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Statistical modelisation and analisys of the price formation through the scales

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Relu Adrian Iuga. Statistical modelisation and analisys of the price formation through the scales. General Mathematics [math.GM]. Université Paris-Est, 2014. English. NNT : 2014PEST1090 . tel-01130682

HAL Id: tel-01130682

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Submitted on 12 Mar 2015

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UNIVERSITÉ — — PARIS-EST

THÈSE DE DOCTORAT DE L'UNIVERSITÉ PARIS-EST

École doctorale MSTIC

Spécialité Mathématiques

Analyse et modélisation du processus de formation de prix à travers les échelles. Market Impact

Adrian IUGA

*Thèse dirigée par **Emmanuel Bacry** et **Marc Hoffmann**,*

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Analyse et modélisation du processus de formation de prix à travers les échelles. Market Impact

Résumé :

Le développement des marchés électroniques organisés induit une pression constante sur la recherche académique en finance. L'impact sur le prix d'une transaction boursière portant sur une grande quantité d'actions sur une période courte est un sujet central. Contrôler et surveiller l'impact sur le prix est d'un grand intérêt pour les praticiens, sa modélisation est ainsi devenue un point central de la recherche quantitative de la finance. Historiquement, le calcul stochastique s'est progressivement imposé en finance, sous l'hypothèse implicite que les prix des actifs satisfont à des dynamiques diffusives. Mais ces hypothèses ne tiennent pas au niveau de la "formation des prix", c'est-à-dire lorsque l'on se place dans les échelles fines des participants de marché. Des nouvelles techniques mathématiques issues de la statistique des processus ponctuels s'imposent donc progressivement. Les observables (prix traité, prix milieu) apparaissent comme des événements se réalisant sur un réseau discret, le carnet d'ordre, et ceci à des échelles de temps très courtes (quelques dizaines de millisecondes). L'approche des prix vus comme des diffusions browniennes satisfaisant à des conditions d'équilibre devient plutôt une description macroscopique de phénomènes complexes issus de la formation des prix.

Dans un premier chapitre, nous passons en revue les propriétés des marchés électroniques. Nous rappelons la limite des modèles diffusifs et introduisons les processus de Hawkes. En particulier, nous faisons un compte rendu de la recherche concernant le market impact et nous présentons les avancées de cette thèse.

Dans une seconde partie, nous introduisons un nouveau modèle d'impact à temps continu et espace discret en utilisant les processus de Hawkes. Nous montrons que ce modèle tient compte de la microstructure des marchés et est capable de reproduire des résultats empiriques récents comme la concavité de l'impact temporaire.

Dans le troisième chapitre, nous étudions l'impact d'un grand volume d'action sur le processus de formation des prix à l'échelle journalière et à une plus grande échelle (plusieurs jours après l'exécution). Par ailleurs, nous utilisons notre modèle pour mettre en avant des nouveaux faits stylisés découverts dans notre base de données.

Dans une quatrième partie, nous nous intéressons à une méthode non-paramétrique d'estimation pour un processus de Hawkes unidimensionnel. Cette méthode repose sur le lien entre la fonction d'auto-covariance et le noyau du processus de Hawkes. En particulier, nous étudions les performances de cet estimateur dans le sens de l'erreur quadratique sur les espaces de Sobolev et sur une certaine classe contenant des fonctions "très" lisses.

Mots-clés : processus ponctuel, processus de Hawkes, processus stationnaire, market impact, loi en racine carrée, estimation non-paramétrique, discrétisation des processus stochastiques.

Analysis and modeling of price formation process across the scales. Market Impact

Summary:

The development of organized electronic markets induces a constant pressure on academic research in finance. A central issue is the market impact, *i.e.* the impact on the price of a transaction involving a large amount of shares over a short period of time. Monitoring and controlling the market impact is of great interest for practitioners; its modeling and has thus become a central point of quantitative finance research.

Historically, stochastic calculus gradually imposed in finance, under the assumption that the price satisfies a diffusive dynamic. But this assumption is not appropriate at the level of “price formation”, *i.e.* when looking at the fine scales of market participants, and new mathematical techniques are needed as the point processes. The price (last trade, mid-price) appears as events on a discrete network, the order book, at very short time scales (milliseconds). The Brownien motion becomes rather a macroscopic description of the complex price formation process.

In the first chapter, we review the properties of electronic markets. We recall the limit of diffusive models and introduce the Hawkes processes. In particular, we make a review of the market impact research and present this thesis advanced.

In the second part, we introduce a new model for market impact model at continuous time and living on a discrete space using process Hawkes. We show that this model that takes into account the market microstructure and it is able to reproduce recent empirical results as the concavity of the temporary impact.

In the third chapter, we investigate the impact of large orders on the price formation process at intraday scale and at a larger scale (several days after the meta-order execution). Besides, we use our model to discuss stylized facts discovered in the database.

In the fourth part, we focus on the non-parametric estimation for univariate Hawkes processes. Our method relies on the link between the auto-covariance function and the kernel process. In particular, we study the performance of the

estimator in squared error loss over Sobolev spaces and over a certain class containing “very” smooth functions.

Keywords: point processes, Hawkes processes, stationary processes, market impact, square-root law, non-parametric estimation, discretization of stochastic processes.

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Remerciements

Tout d'abord, je remercie mes directeurs de thèse, Emmanuel Bacry et Marc Hoffmann, pour leur disponibilité, pour avoir partagé leurs connaissances avec moi, pour m'avoir inculqué un plus grand sens de la rigueur et pour leur exigence. Je leur suis reconnaissant pour leur soutien personnel et académique, pour leur compréhension et leur patience à certains moments où je n'aurais pas été si gentil avec moi-même.

Je tiens à remercier Charles-Albert Lehalle, pour son constant intérêt pour ma thèse. Après avoir été mon encadrant de stage de master, il m'a présenté à Emmanuel et Marc, et m'a fourni des données afin de tester mes modèles. Sans son aide ma carrière aurait été bien différente. Je voudrais remercier Matthieu Lasnier qui a rendu concrète ma collaboration avec Cheuvreux en me fournissant toutes les données dont j'ai eu besoin et avec qui j'ai pu avoir de nombreuses discussions sur la microstructure des marchés financiers.

Je remercie les membres du Laboratoire de Statistiques du CREST, pour la bonne ambiance qu'ils ont fait régner dans l'équipe, en particulier ceux que j'y ai longtemps côtoyés : Julyan Arbel, Pierre Bellec, Patrice Bertail, Victor-Emmanuel Brunel, Nicolas Chopin, Arnak Dalayan, Eric Gautier, Mathieu Gerber, Adélaïde Olivier, James Ridgway, Judith Rousseau, Robin Ryder, Silviu Vlasceanu. Merci également aux membres du laboratoire CEREMADE, en particulier aux post-doctorants : Valère Bitseki-Penda, Julien Grepat, Didier Jesslé, Omar Lazar.

Pour s'être, sans excès, intéressés à ma thèse, pour m'avoir aidé à passer agréablement ces dernières années, je voudrais remercier mes amis : Olivier Colier, Li Fang, Florin Neacsu, Adrian et Corina Nestor, Virgil Petrea, Vlad Popescu, Simona Saba, Ruxandra Taleanu, Carlyne Taieb, Iulian Tugurlan, Silviu Vlasceanu. Pour m'avoir aidé à la rédaction en anglais, un grand merci à Laurentiu Ionascu.

Je tiens à remercier aussi les professeurs de mathématiques qui ont su m'enseigner la beauté de cette science : à Eugen Alexandrov pour la rigueur qu'il m'a enseignée dès le collège et pour les émotions que nous avons partagées lors de mes premières olympiades nationales ; à Popoiu Dan pour sa pédagogie hors pair et pour son soutien tout au long des années, il est pour moi mon deuxième père ; à Constantin Tanasescu pour les plus belles solutions à des problèmes élémentaires, sans son aide je n'aurais pas participé aux olympiades internationales de mathématiques.

Pour avoir patiemment écouté mes plaintes et m'avoir soutenu en toutes circonstances, je remercie Agnès Deraix, Alain Deraix, Annie Deraix et Valeriu Lacatusu. Vous êtes ma deuxième famille depuis que je suis arrivé en France.

Enfin, je voudrais remercier ma famille pour leur amour continu et leur soutien, particulièrement ma mère qui a su cultiver ma prédisposition pour les mathématiques et ma soeur, Madalina, qui a été à mes côtés dans les moments les plus difficiles. Multumesc!

Context and thesis results

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1.1 PRICE FORMATION PROCESS: CONTINUOUS DOUBLE AUCTION IN ELECTRONIC MARKETS

Technological innovation has completely transformed the financial market. Nearly half of the world's stock exchanges are now organized in electronic markets such as NYSE (New York Stock Exchange), LSE (London Stock Exchange), TSE (Tokyo Stock Exchange), etc. which have largely replaced the traditional floor-based trading. In these markets, the price is formed according to the continuous double auction process. The auction is called "continuous" because market participants (*traders* or *agents*) can submit or remove their preferences (or *orders*) at any moment in a given daily time window. The auction is called "double" because traders can both buy and sell a particular asset (or *share*). As in a conventional auction, a trader informs the rest of the participants of his preference for a particular asset. If he wants to buy, he announces to the community the price he is willing to pay for a given quantity. This price is called his *bid* price. Similarly, if he wants to sell, he announces to the community the price he is willing to accept for a given quantity. This price is called his *ask* price. When two market participants agree on a price, a transaction (or *trade*) occurs. Obviously, if two traders agree on a price, but one intends to buy a quantity Q_1 of shares and the other is willing to sell a quantity Q_2 of shares, the transaction occurs for only $\min(Q_1, Q_2)$ shares. If a trader has a bid price which is higher than another trader's ask price (which is very unlikely to happen), a trade will occur at some price they both agree on and both their orders vanish from the market. Let us point out that at any moment a trader can withdraw his proposal. If no further agreements can be made, the traders wait for someone to engage with them. Such a mechanism naturally divides the participants in two separate categories: buyers (or *bid side*) and sellers (or *ask side*). Naturally, buyers want to purchase shares at prices smaller than the ones proposed by sellers. This allows to define the *best bid* price and the *best ask* price: the maximum buy price and respectively the minimum sell price. The difference between the best ask price and the best bid price is then strictly greater than zero and is called the *bid-ask spread*. The average between the best ask price and the best bid price is called the *mid-price*.

The limit order book

In an electronic market, all the information concerning the traders' preferences is handled electronically. The exchange keeps all participants up to date with market conditions through the *Limit Order Book* (LOB). When a trader wants to buy a quantity of shares q at the maximum price P , he sends what is called a *buy limit order* of quantity q and price P . Similarly, when a trader wants to sell a quantity of shares q at the minimum price P , he sends what is called a *sell limit order* of quantity q and price P . All traders' preferences put together form the Limit Order Book. Any trader authenticated in the electronic market has access to the LOB. He can see, for a given price P , the total volume Q available for trading at that price level. However, he does not have access to the identities of the traders behind those orders or to how many different limit orders compose the total available volume Q . If a new limit order of quantity q_0 arrives at the existing price P , it gets added to the LOB and the available volume



Figure 1.1 – Example of limit order book. On the left there are the buy orders (bid side), the best bid represents the highest buy price. On the right there are the sell orders (ask side), the best ask represents the highest buy price. The bid-ask spread the difference between the best ask price and the best bid price.

at that price becomes $Q + q_0$. At any moment, a trader can withdraw his limit order or a part of it through what is called a *cancel order*; his limit order is then removed from the LOB.

The tick

A trader who decides to send a limit order does not have complete freedom for the choice of the price. Indeed, the electronic market fixes a price grid on which traders can place their orders. The grid step is the smallest difference between two prices and it is called *tick*. In some markets, the tick value depends on the price. For example, stocks trading on the Euronext in Paris with price smaller than 9.999€ have a tick value of 0.001€, while all stocks above 10€ have a tick value equal to 0.005€.

Market Orders and Order Matching Algorithm

In addition to placing limit orders and cancel orders, there are other ways traders can interact with the LOB. More precisely, at any moment, a participant can decide to buy (or sell) a certain quantity of shares regardless of their price. This is called a *market order*. A market order generates immediate transactions at the best available prices: the entire number of shares are bought (or sold), even if it means matching several limit orders in order to satisfy it. For example, if the volume available at the best ask is not enough to fill an ask market order, the market order is filled with part of the volume existing at the second best ask price or even third best ask price and so on.

It is interesting to note that traders sometimes choose to send a buy limit order at a price equal to or higher than the best ask, instead of a buy market order (likewise, they place sell limit orders at prices equal to or less than the best bid, instead of sell market orders). In this case, only ask limit orders whose prices are equal to or less than the buy limit price are executed (similarly, only bid limit orders whose prices are equal to or higher than the sell limit price are filled). If any volume of the buy (sell) limit order is left, it will be placed on the LOB becoming the new best bid (ask). From outside, it is nearly impossible to distinguish these kinds of orders from market orders. This type of order is

called *marketable limit order*. Since an immediate transaction occurs after a marketable limit order, the two types of orders are mixed up and called market order.

The dynamic of the LOB resembles a queuing system where limit orders wait to be executed against market orders or to be canceled. Most of the markets respect the FIFO rule: *First In First Out*. This means that there is a time priority: earlier orders are executed before later orders. To clarify, even if several traders have similar bid or ask prices, the one who made his preference known first will have his orders executed before the others. When a market order needs more than one limit order to be entirely filled, the oldest limit orders are executed first. For example, suppose that at the best ask price of 25,05€, there are 200 shares placed by trader *A* and 300 shares submitted by trader *B* after *A*. Now suppose a buy market order of 300 shares is submitted to the limit order book. The limit order for 200 shares by trader *A* will be executed first, because it is at the front of the queue at the best ask. Then, 100 shares of the order with 300 total shares by trader *B* will be traded, since it was second in the queue. 200 shares of the 300 share order remain in the order book at the price 25,05€.

If the market order's size is greater than the available volume at the best price, the surplus will be executed at worse price levels until it is completed. More precisely, once all the available volume at the best price is consumed by a market order, the matching continues with the oldest order of the immediately higher price, which has become the new best price.

Note that a cancel order which removes only a part of an existing limit order maintains the time priority of the original order and it can only lower its size. This is reasonable because, if size increases were allowed, traders with orders at the highest time priority for a price level could perpetually increase the size of their order, preventing others from being able to transact stock using limit orders at that price level.

Let us remark that this version of the Order Matching Algorithm is adopted by most of the electronic markets. There are few exceptions - let us mention the Chicago Mercantile Exchange where orders at same price are executed proportionally to their quantity.

Now that we have presented the most important mechanisms of the price formation process, we discuss the notion of market impact.

The concept of market impact

Without loss of generality, we assume in this section that the share price at time t is the mid-price (i.e. the average between the best ask price and the best bid price), noted p_t . Suppose that a market participant wants to buy a quantity Q of shares. Therefore, he sends a market order of volume Q at time t_0 . If Q is lower than the volume available at the best ask, the price will not change after the transaction. If the volume Q is greater or equal than the volume available at the best ask, the price at time t_{0+} will increase by at least a half tick. Thus, on average, a buy order pushes the price up. Similarly, on average, a sell order drives the price down. This price change can be explained by standard eco-

conomic theory: an increase in demand should increase prices, while an increase in supply should decrease prices. Accordingly, a transaction has a direct impact on the market price of the asset. This last effect is known as *market impact*. Note that a limit order or even a cancel order could have a market impact, since these kinds of orders can also change the price.

Liquidity, meta-orders and optimal trading

The liquidity of a market, as well as the market impact, is another term linked to the transaction cost. A simple way to define a liquid asset is the easiness of selling and buying non-negligible quantities of shares. Indicators of a liquid asset are: fast arrivals of new limit orders, small bid-ask spread, no important gap in the limit order book. The importance of liquidity is very high for investors who decide to invest in some asset by buying a significant quantity of shares and hold them for a few months (years) before selling. How much will they pay to buy the shares once they make the investment decision? First of all, an investor cannot buy them all at once, because there are not enough available shares in the LOB. Thus, the original large order (or *meta-order*) must be split in slices (or *child-orders*) and executed incrementally. In this case, there is a hard reality: their second buy trade is on average more expensive than the first because of their own impact on the price. We might think it suffices to delay two consecutive child orders long enough so the first has no impact on the second. However, it may take days to complete the meta-order and during this period the price can change in an adverse way. Thus, when slicing a meta-order, there are some optimizations to do by taking into account market risk, market impact and market liquidity. This branch of quantitative finance is called *optimal trading* (see [90] for more details).

As we stress in a previous paragraph, the electronic markets provide updates of the limit order book at very high frequency, which could be used in order to make profit from good prices. Since the volume of information received is tremendous and the limit order book gets updated very often (in much less than a second in liquid markets), a human trader is not able to take all into account. Furthermore, a human trader should react very quickly when he thinks there is an opportunity. This is why trading algorithms are nowadays widely used among financial agents. This type of algorithmic trading uses mathematical tools and computer algorithms to rapidly trade shares and is known as *high frequency trading* (HFT). In order to optimize the trading strategy to minimize such costs, it is necessary to understand the market impact. Monitoring and controlling market impact is one of the most actively researched topics within trading firms; its modeling and estimation has thus become a central point of the quantitative finance research.

1.2 MEAN REVERSION AND LIMITS OF DIFFUSIVE MODELS

In the early 1900s, Louis Bachelier was the first to model the price of a financial asset as a random market. In his thesis, *Theory of speculation*, he continued the previous work of Robert Brown (the famous biologist) and Einstein on

Brownian motion. This work has remained unknown for several decades, until Kolmogorov quoted him several times. After the Second World War, the world of finance has begun to discover the richness of Bachelier's work in financial mathematics thanks to Benoît Mandelbrot.

From the 1960s, the Black-Scholes-Merton model has emerged in finance. In this model, it is not the price increments that are supposed as being Gaussian, but their returns. Thus, stochastic calculus gradually obtrudes in finance via the notion of risk replication under the assumption that asset prices satisfy a diffusive dynamic (from the Brownian diffusion to the most general model of Itô's semi-martingale). The diffusion hypothesis comes as a consequence of the following argument: if the next value of a financial asset was predictable by studying the historical values, then this predictability would have been exploited to extinction. This argument is often known as the assumption of *absence of arbitrage opportunity*.

In this section, I will show some stylized facts for the price formation process that are in conflict with the diffusive assumptions. For this I use high-frequency data from CA Cheuvreux. Before going further, I will briefly describe this database.

Database

The database contains all the single-day meta-orders (nearly 300.000) executed by CA Cheuvreux from January 2010 to December 2010 in the European equity markets and all transactions on the stock from the day the meta-order was executed. For each meta-order I have access to:

- the exact time (with millisecond precision) of the beginning of the meta-order,
- the exact duration of the meta-order,
- the total volume of the meta-order,
- the side of the execution (ask or bid),
- the volume, the price and the execution time of each child-order,
- the market place where the meta-order was executed,
- an "identifier" allowing to determine the adjacent stock,
- the type of algorithm used for execution (see below).

For each adjacent stock I have access to:

- the price in each moment of the trading day,
- the price, the volume, the side and the exact time of each transaction,
- the first limit of the limit order book (i.e. the price and volume of the best ask and the best bid),
- the tick size.

In this entire section, $P(t)$ will represent the evolution in time of a stock price during the trading day. More precisely, $P(t)$ will stand for the last transaction price that occurred before time t . A trading day lasts 8 hours and a half. In order to simplify expressions, let us note $t = 0$ the beginning of the trading day and $T = 8h30$ the end of the trading day. The price $P(t)$ lives on a tick grid and jumps at discrete random times. In the next figure, one can observe the evolution of the price of Deutsche Bank AG shares on 10 January 2010. There are 25527 trades during the entire trading day, or nearly 50 trades per minute. Even

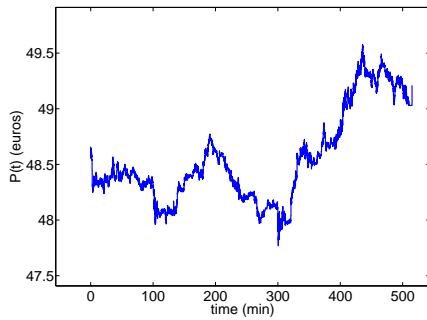


Figure 1.2 – Evolution of the price of Deutsche Bank AG on 10 January 2010 - entire trading day

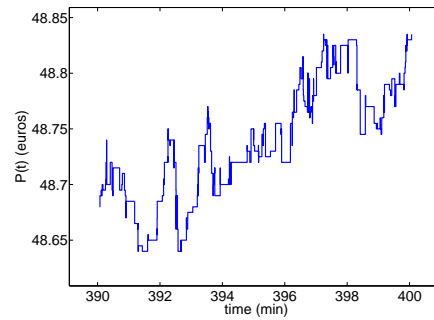


Figure 1.3 – Evolution of the price of Deutsche Bank AG on 10 January 2010 from 3:35 PM to 3:40 PM

if the price seems to be diffusive on the macroscopic scale (Fig. 1.2), at the microscopic scale the reality is totally different (Fig. 1.3).

Volatility and signature plot

A well known property of a Brownian motion $W(t)$ of volatility σ is that:

$$\mathbb{E}[(W(t+\tau) - W(t))^2] = \sigma\tau, \forall \tau > 0. \quad (1.1)$$

This property gives a simple estimator of σ whatever the sampling τ :

$$\hat{\sigma} = \frac{1}{T} \sum_{i=1}^{T/\tau} (W(i\tau) - W((i-1)\tau))^2. \quad (1.2)$$

In particular, this estimator converges when τ tends to 0. Thus, if $P(t)$ is a Brownian motion, then the so-called realized volatility function $V(\tau)$ will be flat:

$$\hat{V}(\tau) = \sum_{i=1}^{T/\tau} (P(i\tau) - P((i-1)\tau))^2. \quad (1.3)$$

Figure 1.4 shows the realised volatility $\hat{V}(\tau)$ for the Deutsche Bank AG stock price on 10 January 2010. This behavior is different from what one would expect if the data were sampled from a Brownian diffusion. We notice an increase of the observed daily volatility when one goes from large to small scales, *i.e.* when $\tau \rightarrow 0$.

Mean reversion

One explanation for the signature plot function is the well-known mean reverting effect of price. We are just going to highlight this effect without going in deeper elucidations (see [90] for more). Even so, let us point out an intuitive enlightenment: the mean reverting effect adds additional volatility when the sampling rate is low.

Let $tick$ denote the tick-size and $k \in \mathbb{N}^*$. We say a k -jump occurs on the price between times t_1 and t_2 if $|P(t_1) - P(t_2)| \geq k * tick$. For $k \in \mathbb{N}^*$, we define a sequence of stopping times $(\tau_n^{(k)})_{n \geq 0}$ such that between $\tau_n^{(k)}$ and $\tau_{n+1}^{(k)}$ a k -jump occurs on the price:

$$- \tau_0^{(k)} = 0,$$

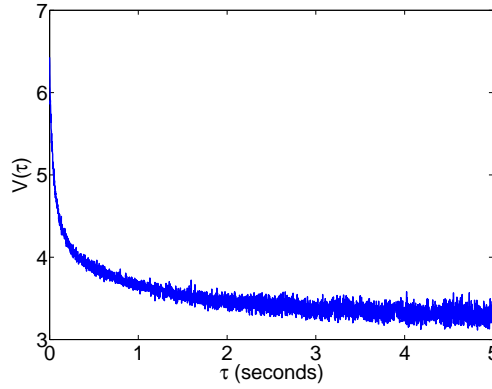


Figure 1.4 – Signature-plot $\hat{V}(\tau)$ (1.3) of Deutsche Bank AG on 10 January 2010.

- For $n \geq 1$, while $A_n^{(k)} = \{t \in [\tau_{n-1}^{(k)}, T] : |P(t) - P(\tau_{n-1}^{(k)})| \geq k * tick\}$ is non-empty:

$$\tau_n^{(k)} = \inf_t A_n^{(k)}.$$

- Once $A_n^{(k)}$ is empty, we stop and $N(k)$ will denote the number of total k -jumps of the trading day.

If the price $P(t)$ was a classical random walk on the tick grid:

$$\mathbb{P}\left(P_{\tau_{n+1}^{(k)}} - P_{\tau_n^{(k)}} \geq k * tick\right) = \mathbb{P}\left(P_{\tau_{n+1}^{(k)}} - P_{\tau_n^{(k)}} \leq -k * tick\right) = 1/2 \quad (1.4)$$

and

$$\mathbb{P}\left(P_{\tau_{n+1}^{(k)}} - P_{\tau_n^{(k)}} \geq k * tick \mid P_{\tau_n^{(k)}} - P_{\tau_{n-1}^{(k)}} \geq k * tick\right) = 1/2, \quad (1.5a)$$

$$\mathbb{P}\left(P_{\tau_{n+1}^{(k)}} - P_{\tau_n^{(k)}} \geq k * tick \mid P_{\tau_n^{(k)}} - P_{\tau_{n-1}^{(k)}} \leq -k * tick\right) = 1/2. \quad (1.5b)$$

The first equation (1.4) translates the symmetry of the jumps. The last two equations (1.5a and 1.5b) translate the Markov nature of the random walk: the next jump depends only on the current state and not on the sequence of events that precede it. In order to test the last assumptions (1.5a and 1.5b), let us denote $N^{(a)}(k)$ the number of *alternations*, i.e. the number of k -jumps whose direction is opposite to the one of the preceding k -jump, $N^{(a)}(k) = card(A^{(a)}(k))$ where:

$$A^{(a)}(k) = \{n \in [1, N(k) - 1] : \left(P_{\tau_n^{(k)}} - P_{\tau_{n-1}^{(k)}}\right) \left(P_{\tau_{n+1}^{(k)}} - P_{\tau_n^{(k)}}\right) < 0\}.$$

In a similar way we define $N^{(c)}(k)$ the number of *continuations*, i.e. the number of k -jumps whose direction is the same as the one of the preceding k -jump. Let us remark, that $N^{(a)}(k) + N^{(c)}(k) = N(k)$.

We define the probability of “mean-reverting” after a k -jump as

(1.6)

$$p_k = \frac{N^{(a)}(k)}{N(k)}.$$

If the price were a random walk, then $p_k = 1/2$. When $p_k > 1/2$, the price is mean reverting, since there is a higher probability of return. When $p_k < 1/2$, we are in the presence of a trend. Figures 1.5,1.6,1.7 and 1.8 shows the histogram of the

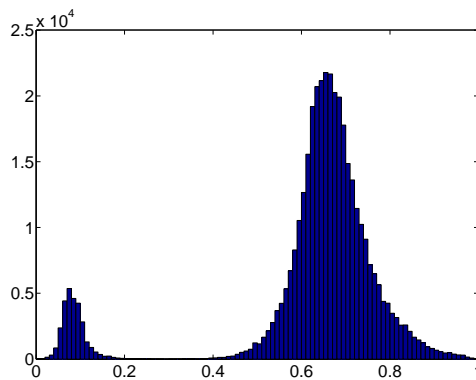


Figure 1.5 – The histogram of the probability p_1 (1.6) of “mean-reverting” after a 1-jump

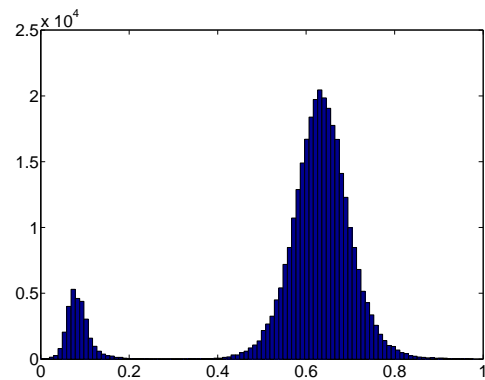


Figure 1.6 – The histogram of the probability p_3 (1.6) of “mean-reverting” after a 3-jump

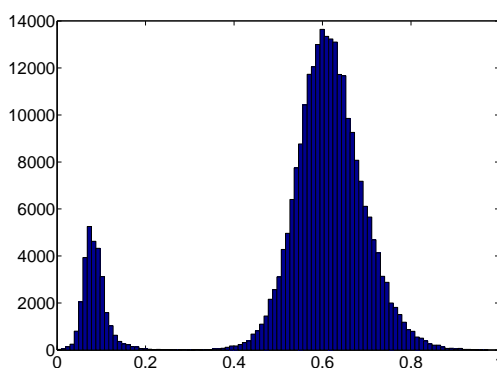


Figure 1.7 – The histogram of the probability p_5 (1.6) of “mean-reverting” after a 5-jump

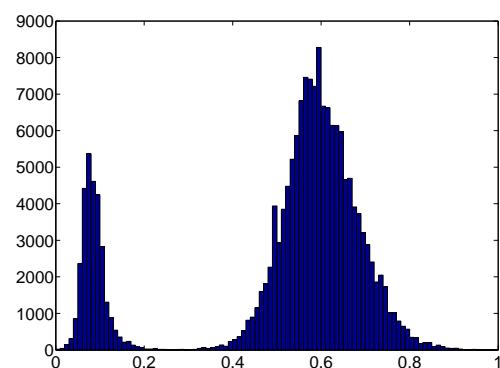


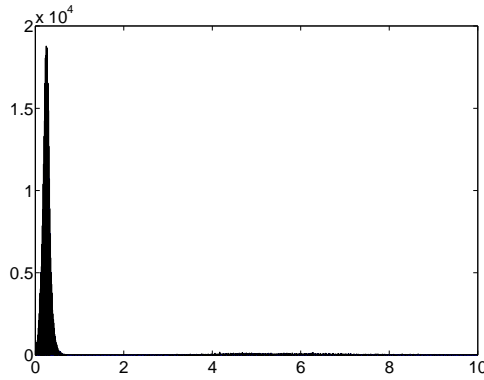
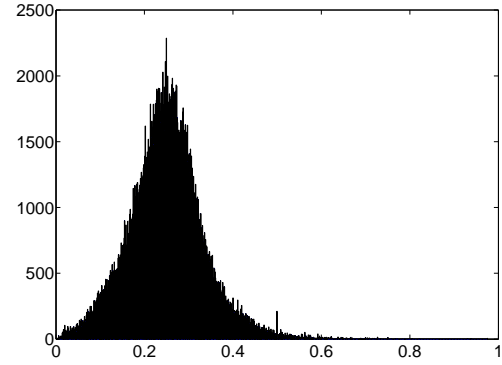
Figure 1.8 – The histogram of the probability p_{10} (1.6) of “mean-reverting” after a 10-jump

mean reverting probability p_k for $k = 1, 3, 5, 10$. We obtain similar pictures for every $k \in \{1, 2, \dots, 20\}$.

We observe two very different regimes for all k . At the right, a mean-reversion regime with an important peak centered around $p_k \approx 0.65$ (slightly different for each k). At the left, a trend-following regime with a less important peak centered around $p_k \approx 0.1$. A qualitative explanation is given by the *fundamental value* theory. Let us be more specific: economists often argue that there are some informed traders who know with high precision the *fundamental value*, i.e. the intrinsic value of a company. Those traders buy shares when the stock is underpriced and sell shares when the stock is overpriced. They make profit because they are informed and through their impact they are mean-reverting the price toward the *fundamental value*. In such a framework, the *fundamental value* of a stock can only change as a result of unpredictable news. When news concerning the asset appears, the informed traders successfully update the new fundamental value, they trade accordingly, price converges toward its new equilibrium value and the process repeats over and over again.

This explains why most of the time the well-known stylized fact of mean reversion is observed (right peak). From time to time, the price goes in one direction and the presence of news could be assumed.

The distribution of p_1 can be also interpreted through the *uncertainty zones*

Figure 1.9 – The histogram of $\hat{\eta}$ (Eq. 1.7)Figure 1.10 – Zoom on the peak of the histogram of $\hat{\eta}$ (Eq. 1.7)

model [122] of Robert and Rosenbaum. Let us briefly describe the model. In this framework, the efficient price $X(t)$ follows a continuous semi-martingale dynamic and cannot be observed directly. Instead, we observe a noisy version $\tilde{X}(t)$ which represent the last traded price value. The uncertainty zones is denides as two bands around the efficient price values with width $2\eta\alpha$, where $0 < \eta = 1$ and α is the tick value of the asset. Let t be any given time and $\tilde{X}(t)$ the associated last traded price value. Let τ_t be the first time after t where $X(t)$ hits the uncertainty around $\tilde{X}(t)$. At time τ_t we say we have a new transaction of price $\tilde{X}(t) + \alpha$ if $X(t)$ hits the upper band or $\tilde{X}(t) - \alpha$ if $X(t)$ hits the lower band.

When $\eta < 1/2$, the model reproduces the mean reversion effect and the signature-plot stylized fact (see Fig. 1.4). Moreover, in practice, the estimated values of η are found to be smaller than $1/2$.

The authors proved in [?] that as the tick value goes to zero, a consistent estimator of η si given by:

$$\hat{\eta} = \frac{N^{(c)(1)}}{2N^{(a)(1)}}, \quad (1.7)$$

or in term of p_1 :

$$\hat{\eta} = 0.5 \left(\frac{1}{p_1} - 1 \right).$$

Next figures (Fig 1.9 and 1.10) show the histogram of η . We observe the right peak from Figure 1.5 centered around $p_1 \approx 0.1$, is transformed in a much less visible “cloud” around the value $\hat{\eta} \approx 4.5$ (see Fig. 1.9). The second Figure 1.10 shows a zoom of the peak observed in precedent histogram. It is centered around $\hat{\eta} \approx 0.3$ and it corresponds to the right peak from Figure 1.5 (which is centered around $p_i \approx 0.6$).

Our empirical findings are consistent with *uncertainty zones* model with $\eta < 1/2$. The cases where $\hat{\eta} > 0$.

First conclusions

As we explain in Section 1, the development of electronic markets induced a constant pressure on academic research in finance. Historically, stochastic calculus gradually imposed in finance after 1960 thanks to the Black-Scholes-Morton model. Starting from this point, several models have been successively

introduced under the implicit assumption that asset prices satisfy a diffusive dynamic. Let me remind some of the most known models: the Vasicek model [131], Heston model [76], the Cox-Ingersoll-Ross model [42].

As we empirically proved in the present section, these assumptions are not appropriate at the level of price formation. Thus, other mathematical tools are needed.

As events occur, the price appears on a discrete grid, the limit order book, and this at very short time scales. Thus, the price dynamic is closely related to the limit order book dynamic. The price seen as a Brownian diffusion satisfying some equilibrium conditions (under the hypothesis of the absence of arbitrage opportunity) becomes rather a macroscopic description of the complex price formation process. At the microscopic level, the price must be rather seen as a complex stochastic process living continuously in a discrete space. The next section is devoted to new approaches to price modeling and in particular to the Hawkes model for microstructure introduced by Bacry *et al.* in [17], which will be one of the key point of this thesis.

1.3 OTHERS APPROACHES TO PRICE MODELING. THE HAWKES MODEL FOR MICROSTRUCTURE.

Other approaches to price modeling

We can distinguish three levels for price modeling:

1. The ***microscopic level***: the ultimate level of price formation, describing the dynamic of the limit order book in continuous time and discrete space. This is the most reasonable but also the most difficult.
2. The ***intermediate level***: at this level the price becomes a projection/function (last price, mid-price, etc.) of the limit order book but remains a point process whose dynamic can be described.
3. The ***macroscopic level***: the price is seen as a Brownian diffusion (or an avatar of the Brownian diffusion); it is the historical approach.

Without being exhaustive, let us remind some approaches for price modeling in the light of the three levels emphasized above.

One of the most known models for the limit order book is the “zero intelligence” model [55]. This is a simple model describing rigorously the statistical mechanics of order placement, price formation and accumulation of revealed supply and demand. In the same framework, Smith *et al.* [125] used a mean-field approach to study the properties of the limit order book, under the assumption of independent Poisson order flows. Remarkably, zero-intelligence models correctly predict important statistical properties of the order book, even if some assumptions are not fully believable. But this approach is not very operational from a mathematical perspective: it is difficult to describe the price at the *intermediate level* or to prove that the price diffuses at the *macroscopic level*. Nevertheless, numerical experiments uphold the last part.

The simplicity of these models, even if the Poisson assumption is quite inconsistent with empirical observations, generated numerous developments. Let

us cite the work of Cont [40, 39], where the LOB is seen as a high-dimensional queuing system and orders arrive and depart randomly. The authors compute the probabilities of various events, such as the probability of a price increase or the distribution of the duration until the next price move, conditional on the state of the order book. Stability conditions for the system are studied by Abergel and Jedidi [1]. The authors prove that the rescaled-centered price process described at *microscopic level* converges to a Brownian motion on the *macroscopic level*.

In order to replace the independent Poisson framework, a more realistic assumption is to consider the order flows as dependent point processes. Hewlett in [77] and Large in [89] are modeling the order arrival by Hawkes self-exciting point processes. These types of models are interesting since they allow the reproduction of some clustering properties.

Concerning the *intermediate level*, where the important factor is the microstructure noise modeling, let us first cite the seminal work of Gloter and Jacod [64, 65] where the authors used the concept of latent price. In this framework, the price $P(t)$ (which follows a Brownian motion) is latent, in the sense that it cannot be observed directly. Instead, we observe a noisy version $\tilde{P}(t)$ of $P(t)$. From a practical point of view, given a sampling scale τ , we rather observe:

$$\tilde{P}(n\tau) = P(n\tau) + \xi_{n,\tau},$$

where $\xi_{n,\tau}$ is the microstructure noise term with zero mean. This model has raised a large interest in the econometrician and statistician communities, see [136, 106, 21, 22, 78, 105] and the references therein. Whereas this model reproduces the microstructure noise effects at the scale of a few minutes, the realised volatility $\hat{V}(\tau)$ as defined in the previous section (1.3) explodes when τ goes to 0. Coherent transition from the intermediate level to the macroscopic level was for long time unenlightened. Recently, Bacry *et al.* [17] have proposed a Hawkes process model for the price, reproducing the classic microscopic feature (microstructure effect, Epps effect) and diffusing to a Brownian motion at the macroscopic level [16]. Within this model, the price dynamic depends on a kernel φ which must satisfy a \mathcal{L}^1 condition, namely $\|\varphi\|_1 < 1$ (see the next section for more details). Starting from empirical measures of $\|\varphi\|_1$, which are close to 1, Jaisson et Rosenbaum [86] studied the dynamic of nearly unstable Hawkes processes where the \mathcal{L}^1 -norm of the kernel is close to unitary and showed they asymptotically behave like integrated Cox-Ingersoll-Ross models (see [42]) at the macroscopic level.

Hawkes price model for microstructure

General Hawkes process

Self-exciting processes were first introduced by Alan Hawkes [74, 73] in order to reproduce the ripple effects generated after the occurrence of an earthquake [133]. They are intuitively similar to Poisson processes, but unlike ordinary Poisson processes, the intensity of Hawkes processes is stochastic and depends upon their own historic events. Gradually, the model has been used by scientists from different areas such as biology [68, 112, 115], genome analysis

[121], neurology [129, 119, 120], seismology [4, 132, 75, 87, 110], social behavior [29, 43, 123, 138] and epidemiology [135], to name but a few.

Nowadays, Hawkes processes are widely used in finance. Because current transactions, which are discrete events, cause future trades [118], self-exciting processes successfully engendered many applications. Without being exhaustive, let us give some examples: microstructure dynamics [17], order arrival [77, 25], market impact [18, 19], financial price modelling across scales [16, 86], volatility clustering [50], price co-jumps [30, 99], limit order book modeling [137, 127, 89, 126, 2] among many other.

In practice, we observe a multivariate counting process $N(t) = (N_1(t), N_2(t), \dots, N_d(t))_{(t \geq 0)}$, each component N_i representing the number of events of the i -th component of the system during the period $[0, t]$. Under relatively weak general assumptions¹, the multivariate counting process N is characterised by its intensity process $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_d(t))$ defined by:

$$\mathbb{P}(N_i \text{ has a jump in } [t, t + dt] | \mathcal{F}_t) = \lambda_i(t) dt, \quad \forall i \in \{1, 2, \dots, d\},$$

where \mathbb{P} stands for probability and \mathcal{F}_t is the sigma-field generated by $\{N_1(s), N_2(s), \dots, N_d(s)\}_{0 \leq s \leq t}$ up to present time t . According to Jacod [85], for all $i \in \{1, 2, \dots, d\}$ the process:

$$N_i(t) - \int_0^t \lambda_i(s) ds$$

is an (\mathcal{F}_t) -martingale. This implies that the law of the d -dimensional process $(N_1(t), N_2(t), \dots, N_d(t))$ is characterised by $(\lambda_1(t), \lambda_2(t), \dots, \lambda_d(t))$.

Definition 1 (General Hawkes process). As pointed out at the beginning of the section, the particularity of the Hawkes process is the past-dependence of the intensity $\lambda(t)$, more precisely:

$$\lambda_i(t) = h_i \left(\sum_{j=1}^d \int_0^t \varphi_{ji}(t-s) dN_j(s) \right), \quad \forall i \in \{1, 2, \dots, d\},$$

where the causal functions $\varphi_{ji} : [0, \infty) \rightarrow \mathbb{R}$ model how N_j acts on N_i and h_i are nonnegative functions.

Definition 2 (Linear Hawkes process). As first introduced by Alan Hawkes [74, 73], the most popular example is for linear functions $h_i(x) = \mu_i + x$, with μ_i positive real numbers:

$$\lambda_i(t) = \mu_i + \sum_{j=1}^d \int_0^t \varphi_{ji}(t-s) dN_j(s), \quad \forall i \in \{1, 2, \dots, d\}, \quad (1.8)$$

Or in its matrix form:

$$\lambda(t) = \mu + \int_0^t \Phi(t-s) dN(s), \quad (1.9)$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_d)$ is a vector form \mathbb{R}_+^d and $\Phi(t) = (\varphi_{ij}(t))_{1 \leq i, j \leq d}$.

¹ For a complete mathematical study of Hawkes and more general point processes we refer the interested reader to the Daley and Vera-Jones' book [45].

In the degenerate case where $\varphi_{ji} = 0$, we retrieve a basic Poisson process. Some conditions on the integrability of Φ give a non-explosion criterion [16]:

$$\textit{non explosion-criterion} : \int_0^t \Phi(s) ds < \infty \text{ componentwise, } \forall t > 0. \quad (1.10)$$

The multivariate Hawkes process can be shown to admit a version with stationary increments under the next condition:

$$\textit{stationary condition} : \textit{the spectral radius of the matrix } K = \int_0^\infty \Phi(t) dt \textit{ is smaller than 1.} \quad (1.11)$$

One reason for its popularity is that due to the simple conditional intensity interpretation, the Hawkes process can be seen as a cluster process. Indeed, any new point N_i can be of two types: “immigrant” - without existing parents in the process, arriving with rate μ_i , or “offspring” - produced by previously existing points. Consequently, each immigrant produces descendants whose numbers in successive generations constitute a Galton-Watson branching process (see [13] for a general introduction to branching processes).

Since self-exciting processes are quite popular in different scientific areas, they have long been studied in probability theory: see for exemple the books of Daley-Vere-Jones [45], Brémaud-Massoulié [36, 34, 35] or the recent research of Zhu [141, 139, 140]. Recent studies (see [59, 47]) investigate the case of large d , when the number of components may become increasingly big or possibly infinite. With regard to the simulation of Hawkes processes, we refer the interesting reader to [102, 103, 62, 46].

From a statistical inference point of view, one of the first studies belongs to Ogata [111] and Ozaki [113], who investigated the maximum likelihood estimator for some classes of functions, like exponential and power laws. For more than two decades, these methods have been used for most of the applications. In 2010, Reynaud-Bouret and Schbath [121] employed a penalized projection method to estimate φ in the univariate Hawkes model. The theoretical estimator is adaptive for Hölderian functions with regularity $(1/2, 1]$, under the hypothesis that the decay kernel has compact support. In 2011, Lewis and Mohler [97] used a maximum penalized likelihood estimation to simultaneously approximate the background rate and the decay kernel of a multivariate model and they numerically studied the convergence rate of the algorithm. Same year (i.e. 2011), Bacry *et al.* [15] proposed another non-parametric estimation method for multivariate symmetric Hawkes processes, based on Fourier computations and Bartlett specter theory and without investigating the convergence speed. The last chapter of my thesis will be dedicated to this aspect. Recently, Bacry *et al.* [20] propounded a non-parametric estimation method for general multivariate Hawkes processes based on the explicit resolution of a Wiener-Hopf system using Gaussian quadrature method. This last method is a generalization of the first one proposed in [15].

Hawkes price model

Following [17], we represent the asset price $P(t)$ as the difference between the positive and negative jumps:

$$P(t) = N^+(t) - N^-(t),$$

where $N^+(t)$ and $N^-(t)$ are two point processes representing respectively the positive and negative jumps of the price over a time interval $[0, T]$. In order to introduce microstructure, Bacry *et al.* [17] used multivariate Hawkes processes. More precisely, they use a bivariate Hawkes process $(N^+(t), N^-(t))$ such that:

$$\lambda^+(t) = \mu + \int_0^t \varphi(t-s) dN^-(s), \quad (1.12a)$$

$$\lambda^-(t) = \mu + \int_0^t \varphi(t-s) dN^+(s), \quad (1.12b)$$

where μ is a positive number called *exogenous intensity* and φ is a causal positive function, i.e. $\varphi : [0, \infty) \rightarrow \mathbb{R}_+$, called *kernel*.

The interpretation of this model is very simple: the more $P(t)$ goes up, the greater the intensity $\lambda^-(t)$ and conversely, the more $P(t)$ goes down, the greater the intensity $\lambda^+(t)$, which translates the mean reversion. As we stress in the previous section, the bivariate Hawkes process $(N^+(t), N^-(t))$ can be shown to be well defined and to admit a version with stationary increments under the stability condition:

$$\|\varphi\|_1 = \int_0^\infty \varphi(t) dt < 1. \quad (1.13)$$

Exponential decay As literature showed us, due to the simplest estimation [113] and simulation [102], a special case is when the kernel φ is exponential:

$$\varphi(t) = \alpha e^{-\beta t} 1_{t \geq 0} \quad 0 < \alpha, \beta, \quad (1.14)$$

such that

$$\|\varphi\|_1 = \frac{\alpha}{\beta} < 1.$$

In this case, one can show (see [17]) that the signature plot (as defined in (1.3)) has the form:

$$V(\tau) = \frac{2\mu}{1 - \alpha/\beta} \left(\frac{1}{(1 + \alpha/\beta)^2} + \left(1 - \frac{1}{(1 + \alpha/\beta)^2} \right) \frac{1 - e^{-(\alpha+\beta)\tau}}{(\alpha + \beta)\tau} \right). \quad (1.15)$$

The next figure (Fig. 1.11) shows a theoretical signature plot for the case $\mu = 1$, $\alpha = 2$ and $\beta = 3$, which recalls the empirical signature plot (Fig. 1.4).

1.4 MARKET IMPACT

Generally, the *market impact* is the link between an incoming order and future price changes (see [33]). It seems obvious that a buy (sell) trade should push the price up (down) and it is easily demonstrated empirically. Thus, market impact tends to increase the execution cost of a strategy. Indeed, the second buy order is, on average, more expensive than the first, the third more expensive than the second and so on. Monitoring and controlling market impact is one of the most active areas of research within trading firms; its modeling and estimation has thus become a central point of quantitative finance research.

For practitioners, a particular importance is attached to the execution of big orders whose volume is much larger than the available liquidity in the limit order book. A common strategy consists in splitting the big order in small orders

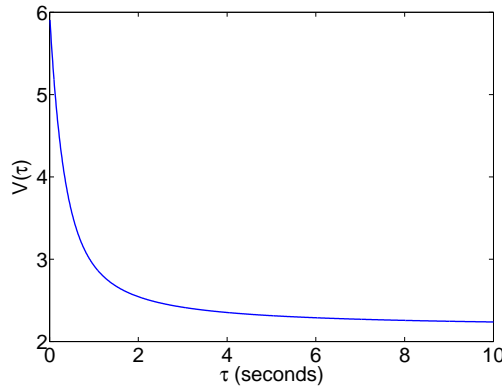


Figure 1.11 – Theoretical signature plot $V(\tau)$ for exponential kernel φ (1.15) with $\mu = 1$, $\alpha = 2$ and $\beta = 3$.

(*child orders* or *atomic orders*) and executing it incrementally. This sequence of trades is commonly called *meta-order* or *hidden order*. Nowadays, algorithmic high-frequency trading is the new way of executing those meta-orders. The investors use algorithmic trading, relying heavily on smart technologies and mathematical methods in order to provide efficient execution. Analyzing how the market impact is formed and the path leading to the completion of the meta-order is important for optimal execution [9, 7, 6, 8, 28, 108].

The market impact literature starts with the seminal work of Kyle [88] in the early 80's. The author showed that in order to achieve optimal execution an informed trader who has access to private information about the future value of the price will execute orders incrementally, at a constant speed, until the price reaches the target value. The informed trader slowly reveals the “true” price to the rest of the market; at the end everyone discovers the final price and there is no reversion. Within the setting of the Kyle model, the informed trader and other noise traders submit orders that are cleared by a market maker every step $\Delta\tau$. At each step, the price increment is:

$$\Delta P = \lambda \epsilon \nu,$$

where ν is the volume, ϵ is the sign of the order ($\epsilon = 1$ for buy order and $\epsilon = -1$ for sell order) and λ is a constant linked to the impact and considered to be inversely proportional to the liquidity of the market. The price change between $t = 0$ and $T = N\Delta\tau$ is:

$$P(T) = P(0) + \sum_{k=0}^{N-1} \Delta P(k\Delta\tau) = P(0) + \lambda \sum_{k=0}^{N-1} \epsilon_k \nu_k \quad (1.16)$$

Thus, in Kyle's framework, the market impact is both linear in the traded volume and permanent in time. Starting from a principle of *no-quasi arbitrage*, Huberman and Stanzl [81] show that the linear permanent market impact model is necessary in order to not allow price manipulations.

However, the Kyle model is not consistent with empirical studies that give a strictly concave volume dependence. Large trading firms can measure the price impact of their own trades; despite the diversity of market participants, trading strategies, execution style or execution time, a very similar concave power law is

reported. The literature on this topic is often rooted to the Barra Market Impact Model Handbook [23], who claimed that the price impact of a meta-order of volume V is remarkably well given by the formula:

$$C\sigma\sqrt{\frac{V}{V_d}}, \quad (1.17)$$

where V_d is the traded volume per unit time, σ is the volatility of the stock (in daily units) and C is a constant of order unity. This empirical result is referred as the *square-root law*. This “law” has been widely used in practice to estimate the pre-trade transactions cost. The fame of this formula lies in its simplicity; in particular, we notice that it does not depend on the execution trading time. Let us cite other empirical investigations concerning the volume dependence of market impact. Since access to data is often very difficult and some special agreement must be reached with the trading firms, there are rather few published research papers. The published studies give slightly different exponents for the power-law. Using US stock trade orders executed by Citigroup Equity Trading, Almgren *et al.* [10] find an exponent close to 0.7. Bouchaud *et al.* [118] find an exponent close to 0.5 for small tick contracts and 0.6 for large tick contracts, using data from Capital Fund Management.

Autocorrelation of trades sign and propagator model

One explanations for the concavity of the impact function is the persistence of the order flow (see [118]). Under the Kyle model (1.16), the sign of the trades must be uncorrelated if the price is to follow an unpredictable random walk. However, empirical studies shows that the autocorrelation function of the sign of the trades ϵ_k has a very slow decay. More precisely, one considers the function:

$$S(n) = \mathbb{E}[\epsilon_k \epsilon_{k+n}].$$

Under the assumption $\mathbb{E}[\epsilon_k] = 0$ (which is verified empirically), if trades were random, one should observe that $S(n)$ decays quickly to zero. On the contrary, this is not observed empirically. The autocorrelation function $S(n)$ decays slowly to zero as a power-law of n with negative exponent:

$$S(n) \approx S(0)n^{-\gamma}, S(0) > 0.$$

The value of γ is stock dependent, but always smaller than one, $0 < \gamma < 1$. Thus, the autocorrelation function is not summable and one says that the order flow has a long memory. A plausible explanation is the strategic behavior of large investors who split their orders into many small pieces and execute them incrementally (see [98]). More precisely, a large investor who decides to buy an important number of shares cannot just state his order in the limit order book because it is unlikely that there will be enough sellers to accommodate him, and even if there were, unveiling the intention to buy a large quantity of shares would greatly push the price in the adverse direction.

In order to obtain a long memory property of the order flow and to have a diffusive price, Bouchaud *et al.* [118] proposed a price impact model called the *propagator model*:

$$P(n) = \sum_{k \leq n} G(n-k)\epsilon_k f(v_k) + \eta_k, \quad (1.18)$$

where η_k is a white noise, f is a function describing the volume dependence of a single trade's impact, also called *instantaneous market impact function*, and G accounts for the temporal dependence of the market order's impact.

This model provides a different vision from that of Kyle, where the market impact was linear in volume and permanent in time. In the framework of the *propagator model* (1.18), the market impact of a trade of volume v has an instantaneous impact proportional to $f(v)$ and afterwards, the impact decays in time. Thus, an assumption about the separability of the market impact in a factorized form is implicitly made: one factor depending on volume and the other depending on time. This has been shown empirically by Bouchaud *et al.* in [117].

The *propagator model* was the starting point of numerous articles that deal with the following questions: what is the shape of the *instantaneous market impact function* $f(v)$ and that of the decay kernel G , is the market impact permanent or not and how do those quantities affect the impact of a meta-order.

Instantaneous market impact function

The market impact of a *single* trade was empirically found to be strongly concave in volume. Using the 1.000 largest firms on the New York Stock Exchange in 1995-1998, Lillo *et. al* discovered that $f(v) \sim v^\delta$ with an exponent δ varying from roughly 0.5 for small transactions in higher-capitalization stocks, to about 0.2 for larger transactions in lower-capitalization stocks. Hompan [79] arrived at similar results using data from the Paris Stock Exchange. Bouchaud *et. al.*, using French and British stocks, found a logarithmic rather than a power-law dependence of the price impact of an order. One of the explanations is the fact that a market order rarely consumes more than the volume available at the best price. For example, on 22 January 2010, on the FCAM stock (F&C Asset Management), among the 725 trades on the trading day, 184 (or 25%) were executed against exactly the volume available at the best price, 442 (or 61%) trades consumed only a part of the volume available and only 99 (or 14%) of the trades consumed more than the volume available. Thus, a lot of trades do not change the price.

Remark Single order vs. meta-order

There is a very important difference between the market impact of a *single* trade, $f(v)$, and the market impact of a meta-order. As we stressed before, the market impact of a meta-order of size V is "known" to be proportional to \sqrt{V} . Thus, the market impact of the first part $V/2$ is more important than the second part. Generally, the impact of the trades becomes less and less important as the execution of the meta-order evolves. Therefore, even orders belonging to the same investor have different market impacts.

Decay kernel G

In qualitative finance literature three types of decay kernel are considered: linear, exponential and power-law. Almgren *et. al* in [10] consider the function G to be decreasing instantaneously as a linear function. Obizhaeva and Wang in [108] assume that the kernel G acts as an exponential decay, in order to find an optimal trading strategy. Using the slow decay of the autocorrelation of trade signs and the principle that price changes should be unpredictable, Bouchaud *et. al* showed that the kernel G must also decay as a power-law with exponent $\beta \approx (1 - \gamma)/2$. Using the principle of *no-quasi arbitrage* of Huberman and Stanzl

[81] and a continuous version of the *propagator model* (1.18), Gatheral [60] argues that exponential decay is possible only if the instantaneous market impact function $f(v)$ is linear in v , which is not verified empirically. In the latter, it is shown that the power-law decay of the kernel G is compatible with the instantaneous market impact power-law, $f(v) \sim v^\delta$, only under the condition $\delta + \beta \geq 1$.

Permanent vs. non-permanent impact

Concerning permanent market impact, there seems to be some disagreement in the literature; two different opinions have emerged in quantitative finance. From the point of view of Bouchaud *et al.* [118, 33], the permanent market impact is zero. He argues that since the order flow is highly autocorrelated ($\gamma < 1$), a permanent market impact leads to trends in the market, which is not consistent with the unpredictability of price changes observed empirically. In the opinion of Farmer *et al.* [104], the permanent market impact is non-negligible - they empirically found that its value is equal to roughly 0.5–0.7 of the instant market impact. Recently, Bershova and Rakhlin [27] found a permanent market impact equal to roughly 2/3 of the instant market impact. In a later paper, Bouchaud, Farmer and Lillo [32] argued that these two seemingly incompatible pictures are equivalent under the assumption that the order flow follows a long memory FARIMA process (see [26] for definition). In other words, Farmer&al. [104] consider that the market impact is history dependent, whereas Bouchaud sees the market impact as *the difference between the observed price moves and what it would have been without this specific order*, as defined in Barra Market Impact Model Handbook [23]. Recently, Gomes and Woelbroeck [66], using data that allowed them to identify different trades, have empirically showed that there is no permanent impact.

Latent order book

An intuitive idea in order to explain the square-root law is to link the market impact to the shape of the limit order book. If the available volume on the limit order book were growing linearly with the distance from the best price, then the market impact of an order of volume V executed instantaneously would be proportional to \sqrt{V} . More precisely, suppose the available volume for buys² at price P , $v(P)$, is a linear function $c(P - P_0)$ (where P_0 is the best sell price). If a buy order of volume V moves the price by a value MI , then:

$$V = \int_{P_0}^{P_0+MI} v(P) dP,$$

so the market impact is a square-root:

$$MI \approx \sqrt{cV/2}.$$

Unfortunately, this is not what we observe in practice. Indeed, the limit order book has a roughly linear shape only for the first few limits closest to the best price, after, the linear shape is no longer maintained and actually decreases [117].

However, a qualitative explanation for the square-root law is the linearity (or V-shape) of the *latent order book*. More precisely, agents place limit orders only

2. Similar reasoning for the impact of a sell order gives an identical result.

when the actual price is close enough to their vision of the price. From a statical point of view, the *latent order book* consists in orders which are not visible in the order book and only reveal themselves when the price moves closer to their limit price. Toth *et al.* successfully tested the V-shape of the *latent order book* in [128], providing another ingredient to the explanation of the square-root impact law.

Informational vs. mechanical impact

The quantitative finance literature comprises two visions about why meta-orders impact the price: an informational vision, which is favored by economists, and a mechanical vision, shared by econophysicists:

– *Informational impact*

One vision refers to market impact as the way information is conveyed to the market. In this framework, large investors react to new information by trying to successfully forecast price movements. Thus, they update their expectation by using meta-orders, which leads to a new global equilibrium resulting in new price levels. According to this vision, meta-orders reveal the *fundamental value*³ of the price but do not really provoke it. As stated in Hasbrouck [70]: “orders do not impact prices, it is more accurate to say that orders forecast prices”. In particular, if the market participants were to know a trade is coming from a “noise trader” (i.e. no information in his trade), the trade should have no price impact. This vision of the market impact is derived from the neoclassical economic principle that price is always in equilibrium and only new and unexpected information change it.

In Kyle’s model [88], it is shown how prices are driven to their new level through the execution of a meta-order by an informed trader. As we pointed out before, in this framework the market impact is linear in volume and there is no reversion.

Farmer *et al.* [53] consider the case where a large number of investors receive the same information sign α and generate meta-orders drawn from a given distribution $p(\alpha)$. Liquidity providers competitively generate bids and offers to maximize their profits, while noise traders render the market uncertain. Market makers know the distribution $p(\alpha)$, but, because of noise traders, they do not know with certainty whether a meta-order is present, and if it is, they do not know its size. The authors show that these assumptions lead to a Nash equilibrium where meta-order sizes reflect information. This theory is called *fair pricing* because the average price paid to execute a meta-order is roughly equal to the post-reversion price.

– *Mechanical impact*

On the other side, the mechanical vision describes price moves as the result of antagonist forces: a sell pressure driving prices down and a buy pressure driving prices up. The market impact of a meta-order is created by the interaction between the child orders and the limit order book, which leads to a long term imbalance in supply and demand. In this framework, an external observer sees price changes **as if** all traders are acting randomly. The adepts of this vision have a phenomenological ap-

3. See page 9 for definition.

proach: like physicists, they strive to describe the behavior of the price dynamic through consistent quantitative laws.

The finance literature where a mechanical vision of market impact is assumed, often implicitly, is too big to be included here, since each author has his various objectives. Far from being exhaustive, let us cite a few examples of the main topics studied: order book dynamic [55, 125, 39, 40, 80, 128], order arrival [77, 89], meta-order volume distribution [116, 57, 130], impact of order book events [49, 38], microstructure dynamics [17, 18, 64, 65, 122].

As “yin and yang”, these two antagonist visions are complementary: no single one can entirely describe the complexity of price formation, but if used together one can have a large clearer picture of what happens. In Bouchaud’s words: “reality should lie somewhere in the middle”. For example, using the *fair pricing* theory and the empirical observations of Plerou [116] and Gabaix [57] concerning the Pareto distribution of meta-order sizes, Farmer *et al.* [53] provide another explanation for the *square-root* law. Moreover, in the latter, the permanent impact is 2/3 of the peak impact. Recently, Bershova and Rakhlin [27] have found a similar Pareto distribution of the meta-order size and have showed that the fair pricing condition is rather well empirically verified.

An intermediate position is the one of Gomes and Waelbroeck [66], which observe that after several days, the impact for “informed traders” is larger than the one for “uninformed traders”, which tends to be null. In order to empirically prove this result, the authors used brokerage data where clients revealed their reasons for trading, allowing them to distinguish between “informed traders” and “uninformed traders”. At intraday time scales, the impact functions of “informed traders” and “uninformed traders” were found to be very close. Thus, the mechanical vision is more appropriate at the intraday scale, whereas the informational vision is more appropriate at a larger scale (days, weeks).

1.5 THESIS RESULTS

I introduce in this section the thesis results:

- In section 5.1 we propose a new structural model for market impact reproducing the reaction of market makers in front of an order or a series of correlated orders (meta-order) using mutually exciting point process approach. We show the model successfully reproduces important style facts of the market impact: concave shape of transient market impact and convex shape of the decay.
- In section 5.2 we show the model successfully fits the real data. Exploring the database, give me the opportunity to point out some new stylized fact regarding the market impact.
- In section 5.3 we are investigate the non-parametric estimation of the kernel shape in a univariate Hawkes processes proposed by Bacry *et al.* [15]. More precisely, we study the performance of the estimator for the \mathcal{L}^2 loss over Sobolev spaces and some class containing “very” smooth functions.

A new market impact model using self-exciting point process

Price model

As presented in Section 1.3, following Bacry *et al.* [17], we represent the price as the difference between the sum of positive and negative jumps:

$$P(t) = N^+(t) - N^-(t),$$

where $N = (N^+, N^-)$ is a bivariate Hawkes process defined by the two stochastic intensities λ^+ and λ^- :

$$\begin{aligned}\lambda_t^+ &= \mu + \int_0^t \phi(t-s) dN_s^-, \\ \lambda_t^- &= \mu + \int_0^t \phi(t-s) dN_s^+.\end{aligned}$$

Market impact model

We consider the next definition for market impact given in Barra Market Impact Model Handbook [23]:

Market impact denotes the difference between the expected change of price of a given trade (or a series of trade) of given size(s) and sign(s) and the expected change of market price in absence of this specific transaction.

The market impact of a trade or a series of trades that occur at a certain rate between t_0 and $t_0 + T$, for some $T > 0$, will be modeled by a smooth perturbation \mathbb{Q} of the historical probability \mathbb{P} :

$$\mathbb{E}_{\mathbb{Q}}[P_t - P_{t_0}] \text{ for every } t \geq t_0$$

or, in other words, the expected price change $P_t - P_{t_0}$ under \mathbb{Q} , *i.e.* from the point of view of the trader who has prior knowledge that a certain series of aggressive orders will be launched over $[t_0, t_0 + T]$.

For $t_0, T > 0$, let $\mathcal{S}(t_0, T)$ denote the set of functions

$$\mathbf{h} : [0, \infty) \rightarrow [0, \infty) \times [0, \infty)$$

such that $\mathbf{h}(t) = 0$ if $t \notin [t_0, t_0 + T]$ and such that (componentwise)

$$\int_{[t_0, t_0+T]} \mathbf{h}(t) dt < \infty.$$

Definition 3. The market impact measure associated to $\mathbf{h} \in \mathcal{S}(t_0, T)$ is the probability measure

$$\mathbb{P}_{\mathbf{h}} = L_t(\mathbf{h})\mathbb{P} \text{ on } \mathcal{F}_t,$$

where

$$L_t(\mathbf{h}) = \exp\left(\int_0^t \log\left(1 + \frac{h^+(s)}{\lambda_s^+}\right) dN_s^+ - h^+(s) ds + \int_0^t \log\left(1 + \frac{h^-(s)}{\lambda_s^-}\right) dN_s^- - h^-(s) ds\right) \quad (1.19)$$

for $\mathbf{h} = (h^+, h^-)$ and where $\boldsymbol{\lambda} = (\lambda^+, \lambda^-)$ is the stochastic intensity of $N = (N^+, N^-)$ under \mathbb{P} .

Definition 4. The market impact function of $\mathbf{h} \in \mathcal{S}(t_0, T)$ is the function

$$t \rightsquigarrow \mathcal{M}_{\mathbf{h}}(t) = \mathbb{E}_{\mathbb{P}_{\mathbf{h}}}[P_t - P_{t_0}] \text{ for every } t \geq t_0.$$

The transient market impact function of \mathbf{h} is the restriction of the market impact function of \mathbf{h} over $[t_0, t_0 + T]$. The decay market impact function of \mathbf{h} is the restriction of the market impact function of \mathbf{h} over $[t_0 + T, \infty)$. The permanent impact of \mathbf{h} is the limit (if it exists)

$$\mathcal{M}_{\mathbf{h}}(\infty) = \lim_{t \rightarrow \infty} \mathcal{M}_{\mathbf{h}}(t).$$

Using a classical result of the theory of point processes obtain via the Girsanov formula, for any $\mathbf{h} \in \mathcal{S}(t_0, T)$, by 1.19, under $\mathbb{P}_{\mathbf{h}}$, we have

$$\begin{aligned}\lambda_t^+ &= \mu + h^+(t) + \int_0^t \varphi(t-s) dN_s^-, \\ \lambda_t^- &= \mu + h^-(t) + \int_0^t \varphi(t-s) dN_s^+.\end{aligned}$$

This representation enables to obtain an explicit formula for the market impact $\mathcal{M}_{\mathbf{h}}(t)$ ⁴.

Theorem 1. *Work under the stationary condition for Hawkes processes (1.11) which, in our case, is translated by $\|\varphi\|_{\mathcal{L}^1} < 1$ ⁵. For every $\mathbf{h} \in \mathcal{S}(t_0, T)$, we have*

$$\mathcal{M}_{\mathbf{h}}(t) = G_{\mathbf{h}}(t) - \int_{t_0}^t \kappa(t-s) G_{\mathbf{h}}(s) ds, \quad t \geq t_0, \quad (1.20)$$

where

$$G_{\mathbf{h}}(t) = \int_{t_0}^t (h^+(s) - h^-(s)) ds, \quad \forall t \geq t_0$$

and

$$\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{*n}, \quad (1.21)$$

where $f \star g(t) = \int_0^\infty f(t-s)g(s)ds$ denotes the usual convolution product on $L^1([0, \infty), dt)$, $\varphi^{*(1)} = \varphi$ and $\varphi^{*(n)} = \varphi^{*(n-1)} \star \varphi$.

Single trade model

Since a buy order of volume v arriving at time t_0 could instantaneously change the price and following some previous studies [18], it is reasonable to consider that h^+ is “purely” impulsive, i.e.,

$$h^+(t) = f(v)\delta_t(t_0) \quad (1.22)$$

where δ_t stands for the Dirac distribution and

$$f : [0, \infty) \rightarrow [0, \infty)$$

models the influence of the volume v . In practice, a buy order at time t_0 encourages an extra-activity on the bid side too (see Bouchaud *et al.* [118]). Thus, we consider that the market reacts to the newly arrived order as if it triggered an upward jump. Doing so leads to the choice:

$$h^-(t) = cf(v)\varphi(t - t_0). \quad (1.23)$$

Before going further, one remark that the impact of selling order can be modeled using the exact same principles by interchanging h^+ and h^- . Let us point out, $\mathbf{h} = (h^+, h^-) \notin \mathcal{S}(t_0, T)$ for any $T > 0$. In order to circumvent this difficulty, we define the market impact of \mathbf{h} as the limit market impact of a series of function \mathbf{h}_T converging to \mathbf{h} ⁶. Applying Theorem 1 we have an explicit representation (see Chapter 2 for proof):

4. The proof in Chapter 2 Section 2.7

5. Where $\|\cdot\|_{\mathcal{L}^1}$ denotes the usual norm one on $L^1([0, \infty), dt)$

6. More details can be found in Chapter 2 Section 2.5.

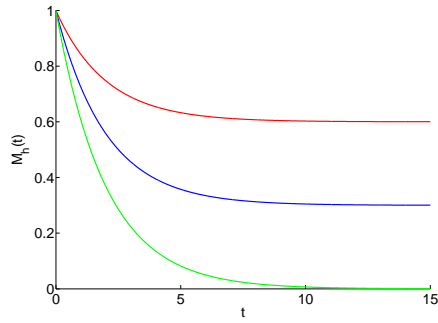


Figure 1.12 – Market Impact of one trade (exponential decay)

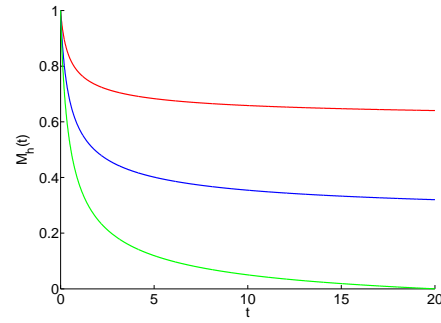


Figure 1.13 – Market Impact of a continuous execution (exponential decay)

Theorem 2. Assume that (\mathbf{h}) is specified by (1.22) and d (1.23). Then the market impact function of a single trade is given by:

$$\mathcal{M}_{\mathbf{h}}(t) = f(v) \left(1 - (1+c) \int_0^{t-t_0} \kappa(s) ds \right), \quad t \geq t_0. \quad (1.24)$$

Under the assumption

$$(A1) \quad \varphi(t) \geq \varphi^{(*2)}(t), \quad \forall t \geq 0,$$

the market impact function $\mathcal{M}_{\mathbf{h}}(t)$ is decreasing on $[t_0, \infty)$. Moreover, the permanent impact of an instantaneous trade is given by

$$\mathcal{M}_{\mathbf{h}}(\infty) = f(v) \frac{1 - c \|\varphi\|_{L^1}}{1 + \|\varphi\|_{L^1}} \quad (1.25)$$

and the following bracketing property holds:

$$\mathcal{M}_{\mathbf{h}}(t) \geq 0 \text{ if and only if } c \leq \frac{1}{\|\varphi\|_{L^1}} \quad (1.26)$$

and for $c = 1/\|\varphi\|_{L^1}$ the permanent market impact is zero, $\mathcal{M}_{\mathbf{h}}(\infty) = 0$. Finally, under the assumption:

$$(A2) \quad \kappa(t) \text{ is decreasing,}$$

the market impact function $\mathcal{M}_{\mathbf{h}}(t)$ is convex over $[t_0, \infty)$.

the parameter c give us the some flexibility in order to obtain different shapes for the market impact function (see Fig. 1.14 and Fig. 1.15). Its interpretation will be discussed in Chapter 2.

The market impact of a continuous strategy

Let us consider the case of the market impact of a meta-order starting at time t_0 , lasting a period T and corresponding to a continuous flow of buying⁷ orders with a trading rate r_t supported by $[t_0, t_0 + T]$ ($r_t \neq 0$ only for $t \in [t_0, t_0 + T]$). It seems natural to consider the class of impact functions $\mathbf{h}(t) \mathcal{S}(t_0, T)$ of the form:

$$\mathbf{h}^c(t) = \left(\tilde{f}(r_t), c \int_{t_0}^t \tilde{f}(r_s) \varphi(t-s) ds \right). \quad (1.27)$$

Seeing the impact of $\tilde{f}(r_s) ds$ as the infinitesimal impact of a buy order of volume $r dt$, the previous model it is a superposition of single trade model.

7. The impact of selling meta-order can be modeled using the exact same principles

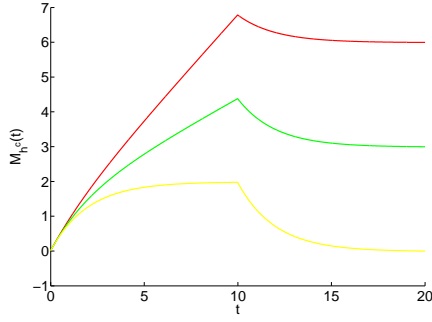


Figure 1.14 – Market Impact of one trade (exponential decay)

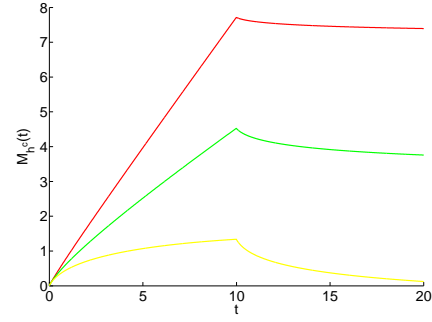


Figure 1.15 – Market Impact of a continuous execution (exponential decay)

Theorem 3. The market impact of an continuous strategy r_t is given by

$$\mathcal{M}_{\mathbf{h}^c}(t) = \int_{t_0}^t \tilde{f}(r_s) H(t-s) ds, \quad t \geq t_0, \quad (1.28)$$

where $H(t) = 1 - (1+c) \int_{t_0}^t \kappa(s) ds$. Under the assumption that \tilde{f} is bounded, the permanent market impact writes

$$\mathcal{M}_{\mathbf{h}^c}(\infty) = \frac{1-C}{\|\varphi\|_{L^1}} \|\tilde{f}(r)\|_{L^1}. \quad (1.29)$$

In the particular case of a constant strategy, $r_t = r_0, \forall t \in [t_0, t_0 + T]$, we have:

$$\mathcal{M}_{\mathbf{h}^c}(t) = \tilde{f}(r_0) 1_{[0, T]} \star H(t), \quad t \geq t_0. \quad (1.30)$$

As a consequence, when $c \leq \frac{1}{\|\varphi\|_{L^1}}$, under the Assumptions (A1) and (A2), the function $t \rightsquigarrow \mathcal{M}_{\mathbf{h}^c}(t)$ is increasing and concave over $[t_0, t_0 + T]$ and decreasing and convex over $[t_0 + T, \infty)$.

The figures 1.14 and 1.15 and represent the market impact function of a constant strategy $r_t = r_0, \forall t \in [0, T]$ with exponential and respectively power-law kernel for different values of c . For the exponential kernel 1.14 we took $\varphi(t) = \alpha e^{-\beta t}$ with $\alpha = 0.2$ and $\beta = 0.3$. For the power-law kernel 1.15 we took $\kappa = \alpha(\delta + t)^{-\beta}$ with $\alpha = 0.1, \beta = 1.5$ and $\delta = 0.25$. The trading rate have no influence on the shape of the transient market impact, it is just a multiplicative constant. We took $\tilde{f}(r_0) = 1$ and $T = 10$.

Market impacts and the life cycle of investors orders

Empirical market impact definition and latent price

The *market impact curve* of the meta-order ω quantifies the magnitude of the (relative) price variation which is **due to the meta-order** ω between the starting time of the meta-order $t_0(\omega)$ and the current time t . Theoretically, in order to estimate this curve one should have access to the price variation that would have occurred if the meta-order was not sent. In the following we shall refer to that latter price as the *latent price*.

Let $\Delta P_t(\omega)$ be a proxy for the realized price variation between time $t_0(\omega)$ and time $t_0(\omega) + t$ and let $\Delta P_t^{(lat)}(\omega)$ be a proxy for the corresponding latent price variation. One can then define the price variation due to the meta-order ω :

$$\Delta P_t^{(\eta)}(\omega) = \Delta P_t(\omega) - \Delta P_t^{(lat)}(\omega), \quad (1.31)$$

When the meta-order is a buy (resp. sell) meta-order, one expects this price to go up (resp. down), consequently it is very natural to define the impact as

$$\eta_s(\omega) = \epsilon(\omega) \Delta P_{sT(\omega)}^{(\eta)}, \quad s \geq 0, \quad (1.32)$$

where $T(\omega)$ represents the total duration of the meta-order ω and $\epsilon(\omega)$ stands for the sign of the meta-order: +1 for a buy and -1 for a sell. Let us remark that, for synchronicity convenience, we have rescaled the time in order that the time $s = 1$ always corresponds to the ending-time of the meta-order. All along this section, we will use the so-called *return proxy*⁸ defined by (1.33) in which P_t corresponds to the last traded price:

$$\Delta P_t^{(ret)}(\omega) = \frac{P_{t_0(\omega)+t} - P_{t_0(\omega)}}{P_{t_0(\omega)}} \quad (1.33)$$

Finally, we introduce the following terminology in order to address the different parts of the market impact curve:

- $\eta_{s=1}(\omega)$ as the *temporary market impact*, i.e., the impact at the end of the meta-order,
- $\{\eta_s(\omega)\}_{0 \leq s \leq 1}$ as the *transient market impact curve*, the impact curve during the execution of the meta-order,
- $\{\eta_s(\omega)\}_{1 \leq s}$ as the *decay market impact curve*, the impact curve after the execution of the meta-order and
- $\eta_{s \gg 1}(\omega)$ as the *permanent impact*, where $s \gg 1$ refers to an extraday time limit, sufficiently far from the ending-time of ω (more precise definition is given in Chapter 3 Section 3.7).

In the following, we shall compute the estimation of the impact through

$$\hat{\eta}_s = \frac{1}{\#\Omega} \sum_{\omega \in \Omega} \Delta P_{sT(\omega)}(\omega), \quad (1.34)$$

where $\#\Omega$ denotes the total number of available meta-orders.

When one studies the influence on the impact of a factor $X(\omega)$ (e.g., the daily participation rate or trading time), we shall condition the impact estimation by the fact that the value of $X(\omega)$ belongs to an interval I , i.e.,

$$\hat{\eta}_s(\omega \mid X(\omega) \in I) = \frac{1}{M(I)} \sum_{\omega, X(\omega) \in I} \Delta P_{sT(\omega)}(\omega) \quad (1.35)$$

where $M(I)$ is the number of meta-orders $\omega \in \Omega$ (or $\in \Omega^{(H)}$ or $\in \Omega^{(F)}$, when specified) such that $X(\omega) \in I$.

Why is the market impact estimation so difficult

There are mainly two reasons why estimation is very difficult and we will put them both in evidence.

- One has not access to the latent relative price $\Delta P_t^{(lat)}(\omega)$. The only observable quantities that can be used for estimating the impact are the relative price variations $\Delta P_t(\omega)$
- The variance of the impact is very large as compared to its mean, i.e., the signal noise/ratio is very low.

8. In Chapter 3 we show that other proxys give very similar market impact curves.

In order to circumvent these difficulties, most academic works follow the same path:

1. They (most of the time implicitly) :
 - i. either make the assumption that the time $t_1(\omega)$ at which the meta-order ω is placed is independent from the latent relative price $\Delta P_t^{(lat)}(\omega)$,
 - ii. or consider that the eventual dependence do not affect significantly the results obtained when using $\Delta P(\omega)$ as a proxy for $\Delta P^{(\eta)}(\omega)$ in the impact definition (3.3)
2. They compute some averages of $\eta_t(\omega)$ on a large number of (more or less "comparable") meta-orders ω which leads to reducing the variance of the "noise" term $\Delta P_t^{(lat)}(\omega)$ (in the case of assumption 1.i.) as well as the variance of the impact itself.

The assumption 1.i. in most cases does not hold : many agents (e.g. trend-followers) place their meta-orders at times where some particular patterns are seen in the latent price. Actually, it does not seem to be a real problem if one is "only" interested in characterizing the dependence of the impact curve on some factors, as it is generally the case when studying the temporary market impact of the transient/decay market impact curve. In that case, the alternative assumption 1.ii. seems reasonable. However, for addressing questions relative to the presence or not of permanent impact, i.e., for estimating "absolute" levels of impact, this assumption is clearly not acceptable.

Figure 1.16 shows a typical market impact curve. We used all meta-orders ω such that $t_0(\omega) + 2T(\omega)$ is a time that takes place before the closing time of the corresponding asset, i.e., 61.671 meta-orders. We observe that during the execution, i.e., η_s , $0 \leq s \leq 1$, the price is pushed in the adverse direction making it less attractive as time goes by. The price reaches the maximum distance from the arrival price at the end of the execution. The difference between the two is the so-called temporary impact. After the execution a reversal effect is noticed. This is the relaxation part when the price converges back toward its future permanent level.

In practice, we have sample our estimations on 201 points using a uniform sampling grid for s : $s_i = i/100$, $\forall i \in [0;200]$ and compute the estimation $\hat{\eta}_{0 \leq s \leq 2}$ as defined by (1.34). We express $\hat{\eta}$ in basis points (bp). The next figures (i.e., 1.17, 1.18 and 1.19) show how important is the variance. Figure 1.17 represents the ratio mean/variance for the precedent market impact curve, i.e., $\hat{\eta}/\mathcal{V}(\eta_s(\omega))$. In figure 1.18 is represented the 25% and 75% quantile for the market precedent impact curve. In figure 1.19 we can see the 95% confidence interval for the market impact curve. In order to build this confidence interval, we make the assumption that the time $t_0(\omega)$ at which the meta-order ω is placed is independent from the latent relative price $\Delta P_t^{(lat)}(\omega)$,

$$\eta_s = \eta_s(\omega) + \zeta_s(\omega), \forall s \in [0,2],$$

where $\zeta_s(\omega)$ are supposed to be independent and identically distributed random variables $\forall s \in [0,2]$. Thus, the 95% confidence interval market impact functions write:

$$[\hat{\eta}_s \pm 1.96\sqrt{\mathcal{V}(\eta_s(\omega))}/\sqrt{61.671}]. \quad (1.36)$$

As we mentioned before, we take $\Delta P_t(\omega)$ instead of $\Delta P_t^{(lat)}(\omega)$ in order to build our market impact curves (eq. 1.34). Our next figures 1.20 and 1.21 show

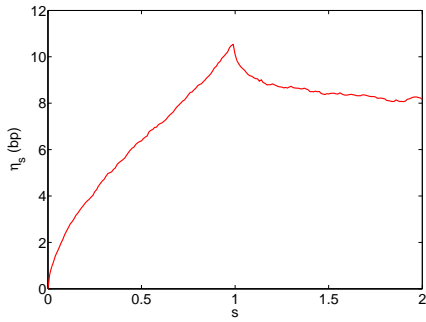


Figure 1.16 – Typical market impact curve

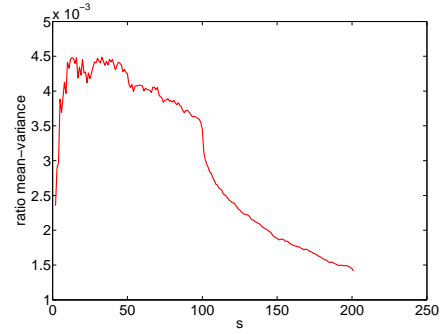


Figure 1.17 – Ratio mean-variance $\hat{\eta}/V(\eta_s(\omega))$ for the market impact curve from Figure 1.16.

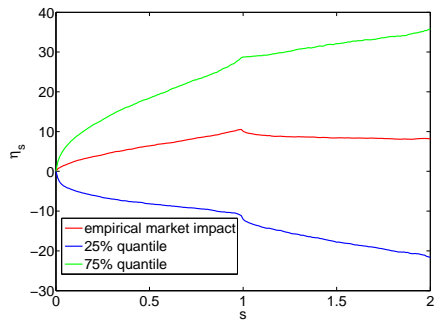


Figure 1.18 – The 25% and 75% quantile for the market impact curve from Figure 1.16.

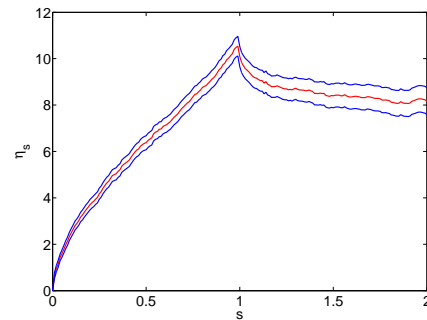


Figure 1.19 – The 95% confidence interval (see eq. (1.36)) for the market impact curve from Figure 1.16.

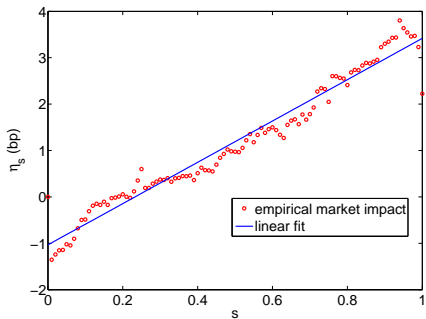


Figure 1.20 – Market impact curve with very low trading speed

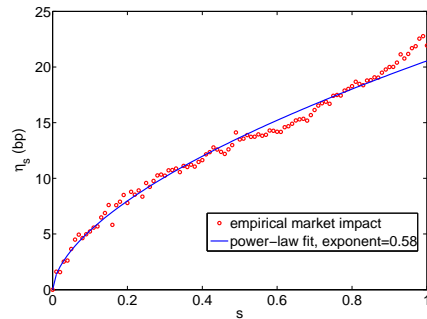


Figure 1.21 – Market impact curve with high low trading speed

that $\Delta P_t^{(lat)}(\omega)$ can be non-negligible. The first figure 1.20, represent the market impact curve for meta-orders with an total duration $T \in [30, 60]$ (T expressed in minutes) and a very low trading speed $\dot{v} = r_D/T$, where $r_D(\omega)$ represent the daily participation,

$$r_D(\omega) = \frac{\text{total volume of the meta-order } \omega}{\text{the daily volume executed on the market}}.$$

The second figure 1.21 represents the market impact curve for meta-orders with an total duration $T \in [30, 60]$ and a very high trading speed. Both market impact curves are built with approximately 2.000 meta-orders.

We observe that the market impact curve where the trading speed is very low the transient market impact is almost linear in time and the ending point is

about 4 basis points. The second market impact curve where the *trading speed* is very high is also very concave and the ending point is about 20 basis points. The first group of meta-orders are barely perceptible on the market since the *trading speed* whereas the second group of meta-orders are easily perceptible on the market by others traders. The most choking part is the linearity of the first group of meta-orders since all empirical studies found a concave transient market impact curve (see [27, 104]). The only explanation is that in the first case the latent relative price $\Delta P_t^{(lat)}(\omega)$ it is non-negligible, even more important than the $\Delta P_t^{(\eta)}(\omega)$.

The temporary market impact

The temporary market impact has been mainly studied in the quantitative finance literature [10, 60, 51, 27, 101, 88, 72, 54, 49]. One common point of these studies is that the temporary market impact of a meta-order ω of size $v(\omega)$ includes three main components :

- A component reflecting the *size of the meta-order*, resized by something reflecting the volume in the order books of the traded security. The daily participation $r_D(\omega)$ should capture most of the dynamics of this component. This size has to be rebased using the daily volume V_D or the traded market volume during the meta-order V , leading to two potentially explanatory variables $r_D := v/V_D$ or $r := v/V$, where v is the meta-order volume.
- A component rendering the *uncertainty on the value of the traded underlying* during the meta-order. The volatility during the meta-order $\sigma(\omega)$ (defined by $\sigma(\omega) = \frac{\sigma_D(\omega)}{\sqrt{T(\omega)}}$) is a typical measure for this, where σ_D stands for the Garman-Klass annualized daily volatility.
- And a last component that captures the *information leakage* generated by the meta-order, a good proxy being the duration $T(\omega)$.

We accordingly define the four potential explanatory variables $X_1 = r_D(\omega)$, $X_2 = r(\omega)$, $X_3 = \sigma(\omega)$ and $X_4 = T(\omega)$. Let us point out that all authors found multiplicative relations between each of these components and their corresponding factor, so we expect a linear dependence of the temporary market impact $\eta_{s=1}(\omega)$ on the logarithm of these factors.

We tested the daily participation r_D as first variable, since it has been identified as significant by other papers. It means we fit equation:

$$\eta_1(\omega) = a \cdot r_D(\omega)^\gamma + \epsilon(\omega) W_T,$$

and found an exponent $\gamma \simeq 0.449$ using the L^2 distance and a lower exponent (around 0.40) using the LASSO one. Table 1.1 gives the results of this regression and fits of other explanatory variables.

The transient market impact curve

Concavity and execution duration. We shall now study the influence of the duration T of the meta-order on the transient market impact. We chose meta-orders with the trading rate $r_D \in I_r = [1\%, 3\%]$ and we study studied $\hat{\eta}_{s \leq 1}(\omega \mid T(\omega) \in I_r, r_D(\omega) \in I_r)$ as a function of the interval I_T . We chose 6 intervals for I_T such that the number of meta-orders in each interval is approximately the same (the duration are expressed in minutes): $T \in [3, 15]$, $T \in [15, 30]$,

Regression	parameter	value (log-log)	value (L2)	value (LASSO)
(R,0)	Daily participation	0.542	0.449	0.400
	Trading rate	0.435	0.330	0.426
(R,2)	Daily participation	0.529	0.529	0.529
	volatility	0.961	0.961	0.961
(R,3)	Daily participation	0.401	<i>nan</i>	0.401
	Trading rate	0.285	<i>nan</i>	0.285
(R,4)	Trading rate	0.317	0.317	0.317
	volatility	0.878	0.878	0.878
(R,5)	Daily participation	0.593	0.541	0.593
	T	-0.230	-0.347	-0.230
(R,6)	Trading rate	0.319	<i>nan</i>	0.319
	spread	0.571	<i>nan</i>	0.571
(R,7)	Daily participation	0.438	0.438	<i>nan</i>
	spread	0.276	0.276	<i>nan</i>
(R,8)	Trading rate	0.369	0.561	0.453
	T	0.152	0.241	0.229

Table 1.1 – Main results for studying the influence of the different factors on the temporary market impact.

$T \in [30, 60]$, $T \in [60, 90]$, $T \in [90, 300]$ and $T \in [300, 510]$, each containing around 6.000 meta-order occurrences. We then compute

$$\hat{\eta}_{s \leq 1}(\omega \mid T(\omega) \in I_T, r_D(\omega) \in I_r) = \frac{1}{M(I_T, I_r)} \sum_{\omega, T(\omega) \in I_T, r_D(\omega) \in I_r} \Delta P_{sT(\omega)}(\omega), \quad (1.37)$$

where $M(I_T, I_r)$ the number of meta-orders ω such that $T(\omega) \in I$ and $r_D(\omega) \in I_r$. In order to point out the different regimes, on each so-obtained graph, we performed the power-law fit

$$\hat{\eta}_{s \leq 1}(\omega \mid T(\omega) \in I, r_D(\omega) \in I_r) \propto s^{\beta^{(tr)}} \quad (1.38)$$

leading to an estimation of the power-law exponent $\beta^{(tr)}$.

We observe that market impact is actually a multi-regime process. The first five plots of Fig. 1.22 - 1.26 show clearly that, for a fixed participation rate when the duration of a meta-order decreases

- the transient market impact of a meta-order increases and
- the curvature decreases leading to an almost linear transient market impact for small durations.

Fig. 1.26 seems to show that a kind saturation is reached before the end of the meta-order. Actually Fig. 1.27 surprisingly shows that when the duration T

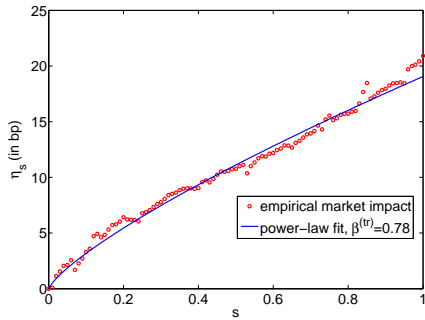


Figure 1.22 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [3, 15]$, $\beta^{(tr)} \approx 0.80$

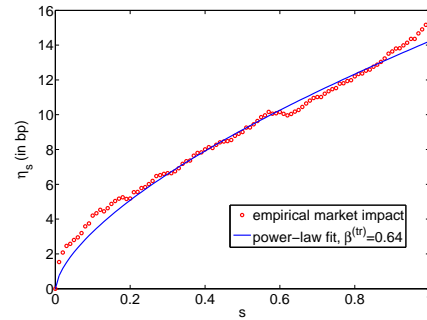


Figure 1.23 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [15, 30]$, $\beta^{(tr)} \approx 0.66$

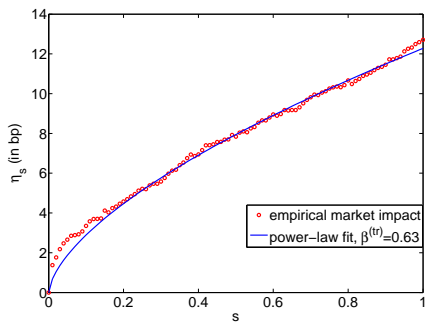


Figure 1.24 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [30, 60]$, $\beta^{(tr)} \approx 0.63$

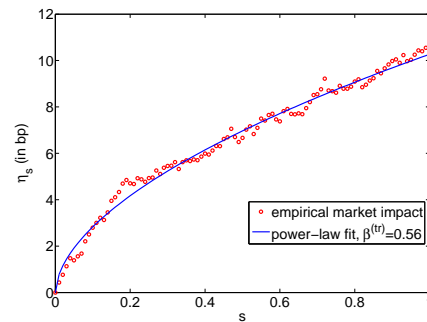


Figure 1.25 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [60, 90]$, $\beta^{(tr)} \approx 0.56$

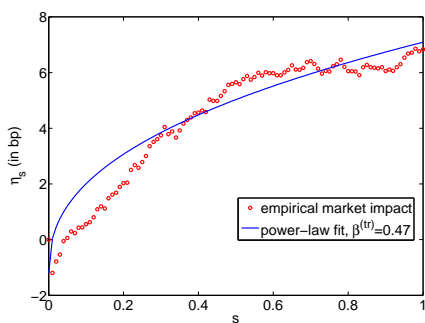


Figure 1.26 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [90, 300]$, $\beta^{(tr)} \approx 0.53$

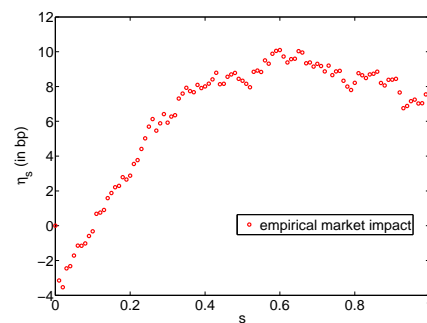


Figure 1.27 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [300, +\infty)$

becomes very large, the market impact curve starts decaying *before* the end of the meta-order.

Market prediction of meta-order sizes. In this paragraph, we want to study whether the market has or has no precise insights about the total size of a given meta-order before the end of its execution (apart of course from the unconditional distribution of the meta-order sizes). In order to do so, we consider all the meta-orders with a given *trading speed* $\dot{v} = r_D/T$. Of course, these meta-orders correspond to different execution durations $T(\omega)$, and their volume $v(\omega)$ is basically proportional to their execution duration: $v(\omega) = T(\omega) \dot{v}/V_D$. If the market does not have precise insights about $v(\omega)$ (i.e., about $T(\omega)$) then, at a given time t_0 , there is no way it can differentiate between two meta-orders ω_1 and ω_2 such that $\dot{v}(\omega_1) = \dot{v}(\omega_2) = \dot{v}$ and $T(\omega_1) > t_0$, $T(\omega_2) > t_0$. Consequently the corresponding transient market impact curves should look alike on the time interval $[0, t_0]$.

This assertion above can be translated into the fact that the restrictions on $t \in [0, t_0]$ of $\eta_{t/T(\omega_1)}(\omega_1)$ and of $\eta_{t/T(\omega_2)}(\omega_2)$ should be very close. In order to test this assertion, we choose five groups of meta-orders \mathcal{A}_i ($i = 1, \dots, 5$) such that:

$$\mathcal{A}_i = \left\{ \omega \in \Omega^{(tr)} : r_D \omega \in [2^{i-1} r_0, 2^i r_0) \text{ and } T(\omega) \in [2^{i-1} T_0, 2^i T_0) \right\}, \quad (1.39)$$

where $r_0 = 0.25$ and $T_0 = 5$ minutes. Thus, all the selected meta-orders correspond, in a good approximation, to the same trading speed $\dot{v} = r_0/T_0 = 0.833 * 10^{-4} s^{-1}$. Moreover \mathcal{A}_{i+1} ($\forall i \in [2, 5]$) corresponds to meta-orders with durations twice as large as those of \mathcal{A}_i . We then compute:

$$\hat{\eta}_{s \leq 1}^{(i)}(\omega \in \mathcal{A}_i) = \frac{1}{M(\mathcal{A}_i)} \sum_{\omega \in \mathcal{A}_i} \Delta P_{sT(\omega)}(\omega), \quad (1.40)$$

where $M(\mathcal{A}_i)$ is the number of meta-orders in \mathcal{A}_i . For each $i = 1, \dots, 4$, Fig. 1.28 - 1.31 show $\hat{\eta}_{s \leq 1}^{(i)}$ with the first half of $\hat{\eta}_{2s \leq 2}^{(i+1)}$. One can see that, in each of the subplots, the two market impact curves are very close, indicating that the market basically does not anticipate the size of the corresponding meta-orders. Each subplot corresponds to $\hat{\eta}_{s \leq 1}^{(i)}$ and $\hat{\eta}_{2s \leq 2}^{(i+1)}$ for $s \in [0, 1]$ (see (1.40)). Top-left (resp. top-right) subplot corresponds to $i = 1$ (resp. $i = 2$) and bottom-left (resp. bottom-right) subplot corresponds to $i = 3$ (resp. $i = 4$). In each of the subplots, the two curves are very close, indicating that the market basically does not anticipate the meta-order size.

The decay market impact curve

The existing empirical literature of decay meta-orders market impact is limited ([104], [27]) since the difficulty of obtaining data is very high. In the first study, Moro *et al.* are the first showing a decay of the impact to a level roughly equal to 0.5 – 0.7 of its highest peak. In the second study, Bershova and Rakhlin show the decay is a two-regime process: slow initial power decay followed by a faster relaxation. Following the same lines as before, we compute the estimation (see (1.37))

$$\hat{\eta}_{s \leq 2}(\omega \mid T(\omega) \in I_T, r_D(\omega) \in I_r) = \frac{1}{M(I_T, I_r)} \sum_{\omega, T(\omega) \in I_T, r_D(\omega) \in I_r} \Delta P_{sT(\omega)}(\omega). \quad (1.41)$$

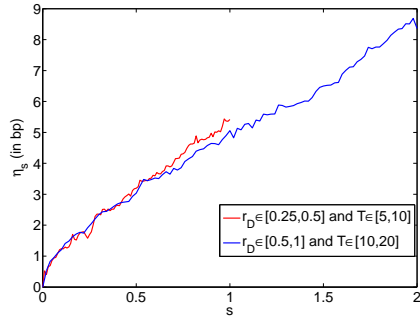


Figure 1.28 – The market impact curves $\hat{\eta}_{s \leq 1}^{(1)}$ and $\hat{\eta}_{2s \leq 2}^{(2)}$

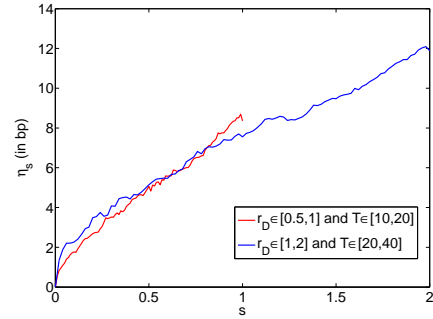


Figure 1.29 – The market impact curves $\hat{\eta}_{s \leq 1}^{(2)}$ and $\hat{\eta}_{2s \leq 2}^{(3)}$

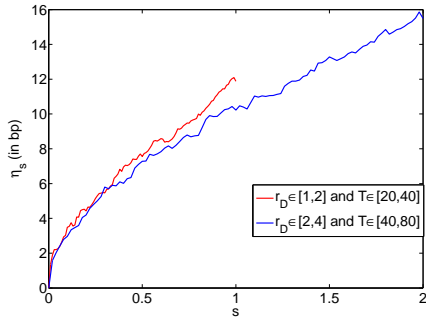


Figure 1.30 – The market impact curves $\hat{\eta}_{s \leq 1}^{(3)}$ and $\hat{\eta}_{2s \leq 2}^{(4)}$

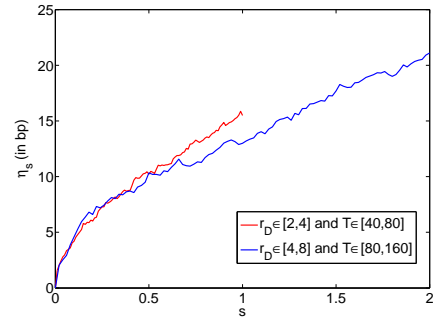


Figure 1.31 – The market impact curves $\hat{\eta}_{s \leq 1}^{(4)}$ and $\hat{\eta}_{2s \leq 2}^{(5)}$

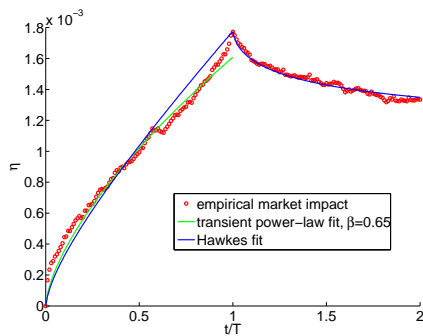


Figure 1.32 – Transient and decay market impact curve estimations $\eta_{s \leq 2}$ for $r_d \in I_R = [1\%, 3\%]$ and $T \in I_T = [15, 30]$.

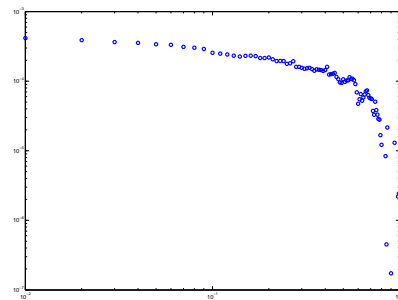


Figure 1.33 – Log-log plot du decay market impact curve estimations $\eta_{s \leq 2}$ for $r_d \in I_R = [1\%, 3\%]$ and $T \in I_T = [15, 30]$.

No permanent market impact at daily scale

As we explained in Section 1.4, the debate permanent/non-permanent market impact is not closed. In the picture of [53] and of [27] permanent market impact is important and roughly equals to 2/3 of the temporary impact on an order by order basis. In the picture of [33], there is no such thing as permanent impact. The author argues that what is called permanent market impact is a consequence of the long memory of the sign of the meta-orders flow. This picture is incompatible with the permanent market impact hypothesis because long memory of order flow would result in trending stock prices and thus contradicting market efficiency. We pretend here to conciliate the two positions of [53] and [33] even if at first sight they seem pretty much incompatible. In [53] the long memory of the order flow is not taken in consideration and thus it is shown that the so-called total effect is non-negligible. On the other hand, in [33] it is taken into consideration and the total effect is described as an effect of other correlated meta-orders being executed on the same side. Eventually the relaxation taking place at the end of the meta-order execution is an averaging effect. As previously stated, we consider that the permanent market impact is the additional price movement after the execution of a meta-order over the price that would be in the absence of the meta-order.

In [66], the authors know some of their meta-orders are “cash trades” (i.e. without any information on price moves; in their case $\mathbb{E}(\epsilon(\omega)W_t) = 0$, to come back to equation (3.4)), and find these cash trades have no permanent impact. They perform the same analysis of the subset of their meta-orders, and find they have a permanent effect, obtaining curves similar to the red one of Figure 1.34.

We intend to remove the informational part of the price move by assuming the all our meta-orders are coming from large institutional investors whiwh are trying to “capture some β ”, in the sense of the CAPM. The reader should keep in mind that in [37], authors succeeding in removing most of the information content of their meta-orders, using their anticipation of the α made by Capital Fund Management at the initiation of the meta-order.

We have a similar approach, adding the assumption the initiation of meta-orders by institutional investors in our database has been caused by an anticipation of the β of market moves.

Our results are compatible with Capital Fund Management meta-orders cleaned from α and with the cash trades of [66]: we found no permanent effect once the informational content associated to the meta-orders. Our methodology is described in Chapter 3 Section 3.7.

Estimation of Hawkes kernel

Recently, Bacry *et al.* [15] proposed a non-parametric estimation method for multivariate symmetric Hawkes processes, based on Fourier computations and Bartlett specter theory. In the following, we are investigating the convergence speed of this method in the case of an univariate counting process, i.e. such that its intensity λ_t , satisfies:

$$\lambda_t = \mu + \int_{-\infty}^t \varphi(t-s) dN_s, \quad (1.42)$$

where $\mu > 0$ is the exogenous intensity and φ is a positive function with support on $(0, \infty)$ called decay kernel.

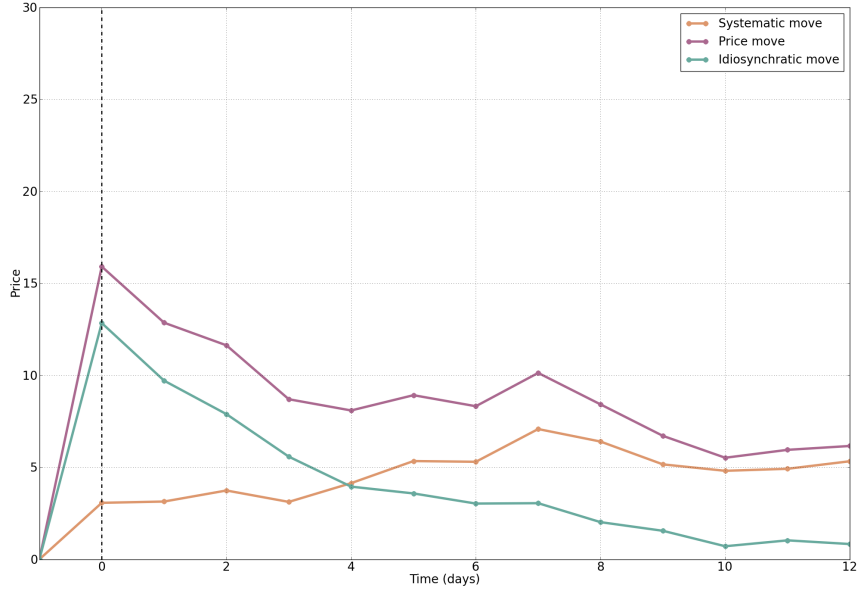


Figure 1.34 – Post-execution profile without the impact of other meta-orders. Price moves are considered relatively to close price the day before execution (units = basis points).

Notation

Let us consider a univariate Hawkes process $(N_t)_{t \in \mathbb{R}_+}$ with stationary increments described by (1.42). The conditional intensity $(\lambda_t)_{t \in \mathbb{R}_+}$ has itself stationary increments with mean:

$$\Lambda = \mathbb{E}[\lambda_t].$$

Taking expectation of both sides of (1.42) allow us to compute Λ under the stationary hypothesis:

$$\Lambda = \mu + \ell \Lambda.$$

This equation has a meaningful solution only under the condition, $\ell < 1$:

$$\Lambda = \frac{\mu}{1 - \ell}.$$

The estimation procedure of φ proposed by Bacry *et al.* in [15] is based on the empirical computation of the auto-covariance function of the Hawkes process at scale h and lag t , $v^{(h)}(t)$:

$$v^{(h)}(t) = \frac{1}{h} \text{Cov}(N_u - N_{u-h}, N_{u+t} - N_{u+t-h}),$$

hich does not depend on u because the increments of N_t are stationary. The authors showed in the later cited paper that $v^{(h)}$ can be expressed as a function of φ and $g^{(h)}$, where $g^{(h)}(t) = (1 - |t|/h)^+$:

$$v^{(h)}(t) = \Lambda g^{(h)}(t) + \Lambda g^{(h)} \star \psi(t) + \Lambda g^{(h)} \star \tilde{\psi}(t) + \Lambda g^{(h)} \star \tilde{\psi} \star \psi(t), \quad (1.43)$$

where \star stands for the convolution product, $\tilde{\psi}(t) = \psi(-t)$ and ψ is defined as:

$$\psi(t) = \sum_{n=1}^{\infty} \varphi^{(\star n)}(t) \quad (1.44)$$

where $\varphi^{(\star n)}$ refers to the n -th auto-convolution of φ . In the Fourier domain (1.43) and (1.44) become:

$$\mathcal{F}[v^{(h)}] = \Lambda \mathcal{F}[g^{(h)}] |1 + \mathcal{F}[\psi]|^2 \quad (1.45)$$

and

$$\mathcal{F}[\psi] = \frac{\mathcal{F}[\varphi]}{1 - \mathcal{F}[\varphi]}. \quad (1.46)$$

where $\mathcal{F}[f]$ is the Fourier transform of f :

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} e^{-i\omega t} f(t) dt, \quad \forall \omega \in \mathbb{R} \quad (1.47)$$

Knowing $v^{(h)}$, we are able to compute $|1 + \mathcal{F}[\psi]|^2$ since the function $g^{(h)}$ is known (see equation 1.45). The central point of the estimation is the Paley-Wiener theorem which allows to recover $\mathcal{F}[\psi]$. But in the Fourier domain, $\mathcal{F}[\psi]$ corresponds to a unique $\mathcal{F}[\varphi]$ (through equation 1.46). Knowing $\mathcal{F}[\varphi]$ one can easily find our decay kernel φ .

The Paley-Wiener theorem allows to recover a filter knowing the filter's amplitude under some hypotheses (see [114] for more details). In our case, we can retrieve $1 + \mathcal{F}[\psi]$ knowing $R = |1 + \mathcal{F}[\psi]|$ and using the Hilbert transform:

$$1 + \mathcal{F}[\psi] = Re^{-i\mathcal{H}[\log R]},$$

where the Hilbert transform, $\mathcal{H}[\cdot]$, is defined as:

$$\mathcal{H}[f](t) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|u| > \epsilon} \frac{f(t-u)}{u} du, \quad \forall f \in \mathcal{S}(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space.

Construction of the estimator

Assume that we have the access of the all jump times t_1, t_2, \dots, t_n of the Hawkes process on the interval $[0, 2T]$. Let us summarize the different steps for the decay kernel estimation:

1. Estimate the average intensity Λ :

$$\hat{\Lambda}_T = \frac{N_T}{T}. \quad (1.48)$$

and set $\hat{\Lambda}_T^\star = \max(\mu_0/(1 - \ell_0), \hat{\Lambda}_T)$.

2. Set $h > 0$ "small enough" and estimate the auto-covariation function $v^{(h)}(t)$ for $t \in [0, \alpha_T]$ (α_T will be optimally chosed):

$$\hat{v}_T^{(h)}(t) = \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} (\Delta N(ih) - h\hat{\Lambda}_T^\star)(\Delta N(ih+t) - h\hat{\Lambda}_T^\star), \quad (1.49)$$

where $\Delta N(ih) = N(ih) - N((i-1)h)$. For $t \in [-\alpha_T, 0]$ put $\hat{v}_T^{(h)}(t) = \hat{v}_T^{(h)}(-t)$ and for $t \notin [-\alpha_T, \alpha_T]$ put $\hat{v}_T^{(h)}(t) = 0$. We will need the mass of $v^{(h)}$ to be concentrated on $[-\alpha_T, \alpha_T]$. Since $v^{(h)}$ has an exponential decrease, α_T will be a function of $\log T$.

3. Decompose the function $\hat{v}_T^{(h)} - \hat{\Lambda}_T^\star g^{(h)}$ on a Fourier base $(f_n)_{n \geq 0}$ on the interval $[-\alpha_T, \alpha_T]$.

$$\hat{c}_{n,T}^{(h)} = \frac{1}{2\alpha_T} \int_{-\alpha_T}^{\alpha_T} (\hat{v}_T^{(h)}(t) - \hat{\Lambda}_T^\star g^{(h)}(t)) f_n(t) dt, \quad (1.50)$$

where $f_n(t) = \exp(-i2\pi nt/\alpha_T)$.

4. Approximate the Fourier transform of $v^{(h)}$ by

$$\mathcal{F}[\hat{v}_T^{(h)}] = \hat{\Lambda}_T^* \mathcal{F}[g^{(h)}] + \sum_{-L}^L \hat{c}_{n,T}^{(h)} \mathcal{F}[f_n],$$

where L is a positive integer which will be optimally chose later on.

5. Set $\hat{R}_T^{*(h)} = \max\{\hat{R}_T^{(h)}, 1/(1 + \ell_0)\}$, where:

$$\hat{R}_T^{(h)}(\omega) = \begin{cases} \sqrt{\frac{|\mathcal{F}[\hat{v}_T^{(h)}](\omega)|}{\hat{\Lambda}_T^* \mathcal{F}[g^{(h)}](\omega)}}, & \text{if } |\omega| \leq \frac{\pi}{h} \\ 1 & \text{if } |\omega| > \frac{\pi}{h} \end{cases} \quad (1.51)$$

6. Compute the Hilbert transform of $\log(\hat{R}_T^{*(h)})$:

$$\hat{\xi}_T^{(h)} = \mathcal{H}[\log \hat{R}_T^{*(h)}] \quad (1.52)$$

7. Compute the Fourier transform of φ using the Paley-Wiener theorem:

$$\mathcal{F}[\hat{\varphi}_T^{(h)}] = 1 - \frac{e^{i\hat{\xi}_T^{(h)}}}{\hat{R}_T^{*(h)}} \quad (1.53)$$

8. Invert $\mathcal{F}[\hat{\varphi}_T^{(h)}]$:

$$\hat{\varphi}_T^{(h)} = \mathcal{F}^{-1}\left[1 - \frac{e^{i\hat{\xi}_T^{(h)}}}{\hat{R}_T^{*(h)}}\right]$$

From a theoretical point of view, the estimation method works for any h . But $\mathcal{F}[g](\omega) = (4/\omega^2 h) \sin^2(\omega h/2)$ cancels for all $\omega = 2n\pi/h, n \in \mathbb{Z}, n \neq 0$ and it is not straightforward how to invert (4.10). So, as long as h is “small enough”, we compute \hat{R} only on the $[-\pi/h, \pi/h]$ interval (step 5 of our estimator construction). Consequently, we need to have the mass of $|1 + \mathcal{F}[\psi]|$ concentrated on the $[-\pi/h, \pi/h]$ interval, so we need h “small enough”. In order to obtain \hat{R}_T^* in equation (4.21), we divide by $\mathcal{F}[g^{(h)}]$, which is close to h on the $[-\pi/h, \pi/h]$ interval. So we need h “large enough”. The last two constraints on h forced us to choose h in order to obtain our best convergence rate.

Main results

We describe the smoothness of a function by the number of times it is differentiable. In order to extend the notion of differentiability to non-integer values, we use Sobolev spaces on \mathbb{R} defined by:

$$\mathcal{W}^s = \left\{ f \in \mathcal{L}^2(\mathbb{R}), \int_{\mathbb{R}} (1 + \omega^2)^s |\mathcal{F}[f](\omega)|^2 d\omega < \infty \right\},$$

equipped with the norm:

$$\|f\|_{\mathcal{W}^s}^2 = \int_{\mathbb{R}} (1 + \omega^2)^s |\mathcal{F}[f](\omega)|^2 d\omega.$$

For technical reasons, we are not able to accurately control the behavior of the estimator if μ tends to 0 or infinity, but also if ℓ tends to 1 or $\|\varphi\|_\infty$ tends to infinity. In such cases, the number of point in the process is either exploding or vanishing. Since our estimator of $v^{(h)}$ will be computed only on a finite interval I (from practical viewpoint), we need an assumption on φ ensuring the mass of

$v^{(h)}$ on the interval I^c is unimportant. This could be done by assuming φ has a compact support. Let us define for all positive real numbers $0 < \mu_0 < \mu_1$, $0 < p_0 < 1$, ϕ and A the following subsets of $\mathbb{R} \times \mathcal{W}^s$ and $\mathbb{R} \times \mathcal{A}^{\gamma,r}$:

$$\mathcal{W}_{\mu_0, \mu_1, \ell_0, \phi, A}^s = \left\{ (\mu, f) \in [\mu_0, \mu_1] \times \mathcal{W}^s : \text{supp}(f) \subset [0, A], \int_0^A f(t) dt \leq \ell_0, f(t) \leq \phi \ \forall t \in [0, A] \right\}.$$

Theorem 4. *We work under the assumption $\|\varphi\|_{\mathcal{L}^1} < 1$. If (μ, φ) belongs to $\mathcal{W}_{\mu_0, \mu_1, \ell_0, \phi, A}^s$, our estimator $\hat{\varphi}_T^{(h)}$ of the decay kernel φ satisfies:*

$$\mathbb{E}[\|\varphi - \hat{\varphi}_T^{(h)}\|_2^2] \lesssim T^{-\frac{2s-1}{2s+1}},$$

where \lesssim means inequality up to a constant that depends on $A, \mu_0, \mu_1, \ell_0, \phi$, and M only.

One can observe that the convergence rate is not optimal. This is a shortcoming of the estimation method. The problem appears in step 5, when we divide by $\mathcal{F}[g]$. As we stressed before, since $\mathcal{F}[g]$ cancels for all $\omega = 2n\pi/h, n \in \mathbb{Z}, n \neq 0$, we compute quantity \hat{R} only on the interval $[-\pi/2h, \pi/2h]$. This gives us our worse bound L/Th which derived from the variance of the coefficients \hat{c}_n . We can show that $\mathbb{E}[|\hat{c}_n - c_n|^2] \lesssim h/T$ and this is the best we can hope. Thus, $\mathbb{E}[|\hat{v}(t) - v(t)|^2]$ has, at least, an order of h/T . In this condition, it is hard to believe the inequality $\mathbb{E}[|\hat{c}_n - c_n|^2] \lesssim h/T$ is suboptimal.

At the end of 2000, B.Y. Levit was interested in the estimation of “very” regular functions, the $\mathcal{A}^{\gamma,r}$ class containing all functions f such that:

$$\|f\|_{\mathcal{A}}^2 := \int_{\mathbb{R}} e^{2|\gamma t|^r} |\mathcal{F}[f](t)|^2 dt < \infty.$$

In a series of papers (see for example [12], [11]), he showed that under different statistical models, the convergence rate of estimators is the best we can hope, T^{-1} (where T is the asymptotic parameter). From this point of view, our estimation method is good enough:

Theorem 5. *If $\varphi \in \mathcal{A}_{\mu_0, \mu_1, \ell_0, \phi, A}^{\gamma,r}$ where*

$$\mathcal{A}_{\mu_0, \mu_1, \ell_0, \phi, A}^{\gamma,r} = \left\{ (\mu, f) \in [\mu_0, \mu_1] \times \mathcal{A}^{\gamma,r} : \text{supp}(f) \subset [0, A], \int_0^A f(t) dt \leq \ell_0, f(t) \leq \phi \ \forall t \in [0, A] \right\},$$

our estimator $\hat{\varphi}_T^{(h)}$ of the decay kernel φ satisfies:

$$\mathbb{E}[\|\varphi - \hat{\varphi}_T^{(h)}\|_2^2] \lesssim \frac{1}{\gamma^2} (\log T)^{2/r} T^{-1},$$

where \lesssim means inequality up to a constant that depends on $A, \mu_0, \mu_1, \ell_0, \phi$, and M only.

2 **Market impact and
microstructure: a point
process approach**

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2.1 INTRODUCTION

The increasing proportion of financial transaction taking place in electronic markets has engendered huge changes. In stock (or future) markets, human brokers (dealers) were replaced by the limit order book (LOB) where participants can place limit, market or cancel order. The dynamic of the LOB resembles a queuing system where limit order wait to be executed against market orders or to be cancel (see Chapter 1 Section 1.1 for detail). One of the central issue of the quantitative research is the modeling and the analysis of the volume pressures effects on the price, namely the *market impact* (see [10, 27, 60, 104, 128, 32, 52] and the references therein).

It seems obvious that a buy (sell) trade should push the price up (down) and it is easily demonstrated empirically. Thus, market impact tends to increase the execution cost of a strategy. Indeed, the second buy order is, on average, more expensive than the first, the third more expensive than the second and so on. Controlling the market impact is essential for practitioners in order to limit execution costs (see [90]), particularly for important investors whose volume is much larger than the available liquidity in the LOB. A common strategy consists in splitting in small orders and executes them incrementally. This sequence of small orders is commonly called *meta-order*. Nowadays, algorithmic high-frequency trading is the new way of executing those meta-orders.

The literature on this topic is often rooted to the seminal work of Kyle [88]. The author showed that the market impact is both linear in the traded volume and permanent in time. However, the Kyle's model is not consistent with empirical studies that, despite the diversity of market participants, trading strategies, execution style or execution time, report a strictly concave volume dependence, *i.e.* the so-called *square-root law* (see [23, 10, 118]).

A real breakthrough in the market impact understanding is the “propagator model”¹ of Bouchaud *et al.* [118, 32]:

$$P(n) = \sum_{k \leq n} G(n-k) \epsilon_k f(v_k) + \eta_k, \quad (2.1)$$

where P is the price, η_k is a white noise, f is a function describing the volume dependence of a single trade's impact, also called *instantaneous market impact function*, and G accounts for the temporal dependence of the market order's impact. This model was the starting point of numerous works that deal with the following questions: what is the shape of the *instantaneous market impact function* $f(v)$ (see[79]) and that of the decay kernel G (see [81, 60]), the market impact is permanent or not (see [118, 33, 27, 104, 66]).

Nonetheless, the “propagator model” has some inconveniences. First, the price does not evolve in real time (*i.e.* physical time), but in trading time (*i.e.* n accounts for the n -th trade). Thus, the model is rather difficult to be used in high frequency trading. Moreover, the nature of the white noise η_k is not well defined and estimating the volatility is not straightforward.

Even if the market impact time dependence presents a real interest in term of optimal execution, few empirical studies exist in the quantitative finance literature. In 2009, Moro *et al.* [104] showed the market impact grows in time according to a power-law while trading and, after the order is finished, it reverts to

1. The interested reader could find in [60] a continuous version of the “propagator model”.

a level of about 0.5 – 0.7 of its value at its peak (see Fig. 2.1). We observe that during the execution, the price is pushed in the adverse direction becoming less attractive as time goes by. The price reaches the maximum distance from the arrival price at the end of the execution. The difference between the two is generally refers as *temporary impact*. After the execution a reversal effect is noticed. This is the *decay* part when the price converges back toward its future permanent level. The phenomenon is called *permanent impact*. Recently, Bershova and Rakhlin [27] found that the temporary market impact is on average a square root of trade duration and the permanent market impact to be roughly 0.66 of the temporary impact as predict by the *faire price* condition (see [53] for more details).

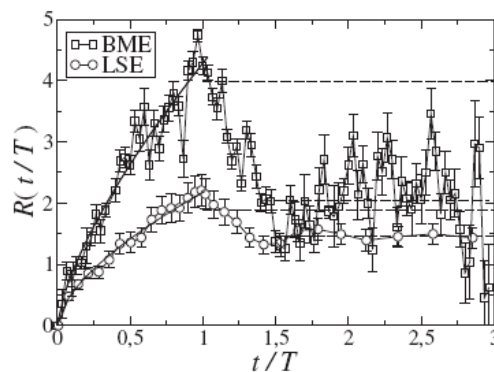


Figure 2.1 – Market impact time dependence from Moro et al. [104].

There are two visions about why orders impact the price. The *informational* vision, which is favored by economists, refers to market impact as the way information is conveyed to the market. In this framework, orders reveal the fundamental value of the price but do not really provoke it (see [70, 66, 53]). In this paper we share a *mechanical* vision where price moves are the result of two antagonist forces: a sell pressure driving prices down and a buy pressure driving prices up. The adepts of this vision have a phenomenological approach: like physicists, they strive to describe the behavior of the price dynamic through consistent quantitative laws and models [17, 18, 39, 3, 38, 55, 116, 32, 118, 117, 86, 122].

The goal of this work is to propose a continuous time model for the market impact using a point process approach [45]. This is a natural framework since price lives on a tick grid and jumps at discrete random times (see [55, 127, 77, 25, 18, 16, 15] for others studies where a point process approach is used in quantitative finance). A special class of point processes is the self exciting (Hawkes) processes. Those processes are intuitively similar to Poisson processes, but unlike the ordinary Poisson processes, the intensity of Hawkes processes is stochastic and depends upon their own historic events. Because current transactions cause future trades (see [118]), the self exciting processes naturally recreate the microstructure price dynamic (see [17]). The Hawkes framework allows us to propose a market impact model living on the same space as the price and accounting for empirical findings as Fig. 2.1.

Let us insist on the fact that market impact model living on the same space as the price. Indeed, in various papers concerning the optimal execution, the price model and market impact model are exogenous (see [9, 6, 124, 84]). In our

framework, the price and the market impact models reside in the same space and interact through the Hawkes process. This really is a natural framework for those who are interested in optimal execution.

2.2 TICK-BY-TICK PRICE MODELS

We consider price modelling at the level of tick-by-tick data. Let

$$P = (P_t)_{t \geq 0}$$

denote the process of price levels at which buy or sell transactions of a given asset occur. We start with $P_0 = 0$ and the process P has values in \mathbb{Z} (tick units), implicitly defined on a rich enough filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

We assume that P is piecewise continuous, with jumps of ± 1 tick². Thus we have the decomposition

$$P_t = N_t^+ - N_t^-, \quad t \geq 0$$

where $N = (N^+, N^-)$ is a bivariate point process,³ each coordinate representing the cumulated sum of upward or downward jumps respectively. We associate to N its stochastic intensity $\lambda = (\lambda^+, \lambda^-)$, *i.e.* the unique \mathcal{F}_t -predictable process defined⁴ by

$$\mathbb{P}(N^\pm \text{ has a jump in } [t, t + dt] | \mathcal{F}_t) = \lambda_t^\pm dt$$

where \mathcal{F}_t is the sigma-field generated by N up to time t and \mathbb{P} denotes the law of $N = (N^+, N^-)$. In the paper, we shall always impose a certain *stability property*, which ensures that P behaves like a Brownian diffusion under \mathbb{P} when correctly scaled macroscopically.

Definition 5. The law \mathbb{P} of $N = (N_t)_{t \geq 0}$ is macroscopically stable if $(\tau^{-1/2}(N_{\tau v}^+ - N_{\tau v}^-), v \in [0, 1])$ converges in law to $(\sigma W_v, v \in [0, 1])$ as $\tau \rightarrow \infty$, where $(W_v)_{v \in [0, 1]}$ is a standard Brownian motion and $\sigma^2 > 0$.

The simplest example is given by $\lambda_t^\pm = \mu$ for all $t \geq 0$ and for some $\mu > 0$. In that case, P is a compound Poisson process with intensity 2μ and symmetric Bernoulli jumps $\{-1, 1\}$. In particular, $P = N^+ - N^-$ has independent increments and $\mathbb{E}[P_t] = 0$. The macroscopical stability is obtained with $\sigma^2 = 2\mu$.

2.3 MARKET IMPACT

Following Barra Market Impact Model Handbook [23], market impact denotes the difference between the expected change of price of a given trade (or a series of trade) of given size(s) and sign(s) and the expected change of market price in absence of this specific transaction. However, such a notion has to be handled with care if practical measurements are sought.

In this chapter, we consider the point of view of an agent who sends to the market a trade (or a series of trades) at a certain rate between t_0 and $t_0 + T$, for some $T > 0$. This may correspond to a trader that will launch aggressive orders

2. A fairly accurate approximation for liquid securities.

3. assumed to have no common jumps

4. See the book Of Jacod and Shiryaev [?] for more details

starting at time t_0 and for a certain duration T . The action of the trader will be modeled by a smooth perturbation \mathbb{Q} of the historical probability \mathbb{P} . The market impact can then be measured as

$$\mathbb{E}_{\mathbb{Q}}[P_t - P_{t_0}] \text{ for every } t \geq t_0$$

or, in other words, the expected price change $P_t - P_{t_0}$ under \mathbb{Q} , *i.e.* from the point of view of the trader who has prior knowledge that a certain series of aggressive orders will be launched over $[t_0, t_0 + T]$. In practice moreover, thanks to this privileged information it will be possible to approximate the expectation of the price change $P_t - P_{t_0}$ under \mathbb{Q} empirically.

For $t_0, T > 0$, let $\mathcal{S}(t_0, T)$ denote the set of bounded functions

$$\mathbf{h} : [0, \infty) \rightarrow [0, \infty) \times [0, \infty)$$

such that $\mathbf{h}(t) = 0$ if $t \notin [t_0, t_0 + T]$ and such that (componentwise)

$$\int_{[t_0, t_0 + T]} \mathbf{h}(t) dt < \infty.$$

Definition 6. The market impact measure associated to $\mathbf{h} \in \mathcal{S}(t_0, T)$ is the probability measure

$$\mathbb{P}_{\mathbf{h}} = L_t(\mathbf{h})\mathbb{P} \text{ on } \mathcal{F}_t,$$

where

$$L_t(\mathbf{h}) = \exp\left(\int_0^t \log\left(1 + \frac{h^+(s)}{\lambda_s^+}\right) dN_s^+ - h^+(s) ds + \int_0^t \log\left(1 + \frac{h^-(s)}{\lambda_s^-}\right) dN_s^- - h^-(s) ds\right) \quad (2.2)$$

for $\mathbf{h} = (h^+, h^-)$ and where $\boldsymbol{\lambda} = (\lambda^+, \lambda^-)$ is the stochastic intensity of $N = (N^+, N^-)$ under \mathbb{P} .

The previous definition is nothing more than a change of probability of ‘‘Girsanov type’’ for point processes (see for example Chapter 13 of Daley and Vere-Jones [45] or Chapter 19 of Lipster and Shiryaev [100]). Thus, the dynamic of our stochastic processes $N = (N^+, N^-)$ changes when the original probability \mathbb{P} is changed to an equivalent probability measure $\mathbb{P}_{\mathbf{h}}$. In Section 2.4 we describe the dynamic of N under $\mathbb{P}_{\mathbf{h}}$ in the special case of a Hawkes process.

Definition 7. The *market impact function* of $\mathbf{h} \in \mathcal{S}(t_0, T)$ is the function

$$t \rightsquigarrow \mathcal{M}_{\mathbf{h}}(t) = \mathbb{E}_{\mathbb{P}_{\mathbf{h}}}[P_t - P_{t_0}] \text{ for every } t \geq t_0.$$

The *transient market impact function* of \mathbf{h} is the restriction of the market impact function of \mathbf{h} over $[t_0, t_0 + T]$. The *decay market impact function* of \mathbf{h} is the restriction of the market impact function of \mathbf{h} over $[t_0 + T, \infty)$. The *permanent impact* of \mathbf{h} is the limit (if it exists)

$$\mathcal{M}_{\mathbf{h}}(\infty) = \lim_{t \rightarrow \infty} \mathcal{M}_{\mathbf{h}}(t).$$

Proposition 1. For every $\mathbf{h} \in \mathcal{S}(t_0, T)$, we have

$$H(\mathbb{P}, \mathbb{P}_{\mathbf{h}}) < \infty, \quad (2.3)$$

where $H(\mathbb{P}, \mathbb{Q}) = \mathbb{E}_{\mathbb{P}}\left[\log \frac{d\mathbb{P}}{d\mathbb{Q}}\right] \leq \infty$ denotes the Kullback-Leibler divergence or entropy between the two probability measures \mathbb{P} and \mathbb{Q} .

The proof of this result is straightforward since \mathbf{h} has compact support and $\lambda_t^\pm \geq \mu, \forall t \geq 0$.

Remark 1. The interpretation of the entropy property (2.3) can be the following: there is no event \mathcal{A} such that the market impact of \mathbf{h} can be detected by observing the occurrence or non-occurrence of \mathcal{A} . In a statistical oriented language, there is no possibility to test \mathbb{P} against $\mathbb{P}_{\mathbf{h}}$ perfectly.

2.4 MARKET IMPACT AND MICROSTRUCTURE

Modeling microstructure

In this paragraph we incorporate microstructure effects following the price model introduced by Bacry *et al.* [17]. For most liquid assets, an upward jumps is more likely followed by a downward jump and vice-versa. Phenomenologically, this can be reproduced as follows: take $\boldsymbol{\lambda} = (\lambda^+, \lambda^-)$ under \mathbb{P} as the intensity of a bivariate Hawkes process specified by

$$\begin{aligned}\lambda_t^+ &= \mu + \int_0^t \phi(t-s) dN_s^-, \\ \lambda_t^- &= \mu + \int_0^t \phi(t-s) dN_s^+, \end{aligned}$$

for a *causal* kernel

$$\phi: \mathbb{R} \rightarrow [0, \infty), \quad \phi(t) = 0 \text{ if } t \notin [0, \infty)$$

that accounts for market manipulation at small scales: baseline Poisson arrivals with intensity μ for both upward and downward jumps are perturbed as follows: an upward jump recorded by N^+ (respectively, a downward jump recorder by N^-) automatically increases the instantaneous probability λ^- (respectively λ^+) of a downward jump (respectively upward jump) by the effect of $\phi \star dN^-$ (respectively, $\phi \star dN^+$). This model creates microstructure effects, *i.e.* it has the ability to reproduce important empirical stylized facts such as a strong microscopic mean reversion and Epps effect (we refer the interested reader to [17] for more details about the construction and the properties of the model) and does not explode as soon as $\phi(t)$ is locally integrable (see [16] for detail):

Assumption 1.

$$\int_0^t \phi(s) ds < \infty \text{ for all } t \geq 0.$$

Furthermore, the process $N = (N^+, N^-)$ has a stationary version under the more restrictive condition:

Assumption 2.

$$\|\phi\|_{L^1} = \int_0^\infty \phi(t) dt < 1. \quad (2.4)$$

In this framework, the price is confined to live on a tick grid but preserves a standard Brownian diffusion behaviour on large scales. Moreover, this approach enables to track the macroscopical variance in rigorous mathematical terms.

Theorem 6. *Work under Assumption 1. Then there exists a unique probability measure \mathbb{P} on a rich enough probability space such that under \mathbb{P} , the process $N = (N^+, N^-)$ has stochastic intensity λ specified as above. If moreover Assumption 2.4 is verified and*

$$\int_0^\infty t^{1/2} \varphi(t) dt < \infty,$$

then \mathbb{P} is macroscopically stable with variance

$$\sigma^2 = \frac{2\mu}{(1 - \|\varphi\|_{L^1})(1 + \|\varphi\|_{L^1})^2}.$$

This property and more general results on scaling limits for Hawkes processes can be found in Bacry et al. [16]. Following the guidelines of the proof exposed in the last cited paper and the fact that \mathbf{h} has support compact, we also have (see Section 2.7 for a sketch of proof):

Proposition 2. *Work in the same settings as in Theorem 6. Then $\mathbb{P}_{\mathbf{h}}$ is macroscopically stable with same variance σ^2 as in previous theorem.*

Microstructure and market impact

For any $\mathbf{h} = (h^+, h^-) \in \mathcal{S}(t_0, T)$, by 2.2, under $\mathbb{P}_{\mathbf{h}}$, we have

$$\begin{aligned} \lambda_t^+ &= \mu + h^+(t) + \int_0^t \varphi(t-s) dN_s^-, \\ \lambda_t^- &= \mu + h^-(t) + \int_0^t \varphi(t-s) dN_s^+. \end{aligned}$$

This is a classical result of the theory of point processes obtain via the Girsanov formula (see Daley and Vere-Jones [45]). The previous representation enables to obtain an explicit formula for the market impact $\mathcal{M}_{\mathbf{h}}(t)$ (see Section 2.7 for the proof).

Theorem 7 (General formula for market impact). *Work under Assumption 2.4. For every $\mathbf{h} \in \mathcal{S}(t_0, T)$, we have*

$$\mathcal{M}_{\mathbf{h}}(t) = G_{\mathbf{h}}(t) - \int_{t_0}^t \kappa(t-s) G_{\mathbf{h}}(s) ds, \quad t \geq t_0, \quad (2.5)$$

where

$$G_{\mathbf{h}}(t) = \int_{t_0}^t (h^+(s) - h^-(s)) ds, \quad \forall t \geq t_0$$

and

$$\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{(\star n)}, \quad (2.6)$$

where $f \star g(t) = \int_0^\infty f(t-s)g(s)ds$ denotes the usual convolution product on $L^1([0, \infty), dt)$, $\varphi^{(\star 1)} = \varphi$ and $\varphi^{(\star n)} = \varphi^{(\star n-1)} \star \varphi$.

Remark 2. Equation 2.6 reminds the renewal theory. This is normal because, in order to find 2.5, we solve a renewal equation (see section 2.7).

2.5 INSTANTANEOUS MARKET IMPACT

In this section, we specify two forms for $\mathbf{h} \in \mathcal{S}(t_0, T)$ introducing moreover a volume variable throughout a trading strategy. By means of Theorem 7, we investigate further classical empirical properties of market impact and their link with microstructure as given by our model.

Instantaneous one-sided market impact

We consider the toy example of measuring the effect of a single buy aggressive order (for instance) of volume v (measured in number of shares say) at time $t = t_0$ under $\mathbb{P}_{\mathbf{h}}$. This formally corresponds to taking for the impact function⁵ $\mathbf{h} = \mathbf{h}_1^i = (h^+, h^-)$ with

$$\begin{cases} h^+(t) = f(v)\delta_{t_0}(t) \\ h^-(t) = 0, \end{cases}$$

where δ_{t_0} denotes the Dirac mass at point t_0 and

$$f : [0, \infty) \rightarrow [0, \infty)$$

models the influence of the volume v . However, $\mathbf{h}_1^i \notin \mathcal{S}(t_0, T)$ for any $T > 0$. In order to circumvent this difficulty, we introduce the family of impact functions $\mathbf{h} = \mathbf{h}_T^1 = (h_{1,T}^+, h_{1,T}^-)$ defined by

$$\begin{cases} h_{1,T}^+(t) = \frac{f(v)}{T} \rho\left(\frac{t-t_0}{T}\right), & T > 0 \\ h_{1,T}^-(t) = 0, \end{cases} \quad (2.7)$$

where

$$\rho : \mathbb{R} \rightarrow [0, \infty), \quad \rho(t) = 0 \text{ if } t \notin [0, \infty)$$

is a *causal* kernel such that

$$\int_0^\infty \rho(t) dt = 1.$$

We are interested in $\mathcal{M}_{\mathbf{h}_T^1}(t)$ in the limit $T \rightarrow 0$. Abusing notation slightly, we set

$$\mathcal{M}_{\mathbf{h}_1^i}(t) = \limsup_{T \rightarrow 0} \mathcal{M}_{\mathbf{h}_T^1}(t) \quad (2.8)$$

and refer to the *one-sided market impact of an instantaneous trade*.

Remark 3. Although $\mathcal{M}_{\mathbf{h}_T^1}(t)$ is implicitly specified through ρ , we will see that the *market impact of an instantaneous trade* $\mathcal{M}_{\mathbf{h}_1^i}(t)$ is independent of the choice of ρ .

Applying Theorem 7 one can prove (see Section 2.7) that:

Theorem 8 (One-sided market impact of an instantaneous trade). *Assume that $(\mathbf{h}_T^1, T > 0)$ is specified by (2.7). Then the market impact function of an instantaneous trade according to (2.8) is given by*

$$\mathcal{M}_{\mathbf{h}_1^i}(t) = f(v) \left(1 - \int_0^{t-t_0} \kappa(s) ds\right), \quad t \geq t_0 \quad (2.9)$$

and does not depend on ρ . Moreover, under the assumption

5. The index 1 is from one-sided and i from instantaneous

$$(A1) \quad \varphi(t) \geq \varphi^{(*2)}(t), \forall t \geq 0,$$

the market impact function $\mathcal{M}_{h_1^i}(t)$ is decreasing on $[t_0, \infty)$ and the permanent impact of an instantaneous trade is given by

$$\mathcal{M}_{h_1^i}(\infty) = f(v) \frac{1}{1 + \|\varphi\|_{L^1}} \quad (2.10)$$

and the following bracketing property holds:

$$\frac{1}{2} \mathcal{M}_{h_1^i}(t) < \mathcal{M}_{h_1^i}(\infty) \leq \mathcal{M}_{h_1^i}(t) \quad (2.11)$$

for every $t \geq t_0$. Finally, under the assumption

$$(A2) \quad \kappa(t) \text{ is decreasing,}$$

the market impact function $\mathcal{M}_{h_1^i}(t)$ is convex over $[t_0, \infty)$.

Remark on the volume-time dependence. The f function corresponds to the *instantaneous market impact function* from the *propagator model* (2.1). As empirical found in [117] and used by others authors before us (see [60, 118]), we suppose the market impact can be separated in a factorized form: one depending on volume and other depending only on time.

Remark on Assumptions (A1). The assumption $\varphi \geq \varphi^{(*2)}$ ensures that κ is positive (see Section 2.7). This is necessary for the market impact function, $\mathcal{M}_{h_1^i}(t)$, to be decreasing in time. Let us point out that even if φ is positive and decreasing, we can not be sure that κ is non-negative. For example, if φ has compact support then κ is negative for some $t > 0$.

Remark on Assumptions (A2). There is no simple condition on φ such that κ is decreasing. Nevertheless, this assumption is verified for some natural φ as we will see later on. Let us point out that the reciprocal is true: if κ is positive and decreasing then φ is decreasing. This last result is an outcome of the “inverse” of equation (2.6) (see Section 2.7 for the proof):

$$\varphi = \sum_{n=1}^{\infty} \kappa^{(*n)}.$$

Some particular cases for kernel φ

Exponential kernel decay. A slightly annoying fact is that the explicit formula in (2.9) depends on $\kappa = \sum_{n \geq 1} (-1)^{n-1} \varphi^{(*n)}$ which is not readily computable for a given φ , except in the case where φ is exponential. In that case, we have

Corollary 1 (One sided instantaneous market impact with exponential kernel). For microstructure kernels φ with exponential decay

$$\varphi(t) = \alpha \exp(-\beta t) \mathbf{1}_{[0, \infty)}(t), \quad 0 < \alpha < \beta,$$

the Assumptions 2.4, (A1) and (A2) are satisfied and we have (taking $t_0 = 0$ to ease further notation)

$$\mathcal{M}_{h_1^i}(t) = f(v) \left(\frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} e^{-(\alpha + \beta)t} \right). \quad (2.12)$$

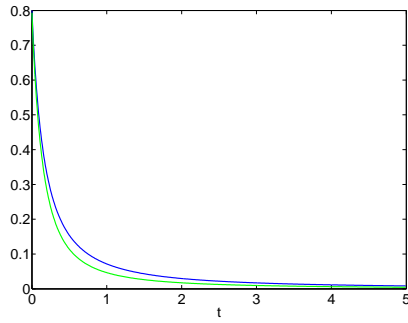


Figure 2.2 – The functions φ (in blue), $\varphi(t) = 0.1 * (0.25+t)^{-1.5}$ and κ (in green).

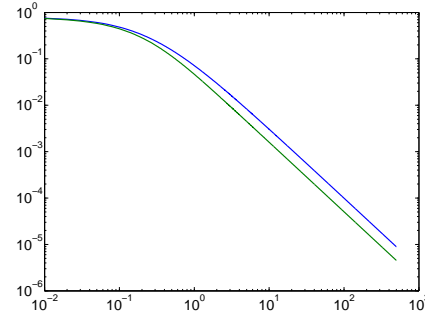


Figure 2.3 – Plot log-log of functions φ (in blue) and κ (in green).

Power-law kernel decay. Beyond this specific case, it seems hopeless to obtain simple formulas for the instantaneous market impact. However, there is an explicit connection between φ and κ in terms of their asymptotic decay, which enables to exploit Theorem 8 further.

Lemma 1 (Asymptotic connection between φ and κ). *Work under Assumptions 2.4 and (A1). Let $p \in [0, 1]$. Then $\int_{[0, \infty)} t^p \kappa(t) dt < \infty$ if and only if $\int_{[0, \infty)} t^p \varphi(t) dt < \infty$.*

As a consequence, if we specify φ with a power-law decay with exponent β , in the sense that

$$\sup \left\{ b, : \int_1^\infty \varphi(t) t^{b-1} dt < \infty \right\} = \beta,$$

then an analogous result holds true for κ thanks to Lemma 1. Let us point out that if $\varphi(t) \sim t^{-\beta}$, $\beta > 1$, when t tends to infinity⁶, we cannot prove a similar property for κ without additional assumption on φ . Nevertheless, in the special case when φ writes in *canonical form*:

$$\varphi(t) = \alpha(\delta + t)^{-\beta} \mathbf{1}_{t \geq 0}, \beta \in (1, 2), \quad (2.13)$$

such that $\|\varphi\|_{L^1} < 1$, we empirically verified that φ and κ have same equivalent to infinity. The next figures (Fig. 2.2 and 2.3) represent an example of functions φ and $\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{(*n)}$ for $\alpha = 0.1$, $\beta = -1.5$ and $\delta = 0.25$.

Corollary 2 (One-sided instantaneous market impact with power-law kernel). *Work in the same setting as in Theorem 8. For microstructure kernels φ such that $\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{(*n)}$ has power-law decay of the form*

$$\kappa(t) = \alpha(\gamma + t)^{-\beta} \mathbf{1}_{[0, \infty)}(t), \alpha, \beta, \gamma > 0, \beta > 1,$$

we have (taking $t_0 = 0$ to ease further notation)

$$\mathcal{M}_{\mathbf{h}_1^i}(t) = f(v) \left(\frac{1}{1 + \alpha \gamma^{-\beta+1} / (\beta - 1)} + \frac{\alpha}{\beta - 1} (\gamma + t)^{-\beta+1} \right). \quad (2.14)$$

Remark 4. We observe that the market impact function, $\mathcal{M}_{\mathbf{h}_1^i}(t)$ (eq. 2.14), decreases when $t \rightarrow \infty$ toward the permanent market impact, $\mathcal{M}_{\mathbf{h}_1^i}(\infty)$, as $t^{-\beta+1}$, i.e.

$$\left(\mathcal{M}_{\mathbf{h}_1^i}(t) - \mathcal{M}_{\mathbf{h}_1^i}(\infty) \right) \sim_{t \rightarrow \infty} t^{-\beta+1}.$$

We expect this remains true whenever the kernel φ has a canonical form with exponent β .

6. In this case φ has a power-law decay with exponent β according to our definition.

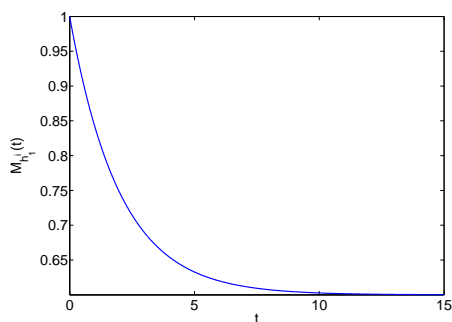


Figure 2.4 – Market impact of one trade with exponential decay kernel (2.12)

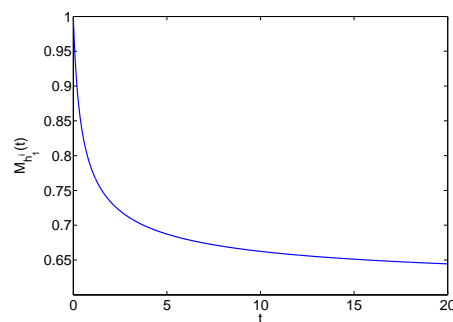


Figure 2.5 – Market impact of one trade with power-law decay (2.14)

The following figures represent the instantaneous one-side market impact function with exponential and respectively power-law kernel. For the exponential kernel 2.4 we took $\varphi(t) = \alpha e^{-\beta t}$ with $\alpha = 0.2$ and $\beta = 0.3$. For the power-law kernel 2.5 we took $\kappa = \alpha(\delta + t)^{-\beta}$ with $\alpha = 0.1$, $\beta = 1.5$ and $\delta = 0.25$. Let us point out that the volume v have no influence on the shape of the market impact function, it is just a multiplicative constant. For practical reasons, we took $f(v) = 1$.

Instantaneous two-sided market impact

Even at a toy modelling level, a slightly annoying fact in our example of Section 2.5 is that the choice $\mathbf{h}_1^i = (h^+, h^-) = (f(v)\delta_{t_0}(dt), 0)$ (or rather its modified version $\mathbf{h}_T^1(t)$ given by (2.7)) for an instantaneous buy order ignores any specific reaction of the market on the downward change of price, since $h^-(t) \equiv 0$. In particular, it is no surprise that we exhibit permanent price impact in the sense that $\mathcal{M}_{\mathbf{h}_1^i}(\infty) \neq 0$. In our restricted context of instantaneous price impact, this drawback can easily be circumvented by allowing $h^-(t) \neq 0$, enabling thus to obtain a new flexibility with respect to permanent market impact.

Our price formation model acts right above the limit order book: we aggregate the effect of agents by two coupled random intensities for the upward and downward changes of price. So far, we have modeled the activity of a new buy by a significant perturbation of the stochastic intensity of upward change of prices (and consistently have obtained a significant permanent market impact). In practice, a buy order at time t_0 encourages an extra-activity on the bid side too, (see for example [49, 3]). A simple, intuitive and explicit way to integrate microstructure effects in the price formation process can be the following: right after time t_0 , a buy order (modeled⁷ by $h^+(t) = f(v)\delta_{t_0}(dt)$) acts on the stochastic intensity λ^- of downward change of prices as if a certain intensity of upward jumps had occurred, leading to

$$\lambda_t^- = \mu + \int_0^t \varphi(t-s)(dN_s^+ + c\delta_{t_0}(ds)),$$

where φ is the mutually exciting kernel accounting for microstructure and $c \geq 0$ is a constant to be specified, possibly depending on the volume function $f(v)$ that we shall further parameterise as

$$c = f(v) \frac{C}{\|\varphi(t)\|_{L^1}} \text{ for some } C \geq 0. \quad (2.15)$$

7. or rather its mollified version 2.7.

We are thus led to consider the class of impact functions $\mathbf{h} \in \mathcal{S}(t_0, T)$ of the form $\mathbf{h} = \mathbf{h}_2^i = (h^+, h^-)$ with

$$\begin{cases} h^+(t) = f(v)\delta_{t_0}(dt) \\ h^-(t) = f(v)\frac{C}{\|\varphi\|_{L^1}}\varphi(t-t_0), \end{cases} \quad (2.16)$$

As in previous section, $\mathbf{h}_2^i \notin \mathcal{S}(t_0, T)$ ⁸ for any $T > 0$. In order to circumvent this difficulty, we introduce the family of impact functions $\mathbf{h} = \mathbf{h}_T^2 = (h_{2,T}^+, h_{2,T}^-)$, $T > 0$ defined by

$$\begin{cases} h_{2,T}^+(t) = \frac{f(v)}{T}\rho\left(\frac{t-t_0}{T}\right) \\ h_{2,T}^-(t) = f(v)\frac{C}{\|\varphi\|_{L^1}}\varphi(t-t_0), \end{cases} \quad (2.17)$$

where ρ is a positive *causal* kernel with integral 1. Similarly, we set

$$\mathcal{M}_{\mathbf{h}_2^i}(t) = \limsup_{T \rightarrow 0} \mathcal{M}_{\mathbf{h}_T^2}(t) \quad (2.18)$$

and refer to the *two-sided market impact of an instantaneous trade*.

Discussion on the parameter C . The parameter C is very intuitive and it quantifies the ratio of mean-reversion reaction (i.e., the downward impact) and of the “trend-following” reaction (the upward impact). Indeed the L^1 norm of the upward reaction to an impulsive buying order is $f(v)\|h^+\|_1 = f(v)\|\delta\|_{L^1} = f(v)$, whereas the downward reaction to the same order is $f(v)\|h^-\|_1 = f(v)\|\varphi\|_{L^1}C/\|\varphi\|_{L^1} = Cf(v)$. Thus, one can distinguish 3 cases of interests.

- $C = 0$: no mean-reversion reaction. We have seen in the previous section that in this case, the permanent market impact is very important, at least one half of the instantaneous market impact $\mathcal{M}_{\mathbf{h}_1^i}(t_0)$.
- $C = 1$: the mean-reversion reaction is as “strong” (in terms of the norm $\|\cdot\|_{L^1}$) as the trend-following one. So we expect the two to compensate asymptotically, *i.e.*, we expect the permanent market impact to be 0 (see Theorem 9 for confirmation),
- $C \in]0, 1[$: the mean-reversion reaction is not zero but strictly smaller than the trend-following reaction, *i.e.*, we expect the permanent market impact to be strictly positive but not so important as in the $C = 0$ case.

Applying Theorem 7 we have an explicit representation of the effect of the component $h^-(t) = c\varphi(t-t_0)$ (see Section 2.7 for the proof):

Theorem 9 (Two-sided market impact of an instantaneous trade). *Assume that $(\mathbf{h}_T^2, T > 0)$ is specified by 2.17. Then the market impact function of an instantaneous trade is given by*

$$\mathcal{M}_{\mathbf{h}_2^i}(t) = f(v)\left(1 - \left(1 + \frac{C}{\|\varphi\|_{L^1}}\right) \int_0^{t-t_0} \kappa(s)ds\right), \quad t \geq t_0 \quad (2.19)$$

and does not depend on ρ . Under the Assumption (A1), the market impact function $\mathcal{M}_{\mathbf{h}_2^i}(t)$ is decreasing on $[t, \infty)$. Moreover, the permanent impact of an instantaneous trade is given by

$$\mathcal{M}_{\mathbf{h}_2^i}(\infty) = f(v)\frac{1-C}{1+\|\varphi\|_{L^1}} \quad (2.20)$$

8. Index 2 for two-sided and i for instantaneous

and the following bracketing property holds:

$$\mathcal{M}_{h_2^i}(t) \geq 0 \text{ if and only if } C \leq 1. \quad (2.21)$$

In particular, for $C = 1$ the permanent market impact is zero, $\mathcal{M}_{h_2^i}(\infty) = 0$. Finally, under the Assumption (A2), the market impact function $\mathcal{M}_{h_2^i}(t)$ is convex over (t_0, ∞) .

Remark on the permanent/non-permanent market impact condition. In the general framework of Theorem 7 the permanent market impact is zero as soon as

$$\|h^+\|_{L^1} = \|h^-\|_{L^1},$$

i.e. the intensities λ^+ and λ^- are disturbed by functions, h^+ and respectively h^- , having same mass. This result have a very intuitive interpretation: the permanent market impact is zero if the “activities” on the buy and respectively sell side are similar. Moreover, the market impact function $\mathcal{M}_{h_2^i}(t)$ is positive for all $t \geq t_0$ if and only if the mass put on λ^- is less important than the mass put on λ^+ , i.e.

$$\|h^+\|_{L^1} \geq \|h^-\|_{L^1}.$$

Some particular cases for kernel φ

Corollary 3 (Two-sided instantaneous market impact with exponential kernel). For microstructure kernels φ with exponential decay

$$\varphi(t) = \alpha \exp(-\beta t) \mathbf{1}_{[0, \infty)}(t), \quad 0 < \alpha < \beta,$$

the Assumptions 2.4, (A1) and (A2) are satisfied and we have (taking $t_0 = 0$ to ease further notation)

$$\mathcal{M}_{h_2^i}(t) = f(v) \left(\frac{\beta(1-C)}{\alpha + \beta} + \frac{C\beta + \alpha}{\alpha + \beta} e^{-(\alpha + \beta)t} \right). \quad (2.22)$$

Corollary 4 (Two-sided instantaneous market impact with power-law kernel). Work in the same setting as in Theorem 9. For microstructure kernels φ such that $\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{(\star n)}$ has power-law decay of the form

$$\kappa(t) = \alpha(\gamma + t)^{-\beta} \mathbf{1}_{[0, \infty)}(t), \quad \alpha, \beta, \gamma > 0, \quad \beta > 1,$$

we have (taking $t_0 = 0$ to ease further notation)

$$\mathcal{M}_{h_2^i}(t) = f(v) \left(\frac{1-C}{1 + \alpha\gamma^{-\beta+1}/(\beta-1)} + \left(1 + \frac{C}{\alpha\gamma^{-\beta+1}/(\beta-1)}\right) \frac{\alpha}{\beta-1} (\gamma + t)^{-\beta+1} \right). \quad (2.23)$$

Remark 5. As in the previous case of an one-side market impact, the market impact function, $\mathcal{M}_{h_2^i}(t)$ (eq. 2.23), decreases when $t \rightarrow \infty$ toward the permanent market impact, $\mathcal{M}_{h_2^i}(\infty)$, as $t^{-\beta+1}$, i.e.

$$\left(\mathcal{M}_{h_2^i}(t) - \mathcal{M}_{h_2^i}(\infty) \right) \sim_{t \rightarrow \infty} t^{-\beta+1}.$$

We expect this remains true whenever the kernel φ has a canonical form with exponent β .

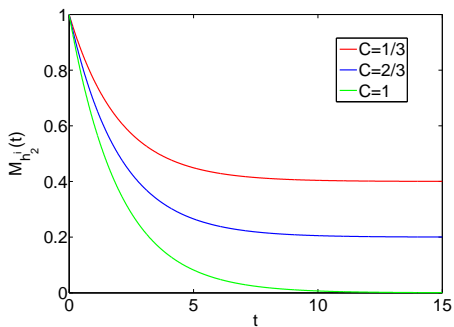


Figure 2.6 – Market impact of one trade with exponential decay kernel (2.22) and different ration C of mean-reversion/trend-following (2.15)

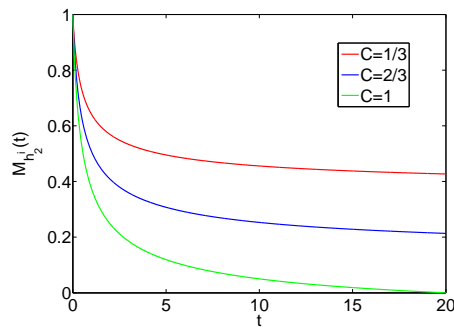


Figure 2.7 – Market impact of one trade with power-law decay (2.23) and different ration C of mean-reversion/trend-following (2.15)

The following figures represent the instantaneous two-sided market impact function with exponential and respectively power-law kernel for different values of C . For the exponential kernel (Fig. 2.6) we took $\varphi(t) = \alpha e^{-\beta t}$ with $\alpha = 0.2$ and $\beta = 0.3$. For the power-law kernel (Fig. 2.7) we took $\kappa = \alpha(\delta + t)^{-\beta}$ with $\alpha = 0.1$, $\beta = 1.5$ and $\delta = 0.25$. As in previous section, we took $f(v) = 1$.

2.6 MARKET IMPACT OF A CONTINUOUS STRATEGY

In this section, we consider the case of the market impact of a meta-order starting at time t_0 , lasting a period T and corresponding to a continuous flow of buying⁹ orders with a trading rate r_t supported by $[t_0, t_0 + T]$ ($r_t \neq 0$ only for $t \in [0, T]$). A particular interest will be attached for the case with a constant trading rate. As stressed in Section 2.5, it seems natural to have a response function $h^- \neq 0$. Thus, we consider the class of impact functions $\mathbf{h} = \mathbf{h}^c = (h^+, h^-) \in \mathcal{S}(t_0, T)$ of the form:

$$\begin{cases} h^+(t) = \tilde{f}(r_t) \\ h^-(t) = \frac{C}{\|\varphi\|_{L^1}} \int_{t_0}^t \tilde{f}(r_s) \varphi(t-s) ds. \end{cases} \quad (2.24)$$

Seeing the impact of $\tilde{f}(r_s) ds$ as the infinitesimal impact of a buy order of volume $r dt$, the previous model it is a superposition of the instantaneous two-sided impact model. Let us point out that the case with no response function (i.e. $h^- \equiv 0$) is simply obtain by taking $C = 0$.

Remark on the function \tilde{f} . Within this model, we no longer have a function of volume, but a function of the trading rate r_t . In order to mark this difference, we chose to denote it \tilde{f} instead of f as in previous cases. For the rest of this chapter, we will suppose that \tilde{f} is positive and bounded. This seems a natural assumption since, in practice, the market impact of a buying strategy pushes, in expectation, the price up and is never unbounded. Moreover, this assumption guarantees that $\mathbf{h} \in \mathcal{S}(t_0, T)$.

By the means of Theorem 7, one can prove (see Section 2.7) that :

9. The impact of selling meta-order can be modeled using the exact same principles

Theorem 10 (Two-sided market impact of a continuous strategy). *The market impact of a continuous strategy r_t is given by*

$$\mathcal{M}_{\mathbf{h}^c}(t) = \int_{t_0}^t \tilde{f}(r_s) H(t-s) ds, \quad t \geq t_0, \quad (2.25)$$

where $H(t) = 1 - \left(1 + \frac{C}{\|\varphi\|_{L^1}}\right) \int_{t_0}^t \kappa(s) ds$. Under the assumption that \tilde{f} is bounded, the permanent market impact writes

$$\mathcal{M}_{\mathbf{h}^c}(\infty) = \frac{1-C}{\|\varphi\|_{L^1}} \|\tilde{f}(r)\|_{L^1}. \quad (2.26)$$

In the particular case of a constant strategy, $r_t = r_0, \forall t \in [t_0, t_0 + T]$, we have:

$$\mathcal{M}_{\mathbf{h}^c}(t) = \tilde{f}(r_0) \mathbf{1}_{[0, T]} \star H(t), \quad t \geq t_0. \quad (2.27)$$

As a consequence, when $C \leq 1$, under the Assumptions (A1) and (A2), the function $t \mapsto \mathcal{M}_{\mathbf{h}^c}(t)$ is increasing and concave over $[t_0, t_0 + T]$ and decreasing and convex over $[t_0 + T, \infty)$.

Some particular cases for kernel φ

Corollary 5 (Two-sided continuous market impact with exponential kernel).

In the case of a constant strategy, for microstructure kernels φ with exponential decay

$$\varphi(t) = \alpha \exp(-\beta t) \mathbf{1}_{[0, \infty)}(t), \quad 0 < \alpha < \beta,$$

we have (taking $t_0 = 0$ to ease further notation)

– if $t \in [0, T]$:

$$\mathcal{M}_{\mathbf{h}^c}(t) = \tilde{f}(r_0) \left(\frac{\beta(1-C)}{\alpha + \beta} t + \frac{C\beta + \alpha}{(\alpha + \beta)^2} e^{-(\alpha + \beta)t} \right) \quad (2.28)$$

– if $t > T$

$$\mathcal{M}_{\mathbf{h}^c}(t) = \tilde{f}(r_0) \left(\frac{\beta(1-C)}{\alpha + \beta} T + \frac{C\beta + \alpha}{(\alpha + \beta)^2} (e^{-(\alpha + \beta)(t-T)} - e^{-(\alpha + \beta)t}) \right). \quad (2.29)$$

Corollary 6 (Two-sided continuous market impact with power-law kernel).

*Work in the same setting as in Theorem 10. In the case of a constant strategy, for microstructure kernels φ such that $\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{(*n)}$ has power-law decay of the form*

$$\kappa(t) = \alpha(\gamma + t)^{-\beta} \mathbf{1}_{[0, \infty)}(t), \quad \alpha, \beta, \gamma > 0, \quad \beta > 1$$

such that Assumption 2.4 is satisfied, we have (taking $t_0 = 0$ to ease further notation)

– if $t \in [0, T]$:

$$\mathcal{M}_{\mathbf{h}^c}(t) = \tilde{f}(r_0) \left(\frac{1-C}{1 + \alpha\gamma^{-\beta+1}/(\beta-1)} t + \left(1 + \frac{C}{\alpha\gamma^{-\beta+1}/(\beta-1)} \right) \frac{\alpha}{(\beta-1)(2-\beta)} (\delta+t)^{2-\beta} \right) \quad (2.30)$$

– if $t > T$

$$\mathcal{M}_{\mathbf{h}^c}(t) = \tilde{f}(r_0) \frac{1-C}{1 + \alpha\gamma^{-\beta+1}/(\beta-1)} T + \tilde{f}(r_0) \left(1 + \frac{C}{\alpha\gamma^{-\beta+1}/(\beta-1)} \right) \frac{\alpha}{(\beta-1)(2-\beta)} \left((\delta+t+T)^{-\beta+2} - (\delta+t)^{-\beta+2} \right) \quad (2.31)$$

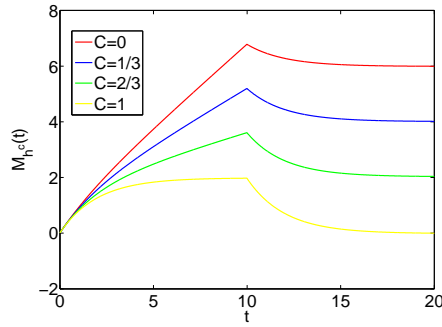


Figure 2.8 – Market impact of a continuous strategy with exponential decay kernel (2.28) and (2.29) for different ration C of mean-reversion/trend-following (2.15)

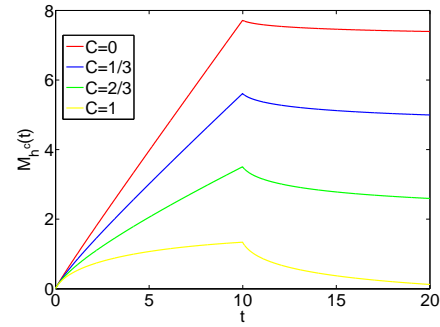


Figure 2.9 – Market impact of a continuous strategy with power-law decay kernel (2.30) and (2.31) for different ration C of mean-reversion/trend-following (2.15)

Remark on the case of the power-law kernel decay We work in the case when $\beta \in (1, 2)$. Before going further, let us point out that several recent empirical results ([15] and [69]) seem to show that the Hawkes kernel $\varphi(t)$ decays as $t^{-\beta}$, $\beta > 1$. Both studies found the exponent in the interval $[1, 1.5]$.

Decay. As in the previous cases, the market impact function, $\mathcal{M}_{h^c}(t)$ (eq. 2.31), decreases when $t \rightarrow \infty$ toward the permanent market impact, $\mathcal{M}_{h^c}(\infty) = \tilde{f}(r_0) \frac{1-C}{1+\alpha\gamma^{-\beta+1}/(\beta-1)} T$, as $t^{-\beta+1}$, i.e.

$$\left(\mathcal{M}_{h^c}(t) - \mathcal{M}_{h^c}(\infty) \right) \sim_{t \rightarrow \infty} t^{-\beta+1}.$$

Transient market impact. In the case when the execution time T is very large, i.e. $T \rightarrow \infty$, the transient market impact (eq. 2.31) can have two asymptotical behaviors:

- $C = 1$ (corresponding to no permanent impact): the transient market impact curve has asymptotically a power-law behave, in the sense that

$$\mathcal{M}_{h^c}(T) \sim T^{2-\beta} \text{ when } T \rightarrow \infty,$$

- $C < 1$: the transient curve has asymptotically a linear behave in the sense that

$$\mathcal{M}_{h^c}(T) \sim T \text{ when } T \rightarrow \infty.$$

The following figures represent the market impact function of a constant strategy $r_t = r_0, \forall t \in [0, T]$ with exponential and respectively power-law kernel for different values of C . For the exponential kernel 2.8 we took $\varphi(t) = \alpha e^{-\beta t}$ with $\alpha = 0.2$ and $\beta = 0.3$. For the power-law kernel 2.9 we took $\kappa = \alpha(\delta + t)^{-\beta}$ with $\alpha = 0.1, \beta = 1.5$ and $\delta = 0.25$. The trading rate have no influence on the shape of the transient market impact, it is just a multiplicative constant. We took $\tilde{f}(r_0) = 1$ and $T = 10$. Let us remark that, when $C > 0$, the transient market impact “seemms” to be linear even if we rigorously proved it is concave. This is explained by the ratio mean-reversion/trend-following which is too small.

2.7 PROOFS

Proof of Theorem 7

This is a consequence of a more general result on Hawkes Processes:

Theorem 11. *Let (N_1, \dots, N_d) be a d -multivariate Hawkes process defined by his intensity $\lambda = (\lambda_1, \dots, \lambda_d)$:*

$$\lambda(t) = \mu(t) + \int_{[0,t)} \Phi(t-s) dN(s) \quad (2.32)$$

where $\mu = (\mu_1, \dots, \mu_d)$ and $\Phi = (\varphi_{ij})_{1 \leq i, j \leq d}$ are vectors containing functions from \mathbb{R}_+ to \mathbb{R}_+ .

Under the assumptions:

- $\int_0^t \mu(s) ds < \infty, \forall t > 0$,
- the spectral radius of the matrix $K = \int_0^\infty \Phi(t) dt$ is inferior to 1, $\rho(K) < 1$,

we have

$$\mathbb{E}[N_t] = \nu(s) + \int_0^t \Psi(t-s) \nu(s) ds, \quad (2.33)$$

where $\Phi^{(\star n)} = \Phi \star \dots \star \Phi$ (with n terms on the right hand-side) and where the convolution product of two matrices $A(t) = \{a_{ij}(t)\}$ and $B(t) = \{b_{ij}(t)\}$ is defined as the matrix $C(t) = \{c_{ij}(t)\}$, such that

$$c_{ij}(t) = \sum_k a_{ik} \star b_{kj}.$$

.

Proof. Let us first remark the elements of the matrix Ψ are in L^1 . Indeed, by induction we have, $\int_0^\infty \Phi^{(\star n)}(t) dt = K^n$ and since $\rho(K) < 1$, the series $\psi = \sum_{n \geq 1} K^n$ is finite component-wise, thus one gets:

$$\int_0^\infty \Psi(t) dt = K(Id - K)^{-1}. \quad (2.34)$$

We will now show that:

$$\mathbb{E}[N_t] = \int_0^t \mu(s) ds + \mathbb{E} \left[\int_0^t \Phi(t-s) N_s ds \right] \quad \forall t > 0. \quad (2.35)$$

Using that $N(t) - \int_0^t \lambda(s) ds$ is a (\mathcal{F}_t) -martingale (where $(\mathcal{F}_t)_{t \geq 0}$ be the σ -algebra generated by the random variables $N_i(s); s \leq t; 1 \leq i \leq d$), we have:

$$\mathbb{E}[N_t] = \mathbb{E} \left[\int_0^t \lambda(s) ds \right] = \mathbb{E} \left[\int_0^t \mu(s) ds \right] + \mathbb{E} \left[\int_0^t ds \int_0^s \Phi(s-u) dN_u \right].$$

But, by Fubini's theorem:

$$\int_0^t ds \int_0^s \Phi(s-u) dN_u du = \int_0^t \left(\int_s^t \Phi(t-u) dt \right) dN_u = \int_0^t \left(\int_0^{t-s} \Phi(s) ds \right) dN_u.$$

We denote $F(t) = \int_0^t \Phi(s) ds$ and using an integration by parts we have:

$$\int_0^t F(t-s) dN_s = \left[F(t-s) N(s) \right]_0^t + \int_0^t \Phi(t-s) N_s ds =$$

$$= F(0)N_t - F(t)N_0 + \int_0^t \Phi(t-s)N_s ds = \int_0^t \Phi(t-s)N_s ds.$$

So we obtain:

$$\mathbb{E}[N_t] = \mathbb{E}\left[\int_0^t \mu(s)ds\right] + \mathbb{E}\left[\int_0^t \Phi(t-s)N_s ds\right].$$

Using once again Fubini's theorem:

$$\mathbb{E}[N_t] = \int_0^t \mu(s)ds + \int_0^t \Phi(t-s)\mathbb{E}[N_s] \quad (2.36)$$

This is a classical renewal equation and the solution is given by (3.31). The interested reader can find more on renewal theory on the book of David Cox ([41]).

□

Let us now prove Theorem 7. We work under probability \mathbb{P}_h as defined in 2.2. We recall that under \mathbb{P}_h the intensities of (N^+, N^-) write:

$$\begin{aligned} \lambda_t^+ &= \mu + h^+(t) + \int_0^t \varphi(t-s)dN_s^-, \\ \lambda_t^- &= \mu + h^-(t) + \int_0^t \varphi(t-s)dN_s^+. \end{aligned}$$

In order to ease further notation we take $t_0 = 0$. We apply the previous theorem in the particular case of the 2-dimensional Hawkes process (N^+, N^-) with $\Phi = \begin{pmatrix} 0 & \varphi \\ \varphi & 0 \end{pmatrix}$ and we successively compute:

- $\Phi^{*n} = \begin{pmatrix} 0 & \varphi^{*n} \\ \varphi^{*n} & 0 \end{pmatrix}$ if n is even and $\Phi^{*n} = \begin{pmatrix} \varphi^{*n} & 0 \\ 0 & \varphi^{*n} \end{pmatrix}$ if n is odd.
- $v(t) = (v^+(t), v^-(t)) = (t\mu + \int_0^t (f(r) \star h^+)(u)du, t\mu + \int_0^t (f(r) \star h^-)(u)du)$
- $\mathcal{M}_h(t) = \mathbb{E}_{\mathbb{P}_h}[N_t^+ - N_t^-] = (v^+(t) - v^-(t)) - \kappa \star (v^+ - v^-)(t)$.

This proves Eq. 2.5 since $v^+ - v^- = f(r) \star G$.

Proof of Lemma 1

We recall that $\kappa = \sum_{n=1}^{\infty} (-1)^{n+1} \varphi^{(\star n)}$. Which in the Fourier domain becomes:

$$\hat{\kappa}(\omega) = \sum_{n=1}^{\infty} (-1)^{n+1} \hat{\varphi}^n(\omega) = \frac{\hat{\varphi}(\omega)}{1 + \hat{\varphi}(\omega)},$$

where \hat{f} stands for the standard Fourier transform of f :

$$\hat{f}(\omega) = \int_{\mathbb{R}} \exp(-it\omega) f(t) dt.$$

By reversing the previous formula we get:

$$\hat{\varphi}(\omega) = \frac{\hat{\kappa}(\omega)}{1 - \hat{\kappa}(\omega)} = \sum_{n=1}^{\infty} \hat{\kappa}^n(\omega).$$

Returning in the time-domain we have:

$$\varphi = \sum_{n=1}^{\infty} \kappa^{(\star n)}. \quad (2.37)$$

Let $p \in [0, 1]$ and assume that $\int_0^\infty t^p \varphi(t) dt < \infty$. Since we are under Assumption **(A1)**, i.e. $\varphi \geq \varphi^{(\star 2)}$, $\kappa(t) \geq 0$. Indeed,

$$\kappa = \varphi - \varphi^{(\star 2)} + \left(\varphi - \varphi^{(\star 2)} \right) \star \sum_{n \geq 1} \varphi^{(\star 2n)}. \quad (2.38)$$

Since κ is positive, it is easy to see that $\kappa^{(\star n)} \geq 0$. So $0 \leq \kappa(t) \leq \varphi(t)$ and than $\int_0^\infty t^p \psi_d(t) dt < \infty$ is obvious.

Now, we will assume that $\int_0^\infty t^p \kappa(t) dt < \infty$ and we will show that $\int_0^\infty t^p \varphi(t) dt < \infty$.

Let $I_n = \int_0^\infty t^p \kappa^{(\star n)}(t) dt$, $\forall n \geq 1$, and $\int_0^\infty \kappa(t) dt = x$, $x \in \mathbb{R}_+$. By the concavity of the function $t \rightarrow t^p$, $p \in [0, 1]$, we get:

$$\begin{aligned} I_{n+1} &= \int_0^\infty t^p \left(\int_0^t \kappa(t-s) \kappa^{(\star n)}(s) ds \right) dt = \int_0^\infty \left(\int_0^\infty (t+s)^p \kappa(t) dt \right) \kappa^{(\star n)}(s) ds \\ &\leq \int_0^\infty \left(\int_0^\infty (t^p + s^p) \kappa(t) dt \right) \kappa^{(\star n)}(s) ds = \int_0^\infty (I_1 + x s^p) \kappa^{(\star n)}(s) ds = x^n I_1 + x I_n. \end{aligned}$$

Therefore for all integer N :

$$\sum_{n=1}^N I_n \leq I_1 + \left(\sum_{n=1}^{N-1} x^n \right) I_1 + x \sum_{n=1}^{N-1} I_n.$$

Since $0 \leq \kappa(t) \leq \varphi(t)$, $\forall t \geq 0$, we have $x \leq \|\varphi\|_{\mathcal{L}^1} < 1$ and we obtain

$$\sum_{n=1}^{N-1} I_n \leq \frac{I_1}{(1-x)^2}$$

Thus, for $N \rightarrow \infty$:

$$\int_0^\infty t^p \varphi(t) dt = \sum_{n=1}^\infty I_n \leq \frac{I_1}{(1-x)^2} < \infty.$$

Proof of Theorem 8

Let $\mathbf{h}_T^1 = (h_{1,T}^+, h_{1,T}^-)$ be as specified by (2.7):

$$\begin{cases} h_{1,T}^+(t) = \frac{f(v)}{T} \rho\left(\frac{t-t_0}{T}\right), & T > 0 \\ h_{1,T}^-(t) = 0, \end{cases}$$

where

$$\rho: \mathbb{R} \rightarrow [0, \infty), \quad \rho(t) = 0 \text{ if } t \notin [0, \infty)$$

is a *causal* kernel such that

$$\int_0^\infty \rho(t) dt = 1.$$

We apply Theorem 7. We write:

$$G_{\mathbf{h}_T^1}(t) = f(v) T^{-1} \int_{t_0}^t \rho(T^{-1}(s-t_0)) ds = f(v) \int_0^{(t-t_0)T^{-1}} \rho(u) du.$$

We have:

$$\limsup_{T \rightarrow 0} G_{\mathbf{h}_T^1}(t) = f(v), \quad \forall t > t_0.$$

Since all quantities are positive, we can invert limit and integral and we obtain:

$$\mathcal{M}_{h_1^i}(t) = \limsup_{T \rightarrow 0} \int_{t_0}^t \kappa(t-s) h_{h_1^i}(s) ds = \int_{t_0}^t \kappa(t-s) f(v) ds = f(v) \int_0^{t-t_0} \kappa(s) ds.$$

Under the Assumption **(A1)**, the function κ is non-negative (see 2.38) and we obtain the permanent market impact by a taking $t \rightarrow \infty$ in previous equation:

$$\mathcal{M}_{h_1^i}(\infty) = f(v)(1 - \|\kappa\|_{L^1}).$$

We recall the definition of κ 2.6:

$$\kappa = \sum_{n \geq 1} (-1)^{n+1} \varphi^{(\star n)}.$$

By integration we get:

$$\|\kappa\|_{L^1} = \sum_{n \geq 1} (-1)^{n+1} \|\varphi\|_{L^1}^n = \frac{\|\varphi\|_{L^1}}{1 + \|\varphi\|_{L^1}}. \quad (2.39)$$

Which allows us to find:

$$\mathcal{M}_{h_1^i}(\infty) = f(v) \frac{1}{1 + \|\varphi\|_{L^1}}.$$

Under the assumption **(A2)**, the function κ is non-negative, the market impact function $\mathcal{M}_{h_1^i}(t)$ is decreasing on $[t_0, \infty)$ and the next bracketing property is easy to check since $\|\varphi\|_{\mathcal{L}^1} < 1$:

$$\frac{1}{2} \mathcal{M}_{h_1^i}(t) < \mathcal{M}_{h_1^i}(\infty) \leq \mathcal{M}_{h_1^i}(t).$$

Under the Assumption **(A2)**, κ is decreasing and the market impact function $\mathcal{M}_{h_1^i}(t)$ is convex over $[t_0, \infty)$ since its derivative on this interval is equal to $-\kappa(t-t_0)$ which is an increasing function.

Proof of Theorem 9

Let $h_T^2 = (h_{2,T}^+, h_{2,T}^-)$ be as specified by (2.17):

$$\begin{cases} h_{2,T}^+(t) = \frac{f(v)}{T} \rho\left(\frac{t-t_0}{T}\right) \\ h_{2,T}^-(t) = f(v) \frac{C}{\|\varphi(t)\|_{\mathcal{L}^1}} \varphi(t-t_0), \end{cases}$$

where

$$\rho: \mathbb{R} \rightarrow [0, \infty), \quad \rho(t) = 0 \text{ if } t \notin [0, \infty)$$

is a *causal* kernel such that

$$\int_0^\infty \rho(t) dt = 1.$$

We apply Theorem 7. We denote $C_1 = \frac{C}{\|\varphi(t)\|_{\mathcal{L}^1}}$ to ease the notation:

$$G_{h_T^2}(t) = f(v) \int_{t_0}^t (T^{-1} \rho(T^{-1}(s-t_0)) - C_1 \varphi(s-t_0)) ds = f(v) \int_0^{(t-t_0)T^{-1}} \rho(u) du - C_1 \int_0^{t-t_0} \varphi(s) ds.$$

We have

$$\limsup_{T \rightarrow 0} G_{h_T^2}(t) := G_{h_2^i}(t) = f(v) \left(1 - C_1 \int_0^{t-t_0} \varphi(s) ds \right) \forall t > t_0.$$

Since ρ is positive, the previous limit is monotone. Thus, by the monotone convergence theorem we have

$$\mathcal{M}_{h_2^i}(t) = G_{h_2^i}(t) - \int_{t_0}^t \kappa(t-s)G_{h_2^i}(s)ds.$$

We observe $\mathcal{M}_{h_2^i}(t_0) = f(v)$ before derivate last equation for $t > t_0$:

$$\mathcal{M}'_{h_2^i}(t) = G'_{h_2^i}(t) - G'_{h_2^i} \star \kappa(t) = f(v) \left(-C_1 \varphi(t-t_0) - C_1 \kappa(t-t_0) + C_1 \kappa \star \varphi(t-t_0) \right).$$

Let us recall

$$\kappa = \sum_{n \geq 1} \varphi^{(\star n)} = \varphi - \varphi \star \sum_{n \geq 1} \varphi^{(\star n)} = \varphi - \varphi \star \kappa.$$

So we obtain:

$$\mathcal{M}'_{h_2^i}(t) = -f(v)(1+C_1)\kappa(t-t_0).$$

This allows us to find:

$$\mathcal{M}_{h_2^i}(t) = f(v)(1 - (1+c) \int_0^{t-t_0} \kappa(s)ds).$$

We obtain the permanent market impact by a taking $t \rightarrow \infty$ in previous equation and using the link between the L^1 norms of φ and κ (eq. 2.39):

$$\mathcal{M}_{h_2^i}(\infty) = f(v)(1 - (1+C_1)\|\kappa\|_{L^1}) = f(v) \left(1 - (1+C_1) \frac{\|\varphi\|_{L^1}}{1+\|\varphi\|_{L^1}} \right) = f(v) \frac{1-C}{1+\|\varphi\|_{L^1}}.$$

Thus, under the under Assumption **(A1)**, the market impact function is decreasing and we have a positive market impact if and only if $C \leq 1$. When $C = 1$, we clearly have no permanent market impact, $\mathcal{M}_{h_2^i}(\infty) = 0$.

As in the case of instantaneous one-sided market impact (Theorem 8), the market impact function $\mathcal{M}_{h_2^i}$ is convex under the **(A2)**.

Proof of Theorem 10

As before, this is a direct application of Theorem 7. Let $\mathbf{h}^c = (h^+, h^-)$ be as specified by (2.24):

$$\begin{cases} h^+(t) = \tilde{f}(r_t) \\ h^-(t) = \frac{C}{\|\varphi\|_{\mathcal{L}^1}} \int_{t_0}^t \tilde{f}(r_t) \varphi(t-s)ds. \end{cases}$$

In order to shorten the proof, we remark that formally we have:

$$G_{\mathbf{h}^c} = \tilde{f}(r) \star G_{h_2^i}.$$

Thus, using Theorem 9, equation (2.19), we have:

$$\mathcal{M}_{\mathbf{h}^c}(t) = \frac{\tilde{f}(r)}{f(v)} \star H(t),$$

where

$$H(t) = \frac{1}{f(v)} \mathcal{M}_{h_2^i}(t) = 1 - (1+C_1) \int_{t_0}^t \kappa(s)ds,$$

which proves (2.25). Equation (2.27) is a direct consequence of this equation when the rate r_t is constant.

As in Theorem 9 we have:

$$\lim_{t \rightarrow \infty} H(t) = \frac{1 - C}{1 + \|\varphi\|_{L^1}} := \ell.$$

Putting $\tilde{H}(t) = H(t) - \ell$, we have for $\theta > t_0 + T$:

$$\mathcal{M}_{h^c}(t) = \int_{t_0}^{t_0+T} \tilde{f}(r_s) H(t-s) ds = \ell \int_{t_0}^{t_0+T} \tilde{f}(r_s) ds + \int_{t_0}^{t_0+T} \tilde{f}(r_s) \tilde{H}(t-s) ds.$$

Under the assumption that \tilde{f} is bounded, the second term converges to zero as t tends to infinity

$$\lim_{t \rightarrow \infty} \int_{t_0}^{t_0+T} \tilde{f}(r_s) \tilde{H}(t-s) ds = 0,$$

which proves (2.26).

Let us consider the case of a constant strategy, $r_t = r_0$. For $t \in [t_0, t_0 + T]$ we have

$$\mathcal{M}_{h^c}(t) = \tilde{f}(r_0) \int_{t_0}^t H(t-s) ds = \tilde{f}(r_0) \int_0^{t-t_0} H(s) ds.$$

When $C \leq 1$ we have $H(t) \geq 0$, $\forall t \geq 0$, $\mathcal{M}_{h^c}(t)$ is increasing on $t \in [t_0, t_0 + T]$. Since $\mathcal{M}'_{h^c}(t) = \tilde{f}(r_0) H(t-t_0) > 0 \forall t \in [t_0, t_0 + T]$, the transient market impact is convex, i.e. \mathcal{M}_{h^c} convex on $[t_0, t_0 + T]$.

For $t > t_0 + T$, $r_t = 0$ and we have:

$$\mathcal{M}_{h^c}(t) = \tilde{f}(r_0) \int_{t_0}^{t_0+T} H(t-s) ds = \tilde{f}(r_0) \int_{t-t_0-T}^{t-t_0} H(s) ds.$$

Thus

$$\mathcal{M}'_{h^c}(t) = \tilde{f}(r_0) \left(H(t-t_0) - H(t-t_0-T) \right).$$

But H is decreasing under assumption **(A1)** since $H'(t) = -(1 + C_1)\kappa(t) < 0$. Thus $H(t-t_0) - H(t-t_0-T) < 0$ and \mathcal{M}_{h^c} is decreasing on $(t_0 + T, \infty)$. Differentiating a second time \mathcal{M}_{h^c} we get:

$$\mathcal{M}''_{h^c}(t) = \tilde{f}(r_0) \left(H'(t-t_0) - H'(t-t_0-T) \right) = -(1 + C_1) \tilde{f}(r_0) \left(\kappa(t-t_0) - \kappa(t-t_0-T) \right).$$

Under the assumption **(A2)**, i.e. κ is decreasing, $\mathcal{M}''_{h^c}(t) > 0$. This finishes the proof.

Proof of Proposition 2

We can prove Proposition 2 following the guidelines of the proof of Theorem 6 from Bacry *et al.* [17]. In the last cited paper it is proved a more general result:

Theorem 12. Let (N_1, \dots, N_d) be a d -multivariate Hawkes process defined by his intensity $\lambda = (\lambda_1, \dots, \lambda_d)$:

$$\lambda(t) = \mu + \int_{[0,t)} \Phi(t-s) dN(s) \quad (2.40)$$

where $\mu = (\mu_1, \dots, \mu_d)$ and $\Phi = (\varphi_{ij})_{1 \leq i, j \leq d}$ are vectors containing functions from \mathbb{R}_+ to \mathbb{R}_+ . Under the assumptions:

- the spectral radius of the matrix $K = \int_0^\infty \Phi(t) dt$ is inferior to 1, $\rho(K) < 1$,
- $\int_0^\infty \Phi(t) t^{1/2} dt < \infty$ componentwise,

the process

$$\sqrt{T} \left(\frac{1}{T} N_{Tv} - v(Id - K)^{-1} \mu \right) \quad v \in [0, 1]$$

converge in law for the Skorokod topology to

$$(Id - K)^{-1} \Sigma^{1/2} W_v, \quad v \in [0, 1],$$

where $(W_v)_{v \in [0, 1]}$ is a standard d -dimensional Brownian motion and Σ is the diagonal matrix such that $\Sigma_{ii} = ((Id - K)^{-1} \mu)_i$.

Theorem 2 is obtained as a consequence of this result for the special case $d = 2$ and $\Phi = \begin{pmatrix} 0 & \varphi \\ \varphi & 0 \end{pmatrix}$. In the following we give the sketch of the proof for

Theorem 13. Let $N^{(h)} = (N_1^{(h)}, \dots, N_d^{(h)})$ be a d -multivariate Hawkes process defined by his intensity $\lambda^{(h)} = (\lambda_1^{(h)}, \dots, \lambda_d^{(h)})$:

$$\lambda^{(h)}(t) = \mu(t) + \int_{[0, t)} \Phi(t-s) dN^{(h)}(s) \quad (2.41)$$

where $\mu(t) = \mu + h(t)$, where h is a positive function with support in $[a, b]$ (i.e. $h(t) = 0, \forall t \notin [a, b]$). Under the assumptions:

- the spectral radius of the matrix $K = \int_0^\infty \Phi(t) dt$ is inferior to 1, $\rho(K) < 1$,
- $\int_0^\infty \Phi(t) t^{1/2} dt < \infty$ componentwise,
- h bounded and $\int_a^b h(s) ds < \infty$ componentwise,

the process

$$\sqrt{T} \left(\frac{1}{T} N_{Tv}^{(h)} - v(Id - K)^{-1} \mu \right) \quad v \in [0, 1]$$

converge in law for the Skorokod topology to

$$(Id - K)^{-1} \Sigma^{1/2} W_v, \quad v \in [0, 1],$$

where $(W_v)_{v \in [0, 1]}$ is a standard d -dimensional Brownian motion and Σ is the diagonal matrix such that $\Sigma_{ii} = ((Id - K)^{-1} \mu)_i$.

The proof of Theorem 12 is technical. The central idea of the proof is the next decomposition:

$$X_t := N_t - \mathbb{E}[N_t] = M_t + \int_0^t \Psi(t-s) M_s ds, \quad (2.42)$$

where $(M_t)_{t \geq 0}$ is the martingale

$$M_t = N_t - \int_0^t \lambda_s ds.$$

Exact the same identity holds for $N^{(h)}$, the proof is identical as for N :

$$X_t^{(h)} := N_t^{(h)} - \mathbb{E}[N_t^{(h)}] = M_t^{(h)} + \int_0^t \Psi(t-s) M_s^{(h)} ds, \quad (2.43)$$

where $(M_t^{(h)})_{t \geq 0}$ is the martingale

$$M_t^{(h)} = N_t^{(h)} - \int_0^t \lambda_s^{(h)} ds.$$

In the following, we give the steps of the proof of Theorem 12 and we point out where are changes in order to prove Theorem 13. Before going further, let us explain why the stability condition, *i.e.* the convergence after rescaling, holds also for $N^{(h)}$. We disturb the intensity λ by a function h which is integrable and with compact support. It is natural that after rescaling this quantity vanish when the “limit” parameter T tends to infinity.

Theorem 12 is the consequence of the next two theorems:

Theorem 14. *Work in the same setting as in Theorem 12. We have:*

$$\sup_{v \in [0,1]} \|T^{-1}N_{Tv} - v(Id - K)^{-1}\mu\| \rightarrow 0 \text{ as } T \rightarrow \infty$$

almost-surely and in $L^2(\mathbb{P})$.

and

Theorem 15. *Work in the same setting as in Theorem 12. We have:*

$$\sqrt{T}(N_{Tv} - \mathbb{E}[N_{Tv}]), \quad v \in [0, 1]$$

converge in law for the Skorokod topology to

$$(Id - K)^{-1}\Sigma^{1/2}W_v, \quad v \in [0, 1].$$

Theorem 14 is proved using two lemmas:

Lemma 2. *Let $p \in [0, 1]$ and assume $\int_0^\infty t^p \Phi(t) dt < \infty$ componentwise. Then*

$$T^p (T^{-1}\mathbb{E}[N_{Tv}] - v(Id - K)^{-1}\mu) \rightarrow 0 \text{ as } T \rightarrow \infty$$

uniformly in $v \in [0, 1]$.

Lemma 3. *There exist a constant $c(\mu, \Phi)$ such that for all $t, \Delta \geq 0$:*

$$\mathbb{E}[\sup_{t \leq s \leq t+\Delta} \|M_s - M_t\|^2] \leq c(\mu, \Phi)\Delta.$$

In order to prove a similar result as Lemma 2 for $N^{(h)}$, we use Theorem 11. We obtain:

$$\begin{aligned} \mathbb{E}[N_T^{(h)}] &= v(Tv) + \int_0^{Tv} \Psi(Tv - s)v(s)ds = \\ &= \mu Tv + \int_0^{Tv} h(s)ds + \left(\int_0^{Tv} \Psi(Tv - s)sds \right) \mu + \int_0^{Tv} \Psi(Tv - s)h(t - s)ds. \end{aligned}$$

But h has compact support, is integrable and bounded, so

$$\frac{1}{T^{1-p}} \left(\int_0^{Tv} h(s)ds + \int_0^{Tv} \Psi(Tv - s)h(t - s)ds \right) \rightarrow 0 \text{ as } T \rightarrow \infty.$$

Since $\mathbb{E}[N_T v] = \mu Tv + \left(\int_0^{Tv} \Psi(Tv - s)sds \right) \mu$ (obtain by taking $h \equiv 0$ in Theorem 11), we have same result as in Lemma 2 for $N^{(h)}$.

The proof of Lemma 3 is based on the Doob inequality and Theorem 11. In order to show the same property for $M^{(h)}$ we use the same technics as in Bacry *et al.* [16]

$$\mathbb{E}[\sup_{t \leq s \leq t+\Delta} \|M_s^{(h)} - M_t^{(h)}\|^2] \leq 4 \sum_{i=1}^d \mathbb{E}[(M_{i,t+\Delta}^{(h)} - M_{i,t}^{(h)})^2].$$

But $\mathbb{E}[(M_{i,t+\Delta}^{(h)} - M_{i,t}^{(h)})^2] = \mathbb{E}[N_{i,t+\Delta}^{(h)} - N_{i,t}^{(h)}]$ since the quadratic variation of the martingale $(M_{i,t}^{(h)})_{t \geq 0}$ is the process $(N_{i,t}^{(h)})_{t \geq 0}$. Since h is bounded and integrable, we can conclude as in [17].

The proof of Theorem 15 is based on the next lemma:

Lemma 4. *The martingale $M^{(T)} := (T^{-1/2}M_{Tv})_{v \in [0,1]}$ converge in law for the Skorokod topology to $(Id - K)^{-1}\Sigma^{1/2}W_v$.*

In order to prove the same result holds for $M^{(T,h)} := (T^{-1/2}M_{Tv}^{(h)})_{v \in [0,1]}$, we remark (as in [17]) that:

- $[M_i^{(T,h)}, M_j^{(T,h)}]_v = 0$ for $i \neq j$ since the processes $N_i^{(h)}$, $1 \leq i \leq d$, are supposed to have no common jump.
- $[M_i^{(T,h)}, M_i^{(T,h)}]_v = \frac{1}{T}N_i^{(h)} \rightarrow \Sigma_{ii}$ in $L^2(\mathbb{P})$ by Theorem 14 for $N^{(h)}$.

The rest of the proof is based on the decomposition (2.42) and Lemma . Since similar properties are satisfied by $N^{(h)}$, we finish the sketch of proof for Theorem 13.

Market impacts and the life cycle of investors orders

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3.1 INTRODUCTION

Interactions between the Price Formation Process (PFP) and market microstructure are of paramount importance: since regulatory evolution (Reg NMS in the US and MiFID 1 and now MiFID 2 in Europe, promoting competition among exchanges via an extensive use of electronic order books) and the 2008 liquidity crisis, academics and market participants try to understand how to align the order book dynamics with the interests of final investors.

The market impact of large orders (or *meta-orders*) is at the middle of the discussion, since market microstructure evolution should target to decrease it. That for, one should be clear about what is the market impact of such large meta-orders. On the one hand it seems reasonable the information processed by investors, drives their decisions. Hence the price should move during the execution of their meta-orders, in a detrimental decision (*i.e.* up if they buy, or down if they sell). On the other hand, large orders lead to liquidity imbalance, mechanically moving the price in a detrimental decision. Part of this mechanical move generates trading costs. And trading costs prevent investors to initiate some trades, or at least decreases the performance of their trades.

Controlling the market impact is essential for investors whose volume is much larger than the available liquidity in the limit order book. A common strategy consists in splitting in small orders (*child order* or *atomic order*) and execute it incrementally. Nowadays, algorithmic high-frequency trading is the new way of executing those meta-orders.

The gap between informational move and mechanical reaction to trading pressure can be empirically investigated for few years only, because of the recent availability of large databases of investor's meta-orders. Very specific databases are needed to give an empirical answer to the nature of market impact: accurate market data are not enough, a clear identification and time stamping of meta-orders is needed too.

In this paper we report measures made on a database coming from the brokerage arm of a large European investment bank, whose trading flow was around 5% of investors' flow at the time the meta-orders have been recorded. As it is detailed in Section 3.2, this database is made of very coherent meta-orders in terms of trading style, trading universe and market context.

This paper provides information on the impact of large orders on the PFP at many scales:

- at the scale of each meta-order: the *temporary impact* and its relationship with explanatory variables are documented;
- at a lower time scale: the price reaction to trading pressure (*i.e.* the *transient impact*) is measured, and the relaxation of prices once meta-order ends (*i.e.* the *impact decay*) is also studied.
- at a larger scale: we zoom out days after the meta-order ends to document the *permanent impact* in an attempt to disentangle it from the informational price move.

Besides, we use the model introduced in Chapter 2 to discuss stylized facts discovered in the database. It is based on Hawkes processes, since they have demonstrated in previous papers to be well suited for order book dynamics modeling [89, 137, 127, 56, 134].

3.2 DEFINITIONS, DATABASE AND MARKET IMPACT ESTIMATION PRINCIPLES

Basic definitions

Let Ω be the set of all the available meta-orders which are entirely executed in a single day. Each meta-order is made of several transactions, i.e., several *atomic orders*. Let us consider $\omega \in \Omega$ a meta-order executed on stock S and on a given day d . It is characterized by several variables :

- $\epsilon(\omega)$: the sign of the meta-order: +1 for a buy and -1 for a sell,
- $N(\omega)$: its number of *atomic orders*,
- $v_i(\omega)$: the volume of its i^{th} atomic order ($i \in [1, N(\omega)]$),
- $v(\omega) = \sum_{i=1}^{N(\omega)} v_i(\omega)$: its total volume,
- $t_i(\omega)$: the time of its i^{th} atomic order,
- $T(\omega) = t_{N(\omega)} - t_1$: its total duration,
- $V_D(\omega)$: the daily volume executed on the market, on the day d and on the stock S
- $n(\omega)$: the number of transactions, on the day d and on the stock S
- $V(\omega)$: the volume exchanged on the market, on the stock S , during the duration of ω , i.e., during the time-interval $[t_1, t_1 + T(\omega)]$,
- $r(\omega) = v(\omega) / V(\omega)$: its trading rate.
- $r_D(\omega) = v(\omega) / V_D(\omega)$: its daily participation,
- $\psi(\omega)$ stands for daily average spread of the stock S and on the day d . It is computed using average on the transactions, weighted by their volumes and expressed in basis points,
- $\sigma_D(\omega)$: the Garman-Klass annualized daily volatility of the stock S and on the day d .
- $\sigma(\omega) = \frac{\sigma_D(\omega)}{\sqrt{T(\omega)}}$: the volatility during the meta-order (deduced from the daily volatility just above).

The main meta-order database Ω

The database is made of nearly 300.000 meta-orders, selected from a huge database of around 400.000. The selected meta-orders have been traded electronically by this large broker during year 2010 on European markets. We built three databases from the original 400.000 once to have as much as possible orders for each time scale (see Table 3.1):

- For intraday studies, we only kept orders traded by trading algorithms which trading speed is as much as possible independent from the market conditions, to avoid sudden accelerated trading rates to have a hidden influence of the price moves. Hence we kept VWAP, PoV and only few Implementation Shortfall instances (see Appendix 3.8 for details).
- We kept orders large enough to protect our data from noise.
- For small time scale study, and especially for the impact decay, we needed the meta-order to stop halfway to the market close, to observe prices relaxation long enough between the end of the meta-order and the close.

We end up with four main databases: $\Omega^{(te)}$ to study temporary impact, $\Omega^{(tr)}$ to study transient impact and impact decay, and $\Omega^{(de)}$ to study decay and $\Omega^{(day)}$ to study daily effects. In the next paragraph, we clearly specify the filters we apply

in order to obtain those different databases. Table 3.1 briefly recalls those filters. In Appendix 3.8 one finds statistics concerning the different databases.

Database for intraday market impact $\Omega^{(day)}$

Often, a large single-day meta-order is split in several pieces and recorded in Ω as different meta-orders. Putting together these pieces we obtain our intraday data-base $\Omega^{(day)}$. This data-base contains 299.824 meta-orders and it will be used in section 3.3 for the permanent market impact.

Database for temporary market impact $\Omega^{(te)}$

We selected a subsample of the intraday database $\Omega^{(day)}$ containing meta-orders with minimum 10 atomic orders and removing those that are not consistent with the market data (volume and time order arrival). We finally obtain 157.061 different occurrences of meta-orders. This data-base is used for the temporary market impact (section 3.3).

Database for transient market impact $\Omega^{(tr)}$

In order to eliminate some extreme effects and to stabilize the results, we removed from previous database $\Omega^{(te)}$, meta-orders with extreme characteristics. More precisely, we built a new database $\Omega^{(tr)}$ by selecting the meta-orders ω of $\Omega^{(te)}$ such that:

- $T(\omega) > 3$ minutes,
- $r(\omega) \in [3\%, 40\%]$,
- $r_D(\omega) \in [0.1\%, 20\%]$ and
- $n(\omega) \geq 500$ (which removed illiquid stocks).

This database is used for the transient market impact (section 3.4) and contains $M = 92.100$ trades out of the original 157.061 meta-orders.

Database for market impact decay $\Omega^{(de)}$

In order to estimate the market impact decay curve (section 3.5), we need to subsample the database $\Omega^{(tr)}$ selecting only the meta-orders of which do not end too late in the day. Indeed, we need to have enough time after the ending-time of the meta-order in order to study the decay of the impact. More precisely, we built a new database $\Omega^{(de)}$ by selecting the meta-orders ω of $\Omega^{(tr)}$ such that

- $t_0(\omega) + 2T(\omega)$ is a time that takes place before the closing time of the corresponding asset.

The final sample contains 61.671 meta-orders out of the original 92.100 meta-orders in $\Omega^{(tr)}$.

Market impact definitions and Latent price

The *market impact curve* of the meta-order ω quantifies the magnitude of the (relative) price variation which is **due to the meta-order** ω between the starting time of the meta-order $t_1(\omega)$ and the current time t . Theoretically, in order to estimate this curve one should have access to the price variation that would have occurred if the meta-order was not sent. In the following we shall refer to that latter price as the *latent price*.

Databases Ω	Filters	Remaining meta-orders
Original database Ω	\emptyset	398.812
Intraday database $\Omega^{(day)}$	smooth execution condition	299.824
Database for temporary $\Omega^{(te)}$	10 atomic orders minimum consistency with HF market data	191.324 157.061
Database for transient $\Omega^{(tr)}$	$n \geq 500$	150.100
	$T > 3$ minutes	134.529
	$r \in [3\%, 40\%]$	94.818
	$r_D \in [0.1\%, 20\%]$	92.100
Daily data-base for transient $\Omega^{(de)}$	$t_0(\omega) + 2T(\omega)$	61.671

Table 3.1 – *Different databases and filters used to obtain them*

Let $\Delta P_t(\omega)$ be a proxy for the realized price variation between time $t_1(\omega)$ and time $t_1(\omega) + t$. Like all other authors, we assume the market impact of a meta-order ω is additive, meaning the price move between $t_1(\omega)$ and time $t_1(\omega) + t$ is the sum of a mechanical impact $\Delta P_t^{(\eta)}(\omega)$ due to the trading pressure of the meta-order and an exogenous component W_t (the latent price):

$$\Delta P_t(\omega) = \Delta P_t^{(\eta)}(\omega) + W_t, \quad (3.1)$$

where by definition

$$\Delta P_t^{(\eta)}(\omega) = \Delta P_t(\omega) - W_t, \quad (3.2)$$

corresponds to the price variation due to the meta-order ω . When the meta-order is a buy (resp. sell) meta-order, one expects this price to go up (resp. down), consequently it is very natural to define the impact as

$$\eta_s(\omega) = \varepsilon(\omega) \Delta P_{sT(\omega)}^{(\eta)}, \quad s \geq 0, \quad (3.3)$$

in which, for synchronicity convenience, we have rescaled the time in order that the time $s = 1$ always corresponds to the ending-time of the meta-order. Let us point out that the choice for the proxy is discussed later in this section. As explained later on, we shall use all along the paper, the so-called *return proxy* defined by (3.8) in which P_t corresponds to the last traded price.

The market impact $\eta(\omega)$ is thus not directly observable; it is “noised” by price moves more linked to market moves than to the trading flow generated by ω . Nevertheless it cannot be excluded that W_t is 100% independent of the decision the investor took initiating the trade. Like it has been said previously, the investor share information at the root of his decision with other investors, and part of W_t independent from the *trading flow* generated by ω , is not independent to the *decision* that generated ω .

This “informational pollution” of market impact measurement is eluded in some papers (like [10]), and discussed in others (like [104, 66]). In this paper, we will neglect it for intraday measurements (Sections 3.3, 3.4 and 3.5), but take it into account for daily estimates (Section 3.7).

Finally, we introduce the following terminology in order to address the different parts of the market impact curve :

- $\eta_{s=1}(\omega)$ as the *temporary market impact*, i.e., the impact at the end of the meta-order,
- $\{\eta_s(\omega)\}_{0 \leq s \leq 1}$ as the *transient market impact curve*, the impact curve during the execution of the meta-order,

- $\{\eta_s(\omega)\}_{1 \leq s}$ as the *decay market impact curve*, the impact curve after the execution of the meta-order and
- $\eta_{s \gg 1}(\omega)$ as the *permanent effect*. What we exactly mean by the limit $s \gg 1$ will be made clearer in Section 3.7. For now, it is sufficient for the reader to know that it refers to an extraday time limit, sufficiently far from the ending-time of ω , so that its impact can be considered as constant.

Market impact estimation principles - First estimations

Estimation via averaging. Usual market impact estimations use an “averaging” of equation (3.3) over “slices” of a variable of interest. Thus, we shall compute the estimation of the impact through

$$\hat{\eta}_s = \frac{1}{\#\Omega} \sum_{\omega \in \Omega} \Delta P_{sT(\omega)}(\omega) := \mathbb{E}[\Delta P_{sT(\omega)}(\omega)], \quad (3.4)$$

where $\#\Omega$ denotes the total number of available meta-orders. This approach relies on the assumption that the “exogenous market moves” W_{sT} cancels once averaged. Meaning that as a random variable, they are centered and with finite variance (*i.e.* the central limit theorem can be applied).

Studying the influence of a factor. Following the same lines, when one studies the influence on the impact of a factor $X(\omega)$ (e.g., the daily participation rate $X(\omega) = r_D(\omega)$), we shall condition the impact estimation by the fact that the value of $X(\omega)$ belongs to an interval I , *i.e.*,

$$\hat{\eta}_s(\omega | X(\omega) \in I) = \mathbb{E}[\Delta P_{sT(\omega)}(\omega) | X(\omega) \in I]. \quad (3.5)$$

Note that in this case, if we want the central limit theorem to be applied, it is needed $\mathbb{E}(\epsilon(\omega)W_T(\omega)|X(\omega) \in I) = 0$ and not $\mathbb{E}(W_T(\omega)) = 0$. This is a strong supposition and is not always verified in practice.

If an investor systematically buy or sell large quantities when he has reliable information on future price moves, this assumption will not hold. It will remain that $\hat{\eta}_s$ is the sum of the “pure” market impact and an “informational bias”. We will not try to remove this bias in the intraday studies.

Coming back to the averaging approach, note that averaging thanks to quantiles of one explanatory variable allows to identify a dependence with this variable only. To be able to regress the market impact on three variables, one will need to use K quantiles for each variable, *i.e.* averaging on K^3 subsets. The number of meta-orders used to estimate one $\hat{\eta}_T(k_1, k_2, k_3)$ will be divided by K^3 .

Last but not least, if linear regressions are used to explain $\hat{\eta}$ by the averaging variables (or any nonlinear transform of them), the usual statistical measures of regression quality do not have the same meaning. The “ R^2 ” for instance can be very high, even if the relation between the impact and the regressors is very noisy (since the noise would have been decreased by the averaging across quantiles).

This averaging methodology is nevertheless useful to draw figures (like Figure 3.1) and obtain qualitative results. It is used through this paper.

Direct regression. An alternative approach is to fit directly an explicit model on equation (3.5). Say we want to fit the parameters of a power law linking the market impact to the daily participation of the meta-order. It leads to the following parametric version of (3.3):

$$\eta_s = a \cdot X(\omega)^\gamma + \epsilon(\omega)W_{sT}. \quad (3.6)$$

As far as we assume once more the exogenous prices moves are independent enough from ω to be averaged, we can select a distance $d(\cdot, \cdot)$ in the space of returns ΔP and use any minimization method to obtain “fitted” parameters (a^*, γ^*) :

$$(a^*, \gamma^*) = \underset{(a, \gamma)}{\operatorname{argmin}} \mathbb{E}_\omega d(\Delta P_T(\omega), a \cdot R_T(\omega)^\gamma + \epsilon(\omega)W_T)^2. \quad (3.7)$$

If $\mathbb{E}(\epsilon(\omega)W_T) \neq 0$ then a^* and γ^* will be biased.

This approach uses all the points in the database (and not their averaged version), and thus produces more accurate results. It nevertheless relies on the same independence between meta-order initiation and market moves to be unbiased.

Gaussianity of the market moves W_t is not needed, and it is possible to “take care of it” by selecting an appropriate distance d . In this paper we use the usual L^2 distance, and the LASSO¹ one (more compatible with fat tails and rare but intense events).

The usual quality metrics (like the “ R^2 ”) will be more reliable but worst (since the signal/noise ratio for market impact estimation is very low). Notice the meaning of the residuals of such a regression (i.e. ΔP minus its model) is straightforward: it is the remaining price move once our estimation of the market impact has been removed. They allow to test the dependence of the remaining moves to another variable. This is not accessible to the averaging approach.

Choice of the proxy - First estimations. When averaging on meta-orders of different sizes and different durations, as done above, it seems natural to perform some kind of rescaling on time and price, so that market impact curves are somewhat homogeneous. Rescaling in time has already been taken care by the definition (3.3) (indeed, the ending time of any meta-order ω always correspond to $s = 1$). Rescaling of the price should be taken care by the choice of the proxy $\Delta P_t(\omega)$ of the relative price variation during the time period $[t_1(\omega), t_1(\omega) + t]$. Mainly three proxies (or combination of these three proxies) are used in the academic literature (in the following P_t refers indifferently to the last traded price or the mid-price) :

- The *return proxy*

$$\Delta P_t^{(ret)}(\omega) = \frac{P_{t_1(\omega)+t} - P_{t_1(\omega)}}{P_{t_1(\omega)}} \quad (3.8)$$

- The *spread relative proxy*

$$\Delta P_t^{(spread)}(\omega) = \frac{P_{t_1(\omega)+t} - P_{t_1(\omega)}}{\psi(\omega)} \quad (3.9)$$

- The *log-return proxy*

$$\Delta P_t^{(ln)}(\omega) = \ln(P_{t_1(\omega)+t})(\omega) - \ln(P_{t_1(\omega)})(\omega) \quad (3.10)$$

1. See [71] for more detail on LASSO method

Thus for instance, the return proxy is used by Almgren [10] and by Bershova and Rakhlin [27]. In order to obtain more heterogeneous prices, some authors (Lillo and Farmer [104]) prefer to combine the log-return and the spread relative proxies : they choose to divide the log-return proxy by $\psi(\omega)$. In Fig. 3.1, we show empirically that whatever the choice for this proxy and whatever the choice for the price P_t (either last traded price or mid-price), the average impact curve "looks" very similar. In the remaining of the paper, we choose to use the last-

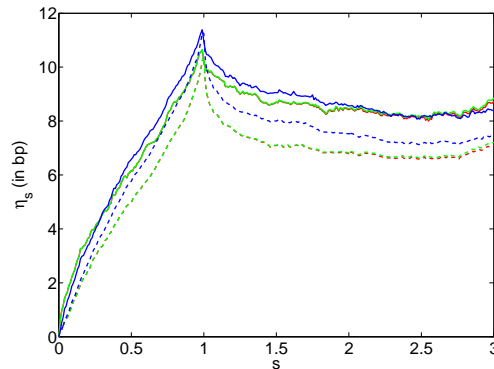


Figure 3.1 – The average market impact curve (3.3) using different proxy for the relative price variation $\Delta P_t(\omega)$ and different prices P_t . Each curve corresponds to a color (blue, red or green) and a type of line (either solid or dashed). The type of line codes the price P_t used : dashed-lines for the last traded price and solid-lines for the mid-price. The color of the line codes the relative price variation proxy used: red for the return proxy (3.8), blue for the spread relative proxy (3.9) and green for the log-return proxy (3.10).

traded price along with the return proxy.

3.3 THE TEMPORARY MARKET IMPACT

Selection of the factors

The temporary market impact has been mainly studied from three viewpoints:

- as the main source of trading costs. The obtained model can be then used in an optimal trading scheme (see [10], [60] and [91] have been written by authors involved in optimal trading: [9], [61] and [31]), or used by an investment firm to understand its trading costs (like in [51], [27] or [101] written by author involved in investment firms).
- It can be viewed as an important factor of price discovery and studied as such, often by economists, like the seminal work of Kyle [88] or later in [72] or [54].
- Last but not least, statistical tools have been built to be able to estimate the temporary market impact at the scale of one trade (see [17] or [49]). The implicit conditioning of such "atomic" orders by meta-orders is sometimes discussed in such papers, but it is not their main goal.

One common point of these studies is that the temporary market impact of a meta-order ω of size $\nu(\omega)$ includes three main components :

- A component reflecting the *size of the meta-order*, resized by something reflecting the volume in the order book of the traded security. The daily

participation $r_D(\omega)$ should capture most of the dynamics of this component. This size has to be rebased using the daily volume V_D or the traded market volume during the meta-order V , leading to two potentially explanatory variables: the daily participation $r_D = v/V_D$ or the trading rate $r = v/V$.

- A component rendering the *uncertainty on the value of the traded underlying* during the meta-order. The volatility during the meta-order $\sigma(\omega)$ (defined by $\sigma(\omega) = \frac{\sigma_D(\omega)}{\sqrt{T(\omega)}}$) is a typical measures for this.
- And a last component that captures the *information leakage* generated by the meta-order, a good proxy being the duration $T(\omega)$.

We accordingly define the four potential explanatory variables $X_1 = r_D(\omega)$, $X_2 = r(\omega)$, $X_3 = \sigma(\omega)$ and $X_4 = T(\omega)$. Let us point out that all authors found multiplicative relations between each of these components and their corresponding factor, so we expect a linear dependence of the temporary market impact $\eta_{s=1}(\omega)$ on the logarithm of these factors.

As an example, the left plot of Fig. 3.2 displays (using log-log scales) the scatter plot of $\eta_{s=1}(\omega)$ versus the daily participation $X_1(\omega) = r_D(\omega)$ when ω is varying (meta-orders for which $\eta_{s=1}(\omega)$ is negative have been discarded). It clearly shows the influence of r_D on the impact: the higher the daily participation rate the higher the impact. In the following Section, we explain our methodology for studying the regressing the impact on the different factors mentioned above.

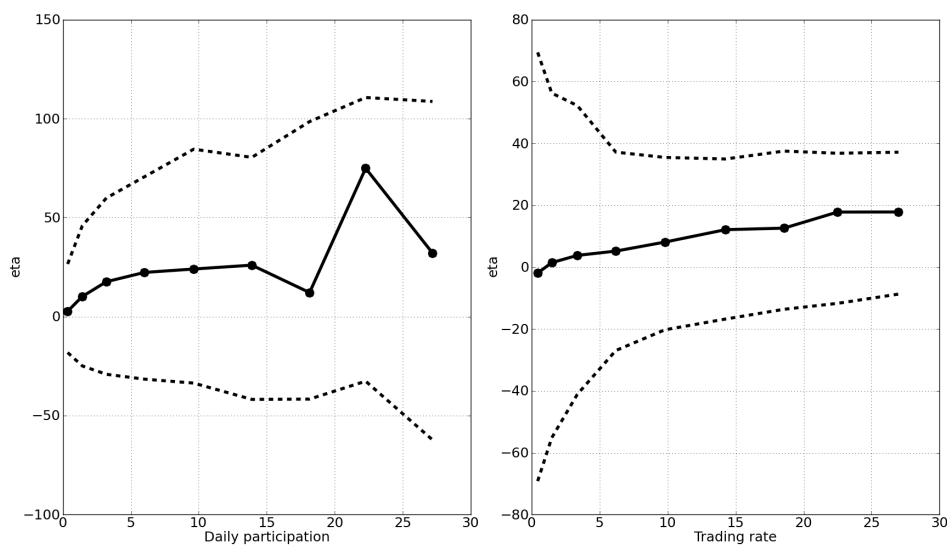


Figure 3.2 – The estimated market impact $\hat{\eta}_{s=1}(\omega)$ as a function of the traded volume normalized by the daily volume (left: $X_1 = r_D$) or traded market volume during the meta-order (right: $X_1 = r$). Each point is the average of one decile of the X variable, dotted lines are 25% and 75% quantiles, showing the amplitude of market moves.

Numerical results

To study temporary market impact and its dependence to explanatory variables, we followed the direction estimation approach described in Section 3.2. We tested the daily participation r_D as first variable, since it has been identified as significant by other papers. It means we fit equation:

$$\eta_1(\omega) = a \cdot r_D(\omega)^Y + \epsilon(\omega)W_T,$$

and found an exponent $\gamma \approx 0.449$ using the L^2 distance and a lower exponent (around 0.40) using the LASSO one. As a reference the log-log regression on 100 bins gives an estimated power of 0.542. Table 3.2 gives all results.

The implication of the difference between the two exponents is that the joint distribution of η_1 and r_D is skewed to large values of η_1 . The L^2 distance has no other choice than to render this skewness by setting a high value to γ , while the LASSO one focuses on the center of the distribution. The source of this skewness could stem from an informational effect as a dependence between ϵ (the sign of the meta-order) and W_T (the market move). But we do not have enough elements to conclude.

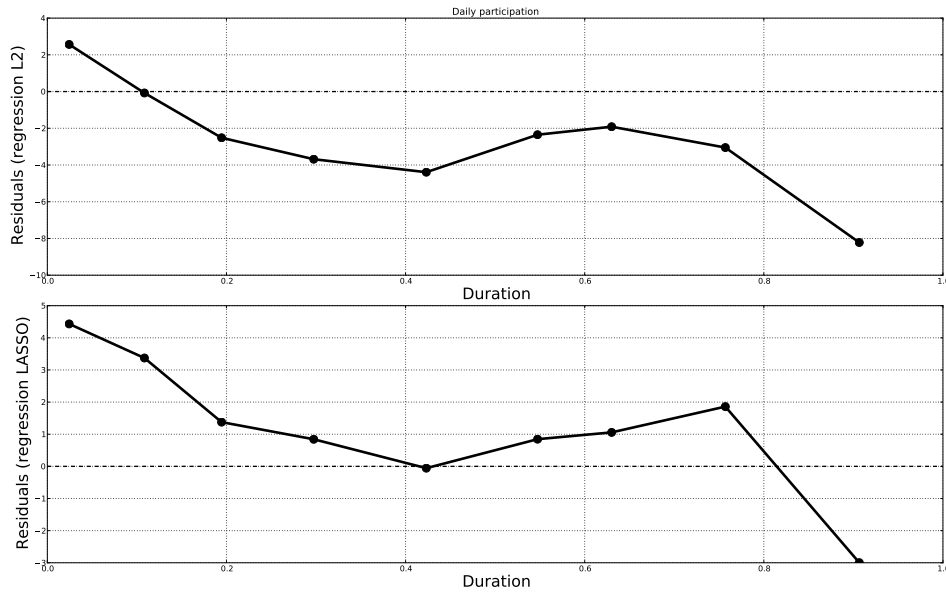


Figure 3.3 – Trace of the duration on the residuals of a regression of $\hat{\eta}$ on the daily participation. Top: using an L^2 metric, bottom: using a LASSO metric.

Once this power coefficient is fit, the effect of any other variable X can be explored and shown by averaging the residuals over quantiles of X . We will name the associated chart the *trace of X on the residuals*. Figure 3.3 shows the trace of the duration of the meta-orders on the residuals. A negative slope is obvious. Moreover, one can notice the bump around 0.4 days (meaning 6 hours). We suspect it stems from the opening of NY markets, 6 hours after the opening of European ones. At the open of US markets, volatility is higher and the dependence between the side of the meta-order and the market move can be higher too.

To confirm the dependence in T , we fit a power law on the trading rate r instead of the daily participation r_D :

$$\eta_1(\omega) = a \cdot r(\omega)^\gamma + \epsilon(\omega)W_T.$$

We found respectively power 0.43, 0.33 and 0.42 for loglog regression, L^2 minimization and LASSO minimization. The trace of T on the residuals on this regression (see Figure 3.4) exhibits a positive slope, confirming the way duration T can be used in combination to daily participation or trading rate to improve the modeling of impact.

We performed direct fit of other explanatory variables and found the usual dependencies in σ and ψ (see Table 3.2).

Regression	parameter	value (log-log)	value (L2)	value (LASSO)
(R,0)	Daily participation	0.542	0.449	0.400
	Trading rate	0.435	0.330	0.426
(R,2)	Daily participation	0.529	0.529	0.529
	volatility	0.961	0.961	0.961
(R,3)	Daily participation	0.401	<i>nan</i>	0.401
	Trading rate	0.285	<i>nan</i>	0.285
(R,4)	Trading rate	0.317	0.317	0.317
	volatility	0.878	0.878	0.878
(R,5)	Daily participation	0.593	0.541	0.593
	T	-0.230	-0.347	-0.230
(R,6)	Trading rate	0.319	<i>nan</i>	0.319
	spread	0.571	<i>nan</i>	0.571
(R,7)	Daily participation	0.438	0.438	<i>nan</i>
	spread	0.276	0.276	<i>nan</i>
(R,8)	Trading rate	0.369	0.561	0.453
	T	0.152	0.241	0.229

Table 3.2 – Main results for studying the influence of the different factors on the temporary market impact.

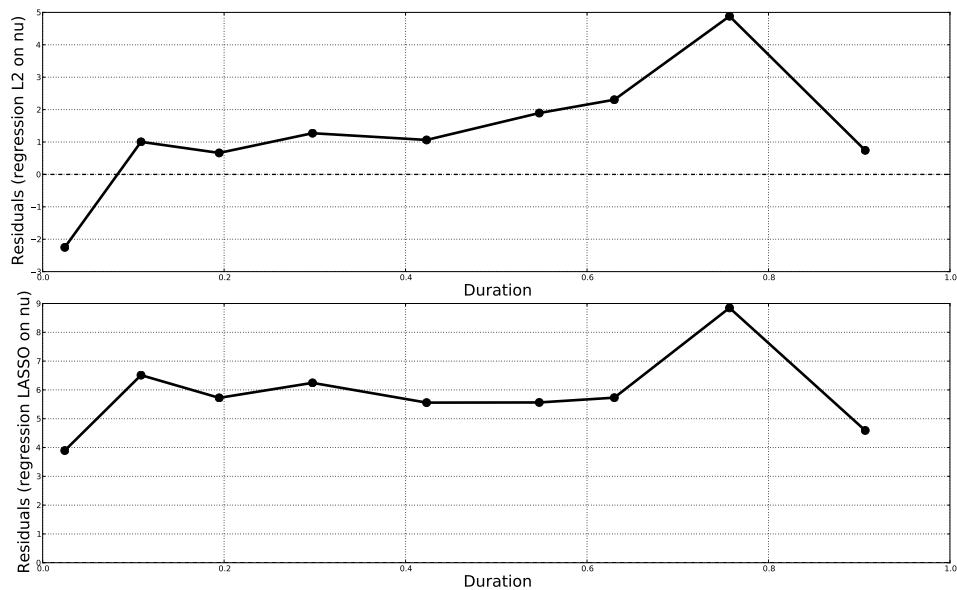


Figure 3.4 – Trace of the duration on the residuals of a regression of $\hat{\eta}$ on the trading rate. Top: using an L^2 metric, bottom: using a LASSO metric.

3.4 THE TRANSIENT MARKET IMPACT CURVE

A well known stylized fact: concavity of transient market impact

Previous section confirms, as many other empirical studies before us, that the temporary market impact of a meta-order of size ν is proportional to $\sqrt{\nu}$ (as predicted by the so-called square-root law). Thus, we expect the temporary market impact of the first half of the execution (the first $\nu/2$ contracts) to be more important than the one of the second half (the last $\nu/2$ contracts). Generalizing this argument to any portion of the meta-order, we expect the transient market impact curve to be a concave function of the time. The first empirical study confirming this intuition is due to Moro *et al.* [104], lately, it has also been confirmed by the work of Bershova and Rakhlin [27]. In both cases, behavior close to power-laws were found. Let us point out that the *latent order book model* of [128] can be seen as a possible qualitative explanation of this well established stylized fact. In this model the agents place limit orders only when the price is close enough to their vision of the price. Thus, more and more liquidity is revealed as the price is trending, it results in "slowing down" this trend.

In this section we confirm that the transient market impact curve is concave. Apart from this well known stylized fact, we study the link between the curvature of the transient market impact curve and the execution duration and the market prediction of meta-order sizes.

Numerical results

In this Section, we study the transient market impact $\eta_{s \leq 1}$. In practice, we sample our estimations on 100 points using a uniform sampling grid for s : $s_i = i/100, \forall i \in [0, 100]$ and compute the estimation $\hat{\eta}_{s \leq 1}$ as defined by (3.4) (in which Ω is replaced by $\Omega^{(tr)}$ defined at the beginning of this section). Fig. 3.5 illustrates this computation and shows that a power law behavior

$$\hat{\eta}_{s \leq 1} \propto s^{\beta^{(tr)}} \quad (3.11)$$

with $\beta^{(tr)} = 0.64$ provides an excellent fit (the (tr) subscript in $\beta^{(tr)} = 0.64$ stands for *transient*). Our empirical findings are compatible with Moro *et al.* [104] which found an exponent equal to 0.62 for meta-orders executed on London Stock Exchange (LSE) and 0.71 for meta-orders executed on Spanish Stock Exchange (BME, Bolsas y Mercados Españoles). Moreover, the market impact proxy used by the authors is similar to ours.

Concavity and execution duration. We shall now study the influence of the duration T of the meta-order on the transient market impact. In order to avoid spurious effects of other factors, we chose to work on the subset of meta-orders which correspond to a given range I_r of daily participation. We have checked that the so-obtained results do not significantly change when one changes I_r . Thus, in the following, we fixed $r_D \in I_r = [1\%, 3\%]$, selecting in this way 31.105 meta-orders. We then compute

$$\hat{\eta}_{s \leq 1}(\omega \mid T(\omega) \in I_T, r_D(\omega) \in I_r) = \frac{1}{M(I_T, I_r)} \sum_{\omega, T(\omega) \in I_T, r_D(\omega) \in I_r} \Delta P_{sT(\omega)}(\omega), \quad (3.12)$$

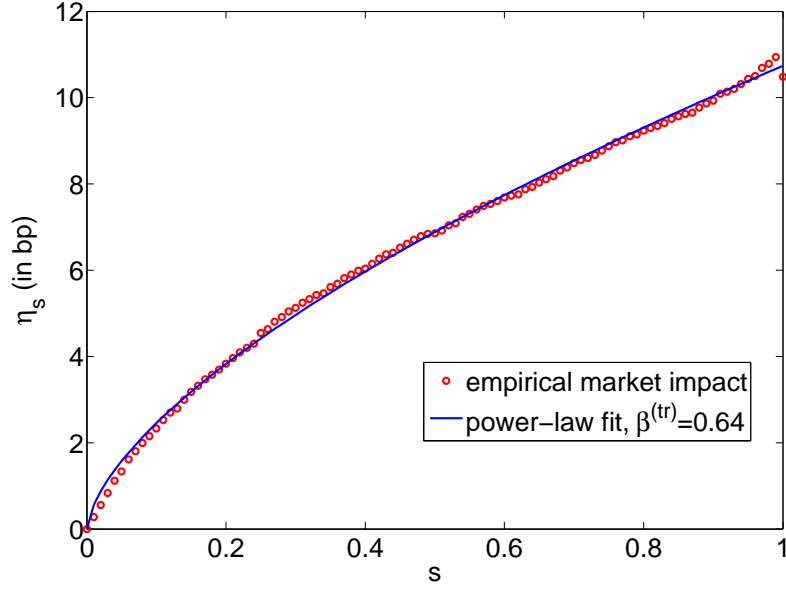


Figure 3.5 – The estimated transient market impact curve $\hat{\eta}_{s \leq 1}$ as defined by (3.4) (in which Ω is replaced by $\Omega^{(tr)}$ as defined in Section 3.2). A power law behavior $\hat{\eta}_{s \leq 1} \propto s^{\beta^{(tr)}}$ with $\beta^{(tr)} = 0.64$ provides an excellent fit.

where I_T is a range for the duration T and $M(I_T, I_r)$ the number of meta-orders ω such that $T(\omega) \in I$ and $r_D(\omega) \in I_r$. We studied $\hat{\eta}_{s \leq 1}(\omega \mid T(\omega) \in I_r, r_D(\omega) \in I_r)$ as a function of the interval I_T , choosing different intervals for I_T such that the number of meta-orders in each interval is approximately the same. More precisely we chose the 6 intervals (the duration are expressed in minutes) : $T \in [3, 15]$, $T \in [15, 30]$, $T \in [30, 60]$, $T \in [60, 90]$, $T \in [90, 300]$ and $T \in [300, 510]$, each containing around 6.000 meta-order occurrences. Fig. 3.6 - Fig. 3.11 show the transient market impact for each of these 6 groups. In order to point out the different regimes, on each so-obtained graph, we performed the power-law fit

$$\hat{\eta}_{s \leq 1}(\omega \mid T(\omega) \in I, r_D(\omega) \in I_r) \propto s^{\beta^{(tr)}} \quad (3.13)$$

leading to an estimation of the power-law exponent $\beta^{(tr)}$. The power-law fit is obtained by linear regression on a log-log representation. The corresponding log-log plots (for the first 4 figures, Fig. 3.6 - Fig. 3.9) can be seen on Fig. 3.12 - Fig. 3.15: the power law fit seems pretty accurate. In order to test the significance of our results we use bootstrap regressions drawing randomly 500 times 80% of the available $M(I_T)$ points (see [63] for references on bootstrap). Table 3.3 gives the estimated transient market impact (3.12) as a function of the renormalized time s for fixed participation $r_D \in I_r = [1\%, 3\%]$ and for different duration intervals I_T . On Fig. 3.6 - Fig. 3.10, we see that the larger T the larger the curvature of the transient market impact and the smaller the temporary market impact. On figure 3.11 decay seems to happen before the end of market order.

We observe that market impact is actually a multi-regime process. The first five figures, *i.e.* Fig. 3.6 - Fig. 3.10, show clearly that, for a fixed participation rate (as we already mentioned changing the participation rate interval does not affect the results), when the duration of a meta-order decreases

- the transient market impact of a meta-order increases and

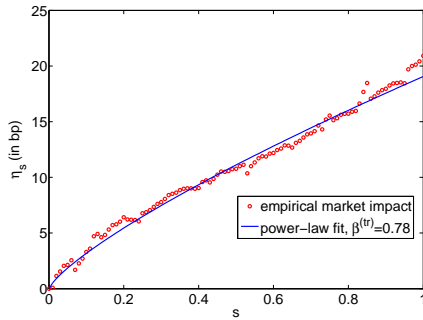


Figure 3.6 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [3, 15]$, $\beta^{(tr)} \approx 0.80$

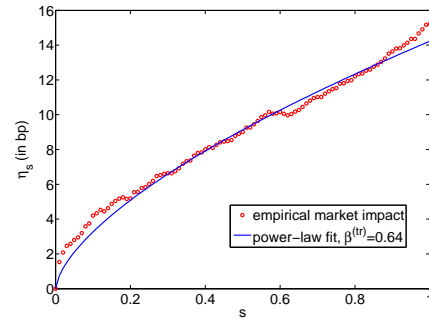


Figure 3.7 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [15, 30]$, $\beta^{(tr)} \approx 0.66$

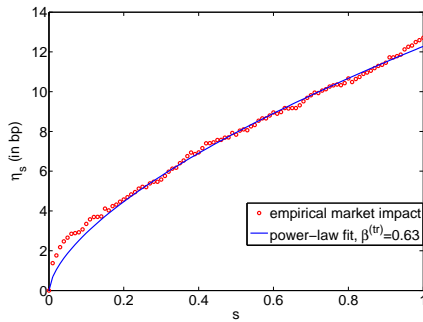


Figure 3.8 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [30, 60]$, $\beta^{(tr)} \approx 0.63$

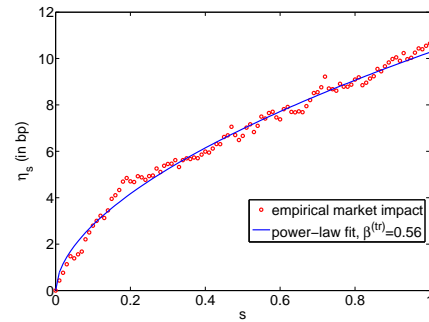


Figure 3.9 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [60, 90]$, $\beta^{(tr)} \approx 0.56$

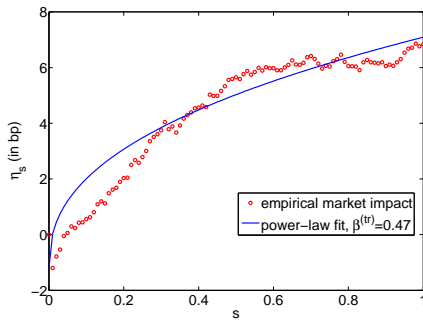


Figure 3.10 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [90, 300]$, $\beta^{(tr)} \approx 0.53$

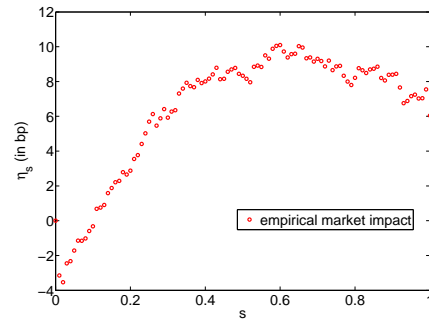


Figure 3.11 – Transient market impact curve with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [300, +\infty)$

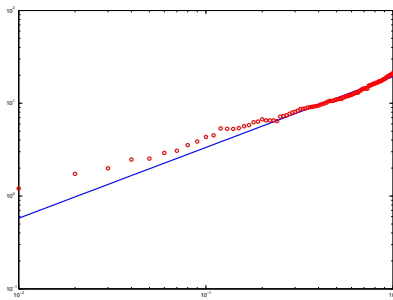


Figure 3.12 – Log-log plot of market impact curve for Fig. 3.6

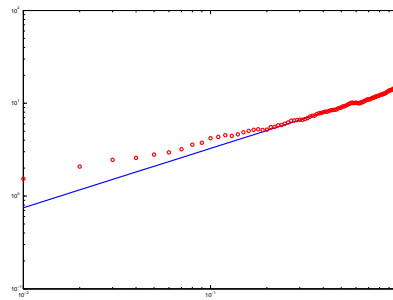


Figure 3.13 – Log-log plot of market impact curve for Fig. 3.7

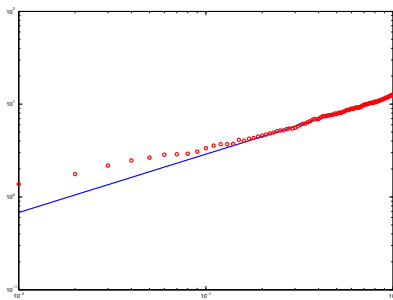


Figure 3.14 – Log-log plot of market impact curve for Fig. 3.8

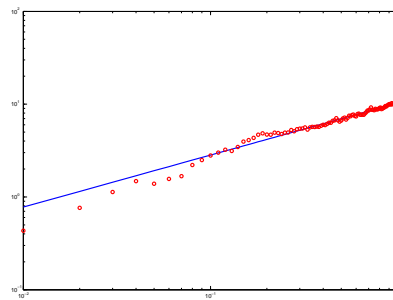


Figure 3.15 – Log-log plot of market impact curve for Fig. 3.9

- the curvature decreases leading to an almost linear transient market impact for small durations.

Thus, when executed faster, a meta-order seems to have a stronger and more linear impact. These results are rather intuitive: when a meta-order has a short duration, the market has hardly the time to "digest" it resulting into a strong linear impact (not time for relaxation). However, Fig. 3.10 seems to show that a kind saturation is reached before the end of the meta-order. Actually Fig. 3.11 surprisingly shows that when the duration T becomes very large, the market impact curve starts decaying *before* the end of the meta-order. From our knowledge, this is the first study pointing out this effect. One possible interpretation can be found within the *fair price* theory ([53]). Indeed, within this framework, the market ecology is explained by the perfect competition between long-term institutional investors who act on a common informational signal and generate meta-orders following a distribution p_N and market makers supposed very competitive and knowing the p_N distribution. As suggested in [48], this perfect competition is a utopia and an asymmetry on information is more realistic. In this case, market makers try to guess the p_N distribution. The market makers, as any others traders, use predictive tools based on statistical models detecting trends which are obviously noisy and even biased. When the execution time T of a meta-order is large enough, the market makers stop trading because T do not match with p_N prediction or, even if they perfectly estimate p_N , they are aware of the sensibility of their statistical tools.

Market prediction of meta-order sizes. In this paragraph, we want to study whether the market has or has no precise insights about the total size of a given meta-order before the end of its execution (apart of course from the uncondi-

Execution time	Mean	Q5%	Q25%	Q25%	Q75%	Q95%
$T = [3, 15]$	0.80	0.76	0.78	0.80	0.82	0.85
$T = [15, 30]$	0.66	0.62	0.65	0.66	0.68	0.70
$T = [30, 60]$	0.62	0.58	0.60	0.62	0.64	0.66
$T = [60, 90]$	0.55	0.49	0.52	0.56	0.58	0.62
$T = [90, 300]$	0.54	0.48	0.52	0.55	0.57	0.62

Table 3.3 – Statistics on the distribution of the power-law exponent $\beta^{(tr)}$ of the transient market impact estimation of meta-orders with a participation rate $r_D \in I_r = [1\%, 3\%]$ (see (3.13)). The exponent is estimated using log-log regression conditioned on different duration intervals. The larger T the larger the curvature of the transient market impact and the smaller the temporary market impact (see Fig. 3.6 - Fig. 3.10).

tionnal distribution of the meta-order sizes). In order to do so, we consider all the meta-orders with a given trading speed $\dot{v} = r_D/T$. Of course, these meta-orders correspond to different execution durations $T(\omega)$, and their volume $v(\omega)$ is basically proportional to their execution duration : $v(\omega) = T(\omega)\dot{v}/V_D$. If the market does not have precise insights about $v(\omega)$ (i.e., about $T(\omega)$) then, at a given time t_0 , there is no way it can differentiate between two meta-orders ω_1 and ω_2 such that $\dot{v}(\omega_1) = \dot{v}(\omega_2) = \dot{v}$ and $T(\omega_1) > t_0$, $T(\omega_2) > t_0$. Consequently the corresponding transient market impact curves should look alike on the time interval $[0, t_0]$.

This assertion above can be translated into the fact that the restrictions on $t \in [0, t_0]$ of $\eta_{t/T(\omega_1)}(\omega_1)$ and of $\eta_{t/T(\omega_2)}(\omega_2)$ should be very close. In order to test this assertion, we choose five groups of meta-orders \mathcal{A}_i ($i = 1, \dots, 5$) such that:

$$\mathcal{A}_i = \left\{ \omega \in \Omega^{(tr)} : r_D \omega \in [2^{i-1} r_0, 2^i r_0) \text{ and } T(\omega) \in [2^{i-1} T_0, 2^i T_0) \right\}, \quad (3.14)$$

where $r_0 = 0.25$ and $T_0 = 5$ seconds. Thus, all the selected meta-orders correspond, in a good approximation, to the same trading speed $\dot{v} = r_0/T_0 = 0.05s^{-1}$. Moreover \mathcal{A}_{i+1} ($\forall i \in [2, 5]$) corresponds to meta-orders with durations twice as large as those of \mathcal{A}_i . We then compute:

$$\hat{\eta}_{s \leq 1}^{(i)}(\omega \in \mathcal{A}_i) = \frac{1}{M(\mathcal{A}_i)} \sum_{\omega \in \mathcal{A}_i} \Delta P_{sT(\omega)}(\omega), \quad (3.15)$$

where $M(\mathcal{A}_i)$ is the number of meta-orders in \mathcal{A}_i . For each $i = 1, \dots, 4$, Fig. 3.16 - Fig. 3.19 show $\hat{\eta}_{s \leq 1}^{(i)}$ with the first half of $\hat{\eta}_{2s \leq 2}^{(i+1)}$. One can see that, in each of the subplots, the two market impact curves are very close, indicating that the market basically does not anticipate the size of the corresponding meta-orders. Let us point out that the same results would be obtained when changing r_0 and/or T_0 . Each subplot corresponds to $\hat{\eta}_{s \leq 1}^{(i)}$ and $\hat{\eta}_{2s \leq 2}^{(i+1)}$ for $s \in [0, 1]$ (see (3.15)). Top-left (resp. top-right) subplot corresponds to $i = 1$ (resp. $i = 2$) and bottom-left (resp. bottom-right) subplot corresponds to $i = 3$ (resp. $i = 4$). In each of the subplots, the two curves are very close, indicating that the market basically does not anticipate the meta-order size.

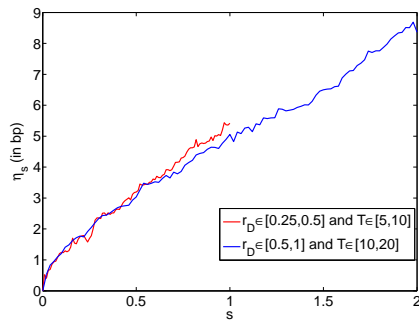


Figure 3.16 – The market impact curves $\hat{\eta}_{s \leq 1}^{(1)}$ and $\hat{\eta}_{2s \leq 2}^{(2)}$

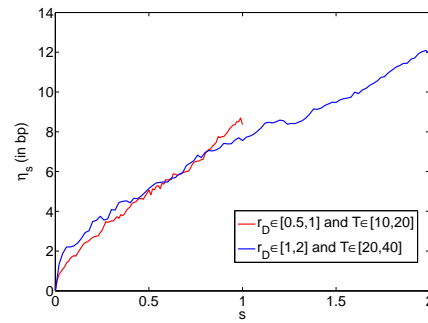


Figure 3.17 – The market impact curves $\hat{\eta}_{s \leq 1}^{(2)}$ and $\hat{\eta}_{2s \leq 2}^{(3)}$

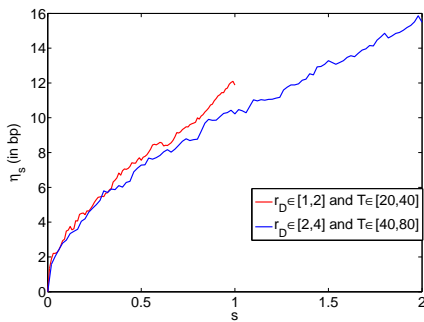


Figure 3.18 – The market impact curves $\hat{\eta}_{s \leq 1}^{(3)}$ and $\hat{\eta}_{2s \leq 2}^{(4)}$

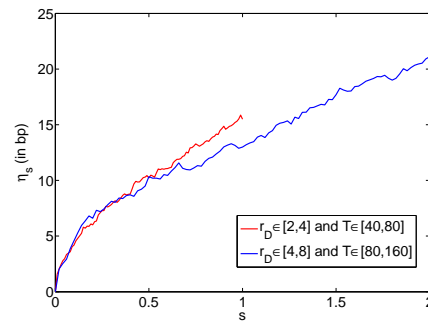


Figure 3.19 – The market impact curves $\hat{\eta}_{s \leq 1}^{(4)}$ and $\hat{\eta}_{2s \leq 2}^{(5)}$

3.5 THE DECAY MARKET IMPACT CURVE

A well known stylized fact : convexity of decay market impact

During the execution of the meta-order the price is pushed in the adverse direction making it less attractive as time goes by reaching higher level (temporary impact) at the end of the execution. After the execution a reversal effect is expected as seen in Fig. 3.1. This is the decay part when the price converges back toward its future permanent level. The full execution of the meta-order has created some imbalance in the limit order book, this imbalance is "taken care" by the market after the execution of the meta-order and ultimately, equilibrium (balance order book) is reached at a new price, the dynamics which conducts to this new price corresponds to a convex curve of the decay market impact $\eta_{s \geq 1}$.

The existing empirical literature of decay meta-orders market impact is limited ([104], [27]) since the difficulty of obtaining data is very high. In the first study, Moro *et al.* are the first showing a decay of the impact to a level roughly equal to 0.5 – 0.7 of its highest peak. In the second study, Bershova and Rakhlin show the decay is a two-regime process: slow initial power decay followed by a faster relaxation.

In this section we confirm that the transient market impact curve is convex and that it seems to have a slow initial regime.

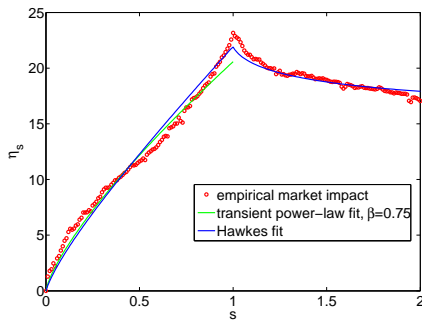


Figure 3.20 – Transient and decay market impact curves with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [3, 15]$,

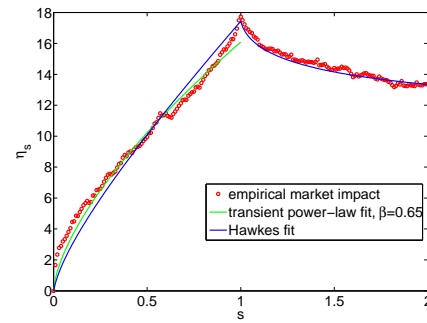


Figure 3.21 – Transient and decay market impact curves with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [15, 30]$,

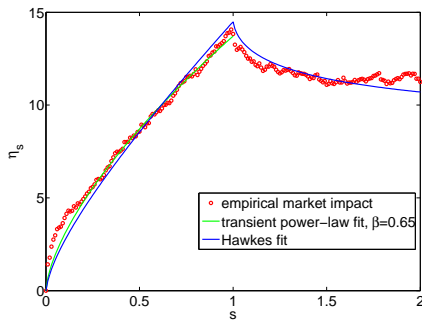


Figure 3.22 – Transient and decay market impact curves with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [30, 60]$,

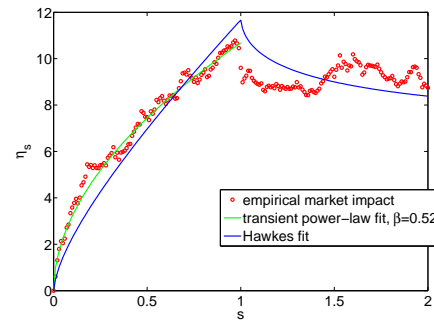


Figure 3.23 – Transient and decay market impact curves with $r_D \in I_r = [1\%, 3\%]$, $T \in I_T = [60, 90]$,

Numerical results.

To avoid spurious effects, we did not want to use extraday data in these estimations. Consequently, the larger one chooses K , the smaller number of meta-orders are available for estimation. We chose to follow [104] and fixed $K = 2$. Following the same lines as in previous section, we compute the estimation (see (3.12))

$$\hat{\eta}_{s \leq 2}(\omega \mid T(\omega) \in I_T, r_D(\omega) \in I_r) = \frac{1}{M(I_T, I_r)} \sum_{\omega, T(\omega) \in I_T, r_D(\omega) \in I_r} \Delta P_{sT(\omega)}(\omega). \quad (3.16)$$

Fig. 3.20 - Fig. 3.23 show such estimations (of both transient and decay market impact curves) for the same ranges I_T and I_r as the ones used in Fig. 3.6 - Fig. 3.9. Thus the transient parts of Fig. 3.20 - Fig. 3.23 are (respectively) almost the same as the curves displayed in Fig. 3.20 - Fig. 3.23. The difference between the lies in the fact that for the Fig. 3.20 - Fig. 3.23 we used meta-orders from $\Omega^{(de)}$ database and for Fig. 3.20 - Fig. 3.23 we used meta-orders from $\Omega^{(tr)}$ database. Power-law fit of the transient part (exponent $\beta^{(tr)}$ in (3.13)) are shown. Fits with the Hawkes model I-HIM (see Section 3.6) are also displayed.

Log-log plots of the corresponding decay market impacts are shown on Fig. 3.24 - Fig. 3.27. More precisely, these plots display, using logarithm scales, $\eta_s - \log \eta_K$ as a function of $s - 1$ for $s \in (1, K]$ (let us recall that we chose $K = 2$). They clearly show that the decay is much slower at the very beginning (i.e., right after the end of the execution of the meta-order). We have checked that changing the daily trading rate range I_r does not affect his result. This is a very stable result that confirms the results obtained previously by [27] and [66].

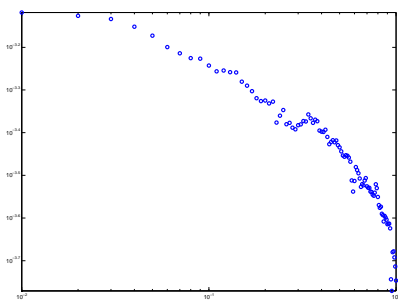


Figure 3.24 – Log-log plot of the decay market impact curve for Fig. 3.20

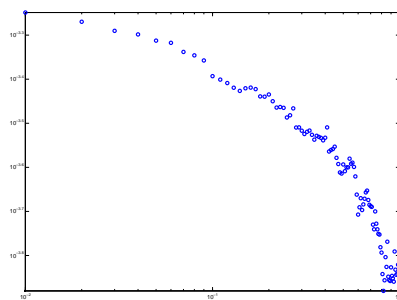


Figure 3.25 – Log-log plot of the decay market impact curve for Fig. 3.21

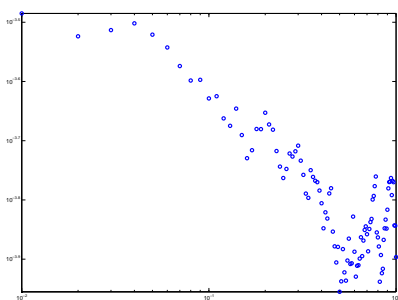


Figure 3.26 – Log-log plot of the decay market impact curve for Fig. 3.22

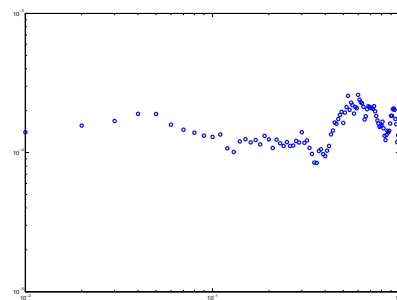


Figure 3.27 – Log-log plot of the decay market impact curve for Fig. 3.23

We tried to get more insights about this change of regime: Is it power-law in the first part? In the second part? How does the time-scale at which the change regime appears depends on T or r_D ? ... Unfortunately, the noise on the estimation is too large; it prevented us from answering any of these questions precisely.

3.6 THE HIM TRANSIENT AND DECAY MARKET IMPACT MODELS USING HAWKES PROCESSES

Hawkes based models for microstructure

Hawkes processes have already been proved successful for modeling high frequency financial time-series (see [17, 118, 18, 77, 5]). Hawkes processes are point processes with a stochastic intensity which depends on the past of the process.

Following [17], we consider the following price model. Let P_t be a proxy for the high-frequency price of an asset (e.g., last-traded price, mid-price, ...). For the sake of simplicity, we shall not consider the size of the jumps in the price and consider that they are only of size 1. Let (J_t^+, J_t^-) be the point processes representing respectively upward and downward jumps of P_t .

$$P_t = J_t^+ - J_t^-. \quad (3.17)$$

Let λ^+ and λ^- the respective intensities of (J_t^+, J_t^-) . It is well known that at microstructure level, the price is highly mean reverting (at least for large tick-size

assets). It has been shown ([17]) that this mean-reversion property is well mimicked using a 2-dimensional Hawkes process using only a single "cross" kernel $\varphi(t)$:

$$\lambda_t^+ = \mu + \varphi \star dJ_s^- \quad \text{and} \quad \lambda_t^- = \mu + \varphi \star dJ_s^+ \quad (3.18)$$

where $\varphi(t)$ is a causal (i.e., supported by \mathbb{R}^+), positive function and where \star stands for the convolution product $\varphi \star dJ_t = \int_{-\infty}^t \varphi(t-s)dJ(t)$. The mean reversion property reads clearly from these last two equations : the more P_t goes up (resp. down), the greater the intensity λ_t^- (resp. λ_t^+) will be. A criteria for the price increments and the intensities to be stationary is given by $\|\varphi\|_1 < 1$, where $\|\cdot\|_1$ denotes the $\mathcal{L}^1(\mathbb{R})$ norm (for a complete mathematical study of Hawkes process, see [45]).

The Hawkes Impact Model (HIM) for market impact of a meta-order

We model the impact of a meta-order starting at time t_0 , ending at time $t_0 + T$ and corresponding to a continuous flow of buying² orders with a trading rate r_t supported by $[t_0, t_0 + T]$ ($r_t \neq 0$ only for $t \in [t_0, t_0 + T]$) by a perturbation of the intensities.

For the sake of simplicity, we will follow the microstructure model above and just consider mean-reversion reaction of the market (e.g., [18], [44]). Let us point out that this is clearly not a realistic hypothesis if one is interested in mimicking precisely the microstructure. However, this is not our goal. In this section, we want to build a structural model that allows to explain the main dynamics of the market impact curve. In the same line as Bouchaud [118], Gatheral [60] and [18], we shall build a *linear* model, in the sense that the impact of the meta-order is nothing but the sum of the impact of its children order.

The model HIM. This model consists in replacing (3.18) by the two equations :

$$\lambda_t^+ = \mu + \varphi \star dJ_t^- + \int_{t_0}^t f(r_s)g^+(s-t_0)ds \quad \text{and} \quad \lambda_t^- = \mu + \varphi \star dJ_t^+ + \int_{t_0}^t f(r_s)g^-(s-t_0)ds, \quad (3.19)$$

where $f(r_s)ds$ (with $f(0) = 0$) codes the infinitesimal impact of a buy order of volume $r_s ds$. The f function corresponds to the *instantaneous* impact function and g^+ and g^- are the *impact kernel* functions. As empirical found in [117] and used by others authors before us ([60, 118]), we suppose the market impact can be separated in a factorized form: one depending on volume (or volume per time) and the other depending only on time.

The impulsive-HIM (I-HIM) model: a particular choice for the kernels. Following [18], it is reasonable to consider that the only "upward" impact of a single buying order is instantaneous, i.e., either the corresponding order ate up the whole first limit (in which case there is an instantaneous jump in the price) or it did not (in which case a limit order fills up the missing volume). This case corresponds to consider that g^+ is "purely" impulsive, i.e.,

$$g^+(t) = g_i^+(t) = \delta_t(t), \quad (3.20)$$

where $\delta(t)$ stands for the Dirac distribution. As for the "downward" impact, we shall consider that the market reacts to the newly arrived order as if it triggered

2. The impact of selling meta-order can be modeled using the exact same principles

an upward jump. Doing so leads to the choice

$$g^-(t) = g_i^-(t) = C \frac{\varphi(t)}{\|\varphi(t)\|_{L^1}}, \quad (3.21)$$

where $C > 0$ is a very intuitive parameter that quantifies the ratio of mean-reversion reaction (i.e., the downward impact) and of the "trend-following" reaction (the upward impact). Indeed the L^1 norm of the upward reaction to an impulsive buying order is $f(r_0)\|g_i^+\|_{L^1} = f(r_0)\|\delta\|_{L^1} = f(r_0)$, whereas the downward reaction to the same order is $f(r_0)\|g_i^-\|_{L^1} = f(r_0)\|\varphi\|_{L^1} C / \|\varphi\|_{L^1} = C f(r_0)$. Thus, one can distinguish 3 cases of interests.

- $C = 0$: no mean-reversion reaction,
- $C = 1$: the mean-reversion reaction is as "strong" (in terms of the norm $\|\cdot\|_{L^1}$) as the trend-following one. So we expect the two to compensate asymptotically, i.e., we expect the permanent market impact to be 0 (see Eq. (3.26) of Proposition 3 for confirmation),
- $C \in (0, 1)$: the mean-reversion reaction is not zero but strictly smaller than the trend-following reaction.

Thus the model I-HIM corresponds to the equations

$$\lambda_t^+ = \mu + \varphi \star dJ_t^- + f(r_t) \quad \text{and} \quad \lambda_t^- = \mu + \varphi \star dJ_t^+ + C \int_{t_0}^t f(r_s) \varphi(s - t_0) ds, \quad (3.22)$$

where C is a positive constant that controls the "downward" reaction of the market.

Market impact curve within HIM

According to our definition of market impact: *the difference between the observed price moves and what it would have been without this specific order*, within HIM, the market impact of a meta-order writes :

$$\eta_t = \mathbb{E}[P_t]. \quad (3.23)$$

Let us point out that, for convenience purposes, we have used physical time in this latest definition instead of renormalized time (which is convenient for estimation purpose see (3.3)).

Then, one can prove (see Appendix 3.8) that :

Proposition 3. (Transient, decay curves and permanent impact)

In the framework of the model HIM (3.19), for all $t \geq t_0$ (t_0 is the starting time of the meta-order), one has:

$$\eta_t = \int_{t_0}^{\infty} f(r_s) \left(G(t-s) - (\kappa \star G)(t-s) \right) ds, \quad (3.24)$$

where

- $G(t) = \int_0^t (g^+(u) - g^-(u)) du$
- $\kappa = \sum_{n=1}^{\infty} (-1)^{n-1} \varphi^{(\star n)}$, where $\varphi^{(\star 1)} = \varphi$ and $\varphi^{(\star n)} = \varphi^{(\star n-1)} \star \varphi$.

In the case of the model I-HIM (3.22), this formula gives

$$\eta_t = \int_{t_0}^t f(r_s) H(t-s) ds, \quad t \geq t_0, \quad (3.25)$$

where $H(t) = 1 - (1 + C/\|\varphi\|_{L^1}) \int_0^t \kappa(s) ds$. Moreover, under the assumption that f is bounded, the permanent market impact writes:

$$\mathcal{M}_{H^c}(\infty) = \frac{1 - C}{\|\varphi\|_{L^1}} \|f(r)\|_{L^1}. \quad (3.26)$$

Let us point out that several recent empirical results ([15] and [69]) seem to show that the Hawkes kernel φ decays as a power-law. Both studies found the exponent in the interval $[-1.5, -1]$. The following corollary shows that, in the framework of the model I-HIM, and in the case of a constant rate strategy, then if φ is power-law then the market impact curve asymptotically decays (to the limit permanent impact) as a power-law, with an exponent which is related to the exponent of φ . More precisely:

Corollary 7. *In the framework of the model I-HIM, let us consider a constant rate strategy, i.e., $r_t = r$, $\forall t \in [t_0, t_0 + T]$ and $r_t = 0$ otherwise. Assume that φ is such that*

- $\varphi \geq \varphi^{(\star 2)}$ and,
- $\exists K > 0$, $\lim_{t \rightarrow \infty} \varphi(t) t^{-\beta^{(\phi)}} = K$, with $\beta^{(\phi)} \in]-2, -1[$.

Then, the market impact curve decays asymptotically as a power-law with exponent $\beta^{(\phi)} + 1$, in the sense that

$$\sup \left\{ \beta, : \int_1^\infty \eta_t t^{\beta-1} dt < \infty \right\} = \beta^{(\phi)} + 1. \quad (3.27)$$

The proof of this Corollary can be found in Appendix 3.8.

Back to real data

Let us consider the case of a meta-order of fixed size ν and executed on a period of time T . In the framework of the model I-HIM, for a constant trading rate $r_t = r_0 = \nu/T$, Eq. (3.24) becomes:

$$\eta_t = f(r_0) (1_{[t_0, t_0+T]} \star H)(t). \quad (3.28)$$

Let us point out that the trading rate r_0 has no influence on the shape of the transient market impact, it is just a multiplicative constant. In the following we want to use the model I-HIM to reproduce the transient and the decay market impact curves obtained in Sections 3.4 and 3.5 and displayed in Fig 3.20-Fig 3.23.

Following [15] and [69], we choose to use a power-law kernel

$$\varphi(t) = \alpha(\gamma + t)^\beta.$$

Thus, apart from the instantaneous impact function f (which is basically responsible for a rescaling of the whole market impact curve), there are 4 parameters left in the I-HIM model, namely, α , β , γ and C . The parameters α and β are responsible for the endogenous mean-reverting activity of the market itself. As pointed out previously, the parameter C encodes the proportion of mean-reversion in the impact.

Estimation is performed simultaneously on the four market impact curves $\{\hat{\eta}_s^{(i)}\}_{1 \leq i \leq 4}$ (for $s \in [0, 2]$) displayed on Fig. 3.20 - Fig. 3.23. The parameters are α and β (these two parameters are shared by all the curves) and the parameters $\{C^{(i)}\}_{1 \leq i \leq 4}$. Estimation follows the three following principles

- We constrain the estimation to fit the transitory market impact $\hat{\eta}_{s=1}^{(i)}$ for each curve $i \in \{1, \dots, 4\}$.
- Each curve $\hat{\eta}_s^{(i)}$ is as a function of the renormalized time s whereas the model gives the market impact η_t (Eq. (3.28)) as a function of the physical time, rescaling in time must be performed independently on each curve. The corresponding rescaling parameter T_i has been chosen to be the largest duration in the corresponding time ranges, i.e., $T_1 = 15$ min for curve in Fig. 3.20, $T = 30$ min for curve in Fig. 3.21, $T = 60$ min for curve in Fig. 3.22 and $T = 90$ min for curve in Fig. 3.23.
- To account for the instantaneous impact function $f(\cdot)$, each curve is rescaled independently

Thus, the estimation procedure sums up in

$$(\hat{\alpha}, \hat{\beta}, \hat{C}_i) = \underset{(\alpha, \beta, C_i)}{\operatorname{argmin}} \sum_i \int_0^2 \left(\frac{\hat{\eta}_1^{(i)}}{\eta_{sT_i}} \eta_{sT_i} - \hat{\eta}_s^{(i)} \right)^2 ds.$$

The value we find for $\hat{\beta}$ is close to -1.5 and for α we find a value such that $\|\varphi\|_{L^1} \approx 0.8456$ which is rather close to the critical value 1. These results are in good agreement with the works [15] and [69]. For the parameters $\{C_i\}_{1 \leq i \leq 4}$, we find $\hat{C}_1 \approx 0.5$, $\hat{C}_2 \approx 0.70$, $\hat{C}_3 \approx 0.80$ and $\hat{C}_4 \approx 0.85$.

The so-obtained fits are shown in Fig. 3.20 - Fig. 3.23. One can see that the model I-HIM is able to reproduce precisely the shapes of both transient and decay market impact curves. Moreover, it also reproduces the dependence on T of the curvature of the transient market impact, i.e., the smaller T , the more linear the transient market impact.

3.7 THE IMPACT OF THE TYPE OF STRATEGY ON THE PERMANENT IMPACT

Positioning

So far very few papers, even if the permanent effect has been extensively studied, addressed this subject from the perspective of the daily scale. However one distinguishes two clear different points of view about the nature of the phenomenon at the origin of permanent impact. A mechanical vision shared by number of econophysists and an informational vision which is further favoured by economists:

- In a purely mechanical vision price moves are described as the point of equilibrium between two antagonist forces: a sell pressure driving prices down and a buy pressure driving prices up. The effect of these forces on price dynamic follows a law that the econophysist strive to describe. Eventually the price at time t is the sum of all the effects since the beginning. See [58] and [33].
- On the other side, the informational vision says stock prices move because new information is made available to market participants. According to this new information investors update their expectations changing their offer and demand which leads to a new global equilibrium resulting in new price levels. In a seminal paper, [88], Kyle shows how prices are

driven to their new level through the execution of a meta-order by an informed trader. The informed trader is constantly adjusting her trading to her observation of the price in real time: she increases or reduces pressure whether the price is too far or not from the targeted price.

An important contribution has been made in [66] to the question of the *permanent effect* at the daily scale. In the paper the authors have at their disposal a buy side database of meta-orders. The specificity of such a database is to give the information about the nature of the decision at the root of the execution, thus giving additional controls to separate purely impact moves from others. They separated meta-orders into two groups:

- Those triggered by additional or cancelled subscriptions, thus triggered by heterogeneous and relatively exogenous to the market information. Those meta-orders are called “uninformed trades”.
- The rest, essentially meta-orders coming from portfolio rebalancing, is designated the set of “informed trades”.

Waelbroeck and Gomes [66] show that only a daily basis (from execution until 60 days after) informed trades have permanent effect but uninformed trades do not.

The purpose of the paper is not to settle for one or the other debaters. We believe that there is no rule; the mechanical - informational dual vision about the nature of permanent effect renders a good picture of the phenomenon. As every dual paradigm, this is a principle well known by physicists, taken separately the two pure concepts fail to give a satisfying picture of the whole phenomenon.

Focus on permanent effect. In the problem of permanent effect the informational versus mechanical debate boils down to understand the price reversal observed after the execution. Is the price reversal:

- the *temporary effect* of liquidity providers, counterparts of the meta-order, who are taking profit thus applying a pressure on the price in the opposite direction. Let us imagine a strung bow releasing all the tension accumulated during the execution (Mechanical explanation);
- the convergence of the price to its new Walrasian equilibrium. If the price does not revert until its initial position it means that the market integrated new information establishing the equilibrium to a new level (informational explanation)?

To be really fair in the informational versus mechanical debate one should recognize that the existence of permanent effect is still debated among mechanists. The two following pictures have emerged:

- In the picture of [53] and of [27] permanent market impact is important and roughly equals to 2/3 of the temporary impact on an order by order basis. This is the fair pricing hypothesis.
- In the picture of [33], there is no such thing as permanent impact. The author argues that what is called permanent market impact is a consequence of the long memory of the sign of the meta-orders flow. This picture is incompatible with the permanent market impact hypothesis because long memory of order flow would result in trending stock prices and thus contradicting market efficiency.

In this paper we pretend to conciliate the two positions of [53] and [33] even if at first sight they seem pretty much incompatible. In [53] the long memory of the order flow is not taken in consideration and thus it is shown that the so-

called total effect is non-negligible. On the other hand, in [33] it is taken into consideration and the total effect is described as an effect of other correlated meta-orders being executed on the same side. Eventually the relaxation taking place at the end of the meta-order execution is an averaging effect. In this paper, as previously stated, we consider that the permanent market impact is the additional price movement after the execution of a meta-order over the price that would be in the absence of the meta-order. This explains why we prefer the term of *permanent effect* instead of permanent market impact.

Our methodology. To go one step further, we will try to remove as much informational effect as possible from the price move. Since we use the database of an executing broker, we do not know explicitly what triggered the creation of the meta-orders.

In [66], the authors know some of their meta-orders are “cash trades” (i.e. without any information on price moves; in their case $\mathbb{E}(\epsilon(\omega)W_t) = 0$, to come back to equation (3.4)), and find these cash trades have no permanent impact. They perform the same analysis of the subset of their meta-orders, and find they have a permanent effect, obtaining curves similar to the red one of Figure 3.30.

In this paper, we intend to remove the informational part of the price move by assuming the client of this marge broker, being almost all large institutional investors, are trying to “capture some β ”, in the sense of the CAPM. The reader should keep in mind that in [37], authors succeeding in removing most of the information content of their meta-orders, using their anticipation of the α made by Capital Fund Management at the initiation of the meta-order.

We have here a similar approach, adding the assumption the initiation of meta-orders by institutional investors in our database has been caused by an anticipation of the β of market moves.

Our results are compatible with Capital Fund Management meta-orders cleaned from α and with the cash trades of [66]: we found no permanent effect once the informational content associated to the meta-orders.

To conciliate results of [53] and cash trades of [66] or Capital Fund Management meta-orders of [37], it is enough to assume the mix of informed meta-orders and non informed ones in the databases available by now to academics is such that the remaining price move a dozen of days after a decision is around 2/3 of the temporary impact. This assumption can be supported by the natural selection argument of [66] : portfolio managers learn to select trades which expected return at the time scale of investment committees are largest than their trading costs.

Debiasing price moves of the temporary of meta-orders trading during the post-execution period

When studying the average profile of the post-execution price moves particular attention should be paid to the presence of autocorrelations in the meta-order flow. Indeed the temporary impact of correlated meta-orders executed over the post-execution can be considered as the permanent effect of the meta-orders executed on the execution day. In [52] the authors discuss the issue of price efficiency and of order flow correlations. [66] recognizes the presence of autocorrelations in the flow of meta-order and uses a simple market impact model to withdraw the impact of the meta-orders traded over the post-

execution period. Figure 3.28 shows the autocorrelations of the market participation rate for the meta-orders of our database (bootstrapped quartiles and median). One clearly sees that the autocorrelation persists beyond 20 days after the execution day. Removing the effect of the autocorrelations of the meta-order

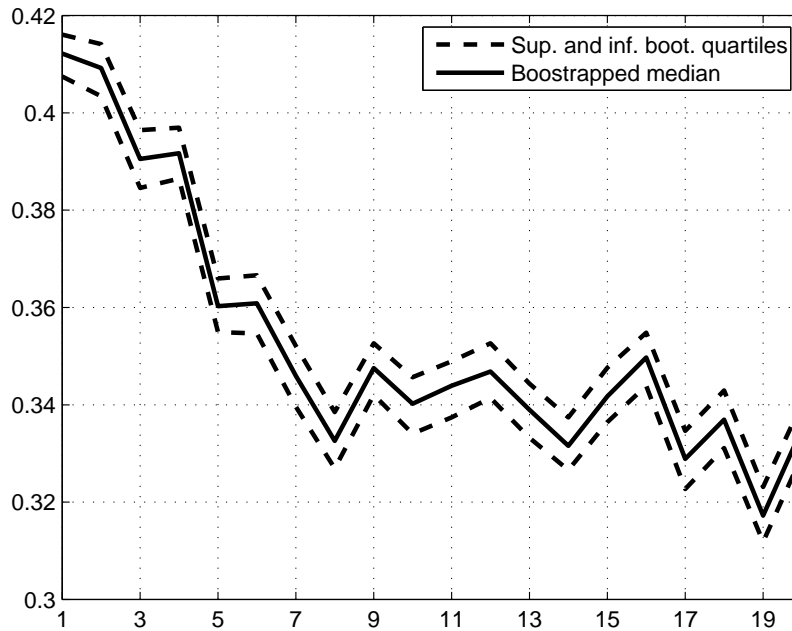


Figure 3.28 – Autocorrelations of the market participation of meta-orders with lags in $\{1, \dots, 20\}$. The dashed curves represent the first and the last bootstrapped quartiles, the solid curve is the bootstrapped median.

flow in order to get an unbiased long term impact picture is not straight forward. The temporary impact of meta-orders on the price moves turns out to be far from linear. In the paper we followed the methodology introduced in [66], namely we fitted a square root model on the temporary impact of aggregated signed participation rates. In concrete terms, for each stock of our universe and any date we defined the unbiased returns by subtracting the market impact related to the aggregated signed participation rate of all meta-orders that were being executed.

To compute statistics, we need to use the day $d(\omega)$ during which the meta-order ω has been traded. Notice that, in the rare cases we have multiple meta-orders the same day on the same stock, we summarize them in one “synthetic” meta-order, which volume is the sum of their signed volumes.

As previously stated figure 3.28 shows the autocorrelations of the process $v(d)$, showing the occurrence of one meta-order on day d is positively correlated with the issuance of another meta-order on the same stock the day after, etc.

Following [66] we decided to remove part of this effect by removing the expected market impact associated to $d + n$ days thanks to a rough version of the models obtained in the previous section:

- we fit a power law on the market impact observed in the daily database,
- we apply it to the price moves observed at $d + n$, using $v(d + n)$ to deduce the expected impact.

The meaning of such a “cleaning” needs to be commented. Assume the following initial event: information affecting the value of a stock is accessible to investors. They will not simultaneously take it into account in their portfolios. Each of them has constraints and processes different enough so that their decision will span several days.

Each time an investor take a decision accordingly to this information, he impacts the price, mechanically ; there should be an informational component to the price move too, as a consequence to the initial event.

The rough market impact model built on the daily data meant to capture most of the mechanical effect, and may be some part of the informational one. Removing it from the data targets to observe the future of the price given the meta-order traded on day d , as independently as possible to the subsequent trades.

Systematic + idiosyncratic decomposition

For each meta-order we define a CAPM-like decomposition into a systematic and an idiosyncratic component centred on the execution day over a 41 days period (20 days before and 20 days after),

$$\forall s \in \{d-20, \dots, d, \dots, d+20\}, \quad \log(P_s) - \log(P_{s-1}) = \beta(\omega)(\log(I_s) - \log(I_{s-1})) + \Delta \text{Idio. Component}_s, \quad (3.29)$$

where d is the date of meta-order execution, P_s is the stock's close price on date s , and $\beta(\omega)$ implicitly designates the beta of the traded stock on the period from $d-20$ to $d+20$.

We use the reference index for each stock, which price is noted I_s on day s .

- We define the idiosyncratic component as the cumulative sum of the $\Delta(\text{Idio. Component}_s)$ from (3.7):

$$\sum_{k=d}^s \Delta(\text{Idio. Component}_k) = \text{Idio. Component}_s - \text{Idio. Component}_{d-1};$$

- Similarly we define $\log(I_s) - \log(I_{d-1})$ as the systematic component.

The systematic, the idiosyncratic components and the “total” are considered relatively to the close prices one day before the execution, $d-1$. This cumulative version of (3.7) is also called the post-execution profile and is a measure of the *permanent effect*.

Remind here our main assumption is the investors initiating the meta-orders target an exposure in β (i.e. to the systematic component in our sense). The informational effect should then be capture by this component, and the dynamics of the mechanical part of his impact should be seen on its idiosyncratic component.

Result analysis and figures

The two figures 3.29 and 3.30 present the idiosyncratic, the systematic and the total post-execution profiles of price moves. Figure 3.29 shows “not-yet-debiased” post-execution profiles. The observed price jump between day 0 and day 1 is the daily market impact. This jump is visible on the idiosyncratic and the systematic components. Over the period extended from day 1 until day 20

the prices are slowly trending in the same direction as the jump on the execution day and no reversal is seen at all. This holds true for idiosyncratic and systematic components as well. Figure 3.30 shows the debiased post-execution profile. The market impact of the meta-orders executed the day after the execution day has been removed. Over the post-execution period the price is converging back to a level lower than the one reached on execution day. The idiosyncratic post-execution profile is even reaching its level before the execution before day 20. Thus the remaining *permanent effect* 20 days after the execution is entirely explained by the systematic component that is to say by the average level of the market.

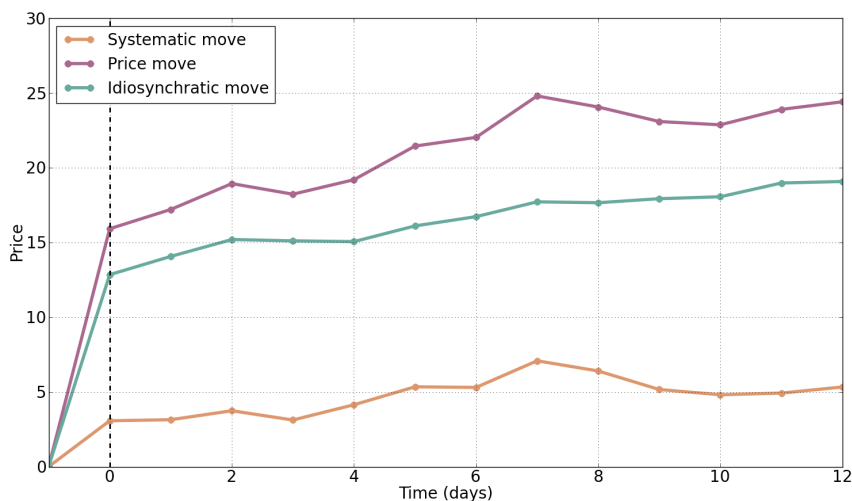


Figure 3.29 – Post-execution profile relatively to close price on the day before execution (units = basis points). Idiosyncratic component + Systematique component = Total component.

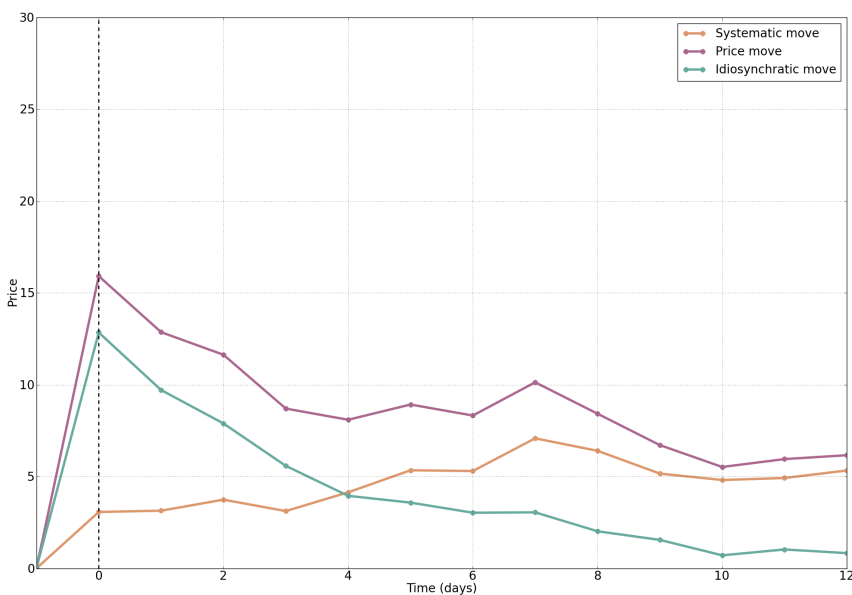


Figure 3.30 – Post-execution profile without the impact of other meta-orders. Price moves are considered relatively to close price the day before execution (units = basis points). Idiosyncratic component + Systematique component = Total component.

3.8 APPENDIX

Proofs**Proof of Proposition 3**

This is a consequence of a more general result on Hawkes Processes:

Theorem 16. *Let (N_1, \dots, N_d) be a d -multivariate Hawkes process defined by his intensity $\lambda = (\lambda_1, \dots, \lambda_d)$:*

$$\lambda_t = \mu(t) + \int_{[0,t)} \Phi(t-s) dN_s \quad (3.30)$$

where $\mu(t) = (\mu_1(t), \dots, \mu_d(t))$ is a vector of functions from \mathbb{R}_+ to \mathbb{R}_+ and $\Phi(t) = (\varphi_{ij}(t))_{1 \leq i, j \leq d}$ is a matrix containing functions from \mathbb{R}_+ to \mathbb{R}_+ .

Under the assumptions:

- $\int_0^t \mu(s) ds < \infty \quad \forall t > 0$,
- the spectral radius of the matrix $K = \int_0^\infty \Phi(t) dt$ is inferior to 1, $\rho(K) < 1$,

we have

$$\mathbb{E}[N_t] = h(s) + \int_0^t \Psi(t-s) h(s) ds, \quad (3.31)$$

where $\Psi = \sum_{n \geq 1} \Phi^{(\star n)}$ and $h(s) = \int_0^s \mu(s) ds$, where $\Phi^{(\star n)} = \Phi \star \dots \star \Phi$ (with n terms on the right hand-side) and where the convolution product of two matrices $A(t) = \{a_{ij}(t)\}$ and $B(t) = \{b_{ij}(t)\}$ is defined as the matrix $C(t) = \{c_{ij}(t)\}$, such that

$$c_{ij}(t) = \sum_k a_{ik} \star b_{kj}.$$

Proof. Let us first remark the elements of the matrix Ψ are in L^1 . Indeed, by induction we have, $\int_0^\infty \Phi^{(\star n)}(t) dt = K^n$ and since $\rho(K) < 1$, the series $\psi = \sum_{n \geq 1} K^n$ is finite component-wise, thus one gets:

$$\int_0^\infty \Psi(t) dt = K(I - K)^{-1}. \quad (3.32)$$

We will now show that:

$$\mathbb{E}[N_t] = \int_0^t \mu(s) ds + \mathbb{E} \left[\int_0^t \Phi(t-s) N_s ds \right] \quad \forall t > 0 \quad (3.33)$$

Using that $N_t - \int_0^t \lambda_s ds$ is a (\mathcal{F}_t) -martingale (where $(\mathcal{F}_t)_{t \geq 0}$ is the σ -algebra generated by the random variables $N_s^i; s \leq t; 1 \leq i \leq d$), we have:

$$\mathbb{E}[N_t] = \mathbb{E} \left[\int_0^t \lambda_s ds \right] = \mathbb{E} \left[\int_0^t \mu(s) ds \right] + \mathbb{E} \left[\int_0^t ds \int_0^s \Phi(s-u) dN_u \right].$$

But, by Fubini's theorem:

$$\int_0^t ds \int_0^s \Phi(s-u) dN_u du = \int_0^t \left(\int_s^t \Phi(t-u) dt \right) dN_u = \int_0^t \left(\int_0^{t-s} \Phi(s) ds \right) dN_u.$$

We denote $F(t) = \int_0^t \Phi(s) ds$ and using an integration by parts we have:

$$\int_0^t F(t-s) dN_s = \left[F(t-s) N_s \right]_0^t + \int_0^t \Phi(t-s) N_s ds =$$

$$= F(0)N_t - F(t)N_0 + \int_0^t \Phi(t-s)N_s ds = \int_0^t \Phi(t-s)N_s ds.$$

So we obtain:

$$\mathbb{E}[N_t] = \mathbb{E}\left[\int_0^t \mu(s)ds\right] + \mathbb{E}\left[\int_0^t \Phi(t-s)N_s ds\right].$$

Using once again Fubini's theorem:

$$\mathbb{E}[N_t] = \int_0^t \mu(s)ds + \int_0^t \Phi(t-s)\mathbb{E}[N_s] \quad (3.34)$$

This is a classical renewal equation and the solution is given by (3.31). The interested reader can find more on renewal theory on the book of David Cox ([41]). □

Let us now prove the first part of Proposition 3. In order to ease further notation we take $t_0 = 0$. We apply the previous theorem in the particular case of the 2-dimensional Hawkes process (J^+, J^-) with $\Phi = \begin{pmatrix} 0 & \varphi \\ \varphi & 0 \end{pmatrix}$ and we successively compute:

$$- \Phi^{\star n} = \begin{pmatrix} 0 & \varphi^{\star n} \\ \varphi^{\star n} & 0 \end{pmatrix} \text{ if } n \text{ is even and } \Phi^{\star n} = \begin{pmatrix} \varphi^{\star n} & 0 \\ 0 & \varphi^{\star n} \end{pmatrix} \text{ if } n \text{ is odd.}$$

$$- h(t) = (h_1(t), h_2(t)) = (t\mu + \int_0^t (f(r) \star g^+)(u)du, t\mu + \int_0^t (f(r) \star g^-)(u)du)$$

$$- \eta_t = \mathbb{E}[J_t^+ - J_t^-] = (h_1(t) - h_2(t)) - \kappa \star (h_1 - h_2)(t).$$

This proves Eq. 3.24 since $h_1 - h_2 = f(r) \star G$.

In the case of I-HIM, we set $H(t) = G(t) - G \star \kappa(t)$ and $C_1 = C/\|\varphi\|_1$. Let us compute the derivative of H for $t > 0$:

$$H'(t) = G'(t) - G' \star \kappa(t) = -C_1\varphi(t) - (\delta - C_1\varphi) \star \kappa(t) = -C_1\varphi(t) - \kappa(t) + C_1\varphi \star \kappa(t)$$

Using:

$$\varphi \star \kappa = \varphi \star \sum_{n=1}^{\infty} (-1)^{n+1} \varphi^{\star n} = \sum_{n=2}^{\infty} (-1)^n \varphi^{\star n} = \varphi - \kappa,$$

we obtain

$$H'_t = -(1 + C_1)\kappa(t).$$

This allows to find $H(t) = 1 - (1 + C_1) \int_0^t \kappa(s)ds$ which proves (3.25). Eq. (3.26) is a direct consequence of this equation when the rate r_t is constant.

Proof of Corollary 7

This Corollary is a direct consequence of the following Lemma (which links the power-law exponent of φ with the one of κ) and of the expression (3.25).

Lemma 5. *Let $p \in [0, 1]$. Under the assumption $\varphi \geq \varphi^{\star 2}$, $\int_{[0, \infty)} t^p \varphi(t)dt < \infty$ if and only if $\int_{[0, \infty)} t^p \kappa(t)dt < \infty$.*

Proof. We recall that $\kappa = \sum_{n=1}^{\infty} (-1)^{n+1} \varphi^{\star n}$. Which in the Fourier domain becomes:

$$\hat{\kappa}(\omega) = \sum_{n=1}^{\infty} (-1)^{n+1} \hat{\varphi}^n(\omega) = \frac{\hat{\varphi}(\omega)}{1 + \hat{\varphi}(\omega)},$$

By inverting the previous formula we get:

$$\hat{\varphi}(\omega) = \frac{\hat{\kappa}(\omega)}{1 - \hat{\kappa}(\omega)} = \sum_{n=1}^{\infty} \hat{\kappa}^n(\omega).$$

Returning in the time-domain we have:

$$\varphi = \sum_{n=1}^{\infty} \kappa^{(\star n)}. \quad (3.35)$$

Let us remark the assumption $\varphi \geq \varphi^{(\star 2)}$ ensures that κ is positive. We are ready for the proof

- Let us assume that $\int_0^{\infty} t^p \varphi(t) dt < \infty$.
Since $\kappa(t) \geq 0$, it is easy to see that $\kappa^{(\star n)} \geq 0$. So $0 \leq \kappa(t) \leq \varphi(t)$ and than $\int_0^{\infty} t^p \kappa(t) dt < \infty$ is evident.
- Let us assume that $\int_0^{\infty} t^p \kappa(t) dt < \infty$.
Let $I_n = \int_0^{\infty} t^p \kappa^{(\star n)}(t) dt$, $\forall n \geq 1$, and $\int_0^{\infty} \kappa(t) dt = c$, $c \in \mathbb{R}_+$. Using that the function t^p is concave for $p \in [0, 1]$, we get:

$$\begin{aligned} I_{n+1} &= \int_0^{\infty} t^p \left(\int_0^t \kappa(t-s) \kappa^{(\star n)}(s) ds \right) dt = \int_0^{\infty} \left(\int_0^{\infty} (t+s)^p \kappa(t) dt \right) \kappa^{(\star n)}(s) ds \\ &\leq \int_0^{\infty} \left(\int_0^{\infty} (t^p + s^p) \kappa(t) dt \right) \kappa^{(\star n)}(s) ds = \int_0^{\infty} (I_1 + c s^p) \kappa^{(\star n)}(s) ds = c^n I_1 + c I_n. \end{aligned}$$

Therefore for all integer N :

$$\sum_{i=1}^N I_n \leq I_1 + \left(\sum_{n=1}^{N-1} c^n \right) I_1 + c \sum_{n=1}^{N-1} I_n.$$

And we easily obtain:

$$\sum_{n=1}^{N-1} c^n \leq \frac{I_1}{(1-c)^2}$$

Thus, for $N \rightarrow \infty$:

$$\int_0^{\infty} t^p \varphi(t) dt = \sum_{n=1}^{\infty} I_n \leq \frac{I_1}{(1-c)^2} < \infty$$

□

Trading algorithms

All meta-orders of the database have been traded using different trading algorithms. There are mainly three:

- a PoV (i.e. Percentage of Volume) is a trading algorithm for which the volumes of the transactions stays in a narrow band (of width of the order of a few percents) around a constant chosen as a fixed fraction of the estimated daily volume. Thus, "smart" limit orders are sent as long as the integrated volume is in the band. When it is no longer the case, these limit orders are canceled and market orders are sent instead.
- a VWAP (i.e. Volume Weighted Average Price) is a trading algorithm parameterized by a start time and an end time, which tries to make the integrated transaction volume to be as close as possible to the *average intraday volume curve* of the traded security (i.e. the *U-shaped pattern* on the US stocks for instance, see [90] Chapter 2.1. for details about fixing auctions and intraday volume curves around the world). It means that the transaction volumes are higher (resp. lower) during the period of high (resp. low) averaged activity.
- a IS (Implementation Shortfall) is the typical implementation of an Almgren-Chriss like algorithm.

For more details about trading algorithms, see Chapter 3.3 of [90].

The structural difference between the different trading algorithms

The relationship between $r(\omega)$ (or $r(\omega)$) and $T(\omega)$ are very different depending on the trading algorithm used for executing ω . Tables 3.4, 3.5 and 3.6 give some basic statistics on the main characteristics of the meta-orders corresponding to each trading algorithm (see Section 3.2 for notations).

Characteristic	Mean	Std	Q5%	Q25%	Q75%	Q95%	Kurtosys	Skewness
$N(\omega)$	74.5	108.9	11	20	84	247	6.2	79.9
$r(\omega)$	2.33%	3.75%	0.07%	0.31%	2.62%	9.65%	3.55	20.53
$\sigma(\omega)$	32.86%	73.18%	11.59%	18.89%	38.58%	69.12%	72.9	5918
$\psi(\omega)$	0.231	1.496	0.003	0.013	0.132	1.034	37.1	1797
$T(\omega)(inmn)$	62.6	92	0.95	6.61	74.38	277	2.42	9.07
$r(\omega)$	20.39%	11.31%	2.97%	13.12%	26.18%	36.3%	1.24	8.15

Table 3.4 – Statistics on the main characteristics of the meta-orders for PoV algorithms (see Section 3.2 for notation and 3.2 for algorithm description). Qp% stands for the p-quantile of the distribution.

Characteristic	Mean	Std	Q5%	Q25%	Q75%	Q95%	Kurtosys	Skewness
$N(\omega)$	78	131	11	19	83	276	7.18	108
$r(\omega)$	2%	3.16%	0.07%	0.292%	2.184%	7.74%	4.97	40.41
$\sigma(\omega)$	27.30%	44.45%	10.95%	16.99%	32.23%	54.61%	109	15120
$\psi(\omega)$	0.334	1.054	0.003	0.014	0.3	1.441	28.64	1485
$T(\omega)(inmn)$	250	131	24	129	330	450	-0.29	2.1
$r(\omega)$	5.36%	8.05%	0.2%	0.82%	6.24%	22.03%	3.28	17.86

Table 3.5 – Statistics on the main characteristics of the meta-orders for VWAP algorithm (see Section 3.2 for notation and 3.2 for algorithm description). Qp% stands for the p-quantile of the distribution.

Characteristic	Mean	Std	Q5%	Q25%	Q75%	Q95%	Kurtosys	Skewness
$N(\omega)$	32	35	10	14	38	93	4.97	48.14
$r(\omega)$	1.04%	1.58%	0.07%	0.216%	1.25%	3.75%	5.69	64.73
$\sigma(\omega)$	25.72%	13.81%	10.79%	16.67%	30.80%	49.21%	2.85	21.92
$\psi(\omega)$	0.409	0.675	0.004	0.022	0.525	1.643	3.76	27.53
$T(\omega)(inmn)$	24	33	1.76	7	28.3	71.2	4.61	32.16
$r(\omega)$	11.76%	9%	4.26%	6.39%	13%	30.5%	2.71	13.24

Table 3.6 – Statistics on the main characteristics of the meta-orders for IS algorithm (see Section 3.2 for notations and 3.2 for algorithm description). Qp% stands for the p-quantile of the distribution.

Other statistics

Table 3.7 shows the distribution of meta-orders function of the stock exchange where the meta-orders are executed and Table 3.8 gives some basic statistics on the main characteristics of those meta-orders (see Section 3.2 for notations)

Stock Exchange	$\Omega^{(in)}$	$\Omega^{(te)}$	$\Omega^{(tr)}$	$\Omega^{(de)}$
Amsterdam	4.89%	5.43%	5.18%	5.65%
Frankfurt	13.34%	13.27%	13.37%	14.54%
London	25.84%	25.56%	24.13%	21.62%
Madrid	4.71%	5.02%	5.05%	5.21%
Milan	6.08%	5.98%	5.34%	5.120%
Paris	22.64%	26.04%	27.94%	29.90%
Others	22.50%	18.67%	18.95%	17.92%

Table 3.7 – *Meta-orders distribution on different European Stock Exchanges for different databases*

Characteristic	Database	Mean	Q5	Q25	Q50	Q75	Q95
N	$\Omega^{(day)}$	73	11	19	50	80	249
	$\Omega^{(te)}$	77	11	20	40	87	261
	$\Omega^{(tr)}$	100	13	28	57	117	326
	$\Omega^{(de)}$	78	12	24	47	94	248
T (min)	$\Omega^{(day)}$	130	1.6	13.4	52	262	415
	$\Omega^{(te)}$	122	1.1	11.1	49	239	410
	$\Omega^{(tr)}$	85	4.5	12.8	33	102	339
	$\Omega^{(de)}$	40	4.1	9.8	23	56	140
r_D	$\Omega^{(day)}$	2.12%	0.07%	0.30%	0.73%	2.33%	8.69%
	$\Omega^{(te)}$	1.78%	0.06%	0.25%	0.68%	1.89%	7.19%
	$\Omega^{(tr)}$	2.09%	0.17%	0.48%	1.15%	2.56%	7.21%
	$\Omega^{(de)}$	1.20%	0.14%	0.32%	0.70%	1.51%	4%
r	$\Omega^{(day)}$	14.25%	0.43%	3.35%	12.11%	22.57%	34.12%
	$\Omega^{(te)}$	13.82%	0.38%	3.12%	11.48%	21.93%	34%
	$\Omega^{(tr)}$	16.01%	3.85%	7.98%	15.23%	22.76%	32.41%
	$\Omega^{(de)}$	16.99%	2.45%	9.36%	17%	24.13%	32.97%
σ_D	$\Omega^{(day)}$	30.32%	11.27%	17.94%	24.89%	35.56%	63.18%
	$\Omega^{(te)}$	28.10%	10.65%	16.91%	23.69%	33.50%	58.98%
	$\Omega^{(tr)}$	27.89%	10.68%	16.89%	23.62%	33.388%	58.35%
	$\Omega^{(de)}$	28.92%	10.97%	17.38%	24.42%	34.50%	60.76%
ψ (bp)	$\Omega^{(day)}$	15.56	3.57	6.27	10.29	16.83	44.92
	$\Omega^{(te)}$	12.08	3.41	5.57	9.01	13.58	31.97
	$\Omega^{(tr)}$	11.98	3.57	6.16	9.73	14.17	28.85
	$\Omega^{(de)}$	1.075	3.40	5.47	8.64	12.705	25.75

Table 3.8 – *Statistics on the main characteristics distribution of meta-orders for different databases*

Non-parametric estimation of Hawkes kernel decay

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4.1 INTRODUCTION

Self-exciting processes were first introduced by Alan Hawkes [74, 73] in order to reproduce the ripple effects generated after the occurrence of an earthquake [133]. They are intuitively similar to Poisson processes, but unlike ordinary Poisson processes, the intensity of Hawkes processes is stochastic and depends upon their own historic events. Gradually, the model has been used by scientists from different areas such as biology [68, 112, 115], genome analysis [121], neurology [129, 119, 120], seismology [4, 132, 75, 87, 110], social behavior [29, 43, 123, 138] and epidemiology [135], to name but a few.

Nowadays, Hawkes processes are widely used in finance. Because current transactions, which are discrete events, cause future trades [118], self-exciting processes successfully engendered many applications. Without being exhaustive, let us give some examples: microstructure dynamics [17], order arrival [77, 25], market impact [18, 19], financial price modeling across scales [16, 86], volatility clustering [50], price co-jumps [30], limit order book modeling [137, 127, 89, 126] among many other.

Every counting process $(N_t)_{t \in \mathbb{R}_+}$ can be entirely defined by his intensity $(\lambda_t)_{t \in \mathbb{R}_+}$:

$$\mathbb{P}(N \text{ has a jump in } [t, t + dt] | \mathcal{F}_t) = \lambda_t dt,$$

where \mathbb{P} stands for probability and \mathcal{F}_t is the sigma-field generated by N up to present time t . The most basic Hawkes model is a univariate counting process such that its intensity λ_t , satisfies:

$$\lambda_t = \mu + \int_{-\infty}^t \varphi(t-s) dN_s, \quad (4.1)$$

where $\mu > 0$ is the exogenous intensity and φ is a positive function with support on \mathbb{R}_+ called decay kernel. If $\varphi \equiv 0$, we have a basic Poisson process. The univariate Hawkes process can be shown to be well defined and to admit a version with stationary increments under the stability condition:

$$\textbf{Assumption 1} \quad \ell = \|\varphi\|_1 = \int_0^\infty \varphi(t) dt < 1 \quad (4.2)$$

where $\|\cdot\|_1$ denote the usual $\mathcal{L}^1(\mathbb{R})$ -norm. For a complete mathematical study of Hawkes and more general point processes we refer the interested reader to the Daley and Vera-Jones' book [45].

Since self-exciting processes are quite popular in different scientific areas, they have long been studied in probability theory: see for example the books of Daley-Vere-Jones [45], Brémaud-Massoulié [36, 34, 35] or the recent research of Zhu [141, 139, 140]. Recent studies (see [59, 47]) investigate the case of large d , when the number of components may become increasingly big or possibly infinite. With regard to the simulation of Hawkes processes, we refer the interesting reader to [102, 103, 62, 46].

From a statistical inference point of view, one of the first studies belongs to Ogata [111] and Ozaki [113], who investigated the maximum likelihood estimator for some classes of functions, like exponential and power laws. For more than two decades, these methods have been used for most of the applications.

In 2010, Reynaud-Bouret and Schbath [121] employed a penalized projection method to estimate φ in the univariate Hawkes model. The theoretical estimator is adaptive for Hölderian functions with regularity $(1/2, 1]$, under the hypothesis that the decay kernel has compact support. In 2011, Lewis and Mohler [97] used a maximum penalized likelihood estimation to simultaneously approximate the background rate and the decay kernel of a multivariate model and they numerically studied the convergence rate of the algorithm. Same year (i.e. 2011), Bacry *et al.* [15] proposed another non-parametric estimation method for multivariate symmetric Hawkes processes, based on Fourier computations and Bartlett specter theory and without investigating the convergence speed. The aim of this chapter will be dedicated to this aspect. Recently, Bacry *et al.* [20] propounded a non-parametric estimation method for general multivariate Hawkes processes based on the explicit resolution of a Wiener-Hopf system using Gaussian quadrature method.

This chapter is organized as follows. In the next section we introduce the univariate version of a Hawkes processes and we set some notations. In section 2, we present the framework of the article. We provide also the estimation method and our main results. In section 3, we establish some properties of the martingale representation of the Hawkes processes and the auto-covariance function. Section 4 is dedicated to the proof of the main results.

Notation and definitions

Let us consider a univariate Hawkes process $(N_t)_{t \in \mathbb{R}_+}$ with stationary increments described by (4.1). The conditional intensity $(\lambda_t)_{t \in \mathbb{R}_+}$ has itself stationary increments with mean:

$$\Lambda = \mathbb{E}[\lambda_t]. \quad (4.3)$$

Or, using Hawkes notation:

$$\Lambda = \frac{\mathbb{E}[dN_t]}{dt}.$$

Taking expectation of both sides of (4.1) allow us to compute Λ under the stationary hypothesis:

$$\Lambda = \mu + \ell \Lambda.$$

This equation has a meaningful solution only under the condition, $\ell < 1$:

$$\Lambda = \frac{\mu}{1 - \ell}. \quad (4.4)$$

Inspired by Bartlett's work ([24]), Hawkes point out the self-exciting point process is uniquely defined by its first-order statistics (i.e. the expectation Λ of its conditional intensity $\lambda(t)$ (4.4)) and the second-order statistic (or covariance density):

$$v(\tau) = \frac{\mathbb{E}[dN_{t+\tau}dN_t]}{(dt)^2} - \Lambda^2, \quad (4.5)$$

which does not depends on t under the stationarity assumption. Indeed, he proved (see [74, 73]) that φ and v are related thanks to the equation:

$$v(\tau) = \Lambda\varphi(\tau) + (\varphi \star v)(\tau) \quad \forall \tau > 0 \quad (4.6)$$

This is a Wiener-Hopf equation, which has unique solution on φ under some hypothesis on ν (see [107]). Knowing the φ kernel allows to recover μ through (4.4).

From a practical point of view, the covariance density ν cannot be estimated directly. Nevertheless, a similar quantity can be computed by discretization.

Definition. We define the (normalized) auto-covariance function of the Hawkes process at scale h and lag t , by:

$$\begin{aligned} v^{(h)}(t) &= \frac{1}{h} \text{Cov}(N_u - N_{u-h}, N_{u+t} - N_{u+t-h}) \\ &= \frac{1}{h} \mathbb{E} \left[\left(\int_0^h dN_s - \Lambda h \right) \left(\int_t^{t+h} dN_s - \Lambda h \right) \right]. \end{aligned} \quad (4.7)$$

The second identity does not depend on u because the increments of N_t are stationary. This quantity is central to our estimation method and can be easily measured using empirical means. The estimation method proposed by Bacry *et al.* in [15] is based on the following theorem:

Theorem 17. *Let $g^{(h)}(t) = (1 - |t|/h)^+$. Then $v^{(h)}$ can be expressed as a function of φ and $g^{(h)}$:*

$$v^{(h)}(t) = \Lambda g^{(h)}(t) + \Lambda g^{(h)} \star \psi(t) + \Lambda g^{(h)} \star \tilde{\psi}(t) + \Lambda g^{(h)} \star \tilde{\psi} \star \psi(t), \quad (4.8)$$

where \star stands for the convolution product, $\tilde{\psi}(t) = \psi(-t)$ and ψ is defined as:

$$\psi(t) = \sum_{n=1}^{\infty} \varphi^{(\star n)}(t) \quad (4.9)$$

where $\varphi^{(\star n)}$ refers to the n -th auto-convolution of φ .

Corollary 8. *In the Fourier domain (4.8) and (4.9) become:*

$$\mathcal{F}[v^{(h)}] = \Lambda \mathcal{F}[g^{(h)}] |1 + \mathcal{F}[\psi]|^2 \quad (4.10)$$

and

$$\mathcal{F}[\psi] = \frac{\mathcal{F}[\varphi]}{1 - \mathcal{F}[\varphi]}. \quad (4.11)$$

where $\mathcal{F}[f]$ is the Fourier transform of f :

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} e^{-i\omega t} f(t) dt, \quad \forall \omega \in \mathbb{R} \quad (4.12)$$

The estimation procedure of φ proposed by Bacry *et al.* in [15] is based on the empirical computation of $v^{(h)}$ and after that of $\mathcal{F}[v^{(h)}]$. Thus we are able to compute $|1 + \mathcal{F}[\psi]|^2$ since the function $g^{(h)}$ is known (see equation 4.10). The central point of the estimation is the Paley-Wiener theorem which allows to recover $\mathcal{F}[\psi]$. But in the Fourier domain, $\mathcal{F}[\psi]$ corresponds to a unique $\mathcal{F}[\varphi]$ (through equation 4.11). Knowing $\mathcal{F}[\varphi]$ one can easily find our decay kernel φ .

The Paley-Wiener theorem allows, under some hypotheses, to recover a filter knowing the filter's amplitude¹. Thus, we can retrieve $1 + \mathcal{F}[\psi]$ knowing $R = |1 + \mathcal{F}[\psi]|$ using the Hilbert transform:

$$1 + \mathcal{F}[\psi] = \text{Re} e^{-i\mathcal{H}[\log R]},$$

1. For more details regarding Paley-Wiener theorem see [114]. We do not argue why this theorem can be applied in our case, the interested reader is encouraged to see the original paper of Bacry *et al.* [15].

where the Hilbert transform, $\mathcal{H}[\cdot]$, is defined as:

$$\mathcal{H}[f](t) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|u| > \epsilon} \frac{f(t-u)}{u} du, \quad \forall f \in \mathcal{S}(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space. The definition can be extended to any space $f \in \mathcal{L}^p(\mathbb{R})$. Thanks to the Riesz theorem ([67]), we know the Hilbert transform on \mathcal{L}^p is a bounded linear operator for any $p \in (1, \infty)$. Moreover, the Hilbert transform has a straight link with the Fourier transform via

$$\mathcal{F}[\mathcal{H}[f]](\omega) = -i \operatorname{sgn}(\omega) \mathcal{F}[f](\omega),$$

where $\operatorname{sgn}(\cdot)$ is the ε function. An immediate consequence of this link is that $\mathcal{H}[\cdot]$ is an isometry on $\mathcal{L}^2(\mathbb{R})$. Furthermore, the last formula can be used for practical computations.

We would like to make some remarks about the quantity R . Using (4.9) we can express R as a function of φ :

(4.13)

$$R = \frac{1}{|1 - \mathcal{F}[\varphi]|}.$$

Furthermore, R does not depend on h , it is bounded:

$$\frac{1}{(1 + \ell)} \leq R(\omega) \leq \frac{1}{(1 - \ell)}, \quad \forall \omega \in \mathbb{R}, \quad (4.14)$$

and R^2 is equal (up to a constant, $\Lambda/2\pi$) to the spectral density (or Bartlett spectrum) of the Hawkes process $(N_t)_{t \in \mathbb{R}}$. The interested reader shall find in [35, 34] more information about the spectral density.

4.2 FRAMEWORK AND MAIN RESULTS

Framework

In non-parametric statistics, classes of functions are in general described by some smoothness parameters. In this paper the smoothness parameters refer to the decrease of the Fourier transform.

Sobolev spaces

We describe the smoothness of a function by the number of times it is differentiable. In order to extend the notion of differentiability to non-integer values, we use Sobolev spaces on \mathbb{R} defined by:

$$\mathcal{W}^s = \left\{ f \in \mathcal{L}^2(\mathbb{R}), \int_{\mathbb{R}} (1 + \omega^2)^s |\mathcal{F}[f](\omega)|^2 d\omega < \infty \right\},$$

equipped with the norm:

$$\|f\|_{\mathcal{W}^s}^2 = \int_{\mathbb{R}} (1 + \omega^2)^s |\mathcal{F}[f](\omega)|^2 d\omega.$$

Proposition 4. *If $\varphi \in \mathcal{W}^s$ then ψ is also in \mathcal{W}^s .*

Proof. Obviously $\psi \in \mathcal{L}^1(\mathbb{R})$ because $\|\psi\|_1 = \ell/(1-\ell)$. We rewrite equation (4.9) as follows:

$$\psi = \varphi + \varphi \star \psi. \quad (4.15)$$

Using Young's inequality we have $\|\varphi \star \psi\|_2 \leq \|\psi\|_1 \|\varphi\|_2$, so $\psi \in \mathcal{L}^2(\mathbb{R})$. We find $\psi \in \mathcal{W}^s$ thanks to the link between φ and ψ in the Fourier domain (4.11):

$$\int_{\mathbb{R}} (1 + \omega^2)^s |\mathcal{F}[\psi](\omega)|^2 d\omega \leq \int_{\mathbb{R}} (1 + \omega^2)^s |\mathcal{F}[\varphi](\omega)|^2 / (1 - \ell)^2 d\omega < \infty.$$

□

Remark 6. We want to emphasize that $g^{(h)} \in \mathcal{W}^s$ if and only if $0 < s < 3/2$. This is obvious after we use the definition of the Sobolev spaces and $\mathcal{F}[g^{(h)}](\omega) = 4 \sin^2(h\omega/2)/(h\omega^2)$. So, for $s \geq 3/2$, $v^{(h)} \notin \mathcal{W}^s$. Even if φ is "very" regular (its Fourier transform decreases fast or very fast), $v^{(h)}$ is not. Nevertheless, since $g^{(h)} \in \mathcal{W}^1$, $v^{(h)} - \Lambda g^{(h)} \in \mathcal{W}^{s+1}$. This is a natural consequence of the previous proposition (4).

The $\mathcal{A}^{\gamma,r}$ class

The $\mathcal{A}^{\gamma,r}$ class is the collection of all continuous functions f such that:

$$\|f\|_{\mathcal{A}}^2 := \int_{\mathbb{R}} e^{2|\gamma t|^r} |\mathcal{F}[f](t)|^2 dt < \infty \quad (4.16)$$

For all $(\gamma, r) \in \mathbb{R}_+^2$ the functions $f \in \mathcal{A}^{\gamma,r}$ are infinitely differentiable. Each of those parameters affects the smoothness, so the accuracy of the best non-parametric estimator, in its own way. One can verify that $f(\cdot) \in \mathcal{A}^{\gamma,r}$ if and only if $\frac{1}{\gamma} f(\frac{\cdot}{\gamma}) \in \mathcal{A}^{\gamma,r}$. We can refer of γ as a scale parameter. When γ is close to zero, $\frac{1}{\gamma} f(\frac{\cdot}{\gamma})$ is close to a Dirac distribution. From this point of view, the the biggest γ is, the smoother are the functions of the $\mathcal{A}^{\gamma,r}$ class. The parameter r has a great impact on the function type, since for $r > 1$, the functions in $\mathcal{A}^{\gamma,r}$ class admit analytic continuation into the whole complex plane. If $r = 1$ the functions in $\mathcal{A}^{\gamma,r}$ admit bounded analytic continuation into the strip $\{z = x + iy, x^2 + y^2 < \gamma^2\}$ of the complex plane (the scale parameter γ affects once again the smoothness of the class). If $0 < r < 1$, the functions $f \in \mathcal{A}^{\gamma,r}$ do not have other interesting properties except for the fact that they are infinitely differentiable. Nevertheless, we can say their smoothness increases with r . Ibragimov and Has'miskii [82, 83] were the first to be interested in this class of function. In the 90', Lepski and Levit studied the adaptive estimation of the parameters γ and r [92, 93, 94, 14, 95, 96].

Proposition 5. *If $\varphi \in \mathcal{A}^{\gamma,r}$ than ψ is also in $\mathcal{A}^{\gamma,r}$.*

The proof is straightforward once we use the link between φ and ψ in the Fourier domain (4.11).

Remark 7. As in the case of Sobolev spaces (see Remark 6), $v^{(h)} \notin \mathcal{A}^{\gamma,r}$, but $v^{(h)} - \Lambda g^{(h)} \in \mathcal{A}^{\gamma,r}$.

Framework

For technical reasons, we are not able to accurately control the behavior of the estimator if μ tends to 0 or infinity, but also if ℓ tends to 1 or $\|\varphi\|_{\infty}$ tends to

infinity. In such cases, the number of point in the process is either exploding or vanishing. Since our estimator of $v^{(h)}$ will be computed only on a finite interval I (from practical viewpoint), we need an assumption on φ ensuring the mass of $v^{(h)}$ on the interval I^c is unimportant. This could be done by assuming φ has a compact support. Let us define for all positive real numbers $0 < \mu_0 < \mu_1$, $0 < \ell_0 < 1$, ϕ and A the following subsets of $\mathbb{R} \times \mathcal{W}^s$ and $\mathbb{R} \times \mathcal{A}^{Y,r}$:

$$\mathcal{W}_{\mu_0, \mu_1, \ell_0, \phi, A}^s = \left\{ (\mu, f) \in [\mu_0, \mu_1] \times \mathcal{W}^s : \text{supp}(f) \subset [0, A] \text{ and} \right. \\ \left. \int_0^A f(t) dt \leq \ell_0, 0 \leq f(t) \leq \phi \quad \forall t \in [0, A] \right\}.$$

$$\mathcal{A}_{\mu_0, \mu_1, \ell_0, \phi, A}^{Y,r} = \left\{ (\mu, f) \in [\mu_0, \mu_1] \times \mathcal{A}^{Y,r} : \text{supp}(f) \subset [0, A] \text{ and} \right. \\ \left. \int_0^A f(t) dt \leq \ell_0, 0 \leq f(t) \leq \phi \quad \forall t \in [0, A] \right\}.$$

Let us prove that the hypothesis that φ has a compact support guarantees the concentration of the mass of $v^{(h)}$ on a known compact interval:

Proposition 6. *If $\varphi \in \mathcal{W}_{\mu_0, \mu_1, \ell_0, \phi, A}^s$ or $\varphi \in \mathcal{A}_{\mu_0, \mu_1, \ell_0, \phi, A}^{Y,r}$ then ψ has an exponential decreasing. More precisely, there are two constants $a > 0$ and $C > 0$ depending on A , ℓ_0 and ϕ such that:*

$$v^{(h)}(t) \leq C \exp(-at), \quad \forall t \geq 0. \quad (4.17)$$

Proof. We recall the ψ definition (4.9):

$$\psi(t) = \sum_{n \geq 1} \varphi^{(\star n)}(t).$$

Since $0 \leq f(t) \leq \phi$, $\forall t \in [0, A]$, we have for all $t \geq 0$:

$$\varphi^{(\star 2)}(t) = \int_0^t \varphi(t-s)\varphi(s) ds \leq \phi \int_0^t \varphi(s) ds \leq \phi \ell_0.$$

By induction we obtain

$$\varphi^{(\star n)}(t) \leq \phi \ell_0^{n-1}.$$

By another simple induction, it is easy to see that $\text{supp}(\varphi^{(\star n)}) \subset [0, nA]$. Putting together the two last remarks, we have for all $t \geq 0$:

$$\psi(t) = \sum_{n \geq \lfloor t/A \rfloor} \varphi^{(\star n)}(t) \leq \sum_{n \geq \lfloor t/A \rfloor} \phi \ell_0^{n-1}.$$

Thus

$$\psi(t) \leq \frac{\phi}{1-\ell_0} \ell_0^{\lfloor t/A \rfloor - 1} = \frac{\phi}{1-\ell_0} \exp\left(-\ln(1/\ell_0)(\lfloor t/A \rfloor - 1)\right) \leq \\ \leq \frac{\phi}{1-\ell_0} \exp\left(-\ln(1/\ell_0)(t/A - 2)\right).$$

Finally we obtain:

$$\psi(t) \leq C \exp(-at),$$

where $C = \frac{\phi}{(1-\ell_0)\ell_0^2}$ and $a = \frac{\ln(1/\ell_0)}{A}$. \square

Corollary 9. *If $\varphi \in \mathcal{W}_{\mu_0, \mu_1, \ell_0, \phi, A}^s$ or $\varphi \in \mathcal{A}_{\mu_0, \mu_1, \ell_0, \phi, A}^{Y,r}$ then $v^{(h)}$ has an exponential decreasing.*

This is a direct consequence of previous Proposition and Definition (4.8) since $g^{(h)}$ has a compact support.

Construction of the estimator

Assume that we have the access of the all jump times t_1, t_2, \dots, t_n of the Hawkes process on the interval $[0, 2T]$. Let us summarize the different steps for the decay kernel estimation:

1. Estimate the average intensity Λ :

$$\hat{\Lambda}_T = \frac{N_T}{T}. \quad (4.18)$$

and set $\hat{\Lambda}_T^* = \max(\mu_0/(1 - \ell_0), \hat{\Lambda}_T)$.

2. Set $h > 0$ "small enough" and estimate the auto-covariation function $v^{(h)}(t)$ for $t \in [0, \alpha_T]$ (α_T will be optimally chosed):

$$\hat{v}_T^{(h)}(t) = \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} (\Delta N(ih) - h\hat{\Lambda}_T^*)(\Delta N(ih+t) - h\hat{\Lambda}_T^*), \quad (4.19)$$

where $\Delta N(ih) = N(ih) - N((i-1)h)$. For $t \in [-\alpha_T, 0]$ put $\hat{v}_T^{(h)}(t) = \hat{v}_T^{(h)}(-t)$ and for $t \notin [-\alpha_T, \alpha_T]$ put $\hat{v}_T^{(h)}(t) = 0$. We will need the mass of $v^{(h)}$ to be concentrated on $[-\alpha_T, \alpha_T]$. Since $v^{(h)}$ has an exponential decrease, α_T will be a function of $\log T$.

3. Decompose the function $\hat{v}_T^{(h)} - \hat{\Lambda}_T^* g^{(h)}$ on a Fourier base $(f_n)_{n \geq 0}$ on the interval $[-\alpha_T, \alpha_T]$.

$$\hat{c}_{n,T}^{(h)} = \frac{1}{2\alpha_T} \int_{-\alpha_T}^{\alpha_T} (\hat{v}_T^{(h)}(t) - \hat{\Lambda}_T^* g(t)) f_n(t) dt, \quad (4.20)$$

where $f_n(t) = \exp(-i2\pi nt/\alpha_T)$.

4. Approximate the Fourier transform of $v^{(h)}$ by

$$\mathcal{F}[\hat{v}_T^{(h)}] = \hat{\Lambda}_T^* \mathcal{F}[g^{(h)}] + \sum_{-L}^L \hat{c}_{n,T}^{(h)} \mathcal{F}[f_n],$$

where L is a positive integer which will be optimally chose later on.

5. Set $\hat{R}_T^{*(h)} = \max\{\hat{R}_T^{(h)}, 1/(1 + \ell_0)\}$, where:

$$\hat{R}_T^{(h)}(\omega) = \begin{cases} \sqrt{\frac{|\mathcal{F}[\hat{v}_T^{(h)}](\omega)|}{\hat{\Lambda}_T^* \mathcal{F}[g^{(h)}](\omega)}}, & \text{if } |\omega| \leq \frac{\pi}{h} \\ 1 & \text{if } |\omega| > \frac{\pi}{h} \end{cases} \quad (4.21)$$

6. Compute the Hilbert transform of $\log(\hat{R}_T^{*(h)})$:

$$\hat{\xi}_T^{(h)} = \mathcal{H}[\log \hat{R}_T^{*(h)}] \quad (4.22)$$

7. Compute the Fourier transform of φ using the Paley-Wiener theorem:

$$\mathcal{F}[\hat{\varphi}_T^{(h)}] = 1 - \frac{e^{i\hat{\xi}_T^{(h)}}}{\hat{R}_T^{*(h)}} \quad (4.23)$$

8. Invert $\mathcal{F}[\hat{\varphi}_T^{(h)}]$:

$$\hat{\varphi}_T^{(h)} = \mathcal{F}^{-1}\left[1 - \frac{e^{i\hat{\xi}_T^{(h)}}}{\hat{R}_T^{*(h)}}\right]$$

Obviously, our procedure depends on the scale parameter h and the asymptotic parameter T . As soon as the notation do not lead to confusion, we will not “write” the dependence on h and T . For the rest of the paper, we will write $\hat{\Lambda}^*$ instead of $\hat{\Lambda}_T^*$, \hat{v} instead of $\hat{v}_T^{(h)}$, g instead of $g^{(h)}$ and so on.

Discussion on h

From a theoretical point of view, the estimation method works for any h . But $\mathcal{F}[g](\omega) = (4/\omega^2 h) \sin^2(\omega h/2)$ cancels for all $\omega = 2n\pi/h, n \in \mathbb{Z}, n \neq 0$ and it is not straightforward how to invert (4.10). So, as long as h is “small enough”, we compute \hat{R} only on the $[-\pi/h, \pi/h]$ interval (step 5 of our estimator construction). Consequently, we need to have the mass of $|1 + \mathcal{F}[\psi]|$ concentrated on the $[-\pi/h, \pi/h]$ interval, so we need h “small enough”. In order to obtain \hat{R}_T^* in equation (4.21), we divide by $\mathcal{F}[g^{(h)}]$, which is close to h on the $[-\pi/h, \pi/h]$ interval. So we need h “large enough”. The last two constraints on h forced us to choose h in order to obtain our best convergence rate.

Remark on estimation method

As we stressed before, the function g is not “so smooth”, *i.e.* $g \notin \mathcal{W}^s$ for $s > 3/2$ (or in $\mathcal{A}^{\gamma, r}$). Thus, the step 3 is important since the function g is known and $v - \Lambda g$ verified the same conditions of “smoothness” as φ . We use the linearity of Fourier transform and we integrate what we have remove in the step 3, *i.e.* Λg , in the next step. The remaining of our estimation method coincides avec the one of Bacry *et al.* [15].

Main results

Let us introduce some notation:

- For $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ and a positive number a we denote by $D_a(f) = \|f\|_2^2 - \int_{-a}^a f^2(t) dt$.
- For a function $f \in \mathcal{L}^2(\mathbb{R})$, a positive integer N we denote by $B_{N,a}(f) = \sum_{n \geq N} |c_n|^2$, where c_n are the Fourier coefficients of the function $f \mathbf{1}_{[-a,a]}$.

With this notation we have:

Theorem 18. *We work under the Assumption 1. If (μ, φ) belongs to $\mathcal{W}_{\mu_0, \mu_1, \ell_0, \phi, A}^s$ our estimator $\hat{\varphi}_T^{(h)}$ of the decay kernel φ satisfies:*

$$\mathbb{E}[\|\varphi - \hat{\varphi}_T^{(h)}\|_2^2] \lesssim T^{-1} \vee \frac{L}{Th} \vee \frac{B_{L,\alpha}(v - \Lambda g)}{h^2} \vee \frac{D_\alpha(v)}{h^2} \vee D_{\frac{\pi}{2h}}(\mathcal{F}[\psi]),$$

where \lesssim means inequality up to a constant that depends on $A, \mu_0, \mu_1, \ell_0, \phi$, and M only.

Remark 8. The hypothesis of compact support of φ can be replaced by a hypothesis of power law decrease: $\varphi(t) \leq t^{-k}$, for $t \geq t_0$ and $k \geq 1$. We find similar bounds and the proof is nearly the same, it suffices to decompose $v^{(h)}$ in a wavelet basis of \mathcal{L}^2 which verifies that the \mathcal{L}^1 -norm of the functions of the basis is uniformly bounded by a constant $M > 0$. For computation’s simplicity, we work under the compact support hypothesis.

Corollary 10. If $\varphi \in \mathcal{W}^s_{\mu_0, \mu_1, \ell_0, \phi, A}$, for $\alpha = \frac{2A}{-\log \ell} \log T$, $h = L^{-1} = T^{-\frac{1}{2s+1}}$ we have:

$$\mathbb{E}[\|\varphi - \hat{\varphi}_T^{(h)}\|_2^2] \lesssim T^{-\frac{2s-1}{2s+1}},$$

where \lesssim means inequality up to a constant that depends on $A, \mu_0, \mu_1, \ell_0, \phi$, and M only.

Proof. Since $\psi \in \mathcal{W}^s$ (**Proposition 2.1**), $D_{\frac{\pi}{2h}}(\mathcal{F}[\psi]) \lesssim h^{2s} = T^{-\frac{2s}{2s+1}}$.

We bound the term $D_\alpha(v)$ using the exponential decrease of $v^{(h)}$ (**Proposition 2.3**). Taking $\alpha = \frac{2A}{-\log \ell} \log T$ we have

$$D_\alpha(v) \lesssim T^{-2}.$$

Furthermore $g \in \mathcal{W}^1$, so $v - \Lambda g \in \mathcal{W}^{s+1}$ and we have:

$$B_{L,\alpha}(v - \Lambda g) \lesssim L^{-(2s+2)} = T^{-\frac{2s+2}{2s+1}}.$$

Let us remember that $B_{N,\alpha}(f)$, is a sum of Fourier coefficients of the function $f 1_{[-a,a]}$. So, the last inequality must be detailed. Let us note $d_n(f)$, $n \in \mathbb{Z}$ the Fourier coefficients of function f .

$$\begin{aligned} B_{L,\alpha}(v - \Lambda g) &= \sum_{n \geq L} |d_n((v - \Lambda g) 1_{[-a,a]})|^2 \leq \\ &2 \sum_{n \geq L} |d_n((v - \Lambda g))|^2 + 2 \sum_{n \geq L} |d_n((v - \Lambda g) 1_{(-\infty, -a) \cup (a, \infty)})|^2. \end{aligned}$$

We can use that $v - \Lambda g \in \mathcal{W}^{s+1}$ to bound the first term. Since $v - \Lambda g$ has an exponential decrease (consequence of **Proposition 2.1**), the second term can be bound by

$$2\|(v - \Lambda g) 1_{(-\infty, -a) \cup (a, \infty)}\|_2^2 \lesssim T^{-2}.$$

The proof is now complet. \square

Corollary 11. If $\varphi \in \mathcal{A}^{\gamma,r}_{\mu_0, \mu_1, \ell_0, \phi, A}$, for $\alpha = \frac{2A}{-\log \ell} \log T$, $h = L^{-1} = \gamma^{-1} (\log T)^{1/r}$ we have:

$$\mathbb{E}[\|\varphi - \hat{\varphi}_T^{(h)}\|_2^2] \lesssim \frac{1}{\gamma^2} (\log T)^{2/r} T^{-1},$$

where \lesssim means inequality up to a constant that depends on $A, \mu_0, \mu_1, \ell_0, \phi$, and M only.

Proof. As in the previous case, α is chosen such as $D_\alpha(v) \lesssim T^{-2}$. Since $\psi \in \mathcal{A}^{\gamma,r}$ (**Proposition 2.12**), $D_{\frac{\pi}{2h}}(\mathcal{F}[\psi]) \lesssim T^{-2}$. We recall that $g \in \mathcal{W}^1$, so $v - \Lambda g \in \mathcal{A}^{\gamma,r}$, which give us $B_{L,\alpha}(v - \Lambda g) \lesssim T^{-1}$. The proof ends by putting all those estimates together. \square

Comments At the end of 2000, B.Y. Levit was interested in the estimation of “very” regular functions (for example the $\mathcal{A}^{\gamma,r}$ class). In a series of papers (see for example [12], [11]), he showed that under different statistical models, the convergence rate of estimators is the best we can hope, T^{-1} (where T is the asymptotic parameter). From this point of view, our estimation method is good enough.

In the case where the φ function belongs to a larger space, \mathcal{W}^s in our case, the convergence rate is not optimal. This is a shortcoming of the estimation

method. The problem appears in step 5, when we divide by $\mathcal{F}[g]$. As we stressed before, since $\mathcal{F}[g]$ cancels for all $\omega = 2n\pi/h, n \in \mathbb{Z}, n \neq 0$, we compute quantity \hat{R} only on the interval $[-\pi/2h, \pi/2h]$. This give us our worse bound L/Th which derived from the variance of the coefficients \hat{c}_n . We show in **Proposition 4.3** that $\mathbb{E}[|\hat{c}_n - c_n|^2] \lesssim h/T$ and this is the best we can hope. Indeed, take step 1 of our estimator and rewrite:

$$\hat{v}(t) = \frac{1}{T} \sum_{i=1}^{T/h} Z_i,$$

where $Z_i = (\Delta N(ih) - h\hat{\Lambda}_T^*)(\Delta N(ih+t) - h\hat{\Lambda}_T^*)$. The variables Z_i are positive correlated and we show latter on that $\mathbb{E}[Z_i^2]$ has the order of h^2 . This is consistent with the fact that Z_i is the product of two martingale increments of length h . We can easily see:

$$\mathbb{E}[|\hat{v}(t) - v(t)|^2] \geq \frac{1}{T^2} \sum_{i=1}^{T/h} \mathbb{E}[Z_i^2].$$

Thus, $\mathbb{E}[|\hat{v}(t) - v(t)|^2]$ has, at least, an order of h/T . In this condition, it is hard to believe the inequality $\mathbb{E}[|\hat{c}_n - c_n|^2] \lesssim h/T$ is suboptimal.

Numerical illustrations

Let us illustrate the estimation method as defined previously using simulated Hawkes processes. All the simulations have performed with the thinning algorithm described in [109]. We consider a power-law decay kernel:

$$\varphi(t) = \delta(t + \gamma)^\beta 1_{t \geq 0},$$

with $\beta < -1$ in order to satisfy $\varphi \in \mathcal{L}^1(\mathbb{R})$. We choosed $delta = 0.1, \gamma = 0.16$ and $\beta = -1.2$. In this case we have $\|\varphi\|_1 = \frac{\delta}{-(\beta+1)} \gamma^{\beta+1} \approx 0.72 < 1$. The simulated process contains around 35.000 jumps for $T \approx 10000$ seconds.

We estimate $v^{(h)}$ from the realization of a Hawkes process $(N_t)_{t \geq 0}$. We then strictly follow the method described in previous section in order to estimate the decay kernel φ . Let us point out the importance of the step 3 where, before decomposing \hat{v} in a Fourier base we remove the quantity Λg . The reason is the g function is not smooth enough. Figure 4.1 shows the importance of this step. Indeed, we observe that around $t \approx 0.5$ the auto-covariance function $v^{(h)}$ is not “so-smooth”.

Let us show the φ estimation. One can see that, up to an additive noise, the estimated kernel fits well the real one:

4.3 PREPARATION FOR THE PROOF

Martingale representation

Let $(M_t)_{t \geq 0}$ be the (\mathcal{F}_t) -martingale compensated process $(N_t)_{t \geq 0}$ defined by:

$$M_t = N_t - \int_0^t \lambda_t dt. \quad (4.24)$$

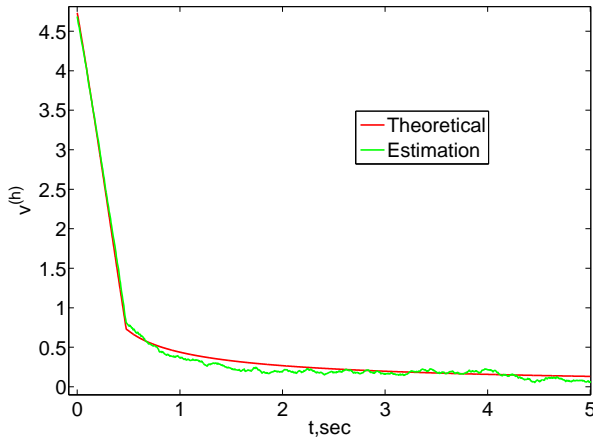


Figure 4.1 – The auto-covariance function $v^{(h)}$ (4.7) and its estimation $\hat{v}^{(h)}$

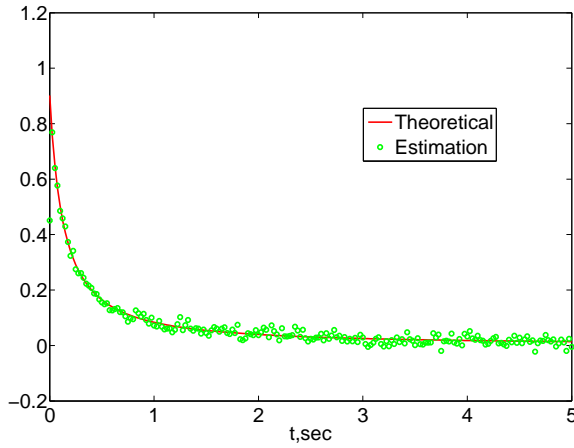


Figure 4.2 – Non parametric estimation of one dimensional Hawkes process using method described in Section 4.2 from a unique realization with 35.000 jumps with power-law kernel $\varphi(t) = \delta(t + \gamma)^\beta 1_{t \geq 0}$, $\delta = 0.1$, $\gamma = 0.16$ and $\beta = -1.2$.

This identity can be rewritten in its differential form : $dN_t = dM_t - \lambda_t dt$. It can be shown (see [16]) that

$$\lambda_t = \Lambda + \psi \star dM_t \quad (4.25)$$

where $\psi \star dM_t = \int_{\mathbb{R}} \psi(t-s) dM_s$. These two identities allow us to prove the next lemma:

Lemma 6. Let $t_1 \leq t_2 \leq t_3 \leq t_4$. Then we have:

1. If $t_1 \neq t_2$, $\mathbb{E}[dM_{t_1} dM_{t_2}] = 0$;
2. $\mathbb{E}[dM_{t_1}^2] = \Lambda dt_1$;
3. If $t_3 < t_4$, $\mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3} dM_{t_4}] = 0$;
4. If $t_1 < t_2 = t_3 = t_4$, $\mathbb{E}[dM_{t_1} dM_{t_2}^3] = \Lambda \psi(t_2 - t_1) dt_1 dt_2$;
5. If $t_1 = t_2 = t_3 = t_4$, $\mathbb{E}[dM_{t_1}^4] = \Lambda dt_1$;
6. If $t_1 < t_2 < t_3 = t_4$,

$$\mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3}^2] = \Lambda \psi(t_3 - t_2) \psi(t_2 - t_1) dt_1 dt_2 dt_3;$$

7. If $t_1 = t_2 < t_3 = t_4$,

$$\mathbb{E}[dM_{t_1}^2 dM_{t_3}^2] = \left(\Lambda^2 + \Lambda\psi(t_2 - t_1) + \Lambda\psi \star \tilde{\psi}(t_2 - t_1) \right) dt_1 dt_2.$$

Proof. 1. It is only the consequence of the fact that $(M_t)_{t \geq 0}$ is a martingale.

2. Using (4.24) we have: $\mathbb{E}[dM_{t_1}^2] = \mathbb{E}[dN_{t_1}^2] - 2\mathbb{E}[dN] \lambda_{t_1} dt_1 + \lambda_{t_1}^2 dt_1^2$. Keeping only first order terms in dt_1 :

$$\mathbb{E}[dM_{t_1}^2] = \mathbb{E}[dN_{t_1}^2] = \mathbb{E}[dN_{t_1}] = \Lambda dt_1.$$

3. Similar to point 1.

4. Using the notation $\mathbb{E}_a[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_a]$, we have:

$$\mathbb{E}[dM_{t_1} dM_{t_2}^3] = \mathbb{E}[dM_{t_1} \mathbb{E}_{t_2^-}[dM_{t_2}^3]].$$

We keep only first order terms in dt_2 :

$$\mathbb{E}_{t_2^-}[dM_{t_2}^3] = \lambda_{t_2} dt_2.$$

This allows us to find:

$$\mathbb{E}[dM_{t_1} dM_{t_2}^3] = \mathbb{E}[dM_{t_1} \lambda_{t_2} dt_2] = \mathbb{E}[dM_{t_1} \psi \star dM_{t_2}] dt_2 = \mathbb{E}\left[\int \psi(t_2 - s) dM_s dM_{t_1} dt_2\right].$$

Using point 1, $\mathbb{E}[dM_s dM_{t_1}] \neq 0$ only for $s = t_1$. So we finally obtain:

$$\mathbb{E}[dM_{t_1} dM_{t_2}^3] = \Lambda\psi(t_2 - t_1) dt_1 dt_2.$$

5. Same computations as in point 2:

$$\mathbb{E}[dM_{t_1}^4] = \mathbb{E}[dN_{t_1}^4] = \mathbb{E}[dN_{t_1}] = \Lambda dt_1.$$

6. Like the previous case, we first have:

$$\mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3}^2] = \mathbb{E}[dM_{t_1} dM_{t_2} \lambda_{t_3} dt_3] = \mathbb{E}\left[\int_{\mathbb{R}} \psi(t_3 - s) dt_3 dM_{t_1} dM_{t_2} dM_s\right].$$

Since $\mathbb{E}[dM_{t_1} dM_{t_2} dM_s] = 0$ for $s \neq t_2$, we deduce:

$$\mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3}^2] = \psi(t_3 - t_2) dt_3 \mathbb{E}[dM_{t_1} dM_{t_2}^2].$$

Using the same technics as before: $\mathbb{E}[dM_{t_1} dM_{t_2}^2] = \Lambda\psi(t_2 - t_1) dt_1 dt_2$, which allows us to complete the proof.

7. First we have: $\mathbb{E}[dM_{t_1}^2 dM_{t_2}^2] = \Lambda dt_2 \mathbb{E}[dM_{t_1}^2] + \mathbb{E}[dM_{t_1}^2 \psi \star dM_{t_2}] dt_2$. Using point 2, $\mathbb{E}[dM_{t_1}^2] = \Lambda dt_1$. It remains to calculate:

$$\mathbb{E}[dM_{t_1}^2 \psi \star dM_{t_2}] = \mathbb{E}\left[\int_{\mathbb{R}} \psi(t_2 - s) dM_{t_1}^2 dM_s\right].$$

The only non-trivial case is for $s \leq t_1$. If $s = t_1$, $\mathbb{E}[dM_{t_1}^3] = \Lambda dt_1$ (similar point 1). When $s < t_1$:

$$\mathbb{E}[dM_s dM_{t_1}^2] = \mathbb{E}[dM_s \lambda_{t_1} dt_1] = \Lambda\psi(t_1 - s) dt_1 ds.$$

Putting all together:

$$\begin{aligned} \mathbb{E}[dM_{t_1}^2 \psi \star dM_{t_2}] &= \Lambda\psi(t_2 - t_1) dt_1 dt_2 + \Lambda \int_{s < t_1} \psi(t_2 - s) \psi(t_1 - s) dt_1 ds \\ &= \left(\Lambda^2 + \Lambda\psi(t_2 - t_1) + \Lambda\psi \star \tilde{\psi}(t_2 - t_1) \right) dt_1 dt_2. \end{aligned}$$

□

Convergence speed of auto-covariance function

Let $X_t = N_t - \mathbb{E}[N_t]$. We know (see [16]) that:

$$X_t = M_t + \psi \star M(t) = M_t + Y_t,$$

where $\psi \star M(t) = \int_{\mathbb{R}} \psi(t-s)M_s ds$. This decomposition is very interesting for us: once we have found a property verified by the martingale M_t , we will prove that Y_t and X_t satisfy the same property.

Let us look more closely at the auto-covariance estimator \hat{v} . Using $\mathbb{E}[\Delta N(ih)] = \mathbb{E}[\Delta N(ih+t)] = \Lambda h$, we can rewrite (4.8):

$$\begin{aligned} \hat{v}(t) &= \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} \left(\Delta X(ih) + (\Lambda - \hat{\Lambda}^*)h \right) \left(\Delta X(ih+t) + (\Lambda - \hat{\Lambda}^*)h \right) \\ &= \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} \Delta X(ih) \Delta X(ih+t) + h(\Lambda - \hat{\Lambda}^*) \frac{X_{T+t} - X_t}{T} + h(\Lambda - \hat{\Lambda}^*) \frac{X_T}{T} + h(\Lambda - \hat{\Lambda}^*)^2. \end{aligned}$$

We finally find:

$$\hat{v}^{(h)}(t) = v^*(t) + v_{\Lambda}(t) + c_{\Lambda}, \quad (4.26)$$

where:

$$\begin{aligned} v^*(t) &= \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} \Delta X(ih) \Delta X(ih+t), \\ v_{\Lambda}(t) &= h(\Lambda - \hat{\Lambda}^*) \left(\frac{X_{T+t} - X_t}{T} + \frac{X_T}{T} \right), \\ c_{\Lambda} &= h(\hat{\Lambda} - \hat{\Lambda}^*)^2. \end{aligned}$$

Let

$$v^{(h)}(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$$

and

$$v^*(t) = \hat{v}_1(t) + \hat{v}_2(t) + \hat{v}_3(t) + \hat{v}_4(t),$$

with:

$$\begin{aligned} v_1(t) &= \Lambda g^{(h)}(t) \text{ and } \hat{v}_1(t) = \sum_{i=1}^{\lfloor T/h \rfloor} \Delta M(ih) \Delta M(ih+t), \\ v_2(t) &= \Lambda g^{(h)} \star \psi(t) \text{ and } \hat{v}_2(t) = \sum_{i=1}^{\lfloor T/h \rfloor} \Delta M(ih) \Delta Y(ih+t), \\ v_3(t) &= \Lambda g^{(h)} \star \tilde{\psi}(t) \text{ and } \hat{v}_3(t) = \sum_{i=1}^{\lfloor T/h \rfloor} \Delta Y(ih) \Delta M(ih+t), \\ v_4(t) &= \Lambda g^{(h)} \star \psi(t) \star \tilde{\psi}(t) \text{ and } \hat{v}_4(t) = \sum_{i=1}^{\lfloor T/h \rfloor} \Delta Y(ih) \Delta Y(ih+t). \end{aligned}$$

Bacry *et al.* [16] proved that for every t , $\hat{v}_i(t) - v_i(t) \rightarrow 0$ in $\mathcal{L}^2(P)$ as $T \rightarrow \infty$, $\forall i \in \{1, 2, 3, 4\}$. Our next result indicates the convergence rate of $\hat{v}_1(t)$. Similar results can be proved for $\hat{v}_i(t)$, $i \in \{2, 3, 4\}$, but we don't need them in this paper.

Proposition 7. *Let $0 < h \leq 1$.*

1. *For any $t \geq h$, we have:*

$$\mathbb{E} \left[|\hat{v}_1(t) - v_1(t)|^2 \right] \lesssim \frac{h}{T}.$$

2. For any $0 \leq t < h$, we have:

$$\mathbb{E}\left[|\hat{v}_1(t) - v_1(t)|^2\right] \lesssim \frac{1}{T}.$$

Proof. 1. In this case, $t \geq h$, $v_1(t) = 0$. Introduce $Z_i = \Delta M(ih)\Delta M(ih + t)$, $i = 1, \dots, \lfloor T/h \rfloor$, the Z_i are centered and uncorrelated random variables. Indeed, the increments $\Delta M(ih)$ and $\Delta M(ih + t)$ do not “intersect” and the martingale property allows to find $\mathbb{E}[Z_i Z_j] = 0$ for $i \neq j$. Since M_t has stationary increments:

$$\mathbb{E}\left[|\hat{v}_1(t)|^2\right] = \frac{1}{T^2} \lfloor T/h \rfloor \mathbb{E}[Z_1^2].$$

Let us compute:

$$\mathbb{E}[Z_1^2] = \int_0^h \int_0^h \int_t^{t+h} \int_t^{t+h} \mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3} dM_{t_4}].$$

The expectation is non-trivial only when $t_3 = t_4$. Since $t > h$, $t_3 > t_1$ and $t_3 > t_2$, we are within point 6 and 7 of the previous lemma:

$$\begin{aligned} \mathbb{E}[Z_1^2] &= \int_0^h \int_t^{t+h} \mathbb{E}[dM_{t_1}^2 dM_{t_3}^2] + 2 \int_0^h \int_{t_1}^h \int_t^{t+h} \mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3}^2] = \\ &\int_0^h \int_t^{t+h} (\Lambda^2 + \Lambda\psi(t_3 - t_1) + \Lambda\psi \star \tilde{\psi}(t_3 - t_1)) dt_1 dt_3 + \\ &2 \int_0^h \int_{t_1}^h \int_t^{t+h} \Lambda\psi(t_3 - t_2)\psi(t_2 - t_1) dt_1 dt_2 dt_3. \end{aligned}$$

First term can be bounded directly by $h^2(\Lambda^2 + \Lambda\|\psi\|_\infty + \Lambda\|\psi \star \tilde{\psi}\|_\infty)$. In order to bound the second term we use $\int_t^{t+h} \psi(t_3 - t_2) dt_3 \leq \|\psi\|_1$. It follows that:

$$\mathbb{E}[Z_1^2] \leq h^2(\Lambda^2 + \Lambda\|\psi\|_\infty + 3\Lambda\|\psi\|_\infty\|\psi\|_1), \quad (4.27)$$

which completes the first part of the proof.

2. Let us decompose $\hat{v}_1(t) = S_1 + S_2 + S_3$, with:

$$\begin{aligned} S_1 &= \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} \left(M_{(i-1)h+t} - M_{(i-1)h} \right) \Delta M(ih + t), \\ S_2 &= \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} \left(M_{ih} - M_{(i-1)h+t} \right) \left(M_{ih+t} - M_{ih} \right), \\ S_3 &= \frac{1}{T} \sum_{i=1}^{\lfloor T/h \rfloor} \left(M_{ih} - M_{(i-1)h+t} \right)^2. \end{aligned}$$

Since the increments in S_1 and S_2 are disjoint, $\mathbb{E}[S_1] = \mathbb{E}[S_2] = 0$. In the same way as before we bound $\mathbb{E}[S_1^2] \lesssim h/T$, $\mathbb{E}[S_2^2] \lesssim h/T$. Using the first 2 points of the previous lemma, $\mathbb{E}\left[\left(M_{ih} - M_{(i-1)h+t}\right)^2\right] = \Lambda(h-t)$, so $\mathbb{E}[S_3] = v_1(t)$. It remains to bound $\mathbb{E}[(S_3 - v_1(t))^2]$. Let us compute $\mathbb{E}[A_i^4]$ and $\mathbb{E}[A_i^2 A_j^2]$, where $A_i = M_{ih} - M_{(i-1)h+t}$.

- In order to simplify the expressions we write $l = h - t$. Using the stationarity of the increments of M , we have:

$$\begin{aligned} \mathbb{E}[A_i^4] &= \int_0^l \int_0^l \int_0^l \int_0^l \mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3} dM_{t_4}] = \\ & \int_0^l \mathbb{E}[dM_{t_1}^4] + 4 \int_0^l \int_{t_1}^l \mathbb{E}[dM_{t_1} dM_{t_2}^3] + 12 \int_0^l \int_{t_1}^l \mathbb{E}[dM_{t_1}^2 dM_{t_2}^2] + \\ & + 24 \int_0^l \int_{t_1}^l \int_{t_2}^l \mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3}^2]. \end{aligned}$$

The first term is equal to Λl .

The second term is equal to $\int_0^l \int_0^l \Lambda \psi(t_2 - t_1) dt_1 dt_2$ and can be bounded by $l\Lambda \|\psi\|_1$.

The third and fourth term can be bounded by $l^2(\Lambda^2 + \Lambda \|\psi\|_\infty + \Lambda \|\psi\|_1 \|\psi\|_\infty)$, and $l^2 \Lambda \|\psi\|_1 \|\psi\|_\infty$ (similar (4.27)).

In conclusion:

$$\mathbb{E}[Y_i^4] \lesssim l. \quad (4.28)$$

- *Term* $\mathbb{E}[A_i^2 A_j^2]$. Let $i < j$:

$$\mathbb{E}[A_i^2 A_j^2] = \int_{(i-1)h+t}^{ih} \int_{(i-1)h+t}^{ih} \int_{(j-1)h+t}^{jh} \int_{(j-1)h+t}^{jh} \mathbb{E}[dM_{t_1} dM_{t_2} dM_{t_3} dM_{t_4}].$$

The expectation is non-trivial only when $t_3 = t_4$. We find ourselves in the case of the previous lemma, point 6 and 7:

$$\begin{aligned} \mathbb{E}[A_i^2 A_j^2] &= \int_{(i-1)h+t}^{ih} \int_{(j-1)h+t}^{jh} (\Lambda^2 + \Lambda \psi(t_3 - t_1) + \Lambda \psi \star \tilde{\psi}(t_3 - t_1)) dt_1 dt_3 + \\ & 2 \int_{(i-1)h+t}^{ih} \int_{t_1}^{ih} \int_{(j-1)h+t}^{jh} \Lambda \psi(t_3 - t_2) \psi(t_2 - t_1) dt_1 dt_2 dt_3 = \\ & \Lambda^2 l^2 + \Lambda(s_{1,k} + s_{2,k} + s_{3,k}), \end{aligned}$$

where $k = |j - i|$ and:

$$\begin{aligned} s_{1,k} &= \int_0^l \int_{kh}^{kh+l} \psi(t_3 - t_1) dt_1 dt_3, \\ s_{2,k} &= \int_0^l \int_{kh}^{kh+l} \psi \star \tilde{\psi}(t_3 - t_1) dt_1 dt_3, \\ s_{3,k} &= 2 \int_0^l \int_{t_1}^l \int_k^l h^{kh+l} \psi(t_3 - t_2) \psi(t_2 - t_1) dt_1 dt_2 dt_3. \end{aligned}$$

Setting $N = \lfloor T/h \rfloor$, we have:

$$\begin{aligned} \mathbb{E}[(S_3^2 - v_1(t))^2] &= \frac{1}{T^2} \sum_{k=1}^N \mathbb{E}[A_k^4] + \frac{1}{T^2} \sum_{i \neq j} \mathbb{E}[A_i^2 A_j^2] - v_1^2(t) \\ &\leq \frac{1}{T^2} \sum_{k=1}^N \mathbb{E}[A_k^4] + \frac{1}{T^2} N(N-1) \Lambda^2 l^2 - v_1^2(t) + \frac{\Lambda}{T^2} \sum_{k=1}^N (N-k)(s_{1,k} + s_{2,k} + s_{3,k}). \end{aligned}$$

First term is bounded by $\lesssim \frac{Nl}{T^2} \leq \frac{1}{T}$ (because $l \leq h$). The second term is bound by $\Lambda^2 l^2 \left| \frac{N(N-1)}{T^2} - \frac{1}{h^2} \right| = \frac{\Lambda^2 l^2}{Th} \leq \frac{\Lambda^2 h}{T}$. It suffices to bound

$$\frac{1}{T^2} \sum_{k=1}^N (N-k) s_{1,k},$$

because the same technics can be used for the other two terms. Let us note $\Psi(t) = \int_t^\infty \psi(s) ds$ and $a_k = \int_0^{kh} \Psi(s) ds$. Since ψ is a positive function, Ψ is a positive and decreasing function. We deduce

$$s_{1,k} \leq \int_0^h \int_{kh}^{(k+1)h} \psi(t_3 - t_1) dt_1 dt_3 = \int_0^h \left(\Psi(kh - t_1) - \Psi((k+1)h - t_1) \right) dt_1 = -a_{k-1} + 2a_k + a_{k+1}.$$

But

$$\sum_{k=1}^N (N-k) s_{1,k} = Na_1 - a_N \leq Na_1.$$

Using $a_1 \leq h \|\psi\|_\infty$, we finally obtain:

$$\frac{1}{T^2} \sum_{k=1}^N (N-k) s_{1,k} \lesssim \frac{1}{T}.$$

The proof is now complete. \square

4.4 PROOF OF THE THEOREM

Proposition 8.

$$\mathbb{E}[|\Lambda - \hat{\Lambda}|^2] \lesssim \frac{1}{T}. \quad (4.29)$$

Proof. Clearly $\mathbb{E}[|\Lambda - \hat{\Lambda}|^2] = |\mathbb{E}[N_T^2/T^2] - \Lambda^2|$. The previous lemma helps us to compute:

$$\begin{aligned} \mathbb{E}[N_T^2] &= \mathbb{E}\left[\int_0^T \int_0^T \lambda_{t_1} \lambda_{t_2} dt_1 dt_2\right] = \\ &= \mathbb{E}\left[\int_0^T \int_0^T \left(\Lambda^2 + \Lambda \psi \star dM_{t_1} + \Lambda \psi \star dM_{t_2} + \psi \star dM_{t_1} \psi \star dM_{t_2}\right) dt_1 dt_2\right]. \end{aligned}$$

The first term gives $\Lambda^2 T^2$, second and third term are trivial. The last term gives $2\Lambda \int_0^T \int_{t_1}^T \psi \star \tilde{\psi}(t_2 - t_1) dt_1 dt_2$. We obtain:

$$\mathbb{E}[|\Lambda - \hat{\Lambda}|^2] = \frac{2\Lambda}{T^2} \int_0^T \int_{t_1}^T \psi \star \tilde{\psi}(t_2 - t_1) dt_1 dt_2.$$

Integrating with respect to t_2 leads to:

$$\mathbb{E}[|\Lambda - \hat{\Lambda}|^2] \leq \frac{2\Lambda}{T^2} \int_0^T \|\psi \star \tilde{\psi}\|_1 dt_1.$$

Thus we obtain:

$$\mathbb{E}[|\Lambda - \hat{\Lambda}|^2] \leq \frac{2\Lambda \|\psi\|_1^2}{T},$$

which finishes the proof. \square

Proposition 9. For any $T \geq 1$ we have:

$$\mathbb{E}[X_T^4] \lesssim T^2. \quad (4.30)$$

Proof. Let $0 \leq t \leq T$. We will begin by proving $\mathbb{E}[M_t^4] \lesssim T^2$. The computation of $\mathbb{E}[M_t^4]$ is similar to (4.28), so we find the upper bound:

$$\mathbb{E}[M_t^4] \lesssim t + t^2 \lesssim T^2.$$

Let us prove that Y_T satisfies the same kind of inequality:

$$\mathbb{E}[Y_t^4] = \int_0^T \int_0^T \int_0^T \int_0^T \psi(T-t_1)\psi(T-t_2)\psi(T-t_3)\psi(T-t_4)\mathbb{E}[M_{t_1}M_{t_2}M_{t_3}M_{t_4}]dt_1dt_2dt_3dt_4.$$

Using Cauchy inequality we have:

$$\mathbb{E}[M_{t_1}M_{t_2}M_{t_3}M_{t_4}] \leq \left(\mathbb{E}[M_{t_1}^4]\mathbb{E}[M_{t_2}^4]\mathbb{E}[M_{t_3}^4]\mathbb{E}[M_{t_4}^4]\right)^{1/4} \lesssim T^2.$$

It follows that:

$$\mathbb{E}[Y_t^4] \lesssim \|\psi\|_1^4 T^2.$$

In order to finish the proof, we observe that: $X_T^4 \leq 4M_t^4 + 4Y_T^4$. \square

Proposition 10. For all $n \in \mathbb{N}$, we have:

$$\mathbb{E}[|c_n - \hat{c}_n|^2] \lesssim \frac{h}{T}, \quad (4.31)$$

where

$$c_n = \frac{1}{\alpha} \int_0^\alpha (v(t) - v_1(t))f_n(t)dt.$$

Proof. Let $\hat{c}_n = \hat{c}_{0,n} + \hat{c}_{1,n} + \hat{c}_{2,n} + \hat{c}_{3,n} + \hat{c}_{4,n} + \hat{c}_{5,n} + \hat{c}_{6,n}$, where:

$$\hat{c}_{0,n} = \frac{1}{\alpha} \int_0^\alpha (v_1(t) - \hat{\Lambda}^* g(t))f_n(t)dt,$$

$$c_{0,n} = \frac{1}{\alpha} \int_0^\alpha v^{(h)}(t)f_n(t)dt,$$

$$\hat{c}_{i,n} = \frac{1}{\alpha} \int_0^\alpha v_i^*(t)f_n(t)dt, \forall i \in \{1, 2, 3, 4\},$$

$$\hat{c}_{5,n} = \frac{1}{\alpha} \int_0^\alpha v_\Lambda(t)f_n(t)dt,$$

$$\hat{c}_{6,n} = \frac{1}{\alpha} \int_0^\alpha c_\Lambda f_n(t)dt.$$

We plan to use the following inequality

$$|c_n - \hat{c}_n|^2 \lesssim |\hat{c}_{0,n}|^2 + |c_{1,n} - \hat{c}_{1,n}|^2 + |c_{2,n} - \hat{c}_{2,n}|^2 + |c_{3,n} - \hat{c}_{3,n}|^2 + |c_{4,n} - \hat{c}_{4,n}|^2 + |\hat{c}_{5,n}|^2 + |\hat{c}_{6,n}|^2,$$

and bound each term separately, where:

$$c_{i,n} = \frac{1}{\alpha} \int_0^\alpha v_{i,n}(t)f_n(t)dt.$$

- Since $v_1 = \Lambda g$, $|\hat{c}_{0,n}|^2 = (\Lambda^* - \Lambda)^2 / \alpha^2 \left(\int_0^\alpha g(t) f_n(t) dt \right)^2$. Using **Proposition 4.1** we obtain:

$$|\hat{c}_{0,n}|^2 \lesssim \frac{h^2}{T}.$$

- Let us first remark

$$c_{1,n} - \hat{c}_{1,n} = \frac{1}{\alpha} \int_0^\alpha (v_1(t) - \hat{v}_1(t)) f_n(-t) dt.$$

By the triangle inequality:

$$|c_{1,n} - \hat{c}_{1,n}|^2 \lesssim \frac{1}{\alpha^2} \left(\int_h^\alpha \hat{v}_1(t) f_n(t) dt \right)^2 + \frac{1}{\alpha^2} \left(\int_0^h (v_1(t) - \hat{v}_1(t)) f_n(t) dt \right)^2.$$

We will bound the two terms separately. Using the Cauchy inequality and **Proposition 3.1.1**, we successively have:

$$\begin{aligned} \mathbb{E} \left[\left(\int_h^\alpha \hat{v}_1(t) f_n(t) dt \right)^2 \right] &= \mathbb{E} \left[\int_h^\alpha \int_h^\alpha \hat{v}_1(t_1) \hat{v}_1(t_2) f_n(t_1) f_n(t_2) dt_1 dt_2 \right] \\ &\leq \int_h^\alpha \int_h^\alpha \mathbb{E} [\hat{v}_1(t_1) \hat{v}_1(t_2)] |f_n(t_1) f_n(t_2)| dt_1 dt_2 \\ &\leq \int_h^\alpha \int_h^\alpha \mathbb{E} [\hat{v}_1^2(t_1)]^{1/2} \mathbb{E} [\hat{v}_1^2(t_2)]^{1/2} |f_n(t_1) f_n(t_2)| dt_1 dt_2 \\ &\lesssim \frac{h}{T} \int_h^\alpha \int_h^\alpha |f_n(t_1) f_n(t_2)| dt_1 dt_2 \leq \frac{h\alpha^2}{T}. \end{aligned}$$

We reproduce the steps use before and **Proposition 3.1.2** to deduce:

$$\mathbb{E} \left[\left(\int_0^h (v_1(t) - \hat{v}_1(t)) f_n(t) dt \right)^2 \right] \lesssim \frac{1}{T} \left(\int_0^h |f_n(t)| dt \right)^2 = \frac{h^2}{T}.$$

Putting all together, we finally obtain

$$\mathbb{E} [|c_{1,n} - \hat{c}_{1,n}|^2] \lesssim \frac{h}{T}.$$

- Set $M_t = 0$ and $\psi(t) = 0$ for $t \leq 0$. We first have

$$Y_t = \int_0^\infty \psi(s) M_{t-s} ds,$$

therefore

$$\Delta Y(ih + t) = \int_0^\infty \psi(s) \Delta M(ih + t - s),$$

hence

$$v_2^*(t) = \int_0^\infty \psi(s) v_1^*(t - s) ds.$$

Second, we now write

$$\begin{aligned} c_{2,n} - \hat{c}_{2,n} &= \frac{1}{\alpha} \int_0^\alpha f_n(t) dt \int_0^\infty ds \psi(s) (v_1(t - s) - v_1^*(t - s)) = \\ &= \frac{1}{\alpha} \int_0^\infty ds \int_s^\alpha dt f_n(t) \psi(s) (v_1(t - s) - v_1^*(t - s)) = \\ &= \frac{1}{\alpha} \int_0^\infty ds \int_s^{s+h} dt f_n(t) \psi(s) (v_1(t - s) - v_1^*(t - s)) + \end{aligned}$$

$$+ \frac{1}{\alpha} \int_0^\infty ds \int_{s+h}^\alpha dt f_n(t) \psi(s) (v_1(t-s) - v_1^*(t-s))$$

We reproduce the steps we use before for the previous term, except we have double integral. Nevertheless, the path of demonstration remains the same: **Proposition 3.1** and Cauchy's inequality. We will show only the line of proof for bounding the first integral.

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^\infty ds \int_s^{s+h} dt \psi(s) (v_1(t-s) - v_1^*(t-s)) \right)^2 \right] \leq \\ & \iint_0^\infty \iint_s^{s+h} \mathbb{E}[(v_1(t_1-s_1) - v_1^*(t_1-s_1))(v_1(t_2-s_2) - v_1^*(t_2-s_2))] \\ & \quad \psi(s_1) \psi(s_2) |f_n(t_1) f_n(t_2)| ds_1 ds_2 dt_1 dt_2 \leq \\ & \iint_0^\infty \iint_s^{s+h} \mathbb{E}[(v_1(t_1-s_1) - v_1^*(t_1-s_1))^2]^{1/2} \mathbb{E}[(v_1(t_2-s_2) - v_1^*(t_2-s_2))^2]^{1/2} \\ & \quad \psi(s_1) \psi(s_2) ds_1 ds_2 dt_1 dt_2 \lesssim \\ & \iint_0^\infty \iint_s^{s+h} \frac{h}{T} \psi(s_1) \psi(s_2) ds_1 ds_2 dt_1 dt_2 = \frac{h^3 \|\psi\|_1^2}{T} \lesssim \frac{h^3}{T}. \end{aligned}$$

Similar technics are sufficient to bound the second integral and to obtain:

$$\mathbb{E}[|c_{2,n} - \hat{c}_{2,n}|^2] \lesssim \frac{h}{T}.$$

Thus, this term has also the right order.

- We proceed likewise for the terms $\mathbb{E}[|c_{i,n} - \hat{c}_{i,n}|^2]$, $i \in \{3, 4\}$.
- The arguments needed here are quite similar to those used before:

$$\begin{aligned} \mathbb{E}[|\hat{c}_{5,n}|^2] &= \frac{h^2}{\alpha^2 T^2} \mathbb{E} \left[(\Lambda - \hat{\Lambda}^*)^2 \left(\int_0^\alpha (X_{T+t} - X_t + X_T) f_n(t) dt \right)^2 \right] \\ &\leq \frac{2h^2}{\alpha^2 T^2} \mathbb{E} \left[(\Lambda - \hat{\Lambda}^*)^2 \left(\int_0^\alpha (X_{T+t} - X_t) f_n(t) dt \right)^2 \right] + \frac{2h^2}{\alpha^2 T^2} \mathbb{E} \left[(\Lambda - \hat{\Lambda}^*)^2 X_T^2 \left(\int_0^\alpha f_n(t) dt \right)^2 \right] \end{aligned}$$

For the second term we use $|\Lambda - \hat{\Lambda}^*| \leq |\Lambda - \hat{\Lambda}| = |X_T|/T$ and **Proposition 4.2** in order to find:

$$\frac{2h^2}{\alpha^2 T^2} \mathbb{E} \left[(\Lambda - \hat{\Lambda}^*)^2 X_T^2 \left(\int_0^\alpha f_n(t) dt \right)^2 \right] \lesssim \frac{h^2}{T^2}$$

For the second term we successively use the Cauchy's inequality:

$$\begin{aligned} & \frac{2h^2}{\alpha^2 T^2} \mathbb{E} \left[(\Lambda - \hat{\Lambda}^*)^2 \left(\int_0^\alpha (X_{T+t} - X_t) f_n(t) dt \right)^2 \right] \\ & \leq \frac{h^2}{\alpha^2 T^2} \mathbb{E}[(\Lambda - \hat{\Lambda})^4]^{1/2} \mathbb{E} \left[\left(\int_0^T (X_{T+t} - X_t) f_n(t) dt \right)^4 \right]^{1/2} \\ & \leq \frac{h^2}{\alpha^2 T^2} \mathbb{E}[(\Lambda - \hat{\Lambda})^4]^{1/2} \left(\iiint_0^\alpha \prod_{i=1}^4 \mathbb{E}[(X_{T+t_i} - X_{t_i})^4]^{1/4} |f_n(t_1) f_n(t_2) f_n(t_3) f_n(t_4)| dt \right)^{1/2} \end{aligned}$$

We remind that $(X_t)_{(t \geq 0)}$ has stationary increments and we use same inequalities as for the first term before to conclude:

$$\mathbb{E}[|\hat{c}_{5,n}|^2] \lesssim \frac{h^2}{T^2}.$$

Thus, this term has also the right order.

- Using **Proposition 4.1** we have

$$\mathbb{E}[|\hat{c}_{6,n}|^2] \leq h^2 \mathbb{E}[(\Lambda - \hat{\Lambda}^*)^2 (\hat{\Lambda} - \hat{\Lambda}^*)^2].$$

By the definition of $\hat{\Lambda}^*$, $|\hat{\Lambda} - \hat{\Lambda}^*| \leq \mu_0$, and we deduce:

$$\mathbb{E}[|\hat{c}_{6,n}|^2] \lesssim \frac{h^2}{T}.$$

This last term have also the right order.

The conclusion follows. \square

Proposition 11. *If $\|R - \hat{R}^*\|_2 \leq \epsilon$ than $\|\varphi - \hat{\varphi}\|_2 \lesssim \epsilon$.*

Proof. Thanks to Plancherel's theorem, it remains to show $\|\mathcal{F}[\varphi] - \mathcal{F}[\hat{\varphi}]\|_2 \lesssim \epsilon$.

Let us remind:

$$\mathcal{F}[\varphi](\omega) = 1 - \frac{1}{R(\omega)} e^{i\mathcal{H}[\log R](\omega)} = 1 - \frac{\cos \xi(\omega)}{R(\omega)} - i \frac{\sin \xi(\omega)}{R(\omega)},$$

where $\xi(\omega) = \mathcal{H}[\log R](\omega)$.

$$\begin{aligned} |\mathcal{F}[\hat{\varphi}](\omega) - \mathcal{F}[\varphi](\omega)|^2 &= \left(\frac{\cos \hat{\xi}(\omega)}{\hat{R}^*(\omega)} - \frac{\cos \xi(\omega)}{R(\omega)} \right)^2 + \left(\frac{\sin \hat{\xi}(\omega)}{\hat{R}^*(\omega)} - \frac{\sin \xi(\omega)}{R(\omega)} \right)^2 = \\ &= \frac{1}{R^2(\omega)} + \frac{1}{\hat{R}^{*2}(\omega)} - \frac{2 \cos(\xi(\omega) - \hat{\xi}(\omega))}{R(\omega) \hat{R}^*(\omega)} = \\ &= \frac{1}{R^2(\omega)} + \frac{1}{\hat{R}^{*2}(\omega)} + \frac{2}{R(\omega) \hat{R}^*(\omega)} - \frac{2}{R(\omega) \hat{R}^*(\omega)} - \frac{2 \cos(\xi(\omega) - \hat{\xi}(\omega))}{R(\omega) \hat{R}^*(\omega)} = \\ &= \frac{(R(\omega) - \hat{R}^*(\omega))^2}{R^2(\omega) \hat{R}^{*2}(\omega)} + \frac{4 \sin^2\left(\frac{\xi(\omega) - \hat{\xi}(\omega)}{2}\right)}{R(\omega) \hat{R}^*(\omega)} \end{aligned}$$

We use the boundness of R (4.14), the definition of \hat{R}^* (4.21) and the basic inequality $|\sin x| \leq |x|$, to deduce:

$$\|\mathcal{F}[\hat{\varphi}] - \mathcal{F}[\varphi]\|_2 \leq (1 + \ell_0)^4 \|R - \hat{R}^*\|_2 + (1 + \ell_0)^2 \|\mathcal{H}[\log \frac{R}{\hat{R}^*}]\|_2.$$

Since $\|\mathcal{H}[f]\|_2 = \|f\|_2, \forall f \in L_2(\mathbb{R})$ (see [67] for more technical precisions), it remains to bound $\|\log \frac{R}{\hat{R}^*}\|_2$. Let $M = \max(R, \hat{R}^*)$ and $m = \min(R, \hat{R}^*)$:

$$\begin{aligned} \left| \log \frac{R}{\hat{R}^*} \right| &= \log \frac{M(\omega)}{m(\omega)} = \log \left(\frac{M(\omega) - m(\omega)}{m(\omega)} + 1 \right) \leq \frac{M(\omega) - m(\omega)}{m(\omega)} \\ &\leq (1 + \ell_0)(M(\omega) - m(\omega)) = (1 + \ell_0) |R(\omega) - \hat{R}^*(\omega)|. \end{aligned}$$

This allow us to find $\|\mathcal{H}[\log \frac{R}{\hat{R}^*}]\|_2 \lesssim \|R - \hat{R}^*\|_2$, which finishes the proof. \square

Proof of Theorem 2.1

Proof. We have:

$$v = \Lambda g + \sum_{n \in \mathbb{Z}} c_n f_n + v_{[-\infty, -\alpha] \cup [\alpha, \infty]},$$

$$\hat{v} = \hat{\Lambda}^* g + \sum_{-L}^L \hat{c}_n f_n.$$

The triangular inequality helps us to find:

$$\|\mathcal{F}[v] - \mathcal{F}[\hat{v}]\|_2^2 = \|v - \hat{v}\|_2^2 \leq 4 \left((\Lambda - \hat{\Lambda}^*)^2 \|g\|_2^2 + \sum_{n=-L}^L |c_n - \hat{c}_n|^2 + \sum_{|n|>L} |c_n|^2 + \|v_{[-\infty, -\alpha] \cup [\alpha, \infty]}\|_2^2 \right).$$

Concerning the first term, we have $\|g\|_2^2 = 2h/3$ (direct computation) and using **Proposition 4.1** we obtain:

$$\mathbb{E}[(\Lambda - \hat{\Lambda}^*)^2 \|g\|_2^2] \lesssim \frac{h}{T}.$$

The bound for the second term is a direct consequence of the **Proposition 4.3**:

$$\mathbb{E} \left[\sum_{n=-L}^L |c_n - \hat{c}_n|^2 \right] \lesssim \frac{Lh}{T}.$$

With the notation introduced in section 2.3, we have:

$$\|\mathcal{F}[v] - \mathcal{F}[\hat{v}]\|_2 \lesssim \frac{Lh}{T} \vee B_{L,\alpha}(v - \Lambda g) \vee D_\alpha(v). \quad (4.32)$$

According to **Proposition 4.4** it remains to control $\mathbb{E}[|R - \hat{R}^*|_2^2]$. We recall $\hat{R}^* = 1$ on the interval $J = \mathbb{R} \setminus [-\pi/2h, \pi/2h]$. We use the fact that $R = |1 + \mathcal{F}[\psi]|$ to obtain:

$$\|(1 - R(\omega))1_J(\omega)\|_2^2 \leq \|\mathcal{F}[\psi]1_J(\omega)\|_2^2 = D_{\frac{\pi}{2h}}(\mathcal{F}[\psi]). \quad (4.33)$$

On the interval $J^c = [-\pi/2h, \pi/2h]$ we have:

$$|R(\omega) - \hat{R}^*(\omega)| = \left| \frac{\sqrt{\mathcal{F}[v](\omega)}}{\sqrt{\Lambda \mathcal{F}[g](\omega)}} - \frac{|\sqrt{\mathcal{F}[\hat{v}](\omega)}|}{\sqrt{\hat{\Lambda}^* \mathcal{F}[g](\omega)}} \right| \leq I(\omega) + II(\omega),$$

where

$$I(\omega) = \left| \frac{\sqrt{\mathcal{F}[v](\omega)}}{\sqrt{\Lambda \mathcal{F}[g](\omega)}} - \frac{\sqrt{\mathcal{F}[v](\omega)}}{\sqrt{\hat{\Lambda}^* \mathcal{F}[g](\omega)}} \right|$$

and

$$II(\omega) = \left| \frac{\sqrt{\mathcal{F}[v](\omega)}}{\sqrt{\hat{\Lambda}^* \mathcal{F}[g](\omega)}} - \frac{\sqrt{\mathcal{F}[\hat{v}](\omega)}}{\sqrt{\hat{\Lambda}^* \mathcal{F}[g](\omega)}} \right|.$$

Let us bound the two terms separately:

– *The term I:*

$$I(\omega) = \frac{\sqrt{\mathcal{F}[v](\omega)}}{\sqrt{\mathcal{F}[g](\omega)}} \frac{\Lambda - \hat{\Lambda}^*}{\sqrt{\Lambda \hat{\Lambda}^* (\sqrt{\Lambda} + \sqrt{\hat{\Lambda}^*})}} = R(\omega) \frac{\Lambda - \hat{\Lambda}^*}{\sqrt{\hat{\Lambda}^* (\sqrt{\Lambda} + \sqrt{\hat{\Lambda}^*})}}.$$

Using **Proposition 4.1** we obtain

$$\mathbb{E}[|I|_2^2] \leq \frac{1}{(1 - \ell_0)^2} \mathbb{E} \left[\frac{(\Lambda - \hat{\Lambda}^*)^2}{\hat{\Lambda}^* (\Lambda + \hat{\Lambda}^*)} \right] \lesssim \frac{1}{T} \quad (4.34)$$

– *The term II:*

First, we have

$$II(\omega) = \frac{|\mathcal{F}[v](\omega) - \mathcal{F}[\hat{v}](\omega)|}{\sqrt{\hat{\Lambda}^* \mathcal{F}[g](\omega)} \left(\sqrt{\mathcal{F}[v](\omega)} + \sqrt{\mathcal{F}[\hat{v}](\omega)} \right)}$$

$$\leq \frac{|\mathcal{F}[v](\omega) - \mathcal{F}[\hat{v}](\omega)|}{\sqrt{\hat{\Lambda}^* \mathcal{F}[g](\omega) \sqrt{\mathcal{F}[v](\omega)}}} = \frac{|\mathcal{F}[v](\omega) - \mathcal{F}[\hat{v}](\omega)|}{R(\omega) \sqrt{\hat{\Lambda}^* \Lambda \mathcal{F}[g](\omega)}}$$

Basic trigonometric inequalities show us that $\mathcal{F}[g](\omega) > h/4, \forall \omega \in [-\pi/h, \pi/h]$. We remind $R(\omega) \geq 1/(1 + \ell_0)$ (4.14). Using (4.32), we finally obtain:

$$\mathbb{E}[\|II\|_2^2] \lesssim \frac{1}{h^2} \left(\frac{Lh}{T} \vee B_{L,\alpha}(v - \Lambda g) \vee D_\alpha(v) \right). \quad (4.35)$$

Putting together (4.33), (4.34) and (4.35) we obtain:

$$\mathbb{E}[\|R - \hat{R}^*\|_2^2] \lesssim T^{-1} \vee \frac{L}{Th} \vee \frac{B_{L,\alpha}(v - \Lambda g)}{h^2} \vee \frac{D_\alpha(v)}{h^2} \vee D_{\frac{\pi}{2h}}(\mathcal{F}[\psi]).$$

Proposition 4.4 completes the proof of the theorem. \square



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