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## Autour de quelques processus à accroissements stationnaires et autosimilaires

Benjamin Arras

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Benjamin Arras. Autour de quelques processus à accroissements stationnaires et autosimilaires. Autre. Ecole Centrale Paris, 2014. Français. NNT : 2014ECAP0060 . tel-01130239

**HAL Id: tel-01130239**

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ECOLE CENTRALE DES  
ARTS  
ET MANUFACTURES  
“ECOLE CENTRALE PARIS”

**Thèse**  
**préparée par Benjamin Arras**  
**pour l’obtention du grade de Docteur**

Spécialité: Mathématiques  
Laboratoire d’accueil: Mathématiques Appliquées aux Systèmes (MAS)

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**Autour de quelques processus à accroissements  
stationnaires et autosimilaires**

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N° d’ordre : 2014ECAP0061

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## Remerciements

Tout d'abord, je remercie mon directeur de thèse, Jacques Lévy-Véhel, pour m'avoir initié aux théories des ondelettes et des distributions stochastiques de Hida. Je le remercie également de m'avoir laissé beaucoup de liberté dans mes recherches au cours de ces trois années de travail. J'exprime de chaleureux et sincères remerciements à Mihai Gradinaru et à Ciprian Tudor pour avoir accepté la lourde tâche de rapporter mon manuscrit. Leurs présences dans mon jury, ainsi que celles de Massimiliano Gubinelli, d'Ivan Nourdin et de Frederi Viens m'honorent véritablement. A différents instants de ces trois années, leurs travaux respectifs ont influencé de près comme de loin mes réflexions scientifiques.

Je pense que je n'en serai pas là aujourd'hui sans les conseils avisés d'Erick Herbin. Grâce à lui, j'ai redécouvert les mathématiques à Centrale, ai appris la rigueur des probabilités et ai poursuivi l'étude de celles-ci en master. Sans lui, je n'aurais jamais continué dans cette voie. C'est pourquoi, je te remercie beaucoup, Erick, pour les nombreuses discussions échangées et les nombreux conseils prodigués. Ces années de thèse auraient été sûrement plus difficiles sans l'environnement joyeux du laboratoire MAS et tout particulièrement de certaines personnes le constituant. Je remercie donc avec plaisir mes anciens compagnons de bureau, Alexandre et Paul, et des bureaux voisins, Blandine, Benoit, Gautier, Ludovic, Marion et Pierre-André. Je remercie également Sylvie pour son humour décapant et sa franchise à toute épreuve. Je souhaiterais en particulier remercier Pauline pour ses conseils éclairés concernant le monde de la recherche et pour les discussions que nous avons pue partager.

Ces trois années de thèse ne se réduisent bien évidemment pas à l'environnement de travail. Je n'aurai sûrement pas terminé cette thèse sans le soutien de mes amis et de ma famille. Ainsi, je remercie, avec joie, Jacques avec qui c'est toujours un plaisir d'échanger politiquement, littérairement, scientifiquement et culturellement, et Sarah et Thibaut pour les nombreuses années d'amitié qui nous lient. Je remercie profondément ma soeur et mes parents pour leur présence et leur soutien inconditionnels, quels que furent mes choix de vie parfois difficiles. Je ne saurai exprimer par de simples mots toute la gratitude que j'ai pour ma mère et mon père.

Enfin, je souhaite remercier avec tendresse, Charlotte: ta rencontre a profondément marqué mes deux dernières années de thèse et ta présence à mes côtés ainsi que ta personnalité joyeuse et rare ont éclairé mon caractère ténébreux, mon moral souvent défaitiste et mes doutes récurrents. Les beaux moments que nous avons partagés furent des repères importants au cours des ses trois années parfois difficiles. J'espère qu'ils seront encore nombreux !



**Résumé:** Dans ce travail de thèse, nous nous intéressons à certaines propriétés d'une classe de processus stochastiques à accroissements stationnaires et autosimilaires. Ces processus sont représentés par des intégrales multiples de Wiener-Itô. Dans le premier chapitre, nous étudions les propriétés géométriques des trajectoires de ce type de processus. En particulier, nous obtenons un développement en ondelettes presque-sûr. Celui-ci permet alors de trouver une borne supérieure pour le module de continuité uniforme, une borne supérieure pour le comportement asymptotique du processus et un résultat presque-sûr concernant les coefficients ponctuel et local de Hölder. De plus, nous obtenons des bornes inférieures et supérieures pour les dimensions de Hausdorff du graphe et de l'image des versions multidimensionnelles anisotropes de la classe de processus considérée. Dans le deuxième et le troisième chapitre de cette thèse, nous nous intéressons au calcul différentiel stochastique relatif au processus de Rosenblatt. À l'aide de la théorie des distributions de Hida, nous définissons une intégrale stochastique par rapport au processus de Rosenblatt. Nous obtenons une formule d'Itô pour certaines fonctionnelles du processus de Rosenblatt. Nous calculons explicitement la variance de l'intégrale stochastique par rapport au processus de Rosenblatt pour une classe spécifique d'intégrandes aléatoires. Enfin, nous comparons l'intégrale introduite avec d'autres définitions utilisées dans la littérature et procédons à une étude fine des termes résiduels faisant le lien entre ces différentes définitions.

**Mots-clefs:** Calcul stochastique, Développement en ondelettes, Dimension de Hausdorff, Intégrale multiple de Wiener-Itô, Processus autosimilaires, Processus de Rosenblatt, Régularité Höldérienne, Théorie des distributions stochastiques de Hida.

**Summary:** In this PhD thesis, we are concerned with some properties of a class of self-similar stochastic processes with stationary increments. These processes are represented by multiple Wiener-Itô integrals. In the first chapter, we study geometric properties of the sample path of this type of processes. Specifically, we obtain an almost sure wavelet expansion which, in turn, allows us to compute an upper bound for the uniform modulus of continuity, an upper bound for the asymptotic growth at infinity of the processes and the almost sure values of the pointwise and local Hölder exponents at any points. Moreover, we obtain lower and upper bounds for the Hausdorff dimensions of the graph and the image of multidimensional anisotropic versions of the class of processes previously considered. In the second and in the third chapters, we are interested in the stochastic calculus with respect to the Rosenblatt process. Using Hida distributions theory, we define a stochastic integral with respect to the Rosenblatt process. We obtain an Itô formula for some functional of the Rosenblatt process. We compute explicitly the variance of the stochastic integral with respect to the Rosenblatt process for a specific class of stochastic integrands. At last, we compare the considered integral with other definitions used in the literature and provide a careful analysis of the residual terms linking the different definitions of integrals.

**Key words:** Hölder regularity, Hausdorff dimension, Multiple Wiener-Itô integral, Self-similar processes, Rosenblatt process, Stochastic calculus, Wavelet expansion, White noise distribution theory.





# Introduction

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## 1 Régularité des trajectoires de processus stochastiques.

En analyse, la première rencontre avec la notion de régularité se fait à travers celle de continuité. La question naturelle qui suit ce concept, permettant un approfondissement de l'étude des fonctions, est celle de la dérivabilité. Néanmoins, dès la fin du XIX<sup>ème</sup> siècle, des mathématiciens tel que K. Weierstrass ont introduit et débuté l'étude de fonctions continues partout mais dérivables nulle part. Ces objets, qualifiés de pathologiques par la communauté mathématique du début du XX<sup>ème</sup> siècle, ont rencontré un essor particulièrement important sous l'influence du mathématicien B. Mandelbrot par l'introduction du concept d'ensemble fractal. En effet, l'image et le graphe de ce type de fonctions sont des exemples de fractales. Ainsi, il a fallu développer de nouveaux outils d'analyse afin d'étudier de façon quantitative la régularité des trajectoires de ces fonctions. Dans ce contexte de recherche, sont apparus l'espace de Hölder ponctuel et l'exposant ponctuel de Hölder d'une fonction. Ils sont définis comme suit:

**Définition 1.1:** Soit  $\alpha > 0$  et  $t_0 \in \mathbb{R}$ . La fonction  $f : \mathbb{R} \rightarrow \mathbb{R}$  appartient à l'espace  $C^\alpha(t_0)$  s'il existe  $C > 0$ ,  $\delta > 0$  et un polynôme  $P$  de degré au plus  $\lfloor \alpha \rfloor$  tels que:

$$\forall |t - t_0| \leq \delta, |f(t) - P(t)| \leq C|t - t_0|^\alpha.$$

De plus, l'exposant ponctuel de Hölder de  $f$  en  $t_0$  est défini par:

$$\alpha_f(t_0) = \sup\{\alpha > 0; f \in C^\alpha(t_0)\}.$$

Lorsque cet exposant est compris entre  $(0, 1)$ , il admet la représentation utile suivante:

$$\alpha_f(t_0) = \sup \left\{ \alpha > 0; \limsup_{\rho \rightarrow 0} \frac{|f(t_0 + \rho) - f(t_0)|}{|\rho|^\alpha} < \infty \right\}.$$

De façon analogue à cette représentation, un autre exposant de régularité, l'exposant local de Hölder, peut s'avérer complémentaire dans l'étude de la régularité d'une fonction continue mais

nulle part dérivable. Il est défini par:

$$\tilde{\alpha}_f(t_0) = \sup \left\{ \alpha > 0; \limsup_{\rho \rightarrow 0} \sup_{t,s \in \mathcal{B}(t_0, \rho)} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty \right\}.$$

L'exemple paradigmatique de fonction pour laquelle ces exposants de régularité diffèrent est la fonction dite de "chirp". Pour  $\alpha \in (0, 1)$  et  $\beta > 0$ , elle est définie par:

$$\forall x \in \mathbb{R}, f_{ch}(x) = |x|^\alpha \sin \left( \frac{1}{|x|^\beta} \right).$$

Cette fonction possède un comportement local non-trivial en 0. En effet, on a le résultat suivant:

$$\alpha_{f_{ch}}(0) = \alpha, \tilde{\alpha}_{f_{ch}}(0) = \frac{\alpha}{1 + \beta}.$$

Parallèlement aux développements d'outils analytiques tels que les exposants de régularité, le XXième siècle a vu l'émergence et le développement de la théorie des probabilités et notamment des processus stochastiques. Le représentant central de cette théorie est incontestablement le mouvement brownien. Observé empiriquement pour la première fois par le botaniste R. Brown, puis introduit plus tardivement en physique par A. Einstein [35],[34], cet objet n'a été défini mathématiquement qu'en 1923 par N. Wiener [120]:

**Définition 1.2:** Le mouvement brownien standard, noté  $\{B_t; t \in \mathbb{R}_+\}$ , est l'unique processus gaussien centré de fonction de covariance donnée par:

$$\forall s, t \in \mathbb{R}_+, \mathbb{E}[B_t B_s] = s \wedge t.$$

Les trajectoires du mouvement brownien présentent des propriétés similaires à celles de la fonction de Weierstrass. Notamment, elles sont continues partout mais dérivables nulle part. De plus, concernant les exposants de Hölder ponctuel et local, le résultat suivant a été démontré ([83]):

$$p.s. \forall t \in \mathbb{R}_+, \alpha_B(t) = \tilde{\alpha}_B(t) = \frac{1}{2}.$$

Malgré son caractère central et son apparition naturelle comme processus limite de certains phénomènes discrets aléatoires indépendants (ou faiblement dépendants), le mouvement brownien appartient à une classe plus générale de processus à savoir les mouvements browniens fractionnaires (notés mBf dans la suite). Cette classe est paramétrée par un réel  $H \in (0, 1)$  appelé l'exposant de Hurst. Ces processus sont définis comme suit:

**Définition 1.3:** Soit  $H \in (0, 1)$ . Le mouvement brownien fractionnaire de paramètre  $H$ , noté  $\{B_t^H : t \in \mathbb{R}_+\}$ , est l'unique processus gaussien centré de fonction de covariance donnée

par:

$$\forall s, t \in \mathbb{R}_+, \mathbb{E} [B_t^H B_s^H] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}].$$

On remarque que pour  $H = 1/2$ , on retrouve le mouvement brownien standard. Originellement introduit par A. Kolmogorov en 1940, [57], afin de rendre compte des phénomènes de turbulence dans les fluides, cette classe de processus ne fut démocratisée que bien plus tard par les travaux de B. Mandelbrot et J. Van Ness, [75]. De même que les trajectoires du mouvement brownien, celles du mBf sont continues partout mais dérivables nulle part et le résultat suivant a été démontré [11],[42],[17]:

$$p.s. \forall t \in \mathbb{R}_+, \alpha_{B^H}(t) = \tilde{\alpha}_{B^H}(t) = H.$$

Afin d'approfondir l'étude des trajectoires du mouvement brownien et du mouvement brownien fractionnaire, d'autres outils plus fins que les exposants de Hölder peuvent être employés. Puisque les trajectoires de ces deux processus sont continues sur un compact, elles y sont en particulier uniformément continues. Ainsi, il existe des fonctions croissantes positives (aléatoires à priori), notées  $\phi_H$ , telles que:

$$p.s. \lim_{h \rightarrow 0^+} \phi_H(h) = 0,$$

$$p.s. \limsup_{h \rightarrow 0^+} \sup_{0 \leq t \leq 1-h} \frac{|B_{t+h}^H - B_t^H|}{\phi_H(h)} \leq 1.$$

Des résultats bien plus précis furent démontrés par P. Lévy (1937, [64]) pour les trajectoires du mouvement brownien, par M. B. Marcus pour les trajectoires du mBf avec  $H \in [0, 1/2]$  (1968, [76]), et par D. Khoshnevisan et Z. Shi (2000, [56]), pour le mouvement brownien fractionnaire avec  $H \in ]1/2, 1[$ . En effet, on a:

$$p.s. \limsup_{h \rightarrow 0^+} \sup_{0 \leq t \leq 1-h} \frac{|B_{t+h}^H - B_t^H|}{\sqrt{2h^{2H} \log \frac{1}{h}}} = 1.$$

De plus, le comportement local en tout point des trajectoires de ces processus est décrit de façon précise par le module de continuité local suivant:

$$\forall t \in \mathbb{R}_+, p.s. \limsup_{h \rightarrow 0^+} \frac{|B_{t+h}^H - B_t^H|}{\sqrt{2h^{2H} \log(\log(\frac{1}{h}))}} = 1.$$

Un tel résultat fut démontré par A. Khinchin (1933, [55]) pour le mouvement brownien et par S. Orey (1972, [91]) pour le mouvement brownien fractionnaire. Il est important de noter que ce résultat sur le comportement local des trajectoires se déduit de l'asymptotique de celles-ci en temps long par inversion du temps et par la stationnarité des accroissements du mBf.

Différentes méthodes peuvent être employées afin de démontrer ces résultats de régularité. L'une d'entre elles tire son origine de la construction du mouvement brownien par P. Lévy ([64] et [83] pour une présentation moderne des résultats). Cette construction s'apparente, en effet, à une décomposition en ondelettes du mouvement brownien dans la base des fonctions de Haar de  $L^2([0; 1])$ . Avant d'aller plus loin dans la description des résultats obtenus par cette méthode, un bref rappel concernant la théorie des ondelettes et son lien avec la régularité ponctuelle de fonction est nécessaire (voir [80] et [28] pour plus de détails). Une base d'ondelettes de classe  $N \geq 0$  est une base orthonormale de  $L^2(\mathbb{R})$  (ou de  $L^2([0; 1])$ ) définie de la façon suivante:

- Il existe une fonction,  $\psi$ , telle que elle-même et ses dérivées d'ordre  $N$  soient dans  $L^\infty(\mathbb{R})$ .
- Pour tout  $0 \leq n \leq N$  et pour tout  $p \in \mathbb{N}$ ,  $\sup_{x \in \mathbb{R}} (1 + |x|)^p |\psi^{(n)}(x)| < \infty$ .
- Pour tout  $0 \leq n < N$ ,  $\int_{\mathbb{R}} x^n \psi(x) dx = 0$ .
- La collection de fonctions  $\{\psi_{j,k} = 2^{j/2} \psi(2^j \cdot - k); j \in \mathbb{Z}, k \in \mathbb{Z}\}$  forme une base orthonormée de  $L^2(\mathbb{R})$ .

Il existe différentes bases orthonormées d'ondelettes vérifiant des propriétés supplémentaires souvent cruciales suivant le problème considéré. Ainsi, l'élément  $\psi$ , de la base de Lemarié-Meyer appartient à l'espace de Schwartz, des fonctions  $C^\infty(\mathbb{R})$  et à décroissance rapide, et l'élément  $\psi$  des ondelettes de Daubechies possède une certaine régularité prescrite et est à support compact. Concernant la régularité ponctuelle de fonction, le résultat suivant fut démontré par S. Jaffard ([48], voir aussi [50]):

**Théorème 1.4:** Soit  $\alpha > 0$  et  $t_0 \in \mathbb{R}$ . Soit  $f \in C^\alpha(t_0)$ . Alors, il existe une constante  $C > 0$  telle que:

$$|\langle f; \psi_{j,k} \rangle| \leq C 2^{-(\frac{1}{2} + \alpha)j} (1 + |2^j t_0 - k|)^\alpha.$$

Réciproquement, si l'inégalité précédente est vérifiée et s'il existe  $\epsilon > 0$  telle que  $f \in C^\epsilon(\mathbb{R})$  (l'espace des fonctions uniformément  $\epsilon$ -Höldériennes), alors il existe un polynôme,  $P$ , de degré au plus  $[\alpha]$ ,  $\delta > 0$  et  $C > 0$  tels que:

$$\forall |t - t_0| < \delta, |f(t) - P(t)| \leq C |t - t_0|^\alpha |\log(|t - t_0|)|.$$

Il est important de remarquer que presque toutes les trajectoires du mouvement brownien fractionnaire appartiennent à un espace Höldérien uniforme. De façon plus précise, les invariances statistiques du mBf telles que la  $H$ -autosimilarité et la stationnarité des accroissements assurent, par le critère de continuité de Kolmogorov, que ce processus admet une modification dont les trajectoires sont presque-sûrement  $H - \eta$  Höldériennes, pour tout  $0 < \eta < H$ , sur tout compact de  $\mathbb{R}_+$ .

La décomposition en ondelettes du mouvement brownien fractionnaire fut introduite par Meyer et al. ([81]) et sous une forme légèrement différente par Benassi et al. [20]. Plus précisément, plusieurs décompositions sont présentées dans [81], l'une d'entre elles étant particulièrement élégante et utile pour l'étude de la régularité des trajectoires du mouvement brownien fractionnaire. Elle correspond à une généralisation directe de la construction du mouvement brownien de P. Lévy.

**Proposition 1.5:** Soit  $H \in (0, 1)$  et  $K \subset \mathbb{R}$  compact. Presque-sûrement, on a:

$$\forall t \in K, B_t^H = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} [\psi_H(2^j t - k) - \psi_H(-k)] \epsilon_{j,k},$$

où la série à droite de l'égalité converge uniformément sur  $K$ ,  $\{\epsilon_{j,k}, (j, k) \in \mathbb{Z} \times \mathbb{Z}\}$  est une collection de variables aléatoires gaussiennes standards indépendantes et  $\psi_H$  est définie par:

$$\forall t \in \mathbb{R}, \psi_H(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{F}(\psi)(\xi)}{(i\xi)^{H+\frac{1}{2}}} e^{it\xi} d\xi,$$

où  $\mathcal{F}$  est l'opérateur de transformée de Fourier et  $\psi$  l'ondelette mère de la base d'ondelettes de Lemarié-Meyer.

Néanmoins, l'étude de la régularité des trajectoires à l'aide de cette décomposition en ondelettes n'a pas été faite dans [81], les auteurs s'intéressant plutôt aux différentes représentations possibles du mBf. En revanche, les auteurs de l'article [20] ont véritablement initié l'étude de la régularité en calculant les modules de continuité uniforme et les lois du logarithme itéré pour une large classe de champs aléatoires gaussiens, à partir d'une décomposition en ondelettes adéquate. Cette méthode a alors été utilisée afin d'étudier différentes généralisations du mouvement brownien fractionnaire.

En particulier, dans la première partie de [16], les auteurs ont étudié la décomposition en ondelettes du drap brownien fractionnaire. Ils en ont tiré plusieurs résultats de régularité: un module de continuité uniforme du drap brownien fractionnaire ainsi que le comportement à l'infini et la non-dérivabilité presque-sûre de ce champs aléatoire.

Dans [14], les auteurs ont considéré le drap stable fractionnaire linéaire, généralisation non-gaussienne du drap brownien fractionnaire. A partir d'une décomposition en ondelettes, les auteurs ont obtenu des bornes supérieures fines pour le module de continuité uniforme, pour le comportement à l'infini et pour le comportement autour des axes cartésiens.

De plus, dans [11], les auteurs ont considéré une classe de processus appelés les processus multifractionnaires généralisés à exposant aléatoire. Cet article fait suite aux travaux de A.

Ayache et J. Lévy-Véhel ([13], [12]) et A. Ayache ([10]) sur les mouvements browniens multifractionnaires généralisés. Sommairement, l'idée de leurs constructions est la suivante: on considère, par exemple, la décomposition en ondelettes du mouvement brownien fractionnaire sur tout compact et on remplace, d'une certaine façon, l'exposant de Hurst,  $H$ , par une suite de fonctions Lipschitziennes,  $\{H_n(\cdot)\}$  (ou par une suite de processus admissibles) de sorte que le processus obtenu soit bien défini, à trajectoires continues et étende naturellement le mouvement brownien fractionnaire. L'important résultat de régularité ponctuelle suivant a alors été démontré dans [11]:

$$p.s. \forall t \in \mathbb{R}, \alpha_Z(t) = \liminf_{n \rightarrow +\infty} H_n(t),$$

où  $\{Z_t\}$  désigne le processus multifractionnaire généralisé associé à la suite  $\{H_n\}$ . Il est important de noter que, de ce résultat général de régularité ponctuelle, se déduit le résultat uniforme concernant les exposants de Hölder ponctuel et local du mouvement brownien fractionnaire énoncé plus haut.

De surcroît, les séries aléatoires d'ondelettes ont été utilisées par S. Jaffard dans [49] et par J.M. Aubry et S. Jaffard dans [8] afin de construire des exemples de processus stochastiques exhibant un caractère multifractale. De façon plus précise, certains exposants de régularité, capturant des comportements plus fins que les exposants ponctuel et local, sont considérés dans ces deux articles. Ces exposants varient le long des trajectoires des séries aléatoires d'ondelettes et leurs ensembles de niveau sont des fractales dont la dimension de Hausdorff est non-triviale.

Enfin, dans [19], la construction de P. Lévy du mouvement brownien a été utilisée afin de modéliser des processus dits autorégulés. Ces processus possèdent un exposant ponctuel de Hölder variant le long de leurs trajectoires et dont la valeur en un point dépend de l'amplitude du processus en ce point. En particulier, sous certaines hypothèses concernant le processus autorégulé, les auteurs ont obtenu le résultat suivant (voir section 4 de [19]):

$$p.s. \forall t \in \mathbb{R}_+, \alpha_X(t) = \tilde{\alpha}_X(t) = g(X_t),$$

où  $g$  est une fonction  $C^1(\mathbb{R})$  à valeurs dans  $(0, 1)$  intervenant dans la construction de  $\{X_t\}$ .

En outre, d'autres extensions du mouvement brownien fractionnaire existent. Celles-ci font appel aux intégrales multiples de Wiener-Itô. Introduites par N. Wiener en 1938 [121] sous le terme de chaos polynomial, les intégrales multiples de Wiener-Itô furent définies et étudiées par K. Itô en 1951 dans [47]. Comme l'intégrale de Wiener simple, elles sont définies à partir d'une procédure limite et d'une isométrie entre l'espace des fonctions symétriques de carré sommable sur  $\mathbb{R}^n$  et l'ensemble des variables aléatoires à variance finie mesurables par rapport à la tribu engendrée par un mouvement brownien standard. Les processus généralisant le mouvement brownien fractionnaire aux chaos de Wiener apparaissent comme objets limites dans

des théorèmes généraux de la limite non-centrale. En effet, R. L. Dobrushin et P. Major ([32]) ainsi que M. Taqqu ([108],[109]) ont montré le résultat suivant:

**Theorem 1.6:** Soient  $H \in (1/2, 1)$  et  $d \in \mathbb{N}^*$ . Soit  $\{\xi_n, n \in \mathbb{Z}\}$  une suite gaussienne stationnaire centrée de variance unitaire telle que:

$$\mathbb{E}[\xi_n \xi_0] = n^{\frac{2H-2}{d}}.$$

Soit  $g$  une fonction mesurable telle que  $\mathbb{E}[g(\xi_0)] = 0$ ,  $\mathbb{E}[g(\xi_0)^2] < \infty$  et:

$$g(x) = \sum_{j=d}^{+\infty} c_j H_j(x),$$

où  $H_j$  est le  $j$ -ième polynôme de Hermite et  $c_j = 1/(j!2\pi) \int_{\mathbb{R}} g(x) H_j(x) \exp(-x^2/2) dx$ . Alors, on a, pour tout  $(t_1, \dots, t_p) \in (\mathbb{R}_+)^p$ :

$$\left( \frac{1}{n^H} \sum_{i=1}^{\lfloor nt_1 \rfloor} g(\xi_i), \dots, \frac{1}{n^H} \sum_{i=1}^{\lfloor nt_p \rfloor} g(\xi_i) \right) \xrightarrow[n \rightarrow +\infty]{(d)} \left( I_d(h_{t_1}^{H,d}), \dots, I_d(h_{t_p}^{H,d}) \right),$$

avec,

$$I_d(h_t^{H,d}) = \int_{\mathbb{R}^d} \underbrace{\left( c(H, d) \int_0^t \prod_{j=1}^d (s - x_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{d}\right)} ds \right)}_{=h_t^{H,d}} dB_{x_1} \dots dB_{x_d}$$

où  $\{B_x, x \in \mathbb{R}\}$  est un mouvement brownien indexé par  $\mathbb{R}$  et  $c(H, d)$  est une constante strictement positive telle que  $\mathbb{E}[(\int_{\mathbb{R}^d} h_1^{H,d}(x_1, \dots, x_d) dB_{x_1} \dots dB_{x_d})^2] = 1$ .

Les objets limites ainsi obtenus sont appelés processus de Hermite. Pour  $d = 1$ , on retrouve la représentation du mouvement brownien fractionnaire à l'aide d'une intégrale de Wiener simple introduite dans [75]. Pour  $d = 2$ , le processus obtenu est appelé le processus de Rosenblatt. Il est important de noter que ces processus possèdent plusieurs propriétés en commun avec le mBf. En particulier, ils sont à accroissements stationnaires,  $H$ -autosimilaires et leurs trajectoires sont presque toutes  $H$ - $\eta$  Höldériennes, pour tout  $0 < \eta < H$ , sur tout compact. Néanmoins, pour  $d \neq 1$ , ce ne sont plus des processus gaussiens. De plus, alors que le mouvement brownien fractionnaire est l'unique (à une constante multiplicative près) processus gaussien à accroissements stationnaires et  $H$ -autosimilaire, plusieurs noyaux déterministes peuvent être considérés afin de construire, à  $d \neq 1$  fixé, des processus à accroissements stationnaires et  $H$ -autosimilaires mais de lois différentes.

L'étude de la régularité uniforme et locale de ces processus n'a été que partiellement initiée



depuis leur découverte. Dans [82], T. Mori et H. Oodaira ont considéré une large classe de processus  $H$ -autosimilaires et à accroissements stationnaires représentés par:

$$\forall t \in \mathbb{R}_+, X_t^{H,d} = \int_{\mathbb{R}^d} Q_t(x_1, \dots, x_d) dB_{x_1} \dots dB_{x_d},$$

où  $Q_t$  est une fonction symétrique de carré sommable vérifiant les propriétés suivantes:

- $Q_t(x_1, \dots, x_d) = \int_0^t q(s - x_1, \dots, s - x_d) ds,$
- $\forall c > 0, q(cx_1, \dots, cx_d) = c^{H - \frac{d}{2} - 1} q(x_1, \dots, x_d),$
- $\int_{\mathbb{R}^d} |q(x_1, \dots, x_d) q(x_1 + 1, \dots, x_d + 1)| dx_1 \dots dx_d < \infty.$

On remarque que le noyau des processus de Hermite vérifie les propriétés précédentes. Pour cette classe de processus, les auteurs ont obtenu une loi du logarithme itéré fonctionnelle. Celle-ci permet d'obtenir les enveloppes supérieures et inférieures exactes du comportement à l'infini des processus ainsi considérés. L'ordre des fluctuations apparaissant alors dans ce comportement asymptotique est  $h^H (\log \log(h))^{d/2}$ . En revanche, il est important de noter qu'on ne peut pas en déduire le comportement local en 0 contrairement au cas du mBf. L'inversion du temps n'est a priori pas valable pour  $d \geq 2$ .

Dans [106], des résultats partiels de régularité sont présentés concernant des processus autosimilaires ergodiques. En particulier, le résultat suivant concernant les exposants ponctuel et local de processus  $H$ -autosimilaires et à accroissements stationnaires représentés par des intégrales multiples de Wiener-Itô est démontré:

$$\forall t \in \mathbb{R}_+, p.s. \alpha_{X^{H,d}}(t) = \tilde{\alpha}_{X^{H,d}}(t) = H.$$

L'ordre des quantificateurs est ici primordial. Ce résultat s'avère bien moins fort que celui concernant les exposants de régularité du mBf.

Dans [93], V. Pipiras démontre plusieurs développements de type ondelette pour le processus de Rosenblatt. On remarque, en particulier, qu'une représentation analogue à celle présentée plus haut pour le mBf et utile pour l'étude de la régularité des trajectoires n'est pas obtenue dans [93].

Enfin, dans [118], les auteurs obtiennent, par des arguments de chaînage, une borne supérieure concernant le module de continuité uniforme de processus  $H$ -autosimilaires et à accroissements stationnaires représentés par des intégrales multiples de Wiener-Itô. De même que pour le comportement asymptotique à l'infini, l'ordre des fluctuations locales au voisinage de  $0^+$  est  $h^H (\log(1/h))^{d/2}$ .

## 2 Géométrie fractale.

Afin d'approfondir l'étude des propriétés de régularité des processus stochastiques, un examen complémentaire des trajectoires peut se faire à travers le prisme de la géométrie fractale. Au coeur de ce domaine se trouve le concept de dimension fractale ([74]). Introduite en 1919 par F. Hausdorff ([41]) et développée par A. S. Besicovitch, la dimension de Hausdorff est la définition la plus répandue. Elle repose sur la notion de recouvrement d'ensembles et permet de définir une quantité intrinsèque à tout sous-ensemble non vide de  $\mathbb{R}^N$  mesurant, d'une certaine façon, son taux d'occupation de l'espace ambiant. Avant d'aller plus loin, il est essentiel de rappeler quelques définitions et notations (voir [36],[78] pour plus de détails). Pour tout sous-ensemble,  $F$ , non vide de  $\mathbb{R}^N$ , on note,  $|F| = \sup\{\|x - y\|_2, x, y \in F\}$ , le diamètre de  $F$  pour la norme euclidienne dans  $\mathbb{R}^N$ . Pour tout  $\delta > 0$ , on appelle  $\delta$ -recouvrement de  $F$  toute collection dénombrable ou finie d'ensembles,  $\{F_i\}$ , telle que  $F \subset \cup_{i=1}^{\infty} F_i$  et  $|F_i| \leq \delta$  pour tout  $i \in \mathbb{N}^*$ . Pour tout  $s \geq 0$ , on définit la mesure de Hausdorff  $s$ -dimensionnelle de  $F$  par:

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0^+} \inf \left\{ \sum_{i=1}^{+\infty} |F_i|^s, F \subset \bigcup_{i=1}^{\infty} F_i, |F_i| \leq \delta \right\}.$$

On remarque de plus que s'il existe  $s \geq 0$  tel que  $\mathcal{H}^s(F) < \infty$  alors pour tout  $r > s$ , on a  $\mathcal{H}^r(F) = 0$ . Ceci autorise alors la définition suivante pour la dimension de Hausdorff de  $F$ :

$$\dim_{\mathcal{H}}(F) = \inf\{s \geq 0, \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0, \mathcal{H}^s(F) = \infty\}.$$

Sauf sur des exemples simples, le calcul explicite de cette quantité peut s'avérer bien délicat. De façon générale, des bornes supérieure et inférieure peuvent être obtenues par des méthodes bien distinctes.

Dans un premier temps, les ensembles fractales aléatoires auxquels se sont intéressés de nombreux probabilistes sont l'image et le graphe de processus stochastiques ([110], [111] pour les travaux initiaux sur le mouvement brownien et [112],[123] et [124] pour des articles de synthèse plus ou moins récents sur la géométrie fractale de processus stochastiques).

**Définition 2.1:** Soient  $N \geq 1$  et  $\{\mathbb{X}_t, t \in \mathbb{R}_+\}$  un processus multidimensionnel à valeurs dans  $\mathbb{R}^N$ . Soit  $F \subset \mathbb{R}_+$ . On définit l'image et le graphe de  $\{\mathbb{X}_t\}$  sur  $F$  respectivement par:

$$\begin{aligned} R_F(\mathbb{X}) &= \{\mathbb{X}_t, t \in F\} \subset \mathbb{R}^N, \\ Gr_F(\mathbb{X}) &= \{(t, \mathbb{X}_t), t \in F\} \subset \mathbb{R}^{N+1}. \end{aligned}$$

Le résultat suivant concernant le mouvement brownien fractionnaire  $N$ -mutlidimensionnel de paramètre de Hurst  $H \in (0, 1)$  a été démontré par H. McKean dans [79] pour  $H = 1/2$  et par J-P. Kahane dans [54] pour  $H$  quelconque dans  $(0, 1)$ .

**Théorème 2.2:** Soit  $\{\mathbb{B}_t = (B_t^{1,H}, \dots, B_t^{N,H}), t \in \mathbb{R}_+\}$  un mouvement brownien fractionnaire  $N$ -dimensionnel de paramètre de Hurst  $H$ . Soit  $F \subset \mathbb{R}_+$ . On a, presque-sûrement:

$$\begin{aligned} \dim_{\mathcal{H}}(R_F(\mathbb{B})) &= \min \left\{ N, \frac{1}{H} \dim_{\mathcal{H}}(F) \right\}, \\ \dim_{\mathcal{H}}(Gr_F(\mathbb{B})) &= \min \left\{ \frac{1}{H} \dim_{\mathcal{H}}(F), \dim_{\mathcal{H}}(F) + (1 - H)N \right\}. \end{aligned}$$

En outre, un résultat beaucoup plus général concernant les processus multidimensionnels dont les composantes sont des copies indépendantes d'un processus auto-similaire et à accroissements stationnaires fut démontré par Y. Xiao et L. Huoan dans [122].

**Théorème 2.3:** Soit  $\{\mathbb{X}_t = (X_t^1, \dots, X_t^N), t \in \mathbb{R}_+\}$  tel que, pour tout  $i \in \{1, \dots, N\}$ ,  $\{X_t^i\}$  soit un processus auto-similaire de paramètre  $H_i$  et à accroissements stationnaires. On suppose que  $0 < H_1 \leq \dots \leq H_N \leq 1$  et que:

$$\begin{aligned} \forall i \in \{1, \dots, N\}, \exists p_i > \frac{1}{H_i}, \mathbb{E}[|X_1^i|^{p_i}] < +\infty, \\ \exists K > 0, \forall i \in \{1, \dots, N\}, \mathbb{P}(|X_1^i| \leq x) \leq Kx. \end{aligned}$$

Alors, on a, pour tout  $F \subset \mathbb{R}_+$ , presque-sûrement:

$$\begin{aligned} \dim_{\mathcal{H}}(R_F(\mathbb{X})) &= \min \left\{ N, \frac{\dim_{\mathcal{H}}(F) + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right\}, \\ \dim_{\mathcal{H}}(Gr_F(\mathbb{X})) &= \min \left\{ \frac{\dim_{\mathcal{H}}(F) + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N, \dim_{\mathcal{H}}(F) + \sum_{j=1}^N (1 - H_j) \right\}. \end{aligned}$$

Il est important de noter que la première hypothèse, portant sur la finitude de certains moments des variables aléatoires  $X_1^i$ , permet d'obtenir la borne supérieure des égalités précédentes. De même, la seconde hypothèse permet d'obtenir la borne inférieure.

Ainsi, si les composantes de  $\{\mathbb{X}_t\}$  sont des copies indépendantes d'un processus  $H_i$ -autosimilaire et à accroissements stationnaires représenté par une intégrale multiple de Wiener-Itô d'ordre  $d \geq 1$ , la borne supérieure pour l'image et le graphe de  $\{\mathbb{X}_t\}$  sur  $F$  est valable. En effet, comme pour les variables aléatoires gaussiennes, chaque  $X_1^i$  admet des moments de tout ordre. En revanche, la seconde hypothèse s'avère beaucoup plus difficile à vérifier. En effet, sauf pour  $d = 1$  et pour le cas de la distribution de Rosenblatt (loi de la variable aléatoire représentée par un processus de Rosenblatt au temps 1), une telle borne n'est a priori pas établie. Néanmoins, pour le processus de Rosenblatt  $N$ -multidimensionnel, le résultat précédent s'applique donc. Enfin, dans l'article [103], un résultat concernant la dimension de packing (autre dimension fractale complémentaire de celle de Hausdorff) de l'image du processus de Rosenblatt  $N$ -multidimensionnel sur  $F$  est démontré.

### 3 Analyse stochastique.

L'analyse stochastique est un domaine relativement récent regroupant de nombreuses théories liées à l'étude des processus stochastiques. Depuis la fin des années 70, elle a rencontré un développement particulièrement important, notamment grâce aux travaux de P. Malliavin (voir [72] et [73]) et à l'introduction par celui-ci du calcul différentiel qui porte maintenant son nom. En effet, le calcul de Malliavin est un calcul différentiel en dimension infinie permettant une extension dans ce contexte des notions de dérivation, d'espaces de Sobolev et d'intégration. L'exemple paradigmatique d'un tel calcul est celui associé à l'espace de Wiener standard, à savoir  $(C_0([0, 1], \mathbb{R}), H_0^1([0, 1]), \mathcal{W})$ , où  $C_0([0, 1], \mathbb{R})$  désigne l'ensemble des fonctions continues sur  $[0, 1]$ , à valeurs dans  $\mathbb{R}$  et s'annulant en 0,  $H_0^1([0, 1])$  l'ensemble des fonctions absolument continues sur  $[0, 1]$ , s'annulant en 0 et dont la dérivée faible est de carré sommable sur  $[0, 1]$  et  $\mathcal{W}$  la mesure de Wiener sur  $C_0([0, 1], \mathbb{R})$ . Le calcul de Malliavin possède de nombreuses applications en mathématiques ([87], [84]) et constitue un champ encore très actif de recherche. En particulier, il présente des liens étroits avec la théorie de l'intégration stochastique. En effet, depuis les travaux de B. Gaveau et P. Trauber ([38]), les outils du calcul de Malliavin se sont avérés efficaces afin de définir et d'étudier certaines extensions de l'intégrale stochastique par rapport au mouvement brownien ([89], [88]). De façon précise, cet article établit la connexion entre l'intégrale de Skorohod ([104]) et l'opérateur adjoint de la dérivation stochastique associée à l'espace de Wiener standard.

Simultanément aux travaux de P. Malliavin, T. Hida (voir [44] et [43]) a développé un calcul différentiel en dimension infinie appelé le calcul du bruit blanc. Celui-ci possède de nombreuses similarités avec le calcul de Malliavin mais présente néanmoins une différence de points de vue. En effet, l'objet central de cette théorie n'est plus le mouvement brownien, contrairement au calcul de Malliavin, mais sa "dérivée", le bruit blanc. Ainsi, l'espace sous jacent de cette théorie est le triplet suivant  $(S'(\mathbb{R}), L^2(\mathbb{R}), \mu)$ , où  $S'(\mathbb{R})$  désigne l'ensemble des distributions tempérées à valeurs dans  $\mathbb{R}$ ,  $L^2(\mathbb{R})$  l'ensemble des fonctions de carré sommable sur  $\mathbb{R}$  et  $\mu$  la mesure gaussienne sur l'espace conucléaire  $S'(\mathbb{R})$ , appelée mesure bruit blanc. De plus, dans ce cadre théorique, I. Kubo et S. Takenaka ([59]) ont construit une extension de l'intégrale stochastique par rapport au mouvement brownien qui coïncide, dans certains cas, avec l'extension introduite par A. Skorohod et reliée aux opérateurs du calcul de Malliavin.

Avec la popularisation du mouvement brownien fractionnaire, des questions relatives à un calcul stochastique par rapport à celui-ci se sont alors naturellement posées. En effet, sauf pour  $H = 1/2$ , un mBf d'indice de Hurst,  $H$ , n'est ni une semimartingale ni un processus de Markov. Ainsi, la théorie classique de l'intégrale stochastique ne s'applique pas au mBf et de nouveaux outils ont dû être développés afin de construire une intégrale par rapport à celui-ci. De façon naturelle, plusieurs mathématiciens se sont alors tournés vers le calcul différentiel en dimension infinie, que ce soit celui de Malliavin ou celui de Hida, pour répondre à ce problème.

En particulier, dans l'article [30], les auteurs ont initié l'analyse stochastique du mBf à l'aide du calcul de Malliavin sur l'espace de Wiener naturellement associé à ce processus, noté  $(C_0([0, 1], \mathbb{R}), \mathcal{H}_H, \mathcal{W}_H)$ . Ils ont alors défini deux types d'intégrales stochastiques par rapport au mBf à l'aide de l'opérateur adjoint de la dérivation stochastique sur cet espace. Pour  $H > 1/2$  et pour chacune des deux intégrales, ils ont entre autre obtenu les formules d'Itô respectives pour des fonctionnelles  $C^2(\mathbb{R})$  à dérivées bornées de processus d'Itô "fractionnaires".

Suite à ces résultats partiels, une pléthore d'articles a vu le jour sur l'intégrale stochastique par rapport au mouvement brownien fractionnaire, utilisant le calcul de Malliavin (par exemple [3], [25], [4], [26]). Notamment, dans [3], les auteurs ont développé un calcul stochastique pour une large classe de processus gaussiens (dits de Volterra) représentés par:

$$W_t = \int_0^t K(t, s) dB_s,$$

où  $K(., .)$  est un noyau déterministe vérifiant certaines propriétés et  $K(t, .)$  est de carré sommable sur  $[0, t]$ . L'idée fondamentale de cet article est de relier l'opérateur adjoint de la dérivation stochastique associé à  $\{W_t\}$  à celui de  $\{B_t\}$  grâce à la représentation précédente. Ainsi, l'intégrale stochastique par rapport à  $\{W_t\}$  d'un processus aléatoire se ramène à une transformation déterministe de ce processus et à une intégration stochastique par rapport à  $\{B_t\}$ . De plus, pour  $H > 1/4$  et pour toute fonction  $f \in C^2(\mathbb{R})$  telle que:

$$\exists c > 0, 0 < \lambda < \frac{1}{4T^{2H}}, \forall x \in \mathbb{R}, \max \{|f(x)|, |f'(x)|, |f''(x)|\} \leq c \exp(\lambda x^2),$$

la formule d'Itô suivante est vraie:

$$\forall t \in [0, T], f(B_t^H) - f(0) = \int_0^t f'(B_s^H) \delta B_s^H + H \int_0^t f''(B_s^H) s^{2H-1} ds.$$

Dans [21], C. Bender définit une intégrale stochastique par rapport au mouvement brownien fractionnaire (pour tout  $H \in (0, 1)$ ) à l'aide des outils du calcul du bruit blanc. De façon plus précise, l'auteur calcule la dérivée, au sens des distributions de Hida, du mouvement brownien fractionnaire. Ensuite, l'intégrale stochastique par rapport au mBf est alors définie à l'aide de cette dérivée, du produit de Wick et de l'intégrale au sens de Pettis. Ainsi, il obtient une formule d'Itô valable pour tout  $H \in (0, 1)$ , pour tout  $f \in S'(\mathbb{R})$  et pour tout  $0 < a \leq b$ , au sens des distributions de Hida:

$$f(B_b^H) - f(B_a^H) = \int_a^b f'(B_s^H) dB_s^H + H \int_a^b s^{2H-1} f''(B_s^H) ds.$$

Il est important de noter que, dans cet article, la représentation du mBf introduite par B. Mandelbrot et J. Van Ness ([75]) est utilisée alors que, dans [3], la représentation du mBf sur

un compact est employée. Le cas  $H = 1/2$  fut démontré par I. Kubo dans [58].

Enfin, dans [26], les auteurs définissent une intégrale stochastique par rapport au mBf à l'aide d'un opérateur adjoint étendu. Cela leur permet en retour d'obtenir une formule d'Itô pour  $H \in (0, 1/2)$ , pour  $f \in C^2(\mathbb{R})$  et même plus généralement, pour  $f$  convexe.

On note également qu'il existe d'autres approches pour traiter l'intégration stochastique par rapport au mouvement brownien fractionnaire valables suivant les valeurs de  $H$ . On renvoie pour cela à l'article de synthèse [27] et aux articles plus récents [115], [116] traitant du cas multidimensionnel.

L'état actuel de la recherche concernant le calcul stochastique par rapport aux extensions du mBf aux chaos gaussiens d'ordre supérieur à 1 est beaucoup moins avancé. Plusieurs mathématiciens se sont donc naturellement penchés sur la question de définir une intégrale stochastique par rapport aux processus de Hermite (introduits à la section 1). Une première interrogation dans cette direction est d'identifier la classe d'intégrandes déterministes pouvant être intégrée par rapport à ces processus. Les auteurs de [69] ont obtenu le résultat suivant:

**Définition-Théorème 3.1:** Soit  $\mathcal{H} = \{f : \mathbb{R} \rightarrow \mathbb{R}, \int_{\mathbb{R} \times \mathbb{R}} f(u)f(v)|u - v|^{2H-2}dudv < \infty\}$ . On note, pour tout  $f \in \mathcal{H}$ :

$$\|f\|_{\mathcal{H}}^2 = H(2H - 1) \int_{\mathbb{R} \times \mathbb{R}} f(u)f(v)|u - v|^{2H-2}dudv.$$

Alors il existe une isométrie,  $\tilde{X}^{H,d}$ , de  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  dans  $(L^2(\Omega, \mathcal{F}, \mathbb{P}), \|\cdot\|_{L^2(\Omega)})$  telle que:

$$\forall (a_1, \dots, a_p) \in \mathbb{R}^p, t_i < t_{i+1}, f(t) = \sum_{i=1}^p a_i \mathbb{I}_{(t_i, t_{i+1}]}(t), \tilde{X}^{H,d}(f) = \sum_{i=1}^p a_i (\tilde{X}_{t_{i+1}}^{H,d} - \tilde{X}_{t_i}^{H,d}),$$

où  $\tilde{X}_t^{H,d} = \int_{\mathbb{R}^d} h_t^{H,d}(x_1, \dots, x_d) dB_{x_1} \dots dB_{x_d}$ .

Ainsi, la classe d'intégrandes déterministes est identique à celle du mBf pour  $H \in (1/2, 1)$ . Cette classe contient notamment des distributions (voir [94], [52], [53]).

Concernant des intégrandes aléatoires, une analyse fine des intégrales stochastiques par rapport au processus de Rosenblatt a été initiée dans [113] par C. Tudor. Un rappel concernant les résultats obtenus dans cet article est de mise. Tout d'abord, puisque la variation quadratique des processus de Hermite est nulle ( $H > 1/2$ ) et puisque leurs trajectoires sont presque-sûrement  $H - \eta$  Höldériennes (pour tout  $0 < \eta < H$ ), l'auteur remarque que l'approche dite par "régularisation", développée par F. Russo et P. Vallois dans [97] (voir aussi [98]), peut être employée avec succès afin de définir une intégrale stochastique par rapport aux processus de Hermite pour des intégrandes aléatoires suffisamment réguliers. En particulier, pour tout  $f \in C^2(\mathbb{R})$ ,

on a la formule d'Itô-Stratonovich suivante:

$$\forall t \in \mathbb{R}_+, f(\tilde{X}_t^{H,d}) = f(\tilde{X}_0^{H,d}) + \int_0^t f'(\tilde{X}_s^{H,d}) d\tilde{X}_s^{H,d}.$$

Il est important de noter que l'intégrale stochastique précédente est définie de la façon suivante:

$$\int_0^t f'(\tilde{X}_s^{H,d}) d\tilde{X}_s^{H,d} = \lim_{\epsilon \rightarrow 0^+} \text{-ucp} \int_0^t f'(\tilde{X}_s^{H,d}) \frac{\tilde{X}_{s+\epsilon}^{H,d} - \tilde{X}_{s-\epsilon}^{H,d}}{2\epsilon} ds,$$

où ucp désigne l'uniforme convergence en probabilités sur  $[0, T]$ , pour tout  $T > 0$ . Néanmoins, une telle intégrale paraît difficile à manipuler d'un point de vue de la théorie des probabilités. En effet, un calcul explicite de sa moyenne et de sa variance semble ardu a priori. C'est pourquoi, dans [113], l'auteur a défini une intégrale stochastique de type "Skorohod" par rapport au processus de Rosenblatt. De façon plus précise, grâce à l'adjoint de la dérivée seconde stochastique associée au mouvement brownien et à l'aide d'une représentation intégrale du processus de Rosenblatt sous la forme d'une intégrale double de Wiener sur un compact, l'intégrale stochastique d'une certaine classe d'intégrandes aléatoires par rapport au processus de Rosenblatt est définie. Les propriétés de l'opérateur adjoint intervenant dans la définition de cette intégrale rendent alors possible une estimation de ses moments d'ordre  $p > 1$ . Une question alors naturelle est de savoir si un lien existe entre les deux intégrales stochastiques ainsi définies pour une certaine classe d'intégrandes. Le théorème 2 de [113] présente un tel résultat. La différence fondamentale avec les résultats connus pour le mBf pour  $H > 1/2$  (voir [4]) est l'apparition d'un terme, dit de "trace", supplémentaire. Pour le processus de Rosenblatt, ces termes ne sont pas explicites ni leurs conditions d'existence. De plus, une formule d'Itô faisant intervenir l'intégrale de type Skorohod par rapport au processus de Rosenblatt est obtenue. Pour tout  $f \in C^2(\mathbb{R})$  et pour tout  $t \in [0, T]$ , celle-ci s'écrit:

$$f(\tilde{X}_t^{H,2}) = f(\tilde{X}_0^{H,2}) + \int_0^t f'(\tilde{X}_s^{H,2}) \delta \tilde{X}_s^{H,2} + H \int_0^t f''(\tilde{X}_s^{H,2}) s^{2H-1} ds + N_t,$$

où  $\{N_t\}$  est un processus limite explicite uniquement pour des instances particulières de  $f$ , à savoir  $x^2$  et  $x^3$  (voir théorème 4 de [113]). La remarque fondamentale concluant cet article (remarque 8 de [113]) mentionne l'apparition, pour  $f(x) = x^3$ , de la dérivée troisième de  $f$  ainsi que de certains cumulants (d'ordre supérieur à 2) de la distribution de Rosenblatt dans le calcul de  $\{N_t\}$ .

## 4 Organisation et résultats principaux.

Cette thèse comporte trois chapitres. Ceux-ci peuvent se lire de façon indépendante les uns des autres. Le premier chapitre traite des propriétés géométriques des trajectoires de certains pro-

cessus auto-similaires et à accroissements stationnaires représentés par des intégrales multiples de Wiener-Itô d'ordre quelconque. De façon précise, dans la première partie de ce chapitre, une classe particulière de processus est considérée :

$$\forall t \in \mathbb{R}, X_t^\alpha = \int_{\mathbb{R}^d} \left[ \|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}} \right] dB_{x_1} \dots dB_{x_d}$$

où  $\mathbf{t}^* = (t, \dots, t)$ . Il est important de noter que cette classe n'appartient pas à la classe de processus pour laquelle les résultats de l'article [82] s'appliquent. A partir d'une représentation en ondelettes de ce processus (proposition 2.1 du chapitre 1), les résultats suivants sont obtenus:

**Proposition 4.1:** Il existe un espace de probabilité,  $\Omega^*$ , de mesure pleine, une variable aléatoire strictement positive  $A_d$  dont les moments de tout ordre sont finis et une constante,  $b_d > 1$ , tels que:

$$\forall \omega \in \Omega^* \quad \sup_{(s,t) \in [0,1]} \frac{|X_t^\alpha(\omega) - X_s^\alpha(\omega)|}{|t-s|^H (\log(b_d + |t-s|^{-1}))^{\frac{d}{2}}} \leq A_d(\omega).$$

**Proposition 4.2:** Il existe un espace de probabilité,  $\Omega^*$ , de mesure pleine, une variable aléatoire strictement positive  $B_d$  dont les moments de tout ordre sont finis et une constante  $c_d > 3$ , tels que:

$$\forall \omega \in \Omega^* \quad \sup_{t \in \mathbb{R}_+} \frac{|X_t^\alpha(\omega)|}{(1+|t|)^H (\log \log(c_d + |t|))^{\frac{d}{2}}} \leq B_d(\omega).$$

**Théorème 4.3:** Presque-sûrement,

$$\forall t \in (0, 1), \alpha_{X^\alpha}(t) = \tilde{\alpha}_{X^\alpha}(t) = H.$$

Il est important de noter que ce résultat est uniforme sur les trajectoires du processus  $\{X_t^\alpha\}$ . De plus, l'ordre du chaos n'apparaît pas dans la valeur des exposants ponctuel et local.

Dans la deuxième partie de ce chapitre, on s'intéresse à des processus  $N$ -multidimensionnels,  $\{\mathbb{Y}_t^H = (Y_t^{H_1}, \dots, Y_t^{H_N}), t \in \mathbb{R}_+\}$ , dont les composantes sont des copies indépendantes d'un processus auto-similaire et à accroissements stationnaires représenté par une intégrale multiple de Wiener-Itô d'ordre  $d \geq 1$  fixé. Chaque composante admet un exposant d'auto-similarité distinct. Ces exposants vérifient  $1/2 < H_1 < \dots < H_N < 1$ . Le résultat suivant concernant la dimension de Hausdorff du graphe et de l'image de ces processus multidimensionnels est obtenu.



**Théorème 4.4:** Soit  $F \in \mathbb{R}_+$ . Presque-sûrement,

$$\min \left( N; \frac{\dim_{\mathcal{H}} F + \frac{\sum_{j=1}^k (H_k - H_j)}{d}}{H_k}, k = 1, \dots, N \right) \leq \dim_{\mathcal{H}} R_F(\mathbb{Y}^H)$$

$$\leq \min \left( N; \frac{\dim_{\mathcal{H}} F + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right),$$

$$\min \left( \frac{\dim_{\mathcal{H}} F + \frac{\sum_{j=1}^k (H_k - H_j)}{d}}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} F + \sum_{i=1}^N \frac{(1 - H_i)}{d} \right) \leq \dim_{\mathcal{H}} Gr_F(\mathbb{Y}^H)$$

$$\leq \min \left( \frac{\dim_{\mathcal{H}} F + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} F + \sum_{i=1}^N (1 - H_i) \right).$$

Les deuxième et troisième chapitres de cette thèse sont indépendants du premier dans la mesure où ils s'intéressent à des questions relatives au calcul stochastique par rapport au processus de Rosenblatt. Dans le deuxième chapitre, en utilisant la théorie des distributions de Hida, une intégrale stochastique par rapport au processus de Rosenblatt est introduite dont la définition est motivée par celles de l'intégrale bruit blanc ([59, 60]) et de l'intégrale bruit fractionnaire ([21]). De façon précise, la dérivée du processus de Rosenblatt, au sens des distributions de Hida, est calculée (voir lemme 3.4 du chapitre 2). L'intégrale du bruit de Rosenblatt est alors définie par multiplication de Wick de l'intégrande avec cette dérivée et par intégration au sens de Pettis (voir définition-théorème 3.10 du chapitre 2). La formule d'Itô suivante est alors obtenue.

**Théorème 4.5:** Soit  $(a, b) \in \mathbb{R}_+^*$  tel que  $a \leq b < \infty$ . Soit  $F$  une fonction analytique entière de la variable complexe vérifiant:

$$\exists N \in \mathbb{N}, \exists C > 0, \forall z \in \mathbb{C} \quad |F(z)| \leq C(1 + |z|)^N \exp \left( \frac{1}{\sqrt{2}b^H} |\Im(z)| \right).$$

Alors, on a dans  $(S)^*$ :

$$F(\tilde{X}_b^{2,H}) - F(\tilde{X}_a^{2,H}) = \int_a^b F^{(1)}(\tilde{X}_t^{2,H}) \diamond \dot{X}_t^{2,H} dt + \sum_{k=2}^{\infty} \left( H \kappa_k(\tilde{X}_1^{2,H}) \int_a^b \frac{t^{Hk-1}}{(k-1)!} F^{(k)}(\tilde{X}_t^{2,H}) dt \right)$$

$$+ \sum_{k=2}^{\infty} \left( 2^{k-1} \int_a^b F^{(k)}(\tilde{X}_t^{2,H}) \diamond \dot{X}_t^{H,k} dt \right),$$

où,  $\diamond$  désigne le produit de Wick relatif au mouvement brownien,  $\kappa_k(\tilde{X}_1^{2,H})$  désigne le cumulants d'ordre  $k \geq 2$  de la distribution de Rosenblatt et  $\dot{X}_t^{H,k}$  la dérivée au sens des distributions de

Hida du processus  $\{X_t^{H,k}\}$  défini par:

$$X_t^{H,k} = \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{((h_t^{H,2} \otimes_1 h_t^{H,2}) \otimes_1 h_t^{H,2}) \dots \otimes_1 h_t^{H,2}}_{k-1 \times \otimes_1} (x_1, x_2) dB_{x_1} dB_{x_2}.$$

Le symbole  $\otimes_1$  désigne le produit tensoriel contracté d'ordre 1.

Il est important de noter l'apparition dans cette formule d'Itô des cumulants de tout ordre de la distribution de Rosenblatt ainsi que des dérivées de tout ordre de la fonction  $F$ . Cette formule confirme les attentes exprimées dans l'article [113] à la remarque 8.

Dans la quatrième partie de ce chapitre, une comparaison est faite entre l'intégrale du bruit de Rosenblatt associée à une représentation du processus de Rosenblatt en intégrale double de Wiener-Itô sur un compact, notée  $\{Z_t^H, t \in [0, T]\}$ , et l'intégrale de type Skorohod définie dans l'article [113]. Le résultat est le suivant:

**Proposition 4.6:** Soit  $\{\Phi_t; t \in [0; T]\}$  un processus stochastique tel que  $\Phi \in L^2(\Omega; \mathcal{H}) \cap L^2([0, T]; \mathbb{D}^{2,2})$  et  $\mathbb{E}[\int_0^T \int_0^T \|D_{s_1, s_2}^2 \phi\|_{\mathcal{H}}^2 ds_1 ds_2] < \infty$ . Alors,  $\{\Phi_t\}$  est intégrable au sens de Skorohod et  $(S)^*$ -intégrable par rapport au processus de Rosenblatt,  $\{Z_t^H\}_{t \in [0; T]}$ , et on a:

$$\int_0^T \Phi_t \delta Z_t^H = \int_0^T \phi_t \diamond \dot{Z}_t^H dt.$$

$\mathbb{D}^{2,2}$  désigne l'espace de Sobolev-Watanabe-Kree d'ordre  $(2, 2)$  et  $D_{s_1, s_2}^2$  l'opérateur de dérivée seconde stochastique, tous deux associés à un mouvement brownien standard.

Enfin, la variance de l'intégrale stochastique par rapport au processus de Rosenblatt est calculée explicitement pour une certaine classe de processus intégrandes. Des conditions suffisantes sont imposées sur la classe d'intégrandes et notamment sur leur décomposition en chaos de Wiener. Afin d'énoncer précisément le résultat, quelques notations doivent être introduites. Un outil récurrent intervenant dans l'analyse stochastique associée au mouvement brownien fractionnaire est l'intégrale fractionnaire d'ordre  $\beta \in (0, 1)$  sur l'axe réel, dénotée  $I_+^\beta$ . Elle est définie comme suit sur l'espace de Schwartz des fonctions infiniment dérivables et à décroissance rapide:

$$\forall f \in S(\mathbb{R}), \forall x \in \mathbb{R}, I_+^\beta(f)(x) = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}} f(y)(x-y)_+^{\beta-1} dy.$$

Brièvement, on suppose que les noyaux de la décomposition en chaos de Wiener du processus intégrande  $\{\Phi_t, t \in (a, b)\}$  vérifient, pour tout  $m \geq 2$  et pour tout  $t$ :

$$\mathbf{H}_1 : \exists g_m(\cdot, t) \in L^2(\mathbb{R}^m), \mathcal{F}(f_m(\cdot, t))(\xi_1, \dots, \xi_m) = \frac{(-i\xi_{m-1})^{\frac{H}{2}} (-i\xi_m)^{\frac{H}{2}}}{\prod_{i=m-1}^m (1 + |\xi_i|^2)^{\frac{H}{4}}} \mathcal{F}(g_m(\cdot, t))(\xi_1, \dots, \xi_m),$$

où  $\Phi_t = \sum_{m=0}^{+\infty} I_m(f_m(\cdot, t))$  et  $(-i\xi)^{H/2} = |\xi|^{H/2} \exp(-iH \operatorname{sign}(\xi)\pi/4)$ . Le résultat s'énonce alors de la façon suivante:

**Théorème 4.7:** Soit  $(a, b) \subset \mathbb{R}_+$  tels que  $0 \leq a < b < \infty$ . Soit  $\{\Phi_t; t \in (a, b)\}$  un processus stochastique tel que pour tout  $t \in (a, b)$ ,  $\Phi_t \in (L^2)$ . De plus, on suppose que:

$$\forall t \in (a, b), \forall m \geq 2, f_m(\cdot, t) \text{ vérifie } \mathbf{H1},$$

$$\sum_{m=2}^{+\infty} (m+2)! \left( \int_{(a,b)} \|g_m(\cdot; t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}} < +\infty,$$

où  $q = \frac{1}{H}(1 + \epsilon)$  pour un certain  $\epsilon \in (\frac{1-H}{3H-1} \vee (2H-1); 1)$ . Alors, on a:

$$\begin{aligned} \mathbb{E} \left[ \left( \int_{(a,b)} \Phi_t \diamond \dot{X}_t^{2,H} dt \right)^2 \right] &= H(2H-1) \int_{(a,b) \times (a,b)} |t-s|^{2(H-1)} \mathbb{E}[\Phi_t \Phi_s] ds dt \\ &+ 4d(H) \sqrt{\frac{H(2H-1)}{2}} \int_{(a,b) \times (a,b)} |t-s|^{H-1} \mathbb{E}[\tilde{D}_{\delta_s \circ I_+^{\frac{H}{2}}}(\Phi_t) \tilde{D}_{\delta_t \circ I_+^{\frac{H}{2}}}(\Phi_s)] ds dt \\ &+ d(H)^2 \int_{(a,b) \times (a,b)} \mathbb{E}[(\tilde{D}_{\delta_s \circ I_+^{\frac{H}{2}}})^2(\Phi_t) (\tilde{D}_{\delta_t \circ I_+^{\frac{H}{2}}})^2(\Phi_s)] ds dt, \end{aligned}$$

où  $d(H) > 0$  est une constante connue explicitement et  $\tilde{D}_{\delta_s \circ I_+^{H/2}}(\Phi_t)$  et  $(\tilde{D}_{\delta_s \circ I_+^{H/2}})^2(\Phi_t)$  deux éléments de  $L^2((a, b) \times (a, b)) \otimes (L^2)$  coïncidant avec  $D_{\delta_s \circ I_+^{H/2}}(\Phi_t)$  et  $(D_{\delta_s \circ I_+^{H/2}})^2(\Phi_t)$  pour  $\{\Phi_t\}$  plus réguliers.

Il est important de remarquer l'apparition d'un terme faisant intervenir une dérivée seconde stochastique dans la formule précédente, ce qui diffère significativement du cas gaussien.

Dans le dernier chapitre de cette thèse, on étudie le lien entre les intégrales définies par régularisation par rapport au mouvement brownien fractionnaire et par rapport au processus de Rosenblatt et les intégrales bruit fractionnaire et bruit de Rosenblatt. Pour ce faire, on combine des outils issus de la théorie du bruit blanc et du calcul de Malliavin. En particulier, en utilisant les espaces de Sobolev-Watanabe-Kree, notés  $(\mathcal{W}^{r,2})$  pour  $r \geq 0$  dans ce chapitre, on définit des familles d'opérateurs sur ces espaces associées au mBf et au processus de Rosenblatt (voir propositions 2.4 et 2.6 du chapitre 3). Ceux-ci sont notés  $\nabla^{H-1/2}$  pour le mBf et  $\nabla^{H/2}$ ,  $\nabla^{(2),H/2}$  pour le processus de Rosenblatt. Le résultat concernant le mouvement brownien fractionnaire est alors le suivant:

**Proposition 4.8 :** Soit  $\{\Phi_t : t \in (a, b)\}$  un processus stochastique tel que pour tout  $t \in (a, b)$ ,  $\Phi_t \in (\mathcal{W}^{1,2})$ . De plus, on suppose que:

$$\int_a^b \|\Phi_t\|_{(L^2)}^2 dt < +\infty,$$

$$\int_a^b \|\nabla^{H-\frac{1}{2}}(\Phi_t)\|_{L^2(a,b)\otimes(L^2)}^2 dt < +\infty.$$

Alors, on a:

$$\int_a^b \Phi_t \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt \stackrel{(S)^*}{=} \int_a^b \Phi_t \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt + A(H) \int_a^b \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s) \frac{ds}{\epsilon} dt,$$

où  $A(H)$  est une constante strictement positive connue explicitement.

En combinant alors une formule de dérivation des fonctions composées pour l'opérateur  $\nabla^{H-1/2}$  avec certaines fonctionnelles du mBf, on retrouve le résultat bien connu suivant:

**Proposition 4.9:** Soit  $F \in C^1(\mathbb{R})$ . On suppose que pour tout  $t \in (a, b)$ ,  $F(B_t^H) \in (\mathcal{W}^{1,2})$  et que:

$$\forall x \in \mathbb{R}, \max\{F(x), F'(x)\} \leq C e^{\lambda x^2},$$

pour  $C > 0$  et  $\lambda > 0$  tel que  $\lambda < 1/(4b^{2H})$ . Alors, on a:

$$\int_a^b F(B_t^H) d^- B_t^H = (\nabla^{H-\frac{1}{2}})^*(F(B_t^H)) + H \int_a^b t^{2H-1} F'(B_t^H) dt,$$

où l'égalité est vraie dans  $(L^2)$  et presque-sûrement.

Concernant le processus de Rosenblatt, comme dans [113], on note l'apparition d'un second terme de trace. Celui-ci fait intervenir l'opérateur  $\nabla^{(2),H/2}$  et notamment sa restriction à la diagonale de  $(a, b) \times (a, b)$ . Celle-ci n'existe pas nécessairement pour les éléments de  $(\mathcal{W}^{2,2})$ . En effet, pour tout  $\Phi \in (\mathcal{W}^{2,2})$ ,  $\nabla^{(2),H/2}(\Phi)$  appartient à  $L^2((a, b) \times (a, b)) \otimes (L^2)$  (voir proposition 2.6 du chapitre 3). Ainsi, une hypothèse de même nature que  $\mathbf{H}_1$  est suffisante afin de définir une telle restriction, notée  $\tilde{\nabla}^{(2),H/2}$ . Le résultat est alors le suivant:

**Proposition 4.10:** Soit  $\{\Phi_t : t \in (a, b)\}$  satisfaisant les conditions (1), (2) and (3) de la proposition 3.2.4. De plus, on suppose que:

$$\begin{aligned} \int_a^b \|\Phi_t\|_{(L^2)}^2 dt &< \infty, \\ \int_a^b \|\nabla^{\frac{H}{2}}(\Phi_t)\|_{L^2(a,b)\otimes(L^2)}^2 dt &< \infty, \\ \int_a^b \|\tilde{\nabla}^{(2),H/2}(\Phi_t)\|_{L^2((a,b))\otimes(L^2)}^2 dt &< \infty. \end{aligned}$$

Alors,  $\Phi_t \frac{\tilde{X}_{t+\epsilon}^{2,H} - \tilde{X}_t^{2,H}}{\epsilon}$  est  $(S)^*$ -intégrable et on a :

$$\begin{aligned} \int_a^b \Phi_t \frac{\tilde{X}_{t+\epsilon}^{2,H} - \tilde{X}_t^{2,H}}{\epsilon} dt &\stackrel{(S)^*}{=} \int_a^b \Phi_t \diamond \frac{\tilde{X}_{t+\epsilon}^{2,H} - \tilde{X}_t^{2,H}}{\epsilon} dt + 2d(H) \int_a^b \left( \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s,)) \frac{ds}{\epsilon} \right) dt \\ &\quad + d(H) \int_a^b \left( \int_t^{t+\epsilon} \tilde{\nabla}^{(2), \frac{H}{2}}(\Phi_t)(s, s, ) \frac{ds}{\epsilon} \right) dt. \end{aligned}$$

Il est important de noter le caractère algébrique de cette égalité analogue à une identité remarquable. En effet, le premier terme fait intervenir  $(D_{\delta_s \circ I_+^{\frac{H}{2}}}^*)^2$ , le second s'apparente à un terme croisé  $D_{\delta_s \circ I_+^{\frac{H}{2}}}^* \circ \nabla^{H/2}$  et le dernier fait intervenir  $\tilde{\nabla}^{(2), H/2}$ . Ce caractère algébrique est issu de la formule multiplicative du calcul de Malliavin (Lemma 3.11 du Chapitre 2) utilisée dans la preuve de cette proposition.

Enfin, pour  $\Phi_t = F(\tilde{X}_t^{2,H})$ , avec  $F$  une fonction infiniment dérivable sur  $\mathbb{R}$  telle que  $F$  et toutes ses dérivées soient à croissance au plus polynomiale, un calcul explicite des termes de trace apparaissant à la proposition précédente est réalisé. En particulier, on obtient :

**Proposition 4.11:** On a dans  $(S)^*$  :

$$\begin{aligned} &\int_a^b \left( \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(F(\tilde{X}_t^{2,H}))(s, .)) \frac{ds}{\epsilon} \right) dt \xrightarrow{\epsilon \rightarrow 0^+} \\ &\frac{2}{(\Gamma(\frac{H}{2}))^2} \sqrt{\frac{H(2H-1)}{2}} (\nabla)^* \left( \int_a^b (t - .)_+^{\frac{H}{2}-1} I_1(l_{t,t}^H) F'(\tilde{X}_t^{2,H}) dt \right). \end{aligned}$$

où  $I_1(l_{t,t}^H) = \int_{\mathbb{R}} \left( \int_0^t (u-x)_+^{H/2-1} |t-u|^{H-1} du \right) dB_x$ . Et, on a presque-sûrement :

$$\begin{aligned} &d(H) \int_a^b \left( \int_t^{t+\epsilon} \nabla^{(2), \frac{H}{2}}(F(\tilde{X}_t^{2,H}))(s, s, ) \frac{ds}{\epsilon} \right) dt \xrightarrow{\epsilon \rightarrow 0^+} \\ &H \int_a^b t^{2H-1} F'(\tilde{X}_t^{2,H}) dt + \frac{H}{2} \kappa_3(\tilde{X}_1^{2,H}) \int_a^b t^{3H-1} F^{(2)}(\tilde{X}_t^{2,H}) dt \\ &+ \frac{2d(H)}{(\Gamma(\frac{H}{2}))^2} H(2H-1) \int_a^b I_2(e_{t,t}^H(., *)) F^{(2)}(\tilde{X}_t^{2,H}) dt. \end{aligned}$$

où  $I_2(e_{t,t}^H(., *)) = I_2(\int_0^t \int_0^t (u - .)_+^{\frac{H}{2}-1} (v - *)_+^{\frac{H}{2}-1} |u-t|^{H-1} |v-t|^{H-1} dudv)$ .

Il est important de noter que les résultats de ce chapitre sont obtenus à l'aide des représentations du mouvement brownien fractionnaire et du processus de Rosenblatt en intégrale de Wiener (simple et double) issues du théorème de la limite non-centrale 1.6 et non à l'aide des représentations respectives sur un compact.

## Publications et prépublications issues de la thèse

- Benjamin Arras. On a class of self-similar processes with stationary increments in higher order Wiener chaoses. *Stochastic Process. Appl.*, vol. 124, nu. 7, 2415-2441, 2014.
- Benjamin Arras. A white noise approach to stochastic integration with respect to the Rosenblatt process. preprint, 2013. [arXiv:1308.1835](https://arxiv.org/abs/1308.1835)

# Chapter 1

## On a class of self-similar processes with stationary increments in higher order Wiener chaoses.

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Self-similar processes with stationary increments (SSSI processes), *i.e.* processes  $X$  which satisfy:

$$\begin{aligned}\forall c > 0 \quad \{X_{ct} : t \in \mathbb{R}_+\} &\stackrel{(d)}{=} \{c^H X_t : t \in \mathbb{R}_+\} \\ \forall h > 0 \quad \{X_{t+h} - X_h : t \in \mathbb{R}_+\} &\stackrel{(d)}{=} \{X_t : t \in \mathbb{R}_+\}\end{aligned}$$

for some positive  $H$ , have been studied for a long time due to their importance both in theory and in practice. Such processes appear as limits in various normalisation procedures [63, 101, 109]. In addition, they are the only possible tangent processes [37]. In applications, they occur in various fields such as hydrology, biomedicine and image processing. The simplest SSSI processes are simply Brownian motion and, more generally, Lévy stable motions. Apart from these cases, the best known such process is probably fractional Brownian motion (fBm), which was introduced in [57] and popularized by [75]. A construction of SSSI processes that generalizes fBm to higher order Wiener chaoses was proposed in [82]. These processes read

$$\forall t \in \mathbb{R}_+ \quad X_t = \int_{\mathbb{R}^d} h_t(x_1, \dots, x_d) dB_{x_1} \dots dB_{x_d}$$

where  $\{B_x : x \in \mathbb{R}\}$  is a two-sided Brownian motion and where  $h_t$  satisfies:

1.  $h_t \in \hat{L}^2(\mathbb{R}^d)$ , where  $\hat{L}^2(\mathbb{R}^d)$  denotes the space of square-integrable symmetric functions,
2.  $\forall c > 0, \quad h_{ct}(cx_1, \dots, cx_d) = c^{H-\frac{d}{2}} h_t(x_1, \dots, x_d)$ ,
3.  $\forall \rho \geq 0, \quad h_{t+\rho}(x_1, \dots, x_d) - h_t(x_1, \dots, x_d) = h_\rho(x_1 - t, \dots, x_d - t)$ .

Properties 2. and 3. ensure self-similarity and stationarity of the increments of the process. Among the kernels satisfying the above properties, [82] considered:

- $\int_0^t \|\mathbf{s}^* - \mathbf{x}\|_2^{H-\frac{d}{2}-1} ds,$
- $\int_0^t \prod_{j=1}^d (s - x_j)_+^{-(\frac{1}{2} + \frac{1-H}{d})} ds.$

where  $\mathbf{s}^* = (s, \dots, s)$ ,  $\|\cdot\|_2$  is the euclidian norm in  $\mathbb{R}^d$  and  $(x)_+ = \max(0, x)$ .

The second kernel defines a class of processes called *Hermite processes*, which have been and still are the subject of considerable interest [32, 109, 2, 93, 113]. For  $d = 1$ , one recovers fBm (the only SSSI Gaussian process). The process obtained with  $d = 2$  is called the *Rosenblatt* process.

When  $d > 1$ , the processes in the general class defined in [82] are somewhat difficult to analyse, because they are no longer Gaussian. In recent years, the family of Hermite processes, and specially the Rosenblatt one, has been studied in depth from the point of views of estimation [18], stochastic calculus [113], distributional properties [117], wavelet-like decomposition for  $d = 2$  [93], and more. Wavelet-like decompositions, in particular, are useful for investigating the local properties and providing synthesis algorithms.

General results on sample paths properties of ergodic self-similar processes, such as local or pointwise regularity or Hausdorff dimensions were obtained in [106]. These results apply to SSSI processes belonging to higher order Wiener chaoses. Namely, at any  $t \in \mathbb{R}_+$ , almost surely, the pointwise and local Hölder exponents are equal to  $H$  (see section 4 for a definition). Nevertheless, in the case of fBm, a lot more is known, both in the one and multidimensional cases [92, 33, 107, 11, 9]. However, in the non-Gaussian case  $d > 1$ , there is still room for improvements. For instance, almost sure pointwise Hölder exponent at any point is not known for SSSI processes belonging to Wiener chaoses of order  $d \geq 2$ . Of course, by stationarity of increments, self-similarity and the finiteness of every moments of  $X_1$ , one has:

$$\forall p > 1, \quad \mathbb{E}[|X_t - X_s|^p] = |t - s|^{pH} \mathbb{E}[(X_1)^p].$$

By Kolmogorov's lemma,  $X$  has a modification whose sample paths are Hölder-continuous of all exponents smaller than  $H$  (we will always work with such a version in the following), but obtaining the exact almost sure pointwise Hölder exponent at any point is more difficult.

The aim of this article is to study a class of SSSI processes obtained by considering a particular kernel  $h$  satisfying conditions 1., 2., 3. above. The interest of this class is that it allows one to obtain a wavelet-type expansion for all values of  $d$  (and not only for  $d = 1$  and  $d = 2$  as it is the case for Hermite processes). This expansion permits in turn to deduce sharp local regularity results on the sample paths. Our class of processes is defined by the following multiple



Wiener-Itô integral representation:

$$X_t^\alpha = \int_{\mathbb{R}^d} \left[ \|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}} \right] dB_{x_1} \dots dB_{x_d} \quad (1.1)$$

where  $t \in [0, 1]$ ,  $\mathbf{t}^* = (t, \dots, t)$  and  $\alpha = H - 1 + \frac{d}{2}$ . When  $d = 1$ ,  $\{X_t^\alpha\}$  is a fBm. One can think of this class as a natural extension of fractional Brownian motion to higher order Wiener chaoses. Indeed, this class of kernels appears in the definition of the Lévy fractional Brownian field (see [101], chapter 8).

The main results of this article are propositions 3.2 and 3.3 and theorems 3.1 and 4.1. Propositions 3.2 and 3.3 are concerned respectively with the uniform modulus of continuity and the asymptotic behaviour at infinity of the process  $\{X_t^\alpha\}$ . Due to the self-similarity and the stationarity of the increments, the exponent of the power function controlling these behaviours is  $H$ . Nevertheless, due to the non-Gaussianity of Wiener chaoses random variables for  $d \geq 2$ , the sample paths of the process  $\{X_t^\alpha\}$  differ significantly from the ones of fBm since the logarithmic correction is to the power of  $\frac{d}{2}$ . Moreover, theorem 3.1 gives us the almost sure pointwise and local Hölder exponents at any points. It extends a well-known result regarding fBm, namely that, almost surely, for any  $t \in (0, 1)$ , the pointwise and local Hölder exponents at  $t$  are equal to  $H$ . Nevertheless, the techniques in order to obtain this result for the considered class of processes are somewhat more involved since the collection of random variables used to define the wavelet-type expansion of  $\{X_t^\alpha\}$  is not a collection of independent standard normal random variables (for  $d > 1$ ) anymore (see the remark after proposition 2.1). Lemma 3.3 is the key to obtain such a result. It is based on a geometric condition which lets us obtain independence and the Carbery and Wright inequality ([24]). This inequality provides us an estimate for the small ball probability of a random variable from the collection previously mentioned. It appears in the study of the convergence in law for sequences of functionals on the Wiener space (see [85]). Theorem 4.1 is concerned with Hausdorff dimension of the range and the graph of general anisotropic multidimensional processes whose independent components are copies, with different self-similarity exponents, of a SSSI process belonging to a fixed Wiener chaos. Lower bounds and upper bounds are obtained using methods similar to those of [122]. Moreover, regarding the lower bounds, Theorem 2.3 of [122] can not be applied directly to the multidimensional processes under consideration. Indeed, to our knowledge, it is not known (except for the Rosenblatt distribution) whether or not a random variable, represented by a SSSI process, at a fixed time, belonging to a Wiener chaos, admits a continuous and bounded density (see the remark after proposition 4.1). Once again, using an approximation lemma based on the Carbery and Wright inequality (see lemma 4.2), we obtain lower bounds for the Hausdorff dimensions.

The remaining of this work is organized as follows. In the next section, we define precisely our process. In Section 2, we prove the existence of a modification of  $\{X_t^\alpha\}$  which is expressed

as an almost-surely absolutely convergent wavelet-type series. Section 3 studies the pointwise and local Hölder regularity of sample paths and the behaviour at infinity of the process. Finally, we provide in Section 4 general results on the Hausdorff dimension of the range and graphs of multidimensional anisotropic SSSI processes defined by multiple Wiener-Itô integrals.

## 1 Definitions and notations

In this section, we define the class of processes,  $\{X_t^\alpha\}$ , by means of the multiple Wiener-Itô integrals and the Riesz kernel. First of all, let us define the multiple Wiener-Itô integrals. We refer the reader to chapter 1.1.2 of [87]. The multiple Wiener-Itô integrals, denoted by  $I_d$ , is a linear continuous application from  $\hat{L}^2(\mathbb{R}^d)$  to  $L^2(\Omega, \mathcal{G}, \mathbb{P})$ , where  $\mathcal{G}$  is the sigma-field generated by the two-sided Brownian motion. Moreover, it satisfies the following properties:

- For all  $f \in L^2(\mathbb{R}^d)$ ,  $I_d(f) = I_d(\hat{f})$ ,
- For all  $f \in L^2(\mathbb{R}^d)$ ,  $\mathbb{E}[I_d(f)] = 0$ ,
- For all  $f \in L^2(\mathbb{R}^p)$  and  $g \in L^2(\mathbb{R}^q)$ ,

$$\mathbb{E}[I_p(f)I_q(g)] = \begin{cases} 0 & p \neq q \\ p! \langle \hat{f}; \hat{g} \rangle_{L^2(\mathbb{R}^p)} & p = q \end{cases}$$

where  $\hat{f}(x_1, \dots, x_d) = \frac{1}{d!} \sum_{\sigma \in \mathcal{S}_d} f(x_{\sigma(1)}, \dots, x_{\sigma(d)})$  and  $\mathcal{S}_d$  is the set of permutations of  $\{1, \dots, d\}$ .

Let us define the Riesz kernel:

$$k_\alpha(\mathbf{x}) = \frac{1}{\gamma_d(\alpha)} \begin{cases} \|\mathbf{x}\|_2^{\alpha-d} & \alpha - d \neq 0, 2, 4, 6 \\ \|\mathbf{x}\|_2^{\alpha-d} \ln \frac{1}{\|\mathbf{x}\|_2} & \alpha - d = 0, 2, 4, 6 \end{cases}$$

with:

$$\gamma_d(\alpha) = \begin{cases} 2^\alpha \pi^{\frac{d}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})} & \alpha \neq d + 2k, \quad \alpha \neq -2k \\ 1 & \alpha = -2k \\ (-1)^{\frac{d-2}{2}} \pi^{\frac{d}{2}} 2^{\alpha-1} \left(\frac{\alpha-d}{2}\right)! \Gamma(\frac{\alpha}{2}) & \alpha = d + 2k \end{cases}$$

where  $k \in \mathbb{N}$  and  $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ . This kernel occurs in the definition of the Riesz potential which will be used below. Let  $H \in (\frac{1}{2}, 1)$  and set  $\alpha = H + \frac{d}{2} - 1$  which is, thus, in  $\mathbb{R} \setminus \mathbb{N}$ . Consider the following kernel:

$$h_t^\alpha(x_1, \dots, x_d) = k_{\alpha+1}(\mathbf{t}^* - \mathbf{x}) - k_{\alpha+1}(\mathbf{x})$$

where  $\mathbf{t}^* = (t, \dots, t)$ . The kernel  $h_t^\alpha$  is clearly symmetric and it is classical to show that it is a square-integrable function. Moreover, simple calculations show that Properties 2. and 3. are also verified. Thus the process:

$$\forall t \in [0, 1] \quad X_t^\alpha = I_d(h_t^\alpha),$$

is an SSSI process by [82].

Self-similarity and increments stationarity entail that

$$\forall (s, t) \in [0, 1]^2, \quad \mathbb{E}[X_t^\alpha X_s^\alpha] = \frac{1}{2} \mathbb{E}[(X_1^\alpha)^2] [|t|^{2H} + |s|^{2H} - |t - s|^{2H}],$$

*i.e.*  $X_t^\alpha$  has same covariance structure as fBm. In particular, it displays long range dependence. Note that, as mentioned above, Kolmogorov's lemma implies that the process has a modification whose sample paths are almost surely continuous. We will always work with this modification in the sequel. Using a classical patching argument,  $X^\alpha$  can then be defined on the whole of  $\mathbb{R}_+$ .

## 2 Wavelet decomposition

Wavelet decompositions of stochastic processes are a powerful tool for the study of sample paths properties such as regularity. They have been used in particular in [16] and [14] respectively for the cases of fractional Brownian sheet and linear fractional stable sheet. Following the method in these articles, we shall obtain an almost sure wavelet-type expansion of  $\{X_t^\alpha\}$ .

In that view, we use a specific wavelet basis of  $L^2(\mathbb{R}^d)$  (the reader is referred to [80] for an introduction to wavelet theory). For  $d = 1$ , we consider the Lemarié-Meyer wavelet basis, which satisfies the following property:

- the scaling function  $\phi$  and the mother wavelet  $\psi$  are in the Schwartz space,  $S(\mathbb{R})$ .
- $\text{supp}(\mathcal{F}(\phi)) \subset \{\xi \in \mathbb{R} : |\xi| \leq \frac{4\pi}{3}\}$  and  $\text{supp}(\mathcal{F}(\psi)) \subset \{\xi \in \mathbb{R} : \frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}\}$ , where  $\mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}} \phi(x) e^{-ix\xi} dx$ .
- $\{2^{\frac{j}{2}} \psi(2^j x - k) : (j, k) \in \mathbb{Z} \times \mathbb{Z}\}$  is an orthonormal basis of  $L^2(\mathbb{R})$ .
- $\forall \beta \in \mathbb{N} \quad \int_{\mathbb{R}} x^\beta \psi(x) dx = 0$ .

The last property implies that the mother wavelet is in the Lizorkin Space,  $S_0(\mathbb{R})$ . We use the canonical construction of a multidimensional multiresolution analysis from a monodimensional one using tensor products. Let  $E$  denote the set  $\{0, 1\}^d \setminus (0, \dots, 0)$ . For all  $\epsilon \in E$ , we note  $\psi^{(\epsilon)}$  the function defined by:

$$\psi^{(\epsilon)}(\mathbf{x}) = \psi^{(\epsilon_1)}(x_1) \dots \psi^{(\epsilon_d)}(x_d)$$

where  $\psi^{(0)} = \phi$  and  $\psi^{(1)} = \psi$ . Then  $\{\psi_{j,\mathbf{k}}^{(\epsilon)}(\mathbf{x}) = 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}) : (j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E\}$  is

an orthonormal wavelet basis of  $L^2(\mathbb{R}^d)$ . By construction, for all  $\epsilon \in E$ ,  $\psi^{(\epsilon)}$  is in the following Lizorkin type space:

$$S_0(\mathbb{R}^d) = \{\psi \in S(\mathbb{R}^d) : \forall \beta \in \mathbb{N}^d \int_{\mathbb{R}^d} \mathbf{x}^\beta \psi(\mathbf{x}) d\mathbf{x} = 0\}.$$

This space is invariant by the Riesz potential which is defined by (see [100], page 483):

$$\mathcal{I}^\alpha \psi(\mathbf{x}) = \int_{\mathbb{R}^d} k_\alpha(\mathbf{x} - \mathbf{y}) \psi(\mathbf{y}) d\mathbf{y}.$$

For  $\psi$  in  $S_0(\mathbb{R}^d)$ , the following inversion formula holds:

$$\mathcal{I}^\alpha \psi(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \mathbf{x}, \xi \rangle} \frac{\mathcal{F}(\psi)(\xi)}{\|\xi\|_2^\alpha} d\xi.$$

We are now ready to give our wavelet-type expansion for the process  $\{X_t^\alpha\}$ . For this purpose, we proceed as follows:

- We show the existence of a modification of  $\{X_t^\alpha\}$ , denoted by  $\{\tilde{X}_t^\alpha\}$ , which can be expressed as a random wavelet-type series which almost surely converges absolutely at every point  $t \in [0, 1]$ .
- We show that this modification is continuous on  $[0, 1]$ .
- We conclude that these two processes are indistinguishable.

**Proposition 2.1.** *There exists a modification of the process  $\{X_t^\alpha; t \in [0, 1]\}$ , denoted by  $\{\tilde{X}_t^\alpha\}$ , such that:*

$$\forall \omega \in \Omega^* \quad \forall t \in [0, 1] \quad \tilde{X}_t^\alpha(\omega) = \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k})] I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega).$$

**Remark:** In [16], the wavelet expansion of the fractional Brownian sheet is proved by means of the Itô-Nisio theorem (see theorem 2.1.1 in [62]). In our case, we can not use such theorem since the collection  $\{I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}); (j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E\}$  is not a collection of independent random variables. Indeed, for any  $(j, \mathbf{k}, \mathbf{k}', \epsilon, \epsilon')$ , one has:

$$\begin{aligned} \mathbb{E}[I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}) I_d(\psi_{j,\mathbf{k}'}^{(\epsilon')})] &= d! \langle \hat{\psi}_{j,\mathbf{k}}^{(\epsilon)}; \hat{\psi}_{j,\mathbf{k}'}^{(\epsilon')} \rangle \\ &= d! \langle \hat{\psi}_{j,\mathbf{k}}^{(\epsilon)}; \psi_{j,\mathbf{k}'}^{(\epsilon')} \rangle \\ &= \sum_{\sigma \in \mathcal{S}_d} \langle \psi_{j,\mathbf{k}}^{(\epsilon)} \circ \sigma; \psi_{j,\mathbf{k}'}^{(\epsilon')} \rangle \\ &= \sum_{\sigma \in \mathcal{S}_d} \langle \psi_{j,\mathbf{k}_\sigma}^{(\epsilon_\sigma)}; \psi_{j,\mathbf{k}'}^{(\epsilon')} \rangle \end{aligned}$$

where  $\psi \circ \sigma(x_1, \dots, x_d) = \psi(x_{\sigma(1)}, \dots, x_{\sigma(d)})$ ,  $\epsilon_\sigma = (\epsilon_{\sigma(1)}, \dots, \epsilon_{\sigma(d)})$  and similarly for  $\mathbf{k}_\sigma$ . Thus, if

there exists  $\sigma \in \mathcal{S}_d$  for which  $\epsilon_\sigma = \epsilon'$  and  $\mathbf{k}_\sigma = \mathbf{k}'$ , we have  $\mathbb{E}[I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})I_d(\psi_{j,\mathbf{k}'}^{(\epsilon')})] \neq 0$ . For  $d = 3$ , one can choose  $\epsilon = (1, 0, 1)$ ,  $\epsilon' = (1, 1, 0)$ ,  $\mathbf{k} = (1, 2, 3)$  and  $\mathbf{k}' = (1, 3, 2)$  and  $\sigma = \{1, 2, 3\}^{\{1,3,2\}}$ .

In order to show this proposition, we need the following lemma:

**Lemma 2.1.** *Let  $\{f_n\}$  be an orthonormal basis of  $L^2(\mathbb{R}^d)$ . There exists an event  $\Omega^*$  of probability 1 and a strictly positive random variable,  $\tilde{C}_d(\omega)$ , which has all its moments finite such that:*

$$\forall \omega \in \Omega^* \quad \forall n \in \mathbb{N} \quad |I_d(f_n)| \leq \tilde{C}_d(\omega)(\log(e+n))^{\frac{d}{2}}.$$

**Proof:** The proof follows the lines of the one of Lemma 1 in [15]. Let  $a_n = (\log(e+n))^{\frac{d}{2}}$ . Theorem 6.7 of [51] entails that:

$$\exists c_d > 0 : \quad \forall b > 2, \quad \forall n \in \mathbb{N}, \quad \mathbb{P}\left(|I_d(f_n)| \geq ba_n \sqrt{d!} \|\hat{f}_n\|_{L^2(\mathbb{R}^d)}\right) \leq e^{-c_d(ba_n)^{\frac{2}{d}}} = \frac{1}{(e+n)^{c_d b^{\frac{2}{d}}}}.$$

For  $c_d b^{\frac{2}{d}} > 1$ ,  $\sum \frac{1}{(e+n)^{c_d b^{\frac{2}{d}}}} < \infty$ , and the Borel-Cantelli lemma implies that:

$$\exists \Omega^*, \quad \mathbb{P}(\Omega^*) = 1 : \quad \forall \omega \in \Omega^*, \quad \exists n(\omega) \in \mathbb{N} : \quad \forall n \geq n(\omega),$$

$$|I_d(f_n)(\omega)| \leq b(\log(e+n))^{\frac{d}{2}} \sqrt{d!} \|\hat{f}_n\|_{L^2(\mathbb{R}^d)} \leq b(\log(e+n))^{\frac{d}{2}} \sqrt{d!}.$$

Set:

$$\tau(\omega) = \begin{cases} \min\{n \in \mathbb{N} : k \geq n \quad |I_d(f_k)| \leq b(\log(e+k))^{\frac{d}{2}} \sqrt{d!}\} & \omega \in \Omega^* \\ +\infty & \omega \notin \Omega^*, \end{cases}$$

and define:

$$C(\omega) = \begin{cases} \sup_{0 \leq k \leq \tau(\omega)} (|I_d(f_k)|) & \omega \in \Omega^* \\ +\infty & \omega \notin \Omega^*. \end{cases}$$

We compute the moments of any order  $p > 0$  of this random variable. Following the proof of Lemma 1 of [15], one gets:

$$\begin{aligned} \mathbb{E}(|C|^p) &= \sum_{n=0}^{+\infty} \mathbb{E}\left(\sup_{0 \leq k \leq n} (|I_d(f_k)|^p) \chi_{\{\tau(\omega)=n\}}\right) \\ &\leq \mathbb{E}(|I_d(f_0)|^p) + \sum_{n=1}^{+\infty} \sum_{k=0}^n \mathbb{E}(|I_d(f_k)|^p \chi_{\{|I_d(f_{n-1})| > b(\log(e+n-1))^{\frac{d}{2}} \sqrt{d!}\}}) \\ &\leq \mathbb{E}(|I_d(f_0)|^p) + \sum_{n=1}^{+\infty} \sum_{k=0}^n \mathbb{E}(|I_d(f_k)|^{2p})^{\frac{1}{2}} \mathbb{P}(|I_d(f_{n-1})| > b(\log(e+n-1))^{\frac{d}{2}} \sqrt{d!})^{\frac{1}{2}}. \end{aligned}$$

Using Theorems 5.10 and 6.7 of [51], one deduces

$$\mathbb{E}(|C|^p) \leq \mathbb{E}(|I_d(f_0)|^p) + c_p \sum_{n=1}^{+\infty} \frac{n+1}{(e+n-1)^{\frac{c_d b^{\frac{2}{d}}}{2}}},$$

*i.e.*, for  $\frac{c_d b^{\frac{2}{d}}}{2} > 2$ ,

$$\forall p > 0 \quad \mathbb{E}(|C|^p) < +\infty.$$

From this, one deduces that there exists a strictly positive random variable  $\tilde{C}_d(\omega)$ , whose every moments are finite and such that:

$$a.s. \quad \forall n \in \mathbb{N} \quad |I_d(f_n)| \leq \tilde{C}_d(\omega) (\log(e+n))^{\frac{d}{2}} \quad \square$$

An indexing argument shows that the following inequality holds:

$$\begin{aligned} a.s. \quad \forall (j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E \quad |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| &\leq \tilde{C}_{1,d}(\omega) (\log(e+|j| + \|\mathbf{k}\|_1))^{\frac{d}{2}} \\ &\leq \tilde{C}_{1,d}(\omega) (\log(e+|j|))^{\frac{d}{2}} (\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}, \end{aligned}$$

where we have used the two trivial facts:

- the function  $x \rightarrow x^{\frac{d}{2}}$  is increasing on  $\mathbb{R}_+$
- $\forall (x, y) \in \mathbb{R}_+ \quad \log(e+x+y) \leq \log(e+x) \log(e+y)$

Let us now move to the proof of Proposition 2.1. It is similar to that of the wavelet-type expansion of fBm in [81] and in section 4.2 of [9].

**Proof:** For every  $t \in [0, 1]$ ,  $h_t^\alpha$  is in  $\hat{L}^2(\mathbb{R}^d)$ . We can expand it into the wavelet basis  $\{\psi_{j,\mathbf{k}}^{(\epsilon)}\}$ . Using Itô isometry we have, in  $L^2(\Omega, \mathcal{G}, \mathbb{P})$ :

$$X_t^\alpha = \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} \langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}).$$

Let us show that the right hand side converges absolutely for every  $t$  in  $[0, 1]$  on  $\Omega^*$ . It suffices to show that:

$$\forall \omega \in \Omega^* \quad \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} |\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle| |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| < +\infty$$

For this purpose, write:

$$\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle = \int_{\mathbb{R}^d} [k_{\alpha+1}(\mathbf{t}^* - \mathbf{x}) - k_{\alpha+1}(-\mathbf{x})] \psi_{j,\mathbf{k}}^{(\epsilon)}(\mathbf{x}) d\mathbf{x}.$$

The change of variables  $2^j \mathbf{x} - \mathbf{k} = \mathbf{u}$  yields:

$$\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle = 2^{-jH} \int_{\mathbb{R}^d} [k_{\alpha+1}(2^j \mathbf{t}^* - \mathbf{k} - \mathbf{u}) - k_{\alpha+1}(-\mathbf{k} - \mathbf{u})] \psi^{(\epsilon)}(\mathbf{u}) d\mathbf{u}.$$

Split the integral into two terms to get:

$$\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle = 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k})].$$

We need to distinguish between  $j \leq -1$  and  $j \geq 0$ . Since  $\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)}) \in S_0(\mathbb{R}^d)$ , the function  $f$  defined on  $\mathbb{R}$  by:

$$f(u) = \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(\mathbf{u}^* - \mathbf{k})$$

is in  $C^\infty(\mathbb{R})$  (with  $\mathbf{u}^* = (u, \dots, u)$ ). Thus, by the mean value theorem:

$$\exists v \in [0, 2^j t] : \quad \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k}) = 2^j t \frac{df}{du}(v).$$

By the chain rule,

$$\frac{df}{du}(v) = \sum_{i=1}^d \frac{\partial}{\partial u_i} (\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)}))(\mathbf{v}^* - \mathbf{k}),$$

where  $\mathbf{v}^* = (v, \dots, v)$ . Using the rapid decrease of  $\frac{\partial}{\partial u_i} (\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)}))$ , one gets:

$$\forall p > 0 \quad \left| \frac{df}{du}(v) \right| \leq \frac{C_{p,d}}{(1 + \|\mathbf{v}^* - \mathbf{k}\|_2)^p}$$

For  $j \leq -1$ ,  $v \in [0, 2^j t] \subset [0, \frac{1}{2}]$ . Thus, for  $\|\mathbf{k}\|_2$  large enough, the following estimate holds:

$$\forall t \in [0, 1] \quad |\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle| \leq 2^{j(1-H)} \frac{C_{p,d}}{(1 + \|\mathbf{k}\|_2)^p}.$$

By Lemma 2.1,

$$\forall \omega \in \Omega^* \quad \forall (j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E \quad |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)| \leq \tilde{C}(\omega) (\log(e + |j|))^{\frac{d}{2}} (\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}.$$

Thus,

$$\| \langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle \|_{\infty, [0,1]} |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)| \leq C'_{p,d}(\omega) \frac{2^{j(1-H)}}{(1 + \|\mathbf{k}\|_2)^p} (\log(e + |j|))^{\frac{d}{2}} (\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}.$$

Since  $H < 1$ , almost sure uniform convergence for the ‘‘low frequency’’ part  $j < 0$  for  $p$  sufficiently large does hold. The high frequency part  $j \geq 0$  requires a bit more work. Using the rapid decrease at infinity of  $\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})$ , one gets:

$$|\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k})| \leq \frac{C''_{p,d}}{(1 + \|2^j \mathbf{t}^* - \mathbf{k}\|_2)^p}.$$

Thus, for  $j \geq 0$ :

$$|\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle| \leq C''_{p,d} 2^{-jH} \left[ \frac{1}{(1 + \|2^j \mathbf{t}^* - \mathbf{k}\|_2)^p} + \frac{1}{(1 + \|\mathbf{k}\|_2)^p} \right].$$

In order to conclude, we need to estimate the following quantity:

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{(\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}}{(1 + \|2^j \mathbf{t}^* - \mathbf{k}\|_2)^p}.$$

Denote  $[2^j \mathbf{t}]^* = ([2^j t], \dots, [2^j t]) \in \mathbb{Z}^d$  where  $[u]$  is the integer part of the real  $u$ . A change of variable in the previous sum yields:

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{(\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}}{(1 + \|2^j \mathbf{t}^* - \mathbf{k}\|_2)^p} = \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{(\log(e + \|\mathbf{k} + [2^j \mathbf{t}]^*\|_1))^{\frac{d}{2}}}{(1 + \|2^j \mathbf{t}^* - [2^j \mathbf{t}]^* - \mathbf{k}\|_2)^p}.$$

By the triangular inequality,

$$\log(e + \|\mathbf{k} + [2^j \mathbf{t}]^*\|_1) \leq \log(e + \|\mathbf{k}\|_1 + \|[2^j \mathbf{t}]^*\|_1) \leq C' \log(e + \|\mathbf{k}\|_1) \log(e + \|[2^j \mathbf{t}]^*\|_1).$$

Consequently, one obtains:

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{(\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}}{(1 + \|2^j \mathbf{t}^* - \mathbf{k}\|_2)^p} \leq C' (\log(e + \|[2^j \mathbf{t}]^*\|_1))^{\frac{d}{2}} \sup_{\mathbf{z} \in [0,1]^d} \left( \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{(\log(e + \|\mathbf{k}\|_1))^{\frac{d}{2}}}{(1 + \|\mathbf{z} - \mathbf{k}\|_2)^p} \right).$$

Thus, for  $j \geq 0$ :

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} |\langle h_t^\alpha, \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle| |I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})(\omega)| \leq C'''_{p,d}(\omega) 2^{-jH} (\log(e + d2^j t))^{\frac{d}{2}} (\log(e + |j|))^{\frac{d}{2}},$$

which concludes the proof.  $\square$

**Remark:** This wavelet expansion does not extend straightforwardly to every stochastic process represented by  $I_d(h_t)$  where  $h_t$  satisfies properties 1.,2. and 3. Indeed, for Hermite processes, the wavelet coefficients of the kernel  $h_t$  are equal to:

$$\begin{aligned} \langle h_t; \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle &= c(H_0) \int_0^t \int_{\mathbb{R}^d} \left( \prod_{j=1}^d \frac{(s - x_j)_+^{H_0-1}}{\Gamma(H_0)} \right) \psi_{j,\mathbf{k}}^{(\epsilon)}(x_1, \dots, x_d) dx_1 \dots dx_d ds \\ &= c(H_0) \int_0^t (I_+^{H_0} \otimes \dots \otimes I_+^{H_0})(\psi_{j,\mathbf{k}}^{(\epsilon)})(\mathbf{s}^*) ds \\ &= c(H_0) \int_0^t I_+^{H_0}(\psi_{j,k_1}^{(\epsilon_1)})(s) \dots I_+^{H_0}(\psi_{j,k_d}^{(\epsilon_d)})(s) ds \\ &= c(H_0) 2^{-jH} \int_0^{2^j t} I_+^{H_0}(\psi^{(\epsilon_1)})(s - k_1) \dots I_+^{H_0}(\psi^{(\epsilon_d)})(s - k_d) ds \end{aligned}$$



where  $H_0 = \frac{H}{d} + \frac{d-2}{2d}$ ,  $I_+^{H_0}(\psi)(t) = \frac{1}{\Gamma(H_0)} \int_{-\infty}^t (t-s)_+^{H_0-1} \psi(s) ds$  is the fractional integral of Weyl type of order  $H_0$  (see [100]) and  $c(H_0)$  a certain normalizing constant. Take for instance,  $d = 2$ ,  $\epsilon = (1, 0)$  and  $(j, \mathbf{k}) \in \mathbb{Z} \times \mathbb{Z}^2$ . We obtain:

$$\langle h_t; \psi_{j,\mathbf{k}}^{(\epsilon)} \rangle = c(H_0) 2^{-jH} \int_0^{2^j t} I_+^{H_0}(\psi)(s - k_1) I_+^{H_0}(\phi)(s - k_2) ds$$

But the scaling function  $\phi$  is not in the Lizorkin space  $S_0(\mathbb{R})$ . Thus,  $I_+^{H_0}(\phi)$  does not decrease sufficiently fast at infinity. Actually, by Lemma 1 of [93], we have:

$$\exists C > 0 \quad \forall x \in \mathbb{R} \quad |I_+^{H_0}(\phi)(x)| \leq \frac{C}{(1 + |x|)^{1-H_0}}.$$

**Proposition 2.2.** *The two processes  $\{X_t^\alpha; t \in [0, 1]\}$  and  $\{\tilde{X}_t^\alpha; t \in [0, 1]\}$  are indistinguishable.*

**Proof:** We postpone it to the next section where we derive, in proposition 3.2, a uniform modulus of continuity for  $\{\tilde{X}_t^\alpha; t \in [0, 1]\}$ .

### 3 Uniform and Local Regularity of the sample paths

From Proposition 2.1, one sees that the process  $X_t^\alpha$  may be written:

$$X_t^\alpha = X_t^{l,\alpha} + X_t^{h,\alpha},$$

where  $X_t^{l,\alpha}$  is the low frequency part and  $X_t^{h,\alpha}$  the high frequency part, namely:

$$X_t^{l,\alpha} = \sum_{j \in \mathbb{Z}_-} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k})] I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}),$$

$$X_t^{h,\alpha} = \sum_{j \in \mathbb{N}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k})] I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}).$$

In order to study the regularity of the stochastic process, we will show that the low frequency part is  $C^\infty(\mathbb{R})$  and then focus on the regularity of the high frequency part.

**Proposition 3.1.** *Almost surely, the process  $\{X_t^{l,\alpha}\}$  is in  $C^\infty(\mathbb{R})$ .*

**Proof:** Let  $g_{j,\mathbf{k}}$  be defined by:

$$\forall t \in [0, 1] \quad g_{j,\mathbf{k}}(t) = \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}).$$

We need to show that the following random series converges uniformly on  $[0, 1]$  almost surely:

$$\sum_{j \in \mathbb{Z}_-} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} \frac{d^m g_{j,\mathbf{k}}}{dt^m}(t) I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}), \quad m \in \mathbb{N}^*.$$

The chain rule formula entails that:

$$\frac{d^m g_{j,\mathbf{k}}}{dt^m}(t) = 2^{jm} \sum_{(i_1, \dots, i_m) \in \{1, \dots, d\}^m} \frac{\partial^m}{\partial x_{i_1} \dots \partial x_{i_m}} (\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)}))(2^j \mathbf{t}^* - \mathbf{k}).$$

Since  $\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)}) \in S_0(\mathbb{R}^d)$ , one obtains the following estimate:

$$\forall p > 0 \quad \left| \frac{d^m g_{j,\mathbf{k}}}{dt^m}(t) \right| \leq 2^{jm} \frac{C_{p,d,m}}{(1 + \|2^j \mathbf{t}^* - \mathbf{k}\|_2)^p}.$$

Moreover, since  $t \in [0, 1]$  and  $j \leq -1$ ,  $2^j \mathbf{t}^* \in [0, \frac{1}{2}]^d$ . Thus, for  $\|\mathbf{k}\|_2$  sufficiently large, one has:

$$\left| \frac{d^m g_{j,\mathbf{k}}}{dt^m}(t) \right| \leq 2^{jm} \frac{C_{p,d,m}}{(1 + \|\mathbf{k}\|_2)^p}.$$

Finally, since  $H < 1 \leq m$  and using Lemma 2.1, one obtains the almost sure uniform convergence of the random series.  $\square$

The next two results are analogues of Theorems 1 and 2 of [16]. The first one gives a modulus of continuity for the process  $X_t^\alpha$  while the second one deals with the asymptotic behaviour of the process as  $t \rightarrow +\infty$ . Their proofs are also similar.

**Proposition 3.2.** *There exists a strictly positive random variable  $A_d$  of finite moments of any order and a constant,  $b_d > 1$ , such that:*

$$\forall \omega \in \Omega^* \quad \sup_{(s,t) \in [0,1]} \frac{|X_t^\alpha(\omega) - X_s^\alpha(\omega)|}{|t - s|^H (\log(b_d + |t - s|^{-1}))^{\frac{d}{2}}} \leq A_d(\omega).$$

This result should be compared with Remark 2.3 p.166 and the sentence that follows its proof on p.167 of [106], which provide a related (but weaker) result, namely that the local Hölder exponent of  $X^\alpha$  (as well as the ones of all Hermite processes) is equal to  $H$ : the above proposition entails that the local Hölder exponent of  $X^\alpha$  is not smaller than  $H$ , and Theorem 3.4 below will ensure that it is not greater than  $H$ . See also Lemma 5.9 of [103], which provides a similar result for Hermite processes.

**Proof:** Since  $t \rightarrow X_t^{l,\alpha}$  is in  $C^\infty(\mathbb{R})$  almost surely, we need only to deal with the high frequency component  $X_t^{h,\alpha}$ . Let  $\omega \in \Omega^*$  and  $(s, t)$  be in  $[0, 1]$ . There exists a positive integer  $j_0$  such that:

$$2^{-j_0-1} < |t - s| \leq 2^{-j_0}.$$

Write:

$$X_t^{h,\alpha} - X_s^{h,\alpha} = (I) + (II),$$

where

$$(I) = \sum_{j=0}^{j_0} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{s}^* - \mathbf{k})] I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})$$

and

$$(II) = \sum_{j \geq j_0+1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{s}^* - \mathbf{k})] I_d(\psi_{j,\mathbf{k}}^{(\epsilon)}).$$

Let us consider the first sum. Using Lemma 2.1, the smoothness of  $\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})$ , its rapid decrease at infinity and similar upper bounds as in the proof of proposition 2.1 for the high frequency part, there exists a deterministic constant  $b_{1,d} > 1$  such that:

$$|(I)| \leq A_{1,d}(\omega) |t - s| \sum_{j=0}^{j_0} 2^{(1-H)j} (b_{1,d} + j)^{\frac{d}{2}} \leq A'_{1,d}(\omega) |t - s| 2^{(j_0+1)(1-H)} (b_{1,d} + j_0)^{\frac{d}{2}}.$$

Similarly, for the second sum, there exists a deterministic constant  $b_{2,d} > 1$  such that:

$$|(II)| \leq A_{2,d}(\omega) \sum_{j=j_0+1}^{\infty} 2^{-jH} (b_{2,d} + j)^{\frac{d}{2}} \leq A'_{2,d}(\omega) 2^{-j_0 H} (b_{2,d} + j_0)^{\frac{d}{2}}.$$

Putting the results altogether and using the definition of  $j_0$  yields the desired conclusion.  $\square$

**Proposition 3.3.** *There exists a strictly positive random variable  $B_d$  of finite moments of any order and a constant  $c_d > 3$ , such that:*

$$\forall \omega \in \Omega^* \quad \sup_{t \in \mathbb{R}_+} \frac{|X_t^\alpha(\omega)|}{(1 + |t|)^H (\log \log(c_d + |t|))^{\frac{d}{2}}} \leq B_d(\omega).$$

**Proof:** The proof is quite similar to the previous one except that the behaviour at infinity of the process will be governed by the low frequency part. Let  $t$  be a real strictly greater than 1 and let  $j_1$  denote the integer such that:

$$2^{j_1} \leq t < 2^{j_1+1}.$$

Let  $\omega \in \Omega^*$ . We consider the following quantity that we split into three parts:

$$\sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^\epsilon)(-\mathbf{k})] (\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}} = (I) + (II) + (III)$$

where,

$$(I) = \sum_{j=-\infty}^{-(j_1+1)} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^\epsilon)(-\mathbf{k})] (\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}},$$

$$(II) = \sum_{j=-j_1}^{-1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^\epsilon)(-\mathbf{k})] (\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}},$$

$$(III) = \sum_{j=0}^{\infty} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t}^* - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^\epsilon)(-\mathbf{k})] (\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}}.$$

Let us deal first with the high frequency part. We have:

$$|(III)| \leq h_3(t) + h_3(0)$$

where  $h_3$  is the function defined by:

$$h_3(t) = \sum_{j=0}^{\infty} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} |\mathcal{I}^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t}^* - \mathbf{k})| (\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}}.$$

Reasoning as in the proof of proposition 2.1 for the high frequency part, one gets:

$$h_3(t) \leq C_3 \sum_{j=0}^{+\infty} 2^{-jH} (\log(e + j + \|2^j \mathbf{t}^*\|))^{\frac{d}{2}}.$$

Using that:

$$\forall a, b > 0 \quad \log(e + a + b) \leq \log(e + a) + \log(e + b),$$

and, for  $d > 2$ , by a convexity argument:

$$\forall a, b > 0 \quad (a + b)^{\frac{d}{2}} \leq c_d (a^{\frac{d}{2}} + b^{\frac{d}{2}}),$$

one gets:

$$h_3(t) \leq C_{3d} \sum_{j=0}^{+\infty} 2^{-jH} (\log(e + j))^{\frac{d}{2}} + C_{3d} \sum_{j=0}^{+\infty} 2^{-jH} (\log(e + 2^j \|\mathbf{t}^*\|))^{\frac{d}{2}}.$$

Finally, since  $\log(e + 2^j \|\mathbf{t}^*\|) \leq \log(e + 2^j) + \log(e + \|\mathbf{t}^*\|)$ :

$$h_3(t) \leq C'_{3d} (\log(e + \|\mathbf{t}^*\|))^{\frac{d}{2}}.$$

Consider now the second term. One has:

$$|(II)| \leq h_2(t) + h_2(0),$$

where  $h_2$  is the function defined by:

$$h_2(t) = \sum_{j=-j_1}^{-1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} |\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k})| (\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}}.$$

It is clear that:

$$h_2(t) \leq C_2 \sum_{j=1}^{j_1} 2^{jH} (\log(e + j + 2^{-j} \|\mathbf{t}^*\|))^{\frac{d}{2}}.$$

Using the definition of  $j_1$  and elementary computations, one obtains:

$$h_2(t) \leq C_{2,d} (1 + |t|)^H (\log \log(c_d + |t|))^{\frac{d}{2}}.$$

Similarly for  $h_2(0)$ , one has:

$$h_2(0) \leq C'_{2,d} (1 + |t|)^H (\log \log(c_d + |t|))^{\frac{d}{2}}$$

For the last term, we apply the mean value theorem to the function  $f$  defined above to get:

$$|(I)| \leq C_{1,d} \sum_{j=-\infty}^{-(j_1+1)} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{(1-H)j} |t| \frac{(\log(e + |j| + \|\mathbf{k}\|))^{\frac{d}{2}}}{(1 + \|\mathbf{k}\|)^p} \leq C'_{1,d} |t|,$$

and the result follows.  $\square$

**Remark 3.1.** *In order to study the local regularity of the process, we will only need the behaviour at infinity of the high frequency part.*

There exist different tools in order to measure efficiently the roughness of the sample paths of stochastic processes. In the setting of continuous nowhere differentiable Gaussian random processes, one way to study this feature is the Hölderian regularity through the computation of the local and pointwise Hölder exponents (see [9] and references therein). Since the paths of the class of processes studied are nowhere differentiable, they are defined respectively by:

$$\begin{aligned} \tilde{\gamma}_{X^\alpha}(t) &= \sup\{\gamma > 0 : \limsup_{\rho \rightarrow 0^+} \sup_{u,v \in \mathcal{B}(t,\rho)} \frac{|X_u^\alpha - X_v^\alpha|}{|u - v|^\gamma} < +\infty\}, \\ \gamma_{X^\alpha}(t) &= \sup\{\gamma > 0 : \limsup_{\rho \rightarrow 0} \frac{|X_{t+\rho}^\alpha - X_t^\alpha|}{|\rho|^\gamma} < +\infty\}, \end{aligned}$$

where  $\mathcal{B}(t, \rho)$  is the ball centered at  $t$  of radius  $\rho$ . Notice that one always has

$$\tilde{\gamma}_{X^\alpha}(t) \leq \gamma_{X^\alpha}(t).$$

Proposition 3.2 entails that, almost surely:

$$\forall t \in (0, 1) \quad \gamma_{X^\alpha}(t) \geq \tilde{\gamma}_{X^\alpha}(t) \geq H.$$

We will show the converse inequality and obtain the following result:

**Theorem 3.1.** *Almost surely,*

$$\forall t \in (0, 1), \quad \gamma_{X^\alpha}(t) = \tilde{\gamma}_{X^\alpha}(t) = H.$$

This result should be compared with Proposition 2.2 p.163 and Remark 2.1 p.164 of [106], which entail that, for all  $t$ , almost surely, the pointwise Hölder exponent of  $X^\alpha$  (as well as the ones of all Hermite processes) are equal to  $H$ .

To prove the previous theorem, we introduce a random field  $\mathbf{X}_t^\alpha$  that is closely related to  $X_t^\alpha$  and defined as follows:

$$\forall \omega \in \Omega^*, \forall \mathbf{t} \in \mathbb{R}^d \quad \mathbf{X}_t^\alpha(\omega) = \sum_{j \in \mathbb{N}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} [\mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t} - \mathbf{k}) - \mathcal{I}^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k})] I_d(\psi_{j,\mathbf{k}}^\epsilon).$$

It is easily checked that  $\mathbf{X}_t^\alpha$  is well defined and that its sample paths are continuous. Moreover, proceeding as in the proof of Proposition 3.3 for the high frequency part (III), one can prove that:

$$a.s., \forall \mathbf{t} \in \mathbb{R}^d, \quad |\mathbf{X}_t^\alpha| \leq \tilde{C}_d(\omega) (\log(e + \|\mathbf{t}\|))^{\frac{d}{2}}.$$

We obtain the following result regarding the pointwise Hölder exponent of the random field  $\mathbf{X}_t^\alpha$ :

**Proposition 3.4.** *Almost surely,*

$$\forall \mathbf{t} \in (0, 1)^d \quad \gamma_{\mathbf{X}^\alpha}(\mathbf{t}) = H.$$

In order to obtain this result, we need several lemmas. We define the following operator on  $S_0(\mathbb{R}^d)$ :

$$\forall \mathbf{x} \in \mathbb{R}^d \quad D^{\alpha+1}(\psi)(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \mathbf{x}; \xi \rangle} \|\xi\|^{\alpha+1} \mathcal{F}(\psi)(\xi) d\xi.$$

The Fourier transform of the dilated and translated version of this function is equal to:

$$\forall \xi \in \mathbb{R}^d \quad \mathcal{F}(D^{\alpha+1}(\psi)_{j,\mathbf{k}})(\xi) = 2^{-jd} e^{-i2^{-j}\langle \mathbf{k}; \xi \rangle} \mathcal{F}(D^{\alpha+1}(\psi))(2^{-j}\xi),$$

where  $\mathcal{F}(D^{\alpha+1}(\psi))(\xi) = \|\xi\|^{\alpha+1} \mathcal{F}(\psi)(\xi)$ .

**Lemma 3.1.** *Almost surely,*

$$\forall (j, \mathbf{k}, \epsilon) \in \mathbb{N} \times \mathbb{Z}^d \times E \quad I_d(\psi_{j,\mathbf{k}}^\epsilon) = 2^{j(H+d)} \int_{\mathbb{R}^d} \mathbf{X}_t^\alpha D^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t} - \mathbf{k}) dt.$$

**Proof:** For  $\omega \in \Omega^*$  and  $(j, \mathbf{k}, \epsilon) \in \mathbb{N} \times \mathbb{Z}^d \times E$ , we compute the following quantity:

$$A := \int_{\mathbb{R}^d} \mathbf{X}_{t_1, \dots, t_d}^\alpha D^{\alpha+1}(\psi^{(\epsilon)})(2^j t_1 - k_1, \dots, 2^j t_d - k_d) dt_1 \dots dt_d.$$

Since the function  $D^{\alpha+1}(\psi^{(\epsilon)})$  is in  $S_0(\mathbb{R}^d)$ , the asymptotic growth properties of the random field  $\{\mathbf{X}_t^\alpha\}$  allows one to invert the sum and the integral, so that:

$$\begin{aligned} A &= \sum_{j', \mathbf{k}', \epsilon'} 2^{-j'H} \int_{\mathbb{R}^d} \mathcal{I}^{\alpha+1}(\psi^{(\epsilon')})(2^{j'} \mathbf{t} - \mathbf{k}') D^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t} - \mathbf{k}) d\mathbf{t} I_d(\psi_{j', \mathbf{k}'}^{(\epsilon')}) \\ &=: \sum_{j', \mathbf{k}', \epsilon'} A_{j', \mathbf{k}', \epsilon'} I_d(\psi_{j', \mathbf{k}'}^{(\epsilon')}). \end{aligned}$$

Let us compute the inner products  $A_{j', \mathbf{k}', \epsilon'}$ . By Fourier isometry,

$$A_{j', \mathbf{k}', \epsilon'} = 2^{-j'H} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} 2^{-j'd} e^{-i2^{-j'} \langle \mathbf{k}'; \xi \rangle} \mathcal{F}(\mathcal{I}^{\alpha+1}(\psi^{(\epsilon')}))(2^{-j'} \xi) 2^{-jd} e^{+i2^{-j} \langle \mathbf{k}; \xi \rangle} \overline{\mathcal{F}(D^{\alpha+1}(\psi^{(\epsilon)}))(2^{-j} \xi)} d\xi.$$

Using the inversion formula and the definition of the operator  $D^{\alpha+1}$ , one gets:

$$A_{j', \mathbf{k}', \epsilon'} = \frac{2^{-j(H+d)}}{(2\pi)^d} \int_{\mathbb{R}^d} 2^{-j\frac{d}{2}} 2^{-j'\frac{d}{2}} e^{-i2^{-j'} \langle \mathbf{k}'; \xi \rangle} e^{+i2^{-j} \langle \mathbf{k}; \xi \rangle} \mathcal{F}(\psi^{(\epsilon')})(2^{-j'} \xi) \overline{\mathcal{F}(\psi^{(\epsilon)})(2^{-j} \xi)} d\xi.$$

Since:

$$\mathcal{F}(\psi_{j, \mathbf{k}}^{(\epsilon)})(\xi) = 2^{-j\frac{d}{2}} e^{-i2^{-j} \langle \mathbf{k}; \xi \rangle} \mathcal{F}(\psi^{(\epsilon)})(2^{-j} \xi),$$

one finally obtains,

$$A_{j', \mathbf{k}', \epsilon'} = 2^{-j(H+d)} \int_{\mathbb{R}^d} \psi_{j, \mathbf{k}}^{(\epsilon)}(\mathbf{x}) \psi_{j', \mathbf{k}'}^{(\epsilon')}(\mathbf{x}) dx.$$

But  $\{\psi_{j, \mathbf{k}}^{(\epsilon)}\}$  is an orthonormal basis of  $L^2(\mathbb{R}^d)$ , so that:

$$\int_{\mathbb{R}^d} \mathbf{X}_t^\alpha D^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t} - \mathbf{k}) d\mathbf{t} = 2^{-j(H+d)} I_d(\psi_{j, \mathbf{k}}^{(\epsilon)}). \quad \square$$

We want to show that the pointwise Hölder exponent of the random field  $\mathbf{X}_t^\alpha$  is everywhere equal to  $H$  almost surely. As a first step, we will exhibit an event of probability one such that, for any  $\mathbf{t}_0 \in [0, 1]^d$ , there exists a random variable  $I_d(\psi_{j, \mathbf{k}}^{(\epsilon)})$  which is both “large” enough and whose index is “close” to  $\mathbf{t}_0$ . The construction of such an event is similar to lemma 4.1 in [11] but, as mentioned in the introduction, somewhat more involved due to the fact that the collection  $\{I_d(\psi_{j, \mathbf{k}}^{(\epsilon)}); (j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E\}$  is not a collection of independent standard normal random variables for  $d > 1$ . In that view, we will use the following result from [24]:

**Lemma 3.2.** [24, Theorem 8] *There exists an absolute constant  $c > 0$  such that for all polynomial  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  of degree at most  $d$ , all  $1 < q < \infty$ , all log-concave probability measures  $\mu$*

on  $\mathbb{R}^n$  and all  $\alpha > 0$ ,

$$\left( \int_{\mathbb{R}^n} |Q(\mathbf{x})|^{\frac{q}{d}} \mu(d\mathbf{x}) \right)^{\frac{1}{q}} \mu(\{\mathbf{x} \in \mathbb{R}^n \mid |Q(\mathbf{x})| < \alpha\}) \leq cq\alpha^{\frac{1}{d}}.$$

**Lemma 3.3.** *Let  $1 < \beta < 2$ . There is an event  $\Omega^{**}$  of probability one such that for all  $\omega \in \Omega^{**}$ , for all  $\epsilon$  in  $E$ , for all  $\mathbf{t}_0 \in [0, 1]^d$ , and for  $j$  large enough, there is a  $\mathbf{k}_j \in \mathbb{Z}^d$  such that:*

$$|\mathbf{t}_0 - 2^{-j}\mathbf{k}_j| \leq bj^\beta 2^{-j}$$

where  $b > 0$  is a constant independent of  $\omega$  and

$$|I_d(\psi_{j,\mathbf{k}_j}^\epsilon)| > \eta$$

where  $\eta > 0$  is a constant independent of  $\omega$ .

**Proof:** Let  $j$  be a positive integer. For simplicity, we assume  $\epsilon = (1, \dots, 1)$ . The general case follows in a similar way. We consider the dyadic points of order  $j$  of  $(0, 1)^d$  i.e.  $\{\mathbf{k}2^{-j} : \mathbf{k} \in \{1, \dots, 2^j - 1\}^d\}$ . Set  $\Delta_j = \{1, \dots, 2^j - 1\}^d$ . Let us build a covering of  $\Delta_j$ . In that view, define the following sets:

$$Q_j = \{0, \dots, \lfloor \frac{2^j}{j^\beta} \rfloor + 1\}^d,$$

$$\forall \mathbf{q} \in Q_j, D_j^{\mathbf{q}} = \{[\mathbf{j}^\beta]\mathbf{q} + \mathbf{r} : \mathbf{r} \in \{0, \dots, [\mathbf{j}^\beta] - 1\}^d\}.$$

Clearly:

$$\Delta_j \subset \bigcup_{\mathbf{q} \in Q_j} D_j^{\mathbf{q}}.$$

Consider the following event:

$$A_j = \bigcup_{\mathbf{q} \in Q_j} \bigcap_{\mathbf{k} \in \text{diag}(D_j^{\mathbf{q}})} \{|I_d(\psi_{j,\mathbf{k}}^\epsilon)| < \frac{1}{4 \cdot d^d c^d}\}.$$

Where  $\text{diag}(D_j^{\mathbf{q}})$  denotes the diagonal of  $D_j^{\mathbf{q}}$ . We wish to estimate its probability, and in particular its dependence on  $j$ . For this purpose, we first perform a study of the random variables  $I_d(\psi_{j,\mathbf{k}}^\epsilon)$ . Since  $\epsilon = (1, \dots, 1)$ , one has:

$$\psi_{j,\mathbf{k}}^\epsilon = \psi_{j,k_1} \otimes \dots \otimes \psi_{j,k_d}$$

and

$$\hat{\psi}_{j,\mathbf{k}}^\epsilon = \psi_{j,k_1} \hat{\otimes} \dots \hat{\otimes} \psi_{j,k_d}.$$

Let us reorder the previous symmetric tensor product in the following way:

$$\hat{\psi}_{j,\mathbf{k}}^\epsilon = \psi_{j,k_1}^{\hat{\otimes} \gamma_1} \hat{\otimes} \dots \hat{\otimes} \psi_{j,k_l}^{\hat{\otimes} \gamma_l},$$



where  $l \in 1, \dots, d$ ,  $\gamma_1 + \dots + \gamma_l = d$  and, for all  $p, m$  in  $\{1, \dots, l\}$ ,  $k_p \neq k_m$ . Using the formula expressing Wiener-Itô integrals in terms of Hermite polynomials, one obtains:

$$I_d(\psi_{j,\mathbf{k}}^\epsilon) = \prod_{p=1}^l H_{\gamma_p} \left( \int_{\mathbb{R}} \psi_{j,k_p}(u) dB_u \right).$$

The standard normal random variables  $\int_{\mathbb{R}} \psi_{j,k_p}(u) dB_u$  are independent. As a consequence, each  $I_d(\psi_{j,\mathbf{k}}^\epsilon)$  is a polynomial of degree at most  $d$  evaluated at independent standard normal random variables. Moreover, for each  $\mathbf{k}, \mathbf{k}' \in \text{diag}(D_j^q)$ ,  $I_d(\psi_{j,\mathbf{k}}^\epsilon)$  and  $I_d(\psi_{j,\mathbf{k}'}^\epsilon)$  are mutually independent for  $\mathbf{k} \neq \mathbf{k}'$ . Therefore:

$$\mathbb{P}(A_j) \leq \sum_{q \in Q_j} \prod_{\mathbf{k} \in \text{diag}(D_j^q)} \mathbb{P}(\{|I_d(\psi_{j,\mathbf{k}}^\epsilon)| < \frac{1}{4 \cdot d^d c^d}\})$$

Lemma 3.2 with  $q = 2d$  yields

$$\mathbb{P}(\{|I_d(\psi_{j,\mathbf{k}}^\epsilon)| < \frac{1}{4 \cdot d^d c^d}\}) \leq \left(\frac{1}{4}\right)^{\frac{1}{d}} (\sqrt{d!} \|\hat{\psi}_{j,\mathbf{k}}^\epsilon\|_2)^{-\frac{1}{d}}.$$

Using the trivial fact:

$$\sqrt{d!} \|\hat{\psi}_{j,\mathbf{k}}^\epsilon\|_2 \geq 1,$$

one obtains the estimate

$$\mathbb{P}(\{|I_d(\psi_{j,\mathbf{k}}^\epsilon)| < \frac{1}{4 \cdot d^d c^d}\}) \leq \left(\frac{1}{4}\right)^{\frac{1}{d}}$$

which entails that

$$\mathbb{P}(A_j) \leq (\lfloor \frac{2^j}{j^\beta} \rfloor + 2)^d \left(\frac{1}{4}\right)^{\frac{j^{\beta-1}}{d}}.$$

Since  $\beta > 1$ ,  $\sum_{j=1}^{+\infty} \mathbb{P}(A_j) < \infty$ . By the Borel-Cantelli lemma, the following event has probability one:

$$\Omega^{**} = \bigcup_{J=1}^{\infty} \bigcap_{j=J}^{\infty} A_j^c,$$

where  $A_j^c$  denotes the complement of the event  $A_j$ . Let  $\omega \in \Omega^{**}$  and  $\mathbf{t}_0 \in [0, 1]^d$ . There exists  $\tilde{\mathbf{k}}_j \in \Delta_j$  such that  $|\mathbf{t}_0 - \tilde{\mathbf{k}}_j 2^{-j}| \leq \sqrt{d} 2^{-j}$ . Moreover, since  $\bigcup_{q \in Q_j} D_j^q$  is a covering of  $\Delta_j$  there exists  $\tilde{q}_j$  such that  $\tilde{\mathbf{k}}_j \in D_j^{\tilde{q}_j}$ . But for  $j$  large enough, there also exists  $\hat{\mathbf{k}}_j \in D_j^{\tilde{q}_j}$  such that  $|I_d(\psi_{j,\hat{\mathbf{k}}_j})| > \frac{1}{4 \cdot d^d c^d}$ . Write:

$$|\mathbf{t}_0 - 2^{-j} \hat{\mathbf{k}}_j| \leq |\mathbf{t}_0 - \tilde{\mathbf{k}}_j 2^{-j}| + 2^{-j} |\hat{\mathbf{k}}_j - \tilde{\mathbf{k}}_j| \leq b j^\beta 2^{-j},$$

which yields the desired conclusion.  $\square$

We are now ready to prove that the pointwise Hölder exponent of the random field  $\mathbf{X}^\alpha$  is equal to  $H$  almost surely:

**Proof of proposition 3.4:** Assume that there exists  $\omega_0 \in \Omega^{**}$ ,  $\mathbf{t}_0 \in (0, 1)^d$  and  $\epsilon_0 > 0$  such that:

$$\gamma_{\mathbf{X}^\alpha}(\mathbf{t}_0)(\omega_0) \geq H + \epsilon_0.$$

By definition of the pointwise Hölder exponent, for any  $\mathbf{t}$  such that  $\|\mathbf{t} - \mathbf{t}_0\|_2$  is sufficiently small, the following inequality holds:

$$|\mathbf{X}_{\mathbf{t}}^\alpha(\omega_0) - \mathbf{X}_{\mathbf{t}_0}^\alpha(\omega_0)| \leq C(\omega_0) \|\mathbf{t} - \mathbf{t}_0\|_2^{H+\epsilon_0}, \quad (1.2)$$

where  $C(\omega_0) > 0$  is a finite constant. By continuity and growth at infinity of  $\mathbf{X}^\alpha$ , this inequality holds for any  $\mathbf{t} \in \mathbb{R}^d$ . For any  $(j, \mathbf{k}, \epsilon) \in \mathbb{N} \times \mathbb{Z}^d \times E$ ,

$$|I_d(\psi_{j,\mathbf{k}}^\epsilon)(\omega_0)| = 2^{j(H+d)} \left| \int_{\mathbb{R}^d} \mathbf{X}_{\mathbf{t}}^\alpha D^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t} - \mathbf{k}) d\mathbf{t} \right|.$$

Using the fact that  $D^{\alpha+1}(\psi^\epsilon) \in S_0(\mathbb{R}^d)$  and inequality (1.2), one gets:

$$\begin{aligned} |I_d(\psi_{j,\mathbf{k}}^\epsilon)(\omega_0)| &\leq 2^{j(H+d)} \int_{\mathbb{R}^d} |\mathbf{X}_{\mathbf{t}}^\alpha - \mathbf{X}_{\mathbf{t}_0}^\alpha| |D^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t} - \mathbf{k})| d\mathbf{t} \\ &\leq C(\omega_0) 2^{j(H+d)} \int_{\mathbb{R}^d} \|\mathbf{t} - \mathbf{t}_0\|_2^{H+\epsilon_0} |D^{\alpha+1}(\psi^\epsilon)(2^j \mathbf{t} - \mathbf{k})| d\mathbf{t}. \end{aligned}$$

The rapid decrease of  $D^{\alpha+1}(\psi^\epsilon)$  and the the change of variable  $\mathbf{x} = 2^j \mathbf{t} - \mathbf{k}$  yield

$$|I_d(\psi_{j,\mathbf{k}}^\epsilon)(\omega_0)| \leq C'(\omega_0) 2^{jH} \int_{\mathbb{R}^d} \|\mathbf{x} + \mathbf{k} 2^{-j} - \mathbf{t}_0\|_2^{H+\epsilon_0} \frac{1}{(1 + \|\mathbf{x}\|_2)^p} d\mathbf{x}.$$

Since  $(|t| + |s|)^{H+\epsilon_0} \leq C''(|t|^{H+\epsilon_0} + |s|^{H+\epsilon_0})$ , one obtains

$$|I_d(\psi_{j,\mathbf{k}}^\epsilon)(\omega_0)| \leq C'''(\omega_0) 2^{-j\epsilon_0} (1 + \|\mathbf{k} - 2^j \mathbf{t}_0\|_2^{H+\epsilon_0}).$$

This inequality is true for any  $j$  and  $\mathbf{k}$ . Applying it with the  $\mathbf{k}_j$  of Lemma 3.3 yields

$$|I_d(\psi_{j,\mathbf{k}_j}^\epsilon)(\omega_0)| \leq C''''(\omega_0) 2^{-j\epsilon_0} (1 + (bj^\beta)^{H+\epsilon_0}).$$

This right hand side in the above inequality tends to 0 as  $j$  tends to  $+\infty$ . But, by Lemma 3.3,  $|I_d(\psi_{j,\mathbf{k}_j}^\epsilon)(\omega_0)| > \eta$ . Then, ad absurdum,

$$a.s., \forall \mathbf{t} \in (0, 1)^d, \gamma_{\mathbf{X}^\alpha}(\mathbf{t}) = H. \quad \square$$

## 4 Hausdorff Dimensions of SSSI processes represented by multiple Wiener-Itô integrals

In this section, we consider geometric properties of fractal random sets generated by general multidimensional processes whose components are represented by multiple Wiener-Itô integrals of order  $d$  whose kernel verify conditions 1., 2., 3. with self-similar exponent  $H$ . Especially, we provide estimates for the Hausdorff dimension of the range and the graph of such multidimensional processes. For example, these estimates hold for multidimensional versions of  $X^\alpha$  as well as for multidimensional Hermite processes. More precisely, let  $d \geq 1$  and  $N \geq 1$  be two integers. Let  $\{Y_t^H\}$  be the process defined by:

$$Y_t^H = \gamma(H, d)I_d(h_t^H)$$

where  $\gamma(H, d)$  is a normalizing positive constant such that  $\mathbb{E}[|Y_1^H|^2] = 1$  and  $h_t^H$  any symmetric square-integrable kernel satisfying properties 1., 2., 3.

Let  $\frac{1}{2} < H_1 \leq \dots \leq H_N < 1$ . Let  $\{\mathbb{Y}_t^H\}$  be the multidimensional process defined by:

$$\{\mathbb{Y}_t^H\} = \{(Y_t^{H_1}, \dots, Y_t^{H_N}) : t \in \mathbb{R}_+\}$$

where the coordinates are independent copies of the process  $Y_t^H$ . Let  $F \subset \mathbb{R}^N$  be non-empty. We recall briefly the basic frame of the Hausdorff dimension definition (see [36] for further details). For all  $\delta > 0$ , we denote by  $\{U_i\}$ , a  $\delta$ -cover of  $F$ , any countable (or finite) collection of sets verifying:

- $F \subset \bigcup_{i=1}^{\infty} U_i$ ,
- $\forall i \in \mathbb{N}^* \quad 0 \leq \text{diam}(U_i) = \sup_{x, y \in U_i} (|x - y|) \leq \delta$ .

We denote by  $C_\delta(F)$  the set of all  $\delta$ -covers of  $F$ . For  $s > 0$ , we define:

$$\mathcal{H}_\delta^s(F) = \inf_{\{U_i\} \in C_\delta(F)} \left( \sum_{i=1}^{\infty} \text{diam}(U_i)^s \right)$$

and the  $s$ -dimensional Hausdorff measure of  $F$  by:

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F) = \begin{cases} \infty & 0 \leq s < \text{dim}_{\mathcal{H}}(F) \\ 0 & s > \text{dim}_{\mathcal{H}}(F), \end{cases}$$

where  $\text{dim}_{\mathcal{H}}(F)$  is the Hausdorff dimension of  $F$ . It is formally given by:

$$\text{dim}_{\mathcal{H}}(F) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

Moreover, there is a close link between Hölderian regularity and Hausdorff dimension (see proposition 2.3 page 32 of [36]). Specifically, one can obtain upper bound for the Hausdorff dimension of Hölderian transformation of set. This bound depends on the Hölderian exponent of the transformation and the Hausdorff dimension of the initial set. Finally, for the sake of completeness, we recall a useful result concerning potential theoretical methods in order to derive lower bounds for Hausdorff dimension (see chapter 4 of [36]).

**Lemma 4.1.** *Let  $F$  be a Borel subset of  $\mathbb{R}^N$ .*

- *If there exists a measure,  $\mu$ , on  $F$  such that  $0 < \mu(F) < \infty$  and such that:*

$$\int_F \int_F \frac{d\mu(x)d\mu(y)}{\|x - y\|_2^s} < \infty,$$

*for some  $s \geq 0$ , then  $\dim_{\mathcal{H}}(F) \geq s$ .*

- *If  $\mathcal{H}^s(F) > 0$  then there exists a measure  $\mu$  on  $F$  such that  $0 < \mu(F) < \infty$  and such that:*

$$\forall 0 < r < s \quad \int_F \int_F \frac{d\mu(x)d\mu(y)}{\|x - y\|_2^r} < \infty.$$

Let  $E \subset \mathbb{R}_+$  be non-empty. It turns out that the methods developed in [122] apply with few modifications for estimating the Hausdorff dimension of the range  $R_E(\mathbb{Y}^H) := \{\mathbb{Y}^H(t), t \in E\}$  and the graph  $Gr_E(\mathbb{Y}^H) := \{(t, \mathbb{Y}^H(t)), t \in E\}$  of  $\mathbb{Y}^H$  over  $E$ . We shall make use of the following lemma which is similar to lemma 4.3 of [85]:

**Lemma 4.2.** *There exists a strictly positive constant,  $C$ , depending only on  $d$ , such that:*

$$\forall j \in \{1, \dots, N\} \quad \mathbb{P}(|Y_1^{H_j}| \leq x) \leq Cx^{\frac{1}{d}}.$$

**Proof:** We shall use Lemma 3.2 in order to prove this inequality. For all  $j \in \{1, \dots, N\}$ ,  $\gamma(H_j, d)h_1^{H_j} \in L^2(\mathbb{R}^d)$ . If  $\{f_n : n \in \mathbb{N}\}$  is an orthonormal basis of  $L^2(\mathbb{R})$ , then  $\{f_{i_1} \otimes \dots \otimes f_{i_d} : (i_1, \dots, i_d) \in \mathbb{N}^d\}$  is an orthonormal basis of  $L^2(\mathbb{R}^d)$ . Thus, one can approximate  $\gamma(H_j, d)h_1^{H_j}$  by:

$$h_1^{H_j, n} = \sum_{i_1, \dots, i_d=1}^n \langle \gamma(H_j, d)h_1^{H_j}; f_{i_1} \otimes \dots \otimes f_{i_d} \rangle f_{i_1} \otimes \dots \otimes f_{i_d}.$$

By linearity of the multiple Wiener-Itô integral,

$$I_d(h_1^{H_j, n}) = \sum_{i_1, \dots, i_d=1}^n \langle \gamma(H_j, d)h_1^{H_j}; f_{i_1} \otimes \dots \otimes f_{i_d} \rangle I_d(f_{i_1} \otimes \dots \otimes f_{i_d}).$$

The Wiener-Itô theorem entails that  $I_d(f_{i_1} \otimes \dots \otimes f_{i_d})$  is a polynomial of degree at most  $d$  evaluated at independent standard normal random variables. Therefore,

$$I_d(h_1^{H_j, n}) = P_{n,d} \left( \int_{\mathbb{R}} f_1(u) dB_j(u), \dots, \int_{\mathbb{R}} f_n(u) dB_j(u) \right)$$

where  $P_{n,d}$  is a polynomial of degree at most  $d$ . Applying Lemma 3.2 to this quantity with  $q = 2d$  and  $\mu$  the Gaussian probability measure on  $\mathbb{R}^n$  yields that there exists a constant  $C(d) > 0$  such that:

$$\forall n \in \mathbb{N} \quad \forall x > 0 \quad \mathbb{P}(|I_d(h_1^{H_j, n})| \leq x) \leq C(d)x^{\frac{1}{d}} \|h_1^{H_j, n}\|_2^{-\frac{1}{d}}.$$

By the continuity of the multiple Wiener-Itô integral,  $I_d(h_1^{H_j, n}) \rightarrow I_d(\gamma(H_j, d)h_1^{\alpha_j})$  in  $L^2(\Omega)$ . Consider a subsequence such that  $I_d(h_1^{H_j, n})$  converges almost surely to  $I_d(\gamma(H_j, d)h_1^{H_j})$ . Using Fatou's lemma along this subsequence, one obtains:

$$\begin{aligned} \mathbb{P}(I_d(h_1^{H_j}) \leq x) &\leq \liminf_{n \rightarrow +\infty} \mathbb{P}(|I_d(h_1^{H_j, n})| \leq x) \\ &\leq C(d)x^{\frac{1}{d}} \liminf_{n \rightarrow +\infty} \|h_1^{H_j, n}\|_2^{-\frac{1}{d}} \\ &\leq C(d)x^{\frac{1}{d}} \|\gamma(H_j, d)h_1^{H_j}\|_2^{-\frac{1}{d}} \\ &\leq C(d)x^{\frac{1}{d}} \left(\frac{1}{\sqrt{d!}}\right)^{-\frac{1}{d}} \\ &\leq K(d)x^{\frac{1}{d}} \end{aligned}$$

since, we have  $\mathbb{E}[|Y_1^{H_j}|^2] = \gamma(H_j, d)^2 d! \|h_1^{H_j}\|_2^2 = 1$ .  $\square$

Now, we state the main theorem of this section.

**Theorem 4.1.** *Almost surely,*

$$\begin{aligned} \min \left( N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right) &\leq \dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \\ &\leq \min \left( N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right), \\ \min \left( \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} E + \sum_{i=1}^N \frac{(1 - H_i)}{d} \right) &\leq \dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \end{aligned}$$

$$\leq \min \left( \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} E + \sum_{i=1}^N (1 - H_i) \right).$$

The lower bounds of the previous theorem are obtained similarly to Theorem 2.3 of [122] using the approximation lemma 4.2.

**Proposition 4.1.** *Almost surely,*

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \geq \min \left( N; \frac{\dim_{\mathcal{H}} E + \frac{\sum_{j=1}^k (H_k - H_j)}{d}}{H_k}, k = 1, \dots, N \right),$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \geq \min \left( \frac{\dim_{\mathcal{H}} E + \frac{\sum_{j=1}^k (H_k - H_j)}{d}}{H_k}, k = 1, \dots, N, \dim_{\mathcal{H}} E + \sum_{i=1}^N \frac{(1 - H_i)}{d} \right).$$

**Proof:** Assume that  $\dim_{\mathcal{H}} E > 0$  (otherwise there is nothing to prove) and let  $k$  be such that:

$$\frac{\sum_{i=1}^{k-1} H_i}{d} < \dim_{\mathcal{H}} E \leq \frac{\sum_{i=1}^k H_i}{d}$$

Let us compute the quantity:

$$\lambda = \min \left( N; \frac{\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^j (H_j - H_i)}{d}}{H_j}, j = 1, \dots, N \right).$$

Let  $j < k$ . Let:

$$\Delta := \frac{\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^k (H_k - H_i)}{d}}{H_k} - \frac{\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^j (H_j - H_i)}{d}}{H_j}.$$

Then,

$$\begin{aligned} \Delta &= \frac{1}{H_j H_k} \left[ \dim_{\mathcal{H}} E (H_j - H_k) + \frac{\sum_{i=1}^k (H_k - H_i) H_j}{d} - \frac{\sum_{i=1}^j (H_j - H_i) H_k}{d} \right] \\ &= \frac{1}{H_j H_k} \left[ \dim_{\mathcal{H}} E (H_j - H_k) + \frac{\sum_{i=1}^j (H_k - H_j) H_i}{d} + \frac{\sum_{i=j+1}^k (H_k - H_i) H_j}{d} \right]. \end{aligned}$$

Since:

$$\sum_{i=j+1}^k (H_k - H_i) H_j = \sum_{i=j+1}^{k-1} (H_k - H_i) H_j \leq \sum_{i=j+1}^{k-1} (H_k - H_j) H_i,$$

one gets:

$$\frac{\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^k (H_k - H_i)}{d}}{H_k} - \frac{\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^j (H_j - H_i)}{d}}{H_j} \leq \frac{H_k - H_j}{H_j H_k} \left[ -\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^{k-1} H_i}{d} \right]$$

which is strictly less than 0 by definition of  $k$ . One proves in a similar way that this inequality also holds for  $j > k$ . Thus,

$$\lambda = \frac{\dim_{\mathcal{H}} E + \frac{\sum_{i=1}^k (H_k - H_i)}{d}}{H_k}.$$

From this one deduces that  $\frac{k-1}{d} < \lambda \leq \frac{k}{d}$ .

We shall use a classical potential theoretic method [36] to obtain a lower bound on the Hausdorff dimension. More precisely, in the case of the range of  $\{\mathbb{Y}_t^H\}$ , we study the convergence of the following quantity:

$$\int_{R_E(\mathbb{Y}^H) \times R_E(\mathbb{Y}^H)} \frac{d\mu_{\mathbb{Y}^H}(x) d\mu_{\mathbb{Y}^H}(y)}{|x - y|^\gamma}$$

where  $\gamma$  is a strictly positive constant and  $\mu_{\mathbb{Y}^H}$  is the image by  $\mathbb{Y}^H$  of a positive measure,  $\mu$ , on  $E$ , both of them suitably chosen. By a change of variable, the previous quantity is equal to:

$$\int_{E \times E} \frac{d\mu(t) d\mu(s)}{|\mathbb{Y}_t^H - \mathbb{Y}_s^H|^\gamma}.$$

By Fubini's theorem, it suffices to show the convergence of the following integral:

$$\int_{E \times E} \mathbb{E} [|\mathbb{Y}_t^H - \mathbb{Y}_s^H|^{-\gamma}] d\mu(t) d\mu(s).$$

Fix  $\gamma \in (\frac{k-1}{d}, \lambda)$  and set

$$F(x) = \prod_{j=1}^N \mathbb{P}[|Y_1^{H_j}| \leq x_j].$$

One computes:

$$\begin{aligned} \mathbb{E}[|\mathbb{Y}_t^H - \mathbb{Y}_s^H|^{-\gamma}] &= \int_{\mathbb{R}_+^N} [(x_1|t - s|^{H_1})^2 + \dots + (x_N|t - s|^{H_N})^2]^{-\frac{\gamma}{2}} dF(x) \\ &= |t - s|^{-H_1 \gamma} \int_{\mathbb{R}_+^{d-1}} dF_2(x_2) \dots dF_N(x_N) \int_{\mathbb{R}_+} [x_1^2 + \dots + (x_N|t - s|^{H_N - H_1})^2]^{-\frac{\gamma}{2}} dF_1(x_1). \end{aligned}$$

For all  $a > 0$  and all  $\beta > 0$ :

$$J := \int_{\mathbb{R}_+} [x_1^2 + a^2]^{-\frac{\beta}{2}} dF_1(x_1) = \beta \int_{\mathbb{R}_+} x_1 F_1(x_1) [x_1^2 + a^2]^{-\frac{\beta}{2}-1} dx_1.$$

When  $\beta > \frac{1}{d}$ , Lemma 4.2 yields:

$$\begin{aligned} J &\leq \beta C(d) \int_0^{+\infty} [x_1^2 + a^2]^{-\frac{\beta}{2}-1} x_1^{1+\frac{1}{d}} dx_1 \\ &\leq C(d) \left[ a^{-\beta-2} \int_0^a x_1^{1+\frac{1}{d}} dx_1 + \int_a^{+\infty} x_1^{-\beta-1+\frac{1}{d}} dx_1 \right] \\ &\leq C'(\beta, d) a^{-\beta+\frac{1}{d}}. \end{aligned}$$

When  $\beta < \frac{1}{d}$ ,

$$\int_{\mathbb{R}_+} x_1 F_1(x_1) [x_1^2 + a^2]^{-\frac{\beta}{2}-1} dx_1 \leq \int_0^1 x_1^{-\beta-1+\frac{1}{d}} dx_1 + \int_1^{+\infty} x_1^{-\beta-1} dx_1$$

and thus

$$\int_{\mathbb{R}_+} [x_1^2 + a^2]^{-\frac{\beta}{2}} dF_1(x_1) \leq C''(\beta, d).$$

Since  $\gamma \in (\frac{k-1}{d}, \lambda)$  we see that, upon integrating iteratively with respect to  $dF_1(x_1), dF_2(x_2), \dots, dF_k(x_k)$ , the integral

$$\int_{\mathbb{R}_+^N} [(x_1|t-s|^{H_1})^2 + \dots + (x_N|t-s|^{H_N})^2]^{-\frac{\gamma}{2}} dF(x)$$

is not larger than a constant times:

$$|t-s|^{-H_1\gamma - (H_2-H_1)(\gamma-\frac{1}{d}) - \dots - (H_k-H_{k-1})(\gamma-\frac{k-1}{d})} = |t-s|^{-H_k\gamma + \sum_{j=1}^k \frac{H_k-H_j}{d}}.$$

By Lemma 4.1, the condition  $H_k\gamma - \sum_{j=1}^k \frac{H_k-H_j}{d} < \dim_{\mathcal{H}} E$  entails that there exists a positive measure  $\mu$  on  $E$  such that

$$\int_{E \times E} \frac{\mu(ds)\mu(dt)}{|t-s|^{H_k\gamma - \sum_{j=1}^k \frac{H_k-H_j}{d}}} < +\infty.$$

Therefore,

$$\mathbb{E} \left[ \int_{E \times E} \frac{\mu(ds)\mu(dt)}{|\mathbb{Y}_t^H - \mathbb{Y}_s^H|^\gamma} \right] < +\infty$$

*i.e.*

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \geq \gamma \quad a.s.$$

for all  $\gamma \in (\frac{k-1}{d}, \lambda)$ .



The proof of the lower bound for the Hausdorff dimension of the graph of  $\mathbb{Y}_t^H$  over  $E$  follows from the one of Theorem 2.3 in [122] using the upper bound supplied by Lemma 4.2. We omit the details.  $\square$

**Remarks:**

- The proof relies essentially on Lemma 4.2. The lower bound holds for multidimensional Hermite processes of any order  $d$  as well as for multidimensional versions of  $X^\alpha$ .
- Moreover, it is well-known that the density of the Rosenblatt distribution is continuous and bounded ([2], [29]). As a consequence,  $\mathbb{P}(|Y_1^{H_j}| \leq x) \leq Kx$ , which yields the following lower bounds, almost surely:

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \geq \min \left( N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right)$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \geq \min \left( \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N, \dim_{\mathcal{H}} E + \sum_{i=1}^N (1 - H_i) \right)$$

One thus retrieves here the result of Theorem 2.3 of [122].

- Actually, for  $d = 2$ , we have the following equality in law:

$$Y_1^{H_j} \stackrel{(d)}{=} \sum_{n=1}^{\infty} \lambda_{n,H_j} (\eta_n^2 - 1)$$

where  $\{\lambda_{n,H_j}; n \geq 1\}$  satisfying  $|\lambda_{1,H_j}| \geq \dots \geq |\lambda_{n-1,H_j}| \geq |\lambda_{n,H_j}| \geq \dots \geq 0$  are the eigenvalues of the Hilbert-Schmidt operator associated with the  $L^2(\mathbb{R}^2)$ -kernel  $h_1^{H_j}$  and  $\{\eta_n; n \geq 1\}$  is a collection of independent standard normal random variables. A sufficient condition for  $Y_1^{H_j}$  to have a continuous and bounded density is obtained in [46] using Malliavin calculus. The boundedness and the continuity of the density rely on the existence of negative moments for  $\|D(Y_1^{H_j})\|_{L^2(\mathbb{R})}$  where  $D$  is the Malliavin derivative (see theorem 3.1 of [46] and chapter 1.2 of [87] for a definition of  $D$ ). Lemma 7.1. in [46] provides sufficient and necessary conditions for the existence of negative moments for  $\|D(Y_1^{H_j})\|_{L^2(\mathbb{R})}$ . It depends on the number of non-zero eigenvalues in  $\{\lambda_{n,H_j}; n \geq 1\}$ . Thus, for such  $Y_1^{H_j}$ , we have the following lower bounds, almost surely:

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \geq \min \left( N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right)$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \geq \min \left( \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N, \dim_{\mathcal{H}} E + \sum_{i=1}^N (1 - H_i) \right)$$

Upper bounds for the Hausdorff dimensions of the range and the graph are obtained by a direct application of Theorem 3.1 in [122], since we deal with elements of Wiener's chaoses, which thus have finite moments of all order:

**Proposition 4.2.** [122, Theorem 3.1] *Almost surely,*

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \leq \min \left( N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right),$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \leq \min \left( \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} E + \sum_{i=1}^N (1 - H_i) \right).$$

**Remarks:**

- For the  $N$ -dimensional anisotropic Rosenblatt process, we retrieve the known results, almost surely:

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) = \min \left( N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right),$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) = \min \left( \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} E + \sum_{i=1}^N (1 - H_i) \right)$$

- In the isotropic case, the first equality was proven in [103].



## Chapter 2

# A white noise approach to stochastic integration with respect to the Rosenblatt process.

---

Since the nineties, there have been mainly two different approaches in order to develop stochastic calculus for processes which are not semimartingales. The first one is a pathwise Riemann-Stieltjes integration which originates from a stochastic analysis of fractional Brownian motion for  $H > \frac{1}{2}$  (see [66]). Indeed, fractional Brownian motion (fBm), popularized by Mandelbrot (see [75]), is not a semimartingale for  $H \neq \frac{1}{2}$ . This approach which has been extended by Zhäle, using fractional integration by part formula, (see [125]), allows us to obtain an integral with similar properties to the ones of the Stratonovich integral. The second approach is based on tools from the Malliavin calculus and leads us to a stochastic integral close to the Itô one. These techniques were first developed by Decreusefond and Üstünel (see [30]) for fractional Brownian motion and extended by Alos et al. to Gaussian Volterra processes (see [3]). In [21], using white noise distribution theory, the author developed a stochastic calculus with respect to fractional Brownian motion (for every  $H \in (0, 1)$ ) which leads, quite easily, to an Itô formula for generalized functionals of this process.

Moreover, fBm belongs to the family of Hermite processes which appear in non-central limit theorems for processes defined as integrals or partial sums of non-linear functionals of stationary Gaussian sequences with long-range dependence (see [109], [108] and [32]). They admit the following representation, for all  $d \geq 1$ :

$$\forall t > 0 \quad Y_t^{H,d} = c(H_0) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left( \int_0^t \prod_{j=1}^d (s - x_j)_+^{H_0-1} ds \right) dB_{x_1} \dots dB_{x_d}$$

where  $\{B_x : x \in \mathbb{R}\}$  is a two-sided Brownian motion,  $c(H_0)$  is a normalizing constant such that  $\mathbb{E}[|Y_1^{H,d}|^2] = 1$  and  $H_0 = \frac{1}{2} + \frac{H-1}{d}$  with  $H \in (\frac{1}{2}, 1)$ . For  $d = 1$ , one recovers fractional Brownian motion. These processes share many properties with fBm. Namely, they are  $H$ -self-similar processes with stationary increments. They possess the same covariance structure, exhibit long range-dependence and their sample paths are almost-surely  $\delta$ -Hölder continuous, for every  $\delta < H$ . For  $d = 2$ , the process is called the Rosenblatt process by Taqqu (see [108]).

This process has received lots of interest in the past and more recent years. The distributional properties have first been studied by Albin (see [2]) and more recently by Veillette and Taqqu ([117]) and Maejima and Tudor ([71]). A wavelet-type expansion has been established by Pipiras ([93]). Geometric properties of random fractal sets generated by anisotropic multi-dimensional version of the Rosenblatt process have been considered in [103]. The construction of estimators for the self-similarity parameter  $H$  has been investigated by Bardet and Tudor in [18] and by Tudor and Viens in [114].

Finally, stochastic calculus with respect to the Rosenblatt process has been developed by Tudor in [113] from both, the pathwise type calculus and Malliavin calculus points of view. Even if these two approaches are successful in order to define a stochastic integral with respect to the Rosenblatt process, the Malliavin calculus one fails to give an Itô formula for the Rosenblatt process in the divergence sense for general smooth functionals. Indeed, the appearance of two trace terms from a limiting procedure makes the Itô formula in the divergence sense difficult to obtain (see Section 8 of [113] and more specifically Theorem 3). However, by analyzing the peculiar cases of the monomials  $x^2$  and  $x^3$ , the author noticed the appearance of certain non-zero cumulants of the Rosenblatt distribution (see section 2 below for a definition) as well as the appearance of the third derivative of  $x^3$ . Moreover, in Remark 8 of [113], the author expected that, for sufficiently smooth functionals, Itô formula regarding stochastic integration with respect to the Rosenblatt process should involve all the derivatives of such functionals as well as all the non-zero cumulants of the Rosenblatt distribution. Using white noise distribution theory, we define, in this article, a stochastic integral with respect to the Rosenblatt process which leads to an Itô formula for a certain class of functionals of this process confirming the above conjecture. This class of functionals is completely characterized by Paley-Wiener-Schwartz theorems. All the derivatives of these functionals appear in the change of variable formula (see section 3, specifically Remarks 3.13 and 3.15). The non-zero cumulants of the Rosenblatt distribution seem to be the ones responsible for this result (see Theorems 3.14 and 3.16).

Last but not least, we obtain in section 5 of this article (see Theorem 5.8) an explicit formula for the variance of the stochastic integral with respect to the Rosenblatt process. We note the appearance of three terms. The first one is directly linked with the covariance structure of the Rosenblatt process. Since fractional Brownian motion and the Rosenblatt process share the

same covariance function, this term is similar to the one appearing in the definition of the class of deterministic integrands for which one can define a Wiener integral with respect to fBm (see [94]). The second one requires the use of the derivative operator from white noise distribution theory (see Definition-Theorem 1.8 below). This term is somewhat classical in this setting (see Theorem 13.16 of [60]). Finally, the third term emphasizes the non-Gaussian nature of the Rosenblatt process. Since the Rosenblatt process admits a double Wiener-Itô integral representation, the second order derivative operator is involved in the definition of this term. Moreover, this formula holds under two sufficient conditions on the Wiener-Itô expansion of the stochastic integrand process. More specifically, multidimensional fractional integrals of the kernels of the Wiener-Itô expansion are assumed to belong to a certain function space with dominating mixed smoothness (see the preamble before Theorem 5.8 for a definition and hypothesis **H1**). These spaces were introduced by Lizorkin and Nikol'skii in [67] (see [102] for a recent survey of these spaces). This assumption is needed in order to define properly the second order derivative operator applied to the stochastic integrand process. The second condition ensures that one recovers the variance formula for the stochastic integral of the integrand process from the sum of the variance formulae of the stochastic integral of its projections on Wiener chaoses. This result stresses one of the advantages of white noise distribution theory over pathwise methods for the stochastic calculus with respect to the Rosenblatt process. Indeed, as in the fBm case for  $H \in (1/2, 1)$ , one can use Young integration in order to define a stochastic integral with respect to the Rosenblatt process. Nevertheless, as stated in [27] (see Remark 9 page 17 as well as Conclusion 3.3 page 19), statistical properties such as mean or variance of these stochastic integrals seem rather difficult to obtain.

The remaining of this article is organized as follows. In the first section, we introduce the tools from white noise distribution theory needed in order to define a stochastic calculus with respect to the Rosenblatt process. Section 2 is devoted to the study of the translated characteristic function of the Rosenblatt process at time  $t > 0$  in the direction  $\xi \in S(\mathbb{R})$ . The main results of this article are in section 3 where we define the stochastic integral by means of the Rosenblatt noise, Wick product and Pettis integration. Specifically, we obtain Itô formulae for functionals of the Rosenblatt process. In the fourth section, we compare our approach to the one in [113]. Using the finite interval representation of the Rosenblatt process, we show that the Rosenblatt noise integral extends the Skorohod integral defined by Malliavin calculus tools. Finally, we derive a sufficient condition for the Rosenblatt noise integral of a stochastic process  $\{\Phi_t\}$  to be an element of  $(L^2)$  and we compute the variance of this random variable.

# 1 White noise setting.

In this section, we briefly remind the white noise analysis introduced by Hida and al. in [43]. For a good introduction to the theory of white noise, we refer the reader to the book of Kuo [60]. The underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is the space of tempered distributions endowed with the  $\sigma$ -field generated by the open sets with respect to the weak\* topology in  $S'(\mathbb{R})$  and with the infinite dimensional Gaussian measure  $\mu$  whose existence is ensured by the Bochner-Minlos theorem.

For all  $(\phi_1, \dots, \phi_n) \in S(\mathbb{R})$ , the space of  $C^\infty(\mathbb{R})$  functions with rapid decrease at infinity, the vector  $(\langle \cdot; \phi_1 \rangle, \dots, \langle \cdot; \phi_n \rangle)$  is a centered Gaussian random vectors with covariance matrix  $(\langle \phi_i; \phi_j \rangle)_{(i,j)}$ . As it is written in Kuo [60], for any function  $f \in L^2(\mathbb{R})$ , we can define  $\langle \cdot; f \rangle$  as the random variable in  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  obtained by a classical approximation argument and the following isometry:

$$\forall(\phi, \psi) \in S(\mathbb{R})^2 \quad \mathbb{E}[\langle \cdot; \psi \rangle \langle \cdot; \phi \rangle] = \langle \psi; \phi \rangle_{L^2(\mathbb{R})}$$

Thus, for any  $t \in \mathbb{R}$ , we define ( $\mu$ -almost everywhere):

$$B_t(\cdot) = \begin{cases} \langle \cdot; 1_{[0;t]} \rangle & t \geq 0 \\ - \langle \cdot; 1_{[t;0]} \rangle & t < 0 \end{cases}$$

From the isometry property, it follows immediately that  $B_t$  is a Brownian motion on the white noise space and, by the Kolmogorov-Centsov theorem, it admits a continuous modification. Moreover, using the approximation of any function  $f \in L^2(\mathbb{R})$  by step functions, we obtain:

$$\langle \cdot; f \rangle = \int_{\mathbb{R}} f(s) dB_s$$

We note  $\mathcal{G}$ , the sigma field generated by Brownian motion and  $(L^2) = L^2(\Omega, \mathcal{G}, \mathbb{P})$ . By the Wiener-Itô theorem, any functionals  $\Phi \in (L^2)$  can be expanded uniquely into a series of multiple Wiener-Itô integrals:

$$\Phi = \sum_{n=0}^{\infty} I_n(\phi_n)$$

where  $\phi_n \in \hat{L}^2(\mathbb{R}^n)$ , the space of square-summable symmetric functions. Using this theorem and the second quantization operator of the harmonic oscillator operator,  $A = \frac{-d^2}{dx^2} + x^2 + 1$ , Hida and al. introduced the stochastic space of test functions  $(S)$  and its dual, the space of generalized functions  $(S)^*$  or Hida distributions. We refer the reader to pages 18-20 of Kuo's book for an explicit construction. We have the following Gel'fand triple:

$$(S) \subset (L^2) \subset (S)^*$$

We denote by  $\langle\langle ; \rangle\rangle$  the duality bracket between elements of  $(S)$  and  $(S)^*$  which reduces to the classical inner product on  $(L^2)$  for two elements in  $(L^2)$ . Note that  $(S)$  is defined as an equivalence class of test random variables defined  $\mu$ -almost everywhere. Nevertheless, by Theorem 6.5 of [60], each element of  $(S)$  admits a unique continuous version with which we will be working from now on. Furthermore, we note that, for all  $\omega \in S'(\mathbb{R})$ ,  $\delta_\omega$  is an element of  $(S)^*$  (see Chapter 7, Paragraph 7.2, Theorem 7.8 in [60]).

In the context of white noise analysis, the main tool is the  $S$ -transform. It is a functional on  $S(\mathbb{R})$  which characterizes completely the elements in  $(S)^*$  (as well as the strong convergence in  $(S)^*$ ). In the third section, we will see that the  $S$ -transform is fundamental in order to define the stochastic integral with respect to the Rosenblatt process.

**Definition 1.1:** Let  $\Phi \in (S)^*$ . For every function  $\xi \in S(\mathbb{R})$ , we define the  $S$ -transform of  $\Phi$  by:

$$S(\Phi)(\xi) = \langle\langle \Phi; : \exp(\langle ; \xi \rangle) : \rangle\rangle$$

where  $: \exp(\langle ; \xi \rangle) := \exp(\langle ; \xi \rangle - \frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2}) = \sum_{n=0}^{\infty} \frac{I_n(\xi^{\otimes n})}{n!} \in (S)$ .

**Remark 1.2:** For every  $\Phi \in (L^2)$ , we have:

$$S(\Phi)(\xi) = \mathbb{E}[\Phi : \exp(\langle ; \xi \rangle) :] = \mathbb{E}^{\mu_\xi}[\Phi]$$

where  $\mu_\xi$  is the translated infinite dimensional measure defined by:

$$\mu_\xi(d\omega) = \exp(\langle \omega; \xi \rangle - \frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2})\mu(d\omega).$$

Regarding the  $S$ -transform, we have the following properties and results:

**Theorem 1.3:**

1. The  $S$ -transform is injective. If  $\forall \xi \in S(\mathbb{R}), S(\Phi)(\xi) = S(\Psi)(\xi)$  then  $\Phi = \Psi$  in  $(S)^*$ .
2. For  $\Phi, \Psi \in (S)^*$  there is a unique element  $\Phi \diamond \Psi \in (S)^*$  such that for all  $\xi \in S(\mathbb{R})$ ,  $S(\Psi)(\xi)S(\Phi)(\xi) = S(\Phi \diamond \Psi)(\xi)$ . It is called the Wick product of  $\Phi$  and  $\Psi$ .
3. Let  $\Phi_n \in (S)^*$  and  $F_n = S(\Phi_n)$ . Then  $\Phi_n$  converges strongly in  $(S)^*$  if and only if the following conditions are satisfied:
  - $\lim_{n \rightarrow \infty} F_n(\xi)$  exists for each  $\xi \in S(\mathbb{R})$ .
  - There exists strictly positive constants  $K, a$  and  $p$  independent of  $n$  such that:

$$\forall n \in \mathbb{N}, \forall \xi \in S(\mathbb{R}) \quad |F_n(\xi)| \leq K \exp(a\|A^p \xi\|_{L^2(\mathbb{R})}^2)$$

Next, we introduce the Wick tensors. This object allows us to obtain a certain representation for multiple Wiener-Itô integrals on the white noise space which leads to a formula for the translation of second Wiener chaos elements in any direction  $\xi \in S(\mathbb{R})$ . For further details, we



refer the reader to chapter 5 of Kuo's book pages 32-35 [60].

**Definition 1.4:** The trace operator  $\tau$  is the element of  $\hat{S}'(\mathbb{R}^2)$  uniquely defined by:

$$\forall \psi, \phi \in S(\mathbb{R}) \quad \langle \tau; \phi \otimes \psi \rangle = \langle \psi; \phi \rangle$$

**Definition 1.5:** The Wick tensors of any elements  $\omega \in S'(\mathbb{R})$  is defined by:

$$: \omega^{\otimes n} := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} C_{2k}^n (2k-1)!! (-1)^k \omega^{\otimes(n-2k)} \hat{\otimes} \tau^{\otimes k}$$

where  $C_{2k}^n = \frac{n!}{(2k)!(n-2k)!}$ ,  $(2k-1)!! = (2k-1)(2k-3)\dots 3 \cdot 1$  and  $\hat{\otimes}$  is the symmetric tensor product.

**Theorem 1.6:** Let  $f \in \hat{L}^2(\mathbb{R}^n)$ .  $\mu$ -almost everywhere, we have:

$$I_n(f)(\omega) = \langle : \omega^{\otimes n} ; f \rangle$$

**Proof:** See Theorem 5.4 of Kuo's book [60].

**Lemma 1.7:** For all  $f \in \hat{L}^2(\mathbb{R}^2)$  and  $\xi \in S(\mathbb{R})$ , we have:

$$\mu - a.e \quad I_2(f)(\omega + \xi) = I_2(f)(\omega) + 2I_1(\langle f; \xi \rangle)(\omega) + \langle f; \xi^{\otimes 2} \rangle$$

**Proof:** For all  $\omega \in S'(\mathbb{R})$ , we have:

$$\begin{aligned} : \omega^{\otimes 2} &:= \sum_{k=0}^1 \frac{2}{k!(2-2k)!} \frac{(-1)^k}{2^k} \omega^{\otimes(2-2k)} \hat{\otimes} \tau^{\otimes k} \\ &: \omega^{\otimes 2} := \omega^{\otimes 2} - \tau \end{aligned}$$

where the equalities stand in  $S'(\mathbb{R}^2)$ . Thus, we have,

$$\begin{aligned} : (\omega + \xi)^{\otimes 2} &:= (\omega + \xi)^{\otimes 2} - \tau \\ : (\omega + \xi)^{\otimes 2} &:= : \omega^{\otimes 2} : + 2\omega \otimes \xi + \xi^{\otimes 2} \end{aligned}$$

Let  $(f_n)$  be a sequence of functions in  $\hat{S}(\mathbb{R}^2)$  converging to  $f$  in  $\hat{L}^2(\mathbb{R}^2)$ . For each  $n$ , we have:

$$\begin{aligned} \langle : (\omega + \xi)^{\otimes 2} ; f_n \rangle &= \langle : \omega^{\otimes 2} ; f_n \rangle + 2 \langle \omega \otimes \xi; f_n \rangle + \langle f_n; \xi^{\otimes 2} \rangle \\ \langle : (\omega + \xi)^{\otimes 2} ; f_n \rangle &= \langle : \omega^{\otimes 2} ; f_n \rangle + 2 \langle \omega; \langle \xi; f_n \rangle \rangle + \langle f_n; \xi^{\otimes 2} \rangle \end{aligned}$$

Thus, passing to the limit, we obtain  $\mu$ -almost everywhere:

$$I_2(f)(\omega + \xi) = I_2(f)(\omega) + 2I_1(\langle f; \xi \rangle)(\omega) + \langle f; \xi^{\otimes 2} \rangle. \quad \square$$

We end this section by introducing the differential operator and its adjoint. These operators play an important role in the context of white noise integration with respect to Brownian motion (see chapter 13.4 of [60]). Actually, they lead to sufficient condition in order to the white noise integral of a stochastic process to be in  $(L)^2$ . Thus, as we will see in the fifth section, they also lead to sufficient condition for the Rosenblatt noise integral of a stochastic process to be a square integrable random variable. We refer the reader to chapter 9 of [60] for further information concerning these two operators.

**Definition-Theorem 1.8:** Let  $y \in S'(\mathbb{R})$  and  $\Phi \in (S)$ . The operator  $D_y$  is continuous from  $(S)$  into itself and we have:

$$\forall \omega \in S'(\mathbb{R}) \quad D_y(\Phi)(\omega) = \sum_{n=1}^{\infty} n \langle : \omega :^{\otimes n-1}; y \otimes_1 \phi_n \rangle,$$

where we denote by  $\otimes_1$  the contraction of order 1 (see [87]).

**Proof:** See theorem 9.1 of [60].□

**Definition-Theorem 1.9:** Let  $y \in S'(\mathbb{R})$  and  $\Psi \in (S)^*$ . The adjoint operator  $D_y^*$  is continuous from  $(S)^*$  into itself and we have:

$$\forall \xi \in S(\mathbb{R}) \quad S(D_y^*(\Psi))(\xi) = \langle y; \xi \rangle S(\Psi)(\xi) = S(I_1(y) \diamond \Psi)(\xi)$$

where  $I_1(y)$  is a generalized Wiener-Itô integral in  $(S)^*$ . Moreover, we have the following generalized Wiener-Itô decomposition for  $D_y^*(\Psi)$ :

$$D_y^*(\Psi)(\cdot) = \sum_{n=0}^{\infty} \langle : \cdot :^{\otimes n+1}; y \hat{\otimes} \psi_n \rangle$$

**Proof:** See theorem 9.12, 9.13 and the remark following corollary 9.14 in [60].□

## 2 Second order Wiener chaoses and Rosenblatt process.

In this section, we state the definition of the Rosenblatt process and several known properties of the Rosenblatt distribution (the law of the Rosenblatt process at time 1). Specifically, we remind the formulae of its characteristic function and of its cumulants. In [117] and more generally in [77], it is shown that this characteristic function is infinitely divisible. Moreover in [71], the authors show that this distribution is selfdecomposable and unimodal. In this section, we show that the translated characteristic function of the ditribution of the Rosenblatt process at  $t$  is analytic on  $\mathbb{R}$  and admits a certain analytic representation in the neighborhood of the

origin which is fundamental in the derivation of the Itô formula for functionals of the Rosenblatt process.

**Definition 2.1:** The Rosenblatt process is defined by the following double Wiener-Itô integral representation:

$$\forall t \in \mathbb{R}_+ \quad X_t^H = c(H) \int_{\mathbb{R}^2} \left( \int_0^t (s-x_1)_+^{\frac{H}{2}-1} (s-x_2)_+^{\frac{H}{2}-1} ds \right) dB_{x_1} dB_{x_2} \quad (2.1)$$

where  $c(H)$  is a normalizing constant such that  $\mathbb{E}[|X_1^H|^2] = 1$ . In particular, by Remark 2.3 of [70], we have:

$$c(H) = \sqrt{\frac{H(2H-1)}{2}} \frac{1}{\beta(1-H, \frac{H}{2})}.$$

**Remarks 2.2:**

- Moreover, using the well-known formulae:

$$\forall a, b > 0 \quad \beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)},$$

we obtain:

$$c(H) = \sqrt{\frac{H(2H-1)}{2}} \Gamma(H) \frac{\sin(\pi H)}{\sin(\frac{\pi}{2}H)} \frac{1}{(\Gamma(\frac{H}{2}))^2}$$

And thus,

$$X_t^H = d(H) \int_{\mathbb{R}^2} \left( \int_0^t \frac{(s-x_1)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} ds \right) dB_{x_1} dB_{x_2}$$

- As we can see the kernel of the Rosenblatt process verifies the following properties:
  1.  $f_t^H(x_1, x_2) = d(H) \int_0^t \frac{(s-x_1)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} ds$  is in  $\hat{L}^2(\mathbb{R}^2)$ .
  2.  $f_t^H(tx_1, tx_2) = t^{H-1} f_1^H(x_1, x_2)$ .
  3.  $f_{t+h}^H(x_1, x_2) - f_t^H(x_1, x_2) = f_h^H(x_1 - t, x_2 - t)$ .
- It is well known that with each function  $f \in \hat{L}^2(\mathbb{R}^2)$ , we can associate the self-adjoint Hilbert-Schmidt operator,  $T_f$ , defined on  $L^2(\mathbb{R})$  by:

$$\forall g \in L^2(\mathbb{R}) \quad T_f(g)(\cdot) = \int_{\mathbb{R}} f(x_1, \cdot) g(x_1) dx_1$$

For the Rosenblatt process, we denote by  $T_1$  the self-adjoint Hilbert-Schmidt positive operator associated with  $f_1^H$ . Denoting by  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots$  and  $\{e_n\}$  the eigenvalues and the associated eigenvectors of  $T_1$ , we obtain the following representation

for the kernel,  $f_t^H$ , in  $\hat{L}^2(\mathbb{R}^2)$ :

$$f_t^H(\cdot, \cdot) = \sum_{n=1}^{+\infty} \lambda_n t^{H-1} (e_n \otimes e_n) \left( \frac{\cdot}{t}, \frac{\cdot}{t} \right) \quad (2.2)$$

- From this formula, we classically obtain a peculiar representation in law (in  $(L^2)$  and almost surely as well) for the Rosenblatt process at time  $t$ :

$$I_2(f_t^H) \stackrel{(d)}{=} t^H \sum_{n=1}^{+\infty} \lambda_n (X_n^2 - 1)$$

where  $(X_n)$  is a sequence of independent standard normal random variables.

- We end this series of remarks by reminding the formula of the cumulants for any second Wiener chaos element (see [86]):

$$\kappa_1(I_2(f)) = 0 \quad \forall r \geq 2 \quad \kappa_r(I_2(f)) = 2^{r-1} (r-1)! \sum_{n=1}^{+\infty} \lambda_n^r(f)$$

where  $(\lambda_n(f))$  are the eigenvalues of  $T_f$ .

In the computation of the translated characteristic function for the Rosenblatt process at time  $t$ , we need the three following lemmas. The first one is the second part of Lemma 2.1 in [70]. The proof of the second one is quite similar to the calculation of the Rosenblatt distribution's cumulants. The last one is a simple reformulation of Lemma 1.7 using the diagonal representation of the kernel  $f_t^H$ .

**Lemma 2.3:** For every  $s, r > 0$ :

$$\int_{-\infty}^{s \wedge r} (s - x_1)_+^{\frac{H}{2}-1} (r - x_1)_+^{\frac{H}{2}-1} dx_1 = \beta(1 - H, \frac{H}{2}) |s - r|^{H-1}$$

**Lemma 2.4:** For every  $\xi \in S(\mathbb{R})$ , we have:

$$\langle f_t^H; \xi^{\otimes 2} \rangle = d(H) \int_0^t [I_+^{\frac{H}{2}}(\xi)(s)]^2 ds$$

and,

$$\begin{aligned} \langle \underbrace{(\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H)}_{k-1 \times \otimes_1}; \xi^{\otimes 2} \rangle &= d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} \\ &\int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) K_t^{k-2}(s, r) ds dr \end{aligned}$$

where  $I_+^{H/2}(\xi)(x) = 1/(\Gamma(H/2)) \int_{-\infty}^x (x-y)_+^{H/2-1} \xi(y) dy$  is the fractional integral of Weyl's type of order  $H/2$  of  $\xi$  (see [100] chapter 2 section 5.1) and  $K_t^{k-2}(s, r)$  is the sequence of kernels defined by:

$$K_t^0(s, r) = |s-r|^{H-1} \quad K_t^1(s, r) = \int_0^t |s-u|^{H-1} |r-u|^{H-1} du$$

$$\forall k \geq 3, K_t^{k-2}(s, r) = \int_0^t \dots \int_0^t |s-x_1|^{H-1} |x_2-x_1|^{H-1} \dots |x_{k-2}-x_{k-3}|^{H-1} |r-x_{k-2}|^{H-1} dx_1 \dots dx_{k-2}$$

**Remark 2.5:** The type of integrals used in the definition of the sequence of kernels  $(K_t^k(s, r))_k$  intervenes in the calculation of the cumulants of the Rosenblatt distribution (see [117]). Indeed, we have, for all  $r \geq 2$ :

$$\kappa_r(X_1^H) = 2^{r-1}(r-1)! \left( \sqrt{\frac{H(2H-1)}{2}} \right)^r \int_0^1 \dots \int_0^1 |x_1-x_2|^{H-1} \dots |x_r-x_{r-1}|^{H-1} |x_1-x_r|^{H-1} dx_1 \dots dx_r \quad (2.3)$$

And, by Remark 2.2,

$$\sum_{n=1}^{+\infty} \lambda_n^r = \left( \sqrt{\frac{H(2H-1)}{2}} \right)^r \int_0^1 \dots \int_0^1 |x_1-x_2|^{H-1} \dots |x_r-x_{r-1}|^{H-1} |x_1-x_r|^{H-1} dx_1 \dots dx_r \quad (2.4)$$

**Proof:**

Let  $\xi$  be in  $S(\mathbb{R})$ . Note that  $I_+^{\frac{H}{2}}(\xi) \in C^\infty(\mathbb{R})$ . Using Fubini's theorem and the definition of the fractional integral of Weyl's type of order  $\frac{H}{2}$ , we have:

$$\langle f_t^H; \xi^{\otimes 2} \rangle = d(H) \int_0^t [I_+^{\frac{H}{2}}(\xi)(s)]^2 ds$$

Moreover, using Fubini's theorem and Lemma 2.3, we have:

$$\begin{aligned} \langle f_t^H \otimes_1 f_t^H; \xi^{\otimes 2} \rangle &= \int_{\mathbb{R}^2} (f_t^H \otimes_1 f_t^H)(x_1, x_2) \xi(x_1) \xi(x_2) dx_1 dx_2 \\ &= \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} f_t^H(x_1, u) f_t^H(x_2, u) du \right) \xi(x_1) \xi(x_2) dx_1 dx_2 \\ &= (d(H))^2 \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) \left( \int_{\mathbb{R}} \frac{(s-u)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(r-u)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} du \right) ds dr \\ &= (d(H))^2 \frac{\beta(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^2} \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) |s-r|^{H-1} ds dr \\ &= d(H) \sqrt{\frac{H(2H-1)}{2}} \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) |s-r|^{H-1} ds dr \end{aligned}$$

Now, using iteratively Fubini's theorem and Lemma 2.3, we obtain straightforwardly for all  $k \geq 3$ :

$$\begin{aligned} \langle \underbrace{(\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H)}_{k-1 \times \otimes_1}; \xi^{\otimes 2} \rangle &= d(H) \left( \sqrt{\frac{H(2H-1)}{2}} \right)^{k-1} \\ &\int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) K_t^{k-2}(s, r) ds dr. \quad \square \end{aligned}$$

**Lemma 2.6:** For any  $\xi \in S(\mathbb{R})$ :

$$I_2(f_t^H)(\cdot + \xi) \stackrel{(d)}{=} t^H \sum_{n=1}^{+\infty} \lambda_n \left[ X_n^2 - 1 + 2 \langle \xi; \frac{e_n(\cdot/t)}{\sqrt{t}} \rangle X_n \right] + \langle f_t^H; \xi^{\otimes 2} \rangle$$

where  $(X_n)$  is a sequence of independent standard normal random variables.

**Proof:** This is just a direct application of Lemma 1.7 and of the diagonal decomposition of the kernel  $f_t^H$ .  $\square$

**Definition 2.7:** A Rosenblatt distribution is the law of the Rosenblatt process at time 1 whose characteristic function admits the following analytic representation in a neighborhood of the origin:

$$\mathbb{E}[\exp(i\theta X_1^H)] = \exp \left( \frac{1}{2} \sum_{k=2}^{+\infty} \frac{(2i\theta)^k}{k} \left( \sqrt{\frac{(2H-1)H}{2}} \right)^k c_k \right) \quad (2.5)$$

where,

$$c_k = \int_0^1 \dots \int_0^1 |x_1 - x_2|^{H-1} \dots |x_k - x_{k-1}|^{H-1} |x_1 - x_k|^{H-1} dx_1 \dots dx_k$$

**Remark 2.8:**

- Using Cauchy-Schwarz inequality, one can show that  $|c_k| \leq \left( \frac{1}{H(2H-1)} \right)^{\frac{k}{2}}$  ensuring that the former series is convergent in a neighborhood of the origin and thus analytic in this neighborhood. Nevertheless, using a result by Lukacs (see theorem 7.1.1 in [68]), this characteristic function is actually analytic in a horizontal strip of the complex plane containing the real line.
- In [117] and [77] this distribution is shown to be infinitely divisible and, thus, its characteristic function admits a Lévy-Khintchine representation. Using this representation, one can show that the density of the Rosenblatt distribution exists and is infinitely differentiable with all derivative tending to 0 at  $+\infty$  (Corollary 4.3 of [117]).

We are now ready to state the main result of this section.

**Theorem 2.9:** Let  $\xi \in S(\mathbb{R})$  and  $t > 0$ . We have (for  $\theta$  being small enough):

$$\begin{aligned} \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)] &= \exp(i\theta \langle f_t^H; \xi^{\otimes 2} \rangle) \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} \frac{(2i\theta t^H)^k}{k} \left(\sqrt{\frac{(2H-1)H}{2}}\right)^k c_k\right) \\ &\quad \times \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} (2i\theta)^k \|T_t^{\frac{k}{2}}(\xi)\|_{L^2(\mathbb{R})}^2\right) \end{aligned}$$

where  $T_t^{k/2}$  denotes the operator  $T_{f_t^H}$ , associated with  $f_t^H$ , iterated  $k/2$ -times. Moreover, we have:

$$\begin{aligned} \forall k \geq 2 \quad \|T_t^{\frac{k}{2}}(\xi)\|_{L^2(\mathbb{R})}^2 &= \sum_{n=1}^{\infty} (\lambda_n t^H)^k \langle \xi; \frac{e_n(\cdot/t)}{\sqrt{t}} \rangle^2, \\ &= \langle \underbrace{(\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H)}_{k-1 \times \otimes_1}; \xi^{\otimes 2} \rangle. \end{aligned} \quad (2.6)$$

**Remark 2.10:**

- Under the probability  $\mu_\xi$ , the random variable  $X_t^H$  has a non-zero mean and is the sum of two independent random variables with one of them having the law of the Rosenblatt process at time  $t$ .
- Trivially, we obtain the following estimates ensuring the convergence around the origin of the second series in the expression of  $\mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)]$ :

$$\|T_t^{\frac{k}{2}}(\xi)\|_{L^2(\mathbb{R})}^2 \leq \left\| \underbrace{(\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H)}_{k-1 \times \otimes_1} \right\|_{L^2(\mathbb{R}^2)} \|\xi\|_{L^2(\mathbb{R})}^2$$

$$\|T_t^{\frac{k}{2}}(\xi)\|_{L^2(\mathbb{R})}^2 \leq \left( \sum_{n=1}^{\infty} (\lambda_n t^H)^{2k} \right)^{\frac{1}{2}} \|\xi\|_{L^2(\mathbb{R})}^2 = t^{Hk} \left( \sqrt{\frac{(2H-1)H}{2}} \right)^k (c_{2k})^{\frac{1}{2}} \|\xi\|_{L^2(\mathbb{R})}^2$$

**Proof:** Let  $\xi \in S(\mathbb{R})$  and  $t > 0$ . By Lemma 2.6, we have:

$$X_t^H(\cdot + \xi) \stackrel{(d)}{=} t^H \sum_{n=1}^{+\infty} \lambda_n \left[ X_n^2 - 1 + 2 \langle \xi; \frac{e_n(\cdot/t)}{\sqrt{t}} \rangle X_n \right] + \langle f_t^H; \xi^{\otimes 2} \rangle$$

Let us compute the characteristic function of  $X_n^2 - 1 + 2 \langle \xi; \frac{e_n(\cdot/t)}{\sqrt{t}} \rangle X_n$  where  $X_n$  is a standard normal random variable. Let  $\alpha > 0$ ,  $\beta \in \mathbb{R}$  and  $X$  be a centered normal random variable with variance  $\frac{1}{2\alpha}$ . By standard computations, we get the following formulae for the density and the Laplace transform of  $Y = (X + \beta)^2$ :

$$\forall y \in \mathbb{R} \quad f_Y(y) = 1_{(0,+\infty)}(y) \sqrt{\frac{\alpha}{\pi}} \exp(-\alpha\beta^2) \frac{\exp(-\alpha y)}{\sqrt{y}} \cosh(2\alpha\beta\sqrt{y})$$

$$\forall \mu > 0 \quad \mathbb{E}[\exp(-\mu Y)] = \left( \frac{\alpha}{\alpha + \mu} \right)^{\frac{1}{2}} \exp(-\alpha\beta^2) \exp\left(\frac{\alpha^2\beta^2}{\alpha + \mu}\right)$$

Subtracting  $(\frac{1}{2\alpha} + \beta^2)$  from  $Y$  in order to obtain a centered random variable, we have:

$$\forall \mu > 0 \quad \mathbb{E}[\exp(-\mu Y')] = \left( \frac{\alpha}{\alpha + \mu} \right)^{\frac{1}{2}} \exp\left(\frac{\mu}{2\alpha}\right) \exp(-\alpha\beta^2) \exp\left(\frac{\alpha^2\beta^2}{\alpha + \mu}\right) \exp(\mu\beta^2)$$

Putting  $\alpha = \frac{1}{2}$ ,  $\beta_n = \langle \xi; \frac{e_n(\cdot/t)}{\sqrt{t}} \rangle$  and  $Y = (X_n + \beta_n)^2$ , we get:

$$\forall n \in \mathbb{N}^*, \forall \mu > 0 \quad \mathbb{E}[\exp(-\mu\lambda_n(X_n^2 - 1 + 2\beta_n X_n))] = \left( \frac{1}{1 + 2\lambda_n\mu} \right)^{\frac{1}{2}} \exp(\lambda_n\mu) \exp\left(\frac{2\beta_n^2\lambda_n^2\mu^2}{1 + 2\lambda_n\mu}\right)$$

Leading to the following formula for the characteristic function of  $\lambda_n(X_n^2 - 1 + 2\beta_n X_n)$ :

$$\forall \theta \in \mathbb{R} \quad \mathbb{E}[\exp(i\theta\lambda_n(X_n^2 - 1 + 2\beta_n X_n))] = \left( \frac{1}{1 - 2i\lambda_n\theta} \right)^{\frac{1}{2}} \exp(-i\lambda_n\theta) \exp\left(-\frac{2\beta_n^2\lambda_n^2\theta^2}{1 - 2i\lambda_n\theta}\right)$$

Using the independence of the  $X_n$ 's, we have for any  $N \geq 1$ :

$$\begin{aligned} \mathbb{E}[\exp(i\theta t^H \sum_{n=1}^N \lambda_n(X_n^2 - 1 + 2\beta_n X_n))] &= \left( \prod_{n=1}^N \left( \frac{1}{1 - 2i\lambda_n t^H \theta} \right)^{\frac{1}{2}} \exp(-i\lambda_n t^H \theta) \right) \\ &\times \exp\left(-2\theta^2 \sum_{n=1}^N \frac{\beta_n^2 \lambda_n^2 t^{2H}}{1 - 2i\lambda_n t^H \theta}\right). \end{aligned}$$

The first product term converges to the distribution of the Rosenblatt process at time  $t$  as  $N \rightarrow +\infty$ . Using the following estimate:

$$\forall \theta \in \mathbb{R} \quad \left| \frac{\beta_n^2 \lambda_n^2}{1 - 2i\lambda_n t^H \theta} \right| \leq |\beta_n|^2 \lambda_n^2$$

and, (2.6) for  $k = 2$ , we obtain:

$$\begin{aligned} \forall \theta \in \mathbb{R} \quad \mathbb{E}^{\mu^\varepsilon}[\exp(i\theta X_t^H)] &= \exp(i\theta \langle f_t^H; \xi^{\otimes 2} \rangle) \left( \prod_{n=1}^{+\infty} \left( \frac{1}{1 - 2i\lambda_n t^H \theta} \right)^{\frac{1}{2}} \exp(-i\lambda_n t^H \theta) \right) \\ &\times \exp\left(-2\theta^2 \sum_{n=1}^{+\infty} \frac{\beta_n^2 \lambda_n^2 t^{2H}}{1 - 2i\lambda_n t^H \theta}\right) \end{aligned}$$

For  $\theta$  small enough, using (2.4) and (2.6), we get:

$$\prod_{n=1}^{+\infty} \left( \left( \frac{1}{1 - 2i\lambda_n t^H \theta} \right)^{\frac{1}{2}} \exp(-i\lambda_n t^H \theta) \right) = \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} \frac{(2i\theta t^H)^k}{k} \left( \sqrt{\frac{(2H-1)H}{2}} \right)^k c_k\right)$$



$$\exp\left(-2\theta^2 \sum_{n=1}^{+\infty} \frac{\beta_n^2 \lambda_n^2 t^{2H}}{1 - 2i\lambda_n t^H \theta}\right) = \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} (2i\theta)^k \|T_t^{\frac{k}{2}}(\xi)\|^2\right)$$

ensuring the analyticity of the translated characteristic function in a neighborhood of the origin, and, by theorem 7.1.1 of [68], the analyticity in a certain horizontal strip. Thus, this characteristic function is in  $C^\infty(\mathbb{R})$ .

Moreover, using (2.2) and (2.6), we have:

$$\|T_t^{\frac{k}{2}}(\xi)\|_{L^2(\mathbb{R})}^2 = \langle \underbrace{\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H}_{k-1 \times \otimes_1}; \xi^{\otimes 2} \rangle. \square$$

### 3 Stochastic calculus and Itô formula with respect to the Rosenblatt process.

In this section, we define differentiability as well as integrability of stochastic distribution processes in  $(S)^*$ . In particular, we compute the  $(S)^*$  derivative of the Rosenblatt process,  $\dot{X}_t^H$ , named the Rosenblatt noise. Thus, using Wick product and Pettis integration, we define the Rosenblatt noise integral of integrable stochastic distribution process. We end this section with Itô formulae for functionals of the Rosenblatt process.

**Definition 3.1:**

1. Let  $I \subset \mathbb{R}$  be an interval. A mapping  $X : I \rightarrow (S)^*$  is called a stochastic distribution process.
2. A stochastic distribution process  $X$  is said to be differentiable, if  $\lim_{h \rightarrow 0} \frac{X_{t+h} - X_t}{h}$  exists in  $(S)^*$ .

**Remark 3.2:** By the last point of Theorem 1.3, convergence in  $(S)^*$  is ensured by pointwise convergence of the  $S$ -transform and a uniform growth condition.

**Lemma 3.3:** Let  $\xi \in S(\mathbb{R})$  and  $H \in (\frac{1}{2}, 1)$ . Then  $I_+^{\frac{H}{2}}(\xi) \in C^\infty(\mathbb{R})$  and it exists a strictly positive constant  $C_H$  such that:

$$\|I_+^{\frac{H}{2}}(\xi)\|_\infty \leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1)$$

**Proof:** See the proof of Theorem 2.3 in [21].  $\square$

Note that the previous inequality makes  $\delta_t \circ I_+^{\frac{H}{2}}$  a tempered distribution, for all  $t \in \mathbb{R}$ , where  $\delta_t$  denotes the Dirac delta function at time  $t$ .

**Lemma 3.4:** The Rosenblatt process is  $(S)^*$  differentiable and its derivative, the Rosenblatt noise, admits the following generalized double Wiener-Itô integral representation (as an element of  $(S)^*$ ):

$$\forall t > 0 \quad \dot{X}_t^H = d(H)I_2(\delta_t^{\otimes 2} \circ (I_+^{\frac{H}{2}})^{\otimes 2})$$

and its  $S$ -transform is equal to:

$$\forall \xi \in S(\mathbb{R}) \quad S(\dot{X}_t^H)(\xi) = d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2$$

**Proof:** Let  $t > 0$ . By Fubini's theorem, we have:

$$\forall \xi \in S(\mathbb{R}) \quad S(X_t^H)(\xi) = d(H) \int_0^t (I_+^{\frac{H}{2}}(\xi)(s))^2 ds$$

Thus,

$$\forall h > 0 \quad S\left(\frac{X_{t+h}^H - X_t^H}{h}\right)(\xi) = \frac{d(H)}{h} \int_t^{t+h} (I_+^{\frac{H}{2}}(\xi)(s))^2 ds$$

By Lemma 3.3, we obtain the following estimate:

$$\left| S\left(\frac{X_{t+h}^H - X_t^H}{h}\right)(\xi) \right| \leq C_H d(H) (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1)^2$$

Using Theorem 1.3,  $\frac{X_{t+h}^H - X_t^H}{h}$  converges in  $(S)^*$  and the limit is completely characterized by its  $S$ -transform:

$$S(\dot{X}_t^H)(\xi) = d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2$$

By a similar argument than the one used in the representation of the fractional noise (see [21] page 90), we can extend the double Wiener-Itô integral to elements of  $\hat{S}'(\mathbb{R}^2)$ . Consequently, we get the following representation for the Rosenblatt noise:

$$\dot{X}_t^H = d(H)I_2(\delta_t^{\otimes 2} \circ (I_+^{\frac{H}{2}})^{\otimes 2})$$

**Remark 3.5:** Let  $d \geq 1$  and let  $H \in (\frac{1}{2}, 1)$ . Let  $H_0 = \frac{1}{2} + \frac{H-1}{d}$ . We define the Hermite process of order  $d$  by the following multiple Wiener-Itô integral representation:

$$\forall t > 0 \quad Y_t^{H,d} = d(H_0) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left( \int_0^t \prod_{j=1}^d \left( \frac{(s-x_j)_+^{H_0-1}}{\Gamma(H_0)} \right) ds \right) dB_{x_1} \dots dB_{x_d},$$

where  $d(H_0)$  is a normalizing constant such that  $\mathbb{E}[|Y_1^{H,d}|^2] = 1$ . The Hermite processes of order 1 and 2 are respectively fractional Brownian motion and the Rosenblatt process. As previously, we obtain the following formula and representation for the derivative in  $(S)^*$  of the Hermite process of order  $d$ :

$$\forall t > 0 \quad \dot{Y}_t^{H,d} = d(H_0)I_d(\delta_t^{\otimes d} \circ (I_+^{H_0})^{\otimes d}),$$

$$\forall \xi \in S(\mathbb{R}) \quad S(\dot{Y}_t^{H,d})(\xi) = d(H_0)(I_+^{H_0}(\xi)(t))^d.$$

**Definition 3.6:** Let  $k \geq 2$ . For any  $t > 0$ , we define the following sequence of stochastic processes (belonging to the second Wiener chaos):

$$X_t^{H,k} = \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{(\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H)}_{k-1 \times \otimes_1}(x_1, x_2) dB_{x_1} dB_{x_2}$$

Moreover, by Lemma 2.4, the  $S$ -transform of this process is given by:

$$\forall \xi \in S(\mathbb{R}) \quad S(X_t^{H,k})(\xi) = d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) K_t^{k-2}(s, r) ds dr$$

**Lemma 3.7:** For any  $k \geq 2$ , the process  $\{X_t^{H,k} : t > 0\}$  is  $(S)^*$  differentiable and its derivative,  $\dot{X}_t^{H,k}$ , is uniquely defined by the following  $S$ -transform:

$$\begin{aligned} \forall \xi \in S(\mathbb{R}) \quad S(\dot{X}_t^{H,k})(\xi) = & d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} \\ & \times \left[ (H(k-1) + 1) t^{H(k-1)} \int_0^1 \int_0^1 I_+^{\frac{H}{2}}(\xi)(tu) I_+^{\frac{H}{2}}(\xi)(tv) K_1^{k-2}(u, v) dudv \right. \\ & \left. + t^{H(k-1)+1} \int_0^1 \int_0^1 \frac{d}{dt} [I_+^{\frac{H}{2}}(\xi)(tu) I_+^{\frac{H}{2}}(\xi)(tv)] K_1^{k-2}(u, v) dudv \right]. \end{aligned}$$

**Proof:** Let  $(a, b) \in \mathbb{R}_+^*$  such that  $0 < a < b < \infty$ . Let  $t \in (a, b)$  and  $\xi \in S(\mathbb{R})$ . In order to prove the  $(S)^*$  differentiability of the process  $X_t^{H,k}$ , we will proceed as in the proof of the former lemma. Let  $h > 0$  such that  $t+h \in (a, b)$ . We have:

$$S(X_t^{H,k})(\xi) = d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) K_t^{k-2}(s, r) ds dr$$

Using scaling property of the sequence of kernels  $K_t^{k-2}(s, r)$ , we get:

$$S(X_t^{H,k})(\xi) = d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} t^{H(k-1)+1} \int_0^1 \int_0^1 I_+^{\frac{H}{2}}(\xi)(tu) I_+^{\frac{H}{2}}(\xi)(tv) K_1^{k-2}(u, v) dudv$$

Thus, we have:

$$|S(X_{t+h}^{H,k})(\xi) - S(X_t^{H,k})(\xi)| \leq d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} [(I) + (II)]$$

where,

$$(I) = |(t+h)^{H(k-1)+1} - t^{H(k-1)+1}| \left| \int_0^1 \int_0^1 I_+^{\frac{H}{2}}(\xi)((t+h)u) I_+^{\frac{H}{2}}(\xi)((t+h)v) K_1^{k-2}(u, v) dudv \right|$$

$$(II) = t^{H(k-1)+1} \left| \int_0^1 \int_0^1 \left[ I_+^{\frac{H}{2}}(\xi)((t+h)u) I_+^{\frac{H}{2}}(\xi)((t+h)v) - I_+^{\frac{H}{2}}(\xi)(tu) I_+^{\frac{H}{2}}(\xi)(tv) \right] K_1^{k-2}(u, v) dudv \right|$$

Combining Lemma 3.3 and the Mean value theorem, we obtain for (I):

$$(I) \leq (H(k-1) + 1) b^{H(k-1)} C_H^2 (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1)^2 \underbrace{\left( \int_0^1 \int_0^1 K_1^{k-2}(u, v) dudv \right)}_{< \infty} h \quad (2.7)$$

For the second term, we have:

$$(II) \leq 2b^{H(k-1)+1} \|I_+^{\frac{H}{2}}(\xi)\|_\infty \int_0^1 \int_0^1 \left| I_+^{\frac{H}{2}}(\xi)((t+h)u) - I_+^{\frac{H}{2}}(\xi)(tu) \right| K_1^{k-2}(u, v) dudv$$

Using the fact that  $I_+^{\frac{H}{2}}(\xi) \in C^\infty(\mathbb{R})$ , we have as previously:

$$\begin{aligned} (II) &\leq 2b^{H(k-1)+1} \|I_+^{\frac{H}{2}}(\xi)\|_\infty \|I_+^{\frac{H}{2}}(\xi^{(1)})\|_\infty \left( \int_0^1 \int_0^1 K_1^{k-2}(u, v) dudv \right) h \\ (II) &\leq 2b^{H(k-1)+1} C_H^2 (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1) \\ &\quad (\|\xi^{(1)}\|_\infty + \|\xi^{(2)}\|_\infty + \|\xi^{(1)}\|_1) \left( \int_0^1 \int_0^1 K_1^{k-2}(u, v) dudv \right) h \end{aligned} \quad (2.8)$$

By (2.7) and (2.8),  $|S(\frac{X_{t+h}^{H,k} - X_t^{H,k}}{h})(\xi)|$  verifies the uniform growth condition of the last point of Theorem 1.3.

Moreover, the pointwise convergence of  $S(\frac{X_{t+h}^{H,k} - X_t^{H,k}}{h})(\xi)$  as  $h \rightarrow 0^+$  is ensured by the dominated convergence theorem.

Consequently, the process  $X^{H,k}$  is  $(S)^*$  differentiable for any  $k \geq 2$  and its  $(S)^*$ -derivative admits the following  $S$ -transform:

$$\begin{aligned} S(\dot{X}_t^{H,k})(\xi) &= d(H) \sqrt{\frac{H(2H-1)}{2}}^{k-1} \left[ (H(k-1) + 1) t^{H(k-1)} \int_0^1 \int_0^1 I_+^{\frac{H}{2}}(\xi)(tu) I_+^{\frac{H}{2}}(\xi)(tv) K_1^{k-2}(u, v) dudv \right. \\ &\quad \left. + t^{H(k-1)+1} \int_0^1 \int_0^1 \frac{d}{dt} [I_+^{\frac{H}{2}}(\xi)(tu) I_+^{\frac{H}{2}}(\xi)(tv)] K_1^{k-2}(u, v) dudv \right]. \square \end{aligned}$$

As well as differentiability, we can define integrability of a stochastic distribution process in term of the  $S$ -transform.

**Definition 3.8:** A stochastic distribution process  $X : I \rightarrow (S)^*$  is integrable if:

1.  $\forall \xi \in S(\mathbb{R})$ ,  $S(X.)(\xi)$  is measurable on  $I$ .
2.  $\forall \xi \in S(\mathbb{R})$ ,  $S(X.)(\xi) \in L^1(I)$ .

3.  $\int_I S(X_t)(\xi)dt$  is the  $S$ -transform of a certain Hida distribution.

We have the following criterion for integrability of stochastic distribution process.

**Theorem 3.9:** Let  $X : I \rightarrow (S)^*$  be a stochastic distribution process satisfying:

1.  $\forall \xi \in S(\mathbb{R})$ ,  $S(X_t)(\xi)$  is measurable on  $I$ .
2. There is a  $p \in \mathbb{N}$ , a strictly positive constant  $a$  and a non-negative function  $L \in L^1(I)$  such that:

$$\forall \xi \in S(\mathbb{R}) \quad |S(X_t)(\xi)| \leq L(t) \exp(a\|A^p \xi\|_2^2)$$

Then  $X$  is  $(S)^*$ -integrable.

**Proof:** See theorem 13.5 in [60]. $\square$

We are now ready to state the definition of the Rosenblatt noise integral. This definition is motivated by the one of the white noise integral. Indeed, in the case of stochastic integration with respect to Brownian motion in a white noise context, this integral coincides with the classical Itô integral for  $L^2(\Omega \times I)$  nonanticipating integrands (see theorem 13.12 in [60]).

**Definition-Theorem 3.10:** Let  $\{\Phi_t; t \in I\}$  be a  $(S)^*$  stochastic process satisfying the assumptions of Theorem 3.9. Then,  $\Phi_t \diamond \dot{X}_t^H$  is  $(S)^*$  integrable over  $I$  and we define the Rosenblatt noise integral of  $\{\Phi_t\}$  by:

$$\int_I \Phi_t dX_t^H = \int_I \Phi_t \diamond \dot{X}_t^H dt$$

Moreover, we have the following representation:

$$\int_I \Phi_t dX_t^H = \int_I (D^*_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}})^2(\Phi_t) dt$$

**Proof:** Using Lemma 3.4 and Theorem 3.9, one can show that  $\Phi_t \diamond \dot{X}_t^H$  is  $(S)^*$ -integrable over  $I$ . Let us compute the  $S$ -transform of  $\int_I \Phi_t dX_t^H$ .

$$\begin{aligned} \forall \xi \in S(\mathbb{R}) \quad S\left(\int_I \Phi_t dX_t^H\right)(\xi) &= \int_I S(\Phi_t)(\xi) S(\dot{X}_t^H)(\xi) dt \\ &= \int_I S(\Phi_t)(\xi) d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2 dt \end{aligned}$$

Then, using Definition-Theorem 1.9 iteratively, we obtain:

$$S\left(\int_I \Phi_t dX_t^H\right)(\xi) = \int_I S(I_1(\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}) \diamond \Phi_t)(\xi) \sqrt{d(H)} I_+^{\frac{H}{2}}(\xi)(t) dt,$$

$$\begin{aligned}
&= \int_I S(D^*_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}}(\Phi_t))(\xi) \sqrt{d(H)} I_+^{\frac{H}{2}}(\xi)(t) dt, \\
&= \int_I S((D^*_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}})^2(\Phi_t))(\xi) dt. \quad \square
\end{aligned}$$

In the rest of this section, we derive an Itô formula for a certain class of functionals of the Rosenblatt process. In the framework of white noise distribution theory, generalizations of Itô formula have been obtained for Brownian motion (See chapter 13.5 in [60]), for fractional Brownian motion (see [21]), and for convoluted Lévy processes (see [22]). When one wants to derive an Itô formula with respect to a stochastic process, the first step is to identify for which class of functionals this formula will hold. At least, we want the polynomials to belong to this class. So, in the sequel, using the multiplicative formula from Malliavin calculus, we get an Itô formula for  $x^2$  and  $x^3$  in the  $(S)^*$  sense. It appears that the series expansion intervening in the analytic representation around the origin of the logarithm of the translated characteristic function of the Rosenblatt process at time  $t$  plays a fundamental role. Then, using Schwartz's Paley-Wiener theorem, we identify the class of functionals for which this Itô formula is true.

**Lemma 3.11:** Let  $(p, q) \in \mathbb{N}^*$ . Let  $f \in \hat{L}^2(\mathbb{R}^p)$  and  $g \in \hat{L}^2(\mathbb{R}^q)$ . Then, we have:

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! C_r^p C_r^q I_{p+q-2r}(f \otimes_r g)$$

where  $C_r^p = \frac{p!}{r!(p-r)!}$  and  $\otimes_r$  denotes the contraction of order  $r$ .

**Proof:** See proposition 1.1.3 of [87].  $\square$

**Theorem 3.12:** Let  $(a, b) > 0$  such that  $a < b$ . Then in  $(S)^*$ :

$$(X_b^H)^2 - (X_a^H)^2 = 2 \int_a^b X_s^H dX_s^H + b^{2H} - a^{2H} + 4 \int_a^b dX_s^{H,2}$$

and

$$\begin{aligned}
(X_b^H)^3 - (X_a^H)^3 &= 3 \int_a^b (X_s^H)^2 dX_s^H + 6H \int_a^b s^{2H-1} X_s^H ds + 12 \int_a^b X_s^H dX_s^{H,2} + \kappa_3(X_1^H)(b^{3H} - a^{3H}) \\
&\quad + 24 \int_a^b dX_s^{H,3}.
\end{aligned}$$

**Proof:** Let  $t > 0$  and  $\xi \in S(\mathbb{R})$ . Using Lemma 3.11, we get:

$$(X_t^H)^2 = I_4(f_t^H \hat{\otimes} f_t^H) + 4I_2(f_t^H \otimes_1 f_t^H) + 2\|f_t^H\|_{L^2(\mathbb{R}^2)}^2.$$

and,

$$(X_t^H)^3 = I_6(f_t^H \hat{\otimes} f_t^H \hat{\otimes} f_t^H) + 8I_4((f_t^H \hat{\otimes} f_t^H) \otimes_1 f_t^H) + 4I_4((f_t^H \otimes_1 f_t^H) \otimes f_t^H) + 12I_2((f_t^H \hat{\otimes} f_t^H) \otimes_2 f_t^H) \\ + 16I_2((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) + 2\|f_t^H\|_{L^2(\mathbb{R}^2)}^2 I_2(f_t^H) + 8 \langle f_t^H \otimes_1 f_t^H; f_t^H \rangle .$$

Then, using the definition of the  $S$ -transform, we have:

$$S((X_t^H)^2)(\xi) = (\langle f_t^H; \xi^{\otimes 2} \rangle)^2 + 4 \langle f_t^H \otimes_1 f_t^H; \xi^{\otimes 2} \rangle + 2\|f_t^H\|_{L^2(\mathbb{R}^2)}^2$$

$$S((X_t^H)^3)(\xi) = (\langle f_t^H; \xi^{\otimes 2} \rangle)^3 + 8 \langle (f_t^H \hat{\otimes} f_t^H) \otimes_1 f_t^H; \xi^{\otimes 4} \rangle + 4 \langle (f_t^H \otimes_1 f_t^H) \otimes f_t^H; \xi^{\otimes 4} \rangle \\ + 12 \langle (f_t^H \hat{\otimes} f_t^H) \otimes_2 f_t^H; \xi^{\otimes 2} \rangle + 16 \langle (f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H; \xi^{\otimes 2} \rangle \\ + 2\|f_t^H\|_{L^2(\mathbb{R}^2)}^2 \langle f_t^H; \xi^{\otimes 2} \rangle + 8 \langle f_t^H \otimes_1 f_t^H; f_t^H \rangle .$$

By (2.2) and the cumulant's formula, we have:

$$8 \langle f_t^H \otimes_1 f_t^H; f_t^H \rangle = 8 \sum_{n=1}^{+\infty} (\lambda_n t^H)^3 = 8t^{3H} \frac{\kappa_3(X_1^H)}{2^2 2!} = t^{3H} \kappa_3(X_1^H),$$

and

$$2\|f_t^H\|_{L^2(\mathbb{R})}^2 = t^{2H}.$$

Moreover, we have:

$$4 \langle (f_t^H \otimes_1 f_t^H) \otimes f_t^H; \xi^{\otimes 4} \rangle = 4 \langle f_t^H \otimes_1 f_t^H; \xi^{\otimes 2} \rangle \langle f_t^H; \xi^{\otimes 2} \rangle \\ 8 \langle (f_t^H \hat{\otimes} f_t^H) \otimes_1 f_t^H; \xi^{\otimes 4} \rangle = 8 \langle f_t^H; \xi^{\otimes 2} \rangle \|T_t(\xi)\|_{L^2(\mathbb{R})}^2$$

Finally, we get, by Fubini's theorem:

$$12 \langle (f_t^H \hat{\otimes} f_t^H) \otimes_2 f_t^H; \xi^{\otimes 2} \rangle = 12 \langle \langle f_t^H \hat{\otimes} f_t^H; \xi^{\otimes 2} \rangle; f_t^H \rangle .$$

By definition of the symmetric tensor product:

$$\langle f_t^H \hat{\otimes} f_t^H; \xi^{\otimes 2} \rangle = \frac{1}{4!} \sum_{\sigma \in \mathcal{S}_4} \int_{\mathbb{R}^2} f_t^H(x_{\sigma(1)}, x_{\sigma(2)}) f_t^H(x_{\sigma(3)}, x_{\sigma(4)}) \xi^{\otimes 2}(x_1, x_2) dx_1 dx_2$$

The integral inside the sum can only have two values (depending on  $\sigma$ ):

$$\int_{\mathbb{R}^2} f_t^H(x_{\sigma(1)}, x_{\sigma(2)}) f_t^H(x_{\sigma(3)}, x_{\sigma(4)}) \xi^{\otimes 2}(x_1, x_2) dx_1 dx_2 = \begin{cases} \langle f_t^H; \xi^{\otimes 2} \rangle f_t^H & \sigma = (\sigma', \sigma'') \\ T_t(\xi)^{\otimes 2} & \sigma \neq (\sigma', \sigma'') \end{cases}$$

where  $(\sigma', \sigma'') \in (S_2(\{1, 2\}^{\{1,2\}}) \times S_2(\{3, 4\}^{\{3,4\}})) \cup (S_2(\{1, 2\}^{\{3,4\}}) \times S_2(\{3, 4\}^{\{1,2\}}))$ . Thus, we get:

$$\langle f_t^H \hat{\otimes} f_t^H; \xi^{\otimes 2} \rangle = \frac{1}{3} \langle f_t^H; \xi^{\otimes 2} \rangle f_t^H + \frac{2}{3} T_t(\xi)^{\otimes 2}$$

To conclude, we have:

$$12 \langle (f_t^H \hat{\otimes} f_t^H) \otimes_2 f_t^H; \xi^{\otimes 2} \rangle = 4 \langle f_t^H; \xi^{\otimes 2} \rangle \|f_t^H\|_{L^2(\mathbb{R}^2)}^2 + 8 \langle f_t^H; T_t(\xi)^{\otimes 2} \rangle$$

Once again, using (2.2), we obtain:

$$\langle f_t^H; T_t(\xi)^{\otimes 2} \rangle = \langle (f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H; \xi^{\otimes 2} \rangle$$

And so, we obtain the following expression for the S-transform of  $(X_t^H)^3$ :

$$\begin{aligned} S((X_t^H)^3)(\xi) &= (\langle f_t^H; \xi^{\otimes 2} \rangle)^3 + 12 \langle f_t^H; \xi^{\otimes 2} \rangle \|T_t(\xi)\|_{L^2(\mathbb{R})}^2 + 3t^{2H} \langle f_t^H; \xi^{\otimes 2} \rangle \\ &\quad + 24 \langle (f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H; \xi^{\otimes 2} \rangle + t^{3H} \kappa_3(X_1^H) \end{aligned}$$

By differentiation, we get:

$$\begin{aligned} \frac{d}{dt} [S((X_t^H)^3)(\xi)] &= (3 \langle f_t^H; \xi^{\otimes 2} \rangle)^2 + 12 \|T_t(\xi)\|_{L^2(\mathbb{R})}^2 + 3t^{2H} S(\dot{X}_t^H)(\xi) \\ &\quad + (12 \frac{d}{dt} [\|T_t(\xi)\|_{L^2(\mathbb{R})}^2] + 6Ht^{2H-1}) S(X_t^H)(\xi) + 24 \frac{d}{dt} [\langle (f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H; \xi^{\otimes 2} \rangle] \\ &\quad + 3Ht^{3H-1} \kappa_3(X_1^H) \end{aligned}$$

Using the expression of the S-transform of  $(X_t^H)^2$ :

$$\begin{aligned} \frac{d}{dt} [S((X_t^H)^3)(\xi)] &= 3S((X_t^H)^2 \diamond \dot{X}_t^H)(\xi) + 6(Ht^{2H-1} + 2 \frac{d}{dt} [\|T_t(\xi)\|_{L^2(\mathbb{R})}^2]) S(X_t^H)(\xi) \\ &\quad + 6(H \frac{t^{3H-1}}{2} \kappa_3(X_1^H) + 2^2 \frac{d}{dt} [\langle (f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H; \xi^{\otimes 2} \rangle]) \end{aligned}$$

And by Lemma 3.7:

$$\begin{aligned} \frac{d}{dt} [S((X_t^H)^3)(\xi)] &= 3S((X_t^H)^2 \diamond \dot{X}_t^H)(\xi) + (6Ht^{2H-1} + 12S(\dot{X}_t^{H,2})(\xi)) S(X_t^H)(\xi) \\ &\quad + 3Ht^{3H-1} \kappa_3(X_1^H) + 24S(\dot{X}_t^{H,3})(\xi) \end{aligned}$$

In order to conclude, we have to show the  $(S)^*$ -integrability of the distribution processes  $(X_t^H)^2 \diamond \dot{X}_t^H$ ,  $t^{2H-1} X_t^H$  and  $X_t^H \diamond \dot{X}_t^{H,2}$  on  $(a, b)$ . By definition of the Wick product, we have:

$$S((X_t^H)^2 \diamond \dot{X}_t^H)(\xi) = S((X_t^H)^2)(\xi) S(\dot{X}_t^H)(\xi)$$



Using the Chaos expansion of  $(X_t^H)^2$  and Lemma 3.4,

$$S((X_t^H)^2 \diamond \dot{X}_t^H)(\xi) = d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2 \left( (S(X_t^H)(\xi))^2 + 4 \langle f_t^H \otimes_1 f_t^H; \xi^{\otimes 2} \rangle + t^{2H} \right)$$

And by Lemma 2.4,

$$\begin{aligned} S((X_t^H)^2 \diamond \dot{X}_t^H)(\xi) = & d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2 \left[ (d(H) \int_0^t [I_+^{\frac{H}{2}}(\xi)(s)]^2 ds)^2 \right. \\ & \left. + 4d(H) \sqrt{\frac{H(2H-1)}{2}} \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) |s-r|^{H-1} ds dr + t^{2H} \right] \end{aligned}$$

Since  $I_+^{\frac{H}{2}}(\xi) \in C^\infty(\mathbb{R})$ ,  $t \rightarrow S((X_t^H)^2 \diamond \dot{X}_t^H)(\xi)$  is continuous on  $(a, b)$  and then measurable.

Moreover using Lemma 3.3 and the fact that  $\int_0^1 \int_0^1 |s-r|^{H-1} ds dr < \infty$ , we get easily the last point of theorem 3.9 ensuring the  $(S)^*$ -integrability of the distribution process  $(X_t^H)^2 \diamond \dot{X}_t^H$ . We proceed as previously to get the  $(S)^*$ -integrability of the distribution processes  $X_t^H \diamond \dot{X}_t^{H,2}$  and  $t^{2H-1} X_t^H$ .  $\square$

**Remark 3.13:**

- For  $x^3$ , we note the appearance of two terms involving the third derivative of this monomial. As it is written in [113], one cannot hope to obtain an Itô formula which ends with a second derivative term for the Rosenblatt process.
- Before stating the main results, we need an estimate of the radius of convergence of the power series expansion of the logarithm of the translated characteristic function of the Rosenblatt process at time  $t$  in a neighborhood of the origin. Using,  $|c_k| \leq (\frac{1}{H(2H-1)})^{\frac{k}{2}}$  and Theorem 2.9, the power series expansion converges if  $|\sqrt{2\theta}t^H| < 1$ . For  $t \in (a, b)$  where  $0 < a \leq b < \infty$  we obtain the sufficient condition  $|\theta| < \frac{1}{\sqrt{2b^H}}$ .

**Theorem 3.14:** Let  $(a, b) \in \mathbb{R}_+^*$  such that  $a \leq b < \infty$ . Let  $F$  be an entire analytic function of the complex variable verifying:

$$\forall N \in \mathbb{N}, \exists C_N > 0, \forall z \in \mathbb{C} \quad |F(z)| \leq C_N \frac{\exp(\frac{1}{\sqrt{2b^H}} |\Im(z)|)}{(1 + |z|)^N}$$

Then, we have in  $(S)^*$ :

$$\begin{aligned} F(X_b^H) - F(X_a^H) = & \int_a^b F^{(1)}(X_t^H) \diamond \dot{X}_t^H dt + \sum_{k=2}^{\infty} \left( H \kappa_k(X_1^H) \int_a^b \frac{t^{Hk-1}}{(k-1)!} F^{(k)}(X_t^H) dt \right) \\ & + \sum_{k=2}^{\infty} \left( 2^{k-1} \int_a^b F^{(k)}(X_t^H) \diamond \dot{X}_t^{H,k} dt \right). \end{aligned}$$

**Proof:** Let  $(a, b) \in \mathbb{R}_+^*$  such that  $a \leq b < \infty$  and let  $F$  be as in the theorem. First of all, let's compute the  $S$ -transform of  $F(X_t^H)$  for  $t \in (a, b)$ . By definition, we have:

$$\begin{aligned} \forall \xi \in S(\mathbb{R}) \quad S(F(X_t^H))(\xi) &= \mathbb{E}[F(X_t^H) : \exp(\langle \cdot, \xi \rangle) :] = \int_{S'(\mathbb{R})} F((X_t^H)(x)) d\mu_\xi(x) \\ &= \int_{S'(\mathbb{R})} F((X_t^H)(x + \xi)) d\mu(x) \end{aligned}$$

Using the Transfer theorem and the Parseval-Plancherel's theorem, we obtain:

$$S(F(X_t^H))(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}(F)(\theta) \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)] d\theta$$

Moreover, by the Paley-Wiener theorem (see [96], Theorem 9.11), the Fourier transform of  $F$  is in  $C_0^\infty(\mathbb{R})$  and its support is contained in the ball  $\{\theta : |\theta| \leq \frac{1}{\sqrt{2b^H}}\}$ . Thus,

$$\begin{aligned} S(F(X_t^H))(\xi) &= \frac{1}{2\pi} \int_{\text{supp}(\mathcal{F}(F))} \mathcal{F}(F)(\theta) \exp(i\theta \langle f_t^H; \xi^{\otimes 2} \rangle) \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} \frac{(2i\theta t^H)^k}{k} \left(\sqrt{\frac{(2H-1)H}{2}}\right)^k c_k\right) \\ &\quad \times \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} (2i\theta)^k \|T_t^{\frac{k}{2}}(\xi)\|_2^2\right) d\theta \end{aligned}$$

where, we have (by Theorem 2.9):

$$\|T_t^{\frac{k}{2}}(\xi)\|_2^2 = \langle \underbrace{(\dots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \dots \otimes_1 f_t^H)}_{k-1 \times \otimes_1}; \xi^{\otimes 2} \rangle = S(X_t^{H,k})(\xi)$$

We differentiate with respect to  $t$  and get:

$$\begin{aligned} \frac{d}{dt} S(F(X_t^H))(\xi) &= \frac{1}{2\pi} \int_{\text{supp}(\mathcal{F}(F))} \mathcal{F}(F)(\theta) \frac{d}{dt} \left[ \exp(i\theta \langle f_t^H; \xi^{\otimes 2} \rangle) \right. \\ &\quad \left. \times \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} \frac{(2i\theta t^H)^k}{k} \left(\sqrt{\frac{(2H-1)H}{2}}\right)^k c_k\right) \exp\left(\frac{1}{2} \sum_{k=2}^{+\infty} (2i\theta)^k S(X_t^{H,k})(\xi)\right) \right] d\theta. \end{aligned}$$

The differentiation of the translated characteristic function with respect to  $t \in (a, b)$  for  $\theta \in \text{supp}(\mathcal{F}(F))$  gives:

$$\frac{d}{dt} [\mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)]] = (I) + (II) + (III)$$

where,

$$\begin{aligned} (I) &= i\theta S(\dot{X}_t^H)(\xi) \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)], \\ (II) &= \frac{H}{2} \sum_{k=2}^{+\infty} \left[ t^{Hk-1} (2i\theta)^k \left(\sqrt{\frac{(2H-1)H}{2}}\right)^k c_k \right] \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)], \end{aligned}$$

and by Lemma 3.7,

$$(III) = \frac{1}{2} \sum_{k=2}^{+\infty} \left[ (2i\theta)^k S(\dot{X}_t^{H,k})(\xi) \right] \mathbb{E}^{\mu_\xi} [\exp(i\theta X_t^H)].$$

Since we integrate over  $\text{supp}(\mathcal{F}(F))$  strictly contained in  $\{\theta : |\theta| \leq \frac{1}{\sqrt{2b^H}}\}$  where the power series are absolutely convergent, we have:

$$\begin{aligned} \frac{d}{dt} S(F(X_t^H))(\xi) &= S(\dot{X}_t^H)(\xi) \left( \frac{1}{2\pi} \int_{\text{supp}(\mathcal{F}(F))} \mathcal{F}(F)(\theta) i\theta \mathbb{E}^{\mu_\xi} [\exp(i\theta X_t^H)] d\theta \right) \\ &+ \sum_{k=2}^{\infty} \left[ 2^{k-1} H t^{Hk-1} \left( \sqrt{\frac{(2H-1)H}{2}} \right)^k c_k \left( \frac{1}{2\pi} \int_{\text{supp}(\mathcal{F}(F))} \mathcal{F}(F)(\theta) (i\theta)^k \mathbb{E}^{\mu_\xi} [\exp(i\theta X_t^H)] d\theta \right) \right] \\ &+ \sum_{k=2}^{\infty} \left[ 2^{k-1} S(\dot{X}_t^{H,k})(\xi) \left( \frac{1}{2\pi} \int_{\text{supp}(\mathcal{F}(F))} \mathcal{F}(F)(\theta) (i\theta)^k \mathbb{E}^{\mu_\xi} [\exp(i\theta X_t^H)] d\theta \right) \right] \end{aligned}$$

Using  $(i\theta)^k \mathcal{F}(F)(\theta) = \mathcal{F}(F^{(k)})(\theta)$ , the formula for  $S(F(X_t^H))(\xi)$  and (2.3), we obtain:

$$\begin{aligned} \frac{d}{dt} S(F(X_t^H))(\xi) &= S(\dot{X}_t^H)(\xi) S(F^{(1)}(X_t^H))(\xi) + \sum_{k=2}^{\infty} \left[ H \frac{t^{Hk-1}}{(k-1)!} \kappa_k(X_1^H) S(F^{(k)}(X_t^H))(\xi) \right] \\ &+ \sum_{k=2}^{\infty} \left[ 2^{k-1} S(\dot{X}_t^{H,k})(\xi) S(F^{(k)}(X_t^H))(\xi) \right] \end{aligned}$$

To conclude we need to show the  $(S)^*$ -integrability over  $(a, b)$  of the three former terms using Theorem 3.9. From the formula of Lemma 3.7 we get the following estimate:

$$\begin{aligned} |S(\dot{X}_t^{H,k})(\xi)| &\leq C_1(H) \sqrt{\frac{H(2H-1)}{2}}^k \left[ (H(k-1) + 1) t^{H(k-1)} (\|I_+^{\frac{H}{2}}(\xi)\|_\infty)^2 \int_0^1 \int_0^1 K_1^{k-2}(u, v) dudv \right. \\ &\left. + 2t^{H(k-1)+1} \|I_+^{\frac{H}{2}}(\xi)\|_\infty \|I_+^{\frac{H}{2}}(\xi^{(1)})\|_\infty \int_0^1 \int_0^1 K_1^{k-2}(u, v) dudv \right]. \end{aligned} \quad (2.9)$$

By Cauchy-Schwartz inequality, we have:

$$\int_0^1 \int_0^1 K_1^{k-2}(u, v) dudv \leq \left( \frac{1}{2} \frac{2}{(2H-1)H} \right)^{\frac{k}{2}} < \infty$$

Moreover, we have:

$$|S(F^{(k)}(X_t^H))(\xi)| \leq \frac{1}{2\pi} \int_{\text{supp}(\mathcal{F}(F))} |\mathcal{F}(F)(\theta)| |\theta|^k d\theta$$

Writting  $\text{supp}(\mathcal{F}(F)) = [m_F; M_F] \subset \{\theta : |\theta| < \frac{1}{\sqrt{2b^H}}\}$ :

$$|S(F^{(k)}(X_t^H))(\xi)| \leq \frac{1}{2\pi} \|\mathcal{F}(F)\|_{L^1(\mathbb{R})} \max\{|m_F|^k; |M_F|^k\}$$

So the general term of the third series is bounded by:

$$|2^k S(\dot{X}_t^{H,k})(\xi)S(F^{(k)}(X_t^H))(\xi)| \leq C_2(H) \frac{1}{2\pi} (\sqrt{2})^k \|\mathcal{F}(F)\|_{L^1(\mathbb{R})} \max\{|m_F|^k; |M_F|^k\} \\ \left[ (H(k-1) + 1)t^{H(k-1)} (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1)^2 \right. \\ \left. + 2t^{H(k-1)+1} (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1) (\|\xi^{(1)}\|_\infty + \|\xi^{(2)}\|_\infty + \|\xi^{(1)}\|_1) \right].$$

Thus, this term is  $(S)^*$ -integrable over  $(a, b)$  and we have:

$$\int_a^b |2^k S(\dot{X}_t^{H,k})(\xi)S(F^{(k)}(X_t^H))(\xi)| dt \leq C_2(H) \frac{1}{2\pi} (\sqrt{2})^k \|\mathcal{F}(F)\|_{L^1(\mathbb{R})} \max\{|m_F|^k; |M_F|^k\} \\ \left[ (b^{H(k-1)+1} - a^{H(k-1)+1}) (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1)^2 \right. \\ \left. + \frac{2}{H(k-1) + 2} (b^{H(k-1)+2} - a^{H(k-1)+2}) (\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1) (\|\xi^{(1)}\|_\infty + \|\xi^{(2)}\|_\infty + \|\xi^{(1)}\|_1) \right].$$

Since  $\max\{|m_F|; |M_F|\} < \frac{1}{\sqrt{2b^H}}$ , we can invert the sum and the integral. We proceed as previously to prove the  $(S)^*$ -integrability of the two other terms and we obtain the result.  $\square$

**Remark 3.15:**

- The Itô formula obtained stresses the fundamental role of the analytic expansion around the origin of the logarithm of the translated characteristic function. In particular, the non-zero cumulants of the Rosenblatt distribution are crucial.
- Moreover, this result confirms the Itô formula obtained for the monomial  $x^2$  and  $x^3$ . Nevertheless, the polynomials do not belong to the class of functionals for which the Itô formula holds. Thus, we need a generalization of the Paley-Wiener theorem in order to get the result for a bigger class which contains the polynomials. This is the aim of the next theorem.

**Theorem 3.16:** Let  $(a, b) \in \mathbb{R}_+^*$  such that  $a \leq b < \infty$ . Let  $F$  be an entire analytic function of the complex variable verifying:

$$\exists N \in \mathbb{N}, \exists C > 0, \forall z \in \mathbb{C} \quad |F(z)| \leq C(1 + |z|)^N \exp\left(\frac{1}{\sqrt{2b^H}} |\Im(z)|\right)$$

Then, we have in  $(S)^*$ :

$$F(X_b^H) - F(X_a^H) = \int_a^b F^{(1)}(X_t^H) \diamond \dot{X}_t^H dt + \sum_{k=2}^{\infty} \left( H \kappa_k(X_1^H) \int_a^b \frac{t^{Hk-1}}{(k-1)!} F^{(k)}(X_t^H) dt \right) \\ + \sum_{k=2}^{\infty} \left( 2^{k-1} \int_a^b F^{(k)}(X_t^H) \diamond \dot{X}_t^{H,k} dt \right).$$

**Proof:** Let  $(a, b) \in \mathbb{R}_+^*$  such that  $a \leq b < \infty$  and let  $F$  be as in the theorem. As previously, we have to compute the  $S$ -transform of  $F(X_t^H)$  for any  $t \in (a, b)$ . Using Cauchy-Schwartz inequality and the definition of  $F$ , we have:

$$\begin{aligned} \forall \xi \in S(\mathbb{R}) \quad |S(F(X_t^H))(\xi)| &\leq \mathbb{E}^{\mu_\xi}[|F(X_t^H)|] \leq \mathbb{E}[|F(X_t^H)|^2]^{\frac{1}{2}} \mathbb{E}[(\exp(\langle \cdot; \xi \rangle))^2]^{\frac{1}{2}} \\ |S(F(X_t^H))(\xi)| &\leq C \mathbb{E}[(1 + |X_t^H|)^{2N}]^{\frac{1}{2}} \exp\left(\frac{\|\xi\|_2^2}{2}\right) < \infty \end{aligned}$$

since  $X_t^H$  is in the second Wiener's chaos and, by hypercontractivity (see Corollary 2.8.14 in [84]), has finite moments of any order. Thus, we get:

$$S(F(X_t^H))(\xi) = \frac{1}{2\pi} \langle \mathcal{F}(F); \mathbb{E}^{\mu_\xi}[\exp(i.X_t^H)] \rangle$$

where  $\langle \cdot; \cdot \rangle$  has to be understood as the duality bracket since, by Schwartz's Paley-Wiener theorem (see [96], Theorem 9.12),  $\mathcal{F}(F)$  is in  $S'(\mathbb{R})$ , has a compact support contained in the ball  $\{\theta : |\theta| \leq \frac{1}{\sqrt{2b^H}}\}$  and is of order  $N$ . Let  $[m_F; M_F]$  be such that:

$$\text{supp}(\mathcal{F}(F)) \subsetneq [m_F; M_F] \subsetneq \{\theta : |\theta| \leq \frac{1}{\sqrt{2b^H}}\}.$$

Since  $\mathcal{F}(F)$  is a distribution with compact support and since  $\mathbb{E}^{\mu_\xi}[\exp(i.X_t^H)] \in C^\infty(\mathbb{R})$  (by Theorem 2.9), we have:

$$S(F(X_t^H))(\xi) = \frac{1}{2\pi} \langle \mathcal{F}(F); \psi_F(\cdot) \mathbb{E}^{\mu_\xi}[\exp(i.X_t^H)] \rangle$$

where  $\psi_F \in C_0^\infty(\mathbb{R})$ , is equal to 1 in a neighborhood of  $\text{supp}(\mathcal{F})$  and  $\text{supp}(\psi_F) = [m_F; M_F]$ . We differentiate with respect to  $t$  and get as previously:

$$\frac{d}{dt} S(F(X_t^H))(\xi) = \frac{1}{2\pi} \langle \mathcal{F}(F); \psi_F(\cdot) \frac{d}{dt} [\mathbb{E}^{\mu_\xi}[\exp(i.X_t^H)]] \rangle$$

with,

$$\frac{d}{dt} [\mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)]] = (I) + (II) + (III)$$

where,

$$\begin{aligned} (I) &= i\theta S(\dot{X}_t^H)(\xi) \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)], \\ (II) &= \frac{H}{2} \sum_{k=2}^{+\infty} \left[ t^{Hk-1} (2i\theta)^k \left( \sqrt{\frac{(2H-1)H}{2}} \right)^k c_k \right] \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)], \\ (III) &= \frac{1}{2} \sum_{k=2}^{+\infty} \left[ (2i\theta)^k S(\dot{X}_t^{H,k})(\xi) \right] \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)]. \end{aligned}$$

In order to invert the sum and the duality bracket, we need to estimate  $\langle \mathcal{F}(F); \psi_F(\cdot) (i\cdot)^k \mathbb{E}^{\mu_\xi}[\exp(i.X_t^H)] \rangle$  for large values of  $k$  ( $\geq N$ ). Since  $N$  is the order of  $\mathcal{F}(F)$ ,

we have (see chapter 2.3 of [45]):

$$| \langle \mathcal{F}(F); \psi_F(\cdot)(i \cdot)^k \mathbb{E}^{\mu_\xi}[\exp(i X_t^H)] \rangle | \leq C_F \sum_{l=0}^N \sup_{\theta \in [m_F; M_F]} \left| \frac{d^l}{d\theta^l} \left( (i\theta)^k \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)] \right) \right|$$

where  $C_F > 0$ . Using Leibniz's formula, we obtain:

$$| \langle \mathcal{F}(F); \psi_F(\cdot)(i \cdot)^k \mathbb{E}^{\mu_\xi}[\exp(i X_t^H)] \rangle | \leq C_F \sum_{l=0}^N \sum_{m=0}^l C_m^l \times \max\{|m_F|^{k-m}; |M_F|^{k-m}\} k(k-1)\dots(k-m+1) \mathbb{E}^{\mu_\xi}[|X_t^H|^{l-m}]$$

Thus, using  $c_k \leq \left(\frac{1}{H(2H-1)}\right)^{\frac{k}{2}}$ , we obtain the following estimate:

$$t^{Hk-1} 2^k \left( \sqrt{\frac{(2H-1)H}{2}} \right)^k c_k | \langle \mathcal{F}(F); \psi_F(\cdot)(i \cdot)^k \mathbb{E}^{\mu_\xi}[\exp(i X_t^H)] \rangle | \leq C_F \sum_{l=0}^N \sum_{m=0}^l C_m^l \times (\sqrt{2})^k t^{Hk-1} \max\{|m_F|^{k-m}; |M_F|^{k-m}\} k(k-1)\dots(k-m+1) \mathbb{E}^{\mu_\xi}[|X_t^H|^{l-m}]$$

Since  $\max\{|m_F|; |M_F|\} < \frac{1}{\sqrt{2b^H}}$ , we can invert the sum and the duality bracket for (II). Similarly, using (2.9), we can invert the sum and the duality bracket for (III). We get:

$$\begin{aligned} \frac{d}{dt} S(F(X_t^H))(\xi) &= S(\dot{X}_t^H)(\xi) S(F^{(1)}(X_t^H))(\xi) + \sum_{k=2}^{\infty} \left[ H \frac{t^{Hk-1}}{(k-1)!} \kappa_k(X_1^H) S(F^{(k)}(X_t^H))(\xi) \right] \\ &+ \sum_{k=2}^{\infty} \left[ 2^{k-1} S(\dot{X}_t^{H,k})(\xi) S(F^{(k)}(X_t^H))(\xi) \right]. \end{aligned}$$

To prove the  $(S)^*$ -integrability of the three former terms, we proceed as for the proof of theorem 3.13 using the appropriate estimates.  $\square$

**Remark 3.17:**

- Every polynomial verifies the condition of the theorem. Moreover, for such functionals, the sums are finite.
- Using the growth property of  $F$  and the hypercontractivity of the second Wiener chaos elements, we can show that the left-hand side of the equality is in  $(L^2)$ .

## 4 Comparison with other approaches

In [113], the author introduces two definitions of the stochastic integral with respect to the Rosenblatt process. The first one is a pathwise integral using the method of [97] and the second

one is a Skorohod integral using elements of the Malliavin calculus. The Skorohod approach relies on the finite interval representation of the Rosenblatt process. From this representation by means of the double Skorohod integral with respect to Brownian motion, the author defines the stochastic integral with respect to the Rosenblatt process. In this section, we show that this Skorohod integral coincides with the Rosenblatt noise integral defined using the finite interval representation.

**Definition-Theorem 4.1:** The Rosenblatt process  $\{X_t^H\}_{t \in \mathbb{R}}$  is equal in distribution to the process  $\{Z_t^H\}_{t \in [0, T]}$  defined by:

$$\forall t \in [0, T] \quad Z_t^H = c(H) \int_{[0; t]^2} \int_0^t \prod_{j=1}^2 \left(\frac{s}{x_j}\right)^{\frac{H}{2}} (s - x_j)_+^{\frac{H}{2}-1} ds dB_{x_1} dB_{x_2}$$

**Proof:** See theorem 1.1 in [95] or proposition 1 in [113].  $\square$

**Remark 4.2:** It is important to note that even if the two processes have the same finite-dimensional distributions under  $\mu$ , this fact is not true anymore under the probability  $\mu_\xi$  for some  $\xi \in S(\mathbb{R})$ . Indeed, one can show that for  $\xi \in C_0^\infty(\mathbb{R}, \mathbb{R}_+)$  such that  $\text{supp}(\xi) \subset (0, +\infty)$ :

$$\forall t \in [0; T] \quad S(X_t^H)(\xi) \neq S(Z_t^H)(\xi)$$

Using this representation, we can compute the  $S$ -transform of the process  $\{Z_t^H\}$  as well as differentiate it in  $(S)^*$ .

**Proposition 4.3:** The process  $\{Z_t^H\}$  is  $(S)^*$ -differentiable and the  $S$ -transform of its derivative is equal to:

$$\forall \xi \in S(\mathbb{R}) \forall t \in [0, T] \quad S(\dot{Z}_t^H)(\xi) = c(H) \left( \int_0^t \xi(x) \left(\frac{t}{x}\right)^{\frac{H}{2}} (t - x)_+^{\frac{H}{2}-1} dx \right)^2$$

**Proof:** First of all, we need to compute the  $S$ -transform of the process  $\{Z_t^H\}$ . By definition, we have:

$$\forall \xi \in S(\mathbb{R}) \forall t \in [0, T] \quad S(Z_t^H)(\xi) = c(H) \int_{[0; t]^2} \left( \int_0^t \prod_{j=1}^2 \left(\frac{s}{x_j}\right)^{\frac{H}{2}} (s - x_j)_+^{\frac{H}{2}-1} ds \right) \xi(x_1) \xi(x_2) dx_1 dx_2$$

By Fubini's theorem, we have:

$$\begin{aligned} S(Z_t^H)(\xi) &= c(H) \int_0^t \left( \int_0^s \left(\frac{s}{x}\right)^{\frac{H}{2}} (s - x)_+^{\frac{H}{2}-1} \xi(x) dx \right)^2 ds \\ &= c(H) \int_0^t \left( \int_0^s \left(\frac{s}{x}\right)^{\frac{H}{2}} (s - x)_+^{\frac{H}{2}-1} \xi(x) dx \right)^2 ds \end{aligned}$$

By making the change of variable  $x = us$ , we obtain:

$$S(Z_t^H)(\xi) = c(H) \int_0^t s^H \left( \int_0^1 (1-u)_+^{\frac{H}{2}-1} \xi(su) u^{-\frac{H}{2}} du \right)^2 ds$$

Since  $\xi \in S(\mathbb{R})$  and  $H \in (\frac{1}{2}, 1)$ , we get:

$$\begin{aligned} \forall t > 0 \quad \frac{d}{dt}(S(Z_t^H)(\xi)) &= c(H)t^H \left( \int_0^1 (1-u)_+^{\frac{H}{2}-1} \xi(tu) u^{-\frac{H}{2}} du \right)^2 \\ &= c(H) \left( t^{\frac{H}{2}} \int_0^t x^{-\frac{H}{2}} (t-x)_+^{\frac{H}{2}-1} \xi(x) dx \right)^2 \end{aligned}$$

Moreover, for any  $h > 0$  such that  $t+h \leq T$ :

$$\begin{aligned} \left| \frac{S(Z_{t+h}^H)(\xi) - S(Z_t^H)(\xi)}{h} \right| &= c(H) \frac{1}{h} \left| \int_t^{t+h} s^H \left( \int_0^1 (1-u)_+^{\frac{H}{2}-1} \xi(su) u^{-\frac{H}{2}} du \right)^2 ds \right| \\ &\leq c(H) T^H \|\xi\|_\infty^2 \left( \int_0^1 (1-u)_+^{\frac{H}{2}-1} u^{-\frac{H}{2}} du \right)^2 < \infty \end{aligned} \quad (2.10)$$

Thus, using Theorem 1.3, we obtain the result.  $\square$

Similarly to the previous section, we can define a stochastic integral with respect to  $\{Z_t^H\}$ , using the Wick product with the stochastic distribution process  $\{\dot{Z}_t^H\}$  and Pettis integration over  $[0, T]$ .

**Definition-Theorem 4.4:** Let  $\{\Phi_t; t \in [0, T]\}$  be a  $(S)^*$  stochastic process satisfying the assumptions of Theorem 3.9. Then,  $\Phi_t \diamond \dot{Z}_t^H$  is  $(S)^*$  integrable over  $[0, T]$  and:

$$\int_{[0, T]} \Phi_t dZ_t^H = \int_{[0, T]} \Phi_t \diamond \dot{Z}_t^H dt$$

**Proof:** Using estimate (2.10) and Theorem 3.9, one can show that  $\Phi_t \diamond \dot{Z}_t^H$  is  $(S)^*$ -integrable over  $[0, T]$ .  $\square$

In the next proposition, we prove that this integral is an extension of the Skorohod integral defined in [113]. For this purpose, we need to introduce elements of the Malliavin calculus with respect to Brownian motion. Let  $\mathcal{S}$  be the class of smooth random variables of the form:

$$\forall (t_1, \dots, t_n) \in [0, T] \quad F = f(B_{t_1}, \dots, B_{t_n})$$

where  $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ . Then, we define the Malliavin derivative on the class  $\mathcal{S}$  by:

$$\forall t \in [0, T] \quad D_t(F) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B_{t_1}, \dots, B_{t_n}) \mathbf{1}_{[0, t_j]}(t)$$



This operator is unbounded, closable and can be extended to the closure of  $\mathcal{S}$ , denoted  $\mathbb{D}^{k,p}$ , with respect to the norm (see section 1.2 in [87]):

$$\forall F \in \mathcal{S}, k \geq 1, p \geq 2 \quad \|F\|_{k,p}^p = \mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E}[\|D^j(F)\|_{L^2([0,T]^j)}^p]$$

where  $D^j$  is the  $j$ th iteration of the operator  $D$ . For any  $j \geq 1$ ,  $D^j$  takes values in  $L^2(\Omega; L^2([0, T]^j))$ . We denote by  $\delta$  the adjoint operator of  $D$ . It is an unbounded operator from a domain in  $L^2(\Omega; L^2([0, T]))$  to  $L^2(\Omega)$ . An element  $\phi \in L^2(\Omega; L^2([0, T]))$  belongs to the domain of  $\delta$  if and only if:

$$\forall F \in \mathbb{D}^{1,2} \quad |\mathbb{E}[\langle DF; \phi \rangle_{L^2([0;T])}]| \leq C_\phi \sqrt{\mathbb{E}[F^2]}$$

Moreover, we have the following duality relationship:

$$\forall F \in \mathbb{D}^{1,2} \quad \mathbb{E}[\delta(\phi)F] = \mathbb{E}[\langle DF; \phi \rangle_{L^2([0;T])}]$$

For any  $j \geq 1$ , this formula extends to the adjoint operator of  $D^j$ , denoted by  $\delta^j$ , and we have:

$$\forall \phi \in \text{Dom}\delta^j \quad \forall F \in \mathbb{D}^{1,2} \quad \mathbb{E}[\delta^j(\phi)F] = \mathbb{E}[\langle D^j F, \phi \rangle_{L^2([0;T]^j)}]$$

For further properties regarding the multiple integral  $\delta^j$ , we refer the reader to [89]. We are now ready to state the definition of the Skorohod type integral with respect to the Rosenblatt process introduced in [113].

**Definition 4.5:** Let  $\{\Phi_t : t \in [0; T]\}$  be a square integrable stochastic process. Then, we define its Skorohod integral with respect to  $\{Z_t\}_{t \in [0; T]}$  by:

$$\int_0^T \Phi_t \delta Z_t^H = \delta^2(I_H(\Phi))$$

where  $I_H(\Phi)$  is defined by:

$$I_H(\Phi)(s_1, s_2) = c(H) \int_{s_1 \vee s_2}^T \Phi_u \left(\frac{u}{s_1}\right)^{\frac{H}{2}} (u - s_1)^{\frac{H}{2}-1} \left(\frac{u}{s_2}\right)^{\frac{H}{2}} (u - s_2)^{\frac{H}{2}-1} du$$

We state the main result of this section which links the two definitions of integral.

**Proposition 4.6:** Let  $\{\Phi_t; t \in [0; T]\}$  be a stochastic process such that  $\Phi \in L^2(\Omega; \mathcal{H}) \cap L^2([0, T]; \mathbb{D}^{2,2})$  and  $\mathbb{E}[\int_0^T \int_0^T \|D_{s_1, s_2}^2 \Phi\|_{\mathcal{H}}^2 ds_1 ds_2] < \infty$  where

$$\mathcal{H} = \{f : [0; T] \rightarrow \mathbb{R}; \int_0^T \int_0^T f(s)f(t)|t - s|^{2H-2} ds dt < \infty\}.$$

Then,  $\{\Phi_t\}$  is Skorohod integrable and  $(S)^*$ -integrable with respect to the Rosenblatt process,  $\{Z_t^H\}_{t \in [0; T]}$ , and we have:

$$\int_0^T \Phi_t \delta Z_t^H = \int_0^T \Phi_t \diamond \dot{Z}_t^H dt$$

**Proof:** The fact that  $\{\Phi_t\}$  is Skorohod integrable with respect to  $\{Z_t^H\}$  follows from Lemma 1, section 6 in [113]. Let us show that  $\Phi_t \diamond \dot{Z}_t^H$  is  $(S)^*$ -integrable. By definition, we have:

$$\begin{aligned} \forall \xi \in S(\mathbb{R}) \quad |S(\Phi_t \diamond \dot{Z}_t^H)(\xi)| &= |S(\Phi_t)(\xi)| |S(\dot{Z}_t^H)(\xi)| \\ &\leq c(H) t^H \|\xi\|_\infty^2 |\mathbb{E}^\mu[\Phi_t : \exp(\langle \cdot; \xi \rangle) :]| \left( \int_0^1 (1-u)^{\frac{H}{2}-1} u^{-\frac{H}{2}} du \right)^2 \end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain:

$$\begin{aligned} |\mathbb{E}^\mu[\Phi_t : \exp(\langle \cdot; \xi \rangle) :]| &\leq (\mathbb{E}[(\Phi_t)^2])^{\frac{1}{2}} (\mathbb{E}[(\cdot : \exp(\langle \cdot; \xi \rangle) :)^2])^{\frac{1}{2}} \\ &\leq (\mathbb{E}[(\Phi_t)^2])^{\frac{1}{2}} \exp\left(\frac{\|\xi\|_2^2}{2}\right) \end{aligned}$$

Thus, we get the following estimate:

$$|S(\Phi_t \diamond \dot{Z}_t^H)(\xi)| \leq c(H) t^H (\mathbb{E}[(\Phi_t)^2])^{\frac{1}{2}} \|\xi\|_\infty^2 \exp\left(\frac{\|\xi\|_2^2}{2}\right) \left( \int_0^1 (1-u)^{\frac{H}{2}-1} u^{-\frac{H}{2}} du \right)^2$$

Since  $\phi \in L^2([0, T]; \mathbb{D}^{2,2})$ , we have,

$$\int_0^T t^H (\mathbb{E}[(\Phi_t)^2])^{\frac{1}{2}} dt < +\infty$$

So, we conclude by Theorem 3.9. We have to show that the two integrals are equal. Let us compute the  $S$ -transform of the Skorohod integral of  $\{\Phi_t\}$  with respect to  $\{Z_t^H\}$ . By definition,

$$\forall \xi \in S(\mathbb{R}) \quad S\left(\int_0^T \Phi_t \delta Z_t^H\right)(\xi) = \mathbb{E}^\mu[\delta^2(I_H(\phi)) : \exp(\langle \cdot; \xi \rangle) :]$$

By the duality formula, we obtain:

$$\begin{aligned} S\left(\int_0^T \Phi_t \delta Z_t^H\right)(\xi) &= \mathbb{E}^\mu[\langle I_H(\Phi); D^2(\cdot : \exp(\langle \cdot; \xi \rangle) \cdot) \rangle_{L^2([0; T]^2)}] \\ &= \mathbb{E}^\mu[\langle I_H(\Phi); \xi^{\otimes 2} \rangle_{L^2([0; T]^2)} : \exp(\langle \cdot; \xi \rangle) :] \end{aligned}$$

Moreover, for any  $\psi$  such that  $I_H(\psi) \in L^2([0; T]^2)$ ,

$$\langle I_H(\psi); \xi^{\otimes 2} \rangle_{L^2([0; T]^2)} = c(H) \int_{[0; T]^2} \left( \int_{s_1 \vee s_2}^T \psi(u) \prod_{j=1}^2 \left(\frac{u}{s_j}\right)^{\frac{H}{2}} (u - s_j)^{\frac{H}{2}-1} du \right) \xi^{\otimes 2}(s_1, s_2) ds_1 ds_2$$

$$= c(H) \int_0^T \psi(u) \left( \int_0^u \left(\frac{u}{s}\right)^{\frac{H}{2}} (u-s)^{\frac{H}{2}-1} \xi(s) ds \right)^2 du$$

Thus, we get:

$$S\left(\int_0^T \Phi_t \delta Z_t^H\right)(\xi) = \int_0^T \mathbb{E}^\mu[\Phi_t : \exp(\langle \cdot; \xi \rangle) :] S(\dot{Z}_t^H)(\xi) dt = S\left(\int_0^T \Phi_t \diamond \dot{Z}_t^H dt\right)(\xi)$$

And so we are done.  $\square$

## 5 Variance formula

Before stating the theorem, we need to introduce some notations.

**Definition 5.1:** Let  $(p_1, \dots, p_m) \in [1, +\infty)^m$ . The mixed norm space,  $L^{(p_1, \dots, p_m)}(\mathbb{R}^m)$ , is the Banach space of measurable functions on  $\mathbb{R}^m$  with the finite norm:

$$\|f\|_{L^{(p_1, \dots, p_m)}(\mathbb{R}^m)} = \left( \dots \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x_1, \dots, x_m)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \dots dx_m \right)^{\frac{1}{p_m}}.$$

For more details, we refer the reader to Chapter 2.48 in [1]. Furthermore, we need to consider the following multidimensional version of fractional integrals (Chapter 5, Paragraph 24 in [100])

**Definition-Theorem 5.2:** The mixed Liouville-type fractional integrals of order  $\alpha_1, \dots, \alpha_m$ , where  $\alpha_i \in (0, 1)$ , of a function,  $\phi \in S(\mathbb{R}^m)$ , is defined by:

$$I_{+, \dots, +}^{\alpha_1, \dots, \alpha_m}(\phi) = (I_+^{\alpha_1} \otimes \dots \otimes I_+^{\alpha_m})(\phi).$$

Moreover, the linear operator  $I_{+, \dots, +}^{\alpha_1, \dots, \alpha_m}$  is bounded from  $L^{(p_1, \dots, p_m)}(\mathbb{R}^m)$  to  $L^{(q_1, \dots, q_m)}(\mathbb{R}^m)$  if and only if,  $1 < p_i < 1/\alpha_i$  and  $q_i = p_i/(1 - \alpha_i p_i)$ , for every  $i \in \{1, \dots, m\}$ .

**Proof :** See the proof of Theorem 24.1 of [100].  $\square$

**Remark 5.3:** When some of the  $\alpha_i$  are equal to 0, we replace  $I_+^{\alpha_i}$  by the identity operator and the theorem is still true with  $q_i = p_i$  and  $p_i \geq 1$ .

In the sequel, for fixed  $m \geq 2$  and  $H \in (1/2, 1)$ , we have to consider the operators  $I_{+, \dots, +}^{(0, \dots, 0, H/2)}$  and  $I_{+, \dots, +}^{(0, \dots, 0, H/2, H/2)}$  from  $L^2(\mathbb{R}^m)$  to  $L^{(2, \dots, 2, 2/(1-H))}(\mathbb{R}^m)$  and to  $L^{(2, \dots, 2, 2/(1-H), 2/(1-H))}(\mathbb{R}^m)$  respectively. Moreover, we have to consider the restriction of  $I_{+, \dots, +}^{(0, \dots, 0, H/2, H/2)}(f)$  to the hyperplane

$\Delta_m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_{m-1} = x_m\}$  for functions  $f$  not smooth enough. For this purpose, we introduce the function spaces with dominating mixed smoothness which were defined by Lizorkin and Nikol'skii in [67] (see [102] for a recent survey on these spaces and their generalizations).

**Definition 5.4:** For any  $(\alpha_1, \dots, \alpha_m)$  such that  $\alpha_i \geq 0$  for every  $i \in \{1, \dots, m\}$ , we define  $S_2^{(\alpha_1, \dots, \alpha_m)} H(\mathbb{R}^m)$  by:

$$S_2^{(\alpha_1, \dots, \alpha_m)} H(\mathbb{R}^m) = \{f \in S'(\mathbb{R}^m) : \mathcal{F}^{-1}\left(\prod_{i=1}^m (1 + |\xi_i|^2)^{\frac{\alpha_i}{2}} \mathcal{F}(f)\right) \in L^2(\mathbb{R}^m)\}.$$

We note that for any  $\alpha = (\alpha_1, \dots, \alpha_m)$  such that  $\alpha_i \geq 0$ ,  $S_2^{(\alpha_1, \dots, \alpha_m)} H(\mathbb{R}^m)$  is a Hilbert space endowed with the inner product:

$$\forall f, g \in S_2^{(\alpha_1, \dots, \alpha_m)} H(\mathbb{R}^m) \quad \langle f, g \rangle = \int_{\mathbb{R}^m} \left( \prod_{i=1}^m (1 + |\xi_i|^2)^{\alpha_i} \right) \mathcal{F}(f)(\xi) \overline{\mathcal{F}(g)(\xi)} d\xi.$$

These spaces admit several representations, one of them being particularly relevant in our context (see [67]). Indeed, using partial and mixed Liouville derivatives of fractional orders (see Chapter 5, Paragraph 24.2 in [100]), we have:

$$S_2^{(\alpha_1, \dots, \alpha_m)} H(\mathbb{R}^m) = \{f \in L^2(\mathbb{R}^m) : \forall \beta = (\beta_1, \dots, \beta_m), \beta_i = 0 \text{ or } \alpha_i, D_{+, \dots, +}^{(\beta_1, \dots, \beta_m)}(f) \in L^2(\mathbb{R}^m)\}.$$

The mixed Liouville derivative of order  $(\alpha_1, \dots, \alpha_m)$  plays here a dominant role. The trace problem for function spaces with dominating mixed smoothness appears to be much more difficult than for standard isotropic fractional Sobolev spaces. Indeed, function spaces with dominating mixed smoothness are not rotationally invariant in general. For example, trace problems for hyperplanes parallel to coordinate axis are different of trace problems for hyperplanes in oblique position. Nevertheless, for  $m = 2$ , regarding trace problem on the diagonal of  $\mathbb{R}^2$ , we have the following result (see Theorem 5.4 in section 5.1 of [102] and also [119] for the proof):

**Theorem 5.5:** Let  $\alpha_1, \alpha_2$  be strictly positive such that  $\alpha_i < 1/2$  for every  $i = 1, 2$  and  $\alpha_1 + \alpha_2 > 1/2$ . Then, we have:

$$Tr(S_2^{(\alpha_1, \alpha_2)} H(\mathbb{R}^2)) = H^{\alpha_1 + \alpha_2 - \frac{1}{2}}(\mathbb{R}),$$

where  $Tr$  is the trace operator on the diagonal of  $\mathbb{R}^2$  and  $H^{\alpha_1 + \alpha_2 - 1/2}(\mathbb{R})$  is the standard isotropic fractional Sobolev spaces of order  $\alpha_1 + \alpha_2 - 1/2$  (see [1] Chapter 7).

In the following, we are interested in  $S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^m)$ . It is straightforward to check that  $S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^m)$  is isomorphically isometric to  $L^2(\mathbb{R}^{m-2}) \otimes S_2^{(H/2, H/2)} H(\mathbb{R}^2)$ . The

next lemma allows us to consider the restriction of function in  $S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$  to the hyperplane  $\Delta_m$ .

**Lemma 5.6:** There exists a linear and continuous operator  $T$  from  $S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$  to  $S_2^{(0,\dots,0,H-1/2)}H(\mathbb{R}^{m-1})$  such that for any  $f \in S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m) \cap C(\mathbb{R}^m)$ , we have:

$$T(f) = f|_{\Delta_m}.$$

**Proof:** We note  $\tilde{T}$  the operator defined by:

$$\forall \psi \in L^2(\mathbb{R}^{m-2}), \forall \phi \in S_2^{(\frac{H}{2}, \frac{H}{2})}H(\mathbb{R}^2), \tilde{T}(\psi \otimes \phi) = \psi \otimes Tr(\phi),$$

and extended by linearity. Moreover, we have:

$$\|\tilde{T}(\psi \otimes \phi)\|_{L^2(\mathbb{R}^{m-2}) \otimes H^{H-\frac{1}{2}}(\mathbb{R})} \leq C_H \|\psi \otimes \phi\|_{L^2(\mathbb{R}^{m-2}) \otimes S_2^{(\frac{H}{2}, \frac{H}{2})}H(\mathbb{R}^2)},$$

where  $C_H$  is a strictly positive constant depending on  $H$  uniquely. Thus, by a standard argument, we can extend  $\tilde{T}$  to a continuous and linear operator from  $L^2(\mathbb{R}^{m-2}) \otimes S_2^{(H/2, H/2)}H(\mathbb{R}^2)$  to  $L^2(\mathbb{R}^{m-2}) \otimes H^{H-\frac{1}{2}}(\mathbb{R})$ . We note this extension  $\tilde{T}$  and it is exactly the tensor product operator  $E \otimes Tr$  where  $E$  is the identity operator from  $L^2(\mathbb{R}^{m-2})$  to  $L^2(\mathbb{R}^{m-2})$ . Using the linear isometry between  $L^2(\mathbb{R}^{m-2}) \otimes S_2^{(H/2, H/2)}H(\mathbb{R}^2)$  and  $S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$  (and between  $L^2(\mathbb{R}^{m-2}) \otimes H^{H-1/2}(\mathbb{R})$  and  $S_2^{(0,\dots,0,H-1/2)}H(\mathbb{R}^{m-1})$  respectively), we obtain a linear continuous operator  $T$  from  $S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$  to  $S_2^{(0,\dots,0,H-1/2)}H(\mathbb{R}^{m-1})$ . It is straightforward to check that this operator reduces to the restriction operator over  $\Delta_m$  for functions  $f \in S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m) \cap C(\mathbb{R}^m)$  since  $\Delta_m = \mathbb{R}^{m-2} \times \Delta_2$ .  $\square$

In the sequel, we are interested in functions of the form  $g = I_{+, \dots, +}^{(0,\dots,0,H/2,H/2)}(f)$  for some  $f \in L^2(\mathbb{R}^m)$ . However, such a function  $g$  does not belong to  $S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$  even if  $\mathbf{D}_{+, \dots, +}^{(0,\dots,0,H/2,H/2)}(g) \in L^2(\mathbb{R}^m)$  (see Theorem 24.3 in [100]), where  $\mathbf{D}_{+, \dots, +}^{(0,\dots,0,H/2,H/2)}$  is the mixed Marchaud derivative of order  $(0, \dots, 0, H/2, H/2)$ . This is due to the fact that  $I_{+, \dots, +}^{(0,\dots,0,H/2,H/2)}(\cdot)$  maps continuously  $L^2(\mathbb{R}^m)$  into  $L^{(2, \dots, 2, 2/(1-H), 2/(1-H))}(\mathbb{R}^m)$ . Thus, we make the following assumption for the Chaos kernels of the integrand stochastic process which is considered in Theorem 5.8:

$$\mathbf{H1} : \forall m \geq 2, \forall t \in (a, b), I_{+, \dots, +}^{(0,\dots,0,\frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)) \in S_2^{(0,\dots,0,\frac{H}{2}, \frac{H}{2})}H(\mathbb{R}^m).$$

We end this preamble by discussing the hypothesis **H1**.

**Lemma 5.7:** Let  $f \in L^2(\mathbb{R}^m)$  such that  $I_{+, \dots, +}^{(0,\dots,0,H/2,H/2)}(f) \in S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$ . Then,

there exists a function  $g \in L^2(\mathbb{R}^m)$  such that:

$$\mathcal{F}(f) = \frac{(-i\xi_{m-1})^{\frac{H}{2}}(-i\xi_m)^{\frac{H}{2}}}{\prod_{i=m-1}^m(1+|\xi_i|^2)^{\frac{H}{4}}}\mathcal{F}(g).$$

Moreover,

$$\|f\|_{L^2(\mathbb{R}^m)} \leq C_H \|g\|_{L^2(\mathbb{R}^m)},$$

where  $C_H$  is a strictly positive constant independent of  $m$ .

**Proof :** First of all, a function in  $S_2^{(0,\dots,0,H/2,H/2)}H(\mathbb{R}^m)$  admits an integral representation as a convolution of a function in  $L^2(\mathbb{R}^m)$  with the Bessel polypotential kernel  $G_{(0,\dots,0,H/2,H/2)}$  (see [67] and Chapter 5, Paragraph 24.12 of [100] for a definition of this kernel). Thus, we have:

$$\exists g \in L^2(\mathbb{R}^m), I_{+, \dots, +}^{0, \dots, 0, \frac{H}{2}, \frac{H}{2}}(f) = G_{(0, \dots, 0, H/2, H/2)} * g.$$

Let  $V = \cup_{i=1}^m \{(\xi_1, \dots, \xi_m) \in \mathbb{R}^m, \xi_i = 0\}$ . Let  $S_V(\mathbb{R}^m) = \{\phi \in S(\mathbb{R}^m), \forall \alpha \in \mathbb{N}^m, \forall \xi \in V, D^\alpha(\mathcal{F}(\phi))(\xi) = 0\}$ . By Theorem 1 of [99],  $S_V(\mathbb{R}^m)$  is dense in  $L^2(\mathbb{R}^m)$ . Moreover,  $S_V(\mathbb{R}^m)$  is invariant under mixed Liouville-type fractional integration (Chapter 5, Paragraph 24.9 of [100]). Thus, we have, for  $\phi \in S_V(\mathbb{R}^m)$ :

$$\langle I_{+, \dots, +}^{0, \dots, 0, \frac{H}{2}, \frac{H}{2}}(f); \phi \rangle_{L^2(\mathbb{R}^m)} = \langle f, I_{-, \dots, -}^{0, \dots, 0, \frac{H}{2}, \frac{H}{2}}(\phi) \rangle_{L^2(\mathbb{R}^m)},$$

where  $I_-^\alpha(\psi) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} (s-t)_+^{\alpha-1} \psi(s) ds$ . By Parseval-Plancherel Theorem,

$$\langle I_{+, \dots, +}^{0, \dots, 0, \frac{H}{2}, \frac{H}{2}}(f); \phi \rangle = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \mathcal{F}(f)(\xi) \overline{\left( \frac{\mathcal{F}(\phi)(\xi)}{(+i\xi_{m-1})^{\frac{H}{2}}(+i\xi_m)^{\frac{H}{2}}} \right)} d\xi.$$

Since,  $I_{+, \dots, +}^{0, \dots, 0, H/2, H/2}(f) = G_{(0, \dots, 0, H/2, H/2)} * g$  for  $g \in L^2(\mathbb{R}^m)$ , we have:

$$\langle I_{+, \dots, +}^{0, \dots, 0, \frac{H}{2}, \frac{H}{2}}(f); \phi \rangle = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \frac{\mathcal{F}(g)(\xi)}{\prod_{j=m-1}^m (1+|\xi_j|^2)^{\frac{H}{4}}} \overline{\mathcal{F}(\phi)(\xi)} d\xi.$$

Thus, taking  $\mathcal{F}(\phi)(\xi) = \mathcal{F}(\psi)(\xi)[(+i\xi_{m-1})^{H/2}(+i\xi_m)^{H/2}]$  for some  $\psi \in S_V(\mathbb{R}^m)$ , we have:

$$\frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \left( \mathcal{F}(f)(\xi) - \frac{(-i\xi_{m-1})^{\frac{H}{2}}(-i\xi_m)^{\frac{H}{2}}}{\prod_{i=m-1}^m (1+|\xi_i|^2)^{\frac{H}{4}}} \mathcal{F}(g)(\xi) \right) \overline{\mathcal{F}(\psi)(\xi)} d\xi = 0.$$

We conclude by density. Moreover, since  $(-i\xi_{m-1})^{H/2}(-i\xi_m)^{H/2}/(\prod_{i=m-1}^m(1+|\xi_i|^2)^{H/4})$  is a bounded and continuous multiplier (see [105]), we have the following inequality:

$$\|f\|_{L^2(\mathbb{R}^m)} \leq C_H \|g\|_{L^2(\mathbb{R}^m)}. \quad \square$$

Thus, for any  $t \in (a, b)$  and  $m \geq 2$ , we have:

$$\exists g_m(\cdot, t) \in L^2(\mathbb{R}^m), \quad \mathcal{F}(f_m(\cdot, t)) = \frac{(-i\xi_{m-1})^{\frac{H}{2}}(-i\xi_m)^{\frac{H}{2}}}{\prod_{i=m-1}^m(1+|\xi_i|^2)^{\frac{H}{4}}} \mathcal{F}(g_m(\cdot, t)),$$

such that:

$$\|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)} \leq C_H \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}.$$

Moreover, the  $L^2(\mathbb{R}^m)$ -norm of the function  $g_m(\cdot, t)$  is equal to the norm of  $I_{+, \dots, +}^{0, \dots, 0, H/2, H/2}(f_m(\cdot, t))$  as an element of  $S_2^{(0, \dots, 0, H/2, H/2)}H(\mathbb{R}^m)$  (see [67]). We are now ready to state the theorem regarding the variance of the stochastic integral of a random process with respect to the Rosenblatt process.

**Theorem 5.8:** Let  $(a, b) \subset \mathbb{R}_+$  with  $0 \leq a < b < \infty$ . Let  $\{\Phi_t; t \in (a, b)\}$  be a stochastic process such that for all  $t \in (a, b)$ ,  $\Phi_t \in (L^2)$ . Moreover, let us assume that:

$$\begin{aligned} & \forall t \in (a, b), \forall m \geq 2, f_m(\cdot, t) \text{ verifies } \mathbf{H1}, \\ & \sum_{m=2}^{+\infty} (m+2)! \left( \int_{(a,b)} \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}} < +\infty, \end{aligned}$$

where  $\Phi_t = \sum_{m=0}^{+\infty} I_m(f_m(\cdot, t))$  and  $q = \frac{1}{H}(1 + \epsilon)$  for some  $\epsilon \in (\frac{1-H}{3H-1} \vee (2H-1); 1)$ . Thus, we have:

$$\begin{aligned} \mathbb{E}[\left(\int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt\right)^2] &= H(2H-1) \int_{(a,b)} \int_{(a,b)} |t-s|^{2(H-1)} \mathbb{E}[\Phi_t \Phi_s] ds dt \\ &+ 4\sqrt{\frac{H(2H-1)}{2}} \int_{(a,b)} \int_{(a,b)} |t-s|^{H-1} \mathbb{E}[D_{\sqrt{d(H)}\delta_s \circ I_+^{\frac{H}{2}}}(\Phi_t) D_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}}(\Phi_s)] ds dt \\ &+ \int_{(a,b)} \int_{(a,b)} \mathbb{E}[(D_{\sqrt{d(H)}\delta_s \circ I_+^{\frac{H}{2}}}(\Phi_t))^2 (D_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}}(\Phi_s))^2] ds dt. \end{aligned}$$

**Remark 5.9:**

1. Note that the condition  $\sum_{m=2}^{+\infty} (m+2)! \left( \int_{(a,b)} \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{2/q} < +\infty$  ensures the existence of two elements in  $L^2((a,b) \times (a,b)) \otimes (L^2)$  represented by:

$$\lambda_t \otimes \lambda_s \otimes \mu - a.e., \quad \tilde{D}_{\delta_s \circ I_+^{\frac{H}{2}}}(\Phi_t) := \sum_{m=1}^{+\infty} m I_{m-1}(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})})(f_m(\cdot, t))(, s),$$

$$\lambda_t \otimes \lambda_s \otimes \mu - a.e., \quad (\tilde{D}_{\delta_s \circ I_+^{\frac{H}{2}}})^2(\Phi_t) := \sum_{m=2}^{+\infty} m(m-1) I_{m-2}(T(I_{+, \dots, +}^{(0, \dots, \frac{H}{2}, \frac{H}{2})})(f_m(\cdot, t)))(, s).$$

Indeed, we have for the first term, for  $m \geq 2$ :

$$\begin{aligned} \int_{(a,b) \times (a,b)} \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, t))(, s)\|_{L^2(\mathbb{R}^{m-1})}^2 ds dt &\leq C_{a,b,H} \int_{(a,b)} \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, t))\|_{L^{(2, \dots, 2, \frac{2}{1-H})}(\mathbb{R}^m)}^2 dt, \\ &\leq C'_{a,b,H} \int_{(a,b)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^2 dt, \\ &\leq C''_{a,b,H} \int_{(a,b)} \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^2 dt, \end{aligned}$$

and similarly for the second term. Since these elements coincide with  $D_{\delta_s \circ I_+^{H/2}}(\Phi_t)$  and  $(D_{\delta_s \circ I_+^{H/2}})^2(\Phi_t)$  for smoother  $\{\Phi_t\}$ , we denote them by  $D_{\delta_s \circ I_+^{H/2}}(\Phi_t)$  and  $(D_{\delta_s \circ I_+^{H/2}})^2(\Phi_t)$  in the sequel.

2. These conditions are quite similar to the ones of Theorem 13.16 in the case of the white noise integral (see chapter 13 of [60]).
3. The first term appearing in the formula for the variance is classical and is linked to the space of distributions for which we can define the Wiener integrals with respect to the Rosenblatt process (see section 3 of [113]). Actually, since the Rosenblatt process and fractional Brownian motion possess the same covariance structure, this space of distribution is the same than the one for fBm (see [94] for further details).
4. We note the appearance of the second order stochastic derivative which differs significantly from the Gaussian case.

**Proof:** For any  $t \in (a, b)$ , by the Wiener-Itô Theorem, we have:

$$\Phi_t = \sum_{m=0}^{+\infty} I_m(f_m(\cdot, t))$$

Let us fix  $m$ . By the definition of the operator  $D^* \sqrt{d(H)} \delta_t \circ I_+^{\frac{H}{2}}$  and by Pettis integration (in  $(S)^*$  as well as in  $S'(\mathbb{R})$ ), we obtain:

$$\int_{(a,b)} I_m(f_m(\cdot, t)) \diamond \dot{X}_t^H dt = d(H) I_{m+2} \left( \int_{(a,b)} (\delta_t \circ I_+^{\frac{H}{2}})^{\otimes 2} \hat{\otimes} f_m(\cdot, t) dt \right)$$



Let  $\xi \in S(\mathbb{R})$ . We have:

$$\begin{aligned} & \left\langle \int_{(a,b)} (\delta_t \circ I_+^{\frac{H}{2}})^{\otimes 2} \hat{\otimes} f_m(\cdot, t) dt; \xi^{\otimes(m+2)} \right\rangle = \int_{(a,b)} \left\langle (\delta_t \circ I_+^{\frac{H}{2}})^{\otimes 2} \hat{\otimes} f_m(\cdot, t); \xi^{\otimes(m+2)} \right\rangle dt \\ & = \int_{\mathbb{R}^{m+2}} \left( \int_{(a,b)} \frac{(t-x_1)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(x_3, \dots, x_{m+2}, t) dt \right) \xi^{\otimes(m+2)}(x_1, \dots, x_{m+2}) dx_1 \dots dx_{m+2} \end{aligned}$$

Moreover, using Lemma 2.3, we have:

$$\begin{aligned} (d(H))^2 \left\| \int_{(a,b)} \frac{(t-\cdot)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-\cdot)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(\cdot, t) dt \right\|_{L^2(\mathbb{R}^{m+2})}^2 &= \frac{H(2H-1)}{2} \\ & \int_{(a,b)} \int_{(a,b)} |t-s|^{2(H-1)} \langle f_m(\cdot, t); f_m(\cdot, s) \rangle_{L^2(\mathbb{R}^m)} dt ds \end{aligned}$$

which is finite by Hardy-Littlewood-Sobolev inequality (see Theorem 4.3 in [65], with  $n = 1$ ,  $\lambda = 2 - 2H$ ,  $p = r = \frac{1}{H}$ ). Indeed,

$$\begin{aligned} \left| \int_{(a,b) \times (a,b)} |t-s|^{2(H-1)} \langle f_m(\cdot, t); f_m(\cdot, s) \rangle_{L^2(\mathbb{R}^m)} dt ds \right| &\leq \int_{(a,b) \times (a,b)} |t-s|^{2(H-1)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)} \\ & \quad \times \|f_m(\cdot, s)\|_{L^2(\mathbb{R}^m)} dt ds, \\ &\leq C_H \left( \int_{(a,b)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^{\frac{1}{H}} dt \right)^{2H}, \\ &\leq C_{H,a,b,q} \left( \int_{(a,b)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}}, \\ &\leq C_{H,a,b,q} \left( \int_{(a,b)} \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}} < \infty. \end{aligned}$$

Thus,

$$d(H) \int_{(a,b)} (\delta_t \circ I_+^{\frac{H}{2}})^{\otimes 2} \hat{\otimes} f_m(\cdot, t) dt = d(H) \text{sym} \left( \int_{(a,b)} \frac{(t-x_1)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(x_3, \dots, x_{m+2}, t) dt \right)$$

where  $\text{sym}()$  denotes the symmetrization's operation and we have the following estimate ensuring that  $\int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt$  is in  $(L)^2$ :

$$\begin{aligned} \mathbb{E} \left[ \left| \int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt \right|^2 \right] &\leq \frac{H(2H-1)}{2} \sum_{m=0}^{+\infty} (m+2)! \int_{(a,b)} \int_{(a,b)} |t-s|^{2(H-1)} \langle f_m(\cdot, t); f_m(\cdot, s) \rangle_{L^2(\mathbb{R}^m)} dt ds, \\ &\leq \frac{H(2H-1)}{2} \sum_{m=0}^{+\infty} (m+2)! \int_{(a,b)} \int_{(a,b)} |t-s|^{2(H-1)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)} \|f_m(\cdot, s)\|_{L^2(\mathbb{R}^m)} dt ds, \\ &\leq C_H \frac{H(2H-1)}{2} \sum_{m=0}^{+\infty} (m+2)! \left( \int_{(a,b)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^{\frac{1}{H}} dt \right)^{2H}, \end{aligned}$$

$$\leq C_{H,a,b,q} \frac{H(2H-1)}{2} \sum_{m=0}^{+\infty} (m+2)! \left( \int_{(a,b)} \|f_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}} < \infty.$$

Now, we compute precisely the symmetrization of the previous function in order to obtain the exact formula for the variance of  $\int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt$ . By definition, we have:

$$\begin{aligned} \text{sym} \left( \int_{(a,b)} \frac{(t-x_1)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(x_3, \dots, x_{m+2}, t) dt \right) &= \\ \frac{1}{(m+2)!} \int_{(a,b)} \sum_{\sigma \in \mathcal{S}_{m+2}} \left( \frac{(t-x_{\sigma(1)})_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_{\sigma(2)})_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(x_{\sigma(3)}, \dots, x_{\sigma(m+2)}, t) \right) dt & \\ = \frac{1}{(m+2)!} \int_{(a,b)} \left( \sum_{j=1}^{m+2} \frac{(t-x_j)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \sum_{\sigma \in \mathcal{S}_{m+1}^j} \frac{(t-x_{\sigma(1)})_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(x_{\sigma(2)}, \dots, x_{\sigma(m+1)}, t) \right) dt & \end{aligned}$$

where  $\mathcal{S}_{m+1}^j$  is the set of permutations of the set  $\{1, \dots, j-1, j+1, \dots, m+2\}$ .

$$= \frac{1}{(m+2)!} \int_{(a,b)} \left( \sum_{j=1}^{m+2} \sum_{k=1, k \neq j}^{m+2} \frac{(t-x_j)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_k)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \sum_{\sigma \in \mathcal{S}_m^{j,k}} f_m(x_{\sigma(1)}, \dots, x_{\sigma(m+2)}, t) \right) dt$$

where  $\mathcal{S}_m^{j,k}$  is the set of permutations of the set  $\{1, \dots, m+2\} \setminus \{j, k\}$ . But, the kernel  $f_m(\cdot, t)$  is symmetric in the  $m$  first coordinates. Thus,

$$= \frac{1}{(m+2)(m+1)} \int_{(a,b)} \left( \sum_{j=1}^{m+2} \sum_{k=1, k \neq j}^{m+2} \frac{(t-x_j)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_k)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(\underline{x}_{j,k}, t) \right) dt$$

where  $\underline{x}_{j,k}$  is the vector  $(x_1, \dots, x_{m+2})$  without the coordinates  $x_j$  and  $x_k$ . At this point, in order to pursue the computations and to explain the inversion of integrals, we approximate smoothly each  $f_m(\cdot)$  (considered as functions of  $m+1$  variables). We proceed as follows to do so: we use a sequence of functions,  $(g_m^n)$ , in  $S(\mathbb{R}^{m+1})$  such that  $\int_{(a,b)} \|g_m(\cdot, t) - g_m^n(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \xrightarrow{n \rightarrow +\infty} 0$ .

Then, for each  $t \in (a, b)$  and for each  $m \geq 2$ :

$$\forall n \in \mathbb{N}, \mathcal{F}(f_m^n(\cdot, t)) = \frac{(-i\xi_{m-1})^{\frac{H}{2}} (-i\xi_m)^{\frac{H}{2}}}{\prod_{i=m-1}^m (1 + |\xi_i|^2)^{\frac{H}{4}}} \mathcal{F}(g_m^n(\cdot, t)),$$

which ensures that  $f_m^n(\cdot, t) \in L^2(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m)$ . The a posteriori bounds obtained after the computations will allow us to extend the formula of the  $L^2$ -norm of the symmetrization to the non-regularized version of  $f_m(\cdot)$ . We can now compute the  $L^2$ -norm of the symmetrization. We

obtain:

$$= \frac{1}{[(m+2)(m+1)]^2} \sum_{j=1}^{m+2} \sum_{k=1, k \neq j}^{m+2} \sum_{i=1}^{m+2} \sum_{l=1, l \neq i}^{m+2} \int_{(a,b) \times (a,b)} \left( \int_{\mathbb{R}^{m+2}} \frac{(t-x_j)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_k)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_i)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_l)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(\underline{\mathbf{x}}_{j,k}, t) f_m(\underline{\mathbf{x}}_{i,l}, s) d\underline{\mathbf{x}} \right) dt ds$$

To pursue, we need to enumerate the different values of the integral over  $\mathbb{R}^{m+2}$  and to count the number of times such values appear. Since  $j \neq k$  and  $i \neq l$ , we have at most equality between two couples of subscripts, namely,  $i = j$  and  $l = k$  or  $j = l$  and  $i = k$  which leads to  $2(m+2)(m+1)$  possibilities. On the contrary, we can have every subscripts distinct corresponding to  $(m+2)(m+1)m(m-1)$  possibilities. Finally, the  $4(m+2)(m+1)m$  last possibilities correspond to the case where two subscripts are equal and all the others distinct. These three cases lead to three different values for the integral over  $\mathbb{R}^{m+2}$ . Specifically, we obtain:

$$\int_{\mathbb{R}^{m+2}} \frac{(t-x_j)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(t-x_k)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_i)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_l)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} f_m(\underline{\mathbf{x}}_{j,k}, t) f_m(\underline{\mathbf{x}}_{i,l}, s) d\underline{\mathbf{x}} = \begin{cases} \frac{(\beta(1-H, \frac{H}{2}))^2}{(\Gamma(\frac{H}{2}))^4} |t-s|^{2(H-1)} < f_m(\cdot, t); f_m(\cdot, s) >_{L^2(\mathbb{R}^m)} \\ \frac{\beta(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^2} |t-s|^{H-1} < (\delta_s \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, t); (\delta_t \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, s) >_{L^2(\mathbb{R}^{m-1})} \\ < (\delta_s \circ I_+^{\frac{H}{2}}) \otimes_1 ((\delta_s \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, t)); (\delta_t \circ I_+^{\frac{H}{2}}) \otimes_1 ((\delta_t \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, s)) >_{L^2(\mathbb{R}^{m-2})}, \end{cases}$$

and we obtain for the  $L^2$  norm of the symmetrization:

$$\begin{aligned} & \frac{1}{(m+1)(m+2)} \left[ 2 \frac{(\beta(1-H, \frac{H}{2}))^2}{(\Gamma(\frac{H}{2}))^4} \int_{(a,b) \times (a,b)} |t-s|^{2(H-1)} < f_m(\cdot, t); f_m(\cdot, s) >_{L^2(\mathbb{R}^m)} ds dt \right. \\ & + 4m \frac{\beta(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^2} \int_{(a,b) \times (a,b)} |t-s|^{H-1} < (\delta_s \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, t); (\delta_t \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, s) >_{L^2(\mathbb{R}^{m-1})} ds dt \\ & + m(m-1) \int_{(a,b) \times (a,b)} < (\delta_s \circ I_+^{\frac{H}{2}}) \otimes_1 ((\delta_s \circ I_+^{\frac{H}{2}}) \otimes_1 f_m(\cdot, t)); (\delta_t \circ I_+^{\frac{H}{2}}) \otimes_1 ((\delta_t \circ I_+^{\frac{H}{2}}) \otimes_1 \\ & \left. f_m(\cdot, s)) >_{L^2(\mathbb{R}^{m-2})} ds dt \right]. \end{aligned}$$

We want to bound the three former terms in order to invert the sum and the integrals and to obtain the final formula for the variance. We denote the three integrals respectively by (I), (II) and (III). The first one, (I), has already been dealt with. Let us bound the second one:

$$(II) = \int_{(a,b) \times (a,b)} |t-s|^{H-1} < I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, t))(\cdot, s); I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, s))(t) >_{L^2(\mathbb{R}^{m-1})} ds dt,$$

By Cauchy-Schwarz inequality, we obtain:

$$|(II)| \leq \int_{(a,b) \times (a,b)} |t-s|^{H-1} \|I_{+, \dots, +}^{(0, \dots, 0, H/2)}(f_m(\cdot, t))(s)\|_{L^2(\mathbb{R}^{m-1})} \|I_{+, \dots, +}^{(0, \dots, 0, H/2)}(f_m(\cdot, s))(t)\|_{L^2(\mathbb{R}^{m-1})} ds dt.$$

First of all, by Definition-Theorem 5.2 and since  $\left( \int_{(a,b)} \|f_m(\cdot; t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{1/q} < \infty$ , we note that:

$$\left( \int_{(a,b)} \left( \int_{\mathbb{R}} \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, t))(s)\|_{L^2(\mathbb{R}^{m-1})}^{\frac{2}{1-H}} ds \right)^{\frac{q(1-H)}{2}} dt \right)^{\frac{1}{q}} < \infty.$$

Or stated otherwise, the measurable function  $(s, t) \rightarrow \|I_{+, \dots, +}^{(0, \dots, 0, H/2)}(f_m(\cdot, t))(s)\|_{L^2(\mathbb{R}^{m-1})} \mathbf{1}_{(a,b)}(t)$  is an element of  $L^{(2/(1-H), q)}(\mathbb{R}^2)$ . Similarly, using Theorem 2.51 of [1], the measurable function  $(s, t) \rightarrow \|I_{+, \dots, +}^{(0, \dots, 0, H/2)}(f_m(\cdot, s))(t)\|_{L^2(\mathbb{R}^{m-1})} \mathbf{1}_{(a,b)}(s)$  is an element of  $L^{(q, 2/(1-H))}(\mathbb{R}^2)$ . Then, using a generalized version of Hölder inequality in the framework of mixed-norm spaces (see Theorem 2.49 of [1]), the measurable function  $(s, t) \rightarrow \|I_{+, \dots, +}^{(0, \dots, 0, H/2)}(f_m(\cdot, t))(s)\|_{L^2(\mathbb{R}^{m-1})} \times \|I_{+, \dots, +}^{(0, \dots, 0, H/2)}(f_m(\cdot, s))(t)\|_{L^2(\mathbb{R}^{m-1})} \times \mathbf{1}_{(a,b) \times (a,b)}(s, t)$  is an element of  $L^{(r, r)}(\mathbb{R}^2)$  where  $1/r = (1-H)/2 + 1/q$ . Moreover, for  $\epsilon \in ((1-H)/(3H-1) \vee (2H-1); 1)$ , we have  $r > 1/H$ . Thus, by Hölder inequality with  $p^* = r$ , we obtain:

$$|(II)| \leq \left( \int_{(a,b) \times (a,b)} |t-s|^{p(H-1)} ds dt \right)^{\frac{1}{p}} \left( \int_{(a,b) \times (a,b)} \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, t))(s)\|_{L^2(\mathbb{R}^{m-1})}^{p^*} \times \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(f_m(\cdot, s))(t)\|_{L^2(\mathbb{R}^{m-1})}^{p^*} ds dt \right)^{\frac{1}{p^*}}.$$

where,

$$\forall p \in \left(1, \frac{1}{1-H}\right) \int_{(a,b) \times (a,b)} |t-s|^{p(H-1)} ds dt < \infty.$$

Using the generalized version of Hölder inequality for mixed-norm spaces, the permutation inequality for mixed norms, and Definition-Theorem 5.2, we get:

$$|(II)| \leq C_{a,b,H,\epsilon} \left( \int_{(a,b)} \|f_m(\cdot; t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}}.$$

In order to conclude, we need to bound the last integral, (III). We have

$$(III) = \int_{(a,b)} \int_{(a,b)} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))(s); T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, s)))(t) \rangle_{L^2(\mathbb{R}^{m-2})} ds dt.$$

By Cauchy-Schwarz inequality, we obtain:

$$|(III)| \leq \int_{(a,b)} \int_{(a,b)} \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{m-2})} \times \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, s)))(\cdot, t)\|_{L^2(\mathbb{R}^{m-2})} ds dt.$$

By Lemma 5.6, we have:

$$\begin{aligned} \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{m-1})} &\leq \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))\|_{S_2^{(0, \dots, 0, H-1/2)} H(\mathbb{R}^{m-1})} \\ &\leq C_H \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t))\|_{S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^m)} \\ &\leq C'_H \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}. \end{aligned}$$

Thus,

$$\int_{(a,b)} \left( \int_{\mathbb{R}} \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{m-2})}^2 ds \right)^{\frac{q}{2}} dt < \infty.$$

Since  $q = 1/H(1 + \epsilon)$  with  $\epsilon > (1 - H)/(3H - 1) \vee (2H - 1)$ , we have  $q > 2$ . Thus,

$$\begin{aligned} \left( \int_{(a,b) \times (a,b)} \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{m-2})}^2 ds dt \right)^{\frac{1}{2}} &\leq C_{a,b,H,\epsilon} \\ &\times \left( \int_{(a,b)} \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(f_m(\cdot, t)))\|_{L^2(\mathbb{R}^{m-1})}^q dt \right)^{\frac{1}{q}}, \end{aligned}$$

for some  $C_{a,b,H,\epsilon} > 0$ . Using Cauchy-Schwarz inequality, we obtain:

$$|(III)| \leq C_{a,b,H,\epsilon} \left( \int_{(a,b)} \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}}.$$

Multiplying by  $(m + 2)!(d(H))^2$  and using the condition:

$$\sum_{m=0}^{+\infty} (m + 2)! \left( \int_{(a,b)} \|g_m(\cdot, t)\|_{L^2(\mathbb{R}^m)}^q dt \right)^{\frac{2}{q}} < +\infty,$$

we can invert the sum and the integrals. Using the definition of  $D_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}}$  as well as the isometry for multiple Wiener-Itô integrals, we get the following formula for the  $(L^2)$  norm of  $\int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt$ .

$$\begin{aligned} \mathbb{E} \left[ \left( \int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt \right)^2 \right] &= H(2H - 1) \int_{(a,b)} \int_{(a,b)} |t - s|^{2(H-1)} \mathbb{E}[\Phi_t \Phi_s] ds dt \\ &+ 4 \sqrt{\frac{H(2H - 1)}{2}} \int_{(a,b)} \int_{(a,b)} |t - s|^{H-1} \mathbb{E} \left[ D_{\sqrt{d(H)}\delta_s \circ I_+^{\frac{H}{2}}}(\Phi_t) D_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}}(\Phi_s) \right] ds dt \\ &+ \int_{(a,b)} \int_{(a,b)} \mathbb{E} \left[ \left( D_{\sqrt{d(H)}\delta_s \circ I_+^{\frac{H}{2}}}(\Phi_t) \right)^2 \left( D_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}}(\Phi_s) \right)^2 \right] ds dt. \quad \square \end{aligned}$$

## Chapter 3

# From forward integrals to Wick-Itô integrals: the fBm and the Rosenblatt process cases.

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Forward integration is a natural generalization of Itô integration allowing for anticipating integrands. There are basically two approaches in order to define a forward integral with respect to Brownian motion: by means of Wiener analysis (and/or white noise analysis) and by regularization techniques. It was first introduced by M. A. Berger and V. J. Mizel in [23] through a Wiener-Itô expansion of the integrand. Using a simple instance of the multiplication formula from Malliavin calculus, the authors define the forward integral as the sum of the Skorohod integral and a term involving what they call "the  $\beta$ -derivative", an ancestor of the Malliavin derivative. In [88], the whole machinery of Malliavin calculus is used in order to study deeply the stochastic calculus with anticipating integrands. Similarly, in [7], J. Asch and J. Potthoff define in the framework of white noise distribution theory a forward integral with respect to Brownian motion allowing for anticipating integrands. Inspired by the work of H-H. Kuo and A. Russek in [61], the authors prove that this stochastic integral verifies many suitable properties. Namely, they prove that it is linear when the integrand is a constant random variable and that it is well approximated by forward Riemann sums. Moreover, they compute the quadratic variation of a forward stochastic integral process, prove its sample path continuity and obtain an Itô lemma in its usual form. Nevertheless, the definition of this forward integral relies heavily on the existence of a term involving the right stochastic derivative of the integrand.

The regularization technique was developed by F. Russo and P. Vallois in [97]. It is an almost pathwise method allowing to define forward, backward and symmetric integrals with respect to general integrators. This forward integral coincides with the classical Itô integral

when the integrator is a semimartingale and the stochastic integrand verifies certain conditions. Moreover, in the Brownian motion case, thanks to a Wiener analysis point of view, the authors study another type of forward integral. Once again, this integral is the sum of the Skorohod integral and a term involving the stochastic gradient of the integrand. Specifically, it involves a certain type of trace of the stochastic gradient of the integrand (see Theorem 2.1 of [97]). Naturally, they raise the question of the link between these two types of forward integration (see Remark 2.2 of [97]). They are the same when the limiting procedure of the regularization is strengthened.

Regarding fractional Brownian motion, the regularization techniques can be applied readily when  $H > 1/2$ . Indeed, in this case, fBm is a zero quadratic variation process which admits a modification whose sample paths are almost surely  $H - \eta$  Hölder continuous, for any  $0 < \eta < H$ . Therefore, the forward, backward and symmetric integrals with respect to fBm exist and might coincide for peculiar integrands. When  $H < 1/2$ , the regularization techniques must be adapted and new tools are required in order to define properly symmetric integrals with respect to fBm (see [39, 40]). In [4], the explicit link is made between the symmetric integral and the divergence operator associated with fractional Brownian motion for  $H > 1/2$ . In this Gaussian framework, a term involving the stochastic derivative of the integrand appears. In [113], where stochastic integration with respect to the Rosenblatt process is performed, a similar analysis leads to two residual terms. These terms seem difficult to analyse. This is mostly due to the complexity of the kernel of the representation of the Rosenblatt process used in this article.

In this chapter, we intend to study the connection between the forward integrals with respect to fBm ( $H > 1/2$ ) and with respect to the Rosenblatt process and the respective Wick-Itô integrals defined using white noise distribution theory. For this purpose, we introduce in section 2 stochastic derivative operators which are compositions of iterations of the stochastic gradient, denoted  $\nabla^{(k)}$  for  $k \geq 1$ , with mixed Liouville-type fractional integrals of certain orders. Then, for fixed  $\epsilon > 0$ , we compute explicitly, the  $(S)$ -transform of the following Hida distributions:

$$\int_a^b \Phi_t \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt, \\ \int_a^b \Phi_t \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} dt,$$

where  $\{B_t^H\}$  is a fractional Brownian motion,  $\{X_t^H\}$  is a Rosenblatt process and  $\{\Phi_t\}$  a certain stochastic integrand process. It is important to notice that  $\{B_t^H\}$  and  $\{X_t^H\}$  are defined through their classical representation, namely:

$$B_t^H = A(H) \int_{\mathbb{R}} \left( \int_0^t \frac{(s-x)_+^{H-\frac{3}{2}}}{\Gamma(H-\frac{1}{2})} ds \right) dB_x,$$

$$X_t^H = d(H) \int_{\mathbb{R}^2} \left( \int_0^t \frac{(s-x_1)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} ds \right) dB_{x_1} dB_{x_2},$$

where  $A(H)$  and  $d(H)$  are positive constants such that  $\mathbb{E}[|B_1^H|^2] = \mathbb{E}[|X_1^H|^2] = 1$ . Moreover, for fixed  $\epsilon > 0$ , the two previously mentioned Hida distributions appear to be the sum of two terms in the fBm case and of three terms in the Rosenblatt case (see Propositions 3.1.2 and 3.2.6 respectively). Regularized versions of the Wick-Itô integrals appear naturally as well as residual terms involving the stochastic operators introduced in section 2. In the Rosenblatt case, the second residual term is way more difficult to handle than the first one (see the preamble before Proposition 3.2.4). Finally, for specific integrand processes, these terms are computed explicitly. As in chapter 2 and as in [113], the non-zero cumulants of the Rosenblatt distribution appear in the computation of the residual terms associated to the Rosenblatt process.

## 1 Hida distribution and Sobolev-Watanabe-Kree spaces.

In this section, we briefly remind the white noise analysis introduced by Hida and al. in [43]. For a good introduction to the theory of white noise, we refer the reader to the book of Kuo [60]. The underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is the space of tempered distributions endowed with the  $\sigma$ -field generated by the open sets with respect to the weak\* topology in  $S'(\mathbb{R})$  and with the infinite dimensional Gaussian measure  $\mu$  whose existence is ensured by the Bochner-Minlos theorem.

For all  $(\phi_1, \dots, \phi_n) \in S(\mathbb{R})$ , the space of  $C^\infty(\mathbb{R})$  functions with rapid decrease at infinity, the vector  $(\langle \cdot; \phi_1 \rangle, \dots, \langle \cdot; \phi_n \rangle)$  is a centered Gaussian random vectors with covariance matrix  $(\langle \phi_i; \phi_j \rangle)_{(i,j)}$ . As it is written in Kuo [60], for any function  $f \in L^2(\mathbb{R})$ , we can define  $\langle \cdot; f \rangle$  as the random variable in  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  obtained by a classical approximation argument and the following isometry:

$$\forall (\phi, \psi) \in S(\mathbb{R})^2 \quad \mathbb{E}[\langle \cdot; \psi \rangle \langle \cdot; \phi \rangle] = \langle \psi; \phi \rangle_{L^2(\mathbb{R})}$$

Thus, for any  $t \in \mathbb{R}$ , we define ( $\mu$ -almost everywhere):

$$B_t(\cdot) = \begin{cases} \langle \cdot; 1_{[0;t]} \rangle & t \geq 0 \\ - \langle \cdot; 1_{[t;0]} \rangle & t < 0 \end{cases}$$

From the isometry property, it follows immediately that  $B_t$  is a Brownian motion on the white noise space and, by the Kolmogorov-Centsov theorem, it admits a continuous modification.



Moreover, using the approximation of any function  $f \in L^2(\mathbb{R})$  by step functions, we obtain:

$$\langle ; f \rangle = \int_{\mathbb{R}} f(s) dB_s$$

We note  $\mathcal{G}$ , the sigma field generated by Brownian motion and  $(L^2) = L^2(\Omega, \mathcal{G}, \mathbb{P})$ . By the Wiener-Itô theorem, any functionals  $\Phi \in (L^2)$  can be expanded uniquely into a series of multiple Wiener-Itô integrals:

$$\Phi = \sum_{n=0}^{\infty} I_n(\phi_n)$$

where  $\phi_n \in \hat{L}^2(\mathbb{R}^n)$ , the space of square-summable symmetric functions. Using this theorem and the second quantization operator of the harmonic oscillator operator,  $A = \frac{-d^2}{dx^2} + x^2 + 1$ , Hida and al. introduced the stochastic space of test functions  $(S)$  and its dual, the space of generalized functions  $(S)^*$  or Hida distributions. We refer the reader to pages 18-20 of Kuo's book for an explicit construction. We have the following Gel'fand triple:

$$(S) \subset (L^2) \subset (S)^*$$

We denote by  $\langle\langle ; \rangle\rangle$  the duality bracket between elements of  $(S)$  and  $(S)^*$  which reduces to the classical inner product on  $(L^2)$  for two elements in  $(L^2)$ . Note that  $(S)$  is defined as an equivalence class of test random variables defined  $\mu$ -almost everywhere. Nevertheless, by Theorem 6.5 of [60], each element of  $(S)$  admits a unique continuous version with which we will be working from now on. Furthermore, we note that, for all  $\omega \in S'(\mathbb{R})$ ,  $\delta_\omega$  is an element of  $(S)^*$  (see Chapter 7, Paragraph 7.2, Theorem 7.8 in [60]).

In the context of white noise analysis, the main tool is the  $S$ -transform. It is a functional on  $S(\mathbb{R})$  which characterizes completely the elements in  $(S)^*$  (as well as the strong convergence in  $(S)^*$ ).

**Definition 1.1:** Let  $\Phi \in (S)^*$ . For every function  $\xi \in S(\mathbb{R})$ , we define the  $S$ -transform of  $\Phi$  by:

$$S(\Phi)(\xi) = \langle\langle \Phi ; \exp(\langle ; \xi \rangle) \rangle\rangle$$

where  $\exp(\langle ; \xi \rangle) := \exp(\langle ; \xi \rangle - \frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2}) = \sum_{n=0}^{\infty} \frac{I_n(\xi^{\otimes n})}{n!} \in (S)$ .

**Remark 1.2:** For every  $\Phi \in (L^2)$ , we have:

$$S(\Phi)(\xi) = \mathbb{E}[\Phi : \exp(\langle ; \xi \rangle) :] = \mathbb{E}^{\mu_\xi}[\Phi]$$

where  $\mu_\xi$  is the translated infinite dimensional measure defined by:

$$\mu_\xi(d\omega) = \exp(\langle \omega ; \xi \rangle - \frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2}) \mu(d\omega).$$

Regarding the  $S$ -transform, we have the following properties and results:

**Theorem 1.3:**

1. The  $S$ -transform is injective. If  $\forall \xi \in S(\mathbb{R}), S(\Phi)(\xi) = S(\Psi)(\xi)$  then  $\Phi = \Psi$  in  $(S)^*$ .
2. For  $\Phi, \Psi \in (S)^*$  there is a unique element  $\Phi \diamond \Psi \in (S)^*$  such that for all  $\xi \in S(\mathbb{R}), S(\Psi)(\xi)S(\Phi)(\xi) = S(\Phi \diamond \Psi)(\xi)$ . It is called the Wick product of  $\Phi$  and  $\Psi$ .
3. Let  $\Phi_n \in (S)^*$  and  $F_n = S(\Phi_n)$ . Then  $\Phi_n$  converges strongly in  $(S)^*$  if and only if the following conditions are satisfied:
  - $\lim_{n \rightarrow \infty} F_n(\xi)$  exists for each  $\xi \in S(\mathbb{R})$ .
  - There exists strictly positive constants  $K, a$  and  $p$  independent of  $n$  such that:

$$\forall n \in \mathbb{N}, \forall \xi \in S(\mathbb{R}) \quad |F_n(\xi)| \leq K \exp(a \|A^p \xi\|_{L^2(\mathbb{R})}^2)$$

In the sequel, we introduce the differential calculus and Sobolev-Watanabe-Kree spaces on the white noise probability space. For further details, we refer the reader to chapter 9 of [60] and chapter 5 of [43]. First, we define the Gâteaux derivative of elements in  $(S)$  for direction in  $S'(\mathbb{R})$ .

**Definition-Theorem 1.4:** Let  $y \in S'(\mathbb{R})$  and  $\Phi \in (S)$ . The operator  $D_y$  is continuous from  $(S)$  into itself and we have:

$$\forall \omega \in S'(\mathbb{R}) \quad D_y(\Phi)(\omega) = \sum_{n=1}^{\infty} n I_{n-1}(y \otimes_1 \phi_n)(\omega),$$

where we denote by  $\otimes_1$  the contraction of order 1 (see [87]).

**Proof:** See Theorem 9.1 of [60].  $\square$

The next result states that every test random variable is actually infinitely often differentiable in Gâteaux and in Fréchet senses.

**Theorem 1.5:** Let  $\Phi \in (S)$ .  $\Phi$  is infinitely often Gâteaux differentiable in every direction of  $S'(\mathbb{R})$  and infinitely often differentiable in Fréchet sense. Moreover, for every  $k \in \mathbb{N}^*$  and for every  $y_1, \dots, y_k \in (S'(\mathbb{R}))^k$ , we have:

$$D_{y_1} \circ D_{y_2} \circ \dots \circ D_{y_k}(\Phi) = \langle y_1 \otimes y_2 \otimes \dots \otimes y_k; \nabla^{(k)}(\Phi) \rangle,$$

where  $\nabla^{(k)}(\Phi)$  is the  $k$ -th Fréchet derivative of  $\Phi$  and the equality stands in  $(S)$ . In particular,  $\nabla^{(k)}(\Phi) \in \hat{S}(\mathbb{R}^k) \otimes (S)$ .

**Proof:** See Theorems 5.7 and 5.14 of [43].  $\square$

We introduce as well the number operator,  $N$ , on  $S$  in order to define Sobolev-Watanabe-Kree spaces.

**Definition-Theorem 1.6:** Let  $r \geq 0$  and  $\Phi \in (S)$  given by  $\Phi = \sum_{n=0}^{\infty} I_n(\phi_n)$ . We have:

$$N^r \Phi = \sum_{n=0}^{\infty} n^r I_n(\phi_n).$$

Moreover,  $N^r$  is a linear and continuous operator from  $(S)$  into itself.

**Proof:** See Theorem 9.23 in [60].  $\square$

Let  $r \geq 0$  and  $(\mathcal{P}) \subset (S)$  the algebra of polynomial random variables generated by elements of the form  $I_1(\xi)$  with  $\xi \in S(\mathbb{R})$ . We denote by  $(\mathcal{W}^{r,2})$  the completion of  $(\mathcal{P})$  with respect to the norm:

$$\forall \Phi \in (\mathcal{P}), \|\Phi\|_{r,2} = \|(N + E)^{\frac{r}{2}} \Phi\|_{(L^2)},$$

where  $E$  is the identity operator.

**Theorem 1.7:** Let  $k \in \mathbb{N}^*$  and  $r \geq k$ .  $\nabla^{(k)}$  extends to a continuous operator from  $(\mathcal{W}^{r,2})$  into  $\hat{L}^2(\mathbb{R}^k) \otimes (\mathcal{W}^{r-k,2})$ .

**Proof:** See Theorem 5.24 and Corollary 5.25 of [43].  $\square$

We introduce a space of test random variables which is useful when considering functionals of fractional Brownian motion as well as functionals of the Rosenblatt process. For this purpose, we define, for any  $r \in \mathbb{R}$  and any  $p > 1$ ,  $(\mathcal{W}^{r,p})$  by the completion of  $(\mathcal{P})$  with respect to the norm  $\|(N + E)^{r/2} \cdot\|_{(L^p)}$ . We denote by  $(\mathcal{W}^{\infty,\infty})$  the projective limit of the family  $\{(\mathcal{W}^{r,p}), p > 1, r \in \mathbb{R}\}$ . Due to Meyer inequality (see chapter 1.5 of [87]), for every  $k \geq 1$ , the operator  $\nabla^{(k)}$  is continuous from  $(\mathcal{W}^{\infty,\infty})$  into  $(\mathcal{W}^{\infty,\infty})(L^2(\mathbb{R}^k))$ , this space is stable under pointwise multiplication and the following version of the product and chain rules hold.

**Proposition 1.8:** Let  $\Phi, \Psi \in (\mathcal{W}^{\infty,\infty})$  and let  $F$  be an infinitely continuously differentiable function on  $\mathbb{R}$  such that  $F$  and its derivatives have polynomial growth. Then,  $F(\Phi) \in (\mathcal{W}^{\infty,\infty})$  and:

$$\begin{aligned} \nabla(\Phi\Psi) &= \Phi\nabla(\Psi) + \Psi\nabla(\Phi), \\ \nabla(F(\Phi)) &= \nabla(\Phi)F'(\Phi). \end{aligned}$$

**Proof:** This proposition is a consequence of the product and chain rules on  $(\mathcal{P})$  for  $\nabla$  as well as Remark 1 page 78, Proposition 1.5.1 and Proposition 1.5.6 of [87].  $\square$

We end this section by the definition of the adjoint of the Gâteaux derivative  $D_y$  for every  $y \in S'(\mathbb{R})$ .

**Definition-Theorem 1.9:** Let  $y \in S'(\mathbb{R})$  and  $\Psi \in (S)^*$ . The adjoint operator  $D_y^*$  is continuous from  $(S)^*$  into itself and we have:

$$\forall \xi \in S(\mathbb{R}) \quad S(D_y^*(\Psi))(\xi) = \langle y; \xi \rangle S(\Psi)(\xi) = S(I_1(y) \diamond \Psi)(\xi)$$

where  $I_1(y)$  is a generalized Wiener-Itô integral in  $(S)^*$ . Moreover, we have the following generalized Wiener-Itô decomposition for  $D_y^*(\Psi)$ :

$$D_y^*(\Psi)(\cdot) = \sum_{n=0}^{\infty} I_{n+1}(y \hat{\otimes} \psi_n).$$

**Proof:** See Theorem 9.12, 9.13 and the remark following corollary 9.14 in [60].  $\square$

## 2 Stochastic analysis of fractional Brownian motion and of the Rosenblatt process.

In this section, we state the definition of fractional Brownian motion and of the Rosenblatt process. Following [21] and [6], we remind that these processes are  $(S)^*$ -differentiable and compute their  $(S)^*$  derivatives. Then, we define stochastic derivative operators of first and second orders which play a significant role in the trace terms appearing in the relationship between Wick-Itô integral and forward integral with respect to these two processes. Moreover, we compute explicitly the Hilbert space adjoint of the first order stochastic gradient which is linked to Wick-Itô integral with respect to fractional Brownian motion. Therefore, we give a brief introduction to the stochastic integrals with respect to the fractional and the Rosenblatt noises in the Wick-Itô sense and make explicit the aforementioned link with the adjoint operator. In the rest of the article, we fix an interval  $(a, b)$  included in  $\mathbb{R}_+$ .

**Definition 2.1:** For  $H > 1/2$ , we define fractional Brownian motion and the Rosenblatt process by:

$$\forall t \in (a, b), \quad B_t^H = A(H) \int_{\mathbb{R}} \left( \int_0^t \frac{(s-x)_+^{H-\frac{3}{2}}}{\Gamma(H-\frac{1}{2})} ds \right) dB_x,$$

$$X_t^H = d(H) \int_{\mathbb{R}^2} \left( \int_0^t \frac{(s-x_1)_+^{\frac{H}{2}-1} (s-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} ds \right) dB_{x_1} dB_{x_2},$$

where  $A(H)$  and  $d(H)$  are positive constants such that  $\mathbb{E}[|B_1^H|^2] = \mathbb{E}[|X_1^H|^2] = 1$  and defined by:

$$A(H) = \left( \frac{\Gamma(H - \frac{1}{2})H(2H - 1)\Gamma(\frac{3}{2} - H)}{\Gamma(2 - 2H)} \right)^{\frac{1}{2}},$$

$$d(H) = \sqrt{\frac{H(2H - 1)}{2} \frac{(\Gamma(\frac{H}{2}))^2}{\beta(1 - H; \frac{H}{2})}}.$$

**Definition-Theorem 2.2:** Fractional Brownian motion and the Rosenblatt process are  $(S)^*$ -differentiable and their derivatives, the fractional noise,  $\{\dot{B}_t^H\}$ , and the Rosenblatt noise,  $\{\dot{X}_t^H\}$ , admit the following  $(S)$ -transforms:

$$\forall \xi \in S(\mathbb{R}), \quad S(\dot{B}_t^H)(\xi) = A(H)I_+^{H-\frac{1}{2}}(\xi)(t),$$

$$S(\dot{X}_t^H)(\xi) = d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2.$$

where for every  $0 < \alpha < 1$ ,  $I_+^\alpha(\xi)(t) = 1/(\Gamma(\alpha)) \int_{\mathbb{R}} (t-s)_+^{\alpha-1} \xi(s) ds$  is the fractional integral of order  $\alpha$  of  $\xi$  on the real line ([100] chapter 2).

**Proof:** See the proof of Lemma 2.15, Theorem 2.17 and Definition 2.18 of [21] for fractional Brownian motion and Lemma 3.4 of [6] for the Rosenblatt process.

The next Lemma is a technical one allowing us to define the first order stochastic derivative operators associated with fractional Brownian motion and the Rosenblatt process.

**Lemma 2.3:** For every  $\alpha \in (0, 1/2)$  and every  $r \geq 0$ , the operator  $I_+^\alpha \otimes E$  admits a continuous extension from  $L^2(\mathbb{R}) \otimes (\mathcal{W}^{r,2})$  to  $L^2((a,b)) \otimes (\mathcal{W}^{r,2})$ .

**Proof:** We define  $I_+^\alpha \otimes E$  on simple element of  $L^2(\mathbb{R}) \otimes (\mathcal{W}^{r,2})$  by:

$$\forall \phi, \Phi \in L^2(\mathbb{R}) \times (\mathcal{W}^{r,2}), \quad (I_+^\alpha \otimes E)(\phi \otimes \Phi) = I_+^\alpha(\phi) \otimes \Phi.$$

and we extend it by linearity. Moreover, we have:

$$\begin{aligned} \|I_+^\alpha(\phi) \otimes \Phi\|_{L^2((a,b)) \otimes (\mathcal{W}^{r,2})} &= \|I_+^\alpha(\phi)\|_{L^2((a,b))} \|\Phi\|_{(\mathcal{W}^{r,2})}, \\ &\leq C_{a,b,\alpha} \|I_+^\alpha(\phi)\|_{L^{\frac{2}{1-2\alpha}}(\mathbb{R})} \|\Phi\|_{(\mathcal{W}^{r,2})}, \\ &\leq C_{a,b,\alpha} \|\phi\|_{L^2(\mathbb{R})} \|\Phi\|_{(\mathcal{W}^{r,2})}. \end{aligned}$$

since  $I_+^\alpha$  is a continuous operator from  $L^2(\mathbb{R})$  to  $L^{\frac{2}{1-2\alpha}}(\mathbb{R})$  and  $2/(1-2\alpha) > 2$  (Theorem 5.3 of [100]). $\square$

Consequently, we have the following result:

**Proposition 2.4:** Let  $r \geq 1$ ,  $\alpha \in (0; 1/2)$ . There exists a continuous operator, denoted  $\nabla^\alpha$ , from  $(\mathcal{W}^{r,2})$  to  $L^2((a,b)) \otimes (\mathcal{W}^{r-1,2})$  such that:

$$\forall \Phi \in (\mathcal{W}^{r,2}), \lambda \otimes \mu - a.e.(t, \omega) \in (a, b) \times S'(\mathbb{R}), \quad \nabla^\alpha(\Phi)(t, \omega) = \sum_{n=1}^{\infty} n I_{n-1}(\langle \delta_t \circ I_+^\alpha; \phi_n \rangle)(\omega),$$

with  $\Phi = \sum_{n=1}^{\infty} I_n(\phi_n)$ .

**Proof:** From Lemma 2.3 and Theorem 1.7, the operator  $\nabla^\alpha = (I_+^\alpha \otimes E) \circ \nabla$  is continuous from  $(\mathcal{W}^{r,2})$  to  $L^2((a,b)) \otimes (\mathcal{W}^{r-1,2})$ . We only have to prove that the previous equality holds. First of all, notice that, for  $n \geq 1$ , by Definition-Theorem 5.2 of Chapter 2:

$$\begin{aligned} \left( \int_a^b \|\langle \delta_t \circ I_+^\alpha; \phi_n \rangle\|_{L^2(\mathbb{R}^{n-1})}^2 dt \right)^{\frac{1}{2}} &\leq C_{\alpha,a,b} \|I_{+, \dots, +}^{(0, \dots, 0, \alpha)}(\phi_n)\|_{L^{(2, \dots, 2, \frac{2}{1-2\alpha})}(\mathbb{R}^n)}, \\ &\leq C_{\alpha,a,b} \|\phi_n\|_{L^2(\mathbb{R}^n)} < +\infty. \end{aligned}$$

Then,  $I_{n-1}(\langle \delta_{(\cdot)} \circ I_+^\alpha; \phi_n \rangle)(\cdot)$  is an element of  $L^2((a,b)) \otimes (\mathcal{W}^{r-1,2})$ . By a standard argument, one can show that  $\sum_{n=1}^N n I_{n-1}(\langle \delta_{(\cdot)} \circ I_+^\alpha; \phi_n \rangle)(\cdot)$  converges in  $L^2((a,b)) \otimes (\mathcal{W}^{r-1,2})$ , since  $\Phi \in (\mathcal{W}^{r,2})$ , to an element which we denote by  $\sum_{n=1}^{\infty} n I_{n-1}(\langle \delta_{(\cdot)} \circ I_+^\alpha; \phi_n \rangle)(\cdot)$ . Since  $(S)$  is dense in  $(\mathcal{W}^{r,2})$ , there exists a sequence  $(\Phi_n) \in (S)^\mathbb{N}$  such that  $\Phi_n \xrightarrow{n \rightarrow +\infty} \Phi$  in  $(\mathcal{W}^{r,2})$ . By continuity,  $\nabla^\alpha(\Phi_n) \xrightarrow{n \rightarrow +\infty} \nabla^\alpha(\Phi)$  in  $L^2((a,b)) \otimes (\mathcal{W}^{r-1,2})$ . Moreover, for all  $n \in \mathbb{N}$ , since  $\nabla(\Phi_n) \in S(\mathbb{R}) \otimes (S)$  and  $(\delta_t \circ I_+^\alpha) \otimes (\delta_\omega \circ E) \in S'(\mathbb{R}) \otimes (S)^*$ , we have:

$$\begin{aligned} \forall (t, \omega) \in \mathbb{R} \times S'(\mathbb{R}), \quad \nabla^\alpha(\Phi_n)(t, \omega) &= ((I_+^\alpha \otimes E) \circ \nabla)(\Phi_n)(t, \omega), \\ &= \int_{-\infty}^{+\infty} \frac{(t-s)_+^{\alpha-1}}{\Gamma(\alpha)} \nabla(\Phi_n)(s, \omega) ds, \\ &= I_+^\alpha(\nabla(\Phi_n)(\cdot, \omega))(t), \\ &= \langle \delta_t; I_+^\alpha(\nabla(\Phi_n)(\cdot, \omega)) \rangle, \\ &= \langle \delta_t \circ I_+^\alpha; \nabla(\Phi_n)(\cdot, \omega) \rangle, \\ &= D_{\delta_t \circ I_+^\alpha}(\Phi_n)(\omega), \\ &= \sum_{m=1}^{\infty} m I_{m-1}(\langle \delta_t \circ I_+^\alpha; \phi_m^n \rangle)(\omega). \end{aligned}$$

where we use Theorem 1.5 for the next to last equality. Since  $\Phi_n \xrightarrow{n \rightarrow +\infty} \Phi$  in  $(\mathcal{W}^{r,2})$ , one can show that  $\sum_{m=1}^{\infty} m I_{m-1}(\langle \delta_{(\cdot)} \circ I_+^\alpha; \phi_m^n \rangle)(\cdot)$  converges to  $\sum_{m=1}^{\infty} m I_{m-1}(\langle \delta_{(\cdot)} \circ I_+^\alpha; \phi_m \rangle)(\cdot)$  in

$L^2((a, b)) \otimes (\mathcal{W}^{r-1,2})$ . Thus, the equality holds in  $L^2((a, b)) \otimes (\mathcal{W}^{r-1,2})$ .  $\square$

The next Lemma is a technical one needed to define the second order stochastic derivative operator related to the Rosenblatt process.

**Lemma 2.5:** For every  $r \geq 0$ , The operator  $I_+^{H/2} \otimes I_+^{H/2} \otimes E$  admits a continuous extension from  $L^2(\mathbb{R}^2) \otimes (\mathcal{W}^{r,2})$  to  $L^2((a, b) \times (a, b)) \otimes (\mathcal{W}^{r,2})$ .

**Proof:** The proof is similar to one of Lemma 2.3.  $\square$

Consequently, we have the following result:

**Proposition 2.6:** Let  $r \geq 2$ . There exists a continuous operator, denoted  $\nabla^{(2),H/2}$ , from  $(\mathcal{W}^{r,2})$  to  $L^2((a, b) \times (a, b)) \otimes (\mathcal{W}^{r-2,2})$  such that, for every  $\Phi \in (\mathcal{W}^{r,2})$ :

$$\lambda^{\otimes 2} \otimes \mu - a.e. \nabla^{(2),\frac{H}{2}}(\Phi)(s, t, \omega) = \sum_{n=2}^{\infty} n(n-1) I_{n-2}(\langle \delta_s \circ I_+^{\frac{H}{2}} \otimes \delta_t \circ I_+^{\frac{H}{2}}; \phi_n \rangle)(\omega),$$

with  $\Phi = \sum_{n=0}^{\infty} I_n(\phi_n)$ .

**Proof:** The operator  $\nabla^{(2),H/2} = I_+^{\frac{H}{2}} \otimes I_+^{\frac{H}{2}} \otimes E \circ \nabla^{(2)}$  is a continuous operator from  $(\mathcal{W}^{r,2})$  to  $L^2((a, b) \times (a, b); (\mathcal{W}^{r-2,2}))$  by the previous Lemma and Theorem 1.7. The equality is proved similarly to the one of Proposition 2.4 noting that, for  $n \geq 2$ :

$$\begin{aligned} \left( \int_{(a,b) \times (a,b)} \|\langle \delta_s \circ I_+^{\frac{H}{2}} \otimes \delta_t \circ I_+^{\frac{H}{2}}; \phi_n \rangle\|_{L^2(\mathbb{R}^{n-2})}^2 ds dt \right)^{\frac{1}{2}} &\leq C_{H,a,b} \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n)\|_{L^{(2, \dots, 2, \frac{2}{1-H}, \frac{2}{1-H})}(\mathbb{R}^n)}, \\ &\leq C'_{H,a,b} \|\phi_n\|_{L^2(\mathbb{R}^n)} < \infty. \end{aligned}$$

allows to define an element of  $L^2((a, b) \times (a, b); (\mathcal{W}^{r-2,2}))$  as  $I_{n-2}(\langle \delta_{(\cdot)} \circ I_+^{\frac{H}{2}} \otimes \delta_{(\cdot)} \circ I_+^{\frac{H}{2}}; \phi_n \rangle)(\cdot)$ .  $\square$

Next, we briefly remind the definitions of the fractional noise and of the Rosenblatt noise integrals for stochastic integrand processes which are  $(S)^*$ -integrable on  $(a, b)$ .

**Proposition 2.7:** Let  $\{\Phi_t; t \in (a, b)\}$  be a  $(S)^*$  stochastic process such that:

1.  $\forall \xi \in S(\mathbb{R})$ ,  $S(\Phi_t)(\xi)$  is measurable on  $(a, b)$ .
2. There is a  $p \in \mathbb{N}$ , a strictly positive constant  $a$  and a non-negative function  $L \in L^1((a, b))$  such that:

$$\forall \xi \in S(\mathbb{R}), |S(\Phi_t)(\xi)| \leq L(t) \exp(a \|A^p \xi\|_2^2)$$

Then,  $\Phi_t \diamond \dot{B}_t^H$  and  $\Phi_t \diamond \dot{X}_t^H$  are  $(S)^*$  integrable over  $(a, b)$  and we define the fractional noise integral and the Rosenblatt noise integral of  $\{\Phi_t\}$  by:

$$\begin{aligned} \int_{(a,b)} \Phi_t dB_t^H &= \int_{(a,b)} \Phi_t \diamond \dot{B}_t^H dt, \\ \int_{(a,b)} \Phi_t dX_t^H &= \int_{(a,b)} \Phi_t \diamond \dot{X}_t^H dt. \end{aligned}$$

Moreover, we have the following representation:

$$\begin{aligned} \int_{(a,b)} \Phi_t dB_t^H &= \int_{(a,b)} (D_{A(H)\delta_t \circ I_+^{H-\frac{1}{2}}}^*) (\Phi_t) dt, \\ \int_{(a,b)} \Phi_t dX_t^H &= \int_{(a,b)} (D_{\sqrt{d(H)\delta_t \circ I_+^{\frac{H}{2}}}}^*)^2 (\Phi_t) dt. \end{aligned}$$

**Proof:** See Definition-Theorem 3.10 of [6] for the Rosenblatt noise integral and Section 3.3 of [21] for the fractional noise integral.  $\square$

Finally, we compute the Hilbert space adjoint of  $\nabla^\alpha$ , for  $\alpha \in (0; 1/2)$ , and link this operator with the fractional noise integral for a certain class of stochastic integrand processes.

**Proposition 2.8:** Let  $\alpha \in (0, 1/2)$  and  $r \geq 1$ . The operator  $(\nabla^\alpha)^* = \nabla^* \circ (I_-^\alpha(\mathbb{I}_{(a,b)\cdot}) \otimes E)$  is a linear and continuous operator from  $L^2((a, b)) \otimes (\mathcal{W}^{r,2})$  to  $(\mathcal{W}^{r-1,2})$ , where  $I_-^\alpha$  is defined by:

$$\forall \xi \in S(\mathbb{R}), I_-^\alpha(\xi)(t) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} (s-t)_+^{\alpha-1} \xi(s) ds.$$

**Proof:** By Proposition 2.4,  $\nabla^\alpha = (I_+^\alpha \otimes E) \circ \nabla$  is a linear and continuous operator from  $(\mathcal{W}^{r,2})$  to  $L^2((a, b)) \otimes (\mathcal{W}^{r-1,2})$ . Thus, by definition,  $(\nabla^\alpha)^*$  is a linear and continuous operator from  $L^2((a, b)) \otimes (\mathcal{W}^{r-1,2})^*$  to  $(\mathcal{W}^{r,2})^*$ . Moreover,  $(\nabla^\alpha)^*$  is equal to  $\nabla^* \circ (I_+^\alpha \otimes E)^*$ . Let us compute  $(I_+^\alpha \otimes E)^*$ . We have, for every  $s \geq 0$ ,  $f \in L^2((a, b)) \otimes (\mathcal{W}^{s,2})^*$  and  $g \in L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2})$ :

$$\langle (I_+^\alpha \otimes E)^*(f); g \rangle_{(L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2})^*, L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2}))} = \langle f, (I_+^\alpha \otimes E)(g) \rangle_{(L^2((a,b)) \otimes (\mathcal{W}^{s,2})^*, L^2((a,b)) \otimes (\mathcal{W}^{s,2}))}.$$

Assume  $f = f_1 \otimes F_1$  and  $g = g_1 \otimes G_1$ . Then, by relation 5.16 of [100]:

$$\begin{aligned} \langle (I_+^\alpha \otimes E)^*(f); g \rangle_{(L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2})^*, L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2}))} &= \langle f_1; I_+^\alpha(g_1) \rangle_{L^2((a,b))} \langle F_1; G_1 \rangle_{((\mathcal{W}^{s,2})^*, (\mathcal{W}^{s,2}))}, \\ &= \langle I_-^\alpha(\mathbb{I}_{(a,b)} f_1); g_1 \rangle_{L^2(\mathbb{R})} \langle F_1; G_1 \rangle_{((\mathcal{W}^{s,2})^*, (\mathcal{W}^{s,2}))}, \\ &= \langle (I_-^\alpha(\mathbb{I}_{(a,b)\cdot}) \otimes E)(f_1 \otimes F_1); g_1 \otimes G_1 \rangle_{(L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2})^*, L^2(\mathbb{R}) \otimes (\mathcal{W}^{s,2}))}. \end{aligned}$$

Thus,  $(I_-^\alpha(\mathbb{I}_{(a,b)\cdot}) \otimes E)$  and  $(I_+^\alpha \otimes E)^*$  coincide on simple elements of  $L^2((a, b)) \otimes (\mathcal{W}^{s,2})^*$ . Since, both operators are linear and bounded operators on  $L^2((a, b)) \otimes (\mathcal{W}^{s,2})^*$ , they agree on  $L^2((a, b)) \otimes (\mathcal{W}^{s,2})^*$ . Consequently,  $(\nabla^\alpha)^*$  is equal to  $\nabla^* \circ (I_-^\alpha(\mathbb{I}_{(a,b)\cdot}) \otimes E)$  on  $L^2((a, b)) \otimes$



$(\mathcal{W}^{r-1,2})^*$ . Moreover, it is well known that the operator  $\nabla^*$  is a linear and continuous operator from  $L^2(\mathbb{R}) \otimes (\mathcal{W}^{r,2})$  to  $(\mathcal{W}^{r-1,2})$  for any  $r \geq 1$  (see Proposition 5.27 of [43]). This concludes the proof.  $\square$

**Proposition 2.9:** Let  $\alpha \in (0, 1/2)$  and  $\Phi \in L^2((a, b)) \otimes (\mathcal{W}^{1,2})$ . We have:

$$(\nabla^\alpha)^*(\Phi) = \int_a^b D_{\delta_t \circ I_+^\alpha}^*(\Phi_t) dt.$$

**Proof:** Let  $\xi \in S(\mathbb{R})$ .  $(\nabla^\alpha)^*(\Phi) \in (L^2) \subset (\mathcal{W}^{1,2})^*$  and  $: e^{<:\xi>} :$  is in  $(S) \subset (\mathcal{W}^{s,2})$ , for any  $s \geq 0$ . Thus, we have:

$$\begin{aligned} S((\nabla^\alpha(\Phi))^*)(\xi) &= \langle (\nabla^\alpha(\Phi))^*, : e^{<:\xi>} : \rangle_{(L^2)}, \\ &= \langle (\nabla^\alpha(\Phi))^*, : e^{<:\xi>} : \rangle_{((\mathcal{W}^{1,2})^*, (\mathcal{W}^{1,2}))}, \\ &= \langle \Phi, \nabla^\alpha(: e^{<:\xi>} :) \rangle_{L^2((a,b)) \otimes (L^2)}, \\ &= \langle \Phi, I_+^\alpha(\xi) : e^{<:\xi>} : \rangle_{L^2((a,b)) \otimes (L^2)}, \\ &= \int_a^b I_+^\alpha(\xi)(t) \langle \Phi_t, : e^{<:\xi>} : \rangle_{(L^2)} dt, \\ &= \int_a^b S(\Phi_t)(\xi) I_+^\alpha(\xi)(t) dt, \\ &= S\left(\int_a^b D_{\delta_t \circ I_+^\alpha}^*(\Phi_t) dt\right)(\xi). \quad \square \end{aligned}$$

### 3 From forward integrals to Wick-Itô integrals.

In this section, we want to make an explicit connection between forward integrals with respect to fractional Brownian motion and with respect to the Rosenblatt process, built using Russo-Vallois theory of regularisation ([97]), and the Wick-Itô integrals introduced in Proposition 2.7. For this purpose, we adopt the following strategy. First of all, for a certain class of stochastic integrands denoted by  $\{\Phi_t; t \in (a, b)\}$ , we compute the  $(S)$ -transforms of  $\Phi_t(B_{t+\epsilon}^H - B_t^H)$  and of  $\Phi_t(X_{t+\epsilon}^H - X_t^H)$  for every  $\epsilon > 0$  and  $t \in (a, b)$ . Then, we prove the  $(S)^*$ -integrability of the two former terms on  $(a, b)$ . Finally, for peculiar cases, we examine the convergence in  $(S)^*$  of  $\int_a^b \Phi_t(B_{t+\epsilon}^H - B_t^H) dt / \epsilon$  and of  $\int_a^b \Phi_t(X_{t+\epsilon}^H - X_t^H) dt / \epsilon$ .

### 3.1 Fractional Brownian motion.

**Proposition 3.1.1:** Let  $\{\Phi_t : t \in (a, b)\}$  be a stochastic process such that for all  $t \in (a, b)$ ,  $\Phi_t \in (\mathcal{W}^{1,2})$ . Then, we have, for every  $\epsilon > 0$  and  $t \in (a, b)$ :

$$\Phi_t \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} \stackrel{(S)^*}{=} \Phi_t \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} + A(H) \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s, \cdot) \frac{ds}{\epsilon}.$$

**Proof:** Let  $\epsilon > 0$  be small enough such that  $t + \epsilon \in (a, b)$ . Since  $\Phi_t \in (\mathcal{W}^{1,2}) \subset (L^2)$ ,  $\Phi_t = \sum_{n=0}^{\infty} I_n(\phi_n(\cdot, t))$  with  $\phi_n(\cdot, t) \in \hat{L}^2(\mathbb{R}^n)$ . Let us fix  $n \geq 1$ . Using the multiplication formula from Malliavin calculus (see Proposition 1.1.3 of [87]), we obtain:

$$\begin{aligned} I_n(\phi_n(\cdot, t))(B_{t+\epsilon}^H - B_t^H) &= A(H) I_n(\phi_n(\cdot, t)) I_1\left(\int_t^{t+\epsilon} \frac{(s - \cdot)_+^{H-\frac{3}{2}}}{\Gamma(H - \frac{1}{2})} ds\right), \\ &= I_{n+1}(\phi_n(\cdot, t) \otimes g_{t,t+\epsilon}^H) + n I_{n-1}(\phi_n(\cdot, t) \otimes_1 g_{t,t+\epsilon}^H), \end{aligned}$$

where  $g_{t,t+\epsilon}^H(\cdot) = A(H)/\Gamma(H - 1/2) \int_t^{t+\epsilon} (s - \cdot)_+^{H-3/2} ds$ . Then, for any  $\xi \in S(\mathbb{R})$ , we have:

$$S(I_n(\phi_n(\cdot, t))(B_{t+\epsilon}^H - B_t^H))(\xi) = \langle \phi_n(\cdot, t); \xi^{\otimes n} \rangle \langle g_{t,t+\epsilon}^H; \xi \rangle + n \langle \phi_n(\cdot, t) \otimes_1 g_{t,t+\epsilon}^H; \xi^{\otimes n-1} \rangle.$$

By continuity of the scalar product in  $(L^2)$ , we have:

$$S(\Phi_t(B_{t+\epsilon}^H - B_t^H))(\xi) = \sum_{n=0}^{\infty} \left( \langle \phi_n(\cdot, t); \xi^{\otimes n} \rangle \langle g_{t,t+\epsilon}^H; \xi \rangle + n \langle \phi_n(\cdot, t) \otimes_1 g_{t,t+\epsilon}^H; \xi^{\otimes n-1} \rangle \right).$$

We want to compute separately the two infinite series appearing in the right hand side of the previous equality. First of all, we have:

$$\begin{aligned} \sum_{n=0}^{\infty} \langle \phi_n(\cdot, t); \xi^{\otimes n} \rangle \langle g_{t,t+\epsilon}^H; \xi \rangle &= S(\Phi_t)(\xi) S(B_{t+\epsilon}^H - B_t^H)(\xi), \\ &= S(\Phi_t \diamond (B_{t+\epsilon}^H - B_t^H))(\xi). \end{aligned}$$

Moreover, for the second term, we have, by Fubini theorem and Definition-Theorem 5.2 of chapter 2:

$$\begin{aligned} n \langle \phi_n(\cdot, t) \otimes_1 g_{t,t+\epsilon}^H; \xi^{\otimes n-1} \rangle &= A(H) n \int_t^{t+\epsilon} \langle I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(\cdot, t))(s, \cdot); \xi^{\otimes n-1} \rangle ds, \\ &= A(H) \int_t^{t+\epsilon} S(n I_{n-1}(I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(\cdot, t))(s, \cdot))(\xi) ds, \\ &= A(H) \int_t^{t+\epsilon} S(\nabla^{H-\frac{1}{2}}(I_n(\phi_n(\cdot, t)))(s, \cdot))(\xi) ds. \end{aligned}$$

We want to invert the infinite series and the integral over  $(t, t + \epsilon)$ . For any  $n \geq 1$  and  $t \in (a, b)$ , we have:

$$\begin{aligned} \left| \int_t^{t+\epsilon} \langle I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s); \xi^{\otimes n-1} \rangle ds \right| &\leq \int_t^{t+\epsilon} \left| \langle I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s); \xi^{\otimes n-1} \rangle \right| ds, \\ &\leq \|\xi\|_{L^2(\mathbb{R})}^{n-1} \int_t^{t+\epsilon} \|I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s)\|_{L^2(\mathbb{R}^{n-1})} ds, \\ &\leq \|\xi\|_{L^2(\mathbb{R})}^{n-1} \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s)\|_{L^2(\mathbb{R}^{n-1})} ds < +\infty, \end{aligned}$$

since  $I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t)) \in L^{(2, \dots, 2, 1/(1-H))}(\mathbb{R}^n)$  and  $1/(1-H) > 2$  (see Theorem 24.1 of [100]).

Finally, we note that, using Cauchy-Schwarz and Jensen inequalities and Proposition 2.4:

$$\begin{aligned} \sum_{n=1}^{+\infty} \left[ n \|\xi\|_{L^2(\mathbb{R})}^{n-1} \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s)\|_{L^2(\mathbb{R}^{n-1})} ds \right] &\leq \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \times \\ &\left( \sum_{n=1}^{\infty} n!n \left( \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s)\|_{L^2(\mathbb{R}^{n-1})} ds \right)^2 \right)^{\frac{1}{2}}, \\ &\leq (b-a)^{\frac{1}{2}} \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \times \\ &\left( \sum_{n=1}^{\infty} n!n \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s)\|_{L^2(\mathbb{R}^{n-1})}^2 ds \right)^{\frac{1}{2}}, \\ &\leq (b-a)^{\frac{1}{2}} \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \|\nabla^{H-\frac{1}{2}}(\Phi_t)\|_{L^2((a,b)) \otimes (L^2)}, \end{aligned}$$

which is finite since  $\Phi_t \in (\mathcal{W}^{1,2})$ . Thus, we have:

$$\begin{aligned} \sum_{n=1}^{+\infty} n \langle \phi_n(t) \otimes_1 g_{t, t+\epsilon}^H; \xi^{\otimes n-1} \rangle &= A(H) \int_t^{t+\epsilon} \sum_{n=1}^{+\infty} [S(nI_{n-1}(I_{+, \dots, +}^{(0, \dots, 0, H-\frac{1}{2})}(\phi_n(t))(, s)))(\xi)] ds, \\ &= A(H) \int_t^{t+\epsilon} S(\nabla^{H-\frac{1}{2}}(\Phi_t)(s, ))(\xi) ds, \\ &= S(A(H) \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s, ) ds)(\xi), \end{aligned}$$

where we use the square-integrability of  $\|\nabla^{H-\frac{1}{2}}(\Phi_t)(s, )\|_{(L^2)}$  on  $(a, b)$  to justify the last equality. This concludes the proof.  $\square$

**Proposition 3.1.2:** Let  $\{\Phi_t : t \in (a, b)\}$  be a stochastic process such that for all  $t \in (a, b)$ ,  $\Phi_t \in (\mathcal{W}^{1,2})$ . Moreover, assume that:

$$\int_a^b \|\Phi_t\|_{(L^2)}^2 dt < +\infty,$$

$$\int_a^b \|\nabla^{H-\frac{1}{2}}(\Phi_t)\|_{L^2(a,b)\otimes(L^2)}^2 dt < +\infty.$$

Then, we have:

$$\int_a^b \Phi_t \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt \stackrel{(S)^*}{=} \int_a^b \Phi_t \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt + A(H) \int_a^b \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s, ) \frac{ds}{\epsilon} dt.$$

**Proof:** In order to prove the  $(S)^*$ -integrability of  $\Phi_t \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon}$  over  $(a, b)$ , we are going to prove the  $(S)^*$ -integrability of the two terms appearing in Proposition 3.1.1. Let us start with the first term. For every  $\xi \in S(\mathbb{R})$ , we have:

$$\begin{aligned} S\left(\Phi_t \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon}\right)(\xi) &= S(\Phi_t)(\xi) S\left(\frac{B_{t+\epsilon}^H - B_t^H}{\epsilon}\right)(\xi), \\ &= \mathbb{E}^\mu[\Phi_t \exp(\langle \cdot, \xi \rangle - \frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2})] A(H) \int_t^{t+\epsilon} I_+^{H-\frac{1}{2}}(\xi)(s) \frac{ds}{\epsilon}. \end{aligned}$$

By Cauchy-Schwarz inequality and by Lemma 3.3 of Chapter 2, we have:

$$|S\left(\Phi_t \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon}\right)(\xi)| \leq C'_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_{L^1(\mathbb{R})}) \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \mathbb{E}^\mu[\Phi_t^2]^{\frac{1}{2}}.$$

Using the fact that  $\int_a^b \|\Phi_t\|_{L^2}^2 dt < +\infty$ , we have the  $(S)^*$ -integrability of the first term. Let us deal with the second term now. For every  $\xi \in S(\mathbb{R})$ , we have:

$$\begin{aligned} |S\left(\int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s, ) ds\right)(\xi)| &\leq \int_t^{t+\epsilon} |S(\nabla^{H-\frac{1}{2}}(\Phi_t)(s, ))(\xi)| ds, \\ &\leq \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \int_t^{t+\epsilon} \|\nabla^{H-\frac{1}{2}}(\Phi_t)(s, )\|_{(L^2)} ds, \\ &\leq \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \int_a^b \|\nabla^{H-\frac{1}{2}}(\Phi_t)(s, )\|_{(L^2)} ds, \\ &\leq (b-a)^{\frac{1}{2}} \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \|\nabla^{H-\frac{1}{2}}(\Phi_t)\|_{L^2((a,b)\otimes(L^2))} \end{aligned}$$

Using the fact that  $\int_a^b \|\nabla^{H-\frac{1}{2}}(\Phi_t)\|_{L^2((a,b)\otimes(L^2))}^2 dt < +\infty$ , we have the  $(S)^*$ -integrability of the second term. This ends the proof.  $\square$

The next proposition examines the chain rule property for certain functionals of fractional Brownian motion.

**Proposition 3.1.3:** Let  $F$  be in  $C^1(\mathbb{R})$ . Assume that, for every  $t \in (a, b)$ ,  $F(B_t^H) \in (\mathcal{W}^{1,2})$ .

Then, we have, in  $L^2((a, b)) \otimes (L^2)$ :

$$\nabla^{H-\frac{1}{2}}(F(B_t^H))(s, \omega) = \frac{A(H)\beta(H-\frac{1}{2}, 2-2H)}{(\Gamma(H-\frac{1}{2}))^2} \left( \int_0^t |s-r|^{2H-2} dr \right) F'(B_t^H)(\omega).$$

**Proof:** We denote by  $\{H_{n,t^{2H}} : n \in \mathbb{N}\}$  the family of Hermite polynomials of parameter  $t^{2H}$ . Then,  $\{H_{n,t^{2H}}/(\sqrt{n!}t^{Hn}) : n \in \mathbb{N}\}$  is an orthonormal family of  $L^2(\mathbb{R}, \gamma_{t^{2H}}(dx))$  where  $\gamma_{t^{2H}}$  is the centered Gaussian probability measure over  $\mathbb{R}$  with variance  $t^{2H}$ . Since  $\mathbb{E}[(F(B_t^H))^2] < \infty$ , we have:

$$F = \sum_{n=0}^{\infty} c_{n,t^{2H}} \frac{H_{n,t^{2H}}}{\sqrt{n!}t^{Hn}}.$$

where  $c_{n,t^{2H}}$  are the Hermite coefficients of  $F$  with respect to  $\gamma_{t^{2H}}$ . Using Wiener-Itô Theorem, we have:

$$F(B_t^H) = \sum_{n=0}^{\infty} c_{n,t^{2H}} \frac{I_n((g_{0,t}^H)^{\otimes n})}{\sqrt{n!}t^{Hn}}.$$

Let  $n \geq 0$ . Denote by  $d_{n,t^{2H}}$  the Hermite coefficients of  $F'$  with respect to  $\gamma_{t^{2H}}$ . We have:

$$d_{n,t^{2H}} = \int_{\mathbb{R}} F'(x) \frac{H_{n,t^{2H}}(x)}{\sqrt{n!}t^{Hn}} \gamma_{t^{2H}}(dx).$$

Integrating by part, we obtain:

$$\begin{aligned} d_{n,t^{2H}} &= - \int_{\mathbb{R}} F(x) \left[ n H_{n-1,t^{2H}}(x) - \frac{x}{t^{2H}} H_{n,t^{2H}}(x) \right] \frac{\gamma_{t^{2H}}(dx)}{\sqrt{n!}t^{Hn}}, \\ d_{n,t^{2H}} &= \int_{\mathbb{R}} F(x) H_{n+1,t^{2H}}(x) \frac{\gamma_{t^{2H}}(dx)}{\sqrt{(n+1)!}t^{H(n+2)}}, \\ d_{n,t^{2H}} &= \frac{\sqrt{n+1}}{t^H} c_{n+1,t^{2H}}. \end{aligned}$$

Since  $\mathbb{E}[(N+E)^{1/2}F(B_t^H)]^2 < +\infty$ ,  $F' \in L^2(\mathbb{R}, \gamma_{t^{2H}}(dx))$  and we have:

$$\begin{aligned} F' &= \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{t^H} c_{n+1,t^{2H}} \frac{H_{n,t^{2H}}}{\sqrt{n!}t^{Hn}}, \\ F' &= \sum_{n=1}^{\infty} \frac{\sqrt{n}}{t^H} c_{n,t^{2H}} \frac{H_{n-1,t^{2H}}}{\sqrt{(n-1)!}t^{H(n-1)}}. \end{aligned}$$

Thus, by Wiener-Itô Theorem, we obtain:

$$F'(B_t^H) = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{t^H} c_{n,t^{2H}} \frac{I_{n-1}((g_{0,t}^H)^{\otimes n-1})}{\sqrt{(n-1)!}t^{H(n-1)}},$$

$$F'(B_t^H) = \sum_{n=1}^{\infty} n \frac{c_{n,t^{2H}}}{\sqrt{n!t^{Hn}}} I_{n-1}((g_{0,t}^H)^{\otimes n-1}).$$

Moreover, by assumption,  $F(B_t^H) \in (\mathcal{W}^{1,2})$ , consequently, by Proposition 2.4, we have,  $\lambda \otimes \mu$ -a.e:

$$\begin{aligned} \nabla^{H-\frac{1}{2}}(F(B_t^H))(s, \omega) &= \sum_{n=1}^{\infty} n \frac{c_{n,t^{2H}}}{\sqrt{n!t^{Hn}}} I_{n-1}(\langle \delta_s \circ I_+^{H-\frac{1}{2}}; (g_{0,t}^H)^{\otimes n} \rangle)(\omega), \\ \nabla^{H-\frac{1}{2}}(F(B_t^H))(s, \omega) &= I_+^{H-\frac{1}{2}}(g_{0,t}^H)(s) \sum_{n=1}^{\infty} n \frac{c_{n,t^{2H}}}{\sqrt{n!t^{Hn}}} I_{n-1}((g_{0,t}^H)^{\otimes n-1})(\omega), \\ \nabla^{H-\frac{1}{2}}(F(B_t^H))(s, \omega) &= \frac{A(H)\beta(H-\frac{1}{2}, 2-2H)}{(\Gamma(H-\frac{1}{2}))^2} \left( \int_0^t |s-r|^{2H-2} dr \right) F'(B_t^H(\omega)). \square \end{aligned}$$

**Proposition 3.1.4:** Let  $F$  be in  $C^1(\mathbb{R})$ . Assume that, for every  $t \in (a, b)$ ,  $F(B_t^H) \in (\mathcal{W}^{1,2})$ . Then, we have:

$$A(H) \int_a^b \left( \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(F(B_t^H))(s, \cdot) \frac{ds}{\epsilon} \right) dt \xrightarrow[\epsilon \rightarrow 0^+]{a.s.} H \int_a^b t^{2H-1} F'(B_t^H) dt,$$

**Proof:** Using Proposition 3.1.3 as well as standard calculations lead to the convergence of this trace term.  $\square$

**Remark 3.1.5:** Let  $F$  be as in Proposition 3.1.4. In order to apply Proposition 3.1.2 to  $\Phi_t = F(B_t^H)$ , it is sufficient to prove that:

$$\begin{aligned} \int_a^b \|F(B_t^H)\|_{(L^2)}^2 dt &< +\infty, \\ \int_a^b \|\nabla^{H-\frac{1}{2}}(F(B_t^H))\|_{L^2((a,b);(L^2))}^2 dt &< +\infty. \end{aligned}$$

If we assume that there exist  $C > 0$  and  $\lambda > 0$  with  $\lambda < 1/(4b^{2H})$  such that:

$$\forall x \in \mathbb{R}, \max\{F(x), F'(x)\} \leq Ce^{\lambda x^2}.$$

Then, we have:

$$\begin{aligned} \int_a^b \|F(B_t^H)\|_{(L^2)}^2 dt &\leq C \int_a^b \frac{dt}{\sqrt{1-4\lambda t^{2H}}} < \infty, \\ \int_a^b \|\nabla^{H-\frac{1}{2}}(F(B_t^H))\|_{L^2((a,b);(L^2))}^2 dt &= \int_a^b \|I_+^{H-\frac{1}{2}}(g_t^H)\|_{L^2((a,b))}^2 \|F'(B_t^H)\|_{(L^2)}^2 dt, \\ &\leq C_{a,b,H} \int_a^b \frac{t^{2H}}{\sqrt{1-4\lambda t^{2H}}} dt < \infty. \end{aligned}$$

We conclude this subsection by the following convergence result.

**Theorem 3.1.6:** Let  $F$  be in  $C^1(\mathbb{R})$ . Assume that, for every  $t \in (a, b)$ ,  $F(B_t^H) \in (\mathcal{W}^{1,2})$  and that:

$$\forall x \in \mathbb{R}, \max\{F(x), F'(x)\} \leq Ce^{\lambda x^2},$$

for some  $C > 0$  and  $\lambda > 0$  with  $\lambda < 1/(4b^{2H})$ . Then,  $\int_a^b F(B_t^H)(B_{t+\epsilon}^H - B_t^H)dt/\epsilon$  converges in  $(S)^*$  to the following element of  $(L^2)$ :

$$A(H)(\nabla^{H-\frac{1}{2}})^*(F(B^H)) + H \int_a^b t^{2H-1} F'(B_t^H) dt,$$

**Proof:** First of all, let us study the convergence of  $\int_a^b F(B_t^H) \diamond (B_{t+\epsilon}^H - B_t^H) dt/\epsilon$ . For any  $\xi \in S(\mathbb{R})$ , we have:

$$\begin{aligned} S\left(\int_a^b F(B_t^H) \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt\right)(\xi) &= \int_a^b S(F(B_t^H))(\xi) S\left(\frac{B_{t+\epsilon}^H - B_t^H}{\epsilon}\right)(\xi) dt, \\ &= A(H) \int_a^b S(F(B_t^H))(\xi) \int_t^{t+\epsilon} I_+^{H-\frac{1}{2}}(\xi)(s) \frac{ds}{\epsilon} dt. \end{aligned}$$

Since,  $I_+^{H-1/2}(\xi)$  is infinitely differentiable, we have:

$$\lim_{\epsilon \rightarrow 0^+} \int_t^{t+\epsilon} I_+^{H-\frac{1}{2}}(\xi)(s) \frac{ds}{\epsilon} = I_+^{H-\frac{1}{2}}(\xi)(t).$$

Moreover,

$$|S(F(B_t^H))(\xi)| \left| \int_t^{t+\epsilon} I_+^{H-\frac{1}{2}}(\xi)(s) \frac{ds}{\epsilon} \right| \leq A(H) \| : e^{< \cdot, \xi >} : \|_{(L^2)} \|F(B_t^H)\|_{(L^2)} \|I_+^{H-\frac{1}{2}}(\xi)\|_\infty$$

The right hand side of the previous inequality is in  $L^1((a, b))$ . Thus, by Lebesgue dominated convergence theorem,

$$\lim_{\epsilon \rightarrow 0^+} S\left(\int_a^b F(B_t^H) \diamond \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} dt\right)(\xi) = S\left(\int_a^b F(B_t^H) \diamond \dot{B}_t^H dt\right)(\xi).$$

Furthermore, the previous uniform bound allows us to apply the third bullet points of Theorem 1.3. This proves the convergence in  $(S)^*$  of  $\int_a^b F(B_t^H) \diamond (B_{t+\epsilon}^H - B_t^H) dt/\epsilon$  to  $\int_a^b F(B_t^H) \diamond \dot{B}_t^H dt$ . Since,  $F(B_t^H) \in (\mathcal{W}^{1,2})$ , a direct application of Proposition 2.9 leads to:

$$\int_a^b F(B_t^H) \diamond \dot{B}_t^H dt = A(H)(\nabla^{H-\frac{1}{2}})^*(F(B^H)),$$

which makes it an element of  $(L^2)$  by Proposition 2.8. Applying Proposition 3.1.4 and Remark 3.1.5 leads to the result.  $\square$

**Remark 3.1.7:** Actually, one can prove that the convergence of the previous theorem holds in  $(L^2)$ . Thus, using the definition of the forward integral of Russo-Vallois, we obtain the following equality in  $(L^2)$  (and almost surely as well):

$$\int_a^b F(B_t^H) d^- B_t^H = A(H)(\nabla^{H-\frac{1}{2}})^*(F(B^H)) + H \int_a^b t^{2H-1} F'(B_t^H) dt.$$

### 3.2 The Rosenblatt process.

In this subsection, we study the counterpart of Propositions 3.1.1 and 3.1.2 for the Rosenblatt process. As it was first observed in [113] using Malliavin calculus, a second trace term appears when one wants to connect forward and Wick-Itô integrals with respect to the Rosenblatt process. This second trace term is way more delicate to handle and new assumptions are needed to be made on the class of stochastic integrand processes in addition to mere Malliavin differentiability. More specifically, the existence of this term is linked to the restriction on the diagonal of  $(a, b) \times (a, b)$  of  $\nabla^{(2), H/2}(\Phi)$  where  $\Phi$  is at least in  $(\mathcal{W}^{2,2})$ . Based on the Wiener-Itô expansion of  $\Phi$  and Proposition 2.6, we have:

$$\lambda^{\otimes 2} \otimes \mu - a.e.(s, t, \omega), \quad \nabla^{(2), \frac{H}{2}}(\Phi)(s, t, \omega) = \sum_{n=2}^{\infty} n(n-1) I_{n-2}(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n)(\cdot, s, t))(\omega),$$

Thus, one way to ensure the existence of such a restriction is to impose regularity assumptions upon  $I_{+, \dots, +}^{(0, \dots, 0, H/2, H/2)}(\phi_n)$  for each  $n \geq 2$ . For this purpose, we introduce, in this subsection, the function spaces with dominating mixed smoothness which were defined by Lizorkin and Nikol'skii in [67] (see [102] for a recent survey on these spaces and their generalizations).

**Definition 3.2.1:** For any  $(\alpha_1, \dots, \alpha_n)$  such that  $\alpha_i \geq 0$  for every  $i \in \{1, \dots, n\}$ , we define  $S_2^{(\alpha_1, \dots, \alpha_n)} H(\mathbb{R}^n)$  by:

$$S_2^{(\alpha_1, \dots, \alpha_n)} H(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \mathcal{F}^{-1}\left(\prod_{i=1}^n (1 + |\xi_i|^2)^{\frac{\alpha_i}{2}} \mathcal{F}(f)\right) \in L^2(\mathbb{R}^n)\}.$$

Moreover,  $S_2^{(\alpha_1, \dots, \alpha_n)} H(\mathbb{R}^n)$  becomes a Hilbert space when it is endowed with the inner product:

$$\forall f, g \in S_2^{(\alpha_1, \dots, \alpha_n)} H(\mathbb{R}^n) \quad \langle f, g \rangle = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n (1 + |\xi_i|^2)^{\alpha_i}\right) \mathcal{F}(f)(\xi) \overline{\mathcal{F}(g)(\xi)} d\xi.$$



The trace problem for function spaces with dominating mixed smoothness appears to be much more difficult than for standard isotropic fractional Sobolev spaces. Indeed, function spaces with dominating mixed smoothness are not rotationally invariant in general. For example, trace problems for hyperplane parallel to coordinate axis are different of trace problems for hyperplane in oblique position. Nevertheless, for  $n = 2$ , regarding trace problem on the diagonal of  $\mathbb{R}^2$ , we have the following result (see Theorem 5.4 in section 5.1 of [102] and also [119] for the proof):

**Lemma 3.2.2:** Let  $\alpha_1, \alpha_2$  be strictly positive such that  $\alpha_i < 1/2$  for every  $i = 1, 2$  and  $\alpha_1 + \alpha_2 > 1/2$ . Then, we have:

$$Tr(S_2^{(\alpha_1, \alpha_2)} H(\mathbb{R}^2)) = H^{\alpha_1 + \alpha_2 - \frac{1}{2}}(\mathbb{R}),$$

where  $Tr$  is the trace operator on the diagonal of  $\mathbb{R}^2$  and  $H^{\alpha_1 + \alpha_2 - 1/2}(\mathbb{R})$  is the standard isotropic fractional Sobolev spaces of order  $\alpha_1 + \alpha_2 - 1/2$  (see [1] chapter 7).

Now, we consider a specific case of spaces with dominating mixed smoothness closely related to the mixed Liouville derivative of fractional order  $(0, \dots, 0, H/2, H/2)$  (see chapter 5, paragraph 24.2 in [100] for a definition). Namely, we consider  $S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^n)$ . This space is isomorphically isometric to  $L^2(\mathbb{R}^{n-2}) \otimes S_2^{(H/2, H/2)} H(\mathbb{R}^2)$ . Then, we have the following trace result:

**Lemma 3.2.3:** It exists a linear and continuous operator  $T$  from  $S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^n)$  to  $S_2^{(0, \dots, 0, H-1/2)} H(\mathbb{R}^{n-1})$  such that for any  $f \in S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ , we have:

$$T(f) = f|_{\Delta_n}.$$

where  $\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = x_{n-1}\}$ .

**Proof:** See the proof of Lemma 5.6 in Chapter 2.  $\square$

Similarly as in Chapter 2, assuming that  $I_{+, \dots, +}^{(0, \dots, 0, H/2, H/2)}(\phi_n) \in S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^n)$  for each  $n \geq 2$  induces the following structure on  $\phi_n \in \hat{L}^2(\mathbb{R}^n)$ :

$$\mathcal{F}(\phi_n) = \frac{(-i\xi_{n-1})^{\frac{H}{2}} (-i\xi_n)^{\frac{H}{2}}}{\prod_{i=n-1}^n (1 + |\xi_i|^2)^{\frac{H}{4}}} \mathcal{F}(\psi_n),$$

for some  $\psi_n \in L^2(\mathbb{R}^n)$  and we have the following inequality:

$$\|\phi_n\|_{L^2(\mathbb{R}^n)} \leq C_H \|\psi_n\|_{L^2(\mathbb{R}^n)}.$$

for some  $C_H > 0$  independent of  $n$ . We say that  $\phi_n$  verifies **H1** when  $\phi_n$  is as above (see as well Chapter 2 Section 5 for further details).

**Proposition 3.2.4:** Let  $\{\Phi_t : t \in (a, b)\}$  be a stochastic process such that:

$$\forall t \in (a, b), \Phi_t \in (\mathcal{W}^{2,2}), \quad (3.1)$$

$$\forall n \geq 2, \forall t \in (a, b), \phi_n(\cdot, t) \text{ verifies } \mathbf{H1}, \quad (3.2)$$

$$\sum_{n=2}^{\infty} n(n-1)n! \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{n-2})}^2 ds < \infty. \quad (3.3)$$

Thus, for all  $t \in (a, b)$  and for all  $\epsilon > 0$  such that  $t + \epsilon \in (a, b)$ , we have:

$$\begin{aligned} \Phi_t \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} \stackrel{(S)^*}{=} \Phi_t \diamond \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} + 2d(H) \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot)) \frac{ds}{\epsilon} \\ + d(H) \int_t^{t+\epsilon} \nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot) \frac{ds}{\epsilon}. \end{aligned}$$

**Remark 3.2.5:** The last assumption ensures the existence of an element of  $L^2((a, b)) \otimes (L^2)$  defined by the following Wiener-Itô expansion, for all  $t \in (a, b)$ :

$$\lambda_s \otimes \mu - a.e., \tilde{\nabla}^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot) := \sum_{n=2}^{\infty} n(n-1) I_{n-2}(T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)).$$

Since, it coincides with  $\nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot)$  for smoother  $\{\Phi_t\}$ , we denote it  $\nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot)$  in the sequel.

**Proof:** Let  $\epsilon > 0$  be small enough such that  $t + \epsilon \in (a, b)$ . Let us fix  $n \geq 2$ . Using the multiplication formula from Malliavin calculus, we obtain:

$$\begin{aligned} I_n(\phi_n(\cdot, t))(X_{t+\epsilon}^H - X_t^H) &= I_n(\phi_n(\cdot, t))I_2(f_{t, t+\epsilon}^H) \\ &= I_{n+2}(\phi_n(\cdot, t) \hat{\otimes} f_{t, t+\epsilon}^H) + 2n I_n(\phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H) + n(n-1) I_{n-2}(\phi_n(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H), \end{aligned}$$

where  $f_{t, t+\epsilon}^H(x_1, x_2) = d(H)/(\Gamma(H/2))^2 \int_t^{t+\epsilon} (s-x_1)_+^{\frac{H}{2}-1} (s-x_2)_+^{\frac{H}{2}-1} ds$ . Then, for any  $\xi \in S(\mathbb{R})$ , we have:

$$\begin{aligned} S(I_n(\phi_n(\cdot, t))(X_{t+\epsilon}^H - X_t^H))(\xi) &= \langle \phi_n(\cdot, t) \hat{\otimes} f_{t, t+\epsilon}^H; \xi^{\otimes n+2} \rangle + 2n \langle \phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H; \xi^{\otimes n} \rangle \\ &\quad + n(n-1) \langle \phi_n(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H; \xi^{\otimes n-2} \rangle \end{aligned}$$

By continuity of the scalar product in  $(L^2)$ , we have:

$$\begin{aligned} S(\Phi_t(X_{t+\epsilon}^H - X_t^H))(\xi) &= \sum_{n=0}^{\infty} \left( \langle \phi_n(\cdot, t) \hat{\otimes} f_{t, t+\epsilon}^H; \xi^{\otimes n+2} \rangle + 2n \langle \phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H; \xi^{\otimes n} \rangle \right. \\ &\quad \left. + n(n-1) \langle \phi_n(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H; \xi^{\otimes n-2} \rangle \right) \end{aligned}$$

We want to compute separately the three infinite series and to invert the sums and the integrals. Let  $n \in \mathbb{N}^*$ .

$$\begin{aligned}
\langle \phi_n(\cdot, t) \hat{\otimes} f_{t, t+\epsilon}^H; \xi^{\otimes n+2} \rangle &= \langle \phi_n(\cdot, t); \xi^{\otimes n} \rangle \langle f_{t, t+\epsilon}^H; \xi^{\otimes 2} \rangle \\
&= d(H) \langle \phi_n(\cdot, t); \xi^{\otimes n} \rangle \int_t^{t+\epsilon} (I_+^{\frac{H}{2}}(\xi)(s))^2 ds \\
&= d(H) S(I_n(\phi_n(\cdot, t)))(\xi) \int_t^{t+\epsilon} (I_+^{\frac{H}{2}}(\xi)(s))^2 ds
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} \langle \phi_n(\cdot, t) \hat{\otimes} f_{t, t+\epsilon}^H; \xi^{\otimes n+2} \rangle &= d(H) \sum_{n=0}^{\infty} S(I_n(\phi_n(\cdot, t)))(\xi) \int_t^{t+\epsilon} (I_+^{\frac{H}{2}}(\xi)(s))^2 ds \\
&= S(\Phi_t)(\xi) S(X_{t+\epsilon}^H - X_t^H)(\xi) = S(\Phi_t \diamond (X_{t+\epsilon}^H - X_t^H))(\xi).
\end{aligned}$$

Moreover,

$$\begin{aligned}
2n \langle \phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H; \xi^{\otimes n} \rangle &= 2n \int_{\mathbb{R}^n} (\phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H)(x_1, \dots, x_n) \xi^{\otimes n}(x_1, \dots, x_n) dx_1 \dots dx_n \\
&= 2n \int_{\mathbb{R}^{n+1}} \phi_n(x_1, \dots, x_{n-1}, u, t) f_{t, t+\epsilon}^H(x_n, u) \xi^{\otimes n}(x_1, \dots, x_n) du dx_1 \dots dx_n \\
&= 2nd(H) \int_{\mathbb{R}^{n+1}} \phi_n(x_1, \dots, x_{n-1}, u, t) \left( \int_t^{t+\epsilon} \frac{(s-u)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} \frac{(s-x_n)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})} ds \right) \\
&\quad \times \xi^{\otimes n}(x_1, \dots, x_n) du dx_1 \dots dx_n
\end{aligned}$$

Using  $\langle |\phi_n(\cdot, t)| \otimes_1 f_{t, t+\epsilon}^H; |\xi|^{\otimes n} \rangle \leq \|\xi\|_{L^2(\mathbb{R})}^n \|\phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H\|_{L^2(\mathbb{R}^n)} < \infty$ , by Fubini Theorem, we have:

$$\begin{aligned}
2n \langle \phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H; \xi^{\otimes n} \rangle &= 2nd(H) \int_t^{t+\epsilon} I_+^{\frac{H}{2}}(\xi)(s) \langle I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s); \xi^{\otimes n-1} \rangle_{L^2(\mathbb{R}^{n-1})} ds, \\
&= 2d(H) \int_t^{t+\epsilon} S(nI_{n-1}(\delta_s \circ I_+^{\frac{H}{2}} \otimes_1 \phi_n(\cdot, t)))(\xi) I_+^{\frac{H}{2}}(\xi)(s) ds, \\
&= 2d(H) \int_t^{t+\epsilon} S(D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(I_n(\phi_n(\cdot, t)))(s, \cdot)))(\xi) ds.
\end{aligned}$$

We want to invert the infinite sum and the integral over  $(t, t+\epsilon)$ . For any  $n \geq 1$  and  $t \in (a, b)$ , we have, by Theorem 24.1 of [100]:

$$\left( \int_{\mathbb{R}} \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s)\|_{L^2(\mathbb{R}^{n-1})}^{\frac{2}{1-H}} ds \right)^{\frac{1-H}{2}} < \infty,$$

By Hölder inequality, for all  $p \in (1, \frac{2}{1-H})$ :

$$\int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s)\|_{L^2(\mathbb{R}^{n-1})}^p ds < \infty,$$

Since  $\frac{2}{1-H} \in (4; +\infty)$ , this is true for  $p = 1$  and  $p = 2$ . Using  $\|I_+^{\frac{H}{2}}(\xi)\|_\infty \leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_{L^1(\mathbb{R})})$  and Cauchy-Schwarz inequality, we obtain the following bound:

$$\begin{aligned} \int_t^{t+\epsilon} |I_+^{\frac{H}{2}}(\xi)(s)| < I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s); \xi^{\otimes n-1} >_{L^2(\mathbb{R}^{n-1})} ds &\leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_{L^1(\mathbb{R})}) \|\xi\|_{L^2(\mathbb{R})}^{n-1} \\ &\times \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s)\|_{L^2(\mathbb{R}^{n-1})} ds. \end{aligned}$$

Finally, we note that by Jensen Inequality and Proposition 2.4:

$$\begin{aligned} \sum_{n=0}^{+\infty} \left[ n \left( \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s)\|_{L^2(\mathbb{R}^{n-1})} ds \right) \|\xi\|_{L^2(\mathbb{R})}^{n-1} \right] &\leq \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \\ &\times \left( \sum_{n=0}^{\infty} n!n \left( \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s)\|_{L^2(\mathbb{R}^{n-1})} ds \right)^2 \right)^{\frac{1}{2}}, \\ &\leq (b-a)^{\frac{1}{2}} \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \\ &\times \left( \sum_{n=0}^{\infty} n!n \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(\cdot, s)\|_{L^2(\mathbb{R}^{n-1})}^2 ds \right)^{\frac{1}{2}}, \\ &\leq (b-a)^{\frac{1}{2}} \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \|\nabla^{\frac{H}{2}}(\Phi_t)\|_{L^2((a,b) \otimes L^2)}, \end{aligned}$$

which is finite since  $\Phi_t \in (\mathcal{W}^{2,2}) \subset (\mathcal{W}^{1,2})$ . Consequently, we obtain:

$$\begin{aligned} \sum_{n=0}^{\infty} 2n \langle \phi_n(\cdot, t) \otimes_1 f_{t, t+\epsilon}^H; \xi^{\otimes n} \rangle &= 2d(H) \int_t^{t+\epsilon} I_+^{\frac{H}{2}}(\xi)(s) \sum_{n=0}^{\infty} \left( S(\nabla^{\frac{H}{2}}(I_n(\phi_n(\cdot, t)))(s, \cdot))(\xi) \right) ds, \\ &= 2d(H) \int_t^{t+\epsilon} I_+^{\frac{H}{2}}(\xi)(s) S(\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot))(\xi) ds, \\ &= 2d(H) S\left( \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot)) ds \right)(\xi). \end{aligned}$$

where the last equality is justified by the following bound and the fact that  $\|\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot)\|_{(L^2)}$  is square-integrable on  $(a, b)$ :

$$|I_+^{\frac{H}{2}}(\xi)(s) S(\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot))(\xi)| \leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_1) \exp\left(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2\right) \|\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot)\|_{(L^2)}.$$

To conclude, we have to identify the last term. Since for  $n \geq 2$  and for  $t \in (a, b)$ ,  $\phi_n(\cdot, t)$  verifies **H1**, there exists a  $L^2(\mathbb{R}^n)$  function,  $\psi_n(\cdot, t)$ , such that:

$$\mathcal{F}(\phi_n(\cdot, t)) = \frac{(-i\xi_{n-1})^{\frac{H}{2}}(-i\xi_n)^{\frac{H}{2}}}{\prod_{k=n-1}^n(1 + |\xi_k|^2)^{\frac{H}{4}}} \mathcal{F}(\psi_n(\cdot, t)).$$

Let  $(\psi_n^p(\cdot, t))_p$  be a sequence of functions in  $S(\mathbb{R}^n)$  which converges to  $\psi_n(\cdot, t)$  in  $L^2(\mathbb{R}^n)$  when  $p$  tends to infinity. Then, the sequence of  $C^\infty(\mathbb{R}^n)$  functions,  $(\phi_n^p(\cdot, t))_p$ , defined by:

$$\forall p \in \mathbb{N}, \mathcal{F}(\phi_n^p(\cdot, t)) = \frac{(-i\xi_{n-1})^{\frac{H}{2}}(-i\xi_n)^{\frac{H}{2}}}{\prod_{k=n-1}^n(1 + |\xi_k|^2)^{\frac{H}{4}}} \mathcal{F}(\psi_n^p(\cdot, t)),$$

converges to  $\phi_n(\cdot, t)$  in  $L^2(\mathbb{R}^n)$ . By Fubini Theorem, for all  $p$  in  $\mathbb{N}$ , we have:

$$\begin{aligned} n(n-1) \langle \phi_n^p(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H; \xi^{\otimes n-2} \rangle &= d(H)n(n-1) \int_t^{t+\epsilon} \langle I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t))(s, s); \xi^{\otimes n-2} \rangle ds, \\ &= d(H)n(n-1) \int_t^{t+\epsilon} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t)))(s, s); \xi^{\otimes n-2} \rangle ds. \end{aligned}$$

where  $T$  is the trace operator over  $\Delta_n$  defined in Lemma 3.2.3. The left-hand side converges to  $n(n-1) \langle \phi_n(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H; \xi^{\otimes n-2} \rangle$  due to the continuity of the scalar product and of the contraction operator of order 2. Let us show that the right hand side converges to  $d(H)n(n-1) \int_t^{t+\epsilon} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(s, s); \xi^{\otimes n-2} \rangle ds$  when  $p$  tends to infinity. We have, for all  $p \in \mathbb{N}$ :

$$\begin{aligned} \left| \int_t^{t+\epsilon} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t)))(s, s); \xi^{\otimes n-2} \rangle ds \right| &\leq \int_t^{t+\epsilon} \|\xi\|_{L^2(\mathbb{R})}^{n-2} \\ &\quad \times \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t)))(s)\|_{L^2(\mathbb{R}^{n-2})} ds, \\ &\leq \|\xi\|_{L^2(\mathbb{R})}^{n-2} \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t)))(s)\|_{L^2(\mathbb{R}^{n-2})} ds. \end{aligned}$$

Moreover, we have:

$$\begin{aligned} \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t)))\|_{L^2(\mathbb{R}^{n-1})} &\leq \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t)))\|_{S_2^{(0, \dots, 0, H-1/2)} H(\mathbb{R}^{n-1})} \\ &\leq C_H \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t))\|_{S_2^{(0, \dots, 0, H/2, H/2)} H(\mathbb{R}^n)} \\ &\leq C'_H \|\psi_n^p(\cdot, t) - \psi_n(\cdot, t)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Thus,

$$\left| \int_t^{t+\epsilon} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n^p(\cdot, t) - \phi_n(\cdot, t)))(s, s); \xi^{\otimes n-2} \rangle ds \right| \leq C'_H \|\xi\|_{L^2(\mathbb{R})}^{n-2} (b-a)^{\frac{1}{2}} \|\psi_n^p(\cdot, t) - \psi_n(\cdot, t)\|_{L^2(\mathbb{R}^n)}$$

Then, by passing to the limit, we obtain, for  $n \geq 2$  and for  $t \in (a, b)$ :

$$n(n-1) \langle \phi_n(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H; \xi^{\otimes n-2} \rangle = d(H)n(n-1) \int_t^{t+\epsilon} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s); \xi^{\otimes n-2} \rangle ds.$$

Finally, by the previous bound, we have:

$$\begin{aligned} d(H)n(n-1) \left| \int_t^{t+\epsilon} \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s); \xi^{\otimes n-2} \rangle ds \right| &\leq d(H)n(n-1) \|\xi\|_{L^2(\mathbb{R})}^{n-2} \\ &\quad \times \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{n-2})} ds. \end{aligned}$$

Using Jensen inequality and the last assumption of the proposition, we have:

$$\begin{aligned} \sum_{n=2}^{+\infty} \left[ n(n-1) \left( \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{n-2})} ds \right) \|\xi\|_{L^2(\mathbb{R})}^{n-2} \right] &\leq \exp\left(\frac{1}{2} \|\xi\|_{L^2(\mathbb{R})}^2\right) \times \\ &\quad \left( \sum_{n=2}^{\infty} n(n-1)n! \left( \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{n-2})} ds \right)^2 \right)^{\frac{1}{2}}, \\ \sum_{n=2}^{+\infty} \left[ n(n-1) \left( \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{n-2})} ds \right) \|\xi\|_{L^2(\mathbb{R})}^{n-2} \right] &\leq (b-a)^{\frac{1}{2}} \exp\left(\frac{1}{2} \|\xi\|_{L^2(\mathbb{R})}^2\right) \times \\ &\quad \left( \sum_{n=2}^{\infty} n(n-1)n! \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s)\|_{L^2(\mathbb{R}^{n-2})}^2 ds \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

To conclude, we obtain:

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) \langle \phi_n(\cdot, t) \otimes_2 f_{t, t+\epsilon}^H; \xi^{\otimes n-2} \rangle &= d(H) \int_t^{t+\epsilon} \left( \sum_{n=2}^{\infty} n(n-1) \right. \\ &\quad \left. \times \langle T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})}(\phi_n(\cdot, t)))(\cdot, s); \xi^{\otimes n-2} \rangle \right) ds, \\ &= d(H) \int_t^{t+\epsilon} \left( \sum_{n=2}^{\infty} S(\nabla^{(2), \frac{H}{2}}(I_n(\phi_n(\cdot, t)))(s, s, \cdot))(\xi) \right) ds, \\ &= d(H) \int_t^{t+\epsilon} S(\nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot))(\xi) ds, \\ &= d(H) S\left( \int_t^{t+\epsilon} \nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot) ds \right)(\xi). \end{aligned}$$

where the last equality is obtained by the last assumption of the proposition.  $\square$

**Proposition 3.2.6:** Let  $\{\Phi_t : t \in (a, b)\}$  be a stochastic process verifying assumptions (3.1),

(3.2) and (3.3) of Proposition 3.2.4. Moreover, let us assume that:

$$\begin{aligned} \int_a^b \|\Phi_t\|_{(L^2)}^2 dt &< \infty, \\ \int_a^b \|\nabla^{\frac{H}{2}}(\Phi_t)\|_{L^2(a,b)\otimes(L^2)}^2 dt &< \infty, \\ \sum_{n=2}^{\infty} n(n-1)n! \int_a^b \left( \int_a^b \|T(I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2}, \frac{H}{2})})(\phi_n(\cdot, t))\|_{L^2(\mathbb{R}^{n-2})}^2 ds \right) dt &< \infty. \end{aligned}$$

Then,  $\Phi_t \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon}$  is  $(S)^*$  integrable and we have:

$$\begin{aligned} \int_a^b \Phi_t \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} dt \stackrel{(S)^*}{=} \int_a^b \Phi_t \diamond \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} dt + 2d(H) \int_a^b \left( \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s,)) \frac{ds}{\epsilon} \right) dt \\ + d(H) \int_a^b \left( \int_t^{t+\epsilon} \nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, ) \frac{ds}{\epsilon} \right) dt. \end{aligned}$$

**Proof:** In order to prove the  $(S)^*$ -integrability of  $\Phi_t \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon}$  over  $(a, b)$ , we are going to prove the  $(S)^*$ -integrability of the three terms appearing in Proposition 3.2.4. Let us start with the first term. For every  $\xi \in S(\mathbb{R})$ , we have:

$$\begin{aligned} S(\Phi_t \diamond \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon})(\xi) &= S(\Phi_t)(\xi) S(\frac{X_{t+\epsilon}^H - X_t^H}{\epsilon})(\xi), \\ &= \mathbb{E}^\mu[\Phi_t \exp(\langle \cdot, \xi \rangle - \frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2})] d(H) \int_t^{t+\epsilon} (I_+^{\frac{H}{2}}(\xi)(s))^2 \frac{ds}{\epsilon}. \end{aligned}$$

By Cauchy-Schwarz inequality and  $\|I_+^{\frac{H}{2}}(\xi)\|_\infty \leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_{L^1(\mathbb{R})})$ , we have:

$$\begin{aligned} |S(\Phi_t \diamond \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon})(\xi)| &\leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_{L^1(\mathbb{R})})^2 \exp(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2) \mathbb{E}^\mu[\Phi_t^2]^{\frac{1}{2}}, \\ &\leq C_H(\|\xi\|_\infty + \|\xi^{(1)}\|_\infty + \|\xi\|_{L^1(\mathbb{R})})^2 \exp(\frac{1}{2}\|\xi\|_{L^2(\mathbb{R})}^2) \left( \sum_{n=0}^{\infty} n! \|\phi_n(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using the fact that  $\int_a^b \|\Phi_t\|_{(L^2)}^2 dt < \infty$  and Theorem 13.5 of [60], we deduce that the first term is  $(S)^*$ -integrable over  $(a, b)$ . Let us deal with the second term now. For every  $\xi \in S(\mathbb{R})$ , we have:

$$S\left(\int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s,)) \frac{ds}{\epsilon}\right)(\xi) = \int_t^{t+\epsilon} S(\nabla^{\frac{H}{2}}(\Phi_t)(s,))(\xi) I_+^{\frac{H}{2}}(\xi)(s) \frac{ds}{\epsilon}.$$

As for the first term, we get:

$$|S\left(\int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s,)) \frac{ds}{\epsilon}\right)(\xi)| \leq C_H \frac{1}{\epsilon} \exp\left(\frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2}\right) (\|\xi\|_\infty + \|\xi\|_{L^1(\mathbb{R})} + \|\xi^{(1)}\|_\infty)$$

$$\begin{aligned}
& \times \int_a^b \mathbb{E}^\mu [(\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot))^2]^{\frac{1}{2}} ds, \\
|S(\int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot)) \frac{ds}{\epsilon})(\xi)| & \leq C_H \frac{1}{\epsilon} \exp\left(\frac{\|\xi\|_{L^2(\mathbb{R})}^2}{2}\right) (\|\xi\|_\infty + \|\xi\|_{L^1(\mathbb{R})} + \|\xi^{(1)}\|_\infty) \\
& \times (b-a)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} nn! \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(s)\|_{L^2(\mathbb{R}^{n-1})}^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the fact that  $\sum_{n=1}^{\infty} nn! \int_a^b \left( \int_a^b \|I_{+, \dots, +}^{(0, \dots, 0, \frac{H}{2})}(\phi_n(\cdot, t))(s)\|_{L^2(\mathbb{R}^{n-1})}^2 ds \right) dt < \infty$  and Theorem 13.5 of [60], we obtain the  $(S)^*$ -integrability of the second term over  $(a, b)$ . We deal with the last term similarly. This ends the proof of the proposition.  $\square$

### 3.2.1 Trace term of order 1

We want to compute explicitly the term defined by:

$$\forall \epsilon > 0, d(H) \int_a^b \left( \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(\Phi_t)(s, \cdot)) \frac{ds}{\epsilon} \right) dt,$$

for specific processes,  $\{\Phi_t\}$ , expressed as functionals of the Rosenblatt process. Namely, let  $F$  be an infinitely continuously differentiable function on  $\mathbb{R}$  such that  $F$  and all its derivatives have polynomial growth. Then, we set, for every  $t \in (a, b)$ ,  $\Phi_t = F(X_t^H)$ . Since,  $X_t^H = I_2(f_t^H)$ ,  $X_t^H \in (\mathcal{W}^{\infty, \infty})$  and so does  $F(X_t^H)$  by Proposition 1.8. The first step is to compute  $\nabla^{H/2}(F(X_t^H))$ .

**Lemma 3.2.1.1:** We have:

$$\nabla^{\frac{H}{2}}(F(X_t^H))(s, \omega) = \frac{2}{\Gamma(\frac{H}{2})} \sqrt{\frac{H(2H-1)}{2}} I_1 \left( \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} |s - u|^{H-1} du \right) F'(X_t^H).$$

**Proof:** By Proposition 1.8 and by definition of the operator  $\nabla^{\frac{H}{2}}$ , we have:

$$\nabla^{\frac{H}{2}}(F(X_t^H))(s, \omega) = \nabla^{\frac{H}{2}}(X_t^H)(s, \omega) F'(X_t^H)(\omega).$$

Moreover, by Proposition 2.4 with  $\alpha = H/2$ , we obtain:

$$\begin{aligned}
\nabla^{\frac{H}{2}}(F(X_t^H))(s, \omega) &= 2I_1(I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s)) F'(X_t^H)(\omega), \\
\nabla^{\frac{H}{2}}(F(X_t^H))(s, \omega) &= \frac{2d(H)}{(\Gamma(\frac{H}{2}))^3} I_1 \left( \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} \left( \int_{\mathbb{R}} (s - x)_+^{\frac{H}{2}-1} (u - x)_+^{\frac{H}{2}-1} dx \right) du \right) F'(X_t^H)(\omega),
\end{aligned}$$



$$\begin{aligned}\nabla^{\frac{H}{2}}(F(X_t^H))(s, \omega) &= \frac{2d(H)\beta(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^3} I_1 \left( \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} |u - s|^{H-1} du \right) F'(X_t^H)(\omega), \\ \nabla^{\frac{H}{2}}(F(X_t^H))(s, \omega) &= \frac{2}{\Gamma(\frac{H}{2})} \sqrt{\frac{H(2H-1)}{2}} I_1 \left( \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} |s - u|^{H-1} du \right) F'(X_t^H)(\omega). \quad \square\end{aligned}$$

**Lemma 3.2.1.2:** Let  $t \in (a, b)$ . Let  $l_{s,t}^H(\cdot) = \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} |s - u|^{H-1} du$ . The stochastic process  $\{I_1(l_{s,t}^H(\cdot)) : s \in [t, +\infty)\}$  is  $(L^2)$  continuous on  $[t, +\infty)$ .

**Proof:** First of all, note that  $l_{s,t}^H(\cdot) \in L^2(\mathbb{R})$  for any  $s \in [t, +\infty)$ . Indeed, we have:

$$\begin{aligned}\int_{\mathbb{R}} (l_{s,t}^H(x))^2 d\lambda_x &= \beta(1-H, \frac{H}{2}) \int_0^t \int_0^t |s-u|^{H-1} |s-v|^{H-1} |u-v|^{H-1} dudv, \\ &\leq \beta(1-H, \frac{H}{2}) \left( \int_0^t \int_0^t |u-v|^{2H-2} dudv \right)^{\frac{1}{2}} \left( \int_0^t \int_0^t |u-s|^{2H-2} |v-s|^{2H-2} dudv \right)^{\frac{1}{2}}, \\ &\leq \beta(1-H, \frac{H}{2}) \left( \frac{t^{2H}}{H(2H-1)} \right)^{\frac{1}{2}} \frac{s^{2H-1} - (s-t)^{2H-1}}{2H-1} < +\infty.\end{aligned}$$

Let  $(s_n) \in [t, +\infty)^{\mathbb{N}}$  such that  $s_n \xrightarrow{n \rightarrow +\infty} s$ . We have:

$$\begin{aligned}\mathbb{E} \left[ \left( I_1(l_{s_n,t}^H(\cdot)) - I_1(l_{s,t}^H(\cdot)) \right)^2 \right] &= \int_{\mathbb{R}} (l_{s_n,t}^H(x))^2 dx + \int_{\mathbb{R}} (l_{s,t}^H(x))^2 dx - 2 \int_{\mathbb{R}} l_{s_n,t}^H(x) l_{s,t}^H(x) dx, \\ &= \beta(1-H, \frac{H}{2}) (K_t^2(s, s) + K_t^2(s_n, s_n) - 2K_t^2(s, s_n)),\end{aligned}$$

where  $K_t^2(s_1, s_2) = \int_0^t \int_0^t |s_2-v|^{H-1} |s_1-u|^{H-1} |u-v|^{H-1} dudv$ . Moreover, by Lebesgue dominated convergence theorem,  $K_t^2(\cdot, \cdot)$  is continuous on  $[t, +\infty) \times [t, +\infty)$ . Thus, we obtain:

$$\mathbb{E} \left[ \left( I_1(l_{s_n,t}^H(\cdot)) - I_1(l_{s,t}^H(\cdot)) \right)^2 \right] \xrightarrow{n \rightarrow +\infty} 0. \quad \square$$

**Proposition 3.2.1.3:** We have:

$$d(H) \int_a^b \left( \int_t^{t+\epsilon} D_{\delta_s \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(F(X_t^H)))(s, \cdot) \frac{ds}{\epsilon} \right) dt \xrightarrow[\epsilon \rightarrow 0^+]{(S)^*} d(H) \int_a^b D_{\delta_t \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(F(X_t^H)))(t, \cdot) dt.$$

**Proof:** Let  $\xi \in S(\mathbb{R})$ . Let  $t \in (a, b)$ . By Lemma 3.2.1.2, the function  $s \mapsto S(I_1(l_{s,t}^H)F'(X_t^H))(\xi)$  is continuous on  $[t, +\infty)$ . Thus, we have:

$$\lim_{\epsilon \rightarrow 0^+} \int_t^{t+\epsilon} S(I_1(l_{s,t}^H)F'(X_t^H))(\xi) I_+^{\frac{H}{2}}(\xi)(s) \frac{ds}{\epsilon} = S(I_1(l_{t,t}^H)F'(X_t^H))(\xi) I_+^{\frac{H}{2}}(\xi)(t).$$

Moreover, for any  $\epsilon > 0$ :

$$\begin{aligned}
\frac{1}{\epsilon} \left| \int_t^{t+\epsilon} S(I_1(l_{s,t}^H)F'(X_t^H))(\xi)I_+^{\frac{H}{2}}(\xi)(s)ds \right| &\leq \|I_+^{\frac{H}{2}}(\xi)\|_\infty \frac{1}{\epsilon} \int_t^{t+\epsilon} |S(I_1(l_{s,t}^H)F'(X_t^H))(\xi)|ds, \\
&\leq \|I_+^{\frac{H}{2}}(\xi)\|_\infty \frac{1}{\epsilon} \int_t^{t+\epsilon} | \langle I_1(l_{s,t}^H); F'(X_t^H) : e^{\langle \cdot; \xi \rangle} \rangle_{(L^2)} | ds, \\
&\leq \|I_+^{\frac{H}{2}}(\xi)\|_\infty \frac{1}{\epsilon} \int_t^{t+\epsilon} \|I_1(l_{s,t}^H)\|_{(L^2)} \|F'(X_t^H) : e^{\langle \cdot; \xi \rangle} : \|_{(L^2)} ds, \\
&\leq (\beta(1-H, \frac{H}{2}))^{\frac{1}{2}} \|F'(X_t^H) : e^{\langle \cdot; \xi \rangle} : \|_{(L^2)} \|I_+^{\frac{H}{2}}(\xi)\|_\infty \frac{1}{\epsilon} \\
&\quad \times \int_t^{t+\epsilon} (K_t^2(s, s))^{\frac{1}{2}} ds, \\
&\leq (\beta(1-H, \frac{H}{2}))^{\frac{1}{2}} \|F'(X_t^H) : e^{\langle \cdot; \xi \rangle} : \|_{(L^2)} \|I_+^{\frac{H}{2}}(\xi)\|_\infty (K_t^2(t, t))^{\frac{1}{2}},
\end{aligned}$$

since  $K_t^2(s, s) \leq K_t^2(t, t)$  for any  $s \in [t; +\infty)$ . Moreover, we have, by Cauchy-Schwarz inequality:

$$\begin{aligned}
\|F'(X_t^H) : e^{\langle \cdot; \xi \rangle} : \|_{(L^2)} &\leq \mathbb{E}[(F'(X_t^H))^4]^{\frac{1}{4}} \mathbb{E}[e^{4\langle \cdot; \xi \rangle - 2\|\xi\|^2}]^{\frac{1}{4}}, \\
&\leq e^{\frac{3}{2}\|\xi\|^2} \mathbb{E}[(F'(X_t^H))^4]^{\frac{1}{4}}.
\end{aligned}$$

By assumption,  $F'$  is an infinitely differentiable function with polynomial growth. Consequently, it exists  $C > 0$  and  $N \in \mathbb{N}^*$  such that:

$$\forall x \in \mathbb{R}, |F'(x)| \leq C(1 + |x|)^N.$$

Then,

$$\begin{aligned}
\|F'(X_t^H) : e^{\langle \cdot; \xi \rangle} : \|_{(L^2)} &\leq C e^{\frac{3}{2}\|\xi\|^2} (1 + 4Nt^H + \sum_{k=2}^{4N} C_k^{4N} \mathbb{E}[|X_t^H|^k])^{\frac{1}{4}}, \\
&\leq C e^{\frac{3}{2}\|\xi\|^2} (1 + 4Nt^H + \sum_{k=2}^{4N} C_k^{4N} (k-1)^k t^{Hk})^{\frac{1}{4}},
\end{aligned}$$

where for the last inequality, we use a hypercontractivity argument (see Corollary 2.8.14 of [84]). Therefore, we have the following bound:

$$\frac{1}{\epsilon} \left| \int_t^{t+\epsilon} S(I_1(l_{s,t}^H)F'(X_t^H))(\xi)I_+^{\frac{H}{2}}(\xi)(s)ds \right| \leq C_H \|I_+^{\frac{H}{2}}(\xi)\|_\infty t^{\frac{3}{2}H - \frac{1}{2}} e^{\frac{3}{2}\|\xi\|^2} (1 + 4Nt^H + \sum_{k=2}^{4N} C_k^{4N} (k-1)^k t^{Hk})^{\frac{1}{4}}.$$

The right hand side of the previous inequality is in  $L^1((a, b))$ . By Lebesgue dominated convergence theorem, we have:

$$\int_a^b \int_t^{t+\epsilon} S(I_1(l_{s,t}^H)F'(X_t^H))(\xi)I_+^{\frac{H}{2}}(\xi)(s) \frac{ds}{\epsilon} dt \xrightarrow{\epsilon \rightarrow 0^+} \int_a^b S(I_1(l_{t,t}^H)F'(X_t^H))(\xi)I_+^{\frac{H}{2}}(\xi)(t) dt.$$

To conclude, we can use the previous bound in order to apply the third point of Theorem 1.3.  $\square$

A direct application of Proposition 2.9 allows us to write:

$$\begin{aligned} d(H) \int_a^b D_{\delta_t \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(F(X_t^H))(t,)) dt &= d(H) (\nabla^{\frac{H}{2}})^* (\nabla^{\frac{H}{2}}(F(X_{\#}^H))(\#, .)), \\ &= \frac{2d(H)}{(\Gamma(\frac{H}{2}))^2} \sqrt{\frac{H(2H-1)}{2}} (\nabla)^* \left( \int_a^b (t - \cdot)_+^{\frac{H}{2}-1} I_1(l_{t,t}^H) F'(X_t^H) dt \right). \end{aligned}$$

**Remark 3.2.1.4:** In the peculiar case where  $F(x) = x$ , we obtain for the previous term:

$$\begin{aligned} 2d(H) \int_a^b D_{\delta_t \circ I_+^{\frac{H}{2}}}^* (\nabla^{\frac{H}{2}}(F(X_t^H))(t,)) dt &= \frac{4d(H)}{(\Gamma(\frac{H}{2}))^2} \sqrt{\frac{H(2H-1)}{2}} (\nabla)^* \left( \int_a^b (t - \cdot)_+^{\frac{H}{2}-1} I_1(l_{t,t}^H) dt \right), \\ &= \frac{4d(H)}{(\Gamma(\frac{H}{2}))^2} \sqrt{\frac{H(2H-1)}{2}} I_2 \left( \int_a^b \left[ \int_0^t (t - \cdot)_+^{\frac{H}{2}-1} (u - *)_+^{\frac{H}{2}-1} |t - u|^{H-1} du \right] dt \right), \\ &= 2(X_b^{H,2} - X_a^{H,2}). \end{aligned}$$

where  $\{X_t^{H,2}\}$  is the stochastic process introduced in Definition 3.6 of Chapter 2.

### 3.2.2 Trace term of order 2

We want to compute explicitly the term defined by:

$$\forall \epsilon > 0, d(H) \int_a^b \left( \int_t^{t+\epsilon} \nabla^{(2), \frac{H}{2}}(\Phi_t)(s, s, \cdot) \frac{ds}{\epsilon} \right) dt,$$

for specific processes,  $\{\Phi_t\} = \{F(X_t^H)\}$ , with  $F$  an infinitely continuously differentiable function on  $\mathbb{R}$  such that  $F$  and all its derivative have polynomial growth. The first step is to compute  $\nabla^{(2), H/2}(F(X_t^H))$ .

**Lemma 3.2.2.1:** For any  $k \geq 0$ , let  $K_t^k(s, r)$  be defined by:

$$\forall s, r \in (t, +\infty), K_t^0(s, r) = |s - r|^{H-1}, K_t^1(s, r) = \int_0^t |s - u|^{H-1} |r - u|^{H-1} du$$

$$\forall k \geq 3, K_t^{k-2}(s, r) = \int_0^t \dots \int_0^t |s - x_1|^{H-1} |x_2 - x_1|^{H-1} \dots |x_{k-2} - x_{k-3}|^{H-1} |r - x_{k-2}|^{H-1} dx_1 \dots dx_{k-2}.$$

Then, we have:

$$d(H) \nabla^{(2), \frac{H}{2}}(F(X_t^H))(s_1, s_2, \omega) = H(2H-1) K_t^1(s_1, s_2) F'(X_t^H) + 4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 K_t^2(s_1, s_2) F^{(2)}(X_t^H)$$

$$+ \frac{2d(H)}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2 \left( \int_0^t \int_0^t (u-\cdot)_+^{\frac{H}{2}-1} (v-*)_+^{\frac{H}{2}-1} |u-s_1|^{H-1} |v-s_2|^{H-1} dudv \right) F^{(2)}(X_t^H).$$

**Proof:** Using Propostion 1.8 and the definition of the operator  $\nabla^{(2), \frac{H}{2}}$ , we have:

$$\nabla^{(2), \frac{H}{2}}(F(X_t^H))(s_1, s_2, \omega) = \nabla^{(2), \frac{H}{2}}(X_t^H)(s_1, s_2, \omega) F'(X_t^H) + \nabla^{\frac{H}{2}}(X_t^H)(s_1, \omega) \nabla^{\frac{H}{2}}(X_t^H)(s_2, \omega) F^{(2)}(X_t^H),$$

Using Proposition 2.4 and Proposition 2.6, we have:

$$\begin{aligned} \nabla^{(2), \frac{H}{2}}(F(X_t^H))(s_1, s_2, \omega) &= 2I_{++}^{(\frac{H}{2}, \frac{H}{2})}(f_t^H)(s_1, s_2) F'(X_t^H) \\ &\quad + 4I_1(I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s_1)) I_1(I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s_2)) F^{(2)}(X_t^H). \end{aligned}$$

By the multiplication formula from Malliavin calculus (Proposition 1.1.3 of [87]), we get:

$$\begin{aligned} \nabla^{(2), \frac{H}{2}}(F(X_t^H))(s_1, s_2, \omega) &= 2I_{++}^{(\frac{H}{2}, \frac{H}{2})}(f_t^H)(s_1, s_2) F'(X_t^H) + 4[I_2(I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s_1)) \hat{\otimes} I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s_2)) \\ &\quad + \langle I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s_1); I_{++}^{(0, \frac{H}{2})}(f_t^H)(\cdot, s_2) \rangle] F^{(2)}(X_t^H), \\ &= 2 \frac{d(H)}{(\Gamma(\frac{H}{2}))^4} \int_0^t \left( \int_{\mathbb{R}} (s_1-x)_+^{\frac{H}{2}-1} (u-x)_+^{\frac{H}{2}-1} dx \right) \left( \int_{\mathbb{R}} (s_2-x)_+^{\frac{H}{2}-1} (u-x)_+^{\frac{H}{2}-1} dx \right) du F'(X_t^H) \\ &\quad + \frac{2}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2 \left( \int_0^t \int_0^t (u-\cdot)_+^{\frac{H}{2}-1} (v-*)_+^{\frac{H}{2}-1} |u-s_1|^{H-1} |v-s_2|^{H-1} dudv \right) F^{(2)}(X_t^H) \\ &\quad + 2 \frac{H(2H-1)}{(\Gamma(\frac{H}{2}))^2} \int_0^t \int_0^t \left( \int_{\mathbb{R}} (u-x)_+^{\frac{H}{2}-1} (v-x)_+^{\frac{H}{2}-1} dx \right) |s_1-u| |s_2-v| dudv F^{(2)}(X_t^H), \\ &= 2 \frac{d(H)\beta^2(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^4} \int_0^t |u-s_1|^{H-1} |u-s_2|^{H-1} du F'(X_t^H) \\ &\quad + \frac{2}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2 \left( \int_0^t \int_0^t (u-\cdot)_+^{\frac{H}{2}-1} (v-*)_+^{\frac{H}{2}-1} |u-s_1|^{H-1} |v-s_2|^{H-1} dudv \right) F^{(2)}(X_t^H) \\ &\quad + 2 \frac{H(2H-1)\beta(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^2} \int_0^t \int_0^t |u-v|^{H-1} |s_1-u|^{H-1} |s_2-v|^{H-1} dudv F^{(2)}(X_t^H). \end{aligned}$$

Thus,

$$\begin{aligned} d(H) \nabla^{(2), \frac{H}{2}}(F(X_t^H))(s_1, s_2, \omega) &= H(2H-1) K_t^1(s_1, s_2) F'(X_t^H) + 4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 K_t^2(s_1, s_2) F^{(2)}(X_t^H) \\ &\quad + \frac{2d(H)}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2 \left( \int_0^t \int_0^t (u-\cdot)_+^{\frac{H}{2}-1} (v-*)_+^{\frac{H}{2}-1} |u-s_1|^{H-1} |v-s_2|^{H-1} dudv \right) F^{(2)}(X_t^H). \quad \square \end{aligned}$$

**Remark 3.2.2.2:**

- As Mentionned previously, note that for any  $t \in (a, b)$ , we have:

$$K_t^1(t, t) = \int_0^t |t-u|^{H-1} |t-u|^{H-1} du = \frac{t^{2H-1}}{2H-1} < \infty,$$

$$\begin{aligned}
K_t^2(t, t) &\leq \left( \int_0^t \int_0^t |x_1 - x_2|^{2H-2} dx_1 dx_2 \right)^{\frac{1}{2}} \left( \int_0^t \int_0^t |x_1 - t|^{2H-2} |x_2 - t|^{2H-2} dx_1 dx_2 \right)^{\frac{1}{2}}, \\
&\leq \left( \frac{t^{2H}}{H(2H-1)} \right)^{\frac{1}{2}} \frac{t^{2H-1}}{2H-1} = \frac{t^{3H-1}}{\sqrt{H(2H-1)}(2H-1)} < \infty.
\end{aligned}$$

- Moreover, by Lebesgue dominated convergence theorem, one can show that  $K_t^1(\cdot, \cdot)$  and  $K_t^2(\cdot, \cdot)$  are continuous on  $[t, +\infty) \times [t, +\infty)$ .
- Finally, for any  $s \in [t, +\infty)$ , we have:

$$K_t^2(s, s) \leq K_t^2(t, t).$$

**Lemma 3.2.2.3:** We have:

$$\begin{aligned}
&H(2H-1) \int_a^b \left( \int_t^{t+\epsilon} K_t^1(s, s) \frac{ds}{\epsilon} \right) F'(X_t^H) dt \xrightarrow[\epsilon \rightarrow 0^+]{a.s.} H \int_a^b t^{2H-1} F'(X_t^H) dt, \\
&4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b \left( \int_t^{t+\epsilon} K_t^2(s, s) \frac{ds}{\epsilon} \right) F^{(2)}(X_t^H) dt \xrightarrow[\epsilon \rightarrow 0^+]{a.s.} \frac{H}{2} \kappa_3(X_1^H) \int_a^b t^{3H-1} F^{(2)}(X_t^H) dt.
\end{aligned}$$

**Proof:** By the previous remark and Lebesgue dominated convergence theorem, we have:

$$\begin{aligned}
&H(2H-1) \int_a^b \left( \int_t^{t+\epsilon} K_t^1(s, s) \frac{ds}{\epsilon} \right) F'(X_t^H) dt \xrightarrow[\epsilon \rightarrow 0^+]{a.s.} H(2H-1) \int_a^b K_t^1(t, t) F'(X_t^H) dt, \\
&4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b \left( \int_t^{t+\epsilon} K_t^2(s, s) \frac{ds}{\epsilon} \right) F^{(2)}(X_t^H) dt \xrightarrow[\epsilon \rightarrow 0^+]{a.s.} 4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b K_t^2(t, t) F^{(2)}(X_t^H) dt.
\end{aligned}$$

By scaling property of the kernels, we have:

$$\begin{aligned}
&H(2H-1) \int_a^b K_t^1(t, t) F'(X_t^H) dt = H(2H-1) \int_0^1 (1-x)^{2H-2} dx \int_a^b t^{2H-1} F'(X_t^H) dt, \\
&4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b K_t^2(t, t) F^{(2)}(X_t^H) dt = 4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 K_1^2(1, 1) \int_a^b t^{3H-1} F^{(2)}(X_t^H) dt.
\end{aligned}$$

Moreover, we note that:

$$K_1^2(1, 1) = H \int_0^1 \int_0^1 \int_0^1 |x_1 - x_2|^{H-1} |x_2 - x_3|^{H-1} |x_3 - x_1|^{H-1} dx_1 dx_2 dx_3.$$

Thus,

$$\begin{aligned}
&H(2H-1) \int_a^b K_t^1(t, t) F'(X_t^H) dt = H \int_a^b t^{2H-1} F'(X_t^H) dt, \\
&4 \left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b K_t^2(t, t) F^{(2)}(X_t^H) dt = \frac{H}{2} \kappa_3(X_1^H) \int_a^b t^{3H-1} F^{(2)}(X_t^H) dt. \quad \square
\end{aligned}$$

**Lemma 3.2.2.4:** Let  $t \in (a, b)$ . We have:

$$\begin{aligned} & \int_t^{t+\epsilon} I_2\left(\int_0^t \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} (v - *)_+^{\frac{H}{2}-1} |u - s|^{H-1} |v - s|^{H-1} dudv\right) \frac{ds}{\epsilon} \\ & \xrightarrow[\epsilon \rightarrow 0^+]{L^2(\Omega)} I_2\left(\int_0^t \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} (v - *)_+^{\frac{H}{2}-1} |u - t|^{H-1} |v - t|^{H-1} dudv\right). \end{aligned}$$

**Proof:** First of all, note that:

$$\int_0^t \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} (v - *)_+^{\frac{H}{2}-1} |u - t|^{H-1} |v - t|^{H-1} dudv \in L^2(\mathbb{R}^2).$$

Indeed, we have:

$$\int_0^t \int_0^t \int_0^t \int_0^t |u - t|^{H-1} |v - t|^{H-1} |u' - t|^{H-1} |v' - t|^{H-1} |u - u'|^{H-1} |v - v'|^{H-1} dudvdu'dv' = (K_t^2(t, t))^2,$$

which is finite. Let  $\epsilon > 0$ . We have:

$$\begin{aligned} & \mathbb{E}\left[\left(\int_t^{t+\epsilon} I_2(e_{s,t}^H(\cdot, *)) \frac{ds}{\epsilon} - I_2(e_{t,t}^H(\cdot, *))\right)^2\right] = \mathbb{E}\left[\left(\int_t^{t+\epsilon} (I_2(e_{s,t}^H(\cdot, *)) - I_2(e_{t,t}^H(\cdot, *))) \frac{ds}{\epsilon}\right)^2\right], \\ & = \frac{2}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \langle e_{s,t}^H(\cdot, *) - e_{t,t}^H(\cdot, *); e_{s',t}^H(\cdot, *) - e_{t,t}^H(\cdot, *) \rangle_{L^2(\mathbb{R}^2)} ds ds', \end{aligned}$$

where  $e_{s,t}^H(\cdot, *) = \int_0^t \int_0^t (u - \cdot)_+^{\frac{H}{2}-1} (v - *)_+^{\frac{H}{2}-1} |u - s|^{H-1} |v - s|^{H-1} dudv$ . Moreover, we have:

$$\frac{1}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \langle e_{s,t}^H(\cdot, *); e_{s',t}^H(\cdot, *) \rangle_{L^2(\mathbb{R}^2)} ds ds' = \frac{(\beta(1 - H, \frac{H}{2}))^2}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} (K_t^2(s, s'))^2 ds ds'.$$

Thus,

$$\begin{aligned} & \mathbb{E}\left[\left(\int_t^{t+\epsilon} I_2(e_{s,t}^H(\cdot, *)) \frac{ds}{\epsilon} - I_2(e_{t,t}^H(\cdot, *))\right)^2\right] = \frac{2(\beta(1 - H, \frac{H}{2}))^2}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} (K_t^2(s, s'))^2 ds ds' \\ & \quad - \frac{4(\beta(1 - H, \frac{H}{2}))^2}{\epsilon} \int_t^{t+\epsilon} (K_t^2(t, s))^2 ds + 2(\beta(1 - H, \frac{H}{2}))^2 (K_t^2(t, t))^2. \end{aligned}$$

The continuity of  $K_t^2(\cdot, \cdot)$  on  $[t, +\infty) \times [t, +\infty)$  concludes the proof.  $\square$

**Lemma 3.2.2.5:** We have:

$$\int_a^b \left(\int_t^{t+\epsilon} I_2(e_{s,t}^H(\cdot, *)) \frac{ds}{\epsilon}\right) F^{(2)}(X_t^H) dt \xrightarrow[\epsilon \rightarrow 0^+]{L^1(\Omega)} \int_a^b I_2(e_{t,t}^H(\cdot, *)) F^{(2)}(X_t^H) dt.$$

**Proof:** We want to bound the following quantity:

$$(I) = \mathbb{E}\left[\left|\int_a^b \left(\int_t^{t+\epsilon} I_2(e_{s,t}^H(\cdot, *)) \frac{ds}{\epsilon}\right) F^{(2)}(X_t^H) dt - \int_a^b I_2(e_{t,t}^H(\cdot, *)) F^{(2)}(X_t^H) dt\right|\right].$$

By Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} (I) &\leq \int_a^b \mathbb{E} \left[ \left\| \int_t^{t+\epsilon} (I_2(e_{s,t}^H(\cdot, *) - I_2(e_{t,t}^H(\cdot, *))) \frac{ds}{\epsilon} \right\| \left| F^{(2)}(X_t^H) \right| \right] dt, \\ (I) &\leq \int_a^b \left\| \int_t^{t+\epsilon} (I_2(e_{s,t}^H(\cdot, *) - I_2(e_{t,t}^H(\cdot, *))) \frac{ds}{\epsilon} \right\|_{L^2(\Omega)} \|F^{(2)}(X_t^H)\|_{L^2(\Omega)} dt, \end{aligned}$$

In order to conclude, we want to apply the Lebesgue dominated convergence theorem to the right-hand side of the previous inequality. By Lemma 3.2.2.4, we have:

$$\forall t \in (a, b), \lim_{\epsilon \rightarrow 0^+} \left\| \int_t^{t+\epsilon} (I_2(e_{s,t}^H(\cdot, *) - I_2(e_{t,t}^H(\cdot, *))) \frac{ds}{\epsilon} \right\|_{L^2(\Omega)} = 0.$$

Moreover, we have:

$$\begin{aligned} \mathbb{E} \left[ \left( \int_t^{t+\epsilon} (I_2(e_{s,t}^H(\cdot, *) - I_2(e_{t,t}^H(\cdot, *))) \frac{ds}{\epsilon} \right)^2 \right] &\leq \frac{2}{\epsilon} \int_t^{t+\epsilon} \|e_{s,t}^H - e_{t,t}^H\|_{L^2(\mathbb{R}^2)}^2 ds, \\ &\leq \frac{4}{\epsilon} \int_t^{t+\epsilon} \|e_{s,t}^H\|_{L^2(\mathbb{R}^2)}^2 + \|e_{t,t}^H\|_{L^2(\mathbb{R}^2)}^2 ds, \end{aligned}$$

But, we have, for every  $s \in [t, +\infty)$ :

$$\begin{aligned} \|e_{s,t}^H\|_{L^2(\mathbb{R}^2)}^2 &= (\beta(1 - H, \frac{H}{2}))^2 (K_t^2(s, s))^2, \\ K_t^2(s, s) &\leq K_t^2(t, t). \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E} \left[ \left( \int_t^{t+\epsilon} (I_2(e_{s,t}^H(\cdot, *) - I_2(e_{t,t}^H(\cdot, *))) \frac{ds}{\epsilon} \right)^2 \right] &\leq 8(\beta(1 - H, \frac{H}{2}))^2 (K_t^2(t, t))^2, \\ &\leq 8(\beta(1 - H, \frac{H}{2}))^2 t^{6H-2} K_1^2(1, 1). \end{aligned}$$

Moreover, since  $F^{(2)}$  has polynomial growth, we have:

$$\begin{aligned} \|F^{(2)}(X_t^H)\|_{L^2(\Omega)}^2 &= \mathbb{E}[(F^{(2)}(X_t^H))^2], \\ &\leq C \mathbb{E}[(1 + |X_t^H|)^{2N}], \\ &\leq C(1 + 2Nt^H + \sum_{k=2}^{2N} C_k^{2N} \mathbb{E}[|X_t^H|^k]), \\ &\leq C(1 + 2Nt^H + \sum_{k=2}^{2N} C_k^{2N} (k-1)^k t^{Hk}). \end{aligned}$$

The dominating hypothesis is then satisfied. This concludes the proof.  $\square$

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