.4.5a), (6.5.5c)	(6.4.5c), (6.5.5a)	$t^{-3}s^{\delta}$
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1}u_{\hat{\gamma}}Z^{J_2}v_{\hat{k}}$	(6.4.5a), (6.5.1e)	(6.4.5a), (6.5.1e)	(6.4.5c), (6.5.1e)	$t^{-3}s^{-1+2\delta}$

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)	Decreasing rate
$t^{-1}\partial_{\gamma'}Z^{J_1}u_{\hat{\gamma}}Z^{J_2}\partial_{\gamma}u_{\hat{k}}$	(6.4.2a), (6.5.3b)	(6.4.4a), (6.5.3a)	(6.4.4a), (6.5.1a)	$t^{-2}s^{-2+\delta}$
$t^{-1}\partial_{\gamma'}Z^{J_1}u_{\hat{\gamma}}Z^{J_2}\partial_{\gamma}v_{\hat{k}}$	(6.4.2a), (6.5.2a)	(6.4.4a), (6.5.2a)	(6.4.4a), (6.5.2a)	$t^{-3}s^{-1+2\delta}$
$t^{-1}\partial_{\gamma'}Z^{J_1}u_{\hat{\gamma}}Z^{J_2}u_{\hat{k}}$	(6.4.2a), (6.5.5c)	(6.4.4a), (6.5.5c)	(6.4.4a), (6.5.5a)	$t^{-3}s^{\delta}$
$t^{-1}\partial_{\gamma'}Z^{J_1}u_{\hat{\gamma}}Z^{J_2}v_{\hat{k}}$	(6.4.2a), (6.5.1e)	(6.4.4a), (6.5.1e)	(6.4.4a), (6.5.1e)	$t^{-3}s^{-1+2\delta}$

Then we conclude by (7.4.1c).

The estimates on $Z^{J^\dagger} F_{\hat{\gamma}}$ is essentially the same. Recall that $Z^{J^\dagger} F_{\hat{\gamma}}$ is a linear combination of the terms listed in (7.3.3) with I replace by J^\dagger . We write in detail the estimate of the term $Z^{J^\dagger} (\partial_\alpha u_{\hat{\gamma}} \partial_\beta u_{\hat{k}})$, as follows:

$$\begin{aligned}
|Z^{J^\dagger} (\partial_\alpha u_{\hat{\gamma}} \partial_\beta u_{\hat{k}})| &\leq |Z^{J^\dagger} \partial_\alpha u_{\hat{\gamma}}| \cdot |\partial_\beta u_{\hat{k}}| + |\partial_\alpha u_{\hat{\gamma}}| \cdot |Z^{J_1} \partial_\beta u_{\hat{k}}| \\
&\quad + \sum_{\substack{|J_1|=1, |J_2|=1, \\ J_2+J_3=J^\dagger}} |Z^{J_2} \partial_\alpha u_{\hat{\gamma}}| \cdot |Z^{J_3} \partial_\beta u_{\hat{k}}| \\
&\leq CC_1 \epsilon t^{-1/2} s^{-1+\delta} \cdot CC_1 \epsilon t^{-1/2} s^{-1} \\
&\quad + CC_1 \epsilon t^{-1/2} s^{-1} \cdot CC_1 \epsilon t^{-1/2} s^{-1+\delta} \\
&\quad + CC_1 \epsilon t^{-1/2} s^{-1} \cdot CC_1 \epsilon t^{-1/2} s^{-1} \\
&\leq C(C_1 \epsilon)^2 t^{-1} s^{-2+\delta} \\
&\leq C(C_1 \epsilon)^2 t^{-3/2} s^{-1+\delta}.
\end{aligned}$$

Here for the three partition of $J^\dagger = J_1 + J_2$, the L^∞ estimates use are: when $|J_1| = 0$ and $|J_2| \leq |J^\dagger|$, we apply (6.5.3a) and (6.5.1a); when $|J_1| = |J_2| = 1$, we apply (6.5.3a) and (6.5.3a); when $|J_1| = |J^\dagger|$ and $|J_2| \leq 0$, we apply (6.5.1a) and (6.5.3a).

For the remaining terms we will just list out for each partition of J_1 the L^∞ estimates used, which is again a list (decreasing rate modulo $C(C_1 \epsilon)^2$):

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)	Decreasing rate
$\partial_\alpha u_{\hat{\gamma}} \partial_\beta u_{\hat{k}}$	(6.5.1a), (6.5.3a)	(6.5.3a), (6.5.3a)	(6.5.3a), (6.5.1a)	$t^{-1} s^{-2+\delta}$
$\partial_\alpha v_{\hat{\gamma}} \partial_\beta w_k$	(6.5.2a), (6.5.1a)	(6.5.2a), (6.5.1a)	(6.5.2a), (6.5.1a)	$t^{-2} s^{-1+2\delta}$
$v_{\hat{k}} \partial_\alpha w_j$	(6.5.1e), (6.5.1a)	(6.5.1e), (6.5.1a)	(6.5.1e), (6.5.1a)	$t^{-2} s^{-1+2\delta}$
$v_{\hat{\gamma}} v_{\hat{k}}$	(6.5.1e), (6.5.1e)	(6.5.1e), (6.5.1e)	(6.5.1e), (6.5.1e)	$t^{-3+2\delta}$

Combined with the condition $\delta < 1/6$ and the fact that $s \leq Ct \leq Cs^2$, the estimate (7.4.1a) is proved. On the other hand, the estimate of $R(Z^{J_1} u_{\hat{\gamma}})$ is a direct application of (6.4.5a). \square

Proof of Lemma 7.7. The proof is essentially the same to that of (7.7) while the only differences are the inequalities used on each term and each partition of index. We omit the details but present the inequalities used, as follows.

For the estimates of $Q_{G_{\hat{\gamma}}}(J, w, \partial w, \partial \partial w)$, the inequalities used are listed

out in the following list:

Product	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$u_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.5.5c), (6.5.4c)	(6.5.5c), (6.5.4b)	$t^{-3} s^{\delta/2}$
$v_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.5.1f), (6.5.4c)	(6.5.1f), (6.5.4b)	$t^{-3} s^{-1+\delta}$
$u_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.5.5c), (6.5.4c)	(6.5.5c), (6.5.4b)	$t^{-3} s^{\delta/2}$
$v_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.5.1f), (6.5.4c)	(6.5.1f), (6.5.4b)	$t^{-3} s^{-1+\delta}$
$\underline{\partial}_\gamma v_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.5.2b), (6.5.4c)	(6.5.2b), (6.5.4b)	$t^{-3} s^{-1+\delta}$
$\underline{\partial}_\gamma u_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.5.3a), (6.5.4c)	(6.5.3a), (6.5.4b)	$t^{-2} s^{-2+\delta/2}$
Product	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$\underline{\partial}_\gamma u_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.5.3a), (6.5.4c)	(6.5.3a), (6.5.4b)	$t^{-2} s^{-2+\delta/2}$
$\underline{\partial}_\gamma v_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.5.2b), (6.5.4c)	(6.5.2b), (6.5.4b)	$t^{-3} s^{-1+\delta}$
$v_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.5.1f), (6.5.2b)	(6.5.1f), (6.5.2b)	$t^{-3} s^\delta$
$\partial_\gamma u_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.5.3a), (6.5.2b)	(6.5.3a), (6.5.2b)	$t^{-2} s^{-1+\delta/2}$
$\partial_\gamma v_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.5.2b), (6.5.2b)	(6.5.2b), (6.5.2b)	$t^{-3} s^\delta$
terms	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$t^{-1} Z^{J_2} u_{\hat{i}} Z^{J_3} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.5.5c), (6.5.3a)	(6.5.5c), (6.5.3a)	t^{-3}
$t^{-1} Z^{J_2} v_{\hat{i}} Z^{J_3} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.5.1f), (6.5.3a)	(6.5.1f), (6.5.3a)	$t^{-3} s^{-1+\delta/2}$
$t^{-1} Z^{J_2} \partial_\gamma u_{\hat{i}} Z^{J_3} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.5.3a), (6.5.3a)	(6.5.3a), (6.5.3a)	$t^{-2} s^{-2}$
$t^{-1} Z^{J_2} \partial_\gamma v_{\hat{i}} Z^{J_3} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.5.1b), (6.5.3a)	(6.5.1b), (6.5.3a)	$t^{-3} s^{-1+\delta/2}$
For the term $Q_{T_{\hat{i}}}(J, w, \partial w, \partial \partial w)$:			
Product	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1} u_{\hat{i}} Z^{J_2} \partial_\gamma u_{\hat{k}}$	(6.4.5b), (6.5.3a)	(6.4.5c), (6.5.3a)	$t^{-2} s^{-2+\delta/2}$
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1} u_{\hat{i}} Z^{J_2} \partial_\gamma v_{\hat{k}}$	(6.4.5b), (6.4.3b)	(6.4.5c), (6.4.3b)	$t^{-3} s^{-1+\delta}$
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1} u_{\hat{i}} Z^{J_2} u_{\hat{k}}$	(6.4.5b), (6.5.5c)	(6.4.5c), (6.5.5a)	$t^{-3} s^\delta$
$\underline{\partial}_a \underline{\partial}_\beta Z^{J_1} u_{\hat{i}} Z^{J_2} v_{\hat{k}}$	(6.4.5b), (6.5.1f)	(6.4.5c), (6.5.1f)	$t^{-3} s^{-1+\delta}$
Product	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$t^{-1} \partial_{\gamma'} Z^{J_1} u_{\hat{j}} Z^{J_2} \partial_\gamma u_{\hat{k}}$	(6.4.4a), (6.5.3a)	(6.4.4a), (6.5.3a)	$t^{-2} s^{-2}$
$t^{-1} \partial_{\gamma'} Z^{J_1} u_{\hat{j}} Z^{J_2} \partial_\gamma v_{\hat{k}}$	(6.4.4a), (6.5.2b)	(6.4.4a), (6.5.2b)	$t^{-3} s^{-1+\delta/2}$
$t^{-1} \partial_{\gamma'} Z^{J_1} u_{\hat{j}} Z^{J_2} u_{\hat{k}}$	(6.4.4a), (6.5.5c)	(6.4.4a), (6.5.5c)	t^{-3}
$t^{-1} \partial_{\gamma'} Z^{J_1} u_{\hat{j}} Z^{J_2} v_{\hat{k}}$	(6.4.4a), (6.5.1f)	(6.4.4a), (6.5.1f)	$t^{-2} s^{-1+\delta/2}$
For the term $F_{\hat{i}}$:			
Product	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$\partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}}$	(6.5.3a), (6.5.3a)	(6.5.3a), (6.5.3a)	$t^{-1} s^{-2}$
$\partial_\alpha v_{\hat{j}} \partial_\beta w_{\hat{k}}$	(6.5.2b), (6.5.1b)	(6.5.2b), (6.5.1b)	$t^{-2} s^{-1+\delta/2}$
$v_{\hat{k}} \partial_\alpha w_{\hat{j}}$	(6.5.1f), (6.5.1b)	(6.5.1f), (6.5.1b)	$t^{-2} s^{-1+\delta/2}$
$v_{\hat{j}} v_{\hat{k}}$	(6.5.1f), (6.5.1f)	(6.5.1f), (6.5.1f)	$t^{-3+\delta}$

On the other hand, the estimate of $R(Z^J u)$ is a direct result of (6.4.5b). \square

Proof of Lemma 7.8. The proof is essentially the same as the proof of Lemma 7.6 but much easier. We analyze first the estimate on $|Q_{G\hat{\gamma}}(w, \partial w, \partial\partial w)|$. This is a direct application of the L^∞ estimates established earlier on the terms listed in (7.3.2) with $I = 0$. As done in the proof of Lemma 7.6, this gives the following list (Decreasing rate modulo $C(C_1\epsilon)^2$):

terms	(0, 0)	Decreasing rate
$u_{\hat{\gamma}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.5c), (6.5.4c)	t^{-3}
$v_{\hat{\gamma}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.1e), (6.5.4c)	$t^{-3}s^{-1+\delta}$
$u_{\hat{\gamma}}\partial_t\underline{\partial}_a u_{\hat{\gamma}}$	(6.5.5c), (6.5.4c)	t^{-3}
$v_{\hat{\gamma}}\partial_t\underline{\partial}_a u_{\hat{\gamma}}$	(6.5.1e), (6.5.4c)	$t^{-3}s^{-1+\delta}$
$\underline{\partial}_\gamma u_{\hat{\gamma}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.3a), (6.5.4c)	$t^{-2}s^{-2}$
$\underline{\partial}_\gamma v_{\hat{\gamma}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.2a), (6.5.4c)	$t^{-3}s^{-1+\delta}$

terms	(0, 0)	Decreasing rate
$\underline{\partial}_\gamma u_{\hat{\gamma}}\partial_t\underline{\partial}_a u_{\hat{\gamma}}$	(6.5.3a), (6.5.4c)	$t^{-2}s^{-2}$
$\underline{\partial}_\gamma v_{\hat{\gamma}}\partial_t\underline{\partial}_a u_{\hat{\gamma}}$	(6.5.2a), (6.5.4c)	$t^{-3}s^{-1+\delta}$
$v_{\hat{\gamma}}\partial_\alpha\underline{\partial}_\beta v_{\hat{\gamma}}$	(6.5.1e), (6.5.2a)	$t^{-3}s^{2\delta}$
$\partial_\gamma u_{\hat{\gamma}}\partial_\alpha\underline{\partial}_\beta v_{\hat{\gamma}}$	(6.5.3a), (6.5.2a)	$t^{-2}s^{-1+\delta}$
$\partial_\gamma v_{\hat{\gamma}}\partial_\alpha\underline{\partial}_\beta v_{\hat{\gamma}}$	(6.5.2a), (6.5.2a)	$t^{-3}s^{2\delta}$

The following four terms

$$\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}, \quad \partial_\alpha \Psi_\beta^{\beta'} v_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}, \quad \partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma u_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}, \quad \partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma v_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}$$

are estimated by taking into account of the additional decay supplied by the factor $|\Psi_\beta^{\beta'}| \leq Ct^{-1}$. The inequalities used for each term are listed as follows:

terms	(0, 0)	Decreasing rate
$t^{-1} u_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}$	(6.5.5c), (6.5.3a)	t^{-3}
$t^{-1} v_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}$	(6.5.1e), (6.5.3a)	$t^{-3}s^{-1+\delta}$
$t^{-1} \partial_\gamma u_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}$	(6.5.3a), (6.5.3a)	$t^{-2}s^{-2}$
$t^{-1} \partial_\gamma v_{\hat{\gamma}} \underline{\partial}_{\beta'} u_{\hat{\gamma}}$	(6.5.2a), (6.5.3a)	$t^{-3}s^{-1+\delta}$

The estimates of Q_{T_i} are similar. We establish the following list and omits the details:

Product	(0, 0)	Decreasing rate
$\partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.3a), (6.5.4c)	$t^{-2}s^{-2}$
$\partial_\gamma v_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.2a), (6.5.4c)	$t^{-3}s^{-1+\delta}$
$u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.5c), (6.5.4c)	t^{-3}
$v_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{\gamma}}$	(6.5.1e), (6.5.4c)	$t^{-3}s^{-1+\delta}$

Product	(0, 0)	Decreasing rate
$t^{-1}\partial_\gamma u_{\widehat{k}} \partial_{\gamma'} u_{\widehat{j}}$	(6.5.3a), (6.5.3a)	$t^{-2}s^{-2}$
$t^{-1}\partial_\gamma v_{\widehat{k}} \partial_{\gamma'} u_{\widehat{j}}$	(6.5.2a), (6.5.3a)	$t^{-3}s^{-1+\delta}$
$t^{-1}u_{\widehat{k}} \partial_{\gamma'} u_{\widehat{j}}$	(6.5.5c), (6.5.3a)	t^{-3}
$t^{-1}v_{\widehat{k}} \partial_{\gamma'} u_{\widehat{j}}$	(6.5.1e), (6.5.3a)	$t^{-3}s^{-1+\delta}$

Then we conclude with (7.4.3c).

The estimate of $F_{\widehat{i}}$ is as follows: recall the structure of $F_{\widehat{i}}$ described by (7.3.3), we need to estimate these terms with $I = 0$. Then as in the proof of Lemma 7.6, the following list is established (with decreasing rate modulo $C(C_1\epsilon)^2$):

Product	(0, 0)	Decreasing rate
$\partial_\alpha u_{\widehat{i}} \partial_\beta u_{\widehat{j}}$	(6.5.3a), (6.5.3a)	$t^{-1}s^{-2}$
$\partial_\alpha v_{\widehat{i}} \partial_\beta w_j$	(6.5.2a), (6.5.1a)	$t^{-2}s^{-1+2\delta}$
$v_{\widehat{i}} \partial_\alpha w_j$	(6.5.1e), (6.5.1a)	$t^{-2}s^{-1+2\delta}$
$v_{\widehat{i}} v_{\widehat{j}}$	(6.5.1e), (6.5.1e)	$t^{-3}s^{2\delta}$

By tacking into account the condition $\delta < 1/6$ and the fact $s \leq Ct$, we conclude with (7.4.3a).

The estimate of $R(u_{\widehat{i}})$ is a direct application of (6.5.4c). \square

We can now prove the main result of this section.

Proposition 7.1. *Let w_i the solution of (1.1.2) and assume that (2.4.5) holds. Then for the wave components, the following decay estimates hold for any $J^\dagger \leq 2$*

$$\sup_{\mathcal{H}_s} (s^3 t^{-1/2} |\partial_t \partial_t Z^{J^\dagger} u_{\widehat{i}}|) \leq CC_1 \epsilon s^\delta \quad (7.4.4a)$$

$$\sup_{\mathcal{H}_s} (s^3 t^{-1/2} |\partial_t \partial_t Z^J u_{\widehat{i}}|) \leq CC_1 \epsilon s^{\delta/2} \quad (7.4.4b)$$

$$\sup_{\mathcal{H}_s} (s^3 t^{-1/2} |\partial_t \partial_t u_{\widehat{i}}|) \leq CC_1 \epsilon. \quad (7.4.4c)$$

and more generally

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta Z^{J^\dagger} u| \leq CC_1 \epsilon s^\delta, \quad (7.4.5a)$$

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta Z^J u| \leq CC_1 \epsilon s^{\delta/2}, \quad (7.4.5b)$$

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta u| \leq CC_1 \epsilon. \quad (7.4.5c)$$

Proof. (7.4.4c) will be firstly proved. Substitute (7.4.3) into (7.2.5) with $|I| = 0$. Recall the convention $|I| < 0 \Rightarrow Z^I = 0$. Then the desired result is proved.

Observe that for (7.4.4b), the case $|J| \leq 0$ is proved by (7.4.4c). For the case $|J| = 1$, we recall (7.2.5):

$$\begin{aligned} |\partial_t \partial_t Z^J u_{\hat{i}}| &\leq CK(t/s)^2 \sum_{\substack{|J_2|=1 \\ \gamma, \hat{j}, i}} (|Z^{J_2} \partial_\gamma w_i| + |Z^{J_2} w_i|) |\partial_t \partial_t u_{\hat{j}}| \\ &\quad + C(t/s)^2 Q_{T_{\hat{i}}}(J_2, w, \partial w, \partial \partial w) + C(t/s)^2 |Q_{G_{\hat{i}}}(J_2, w, \partial w \partial \partial w)| \\ &\quad + C(t/s)^2 |Z^{J_2} F_{\hat{i}}| + C(t/s)^2 |R(Z^{J_2} u_{\hat{i}})| \end{aligned}$$

substitutes (7.4.4c) (estimate on $\partial_t \partial_t u_{\hat{j}}$) and (7.4.2) into (7.2.5) together with the following estimate:

$$\sum_{\substack{\gamma, \hat{j}, \hat{i}, \\ |J| \leq 1}} (|Z^J \partial_\gamma w_i| + |Z^J w_i|) \leq CC_1 \epsilon (t^{-1/2} s^{-1} + t^{-3/2} s + t^{-3/2} s^\delta) \leq CC_1 \epsilon t^{-3/2} s.$$

For (7.4.4a), remark that the case $|J^\dagger| \leq 1$ is guaranteed by (7.4.4b) and (7.4.4c).

The case $|J^\dagger| = 2$ is by substituting (7.4.4b) (with $|J| = 1$), (7.4.4c) and (7.4.1) into (7.2.5) with $|I| = 2$ together with the following estimate which is a direct result of (6.5.1a), (6.5.5c) and (6.5.1e):

$$\begin{aligned} \sum_{\gamma, \hat{j}, \hat{i}} (|Z^{J^\dagger} \partial_\gamma w_i| + |Z^{J^\dagger} w_i|) &\leq CC_1 \epsilon (t^{-1/2} s^{-1+\delta/2} + t^{-3/2} s + t^{-3/2} s^{\delta/2}) \\ &\leq CC_1 \epsilon t^{-3/2} s^{1+\delta/2}. \end{aligned}$$

Then we recall (7.2.5) with $I = J^\dagger$:

$$\begin{aligned} &|\partial_t \partial_t Z^{J^\dagger} u_{\hat{i}}| \\ &\leq CK(t/s)^2 \sum_{\substack{|J_1|+|J_2| \leq 2 \\ |J_1| \leq 1, \gamma, \hat{j}, i}} (|Z^{J_2} \partial_\gamma w_i| + |Z^{J_2} w_i|) |\partial_t \partial_t Z^{J_1} u_{\hat{j}}| \\ &\quad + C(t/s)^2 Q_{T_{\hat{i}}}(J_2, w, \partial w, \partial \partial w) + C(t/s)^2 |Q_{G_{\hat{i}}}(J_2, w, \partial w \partial \partial w)| \\ &\quad + C(t/s)^2 |Z^{J_2} F_{\hat{i}}| + C(t/s)^2 |R(Z^{J_2} u_{\hat{i}})|. \end{aligned}$$

Substitute (7.4.4b) (with $|J| = 1$), (7.4.4c) and (7.4.1),

$$\begin{aligned}
& |\partial_t \partial_t Z^{J^\dagger} u_{\hat{i}}| \\
& \leq CK(t/s)^2 \sum_{\substack{|J_1|+|J_2| \leq 2 \\ |J_1| \leq 1, \gamma, \hat{j}, i}} (|Z^{J_2} \partial_\gamma w_i| + |Z^{J_2} w_i|) |\partial_t \partial_t Z^{J_1} u_{\hat{j}}| + CKC_1 \epsilon t^{1/2} s^{-3+\delta} \\
& \leq CK(t/s)^2 \sum_{\substack{|J_1|+|J_2| \leq 2 \\ |J_1| = 1, \gamma, \hat{j}, i}} (|Z^{J_2} \partial_\gamma w_i| + |Z^{J_2} w_i|) |\partial_t \partial_t Z^{J_1} u_{\hat{j}}| \\
& \quad + CK(t/s)^2 \sum_{\substack{|J_1|+|J_2| \leq 2 \\ |J_1| = 0, \gamma, \hat{j}, i}} (|Z^{J_2} \partial_\gamma w_i| + |Z^{J_2} w_i|) |\partial_t \partial_t Z^{J_1} u_{\hat{j}}| + CKC_1 \epsilon 2t^{1/2} s^{-3+\delta} \\
& \leq CKC_1 \epsilon (t/s)^2 t^{-3/2} s \cdot t^{1/2} s^{-3+\delta/2} + CKC_1 \epsilon (t/s)^2 t^{-3/2} s^{1+\delta/2} \cdot t^{1/2} s^{-3} \\
& \quad + CKC_1 \epsilon t^{1/2} s^{-3+\delta} \\
& \leq CKC_1 \epsilon t s^{-4+\delta/2} + CKC_1 \epsilon t^{1/2} s^{-3+\delta} \leq CKC_1 \epsilon t^{1/2} s^{-3+\delta}.
\end{aligned}$$

The bound (7.4.5) are direct result of (7.4.4) combined with (7.1.2). \square

Then we can give the complete L^∞ estimates of the second-order derivatives.

Proposition 7.2. *Assume that (2.4.5) hold then the following estimates hold for all $|J^\dagger| \leq 2$ and $|J| \leq 1$:*

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta Z^{J^\dagger} u| + \sup_{\mathcal{H}_s} |s^3 t^{-1/2} Z^{J^\dagger} \partial_\alpha \partial_\beta u| \leq CC_1 \epsilon s^\delta, \quad (7.4.6a)$$

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta Z^J u| + \sup_{\mathcal{H}_s} |s^3 t^{-1/2} Z^J \partial_\alpha \partial_\beta u| \leq CC_1 \epsilon s^{\delta/2}, \quad (7.4.6b)$$

$$\sup_{\mathcal{H}_s} |s^3 t^{-1/2} \partial_\alpha \partial_\beta u| \leq CC_1 \epsilon. \quad (7.4.6c)$$

Proof. These estimates are based on (3.3.3) and (7.4.5). \square

7.5 L^2 estimates

The aim of this section is to get the L^2 estimates on $\partial_t \partial_t Z^I u_{\hat{j}}$. As in the last section, the strategy is to make use of the (7.2.5). First of all, we estimate the terms $Q_{T_{\hat{i}}}$, $Q_{G_{\hat{i}}}$, $Z^I F_{\hat{i}}$ and $R(Z^I u_{\hat{i}})$.

Lemma 7.9. *Under the energy assumption (2.4.5), the following estimates hold for all $|I^\dagger| \leq 4$:*

$$\|s Q_{G_{\hat{i}}}(I^\dagger, w, \partial w, \partial \partial w)\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^\delta, \quad (7.5.1a)$$

$$\|sQ_{T\hat{\gamma}}(I^\dagger, w, \partial w, \partial\partial w)\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^\delta, \quad (7.5.1b)$$

$$\|sZ^{I^\dagger} F_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^\delta, \quad (7.5.1c)$$

$$\|sR(Z^{I^\dagger} u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \leq CC_1\epsilon s^\delta. \quad (7.5.1d)$$

Lemma 7.10. *Under the energy assumption (2.4.5), the following estimates hold for all $|I| \leq 3$:*

$$\|sQ_{G\hat{\gamma}}(I, w, \partial w, \partial\partial w)\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{\delta/2}, \quad (7.5.2a)$$

$$\|sQ_{T\hat{\gamma}}(I, w, \partial w, \partial\partial w)\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{\delta/2}, \quad (7.5.2b)$$

$$\|sZ^I F_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{\delta/2}, \quad (7.5.2c)$$

$$\|sR(Z^I u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \leq CC_1\epsilon s^{\delta/2}. \quad (7.5.2d)$$

Lemma 7.11. *Under the energy assumption (2.4.5), the following estimates hold for all $|I^\flat| \leq 2$:*

$$\|sQ_{G\hat{\gamma}}(I^\flat, w, \partial w, \partial\partial w)\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2, \quad (7.5.3a)$$

$$\|sQ_{G\hat{\gamma}}(I^\flat, w, \partial w, \partial\partial w)\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2, \quad (7.5.3b)$$

$$\|sZ^{I^\flat} F_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2, \quad (7.5.3c)$$

$$\|sR(Z^{I^\flat} u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \leq CC_1\epsilon. \quad (7.5.3d)$$

Proof of Lemma 7.9. Let us regard firstly the estimate on $|Q_{G\hat{\gamma}}|$. Recall the structure of $Q_{G\hat{\gamma}}$ expressed by (7.3.2), the estimate turns to be L^2 estimates on each term of (7.3.2) with $I = I^\dagger, |I^\dagger| \leq 4$. We take $u_{\hat{\gamma}} \underline{\partial}_a \partial_t u_{\hat{\gamma}}$ as example and write down in detail the estimate on it:

$$\begin{aligned} & \|sZ^{I^\dagger} (u_{\hat{\gamma}} \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \\ & \leq \sum_{I_1+I_2=I^\dagger} \|s(Z^{I_1} u_{\hat{\gamma}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \\ & = \|s(u_{\hat{\gamma}} \cdot Z^{I^\dagger} \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} + \sum_{\substack{|I_1|=1, \\ I_1+I_2=I^\dagger}} \|s(Z^{I_1} u_{\hat{\gamma}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \\ & \quad + \sum_{\substack{|I_1|=2, \\ I_1+I_2=I^\dagger}} \|s(Z^{I_1} u_{\hat{\gamma}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} + \sum_{\substack{|I_1|=3, \\ I_1+I_2=I^\dagger}} \|s(Z^{I_1} u_{\hat{\gamma}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \\ & \quad + \|s(Z^{I^\dagger} u_{\hat{\gamma}} \cdot \underline{\partial}_a \partial_t u_{\hat{\gamma}})\|_{L^2(\mathcal{H}_s)} \\ & := T_0 + T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Observe now that in term T_k , $|I_2| \leq 4 - k$.

The term T_0 is estimated by (6.5.5c) and (6.3.5a):

$$\begin{aligned} T_0 &= \|s(u_{\widehat{i}} \cdot Z^{I_1} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon \|s(t^{-3/2} s Z^{I_1} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1 \epsilon s^{-1/2} \|s Z^{I_1} \underline{\partial}_a \partial_t u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-1/2+\delta}. \end{aligned}$$

The term T_1 is estimated by (6.5.5c) and (6.3.5a):

$$\begin{aligned} T_1 &= \sum_{\substack{|I_1|=1, \\ I_1+I_2=I^\dagger}} \|s(Z^{I_1} u_{\widehat{i}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1 \epsilon \sum_{\substack{|I_1|=1, \\ I_1+I_2=I^\dagger}} \|s(t^{-3/2} \cdot s Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1 \epsilon s^{-1/2} \sum_{\substack{|I_1|=1, \\ I_1+I_2=I^\dagger}} \|s Z^{I_3} \underline{\partial}_a \partial_t u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-1/2+\delta}. \end{aligned}$$

The term T_2 is estimated by (6.5.5a) and (6.3.5c) (remark that $|I_2| \leq 4 - 2 = 2$):

$$\begin{aligned} T_2 &= \sum_{\substack{|I_1|=2, \\ I_1+I_2=I^\dagger}} \|s(Z^{I_1} u_{\widehat{i}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1 \epsilon \sum_{\substack{|I_1|=2, \\ I_1+I_2=I^\dagger}} \|s(t^{-3/2} s^{1+\delta} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &= CC_1 \epsilon s^{-1/2+\delta} \sum_{\substack{|I_1|=2, \\ I_1+I_2=I^\dagger}} \|s Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-1/2+\delta}. \end{aligned}$$

The term T_3 is estimated by (6.5.5a) and (6.3.5c) (remark that $|I_2| \leq 4 - 3 = 1$):

$$\begin{aligned} T_3 &= \sum_{\substack{|I_1|=3, \\ I_1+I_2=I^\dagger}} \|s(Z^{I_1} u_{\widehat{i}} \cdot Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1 \epsilon \sum_{\substack{|I_1|=3, \\ I_1+I_2=I^\dagger}} \|s(t^{-3/2} s^\delta \cdot s Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \\ &= CC_1 \epsilon s^{-1/2+\delta} \sum_{\substack{|I_1|=3, \\ I_1+I_2=I^\dagger}} \|s Z^{I_2} \underline{\partial}_a \partial_t u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-1/2+\delta}. \end{aligned}$$

T_4 is estimated by (6.3.6a) and (6.5.4c):

$$\begin{aligned} T_4 &= \|s(Z^{I^\dagger} u_{\widehat{i}} \cdot \underline{\partial}_a \partial_t u_{\widehat{j}})\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon \|s Z^{I^\dagger} u_{\widehat{i}} \cdot t^{-3/2} s^{-1}\|_{L^2(\mathcal{H}_s)} \\ &= CC_1 \epsilon s^{-1/2} \|t^{-1} Z^{I^\dagger} u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{-1/2} \|s^{-1} Z^{I^\dagger} u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1 \epsilon)^2 s^{-1/2+\delta}. \end{aligned}$$

So we conclude with

$$\|sZ^{I^\dagger}(u_{\hat{i}}\underline{\partial}_a\partial_t u_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-1/2+\delta}.$$

For the remaining terms, we will not write the details, but give for each partition of $I^\dagger = I_1 + I_2$ the L^2 and L^∞ estimates to be used established later. As in the estimate of $u_{\hat{i}}\underline{\partial}_a\partial_t u_{\hat{j}}$, we denote by $(k, \leq 4 - k)$ the terms with which $|I_1| = k$, $|I_2| \leq 4 - k$. This leads us to the following list:

term	(4, ≤ 0)	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)	(0, ≤ 4)
$u_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.3.6b), (6.5.4c)	(6.5.5a), (6.3.5c)	(6.5.5a), (6.3.5c)	(6.5.5c), (6.3.5a)	(6.5.5c), (6.3.5a)
$v_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.3.1e), (6.5.4c)	(6.5.1e), (6.3.5c)	(6.5.1e), (6.3.5c)	(6.5.1e), (6.3.5a)	(6.5.1e), (6.3.5a)
$u_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.3.6b), (6.5.4c)	(6.5.5a), (6.3.5c)	(6.5.5a), (6.3.5c)	(6.5.5c), (6.3.5a)	(6.5.5c), (6.3.5a)
$v_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.3.1e), (6.5.4c)	(6.5.1e), (6.3.5c)	(6.5.1e), (6.3.5c)	(6.5.1e), (6.3.5a)	(6.5.1e), (6.3.5a)
$\underline{\partial}_\gamma u_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.3.1a), (6.5.4c)	(6.5.1a), (6.3.5c)	(6.5.1a), (6.3.5c)	(6.5.3a), (6.3.5a)	(6.5.3a), (6.3.5a)
$\underline{\partial}_\gamma v_{\hat{i}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}$	(6.3.2a), (6.5.4c)	(6.5.2a), (6.3.5c)	(6.5.2a), (6.3.5c)	(6.5.2a), (6.3.5a)	(6.5.2a), (6.3.5a)

terms	(4, ≤ 0)	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)	(0, ≤ 4)
$\underline{\partial}_\gamma u_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.3.1a), (6.5.4c)	(6.5.1a), (6.3.5c)	(6.5.1a), (6.3.5c)	(6.5.3a), (6.3.5a)	(6.5.3a), (6.3.5a)
$\underline{\partial}_\gamma v_{\hat{i}} \partial_t \underline{\partial}_a u_{\hat{j}}$	(6.3.2a), (6.5.4c)	(6.5.2a), (6.3.5c)	(6.5.2a), (6.3.5c)	(6.5.2a), (6.3.5a)	(6.5.2a), (6.3.5a)
$v_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.3.1e), (6.5.2a)	(6.5.1e), (6.3.2a)	(6.5.1e), (6.3.2a)	(6.5.1e), (6.3.2a)	(6.5.1e), (6.3.2a)
$\partial_\gamma u_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.3.1a), (6.5.2a)	(6.5.1a), (6.3.2a)	(6.5.1a), (6.3.2a)	(6.5.3a), (6.3.2a)	(6.5.3a), (6.3.2a)
$\partial_\gamma v_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.3.2a), (6.5.2a)	(6.5.2a), (6.3.2a)	(6.5.2a), (6.3.2a)	(6.5.2a), (6.3.2a)	(6.5.2a), (6.3.2a)

There are four terms to be estimated differently:

$$\begin{aligned} & sZ^{I^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \quad sZ^{I^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} v_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \\ & sZ^{I^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}), \quad sZ^{I^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma v_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}). \end{aligned}$$

We will use the extra-decay supplied by $|Z^I \Psi_\beta^{\beta'}| \leq C(I)t^{-1}$. Let us take

$$Z^{I^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}})$$

as an example and write its estimate in details:

$$\begin{aligned} & \|sZ^{I^\dagger}(\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \leq \sum_{I_1+I_2+I_3=I^\dagger} \|sZ^{I_3} \Psi_\beta^{\beta'} \cdot Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \leq C \sum_{|I_1|+|I_2| \leq |I^\dagger|} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & = C \sum_{|I_2| \leq |I^\dagger|} \|st^{-1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + C \sum_{\substack{|I_1|=1, \\ |I_2| \leq |I^\dagger|-1}} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \quad + C \sum_{\substack{|I_1|=2, \\ |I_2| \leq |I^\dagger|-2}} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + C \sum_{\substack{|I_1|=3, \\ |I_2| \leq |I^\dagger|-3}} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \quad + C \sum_{|I_1|=4} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & := T_0 + T_1 + T_2 + T_3 + T_4. \end{aligned}$$

The term T_0 by (6.5.5c) and (6.3.1a):

$$\begin{aligned} T_0 & = C \sum_{|I_2| \leq |I^\dagger|} \|st^{-1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \leq CC_1 \epsilon \sum_{|I_2| \leq |I^\dagger|} \|st^{-1} t^{-3/2} s \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & = CC_1 \epsilon \sum_{|I_2| \leq |I^\dagger|} \|t^{-5/2} s^2 \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \leq CC_1 \epsilon s^{-1/2} \sum_{|I_2| \leq |I^\dagger|} \|Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \leq C(C_1 \epsilon)^2 s^{-1/2+\delta}. \end{aligned}$$

The term T_1 is estimated by (6.5.5c) and (6.3.3a)

$$\begin{aligned}
T_1 &= C \sum_{\substack{|I_1|=1, \\ |I_2| \leq |I^\dagger|-1}} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon \sum_{\substack{|I_1|=1, \\ |I_2| \leq |I^\dagger|-1}} \|st^{-1} t^{-3/2} s \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon s^{-1/2} \sum_{\substack{|I_1|=1, \\ |I_2| \leq |I^\dagger|-1}} \|Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq C(C_1 \epsilon)^2 s^{-1/2+\delta}
\end{aligned}$$

The term T_2 is estimated by (6.5.5a) and (6.3.3a)

$$\begin{aligned}
T_2 &= C \sum_{\substack{|I_1|=2, \\ |I_2| \leq |I^\dagger|-2}} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon \sum_{\substack{|I_1|=2, \\ |I_2| \leq |I^\dagger|-2}} \|st^{-1} t^{-3/2} s^{1+\delta} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon s^{-1/2+\delta} \sum_{\substack{|I_1|=2, \\ |I_2| \leq |I^\dagger|-2}} \|Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq C(C_1 \epsilon)^2 s^{-1/2+\delta}
\end{aligned}$$

The term T_3 is estimated by (6.5.5a) and (6.3.3a):

$$\begin{aligned}
T_3 &= C \sum_{\substack{|I_1|=3, \\ |I_2| \leq |I^\dagger|-3}} \|st^{-1} Z^{I_1} u_{\hat{i}} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon \sum_{\substack{|I_1|=3, \\ |I_2| \leq |I^\dagger|-3}} \|st^{-1} t^{-3/2} s^{1+\delta} \cdot Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon s^{-1/2+\delta} \sum_{\substack{|I_1|=3, \\ |I_2| \leq |I^\dagger|-3}} \|Z^{I_2} \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq C(C_1 \epsilon)^2 s^{-1/2+\delta}
\end{aligned}$$

The term T_4 is estimated by (6.3.6b) and (6.5.1a):

$$\begin{aligned}
T_4 &= C \sum_{|I_2|=4} \|st^{-1} Z^{I_2} u_{\hat{i}} \cdot \underline{\partial}_{\beta'} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1 \epsilon \sum_{|I_2|=4} \|st^{-1} Z^{I_2} u_{\hat{i}} \cdot t^{-1/2} s^{-1}\|_{L^2(\mathcal{H}_s)} \\
&= CC_1 \epsilon s^{-1/2} \sum_{|I_2|=4} \|t^{-1} Z^{I_2} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\
&\leq C(C_1 \epsilon)^2 s^{-1/2+\delta/2}.
\end{aligned}$$

For the remaining three terms, we list out the L^2 and L^∞ estimates to be used for each term and each partition of I^\dagger , recall here the notation $(a, \leq b)$ means $|I_1| = a, |I_2| \leq b$:

terms	(4, ≤ 0)	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)	(0, ≤ 4)
$u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.3.6b), (6.5.3a)	(6.5.5a), (6.3.3a)	(6.5.5a), (6.3.3a)	(6.5.5c), (6.3.3a)	(6.5.5c), (6.3.1a)
$v_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.3.1e), (6.5.3a)	(6.5.1e), (6.3.3a)	(6.5.1e), (6.3.3a)	(6.5.1e), (6.3.3a)	(6.5.1e), (6.3.1a)
$\partial_{\gamma} u_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.3.1a), (6.5.3a)	(6.5.1a), (6.3.3a)	(6.5.1a), (6.3.3a)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.1a)
$\partial_{\gamma} v_{\hat{i}} \underline{\partial}_{\beta'} u_{\hat{j}}$	(6.3.2a), (6.5.3a)	(6.5.2a), (6.3.3a)	(6.5.2a), (6.3.3a)	(6.5.2a), (6.3.3a)	(6.5.2a), (6.3.1a)

Finally we conclude with (7.5.1a).

Then we estimate the term $Q_{T\hat{\gamma}}$ which is similar to that of $Q_{G\hat{\gamma}}$. Recall the structure of $Q_{T\hat{\gamma}}$ presented in (7.3.1). We take the term $Z^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}$ as an example:

$$\begin{aligned}
& \sum_{|I_1|+|I_2|\leq|I^\dagger|} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
= & \sum_{|I_2|\leq|I^\dagger|} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + \sum_{\substack{|I_1|+|I_2|\leq|I^\dagger| \\ |I_1|=1}} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_1|+|I_2|\leq|I^\dagger| \\ |I_1|=2}} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_1|+|I_2|\leq|I^\dagger| \\ |I_1|=3}} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{|I_1|\leq|I^\dagger|} \|s\partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
:= & T_0 + T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

We will estimate each partition of I^\dagger .

$$\begin{aligned}
T_0 &= \sum_{|I_2|\leq|I^\dagger|} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq \sum_{|I_2|\leq 4} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq \sum_{|I_2|\leq 4} \|(s/t)Z^{I_2} \partial_\gamma u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \|s(t/s)\underline{\partial}_a \underline{\partial}_\beta u_{\hat{j}}\|_{L^\infty(\mathcal{H}_s)} \leq CC_1\epsilon s^\delta \cdot CC_1\epsilon s^{-3/2} \\
&\leq C(C_1\epsilon)^2 s^{-3/2+\delta},
\end{aligned}$$

here (6.3.1a) and (6.4.5c) are applied.

$$\begin{aligned}
T_1 &\leq \sum_{|I_1|=1, |I_2|\leq 3} \|sZ^{I_2} \partial_\gamma u_{\hat{k}} \underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq \sum_{|I_1|=1, |I_2|\leq 3} \|(s/t)Z^{I_2} \partial_\gamma u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \|s(t/s)\underline{\partial}_a \underline{\partial}_\beta Z^{I_1} u_{\hat{j}}\|_{L^\infty(\mathcal{H}_s)} \\
&\leq CC_1\epsilon \cdot CC_1\epsilon s^{-3/2+\delta} \leq C(C_1\epsilon)^2 s^{-3/2+\delta},
\end{aligned}$$

where (6.3.3a) and (6.4.5a) are applied.

Similarly, the term T_2 , T_3 and T_4 are estimated by applying respectively (6.3.3a)(6.4.5a), (6.5.3a)(6.3.4a), (6.5.3a)(6.3.4a).

For the other terms, we will not write in detail the proof but list out the inequalities to be used on each term and each partition of index $|I_1| + |I_2| \leq$

$|I^\dagger|$, this makes the following list:

Product	$(4, \leq 0)$	$(3, \leq 1)$	$(2, \leq 2)$
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma u_{\hat{k}}$	(6.3.4a), (6.5.3a)	(6.3.4a), (6.5.3a)	(6.3.4c), (6.5.1a)
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma v_{\check{k}}$	(6.3.4a), (6.5.2a)	(6.3.4a), (6.5.2a)	(6.3.4c), (6.5.2a)
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}u_{\hat{k}}$	(6.3.4a), (6.5.5c)	(6.3.4a), (6.5.5c)	(6.3.4c), (6.5.5a)
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}v_{\check{k}}$	(6.3.4a), (6.5.1e)	(6.3.4a), (6.5.1e)	(6.3.4c), (6.5.1e)

Product	$(1, \leq 3)$	$(0, \leq 4)$
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma u_{\hat{k}}$	(6.4.5a), (6.3.3a)	(6.4.5c), (6.3.1a)
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma v_{\check{k}}$	(6.4.5a), (6.3.2a)	(6.4.5c), (6.3.2a)
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}u_{\hat{k}}$	(6.4.5a), (6.3.6c)	(6.4.5c), (6.3.6b)
$s\partial_a\partial_\beta Z^{I_1}u_{\hat{\gamma}}Z^{I_2}v_{\check{k}}$	(6.4.5a), (6.3.1e)	(6.4.5c), (6.3.1e)

Product	$(4, \leq 0)$	$(3, \leq 1)$	$(2, \leq 2)$
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma u_{\hat{k}}$	(6.2.1a), (6.5.3a)	(6.2.3a), (6.5.3a)	(6.2.3a), (6.5.1a)
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma v_{\check{k}}$	(6.2.1a), (6.5.2a)	(6.2.3a), (6.5.2a)	(6.2.3a), (6.5.2a)
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}u_{\hat{k}}$	(6.2.1a), (6.5.5c)	(6.2.3a), (6.5.5c)	(6.2.3a), (6.5.5a)
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}v_{\check{k}}$	(6.2.1a), (6.5.1e)	(6.2.3a), (6.5.1e)	(6.2.3a), (6.5.1e)

Product	$(1, \leq 3)$	$(0, \leq 4)$
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma u_{\hat{k}}$	(6.4.4a), (6.3.3a)	(6.4.4a), (6.3.1a)
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}\partial_\gamma v_{\check{k}}$	(6.4.4a), (6.3.2a)	(6.4.4a), (6.3.2a)
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}u_{\hat{k}}$	(6.4.4a), (6.3.6c)	(6.4.4a), (6.3.6b)
$st^{-1}\partial_{\gamma'}Z^{I_1}u_{\hat{\gamma}}Z^{I_2}v_{\check{k}}$	(6.4.4a), (6.3.1e)	(6.4.4a), (6.3.1e)

Now, we estimate the term $F_{\hat{\gamma}}$ which also is similar: recall the structure of $Z^{I^\dagger}F_{\hat{\gamma}}$ presented in (7.3.3) with I replaced by I^\dagger . As before, we consider the term $Z^{I^\dagger}(\partial_\alpha u_{\hat{\gamma}}\partial_\beta u_{\hat{k}})$ as an example and we write down the details of the analysis. For the other terms, we just give the L^2 and L^∞ estimates to be used for each factor:

$$\begin{aligned}
& \|sZ^{I^\dagger}(\partial_\alpha u_{\hat{\gamma}}\partial_\beta u_{\hat{k}})\|_{L^2(\mathcal{H}_s)} \\
& \leq \|s\partial_\alpha u_{\hat{\gamma}} \cdot Z^{I^\dagger}\partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} + \sum_{\substack{I_1+I_2=I^\dagger \\ |I_1|=1}} \|sZ^{I_1}\partial_\alpha u_{\hat{\gamma}} \cdot Z^{I_2}\partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
& \quad + \sum_{\substack{I_1+I_2=I^\dagger \\ |I_1|=2}} \|sZ^{I_1}\partial_\alpha u_{\hat{\gamma}} \cdot Z^{I_2}\partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
& \quad + \sum_{\substack{I_1+I_2=I^\dagger \\ |I_1|=3}} \|sZ^{I_1}\partial_\alpha u_{\hat{\gamma}} \cdot Z^{I_2}\partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} + \|sZ^{I^\dagger}\partial_\alpha u_{\hat{\gamma}} \cdot \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
& := T_0 + T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

The term T_0 is estimated by (6.5.3a) and (6.3.1a):

$$\begin{aligned}
T_0 &= \|s\partial_\alpha u_{\hat{j}} \cdot Z^{I^\dagger} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \leq CC_1\epsilon \|s \cdot t^{-1/2} s^{-1} \cdot Z^{I^\dagger} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
&= CC_1\epsilon \|s t^{-1/2} s^{-1} \cdot (t/s)(s/t Z^{I^\dagger} \partial_\beta u_{\hat{k}})\|_{L^2(\mathcal{H}_s)} \leq CC_1\epsilon \|s/t Z^{I^\dagger} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
&\leq C(C_1\epsilon)^2 s^\delta.
\end{aligned}$$

The term T_4 is estimated in the same way by exchange the roll of $u_{\hat{j}}$ and $u_{\hat{k}}$.

The terms T_1 and T_3 are estimated by (6.5.3a) and (6.3.3a). We estimate T_1 as follows. The estimate of T_3 is by exchange $u_{\hat{j}}$ and $u_{\hat{k}}$ in the following proof.

$$\begin{aligned}
T_1 &= \sum_{\substack{I_1+I_2=I^\dagger, \\ |I_1|=1}} \|sZ^{I_1} \partial_\alpha u_{\hat{j}} \cdot Z^{I_2} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon \|s t^{-1/2} s^{-1} \cdot (t/s)(s/t) Z^{I_2} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon \|(s/t) Z^{I_2} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^\delta.
\end{aligned}$$

The term T_2 is estimated by (6.5.1a) and (6.3.3):

$$\begin{aligned}
T_2 &= \sum_{\substack{I_1+I_2=I^\dagger, \\ |I_1|=2}} \|sZ^{I_1} \partial_\alpha u_{\hat{j}} \cdot Z^{I_2} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon \|s t^{-1/2} s^{-1+\delta} \cdot (t/s)(s/t) Z^{I_2} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \\
&= CC_1\epsilon s^\delta \|(s/t) Z^{I_2} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^\delta.
\end{aligned}$$

The estimate of other terms are presented in the following list:

Product	(4, ≤ 0)	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)	(0, ≤ 4)
$\partial_\alpha u_{\hat{\gamma}} \partial_\beta u_{\hat{k}}$	(6.3.1a), (6.5.3a)	(6.3.3a), (6.5.3a)	(6.5.1a), (6.3.3a)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.1a)
$\partial_\alpha v_{\hat{\gamma}} \partial_\beta w_k$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)
$v_{\hat{k}} \partial_\alpha w_j$	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.5.1e), (6.3.1a)	(6.5.1e), (6.3.1a)	(6.5.1e), (6.3.1a)
$v_{\hat{k}} v_{\hat{\gamma}}$	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)	(6.5.1e), (6.3.1e)	(6.5.1e), (6.3.1e)

Then we conclude (7.5.1c).

The estimate of $R(Z^I u_{\hat{v}})$ is a direct result of (6.3.5a). □

Proof of Lemma 7.10. The proof is essentially the same to that of Lemma 7.10. The only difference is the inequalities applied on each term and each partition of index. We will list out the relevant inequalities but will skip the details.

For the proof of (7.5.2a), we list out the inequalities in the following list.

Product	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)	(0, ≤ 3)
$u_{\hat{i}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{j}}$	(6.3.6c), (6.5.4c)	(6.3.6c), (6.5.4b)	(6.5.5c), (6.3.5c)	(6.5.5c), (6.3.5b)
$v_{\hat{i}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{j}}$	(6.3.1f), (6.5.4c)	(6.3.1f), (6.5.4b)	(6.5.1f), (6.3.5c)	(6.5.1f), (6.3.5b)
$u_{\hat{i}}\partial_t\underline{\partial}_a u_{\hat{j}}$	(6.3.6c), (6.5.4c)	(6.3.6c), (6.5.4b)	(6.5.5c), (6.3.5c)	(6.5.5c), (6.3.5b)
$v_{\hat{i}}\partial_t\underline{\partial}_a u_{\hat{j}}$	(6.3.1f), (6.5.4c)	(6.3.1f), (6.5.4b)	(6.5.1f), (6.3.5c)	(6.5.1f), (6.3.5b)
$\underline{\partial}_\gamma u_{\hat{i}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{j}}$	(6.3.3a), (6.5.4c)	(6.3.3a), (6.5.4b)	(6.5.3a), (6.3.5c)	(6.5.3a), (6.3.5b)
$\underline{\partial}_\gamma v_{\hat{i}}\underline{\partial}_a\underline{\partial}_\beta u_{\hat{j}}$	(6.3.2b), (6.5.4c)	(6.3.2b), (6.5.4b)	(6.5.2b), (6.3.5c)	(6.5.2b), (6.3.5b)

Product	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)	(0, ≤ 3)
$\underline{\partial}_\gamma u_{\hat{i}}\partial_t\underline{\partial}_a u_{\hat{j}}$	(6.3.3a), (6.5.4c)	(6.3.3a), (6.5.4b)	(6.5.3a), (6.3.5c)	(6.5.3a), (6.3.5b)
$\underline{\partial}_\gamma v_{\hat{i}}\partial_t\underline{\partial}_a u_{\hat{j}}$	(6.3.2b), (6.5.4c)	(6.3.2b), (6.5.4b)	(6.5.2b), (6.3.5c)	(6.5.2b), (6.3.5b)
$v_{\hat{i}}\partial_\alpha\partial_\beta v_{\hat{j}}$	(6.3.1f), (6.5.2b)	(6.3.1f), (6.5.2b)	(6.5.1f), (6.3.2b)	(6.5.1f), (6.3.2b)
$\partial_\gamma u_{\hat{i}}\partial_\alpha\partial_\beta v_{\hat{j}}$	(6.3.3a), (6.5.2b)	(6.3.3a), (6.5.2b)	(6.5.3a), (6.3.2b)	(6.5.3a), (6.3.2b)
$\partial_\gamma v_{\hat{i}}\partial_\alpha\partial_\beta v_{\hat{j}}$	(6.3.2b), (6.5.2b)	(6.3.2b), (6.5.2b)	(6.5.2b), (6.3.2b)	(6.5.2b), (6.3.2b)

The following four terms are estimated by apply the additional decreasing rate supplied by $\partial_\alpha\Phi_\beta^\gamma$:

$$sZ^I(\partial_\alpha\Psi_\beta^{\beta'}u_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}), \quad sZ^I(\partial_\alpha\Psi_\beta^{\beta'}v_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}),$$

$$sZ^I(\partial_\alpha\Psi_\beta^{\beta'}\partial_\gamma u_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}), \quad sZ^I(\partial_\alpha\Psi_\beta^{\beta'}\partial_\gamma v_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}).$$

terms	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)	(0, ≤ 3)
$\partial_\alpha\Psi_\beta^{\beta'}u_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}$	(6.3.6c), (6.5.3a)	(6.3.6c), (6.5.3a)	(6.5.5c), (6.3.3a)	(6.5.5c), (6.3.3a)
$\partial_\alpha\Psi_\beta^{\beta'}v_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}$	(6.3.1f), (6.5.3a)	(6.3.1f), (6.5.3a)	(6.5.1f), (6.3.3a)	(6.5.1f), (6.3.3a)
$\partial_\alpha\Psi_\beta^{\beta'}\partial_\gamma u_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}$	(6.3.3a), (6.5.3a)	(6.3.3a), (6.5.3a)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.3a)
$\partial_\alpha\Psi_\beta^{\beta'}\partial_\gamma v_{\hat{i}}\underline{\partial}_{\beta'}u_{\hat{j}}$	(6.3.2b), (6.5.3a)	(6.3.2b), (6.5.3a)	(6.5.2b), (6.3.3a)	(6.5.2b), (6.3.3a)

For the term $Q_{T\hat{\gamma}}$,

Product	(3, \leq 0)	(2, \leq 1)	(1, \leq 2)	(0, \leq 3)
$s\hat{\partial}_a\hat{\partial}_\beta Z^{J_1}u_{\hat{\gamma}} Z^{J_2}\partial_\gamma u_{\hat{k}}$	(6.3.4b), (6.5.3a)	(6.3.4c), (6.5.3a)	(6.4.5b), (6.3.3a)	(6.4.5c), (6.3.3a)
$s\hat{\partial}_a\hat{\partial}_\beta Z^{J_1}u_{\hat{\gamma}} Z^{J_2}\partial_\gamma v_{\hat{k}}$	(6.3.4b), (6.5.2b)	(6.3.4c), (6.5.2b)	(6.4.5b), (6.3.2b)	(6.4.5c), (6.3.2b)
$s\hat{\partial}_a\hat{\partial}_\beta Z^{J_1}u_{\hat{\gamma}} Z^{J_2}u_{\hat{k}}$	(6.3.4b), (6.5.5c)	(6.3.4c), (6.5.5c)	(6.4.5b), (6.3.6c)	(6.4.5c), (6.3.6c)
$s\hat{\partial}_a\hat{\partial}_\beta Z^{J_1}u_{\hat{\gamma}} Z^{J_2}v_{\hat{k}}$	(6.3.4b), (6.5.1f)	(6.3.4c), (6.5.1f)	(6.4.5b), (6.3.1f)	(6.4.5c), (6.3.1f)

Product	(3, \leq 0)	(2, \leq 1)	(1, \leq 2)	(0, \leq 3)
$st^{-1}\partial_{\gamma'} Z^{J_1}u_{\hat{\gamma}} Z^{J_2}\partial_\gamma u_{\hat{k}}$	(6.2.3a), (6.5.3a)	(6.2.3a), (6.5.3a)	(6.4.4a), (6.3.3a)	(6.4.4a), (6.3.3a)
$st^{-1}\partial_{\gamma'} Z^{J_1}u_{\hat{\gamma}} Z^{J_2}\partial_\gamma v_{\hat{k}}$	(6.2.3a), (6.5.2b)	(6.2.3a), (6.5.2b)	(6.4.4a), (6.3.2b)	(6.5.3a), (6.3.2b)
$st^{-1}\partial_{\gamma'} Z^{J_1}u_{\hat{\gamma}} Z^{J_2}u_{\hat{k}}$	(6.2.3a), (6.5.5c)	(6.2.3a), (6.5.5c)	(6.4.4a), (6.3.6c)	(6.4.4a), (6.3.6c)
$st^{-1}\partial_{\gamma'} Z^{J_1}u_{\hat{\gamma}} Z^{J_2}v_{\hat{k}}$	(6.2.3a), (6.5.1f)	(6.2.3a), (6.5.1f)	(6.4.4a), (6.3.1f)	(6.4.4a), (6.3.1f)

For the estimates on $Z^I F_{\hat{i}}$, the inequalities used are presented in the following list:

Product	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)	(0, ≤ 3)
$\partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}}$	(6.3.3a), (6.5.3a)	(6.3.3a), (6.5.3a)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.3a)
$\partial_\alpha v_{\hat{j}} \partial_\beta w_k$	(6.3.2b), (6.5.1b)	(6.3.2b), (6.5.1b)	(6.5.2b), (6.3.1b)	(6.5.2b), (6.3.1b)
$v_{\hat{k}} \partial_\alpha w_j$	(6.3.1f), (6.5.1b)	(6.3.1f), (6.5.1b)	(6.5.1f), (6.3.1b)	(6.5.1f), (6.3.1b)
$v_{\hat{k}} v_{\hat{j}}$	(6.3.1f), (6.5.1f)	(6.3.1f), (6.5.1f)	(6.5.1f), (6.3.1f)	(6.5.1f), (6.3.1f)

The estimate on $R(Z^I u_{\hat{i}})$ is a direct result of (6.3.5b) \square

Proof of Lemma 7.11. The proof of Lemma 7.11 is essentially the same to that of Lemma 7.9. The only difference is the different regularity. We will not give the details of the estimate and only list the inequalities which will be used for every term and every partition of $I = I_2 + I_3$:

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)
$u_{\hat{i}} \partial_a \partial_\beta u_{\hat{j}}$	(6.3.6c), (6.5.4c)	(6.5.5c), (6.3.5c)	(6.5.5c), (6.3.5c)
$v_{\hat{i}} \partial_a \partial_\beta u_{\hat{j}}$	(6.3.1e), (6.5.4c)	(6.5.1e), (6.3.5c)	(6.5.1e), (6.3.5c)
$u_{\hat{i}} \partial_t \partial_a u_{\hat{j}}$	(6.3.6c), (6.5.4c)	(6.5.5c), (6.3.5c)	(6.5.5c), (6.3.5c)
$v_{\hat{i}} \partial_t \partial_a u_{\hat{j}}$	(6.3.1e), (6.5.4c)	(6.5.1e), (6.3.5c)	(6.5.1e), (6.3.5c)

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)
$\partial_\gamma u_{\hat{i}} \partial_a \partial_\beta u_{\hat{j}}$	(6.3.3a), (6.5.4c)	(6.5.3a), (6.3.5c)	(6.5.3a), (6.3.5c)
$\partial_\gamma v_{\hat{i}} \partial_a \partial_\beta u_{\hat{j}}$	(6.3.2a), (6.5.4c)	(6.5.2a), (6.3.5c)	(6.5.2a), (6.3.5c)
$\partial_\gamma u_{\hat{i}} \partial_t \partial_a u_{\hat{j}}$	(6.3.3a), (6.5.4c)	(6.5.3a), (6.3.5c)	(6.5.3a), (6.3.5c)
$\partial_\gamma v_{\hat{i}} \partial_t \partial_a u_{\hat{j}}$	(6.3.2a), (6.5.4c)	(6.5.2a), (6.3.5c)	(6.5.2a), (6.3.5c)
$v_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.3.1e), (6.5.2a)	(6.5.1e), (6.3.2a)	(6.5.1e), (6.3.2a)
$\partial_\gamma u_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.3.3a), (6.5.2a)	(6.5.3a), (6.3.2a)	(6.5.3a), (6.3.2a)
$\partial_\gamma v_{\hat{i}} \partial_\alpha \partial_\beta v_{\hat{j}}$	(6.3.2a), (6.5.2a)	(6.5.2a), (6.3.2a)	(6.5.2a), (6.3.2a)

There are four terms to be estimated differently:

$$Z^I (\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \partial_{\beta'} u_{\hat{j}}), \quad Z^I (\partial_\alpha \Psi_\beta^{\beta'} v_{\hat{i}} \partial_{\beta'} u_{\hat{j}}),$$

$$Z^I (\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma u_{\hat{i}} \partial_{\beta'} u_{\hat{j}}), \quad Z^I (\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma v_{\hat{i}} \partial_{\beta'} u_{\hat{j}}).$$

As before, these terms are to be estimated by the aid of extra-decay supplied by $|Z^I \Psi_\beta^{\beta'}| \leq C(I)t^{-1}$. We omit the detail but list out the inequalities to be used for each term and each partition of $I = I_2 + I_3$:

Product	(2, 0)	(1, 1)	(0, 2)
$\partial_\alpha \Psi_\beta^{\beta'} u_{\hat{i}} \partial_{\beta'} u_{\hat{j}}$	(6.3.6c), (6.5.3a)	(6.5.5c), (6.3.3a)	(6.5.5c), (6.3.3a)
$\partial_\alpha \Psi_\beta^{\beta'} v_{\hat{i}} \partial_{\beta'} u_{\hat{j}}$	(6.3.1e), (6.5.3a)	(6.5.1e), (6.3.3a)	(6.5.1e), (6.3.3a)
$\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma u_{\hat{i}} \partial_{\beta'} u_{\hat{j}}$	(6.3.3a), (6.5.3a)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.3a)
$\partial_\alpha \Psi_\beta^{\beta'} \partial_\gamma v_{\hat{i}} \partial_{\beta'} u_{\hat{j}}$	(6.3.2a), (6.5.3a)	(6.5.2a), (6.3.3a)	(6.5.2a), (6.3.3a)

And we conclude with (7.5.3a).

Then we turn our attention to the estimates for $Q_{T\hat{\gamma}}$. As before the details are omitted. The inequalities used for each term and each partition of the indices are listed:

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)
$s\underline{\partial}_\alpha\underline{\partial}_\beta Z^{I_1} u_{\hat{\gamma}} Z^{I_2} \partial_\gamma u_{\hat{k}}$	(6.3.4c), (6.5.3a)	(6.3.4c), (6.5.3a)	(6.5.4c), (6.3.3a)
$s\underline{\partial}_\alpha\underline{\partial}_\beta Z^{I_1} u_{\hat{\gamma}} Z^{I_2} \partial_\gamma v_{\hat{k}}$	(6.3.4c), (6.5.2a)	(6.3.4c), (6.5.2a)	(6.4.5c), (6.3.2a)
$s\underline{\partial}_\alpha\underline{\partial}_\beta Z^{I_1} u_{\hat{\gamma}} Z^{I_2} u_{\hat{k}}$	(6.3.4c), (6.5.5c)	(6.3.4c), (6.5.5c)	(6.4.5c), (6.3.6c)
$s\underline{\partial}_\alpha\underline{\partial}_\beta Z^{I_1} u_{\hat{\gamma}} Z^{I_2} v_{\hat{k}}$	(6.3.4c), (6.5.1e)	(6.3.4c), (6.5.1e)	(6.4.5c), (6.3.1e)

Product	(2, ≤ 0)	(1, ≤ 1)	(0, ≤ 2)
$st^{-1} \partial_{\gamma'} Z^{I_1} u_{\hat{\gamma}} Z^{I_2} \partial_\gamma u_{\hat{k}}$	(6.2.3a), (6.5.3a)	(6.2.3a), (6.5.3a)	(6.4.4a), (6.3.3a)
$st^{-1} \partial_{\gamma'} Z^{I_1} u_{\hat{\gamma}} Z^{I_2} \partial_\gamma v_{\hat{k}}$	(6.2.3a), (6.5.2a)	(6.2.3a), (6.5.2a)	(6.4.4a), (6.3.2a)
$st^{-1} \partial_{\gamma'} Z^{I_1} u_{\hat{\gamma}} Z^{I_2} u_{\hat{k}}$	(6.2.3a), (6.5.5c)	(6.2.3a), (6.5.5c)	(6.4.4a), (6.3.6c)
$st^{-1} \partial_{\gamma'} Z^{I_1} u_{\hat{\gamma}} Z^{I_2} v_{\hat{k}}$	(6.2.3a), (6.5.1e)	(6.2.3a), (6.5.1e)	(6.4.4a), (6.3.1e)

The estimate of $Z^I F_{\hat{\gamma}}$ is essentially the same. As before, we omit the details but list out the inequalities to be used:

Product	(2, 0)	(1, 1)	(0, 2)
$\partial_\alpha u_{\hat{\gamma}} \partial_\beta u_{\hat{k}}$	(6.3.3a), (6.5.3a)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.3a)
$\partial_\alpha v_{\hat{\gamma}} \partial_\beta w_{\hat{k}}$	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)
$v_{\hat{k}} \partial_\alpha w_{\hat{j}}$	(6.3.1e), (6.5.1a)	(6.5.1e), (6.3.1a)	(6.5.1e), (6.3.1a)
$v_{\hat{k}} v_{\hat{j}}$	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)	(6.5.1e), (6.3.1e)

The estimate of $R(Z^I u_{\hat{\gamma}})$ is a direct result of (6.3.5c). \square

Now we are ready to prove the second main result of this section.

Proposition 7.3. *Let $u_{\hat{\gamma}}$ be wave components of the C^∞ local solution of (1.1.2) and assume that (2.4.5) holds. Then the following estimates hold for $|I^\dagger| \leq 4$ and $|I^b| \leq 2$:*

$$\|s^3 t^{-2} \partial_t \partial_t Z^{I^\dagger} u_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (7.5.4a)$$

$$\|s^3 t^{-2} \partial_t \partial_t Z^I u_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (7.5.4b)$$

$$\|s^3 t^{-2} \partial_t \partial_t Z^{I^b} u_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon. \quad (7.5.4c)$$

and further more

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^{I^\dagger} u_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (7.5.5a)$$

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^I u_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (7.5.5b)$$

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^{I^b} u_{\hat{\gamma}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon. \quad (7.5.5c)$$

Proof. The proof is a combination of (7.5.1), (7.5.3) with (7.2.5). Let's prove first (7.5.4c).

The proof is an induction. We first prove (7.5.4c) with $|I^b| = 0$ and observe that with the convention $|I| < 0$ implies $Z^I = 0$, (7.2.5) implies (with $|I| = 0$):

$$\begin{aligned} & \|s^3 t^{-2} \partial_t \partial_t u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\ & \leq C \|s Q_{G_{\hat{i}}}\|_{L^2(\mathcal{H}_s)} + C \|s Q_{T_{\hat{i}}}\|_{L^2(\mathcal{H}_s)} + C \|s F_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + C \|s R(u_{\hat{i}})\|_{L^2(\mathcal{H}_s)}. \end{aligned}$$

Then by the group of inequalities (7.5.3),

$$\|s^3 t^{-2} \partial_t \partial_t u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon.$$

Let's suppose that (7.5.4c) holds with $|I^b| \leq m$, we will prove that (7.5.4c) with $|I^b| \leq m + 1$. By (7.2.5),

$$\begin{aligned} & \|s^3 t^{-2} \partial_t \partial_t Z^{I^b} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\ & \leq CK \sum_{\substack{|I_2|+|I_3| \leq |I^b|, \\ |I_2| < |I^b|}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \quad + C \|s Q_{G_{\hat{i}}}(I^b, w, \partial w, \partial \partial w)_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + C \|s Q_{T_{\hat{i}}}(I^b, w, \partial w, \partial \partial w)_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\ & \quad + C \|s R(Z^{I^b} u_{\hat{i}})\|_{L^2(\mathcal{H}_s)} \end{aligned}$$

when $|I^b| \leq m + 1 \leq 2$, we apply (7.5.3):

$$\begin{aligned} & \|s^3 t^{-2} \partial_t \partial_t Z^{I^b} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\ & \leq CK \sum_{\substack{|I_2|+|I_3| \leq |I^b|, \\ |I_2| < |I^b|}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon \\ & \leq CK \sum_{\substack{|I_2|+|I_3| \leq |I^b|, \\ |I_3|=1}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \quad + CK \sum_{\substack{|I_2|+|I_3| \leq |I^b|, \\ |I_3|=2}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon. \end{aligned}$$

Observe now that when $I_3 = 1$, by (6.5.3), (6.5.1c), (6.5.5a) and (6.5.1c):

$$\begin{aligned} |Z^{I_3} \partial_\gamma u_{\hat{i}}| & \leq CC_1 \epsilon t^{-1/2} s^{-1}, & |Z^{I_3} \partial_\gamma v_{\hat{i}}| & \leq CC_1 \epsilon t^{-3/2} s^\delta, \\ |Z^{I_3} u_{\hat{i}}| & \leq CC_1 \epsilon t^{-3/2} s, & |Z^{I_3} v_{\hat{j}}| & \leq CC_1 \epsilon t^{-3/2} s^\delta. \end{aligned}$$

And by the induction assumption ((7.5.4c) for $|I_2| \leq |I^\flat| - 1$):

$$\begin{aligned}
& \sum_{\substack{|I_2|+|I_3| \leq |I^\flat|, \\ |I_3|=1}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& \leq C(C_1 \epsilon) \sum_{\substack{|I_2|+|I_3| \leq |I^\flat| \\ \gamma, \hat{j}, i, |I_3|=1}} \|t^2 s^{-2} (t^{-1/2} s^{-1} + t^{-3/2} s + t^{-3/2} s^\delta) \cdot s^3 t^{-2} \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& \leq C(C_1 \epsilon) \sum_{\substack{|I_2|+|I_3| \leq |I^\flat|, \\ |I_3|=1}} \sum_{\gamma, \hat{j}, i} \|s^3 t^{-2} \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& \leq C(C_1 \epsilon)^2.
\end{aligned}$$

When $|I_3| = 2$, we observe that $|I_2| \leq 0$. Then by (7.4.4c),

$$\begin{aligned}
& \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& \leq CC_1 \epsilon \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot t^{1/2} s^{-3}\|_{L^2(\mathcal{H}_s)} \\
& \leq CC_1 \epsilon \sum_{\gamma, \hat{i}} \|t^{1/2} s^{-2} Z^{I_3} \partial_\gamma u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon \sum_{\gamma, \check{i}} \|t^{1/2} s^{-2} Z^{I_3} \partial_\gamma v_{\check{i}}\|_{L^2(\mathcal{H}_s)} \\
& \quad + CC_1 \epsilon \sum_{\hat{i}} \|t^{1/2} s^{-2} Z^{I_3} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon \sum_{\check{i}} \|t^{1/2} s^{-2} Z^{I_3} v_{\check{i}}\|_{L^2(\mathcal{H}_s)}
\end{aligned}$$

All of these four terms can be bounded by $C(C_1 \epsilon)^2$ by applying (6.3.3), (6.3.2), (6.3.6c) and (6.3.2). So for $|I^\flat| \leq m + 1 \leq 2$, (7.5.4c) is proved. Then by induction, (7.5.4c) is proved for $|I^\flat| \leq 2$.

Then we turn to the proof of (7.5.4b) and observe that in (7.5.4b), the case $|I^\dagger| \leq 2$ is already proved by (7.5.4c). We need only treat the cases $|I^\dagger| = 3$.

When $|I^\dagger| = 3$, we apply (7.2.5):

$$\begin{aligned}
& \|s^3 t^{-2} \partial_t \partial_t Z^{I^\dagger} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\
& \leq CK \sum_{\substack{|I_2|+|I_3| \leq 3, \\ |I_2| < 3}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& \quad + C \|s Q_{G_{\hat{i}}}(I_1, w, \partial w, \partial \partial w)_{\hat{i}}\|_{L^2(\mathcal{H}_s)} + C \|Z^{I_1} F_{\hat{i}}\| + C \|s Z^{I_1} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)}.
\end{aligned}$$

We observe that by (7.5.1),

$$\begin{aligned}
& \|s^3 t^{-2} \partial_t \partial_t Z^{I^\dagger} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\
& \leq CK \sum_{\substack{|I_2|+|I_3| \leq 3, \\ |I_2| < 3}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon s^\delta
\end{aligned}$$

We focus on

$$\begin{aligned}
& \sum_{\substack{|I_2|+|I_3|\leq 3, \\ |I_2|<3}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3}\partial_\gamma w_i| + |Z^{I_3}w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
\leq & \sum_{\substack{|I_2|+|I_3|\leq 3, \\ |I_2|<3}} \sum_{\gamma, \hat{j}, \hat{i}} \|sZ^{I_3}\partial_\gamma u_{\hat{i}} \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_2|+|I_3|\leq 3, \\ |I_2|<3}} \sum_{\gamma, \hat{j}, \check{i}} \|sZ^{I_3}\partial_\gamma v_{\check{i}} \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_2|+|I_3|\leq 3, \\ |I_2|<3}} \sum_{\gamma, \hat{j}, \hat{i}} \|sZ^{I_3}u_{\hat{i}} \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_2|+|I_3|\leq 3, \\ |I_2|<3}} \sum_{\gamma, \hat{j}, \check{i}} \|sZ^{I_3}v_{\check{i}} \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)}.
\end{aligned}$$

For each sum, the possible choice of $(|I_2|, |I_3|)$ are

$$(0, \leq 3), \quad (1, \leq 2), \quad (2, \leq 1).$$

We list out, for each sum, the relevant inequalities for each possible choice $(|I_2|, |I_3|)$. This makes the following list:

Product	$(2, \leq 1)$	$(1, \leq 2)$	$(0, \leq 3)$
$s\partial_t \partial_t Z^{I_1} u_{\hat{j}} Z^{I_2} \partial_\gamma u_{\hat{i}}$	(7.4.4a), (6.3.3a)	(7.4.4a), (6.3.1b)	(7.4.4c), (6.3.3a)
$s\partial_t \partial_t Z^{I_1} u_{\hat{j}} Z^{I_2} \partial_\gamma v_{\check{i}}$	(7.4.4a), (6.3.2a)	(7.4.4a), (6.3.2a)	(7.4.4c), (6.3.2a)
$s\partial_t \partial_t Z^{I_1} u_{\hat{j}} \cdot Z^{I_2} u_{\hat{i}}$	(7.4.4a), (6.3.6c)	(7.4.4a), (6.3.6c)	(7.4.4c), (6.3.6c)
$s\partial_t \partial_t Z^{I_1} u_{\hat{j}} \cdot Z^{I_2} v_{\check{i}}$	(7.4.4a), (6.3.1c)	(7.4.4a), (6.3.1c)	(7.4.4c), (6.3.1c)

Then we conclude with the fact that these four terms can be bounded by $CC_1\epsilon s^\delta$. So (7.5.4a) is proved for $|I^\dagger| = 3$.

When $|I^\dagger| = 4$, we apply the case $|I^\dagger| = 3$ and, more precisely,

$$\|s^3 t^{-2} \partial_t \partial_t Z^{I_1} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (7.5.6)$$

As in the case $|I^\dagger| = 3$, by (7.2.5) and (7.5.1),

$$\begin{aligned}
& \|s^3 t^{-2} \partial_t \partial_t Z^{I^\dagger} u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\
\leq & CK \sum_{\substack{|I_2|+|I_3|\leq 4, \\ |I_2|<3}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3}\partial_\gamma w_i| + |Z^{I_3}w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + CC_1 \epsilon s^\delta
\end{aligned}$$

The first sum is also decomposed into four parts:

$$\begin{aligned}
& \sum_{\substack{|I_2|+|I_3|\leq 4, \\ |I_2|<4}} \sum_{\gamma, \hat{j}, i} \|s(|Z^{I_3} \partial_\gamma w_i| + |Z^{I_3} w_i|) \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
\leq & \sum_{\substack{|I_2|+|I_3|\leq 4, \\ |I_2|<4}} \sum_{\gamma, \hat{j}, \hat{i}} \|sZ^{I_3} \partial_\gamma u_{\hat{i}} \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_2|+|I_3|\leq 4, \\ |I_2|<4}} \sum_{\gamma, \hat{j}, \check{i}} \|sZ^{I_3} \partial_\gamma v_{\check{i}} \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_2|+|I_3|\leq 4, \\ |I_2|<4}} \sum_{\gamma, \hat{j}, \hat{i}} \|sZ^{I_3} u_{\hat{i}} \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
& + \sum_{\substack{|I_2|+|I_3|\leq 4, \\ |I_2|<4}} \sum_{\gamma, \hat{j}, \check{i}} \|sZ^{I_3} v_{\check{i}} \cdot \partial_t \partial_t Z^{I_2} u_{\hat{j}}\|_{L^2(\mathcal{H}_s)}.
\end{aligned}$$

The possible choices of $(|I_2|, |I_3|)$ are

$$(3, \leq 1), \quad (2, \leq 2), \quad (1, \leq 3), \quad (0, \leq 4)$$

Then we list out the inequalities to be used for each term and each choice of $(|I_2|, |I_3|)$:

Product	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)	(0, ≤ 4)
$s\partial_t\partial_t Z^{I_2} u_{\hat{j}} Z^{I_3} \partial_\gamma u_{\hat{i}}$	(7.5.6), (6.5.3a)	(7.4.4a), (6.3.3a)	(7.4.4a), (6.3.3a)	(7.4.4c), (6.3.1a)
$s\partial_t\partial_t Z^{I_2} u_{\hat{j}} Z^{I_3} \partial_\gamma v_{\hat{i}}$	(7.5.6), (6.5.2a)	(7.4.4a), (6.3.2a)	(7.4.4a), (6.3.2a)	(7.4.4c), (6.3.2a)
$s\partial_t\partial_t Z^{I_2} u_{\hat{j}} Z^{I_3} u_{\hat{i}}$	(7.5.6), (6.5.5c)	(7.4.4a), (6.3.6c)	(7.4.4a), (6.3.6c)	(7.4.4c), (6.3.6c)
$s\partial_t\partial_t Z^{I_2} u_{\hat{j}} Z^{I_3} v_{\hat{i}}$	(7.5.6), (6.5.1e)	(7.4.4a), (6.3.1c)	(7.4.4a), (6.3.1c)	(7.4.4c), (6.3.1c)

Then concludes by (7.5.4a).

(7.5.5) are combinations of (7.5.4) and (7.1.1). \square

Now, we derive the complete L^2 estimates of the second-order derivatives.

Lemma 7.12. *Under the assumption (2.4.5), the following L^2 estimates hold for $|I^\dagger| \leq 4, |I^b| \leq 2$*

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^{I^\dagger} u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} + \|s^3 t^{-2} Z^{I^\dagger} \partial_\alpha \partial_\beta u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^\delta, \quad (7.5.7a)$$

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^I u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} + \|s^3 t^{-2} Z^I \partial_\alpha \partial_\beta u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon s^{\delta/2}, \quad (7.5.7b)$$

$$\|s^3 t^{-2} \partial_\alpha \partial_\beta Z^{I^b} u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} + \|s^3 t^{-2} Z^{I^b} \partial_\alpha \partial_\beta u_{\widehat{i}}\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon. \quad (7.5.7c)$$

Proof. These inequalities are based on (3.3.3) and (7.5.5). \square

Chapter 8

Null forms and decay in time

8.1 Estimates independent on the second-order estimates

In this chapter we will give the estimate of the following terms:

$$\mathcal{T}(\partial u_{\hat{i}}, \partial u_{\hat{j}}) := T^{\alpha\beta} \partial_\alpha u_{\hat{i}} \partial_\beta u_{\hat{j}}, \quad [Z^I, A^{\alpha\beta\gamma} \partial_\gamma u_{\hat{i}} \partial_\alpha \partial_\beta] u_{\hat{j}}.$$

where $T^{\alpha\beta}$ and $B^{\alpha\beta}$ are null quadratic forms. The estimates in this section will not apply to the “second-order derivatives” (7.5.7) and (7.4.6).

Remark that for the term $[Z^I, A^{\alpha\beta\gamma} \partial_\gamma u_{\hat{i}} \partial_\alpha \partial_\beta] u_{\hat{j}}$, we can apply (7.5.4c) and (7.4.4c) for better decreasing rate but the following decreasing rate will be sufficient for the main result.

Lemma 8.1. *Assume that the energy assumption (2.4.5e) holds, then the following estimates holds for all $|I| \leq 3$ and $|J| \leq 1$:*

$$\|Z^I \mathcal{T}(\partial u_{\hat{i}}, \partial u_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-3/2}, \quad (8.1.1a)$$

$$\|[Z^I, A^{\alpha\beta\gamma} \partial_\gamma u_{\hat{i}} \partial_\alpha \partial_\beta] u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-3/2+\delta}. \quad (8.1.1b)$$

Proof. The proof of (8.1.1a) is a combination of Lemma 4.2 with (6.5.3) and (6.3.3). By recalling the decomposition of \mathcal{T} presented in Lemma 4.2, we have the following list/

terms	$(3, \leq 0)$	$(2, \leq 1)$	$(1, \leq 2)$	$(0, \leq 3)$
$(s/t)^2 Z^{I_1} \partial_t u_{\hat{\gamma}} Z^{I_2} \partial_t u_{\hat{\gamma}}$	$(6.3.3a), (6.5.3a)$	$(6.3.3a), (6.5.3a)$	$(6.5.3a), (6.3.3a)$	$(6.5.3a), (6.3.3a)$
$Z^{I_1} \underline{\partial}_a u_{\hat{\gamma}} Z^{I_2} \underline{\partial}_\beta u_{\hat{\gamma}}$	$(6.3.3b), (6.5.3a)$	$(6.3.3b), (6.5.3a)$	$(6.5.3b), (6.3.3a)$	$(6.5.3b), (6.3.3a)$
$Z^{I_1} \underline{\partial}_\beta u_{\hat{\gamma}} Z^{I_2} \underline{\partial}_a u_{\hat{\gamma}}$	$(6.3.3a), (6.5.3b)$	$(6.3.3a), (6.5.3b)$	$(6.5.3a), (6.3.3b)$	$(6.5.3a), (6.3.3b)$

This completes the argument.

The estimate on the term $[Z^I, A^{\alpha\beta\gamma}\partial_\gamma u_{\hat{i}}\partial_\alpha\partial_\beta]u_{\hat{j}}$ is also proved by using the inequalities presented in the following list. Remark that due to the structure of commutator, some partition of the index do not exist.

terms	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)
$(s/t)^2 Z^{I_1}\partial_t u_{\hat{k}} Z^{I_2}\partial_t\partial_t u_{\hat{j}}$	(6.3.3a), (6.5.3a)	(6.3.3a), (6.5.1a)	(6.5.3a), (6.3.3a)
$Z^{I_1}\partial_\alpha u_{\hat{k}} Z^{I_2}\partial_\beta\partial_\gamma u_{\hat{j}}$	(6.3.3b), (6.5.3a)	(6.3.3b), (6.5.1a)	(6.5.3b), (6.3.3a)
$Z^{I_1}\partial_\alpha u_{\hat{k}} Z^{I_2}\partial_b\partial_\gamma u_{\hat{j}}$	(6.3.3a), (6.5.4c)	(6.3.3a), (6.5.4a)	(6.5.3a), (6.3.5c)
$Z^{I_1}\partial_\alpha u_{\hat{k}} Z^{I_2}\partial_\beta\partial_c u_{\hat{j}}$	(6.3.3a), (6.5.4c)	(6.3.3a), (6.5.4a)	(6.5.3a), (6.3.5c)
$t^{-1} Z^{I_1}\partial_\alpha u_{\hat{k}}\partial_\beta u_{\hat{j}}$	(6.3.3a), (6.5.3a)	(6.3.3a), (6.5.3a)	(6.5.3a), (6.3.3a)

Finally, we derive the L^∞ estimates of \mathcal{T} by the following list

terms	(1, ≤ 0)	(0, ≤ 1)	Decreasing rate
$(s/t)^2 Z^{J_1}\partial_t u_{\hat{i}} Z^{J_2}\partial_t u_{\hat{j}}$	(6.5.3a), (6.5.3a)	(6.5.3a), (6.5.3a)	t^{-3}
$Z^{J_1}\partial_\alpha u_{\hat{i}} Z^{J_2}\partial_\beta u_{\hat{j}}$	(6.5.3b), (6.5.3a)	(6.5.3b), (6.5.3a)	$t^{-2}s^{-1}$
$Z^{J_1}\partial_\beta u_{\hat{i}} Z^{J_2}\partial_\alpha u_{\hat{j}}$	(6.5.3a), (6.5.3b)	(6.5.3a), (6.5.3b)	$t^{-2}s^{-1}$

□

8.2 Estimates depending on the second-order estimates

In this section we will establish the following estimates of the terms $[Z^I, B^{\alpha\beta}u_{\hat{i}}\partial_\alpha\partial_\beta]u_{\hat{j}}$.

Lemma 8.2. *Assume that the energy assumption (2.4.5e) holds, then the following estimates hold for $|I^\sharp| \leq 5$, $|I^\dagger| \leq 4$ and $|I| \leq 3$:*

$$\| [Z^I, B^{\alpha\beta}u_{\hat{i}}\partial_\alpha\partial_\beta]u_{\hat{j}} \|_{L^2(\mathcal{I}_{s'})} \leq C(C_1\epsilon)^2 s^{-3/2+\delta}. \quad (8.2.1a)$$

Proof. Recall the structure of $[Z^I, B^{\alpha\beta}u_{\hat{i}}\partial_\alpha\partial_\beta]u_{\hat{j}}$ presented in Lemma 4.4. It is a linear combinations of several terms and we need to control each term in each possible partition of the index $|I_1| + |I_2| \leq |I|$. As before, we lists out the inequalities used for each term and each partition of index:

terms	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)
$(s/t)^2 Z^{I_1}u_{\hat{k}} Z^{I_2}\partial_t\partial_t u_{\hat{j}}$	(6.3.6c), (7.4.4c)	(6.3.6c), (7.4.4a)	(6.3.6c), (7.4.4a)
$Z^{I_1}u_{\hat{k}} Z^{I_2}\partial_\alpha\partial_\beta u_{\hat{j}}$	(6.3.6c), (6.5.4c)	(6.3.6c), (6.5.4a)	(6.3.6c), (6.5.4a)
$Z^{I_1}u_{\hat{k}} Z^{I_2}\partial_\alpha\partial_b u_{\hat{j}}$	(6.3.6c), (6.5.4c)	(6.3.6c), (6.5.4a)	(6.3.6c), (6.5.4a)
$t^{-1} Z^{I_1}u_{\hat{k}} Z^{I_1}\partial_\alpha u_{\hat{j}}$	(6.3.6c), (6.5.3a)	(6.3.6c), (6.5.3a)	(6.5.5c), (6.3.3a)

□

8.3 Decay estimates

We are now in a position to prove (2.4.7a),(2.4.7b), (2.4.8a) and (2.4.9a). The proof of (2.4.7a) requires only the L^∞ estimate established in Chapter 6.

Lemma 8.3. *Let $\{w_i\}$ be the local solution of (1.1.2) and suppose that the energy assumption (2.4.5a) and (2.4.5e) hold, then there exists a constant $\epsilon_1 > 0$ depending only on K such that, if $C_1\epsilon$ sufficiently small, then following estimate holds:*

$$\kappa_1^{-1} \sum_i E_{m,c_i}(s, Z^I w_i) \leq \sum_i E_{G,c_i}(s, Z^I w_i) \leq \kappa_1 \sum_i E_{m,c_i}(s, Z^I w_i).$$

Here K is the L^∞ bound of the constant $A_i^{j\alpha\beta\gamma k}$ and $B_i^{j\alpha\beta k}$.

Proof. We note that

$$\begin{aligned} \sum_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| &\leq CK \sum_i (|\partial w_i| + |w_i|) \\ &\leq CK \sum_{\hat{i}, \hat{i}, \alpha} (|\partial_\alpha v_{\hat{i}}| + |\partial_\beta u_{\hat{i}}| + |v_{\hat{i}}| + |u_{\hat{i}}|) \end{aligned}$$

Then by applying (6.5.2a), (6.5.3a), (6.5.1e) and (6.5.5c), and recall that $0 < \delta < 1/6$,

$$\sum_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| \leq CK(t^{-3/2}s + t^{-1/2}s^{-1}).$$

$$\begin{aligned} &\sum_i |E_{G,c_i}(s, w_i) - E_{m,c_i}(s, w_i)| \\ &= \left| 2 \int_{\mathcal{H}_s} (\partial_t w_i \partial_\beta w_j G_i^{j\alpha\beta})(1, -x^a/t) dx - \int_{\mathcal{H}_s} (\partial_\alpha w_i \partial_\beta w_j G_i^{j\alpha\beta}) dx \right| \\ &\leq 2 \int_{\mathcal{H}_s} \left(\sum_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| \right) \cdot \left(\sum_{\alpha,k} |\partial_\alpha w_k|^2 \right) dx \\ &\leq 2CK \int_{\mathcal{H}_s} \sum_i (|\partial w_i| + |w_i|) \cdot \left(\sum_{\alpha,k} |\partial_\alpha w_k|^2 \right) dx \\ &\leq 2CKC_1\epsilon \int_{\mathcal{H}_s} (t^{-1/2}s^{-1} + t^{-3/2}s)(t/s)^2 \cdot \left(\sum_{\alpha,k} |(s/t)\partial_\alpha w_k|^2 \right) dx \\ &\leq CKC_1\epsilon \sum_i E_m(s, w_i) \leq CKC_1\epsilon \sum_i E_{m,c_i}(s, w_i) \end{aligned}$$

Here we take $CKC_1\epsilon \leq 2/3$ with C a universal constant, then the lemma is proved by fixing $\kappa_1 = 3$. \square

The proof of (2.4.7b) consults the energy estimate (6.2.1a).

Lemma 8.4. *Suppose (2.4.5a) and (2.4.5e) hold, then for any $|I^\sharp| \leq 5$ the following estimates is true:*

$$\begin{aligned} & \left| \int_{\mathcal{H}_s} \frac{s}{t} \left((\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^\sharp} w_i \partial_\beta Z^{I^\sharp} w_j - \frac{1}{2} (\partial_t G_i^{j\alpha\beta}) \partial_\alpha Z^{I^\sharp} w_i \partial_\beta Z^{I^\sharp} w_j \right) dx \right| \\ & \leq C(C_1\epsilon)^2 s^{-1+\delta} E_m(s, Z^{I^\sharp} w_i)^{1/2} \leq C(C_1\epsilon)^2 s^{-1+\delta} E_{m,\sigma}(s, Z^{I^\sharp} w_i)^{1/2}. \end{aligned} \quad (8.3.1)$$

Proof. We note that

$$\begin{aligned} |\partial_\alpha G_i^{j\alpha\beta}| & \leq CK \sum_j |\partial_\alpha w_j| + CK \sum_{j,\beta} |\partial_\alpha \partial_\beta w_j| \\ & \leq CK \sum_{\check{j},\alpha,\beta} (|\partial_\alpha v_j| + |\partial_\alpha \partial_\beta v_j|) + CK \sum_{\hat{j},\alpha,\beta} (|\partial_\alpha u_j| + |\partial_\alpha \partial_\beta u_j|) \\ & \leq CC_1\epsilon t^{-1/2} s^{-1}. \end{aligned}$$

By substituting this result into the expression, the first term in left-hand side of (8.3.1) is bounded as follows:

$$\begin{aligned} & \left\| (s/t) (\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^\sharp} w_i \partial_\beta Z^{I^\sharp} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & = \left\| ((t/s) \partial_\alpha G_i^{j\alpha\beta}) (s/t) \partial_t Z^{I^\sharp} w_i (s/t) \partial_\beta Z^{I^\sharp} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & \leq \sum_{j,\beta} \left\| C(C_1\epsilon) t^{1/2} s^{-2} (s/t) \partial_t Z^{I^\sharp} w_i (s/t) \partial_\beta Z^{I^\sharp} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & \leq CC_1\epsilon s^{-1} \sum_{j,\beta} \left\| (s/t) \partial_t Z^{I^\sharp} w_i (s/t) \partial_\beta Z^{I^\sharp} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & \leq CC_1\epsilon s^{-1} \sum_{j,\beta} \left\| (s/t) \partial_\beta Z^{I^\sharp} w_j \right\|_{L^2(\mathcal{H}_s)} \left\| (s/t) \partial_t Z^{I^\sharp} w_i \right\|_{L^2(\mathcal{H}_s)}. \end{aligned}$$

Then we apply (6.2.1a) and find

$$\begin{aligned} & \left\| (s/t) (\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^\sharp} w_i \partial_\beta Z^{I^\sharp} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & \leq C(C_1\epsilon)^2 s^{-1+\delta} \left(\int_{\mathcal{H}_s} |(s/t) \partial_t Z^{I^\sharp} w_i|^2 dx \right)^{1/2} \\ & \leq C(C_1\epsilon)^2 s^{-1+\delta} E_m(s, Z^{I^\sharp} w_i)^{1/2}. \end{aligned}$$

The second term is estimated in the same way and we omit the details. \square

The proof of (2.4.8a) is similar to that of (2.4.7b).

Lemma 8.5. *Under the assumption (2.4.5c), (2.4.5d) and (2.4.5e), the following estimate hold:*

$$\begin{aligned} & \left| \int_{\mathcal{H}_s} \frac{s}{t} \left((\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^\dagger} w_i \partial_\beta Z^{I^\dagger} w_j - \frac{1}{2} (\partial_t G_i^{j\alpha\beta}) \partial_\alpha Z^{I^\dagger} w_i \partial_\beta Z^{I^\dagger} w_j \right) dx \right| \\ & \leq C(C_1\epsilon)^2 s^{-1+\delta/2} E_m(s, Z^{I^\dagger} w_i)^{1/2} \leq C(C_1\epsilon)^2 s^{-1+\delta/2} E_{m,\sigma}(s, Z^{I^\dagger} w_i)^{1/2}. \end{aligned} \quad (8.3.2)$$

Proof. As in the proof of Lemma 8.4:

$$\begin{aligned} & \left\| (s/t) (\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^\dagger} w_i \partial_\beta Z^{I^\dagger} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & \leq \frac{CC_1\epsilon}{s} \sum_{j,\beta} \left\| (s/t) \partial_\beta Z^{I^\dagger} w_j \right\|_{L^2(\mathcal{H}_s)} \left\| (s/t) \partial_t Z^{I^\dagger} w_i \right\|_{L^2(\mathcal{H}_s)} \end{aligned}$$

Then we apply (6.2.1b):

$$\begin{aligned} & \left\| (s/t) (\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^\dagger} w_i \partial_\beta Z^{I^\dagger} w_j \right\|_{L^1(\mathcal{H}_s)} \\ & \leq C(C_1\epsilon)^2 s^{-1+\delta/2} \left(\int_{\mathcal{H}_s} |(s/t) \partial_t Z^{I^\dagger} w_i|^2 dx \right)^{1/2} \\ & \leq C(C_1\epsilon)^2 s^{-1+\delta/2} E_m(s, Z^{I^\dagger} w_i)^{1/2}. \end{aligned}$$

The second term is estimated in the same way and we omit the details. \square

The proof of (2.4.9a) is quite similar but the null structure will be taken into consideration for the sharp decay rate $s^{-3/2+2\delta}$.

Lemma 8.6. *Suppose (2.4.5a) and (2.4.5e) hold, then for any $|I| \leq 3$ the following estimates are true:*

$$\begin{aligned} & \left| \int_{\mathcal{H}_s} \frac{s}{t} \left((\partial_\alpha G_{\hat{i}}^{\hat{j}\alpha\beta}) \partial_t Z^I u_{\hat{i}} \partial_\beta Z^I u_{\hat{j}} - \frac{1}{2} (\partial_t G_{\hat{i}}^{\hat{j}\alpha\beta}) \partial_\alpha Z^I u_{\hat{i}} \partial_\beta Z^I u_{\hat{j}} \right) dx \right| \\ & \leq CK(C_1\epsilon)^2 s^{-3/2+\delta} E_m(s, Z^I u_{\hat{i}})^{1/2}. \end{aligned} \quad (8.3.3)$$

Proof. We decompose the term $G_{\hat{i}}^{\hat{j}\alpha\beta}$ as follows:

$$\begin{aligned} & (s/t) \partial_\alpha G_{\hat{i}}^{\hat{j}\alpha\beta} \partial_\beta Z^I u_{\hat{j}} \partial_t Z^I u_{\hat{i}} \\ & = (s/t) (A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\hat{k}} \partial_\alpha \partial_\gamma v_{\hat{k}} + B_{\hat{i}}^{\hat{j}\alpha\beta\hat{k}} \partial_\alpha v_{\hat{k}}) \partial_\beta Z^I u_{\hat{j}} \partial_t Z^I u_{\hat{i}} \\ & \quad + (s/t) (A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\hat{k}} \partial_\alpha \partial_\gamma u_{\hat{k}} + B_{\hat{i}}^{\hat{j}\alpha\beta\hat{k}} \partial_\alpha u_{\hat{k}}) \partial_\beta Z^I u_{\hat{j}} \partial_t Z^I u_{\hat{i}} \\ & := R_1 + R_2 \end{aligned}$$

The estimate of R_1 is direct. We apply the inequalities (6.5.1e)

$$\begin{aligned}
\|R_1\|_{L^1(\mathcal{H}_s)} &= \|(s/t)(A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\hat{k}}\partial_\alpha\partial_\gamma v_{\hat{k}} + B_{\hat{i}}^{\hat{j}\alpha\beta\hat{k}}\partial_\alpha v_{\hat{k}})(\partial_\beta Z^I u_{\hat{j}}\partial_t Z^I u_{\hat{i}})\|_{L^1(\mathcal{H}_s)} \\
&\leq CK\|(t^{-3/2}s^\delta(t/s))(s/t)\partial_\beta Z^I u_{\hat{j}}(s/t)\partial_t Z^I u_{\hat{i}}\|_{L^1(\mathcal{H}_s)} \\
&\leq CK\|(t^{-1/2}s^{-1+\delta}(s/t)\partial_\beta Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \cdot \|(s/t)\partial_t Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CKC_1\epsilon s^{-3/2+\delta}\|\partial_\beta Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \cdot E_m(s, Z^I u_{\hat{i}})^{1/2}.
\end{aligned}$$

Then we apply (6.2.3a),

$$\|R_1\|_{L^1(\mathcal{H}_s)} \leq CKC_1\epsilon s^{-3/2+\delta} E_m(s, Z^I u_{\hat{i}})^{1/2}. \quad (8.3.4)$$

The estimates on R_2 is more completed. We have the following decomposition:

$$\begin{aligned}
&(s/t)(A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\hat{k}}\partial_\alpha\partial_\gamma u_{\hat{k}} + B_{\hat{i}}^{\hat{j}\alpha\beta\hat{k}}\partial_\alpha u_{\hat{k}})\partial_\beta Z^I u_{\hat{j}} \\
&= (s/t)A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\hat{k}}\partial_\alpha\partial_\gamma u_{\hat{k}}\partial_\beta Z^I u_{\hat{j}} + (s/t)B_{\hat{i}}^{\hat{j}\alpha\beta\hat{k}}\partial_\alpha u_{\hat{k}}\partial_\beta Z^I u_{\hat{j}} \\
&:= (s/t)\mathcal{A}_{\hat{i}}^{\hat{j}\hat{k}}(\partial Z^I u_{\hat{j}}, \partial\partial u_{\hat{k}}) + (s/t)\mathcal{T}_{\hat{i}}^{\hat{j}\hat{k}}(\partial u_{\hat{k}}, \partial Z^I u_{\hat{j}})
\end{aligned}$$

Recall that $\mathcal{A}_{\hat{i}}^{\hat{j}\hat{k}}$ and $\mathcal{T}_{\hat{i}}^{\hat{j}\hat{k}}$ are null forms.

Let us consider first the terms of $\mathcal{A}_{\hat{i}}^{\hat{j}\hat{k}}$. By Lemma 4.5 (with $I = 0$), we have

$$\begin{aligned}
&\mathcal{A}_{\hat{i}}^{\hat{j}\hat{k}}(\partial Z^I u_{\hat{j}}, \partial\partial u_{\hat{k}}) \\
&\leq CK(s/t)^2|\partial_t\partial_t u_{\hat{k}}||\partial_t Z^I u_{\hat{j}}| + CK\Omega_1(0, Z^I u_{\hat{j}}, u_{\hat{k}}) + CKt^{-1}\Omega_2(0, Z^I u_{\hat{j}}, u_{\hat{k}}) \\
&\leq CK(s/t)^2|\partial_t\partial_t u_{\hat{k}}||\partial_t Z^I u_{\hat{j}}| \\
&\quad + CK \sum_{a,\beta,\gamma} |\underline{\partial}_a \underline{\partial}_\beta u_{\hat{k}}| |\underline{\partial}_\gamma Z^I u_{\hat{j}}| + CK \sum_{\alpha,b,\gamma} |\underline{\partial}_\alpha \underline{\partial}_b u_{\hat{k}}| |\underline{\partial}_\gamma Z^I u_{\hat{j}}| \\
&\quad + CK \sum_{\alpha,\beta,c} |\underline{\partial}_\alpha \underline{\partial}_\beta u_{\hat{k}}| |\underline{\partial}_c Z^I u_{\hat{j}}| + CKt^{-1} \sum_{\alpha,\beta} |\partial_\alpha Z^I u_{\hat{j}}| |\partial_\beta u_{\hat{k}}|.
\end{aligned}$$

Each term is estimated as follows:

$$\begin{aligned}
\|(s/t)^2\partial_t\partial_t u_{\hat{k}}\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} &= \|(s/t)\partial_t\partial_t u_{\hat{k}}(s/t)\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon\|t^{-3/2}(s/t)\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon s^{-3/2}\|(s/t)\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2},
\end{aligned}$$

where we applied (6.5.3a) and (6.2.3a):

$$\begin{aligned}
\|\underline{\partial}_a \underline{\partial}_\beta u_{\hat{k}} \underline{\partial}_\gamma Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} &= \|(t/s)\underline{\partial}_a \underline{\partial}_\beta u_{\hat{k}}(s/t)\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon\|(t/s)t^{-3/2}(s/t)\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\
&\leq CC_1\epsilon s^{-3/2}\|(s/t)\partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2},
\end{aligned}$$

where we applied (6.5.4c) and (6.2.3a).

The term $\|\underline{\partial}_\alpha \underline{\partial}_b u_{\hat{k}} \underline{\partial}_\gamma Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)}$ is estimated in the same manner and we omit the details.

$\|\underline{\partial}_\alpha \underline{\partial}_\beta u_{\hat{k}} \underline{\partial}_c Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq \|\underline{\partial}_\alpha \underline{\partial}_\beta u_{\hat{k}}\|_{L^\infty(\mathcal{H}_s)} \|\underline{\partial}_c Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2}$ where (6.5.3a) and (6.2.3b) are applied:

$$\begin{aligned} \|t^{-1} \partial_\alpha Z^I u_{\hat{j}} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} &\leq \|t^{-1} (t/s) \partial_\beta u_{\hat{k}}\|_{L^\infty(\mathcal{H}_s)} \|(s/t) \partial_\alpha Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 s^{-5/2}, \end{aligned}$$

where (6.5.3a) and (6.2.3a) are applied. Then we conclude with

$$\|(s/t) \mathcal{A}_{\hat{i}}^{\hat{j}k}(\partial Z^I u_{\hat{i}}, \partial \partial u_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \leq CK(C_1\epsilon) s^{-3/2}.$$

Then we have

$$\begin{aligned} &\|(s/t) \mathcal{A}_{\hat{i}}^{\hat{j}k}(\partial Z^I u_{\hat{i}}, \partial \partial u_{\hat{j}}) \partial_t Z^I u_{\hat{i}}\|_{L^1(\mathcal{H}_s)} \\ &\leq \|A_{\hat{i}}^{\hat{j}k}(\partial Z^I u_{\hat{i}}, \partial \partial u_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \|(s/t) \partial_t Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 s^{-3/2} E_m(s, Z^I u_{\hat{i}})^{1/2}. \end{aligned} \tag{8.3.5}$$

Next, we consider the terms of $\mathcal{T}_{\hat{i}}^{\hat{j}k}$. By (4.2.1a), $\mathcal{T}_{\hat{i}}^{\hat{j}k}(\partial Z^I u_{\hat{k}}, \partial u_{\hat{j}})$ can be bounded by a linear combination of the terms presented in Lemma 4.2. All the estimates are based on the inequalities (6.5.3a), (6.5.3b), (6.2.1a) and (6.2.1c).

$$\begin{aligned} \|(s/t)^2 \partial_t Z^I u_{\hat{j}} \partial_t u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} &\leq \|(s/t) \partial_t u_{\hat{k}}\|_{L^\infty(\mathcal{H}_s)} \|(s/t) \partial_t Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 s^{-3/2}, \end{aligned}$$

$$\begin{aligned} \|\underline{\partial}_\alpha u_{\hat{k}} \underline{\partial}_\beta Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} &\leq \|(t/s) \underline{\partial}_\alpha u_{\hat{k}}\|_{L^\infty(\mathcal{H}_s)} \|(s/t) \underline{\partial}_\beta Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 s^{-3/2}, \end{aligned}$$

$$\begin{aligned} \|\underline{\partial}_\alpha u_{\hat{k}} \underline{\partial}_b Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} &\leq \|\underline{\partial}_\alpha u_{\hat{k}}\|_{L^\infty(\mathcal{H}_s)} \|\underline{\partial}_b Z^I u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 s^{-3/2}. \end{aligned}$$

Then we conclude with

$$\begin{aligned} \|\mathcal{T}(\partial u_{\hat{k}}, \partial Z^I u_{\hat{j}}) \partial_t Z^I u_{\hat{i}}\|_{L^1(\mathcal{H}_s)} &\leq \|\mathcal{T}(\partial u_{\hat{k}}, \partial Z^I u_{\hat{j}})\|_{L^2(\mathcal{H}_s)} \|(s/t) \partial_t Z^I u_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 E_m(s, Z^I u_{\hat{i}})^{1/2}. \end{aligned} \tag{8.3.6}$$

Combining (8.3.4), (8.3.5) and (8.3.6), we conclude that

$$\|\partial_\alpha (G_{\hat{i}}^{\hat{\alpha}\beta}(w, \partial w)) \partial_t Z^I u_{\hat{i}}\|_{L^1(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2+\delta} E_m(s, Z^I u_{\hat{i}})^{1/2}.$$

Then we conclude that the term $\|\partial_t (G_{\hat{i}}^{\hat{\alpha}\beta}(w, \partial w)) \partial_\alpha Z^I u_{\hat{i}}\|_{L^1(\mathcal{H}_s)}$ is also bounded by $C(C_1\epsilon)^2 s^{-3/2+\delta}$ which proves the desired result. \square

Chapter 9

L^2 estimates of the interaction terms

9.1 L^2 estimates on higher-order interaction terms

In this chapter, we establish three groups of energy estimates mentioned in the proof of Proposition 2.4, that is, (2.4.7c), (2.4.8b), and (2.4.9b) which are derived under the assumption (2.4.5). We emphasize that three groups of inequalities correspond to different decreasing rates in time. The proofs of (2.4.7c) and (2.4.8b) are much easier than the one of (2.4.9b) since, roughly speaking, it does not require the null structure. The sharp decreasing rate in (2.4.7c) is a consequence of the null structure. Interestingly, this is the only place where the null structure is used in a fundamental way.

This chapter is divided into two parts. The first part is devoted to the proof of (2.4.7c) while the second is devoted to that of (2.4.9b).

Here we will establish the following L^2 estimates.

Lemma 9.1. *Under the assumption of (2.4.5), the following estimates hold for all $|I^\sharp| \leq 5$ and $B + 1 \leq s \leq T$*

$$\|Z^{I^\sharp} F_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)s^{-1+\delta}, \quad (9.1.1a)$$

$$\|[Z^I, G_i^{j\alpha\beta}(w, \partial w)\partial_\alpha\partial_\beta]w_j\| \leq C(C_1\epsilon)s^{-1+\delta}. \quad (9.1.1b)$$

Proof. We begin with (9.1.1a). This concerns only the basic L^2 and L^∞ estimates established earlier. Recall that $Z^{I^\sharp} F_i$ is decomposed as follows:

$$Z^{I^\sharp} F_i = Z^{I^\sharp} (P_i^{\alpha\beta jk} \partial_\alpha w_j \partial_\beta w_k) + Z^{I^\sharp} (Q_i^{\alpha j \check{k}} v_{\check{k}} \partial_\alpha w_j) + Z^{I^\sharp} (R_i^{\check{j} \check{k}} v_{\check{j}} v_{\check{k}})$$

Then we see that $Z^{I^\sharp} F_i$ is a linear combination of the following terms:

$$Z^{I^\sharp} (\partial_\alpha u_{\hat{i}} \partial_\beta u_{\hat{j}}), \quad Z^{I^\sharp} (\partial_\alpha v_{\check{i}} \partial_\beta w_j), \quad Z^{I^\sharp} (v_{\check{i}} \partial_\alpha w_j), \quad Z^{I^\sharp} (v_{\check{j}} v_{\check{k}}).$$

Then as done before, we write down, for each term and each partition of $I^\sharp = I_1^\sharp + I_2^\sharp$, the inequalities used in the following two lists (recall the

convention of notation, on each term I_1^\sharp acts on the first factor and I_2^\sharp acts on the second and the symbol $(a, \leq b)$ means $|I_1^\sharp| = a, |I_2^\sharp| \leq b$:

Product	(5, \leq 0)	(4, \leq 1)	(3, \leq 2)
$\partial_\alpha u_{\hat{i}} \partial_\beta u_{\hat{j}}$	(6.3.1a), (6.5.3a)	(6.3.1a), (6.5.3a)	(6.3.3a), (6.5.1a)
$\partial_\alpha v_{\hat{i}} \partial_\beta w_j$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)
$v_{\hat{i}} \partial_\alpha w_j$	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)
$v_{\hat{j}} v_{\hat{k}}$	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)
Product	(2, \leq 3)	(1, \leq 4)	(0, \leq 5)
$\partial_\alpha u_{\hat{i}} \partial_\beta u_{\hat{j}}$	(6.5.1a), (6.3.3a)	(6.5.3a), (6.3.1a)	(6.5.3a), (6.3.1a)
$\partial_\alpha v_{\hat{i}} \partial_\beta w_j$	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)
$v_{\hat{i}} \partial_\alpha w_j$	(6.3.1e), (6.5.1a)	(6.5.1e), (6.3.1a)	(6.5.1e), (6.3.1a)
$v_{\hat{j}} v_{\hat{k}}$	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)	(6.5.1e), (6.3.1e)

From these inequalities, we conclude by (9.1.1a).

Then we turn our attention to (9.1.1b). Recalling the decomposition of $[Z^{I^\sharp}, G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta] w_j$, we find

$$\begin{aligned}
& [Z^{I^\sharp}, G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta] w_j \\
&= [Z^{I^\sharp}, A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k \partial_\alpha \partial_\beta] w_j + [Z^{I^\sharp}, B_i^{j\alpha\beta\check{k}} v_{\check{k}} \partial_\alpha \partial_\beta] w_j \\
&\quad + [Z^{I^\sharp}, B_i^{j\alpha\beta\hat{k}} u_{\hat{k}} \partial_\alpha \partial_\beta] u_{\hat{j}}
\end{aligned} \tag{9.1.2}$$

The first term of the right-hand-side is decomposed as follows:

$$\begin{aligned}
& [Z^{I^\sharp}, A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k \partial_\alpha \partial_\beta] w_j \\
&= \sum_{\substack{I_1^\sharp + I_2^\sharp = I^\sharp \\ |I_2^\sharp| \leq |I^\sharp| - 1}} A_i^{j\alpha\beta\gamma k} Z^{I_1^\sharp} \partial_\gamma w_k Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j + A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k [Z^{I^\sharp}, \partial_\alpha \partial_\beta] w_j.
\end{aligned} \tag{9.1.3}$$

The first term is decomposed as follows:

$$\begin{aligned}
& \sum_{\substack{I_1^\sharp + I_2^\sharp = I^\sharp \\ |I_2^\sharp| \leq |I^\sharp| - 1}} A_i^{j\alpha\beta\gamma k} Z^{I_1^\sharp} \partial_\gamma w_k Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j \\
&= \sum_{\substack{I_1^\sharp + I_2^\sharp = I^\sharp \\ |I_2^\sharp| \leq |I^\sharp| - 1}} \left(A_i^{j\alpha\beta\gamma \hat{k}} Z^{I_1^\sharp} \partial_\gamma u_{\hat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j + A_i^{j\alpha\beta\gamma \check{k}} Z^{I_1^\sharp} \partial_\gamma v_{\check{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j \right).
\end{aligned}$$

So we observe that this term is a linear combination of the following terms with constant coefficients:

$$Z^{I_1^\sharp} \partial_\gamma u_{\hat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\hat{j}}, \quad Z^{I_1^\sharp} \partial_\gamma u_{\hat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta v_{\check{j}}, \quad Z^{I_1^\sharp} \partial_\gamma v_{\check{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j$$

with $|I_1^\sharp| + |I_2^\sharp| \leq |I^\sharp|$ and $I_2^\sharp \leq |I^\sharp| - 1$.

The second term in right-hand-side of (9.1.3) is estimated as follows. We see that it is a linear combination of the following terms with constant coefficients:

$$\partial_\gamma u_{\widehat{k}}[Z^{I^\sharp}, \partial_\alpha \partial_\beta] u_{\widehat{j}}, \quad \partial_\gamma u_{\widehat{k}}[Z^{I^\sharp}, \partial_\alpha \partial_\beta] v_{\widehat{j}}, \quad \partial_\gamma v_{\widehat{k}}[Z^{I^\sharp}, \partial_\alpha \partial_\beta] w_j.$$

Then by the estimates on commutators (3.3.3), these terms are bounded by

$$\begin{aligned} & \sum_{|I_2^\sharp| \leq |I^\sharp| - 1} |\partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} u_{\widehat{j}}|, & \sum_{|I_2^\sharp| \leq |I^\sharp| - 1} |\partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} v_{\widehat{j}}|, \\ & \sum_{|I_2^\sharp| \leq |I^\sharp| - 1} |\partial_\gamma v_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} w_j|. \end{aligned}$$

Then we observe that $[Z^{I^\sharp}, A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k \partial_\alpha \partial_\beta] w_j$ is bounded by a linear combination of the following terms with constant coefficients:

$$\begin{aligned} & Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}, & Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta v_{\widehat{j}}, & Z^{I_1^\sharp} \partial_\gamma v_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j \\ & Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} u_{\widehat{j}}, & Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} v_{\widehat{j}}, & Z^{I_1^\sharp} \partial_\gamma v_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} w_j \end{aligned}$$

with $|I_1^\sharp| + |I_2^\sharp| \leq |I^\sharp|$ and $|I_2^\sharp| \leq 4$.

Then we give the inequalities used for each term and each partition of index:

Product	(5, ≤ 0)	(4, ≤ 1)	(3, ≤ 2)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}$	(6.3.1a), (7.4.6c)	(6.3.1b), (7.4.6b)	(6.3.3a), (7.4.6a)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta v_{\widehat{j}}$	(6.3.1a), (6.5.2a)	(6.3.1b), (6.5.2a)	(6.3.3a), (6.5.2a)
$Z^{I_1^\sharp} \partial_\gamma v_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} u_{\widehat{j}}$	(6.3.1a), (7.4.6c)	(6.3.1b), (7.4.6b)	(6.3.3a), (7.4.6a)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} v_{\widehat{j}}$	(6.3.1a), (6.4.3a)	(6.3.1b), (6.4.3a)	(6.3.3a), (6.4.3a)
$Z^{I_1^\sharp} \partial_\gamma v_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} w_j$	(6.3.1a), (6.4.2a)	(6.3.1a), (6.4.2a)	(6.3.1a), (6.4.2a)

Product	(2, ≤ 3)	(1, ≤ 4)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}$	(6.5.1b), (7.5.7b)	(6.5.3a), (7.5.7a)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta v_{\widehat{j}}$	(6.5.1a), (6.3.2a)	(6.5.1a), (6.3.2a)
$Z^{I_1^\sharp} \partial_\gamma v_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j$	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} u_{\widehat{j}}$	(6.5.1b), (7.5.7b)	(6.5.3a), (7.5.7a)
$Z^{I_1^\sharp} \partial_\gamma u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} v_{\widehat{j}}$	(6.5.1b), (6.2.2a)	(6.5.3a), (6.2.2a)
$Z^{I_1^\sharp} \partial_\gamma v_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} w_j$	(6.5.1a), (6.2.1a)	(6.5.1a), (6.2.1a)

The estimates of the remaining terms of the right-hand-side of (9.1.2) are estimated similarly. By a calculation, we observe that these two terms can be bounded by the combination of the following terms with constant coefficients:

$$\begin{aligned} Z^{I_1^\sharp} v_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta w_j, & \quad Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}, \\ Z^{I_1^\sharp} v_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} w_j, & \quad Z^{I_1^\sharp} u_{\widehat{k}} \partial_\alpha \partial_\beta Z^{I_2^\sharp} u_{\widehat{j}}, \end{aligned}$$

with $|I_1^\sharp| + |I_2^\sharp| \leq |I^\sharp|$ and $|I_2^\sharp| \leq 4$. We will write in detail the estimate for the most difficult term $Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}$:

$$\begin{aligned} & \sum_{\substack{|I_1^\sharp| + |I_2^\sharp| \leq |I^\sharp| \\ |I_2^\sharp| \leq |I^\sharp| - 1}} |Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}| \\ & \leq \sum_{|I_1^\sharp| \leq 5} |Z^{I_1^\sharp} u_{\widehat{k}} \partial_\alpha \partial_\beta u_{\widehat{j}}| + \sum_{|I_1^\sharp| = 4, |I_2^\sharp| \leq 1} |Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}| \\ & \quad + \sum_{|I_1^\sharp| = 3, |I_2^\sharp| \leq 2} |Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}| \\ & \quad + \sum_{|I_1^\sharp| = 2, |I_2^\sharp| \leq 3} |Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}| + \sum_{|I_1^\sharp| = 1, |I_2^\sharp| \leq 4} |Z^{I_1^\sharp} u_{\widehat{k}} Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}| \\ & := T_5 + T_4 + T_3 + T_2 + T_1. \end{aligned}$$

The term T_5 is estimated by applying (6.3.6a) on the first factor and (7.4.6c) on the second factor:

$$\begin{aligned} \|T_5\|_{L^2(\mathcal{H}_s)} &= \sum_{|I_1^\sharp| \leq 5} \|s^{-1} Z^{I_1^\sharp} u_{\widehat{k}} s \partial_\alpha \partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq \|s^{-1} Z^{I_1^\sharp} u_{\widehat{k}}\|_{L^2(\mathcal{H}_s)} \|s \partial_\alpha \partial_\beta u_{\widehat{j}}\|_{L^\infty(\mathcal{H}_s)} \\ &\leq C(C_1 \epsilon)^2 s^\delta \|t^{1/2} s^{-2}\|_{L^\infty(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-1+\delta} \end{aligned}$$

where we observe that in the half-cone \mathcal{K} , $s \leq t \leq C s^2$.

The term T_4 is bounded by applying (6.3.6b) and (7.4.6b):

$$\begin{aligned} \|T_4\|_{L^2(\mathcal{H}_s)} &\leq \sum_{|I_1^\sharp| = 4, |I_2^\sharp| \leq 1} \|s^{-1} Z^{I_1^\sharp} u_{\widehat{k}} s Z^{I_2^\sharp} \partial_\alpha \partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1 \epsilon)^2 s^{\delta/2} \|s t^{1/2} s^{-3+\delta/2}\|_{L^\infty(\mathcal{H}_s)} \\ &\leq C(C_1 \epsilon) s^{-1+\delta}. \end{aligned}$$

The term T_3 is bounded by applying (6.3.6c) and (7.4.6a):

$$\begin{aligned} \|T_3\|_{L^2(\mathcal{H}_s)} &\leq \sum_{|I_1^\sharp|=3, |I_2^\sharp|\leq 2} \|s^{-1}Z^{I_1^\sharp}u_{\widehat{k}}sZ^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)^2 \|st^{1/2}s^{-3+\delta}\|_{L^\infty(\mathcal{H}_s)} \\ &\leq C(C_1\epsilon)s^{-1+\delta}. \end{aligned}$$

The term T_2 is estimated by applying (6.5.5b), (7.5.7b):

$$\begin{aligned} \|T_2\|_{L^2(\mathcal{H}_s)} &\leq \sum_{|I_1^\sharp|\leq 2, |I_2^\sharp|\leq 3} \|s^{-3}t^2Z^{I_1^\sharp}u_{\widehat{k}}s^3t^{-2}Z^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq \sum_{|I_1^\sharp|\leq 2, |I_2^\sharp|\leq 3} \|s^{-3}t^2Z^{I_1^\sharp}u_{\widehat{k}}\|_{L^\infty(\mathcal{H}_s)} \|s^3t^{-2}Z^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1\epsilon \|s^{-3}t^2t^{-3/2}s^{1+\delta/2}\|_{L^\infty(\mathcal{H}_s)} CC_1\epsilon s^{\delta/2} \\ &= C(C_1\epsilon)^2 s^{-1+\delta}. \end{aligned}$$

The term T_1 is bounded by applying (6.5.5c) and (7.5.7a):

$$\begin{aligned} \|T_1\|_{L^2(\mathcal{H}_s)} &\leq \sum_{|I_1^\sharp|\leq 1, |I_2^\sharp|\leq 4} \|s^{-3}t^2Z^{I_1^\sharp}u_{\widehat{k}}s^3t^{-2}Z^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq \sum_{|I_1^\sharp|\leq 1, |I_2^\sharp|\leq 4} \|s^{-3}t^2Z^{I_1^\sharp}u_{\widehat{k}}\|_{L^\infty(\mathcal{H}_s)} \|s^3t^{-2}Z^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}\|_{L^2(\mathcal{H}_s)} \\ &\leq CC_1\epsilon \|s^{-3}t^2t^{-3/2}s\|_{L^\infty(\mathcal{H}_s)} \cdot CC_1\epsilon s^\delta \leq C(C_1\epsilon)^2 s^{-1+\delta}. \end{aligned}$$

For the remaining terms, we list out the inequalities to be used on each term and each partition of index in the following list:

Product	(5, ≤ 0)	(4, ≤ 1)	(3, ≤ 2)
$Z^{I_1^\sharp}v_{\widehat{k}}Z^{I_2^\sharp}\partial_\alpha\partial_\beta w_j$	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)
$Z^{I_1^\sharp}u_{\widehat{k}}Z^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}$	(6.3.6a), (7.4.6c)	(6.3.6b), (7.4.6b)	(6.3.6c), (7.4.6a)
$Z^{I_1^\sharp}v_{\widehat{k}}\partial_\alpha\partial_\beta Z^{I_2^\sharp}w_j$	(6.3.1e), (6.4.2a)	(6.3.1e), (6.4.2a)	(6.3.1e), (6.4.2a)
$Z^{I_1^\sharp}u_{\widehat{k}}\partial_\alpha\partial_\beta Z^{I_2^\sharp}u_{\widehat{j}}$	(6.3.6a), (7.4.6a)	(6.3.6b), (7.4.6b)	(6.3.6c), (7.4.6c)

Product	(2, ≤ 3)	(1, ≤ 4)
$Z^{I_1^\sharp}v_{\widehat{k}}Z^{I_2^\sharp}\partial_\alpha\partial_\beta w_j$	(6.5.1e), (6.3.1b)	(6.5.1e), (6.3.1a)
$Z^{I_1^\sharp}u_{\widehat{k}}Z^{I_2^\sharp}\partial_\alpha\partial_\beta u_{\widehat{j}}$	(6.5.5b), (7.5.7b)	(6.5.5c), (7.5.7a)
$Z^{I_1^\sharp}v_{\widehat{k}}\partial_\alpha\partial_\beta Z^{I_2^\sharp}w_j$	(6.5.1e), (6.2.1a)	(6.5.1e), (6.2.1a)
$Z^{I_1^\sharp}u_{\widehat{k}}\partial_\alpha\partial_\beta Z^{I_2^\sharp}u_{\widehat{j}}$	(6.5.5b), (6.2.1b)	(6.5.5c), (6.2.1a)

□

Then we estimate the source with fourth-order derivatives.

Lemma 9.2. *Under the assumption of (2.4.5), the following estimates hold for all $|I^\dagger| \leq 4$ and $B + 1 \leq s \leq T$*

$$\|Z^{I^\dagger} F_{\hat{v}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)s^{-1+\delta/2}, \quad (9.1.4a)$$

$$\|[Z^{I^\dagger}, G_i^{j\alpha\beta}(w, \partial w)\partial_\alpha\partial_\beta]w_j\| \leq C(C_1\epsilon)s^{-1+\delta/2}. \quad (9.1.4b)$$

Proof. The proof is essentially the same to that of Lemma 9.1. We will not write the proof in details, but we list out the inequalities used for each term and each partition of index.

First we list out the estimate on $Z^{I^\dagger} F_i$:

Product	(4, ≤ 0)	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)	(0, ≤ 4)
$\partial_\alpha u_{\hat{\gamma}} \partial_\beta u_{\hat{\gamma}}$	(6.3.1b), (6.5.3a)	(6.3.3a), (6.5.3a)	(6.3.3a), (6.5.1b)	(6.5.3a), (6.3.3a)	(6.5.3a), (6.3.1b)
$\partial_\alpha v_{\hat{\gamma}} \partial_\beta w_j$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)
$v_{\hat{\gamma}} \partial_\alpha w_j$	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.5.1e), (6.3.1a)	(6.5.1e), (6.3.1a)
$v_{\hat{\gamma}} v_{\hat{\kappa}}$	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)	(6.5.1e), (6.3.1e)

The terms $[Z^{I^\dagger}, B_i^{j\alpha\beta k} w_k \partial_\alpha \partial_\beta] w_j$ are estimated by the inequalities listed in the following list:

Product	(4, ≤ 0)	(3, ≤ 1)	(2, ≤ 2)	(1, ≤ 3)
$Z^{I^\dagger} \partial_\gamma u_{\hat{\kappa}} Z^{I_2^\dagger} \partial_\alpha \partial_\beta u_{\hat{\gamma}}$	(6.3.1b), (7.4.6c)	(6.3.3a), (7.4.6b)	(6.5.1b), (7.5.7c)	(6.5.3a), (7.5.7b)
$Z^{I^\dagger} \partial_\gamma u_{\hat{\kappa}} \partial_\alpha \partial_\beta Z^{I_2^\dagger} u_{\hat{\gamma}}$	(6.3.1b), (7.4.6c)	(6.3.3a), (7.4.6b)	(6.5.1b), (7.5.7c)	(6.5.3a), (7.5.7b)
$Z^{I^\dagger} \partial_\gamma u_{\hat{\kappa}} Z^{I_2^\dagger} \partial_\alpha \partial_\beta v_{\hat{\gamma}}$	(6.3.1b), (6.5.2a)	(6.3.3a), (6.5.2a)	(6.3.3a), (6.5.2a)	(6.5.3a), (6.3.2a)
$Z^{I^\dagger} \partial_\gamma u_{\hat{\kappa}} \partial_\alpha \partial_\beta Z^{I_2^\dagger} v_{\hat{\gamma}}$	(6.3.1b), (6.5.2a)	(6.3.3a), (6.5.2a)	(6.3.3a), (6.5.2a)	(6.5.3a), (6.3.2a)
$Z^{I^\dagger} \partial_\gamma v_{\hat{\kappa}} Z^{I_2^\dagger} \partial_\alpha \partial_\beta w_j$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)
$Z^{I^\dagger} \partial_\gamma v_{\hat{\kappa}} \partial_\alpha \partial_\beta Z^{I_2^\dagger} w_j$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)
$Z^{I^\dagger} v_{\hat{\kappa}} Z^{I_2^\dagger} \partial_\alpha \partial_\beta w_j$	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)
$Z^{I^\dagger} v_{\hat{\kappa}} \partial_\alpha \partial_\beta Z^{I_2^\dagger} w_j$	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)
$Z^{I^\dagger} u_{\hat{\kappa}} Z^{I_2^\dagger} \partial_\alpha \partial_\beta u_{\hat{\gamma}}$	(6.3.6b), (7.4.6c)	(6.3.6c), (7.4.6b)	(6.5.5b), (7.5.7c)	(6.5.5c), (7.5.7b)
$Z^{I^\dagger} u_{\hat{\kappa}} \partial_\alpha \partial_\beta Z^{I_2^\dagger} u_{\hat{\gamma}}$	(6.3.6b), (7.4.6c)	(6.3.6c), (7.4.6b)	(6.5.5b), (7.5.7c)	(6.5.5c), (7.5.7b)

□

9.2 L^2 estimates on third-order terms

In this section we will prove the L^2 estimates of the source term (2.4.9b) with $|I| \leq 3$. We observe that this is the only place where the null structure is taken into consideration.

Lemma 9.3. *Under the assumption of (2.4.5), the following estimates hold for all $|I| \leq 3$:*

$$\|Z^I F_{\hat{i}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2+2\delta}, \quad (9.2.1a)$$

$$\|Z^I G_{\hat{i}}^{\check{\alpha}\beta}(w, \partial w) \partial_\alpha \partial_\beta v_{\check{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2+2\delta}, \quad (9.2.1b)$$

$$\|[Z^I, G_{\hat{i}}^{\hat{\alpha}\beta}(w, \partial w) \partial_\alpha \partial_\beta] u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2+2\delta}. \quad (9.2.1c)$$

Proof. We first prove (9.2.1a) and recall the structure of $Z^I F_{\hat{i}}$:

$$\begin{aligned} Z^I F_{\hat{i}} &= Z^I (P_{\hat{i}}^{\alpha\beta\hat{j}\hat{k}} \partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}}) \\ &\quad + P_{\hat{i}}^{\alpha\beta\hat{j}\hat{k}} Z^I (\partial_\alpha v_{\hat{j}} \partial_\beta u_{\hat{k}}) + P_{\hat{i}}^{\alpha\beta\check{j}\check{k}} Z^I (\partial_\alpha u_{\check{j}} \partial_\beta v_{\check{k}}) + P_{\hat{i}}^{\alpha\beta\check{j}\check{k}} Z^I (\partial_\alpha v_{\check{j}} \partial_\beta v_{\check{k}}) \\ &\quad + Q_{\hat{i}}^{\alpha\check{j}\check{k}} Z^I (v_{\check{k}} \partial_\alpha w_{\check{j}}) + R_{\hat{i}}^{\hat{j}\hat{k}} Z^I (v_{\hat{j}} v_{\hat{k}}). \end{aligned}$$

Remark that the first term in the right-hand-side is a null term. We apply directly (8.1.1):

$$\|P_{\hat{i}}^{\alpha\beta\hat{j}\hat{k}} \partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}}\|_{L^2(\mathcal{H}_s)} \leq C(C_1\epsilon)^2 s^{-3/2}.$$

The remaining terms are linear combinations of the following terms with constant coefficients:

$$Z^I (\partial_\alpha v_{\check{j}} \partial_\beta w_{\check{k}}), \quad Z^I (v_{\check{k}} \partial_\alpha w_{\check{j}}), \quad Z^I (v_{\hat{j}} v_{\hat{k}}).$$

We will not give in detail the estimates of each term, but we list out the inequalities used on each term and each partition:

Product	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)	(0, ≤ 3)
$\partial_\alpha v_{\check{j}} \partial_\beta w_{\check{k}}$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)	(6.5.2a), (6.3.1a)
$v_{\check{k}} \partial_\alpha w_{\check{j}}$	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.5.1e), (6.3.1a)	(6.5.1e), (6.3.1a)
$v_{\hat{j}} v_{\hat{k}}$	(6.3.1e), (6.5.1e)	(6.3.1e), (6.5.1e)	(6.5.1e), (6.3.1e)	(6.5.1e), (6.3.1e)

To prove the second inequality (9.2.1b), we observe that it can be decomposed as follows:

$$Z^I (G_{\hat{i}}^{\check{\alpha}\beta}(w, \partial w) \partial_\alpha \partial_\beta v_{\check{j}}) = A_{\hat{i}}^{\check{\alpha}\beta\gamma k} Z^I (\partial_\gamma w_k \partial_\alpha \partial_\beta v_{\check{j}}) + B_{\hat{i}}^{\check{\alpha}\beta\check{k}} Z^I (v_{\check{k}} \partial_\alpha \partial_\beta v_{\check{j}})$$

So it is to be bounded by a linear combination of the following terms with constant coefficients:

$$Z^I(\partial_\gamma w_k \partial_\alpha \partial_\beta v_{\check{j}}), \quad Z^I(v_{\check{k}} \partial_\alpha \partial_\beta v_{\check{j}}).$$

As before we list out the inequalities used on each term and each partition of index

Product	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)	(0, ≤ 3)
$\partial_\gamma w_k \partial_\alpha \partial_\beta v_{\check{j}}$	(6.3.1a), (6.5.2a)	(6.3.1a), (6.5.2a)	(6.5.1a), (6.3.2a)	(6.5.1a), (6.3.2a)
$v_{\check{k}} \partial_\alpha \partial_\beta v_{\check{j}}$	(6.3.1e), (6.5.2a)	(6.3.1e), (6.5.2a)	(6.5.1e), (6.3.2a)	(6.5.1e), (6.3.2a)

The proof of the inequality (9.2.1c) also consults the null structure. Recall the following decomposition:

$$\begin{aligned} & [Z^I, G_{\hat{i}}^{\hat{j}\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta] u_{\hat{j}} \\ &= [Z^I, A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\check{k}} \partial_\gamma u_{\check{k}} \partial_\alpha \partial_\beta] u_{\hat{j}} + [Z^I, B_{\hat{i}}^{\hat{j}\alpha\beta\check{k}} u_{\check{k}} \partial_\alpha \partial_\beta] u_{\hat{j}} \\ & \quad + [Z^I, A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\check{k}} \partial_\gamma v_{\check{k}} \partial_\alpha \partial_\beta] u_{\hat{j}} + [Z^I, B_{\hat{i}}^{j h \alpha \beta \check{k}} v_{\check{k}} \partial_\alpha \partial_\beta] u_{\hat{j}} \end{aligned}$$

Observing the null structure of the first two terms and applying (8.1.1b) and (8.2.1a), we obtain

$$\begin{aligned} & \|[Z^I, A_{\hat{i}}^{\hat{j}\alpha\beta\gamma\check{k}} \partial_\gamma u_{\check{k}} \partial_\alpha \partial_\beta] u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} + \|[Z^I, B_{\hat{i}}^{\hat{j}\alpha\beta\check{k}} u_{\check{k}} \partial_\alpha \partial_\beta] u_{\hat{j}}\|_{L^2(\mathcal{H}_s)} \\ & \leq C(C_1 \epsilon)^2 s^{-3/2+\delta}. \end{aligned}$$

The remaining two terms are linear combinations of the following term:

$$\begin{aligned} & Z^{I_1} \partial_\gamma v_{\check{k}} Z^{I_2} \partial_\alpha \partial_\beta u_{\hat{j}}, \quad Z^{I_1} v_{\check{k}} Z^{I_2} \partial_\alpha \partial_\beta u_{\hat{j}}, \\ & Z^{I_1} \partial_\gamma v_{\check{k}} \partial_\alpha \partial_\beta Z^{I_2} u_{\hat{j}}, \quad Z^{I_1} v_{\check{k}} \partial_\alpha \partial_\beta Z^{I_2} u_{\hat{j}} \end{aligned}$$

with $|I_1| + |I_2| \leq |I|$ and $|I_2| \leq 2$.

We omit here the details and list out the inequalities applied to each term and each partition of index:

Product	(3, ≤ 0)	(2, ≤ 1)	(1, ≤ 2)
$Z^{I_1} \partial_\gamma v_{\check{k}} Z^{I_2} \partial_\alpha \partial_\beta u_{\hat{j}}$	(6.3.2a), (6.5.1a)	(6.3.2a), (6.5.1a)	(6.5.2a), (6.3.1a)
$Z^{I_1} v_{\check{k}} Z^{I_2} \partial_\alpha \partial_\beta u_{\hat{j}}$	(6.3.1e), (6.5.1a)	(6.3.1e), (6.5.1a)	(6.5.1e), (6.3.1a)
$Z^{I_1} \partial_\gamma v_{\check{k}} \partial_\alpha \partial_\beta Z^{I_2} u_{\hat{j}}$	(6.3.2a), (6.4.2a)	(6.3.2a), (6.4.2a)	(6.5.2a), (6.2.1a)
$Z^{I_1} v_{\check{k}} \partial_\alpha \partial_\beta Z^{I_2} u_{\hat{j}}$	(6.3.1e), (6.4.2a)	(6.3.1e), (6.4.2a)	(6.5.1e), (6.2.1a)

□

Chapter 10

Revisiting scalar wave equations

10.1 Background and statement of the result

In this chapter, we revisit the classical theory concerning the global existence of small energy solutions to the scalar nonlinear wave equation (with $B > 0$ given)

$$\begin{aligned} \square u &= P^{\alpha\beta} \partial_\alpha u \partial_\beta u \\ u|_{\mathcal{H}_{s_0}} &= u_0, \quad \partial_t u|_{\mathcal{H}_{s_0}} = u_1, \end{aligned} \tag{10.1.1}$$

where C^∞ -regular initial data u_0, u_1 are compactly supported in the open ball $\mathcal{B}(0, s_0)$. We recall that $\mathcal{B}(0, s_0)$ is the intersection of the spacelike hypersurface \mathcal{H}_{s_0} and the cone-like region \mathcal{K} . We impose the classical null condition on the bilinear form $P^{\alpha\beta}$, that is,

$$P^{\alpha\beta} \xi_\alpha \xi_\beta = 0 \quad \text{for all } \xi \in \mathbb{R}^4 \text{ satisfying } -\xi_0^2 + \sum_a \xi_a^2 = 0. \tag{10.1.2}$$

The global existence for (10.1.1) is discussed in many textbooks [Sogge (2008)]. Yet, it is interesting to revisit this problem here with the hyperboloidal foliation method. Importantly, we prove that the hyperboloidal energy of the solutions is uniformly bounded (with respect to the hyperbolic time variable), while in the classical framework the associated energy might increase polynomially. Therefore, our method leads to a sharper asymptotic behavior.

The hyperboloidal energy E_m was introduced in (2.1.11) in Chapter 2, while the admissible vector fields $Z \in \mathcal{L}$ were defined in (2.1.7).

Theorem 10.1 (Existence theory for scalar wave equations).

Let $d \geq 3$ be a given integer, then there exists a real ϵ_0 such that if for all $|I| \geq 3$,

$$E_m(s_0, Z^I u) \leq \epsilon \leq \epsilon_0, \tag{10.1.3}$$

then the local-in-time solution u to the Cauchy problem (10.1.1) extends to arbitrarily large times. Furthermore, there exists a constant $C_1 > 0$ such that the following estimates hold for all $|I| \leq 3$

$$E_m(s, Z^I u) \leq C_1 \epsilon \quad (10.1.4)$$

and

$$|\partial_\alpha u(t, x)| \leq C_1 \epsilon t^{-1} (t - r)^{-1/2}. \quad (10.1.5)$$

10.2 Structure of the proof

We rely on a bootstrap argument and, to begin with, we assume that, in a given time interval $[s_0, s_1]$ the local solution satisfies the following estimate:

$$\max_{|I| \leq 3} \{E_m(s, Z^I u)\} \leq C_1 \epsilon, \quad \text{for } s \in [s_0, s_1] \quad (10.2.1)$$

with $s_1 := \sup_{s \geq s_0} \{ \max_{|I| \leq 4} \{E_m(s, Z^I u)\} \leq C_1 \epsilon \}$ supposed to be finite. When $C_1 \geq 1$ and by continuity, we have $s_1 > s_0$. Then we will prove that for suitable ϵ_0 and C_1 , when $\epsilon \leq \epsilon_0$,

$$\max_{|I| \leq 3} \{E_m(s, Z^I u)\} \leq \frac{1}{2} C_1 \epsilon, \quad \text{for } s \in [s_0, s_1]. \quad (10.2.2)$$

This leads to

$$\max_{|I| \leq 3} \{E_m(s_1, Z^I u)\} \leq \frac{1}{2} C_1 \epsilon.$$

Then by continuity, we conclude that

$$s_1 < \sup_{s \geq s_0} \{ \max_{|I| \leq 3} \{E_m(s, Z^I u)\} \leq C_1 \epsilon \}$$

this contradiction leads to the conclusion that $s_1 = +\infty$. In view of the local-in-time existence theory in Theorem 11.2, the solution u extends to all times.

Then our task reduce to proving the following result and the rest of this chapter is devoted to its proof.

Proposition 10.1. *Let u be the local solution of (10.1.1) with initial data satisfying*

$$E_m(s_0, Z^I u) \leq \epsilon \quad (10.2.3)$$

for all $|I| \leq 3$. Then there exists a couple of positive constant (C_1, ϵ_0) such that if it satisfies the estimate (10.2.1) with $\epsilon \leq \epsilon_0$, then the estimate (10.2.2) holds.

10.3 Energy estimate

In this section we are going to estimate $E_m(t, Z^I u)$. The following lemma is a adapted version of Lemma 2.1 in the case of (10.1.1).

Lemma 10.1. *Let u be a local solution associated to (10.1.1) and $|I| \leq 3$ be a multi-index. Then the following estimate holds:*

$$E_m(s, Z^I u) \leq E_m(s_0, Z^I u) + \int_{s_0}^s \|Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u)\|_{L^2(\mathcal{H}_\tau)} d\tau. \quad (10.3.1)$$

Proof. This is a special case of Lemma 2.1 which we can also check directly. We apply to the equation (10.1.1) a product Z^I with $|I| \leq 3$. Recall the commutation relation $[Z^I, \square] = 0$, we find

$$\square(Z^I u) = Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u).$$

Then, we multiply this equation by $\partial_t Z^I u$ and perform the standard calculation of energy estimate. For simplicity we denote by $\tilde{u} =: Z^I u$:

$$\frac{1}{2} \partial_t \left((\partial_t \tilde{u})^2 + \sum_a (\partial_a \tilde{u})^2 \right) - \partial_a (\partial_a \tilde{u} \partial_t \tilde{u}) = Z^I (P^{\alpha\beta} \partial_\alpha \tilde{u} \partial_\beta \tilde{u}) \partial_t \tilde{u}.$$

Integrating this equation in the region $\mathcal{K}_{[s_0, s]}$, we have

$$\begin{aligned} & \int_{\mathcal{K}_{[s_0, s]}} \left(\frac{1}{2} \partial_t \left((\partial_t \tilde{u})^2 + \sum_a (\partial_a \tilde{u})^2 \right) - \partial_a (\partial_a \tilde{u} \partial_t \tilde{u}) \right) dt dx \\ &= \int_{\mathcal{K}_{[s_0, s]}} Z^I (P^{\alpha\beta} \partial_\alpha \tilde{u} \partial_\beta \tilde{u}) \partial_t \tilde{u} dt dx. \end{aligned} \quad (10.3.2)$$

Recall that by Huygens' principle, the solution is supported in the cone \mathcal{K} , then in a neighbourhood of the cone $\{|x| = \tau - 1\} \cap \{s_0 \leq \tau \leq t\}$, $\tilde{u} = 0$. Then by Stokes' formula, the left-hand side reduces to

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{H}_s} \left((\partial_t \tilde{u})^2 + \sum_a (\partial_a \tilde{u})^2, 2\partial_t \tilde{u} \partial_a \tilde{u} \right) \cdot n \, d\sigma \\ & - \frac{1}{2} \int_{\mathcal{H}_{s_0}} \left(|\partial_t \tilde{u}|^2 + \sum_a |\partial_a \tilde{u}|^2, 2\partial_t \tilde{u} \partial_a \tilde{u} \right) \cdot n \, d\sigma, \end{aligned}$$

where n is the (future oriented) unit normal vector to the hyperboloids and $d\sigma$ is the induced Lebesgue measure on the hyperboloids:

$$n = (t^2 + |x|^2)^{-1/2} (t, -x^a), \quad d\sigma = \frac{(t^2 + |x|^2)^{1/2}}{t} dx$$

Then the left-hand-side of (10.3.2) becomes

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{H}_s} \left(|\partial_t \tilde{u}|^2 + \sum_a |\partial_a \tilde{u}|^2 + 2 \frac{x^a}{t} \partial_a \tilde{u} \partial_t \tilde{u} \right) dx \\ & - \frac{1}{2} \int_{\mathcal{H}_{s_0}} \left(|\partial_t \tilde{u}|^2 + \sum_a |\partial_a \tilde{u}|^2 + 2 \frac{x^a}{t} \partial_a \tilde{u} \partial_t \tilde{u} \right) dx, \end{aligned}$$

which is $\frac{1}{2} E_m(s, Z^I u)^2 - \frac{1}{2} E_m(s_0, Z^I u)^2$.

In another hand, recall that in the region $\mathcal{K}_{[s_0, s]}$, the relation of change of variable $\tau = (t^2 - |x|^2)^{1/2}$, then by the relation $dt dx = (\tau/t) d\tau dx$, the right-hand-side of (10.3.2) becomes

$$\int_{s_0}^s d\tau \int_{\mathcal{H}_\tau} (\tau/t) \partial_t u Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u) dx.$$

Then (10.3.2) becomes

$$\frac{1}{2} E_m(s, Z^I u)^2 - \frac{1}{2} E_m(s_0, Z^I u)^2 = \int_{s_0}^s ds \int_{\mathcal{H}_\tau} (\tau/t) \partial_t u Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u) dx. \quad (10.3.3)$$

Derive (10.3.3) with respect to the variable s ,

$$\begin{aligned} E(\tau, Z^I u) \frac{d}{d\tau} E(\tau, Z^I u) &= \int_{\mathcal{H}_\tau} (\tau/t) \partial_t u Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u) dx \\ &\leq \|(\tau/t) \partial_t u\|_{L^2(\mathcal{H}_\tau)} \|Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u)\|_{\mathcal{H}_\tau} \end{aligned}$$

Recall (2.3.2), $E(s, Z^I u) \geq C \|(\tau/t) \partial_t u\|_{L^2(\mathcal{H}_s)}$. Then

$$\frac{d}{ds} E(s, Z^I u) \leq \|Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u)\|_{\mathcal{H}_s}$$

Integrate this with respect to τ on $[s_0, s]$, the desired result is proved. \square

Now, we observe from (10.3.1) that the key toward estimating the energy of $Z^I u$ is the estimate of $\|Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u)\|_{L^2(\mathcal{H}_s)}$. This is the subject of the following sections.

10.4 Basic L^2 and L^∞ estimates

In this section we deduce from the bootstrap assumption (10.2.1) a set of estimates on L^2 and L^∞ type norms on the derivatives of the solution. The proof is immediate by combining the expression of the energy E_m and (10.2.1).

Lemma 10.2 (Basic L^2 estimates). *Suppose that (10.2.1) holds. Then for all $|I| \leq 3$, the following estimate hold for $s \in [s_0, s_1]$:*

$$\|\underline{\partial}_a Z^I u\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (10.4.1a)$$

$$\|(s/t)\underline{\partial}_0 Z^I u\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (10.4.1b)$$

where $C > 0$ is a (universal) constant.

Now, we combine Lemma 10.2 with the estimates on commutators lemma 3.4 and (3.5), the following estimates hold.

Lemma 10.3. *Suppose that (10.2.1) holds. Then for all $|I_1| + |I_2| \leq 3$, the following estimate hold for $s \in [0, T]$:*

$$\|Z^{I_1} \underline{\partial}_a Z^{I_2} u\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (10.4.2a)$$

$$\|Z^{I_1} ((s/t)\underline{\partial}_0 Z^{I_2} u)\|_{L^2(\mathcal{H}_s)} \leq CC_1 \epsilon \quad (10.4.2b)$$

where $C > 0$ is a universal constant.

Then we have the following decay estimates. The proof is immediate by combining (10.4.2) and Lemma 5.1.1.

Lemma 10.4 (Basic L^∞ estimates). *Suppose that (10.2.1) holds. Then for all $|J| \leq 1$, the following estimate hold for $s \in [0, T]$:*

$$\|t^{3/2} \underline{\partial}_a Z^J u\|_{L^\infty(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (10.4.3a)$$

$$\|t^{1/2} s \underline{\partial}_0 Z^J u\|_{L^\infty(\mathcal{H}_s)} \leq CC_1 \epsilon, \quad (10.4.3b)$$

where $C > 0$ is a universal constant.

10.5 Estimate on the interaction term

We now estimate the source term $P^{\alpha\beta} \partial_\alpha u \partial_\beta u$ with the help of (10.4.2) and (10.4.3).

Lemma 10.5. *Assume that the inequalities (10.4.2) and (10.4.3) hold and $P^{\alpha\beta}$ is a null bilinear form. Then the following estimate hold:*

$$\|Z^I (P^{\alpha\beta} \partial_\alpha u \partial_\beta u)\|_{L^2(\mathcal{H}_s)} \leq C(C_1 \epsilon)^2 s^{-3/2}. \quad (10.5.1)$$

Proof. We must combine Proposition 4.2 with (10.4.2) and (10.4.3). From Proposition 4.2, we recall that

$$\begin{aligned}
& |Z^I(P^{\alpha\beta}\partial_\alpha u\partial_\beta u)| \\
& \leq CK(s/t)^2 \sum_{|I_1|+|I_2|\leq|I|} |Z^{I_1}\partial_t u Z^{I_2}\partial_t u| \\
& \quad + CK \sum_{\substack{\alpha,\beta, \\ |I_1|+|I_2|\leq|I|}} \left(|Z^{I_1}\underline{\partial}_\alpha u Z^{I_2}\underline{\partial}_\beta u| + |Z^{I_1}\underline{\partial}_\beta u Z^{I_2}\underline{\partial}_\alpha u| \right) \\
& =: T_1 + T_2,
\end{aligned}$$

where $K = \max_{\alpha,\beta} |P^{\alpha\beta}|$.

To estimate the L^2 norm of each term of T_1 , we remark that $|I_1| + |I_2| \leq |I| \leq 3$ implies that $|I_1| \leq 1$ or $|I_2| \leq 1$. Suppose, without loss of generality, that $|I_2| \leq 1$. Then, we have

$$\begin{aligned}
& \|(s/t)^2 Z^{I_1}\partial_t u Z^{I_2}\partial_t u\|_{L^2(\mathcal{H}_s)} \\
& \leq \|(s/t)\partial_t u\|_{L^2(\mathcal{H}_s)} \|(s/t)t^{-1/2}s^{-1}(t^{1/2}sZ^{I_2}\partial_t u)\|_{L^\infty(\mathcal{H}_s)} \\
& \leq CC_1\epsilon s^{-3/2}CC_1\epsilon \leq C(C_1\epsilon)^2 s^{-3/2}.
\end{aligned}$$

The terms in T_2 are estimated similarly. We remark that when $|I_1| \leq 1$,

$$\begin{aligned}
\|Z^{I_1}\underline{\partial}_\alpha u Z^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} & = \|t^{3/2}Z^{I_1}\underline{\partial}_\alpha u t^{-3/2}(t/s)(s/t)Z^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} \\
& \leq \|t^{3/2}Z^{I_1}\underline{\partial}_\alpha u s^{-3/2}(s/t)Z^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} \\
& = s^{-3/2}\|t^{3/2}Z^{I_1}\underline{\partial}_\alpha u\|_{L^\infty(\mathcal{H}_s)} \|(s/t)Z^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} \\
& \leq C(C_1\epsilon)^2 s^{-3/2}.
\end{aligned}$$

When $|I_2| \leq 1$, we have

$$\begin{aligned}
\|Z^{I_1}\underline{\partial}_\alpha u Z^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} & = \|Z^{I_1}\underline{\partial}_\alpha u t^{-1/2}s^{-1}t^{1/2}sZ^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} \\
& \leq \|Z^{I_1}\underline{\partial}_\alpha u s^{-3/2}t^{1/2}sZ^{I_2}\underline{\partial}_\beta u\|_{L^2(\mathcal{H}_s)} \\
& = s^{-3/2}\|Z^{I_1}\underline{\partial}_\alpha u\|_{L^2(\mathcal{H}_s)} \|t^{1/2}sZ^{I_2}\underline{\partial}_\beta u\|_{L^\infty(\mathcal{H}_s)} \\
& \leq C(C_1\epsilon)^2 s^{-3/2}.
\end{aligned}$$

So we arrive at the desired result. \square

10.6 Conclusion

Proof of Proposition 10.1. Now we combine (10.5.1) and (10.3.1):

$$\begin{aligned} E_m(s, Z^I u) &\leq E_m(s_0, Z^I u) + C(C_1 \epsilon)^2 \int_{s_0}^s \tau^{-3/2} d\tau \\ &\leq E_m(s_0, Z^I u) + C(C_1 \epsilon)^2 \int_1^\infty \tau^{-3/2} d\tau \leq \epsilon + C(C_1 \epsilon)^2. \end{aligned}$$

Then if we take $C_1 > 2$ and $\epsilon \leq \epsilon_0 = \frac{C_1 - 2}{2CC_1^2}$, then

$$E_m(s, Z^I u) \leq \frac{1}{2} C_1 \epsilon,$$

which concludes the argument. \square

The asymptotic profile of the solution is also clear: (10.1.4) has already been proved by Proposition 10.1. To see 10.1.5, we remark that by Lemma 10.4,

$$|\partial_t u| \leq C_1 \epsilon t^{-1/2} s^{-1} = C_1 \epsilon t^{-1} (t - r)^{-1/2}, \quad |\underline{\partial}_a u| \leq C_1 \epsilon t^{-3/2}.$$

By recalling the relation

$$\partial_a u = \underline{\partial}_a u - \frac{x^a}{t} \partial_t u,$$

then (10.1.5) follows.

In view of the discussion at the beginning of Section 10.2, the result stated in Theorem 10.1 is now established.

Chapter 11

The local well-posedness theory

11.1 Construction of the initial data

In this chapter, we establish a local existence theory for the quasi-linear systems

$$\begin{aligned} \square w_i + G_i^{j\alpha\beta}(w, \partial w) \partial_{\alpha\beta} w_j + c_i^2 w_i &= F_i(w, \partial w), \\ w_i(B+1, x) = w_{i0}, \quad \partial_t w_i(B+1, x) &= w_{i1}, \end{aligned} \quad (11.1.1)$$

where

$$\begin{aligned} G_i^{j\alpha\beta}(w, \partial w) &= A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k + B^{j\alpha\beta k} w_k, \\ F_i(w, \partial w) &= P_i^{\alpha\beta jk} \partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha jk} \partial_\alpha w_j w_k + R_i^{jk} w_j w_k. \end{aligned} \quad (11.1.2)$$

These $A_i^{\alpha\beta\gamma j}$, $B^{j\alpha\beta k}$, $P_i^{\alpha\beta jk}$, $Q_i^{\alpha jk}$, R_i^{jk} are constants. To guarantee the hyperbolicity, the following symmetry conditions are assumed:

$$G_i^{j\alpha\beta} = G_j^{i\alpha\beta}, \quad G_i^{j\alpha\beta} = G_i^{j\beta\alpha}. \quad (11.1.3)$$

Initial data are prescribed on the hyperplane $\{t = B+1\}$ and are denoted by $(w_{i0}, w_{i1}) \in \mathbf{H}^{d+1}(\mathbb{R}^3) \times \mathbf{H}^d(\mathbb{R}^3)$. We assume that they satisfy the following smallness condition:

$$\|w_{i0}\|_{\mathbf{H}^{d+1}(\mathbb{R}^3)} \leq \epsilon', \quad \|w_{i1}\|_{\mathbf{H}^d(\mathbb{R}^3)} \leq \epsilon'. \quad (11.1.4)$$

We also assume that w_{i0} and w_{i1} are supported in the ball $\{|x| \leq B\}$. The following local-in-time existence holds which is essentially established in [Sogge (2008)].

Theorem 11.1. *For any integer $d \geq 2p(n) - 1$, there exists a time interval $[B+1, T(\epsilon') + B+1]$ on which the Cauchy problem (11.1.1) admits exactly one solution $w_i = w_i(t, x)$. Furthermore, one has*

$$w_i(t, x) \in C([B+1, T(\epsilon) + B+1], \mathbf{H}^{d+1}(\mathbb{R}^3)), \quad (11.1.5)$$

and there exists a constant $A = A(\epsilon') > 0$ such that

$$\sum_{\substack{i,\alpha,\beta, \\ |\alpha| \leq 2p(n)-1}} \|\partial_\alpha Z^I w_i\|_{L^\infty([B+1, T(\epsilon)], L^2(\mathbb{R}^n))} \leq A(\epsilon'). \quad (11.1.6)$$

Furthermore, one has

$$\lim_{\epsilon' \rightarrow 0^+} T(\epsilon') = +\infty, \quad \lim_{\epsilon' \rightarrow 0^+} A(\epsilon') = 0 \quad (11.1.7)$$

and, if T^* denotes the supremum of all such times T , then either $T^* = +\infty$ or

$$\sum_{|\alpha| \leq (s+3)/2} |\partial^\alpha(t, x)| \notin L^\infty([0, T^*) \times \mathbb{R}). \quad (11.1.8)$$

In addition, let $E_G(T, w_i)$ be the hyperboloidal energy (defined in Section 2.2). For any $C_0 > 1$ and $\epsilon > 0$, there exists some $\epsilon'_0 > 0$ (depending only on the structure of the system) such that for all $\epsilon' \leq \epsilon'_0$

$$\sum_i E_G(B+1, w_i) \leq C_0 \epsilon. \quad (11.1.9)$$

Proof. We only sketch the proof. The local existence and the blow-up criteria are given by Theorem 4.1 (Section 1.4) of [Sogge (2008)]. Then, the profile of $A(\epsilon')$ and $T(\epsilon')$ follows by the construction of the solution in the proof of this theorem. We thus focus on estimating the energy $E_G(B+1, Z^I w_i)$.

We consider the region $V(B) := \{(t, x) : t \geq B+1, t^2 - |x|^2 \leq B+1\} \cap \mathcal{K}$ where $\mathcal{K} = \{|x| \leq t-1, t \geq 0\}$. We observe that in $V(B)$, $t \leq \frac{(B+1)^2+1}{2}$. So we can fix some ϵ'_0 sufficiently small so that $T(\epsilon') \geq \frac{(B+1)^2+1}{2}$. So the local solution is well defined in the region $V(B)$.

To estimate $E_G(B+1, Z^I w_i)$, we choose $\partial_t w_i$ as a multiplier and, by the standard procedure, we derive the energy estimate:

$$\begin{aligned} & E_G(B+1, Z^I w_i) - E_G^*(B+1, Z^I w_i) \\ &= \int_{V(B)} (Z^I F_i(w, \partial w) \cdot \partial_t Z^I w_i - [Z^I, G_i^{j\alpha\beta} \partial_{\alpha\beta}] Z^I w_j \partial_t Z^I w_i) dx dt \\ & \quad + \int_{V(B)} \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t Z^I w_i \partial_\beta Z^I w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha Z^I w_i \partial_\beta Z^I w_j \right) dx dt \\ & \leq CK \sum_{\substack{i,j,k \\ \alpha,\beta}} \int_{B+1}^{\frac{(B+1)^2+1}{2}} |\partial_\alpha Z^I w_i| \cdot |\partial_\beta Z^J w_j| \cdot |\partial_\gamma Z^J w_k| dx dt \\ & \leq CA(\epsilon'). \end{aligned}$$

Here, C is a constant depending only on the structure of the system. We have

$$E_G(B + 1, Z^I w_i) \leq CA(\epsilon') + C\epsilon'.$$

Thus, for any $C_0 > 1$ and $\epsilon > 0$, we can find some ϵ'_0 sufficiently small such that

$$CA(\epsilon') + C\epsilon' \leq C_0\epsilon.$$

□

11.2 Local well-posedness theory in the hyperboloidal foliation

For the convenience of discussion one introduces the following notion of hyperbolic variables. In the cone \mathcal{K} , we introduce the variables

$$\begin{aligned}\bar{x}^0 &:= s = \sqrt{t^2 - |x|^2}, \\ \bar{x}^a &:= x^a.\end{aligned}$$

Then the natural frame associated with these variables is

$$\begin{aligned}\bar{\partial}_0 &= \partial_s = \frac{s}{t} \partial_t, \\ \bar{\partial}_a &= \underline{\partial}_a = \frac{x^a}{t} \partial_t + \partial_a.\end{aligned}$$

By an easy calculation, we express the D'Alembert operator in this frame:

$$\square u = \bar{\partial}_0 \bar{\partial}_0 u + \frac{2\bar{x}^a}{s} \bar{\partial}_0 \bar{\partial}_a u - \sum_a \bar{\partial}_a \bar{\partial}_a u + \frac{3}{s} \underline{\partial}_a u. \quad (11.2.1)$$

The symmetric second-order quasi-linear operator $G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta w_j$ can also be expressed in this frame:

$$\begin{aligned}G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha \partial_\beta w_j \\ = \bar{G}_i^{j\alpha\beta}(w, \bar{\partial} w) \partial_\alpha \partial_\beta w_j + G_i^{j\alpha\beta}(w, \partial w) \partial_\alpha (\Psi_\beta^{\beta'}) \bar{\partial}_{\beta'} w_j\end{aligned} \quad (11.2.2)$$

with

$$\bar{G}_i^{j\alpha\beta} = G_i^{j\alpha'\beta'} \Psi_{\alpha'}^\alpha \Psi_{\beta'}^\beta$$

which is again symmetric with respect to i, j .

With these preparation, we can transform the system (11.1.1)

$$\begin{aligned}\bar{\partial}_0 \bar{\partial}_0 w_i + \bar{G}_i^{j00} \bar{\partial}_0 \bar{\partial}_0 w_j + \frac{\bar{x}^a}{s} \bar{\partial}_0 \bar{\partial}_a w_i + \bar{G}_i^{j0a}(w, \partial w) \bar{\partial}_0 \bar{\partial}_a w_j \\ - \sum_a \bar{\partial}_a \bar{\partial}_a w_i + \bar{G}_i^{jab} \bar{\partial}_a \bar{\partial}_b w_j = \tilde{F}_i(w, \partial w)\end{aligned} \quad (11.2.3)$$

with initial data posed on $s = B + 1$:

$$w_i(B + 1, \cdot) = w'_{i0}, \quad \partial_s w_i(B + 1, \cdot) = w'_{i1}.$$

We observe that this system is again symmetric and that the perturbation terms \bar{G} are small, so this system is again hyperbolic with respect to the variables (s, \bar{x}^α) . Then, by the standard theory of second-order symmetric hyperbolic system (see for example in [Taylor (2011)] (Proposition 3.1 in Chap. 16, Section 3)), the following theorem holds.

Theorem 11.2. *Let k be a integer satisfying $k \geq n + 2$. Let w_{i0} and w_{i1} are C^∞ functions supported in the ball centered at 0 and with radius $\frac{(B+1)^2 - 1}{2}$. Then, the following Cauchy problem*

$$\begin{aligned} \square w_i + G_i^{j\alpha\beta}(w, \partial w) \partial_{\alpha\beta} w_j + c_i^2 w_i &= F_i(w, \partial w), \\ w_i|_{\mathcal{H}_{B+1}} &= w_{i0}, \quad \partial_t w_i|_{\mathcal{H}_{B+1}} = w_{i1} \end{aligned} \quad (11.2.4)$$

with

$$\sum_{\alpha \leq k+1} \|\partial^\alpha w_i\|_{L^2(\mathcal{H}_{B+1})} + \sum_{\alpha \leq k} \|\partial^\alpha \partial_t w_i\|_{L^2(\mathcal{H}_{B+1})} \leq \epsilon \quad (11.2.5)$$

admits a unique local-in-time solution, which satisfies the following bound in the time interval $[B + 1, T(\epsilon) + B + 1]$:

$$\sum_{\alpha \geq k+1} \|\partial^\alpha w_i\|_{L^2 \mathcal{H}_s} \leq A(\epsilon). \quad (11.2.6)$$

Furthermore, if T^* denotes the supremum of all $T(\epsilon)$ (for ϵ fixed), then either $T^* = +\infty$ or

$$\lim_{s \rightarrow T^{*-}} \sum_{\alpha \leq (s+3)/2} \|\partial^\alpha w_i\|_{L^2 \mathcal{H}_s} = +\infty. \quad (11.2.7)$$

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