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Le Ha Vy Nguyen

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Le Ha Vy Nguyen. Stabilité et stabilisation de diverses classes de systèmes fractionnaires et à retards. Analysis of PDEs [math.AP]. Université Paris Sud - Paris XI, 2014. English. NNT : 2014PA112387 . tel-01140955

HAL Id: tel-01140955

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UNIVERSITÉ PARIS-SUD

ÉCOLE DOCTORALE : Sciences et Technologie de l'Information, des
Télécommunications et des Systèmes

Laboratoire des Signaux et Systèmes

DISCIPLINE : Génie informatique, automatique et traitement du signal

THÈSE DE DOCTORAT

soutenue le 09/12/2014

par

Le Ha Vy NGUYEN

<p>STABILITÉ ET STABILISATION DE DIVERSES CLASSES DE SYSTÈMES FRACTIONNAIRES ET À RETARDS</p>
--

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Remerciements

Mes remerciements vont tout d'abord à ma directrice de thèse, Catherine Bonnet, pour m'avoir guidé, encouragé, conseillé pendant mes trois années de thèse. Sa gentillesse, sa générosité et son humeur ont fait que ces années étaient très agréables.

Je remercie les membres du jury d'avoir accepté d'évaluer mon travail de thèse, d'avoir fait des déplacements dont certains sont très longs pour présenter dans ma soutenance et de m'avoir donné des remarques pertinentes pour améliorer mon travail de recherche dans l'avenir. Je remercie en particulier Jonathan Partington et Jean Jacques Loiseau pour leur travail de rapporteur. Je tiens à remercier Laurent Lefèvre pour son soutien qui dure déjà cinq ans depuis mon arrivée en France.

Je tiens à exprimer ma reconnaissance à Alban Quadrat pour ses aides et ses conseils pertinents pour ma recherche.

Mes remerciements vont également au personnel d'Inria et de L2S . Je voudrais remercier les secrétaires Valérie Berthou, Céline Halter, Maëva Jeannot et Maryvonne Giron pour leur travail professionnel qui m'a beaucoup aidé dans les démarches administratives. Je remercie Céline Labrude et Frédéric Desprez de la cellule informatique de L2S qui m'a aidé à résoudre les gros problèmes informatiques que j'ai rencontrés de nombreuses fois.

Je remercie aussi les amis de laboratoire pour avoir partagé avec moi la vie de thésard: Habib, Victor, Sarra, Georgios, Islam, Ali, Yuling, Long, Hugo, Thang, Duy, Bien, Hieu, Dung, Cuong, Linh, Ngoc Anh, Thach, Manh, Tri, Tam. Merci à Ngoc Anh pour les discussions intéressantes lors des déjeuners. Merci à Thach pour les moments agréables en parlant avec lui.

Enfin, mes remerciements vont à ma famille. Merci à mes parents et mon petit frère pour leur soutien et leurs encouragements. Merci à mon mari Chan pour son amour et sa présence. Merci d'être toujours là pour moi.

Résumé

Nous considérons deux classes de systèmes fractionnaires linéaires invariants dans le temps avec des ordres commensurables et des retards discrets. La première est composée de systèmes fractionnaires à entrées multiples et à une sortie avec des retards en entrées ou en sortie. La seconde se compose de systèmes fractionnaires de type neutre avec retards commensurables. Nous étudions la stabilisation de la première classe de systèmes à l'aide de l'approche de factorisation. Nous obtenons des factorisations copremières à gauche et à droite et les facteurs de Bézout associés: ils permettent de constituer l'ensemble des contrôleurs stabilisants. Pour la deuxième classe de systèmes, nous nous sommes intéressés au cas critique où certaines chaînes de pôles sont asymptotiques à l'axe imaginaire. Tout d'abord, nous réalisons une approximation des pôles asymptotiques afin de déterminer leur emplacement par rapport à l'axe. Le cas échéant, des conditions nécessaires et suffisantes de stabilité H_∞ sont données. Cette analyse de stabilité est ensuite étendue aux systèmes à retard classiques ayant la même forme. Enfin, nous proposons une approche unifiée pour les deux classes de systèmes à retards commensurables de type neutre (standards et fractionnaires). Ensuite, la stabilisation d'une sous-classe de systèmes neutres fractionnaires est étudiée. Premièrement, l'ensemble de tous les contrôleurs stabilisants est obtenu. Deuxièmement, nous prouvons que pour une grande classe de contrôleurs fractionnaires à retards il est impossible d'éliminer dans la boucle fermée les chaînes de pôles asymptotiques à l'axe imaginaire si de telles chaînes sont présentes dans les systèmes à contrôler.

Abstract

We consider two classes of linear time-invariant fractional systems with commensurate orders and discrete delays. The first one consists of multi-input single-output fractional systems with output or input delays. The second one consists of single-input single-output fractional neutral systems with commensurate delays. We study the stabilization of the first class of systems using the factorization approach. We derive left and right coprime factorizations and Bézout factors, which are the elements to constitute the set of all stabilizing controllers. For the second class of systems, we are interested in the critical case where some chains of poles are asymptotic to the imaginary axis. First, we approximate asymptotic poles in order to determine their location relative to the axis. Then, when appropriate, necessary and sufficient conditions for H_∞ -stability are derived. This stability analysis is then extended to classical delay systems of the same form and finally a unified approach for both classes of neutral delay systems with commensurate delays (standard and fractional) is proposed. Next, the stabilization of a subclass of fractional neutral systems is studied. First, the set of all stabilizing controllers is derived. Second, we prove that a large class of fractional controllers with delays cannot eliminate in the closed loop chains of poles asymptotic to the imaginary axis if such chains are present in the controlled systems.

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List of Symbols

\mathbb{C}_+	set of complex numbers with positive real part
$\overline{\mathbb{C}_+}$	set of complex numbers with non-negative real part
$\text{card}(\mathcal{L})$	number of entries of the set \mathcal{L}
\mathbb{N}	set of natural numbers (not including zero)
\mathbb{N}_N	set of the first N natural numbers
\mathbb{R}_+	set of positive real numbers
$[x]$	integer part of $x \in \mathbb{R}$.
\mathbb{Z}_+	set of non-negative integers
\mathbb{Z}_+^*	set of positive integers

Chapter 1

Introduction

Contents

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1.1 Motivation

In this work, we address the problems of stability analysis and stabilization of several classes of SISO and MIMO systems. We work in the frequency domain and our aim is to find easy-to-check conditions of stability as well as explicit expressions of controllers having in mind an integration of our results in a software.

We are interested in two major classes of systems regarding applications: delay systems and fractional systems. Both need tools of infinite-dimensional control theory.

In the time domain, fractional models involve derivatives and/or integrals of orders which are not necessarily integers. Similarly, they involve in the frequency domain non (necessarily) integer powers of the Laplace variable s . These models have found applications in many fields, for instance electromagnetics (Westerlund and Ekstam, 1994; Knospe and Zhu, 2011), mechanics (Caputo and Mainardi, 1971; Koh and Kelly, 1990; Vinagre et al., 1998), and biology (Ionescu and De Keyser, 2008; Grahovac and Zigic, 2010). The increasing popularity of fractional models is due to two reasons. First, more physical phenomena have been described using fractional laws and thus system descriptions constructed from these physical laws are also fractional models. Second, for various macroscopic behaviors, fractional models provide models with less parameters and at the same time better fittings to collected data than integer-order models. For more details about fractional calculus and examples, see (Oldham and Spanier, 1974; Podlubny, 1998) and references therein.

In control engineering, a lot of results are available on fractional controllers and their implementation. See for example (Oustaloup et al., 1995; Podlubny, 1999; Chen et al., 2009; Magin et al., 2011) and the references therein. Two well-known types of fractional

controllers are CRONE (Oustaloup et al., 1995) and fractional PID (Podlubny, 1999). Applications of these fractional controllers on benchmark problems have been reported and have been showed to provide better performance than integer-order controllers (Oustaloup et al., 1995; Xue et al., 2006).

With the spreading of fractional systems including both plants and controllers, it is natural to think about fractional systems with delays since delays are commonly encountered in real systems due to unavoidable communication or transfer distances. Furthermore, as reported in the huge literature of classical (integer-order) systems, delays may in certain cases strongly influence the stability of systems. Sometimes delays help to stabilize the systems, but more often they make the systems unstable and even the task of stabilization becomes more difficult. Therefore, delays could be expected to also play such important roles in the field of fractional systems.

There has been a growing interest for studying fractional systems with delays. The question of stability of linear fractional systems with delays has been answered by many authors. In (Hotzel, 1998a), the system described by the transfer function $\frac{1}{(as^\mu + b) + (cs^\mu + d)e^{-sh}}$ ($a, b, c, d, h \in \mathbb{R}, h > 0$) was considered and conditions for BIBO-stability were derived. Since then many other studies have been conducted in the frequency domain and many results have been obtained for fractional systems with arbitrary real orders and with arbitrary positive delays. (Bonnet and Partington, 2002) studied the BIBO-stability of the general class of fractional systems with delays. A more general class of systems was then examined in (Bonnet and Partington, 2001). This has been the most general class of linear fractional systems with delays considered in the literature. In (Chen and Moore, 2002), by using the Lambert function, the authors derived the closed form solution of the characteristic equation of simple fractional systems with one delay. Fractional systems described by delay fractional differential equations were considered in (Deng et al., 2007) and conditions for Lyapunov globally asymptotic stability were derived. Recently, robust BIBO-stability of some classes of fractional systems with delays were studied in (Akbari Moornani and Haeri, 2010, 2011).

All the stability conditions obtained in the aforementioned work concern the location of poles in the complex plane. For delay fractional systems of retarded type, the necessary and sufficient conditions for stability is the familiar one ‘no pole in the closed right half-plane’. In order to check this condition, several numerical methods have been proposed. We can classify these methods into two categories. In the first one, one checks the stability of the system at fixed delays. Such methods were presented in (Hwang and Cheng, 2005, 2006), being based on Cauchy’s integral theorem and the Lambert function respectively. The second category consists of methods which determine the intervals of delay in which the systems are stable. We mention here (Ozturk and Uraz, 1985; Fioravanti et al., 2012; Mesbahi and Haeri, 2013) among others.

The numerical methods mentioned above exclusively deal with fractional systems of retarded type except (Fioravanti et al., 2010) whose method can be applied to some fractional systems of neutral type. As well in the references on the stability analysis cited earlier, while for retarded systems ‘no pole in the closed right half-plane’ is the necessary and sufficient condition for stability, it is only a necessary condition for neutral systems. This can be explained by the complicated locations of poles of neutral systems: infinitely

many isolated poles gather in some vertical strips in the complex plane (Bellman and Cooke, 1963; Hotzel, 1998a; Bonnet and Partington, 2002).

In the simplest case of systems with commensurate fractional orders and with commensurate delays where the above phenomenon reduces to poles asymptotic to vertical lines, attempts were made in (Bonnet and Partington, 2007; Fioravanti et al., 2010) to obtain necessary and sufficient conditions for H_∞ -stability (which is a weaker notion than BIBO-stability) for a class of these systems.

Some works with the same purpose are also available for classical (integer-order) systems of neutral type for which the same difficulty is encountered. In the frequency domain, we have (Bonnet et al., 2011) for H_∞ -stability and (Abusaksaka and Partington, 2014) for BIBO-stability. And in the time domain, (Rabah et al., 2012) considered the asymptotic stability.

In this thesis, we choose to consider linear fractional systems with commensurate fractional orders and commensurate delays using frequency methods. This means their transfer functions are ratios of two quasi-polynomials in $e^{-s\tau}$ and s^α where $\tau > 0$ is the delay and $\alpha > 0$ is the arbitrary order and often takes values in $(0, 1)$.

The choice to consider commensurate quantities has some advantages.

- Commensurate fractional orders are commonly obtained via identification for linear fractional systems. Together with delays they constitute interconnected systems whose models are linear fractional systems with delays. These systems have a similar form to classical delay systems and thus the stability analysis and control might benefit from large collections of tools used for classical ones.
- Delays measured in reality are commensurate. Although the ratios between them may be constants for a short amount of time and likely to vary, the corresponding systems at an instant are quite simple to analyze and hence provide a good starting point for studying the characteristics of the systems.

Although there have been many results concerning stability analysis, the problem of stabilization of fractional systems with delays has just been marginally addressed. In the early work (Hotzel, 1998b), a control strategy involving distributed delays was proposed to control MIMO linear fractional systems with input delays. Also for SISO fractional systems with one input delay, PID controllers were designed in (Özbay et al., 2012). The parametrization of all stabilizing controllers was obtained in (Bonnet and Partington, 2001) for SISO fractional systems of retarded type and in (Bonnet and Partington, 2007) for some SISO fractional systems of neutral type.

To analyze this left wide open area of stabilization of fractional delay systems, we choose the factorization approach to analysis and synthesis problems (Vidyasagar, 1985). With its algebraic nature, this powerful approach allows one to derive the set of all stabilizing controllers which can be used to study various control problems and in particular robust control.

While the tools for applying the factorization approach to finite-dimensional systems are numerous, they are quite limited for infinite-dimensional systems though qualitative results are available (Quadrat, 2006b). In this thesis, by deriving explicit expressions

of coprime factorizations and Bézout factors for some classes of fractional systems with delays, we contribute to the “implementation” phase of the factorization approach for infinite-dimensional systems.

For MIMO systems, the question of parametrization of all stabilizing controllers has been studied by A. Quadrat and K. Mori who are able to derive the set of all stabilizing controllers once one already knows a particular stabilizing controller (Mori, 2002; Quadrat, 2006b). For the particular class of MIMO (integer-order) systems with I/O delays, the idea in (Mirkin and Raskin, 1999; Moelja and Meinsma, 2003) was to reduce the problem to an equivalent finite-dimensional stabilization problem by involving an unstable finite-dimensional system and a stable infinite-dimensional system (FIR filter). Our purpose in this work is to derive explicit expressions of coprime factorizations and Bézout factors of MIMO fractional systems with I/O delays.

1.2 Outline of the thesis

We consider two classes of linear time-invariant fractional systems with discrete delays. The first one consists of MISO fractional systems of commensurate orders with output or input delays. The second one consists of SISO fractional neutral systems with commensurate delays.

This manuscript is divided into 7 chapters.

We study the stabilization of the first class of systems in Chapter 3 using the factorization approach. We derive explicit expressions of left and right coprime factorizations and Bézout factors, which are the elements to constitute the set of all stabilizing controllers.

The second class of systems are examined in Chapters 4, 6, and 7. We are interested in the critical case where these systems have poles asymptotic to the imaginary axis. First, the stability analysis is realized in Chapter 4. This analysis consists of determining location of poles about the imaginary axis via approximation and then deriving necessary and sufficient conditions for H_∞ -stability. The analysis is similar for classical systems of the same form and thus is extended for these systems in Chapter 5. Then in Chapter 6 we present a new method which allows a unified approach to analyze the stability of both fractional and classical delay systems. The new method covers not only cases considered in the two preceding chapters but also all other (unsolved) possible cases. Furthermore, it can be easily programmed in computation software. Next, the question of stabilization is studied in Chapter 7 for a subclass of fractional systems, making use of the stability analysis results and the factorization approach.

To facilitate the understanding of the aforementioned chapters, some preliminaries are given in Chapter 2. As the second class of systems will be studied in several chapters, in order to avoid repetition, we present it in detail in Chapter 2 along with some basic facts.

Finally, we give conclusions and perspectives in Chapter 8.

Chapter 2

Basic results

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2.1 Preliminaries

2.1.1 Stability

The references used for this subsection are (Curtain and Zwart, 1995; Zhou et al., 1995) for H_∞ -stability, and (Desoer and Vidyasagar, 1975) for BIBO-stability.

Definition 2.1. For $1 \leq p < \infty$,

$$L_p[0, \infty) := \{f : [0, \infty) \mapsto \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_0^\infty |f(t)|^p dt < \infty\}.$$

$$L_\infty[0, \infty) := \{f : [0, \infty) \mapsto \mathbb{C} \mid f \text{ is Lebesgue measurable and } \operatorname{ess\,sup}_{t \in [0, \infty)} |f(t)| < \infty\}.$$

Lemma 2.2. $L_p[0, \infty)$ for $1 \leq p < \infty$ are Banach spaces under the norms

$$\|f(t)\|_p := \left(\int_0^\infty |f(t)|^p dt \right)^{1/p}.$$

Lemma 2.3. $L_2[0, \infty)$ is a Hilbert space under the inner product and the induced norm

$$\langle f, g \rangle := \int_0^\infty f^*(t)g(t)dt,$$

$$\|f(t)\|_2 := \sqrt{\langle f, f \rangle}.$$

Lemma 2.4. $L_\infty[0, \infty)$ is a Banach space under the norm

$$\|f(t)\|_\infty := \operatorname{ess\,sup}_{t \in [0, \infty)} |f(t)|.$$

Definition 2.5. A linear continuous-time system defined by a linear operator

$$\Sigma : L_p[0, \infty) \mapsto L_p[0, \infty) \quad \text{for } 1 \leq p \leq \infty$$

is L_p -stable if

$$\|\Sigma\|_p < \infty,$$

where $\|\Sigma\|_p$ is the norm of the operator and is defined by

$$\|\Sigma\|_p := \sup\{\|\Sigma f\|_p \mid f \in L_p[0, \infty), \|f\|_p = 1\} = \sup_{0 \neq f \in L_p[0, \infty)} \frac{\|\Sigma f\|_p}{\|f\|_p}.$$

Roughly speaking, a system is L_2 -stable if it provides an output signal of bounded energy for an input signal of bounded energy.

Similarly, a system is L_∞ -stable if it provides a bounded output signal for a bounded input signal. Hence, L_∞ -stability is also called BIBO-stability.

Let us denote

$$\mathcal{A} := \{h(t) = f(t) + \sum_{i=1}^{+\infty} a_i \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+), a_i \in \mathbb{C}, \sum_{i=0}^{\infty} |a_i| < \infty, t_i \in \mathbb{R}_+, \\ 0 = t_0 < t_1 < \dots\}.$$

Theorem 2.6. *If the impulse response of a linear time-invariant system is in \mathcal{A} , then the system is L_p -stable for $p \in [1, \infty]$.*

Theorem 2.7. *If a linear time-invariant system whose impulse response has vanishing non-atomic singular part is L_∞ -stable, then its impulse response is in \mathcal{A} .*

Due to Theorems 2.6 and 2.7, if we only consider linear time-invariant systems whose impulse response has vanishing non-atomic singular part, then a linear time-invariant system is BIBO-stable if and only if its response impulse is in \mathcal{A} , or its transfer function is in $\hat{\mathcal{A}}$, which is the set of Laplace transforms of functions in \mathcal{A} .

Definition 2.8 (Hardy spaces).

$$H_2(\mathbb{C}_+) := \{f : \mathbb{C}_+ \mapsto \mathbb{C} \mid f \text{ is analytic in } \mathbb{C}_+ \text{ and } \sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 d\omega < \infty\},$$

$$H_\infty(\mathbb{C}_+) := \{f : \mathbb{C}_+ \mapsto \mathbb{C} \mid f \text{ is analytic in } \mathbb{C}_+ \text{ and } \sup_{s \in \mathbb{C}_+} |f| < \infty\}.$$

Lemma 2.9. $H_2(\mathbb{C}_+)$ is a Hilbert space under the inner product and the induced norm

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega)g(j\omega)d\omega,$$

$$\|f\|_2 := \sqrt{\langle f, f \rangle}.$$

Theorem 2.10 (Paley-Wiener theorem). $L_2[0, \infty)$ is isomorphic to $H_2(\mathbb{C}_+)$ under the Laplace transform.

Lemma 2.11. $H_\infty(\mathbb{C}_+)$ is a Banach space under the H_∞ -norm

$$\|f\|_\infty := \sup_{s \in \mathbb{C}_+} |f(s)|.$$

Lemma 2.12. For $f \in H_\infty(\mathbb{C}_+)$,

$$\sup_{s \in \mathbb{C}_+} |f(s)| = \text{ess sup}_{\omega \in \mathbb{R}} |f(j\omega)|.$$

Definition 2.13.

$$L_\infty(j\mathbb{R}) := \{f : j\mathbb{R} \mapsto \mathbb{C} \mid \text{ess sup}_{\omega \in \mathbb{R}} |f(j\omega)| < \infty\}.$$

Theorem 2.14. *If $G \in H_\infty(\mathbb{C}_+)$ and $\hat{u} \in H_2(\mathbb{C}_+)$, then $G\hat{u} \in H_2(\mathbb{C}_+)$. Moreover, the norm of the multiplication operator $\hat{\Sigma} : \hat{u} \mapsto G\hat{u}$, defined by*

$$\|\hat{\Sigma}\| := \sup_{0 \neq \hat{u} \in H_2(\mathbb{C}_+)} \frac{\|G\hat{u}\|_2}{\|\hat{u}\|_2},$$

satisfies

$$\|\hat{\Sigma}\| = \|G\|_\infty.$$

Lemma 2.15. $H_\infty(\mathbb{C}_+)$ is a closed subspace of $L_\infty(j\mathbb{R})$.

Theorem 2.16. If $G \in L_\infty(j\mathbb{R})$, then $G \in H_\infty(\mathbb{C}_+)$ if and only if $G\hat{u} \in H_2(\mathbb{C}_+)$ for all $\hat{u} \in H_2(\mathbb{C}_+)$.

Hence, due to Theorems 2.10 and 2.16, if we only consider linear time-invariant systems whose transfer function is in $L_\infty(j\mathbb{R})$, then a linear time-invariant system is L_2 -stable if and only if its transfer function is in H_∞ . For this reason, L_2 -stability is called H_∞ -stability.

2.1.2 Stabilization

The references for this subsection are (Desoer et al., 1980; Vidyasagar et al., 1982; Vidyasagar, 1985).

We denote \mathcal{S} a commutative (integral) domain with identity and \mathcal{F} the quotient field of \mathcal{S} , i.e.

$$\mathcal{F} := \{a/b \mid a, b \in \mathcal{S}, b \neq 0\}.$$

Remark 2.17. It can be proved easily that \mathcal{F} is a field if every nonzero element of \mathcal{S} is invertible, i.e. \mathcal{S} is a commutative field, which is normally the case for real systems and which is the case here since a/b is understood in the usual way, i.e. $a/b = ab^{-1}$, in the definition of \mathcal{F} . Here is the simple proof.

If $a/b \in \mathcal{F}$ with $a \neq 0$, then $b/a \in \mathcal{F}$, and $(a/b)(b/a) = (b/a)(a/b) = (ab)/(ab) = 1$. Thus every $x \in \mathcal{F}, x \neq 0$ is a unit in \mathcal{F} . Therefore, \mathcal{F} is a field.

For the more general case where \mathcal{S} is not necessarily a field, a detailed construction of \mathcal{F} is given in Appendix A.2, (Vidyasagar, 1985) with a more general meaning of a/b . In this case, \mathcal{F} is still a field.

Remark 2.18. A set of SISO stable linear systems is a commutative (integral) domain with identity. In particular, parallel and cascade connections of stable systems are also stable.

From now on, we consider \mathcal{S} as a set of SISO stable linear systems. Then \mathcal{F} consists of stable and unstable systems.

However, the following basic results are also applicable for other purposes than stabilization as long as the set of desired systems is a commutative (integral) domain with identity.

2.1.2.1 Internal stability

We consider the closed-loop system in Figure 2.1, where G of dimension $n \times m$ is the plant and K of dimension $m \times n$ the controller.

The transfer matrix between $[u_1, u_2]^T$ and $[e_1, e_2]^T$ is $H(G, K)$, i.e.

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = H(G, K) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

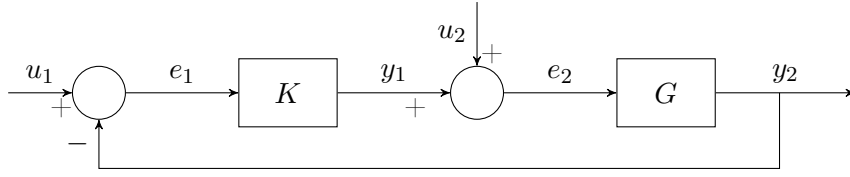


Figure 2.1 – Closed-loop system

which is given by

$$\begin{aligned} H(G, K) &= \begin{bmatrix} I_n - G(I_m + KG)^{-1}K & -G(I_m + KG)^{-1} \\ (I_m + KG)^{-1}K & (I_m + KG)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I_n + GK)^{-1} & -(I_n + GK)^{-1}G \\ K(I_n + GK)^{-1} & I_m - C(I_n + GK)^{-1}G \end{bmatrix} \end{aligned}$$

since $G(I_m + KG)^{-1} = (I_n + GK)^{-1}G$ by basic matrix manipulations.

The transfer matrix between $[u_1, u_2]^T$ and $[y_1, y_2]^T$ is $W(G, K)$, i.e.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = W(G, K) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

which is given by

$$\begin{aligned} W(G, K) &= \begin{bmatrix} 0 & I_n \\ -I_m & 0 \end{bmatrix} (H(G, K) - I_{m+n}) \\ &= \begin{bmatrix} K(I_n + GK)^{-1} & -KG(I_m + KG)^{-1} \\ GK(I_n + GK)^{-1} & G(I_m + KG)^{-1} \end{bmatrix}. \end{aligned} \quad (2.1)$$

Definition 2.19. The closed-loop system given as in Figure 2.1 is internally stable if $H(G, K) \in \mathcal{S}^{(m+n) \times (m+n)}$.

Remark 2.20. Due to (2.1), the closed-loop system is internally stable if and only if $W(G, K) \in \mathcal{S}^{(m+n) \times (m+n)}$. Hence, the closed-loop system is internally stable if and only if all the input/output maps are bounded.

Lemma 2.21 (Vidyasagar, 1985). *If $W(G, K) \in \mathcal{S}^{(m+n) \times (m+n)}$, then $G \in \mathcal{F}^{n \times m}$, $K \in \mathcal{F}^{m \times n}$.*

Proof. Let us denote

$$\begin{aligned} P &= \begin{bmatrix} K & 0 \\ 0 & G \end{bmatrix}, \\ F &= \begin{bmatrix} 0 & I_n \\ -I_m & 0 \end{bmatrix}, \end{aligned}$$

then $W(G, K)$ can be rewritten as

$$W(G, K) = P(I + FP)^{-1},$$

from which, we derive

$$\begin{aligned} P &= W(I_{m+n} - FW)^{-1} \\ &= W \frac{\text{Adj}(I_{m+n} - FW)}{\det(I_{m+n} - FW)}, \end{aligned}$$

where $\text{Adj}(I_{m+n} - FW)$ is the adjoint of $(I_{m+n} - FW)$. Therefore, $P \in \mathcal{F}^{(m+n) \times (m+n)}$, and thus $G \in \mathcal{F}^{n \times m}$, $K \in \mathcal{F}^{m \times n}$. \square

The previous lemma shows that only systems with transfer function with entries in \mathcal{F} can be stabilized with the feedback scheme in Figure 2.1. Hence, from now on we consider plants with transfer function of this kind.

2.1.2.2 Coprime factorizations

Definition 2.22. $N \in \mathcal{S}^{n \times m}$, $D \in \mathcal{S}^{m \times m}$ are right coprime if there exist $X \in \mathcal{S}^{m \times n}$, $Y \in \mathcal{S}^{m \times m}$ such that

$$XN + YD = I_m.$$

Definition 2.23. (N, D) with $N \in \mathcal{S}^{n \times m}$, $D \in \mathcal{S}^{m \times m}$ is a right factorization of $G \in \mathcal{F}^{n \times m}$ if $\det D \neq 0$ and $G = ND^{-1}$.

Remark 2.24. Since \mathcal{S} is commutative, every $G \in \mathcal{F}^{n \times m}$ admits right factorizations. The element (i, j) of G can be written as $g_{ij} = p_{ij}/q_{ij}$ where $p_{ij}, q_{ij} \in \mathcal{S}$. Let us denote $b = \prod_i \prod_j q_{ij} \neq 0$ and A the matrix whose elements are $a_{ij} = bp_{ij}/q_{ij} \in \mathcal{S}$, we have $G = A(bI_m)^{-1}$.

Definition 2.25. (N, D) with $N \in \mathcal{S}^{n \times m}$, $D \in \mathcal{S}^{m \times m}$ is a right coprime factorization (r.c.f.) of $G \in \mathcal{F}^{n \times m}$ if (N, D) is a right factorization of G and N, D are right coprime.

Definition 2.26. $\tilde{N} \in \mathcal{S}^{n \times m}$, $\tilde{D} \in \mathcal{S}^{n \times n}$ are left coprime if there exist $\tilde{X} \in \mathcal{S}^{m \times n}$, $\tilde{Y} \in \mathcal{S}^{n \times n}$ such that

$$\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_n.$$

Remark 2.27. Similarly, we can construct left factorizations of every $G \in \mathcal{F}^{n \times m}$. Indeed, $G = (bI_n)^{-1}A$.

Definition 2.28. (\tilde{N}, \tilde{D}) with $\tilde{N} \in \mathcal{S}^{n \times m}$, $\tilde{D} \in \mathcal{S}^{n \times n}$ is a left factorization (l.c.f.) of $G \in \mathcal{F}^{n \times m}$ if $\det \tilde{D} \neq 0$ and $G = \tilde{D}^{-1}\tilde{N}$.

Definition 2.29. (\tilde{N}, \tilde{D}) with $\tilde{N} \in \mathcal{S}^{n \times m}$, $\tilde{D} \in \mathcal{S}^{n \times n}$ is a left coprime factorization (l.c.f.) of $G \in \mathcal{F}^{n \times m}$ if (\tilde{N}, \tilde{D}) is a left factorization of G and \tilde{N}, \tilde{D} are left coprime.

2.1.2.3 Properties of coprime factorizations

The following properties give an idea of the common features of the definition of coprimeness with the usual definition of coprimeness, i.e. with common factors.

Lemma 2.30. *If $N \in \mathcal{S}^{n \times m}$ and $D \in \mathcal{S}^{m \times m}$ are right coprime, then every common right divisor of N and D , i.e. $U \in \mathcal{S}^{m \times m}$ satisfying $N = N'U$ and $D = D'U$ with $N' \in \mathcal{S}^{n \times m}, D' \in \mathcal{S}^{m \times m}$, is a unit in $\mathcal{S}^{m \times m}$.*

The proof of the lemma makes use of the following result.

Lemma 2.31 (Vidyasagar, 1985, Fact B.1.7). *U is a unit in $\mathcal{S}^{m \times m}$ if and only if $\det U$ is a unit in \mathcal{S} .*

Proof of Lemma 2.30. Since N and D are right coprime, then there exist $X \in \mathcal{S}^{m \times n}, Y \in \mathcal{S}^{m \times m}$ such that $XN + YD = I_m$.

Since $N = N'U$ and $D = D'U$, then $XN'U + YD'U = I_m$ or $(XN' + YD')U = I_m$, thus U has a left-inverse in $\mathcal{S}^{m \times m}$. By taking the determinant of the both sides of the equation, we have $\det(XN' + YD') \det U = 1$. Since \mathcal{S} is a commutative ring, then $\det U$ is a unit in \mathcal{S} , hence U is a unit in $\mathcal{S}^{m \times m}$. \square

Lemma 2.32 (Desoer et al., 1980, Property 2). *If (N, D) is an r.c.f. of $G \in \mathcal{F}^{n \times m}$ and (N_1, D_1) is a right (not necessarily coprime) factorization of G , then there exists $R \in \mathcal{S}^{m \times m}$ such that $N_1 = NR$ and $D_1 = DR$.*

Lemma 2.33. *If (N, D) and (N_1, D_1) are r.c.f.'s of $G \in \mathcal{F}^{n \times m}$, then there exists a unit $U \in \mathcal{S}^{m \times m}$ such that $N_1 = NU$ and $D_1 = DU$.*

Proof. Due to Lemma 2.32, there exists $U \in \mathcal{S}^{m \times m}$ such that $N_1 = NU, D_1 = DU$. Since (N_1, D_1) is an r.c.f., then there exist $X_1 \in \mathcal{S}^{m \times n}, Y_1 \in \mathcal{S}^{m \times m}$ such that $X_1N_1 + Y_1D_1 = I_m$. Then $(X_1N + Y_1D)U = I_m$, and thus $\det U$ is a unit in \mathcal{S} . Therefore, U is a unit in $\mathcal{S}^{m \times m}$. \square

Lemma 2.34. *If (N, D) is an r.c.f. of $G \in \mathcal{F}^{n \times m}$, then (NU, DU) is an r.c.f. of G for all unit $U \in \mathcal{S}^{m \times m}$.*

Proof. There exist $X \in \mathcal{S}^{m \times n}, Y \in \mathcal{S}^{m \times m}$ such that $XN + YD = I_m$, and $U^{-1}XNU + U^{-1}YDU = I_m$. Therefore, (NU, DU) is an r.c.f. of G . \square

From Lemmas 2.33 and 2.34, the following statement is immediate.

Corollary 2.35 ((Vidyasagar et al., 1982, Section II) or (Vidyasagar, 1985, Lemma 8.1.1)). *If (N, D) is an r.c.f. of $G \in \mathcal{F}^{n \times m}$, then all the r.c.f.'s of G are given by (NU, DU) with U is a unit in $\mathcal{S}^{m \times m}$.*

"Thus, an r.c.f. of an element in $\mathcal{F}^{n \times m}$ is unique to within a right associate if an r.c.f. exists. " (Vidyasagar et al., 1982, Section II)

The properties for l.c.f.'s are similar and omitted here.

Lemma 2.36 (Corona theorem, (Vidyasagar, 1985, Lemma 8.1.12)). *Suppose \mathcal{S} is a Banach algebra over \mathbb{C} , with maximal ideal space Ω . Suppose Γ is a dense subset of Ω ,*

and suppose $a_1, \dots, a_n \in \mathcal{S}$. Then there exist $x_1, \dots, x_n \in \mathcal{S}$ such that

$$\sum_{i=1}^n x_i a_i = 1$$

if and only if

$$\inf_{\omega \in \Gamma} \sum_{i=1}^n |\hat{a}_i(\omega)| > 0$$

where \hat{a}_i is the Gelfand transform of a_i .

2.1.2.4 Existence of coprime factorizations

The following lemmas are the conditions for the existence of l.c.f.'s and/or r.c.f.'s of $G \in \mathcal{F}^{n \times m}$.

Lemma 2.37 ((Vidyasagar et al., 1982, Lemma 2.1) or (Vidyasagar, 1985, Lemma 8.1.3)). $G \in \mathcal{F}^{n \times m}$ and (N, D) is a right factorization of G . Then G has an r.c.f. if and only if the left ideal in $\mathcal{S}^{m \times m}$ generated by N and D is a left principal ideal.

Definition 2.38. \mathcal{S} is a Bézout domain if every finitely generated ideal in \mathcal{S} is principal.

Lemma 2.39 (Vidyasagar et al., 1982, Corollary 2.3). If \mathcal{S} is a Bézout domain, then all finitely generated left ideals and right ideals in $\mathcal{S}^{m \times m}$ are principal.

From Lemmas 2.37 and 2.39, the following result is immediate.

Corollary 2.40. If \mathcal{S} is a Bézout domain, then every $G \in \mathcal{F}^{n \times m}$ has right and left coprime factorizations.

Lemma 2.41. H_∞ is not a Bézout domain.

Remark 2.42. The above lemma shows that there exists $G \in H_\infty^{n \times m}$ that does not have left or right or both coprime factorizations.

Definition 2.43 (Complemented matrix). Suppose $A \in \mathcal{S}^{m \times n}$ with $m < n$; then we say that A can be complemented if there exists a unit $U \in \mathcal{S}^{n \times n}$ containing A as a submatrix.

Definition 2.44 (Unimodular row). A row $[a_1, \dots, a_n] \in \mathcal{S}^{1 \times n}$ is a unimodular row if a_1, \dots, a_n together generate \mathcal{S} .

Definition 2.45 (Hermite ring). A ring \mathcal{S} is Hermite if every unimodular row can be complemented.

Lemma 2.46 (Vidyasagar, 1985, Theorem 8.1.23). The following three statements are equivalent:

1. \mathcal{S} is a Hermite ring.
2. If $G \in \mathcal{F}^{n \times m}$ has an r.c.f., then it has an l.c.f.

3. If $G \in \mathcal{F}^{n \times m}$ has an l.c.f., then it has an r.c.f.

Lemma 2.47. *Every Bézout domain is Hermite.*

Lemma 2.48. H_∞ is Hermite.

Proof. The result is immediate from (Vidyasagar, 1985, Lemma B.2.1). \square

Theorem 2.49 (Vidyasagar et al., 1982, Theorem 2.1). *Given \mathcal{S} , suppose that there exists a subring \mathcal{S}_0 of \mathcal{S} and a subset \mathcal{I} of $\mathcal{S}_0 \setminus \{0\}$ such that*

1. \mathcal{I} is closed under multiplication (i.e., $x \in \mathcal{I}, y \in \mathcal{I}$ implies that $xy \in \mathcal{I}$).
2. Every factor in \mathcal{S}_0 of an element of \mathcal{I} belongs to \mathcal{I} (i.e., $x \in \mathcal{S}_0, y \in \mathcal{S}_0, xy \in \mathcal{I}$ implies that $x \in \mathcal{I}, y \in \mathcal{I}$).
3. Whenever $x \in \mathcal{S}_0$ and $y \in \mathcal{I}$, the ideal in \mathcal{S}_0 generated by x and y is principal.

Under these conditions the set of fractions $\mathcal{G} := \{n/d, n \in \mathcal{S}_0, d \in \mathcal{I}\}$ is a subring of \mathcal{F} . Moreover, for every n, m , every element of $\mathcal{G}^{n \times m}$ has both an r.c.f. and an l.c.f.

Theorem 2.50 (Smith, 1989). *If $G \in \mathcal{F}^{n \times m}$ with $\mathcal{S} = H_\infty$, then if G is stabilizable, then G has both l.c.f.'s and r.c.f.'s.*

2.1.2.5 Parametrization of stabilizing controllers

Lemma 2.51 (Vidyasagar et al., 1982, Lemma 3.1). *Suppose $G \in \mathcal{C}_r^{n \times m}, K \in \mathcal{C}_l^{m \times n}$, where $\mathcal{C}_r^{n \times m}$ and $\mathcal{C}_l^{n \times m}$ denote the sets of all $G \in \mathcal{F}^{n \times m}$ that have an r.c.f. and an l.c.f. respectively. Let (N_p, D_p) be any r.c.f. of G , $(\tilde{N}_k, \tilde{D}_k)$ any l.c.f. of K . Under these conditions the pair (G, K) is stable if and only if*

$$\Delta := \tilde{D}_k D_p + \tilde{N}_k N_p$$

is a unit in $\mathcal{S}^{m \times m}$.

Theorem 2.52 (Vidyasagar, 1985, Theorem 8.3.5). *Suppose $G \in \mathcal{F}^{n \times m}$ has an r.c.f. (N, D) and an l.c.f. (\tilde{N}, \tilde{D}) . Select $X \in \mathcal{S}^{m \times n}, Y \in \mathcal{S}^{m \times m}, \tilde{X} \in \mathcal{S}^{m \times n}, \tilde{Y} \in \mathcal{S}^{n \times n}$ such that $XN + YD = I_m, \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_n$. Then*

$$\begin{aligned} S(G) &= \{(Y - R\tilde{N})^{-1}(X + R\tilde{D}) : R \in \mathcal{S}^{m \times n} \text{ and } \det(Y - R\tilde{N}) \neq 0\} \\ &= \{(\tilde{X} + DR)(\tilde{Y} - NR)^{-1} : R \in \mathcal{S}^{m \times n} \text{ and } \det(\tilde{Y} - NR) \neq 0\}. \end{aligned}$$

Remark 2.53. • $(Y - R\tilde{N})$ and $(X + R\tilde{D})$ are left coprime. Indeed, $(Y - R\tilde{N})D + (X + R\tilde{D})N = I_m$ since $XN + YD = I_m$ and $\tilde{D}N = \tilde{N}D$.

- $(\tilde{X} + DR)$ and $(\tilde{Y} - NR)$ are right coprime. Indeed, $\tilde{N}(\tilde{X} + DR) + \tilde{D}(\tilde{Y} - NR) = I_n$ since $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_n$ and $\tilde{D}N = \tilde{N}D$.
- If $\det Y \neq 0$, then a stabilizing controller is given by $K = Y^{-1}X$, which corresponds to $R = 0$.
- If $\det \tilde{Y} \neq 0$, then a stabilizing controller is given by $K = \tilde{X}\tilde{Y}^{-1}$, which corresponds to $\tilde{R} = 0$.

2.1.3 Fractional calculus

Fractional derivatives and integrals are generalizations of classical derivatives and integrals (of integer order) to arbitrary real order. Hence, more precisely, they should be called derivatives and integrals to arbitrary real order.

There exist numerous definitions of fractional differintegrals, which is the short name for fractional derivatives and integrals. We introduce here three definitions which are widely used.

Definition 2.54. The Grünwald-Letnikov differintegral is given by

$${}^{GL}D_t^\mu f(t) := \lim_{h \rightarrow 0} h^{-\mu} \sum_{j=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^j \binom{\mu}{j} f(t - jh).$$

The Riemann-Liouville differintegral is given by

$${}^{RL}D_t^\mu f(t) := \frac{1}{\Gamma(n - \mu)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t - \tau)^{\mu - n + 1}} d\tau$$

where $n - 1 < \mu \leq n$ and $\Gamma(\cdot)$ is the Gamma function.

The Caputo differintegral is given by

$${}^C D_t^\mu f(t) := \frac{1}{\Gamma(n - \mu)} \int_a^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\mu - n + 1}} d\tau$$

where $n - 1 < \mu \leq n$.

While the first two definitions are equivalent (Oldham and Spanier, 1974), the Caputo definition exhibits some differences (Podlubny, 1998). One of the differences is the Laplace transform of the fractional differintegrals.

Theorem 2.55. *The Laplace transform of the Riemann-Liouville and the Caputo differintegrals are respectively given by*

$$\begin{aligned} \mathcal{L}({}^{RL}D_t^\mu f(t)) &= s^\mu \mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^k {}^{RL}D_t^{\mu-k-1} f(t) \Big|_{t=0} \\ \mathcal{L}({}^C D_t^\mu f(t)) &= s^\mu \mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^{\mu-k-1} {}^C D_t^k f(t) \Big|_{t=0} \end{aligned}$$

where $n - 1 < \mu \leq n$.

Before introducing some useful Laplace transform pairs, let us define the Mittag-Leffler function in two parameters (also called the generalized Mittag-Leffler function).

Definition 2.56. The Mittag-Leffler function in two parameters is defined as

$$E_{\mu, \nu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)}, \quad \nu > 0, \nu > 0.$$

Now, for $k \in \mathbb{Z}_+$, we have the following Laplace transforms

$$\mathcal{L}(t^{\mu k + \nu - 1} E_{\mu, \nu}^{(k)}(at^\mu)) = \frac{k! s^{\mu - \nu}}{(s^\mu - a)^{k+1}}.$$

for $\Re(s) > |a|^{1/\mu}$. Then we obtain the following inverse Laplace transforms

$$\mathcal{L}^{-1}\left(\frac{1}{(s^\mu - a)^{k+1}}\right) = \frac{1}{k!} t^{\mu(k+1)-1} E_{\mu, \mu}^{(k)}(at^\mu).$$

2.2 Fractional systems

For the rest of the thesis, we consider systems which are linear and time-invariant.

2.2.1 System descriptions

In the time domain, a fractional linear time-invariant system can be described by a fractional differential equation as follows (Podlubny, 1998)

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \dots + a_0 D^{\alpha_0} y(t) = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \dots + b_0 D^{\beta_0} u(t)$$

where D^μ is the Riemann-Liouville or Caputo derivative with the lower limit $a = 0$; $a_k \in \mathbb{R}$, $\alpha_k > 0$ for $k = 0, \dots, n$; $b_l \in \mathbb{R}$, $\beta_l > 0$ for $l = 0, \dots, m$; $y(t)$ and $u(t)$ are the output and input signals respectively.

In the frequency domain, the above system is described by the following transfer function

$$G(s) := \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \dots + b_0}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \dots + a_0}.$$

In the particular case of commensurate orders, $G(s)$ is of the form

$$G(s) = \frac{\sum_{k=1}^M b_k (s^\mu)^k}{\sum_{k=1}^N a_k (s^\mu)^k}.$$

2.2.2 Examples

Linear fractional models have been used in a lot of domains and many of them were obtained through identification methods. They were showed to fit measured data better than their integer counterparts in requiring less parameters to identify thus reducing computation costs.

In this subsection, we mention some examples of real systems described by linear fractional models. Although the examples will be classified in different application domains, they seem to share some common characteristics which are at the origin of the emergence of fractional models, for instance viscoelasticity, diffusion, and fractal.

2.2.2.1 Circuit

In (Westerlund and Ekstam, 1994), the authors proposed a fractional model for capacitors due to the observation that real capacitors behave according to Curie's empirical law

$$i(t) = \frac{U_0}{h_1 t^\alpha}$$

where U_0 is the constant voltage applied at $t = 0$, h_1 and α are constants, and $\alpha \in (0, 1)$. The fractional capacitor model is

$$Z(s) = \frac{1}{Cs^\alpha}$$

where $Z(s)$ is the impedance of the capacitor and C is a constant which is lightly different from the usually defined capacitance. This model was showed to be more suitable than integer-order models for applications concerning broad frequency bands and high energy losses.

Recently, attempts have been made for fabricating fractional capacitors whose fractional order can be tuned (Cisse Haba et al., 2008; Elshurafa et al., 2013; Sivarama Krishna et al., 2011).

2.2.2.2 Non-laminated electromagnetic suspension systems

In most of electromagnetic suspension systems, laminated ferromagnetic materials are used to make stators and fltors in order to reduce the negative effects of eddy currents. However, non-laminated material is preferred in several applications, for example thrust magnetic bearings in rotating machinery, and thus the effects of eddy currents cannot be neglected.

In (Zhu et al., 2005) and (Zhu and Knospe, 2010), non-laminated electromagnetic suspension systems are first modeled based on physical laws and the obtained transfer functions involves complex functions (hyperbolic tangent and modified Bessel functions). In order to simplify the model for control design purpose, the magnetic reluctance of the system is approximated (Zhu et al., 2005) and has the form

$$R(s) = R^0 + c\sqrt{s},$$

where R^0 is the static reluctance, c is the eddy current coefficient of the stator and the fltor's ferromagnetic material, and s is the Laplace variable.

For systems in current-mode operation with time-varying displacement (Zhu and Knospe, 2010), the transfer function from perturbation current to fltor displacement is

$$\frac{X(s)}{I_p(s)} = \frac{K_i R^0 H(s)}{R(s) - K_x R^0 H(s)}$$

where in the case of a rigid fltor with no mechanical contact

$$H(s) = \frac{1}{m.s^2}$$

with m being the flotor mass. Then the transfer function is

$$\frac{X(p)}{I_p(s)} = \frac{\frac{K_i}{m}}{\frac{c}{R^0} s^{5/2} + s^2 - \frac{K_x}{m}},$$

which is demonstrated to have one real unstable pole (Knospe and Zhu, 2011).

2.2.2.3 Biomedicine and biology

In the field of bioimpedance, electrochemical behaviors of biological materials are modeled using measured impedances of materials over wide ranges of frequencies. A widely used empirical model is the Cole impedance model which is given by

$$Z(s) = R_\infty + \frac{R_1}{1 + s^{\alpha_1} R_1 C_1}$$

where R_∞ , R_1 , C_1 , and α_1 are all positive and $\alpha_1 \in (0, 1)$. Various applications are reported in the survey paper (Freeborn, 2013), including organ tissues, human blood, skull, teeth, fruits and vegetables.

Other modified versions of the Cole model are also used to provide better fittings with experimental data in some cases (Freeborn, 2013). Among them are some applications which potentially require control actions:

- wood tissue whose model is given by

$$Z(s) = \frac{1}{s^{\alpha_1} C_1} + \frac{R_2}{1 + s^{\alpha_2} R_2 C_2},$$

- electrode/tissue interface (in pacemakers for example)

$$Z(s) = R_\infty + \frac{1}{s^{\alpha_1} C_1} + \frac{1}{s^{\alpha_2} C_2},$$

- human respiratory system

$$Z(s) = R + s^\alpha L + \frac{1}{s^\beta C}.$$

2.2.3 Stability analysis

Theorem 2.57 (Matignon, 1998). *A commensurate order system described by a rational transfer function*

$$G(s) = \frac{Q(s^\alpha)}{P(s^\alpha)} \quad \text{for } \Re(s) > a \geq 0$$

where P and Q are two coprime polynomials, $\alpha \in \mathbb{R}_+$, $\alpha \in (0, 1)$ is BIBO-stable if and only if

$$|\arg(\sigma)| > \alpha \frac{\pi}{2},$$

for all $\sigma \in \mathbb{C}$ such that $P(\sigma) = 0$.

Theorem 2.58 (Bonnet and Partington, 2000). *Let G be a strictly proper transfer function given by*

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0} = \frac{Q(s)}{P(s)}$$

where $a_k \in \mathbb{R}$ for $k = 0, \dots, n$, $a_n \neq 0$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$, $b_l \in \mathbb{R}$ for $l = 0, \dots, m$, $0 = \beta_0 < \beta_1 < \cdots < \beta_m < \alpha_n$, P and Q have no common zeros. Then G is BIBO-stable if and only if G has no poles in the closed right half-plane.

2.3 Delay systems

2.3.1 System descriptions

A linear time-delay system can be described by a transfer function of the form

$$G(s) = \frac{t(s) + \sum_{l=1}^M t_l(s) e^{-\beta_l s}}{p(s) + \sum_{k=1}^N q_k(s) e^{-\alpha_k s}}$$

where

- t, t_l for $l = 1, \dots, M$, p and q_k for $k = 1, \dots, N$ are real polynomials;
- $\deg t, \deg t_l \leq \deg p$ in order to have a proper transfer function;
- $\alpha_k, \beta_l \geq 0$ for $k = 1, \dots, N$ and $l = 1, \dots, M$.

According to the degrees of p and q_k , $k = 1, \dots, N$, the system can be of one of three types:

- If $\deg p > \deg q_k$, $k = 1, \dots, N$, then the system is of retarded type.
- If $\deg p \geq \deg q_k$, $k = 1, \dots, N$, and $\deg p = \deg q_k$ for at least one value of k , then the system is of neutral type.
- If $\deg p < \deg q_k$ for at least one value of k , then the system is of advanced type.

2.3.2 Examples

There have been a lot of applications modeled by delay systems of retarded and neutral types. Advanced type systems are rarely used because of their stability properties that we will see later. Here we briefly present some linear neutral systems since they will be one of the objects considered in this thesis.

Linear neutral delay systems are encountered as models of open-loop systems or are obtained in closed-loop systems.

An example of an open-loop system is a lossless transmission line. This example was mentioned in many references, for example (Brayton, 1967; Hale, 1993; Kolmanovskii and Myshkis, 1992), and served as a typical example of neutral delay systems.

An example of neutral systems as models of closed-loop systems is presented in (Niculescu and Brogliato, 1999). The authors described a one-degree-of-freedom prismatic manipulator contacts a one-degree-of-freedom rigid environment. The interaction force is controlled by a PI controller with measurement delay. The closed-loop system is then described by a functional differential equation of neutral type.

2.3.3 Stability analysis

The classification of linear delay systems into retarded, neutral and advanced types is due to their distinct stability properties (Bellman and Cooke, 1963). For systems with commensurate delays, the stability is characterized as follows:

- A retarded system has at most finitely many poles in the closed right half-plane and for poles of large modulus $Re(s) \rightarrow -\infty$. Therefore, retarded systems are BIBO-stable if and only if they have no poles in the closed right half-plane. Furthermore, BIBO-stability is equivalent to H_∞ -stability.
- A neutral system has poles approaching vertical lines. If all these lines are in the open left half-plane, then the system has at most finitely many unstable poles and exhibits the same stability properties as retarded systems. Now, if there is one asymptotic line in the open right half-plane, the system has infinitely many unstable poles and thus is unstable. The last situation where the imaginary axis is one of the asymptotic lines is the most delicate and will partly be considered in this thesis.
- An advanced system has infinitely many unstable poles. In addition, for the poles of large modulus, $Re(s) \rightarrow +\infty$. The system is then unstable.

In the case of incommensurate delays, while the stability characteristics of retarded and advanced systems are the same as above, neutral systems now have poles located in vertical strips.

2.4 Fractional systems with delays

2.4.1 System descriptions

A linear fractional system with delays can be described by a transfer function of the form

$$G(s) = \frac{t(s) + \sum_{l=1}^M t_l(s)e^{-\beta_l s}}{p(s) + \sum_{k=1}^N q_k(s)e^{-\alpha_k s}}$$

where

- t, t_l for $l = 1, \dots, M$, p and q_k for $k = 1, \dots, N$ are real quasi-polynomials involving powers of s of fractional exponent;
- $\deg t, \deg t_l \leq \deg p$ in order to have a proper transfer function;
- $\alpha_k, \beta_l \geq 0$ for $k = 1, \dots, N$ and $l = 1, \dots, M$.

The classification of these systems into three categories (retarded, neutral, and advanced) is similar to that of classical delay systems.

2.4.2 Stability analysis

For linear fractional systems with commensurate delays and with commensurate fractional orders, the stability characterized in the frequency domain is similar to that of classical delay systems (Hotzel, 1998a; Bonnet and Partington, 2002). We present some basic facts here for further use in the next chapters.

A class of (fractional) neutral time-delay systems with commensurate orders and commensurate delays is described by transfer function of the form

$$G(s) = \frac{t(s)}{p(s) + \sum_{k=1}^N q_k(s)e^{-ks\tau}}, \quad (2.2)$$

where

- $\tau > 0$ is the delay,
- t , p , and q_k for all $k \in \mathbb{N}_N$ are real polynomials in s^μ , $0 < \mu \leq 1$,
- $-\pi < \arg(s) < \pi$ in the case where $0 < \mu < 1$ in order to have a single value of s^μ ,
- $\deg p \geq \deg t$, $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$ in order to deal with proper neutral systems.

Here, the degree of a (quasi-)polynomial refers to the degree in s^μ .

Since $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, then for each k we obtain

$$\frac{q_k(s)}{p(s)} = \alpha_k + \frac{\beta_k}{s^\mu} + \frac{\gamma_k}{s^{2\mu}} + \frac{\delta_k}{s^{3\mu}} + \frac{\epsilon_k}{s^{4\mu}} + \mathcal{O}(s^{-5\mu}) \text{ as } |s| \rightarrow \infty. \quad (2.3)$$

The coefficient of the highest degree term of the denominator of the transfer function (2.2) can be written as a multiple of the following polynomial in z

$$\tilde{c}_d(z) = 1 + \sum_{k=1}^N \alpha_k z^k, \quad (2.4)$$

where $z = e^{-s\tau}$. It is called formal polynomial.

Each neutral chain of poles of G is associated to each root r of (2.4) and is first approximated by

$$s_n\tau = \lambda_n + o(1), \quad (2.5)$$

where

$$\lambda_n = -\ln(r) + j2\pi n, \quad n \in \mathbb{Z}, \quad (2.6)$$

as $n \rightarrow \infty$ (Bellman and Cooke, 1963; Hotzel, 1998a; Fioravanti et al., 2010).

As a consequence, the neutral chain of poles asymptotically approaches the vertical line

$$\Re(s) = -\frac{\ln(|r|)}{\tau}. \quad (2.7)$$

If the vertical line is on the right or on the left of the imaginary axis, which happens when $|r| < 1$ or $|r| > 1$, then poles asymptotic to this vertical line are respectively on the right or on the left of the imaginary axis, and then their effects on H_∞ -stability which only depends on their location about the imaginary axis are easily concluded (Bonnet and Partington, 2007; Bonnet et al., 2011).

The next lemma presents properties of the formal polynomial when it has multiple roots.

Lemma 2.59. *Let r be a root of multiplicity $m > 1$ of $f(z) = 1 + \sum_{k=1}^N \alpha_k z^k$, where $\alpha_k \in \mathbb{C}$. Then $\sum_{k=1}^N k^l \alpha_k r^k = 0$ for $l = 1, \dots, m-1$ and $\sum_{k=1}^N k^m \alpha_k r^k \neq 0$.*

Proof. Since $z = r$ is a root of multiplicity m of $f(z) = 1 + \sum_{k=1}^N \alpha_k z^k$, then it is not difficult to see that $z = r$ is also a root of multiplicity m of $f_l(z) = z^l f(z)$ with $l = 1, \dots, m-1$.

For $l = 1$, taking the derivative of $f_1(z) = z + \sum_{k=1}^N \alpha_k z^{k+1}$, we obtain

$$f_1'(z) = 1 + \sum_{k=1}^N \alpha_k z^k + \sum_{k=1}^N k \alpha_k z^k.$$

Since $f_1'(r) = 0$ and $1 + \sum_{k=1}^N \alpha_k r^k = 0$, then $\sum_{k=1}^N k \alpha_k r^k = 0$.

Now, assume that $\sum_{k=1}^N k^l \alpha_k r^k = 0$ for $1 \leq l \leq a$ where $1 \leq a \leq m-1$.

For $l = a+1$, we have

$$f_{a+1}^{(a+1)}(z) = (a+1)! + \sum_{k=1}^N (k+1)(k+2)\dots(k+a+1)\alpha_k z^k.$$

It is not difficult to see that $f_{a+1}^{(a+1)}(r)$ after being expanded contains the term $(a+1)!(1 + \sum_{k=1}^N \alpha_k r^k)$, the terms $\sum_{k=1}^N k^l \alpha_k r^k$ for $1 \leq l \leq a$, which are zeros, and $\sum_{k=1}^N k^{a+1} \alpha_k r^k$. Since $f_{a+1}^{(a+1)}(r) = 0$, we derive $\sum_{k=1}^N k^{a+1} \alpha_k r^k = 0$.

For $l = m$, that is $a = m-1$, since $f_m^m(r) \neq 0$, then $\sum_{k=1}^N k^m \alpha_k r^k \neq 0$. \square

Chapter 3

Stabilization of MISO fractional systems with delays

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3.1 Introduction

The controller synthesis of fractional systems has been of great interest in recent years. However, only few studies have dealt with fractional systems with delays.

Most of the available results concerned SISO systems (Bonnet and Partington, 2002, 2007; Hamamci, 2007; Özbay et al., 2012). In the framework of fractional representation approach to synthesis problems (Vidyasagar, 1985), the works in (Bonnet and Partington, 2002, 2007) derive coprime factorizations of the transfer function of the system and the corresponding Bézout factors. Recently, PID controllers have been designed for a class of fractional time delay systems with only one unstable pole (Özbay et al., 2012).

For MIMO systems, (Curtain et al., 1996) derived coprime factorizations for a large class of infinite-dimensional systems. The factorizations were expressed in terms of operators used in a pseudo state-space representation of the systems.

In this work, we consider MISO fractional systems with delays in inputs or output. The MISO structure, which is a particular and simple MIMO case, might be encountered in communication networks, for example (Quet et al., 2002). With this structure, a class of classical (integer-order) systems with multiple transmission delays was studied in (Bonnet and Partington, 2004) and coprime factorizations and associated Bézout factors over H_∞ were derived. Now for MISO fractional systems with delays, we would like to find explicit expressions of these factors also regarding H_∞ which allow us an immediate application and which benefit from the fact that fractional transfer functions are often obtained by means of frequency identification, for examples (Sabatier et al., 2006; Vinagre et al., 1998). The determination of these factors is the first step for determining the set of all stabilizing controllers.

The chapter is organized as follows. In Section 3.2, the class of systems of interest is presented. The results are stated in Sections 3.3 and 3.4. We give in Section 3.3 explicit expressions of left coprime factorizations and associated Bézout factors over H_∞ of the transfer function of the systems under study. Right coprime factorizations and right Bézout factors for several classes of systems are given in Section 3.4. Some examples are provided to illustrate the results. Finally, Section 3.5 gives conclusions and perspectives.

3.2 A class of MISO fractional time-delay systems

We consider systems described by transfer matrices of the form

$$G(s) = \left[e^{-sh_1} R_1(s^\alpha), \dots, e^{-sh_n} R_n(s^\alpha) \right], \quad (3.1)$$

where

- $0 \leq h_k \in \mathbb{R}$ for $k = 1, \dots, n$ are the delays;
- $\alpha \in \mathbb{R}$, $0 < \alpha < 1$;

- $R_k(s^\alpha) = \tilde{q}_k(s^\alpha)/\tilde{p}_k(s^\alpha)$, where $\tilde{p}_k(s^\alpha)$ and $\tilde{q}_k(s^\alpha)$ are polynomials of integer degree in s^α , $\tilde{p}_k(s^\alpha)$ and $\tilde{q}_k(s^\alpha)$ have no common roots, and $\deg \tilde{p}_k(s^\alpha) \geq \deg \tilde{q}_k(s^\alpha)$ for $k = 1, \dots, n$;
- d_k is the degree in s^α of $\tilde{p}_k(s^\alpha)$;
- s is in the principle branch $\mathbb{C} \setminus \mathbb{R}_-$, that is $\arg(s) \in (-\pi, \pi)$, in order to guarantee a unique value of the transfer function involving s^α with $\alpha \in (0, 1)$.

We refer to poles (resp. roots) in the closed right half-plane $\overline{\mathbb{C}_+}$ as unstable poles (resp. roots).

The following notations will be of intense use later.

Denote

- $p(s^\alpha)$ the lowest common denominator of $R_k(s^\alpha)$ for $k = 1, \dots, n$;
- d the degree in s^α of $p(s^\alpha)$.

Then rational transfer functions $R_k(s^\alpha)$ can be rewritten as

$$R_k(s^\alpha) = \frac{q_k(s^\alpha)}{p(s^\alpha)},$$

where $q_k(s^\alpha)$ are polynomials in s^α .

We can decompose

$$p(s^\alpha) = (s^\alpha)^{m_0} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_i} \right) \left(\prod_{j=1}^{N'} (s^\alpha - c_j)^{m'_j} \right),$$

where

- $b_i \in \mathcal{D} := \{\sigma \in \mathbb{C} \setminus \{0\} \mid -\pi\alpha/2 \leq \text{Arg}(\sigma) \leq \pi\alpha/2\}$,
- $c_j \in \mathbb{C} \setminus \{\mathcal{D} \cup \{0\}\}$,
- $m_0, m_i, m'_j \in \mathbb{Z}_+$ for $i = 1, \dots, N$ and $j = 1, \dots, N'$.

Hence $s_i = b_i^{1/\alpha}$ are the non-zero unstable roots in s of $p(s^\alpha)$.

Similarly, we write

$$\tilde{p}_k(s^\alpha) = (s^\alpha)^{m_{0k}} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ik}} \right) \left(\prod_{j=1}^{N'} (s^\alpha - c_j)^{m'_{jk}} \right),$$

where $m_{0k}, m_{ik}, m'_{jk} \in \mathbb{Z}_+$ for $i = 1, \dots, N, j = 1, \dots, N'$ and $k = 1, \dots, n$. It is obvious that $m_{0k} \leq m_0, m_{ik} \leq m_i$, and $m'_{jk} \leq m'_j$.

3.3 Left coprime factorizations and Bézout factors

In this section, we present left coprime factorizations and Bézout factors for the transfer matrix (3.1).

3.3.1 Left coprime factorizations

Due to the dimension of the transfer matrix, finding a left coprime factorization is straightforward.

Proposition 3.1. *Let $G(s)$ be given by (3.1). Then*

$$\widetilde{M}(s) = \frac{p(s^\alpha)}{(s^\alpha + 1)^d} \text{ and } \widetilde{N}(s) = \frac{1}{(s^\alpha + 1)^d} \left[e^{-sh_1} q_1(s^\alpha), \dots, e^{-sh_n} q_n(s^\alpha) \right] \quad (3.2)$$

is a left coprime factorization over H_∞ of G .

Proof. It is obvious that $\widetilde{M}(s)^{-1} \widetilde{N}(s) = G(s)$.

We see that $\widetilde{M}(s) \in H_\infty$. Also, each component of $\widetilde{N}(s)$ is in H_∞ , and then $\widetilde{N}(s) \in \mathbf{M}(H_\infty)$.

For all roots σ of p , there exists at least one $1 \leq k \leq n$ such that $q_k(\sigma) \neq 0$. Thus $\inf_{s \in \mathbb{C}_+} (\sum_{k=1}^n |\widetilde{N}_k| + |\widetilde{M}|) > 0$ which ensures that $(\widetilde{M}, \widetilde{N})$ is a left coprime factorization over H_∞ of G . \square

3.3.2 Bézout factors

Our objective in this subsection is to propose left Bézout factors corresponding to the left factorization obtained above. It is interesting to note that besides being elements to construct the set of stabilizing controllers using Youla-Kučera parametrization, a pair of left Bézout factors $\widetilde{X}, \widetilde{Y}$ immediately provides us a stabilizing controller $C = \widetilde{Y} \widetilde{X}^{-1}$ (Quadrat, 2006a, Corollary 5).

For the sake of clarity, we consider several cases of systems (3.1) beginning with simple cases before tackling the general one.

3.3.2.1 Systems with one unstable pole for each element of the transfer matrix

The first proposition gives Bézout factors for systems involving at most one unstable pole for each element of the transfer matrix.

Proposition 3.2. *Let $G(s)$ be given by (3.1) with*

$$R_k(s^\alpha) = \frac{a_k}{s^\alpha - \sigma_k} \text{ with } a_k, \sigma_k \in \mathbb{R} \text{ for } k = 1, \dots, n. \quad (3.3)$$

Then Bézout factors corresponding to the left coprime factorization (3.2) are given by

$$\begin{aligned} \widetilde{X}(s) &= \frac{(s^\alpha + 1)^d - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \widetilde{Y}_k}{p(s^\alpha)}, \\ \widetilde{Y}(s) &= \left[\widetilde{Y}_1, \dots, \widetilde{Y}_n \right]^T, \end{aligned}$$

where \tilde{Y}_k for $k = 1, \dots, n$ are constant and satisfy

$$\sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \tilde{Y}_k = (s^\alpha + 1)^d \quad (3.4)$$

at $s = b^{1/\alpha}$ for all $b \in \mathcal{D} \cup \{0\}$.

In the case where all σ_k are positive ($\sigma_k \geq 0$) and distinct, the unique solution of \tilde{Y}_k is given by

$$\tilde{Y}_k = \frac{e^{\sigma_k^{1/\alpha} h_k} (\sigma_k + 1)^n}{q_k(\sigma_k)}. \quad (3.5)$$

In the case where $\sigma_1 = \dots = \sigma_n = \sigma > 0$, then \tilde{Y}_k satisfy the single equation

$$\sum_{k=1}^n e^{-\sigma^{1/\alpha} h_k} a_k \tilde{Y}_k = \sigma + 1. \quad (3.6)$$

Proof. From the left Bézout identity, we obtain

$$\tilde{X}(s) = \tilde{M}^{-1}(1 - \tilde{N}\tilde{Y}) = \frac{(s^\alpha + 1)^d - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \tilde{Y}_k(s)}{p(s^\alpha)}.$$

If we choose $\tilde{Y}(s) = [\tilde{Y}_1(s), \dots, \tilde{Y}_n(s)]^T \in \mathbf{M}(H_\infty)$ such that the numerator of $\tilde{X}(s)$ vanishes at $s = b^{1/\alpha}$ for all $b \in \mathcal{D} \cup \{0\}$, then $\tilde{X}(s)$ is analytic in \mathbb{C}_+ since $s = b^{1/\alpha}$ are also the roots of the denominator of $\tilde{X}(s)$. In this particular case, it suffices to choose constants $\tilde{Y}_1, \dots, \tilde{Y}_n$ such that

$$\sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \tilde{Y}_k = (s^\alpha + 1)^d$$

at $s = b^{1/\alpha}$ for all $b \in \mathcal{D} \cup \{0\}$ to have $\tilde{X}(s) \in H_\infty$.

For all $b \in \mathcal{D} \cup \{0\}$, we see that $q_k(b) = 0$ if $b \neq \sigma_k$. Then it remains to solve

$$\sum_{k:1 \leq k \leq n, \sigma_k = b} e^{-b^{1/\alpha} h_k} q_k(b) \tilde{Y}_k = (b + 1)^d,$$

which gives infinitely many solutions except the case where all σ_k for $k = 1, \dots, n$ are positive and distinct.

Indeed, in that case, $d = n$ and the number of equations is equal to the number of unknowns, which is n . We deduce then the unique solution (3.5).

The case where $\sigma_1 = \dots = \sigma_n = \sigma > 0$ corresponds to $d = 1$. Trivially we have $p(s^\alpha) = s^\alpha - \sigma$ and $q_k(s^\alpha) = a_k$. Thus we have to solve

$$\sum_{k=1}^n e^{-\sigma^{1/\alpha} h_k} a_k \tilde{Y}_k = \sigma + 1,$$

which gives a unique solution if $n = 1$ and infinitely many solutions if $n > 1$. \square

We illustrate the proposition by the next example.

Example 3.1.

$$G(s) = \left[\frac{e^{-s}}{\sqrt{s}}, \frac{e^{-s}}{\sqrt{s-1}} \right]$$

It is easy to see that $p(\sqrt{s}) = \sqrt{s}(\sqrt{s}-1)$, $q_1(\sqrt{s}) = \sqrt{s}-1$, and $q_2(\sqrt{s}) = \sqrt{s}$. Then we obtain

$$\widetilde{M}(s) = \frac{\sqrt{s}(\sqrt{s}-1)}{(\sqrt{s}+1)^2}, \quad \widetilde{N}(s) = \left[\frac{e^{-s}(\sqrt{s}-1)}{(\sqrt{s}+1)^2}, \frac{e^{-s}\sqrt{s}}{(\sqrt{s}+1)^2} \right].$$

This system corresponds to the case of unstable and distinct poles. So we obtain

$$\widetilde{Y}(s) = [-1, 4e]^T, \quad \widetilde{X}(s) = \frac{(\sqrt{s}+1)^2 + e^{-s}(\sqrt{s}-1) - 4e^{1-s}\sqrt{s}}{\sqrt{s}(\sqrt{s}-1)}.$$

3.3.2.2 Systems with constraints on the multiplicity of the pole at zero

The next proposition considers a class of systems which is already general except that the multiplicity of the zero pole is restricted. This allows us to obtain Bézout factors that only contain commensurate powers of s .

Proposition 3.3. *Let $G(s)$ be given by (3.1) and suppose that the multiplicity of the root at zero of $p(s^\alpha)$ verifies the condition $m_0\alpha \leq 1$. Then Bézout factors corresponding to the left coprime factorization (3.2) are given by*

$$\widetilde{X}(s) = \frac{(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s^\alpha)}{p(s^\alpha) u(s^\alpha)}, \quad (3.7)$$

$$\widetilde{Y}(s) = \left[\frac{\mu_1(s^\alpha)}{u(s^\alpha)}, \dots, \frac{\mu_n(s^\alpha)}{u(s^\alpha)} \right]^T, \quad (3.8)$$

where $u(s^\alpha)$ is a polynomial in s^α of degree greater or equal to d whose zeros are stable, and $\mu_k(s^\alpha)$ for $k = 1, \dots, n$ are polynomials in s^α of degree $m_{0k} + \sum_{i=1}^N m_{ik}$ satisfying

$$(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s^\alpha) \sim_{s \rightarrow 0} s^{m_0\alpha}, \quad (3.9)$$

and

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s^\alpha) \right]^{(l)} = 0 \quad (3.10)$$

at $s = b_i^{1/\alpha}$ for $0 \leq l \leq m_i - 1$.

Proof. It is obvious that $\widetilde{Y}(s) \in \mathbf{M}(H_\infty)$.

The condition (3.9) is satisfied if in the development of the denominator of $\widetilde{X}(s)$ around zero the powers of s whose order is smaller than $m_0\alpha$ are canceled. Then this fact gives

m_0 equations of unknown coefficients of $\mu_k(s^\alpha)$. The second condition (3.10) introduces $\sum_{i=1}^N m_i$ equations. Hence in total, there are $m_0 + \sum_{i=1}^N m_i$ equations of $\sum_{k=1}^n m_{0k} + \sum_{k=1}^n \sum_{i=1}^N m_{ik} + n$ unknowns. As $m_0 + \sum_{i=1}^N m_i \leq \sum_{k=1}^n m_{0k} + \sum_{k=1}^n \sum_{i=1}^N m_{ik} + n$, the system of equations admits solutions.

The boundedness of $\tilde{X}(s)$ at zero is assured by (3.9). Then following the same arguments as in the proof of the previous proposition, we can conclude that $X(s) \in H_\infty$.

The left Bézout identity is satisfied. \square

Remark 3.4. When $m_0\alpha > 1$, for the condition (3.9), the development of e^{-sh_k} around zero will contain powers of s , which might not be multiples of s^α . This imposes that μ_k might no longer be polynomials in s^α .

The following example illustrates how to apply the proposition.

Example 3.2.

$$G(s) = \left[\frac{e^{-s}}{s^{1/3} - 1}, \frac{e^{-s}}{s} \right]$$

It is obvious that $p(s^{1/3}) = s(s^{1/3} - 1)$ with degree $d = 4$ in $s^{1/3}$. It has two unstable roots which are $b_0 = 0$ of multiplicity $m_0 = 3$ and $b_1 = 1$ of multiplicity $m_1 = 1$. We have then $q_1(s^{1/3}) = s$, and $q_2(s^{1/3}) = s^{1/3} - 1$.

From Proposition 3.1, we obtain a left coprime factorization as follows

$$\tilde{M}(s) = \frac{s(s^{1/3} - 1)}{(s^{1/3} + 1)^4}, \quad \tilde{N}(s) = \frac{1}{(s^{1/3} + 1)^4} \left[se^{-s}, (s^{1/3} - 1)e^{-s} \right].$$

From Proposition 3.3, we can choose $u(s^{1/3}) = (s^{1/3} + 1)^4$, which has no unstable roots. Also, $\mu_1(s^{1/3})$ and $\mu_2(s^{1/3})$ have the form

$$\begin{aligned} \mu_1(s^{1/3}) &= \beta_{01} + \beta_{11}s^{1/3}, \\ \mu_2(s^{1/3}) &= \beta_{02} + \beta_{12}s^{1/3} + \beta_{22}s^{2/3} + \beta_{32}s. \end{aligned}$$

As $s \rightarrow 0$, the numerator of $\tilde{X}(s)$ is developed as

$$\begin{aligned} &(s^{1/3} + 1)^4 u(s^{1/3}) - e^{-s} q_1(s^{1/3}) \mu_1(s^{1/3}) - e^{-s} q_2(s^{1/3}) \mu_2(s^{1/3}) \\ &= 1 + \beta_{02} + (8 - \beta_{02} + \beta_{12})s^{1/3} + (28 - \beta_{12} + \beta_{22})s^{2/3} + O(s). \end{aligned}$$

The first condition of Proposition 3.3 is satisfied if and only the powers of s whose exponent is smaller than $m_0\alpha$ vanish, thus giving $\beta_{02} = -1$, $\beta_{12} = -9$, and $\beta_{22} = -37$.

From the second condition, at the non-zero unstable pole $s = 1$ of $\tilde{X}(s)$, we must have

$$(s^{1/3} + 1)^4 u(s^{1/3}) - e^{-s} q_1(s^{1/3}) \mu_1(s^{1/3}) - e^{-s} q_2(s^{1/3}) \mu_2(s^{1/3}) = 0.$$

This gives $\beta_{01} + \beta_{11} = 256e$. One possible solution is $\beta_{01} = 256e$, and $\beta_{11} = 0$.

There is no constraint on β_{32} , then we can choose $\beta_{32} = 0$.

In conclusion, we obtain the Bézout factors as follows

$$\begin{aligned}\tilde{Y}(s) &= \left[\frac{256e}{(s^{1/3} + 1)^4}, \frac{-1 - 9s^{1/3} - 37s^{2/3}}{(s^{1/3} + 1)^4} \right], \\ \tilde{X}(s) &= \frac{(s^{1/3} + 1)^8 - e^{1-s}256s - e^{-s}(s^{1/3} - 1)(-1 - 9s^{1/3} - 37s^{2/3})}{s(s^{1/3} - 1)(s^{1/3} + 1)^4}.\end{aligned}$$

3.3.2.3 General case

Now we will tackle the problem of finding Bézout factors of the system (3.1) in its most general configuration.

Proposition 3.5. *Let $G(s)$ be given by (3.1). Then Bézout factors corresponding to the left coprime factorization (3.2) are given by*

$$\tilde{X}(s) = \frac{(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s)}{p(s^\alpha) u(s^\alpha)}, \quad (3.11)$$

$$\tilde{Y}(s) = \left[\frac{\mu_1(s)}{u(s^\alpha)}, \dots, \frac{\mu_n(s)}{u(s^\alpha)} \right]^T, \quad (3.12)$$

where $u(s^\alpha)$ is a polynomial in s^α of degree greater or equal to d whose zeros are stable, and $\mu_k(s)$ for $k = 1, \dots, n$ have the following form

$$\mu_k(s) = \sum_{\substack{\lambda = a + b\alpha < m_0\alpha \\ a, b \in \mathbb{Z}_+}} \beta_{\lambda k} s^\lambda + \sum_{j=m_0}^{m_0 + \sum_{i=1}^N m_{ik}} \beta_{(j\alpha)k} (s^\alpha)^j \quad (3.13)$$

and verify two conditions

(i) as $s \rightarrow 0$

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \right] \sim s^{m_0\alpha},$$

(ii) for each non-zero unstable root $s = b_i^{1/\alpha}$, $i = 1, \dots, N$, of $p(s^\alpha)$

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \right]^{(l)} = 0,$$

where $0 \leq l \leq m_i - 1$.

Proof. It is easy to verify that $\tilde{X}(s)$ and $\tilde{Y}(s)$ satisfy the left Bézout identity.

The degree in s of $\mu_k(s)$ is $(m_0 + \sum_{i=1}^N m_{ik})\alpha \leq (m_0 + \sum_{i=1}^N m_i)\alpha \leq d\alpha$. With the choice of $u(s^\alpha)$, we see that $\tilde{Y}(s) \in \mathbf{M}(H_\infty)$.

Before proving $\tilde{X}(s) \in H_\infty$, we discuss the existence of $\mu_k(s)$ satisfying the two conditions. The first condition implies that the numerator of $\tilde{X}(s)$ has the same or greater order than the denominator near $s = 0$. This in turn implies that in the development of the numerator near zero, all powers of s whose exponent is smaller than $m_0\alpha$ are canceled. Due to the presence of e^{-sh_k} and $\mu_k(s)$, these powers are s^λ with $\lambda = a + b\alpha < m_0\alpha$, $a, b \in \mathbb{Z}_+$. This fact gives a number of equations equal to the number of terms s^λ . In addition, the second condition gives $\sum_{i=1}^N m_i$ equations.

On the other hand, the number of coefficients associated to terms s^λ in $\mu_k(s)$ for $k = 1, \dots, n$ is greater than the number of terms s^λ . Also, the number of coefficients associated to terms $(s^\alpha)^j, j \geq m_0$ is $\sum_{k=1}^n \sum_{i=1}^N m_{ik} + n > \sum_{i=1}^N m_i$. Therefore the number of unknowns is greater than the number of equations. The system of equations thus admits a solution.

We see that $\tilde{X}(s)$ is bounded at ∞ in \mathbb{C}_+ . Moreover, by interpolating the non-zero unstable poles of $\tilde{X}(s)$, the second condition assures that $\tilde{X}(s)$ is analytic in \mathbb{C}_+ . The boundedness of the function at $s = 0$ is satisfied by the first condition. Then $\tilde{X}(s) \in H_\infty$. \square

Remark 3.6. In the case where $\alpha = 1/m$ with $m \in \mathbb{Z}_+ \setminus \{0, 1\}$, we see that λ are multiples of α . Then we obtain an elegant formula of μ_k which only contains the terms in s^α .

Example 3.3.

$$G(s) = \left[\frac{e^{-s}}{(s^{1/2} - 1)^2}, \frac{e^{-2s}}{s^{3/2}} \right]$$

Obviously, $p(s^{1/2}) = s^{3/2}(s^{1/2} - 1)^2$ of degree $d = 5$ in $s^{1/2}$. Its unstable roots are $b_0 = 0$ of multiplicity $m_0 = 3$ and $b_1 = 1$ of multiplicity $m_1 = 2$. We have $q_1(s^{1/2}) = s^{3/2}$, and $q_2(s^{1/2}) = (s^{1/2} - 1)^2$.

The left coprime factorization obtained from Proposition 3.1 is

$$\tilde{M}(s) = \frac{s^{3/2}(s^{1/2} - 1)^2}{(s^{1/2} + 1)^5}, \quad \tilde{N}(s) = \frac{1}{(s^{1/2} + 1)^5} [e^{-s} s^{3/2}, e^{-2s}(s^{1/2} - 1)^2].$$

From Proposition 3.5, we can choose $u(s^{1/2}) = (s^{1/2} + 1)^5$, which has no unstable roots.

The powers of s whose exponent is a linear combination of 1 and $\alpha = 0.5$ and is smaller than $m_0\alpha$ are $1, s^{1/2}, s$. Then $\mu_1(s)$ and $\mu_2(s)$ have the form

$$\begin{aligned} \mu_1(s) &= \beta_{01} + \beta_{(1/2)1} s^{1/2} + \beta_{11} s + \beta_{(3/2)1} s^{3/2} + \beta_{21} s^2 + \beta_{(5/2)1} s^{5/2}, \\ \mu_2(s) &= \beta_{02} + \beta_{(1/2)2} s^{1/2} + \beta_{12} s + \beta_{(3/2)2} s^{3/2}. \end{aligned}$$

As $s \rightarrow 0$, the numerator of $\tilde{X}(s)$ is developed as

$$\begin{aligned} &(s^{1/2} + 1)^5 u(s^{1/2}) - e^{-s} q_1(s^{1/2}) \mu_1(s) - e^{-2s} q_2(s^{1/2}) \mu_2(s) \\ &= 1 - \beta_{02} + (10 + 2\beta_{02} - \beta_{(1/2)2}) s^{1/2} + (45 + \beta_{02} + 2\beta_{(1/2)2} - \beta_{12}) s + O(s^{3/2}). \end{aligned}$$

The first condition of Proposition 3.5 is satisfied if and only if the powers whose exponent is smaller than $m_0\alpha = 3/2$ vanish, thus giving $\beta_{02} = 1$, $\beta_{(1/2)2} = 12$, and $\beta_{12} = 70$.

From the second condition, we have two equations, that is

$$\left[(s^{1/2} + 1)^5 u(s^{1/2}) - e^{-s} q_1(s^{1/2}) \mu_1(s) - e^{-2s} q_2(s^{1/2}) \mu_2(s) \right]^{(l)} = 0$$

at $s = b_1 = 1$ for $l = 0, 1$. This gives

$$\begin{aligned} 1024 - e^{-1}(\beta_{01} + \beta_{(1/2)1} + \beta_{11} + \beta_{(3/2)1} + \beta_{21} + \beta_{(5/2)1}) &= 0, \\ 2560 - e^{-1}(0.5\beta_{01} + \beta_{(1/2)1} + 1.5\beta_{11} + 2\beta_{(3/2)1} + 2.5\beta_{21} + 3\beta_{(5/2)1}) &= 0. \end{aligned}$$

One possible solution is $\beta_{01} = -3072e$, $\beta_{(1/2)1} = 4096e$, and $\beta_{11} = \beta_{(3/2)1} = \beta_{21} = \beta_{(5/2)1} = 0$.

Therefore, we have

$$\begin{aligned} \tilde{X}(s) &= \frac{(s^{1/2} + 1)^{10} - e^{-s} s^{3/2} \mu_1(s) - e^{-2s} (s^{1/2} - 1)^2 \mu_2(s)}{s^{3/2} (s^{1/2} - 1)^2 (s^{1/2} + 1)^5}, \\ \tilde{Y}(s) &= \left[\frac{\mu_1(s)}{(s^{1/2} + 1)^5}, \frac{\mu_2(s)}{(s^{1/2} + 1)^5} \right], \end{aligned}$$

where $\mu_1(s) = -3072e + 4096es^{1/2}$, and $\mu_2(s) = 1 + 12s^{1/2} + 70s$.

3.3.2.4 Minimal form of Bézout factors in the general case

We have seen in Examples 3.2 and 3.3 that we have infinite choices for some coefficients of μ_k for $k = 1, \dots, n$ and so far we have chosen the values for these coefficients such that the orders of μ_k are smallest.

In the next proposition, we present Bézout factors of $G(s)$ in the general case with μ_k for $k = 1, \dots, n$ such that the number of coefficients and the order of μ_k are minimal. The proof justifies in details that a unique solution exists for that form of μ_k and of course solutions exist for the non-minimal forms of μ_k presented in Propositions 3.3 and 3.5.

Before stating the proposition, let us denote

$$k_i := \min\{k \mid k \in \{1, \dots, n\}, m_{ik} = m_i\} \quad \text{for } i = 0, \dots, N, \quad (3.14)$$

$$f_k := \sum_{i \in \{1, \dots, N\}, k_i = k} m_i \quad \text{for } k = 1, \dots, n,$$

$$\mathcal{L}(m_0\alpha) := \{x \in \mathbb{R} \mid x = a + b\alpha < m_0\alpha, a, b \in \mathbb{Z}_+\}. \quad (3.15)$$

Proposition 3.7. *Let $G(s)$ be given by (3.1). Then Bézout factors corresponding to the left coprime factorization (3.2) are given by*

$$\begin{aligned} \tilde{X}(s) &= \frac{(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s)}{p(s^\alpha) u(s^\alpha)}, \\ \tilde{Y}(s) &= \left[\frac{\mu_1(s)}{u(s^\alpha)}, \dots, \frac{\mu_n(s)}{u(s^\alpha)} \right]^T, \end{aligned}$$

where $u(s^\alpha)$ is a polynomial in s^α of degree greater or equal to d whose zeros are stable, and $\mu_k(s)$ for $k = 1, \dots, n$ have the following form

$$\mu_k(s) = \begin{cases} \sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k} s^\lambda + \sum_{j=m_0}^{m_0+f_k-1} \beta_{(j\alpha)k} (s^\alpha)^j & \text{if } k = k_0, \\ \sum_{j=0}^{f_k-1} \beta_{(j\alpha)k} (s^\alpha)^j & \text{if } k \neq k_0, \end{cases}$$

and satisfy

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \right] = O(s^{m_0\alpha}) \quad (3.16)$$

as $s \rightarrow 0$ and

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \right]^{(l)} = 0, \quad (3.17)$$

for each non-zero unstable root $s = b_i^{1/\alpha}$, $i = 1, \dots, N$, of $p(s^\alpha)$ and for $0 \leq l \leq m_i - 1$.

Remark 3.8. If $f_k = 0$, then

$$\mu_k(s) = \begin{cases} \sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k} s^\lambda & \text{if } k = k_0, \\ 0 & \text{if } k \neq k_0. \end{cases}$$

Proof. It is easy to verify that $\tilde{X}(s)$ and $\tilde{Y}(s)$ satisfy the left Bézout identity.

The degree of $\mu_k(s)$ is smaller than or equal to the degree of $u(s^\alpha)$, and so $\tilde{Y}(s) \in \mathbf{M}(H_\infty)$.

We see that $\tilde{X}(s)$ is bounded at ∞ in \mathbb{C}_+ . Moreover, due to (3.17), the numerator of $\tilde{X}(s)$ has the same non-zero unstable roots as the denominator, which assures that $\tilde{X}(s)$ is analytic in \mathbb{C}_+ . The boundedness of the function at $s = 0$ is satisfied by (3.16). Then $\tilde{X}(s) \in H_\infty$.

Now it remains to prove the existence of $\mu_k(s)$ satisfying the two conditions (3.16) and (3.17).

First, we consider the condition (3.16) on the poles at zero. If the system has no zero pole, then the condition is satisfied. Otherwise, the numerator of $\tilde{X}(s)$ can be developed

around zero as follows

$$\begin{aligned}
& (s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \\
&= a_0 + a_1 s^\alpha + \dots + a_{m_0-1} s^{(m_0-1)\alpha} + O(s^{m_0\alpha}) - \sum_{k=1}^n (1 - sh_k + \dots + O(s^{m_0\alpha})) \\
&\quad \times (b_{0k} + b_{1k} s^\alpha + \dots + b_{(m_0-1)k} s^{(m_0-1)\alpha} + O(s^{m_0\alpha})) \left(\sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k} s^\lambda + O(s^{m_0\alpha}) \right) \\
&= a_0 + a_1 s^\alpha + \dots + a_{m_0-1} s^{(m_0-1)\alpha} - \sum_{k=1}^n \left(\sum_{\lambda \in \mathcal{L}(m_0\alpha)} \gamma_{\lambda k} s^\lambda \right) \left(\sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k} s^\lambda \right) \\
&\quad + O(s^{m_0\alpha}).
\end{aligned}$$

The condition imposes that all powers of s whose order is smaller than $m_0\alpha$ are eliminated. Let us denote the elements of $\mathcal{L}(m_0\alpha)$ by λ_j , $j = 0, \dots, N'' - 1$ with $N'' = \text{card}(\mathcal{L}(m_0\alpha))$ and assume that $0 = \lambda_0 < \dots < \lambda_{N''-1}$, then the condition is equivalent to the following matrix equation

$$\sum_{k=1}^n \Gamma_k B_k = \begin{bmatrix} a'_{N''-1} \\ a'_{N''-2} \\ \vdots \\ a'_0 \end{bmatrix}, \quad (3.18)$$

where $\Gamma_k \in \mathbb{R}^{N'' \times N''}$ are upper triangular matrices which contain $\gamma_{\lambda k}$ and whose entries on the main diagonal are all $\gamma_{\lambda_0 k}$; the column vectors B_k contain $\beta_{\lambda k}$; for $j = 0, \dots, N'' - 1$, $a'_j = a_x$ if $\lambda_j = x\alpha$, $x \in \mathbb{Z}_+$ and $a'_j = 0$ otherwise. From the precedent development of the numerator of $\tilde{X}(s)$, note that the coefficients $\gamma_{\lambda k}$ are obtained from the product of $(1 - sh_k + \dots + O(s^{m_0\alpha}))$ and $(b_{0k} + b_{1k} s^\alpha + \dots + b_{(m_0-1)k} s^{(m_0-1)\alpha} + O(s^{m_0\alpha}))$ and in particular $\gamma_{\lambda_0 k} = b_{0k}$. For $k = k_0$ with k_0 defined by (3.14), $b_{0k_0} \neq 0$ since $q_{k_0}(s^\alpha)$ does not have roots at zero. Then $\det \Gamma_{k_0} \neq 0$. And so B_{k_0} admits a unique solution for any values of $\beta_{\lambda k}$ with $\lambda < m_0\alpha$ and $k \in \{1, \dots, n\}$, $k \neq k_0$.

Next, we analyze the second condition (3.17) on non-zero poles.

We first examine the system of equations obtained by replacing s by a non-zero unstable pole $b_i^{1/\alpha}$ and study the existence of $\mu_{k_i}^{(l_2)}(b_i^{1/\alpha})$ for $l_2 = 0, \dots, m_i - 1$ satisfying the equations. The first equation which corresponds to $l = 0$ contains $q_{k_i}(b_i) \mu_{k_i}(b_i^{1/\alpha})$ with $q_{k_i}(b_i) \neq 0$. The second equation, i.e. $l = 1$, contains a linear sum of $q_{k_i}(b_i) \mu_{k_i}(b_i^{1/\alpha})$ and $q_{k_i}(b_i) \mu'_{k_i}(b_i^{1/\alpha})$. Generally, the equation corresponding to the l -th derivative contains a linear sum of $q_{k_i}(b_i) \mu_{k_i}^{(l_2)}(b_i^{1/\alpha})$ with $l_2 = 0, \dots, l - 1$. Therefore, for arbitrary values of $\mu_k^{(l_2)}(b_i^{1/\alpha})$ for $k = 1, \dots, n$, $k \neq k_i$ and $l_2 = 0, \dots, m_i - 1$, the system of m_i equations can be recursively solved for m_i unknowns $\mu_{k_i}^{(l_2)}(b_i^{1/\alpha})$ with $l_2 = 0, \dots, m_i - 1$ and admits a unique solution.

Hence, the second condition introduces in total $\sum_{i=1}^N m_i$ equations. This system of equations has a unique solution for $\mu_{k_i}^{(l_2)}(b_i^{1/\alpha})$ with $i = 1, \dots, N$ and $l_2 = 0, \dots, m_i - 1$ if we choose any values of $\mu_k^{(l_2)}(b_i^{1/\alpha})$ for $i = 1, \dots, N$, $k = 1, \dots, n$, $k \neq k_i$ and $l_2 = 0, \dots, m_i - 1$.

Hence, for each $k \in \{1, \dots, n\}$ and $k \neq k_0$, the coefficients $\beta_{(j\alpha)k}$, $j = 0, \dots, f_k - 1$ of $\mu_k(s)$ satisfy the equations

$$\mu_k^{(l_2)}(b_i^{1/\alpha}) = a_{k,i,l_2}$$

for $i = 1, \dots, N$ such that $k_i = k$ and $l_2 = 0, \dots, m_i - 1$. This is the problem of Hermite interpolation and there exists a unique solution.

For $k = k_0$, the coefficients of $\mu_{k_0}(s)$, i.e. $\beta_{\lambda k_0}$ with $\lambda \in \mathcal{L}(m_0\alpha)$ and $\beta_{(j\alpha)k}$ with $j = m_0, \dots, m_0 + f_k - 1$, satisfy the equations

$$\mu_{k_0}^{(l_2)}(b_i^{1/\alpha}) = a_{k_0,i,l_2}$$

for $i = 1, \dots, N$ such that $k_i = k_0$ and $l_2 = 0, \dots, m_i - 1$. We can write $\mu_{k_0}(s)$ as follows

$$\mu_{k_0}(s) = \nu_{k_0}(s) + s^{m_0\alpha} \eta_{k_0}(s^\alpha)$$

where

$$\begin{aligned} \nu_{k_0}(s) &= \sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k_0} s^\lambda, \\ \eta_{k_0}(s^\alpha) &= \sum_{j=m_0}^{m_0+f_{k_0}-1} \beta_{(j\alpha)k_0} s^{(j-m_0)\alpha}. \end{aligned}$$

For arbitrary values of the coefficients $\beta_{\lambda k_0}$ with $\lambda \in \mathcal{L}(m_0\alpha)$, we can derive the values of $\eta_{k_0}^{(l_2)}(b_i)$ for $i = 1, \dots, N$ such that $k_i = k_0$ and $l_2 = 0, \dots, m_i - 1$. Note that the numbers of unknowns and of equations are the same and are equal to f_{k_0} . This returns to the problem of Hermite interpolation and there exists a unique solution. \square

Remark 3.9. If $m_0\alpha \leq 1$ or $\alpha = 1/m$ with $m \in \mathbb{Z}_+ \setminus \{0, 1\}$, then λ are multiples of α and we obtain an elegant formula of μ_{k_0} which only contains the terms in s^α . More generally, if α is rational, then μ_{k_0} contains powers of s of commensurate exponents.

This can also be achieved if we introduce more coefficients in $\mu_k(s)$, $k = 1, \dots, n$, $k \neq k_0$ than in the forms given in the proposition. More precisely, if we denote x the number of values of $\lambda \in \mathcal{L}(m_0\alpha)$ such that $\lambda \neq b\alpha$, $b \in \mathbb{Z}_+$, then we have to add at least x terms in s^α of higher orders. Then it is possible to choose $\beta_{\lambda k_0} = 0$ for $\lambda \in \mathcal{L}(m_0\alpha)$, $\lambda \neq b\alpha$, $b \in \mathbb{Z}_+$ and solve the system of equations for other coefficients which admit unique or infinitely many solutions.

Remark 3.10. It is enough to choose $u(s^\alpha)$ of degree in s greater or equal to the degree in s of $\mu_k(s)$ for $k = 1, \dots, n$ in order to ensure that $\tilde{Y} \in \mathbf{M}(H_\infty)$.

Remark 3.11. The case studied in Proposition 3.2 is obviously included in Proposition 3.7. However, the expressions of the Bézout factors given in the former are slightly different from those given in the latter. In fact, in Proposition 3.2, $u(s^\alpha)$ is chosen to be of minimal degree as explained in Remark 3.10 while $\mu_k(s)$ for $k = 1, \dots, n$ have more free coefficients.

The same remarks can be stated for Propostions 3.3 and 3.5.

The following example illustrates the case where α is irrational.

Example 3.4.

$$G(s) = \left[\frac{e^{-s}}{s^{\pi/2}(s^{\pi/4} - 1)^2}, \frac{e^{-3s}}{s^{\pi/4} - 1} \right]$$

We have $p(s^{\pi/4}) = s^{\pi/2}(s^{\pi/4} - 1)^2$ with degree $d = 4$ in $s^{\pi/4}$. Its unstable roots are $b_0 = 0$ and $b_1 = 1$ with multiplicity $m_0 = 2$ and $m_1 = 2$ respectively. Obviously, $q_1(s^{\pi/4}) = 1$ and $q_2(s^{\pi/4}) = s^{\pi/2}(s^{\pi/4} - 1)$. Then from Proposition 3.1, we obtain a left coprime factorization as follows

$$\widetilde{M}(s) = \frac{s^{\pi/2}(s^{\pi/4} - 1)^2}{(s^{\pi/4} + 1)^4}, \quad \widetilde{N}(s) = \frac{1}{(s^{\pi/4} + 1)^4} \left[e^{-s}, e^{-3s} s^{\pi/2}(s^{\pi/4} - 1) \right].$$

To complete the expressions of the Bézout factors given in Proposition 3.7, we now choose $u(s^{\pi/4})$ and search for $\mu_1(s)$ and $\mu_2(s)$ by solving the equations imposed by the two conditions (3.16), (3.17).

Here, we choose $u(s^{\pi/4}) = (s^{\pi/4} + 1)^4$. It is easy to see that $\mathcal{L}(m_0\alpha) = \mathcal{L}(\pi/2) = \{0, \pi/4, 1\}$ and $f_1 = 2, f_2 = 0$. Therefore, $\mu_1(s)$ and $\mu_2(s)$ have the forms

$$\begin{aligned} \mu_1(s) &= \beta_{01} + \beta_{(\pi/4)1} s^{\pi/4} + \beta_{11} s + \beta_{(\pi/2)1} s^{\pi/2} + \beta_{(3\pi/4)1} s^{3\pi/4}, \\ \mu_2(s) &= 0. \end{aligned}$$

The numerator of $\widetilde{X}(s)$ is then $(s^{\pi/4} + 1)^8 - e^{-s}\mu_1(s)$.

Its development around zero is

$$(1 - \beta_{01}) + (8 - \beta_{(\pi/4)1})s^{\pi/4} + (\beta_{01} - \beta_{11})s + O(s^{\pi/2}).$$

The condition (3.16) implies that all powers of s with degree smaller than $\pi/2$ vanish, thus leads to $\beta_{01} = 1, \beta_{(\pi/4)1} = 8$, and $\beta_{11} = 1$.

The other coefficients are derived from the condition (3.17), which is represented by

$$\begin{aligned} (s^{\pi/4} + 1)^8 - e^{-s}\mu_1(s) &= 0, \\ [(s^{\pi/4} + 1)^8 - e^{-s}\mu_1(s)]' &= 0 \end{aligned}$$

at $s = 1$. The unique solution of these two equations is $\beta_{(\pi/2)1} = -2(11\pi + 128e\pi - 2 + 512e)/\pi$ and $\beta_{(3\pi/4)1} = 4(3\pi + 128e\pi - 1 + 256e)/\pi$.

Hence, the Bézout factors are

$$\begin{aligned}\tilde{X}(s) &= \frac{(s^{\pi/4} + 1)^8 - e^{-s}\mu_1(s)}{s^{\pi/2}(s^{\pi/4} - 1)^2(s^{\pi/4} + 1)^4}, \\ \tilde{Y}(s) &= \left[\frac{\mu_1(s)}{(s^{\pi/4} + 1)^4}, 0 \right],\end{aligned}$$

where

$$\begin{aligned}\mu_1(s) &= 1 + 8s^{\pi/4} + s - \frac{2(11\pi + 128e\pi - 2 + 512e)}{\pi} s^{\pi/2} \\ &\quad + \frac{4(3\pi + 128e\pi - 1 + 256e)}{\pi} s^{3\pi/4}.\end{aligned}$$

3.4 Right coprime factorizations and Bézout factors

The previous section showed that the systems $G(s)$ under study admit left coprime factorizations over H_∞ , and one of which is given by (3.2). Since H_∞ is a Hermite ring, then by (Quadrat, 2003a, Corollary 4.14), we deduce that there exist right coprime factorizations for $G(s)$.

For our transfer matrices, right coprime factorizations and right Bézout factors are matrices involving more entries than their left counterparts. We will consider two large classes of systems. First, for systems with distinct poles, i.e. $\tilde{p}_k(s^\alpha)$ and $\tilde{p}_{k'}(s^\alpha)$ have no common roots if $k \neq k'$, the matrix $M(s)$ can be simply of diagonal form, which reduces calculation complexity since the inverse matrix is obtained easily. For this class, we will consider three cases ranging from particular to general ones. They were studied in the same order in the previous section for left Bézout factors: systems with at least one unstable pole for each element of the transfer matrix, systems with constraints on the multiplicity of poles at zero, and systems without constraints. Second, for systems with identical poles, the form of the matrix $M(s)$ is much more complicated. Two particular cases are considered as our first attempt: systems with the same pole for all elements of the transfer matrix and systems with one pole for each element.

3.4.1 Distinct poles

3.4.1.1 Systems with one unstable pole for each element of the transfer matrix

We consider the particular case of polynomials \tilde{p}_k of degree one. This class of systems was studied in Proposition 3.2 for left Bézout factors.

Proposition 3.12. *Let $G(s)$ be given by (3.1) with*

$$R_k(s^\alpha) = \frac{a_k}{s^\alpha - \sigma_k} \text{ with } a_k, \sigma_k \in \mathbb{R} \text{ for } k = 1, \dots, n.$$

Suppose that all (zero and non-zero) unstable roots of $\tilde{p}_k(s^\alpha)$ for $k = 1, \dots, n$ are distinct, i.e. $\sigma_k \neq \sigma_{k'}$ for $\sigma_k, \sigma_{k'} \geq 0$, $k \neq k'$. Then a right coprime factorization and associated Bézout factors are given by

$$\begin{aligned} N(s) &= [N_1(s), \dots, N_n(s)], \\ M(s) &= \begin{bmatrix} M_{11}(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \\ X(s) &= \begin{bmatrix} X_{11}(s) & \cdots & X_{1n}(s) \\ \vdots & \ddots & \vdots \\ X_{n1}(s) & \cdots & X_{nn}(s) \end{bmatrix}, \\ Y(s) &= [Y_1(s), \dots, Y_n(s)]^T, \end{aligned}$$

where for $k, k' \in \{1, \dots, n\}$ and $k \neq k'$

$$\begin{aligned} N_k(s) &= \frac{e^{-sh_k} a_k}{s^\alpha + 1}, \\ M_{kk}(s) &= \frac{s^\alpha - \sigma_k}{s^\alpha + 1}, \\ Y_k(s) &= \frac{\mu_k}{u(s^\alpha)} \prod_{\sigma_j \in \mathcal{D} \cup \{0\}, j \neq k} (s^\alpha - \sigma_j), \\ X_{kk}(s) &= \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)}, \\ X_{kk'}(s) &= -Y_k(s) \frac{e^{-sh_{k'}} a_{k'}}{s^\alpha - \sigma_{k'}}, \end{aligned} \tag{3.19}$$

where $u(s^\alpha)$ is a polynomial of degree $(\text{card}(\mathcal{D}) - 1)$ in s^α that has no unstable zeros; μ_k , $k = 1, \dots, n$ are constants and μ_k such that $\sigma_k \geq 0$ are given by

$$\mu_k = \frac{u(\sigma_k)(\sigma_k + 1)e^{\sigma_k^{1/\alpha} h_k}}{a_k \prod_{\sigma_j \in \mathcal{D} \cup \{0\}, j \neq k} (\sigma_k - \sigma_j)}. \tag{3.20}$$

Proof. It is obvious that $N_k(s)$, $M_{kk}(s)$, $Y_k(s)$, $X_{kk'}(s) \in H_\infty$.

$X_{kk}(s)$ in (3.19) can be written as

$$X_{kk}(s) = \frac{u(s^\alpha)(s^\alpha + 1) - \mu_k e^{-sh_k} a_k \prod_{\sigma_j \in \mathcal{D} \cup \{0\}, j \neq k} (s^\alpha - \sigma_j)}{u(s^\alpha)(s^\alpha - \sigma_k)}.$$

If $\sigma_k \geq 0$, then μ_k as in (3.20) makes the numerator vanish at σ_k , thus guaranteeing that $X_{kk}(s) \in H_\infty$. Otherwise, i.e. $\sigma_k < 0$, $X_{kk}(s) \in H_\infty$ with any constant μ_k .

We see also that $G(s) = N(s)M(s)^{-1}$ and that the right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ is verified. \square

The following illustrative example continues Example 3.1.

Example 3.5.

$$G(s) = \left[\frac{e^{-s}}{\sqrt{s}}, \frac{e^{-s}}{\sqrt{s}-1} \right]$$

The right coprime factorization and Bézout factors proposed by Proposition 3.12 are

$$\begin{aligned} N(s) &= \left[\frac{e^{-s}}{\sqrt{s}+1}, \frac{e^{-s}}{\sqrt{s}+1} \right], \\ M(s) &= \begin{bmatrix} \frac{\sqrt{s}}{\sqrt{s}+1} & 0 \\ 0 & \frac{\sqrt{s}-1}{\sqrt{s}+1} \end{bmatrix}, \\ Y(s) &= \left[\frac{1-\sqrt{s}}{\sqrt{s}+1}, \frac{4e\sqrt{s}}{\sqrt{s}+1} \right]^T, \\ X(s) &= \begin{bmatrix} \frac{(\sqrt{s}+1)^2 + (\sqrt{s}-1)e^{-s}}{\sqrt{s}(\sqrt{s}+1)} & \frac{e^{-s}}{\sqrt{s}+1} \\ \frac{-4e^{1-s}}{\sqrt{s}+1} & \frac{(\sqrt{s}+1)^2 - 4e^{1-s}\sqrt{s}}{(\sqrt{s}-1)(\sqrt{s}+1)} \end{bmatrix}. \end{aligned}$$

3.4.1.2 Systems with constraints on the multiplicity of the pole at zero

The systems considered in the next proposition have the same condition on the multiplicity of the root at zero as those considered in Proposition 3.3. For non-zero roots, no condition is imposed, and thus finding Bézout factors by interpolation becomes more difficult than the previous case.

Proposition 3.13. *Let $G(s)$ be given by (3.1). Suppose that $\tilde{p}_k(s^\alpha)$ and $\tilde{p}_{k'}(s^\alpha)$ have no common (zero and non-zero) unstable roots if $k \neq k'$ for $k, k' \in \{1, \dots, n\}$, and suppose that the multiplicity of the root at zero of $p(s^\alpha)$ verifies the condition $m_0\alpha \leq 1$. Then a right coprime factorization and associated Bézout factors are given by*

$$\begin{aligned} N(s) &= [N_1(s), \dots, N_n(s)], \\ M(s) &= \begin{bmatrix} M_{11}(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \\ X(s) &= \begin{bmatrix} X_{11}(s) & \cdots & X_{1n}(s) \\ \vdots & \ddots & \vdots \\ X_{n1}(s) & \cdots & X_{nn}(s) \end{bmatrix}, \\ Y(s) &= [Y_1(s), \dots, Y_n(s)]^T, \end{aligned}$$

where for $k, k' \in \{1, \dots, n\}$ and $k \neq k'$

$$N_k(s) = \frac{e^{-sh_k} \tilde{q}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (3.21)$$

$$M_{kk}(s) = \frac{\tilde{p}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (3.22)$$

$$Y_k(s) = \frac{\mu_k(s^\alpha)}{u(s^\alpha)} \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right),$$

$$X_{kk}(s) = \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)},$$

$$X_{kk'}(s) = -Y_k(s)e^{-sh_{k'}} \frac{\tilde{q}_{k'}(s^\alpha)}{\tilde{p}_{k'}(s^\alpha)},$$

with d_k is the degree of \tilde{p}_k in s^α ; $u(s^\alpha)$ is a polynomial of degree d in s^α that has no unstable zeros; and $\mu_k(s^\alpha)$ are polynomials in s^α of degree $m_{0k} + \sum_{i=1}^N m_{ik}$ satisfying

$$u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \tilde{q}_k(s^\alpha) \mu_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) = O(s^{m_{0\alpha}}) \quad (3.23)$$

as $s \rightarrow 0$ if $\tilde{p}_k(s^\alpha)$ has a root at zero, and for each non-zero unstable root of $\tilde{p}_k(s^\alpha)$, i.e. $s = b_i^{1/\alpha}$ with $m_{ik} \neq 0$ for $i = 1, \dots, N$,

$$\left[u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \tilde{q}_k(s^\alpha) \mu_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) \right]^{(l)} = 0 \quad (3.24)$$

where $l = 0, \dots, m_{ik} - 1$.

Proof. It is obvious that $N_k(s), M_{kk}(s), Y_k(s), X_{kk'}(s) \in H_\infty$.

The two conditions (3.23) and (3.24) guarantee $X_{kk}(s) \in H_\infty$ for $k = 1, \dots, n$. We can find $\mu_k(s^\alpha)$ that satisfies the two conditions. Indeed, if $m_{0k} \neq 0$, the condition (3.23) is satisfied if the first m_{0k} terms, whose order in s^α are $0, \dots, m_{0k} - 1$, in the development around zero of the left expression are zero. Thus (3.23) gives m_{0k} equations, and (3.24) gives $\sum_{i=1}^N m_{ik}$ equations. On the other hand, the number of unknown coefficients of $\mu_k(s)$ is one greater than that of equations and the system of equations admits solutions.

We see also that $G(s) = N(s)M(s)^{-1}$ and the right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ is verified. \square

Remark 3.14. If $m_{0\alpha} > 1$, then e^{-sh_k} needs to be developed to $1 - sh_k + \dots + \mathcal{O}(s^{m_{0\alpha}})$. And in the case where $1/\alpha$ is not an integer, the condition (3.23) is no longer satisfied since others terms, which are polynomials in s^α , cannot compensate s .

The following example continues Example 3.2.

Example 3.6.

$$G(s) = \left[\frac{e^{-s}}{s^{1/3} - 1}, \frac{e^{-s}}{s} \right]$$

The right coprime factorizations obtained by using Proposition 3.13 are

$$N(s) = \left[\frac{e^{-s}}{s^{1/3} + 1}, \frac{e^{-s}}{(s^{1/3} + 1)^3} \right],$$

$$M(s) = \begin{bmatrix} \frac{s^{1/3}-1}{s^{1/3}+1} & 0 \\ 0 & \frac{s}{(s^{1/3}+1)^3} \end{bmatrix}.$$

We choose $u(s^{1/3}) = (s^{1/3} + 1)^4$. Then $Y(s)$ has the form

$$Y(s) = \left[\frac{\mu_1(s^{1/3})s}{(s^{1/3} + 1)^4}, \frac{\mu_2(s^{1/3})(s^{1/3} - 1)}{(s^{1/3} + 1)^4} \right]^T$$

where $\mu_1(s^{1/3}) = \beta_{01} + \beta_{11}s^{1/3}$ and $\mu_2(s^{1/3}) = \beta_{02} + \beta_{12}s^{1/3} + \beta_{22}s^{2/3} + \beta_{32}s$.

The condition (3.23) is only applied for $k = 2$. We develop the left expression around zero as follows

$$\begin{aligned} & (s^{1/3} + 1)^7 - e^{-s}\mu_2(s^{1/3})(s^{1/3} - 1) \\ &= (1 + \beta_{02}) + (7 - \beta_{02} + \beta_{12})s^{1/3} + (21 - \beta_{12} + \beta_{22})s^{2/3} + O(s). \end{aligned}$$

It turns out that all the terms with orders smaller than s in the development have to be zero, thus giving $\beta_{02} = -1$, $\beta_{12} = -8$, and $\beta_{22} = -29$.

Other unknown coefficients are deduced from applying the condition (3.24).

$$(s^{1/3} + 1)^5 - e^{-s}\mu_1(s^{1/3})s = 0$$

at $s = 1$, then $\beta_{01} + \beta_{11} = 32e$. We choose $\beta_{01} = 32e$, $\beta_{11} = 0$, and $\beta_{32} = 0$ in order to reduce the order of $\mu_1(s^{1/3})$ and $\mu_2(s^{1/3})$.

Finally, the right Bézout factors are

$$Y(s) = \left[\frac{32es}{(s^{1/3} + 1)^4}, \frac{-(29s^{2/3} + 8s^{1/3} + 1)(s^{1/3} - 1)}{(s^{1/3} + 1)^4} \right]^T$$

$$X(s) = \begin{bmatrix} \frac{(s^{1/3}+1)^5 - 32e^{1-s}s}{(s^{1/3}-1)(s^{1/3}+1)^4} & \frac{-32e^{1-s}}{(s^{1/3}+1)^4} \\ \frac{(29s^{2/3}+8s^{1/3}+1)e^{-s}}{(s^{1/3}+1)^4} & X_{22}(s) \end{bmatrix}$$

where

$$X_{22}(s) = \frac{(s^{1/3} + 1)^7 + (29s^{2/3} + 8s^{1/3} + 1)(s^{1/3} - 1)e^{-s}}{s(s^{1/3} + 1)^4}.$$

3.4.1.3 General case

We now consider the general form of systems with distinct poles between different elements of the transfer matrix.

Proposition 3.15. *Let $G(s)$ be given by (3.1). Suppose that all (zero and non-zero) unstable roots of $\tilde{p}_k(s^\alpha)$ for $k = 1, \dots, n$ are distinct. Then one right coprime factorization and associated Bézout factors are given by*

$$\begin{aligned} N(s) &= [N_1(s), \dots, N_n(s)], \\ M(s) &= \begin{bmatrix} M_{11}(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \\ X(s) &= \begin{bmatrix} X_{11}(s) & \cdots & X_{1n}(s) \\ \vdots & \ddots & \vdots \\ X_{n1}(s) & \cdots & X_{nn}(s) \end{bmatrix}, \\ Y(s) &= [Y_1(s), \dots, Y_n(s)]^T, \end{aligned}$$

where for $k, k' \in \{1, \dots, n\}$ and $k \neq k'$

$$N_k(s) = \frac{e^{-sh_k} \tilde{q}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (3.25)$$

$$M_{kk}(s) = \frac{\tilde{p}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (3.26)$$

$$Y_k(s) = \frac{\mu_k(s)}{u(s^\alpha)} \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right),$$

$$X_{kk}(s) = \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)},$$

$$X_{kk'}(s) = -Y_k(s) e^{-sh_{k'}} \frac{\tilde{q}_{k'}(s^\alpha)}{\tilde{p}_{k'}(s^\alpha)},$$

with d_k is the degree of \tilde{p}_k in s^α ; $u(s^\alpha)$ is a polynomial of degree d in s^α that has no unstable zeros; and $\mu_k(s)$ have the following form

$$\mu_k(s) = \sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \beta_{\lambda k} s^\lambda + \sum_{j=m_{0k}}^{m_{0k} + \sum_{i=1}^N m_{ik}} \beta_{(j\alpha)k} (s^\alpha)^j$$

and satisfy the following conditions

(i) if $\tilde{p}_k(s^\alpha)$ has a root at zero, as $s \rightarrow 0$

$$u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) = O(s^{m_{0k}\alpha}), \quad (3.27)$$

(ii) for each non-zero unstable root of $\tilde{p}_k(s^\alpha)$, i.e. $s = b_i^{1/\alpha}$ with $m_{ik} \neq 0$ for $i = 1, \dots, N$,

$$\left[u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) \right]^{(l)} = 0 \quad (3.28)$$

where $l = 0, \dots, m_{ik} - 1$.

Proof. It is obvious that $N_k(s), M_{kk}(s), Y_k(s), X_{kk'}(s) \in H_\infty$.

The two conditions (3.27) and (3.28) guarantee $X_{kk}(s) \in H_\infty$ for $k = 1, \dots, n$. We can find $\mu_k(s)$ that satisfies the two conditions. Indeed, (3.27) gives a number of equations which is equal to the number of the terms s^λ , and (3.28) gives $\sum_{i=1}^N m_{ik}$ equations. On the other hand, the number of unknown coefficients of $\mu_k(s)$ is one greater than that of equations. Therefore, for each $k = 1, \dots, n$, the system of equations generally admits solutions.

We see also that $G(s) = N(s)M(s)^{-1}$ and the right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ is verified. \square

Remark 3.16. $u(s^\alpha)$ can be a polynomial of degree d' in s^α , where d' is the number of unstable poles of all $R_k(s^\alpha)$, that is $d' = \sum_{k=1}^n \sum_{i=0}^N m_{ik}$.

3.4.1.4 Minimal form of Bézout factors in the general case

In this context of determining right Bézout factors, the choice of μ_k for $k = 1, \dots, n$ for minimal number of coefficients and minimal order is not quite different from that presented in the preceding proposition.

Proposition 3.17. *Let $G(s)$ be given by (3.1). Suppose that all (zero and non-zero) unstable roots of $\tilde{p}_k(s^\alpha)$ for $k = 1, \dots, n$ are distinct. Then one right coprime factorization and associated Bézout factors are given by*

$$\begin{aligned} N(s) &= [N_1(s), \dots, N_n(s)], \\ M(s) &= \begin{bmatrix} M_{11}(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \\ X(s) &= \begin{bmatrix} X_{11}(s) & \cdots & X_{1n}(s) \\ \vdots & \ddots & \vdots \\ X_{n1}(s) & \cdots & X_{nn}(s) \end{bmatrix}, \\ Y(s) &= [Y_1(s), \dots, Y_n(s)]^T, \end{aligned}$$

where for $k, k' \in \{1, \dots, n\}$ and $k \neq k'$

$$N_k(s) = \frac{e^{-sh_k} \tilde{q}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (3.29)$$

$$M_{kk}(s) = \frac{\tilde{p}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (3.30)$$

$$Y_k(s) = \frac{\mu_k(s)}{u(s^\alpha)} \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right),$$

$$X_{kk}(s) = \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)},$$

$$X_{kk'}(s) = -Y_k(s) e^{-sh_{k'}} \frac{\tilde{q}_{k'}(s^\alpha)}{\tilde{p}_{k'}(s^\alpha)},$$

with d_k is the degree of \tilde{p}_k in s^α ; $u(s^\alpha)$ is a polynomial of degree d in s^α that has no unstable zeros; and $\mu_k(s)$ have the following form

$$\mu_k(s) = \sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \beta_{\lambda k} s^\lambda + \sum_{j=m_{0k}}^{m_{0k} + \sum_{i=1}^N m_{ik} - 1} \beta_{(j\alpha)k} (s^\alpha)^j,$$

satisfying

$$u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) = O(s^{m_{0k}\alpha}) \quad (3.31)$$

as $s \rightarrow 0$ if $\tilde{p}_k(s^\alpha)$ has a root at zero, and for each non-zero unstable root of $\tilde{p}_k(s^\alpha)$, i.e. $s = b_i^{1/\alpha}$ with $m_{ik} \neq 0$ for $i = 1, \dots, N$,

$$\left[u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) \right]^{(l)} = 0 \quad (3.32)$$

where $l = 0, \dots, m_{ik} - 1$.

Proof. It is obvious that $N_k(s), M_{kk}(s), Y_k(s), X_{kk'}(s) \in H_\infty$. The two conditions (3.31) and (3.32) guarantee $X_{kk}(s) \in H_\infty$ for $k = 1, \dots, n$. We see also that $G(s) = N(s)M(s)^{-1}$ and the right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ is verified.

To complete the proof, we prove the existence of $\mu_k(s)$ satisfying the two conditions.

First, we consider the condition (3.31) on zero pole. For $k \in \{1, \dots, n\}$ such that $m_{0k} > 0$, we develop the numerator of $X_{kk}(s)$ around zero as follows

$$\begin{aligned} & u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) \\ = & a_0 + a_1 s^\alpha + \dots + a_{m_{0k}-1} s^{(m_{0k}-1)\alpha} \\ & - \left(\sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \gamma_{\lambda k} s^\lambda \right) \left(\sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \beta_{\lambda k} s^\lambda \right) + O(s^{m_{0k}\alpha}) \end{aligned} \quad (3.33)$$

where

$$\begin{aligned}
u(s^\alpha)(s^\alpha + 1)^{d_k} &= a_0 + a_1 s^\alpha + \dots + a_{m_{0k}-1} s^{(m_{0k}-1)\alpha} + O(s^{m_{0k}\alpha}), \\
e^{-sh_k} \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) &= \sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \gamma_{\lambda k} s^\lambda + O(s^{m_{0k}\alpha}), \\
\mu_k(s) &= \sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \beta_{\lambda k} s^\lambda + O(s^{m_{0k}\alpha}).
\end{aligned} \tag{3.34}$$

Eliminating the powers of s with order smaller than $m_{0k}\alpha$ in the numerator gives the matrix equation

$$\Gamma_k \begin{bmatrix} \beta_{\lambda_{N_k''-1} k} \\ \beta_{\lambda_{N_k''-2} k} \\ \vdots \\ \beta_{\lambda_0 k} \end{bmatrix} = \begin{bmatrix} a'_{N_k''-1} \\ a'_{N_k''-2} \\ \vdots \\ a'_0 \end{bmatrix}, \tag{3.35}$$

where $N_k'' = \text{card}(\mathcal{L}(m_{0k}\alpha))$, $\lambda_j \in \mathcal{L}(m_{0k}\alpha)$, $\lambda_0 < \dots < \lambda_{N_k''-1}$; Γ_k is an upper triangular matrix whose entries on the main diagonal are all $\gamma_{\lambda_0 k}$; $a'_j = a_x$ if $\lambda_j = x\alpha$, $j = 0, \dots, N_k'' - 1$, otherwise $a'_j = 0$. From (3.34), we see that $\lambda_0 k \neq 0$, then Γ_k is invertible and the system of equations admits a unique solution.

Now, we analyze the second condition (3.32) related to non-zero unstable poles. We can write $\mu_k(s)$ as follows

$$\mu_k(s) = \nu_k(s) + s^{m_{0k}\alpha} \eta_k(s^\alpha)$$

where

$$\begin{aligned}
\nu_k(s) &= \sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \beta_{\lambda k} s^\lambda, \\
\eta_k(s^\alpha) &= \sum_{j=m_{0k}}^{m_{0k} + \sum_{i=1}^N m_{ik} - 1} \beta_{(j\alpha)k} s^{(j-m_{0k})\alpha}.
\end{aligned}$$

Then for each $k \in \{1, \dots, n\}$, we can derive the values of $\eta_k^{(l_2)}(b_i)$ for $i \in \{1, \dots, N\}$ such that $m_{ik} > 0$ and $l_2 = 0, \dots, m_{ik} - 1$. If we consider the coefficients $\beta_{(j\alpha)k}$, $j = m_{0k}, \dots, m_{0k} + \sum_{i=1}^N m_{ik} - 1$ as unknowns, then the number of unknowns is the same as the number of equations and is equal to $\sum_{i=1}^N m_{ik}$. This is the problem of Hermite interpolation and there exists a unique solution. \square

Remark 3.18. We cannot eliminate the powers of s with non-commensurate order in $\mu_k(s)$ such that $m_{0k} > 0$ by adding more coefficients as we can in the case of left Bézout factors. In fact, powers of s with order higher than $m_{0k}\alpha$ do not appear in the development up to order $m_{0k}\alpha$ of the numerator of $X_{kk}(s)$ and thus do not affect the matrix equation (3.35).

Here is a numerical example of the use of the proposition.

Example 3.7.

$$G(s) = \left[\frac{e^{-s}}{s^{\pi/2}(s^{\pi/4} - 1)^2}, \frac{e^{-3s}}{s^{\pi/4} - 2} \right]$$

By applying Proposition 3.17 we obtain the right coprime factorizations as follows

$$N(s) = \left[\frac{e^{-s}}{(s^{\pi/4} + 1)^4}, \frac{e^{-3s}}{s^{\pi/4} + 1} \right],$$

$$M(s) = \begin{bmatrix} \frac{s^{\pi/2}(s^{\pi/4}-1)^2}{(s^{\pi/4}+1)^4} & 0 \\ 0 & \frac{s^{\pi/4}-2}{s^{\pi/4}+1} \end{bmatrix}.$$

We choose $u(s^{\pi/4}) = (s^{\pi/4} + 1)^5$, then $Y(s)$ has the form

$$Y(s) = \left[\frac{\mu_1(s)(s^{\pi/4} - 2)}{(s^{\pi/4} + 1)^5}, \frac{\mu_2(s)s^{\pi/2}(s^{\pi/4} - 1)^2}{(s^{\pi/4} + 1)^5} \right]^T$$

where $\mu_1(s) = \beta_{01} + \beta_{(\pi/4)1}s^{\pi/4} + \beta_{11}s + \beta_{(\pi/2)1}s^{\pi/2} + \beta_{(3\pi/4)1}s^{3\pi/4}$ and $\mu_2(s) = \beta_{02}$.

The condition (3.31) is only applied for $k = 1$. We develop the numerator of $X_{11}(s)$ around zero as follows

$$\begin{aligned} & (s^{\pi/4} + 1)^9 - e^{-s}\mu_1(s)(s^{\pi/4} - 2) \\ &= (1 + \beta_{01}) + (9 - \beta_{01} + \beta_{(\pi/4)1})s^{\pi/4} + (\beta_{11} - \beta_{01})s + O(s^{\pi/2}). \end{aligned}$$

It turns out that all the terms with order smaller than $\pi/2$ in the development have to be zero, thus giving $\beta_{01} = -1/2$, $\beta_{(\pi/4)1} = -19/4$, and $\beta_{11} = -1/2$.

Other unknown coefficients are deduced from applying the condition (3.32).

$$\begin{aligned} & (s^{\pi/4} + 1)^9 - e^{-s}\mu_1(s)(s^{\pi/4} - 2) = 0, \\ & [(s^{\pi/4} + 1)^9 - e^{-s}\mu_1(s)(s^{\pi/4} - 2)]' = 0 \end{aligned}$$

at $s = 1$ and

$$(s^{\pi/4} + 1)^6 - e^{-3s}\mu_2(s)s^{\pi/2}(s^{\pi/4} - 1)^2 = 0$$

at $s = 2^{4/\pi}$, then the unique solution of the above equations is $\beta_{(\pi/2)1} = (25\pi + 2560e\pi - 4 + 4096e)/(2\pi)$, $\beta_{(3\pi/4)1} = -(27\pi + 7168e\pi - 8 + 8192e)/(4\pi)$, and $\beta_{02} = (729(e^{2^{4/\pi}})^3)/4$.

Finally,

$$X(s) = \begin{bmatrix} \frac{(s^{\pi/4}+1)^9 - e^{-s}\mu_1(s)(s^{\pi/4}-2)}{s^{\pi/2}(s^{\pi/4}-1)^2(s^{\pi/4}+1)^5} & \frac{-e^{-3s}\mu_2(s)}{(s^{\pi/4}+1)^5} \\ \frac{-e^{-s}\mu_2(s)}{(s^{\pi/4}+1)^5} & \frac{(s^{\pi/4}+1)^6 - e^{-3s}\mu_2(s)s^{\pi/2}(s^{\pi/4}-1)^2}{(s^{\pi/4}-2)(s^{\pi/4}+1)^5} \end{bmatrix}$$

where

$$\begin{aligned} \mu_1(s) &= -\frac{1}{2} - \frac{19}{4}s^{\pi/4} - \frac{1}{2}s + \frac{25\pi + 2560e\pi - 4 + 4096e}{2\pi}s^{\pi/2} \\ &\quad - \frac{27\pi + 7168e\pi - 8 + 8192e}{4\pi}s^{3\pi/4}, \\ \mu_2(s) &= \frac{729(e^{2^{4/\pi}})^3}{4}. \end{aligned}$$

3.4.2 Identical poles

While simple expressions are obtained for systems with distinct poles, much more attention has to be paid for the case of identical poles. The matrix $M(s)$ in diagonal form and $N_k(s)$, $M_{kk}(s)$ in the forms (3.25), (3.26) do not work for the latter case. We deduce from the right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ that for $k, k' \in \{1, \dots, n\}$ and $k \neq k'$

$$X_{kk}(s) = \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)},$$

$$X_{kk'}(s) = -Y_k(s) \frac{N_{k'}(s)}{M_{k'k'}(s)}.$$

In order for $X_{kk'}(s)$ to be in H_∞ , all unstable roots of $M_{k'k'}(s)$ have to be roots of $Y_k(s)$. Consequently, if $M_{kk}(s)$ and $M_{k'k'}(s)$ have a common root, then $X_{kk}(s)$ at that root is infinite, thus $X_{kk}(s) \notin H_\infty$.

3.4.2.1 Systems with one identical pole for each element of the transfer matrix

In the following proposition, we only consider the case where all the \tilde{p}_k , $k = 1, \dots, n$ have the same root. Although the matrix $M(s)$ is no longer diagonal, its inverse can also be easily calculated.

Proposition 3.19. *Let $G(s)$ be given by (3.1) with*

$$h_1 \leq \dots \leq h_n,$$

$$R_k(s^\alpha) = \frac{a_k}{s^\alpha - \sigma}$$

with $a_k, \sigma \in \mathbb{R}$ for $k = 1, \dots, n$. Then a right coprime factorization and associated Bézout factors are given by

$$N(s) = \left[\frac{a_1 e^{-sh_1}}{s^\alpha + 1}, 0, \dots, 0 \right],$$

$$M(s) = \begin{bmatrix} \frac{s^\alpha - \sigma}{s^\alpha + 1} & \frac{-a_2 e^{-s(h_2 - h_1)}}{a_1} & \dots & \frac{-a_n e^{-s(h_n - h_1)}}{a_1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

$$Y(s) = [\beta, 0, \dots, 0]^T,$$

$$X(s) = M^{-1}(s) - Y(s)G(s),$$

where β is given by

$$\beta = \frac{(\sigma + 1)e^{\sigma^{1/\alpha} h_1}}{a_1}, \quad (3.36)$$

and $M^{-1}(s)$ is given by

$$M(s)^{-1} = M^{inv}(s)$$

with

$$M^{inv}(s) := \begin{bmatrix} M_{11}^{inv}(s) & \cdots & M_{1n}^{inv}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}^{inv}(s) \end{bmatrix}, \quad (3.37)$$

$$M_{1k}^{inv}(s) = \frac{a_k e^{-s(h_k - h_1)} (s^\alpha + 1)}{a_1 (s^\alpha - \sigma)} \quad \forall k = 1, \dots, n,$$

$$M_{kk}^{inv}(s) = 1 \quad \forall k = 2, \dots, n,$$

$$M_{kk'}^{inv}(s) = 0 \quad \forall k \neq k', k = 2, \dots, n, k' = 1, \dots, n.$$

Proof. It is easy to verify that the matrix $M^{inv}(s)$ in (3.37) is the inverse of $M(s)$ and that $N(s)M^{-1}(s) = G(s)$. The right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ is clearly satisfied.

It is obvious that $N(s), M(s), Y(s) \in \mathbf{M}(H_\infty)$.

We see that $X_{k'k}(s)$ for $k' = 2, \dots, n$ and $k = 1, \dots, n$ are constants. Now we consider $X_{11}(s)$ and $X_{1k}(s)$ for $k = 2, \dots, n$.

$$X_{11}(s) = \frac{s^\alpha + 1}{s^\alpha - \sigma} - \beta \frac{a_1 e^{-sh_1}}{s^\alpha - \sigma},$$

$$X_{1k}(s) = \frac{a_k e^{-s(h_k - h_1)} (s^\alpha + 1)}{a_1 (s^\alpha - \sigma)} - \beta \frac{a_k e^{-sh_k}}{s^\alpha - \sigma}.$$

With β given by (3.36), the numerators of $X_{11}(s)$ and $X_{1k}(s)$ vanish at $s = \sigma^{1/\alpha}$, which is the unique unstable root of the denominators. Hence, $X(s) \in \mathbf{M}(H_\infty)$. \square

3.4.2.2 More general case

In the following part, we derive right coprime factorizations and Bézout factors for a particular system whose entries (which only have one simple pole) may involve identical poles. To help clarify the demonstration of those results, we will begin with a lemma who derives the inverse of a particular upper triangular matrix.

We consider sparse matrices with some conditions imposed on the entries above the main diagonal: if any entry on the k -th row is non-zero, then all entries on the k -th column must be zeros; if any entry on the k -th column is non-zero, then all other entries on the k -th column as well as those on the k -th row must be zeros.

Lemma 3.20. *Let the upper triangular matrix $M \in \mathbb{R}^{n \times n}$ be given by*

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn} \end{bmatrix},$$

where the entries on the main diagonal are not equal to zero and the entries above the main diagonal satisfy the following conditions

(i) for $k = 1, \dots, n$, if there exists $l' \in \mathbb{Z}$, $l' \in (k, n]$ such that $M_{kl'} \neq 0$ then $M_{lk} = 0$ for $l \in \{1, \dots, k-1\}$,

(ii) for $k = 1, \dots, n$, if there exists $l'' \in \mathbb{Z}$, $l'' \in [1, k)$ such that $M_{l''k} \neq 0$ then $M_{lk} = 0$ for $l \in \{1, \dots, k-1\} \setminus \{l''\}$ and $M_{kl'} = 0$ for $l' \in \{k+1, \dots, n\}$.

Then its inverse is given by

$$M^{-1} = M^{inv}$$

with

$$M^{inv} := \begin{bmatrix} M_{11}^{inv} & \dots & M_{1n}^{inv} \\ \vdots & \ddots & \vdots \\ 0 & \dots & M_{nn}^{inv} \end{bmatrix},$$

where the entries on and above the main diagonal satisfy

$$M_{kk}^{inv} = \frac{1}{M_{kk}}, \quad (3.38)$$

$$M_{kk'}^{inv} = -\frac{M_{kk'}}{M_{kk}M_{k'k'}} \quad (3.39)$$

for $k, k' \in \{1, \dots, n\}$ and $k < k'$.

Proof. It is obvious that the entries below the main diagonal of the product MM^{inv} are all zero, and the entries on the main diagonal are all one.

Now we consider the entries above the main diagonal of the product:

$$(MM^{inv})_{ij} = \sum_{k=1}^n M_{ik}M_{kj}^{inv} = \sum_{k=i}^j M_{ik}M_{kj}^{inv}$$

for $i < j$. Considering $i < k < j$, if $M_{ik} \neq 0$, then $M_{kj} = 0$ under the assumption (ii), and thus $M_{kj}^{inv} = 0$ due to (3.39). Therefore,

$$(MM^{inv})_{ij} = M_{ii}M_{ij}^{inv} + M_{ij}M_{jj}^{inv}.$$

By replacing M_{ij}^{inv} and M_{jj}^{inv} with (3.39) and (3.38) respectively, we obtain $(MM^{inv})_{ij} = 0$. \square

Example 3.8. The following matrix satisfies all the conditions in Lemma 3.20.

$$M = \begin{bmatrix} M_{11} & 0 & M_{13} & 0 & 0 \\ 0 & M_{22} & 0 & M_{24} & M_{25} \\ 0 & 0 & M_{33} & 0 & 0 \\ 0 & 0 & 0 & M_{44} & 0 \\ 0 & 0 & 0 & 0 & M_{55} \end{bmatrix}$$

with $M_{ii} \neq 0$, $i = 1, \dots, 5$, and $M_{13}, M_{24}, M_{25} \neq 0$.

Its inverse is

$$M^{-1} = \begin{bmatrix} \frac{1}{M_{11}} & 0 & -\frac{M_{13}}{M_{11}M_{33}} & 0 & 0 \\ 0 & \frac{1}{M_{22}} & 0 & -\frac{M_{24}}{M_{22}M_{44}} & -\frac{M_{25}}{M_{22}M_{55}} \\ 0 & 0 & \frac{1}{M_{33}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{M_{44}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{M_{55}} \end{bmatrix}.$$

In the following proposition, we consider $G(s)$ with one pole for each of its entries and some entries may have the same pole. To simplify the presentation, we assume that the delays are ordered. A discussion on how to apply the next result to the case of unordered delays will follow the proposition.

Proposition 3.21. *Let $G(s)$ be given by (3.1) with*

$$h_1 \leq \dots \leq h_n,$$

$$R_k(s^\alpha) = \frac{a_k}{s^\alpha - \sigma_k}$$

with $a_k, \sigma_k \in \mathbb{R}$ for $k = 1, \dots, n$. We denote $\mathcal{I}_1 := \emptyset$ and $\mathcal{I}_k := \{j \mid j \in \{1, \dots, k-1\}, \sigma_j = \sigma_k\}$ for $k = 2, \dots, n$. One right coprime factorization and associated Bézout factors are given by

$$N(s) = [N_1(s), \dots, N_n(s)], \quad (3.40)$$

$$M(s) = \begin{bmatrix} M_{11}(s) & \cdots & M_{1n}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \quad (3.41)$$

$$Y(s) = [Y_1(s), \dots, Y_n(s)]^T,$$

$$X(s) = M^{-1}(s) - Y(s)G(s),$$

where for $k, k' \in \{1, \dots, n\}$ and $k' \neq k$

$$N_k(s) = \begin{cases} 0 & \text{if } \mathcal{I}_k \neq \emptyset, \\ \frac{a_k e^{-sh_k}}{s^\alpha + 1} & \text{otherwise,} \end{cases} \quad (3.42)$$

$$M_{kk}(s) = \begin{cases} 1 & \text{if } \mathcal{I}_k \neq \emptyset \\ \frac{s^\alpha - \sigma_k}{s^\alpha + 1} & \text{otherwise,} \end{cases} \quad (3.43)$$

$$M_{k'k}(s) = \begin{cases} -\frac{a_k e^{-s(h_k - h_{k'})}}{a_{k'}} & \text{if } k' = \min \mathcal{I}_k, \\ 0 & \text{otherwise,} \end{cases} \quad (3.44)$$

$$Y_k(s) = \begin{cases} 0 & \text{if } \mathcal{I}_k \neq \emptyset, \\ \frac{\beta_k \hat{p}_k(s^\alpha)}{u(s^\alpha)} & \text{otherwise,} \end{cases}$$

with $u(s^\alpha)$ is a polynomial of degree d in s^α that has no unstable zeros; $\hat{p}_k(s^\alpha) = p(s^\alpha)/(s^\alpha - \sigma_k)$; β_k (for those k such that $\mathcal{I}_k = \emptyset$ and $\sigma_k \geq 0$) are given by

$$\beta_k = \frac{u(\sigma_k)(\sigma_k + 1)e^{\sigma_k^{1/\alpha} h_k}}{a_k \hat{p}'_k(\sigma_k)}, \quad (3.45)$$

β_k for other k can be chosen arbitrarily, and $M^{-1}(s)$ are given by

$$M^{-1}(s) = \begin{bmatrix} M_{11}^{inv}(s) & \cdots & M_{1n}^{inv}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}^{inv}(s) \end{bmatrix} \quad (3.46)$$

where the entries on and above the main diagonal satisfy

$$M_{kk}^{inv} = \frac{1}{M_{kk}},$$

$$M_{kk'}^{inv} = -\frac{M_{kk'}}{M_{kk}M_{k'k'}}$$

for $k, k' \in \{1, \dots, n\}$ and $k < k'$.

Proof. Let us prove that $M(s)$ given by (3.41), (3.43), and (3.44) satisfies the assumptions in Lemma 3.20. Let $k \in \{1, \dots, n\}$, if $M_{k'k} \neq 0$ then due to (3.44) $k' = \min \mathcal{I}_k$, and $M_{k''k} = 0$ for $k'' \neq k'$. Also, since $k' = \min \mathcal{I}_k$, then $k \neq \min \mathcal{I}_{k''}$ for $k'' > k$, and thus $M_{kk''} = 0$. Hence the assumption (ii) in Lemma 3.20 is satisfied. On the other hand, $k' = \min \mathcal{I}_k$ implies that $\mathcal{I}_{k'} = \emptyset$, hence $M_{k''k'} = 0$ for $k'' < k'$. The assumption (i) is then satisfied.

Consequently, due to Lemma 3.20, the inverse of $M(s)$ is given by (3.46).

We now prove that $N(s)M^{-1}(s) = G(s)$.

For $k \in \{1, \dots, n\}$, we have

$$\begin{aligned} (N(s)M(s)^{-1})_k &= \sum_{l=1}^n N_l(s)M_{lk}^{inv}(s) \\ &= \sum_{l=1}^{k-1} N_l(s)M_{lk}^{inv}(s) + N_k(s)M_{kk}^{inv}(s). \end{aligned}$$

- If $\mathcal{I}_k = \emptyset$, then $M_{lk}^{inv}(s) = 0$ for $l = 1, \dots, k-1$, and $(N(s)M(s)^{-1})_k = N_k(s)M_{kk}^{inv}(s) = e^{-sh_k} R_k(s)$.
- If $\mathcal{I}_k \neq \emptyset$, then $M_{lk}^{inv}(s) = 0$ for $l \in \{1, \dots, k-1\} \setminus \{k'\}$ where $k' = \min \mathcal{I}_k$ and $N_k(s) = 0$. Therefore,

$$\begin{aligned} (N(s)M(s)^{-1})_k &= N_{k'}(s)M_{k'k}^{inv}(s) \\ &= -N_{k'}(s) \frac{M_{k'k}(s)}{M_{k'k'}(s)M_{kk}(s)}. \end{aligned}$$

Note that $\mathcal{I}_{k'} = \emptyset$ since $k' = \min \mathcal{I}_k$. By replacing the above terms with appropriate expressions in (3.42), (3.43) and (3.44) and by noting that $\sigma_{k'} = \sigma_k$, we get $(N(s)M(s)^{-1})_k = e^{-sh_k} R_k(s)$.

It is obvious that $N_k(s), M_{kk}(s), M_{k'k}(s), Y_k(s) \in H_\infty$.

Let us now prove that $X(s) \in \mathbf{M}(H_\infty)$.

For $k, k' \in \{1, \dots, n\}$, we have $X_{k'k}(s) = M_{k'k}^{inv}(s) - Y_{k'}(s)e^{-sh_k} R_k(s)$.

- If $\mathcal{I}_{k'} \neq \emptyset$, then $Y_{k'}(s) = 0$, and thus $X_{k'k}(s) = M_{k'k}^{inv}(s)$. Now, for $k' > k$, $M_{k'k}^{inv} = 0$. For $k' = k$, $M_{k'k}^{inv} = 1$. For $k' < k$, from the fact that $\mathcal{I}_{k'} \neq \emptyset$, we deduce that $k' \neq \min \mathcal{I}_k$, thus $M_{k'k}^{inv} = 0$.

- If $\mathcal{I}_{k'} = \emptyset$, then $Y_{k'}(s)$ involves $\hat{p}_{k'}(s^\alpha)$.

- For $k' > k$, the fact that $\mathcal{I}_{k'} = \emptyset$ leads to $\sigma_{k'} \neq \sigma_k$. Therefore,

$$Y_{k'}(s)R_k(s) = \frac{a_k \beta_{k'} \hat{p}_{k'}(s^\alpha)}{(s^\alpha - \sigma_k)u(s^\alpha)} = \frac{a_k \beta_{k'} p(s^\alpha)}{(s^\alpha - \sigma_k)(s^\alpha - \sigma_{k'})u(s^\alpha)}$$

belongs to H_∞ since $(s^\alpha - \sigma_k)(s^\alpha - \sigma_{k'})$ is eliminated by the same term in $p(s^\alpha)$. It is also obvious that $M_{k'k}^{inv} = 0$. Therefore, $X_{k'k}(s) \in H_\infty$.

- For $k' = k$, we have

$$\begin{aligned} X_{k'k'}(s) &= M_{k'k'}^{inv}(s) - Y_{k'}(s)e^{-sh_{k'}}R_{k'}(s) \\ &= \frac{1}{M_{k'k'}(s)} - Y_{k'}(s)e^{-sh_{k'}}R_{k'}(s) \\ &= \frac{s^\alpha + 1}{s^\alpha - \sigma_{k'}} - \frac{\beta_{k'} \hat{p}_{k'}(s^\alpha)}{u(s^\alpha)} \frac{a_{k'} e^{-sh_{k'}}}{s^\alpha - \sigma_{k'}} \\ &= \frac{u(s^\alpha)(s^\alpha + 1) - \beta_{k'} \hat{p}_{k'}(s^\alpha) a_{k'} e^{-sh_{k'}}}{u(s^\alpha)(s^\alpha - \sigma_{k'})}. \end{aligned}$$

If $\sigma_{k'} < 0$, then $X_{k'k'}(s) \in H_\infty$ for all $\beta_{k'}$. If $\sigma_{k'} \geq 0$, since $\beta_{k'}$ given by (3.45) makes the denominators of $X_{k'k'}(s)$ vanish at $s = \sigma_{k'}^{1/\alpha}$, then $X_{k'k'}(s) \in H_\infty$.

- For $k' < k$, if $k' \neq \min \mathcal{I}_k$, together with the fact that $\mathcal{I}_{k'} = \emptyset$ then $\sigma_{k'} \neq \sigma_k$, and thus $Y_{k'}(s)R_k(s) \in H_\infty$. We also have $M_{k'k}^{inv} = 0$, leading to $X_{k'k}(s) \in H_\infty$. In the case where $k' = \min \mathcal{I}_k$, thus $\sigma_{k'} = \sigma_k$, we have

$$\begin{aligned} X_{k'k}(s) &= M_{k'k}^{inv}(s) - Y_{k'}(s)e^{-sh_k}R_k(s) \\ &= -\frac{M_{k'k}(s)}{M_{k'k'}(s)M_{kk}(s)} - Y_{k'}(s)e^{-sh_k}R_k(s) \\ &= \frac{s^\alpha + 1}{s^\alpha - \sigma_{k'}} \frac{a_k e^{-s(h_k - h_{k'})}}{a_{k'}} - \frac{\beta_{k'} \hat{p}_{k'}(s^\alpha)}{u(s^\alpha)} \frac{a_k e^{-sh_k}}{s^\alpha - \sigma_k} \\ &= a_k e^{-sh_k} \frac{u(s^\alpha)(s^\alpha + 1)e^{sh_{k'}} - \beta_{k'} \hat{p}_{k'}(s^\alpha) a_{k'}}{u(s^\alpha)(s^\alpha - \sigma_{k'}) a_{k'}}. \end{aligned}$$

By the same argument as in the case where $k' = k$, we conclude that $X_{k'k}(s) \in H_\infty$.

The right Bézout identity $X(s)M(s) + Y(s)N(s) = I$ is clearly satisfied. \square

Remark 3.22. A transfer matrix G given by (3.1) with the delays of its elements not in order can be transformed to a transfer matrix G_0 with ordered delays by multiplying G by an appropriate permutation matrix P . It is well known that this matrix P is orthogonal and its inverse is P^T . Assume that (M_0, N_0) is a right coprime factorization over H_∞ of G_0 and X_0, Y_0 are the corresponding right Bézout factors. We have then $G = G_0 P^{-1} = N_0 M_0^{-1} P^{-1} = N_0 (P M_0)^{-1}$. It is obvious that $P M_0 \in \mathbf{M}(H_\infty)$. Furthermore,

$X_0P^{-1}PM_0 + Y_0N_0 = I$ and $X_0P^{-1} \in \mathbf{M}(H_\infty)$. Hence, (PM_0, N_0) is a right coprime factorization of G and X_0P^{-1}, Y_0 are the corresponding Bézout factors.

The next example illustrates the proposition.

Example 3.9.

$$G(s) = \left[\frac{e^{-s}}{\sqrt{s}}, \frac{e^{-s}}{\sqrt{s-1}}, \frac{e^{-3s}}{\sqrt{s-1}} \right]$$

From (3.40), (3.41), (3.42), (3.43), and (3.44), we obtain

$$N(s) = \left[\frac{e^{-s}}{\sqrt{s+1}}, \frac{e^{-s}}{\sqrt{s+1}}, 0 \right],$$

$$M(s) = \begin{bmatrix} \frac{\sqrt{s}}{\sqrt{s+1}} & 0 & 0 \\ 0 & \frac{\sqrt{s-1}}{\sqrt{s+1}} & -e^{-2s} \\ 0 & 0 & 1 \end{bmatrix}.$$

The inverse of $M(s)$ is

$$M^{-1}(s) = \begin{bmatrix} \frac{\sqrt{s+1}}{\sqrt{s}} & 0 & 0 \\ 0 & \frac{\sqrt{s+1}}{\sqrt{s-1}} & \frac{e^{-2s}(\sqrt{s+1})}{\sqrt{s-1}} \\ 0 & 0 & 1 \end{bmatrix},$$

which will be used to derive $X(s)$.

The least common denominator of the entries of $G(s)$ is $p(\sqrt{s}) = \sqrt{s}(\sqrt{s-1})$. Then $p'_1(\sqrt{s}) = \sqrt{s-1}$, and $p'_2(\sqrt{s}) = \sqrt{s}$. We choose $u(\sqrt{s}) = (\sqrt{s+1})^2$, which has no unstable poles. We have then

$$Y(s) = \left[\frac{\beta_1(\sqrt{s-1})}{(\sqrt{s+1})^2}, \frac{\beta_2\sqrt{s}}{(\sqrt{s+1})^2}, 0 \right]^T$$

$$X(s) = M^{-1}(s) - Y(s)G(s)$$

$$= \begin{bmatrix} \frac{(\sqrt{s+1})^3 - \beta_1(\sqrt{s-1})e^{-s}}{\sqrt{s}(\sqrt{s+1})^2} & -\frac{\beta_1e^{-s}}{(\sqrt{s+1})^2} & -\frac{\beta_1e^{-3s}}{(\sqrt{s+1})^2} \\ -\frac{\beta_2e^{-s}}{(\sqrt{s+1})^2} & \frac{(\sqrt{s+1})^3 - \beta_2e^{-s}\sqrt{s}}{(\sqrt{s-1})(\sqrt{s+1})^2} & \frac{e^{-2s}(\sqrt{s+1})^3 - \beta_2e^{-3s}\sqrt{s}}{(\sqrt{s-1})(\sqrt{s+1})^2} \\ 0 & 0 & 1 \end{bmatrix}$$

We see that $X_{12}(s), X_{13}(s), X_{21}(s) \in H_\infty$. From (3.45), we obtain $\beta_1 = -1$ and $\beta_2 = 8e$, which make $X_{11}(s), X_{22}(s)$, and $X_{23}(s)$ be in H_∞ respectively.

3.5 Conclusion

In this chapter, we have considered MISO fractional systems with input or output delays. Explicit expressions of a left coprime factorization over H_∞ of the transfer matrices as well as the corresponding Bézout factors are given. Right coprime factorizations

and right Bézout factors are also found for systems with entries of the transfer matrix containing different poles. In the case of identical poles, the right factors are primarily found for some simple classes of systems. Hence, in conclusion, we can have Youla-Kučera parametrization of stabilizing controllers for all systems with distinct poles and a class of systems with identical poles where each element of the transfer matrix involves one pole since for these systems both left and right coprime factorizations and Bézout factors are available.

Determining the right factors of more general systems with identical poles is the objective of a forthcoming work.

Chapter 4

Stability analysis of SISO fractional neutral systems with commensurate delays

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4.1 Introduction

An increasing interest has been paid towards fractional systems with delays, both for stability aspects and also for stabilization problems. For stability aspects, most of the results are obtained in the frequency domain, i.e. by considering the spectrum of the characteristic equation. Note that the characteristic equation is single-valued provided that the Laplace variable s is limited in a sheet of the corresponding Riemann surface, for example, $-\pi < \arg(s) < \pi$.

The stability of fractional delay systems with non-commensurate orders and non-commensurate delays was first investigated in (Bonnet and Partington, 2002) in the sense BIBO. In that paper, the classical stability condition “no poles in the closed right half-plane” is proved to be a necessary and sufficient condition for systems of retarded type and only a necessary condition for neutral systems. Recently, robust BIBO-stability regarding parameter uncertainties was considered in (Akbari Moornani and Haeri, 2010) for some classes of the same systems and necessary and sufficient stability conditions were derived.

For fractional delay systems with commensurate orders and commensurate delays, the characteristics of poles are similar to those of classical systems with delays, i.e. there are infinitely many poles in chains which can be classified in retarded, advanced and neutral types (Hotzel, 1998a; Bonnet and Partington, 2007). For retarded systems, the stabilities in the senses BIBO, $L_1 - L_1$ and H_∞ are equivalent and also share the same usual necessary and sufficient condition “no poles in the closed right half-plane” (Hotzel, 1998a; Bonnet and Partington, 2007). Nevertheless, this is only a necessary condition for neutral systems since, in the critical case where poles approach the imaginary axis, the system may be unstable even though all poles are in the open left half-plane. This interesting phenomenon is also present in classical delay systems.

In such a delicate situation, H_∞ -stability of fractional systems with one delay is studied in (Bonnet and Partington, 2007) where simple necessary and sufficient conditions are derived. Within the same framework, (Fioravanti et al., 2010) studies H_∞ -stability of some classes of fractional systems with commensurate delays and with single chains of poles asymptotic to the imaginary axis, namely there is no more than one neutral chain of poles asymptotic to a set of points on the imaginary axis.

In this chapter, we will extend the work in (Fioravanti et al., 2010) to the case of multiple neutral chains of poles asymptotic to a same set of points on the imaginary axis.

In general, when some chains of poles are asymptotic to the imaginary axis, the idea is that the stability depends on not only the location of poles about the axis but also the magnitude of the transfer function on the axis. This idea is no longer new and is exploited in (Bonnet and Partington, 2007; Fioravanti et al., 2010) for fractional systems and in (Partington and Bonnet, 2004; Bonnet et al., 2009, 2011) for classical systems. The common method is to approximate solutions of high modulus of the characteristic equation, which is a quasi-polynomial involving powers of s and e^s . This approximation then allows one to evaluate the magnitude of the transfer function and to derive stability conditions.

We are interested in the effects of poles of large modulus on the stability and will not pay attention to poles of small modulus. For exact values of these poles, numerical methods such as QPmR (Vyhlidal and Zitek, 2014) and YALTA (Avanessoff et al., 2014) can be used. For their relative location around the imaginary axis, methods for determining crossing frequencies and stability windows such as (Marshall et al., 1992; Fioravanti et al., 2012) can be applied.

The rest of the chapter is organized as follows. In section 4.2, we present the fractional delay system of interest and approximate the characteristic equation around its poles of large modulus. The obtained expression is repeatedly used in the next two sections to determine pole location with respect to asymptotic axes and estimate the magnitude of the characteristic equation on the imaginary axis, which allows one to conclude about H_∞ -stability of the system. Section 4.3 examines single neutral chains of poles while section 4.4 is dedicated to multiple chains. Illustrative examples are given in section 4.5. We conclude the chapter with section 4.6.

4.2 Approximation of the characteristic equation

We consider fractional neutral time-delay systems with transfer function given by (2.2), which is recalled here for easy access. The transfer function is of the form

$$G(s) = \frac{t(s)}{p(s) + \sum_{k=1}^N q_k(s)e^{-ks\tau}}, \quad (4.1)$$

where

- $\tau > 0$ is the delay,
- t , p , and q_k for all $k \in \mathbb{N}_N$ are real polynomials in s^μ ,
- $0 < \mu < 1$, $-\pi < \arg(s) < \pi$ in order to have a single-valued transfer function,
- $\deg p \geq \deg t$, $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$ in order to deal with proper neutral systems.

Note that degrees of the polynomials in this chapter stand for the degrees in s^μ .

For preliminaries regarding these systems, the reader is referred to Subsection 2.4.2.

We already have the first approximation of neutral poles corresponding to a root r of $\tilde{c}_d(z)$ in (2.5). Our objective is to find the next *non-zero* approximation term of these poles, which are denoted by s_n . Let us write

$$s_n\tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

where

$$\nu_{n,1} = \frac{\nu_1}{n^{y_1}}, \quad \nu_1 \neq 0, \quad y_1 > 0, \quad n \in \mathbb{Z}, \quad n \rightarrow \infty.$$

We will see later that $y_1 = \mu$, for example, in certain cases of single chains, but $y_1 = \mu/m$ in certain cases of multiple chains, where m is the multiplicity of r .

We have $\Re(s_n) = [\Re(\lambda_n) + \Re(\nu_{n,1}) + o(n^{-y_1})]/\tau$. Therefore, the sign of $\Re(\nu_{n,1})$ indicates the location of poles of the neutral chain with respect to the asymptotic axis.

Remark 4.1. Note that for a neutral chain of poles relative to a root r

$$\begin{aligned}\Re(\nu_{n,1})_{n>0} &= \frac{\Re(\nu_1)}{n^{y_1}}, \\ \Re(\nu_{n,1})_{n<0} &= \frac{\Re(\nu_1) \cos(y_1\pi) + \Im(\nu_1) \sin(y_1\pi)}{|n|^{y_1}}.\end{aligned}$$

Since the signs of $\Re(\nu_{n,1})_{n>0}$ and $\Re(\nu_{n,1})_{n<0}$, which are determined by the signs of $\Re(\nu_1)$ and $(\Re(\nu_1) \cos(y_1\pi) + \Im(\nu_1) \sin(y_1\pi))$ respectively, may be different, so are the locations around the asymptotic axis of poles of large modulus in the upper and lower half-planes.

Approximation of neutral poles of the system will be derived from the approximation of the characteristic equation around s_n .

Since s_n is a pole of $G(s)$, we have

$$d(s_n) := p(s_n) + \sum_{k=1}^N q_k(s_n) e^{-ks_n\tau} = 0.$$

Dividing both sides by $p(s_n)$, we have

$$1 + \sum_{k=1}^N \frac{q_k(s_n)}{p(s_n)} e^{-ks_n\tau} = 0.$$

As $|s_n| \rightarrow \infty$, using (2.3) leads to

$$1 + \sum_{k=1}^N \left(\alpha_k + \frac{\beta_k}{s_n^\mu} + \frac{\gamma_k}{s_n^{2\mu}} + \frac{\delta_k}{s_n^{3\mu}} + \frac{\epsilon_k}{s_n^{4\mu}} + o(s_n^{-4\mu}) \right) e^{-ks_n\tau} = 0. \quad (4.2)$$

We choose indeed a development of order 4μ which will allow us to analyze in this chapter several cases of interest.

Assume s_n has the form

$$s_n\tau = \lambda_n + \nu_{n,1} + \nu_{n,2} + \dots + \nu_{n,M} + o(n^{-4\mu})$$

with $\nu_{n,i} = \nu_i n^{-y_i}$, $i = 1, \dots, M$ where $\nu_i \neq 0$ and $0 < y_1 < \dots < y_M \leq 4\mu$.

Note that

$$\begin{aligned}e^{-\lambda_n} &= r, \\ e^{-k\nu_{n,i}} &= 1 + \sum_{l=1}^{\lfloor \frac{4\mu}{y_i} \rfloor} \frac{(-1)^l \nu_i^l k^l}{l! n^{ly_i}} + o(n^{-4\mu}).\end{aligned}$$

Thus when n is large enough, (4.2) becomes

$$1 + \sum_{k=1}^N \left(\alpha_k + \frac{\beta_k \tau^\mu}{(j2\pi n)^\mu} (1 + O(n^{-1})) + \frac{\gamma_k \tau^{2\mu}}{(j2\pi n)^{2\mu}} (1 + O(n^{-1})) + \frac{\delta_k \tau^{3\mu}}{(j2\pi n)^{3\mu}} + \frac{\epsilon_k \tau^{4\mu}}{(j2\pi n)^{4\mu}} + o(n^{-4\mu}) \right) r^k \prod_{i=1}^M \left(1 + \sum_{l=1}^{\lfloor \frac{4\mu}{y_i} \rfloor} \frac{(-1)^l \nu_i^l k^l}{l! n^{l y_i}} + o(n^{-4\mu}) \right) = 0$$

and we obtain

$$1 + \sum_{k=1}^N \left(\alpha_k + \frac{\beta_k \tau^\mu}{(j2\pi n)^\mu} (1 + O(n^{-1})) + \frac{\gamma_k \tau^{2\mu}}{(j2\pi n)^{2\mu}} (1 + O(n^{-1})) + \frac{\delta_k \tau^{3\mu}}{(j2\pi n)^{3\mu}} + \frac{\epsilon_k \tau^{4\mu}}{(j2\pi n)^{4\mu}} + o(n^{-4\mu}) \right) r^k \times \left(1 + \sum_{(l_1, \dots, l_M) \in \mathcal{L}(4\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right) k^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} + o(n^{-4\mu}) \right) = 0$$

where $\mathcal{L}(x) := \left\{ (l_1, \dots, l_M) \mid l_i \in \mathbb{Z}_+, \sum_{i=1}^M l_i \geq 1 \text{ and } \sum_{k=1}^M l_i y_i \leq x \right\}$.

After simple computations, we get

$$\frac{d(s_n)}{p(s_n)} = g_1 + g_2 + g_3 + o(n^{-4\mu}) = 0 \quad (4.3)$$

where

$$g_1 = 1 + \sum_{k=1}^N \alpha_k r^k + \frac{\tau^\mu}{(j2\pi n)^\mu} (1 + O(n^{-1})) \sum_{k=1}^N \beta_k r^k + \frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} (1 + O(n^{-1})) \sum_{k=1}^N \gamma_k r^k + \frac{\tau^{3\mu}}{(j2\pi n)^{3\mu}} \sum_{k=1}^N \delta_k r^k + \frac{\tau^{4\mu}}{(j2\pi n)^{4\mu}} \sum_{k=1}^N \epsilon_k r^k, \quad (4.4)$$

$$g_2 = \sum_{(l_1, \dots, l_M) \in \mathcal{L}(4\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \alpha_k r^k k^{\sum_{i=1}^M l_i}, \quad (4.5)$$

and

$$g_3 = \frac{\tau^\mu}{(j2\pi n)^\mu} (1 + O(n^{-1})) \sum_{(l_1, \dots, l_M) \in \mathcal{L}(3\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \beta_k r^k k^{\sum_{i=1}^M l_i} + \frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} (1 + O(n^{-1})) \sum_{(l_1, \dots, l_M) \in \mathcal{L}(2\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \gamma_k r^k k^{\sum_{i=1}^M l_i} + \frac{\tau^{3\mu}}{(j2\pi n)^{3\mu}} \sum_{(l_1, \dots, l_M) \in \mathcal{L}(\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \delta_k r^k k^{\sum_{i=1}^M l_i}. \quad (4.6)$$

In fact, this decomposition into three terms g_1 , g_2 , and g_3 is convenient as in each term it is easy to find the highest order of the development.

Note that for example in g_1 , the highest order in n is $-\mu$ if $\sum_{k=1}^N \beta_k r^k \neq 0$. It is -2μ if $\sum_{k=1}^N \beta_k r^k = 0$ and $\sum_{k=1}^N \gamma_k r^k \neq 0$. To find the highest order in n for g_2 , note that $\sum_{k=1}^N \alpha_k r^k k^{\sum_{i=1}^M l_i} = 0$ if $\sum_{i=1}^M l_i < m$ and it is non-zero if $\sum_{i=1}^M l_i = m$ (see Lemma 2.59). In this case we have $\sum_{i=1}^M l_i y_i \geq m y_1$ and the highest order in n is $-m y_1$. Hence, the highest order of the sum ($g_1 + g_2 + g_3$) may be a function of y_1 .

As $g_1 + g_2 + g_3 + o(n^{-4\mu}) = 0$, the term of highest order of the sum ($g_1 + g_2 + g_3$) is then zero. As we will see in the sequel, this allows us to derive y_1 and ν_1 .

We have already seen that an important role is played by the coefficients $\sum_{k=1}^N \alpha_k r^k$, $\sum_{k=1}^N \beta_k r^k$, \dots . In the following sections, we will derive y_1 and ν_1 for classes of systems which may have some of these coefficients vanishing.

We start with the case of single chains, i.e. $m = 1$, for which the analysis for systems with vanishing or non vanishing coefficients does not differ too much. The analysis in the case of multiple chains, i.e. $m \geq 2$, needs in each case (vanishing or non vanishing coefficients) an appropriate development to get y_1 and ν_1 from the highest order of development in the terms g_1 , g_2 , and g_3 .

4.3 Single chains of poles

To complete the presentation and to facilitate the comparison between the previous and new results, we recall the results presented in (Fioravanti et al., 2010) in the next subsection.

4.3.1 The case where $\sum_{k=1}^N \beta_k r^k \neq 0$

First, the following theorem gives a more precise approximation of roots of large modulus of the characteristic equation than that given in (2.5).

Theorem 4.2. *Let $G(s)$ be a fractional neutral delay system defined by (4.1) and suppose that at least one root of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity one. If such a root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k \neq 0, \quad (4.7)$$

then for large enough $n \in \mathbb{Z}$ poles of the neutral chain relative to r are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-\mu})$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \frac{\tau^\mu \sum_{k=1}^N \beta_k r^k}{(j2\pi n)^\mu \sum_{k=1}^N k \alpha_k r^k}. \quad (4.8)$$

Proof. Under the condition (4.7), the highest order in n of g_1 is $-\mu$. Obviously, the highest order of g_3 is smaller than that of g_2 , which is $-y_1$. Therefore, $y_1 = \mu$ and (4.3) can be rewritten as

$$\frac{\tau^\mu}{(j2\pi n)^\mu} \sum_{k=1}^N \beta_k r^k - \frac{\nu_1}{n^{y_1}} \sum_{k=1}^N k \alpha_k r^k + o(n^{-\mu}) = 0,$$

which completes the proof. \square

Given a more precise approximation of neutral poles of large modulus as above, now our interest is mainly on which side of the vertical line the poles are, in other words, to find out the sign of $\Re(\nu_{n,1})$ for n sufficiently large. This question is particularly important when the asymptotic axis is the imaginary axis.

Recall from Remark 4.1 that for a chain relative to a root of $\tilde{c}_d(s)$, its poles in the upper and lower half-planes may lie on different sides of the asymptotic axis. Fortunately, for the case considered in this subsection, these two parts of the chain may be on the same side. This behavior is characterized in the next two corollaries.

Corollary 4.3. *Let $0 < \mu < 1$, $\nu_{n,1}$ be given by (4.8) and let us denote*

$$K_r = \frac{\sum_{k=1}^N \beta_k r^k}{\sum_{k=1}^N k \alpha_k r^k}. \quad (4.9)$$

Then $\Re(\nu_{n,1}) < 0$ for all $n \in \mathbb{Z}$ if and only if

$$\Re(K_r) < -\tan\left(\frac{\mu\pi}{2}\right) |\Im(K_r)|. \quad (4.10)$$

Proof. Besides K_r , the only term of interest is $J = (jn)^{-\mu}$, as $\text{sgn}(\Re(\nu_{n,1})) = \text{sgn}(\Re(JK_r))$. Since n can be both positive or negative, this term is given by

$$J = \frac{1}{|n|^\mu} \left(\cos\left(\frac{\mu\pi}{2}\right) \mp j \sin\left(\frac{\mu\pi}{2}\right) \right).$$

Multiplying J by K_r and getting its real part leads to

$$\Re(JK_r) = \frac{1}{|n|^\mu} \left(\cos\left(\frac{\mu\pi}{2}\right) \Re(K_r) \pm \sin\left(\frac{\mu\pi}{2}\right) \Im(K_r) \right)$$

from which (4.10) follows from the fact that $0 < \mu < 1$. \square

Some remarks can be drawn from this corollary. First, the condition (4.10) does not depend on the delay τ . This means that for all $\tau > 0$ the chain of poles does not change side with respect to the vertical line in question when the delay varies. Second, the condition (4.10) still holds if r is replaced by its complex conjugate \bar{r} , which is also a root of the formal polynomial $\tilde{c}_d(z)$. Therefore, the chain relative to \bar{r} lies on the same side as the one relative to r .

As K_r is independent of μ , we can reformulate the previous corollary to give the critical value of μ as follows.

Corollary 4.4. *Let $0 < \mu < 1$, $\nu_{n,1}$ be given by (4.8) and its associated K_r by (4.9). Then, if $\Re(K_r) < 0$, all poles of the respective chain asymptotic to the vertical line $\Re(s) = -\ln(|r|)/\tau$ will be on the left of this line if*

$$\mu < \frac{2}{\pi} \arctan \left(-\frac{\Re(K_r)}{|\Im(K_r)|} \right). \quad (4.11)$$

Proof. This follows directly from Corollary 4.3. \square

In the case where $\Re(\nu_{n,1}) = 0$, further analysis is needed to determine the location of poles. However, the procedure is similar to the one given in (Bonnet et al., 2011) and therefore will be omitted.

Now, we are interested in answering the question of stability of $G(s)$ in the H_∞ -sense.

For systems without chains of poles asymptotic to the left of the imaginary axis, the stability can be concluded if there is no poles in the closed right half-plane.

On the other hand, if there exist neutral chains of poles approaching the imaginary axis from the left, we may have to consider the magnitude of the transfer function on the axis in order to answer the question of H_∞ -stability. This is the objective of the proposition below.

Recall that we refer to poles in the closed right half-plane $\overline{\mathbb{C}_+}$ as *unstable poles*.

Proposition 4.5. *Let $G(s)$ be a transfer function given as in (4.1) and suppose that the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has at least one simple root of modulus one, the other roots being of modulus strictly greater than one. We also suppose that every root of modulus one which is denoted by r satisfies (4.7).*

1. *Suppose that $\Re(\nu_{n,1}) < 0$ for all r and that G has no unstable pole of small modulus (which could exist only in a finite number), then G is H_∞ -stable if and only if $\deg p \geq \deg t + 1$.*
2. *If there exists a root r for which $\Re(\nu_{n,1}) = 0$, then the condition $\deg p \geq \deg t + 1$ is necessary for H_∞ -stability.*

Proof. Let $s = s_n + \eta \in j\mathbb{R}$, we have

$$\begin{aligned} \left| p(s) + \sum_{k=1}^N q_k(s) e^{-ks\tau} \right| &\approx |\eta| \left| p'(s_n) + \sum_{k=1}^N (q'_k(s_n) - k\tau q_k(s_n)) e^{-ks_n\tau} \right| \\ &\approx |\eta| |p(s_n)| \left| \frac{p'(s_n)}{p(s_n)} + \sum_{k=1}^N \left(\frac{q'_k(s_n)}{p(s_n)} - k\tau \frac{q_k(s_n)}{p(s_n)} \right) e^{-ks_n\tau} \right| \\ &\approx \tau |\eta| |p(s_n)| \left| \sum_{k=1}^N k\alpha_k r^k \right| \end{aligned}$$

as $n \rightarrow \infty$, $n \in \mathbb{Z}$.

Recall that $\sum_{k=1}^N k\alpha_k r^k$ is non zero by assumption.

If $\Re(\nu_{n,1}) \neq 0$, then η is at least of order $n^{-\mu}$ and a necessary and sufficient condition of H_∞ -stability is that $\deg p \geq \deg t + 1$. If $\Re(\nu_{n,1}) = 0$ the condition is still necessary. \square

The results of this subsection are illustrated later by Example 4.1 in Section 4.5.

In the next section, the same stability analysis will be realized for other cases of systems with single chains of poles, thus completing the analysis for this class of systems.

4.3.2 Other cases

Returning to the approximation of the characteristic equation around poles of a single chain, we see that the terms of highest order are only constituted from those of g_1 and g_2 given in (4.4) and (4.5) respectively since the highest order of g_3 (4.6) is smaller than that of g_2 . While that term of g_2 remains the same, i.e. $(-\nu_{n,1} \sum_{k=1}^N k\alpha_k r^k)$, for all cases of single chains, that of g_1 is decided by its non-zero terms. Some examples are cases where

- $\sum_{k=1}^N \beta_k r^k = 0$ and $\sum_{k=1}^N \gamma_k r^k \neq 0$ (the term of highest order of g_1 is $\tau^{2\mu} \sum_{k=1}^N \gamma_k r^k / (j2\pi n)^{2\mu}$),
- $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N \gamma_k r^k = 0$, and $\sum_{k=1}^N \delta_k r^k \neq 0$ (the term of highest order of g_1 is $\tau^{3\mu} \sum_{k=1}^N \delta_k r^k / (j2\pi n)^{3\mu}$),
- and so on.

Similarly, we easily obtain $\nu_{n,1}$ for the cases above by noting that in the development of the characteristic equation around a pole the coefficient of the highest order is zero. In general,

$$\nu_{n,1} = \left(\frac{\tau}{j2\pi n} \right)^{x_r \mu} K_r \quad (4.12)$$

with K_r a function in r and the coefficients α_k, β_k, \dots in (2.3), and $x_r \in \mathbb{N}$. Note that we get a value of x_r for each root r of multiplicity one of $\tilde{c}_d(z)$ (2.4), where comes the subscript. For example,

- if r satisfies the first case above, i.e. $\sum_{k=1}^N \beta_k r^k = 0$ and $\sum_{k=1}^N \gamma_k r^k \neq 0$, then $x_r = 2$ and $K_r = \sum_{k=1}^N \gamma_k r^k / (\sum_{k=1}^N k\alpha_k r^k)$;
- if r satisfies the second case, i.e. $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N \gamma_k r^k = 0$, and $\sum_{k=1}^N \delta_k r^k \neq 0$, then $x_r = 3$ and $K_r = \sum_{k=1}^N \delta_k r^k / (\sum_{k=1}^N k\alpha_k r^k)$.

Now, as in the previous subsection, we can realize a similar analysis about the location of the chain of poles relative to r . Here, $\text{sgn}(\Re(\nu_{n,1})) = \text{sgn}(\Re(j^{-x_r \mu} K_r))$. Therefore, $\Re(\nu_{n,1}) < 0$ for all $n \in \mathbb{Z}$, i.e. the chain is on the left of the asymptotic axis, if and only if

$$\cos\left(\frac{x_r \mu \pi}{2}\right) \Re(K_r) < -\left| \sin\left(\frac{x_r \mu \pi}{2}\right) \Im(K_r) \right|. \quad (4.13)$$

However, (4.13) does not hold for $x_r \mu = 2k + 1$ for $k \in \mathbb{N}$, $0 < \mu < 1$. In that case, we will be in the same situation as the case $\mu = 1$ (Bonnet et al., 2011), where either

$\Re(\nu_{n,1}) = 0$ for all $n \in \mathbb{Z}$ (when $\Im(K_r) = 0$) or $\Re(\nu_{n,1}) = \pm c \neq 0$ for n positive/negative (when $\Im(K_r) \neq 0$), meaning that respectively more approximation terms are needed or we conclude to have unstable poles.

From all the above analyses about the location of poles of large modulus, the H_∞ -stability condition for systems with single chains asymptotic to the imaginary axis can be restated as follows.

Proposition 4.6. *Let $G(s)$ be a transfer function given as (4.1) and suppose that the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has at least one simple root of modulus one, denoted r , the other roots being of modulus strictly greater than one.*

1. *Suppose that $\Re(\nu_{n,1}) < 0$ for all r and that G has no unstable pole of small modulus, then G is H_∞ -stable if and only if $\deg p \geq \deg t + \max_r \{x_r\}$, where, for each r , $-x_r$ is the order in n of $\nu_{n,1}$.*
2. *If $\Re(\nu_{n,1}) = 0$ for any r , then the condition $\deg p \geq \deg t + \max_r \{x_r\}$ is necessary for H_∞ -stability.*

Proof. As in the proof of Proposition 4.5, let us consider the numerator of $G(s)$ at s on the imaginary axis near poles of the neutral chain relative a root r . Let $s = s_n + \eta \in j\mathbb{R}$, we also have

$$\left| p(s) + \sum_{k=1}^N q_k(s) e^{-ks\tau} \right| \approx \tau |\eta| |p(s_n)| \left| \sum_{k=1}^N k \alpha_k r^k \right|$$

as $n \rightarrow \infty$, $n \in \mathbb{Z}$.

Here, if $\Re(\nu_{n,1}) \neq 0$, then η is at least of order $n^{-x_r\mu}$. □

4.4 Multiple chains of poles

While the stability analysis of single chains under different conditions results in similar conclusions, the stability of multiple chains differs significantly from case to case.

In this section, we do not aim for a complete analysis of general cases but for a large class of systems. This analysis reveals interesting different behaviors.

4.4.1 The case where $m \geq 2$ and $\sum_{k=1}^N \beta_k r^k \neq 0$

Under the same condition, an exhaustive H_∞ -stability analysis for neutral chains relative to roots of multiplicity one of (2.4) has been conducted in Subsection 4.3.1. In this section, multiple chains will be studied. The first step is also to approximate the pole location. However, the analysis based on this approximation ends shortly.

Theorem 4.7. *Let $G(s)$ be a fractional neutral delay system defined by (4.1), and suppose that at least one root of the formal polynomial $\tilde{c}_d(z)$ defined by (2.4) has multiplicity $m > 1$. If for such a root, denoted by r , the condition (4.7) is satisfied, then for large*

enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-\mu/m}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-\mu/m}, \quad (4.14)$$

where

$$\nu_1^m = (-1)^{m+1} \frac{m! \tau^\mu \sum_{k=1}^N \beta_k r^k}{(j2\pi)^\mu \sum_{k=1}^N k^m \alpha_k r^k}. \quad (4.15)$$

Proof. Because of the condition (4.7), the highest order in n of g_1 is $-\mu$, which is obviously higher than that of g_3 . Therefore, in order to vanish the highest order of $d(s_n)/p(s_n)$, those of g_1 and g_2 must be equal. Recall that the highest order of g_2 is $-my_1$. Then $y_1 = \mu/m$ and from (4.3), we obtain

$$\frac{\tau^\mu}{(j2\pi n)^\mu} \sum_{k=1}^N \beta_k r^k + \frac{(-1)^m}{m! n^{my_1}} \nu_1^m \sum_{k=1}^N k^m \alpha_k r^k + o(n^{-\mu}) = 0,$$

then (4.15) holds, which completes the proof. \square

It is interesting to see that, in this case, the order in n of $\nu_{n,1}$ is no longer a multiple of μ as in the cases of single chains but $-\mu/m$.

Also, note that (4.14) and (4.15) are identical to (4.8) for $m = 1$.

Although approximations of poles of single and multiple chains seem to share a similar form, we will show in the next corollary that they have a different position relative to their asymptotic axis.

Corollary 4.8. *Let $G(s)$ be a fractional neutral delay system defined by (4.1). If a root r of multiplicity $m > 1$ of the formal polynomial \tilde{c}_d defined in (2.4) satisfies (4.7), then there exist neutral chains of poles on both sides of the corresponding asymptotic axis $\Re(s) = -\ln(|r|)/\tau$.*

Proof. Under the assumptions, $\nu_{n,1}$ is given by (4.14) and (4.15) for neutral chains relative to r .

Recall from Remark 4.1 that the location of poles of large modulus around the asymptotic axis is decided by the sign of $\Re(\nu_1)$ in the upper half-plane, i.e. $n > 0$, and by the sign of $(\Re(\nu_1) \cos(\mu\pi/m) + \Im(\nu_1) \sin(\mu\pi/m))$ in the lower half-plane, i.e. $n < 0$.

First, we consider $\Re(\nu_1)$. Note that the equation of ν_1^m (4.15) has m distinct roots that are equally distributed on a circle centered at the origin in the complex plane.

If $m \geq 3$, it is obvious that there exist both roots with positive and negative real part.

If $m = 2$, the two roots are symmetric with respect to the origin. Hence, there is always one root with positive real part and the other root with negative real part except for the case of two purely imaginary roots.

In that case, $\Re(\nu_1) = 0$ and $\Im(\nu_1) = \pm c \neq 0$, then $\Re(\nu_1) \cos(\mu\pi/m) + \Im(\nu_1) \sin(\mu\pi/m) = \pm c' \neq 0$ and thus in the lower half-plane there are one chain on the left and one chain on the right of the asymptotic axis. \square

In conclusion, if any multiple root of modulus one of (2.4) satisfies the condition (4.7), then the system is unstable. Clearly, this condition does not depend on τ and μ , with $0 < \mu < 1$.

In the next subsections, we progress in the analysis of the remaining cases and we start in Subsection 4.4.2 with the case of $\sum_{k=1}^N \beta_k r^k = 0$.

4.4.2 The case where $m \geq 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k\beta_k r^k \neq 0$, $\sum_{k=1}^N \gamma_k r^k \neq 0$

In the previous case, all neutral chains relative to the same root r of (2.4) approach the asymptotic axis at the same rate since the corresponding approximation terms have the same order in n . This may no longer occur for the current case as well as for other cases that we will study later.

Theorem 4.9. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that one of the roots of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m > 1$. If this root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k = 0, \quad (4.16)$$

$$\sum_{k=1}^N k\beta_k r^k \neq 0, \quad (4.17)$$

$$\sum_{k=1}^N \gamma_k r^k \neq 0, \quad (4.18)$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $m = 2$, $y_1 = \mu$ and ν_1 satisfies the equation

$$\frac{\nu_1^2}{2} \sum_{k=1}^N k^2 \alpha_k r^k - \frac{\nu_1 \tau^\mu}{(j2\pi)^\mu} \sum_{k=1}^N k \beta_k r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \gamma_k r^k = 0, \quad (4.19)$$

and for $m \geq 3$, (y_1, ν_1) takes m different pair of values below

$$y_1 = \mu, \quad \nu_1 = \frac{\tau^\mu \sum_{k=1}^N \gamma_k r^k}{(j2\pi)^\mu \sum_{k=1}^N k \beta_k r^k}, \quad (4.20)$$

$$y_1 = \frac{\mu}{m-1}, \quad \nu_1^{m-1} = \frac{(-1)^m m! \tau^\mu \sum_{k=1}^N k \beta_k r^k}{(j2\pi)^\mu \sum_{k=1}^N k^m \alpha_k r^k}. \quad (4.21)$$

Proof. From the conditions (4.16)-(4.18), we deduce that the highest orders in n of g_1 , g_2 , and g_3 , which are given by (4.4), (4.5), and (4.6), are -2μ , $-my_1$, and $-\mu - y_1$ respectively.

The following cases may occur in order to eliminate the terms of highest order of the denominator at s_n

$$2\mu = my_1 < \mu + y_1 \quad (4.22)$$

$$2\mu = \mu + y_1 < my_1 \quad (4.23)$$

$$my_1 = \mu + y_1 < 2\mu \quad (4.24)$$

$$my_1 = \mu + y_1 = 2\mu \quad (4.25)$$

The case (4.22) is eliminated as it cannot be satisfied for $m \geq 2$.

The case (4.25) is equivalent to $y_1 = \mu$, $m = 2$ and, from (4.3), we have

$$\frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} \sum_{k=1}^N \gamma_k r^k + \frac{\nu_1^2}{2n^{2\mu}} \sum_{k=1}^N k^2 \alpha_k r^k - \frac{\nu_1 \tau^\mu}{(j2\pi n)^\mu n^\mu} \sum_{k=1}^N k \beta_k r^k + o(n^{-2\mu}) = 0$$

and then (4.19) follows immediately.

When $m > 2$, it is easy to see that both (4.23) and (4.24) are satisfied.

From (4.23), we deduce that $y_1 = \mu$ and thus (4.3) can be rewritten as

$$\frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} \sum_{k=1}^N \gamma_k r^k - \frac{\nu_1 \tau^\mu}{(j2\pi n)^\mu n^\mu} \sum_{k=1}^N k \beta_k r^k + o(n^{-2\mu}) = 0,$$

giving one value ν_1 in (4.20).

Other values of ν_1 are derived from the case (4.24), where $y_1 = \frac{\mu}{m-1}$. In turn, (4.3) becomes

$$\frac{(-1)^m \nu_1^m}{m! n^{my_1}} \sum_{k=1}^N k^m \alpha_k r^k - \frac{\tau^\mu}{(j2\pi n)^\mu} \frac{\nu_1}{n^{y_1}} \sum_{k=1}^N k \beta_k r^k + o(n^{-(\mu+y_1)}) = 0,$$

giving $m - 1$ non-zero values of ν_1 in (4.21). \square

Remark 4.10. A previous version of the above theorem was stated in (Nguyen and Bonnet, 2012). However, the result about $\nu_{n,1}$ for the case $m \geq 3$ was incomplete. Indeed, only the value (4.20) of $\nu_{n,1}$ was given and the others values with different order in n were missing.

As we have seen from (4.20) and (4.21) in the above theorem, due to different order of $\nu_{n,1}$, the chains of poles relative to a multiple root r with $m \geq 3$ approach the asymptotic axis with different rates. An example of such a system is given in Example 4.3 in Section 4.5.

We recognize that ν_1^{m-1} , $m \geq 3$ in (4.21) has the same pattern as ν_1^m , $m \geq 2$ in (4.15), leading to the same conclusion on stability.

Corollary 4.11. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that at least one root r of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m \geq 3$, satisfies (4.16) and (4.17). Then there exist neutral chains of poles on both sides of the asymptotic axis $\Re(s) = -\ln(|r|)/\tau$.*

Proof. The proof is similar to the one of Corollary 4.8. \square

Remark 4.12. The condition (4.18) is omitted in the above corollary. Indeed, whether or not the condition holds does not affect ν_1^{m-1} in (4.21), and thus the existence of a neutral chain on the right of the asymptotic axis. Furthermore, the result in the corollary does not depend on τ and μ , with $0 < \mu < 1$.

Under the conditions in Theorem 4.9, two chains relative to r of multiplicity two may both lie on the left of the asymptotic axis. We will see such a system later in Example 4.3 in Section 4.5. Therefore, the complementary condition to ensure H_∞ -stability of the system in that situation is the objective of the following proposition.

Proposition 4.13. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has at least one root of modulus one of multiplicity two, the other roots being of modulus strictly greater than one. We also suppose that each root of modulus one of $\tilde{c}_d(z)$ satisfies (4.16)-(4.18). If $\Re(\nu_{n,1}) < 0$ and G has no unstable poles of small modulus then G is H_∞ -stable if and only if $\deg p \geq \deg t + 2$.*

Proof. Under the assumptions, all the poles of $G(s)$ are in the open left half-plane. Now, $G(s)$ is H_∞ -stable if and only if $G(s)$ is bounded on the imaginary axis. Therefore, let us consider the magnitude of $G(s)$ on the imaginary axis by first examining its denominator $d(s)$.

Let $s = s_n + \eta_n \in j\mathbb{R}$, where s_n is one of poles of the neutral chain relative to a root r of modulus one and of multiplicity two of $\tilde{c}_d(z)$. Recall that $s_n = (\lambda_n + \nu_1 n^{-\mu})/\tau + o(n^{-\mu})$ and note that $\Re(\lambda_n) = 0$. Since $\Re(\nu_{n,1}) \neq 0$, then η_n is at least of order $n^{-\mu}$. In this case, we can write $\eta_n = \eta n^{-\mu} + o(n^{-\mu})$, and thus $s = [\lambda_n + (\nu_1 + \eta\tau)n^{-\mu}]/\tau + o(n^{-\mu})$, which is of the same form as s_n recalled earlier if we replace $\nu_1' = \nu_1 + \eta\tau$.

Therefore, the developments of the denominator of G around s and s_n are the same. Note that the development of $d(s_n)$ as $|s_n| \rightarrow \infty$ is obtained from (4.3) by collecting terms of highest order of g_1, g_2, g_3 as follows

$$d(s_n) = p(s_n) \left(\frac{f(\nu_1)}{n^{2\mu}} + o(n^{-2\mu}) \right)$$

where $f(\nu_1)$ is the left expression of (4.19). Similarly, $d(s)$ as $|s| \rightarrow \infty$, $s \in j\mathbb{R}$ near s_n is given by

$$d(s) = p(s) \left(\frac{f(\nu_1 + \eta\tau)}{n^{2\mu}} + o(n^{-2\mu}) \right)$$

Now, we will prove that $f(\nu_1 + \eta\tau) \neq 0$. Let us denote $\nu_1^{(1)}$ and $\nu_1^{(2)}$ two roots of $f(\nu_1)$ and first consider $f(\nu_1^{(1)} + \eta\tau)$. We see that $f(\nu_1^{(1)} + \eta\tau) = 0$ if and only if $\nu_1^{(1)} + \eta\tau = \nu_1^{(2)}$,

which is in turn equivalent to

$$\eta = \frac{\nu_1^{(2)} - \nu_1^{(1)}}{\tau}. \quad (4.26)$$

However, this condition cannot be satisfied because

$$\Re(\eta) \neq \frac{\Re(\nu_1^{(2)}) - \Re(\nu_1^{(1)})}{\tau}. \quad (4.27)$$

Indeed, $\Re(\eta) = -\Re(\nu_1^{(1)})/\tau$ since $s \in j\mathbb{R}$ and $\Re(\nu_2^{(2)}) \neq 0$ under the assumption $\Re(\nu_{n,1}) < 0$. Therefore, $f(\nu_1^{(1)} + \eta\tau) \neq 0$. Similarly, we can prove that $f(\nu_1^{(2)} + \eta\tau) \neq 0$. Hence, the order in n^μ of the denominator of $G(s)$ is $\deg p - 2$. \square

While systems considered in the previous subsection were all unstable, we have been able to find in this subsection systems with multiple chains asymptotic to the imaginary axis which are H_∞ -stable. We will then continue our analysis for other cases in order to see in which situation there may exist H_∞ -stable systems.

4.4.3 The case where $m \geq 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k\beta_k r^k = 0$, $\sum_{k=1}^N k^2\beta_k r^k \neq 0$, and $\sum_{k=1}^N \gamma_k r^k \neq 0$

As for the previous cases, pole location is considered first.

Theorem 4.14. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that one of the roots of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m > 1$. If this root, denoted by r , satisfies*

$$\begin{aligned} \sum_{k=1}^N \beta_k r^k &= 0, \\ \sum_{k=1}^N k\beta_k r^k &= 0, \\ \sum_{k=1}^N k^2\beta_k r^k &\neq 0, \end{aligned} \quad (4.28)$$

$$\sum_{k=1}^N \gamma_k r^k \neq 0, \quad (4.29)$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n\tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $2 \leq m \leq 3$, $y_1 = 2\mu/m$ and

$$\nu_1^m = \frac{(-1)^{m+1} m! \tau^{2\mu} \sum_{k=1}^N \gamma_k r^k}{(j2\pi)^{2\mu} \sum_{k=1}^N k^m \alpha_k r^k}, \quad (4.30)$$

for $m = 4$, $y_1 = \mu/2$ and ν_1 satisfies

$$\frac{\nu_1^4}{4!} \sum_{k=1}^N k^4 \alpha_k r^k + \frac{\nu_1^2 \tau^\mu}{2(j2\pi)^\mu} \sum_{k=1}^N k^2 \beta_k r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \gamma_k r^k = 0, \quad (4.31)$$

and for $m \geq 5$, (y_1, ν_1) takes one of m different pairs of values

$$y_1 = \frac{\mu}{2}, \quad \nu_1^2 = -\frac{2\tau^\mu \sum_{k=1}^N \gamma_k r^k}{(j2\pi)^\mu \sum_{k=1}^N k^2 \beta_k r^k}, \quad (4.32)$$

$$y_1 = \frac{\mu}{m-2}, \quad \nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau^\mu \sum_{k=1}^N k^2 \beta_k r^k}{2(j2\pi)^\mu \sum_{k=1}^N k^m \alpha_k r^k}. \quad (4.33)$$

Proof. Under the assumptions, the terms of highest order in n of g_1 , g_2 , and g_3 , which are given by (4.4), (4.5), and (4.6), are -2μ , $-my_1$, and $\max\{-\mu-2y_1, -2\mu-y_1\}$ respectively. Obviously, we just need to compare the first three orders as $-2\mu > -2\mu - y_1$.

To determine y_1 and ν_1 , we proceed similarly to the proof of Theorem 4.9.

The following cases may occur for the highest order of the development of the denominator at s_n

$$\begin{aligned} 2\mu &= my_1 < \mu + 2y_1, \\ 2\mu &= \mu + 2y_1 < my_1, \\ my_1 &= \mu + 2y_1 < 2\mu, \\ 2\mu &= my_1 = \mu + 2y_1. \end{aligned}$$

These cases are respectively equivalent to

$$\begin{aligned} y_1 &= 2\mu/m \quad \text{and} \quad m < 4, \\ y_1 &= \mu/2 \quad \text{and} \quad m > 4, \\ y_1 &= \mu/(m-2) \quad \text{and} \quad m > 4, \\ y_1 &= \mu/2 \quad \text{and} \quad m = 4. \end{aligned}$$

Hence, from (4.3) we obtain respectively

$$\begin{aligned} \frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} \sum_{k=1}^N \gamma_k r^k + \frac{(-1)^m \nu_1^m}{m! n^{my_1}} \sum_{k=1}^N \alpha_k k^m r^k + o(n^{-2\mu}) &= 0, \\ \frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} \sum_{k=1}^N \gamma_k r^k + \frac{\tau^\mu}{(j2\pi n)^\mu} \frac{\nu_1^2}{2! n^{2y_1}} \sum_{k=1}^N \beta_k k^2 r^k + o(n^{-2\mu}) &= 0, \\ \frac{(-1)^m \nu_1^m}{m! n^{my_1}} \sum_{k=1}^N \alpha_k k^m r^k + \frac{\tau^\mu}{(j2\pi n)^\mu} \frac{\nu_1^2}{2! n^{2y_1}} \sum_{k=1}^N \beta_k k^2 r^k + o(n^{-my_1}) &= 0 \\ \frac{\tau^{2\mu}}{(j2\pi n)^{2\mu}} \sum_{k=1}^N \gamma_k r^k + \frac{\nu_1^4}{4! n^{4y_1}} \sum_{k=1}^N \alpha_k k^4 r^k + \frac{\tau^\mu}{(j2\pi n)^\mu} \frac{\nu_1^2}{2! n^{2y_1}} \sum_{k=1}^N \beta_k k^2 r^k + o(n^{-2\mu}) &= 0, \end{aligned}$$

which lead to (4.30), (4.32), (4.33), and (4.31) respectively. \square

Remark 4.15. Note that in the case $2 \leq m \leq 3$, condition (4.28) is not necessary and in the case $m \geq 5$ condition (4.29) is not necessary (and we may conclude as well on the presence of chains of poles in the right half-plane).

Some quick observation leads to the following conclusions on the stability of the system in the current case.

Corollary 4.16. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that at least one root of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m \geq 2$, satisfies $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k \beta_k r^k = 0$ and*

- for $2 \leq m \leq 3$, $\sum_{k=1}^N \gamma_k r^k \neq 0$,
- for $m = 4$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$ and $\sum_{k=1}^N \gamma_k r^k \neq 0$
- for $m \geq 5$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$.

Then there exist neutral chains of poles on both sides of the asymptotic axis $\Re(s) = -\ln(|r|)/\tau$.

Proof. For $2 \leq m \leq 3$ and $m \geq 5$, the proof is similar to that of Corollary 4.8.

Now, we consider the case of $m = 4$. By replacing $\nu_1^2 = x$ in (4.31), we obtain

$$\frac{x^2}{4!} \sum_{k=1}^N k^4 \alpha_k r^k + \frac{x \tau^\mu}{2(j2\pi)^\mu} \sum_{k=1}^N k^2 \beta_k r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \gamma_k r^k = 0$$

Let us denote x_1 and x_2 the two roots of the above equation. Equation (4.31) has at least one value of ν_1 with positive real part except the case where both roots x_1, x_2 are negative. However, we will demonstrate that this case does not exist.

The two roots of the equation satisfy

$$x_1 + x_2 = -\frac{12\tau^\mu \sum_{k=1}^N k^2 \beta_k r^k}{(j2\pi)^\mu \sum_{k=1}^N k^4 \alpha_k r^k} = -\frac{12\tau^\mu}{(j2\pi)^\mu} K_r$$

where $K_r = \sum_{k=1}^N k^2 \beta_k r^k / \sum_{k=1}^N k^4 \alpha_k r^k$.

We consider $r \in \mathbb{R}$ and $r \in \mathbb{C} \setminus \mathbb{R}$.

If r is real, then $x_1 + x_2$ is not real. Therefore, x_1 and x_2 cannot be both real.

If r is not real, then \bar{r} is also a root of (2.4). Denote x'_1 and x'_2 roots corresponding to \bar{r} . Hence, they satisfy

$$x'_1 + x'_2 = -\frac{12\tau^\mu}{(j2\pi)^\mu} \bar{K}_r.$$

Therefore

$$x_1 + x_2 + x'_1 + x'_2 = -\frac{24\tau^\mu}{(j2\pi)^\mu} \Re(K_r),$$

indicating that x_1, x_2, x'_1 , and x'_2 cannot be all real. \square

Corollary 4.16 shows that if (2.4) possesses a multiple root of modulus one satisfying the conditions in Theorem 4.14 then the system is unstable.

4.4.4 The case where $m \geq 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k \beta_k r^k = 0$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$, $\sum_{k=1}^N \gamma_k r^k = 0$, $\sum_{k=1}^N k \gamma_k r^k \neq 0$, and $\sum_{k=1}^N \delta_k r^k \neq 0$

We continue one more step in this kind of development because this case presents an interesting behavior. Besides systems which may be H_∞ -stable, we will encounter here those whose stability may not be concluded with the first approximation. Such systems have not been seen in any of the previous cases with multiple chains.

As usual, we obtain the approximation of roots of large modulus of the characteristic equation in the next theorem.

Theorem 4.17. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that one of the roots of (2.4) has multiplicity $m > 1$. If this root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k = 0, \quad (4.34)$$

$$\sum_{k=1}^N k \beta_k r^k = 0, \quad (4.35)$$

$$\sum_{k=1}^N k^2 \beta_k r^k \neq 0, \quad (4.36)$$

$$\sum_{k=1}^N \gamma_k r^k = 0, \quad (4.37)$$

$$\sum_{k=1}^N k \gamma_k r^k \neq 0, \quad (4.38)$$

$$\sum_{k=1}^N \delta_k r^k \neq 0, \quad (4.39)$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $m = 2$, $y_1 = 3\mu/2$ and

$$\nu_1^2 = \frac{2\tau^{3\mu} \sum_{k=1}^N \delta_k r^k}{(j2\pi)^{3\mu} \sum_{k=1}^N k^2 \alpha_k r^k}, \quad (4.40)$$

for $m = 3$, $y_1 = \mu$ and ν_1 satisfies

$$-\frac{\nu_1^3}{3!} \sum_{k=1}^N k^3 \alpha_k r^k + \frac{\tau^\mu \nu_1^2}{2(j2\pi)^\mu} \sum_{k=1}^N k^2 \beta_k r^k - \frac{\tau^{2\mu} \nu_1}{(j2\pi)^{2\mu}} \sum_{k=1}^N k \gamma_k r^k + \frac{\tau^{3\mu}}{(j2\pi)^{3\mu}} \sum_{k=1}^N \delta_k r^k = 0, \quad (4.41)$$

and for $m \geq 4$, (y_1, ν_1) takes one of m different pairs of values

$$y_1 = \frac{\mu}{m-2}, \quad \nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau^\mu \sum_{k=1}^N k^2 \beta_k r^k}{2(j2\pi)^\mu \sum_{k=1}^N k^m \alpha_k r^k},$$

or $y_1 = \mu$ and ν_1 satisfies

$$\frac{\tau^\mu \nu_1^2}{2(j2\pi)^\mu} \sum_{k=1}^N k^2 \beta_k r^k - \frac{\tau^{2\mu} \nu_1}{(j2\pi)^{2\mu}} \sum_{k=1}^N k \gamma_k r^k + \frac{\tau^{3\mu}}{(j2\pi)^{3\mu}} \sum_{k=1}^N \delta_k r^k = 0.$$

Proof. We deduce that the terms of highest order in n of g_1 , g_2 , and g_3 , which are given by (4.4), (4.5), and (4.6), are -3μ , $-my_1$, and $\max\{-\mu - 2y_1, -2\mu - y_1\}$ respectively. Unlike the previous case where we can actually reduce the number of orders to consider, for this case we cannot omit beforehand any of the two possible orders of g_3 , thus all the four orders above have to be taken into account resulting in more possible cases which may occur among them. However, the same procedure as in the proof of Theorem 4.9 is applied. \square

For $m = 3$, Equation (4.41) may admit all roots with negative real part (see Example 4.4 in Section 4.5). In such systems, the following condition on H_∞ stability may be applied. It is similar to the case in Subsection 4.4.2 and is stated below without proof.

Proposition 4.18. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that (2.4) has at least one root of modulus one of multiplicity three, the other roots being of modulus strictly greater than one. We also suppose that each root of modulus one of (2.4) satisfies (4.34)-(4.39). If $\Re(\nu_{n,1}) \neq 0$ and G has no unstable poles of small modulus then G is H_∞ -stable if and only if $\deg p \geq \deg t + 3$.*

For other multiplicities, except for a special case when $m = 2$, we obtain at least one $\nu_{n,1}$ of positive real part as stated in the next corollary.

Corollary 4.19. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that either of the following conditions is satisfied*

- a root of (2.4) has multiplicity $m \geq 4$ and satisfies (4.34)-(4.39),
- a root of (2.4) has multiplicity $m = 2$ and satisfies (4.34), (4.35), (4.37), and (4.39), and $\mu \neq 2/3$.

Then there exist neutral chains of poles on both sides of the asymptotic axis given by (2.7).

Proof. For $m = 2$ with $\mu \neq 2/3$ and $m \geq 4$, the proof is similar to that of Corollary 4.8. \square

The special case of $m = 2$ and $\mu = 3/2$ is analyzed as follows.

Corollary 4.20. *Let $G(s)$ be a neutral delay system defined by (4.1), and suppose that $\mu = 2/3$. If a root of (2.4) has multiplicity $m = 2$ and satisfies (4.34), (4.35), (4.37), and (4.39), then for this root either $\Re(\nu_{n,1}) = 0$ for all $n \in \mathbb{Z}$ or $\Re(\nu_{n,1}) = \pm c/n$ with $c \neq 0$.*

Proof. With $\mu = 2/3$, then $\nu_{n,1} = \nu_1/n$ and (4.40) becomes

$$\nu_1^2 = -\frac{\tau^2 \sum_{k=1}^N \delta_k r^k}{2\pi^2 \sum_{k=1}^N k^2 \alpha_k r^k}. \quad (4.42)$$

If $\sum_{k=1}^N \delta_k r^k / \sum_{k=1}^N k^2 \alpha_k r^k > 0$, then the two values of ν_1 are purely imaginary. In this case, from Remark 4.1 we have $\Re(\nu_{n,1})_{n < 0} = 0$.

Otherwise, $\Re(\nu_1) = \pm c \neq 0$. □

In the case above, in order to determine the location of the corresponding chain of poles we need to continue the approximation to at least $\nu_{n,2}$.

Remark 4.21. If $m = 2$, then we obtain $\nu_{n,1} = \nu_1/n$ with ν_1 given by (4.40) independently of the conditions (4.36) and (4.38).

4.5 Examples

Example 4.1. (Subsection 4.3.1)

First, let us consider the system with the transfer function given by

$$G_1(s) = \frac{s^{0.5} + 1}{s + (-1.9s + s^{0.5})e^{-s} + (s - s^{0.5} + 0.3)e^{-2s}}.$$

For this system, the fractional order is $\mu = 0.5$ and the delay is $\tau = 1$. It is easy to see that the coefficients of the development $q_k(s)/p(s)$ are $\alpha_1 = -1.9$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = -1$, and thus the formal polynomial is $\tilde{c}_d(z) = 1 - 1.9z + z^2$, which has two complex conjugate roots $r = (19 \pm j\sqrt{39})/20$ of multiplicity $m = 1$. Since $|r| = 1$ for each r , then the asymptotic axis defined by $\Re(s) = -\ln(|r|)/\tau$ is the imaginary axis.

As $\sum_{k=1}^2 \beta_k r^k \neq 0$ for both r , Theorem 4.2 is applied and we obtain $\nu_{n,1} = (-0.1636 + j0.1185)/n^{0.5}$ for $r = (19 + j\sqrt{39})/20$ and $\nu_{n,1} = (-0.1185 + j0.1636)/n^{0.5}$ for $r = (19 - j\sqrt{39})/20$. Therefore, in the upper half-plane, i.e. $n > 0$, the two neutral chains of poles are on the left of the imaginary axis. So are the chains in the lower half-plane since poles of $G(s)$ are symmetric about the real axis, which is due to the fact that the denominator of $G(s)$ is a quasi-polynomial with real coefficients.

The same conclusion about the location of neutral poles can be drawn using Corollary 4.4. The critical value of μ is $\mu_c = (2/\pi) \arctan(-\Re(K_r)/|\Im(K_r)|) = 0.8989$ with $K_r = \sum_{k=1}^2 \beta_k r^k / \sum_{k=1}^2 k \alpha_k r^k$. Recall that μ_c is the same for both r . Since $\mu = 0.5 < \mu_c$, then the two neutral chains of poles relative to r are on the left of the imaginary axis as we can see in Figure 4.1a.

In addition, all poles of small modulus of the system are in the open left half-plane. Then Proposition 4.5 shows that G is H_∞ -stable since $\deg t = \deg p - 1$. Indeed, $G(s)$ is bounded on the imaginary axis, which can be seen in Figure 4.2.

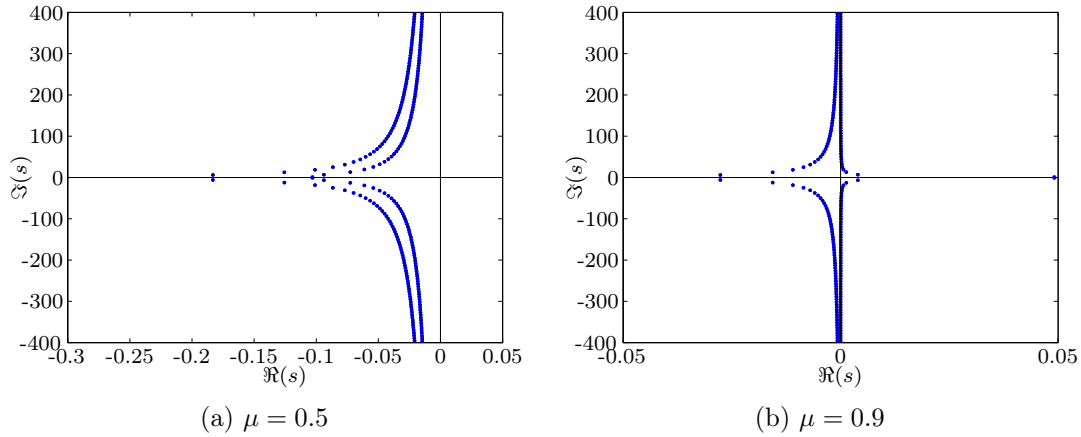


Figure 4.1 – Neutral chains of poles of $G_1(s)$

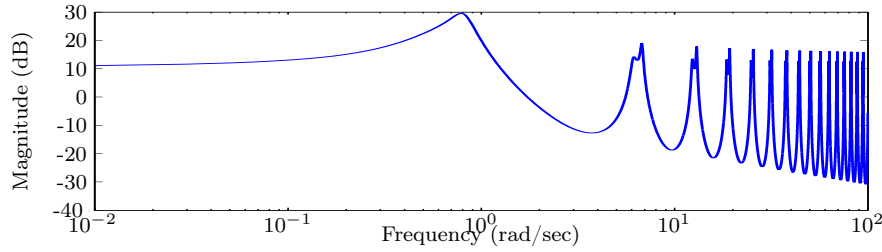


Figure 4.2 – Bode diagram of $G_1(s)$

If $\mu = 0.9 > \mu_c$, then chains of poles go to the right of the imaginary axis (see Figure 4.1b).

Example 4.2. (Subsections 4.4.1 and 4.4.2)

The system is given by

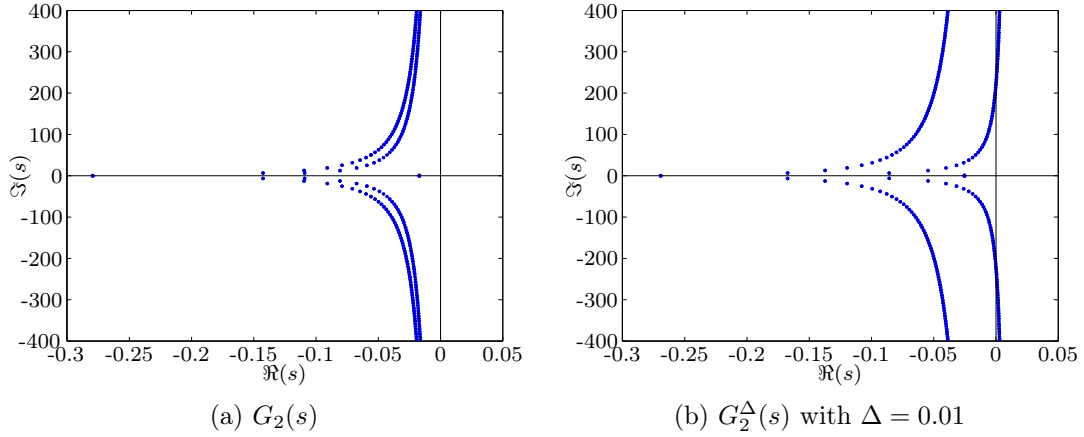
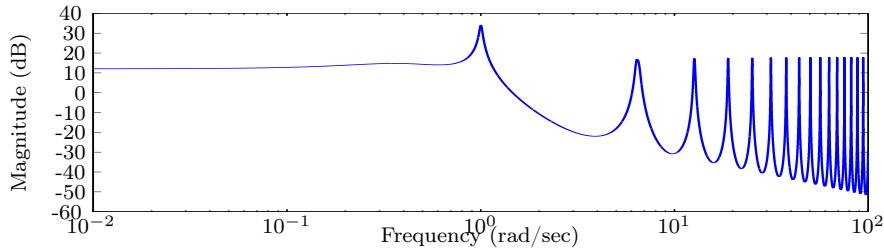
$$G_2(s) = (s + (-2s + s^{0.5} + 0.25)e^{-s} + (s - s^{0.5})e^{-2s})^{-1}.$$

We see that the delay is $\tau = 1$ and the fractional order is $\mu = 0.5$. The polynomial $\tilde{c}_d(z)$ given in (2.4) has root $r = 1$ of multiplicity two, then the system has two chains of poles asymptotic to the imaginary axis. The system satisfies $\sum_{k=1}^2 \beta_k r^k = 0$, $\sum_{k=1}^2 k\beta_k r^k \neq 0$, and $\sum_{k=1}^2 \gamma_k r^k \neq 0$, then Theorem 4.9 is applied. Equation (4.19) has a double root, which gives $\nu_{n,1} = (-0.1410 + 0.1410j)/n^{0.5}$ for $n \rightarrow +\infty$. Therefore, the two neutral chains are on the left of the imaginary axis.

If some parameters of $G_2(s)$ change slightly, the system might fail to satisfy the condition $\sum_{k=1}^2 \beta_k r^k = 0$, and thus is no longer stable due to Corollary 4.8. This remark fits in the following system

$$G_2^\Delta(s) = (s + (-2s + s^{0.5} + 0.25)e^{-s} + (s - (1 + \Delta)s^{0.5})e^{-2s})^{-1}.$$

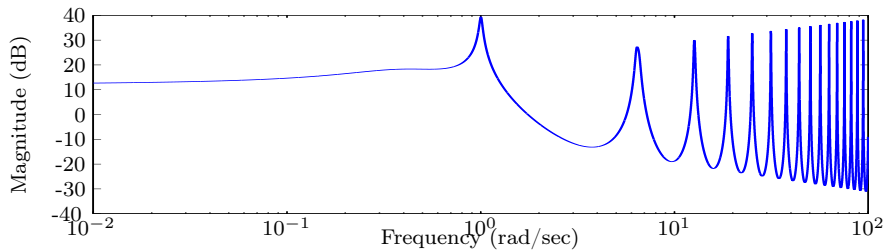
If $\Delta \neq 0$, then $\sum_{k=1}^2 \beta_k r^k \neq 0$, thus Corollary 4.8 states that the system has a chain of poles in the right half-plane.

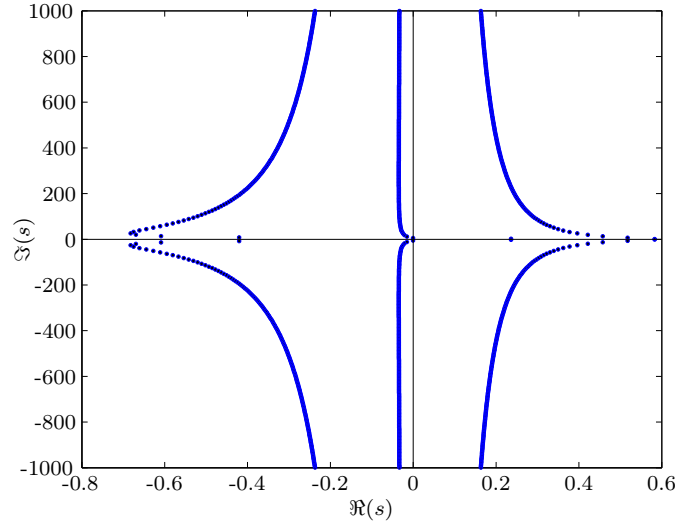
Figure 4.3 – Neutral chains of poles of $G_2(s)$ and $G_2^\Delta(s)$ Figure 4.4 – Bode diagram of $G_2(s)$

We observe the chains of poles of $G_2(s)$ and $G_2^\Delta(s)$ with $\Delta = 0.01$ in Figure 4.3a and 4.3b. The unstable chain of G_2^Δ crosses the imaginary axis from left to right.

Proposition 4.13 shows that $G_2(s)$ is stable in the sense of H_∞ -stability. Indeed, the system does not have unstable poles and is bounded on the imaginary axis (see Figure 4.4). Clearly, the system defined by $(s^{0.5} + 1)G_2(s)$ is unstable since the order of the numerator is too high making the transfer function unbounded on the imaginary axis (see Figure 4.5).

Example 4.3. (Subsection 4.4.2)

Figure 4.5 – Bode diagram of $(s^{0.5} + 1)G_2(s)$

Figure 4.6 – Neutral chains of poles of $G_3(s)$

We consider the system with the transfer function given by

$$G_3(s) = (s^{0.8} + (-3s^{0.8} + 3s^{0.4} + 1)e^{-s} + (3s^{0.8} - 5s^{0.4} + 2)e^{-2s} + (-s^{0.8} + 2s^{0.4} + 3)s^{-3s})^{-1}.$$

Here, we see that $\mu = 0.4$, $\tau = 1$, and $\tilde{c}_d(z) = 1 - 3z + 3z^2 - 1$ with a root $r = 1$ of multiplicity $m = 3$. Therefore, the chains of poles approach the imaginary axis.

Since $\sum_{k=1}^3 \beta_k r^k = 0$, $\sum_{k=1}^3 k\beta_k r^k = -1$, and $\sum_{k=1}^3 \gamma_k r^k = 6$, then Theorem 4.9 is applied. More precisely, since $m = 3$, we obtain from (4.20) and (4.21) three values of $\nu_{n,1}$, which are $(0.2140 + j0.6585)/n^{0.2}$, $(-0.2140 - j0.6585)/n^{0.2}$, and $(-2.3272 + j1.6908)/n^{0.4}$. Therefore, the system has one chain of poles on the right and two chains on the left of the imaginary axis, which are shown in Figure 4.6. It is interesting to note that one chain approaches the imaginary axis faster than the other two, which is due to different orders of $\nu_{n,1}$.

Example 4.4. (Subsections 4.3.1 and 4.4.4)

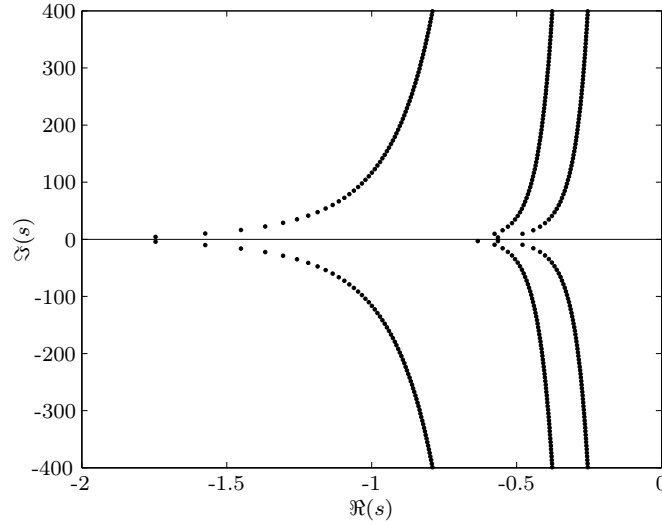
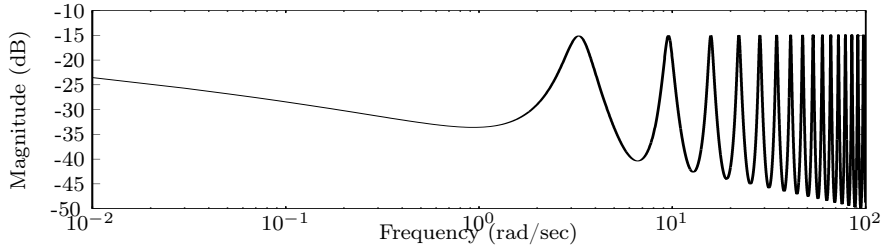
The system is described by the transfer function

$$G_4(s) = \frac{t(s)}{d(s)} \quad (4.43)$$

where the characteristic equation of the system is a product of the characteristic equations of 3 single time-delay systems and is given by

$$\begin{aligned} d(s) &= [(s^{0.2} + 1) + s^{0.2}e^{-s}][(s^{0.2} + 2) + (s^{0.2} - 1)e^{-s}][(s^{0.2} + 3) + (s^{0.2} + 1)e^{-s}] \\ &= s^{0.6} + 6s^{0.4} + 11s^{0.2} + 6 + (3s^{0.6} + 12s^{0.4} + 10s^{0.2} - 1)e^{-s} \\ &\quad + (3s^{0.6} + 6s^{0.4} - 2s^{0.2} - 1)e^{-2s} + (s^{0.6} - s^{0.2})e^{-3s}. \end{aligned}$$

The formal polynomial of this system is $\tilde{c}_d(z) = 1 + 3z + 3z^2 + z^3$ and has $r = -1$ of multiplicity $m = 3$. There are then 3 neutral chains of poles approaching the imaginary

Figure 4.7 – Poles of $G_4(s)$ Figure 4.8 – Bode diagram of $G_4(s)$ with $t(s) = 1$

axis. The conditions in Theorem 4.17 are all satisfied. Therefore, $\nu_{n,1} = \nu_1 n^{-0.2}$ where ν_1 is given by (4.41) and has three values $-0.6585 + 0.2140j$, $-1.3170 + 0.4279j$, and $-1.9756 + 0.6419j$. The upper parts of the chains of poles are then on the left of the imaginary axis and so are the lower parts since poles are symmetric about the real axis.

We obtain the same values of $\nu_{n,1}$ if considering separately each factor of the characteristic equation using the results in Theorem 4.2.

The poles of small modulus are also in the open left half-plane as we can see in Figure 4.7. Therefore, Proposition 4.18 can be applied to determine the necessary and sufficient condition for the system to be H_∞ -stable. The system is H_∞ -stable if and only if $t(s)$ is a constant. Figures 4.8 and 4.9 show the magnitude of the transfer function when $t(s) = 1$ and $t(s) = s^{0.2} + 2$ respectively. The transfer function is bounded in the former case and unbounded in the latter one.

4.6 Conclusion

Fractional delay systems of neutral types where poles approach the imaginary axis is delicate for stability analysis. In this chapter, we answer the stability question in the sense

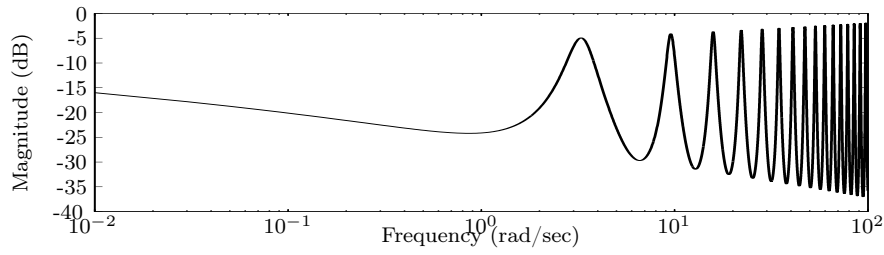


Figure 4.9 – Bode diagram of $G_4(s)$ with $t(s) = s^{0.2} + 2$

of H_∞ -stability for a large class of systems, in particular systems with multiple chains asymptotic to the imaginary axis, and the necessary and sufficient conditions obtained are related not only to the location of poles w.r.t. the imaginary axis but also the relative order between the numerator and the denominator of the transfer function. These results will also be of use to decide on H_∞ -stabilizability of several classes of fractional delay systems by rational or fractional controllers (with delays). The deployed method can be used for other cases which are not examined here. However, it requires time and effort for each particular system.

Chapter 5

Stability analysis of SISO classical neutral systems with commensurate delays

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5.1 Introduction

For the fractional systems considered in the previous chapter, if the fractional order $\mu \in (0, 1)$ is replaced by $\mu = 1$, then they will become classical systems with commensurate delays. The stability of this class of delay systems has been studied intensively (see for instance (Bellman and Cooke, 1963; Richard, 2003; Michiels and Niculescu, 2007) and the references therein). Despite the large literature, the critical case of poles asymptotic to the imaginary axis has not been studied thoroughly and the available results are at the same point as those for fractional systems. More precisely, in the frequency domain (Bonnet et al., 2011) considered the case of single chains and also a particular case of multiple chains where the characteristic equation was a product of characteristic equations of systems with single chains. However the general case of multiple chains has not been addressed. In the time domain, neutral systems with poles approaching the imaginary axis was studied in (Rabah et al., 2012). Sufficient conditions for asymptotic stability were obtained for single chains of poles but the case of multiple chains was still left open.

This chapter aims at extending the work in (Bonnet et al., 2011) to systems of the second case. Based on the similarities with fractional systems, approximations of the pole location for classical systems are obtained from the results in the previous chapter by simply replacing the fractional order $\mu \in (0, 1)$ by $\mu = 1$. Nevertheless, the analysis of these approximations may lead to more conservative conclusions in the case of classical systems. For clarity of presentation, we recall the similar results without proof and only provide proofs for new results.

This chapter has the same organization as the previous chapter. First, we recall the systems to be considered in Section 5.2. The cases of single and multiple chains of poles are studied in Sections 5.3 and 5.4 respectively. Illustrative examples are given in Section 5.5. Finally, in Section 5.6 we conclude the chapter with some final remarks and a discussion about the relation between the results presented here (in the frequency domain) and those in (Rabah et al., 2012) (in the time domain).

5.2 Neutral time-delay systems

We consider neutral systems with commensurate delays whose transfer function is given by (2.2) and is recalled here.

$$G(s) = \frac{t(s)}{p(s) + \sum_{k=1}^N q_k(s)e^{-ks\tau}} = \frac{t(s)}{d(s)}, \quad (5.1)$$

where $\tau > 0$ is the delay, t , p , and q_k for all $k \in \mathbb{N}_N$ are real polynomials, $\deg p \geq \deg t$, $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$ in order to deal with proper neutral systems.

We refer the reader to Subsection 2.4.2 for some basic facts regarding these systems.

Recall that we denote s_n a pole of $G(s)$ relative to a root r of the formal polynomial $\tilde{c}_d(z)$ defined by (2.4) and assume that s_n has the form

$$s_n \tau = \lambda_n + \nu_{n,1} + \nu_{n,2} + \dots + \nu_{n,M} + o(n^{-4}) \quad (5.2)$$

with

$$\lambda_n = -\ln(r) + j2\pi n$$

and

$$\nu_{n,i} = \frac{\nu_i}{n^{y_i}}$$

for $i = 1, \dots, M$ where $\nu_i \neq 0$ and $0 < y_1 < \dots < y_M \leq 4$.

In the next sections, we will first be interested in determining $\nu_{n,1}$ in different cases since the sign of $\Re(\nu_{n,1})$ indicates in which side of the asymptotic axis the poles are. If knowing $\nu_{n,1}$ is not enough to know the location of poles around the asymptotic axis, we will proceed to determine $\nu_{n,2}$. For that purpose we develop $d(s)/p(s)$ at s_n of large modulus as follows

$$\frac{d(s_n)}{p(s_n)} = g_1 + g_2 + g_3 + o(n^{-4}) = 0 \quad (5.3)$$

where

$$\begin{aligned} g_1 = & 1 + \sum_{k=1}^N \alpha_k r^k + \frac{\tau}{j2\pi n} (1 + O(n^{-1})) \sum_{k=1}^N \beta_k r^k + \frac{\tau^2}{(j2\pi n)^2} (1 + O(n^{-1})) \sum_{k=1}^N \gamma_k r^k \\ & + \frac{\tau^3}{(j2\pi n)^3} \sum_{k=1}^N \delta_k r^k + \frac{\tau^4}{(j2\pi n)^4} \sum_{k=1}^N \epsilon_k r^k, \end{aligned} \quad (5.4)$$

$$g_2 = \sum_{(l_1, \dots, l_M) \in \mathcal{L}(4)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \alpha_k r^k k^{\sum_{i=1}^M l_i}, \quad (5.5)$$

and

$$\begin{aligned} g_3 = & \frac{\tau}{j2\pi n} (1 + O(n^{-1})) \sum_{(l_1, \dots, l_M) \in \mathcal{L}(3)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \beta_k r^k k^{\sum_{i=1}^M l_i} \\ & + \frac{\tau^2}{(j2\pi n)^2} (1 + O(n^{-1})) \sum_{(l_1, \dots, l_M) \in \mathcal{L}(2)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \gamma_k r^k k^{\sum_{i=1}^M l_i} \\ & + \frac{\tau^3}{(j2\pi n)^3} \sum_{(l_1, \dots, l_M) \in \mathcal{L}(1)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \delta_k r^k k^{\sum_{i=1}^M l_i}. \end{aligned} \quad (5.6)$$

5.3 Single chains of poles

The chain of poles corresponding to a single root of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) is a single chain.

In this section, we determine the relative location of single chains of poles w.r.t. their asymptotic axes and derive necessary and sufficient conditions for the system to be H_∞ -stable when it is applicable.

5.3.1 The case where $\sum_{k=1}^N \beta_k r^k \neq 0$

This subsection recalls the results obtained in (Bonnet et al., 2011) but presenting them in the framework adopted here (see (5.2)) which indeed allows us to extend some of them in the next subsection.

The next theorem, which approximate poles in a neutral chain, can be obtained from Theorem 4.2 by changing the value of μ to one.

Theorem 5.1. *Let $G(s)$ be a neutral delay system defined by (5.1) and suppose that at least one root of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity one. If such a root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k \neq 0, \quad (5.7)$$

then for large enough $n \in \mathbb{Z}$ poles of the neutral chain relative to r are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-1})$$

where λ_n is given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-1}$$

with

$$\nu_1 = \frac{\tau \sum_{k=1}^N \beta_k r^k}{j2\pi \sum_{k=1}^N k \alpha_k r^k}. \quad (5.8)$$

The sign of the real part of the above approximation is analyzed in the following corollary.

Corollary 5.2. *Let $G(s)$ be a neutral delay system defined by (5.1). Suppose that r is a root of (2.4) of multiplicity one and satisfies (5.7). Then the values of ν_1 relative to r and \bar{r} have either $\Re(\nu_1) = 0$ or $\Re(\nu_1) = \pm c \neq 0$.*

Remark 5.3. For the classical systems considered in this subsection, in contrast with the case of fractional neutral systems, the sign of $\nu_{n,1}$ is not sufficient to detect a chain of poles lying on the left of the asymptotic axis.

When $\Re(\nu_1) = 0$, the next approximation may be needed.

Theorem 5.4. *Let $G(s)$ be a neutral delay system defined by (5.1). Then the neutral poles corresponding to each root r of multiplicity one satisfying (5.7) are approximated by*

$$s_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n^{y_1}} + \frac{\nu_2}{n^{y_2}} \right) + o(n^{-y_2})$$

where λ_n is given by (2.6), ν_1 is given by (5.8), $y_1 = 1$, $y_2 = 2$ and ν_2 is given by

$$\nu_2 = \frac{\tau \ln(r) \sum_{k=1}^N \beta_k r^k + \tau^2 \sum_{k=1}^N \gamma_k r^k - 2\pi^2 \nu_1^2 \sum_{k=1}^N \alpha_k r^k k^2 - j2\pi\tau\nu_1 \sum_{k=1}^N \beta_k r^k k}{(j2\pi)^2 \sum_{k=1}^N \alpha_k r^k k}.$$

For some examples of systems with $\Re(\nu_2) < 0$, see (Bonnet et al., 2011). In addition, some systems there have no unstable poles of small modulus.

However, all poles on the left of the imaginary axis is not sufficient for H_∞ -stability if there are chains of poles asymptotic to the axis. In this scenario, the next theorem provides necessary and sufficient stability condition.

Theorem 5.5. *Let $G(s)$ be a transfer function defined by (5.1) and suppose that $G(s)$ has no unstable pole of small modulus. Suppose also that the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has at least one root of multiplicity one and modulus one which satisfies (5.7). The other roots of $\tilde{c}_d(z)$ are of modulus strictly greater than one. Suppose that $\Re(\nu_2) < 0$. Then G is H_∞ -stable if and only if $\deg p \geq \deg t + 2$.*

5.3.2 Other cases

In the previous subsection, we consider single roots r of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) satisfying the condition $\sum_{k=1}^N \beta_k r^k \neq 0$.

If a root r of multiplicity one satisfies other conditions involving the coefficients α_k, β_k, \dots defined in (2.3), then $\nu_{n,1}$ has the form

$$\nu_{n,1} = \left(\frac{\tau}{j2\pi n} \right)^{x_r} K_r \quad (5.9)$$

with K_r a function in r and α_k, β_k, \dots , and $x_r \in \mathbb{N}$.

Note that we get a value of x_r for each root r of multiplicity one of $\tilde{c}_d(z)$, where comes the subscript. For example, if r satisfies $\sum_{k=1}^N \beta_k r^k = 0$ and $\sum_{k=1}^N \gamma_k r^k \neq 0$, then $x_r = 2$ and $K_r = \sum_{k=1}^N \gamma_k r^k / (\sum_{k=1}^N k \alpha_k r^k)$.

By analyzing $\nu_{n,1}$, we draw the following conclusions.

- If $x_r = 2k - 1$, $k \in \mathbb{N}$, then $\tau^{x_r} / (j2\pi n)^{x_r} \in j\mathbb{R}$. Similar to the case considered previously, for all values of ν_1 relative to r and \bar{r} we have $\Re(\nu_1) = 0$ or $\Re(\nu_1) = \pm c \neq 0$. In the former situation further approximation may be needed while in the latter the system is unstable.
- If x_r is even, then $\text{sgn}(\nu_{n,1}) = \text{sgn}(\Re(K_r))$ when $x_r = 4k$, $k \in \mathbb{N}$ and $\text{sgn}(\nu_{n,1}) = -\text{sgn}(\Re(K_r))$ when $x_r = 4k - 2$, $k \in \mathbb{N}$.

In the case of x_r being even, it may happen that $\Re(\nu_{n,1}) < 0$ and the following proposition can be used to verify whether the system is H_∞ -stable.

Proposition 5.6. *Let $G(s)$ be a transfer function given as (5.1) and suppose that the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has at least one simple root of modulus one, denoted r , the other roots being of modulus strictly greater than one.*

1. Suppose that $\Re(\nu_{n,1}) < 0$ for all n and that G has no unstable pole of small modulus, then G is H_∞ -stable if and only if $\deg p \geq \deg t + \max_r \{y_1\}$, where, for each r , $-y_1$ is the order in n of $\nu_{n,1}$.
2. If $\Re(\nu_{n,1}) = 0$ for any n , then the condition $\deg p \geq \deg t + \max_r \{y_1\}$ is necessary for H_∞ -stability.

5.4 Multiple chains of poles

The chains of poles relative to a multiple root of the formal polynomial $\tilde{c}_d(z)$ defined by (2.4) are called multiple chains. The poles of these chains approach a same set of points on the vertical line defined by $\Re(s) = -\ln(|r|)/\tau$ (2.7).

5.4.1 The case where $m \geq 2$ and $\sum_{k=1}^N \beta_k r^k \neq 0$

The approximation of neutral poles is given in the next theorem. Analyzing this approximation yields the same conclusion as in the fractional case presented in Subsection 4.4.1.

Theorem 5.7. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that at least one root of the formal polynomial $\tilde{c}_d(z)$ defined by (2.4) has multiplicity $m > 1$. If for such a root, denoted by r , the condition*

$$\sum_{k=1}^N \beta_k r^k \neq 0 \quad (5.10)$$

is satisfied, then for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-1/m}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-1/m}, \quad (5.11)$$

where

$$\nu_1^m = (-1)^{m+1} \frac{m! \tau \sum_{k=1}^N \beta_k r^k}{j 2\pi \sum_{k=1}^N k^m \alpha_k r^k}. \quad (5.12)$$

Corollary 5.8. *Let $G(s)$ be a neutral delay system defined by (5.1). If a root r of multiplicity $m > 1$ of the formal polynomial \tilde{c}_d defined in (2.4) satisfies (5.10), then there exist neutral chains of poles on both sides of the corresponding asymptotic axis $\Re(s) = -\ln(|r|)/\tau$.*

5.4.2 The case where $m \geq 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k\beta_k r^k \neq 0$, $\sum_{k=1}^N \gamma_k r^k \neq 0$

The similar class of fractional systems is studied in Subsection 4.4.2. However, the approximation of pole location for the classical systems here leads to different conclusions in some situations.

Theorem 5.9. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that one of the roots of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m > 1$. If this root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k = 0, \quad (5.13)$$

$$\sum_{k=1}^N k\beta_k r^k \neq 0, \quad (5.14)$$

$$\sum_{k=1}^N \gamma_k r^k \neq 0, \quad (5.15)$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $m = 2$, $y_1 = 1$ and ν_1 satisfies the equation

$$\frac{\nu_1^2}{2} \sum_{k=1}^N k^2 \alpha_k r^k - \frac{\nu_1 \tau}{j2\pi} \sum_{k=1}^N k\beta_k r^k + \frac{\tau^2}{(j2\pi)^2} \sum_{k=1}^N \gamma_k r^k = 0, \quad (5.16)$$

and for $m \geq 3$, (y_1, ν_1) takes m different pair of values below

$$y_1 = 1, \quad \nu_1 = \frac{\tau \sum_{k=1}^N \gamma_k r^k}{j2\pi \sum_{k=1}^N k\beta_k r^k}, \quad (5.17)$$

$$y_1 = \frac{1}{m-1}, \quad \nu_1^{m-1} = \frac{(-1)^m m! \tau \sum_{k=1}^N k\beta_k r^k}{j2\pi \sum_{k=1}^N k^m \alpha_k r^k}. \quad (5.18)$$

Corollary 5.10. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that at least one root r of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m \geq 3$ and satisfies (5.13) and (5.14). Then there exist neutral chains of poles on both sides of the asymptotic axis $\Re(s) = -\ln(|r|)/\tau$.*

Remark 5.11. The conclusion in Corollary 5.10 is drawn from analyzing the values of ν_1 given in (5.18) and not from (5.17). Therefore, it does not depend on the condition (5.15) which is omitted in the corollary.

Corollary 5.12. *Let $G(s)$ be a neutral delay system defined by (5.1). Suppose that r is a root of (2.4) of multiplicity $m = 2$ and satisfies (5.13), (5.14) and (5.15). Then either $\Re(\nu_1) = 0$ for all values of ν_1 relative to r and \bar{r} or at least one value of ν_1 has strictly positive real part.*

Proof. Denoting $\nu_1^{(1)}$ et $\nu_1^{(2)}$ roots of (5.16), we have

$$\nu_1^{(1)} + \nu_1^{(2)} = -\frac{j\tau \sum_{k=1}^N k\beta_k r^k}{\pi \sum_{k=1}^N k^2 \alpha_k r^k}.$$

If r is real, then $\nu_1^{(1)} + \nu_1^{(2)} \in j\mathbb{R}$. Thus $\Re(\nu_1^{(1)}) + \Re(\nu_1^{(2)}) = 0$. The conclusion is then obvious.

If $r \in \mathbb{C} \setminus \mathbb{R}$, then \bar{r} is also a root of multiplicity two of 2.4. Denoting $\nu_1^{\prime(1)}$, $\nu_1^{\prime(2)}$ roots of (5.16) corresponding to \bar{r} , we have

$$\nu_1^{(1)} + \nu_1^{(2)} + \nu_1^{\prime(1)} + \nu_1^{\prime(2)} = -\frac{j2\tau}{\pi} \Re \left(\frac{\sum_{k=1}^N k\beta_k r^k}{\sum_{k=1}^N k^2 \alpha_k r^k} \right) \in j\mathbb{R}.$$

Therefore

$$\Re(\nu_1^{(1)}) + \Re(\nu_1^{(2)}) + \Re(\nu_1^{\prime(1)}) + \Re(\nu_1^{\prime(2)}) = 0$$

from which the conclusion is immediately drawn. \square

Remark 5.13. For $m \geq 3$, the system is unstable in both classical and fractional cases. Nevertheless, for $m = 2$, while $\nu_{n,1}$ may allow one to conclude that the chains of poles are on the left of the asymptotic axis in the fractional case, that conclusion is impossible in the classical case. This was the same scenario occurring to single chains of poles in Subsection 5.3.1.

If all values of ν_1 relative to r and \bar{r} are purely imaginary, we do not know yet on which side of the asymptotic axis the corresponding chains of poles lie. If there is no other factor allowing one to conclude that the system is unstable, we need to approximate the poles further in order to reach a conclusion. This is the objective of the next theorem.

Theorem 5.14. *Let $G(s)$ be a neutral delay system defined by (5.1). Suppose that a root r of multiplicity $m = 2$ of the formal polynomial $\tilde{c}_d(z)$ defined by (2.4) satisfies (5.13), (5.14) and (5.15). Suppose also that in the approximation of the neutral poles corresponding to r which is given by*

$$s_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n^{y_1}} \right) + o(n^{-y_1})$$

with λ_n given by (2.6), $y_1 = 1$ and ν_1 given by (5.16), ν_1 satisfies the conditions

$$\begin{aligned} f_n(\nu_1) := & \frac{2\tau^2 \ln(r)}{(j2\pi)^3} \sum_{k=1}^N \gamma_k r^k + \frac{\tau^3}{(j2\pi)^3} \sum_{k=1}^N \delta_k r^k - \frac{\nu_1^3}{3!} \sum_{k=1}^N \alpha_k k^3 r^k + \frac{\tau}{j2\pi} \frac{\nu_1^2}{2} \sum_{k=1}^N \beta_k k^2 r^k \\ & - \frac{\tau \ln(r)}{(j2\pi)^2} \nu_1 \sum_{k=1}^N \beta_k k r^k - \frac{\tau^2}{(j2\pi)^2} \nu_1 \sum_{k=1}^N \gamma_k k r^k \neq 0, \end{aligned} \quad (5.19)$$

$$f_d(\nu_1) := -\nu_1 \sum_{k=1}^N \alpha_k k^2 r^k + \frac{\tau}{j2\pi} \sum_{k=1}^N \beta_k k r^k \neq 0. \quad (5.20)$$

Then the approximation of the neutral poles can be extended to

$$s_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n^{y_1}} + \frac{\nu_2}{n^{y_2}} \right) + o(n^{-y_2})$$

where $y_2 = 2$ and ν_2 is given by

$$\nu_2 = \frac{f_n(\nu_1)}{f_d(\nu_1)}. \quad (5.21)$$

Proof. Our objective is to find the next approximation term ν_2/n^{y_2} of s_n with $\nu_2 \neq 0$ and $y_2 > y_1$. To do that, in (5.3) we develop $1/s_n^p$ (with $p \in \mathbb{N}$) more precisely as follows

$$\frac{1}{s_n^p} = \frac{\tau^p}{(j2\pi n)^p} \left(1 + \frac{p \ln(r)}{j2\pi n} + O(n^{-2}) \right). \quad (5.22)$$

Now we will prove that $y_2 = 2$ is the appropriate value.

If $y_2 < 2$, then the development (5.3) can be rewritten as

$$f_1(\nu_1) - \nu_2 \frac{f_d(\nu_1)}{n^{1+y_2}} + o(n^{-(1+y_2)}) = 0$$

where $f_1(\nu_1)$ is the left expression in (5.16) and $f_1(\nu_1) = 0$. Consequently, $\nu_2 = 0$ which does not satisfy the requirement.

If $y_2 > 2$, then (5.3) becomes

$$f_1(\nu_1) + \frac{f_n(\nu_1)}{n^3} + o(n^{-3}) = 0$$

which cannot happen since $f_1(\nu_1) = 0$ and $f_n(\nu_1) \neq 0$.

Therefore $y_2 = 2$ and (5.3) becomes

$$f_1(\nu_1) + \frac{f_n(\nu_1)}{n^3} - \nu_2 \frac{f_d(\nu_1)}{n^{1+y_2}} + o(n^{-3}) = 0. \quad (5.23)$$

From this, the result is immediate. \square

Remark 5.15. Note that $f_n(\nu_1) = f'_1(\nu_1)$. Then, the assumption $f_n(\nu_1) \neq 0$ implies that ν_1 is not a double root of f_1 , because otherwise $f'_1(\nu_1) = 0$.

There exist systems with $m = 2$ and all purely imaginary ν_1 that have all values of ν_2 with negative real part. Example 5.5 is one such system. In that case, the following criterion allows one to determine the H_∞ -stability of the system.

Theorem 5.16. *Let $G(s)$ be a neutral delay system defined by (5.1) satisfying the following conditions*

- *G has no unstable poles of small modulus.*
- *the formal polynomial (2.4) has roots of modulus one and all these roots, denoted by r , are of multiplicity 2 and satisfies (5.13), (5.14), and (5.15).*
- *the poles associated with each root of (2.4) are approximated by $s_n\tau = \lambda_n + \nu_{n,1} + \nu_{n,2} + o(n^{-2})$ where $\nu_{n,1} = \nu_1 n^{-1}$ with ν_1 given by (5.16) and satisfying (5.19) and (5.20), $\nu_{n,2} = \nu_2 n^{-2}$ with ν_2 given by (5.21) and $\Re(\nu_{n,2}) < 0$.*

Then G is H_∞ -stable if and only if $\deg p \geq \deg t + 3$.

Proof. We consider the module of the denominator of G at a point s on the imaginary axis near a pole s_n relative to a root r of modulus one of (2.4). Let $s = s_n + \eta_n \in j\mathbb{R}$.

Recall that

$$s_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n} + \frac{\nu_2}{n^2} + o(n^{-2}) \right)$$

and $\Re(\lambda_n) = 0$, $\Re(\nu_1) = 0$, and $\Re(\nu_2) < 0$. Therefore η_n is at least of order n^{-2} . In this case, we can write

$$\eta_n = \frac{\eta}{n^2} + o(n^{-2}).$$

Therefore,

$$s = s_n + \eta_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n} + \frac{\nu_2 + \tau\eta}{n^2} + o(n^{-2}) \right). \quad (5.24)$$

We see that s has the same form as s_n if we replace $\nu_2' = \nu_2 + \tau\eta$ in the above expression of s . Therefore, the approximation of $d(s)$ as $|s| \rightarrow \infty$ is similar to that of $d(s_n)$ as $|s_n| \rightarrow \infty$.

We can rewrite the approximation (5.3) of $d(s_n)$ as $|s_n| \rightarrow \infty$ as

$$d(s_n) = p(s_n) \left(\frac{f_1(\nu_1)}{n^2} + \frac{f_2(\nu_1, \nu_2)}{n^3} + o(n^{-3}) \right)$$

where $f_1(\nu_1)$ is the left expression in (5.16) and $f_2(\nu_1, \nu_2) = f_n(\nu_1) - \nu_2 f_d(\nu_1)$ with $f_n(\nu_1)$ and $f_d(\nu_1)$ given by (5.19) and (5.20) respectively. Similarly, we obtain the approximation of $d(s)$ as $|s| \rightarrow \infty$ as follows

$$d(s) = p(s) \left(\frac{f_1(\nu_1)}{n^2} + \frac{f_2(\nu_1, \nu_2 + \tau\eta)}{n^3} + o(n^{-3}) \right).$$

Note that $f_1(\nu_1) = 0$. Now we prove that $f_2(\nu_1, \nu_2 + \tau\eta) \neq 0$. From (5.23) we see that ν_2 is the only root of $f_2(\nu_1, \cdot)$. Consequently, if $f_2(\nu_1, \nu_2 + \tau\eta) = 0$ then $\eta = 0$. However, this cannot happen. Indeed, since $s \in j\mathbb{R}$, then from (5.24) we derive that $\Re(\nu_2 + \tau\eta) = 0$ and thus $\Re(\eta) = -\Re(\nu_2)/\tau \neq 0$. Therefore, $f_2(\nu_1, \nu_2 + \tau\eta) \neq 0$. Hence the order of the denominator of $G(s)$ is $n^{-(d_0-3)}$, where d_0 is the degree of $p(s)$. \square

5.4.3 The case where $m \geq 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k \beta_k r^k = 0$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$, and $\sum_{k=1}^N \gamma_k r^k \neq 0$

For this class of classical systems, both the approximation of pole location and the conclusions drawn from it are the same as those of fractional systems satisfying the same conditions (see Subsection 4.4.3) except for $m = 2$.

Theorem 5.17. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that one of the roots of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m > 1$. If this root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k = 0, \quad (5.25)$$

$$\sum_{k=1}^N k \beta_k r^k = 0, \quad (5.26)$$

$$\sum_{k=1}^N k^2 \beta_k r^k \neq 0, \quad (5.27)$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $2 \leq m \leq 3$, $y_1 = 2/m$ and

$$\nu_1^m = \frac{(-1)^{m+1} m! \tau^2 \sum_{k=1}^N \gamma_k r^k}{(j2\pi)^2 \sum_{k=1}^N k^m \alpha_k r^k}, \quad (5.28)$$

for $m = 4$, $y_1 = 1/2$ and ν_1 satisfies

$$\frac{\nu_1^4}{4!} \sum_{k=1}^N k^4 \alpha_k r^k + \frac{\nu_1^2 \tau}{j4\pi} \sum_{k=1}^N k^2 \beta_k r^k + \frac{\tau^2}{(j2\pi)^2} \sum_{k=1}^N \gamma_k r^k = 0, \quad (5.29)$$

and for $m \geq 5$, (y_1, ν_1) takes one of m different pairs of values

$$y_1 = \frac{1}{2}, \quad \nu_1^2 = -\frac{2\tau \sum_{k=1}^N \gamma_k r^k}{j2\pi \sum_{k=1}^N k^2 \beta_k r^k}, \quad (5.30)$$

$$y_1 = \frac{1}{m-2}, \quad \nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau \sum_{k=1}^N k^2 \beta_k r^k}{j4\pi \sum_{k=1}^N k^m \alpha_k r^k}. \quad (5.31)$$

The same stability results as in the case of fractional systems are obtained for $m \geq 3$.

Corollary 5.18. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that at least one root of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) has multiplicity $m \geq 3$, satisfies $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k \beta_k r^k = 0$ and*

- for $m = 3$, $\sum_{k=1}^N \gamma_k r^k \neq 0$,
- for $m = 4$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$ and $\sum_{k=1}^N \gamma_k r^k \neq 0$
- for $m \geq 5$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$.

Then there exist neutral chains of poles on both sides of the asymptotic axis $\Re(s) = -\ln(|r|)/\tau$.

In the next corollary, we show that when $m = 2$, in most cases we will have chains of poles on both sides of the asymptotic axis.

Corollary 5.19. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that r is a root of multiplicity $m = 2$ of the formal polynomial $\tilde{c}_d(z)$ defined in (2.4) and r satisfies (5.25), (5.26), and (5.27). Then either $\Re(\nu_1) = 0$ for all values of ν_1 corresponding to r and \bar{r} or $\Re(\nu_1) = \pm c \neq 0$.*

Proof. When $m = 2$, (5.28) becomes

$$\nu_1^2 = \frac{\tau^2}{2\pi^2} K_r \quad (5.32)$$

where $K_r = \sum_{k=1}^N \gamma_k r^k / \sum_{k=1}^N k^2 \alpha_k r^k$.

If $K_r < 0$, then $\Re(\nu_1) = 0$ for two values of ν_1 . If r is not real, then \bar{r} is also a root of $\tilde{c}_d(z)$ and $K_{\bar{r}} = K_r < 0$. Hence the two values of ν_1 relative to \bar{r} also has $\Re(\nu_1) = 0$.

In other cases of K_r , it is obvious that $\Re(\nu_1) = \pm c \neq 0$ for ν_1 relative to r and \bar{r} . \square

As usual, in the case where $\Re(\nu_1) = 0$, further approximation may be needed.

5.4.4 The case where $m \geq 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k \beta_k r^k = 0$, $\sum_{k=1}^N k^2 \beta_k r^k \neq 0$, $\sum_{k=1}^N \gamma_k r^k = 0$, $\sum_{k=1}^N k \gamma_k r^k \neq 0$, and $\sum_{k=1}^N \delta_k r^k \neq 0$

The similar class of fractional systems considered in Subsection 4.4.4 has been showed to include systems whose stability cannot be determined with $\nu_{n,1}$ alone. This behavior is rare for fractional systems but is quite common for classical ones as we have seen in many of the cases considered so far in this chapter.

Interestingly, we will see in this subsection that while some fractional systems exhibit that behavior, the classical counterparts do not and vice versa.

Theorem 5.20. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that one of the roots of (2.4) has multiplicity $m > 1$. If this root, denoted by r , satisfies*

$$\sum_{k=1}^N \beta_k r^k = 0, \quad (5.33)$$

$$\sum_{k=1}^N k \beta_k r^k = 0, \quad (5.34)$$

$$\sum_{k=1}^N k^2 \beta_k r^k \neq 0, \quad (5.35)$$

$$\sum_{k=1}^N \gamma_k r^k = 0, \quad (5.36)$$

$$\sum_{k=1}^N k \gamma_k r^k \neq 0, \quad (5.37)$$

$$\sum_{k=1}^N \delta_k r^k \neq 0, \quad (5.38)$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.6) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $m = 2$, $y_1 = 3/2$ and

$$\nu_1^2 = \frac{2\tau^3 \sum_{k=1}^N \delta_k r^k}{(j2\pi)^3 \sum_{k=1}^N k^2 \alpha_k r^k},$$

for $m = 3$, $y_1 = 1$ and ν_1 satisfies

$$-\frac{\nu_1^3}{3!} \sum_{k=1}^N k^3 \alpha_k r^k + \frac{\tau \nu_1^2}{j4\pi} \sum_{k=1}^N k^2 \beta_k r^k - \frac{\tau^2 \nu_1}{(j2\pi)^2} \sum_{k=1}^N k \gamma_k r^k + \frac{\tau^3}{(j2\pi)^3} \sum_{k=1}^N \delta_k r^k = 0, \quad (5.39)$$

and for $m \geq 4$, (y_1, ν_1) takes one of m different pairs of values

$$y_1 = \frac{1}{m-2}, \quad \nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau \sum_{k=1}^N k^2 \beta_k r^k}{j4\pi \sum_{k=1}^N k^m \alpha_k r^k},$$

or $y_1 = 1$ and ν_1 satisfies

$$\frac{\tau \nu_1^2}{j4\pi} \sum_{k=1}^N k^2 \beta_k r^k - \frac{\tau^2 \nu_1}{(j2\pi)^2} \sum_{k=1}^N k \gamma_k r^k + \frac{\tau^3}{(j2\pi)^3} \sum_{k=1}^N \delta_k r^k = 0.$$

For multiplicities other than $m = 3$, we obtain at least one $\nu_{n,1}$ of positive real part as stated in the next corollary.

Corollary 5.21. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that at least one root of (2.4) has multiplicity $m = 2$ or $m \geq 4$ and satisfies (5.33)-(5.38). Then there exist neutral chains of poles on both sides of the asymptotic axis given by (2.7).*

Now, we analyze the location of poles for $m = 3$.

Corollary 5.22. *Let $G(s)$ be a neutral delay system defined by (5.1), and suppose that at least one root of (2.4) has multiplicity $m = 3$ and satisfies (5.33)-(5.38). Then either $\Re(\nu_1) = 0$ for all values of ν_1 relative to r and \bar{r} or at least one value of ν_1 has strictly positive real part.*

Proof. If we denote $\nu_1^{(i)}$, $i = 1, 2, 3$ the values of ν_1 , then from (5.39) we deduce that

$$\nu_1^{(1)} + \nu_1^{(2)} + \nu_1^{(3)} = \frac{3\tau}{j2\pi} K_r$$

with $K_r = (\sum_{k=1}^N \alpha_{1,k} k^2 r^k) / (\sum_{k=1}^N \alpha_{0,k} k^3 r^k)$.

If r is real, then

$$\Re(\nu_1^{(1)} + \nu_1^{(2)} + \nu_1^{(3)}) = 0.$$

If $r \in \mathbb{C} \setminus \mathbb{R}$, then \bar{r} is also a root of the formal polynomial. Denoting $\nu_1'^{(i)}$, $i = 1, 2, 3$ the values of ν_1 for \bar{r} , then $\nu_1^{(1)} + \nu_1^{(2)} + \nu_1^{(3)} + \nu_1'^{(1)} + \nu_1'^{(2)} + \nu_1'^{(3)} = 6\tau \Re(K_r) / (j2\pi)$, and thus

$$\Re(\nu_1^{(1)} + \nu_1^{(2)} + \nu_1^{(3)} + \nu_1'^{(1)} + \nu_1'^{(2)} + \nu_1'^{(3)}) = 0. \quad \square$$

Remark 5.23. At the approximation concerning $\nu_{n,1}$, we cannot determine stable classical systems while it may be possible for fractional ones. The same situation is encountered in the cases $m = 1$, $\sum_{k=1}^N \beta_k r^k \neq 0$ (see Subsection 5.3.1) and $m = 2$, $\sum_{k=1}^N \beta_k r^k = 0$, $\sum_{k=1}^N k \beta_k r^k \neq 0$, $\sum_{k=1}^N \gamma_k r^k \neq 0$ (see Subsection 5.4.2).

However, this phenomenon is not general, for instant it does not happen for the classical system in Example 5.6.

When $\Re(\nu_1) = 0$ for r and \bar{r} , we can determine $\nu_{n,2}$ by similar arguments to those of Theorem 5.14.

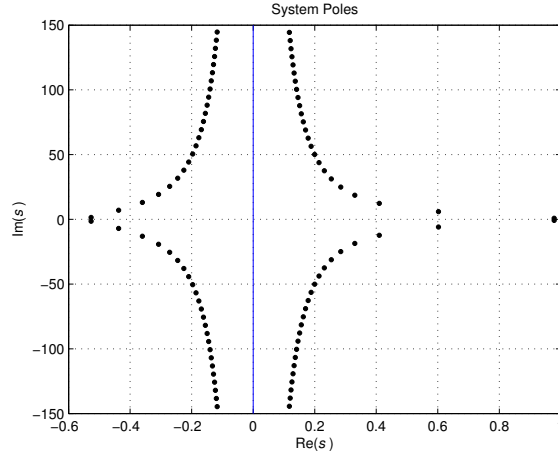
5.5 Examples

Example 5.1. (Subsection 5.4.1)

$$G_1(s) = \frac{1}{s^2 + (-2s^2 - s + 5)e^{-s} + (s^2 - 3s)e^{-2s}}.$$

We have $\alpha_1 = -2$, $\beta_1 = -1$, $\gamma_1 = 5$, $\alpha_2 = 1$, $\beta_2 = -3$, $\gamma_2 = 0$. Then

$$\tilde{c}_d(z) = 1 - 2z + z^2,$$

Figure 5.1 – Neutral chains of poles of $G_1(s)$

which has root $r = 1$ with multiplicity $m = 2$.

Since $\sum_{k=1}^2 \beta_k r^k = -4$, then by applying Theorem 5.7 we obtain $\nu_{n,1} = (-0.5642 + 0.5642j)/n^{1/2}$ and $\nu_{n,1} = (0.5642 - 0.5642j)/n^{1/2}$ for $n \in \mathbb{Z}$, $n \rightarrow \infty$. This implies that in the upper half-plane there is one neutral chain of poles on the left of the imaginary axis and another one on the right. This can be seen in Figure 5.1, which is given by the QPmR algorithm (Vyhlidal and Zitek, 2014). Therefore, $G_1(s)$ is unstable.

This fact can be deduced directly from Corollary 5.10.

Example 5.2. (Subsection 5.4.1)

$$G_2(s) = \frac{1}{s - 10e^{-s} + (3s - 3)e^{-2s} + 7e^{-3s} + (3s + 1)e^{-4s} - 4e^{-5s} + (s - 8)e^{-6s}}.$$

We have $\tilde{c}_d(z) = 1 + 3z^2 + 3z^4 + z^6$. It has roots $r = j$ and $r = -j$ both of multiplicity $m = 3$. Since $\sum_{k=1}^3 \beta_k r^k \neq 0$, then $G_2(s)$ is unstable from Corollary 5.10.

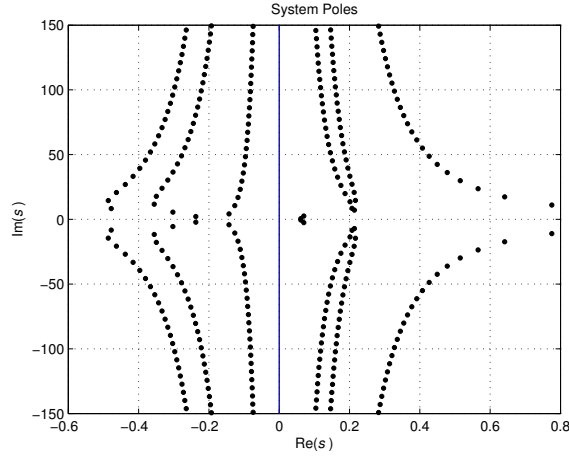
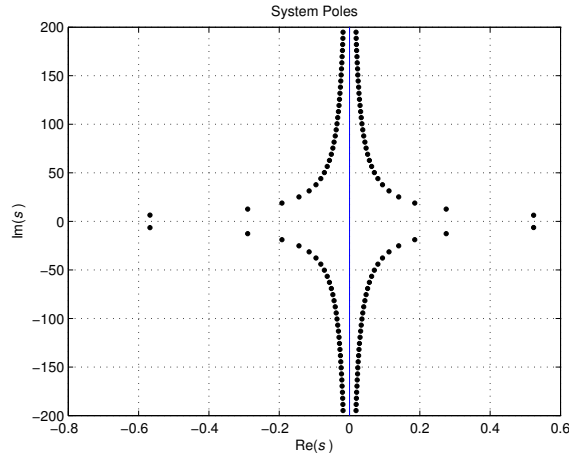
Moreover, by Theorem 5.7, we obtain for $r = j$ two values of ν_1 of negative real part and one of positive real part, and for $r = -j$ one of negative real part and two of positive real part. Therefore, there are three chains of poles in each half-plane (see Figure 5.2).

Example 5.3. (Subsection 5.4.2)

$$G_3(s) = \frac{1}{s^2 + (-2s^2 + s + 10)e^{-s} + (s^2 - s + 3)e^{-2s}}.$$

$\tilde{c}_d(z)$ has root $r = 1$ with multiplicity $m = 2$.

Since $\sum_{k=1}^2 \beta_k r^k = 0$, $\sum_{k=1}^2 k\beta_k r^k = -1$, and $\sum_{k=1}^2 \gamma_k r^k = 13$ then Theorem 5.9 is applied. Resolving (5.16), we obtain $\nu_{n,1} = (-0.5683 - 0.0796j)/n$ and $\nu_{n,1} = (0.5683 - 0.0796j)/n$. There is one neutral chain of poles on the right of the imaginary axis (see Figure 5.3), thus $G_3(s)$ is unstable.

Figure 5.2 – Neutral chains of poles of $G_2(s)$ Figure 5.3 – Neutral chains of poles of $G_3(s)$

Example 5.4. (Subsections 5.3.1 and 5.4.2)

The following particular system allows the application of Theorem 5.1 and Theorem 5.4, as well as Theorem 5.9 and Theorem 5.14.

$$G_4(s) = \frac{1}{(s + 3e^{-s} + (-s + 5)e^{-2s})(s + 1 + (-s + 2)e^{-s})}.$$

For the first quasi-polynomial in the denominator, $\tilde{c}_d(z)$ corresponding to the first has roots $r_1 = 1$ and $r_2 = -1$ both of multiplicity one. For the second one, $\tilde{c}_d(z)$ has one root $r_3 = 1$ also of multiplicity one. Therefore, by applying Theorem 5.1 for each quasi-polynomial, we obtain $\nu_{n,1}^{(1)} = 0.6366j/n$, $\nu_{n,1}^{(2)} = 0.1592j/n$, and $\nu_{n,1}^{(3)} = 0.4775j/n$ respectively relating to r_1 , r_2 , and r_3 . As the real part of $\nu_{n,1}^{(1)}$, $\nu_{n,1}^{(2)}$, and $\nu_{n,1}^{(3)}$ are

zero, we continue to calculate $\nu_{n,2}^{(1)}$, $\nu_{n,2}^{(2)}$, and $\nu_{n,2}^{(3)}$ by using Theorem 5.4. We obtain $\nu_{n,2}^{(1)} = 0.2533/n^2$, $\nu_{n,2}^{(2)} = (0.0633 + 0.0796j)/n^2$, and $\nu_{n,2}^{(3)} = 0.0380/n^2$.

On the other hand, the quasi-polynomial obtained by expanding the denominator has $\tilde{c}_d(z)$ with root $r = 1$ of multiplicity $m = 2$ and root $r = -1$ of multiplicity $m = 1$. For $r = 1$, since $\sum_{k=1}^3 \beta_k r^k = 0$, we use Theorem 5.9 and then Theorem 5.14, which give identical results to Theorem 5.1 and Theorem 5.4.

Example 5.5. (Subsection 5.4.2)

The transfer function of the system is given by

$$G_5(s) = \frac{1}{s^3 + (-2s^3 + s^2 - 10s + 5)e^{-s} + (s^3 - s^2 + 3s + 1)e^{-2s}}.$$

The formal polynomial $\tilde{c}_d(z)$ has one root $r = 1$ with multiplicity $m = 2$. This root satisfies $\sum_{k=1}^2 \beta_k r^k = 0$, $\sum_{k=1}^2 k\beta_k r^k \neq 0$, and $\sum_{k=1}^2 \gamma_k r^k \neq 0$. Therefore, Theorem 5.9 can be applied and thus ν_1 has the values $\nu_1^{(1)} = -0.3490j$ and $\nu_1^{(2)} = 0.5081j$.

Since the values of ν_1 are all purely imaginary, we need to determine ν_2 using Theorem 5.14. We obtain $\nu_2 = -0.0140$ for $\nu_1^{(1)} = -0.3490j$ and $\nu_2 = -0.0493$ for $\nu_1^{(2)} = 0.5081j$. Hence, the two neutral chains of poles are on the left of the imaginary axis.

Example 5.6. (Subsection 5.3.2)

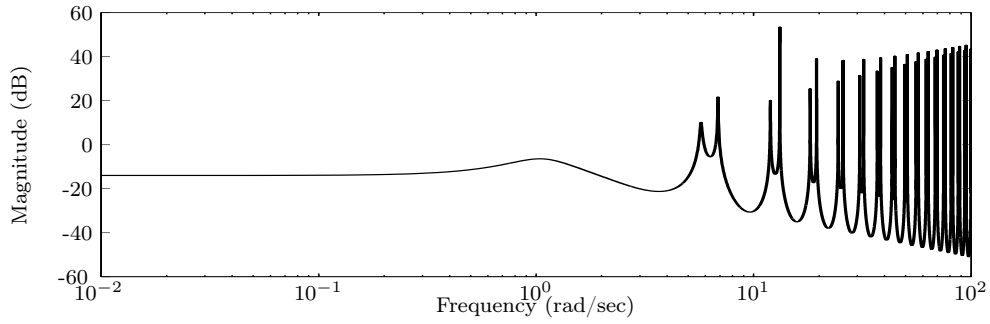
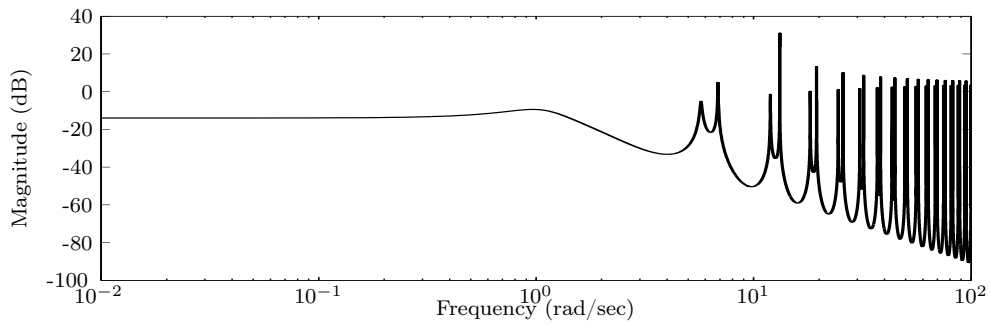
A system is described by the transfer function

$$G_6(s) = \frac{t(s)}{s^2 + 2s + 3 - (1.6s^2 + 3.2s + 2)e^{-s} + (s^2 + 2s + 4)e^{-2s}}.$$

The formal polynomial is $\tilde{c}_d(z) = 1 - 1.6z + z^2$, having two complex conjugate roots $r = 0.8 \pm j\sqrt{2.44}/2$ whose absolute values are 1. The system then has two neutral chains of poles asymptotic to the imaginary axis.

For each root r , the conditions $\sum_{k=1}^2 \beta_k r^k = 0$, and $\sum_{k=1}^2 \gamma_k r^k \neq 0$ are satisfied, then $\nu_{n,1}$ is given by (5.9). We can write $\nu_{n,1} = \nu_1/n^{x_r}$ where $x_r = 2$ and ν_1 is equal to $-0.0127 + 0.0760j$ for $r = 0.8 + j\sqrt{2.44}/2$ or $-0.0127 + 0.0760j$ for $r = 0.8 - j\sqrt{2.44}/2$. Consequently, the two neutral chains approach the imaginary axis from the left side.

Proposition 5.6 indicates that a necessary condition for $G_6(s)$ to be H_∞ -stable is that $\deg t \leq \deg p - \max_r \{x_r\}$. Since $x_r = x_{\bar{r}} = 2$, the condition is $\deg t \leq \deg p - 2$. It is not satisfied if $t(s) = s + 1$ and thus the system is unstable. Figure 5.4 shows that the magnitude of the transfer function increases with increasing frequencies. On the other hand, the condition is satisfied if $t(s) = 1$. In Figure 5.5, we see that the magnitude of the transfer function tends to a constant as the frequencies increase and thus is bounded. However, the system has unstable poles of small modulus, which can be seen using QPmR algorithm (Vyhlidal and Zitek, 2014), and then is unstable.

Figure 5.4 – Bode diagram of $G_6(s)$ with $t(s) = s + 1$ Figure 5.5 – Bode diagram of $G_6(s)$ with $t(s) = 1$

5.6 Conclusion

Before proceeding to the conclusion remarks, we analyze the stability of an example using the results stated in this chapter and those in (Rabah et al., 2012). A system is described by

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \dot{z}(t-1) + \begin{bmatrix} -b & 0 \\ 0 & -b \end{bmatrix} z(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t), \\ y(t) &= [c_1 \quad c_2] z(t) \end{aligned}$$

where $z \in \mathbb{C}^2$, $t \geq 0$, $b, b_1, b_2, c_1, c_2 \in \mathbb{R}$, and $b > 0$. Without the input $u(t)$ and the output $y(t)$, this system was considered in (Rabah et al., 2012, Section 5.1) and was demonstrated to be asymptotically stable. To applied the results in this chapter, we determine the transfer function of the system:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 c_1 + b_2 c_2}{s + b + s e^{-s}}.$$

As proved in (Rabah et al., 2012, Section 5.1), the denominator of $G(s)$ has no unstable root. Theorem 5.5 can be applied and shows that $G(s)$ is H_∞ -unstable.

Now, let us state some final remarks.

The results on approximating poles of neutral chains for classical delay systems are the same as those of fractional systems studied in Chapter 4 provided that the fractional

order μ is replaced by $\mu = 1$. For classical systems as well as for fractional systems, in the case of multiple chains of poles, these approximations reveal various patterns of chains approaching their asymptotic axis. They may approach the axis with the same rate (i.e. the approximations have the same order) or with different rates (i.e. the approximations have different orders).

Analyzing these approximations in order to determine the pole location about the asymptotic axis leads to different results in comparison with fractional systems in certain cases. The phenomenon observed in these cases is that while for fractional systems stable chains of poles may be indicated by the first approximation, for classical systems we need higher approximations to detect such chains. Nevertheless, as shown in the last example, this phenomenon is not general.

The analysis also leads to an important observation that most of the classical systems in the considered classes have chains of poles on the right of the corresponding asymptotic axes. For systems with chains of poles asymptotic to the imaginary axis, this implies that most of those systems are unstable.

The analysis procedure as well as the diverse analysis techniques presented through various cases in the previous and current chapters could be systematically applied to other cases not considered here. However, some efforts are required. To make the procedure easier is the objective of the next chapter where some common results in the previous and current chapters will be generalized to all possible cases.

Chapter 6

Stability analysis of SISO classical and fractional neutral systems with commensurate delays

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6.1 Introduction

Stability analysis has been made in Chapters 4 and 5 for some classes of (fractional) neutral systems with commensurate delays and with chains of poles asymptotic to the

imaginary axis. For each class of systems, the analysis procedure was to approximate poles of large modulus, to examine the approximations and to give necessary and sufficient conditions for H_∞ -stability when it was appropriate.

Although the conclusions on the location of chains of poles about the imaginary axis were different for different classes of systems, the tools used for approximating poles in neutral chains remained the same. However, establishing the results became more complicated when the classes of systems were defined with more conditions on the coefficients.

To overcome this difficulty, we provide in this chapter new results which generalize those of the previous chapters and which can be easily implemented in computation software. They cover both classical and fractional systems in almost every configuration.

The chapter is organized as follows. Section 6.2 presents the (fractional) neutral delay systems of interest. The main results concerning the location of poles and stability conditions are presented in Sections 6.3 and 6.4 respectively. These results are compared with those presented in Chapters 5 and 4 in Section 6.5. The chapter is then concluded by Section 6.6.

6.2 A class of (fractional) neutral time-delay systems

We consider (fractional) neutral time-delay systems with transfer function of the form

$$G(s) = \frac{t(s)}{p(s) + \sum_{k=1}^N q_k(s)e^{-ks\tau}}, \quad (6.1)$$

where

- $\tau > 0$ is the delay,
- t , p , and q_k for all $k \in \mathbb{N}_N$ are real polynomials in s^μ ,
- $0 < \mu \leq 1$,
- $-\pi < \arg(s) < \pi$ in the case where $0 < \mu < 1$ in order to have a single value of s^μ ,
- $\deg p \geq \deg t$, $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$ in order to deal with proper neutral systems.

Here, the degree of a (quasi-)polynomial refers to the degree in s^μ .

Note that with $\mu \in (0, 1]$, the systems defined by (6.1) encompass those studied in both Chapters 4 and 5. Some basic characteristics of these systems are described in Subsection 2.4.2.

In this chapter, for the purpose of developing more general results, we change some notations compared to Chapters 4 and 5. In the development of $q_k(s)/p(s)$ for $k = 1, \dots, N$ as $|s| \rightarrow \infty$, the coefficients corresponding to the terms $s^{-l\mu}$ for $l \in \mathbb{Z}_+$ are now denoted by $\alpha_{l,k}$. Hence $\alpha_{0,k}$, $\alpha_{1,k}$, $\alpha_{2,k}$, $\alpha_{3,k}$, and $\alpha_{4,k}$ replace respectively α_k , β_k , γ_k , δ_k , and ϵ_k

in the previous chapters. The development of $q_k(s)/p(s)$ can be rewritten as

$$\frac{q_k(s)}{p(s)} = \alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}}{s^{l\mu}} + \mathcal{O}(s^{-(M'+1)\mu}) \quad (6.2)$$

where $M' \in \mathbb{Z}_+$ and can be arbitrarily large. The formal polynomial is now

$$\tilde{c}_d(z) = 1 + \sum_{k=1}^N \alpha_{0,k} z^k, \quad (6.3)$$

where $z = e^{-s\tau}$.

6.3 Location of neutral poles

As we have seen in Subsection 2.4.2, to each root r of the formal polynomial $\tilde{c}_d(z)$ corresponds a chain of poles of neutral type. The approximation of these poles given in (2.5) only indicates the vertical line to which the pole chain approaches. To determine the position of the chain around the asymptotic axis, similarly to the two previous chapters, we examine in this section a more precise approximation of neutral poles of the form

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}) \quad (6.4)$$

with

$$\nu_{n,1} = \frac{\nu_1}{n^{y_1}}, \quad \nu_1 \neq 0, y_1 > 0, n \in \mathbb{Z}, n \rightarrow \infty.$$

In other words, we determine the next *non-zero* approximation term when it is appropriate. Such an approximation term does not exist if the neutral poles are precisely $s_n = \lambda_n/\tau$.

Except that special case, $\nu_{n,1}$ exists and the sign of $\Re(\nu_1/n^{y_1})$ then shows on which side of the asymptotic axis the poles are. Note that the sign may change for positive and negative n . Hence, the upper and lower parts of a poles chain may lie on different sides of the asymptotic axis.

Here, remark that we do not fix a value of y_1 beforehand but look for y_1 such that $\nu_1 \neq 0$. This ensures that the approximation gives some new information about the location of poles. The only case where the information is not useful is when $\Re(\nu_1/n^{y_1}) = 0$ and we may need to approximate further to know the location of poles about the asymptotic axis.

Before presenting the main results about the location of poles around the asymptotic axis, we define some notions which will be of use.

- For a root r of $\tilde{c}(z)$,

$$AB(r) = \{(a, b) \in \mathbb{Z}_+^2 : a + b \neq 0, \sum_{k=1}^N \alpha_{a,k} k^b r^k \neq 0\}. \quad (6.5)$$

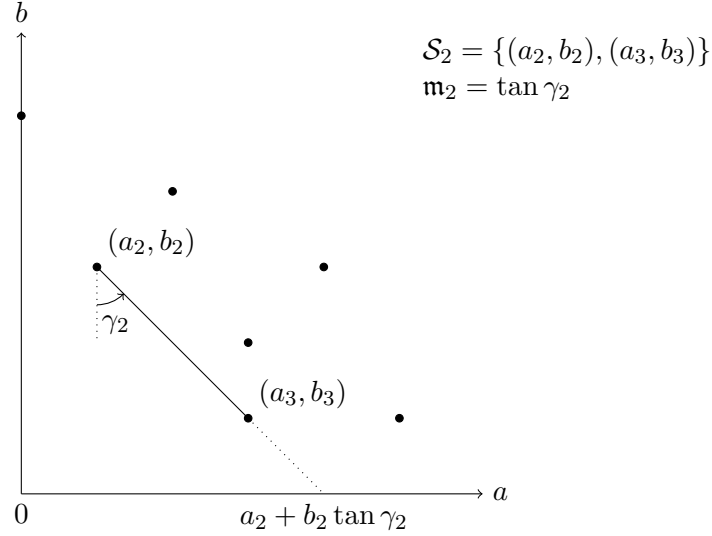


Figure 6.1 – A lower left boundary segment of a set of points in the plane

- \mathcal{S} denotes a subset of $AB(r)$ such that $n(\mathcal{S}) \geq 2$ and there exists $\mathbf{m} > 0$ such that $a + b\mathbf{m} = a' + b'\mathbf{m} \forall (a, b), (a', b') \in \mathcal{S}$ and $a + b\mathbf{m} < a'' + b''\mathbf{m} \forall (a'', b'') \in AB(r) \setminus \mathcal{S}$. We will call \mathcal{S} a *lower left boundary segment* of $AB(r)$.
- \mathbf{m} defined as above for each \mathcal{S} is obviously unique and we call it the *slope* of the segment.
- $\mathfrak{S}(AB(r))$ denotes the set of all lower left boundary segments of $AB(r)$.

A lower left boundary segment is illustrated in Figure 6.1. Note that if we denote \mathbf{m}_2 the slope of the segment then $\mathbf{m}_2 = \tan \gamma_2$ with γ_2 presented in the figure.

The approximation of neutral chains of poles is the objective of the next theorem.

Theorem 6.1. *Let $G(s)$ be a neutral delay system defined by (6.1) and r a root of multiplicity m of the formal polynomial $\tilde{c}_a(s)$ defined by (6.3). With $\alpha_{a,k}$ defined as in (6.2), let us define*

$$C(a, b, \nu) := \frac{\tau^{a\mu}}{(j2\pi)^{a\mu}} \frac{(-1)^b \nu^b}{b!} \sum_{k=1}^N \alpha_{a,k} k^b r^k, \quad (6.6)$$

$$B(\mathcal{S}) := \left\{ (\nu, y) : \nu \text{ is a non-zero root of } \sum_{(a,b) \in \mathcal{S}} C(a, b, \nu) = 0, y = \mathbf{m}\mu \right\}. \quad (6.7)$$

Let us denote n_1 the number of chains of poles relative to r with poles $s_n = \lambda_n/\tau$ where $n \in \mathbb{Z}$, $n \rightarrow \infty$ and λ_n is given by (2.6). Then poles of the other neutral chains corresponding to r are approximated by

$$s_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n^{y_1}} \right) + o(n^{-y_1}) \quad (6.8)$$

where for each chain of poles (ν_1, y_1) takes one of the $m - n_1$ values (counting multiplicity) given by

$$(\nu_1, y_1) \in \bigcup_{S \in \mathfrak{S}(AB(r))} B(S).$$

Proof. Denote s_n a pole of $G(s)$, then

$$d(s_n) := p(s_n) + \sum_{k=1}^N q_k(s_n) e^{-ks_n\tau} = 0.$$

Dividing both sides by $p(s_n)$, we have

$$\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^N \frac{q_k(s_n)}{p(s_n)} e^{-ks_n\tau} = 0.$$

As $|s_n| \rightarrow \infty$, using (6.2) leads to

$$\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^N \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}}{s^{l\mu}} + \mathcal{O}(s^{-(M'+1)\mu}) \right) e^{-ks_n\tau} = 0$$

where $M' \in \mathbb{Z}_+ \setminus \{0\}$.

Assume s_n has the form

$$s_n\tau = \lambda_n + \nu_{n,1} + \nu_{n,2} + \dots + \nu_{n,M} + o(n^{-M'\mu})$$

with $\nu_{n,i} = \nu_i n^{-y_i}$, $i = 1, \dots, M$ where $\nu_i \neq 0$ and $0 < y_1 < \dots < y_M \leq M'\mu$.

Note that

$$e^{-\lambda_n} = r,$$

$$e^{-k\nu_{n,i}} = 1 + \sum_{l=1}^{\lfloor \frac{M'\mu}{y_i} \rfloor} \frac{(-1)^l \nu_i^l k^l}{l! n^{ly_i}} + o(n^{-M'\mu}) \text{ with } l \in \mathbb{Z}_+ \setminus \{0\}.$$

Thus when n is large enough

$$\begin{aligned} \frac{d(s_n)}{p(s_n)} &= 1 + \sum_{k=1}^N \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k} \tau^{l\mu}}{(j2\pi n)^{l\mu}} (1 + \mathcal{O}(n^{-1})) + o(n^{-M'\mu}) \right) r^k \\ &\times \prod_{i=1}^M \left(1 + \sum_{l=1}^{\lfloor \frac{M'\mu}{y_i} \rfloor} \frac{(-1)^l \nu_i^l k^l}{l! n^{ly_i}} + o(n^{-M'\mu}) \right) = 0 \end{aligned}$$

and we obtain

$$1 + \sum_{k=1}^N \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k} \tau^{l\mu}}{(j2\pi n)^{l\mu}} (1 + O(n^{-1})) + o(n^{-M'\mu}) \right) r^k \\ \times \left(1 + \sum_{(l_1, \dots, l_M) \in \mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right) k^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} + o(n^{-M'\mu}) \right) = 0,$$

where $\mathcal{L}(x) := \left\{ (l_1, \dots, l_M) : l_i \in \mathbb{Z}_+, \sum_{i=1}^M l_i \geq 1, \text{ and } \sum_{i=1}^M l_i y_i \leq x \right\}$.

By simple calculations, we obtain

$$1 + \sum_{k=1}^N r^k \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k} \tau^{l\mu}}{(j2\pi n)^{l\mu}} (1 + O(n^{-1})) \right. \\ \left. + \alpha_{0,k} \sum_{(l_1, \dots, l_M) \in \mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right) k^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \right. \\ \left. + \sum_{(l, l_1, \dots, l_M) \in \mathcal{HL}(M'\mu)} \frac{\alpha_{l,k} \tau^{l\mu}}{(j2\pi n)^{l\mu}} (1 + O(n^{-1})) \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right) k^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \right. \\ \left. + o(n^{-M'\mu}) \right) = 0,$$

where $\mathcal{HL}(x) := \left\{ (l, l_1, \dots, l_M) : l \in \mathbb{Z}_+ \setminus \{0\}, l_i \in \mathbb{Z}_+, \sum_{i=1}^M l_i \geq 1, \text{ and } l\mu + \sum_{i=1}^M l_i y_i \leq x \right\}$, and then

$$1 + \sum_{k=1}^N \alpha_{0,k} r^k + \sum_{l=1}^{M'} \frac{\tau^{l\mu}}{(j2\pi n)^{l\mu}} (1 + O(n^{-1})) \sum_{k=1}^N \alpha_{l,k} r^k \\ + \sum_{(l_1, \dots, l_M) \in \mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \alpha_{0,k} r^k k^{\sum_{i=1}^M l_i} \\ + \sum_{(l, l_1, \dots, l_M) \in \mathcal{HL}(M'\mu)} \frac{\tau^{l\mu}}{(j2\pi n)^{l\mu}} (1 + O(n^{-1})) \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i} \right)}{\left(\prod_{i=1}^M l_i! \right) n^{\sum_{i=1}^M l_i y_i}} \sum_{k=1}^N \alpha_{l,k} r^k k^{\sum_{i=1}^M l_i} \\ + o(n^{-M'\mu}) = 0. \tag{6.9}$$

Since $1 + \sum_{k=1}^N \alpha_{0,k} r^k = 0$, then the highest order in n of the above development has the form $-(a\mu + by_1)$ where (a, b) belongs to $AB(r)$ defined in (6.5) and there exists $y_1 > 0$ such that $a\mu + by_1 \leq a'\mu + b'y_1 \forall (a', b') \in AB(r)$.

If the equality only happens for $(a', b') = (a, b)$, then the term of highest order is $C(a, b, \nu_1)/n^{a\mu + by_1}$ and thus $\nu_1 = 0$, which does not satisfy the assumptions.

The equality holds for point(s) other than (a, b) if (a, b) belongs to a segment $\mathcal{S} \in \mathfrak{S}(AB(r))$ and $y_1 = \mathfrak{m}\mu$ with \mathfrak{m} be the slope of \mathcal{S} .

Now, consider a segment \mathcal{S} . By definition, $a\mu + by_1 = a'\mu + b'y_1 \forall (a, b), (a', b') \in \mathcal{S}$ and $a\mu + by_1 > a''\mu + b''y_1 \forall (a'', b'') \in AB(r) \setminus \mathcal{S}$. Hence, the term of highest order is $\sum_{(\alpha, \beta) \in \mathcal{S}} C(\alpha, \beta, \nu_1) / n^{\alpha\mu + \beta y_1}$ with $C(\alpha, \beta, \nu_1)$ defined in (6.6). Note that due to (6.9) this term is zero, which allows us to derive ν_1 . Since there are different $(\alpha, \beta) \in \mathcal{S}$, we obtain some non-zero values of ν_1 . \square

Now we will discuss how to construct all the lower left segments of the set $AB(r)$.

First, we mention two important points of $AB(r)$ which limits a subset of $AB(r)$ containing the lower left segments. The first point is $(0, m)$. This point belongs to $AB(r)$ since $\sum_{k=1}^N \alpha_{0,k} k^m r^k \neq 0$ (see Lemma 2.59). The second point, denoted by (a_L, b_L) , is the leftmost point among the lowest points of $AB(r)$, i.e.

$$\begin{aligned} b_L &= \min\{b \mid (a, b) \in AB(r)\} \\ a_L &= \min\{a \mid (a, b_L) \in AB(r)\}. \end{aligned} \tag{6.10}$$

The lower left segments of $AB(r)$ then belong to the subset $A_L^m = \{(a, b) \in AB(r) \mid a \leq a_L, b \leq m\}$ (see Figure 6.2). Indeed, if $(a, b) \in AB(r)$ and $a > a_L$, then $a + bm > a_L + b_L m$ for all $m > 0$ since $b \geq b_L$ by definition. If $(a, b) \in AB(r)$ and $b > m$, then $a + bm > am$ for all $m > 0$ since $a \geq 0$ by definition.

The subset A_L^m has finite points and thus its convex hull is a convex polygon (De Berg et al., 2008). The vertices of this polygon are points in A_L^m and the line containing each of its edges defines a closed half-plane containing all the points of A_L^m . There is no other line containing two points of A_L^m with such a characteristic.

Therefore, by definition, the points of a lower left segment of $AB(r)$ belong to an edge of the convex hull of A_L^m and two of them are vertices of the hull.

There exist numerous algorithms for determining the points of a finite set in \mathbb{R}^2 which are on the boundary of its convex hull (De Berg et al., 2008). Among them, we can pick up points belonging to lower left segments.

The above discussion indicates that we need to know the points $(0, m)$ and (a_L, b_L) before using convex hull algorithms to determine the lower left segments. In the rest of this section, we present a method to find (a_L, b_L) numerically.

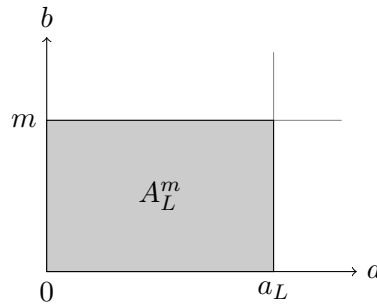


Figure 6.2 – The subset A_L^m of $AB(r)$ which contains all lower left segments of $AB(r)$

First, remark that $b_L = n_1$ with n_1 be the number of chains of poles with $s_n = \lambda_n/\tau$. Indeed, $m - n_1$ is the total number of non-zero values of ν_1 . This number is also equal to $(\max\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))} \mathcal{S}\} - \min\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))} \mathcal{S}\})$ since the number of non-zero values of ν_1 for each $\mathcal{S} \in \mathfrak{S}(AB(r))$ is $(\max\{b \mid (a, b) \in \mathcal{S}\} - \min\{b \mid (a, b) \in \mathcal{S}\})$ and the segments in $\mathfrak{S}(AB(r))$ are interconnected. Also note that $\max\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))} \mathcal{S}\} = m$ and $\min\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))} \mathcal{S}\} = b_L$.

The next lemma provides a tool to derive the number of chains of poles with $s_n = \lambda_n/\tau$.

Lemma 6.2. *Let $G(s)$ be a neutral delay system defined by (6.1). Its denominator can be written as*

$$D(s, z) = p(s) + \sum_{k=1}^N q_k(s) z^k, \quad z = e^{-ks\tau}.$$

Let us denote by r a root of multiplicity m of $\tilde{c}_d(z)$ defined by (2.4). The following statements are equivalent:

(i) $D(s, z)$ has n_1 identical chains of poles s_n on the asymptotic axis corresponding to r with $s_n = \lambda_n/\tau$ where $n \in \mathbb{Z}$ and λ_n is given by (2.6).

(ii)

$$\left. \frac{d^b D(s, z)}{ds^b} \right|_{z=r} \equiv 0, \quad b = 0, \dots, n_1 - 1, \quad (6.11)$$

$$\left. \frac{d^{n_1} D(s, z)}{ds^{n_1}} \right|_{z=r} \neq 0, \quad (6.12)$$

where $d^0 D(s, z)/ds^0 = D(s, z)$.

Proof. (i) \implies (ii):

The fact that s_n for $n \in \mathbb{Z}$ are roots of multiplicity n_1 of $D(s, z)$ is equivalent to

$$\left. \frac{d^b D(s, z)}{ds^b} \right|_{\substack{s=s_n \\ z=e^{-s_n\tau}=r}} = 0, \quad b = 0, \dots, n_1 - 1, \quad (6.13)$$

$$\left. \frac{d^{n_1} D(s, z)}{ds^{n_1}} \right|_{\substack{s=s_n \\ z=e^{-s_n\tau}=r}} \neq 0. \quad (6.14)$$

Equations (6.13) show that $\left. \frac{d^b D(s, z)}{ds^b} \right|_{z=r} = 0$, $b = 0, \dots, n_1 - 1$ have infinitely many roots s_n , $n \in \mathbb{Z}$. This implies (6.11) since $\left. \frac{d^b D(s, z)}{ds^b} \right|_{z=r}$ are polynomials in s^μ , $\mu \in (0, 1]$. Otherwise they would have a finite number of roots.

It is obvious that (6.14) implies (6.12).

(ii) \implies (i):

From (6.11), we deduce that s_n , $n \in \mathbb{Z}$ are roots of $\frac{d^b D(s,z)}{ds^b} \Big|_{z=r}$, $b = 0, \dots, n_1 - 1$. Furthermore, s_n are roots of $e^{-s\tau} = r$. Therefore, s_n are roots of $\frac{d^b D(s,z)}{ds^b}$.

On the other hand, due to (6.12), the polynomial $\frac{d^{n_1} D(s,z)}{ds^{n_1}} \Big|_{z=r}$ has a finite number of roots and its roots are bounded. Therefore, there exists $N_1 \in \mathbb{Z}_+$ such that for $|n| > N_1$ s_n are not roots of $\frac{d^{n_1} D(s,z)}{ds^{n_1}} \Big|_{z=r}$ and thus are not roots of $\frac{d^{n_1} D(s,z)}{ds^{n_1}}$.

Hence, we conclude that s_n , $n \in \mathbb{Z}$ are roots of multiplicity n_1 of $D(s, z)$. \square

After determining b_L using the previous lemma, we can determine a_L by running a loop to find the smallest value of a such that $\sum_{k=1}^N \alpha_{a,k} k^{b_L} r^k \neq 0$.

6.4 Stability

In this section, we study whether or not a system is H_∞ -stable based on the approximation of poles obtained in the preceding section. Here, we are only interested in systems with neutral chains asymptotic to the imaginary axis.

The next theorem provides quick tests on the instability of the systems. It does not even require to know $\nu_{n,1}$.

Theorem 6.3. *Let $G(s)$ be a neutral delay system defined by (6.1), and suppose that the formal polynomial $\tilde{c}_d(z)$ defined in (6.3) has roots of modulus one. If for such a root, denoted by r , there exists $\mathcal{S} \in \mathfrak{S}(AB(r))$ with $AB(r)$ defined in (6.5) such that $n(\mathcal{S}) = 2$ and either of the following conditions holds for $(a_1, b_1), (a_2, b_2) \in \mathcal{S}, b_1 > b_2$*

- $b_1 - b_2 \geq 3$,
- $b_1 - b_2 = 2$, and $(a_2 - a_1)\mu \neq 2k, k \in \mathbb{Z}_+ \setminus \{0\}$,

then the system is unstable.

Proof. ν_1 of entries of $B(\mathcal{S})$ defined by (6.7) are given by

$$\nu_1^{b_1 - b_2} = -\frac{\tau^{(a_2 - a_1)\mu}}{(j2\pi)^{(a_2 - a_1)\mu}} \frac{(-1)^{(b_2 - b_1)} b_1! \sum_{k=1}^N \alpha_{a_2, k} k^{b_2} r^k}{b_2! \sum_{k=1}^N \alpha_{a_1, k} k^{b_1} r^k}.$$

It is easy to see that for $b_1 - b_2 \geq 3$ there exists at least one value of ν_1 with positive real part.

This is also the case for $b_1 - b_2 = 2$ if $\nu_1^2 \in \mathbb{C} \setminus \mathbb{R}_-$. Let us denote

$$K_r = \frac{\sum_{k=1}^N \alpha_{a_2, k} k^{b_2} r^k}{\sum_{k=1}^N \alpha_{a_1, k} k^{b_1} r^k}.$$

From now on we only consider positive n since poles are symmetric w.r.t. the real axis.

If $r \in \mathbb{R}$, then $K_r \in \mathbb{R}$. However, if $(a_2 - a_1)\mu \neq 2k, k \in \mathbb{Z}_+ \setminus \{0\}$ then $j^{(a_2 - a_1)\mu} = e^{j \frac{(a_2 - a_1)\mu\pi}{2}} \in \mathbb{C} \setminus \mathbb{R}$, thus leading to $\nu_1^2 \in \mathbb{C} \setminus \mathbb{R}$. This indicates that the two values of ν_1

have non-zero real parts. Since they are symmetric w.r.t. the origin then one of them has positive real part, which implies that the system is unstable.

If $r \in \mathbb{C} \setminus \mathbb{R}$, then \bar{r} is also a root of the polynomial $\tilde{c}_d(z)$ (6.3). Denote $\nu_{1(r)}$ and $\nu_{1(\bar{r})}$ the values of ν_1 relative to r and \bar{r} respectively. We obtain thus

$$\nu_{1(r)}^2 + \nu_{1(\bar{r})}^2 = -\frac{\tau^{(a_2-a_1)\mu}}{(j2\pi)^{(a_2-a_1)\mu}} \frac{(-1)^{(b_2-b_1)} b_1!}{b_2!} 2\Re(K_r),$$

which is not real. It turns out that either $\nu_{1(r)}^2$ or $\nu_{1(\bar{r})}^2$ is not real, thus giving at least one value of ν_1 with positive real part. \square

Several unstable systems that do not fit in those described in the previous theorem can be found in Chapters 4 and 5. We were able to conclude about the instability of those systems by using other analyses.

In the favorable case where neutral chains approach the imaginary from the left, the next theorem presents other conditions for the system to be H_∞ -stable.

To facilitate the proof of the theorem, we first state a primarily result.

Lemma 6.4. *Suppose that $\mathfrak{S}(AB(r)) \neq \emptyset$. Let $\mathcal{S}_L \in \mathfrak{S}(AB(r))$ be the segment containing (a_L, b_L) and \mathbf{m}_L the slope of the segment. Then for all $\mathcal{S} \in \mathfrak{S}(AB(r))$, every point $(a, b) \in \mathcal{S}$ satisfies $a + \mathbf{b}m \leq a_L + b_L \mathbf{m}_L$.*

Proof. Let $\mathcal{S} \in \mathfrak{S}(AB(r))$. We consider $(a, b) \in \mathcal{S}$ and $(a, b) \neq (a_L, b_L)$. By definition, $a + \mathbf{b}m \leq a_L + b_L \mathbf{m}$, which leads to $\mathbf{m} \leq (a_L - a)/(b - b_L)$ since $b > b_L$.

Also by definition, $a_L + b_L \mathbf{m}_L \leq a + \mathbf{b}m_L$, which leads to $\mathbf{m}_L \geq (a_L - a)/(b - b_L)$.

Therefore, $\mathbf{m} \leq \mathbf{m}_L$, and thus $a + \mathbf{b}m \leq a_L + b_L \mathbf{m} \leq a_L + b_L \mathbf{m}_L$. \square

Theorem 6.5. *Let $G(s)$ be a neutral delay system defined by (6.1), and suppose that G has no unstable poles of small modulus and no chain of poles on the imaginary axis. Also suppose that the formal polynomial $\tilde{c}_d(z)$ defined in (6.3) has roots of modulus one, denoted by r , and that all values of ν_1 relative to each r satisfy $\Re(\nu_1) < 0$ where ν_1 is defined by (6.8). Then G is H_∞ -stable if and only if $\deg p \geq \deg t + x$ where $x = \max_r \{a_L\}$ with (a_L, b_L) defined as in (6.10) (it is the leftmost point among the lowest points of $AB(r)$).*

Proof. Since G has poles approaching the imaginary axis, then $|G(s)|_{s \in j\mathbb{R}}$ is large near these asymptotic poles.

Let us consider the denominator of G at a point s on the imaginary axis near an asymptotic pole relative to a root r of modulus one of $\tilde{c}_d(z)$. We can write $s = s_n + \eta_n \in j\mathbb{R}$, where s_n is one of the poles of the neutral chain relative to r . Recall that $s_n = (\lambda_n + \nu_1 n^{-y_1})/\tau + o(n^{-y_1})$. Since $\Re(\nu_1) \neq 0$, then η_n is at least of order $(-y_1)$ and has the form $\eta_n = \eta n^{-y_1} + o(n^{-y_1})$. We can then write

$$s = \frac{\lambda_n}{\tau} + \frac{\nu_1 + \eta\tau}{\tau n^{y_1}} + o(n^{-y_1}). \quad (6.15)$$

Note that s is of the same form as s_n if we denote $\nu'_1 = \nu_1 + \eta\tau$.

Therefore, the developments of the denominator of G around s as $|s| \rightarrow \infty$ and around s_n as $|s_n| \rightarrow \infty$ are the same. Recall from (6.9) and the discussion that follows the equation that the development of $d(s_n)$ as $|s_n| \rightarrow \infty$ is

$$d(s_n) = p(s_n) \left(\frac{f_i(\nu_1)}{n^{(a+b\mathbf{m})\mu}} + o(n^{-(a+b\mathbf{m})\mu}) \right),$$

where $(a, b) \in \mathcal{S}$ for each $\mathcal{S} \in \mathfrak{S}(AB(r))$, \mathbf{m} is the slope of \mathcal{S} , and $f_i(\nu_1) = \sum_{(a,b) \in \mathcal{S}} C(a, b, \nu_1)$. Hence, the development of $d(s)$ for $s \in j\mathbb{R}$ near s_n is

$$d(s) = p(s) \left(\frac{f_i(\nu_1 + \eta\tau)}{n^{(a+b\mathbf{m})\mu}} + o(n^{-(a+b\mathbf{m})\mu}) \right).$$

Since $s \in j\mathbb{R}$, then (6.15) shows that $\Re(\nu_1 + \eta\tau) = 0$, and thus $f_i(\nu_1 + \eta\tau) = f_i(j\Im(\nu_1 + \eta\tau))$. Since every root of $f_i(\nu_1)$ has strictly negative real part, then $f_i(j\Im(\nu_1 + \eta\tau)) \neq 0$. Hence, the order of the denominator of $G(s)$ is $n^{(d_0 - a - b\mathbf{m})\mu}$ where $d_0 = \deg p$.

Under the assumption that G has no chains of poles on the imaginary axis, the leftmost lowest point of $AB(r)$ is $(a_L, 0)$. Due to Lemma 6.4, $(a+b\mathbf{m})\mu \leq a_L\mu$ for all $\mathcal{S} \in \mathfrak{S}(AB(r))$. Then the lowest order of the denominator of $G(s)$ for $s \in j\mathbb{R}$ near s_n relative to r is $n^{(d_0 - a_L)\mu}$.

For all roots r of \tilde{c}_d , the lowest order of the denominator of $G(s)$ on the imaginary axis is $n^{(d_0 - x)\mu}$ with $x = \max_r \{a_L\}$. \square

6.5 Comparison with previous results

Now we apply Theorems 6.1, 6.3 and 6.5 to examine the classes of systems considered in Chapter 4 and 5. The results obtained here are the same as those obtained in the previous chapters. Note that we consider fractional and classical systems at the same time, that is $\mu \in (0, 1]$.

At the end of this section, we will summarize the stability results of all these classes of systems pointing when the method of this chapter allows one to conclude more quickly: we will see that Theorem 6.3 can be used to conclude in many situations.

Recall that in this chapter we make some changes of notation. In comparison with Chapters 4 and 5, $\alpha_{0,k} = \alpha_k$, $\alpha_{1,k} = \beta_k$, $\alpha_{2,k} = \gamma_k$, and $\alpha_{3,k} = \delta_k$.

6.5.1 The case where $m = 1$ and $\sum_{k=1}^N \alpha_{1,k} r^k \neq 0$

This case was considered in Subsection 4.3.1 in Chapter 4 and in Subsection 5.3.1 in Chapter 5.

Since $\sum_{k=1}^N \alpha_{1,k} r^k \neq 0$, then $(1, 0) \in AB(r)$. Recall from our discussion after Theorem 6.1 that $(0, m) \in AB(r)$. It is then easy to see that $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, 1), (1, 0)\}$ (see Figure 6.3). Therefore, Theorem 6.1 shows that

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = -\nu_1 \sum_{k=1}^N \alpha_{0,k} k r^k + \frac{\tau^\mu}{(j2\pi)^\mu} \sum_{k=1}^N \alpha_{1,k} r^k = 0 \quad \text{and} \quad y_1 = \mathbf{m}_1 \mu$$

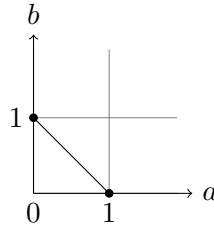


Figure 6.3 – The lower left boundary segment of $AB(r)$ in the case where $m = 1$ and $\sum_{k=1}^N \alpha_{1,k} r^k \neq 0$

which give respectively

$$\nu_1 = \frac{\tau^\mu \sum_{k=1}^N \alpha_{1,k} r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k} k r^k} \quad \text{and} \quad y_1 = \mu.$$

This result is identical to the one obtained in Theorems 4.2 and 5.1.

Some fractional systems in the class of systems considered in this subsection may have all chains of poles asymptotic to the imaginary axis from the left side. If

- these pole chains correspond to the roots of modulus one of the formal polynomials $\tilde{c}_d(z)$ that satisfy the conditions in this subsection,
- other roots of $\tilde{c}_d(z)$ are of modulus greater than one,
- and the system has no unstable poles of small modulus,

then due to Theorem 6.5 the necessary and sufficient condition for the system to be H_∞ -stable is $\deg t \leq \deg p - 1$ as for every root of modulus one of $\tilde{c}_d(z)$ the leftmost lowest point is $(1, 0)$. The same condition was obtained in Proposition 4.5.

6.5.2 The case where $m = 1$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, and $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$

This case was studied in Subsection 4.3.2 in Chapter 4 and in Subsection 5.3.2 in Chapter 5.

We have $(2, 0) \in AB(r)$ since $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$ and $(1, 0) \notin AB(r)$ since $\sum_{k=1}^N \alpha_{1,k} r^k = 0$. Also, $(0, m) \in AB(r)$ with $m = 1$. Then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, 1), (2, 0)\}$ (see Figure 6.4). Due to Theorem 6.1, we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = -\nu_1 \sum_{k=1}^N \alpha_{0,k} k r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \alpha_{2,k} r^k = 0 \quad \text{and} \quad y_1 = \mathbf{m}_1 \mu,$$

which gives

$$\nu_1 = \frac{\tau^{2\mu} \sum_{k=1}^N \alpha_{2,k} r^k}{(j2\pi)^{2\mu} \sum_{k=1}^N \alpha_{0,k} k r^k} \quad \text{and} \quad y_1 = 2\mu. \quad (6.16)$$

The same result was obtained in Subsections 4.3.2 and 5.3.2.

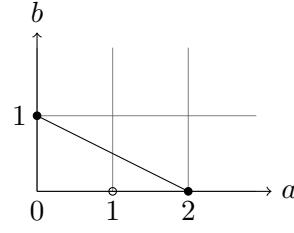


Figure 6.4 – The lower left boundary segment of $AB(r)$ in the case where $m = 1$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, and $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$. The black and white dots represent respectively points in $AB(r)$ and points not in $AB(r)$.

For fractional systems whose formal polynomial $\tilde{c}_d(z)$ has single roots of modulus one satisfying the conditions in this subsection and other roots of modulus greater than one, if other conditions in Theorem 6.5 are satisfied, then the system is H_∞ -stable if and only if $\deg t \leq \deg p - 2$. This is easy to derive since for all roots r of modulus one of $\tilde{c}_d(z)$, the leftmost lowest point of $AB(r)$ is $(2, 0)$. We obtained the same condition in Proposition 4.6.

6.5.3 The case where $m \geq 2$ and $\sum_{k=1}^N \alpha_{1,k} r^k \neq 0$

This case was studied in Subsection 4.4.1 in Chapter 4 and in Subsection 5.4.1 in Chapter 5.

Since $\sum_{k=1}^N \alpha_{1,k} r^k \neq 0$, then $(1, 0) \in AB(r)$. It is also known that $(0, m) \in AB(r)$ and $(0, m') \notin AB(r)$ for $m' < m$. Hence, it is easy to see that $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, m), (1, 0)\}$ (see Figure 6.5).

From Theorem 6.1, we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = \frac{(-1)^m \nu_1^m}{m!} \sum_{k=1}^N \alpha_{0,k} k^m r^k + \frac{\tau^\mu}{(j2\pi)^\mu} \sum_{k=1}^N \alpha_{1,k} r^k = 0 \quad \text{and} \quad y_1 = m_1 \mu$$

thus

$$\nu_1^m = \frac{(-1)^{m+1} m! \tau^\mu \sum_{k=1}^N \alpha_{1,k} r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{and} \quad y_1 = \frac{\mu}{m},$$

which verifies Theorems 4.7 and 5.7.

Theorem 6.5 shows that

- if $m = 2$, we have $n(\mathcal{S}_1) = 2$, $b_1 - b_2 = m - 0 = 2$ and $(a_2 - a_1)\mu = \mu \neq 2k$, $k \in \mathbb{Z}_+ \setminus \{0\}$, $\mu \in (0, 1]$, then the system is unstable,
- if $m \geq 3$, we have $n(\mathcal{S}_1) = 2$ and $b_1 - b_2 = m - 0 = m \geq 3$, then the system is unstable.

These are also the conclusions of Corollaries 4.8 and 5.8.

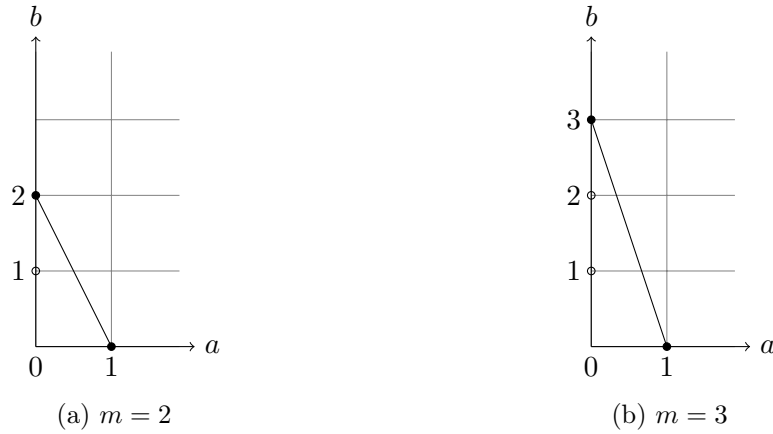


Figure 6.5 – The lower left boundary segment of $AB(r)$ in the case where $m \geq 2$ and $\sum_{k=1}^N \alpha_{1,k} r^k \neq 0$

6.5.4 The case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k \neq 0$, and $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$

This case was studied in Subsection 4.4.2 in Chapter 4 and in Subsection 5.4.2 in Chapter 5.

Due to the above conditions, $(1, 0) \notin AB(r)$ and $(1, 1), (2, 0) \in AB(r)$. It is also known that $(0, m) \in AB(r)$.

- If $m = 2$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, 2), (1, 1), (2, 0)\}$ (see Figure 6.6). Therefore, from Theorem 6.1 we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = \frac{\nu_1^2}{2} \sum_{k=1}^N \alpha_{0,k} k^2 r^k - \frac{\tau^\mu}{(j2\pi)^\mu} \nu_1 \sum_{k=1}^N \alpha_{1,k} k r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \alpha_{2,k} r^k = 0$$

and $y_1 = \mathbf{m}_1 \mu = \mu$. Identical results were presented in Theorems 4.9 and 5.9.

For fractional systems which have no unstable poles but have neutral chains of poles approaching the imaginary axis, if all these chains are relative to double roots of the formal polynomial $\tilde{c}_d(z)$ that satisfy the conditions in this subsection, then from Theorem 6.5 these systems are stable in the sense H_∞ if and only if $\deg t \leq \deg p - \max_r \{a_L\}$ where $\max_r \{a_L\} = 2$ since the leftmost lowest point of $AB(r)$ is $(a_L, b_L) = (2, 0)$ for all r being a root of modulus one of $\tilde{c}_d(z)$. The same stability condition was obtained in Proposition 4.13.

- If $m \geq 3$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1, \mathcal{S}_2\}$ with $\mathcal{S}_1 = \{(0, m), (1, 1)\}$ and $\mathcal{S}_2 = \{(1, 1), (2, 0)\}$ (see Figure 6.6). Therefore,

$$\begin{aligned} \sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) &= 0 \quad \text{and} \quad y_1 = \mathbf{m}_1 \mu, \\ \sum_{(a,b) \in \mathcal{S}_2} C(a, b, \nu_1) &= 0 \quad \text{and} \quad y_1 = \mathbf{m}_2 \mu, \end{aligned}$$

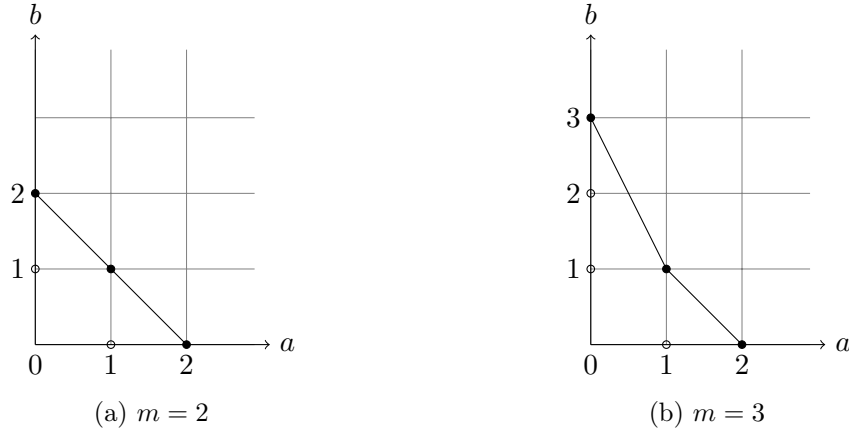


Figure 6.6 – The lower left boundary segment of $AB(r)$ in the case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k \neq 0$, and $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$

which are respectively equivalent to

$$\nu_1^{m-1} = \frac{(-1)^m m! \tau^\mu \sum_{k=1}^N \alpha_{1,k} k r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{and} \quad y_1 = \frac{\mu}{m-1},$$

$$\nu_1 = \frac{\tau^\mu \sum_{k=1}^N \alpha_{2,k} r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{1,k} k r^k} \quad \text{and} \quad y_1 = \mu.$$

These results are the same as those showed in Theorems 4.9 and 5.9.

If $m = 3$, we have $n(\mathcal{S}_1) = 2$, $b_1 - b_2 = m - 1 = 2$ and $(a_2 - a_1)\mu = (1 - 0)\mu = \mu \neq 2k, k \in \mathbb{Z}_+ \setminus \{0\}$, then the system is unstable.

If $m \geq 4$, we have $n(\mathcal{S}_1) = 2$ and $b_1 - b_2 = m - 1 \geq 3$, then the system is unstable.

In Chapter 5, we derive the same conclusions about the stability of the system (see Corollary 5.10).

6.5.5 The case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$, and $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$

This case was studied in Subsection 4.4.3 in Chapter 4 and in Subsection 5.4.3 in Chapter 5.

The above conditions imply that $(1, 0), (1, 1) \notin AB(r)$ and $(1, 2), (2, 0) \in AB(r)$.

- If $2 \leq m \leq 3$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, m), (2, 0)\}$ (see Figure 6.7). From Theorem 6.1, we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{and} \quad y_1 = m_1 \mu,$$

giving

$$\nu_1^m = \frac{(-1)^{m+1} m! \tau^{2\mu} \sum_{k=1}^N \alpha_{2,k} r^k}{(j2\pi)^{2\mu} \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{and} \quad y_1 = \frac{2\mu}{m}.$$

Theorems 4.14 and 5.17 presented the same results.

If $m = 2$, we have $n(\mathcal{S}_1) = 2$, $b_1 - b_2 = m - 0 = 2$, and $(a_2 - a_1)\mu = (2 - 0)\mu = 2\mu$. If $\mu \in (0, 1)$ then $2\mu \neq 2k, k \in \mathbb{Z}_+ \setminus \{0\}$, and thus Theorem 6.5 shows that the system is unstable. (The same conclusion was obtained in Corollary 4.16.) If $\mu = 1$, further analyses are needed. (These analyses were realized in Corollary 5.19.)

If $m = 3$, we have $n(\mathcal{S}_1) = 2$ and $b_1 - b_2 = m - 0 = 3$, then the system is unstable.

- If $m = 4$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, 4), (1, 2), (2, 0)\}$ (see Figure 6.7). From Theorem 6.1, we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{and} \quad y_1 = \mathbf{m}_1\mu,$$

and thus

$$\frac{\nu_1^4}{4!} \sum_{k=1}^N \alpha_{0,k} k^4 r^k + \frac{\nu_1^2 \tau^\mu}{2(j2\pi)^\mu} \sum_{k=1}^N \alpha_{1,k} k^2 r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \alpha_{2,k} r^k = 0 \quad \text{and} \quad y_1 = \frac{\mu}{2}.$$

Theorems 4.14 and 5.17 presented the same results.

Theorem 6.3 cannot be applied here and we have to study the sign of $\Re(\nu_{n,1})$ as in Corollaries 4.16 and 5.18, which showed that the system is unstable.

- If $m \geq 5$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1, \mathcal{S}_2\}$ with $\mathcal{S}_1 = \{(0, m), (1, 2)\}$ and $\mathcal{S}_2 = \{(1, 2), (2, 0)\}$ (see Figure 6.7). From Theorem 6.1, we obtain

$$\begin{aligned} \sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) &= 0 \quad \text{and} \quad y_1 = \mathbf{m}_1\mu, \\ \sum_{(a,b) \in \mathcal{S}_2} C(a, b, \nu_1) &= 0 \quad \text{and} \quad y_1 = \mathbf{m}_2\mu, \end{aligned}$$

which are respectively equivalent to

$$\begin{aligned} \nu_1^{m-2} &= \frac{(-1)^{m+1} m! \tau^\mu \sum_{k=1}^N \alpha_{1,k} k^2 r^k}{2(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{and} \quad y_1 = \frac{\mu}{m-2}, \\ \nu_1^2 &= -\frac{2\tau^\mu \sum_{k=1}^N \alpha_{2,k} r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{1,k} k^2 r^k} \quad \text{and} \quad y_1 = \frac{\mu}{2}. \end{aligned}$$

Theorems 4.14 and 5.17 presented the same results.

We have $n(\mathcal{S}_1) = 2$ and $b_1 - b_2 = m - 2 \geq 3$, then Theorem 6.5 shows that the system is unstable. This conclusion was also obtained in Corollaries 4.16 and 5.18.

6.5.6 The case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$, $\sum_{k=1}^N \alpha_{2,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{2,k} r^k \neq 0$, and $\sum_{k=1}^N \alpha_{3,k} r^k \neq 0$

This case was studied in Subsection 4.4.4 in Chapter 4 and in Subsection 5.4.4 in Chapter 5.

From the conditions, we deduce that $(1, 0), (1, 1), (2, 0) \notin AB(r)$ and $(1, 2), (2, 1), (3, 0) \in AB(r)$. We also have $(0, m) \in AB(r)$.

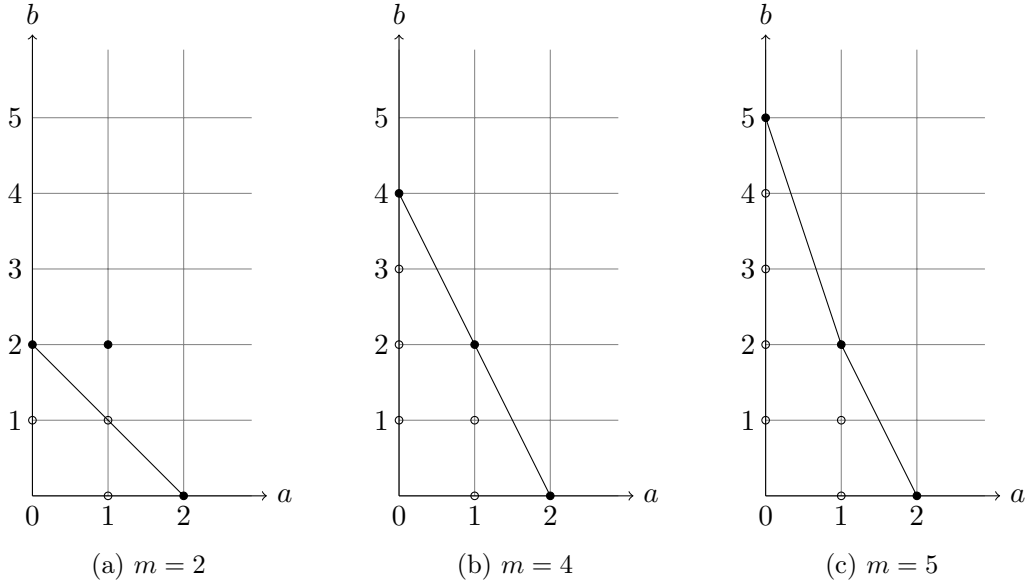


Figure 6.7 – The lower left boundary segment of $AB(r)$ in the case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$, and $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$

- If $\bar{m} = 2$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, 2), (3, 0)\}$ (see Figure 6.8). From Theorem 6.1, we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{and} \quad y_1 = \mathbf{m}_1 \mu,$$

giving

$$\nu_1^2 = -\frac{2\tau^{3\mu} \sum_{k=1}^N \alpha_{3,k} r^k}{(j2\pi)^{3\mu} \sum_{k=1}^N \alpha_{0,k} k^2 r^k} \quad \text{and} \quad y_1 = \frac{3\mu}{2}.$$

This approximation is the same as the one provided in Theorems 4.17 and 5.20. We have $n(\mathcal{S}_1) = 2$, $b_1 - b_2 = 2 - 0 = 2$, and $(a_2 - a_1)\mu = (3 - 0)\mu = 3\mu$. If $\mu \in (0, 1]$ and $\mu \neq 2/3$ then Theorem 6.5 shows that the system is unstable. The same conclusion was drawn in Corollaries 4.19 and 5.21.

- If $m = 3$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0, 3), (1, 2), (2, 1), (3, 0)\}$ (see Figure 6.8). From Theorem 6.1, we obtain

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{and} \quad y_1 = \mathbf{m}_1 \mu$$

leading to

$$-\frac{\nu_1^3}{3!} \sum_{k=1}^N \alpha_{0,k} k^3 r^k + \frac{\nu_1^2 \tau^\mu}{2(j2\pi)^\mu} \sum_{k=1}^N \alpha_{1,k} k^2 r^k - \frac{\nu_1 \tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \alpha_{2,k} k r^k + \frac{\tau^{3\mu}}{(j2\pi)^{3\mu}} \sum_{k=1}^N \alpha_{3,k} r^k = 0 \quad (6.17)$$

and $y_1 = \mu$. This approximation is the same as the one provided in Theorems 4.17 and 5.20.

Theorem 6.3 cannot be applied in this case. For classical systems, the sign of $\Re(\nu_{n,1})$ was analyzed in Corollary 5.22.

There exist fractional systems without unstable poles and with chains of poles asymptotic to the imaginary axis. If these chains correspond to the triple roots of the formal polynomial $\tilde{c}_d(z)$ that satisfy the conditions in this subsection, then Theorem 6.5 shows that the necessary and sufficient for the H_∞ -stability of the systems is $\deg t \leq \deg p - \max_r \{a_L\}$ where $\max_r \{a_L\} = 3$ since $a_L = 3$ for all the roots of modulus one of $\tilde{c}_d(z)$. This stability condition was also obtained in Proposition 4.18.

- If $m \geq 4$, then $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1, \mathcal{S}_2\}$ with $\mathcal{S}_1 = \{(0, m), (1, 2)\}$ and $\mathcal{S}_2 = \{(1, 2), (2, 1), (3, 0)\}$ (see Figure 6.8). From Theorem 6.1, we obtain

$$\begin{aligned} \sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{and} \quad y_1 = \mathbf{m}_1 \mu, \\ \sum_{(a,b) \in \mathcal{S}_2} C(a, b, \nu_1) = 0 \quad \text{and} \quad y_1 = \mathbf{m}_2 \mu, \end{aligned}$$

which are respectively equivalent to

$$\nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau^\mu \sum_{k=1}^N \alpha_{1,k} k^2 r^k}{2(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{and} \quad y_1 = \frac{\mu}{m-2},$$

and

$$\frac{\nu_1^2 \tau^\mu}{2(j2\pi)^\mu} \sum_{k=1}^N \alpha_{1,k} k^2 r^k - \frac{\nu_1 \tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \alpha_{2,k} k r^k + \frac{\tau^{3\mu}}{(j2\pi)^{3\mu}} \sum_{k=1}^N \alpha_{3,k} r^k = 0$$

and $y_1 = \mu$. These approximations are the same as those provided in Theorems 4.17 and 5.20.

We have $n(\mathcal{S}_1) = 2$. If $m = 4$, then $b_1 - b_2 = 4 - 2 = 2$ and $(a_2 - a_1)\mu = (1 - 0)\mu = \mu \neq 2k$ for all $\mu \in (0, 1]$ and $k \in \mathbb{Z}_+ \setminus \{0\}$. Hence the system is unstable due to Theorem 6.5.

If $m \geq 5$, then $b_1 - b_2 = m - 2 \geq 3$, and thus the system is unstable due to the same theorem.

The same conclusions were drawn in Corollaries 4.19 and 5.21.

6.5.7 Summary of previous results

Table 6.1 summaries the stability results of the classes of systems that are already studied in the two precedent chapters and reconsidered in this section. The comments in the table should be understood as follows:

- May be stable: There exist stable systems belonging to the considered class.
- $\Re(\nu_1) = 0$ or unstable: There are two possibilities for chains of poles relative to r and \bar{r} . First, $\Re(\nu_1) = 0$ for all values of ν_1 corresponding to r and \bar{r} . In that case, the next approximation is needed to determine the location of poles around the asymptotic axis. Second, the system is unstable.

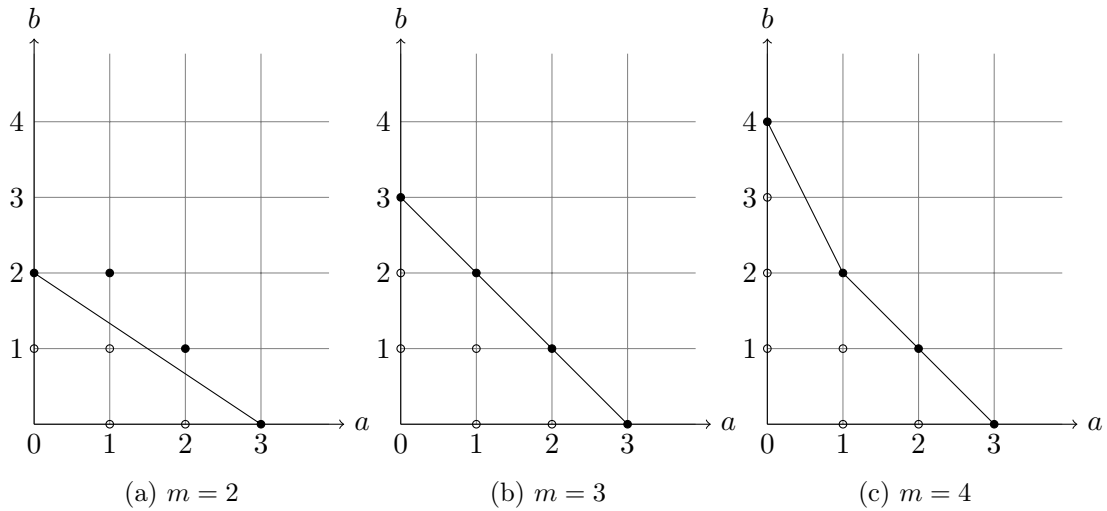


Figure 6.8 – The lower left boundary segment of $AB(r)$ in the case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$, $\sum_{k=1}^N \alpha_{2,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{2,k} r^k \neq 0$, and $\sum_{k=1}^N \alpha_{3,k} r^k \neq 0$

- Unstable (due to Theorem 6.3): All the systems are unstable due to Theorem 6.3.
- Unstable: All the systems are unstable. This conclusion is not due to Theorem 6.3.

6.6 Conclusion

In this chapter we have considered the H_∞ -stability of (fractional) neutral systems with commensurate delays and chains of poles asymptotic to the imaginary axis. More precisely, we have studied the location of these chains of poles around the axis and the boundedness of the transfer function on the axis. The new results generalize those presented in Chapters 4 and 5. They concern both classical and fractional systems and cover all possible cases, some of which were studied separately in the previous chapters. The analysis allows one to reach stability conclusions in a lot of cases except when the location of poles about the axis cannot be determined from the approximation provided and further analyses may be then needed.

	$0 < \mu < 1$	Ssec	$\mu = 1$	Ssec
$m = 1, \sum_{k=1}^N \alpha_{1,k} r^k \neq 0$	may be stable	4.3.1, 6.5.1	$\Re(\nu_1) = 0$ or un- stable	5.3.1, 6.5.1
$m \geq 2, \sum_{k=1}^N \alpha_{1,k} r^k \neq 0$	unstable (due to Theorem 6.3)	4.4.1, 6.5.3	unstable (due to Theorem 6.3)	5.4.1, 6.5.3
$m = 2, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k \neq 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$	may be stable	4.4.2, 6.5.4	$\Re(\nu_1) = 0$ or un- stable	5.4.2, 6.5.4
$m \geq 3, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k \neq 0$	unstable (due to Theorem 6.3)		unstable (due to Theorem 6.3)	
$m = 2, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$	unstable (due to Theorem 6.3)	4.4.3, 6.5.5	$\Re(\nu_1) = 0$ or un- stable	5.4.3, 6.5.5
$m = 3, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$	unstable (due to Theorem 6.3)		unstable (due to Theorem 6.3)	
$m = 4, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$	unstable		unstable	
$m \geq 5, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$	unstable (due to Theorem 6.3)		unstable (due to Theorem 6.3)	
$m = 2$ and $\mu \neq 3/2,$ $\sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k = 0,$ $\sum_{k=1}^N \alpha_{3,k} r^k \neq 0$	unstable (due to Theorem 6.3)	4.4.4, 6.5.6	unstable (due to Theorem 6.3)	5.4.4, 6.5.6
$m = 2$ and $\mu = 3/2,$ $\sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k = 0,$ $\sum_{k=1}^N \alpha_{3,k} r^k \neq 0$	$\Re(\nu_1) = 0$ or un- stable		unstable (due to Theorem 6.3)	
$m = 3, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{2,k} r^k \neq 0,$ $\sum_{k=1}^N \alpha_{3,k} r^k \neq 0$	may be stable		$\Re(\nu_1) = 0$ or un- stable	
$m = 4, \sum_{k=1}^N \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{1,k} r^k = 0,$ $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0,$ $\sum_{k=1}^N \alpha_{2,k} r^k = 0,$ $\sum_{k=1}^N k \alpha_{2,k} r^k \neq 0,$ $\sum_{k=1}^N \alpha_{3,k} r^k \neq 0$	unstable (due to Theorem 6.3)		unstable (due to Theorem 6.3)	

Table 6.1 – Classes of systems considered in the literature

Chapter 7

Stabilization of SISO fractional neutral systems with commensurate delays

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7.1 Introduction

Fractional systems are systems involving non-integer derivatives in the time domain and thus containing power of non-integer order of the Laplace variable s (i.e. s^μ with μ non-integer) in the frequency domain. Such models appear in many engineering fields since they describe certain systems better than their integer counterpart, see for example (Hilfer, 2000) and references therein.

As delays are usually encountered in real-life situations, fractional systems with delays have been of increasing interests in the past ten years. Stability of this kind of systems has been studied in several publications such as (Hotzel, 1998a; Bonnet and Partington, 2002, 2007; Bonnet et al., 2011; Hwang and Cheng, 2006; Fioravanti et al., 2011; Akbari Moornani and Haeri, 2010, 2011). However, not many results are available for stabilization. Some references are (Bonnet and Partington, 2007) on H_∞ -stabilization, (Si-Ammour et al., 2009) on sliding mode control, (Hamamci, 2007; Hamamci and Koksal, 2010) on stabilization of dead-time fractional systems, and (Lazarević, 2011) on finite-time stabilization.

In (Bonnet and Partington, 2007), a rather complete study from stability to stabilization is realized for a class of neutral fractional systems with one delay. The main result of this study is a parametrization of all stabilizing controllers. However, the problem of finding a parametrization for systems with large poles clustering the imaginary axis remained unsolved. Later, the stability of neutral fractional systems with commensurate delays and with large poles asymptotic to the imaginary axis has been studied in (Fioravanti et al., 2010). In this chapter we apply these stability results to examine some stabilizability properties of fractional neutral systems with commensurate delays and an infinite number of unstable poles. We find that a majority of these systems cannot be stabilized by the class of rational fractional controllers of commensurate order except systems with the lowest degree. The systems considered in (Bonnet and Partington, 2007) belong to this exception. A parametrization of stabilizing controllers is derived for these systems.

Remark that some systems we consider (those with infinitely many unstable poles) belong to the class considered in (Gümüşsoy and Özbay, 2004). However the parametrization of all stabilizing controllers they proposed requires an inner/outer factorization of the plant.

By using the parametrization previously derived, we demonstrate that for a large class of stabilizing controllers, the closed-loop system still has chains of poles asymptotic to the imaginary axis, which makes the stabilization sensitive to some parameter changes.

The rest of the chapter is organized as follows. In Section 7.2, we examine the stabilizability of neutral fractional systems with commensurate delays and with chains of poles in the right half-plane. In Section 7.3, we obtain a parametrization of all stabilizing controllers for systems with only one delay, allowing here chains of poles clustering the imaginary axis from the left or the right hand side. These systems with one delay continue to be considered in Section 7.4 where we study the stability of the closed-loop system with a large class of stabilizing controllers. Finally, we conclude the chapter by Section 7.5.

7.2 Stabilizability properties of fractional systems with commensurate delays

We study the H_∞ -stabilization of fractional time-delay systems of neutral type with commensurate delays given as follows

$$G(s) = \frac{1}{p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau}} \quad (7.1)$$

where $\tau > 0$ is the delay, p and q_k , $k \in \mathbb{N}_N$ are real polynomials in s^μ , $\mu \in (0, 1)$, $\deg p \geq 1$, $\deg p \geq \deg q_k$, and there is at least one q_k , $k \in \mathbb{N}_N$ such that $\deg p = \deg q_k$. Here the degree is interpreted as the degree in s^μ and so is an integer. In order to avoid multi-valued functions, we consider s in the Riemann sheet such that $-\pi < \arg s < \pi$.

This class of systems obviously belongs to those described by (2.2).

Let us consider controllers of the form

$$K(s) = \frac{N(s)}{D(s)} \quad (7.2)$$

where N and D are real polynomials in s^μ , $\mu \in (0, 1)$. These controllers are called rational fractional controllers of commensurate order. From (Partington and Bonnet, 2004, Lemma 4.1), we know that $\deg N \leq \deg D$ if $K(s)$ stabilizes $G(s)$ in the sense H_∞ . Suppose that $N(s)$ and $D(s)$ do not have common zeros, and $N(s)$ does not have common zeros with the denominator of $G(s)$.

The closed-loop $[G, K]$ is stable if and only if the following transfer functions are stable

$$\frac{1}{1 + G(s)K(s)} = \frac{D(s)(p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau})}{D(s)(p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau}) + N(s)}, \quad (7.3)$$

$$\frac{G(s)}{1 + G(s)K(s)} = \frac{D(s)}{D(s)(p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau}) + N(s)}, \quad (7.4)$$

$$\frac{K(s)}{1 + G(s)K(s)} = \frac{N(s)(p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau})}{D(s)(p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau}) + N(s)}. \quad (7.5)$$

Under the assumptions about the zeros of $N(s)$ and $D(s)$, the transfer functions (7.3), (7.4), and (7.5) do not have zero cancellation between the numerator and the denominator.

If the formal polynomial $\tilde{c}_d(z)$ defined by (2.4) has a root r with $|r| < 1$, then due to (2.7), the chain of poles relative to r is asymptotic to a vertical line lying in the open right half-plane. Thus this chain has infinitely many poles in the open right half-plane and the system is unstable. The stabilization of such systems under controllers of the form (7.2) is examined in the following proposition.

Proposition 7.1. *Let G be given as in (7.1). If $\tilde{c}_d(z)$ has roots of modulus strictly smaller than one, then G cannot be stabilized by a controller given as in (7.2).*

Proof. Since $\deg N \leq \deg D$ and $\deg p \geq 1$, the denominator of the closed-loop transfer function (7.4) also has the formal polynomial $\tilde{c}_d(z)$ with roots of modulus strictly smaller than one. \square

Another situation where G has an infinite number of poles in the open right half-plane is when G has chains of neutral poles approaching the imaginary axis from the right and the other neutral chains asymptotic to vertical lines in the open left half-plane. The next proposition addresses this class of systems.

Proposition 7.2. *Let G be given by (7.1). Suppose that the polynomial $\tilde{c}_d(z)$ has roots of modulus one of multiplicity one and that the other roots are of modulus greater than one. Suppose also that at least one root of modulus one of $\tilde{c}_d(z)$, denoted r , satisfies*

$$\Re(\nu_{n,1}) > 0 \quad (7.6)$$

where

$$\nu_{n,1} = \frac{\tau^\mu \sum_{k=1}^N \beta_k r^k}{(2jn\pi)^\mu \sum_{k=1}^N k\alpha_k r^k}.$$

Then G can be stabilized by controllers of the form (7.2) only if $\deg p = 1$.

Proof. Recall from Theorem 4.2 that the poles of large modulus corresponding to r , denoted by s_n , are approximated by

$$s_n \tau = -\ln(r) + 2jn\pi + \nu_{n,1} + \mathcal{O}(n^{-2\mu})$$

Under the assumption (7.6), we see that G has infinitely many poles in the open right half-plane.

Let us examine the denominator of the transfer functions of the closed-loop which is

$$D(s)(p(s) + \sum_{k=1}^N q_k(s)e^{-sk\tau}) + N(s) = D(s)p(s) + N(s) + \sum_{k=1}^N D(s)q_k(s)e^{-sk\tau}.$$

We consider the development at infinity of

$$\frac{D(s)q_k(s)}{D(s)p(s) + N(s)} = \bar{\alpha}_k + \frac{\bar{\beta}_k}{s^\mu} + o(s^{-\mu}).$$

As $\deg p \geq 1$ and $\deg D \geq \deg N$, we have that $\bar{\alpha}_k = \alpha_k$ where α_k is a coefficient of the development of $q_k(s)/p(s)$ as $|s| \rightarrow \infty$ given in (2.3). Now, if $\deg p > 1$, we also have that $\bar{\beta}_k = \beta_k$ where β_k is also defined in (2.3). In this case, the closed-loop has an infinite number of unstable poles and thus cannot be H_∞ -stable. \square

Remark 7.3. The systems considered in Proposition 7.2 are not the only ones with chains of poles approaching the imaginary axis from the right. This may also happen to systems with $\Re(\nu_{n,1}) = 0$ but this case needs further analysis as described in (Bonnet et al., 2011).

7.3 Parametrization of the set of stabilizing controllers in a particular case

The simplest systems described by (7.1) and with $\deg p = 1$ are systems with one delay. They have been studied in (Bonnet and Partington, 2007). For such systems with transfer function given by

$$G(s) = \frac{1}{(as^\mu + b) + (cs^\mu + d)e^{-s\tau}}, \quad (7.7)$$

where $a, b, c, d \in \mathbb{R}$, $a > 0$, $|a| = |c|$, and $\mu \in (0, 1)$, fractional PI controllers have been obtained.

These controllers are the starting point to obtain a parametrization of all stabilizing controllers, which is the main result of this section. Before stating the main result, we will recall the results on fractional PI controllers in (Bonnet and Partington, 2007). First, to simplify its presentation, we derive the opposite condition to Remark 4.1 in (Bonnet and Partington, 2007).

Lemma 7.4. $a_2 z^2 + a_1 z + a_0 = 0$ with $a_2, a_1, a_0 \in \mathbb{R}$, $a_2 > 0$ has all roots in $\{z \in \mathbb{C} \setminus \{0\} : |\text{Arg}(z)| > \mu\pi/2\}$ with $\mu \in (0, 1)$ if and only if $a_0 > 0$ and $a_1 > -2\sqrt{a_0 a_2} \cos(\mu\pi/2)$.

Proof. The equation has two strictly negative roots if and only if

$$\begin{aligned} & \begin{cases} \Delta = a_1^2 - 4a_0a_2 \geq 0 \\ a_1 > 0 \\ a_0 > 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \Delta = (a_1 - 2\sqrt{a_0a_2})(a_1 + 2\sqrt{a_0a_2}) \geq 0 \\ a_1 > 0 \\ a_0 > 0 \end{cases} \\ \Leftrightarrow & \begin{cases} a_1 \geq 2\sqrt{a_0a_2} \\ a_0 > 0 \end{cases} \end{aligned}$$

The equation has two complex conjugate roots, denoted $re^{\pm j\phi}$, with $\phi \in (\mu\pi/2, \pi) \cup (-\pi, -\mu\pi/2)$ if and only if

$$\begin{aligned} & \begin{cases} \Delta = a_1^2 - 4a_0a_2 < 0 \\ a_0 > 0 \\ \cos \phi = -\frac{a_1}{2\sqrt{a_0a_2}} < \cos\left(\frac{\mu\pi}{2}\right) \end{cases} \\ \Leftrightarrow & \begin{cases} a_1 < 2\sqrt{a_0a_2} \\ a_0 > 0 \\ a_1 > -2\sqrt{a_0a_2} \cos(\mu\pi/2) \end{cases} \end{aligned}$$

□

We now recall the characterization of H_∞ -stabilizing fractional PI controllers of systems (7.7).

Proposition 7.5 (Proposition 4.1 (Bonnet and Partington, 2007)). *Let G be given by (7.7) and $K(s) = k_p + k_i/s^\mu$ with $k_p, k_i \in \mathbb{R}$.*

1. *Let $a = c$. If k_p and k_i satisfy $\frac{b+d+k_p}{a+c} > -2\sqrt{\frac{k_i}{a+c}} \cos\left(\frac{\mu\pi}{2}\right)$ and $k_i > 0$ then K stabilizes G when $\tau = 0$.
Moreover, if $a(b+k_p-d) \cos(\frac{\mu\pi}{2}) > 0$, then K stabilizes G for small τ .
If k_p and k_i satisfy also $(b+k_p)^2 + 2ak_i \cos(\mu\pi) - d^2 > 0$ and $k_i(b+k_p) \cos(\frac{\mu\pi}{2}) > 0$ then K stabilizes G for all τ .*
2. *Let $a = -c$. If $k_i(b+k_p+d) > 0$, then K stabilizes G when $\tau = 0$.
Moreover, if $a(b+k_p+d) \cos(\frac{\mu\pi}{2}) > 0$ then K stabilizes G for small τ .
If k_p and k_i satisfy also $(b+k_p)^2 + 2ak_i \cos(\mu\pi) - d^2 > 0$ and $k_i(b+k_p) \cos(\frac{\mu\pi}{2}) > 0$ then K stabilizes G for all τ .*

Remark 7.6. The system G considered in Proposition 7.5 may have infinitely many poles in the open right or left half-plane.

In the following examples, we observe the change of side of the chain of poles from open-loop to closed-loop. We have $a = c$ for the first example and $a = -c$ for the second.

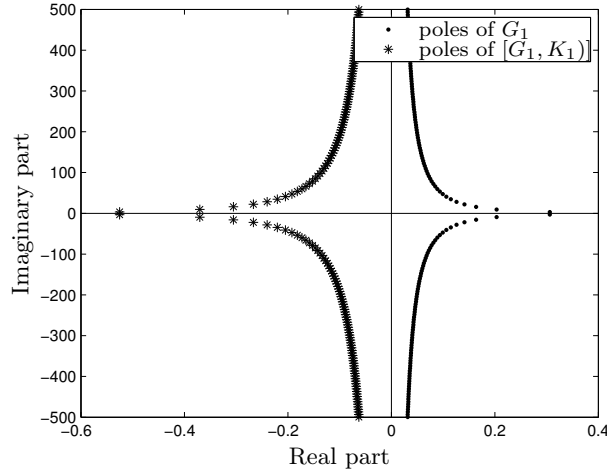


Figure 7.1 – Poles of $G_1(s)$ and of the closed-loop system $[G_1(s), K_1(s)]$

Example 7.1.

$$G_1(s) = \frac{1}{(s^{1/2} + 1) + (s^{1/2} + 2)e^{-s}},$$

$$K_1(s) = 3 + \frac{2}{s^{1/2}}. \quad (7.8)$$

The poles of the open-loop and of the closed-loop systems computed by QPmR algorithm (Vyhlidal and Zitek, 2014) are showed in Figure 7.1.

Example 7.2.

$$G_2(s) = \frac{1}{(s^{1/2} - 3) + (-s^{1/2} + 1)e^{-s}},$$

$$K_2(s) = 5 + \frac{1}{s^{1/2}}.$$

Given $K_0(s)$, a stabilizing controller of the system (7.7), we can directly obtain a parametrization of all stabilizing controllers without finding coprime factorizations by using (Quadrat, 2003b, Theorem 2).

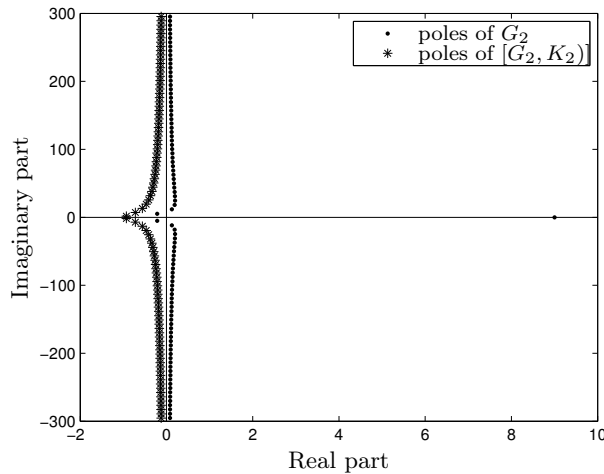
Proposition 7.7. *Let $G(s)$ be given as in (7.7). A parametrization with two degrees of freedom of all H_∞ -stabilizing controllers of $G(s)$ is given by*

$$\frac{-T(s^\mu R + T) + (s^{2\mu} Q_1 + Q_2 T^2) R}{s^\mu (s^\mu R + T) + (s^{2\mu} Q_1 + Q_2 T^2)} \quad (7.9)$$

where $Q_1, Q_2 \in H_\infty$ are two free parameters,

$$R(s) = (as^\mu + b) + (cs^\mu + d)e^{-s\tau},$$

$$T(s) = k_p s^\mu + k_i,$$

Figure 7.2 – Poles of $G_2(s)$ and of the closed-loop system $[G_2(s), K_2(s)]$

$k_i > 0$ and k_p satisfy

$$\begin{aligned} \frac{b+d+k_p}{a+c} &> -2\sqrt{\frac{k_i}{a+c}} \cos\left(\frac{\mu\pi}{2}\right) \quad \text{for } a=c, \\ k_i(b+d+k_p) &> 0 \quad \text{for } a=-c, \\ (a(b+k_p)-cd) \cos\left(\frac{\mu\pi}{2}\right) &> 0, \\ (b+k_p)^2 + 2ak_i \cos(\mu\pi) - d^2 &> 0, \\ k_i(b+k_p) \cos\left(\frac{\mu\pi}{2}\right) &> 0. \end{aligned}$$

Proof. From Proposition 7.5, we have that the fractional PI controller given by $K_0(s) = k_p + k_i/s^\mu$ stabilizes $G(s)$. Denoting

$$\begin{aligned} A(s) &= \frac{1}{1+G(s)K_0(s)} \\ B(s) &= -\frac{K_0(s)}{1+G(s)K_0(s)}, \end{aligned}$$

we have that $A(s)$ and $B(s)$ satisfy $0 \neq A(s), B(s) \in H_\infty$, $A(s) - B(s)G(s) = 1$ and $A(s)G(s) \in H_\infty$. Then Theorem 2 in (Quadrat, 2003b) shows that a parametrization of all stabilizing controllers of $G(s)$ in the sense of H_∞ is given by

$$\frac{B + Q_1A^2 + Q_2B^2}{A + Q_1A^2G + Q_2B^2G}$$

where $Q_1, Q_2 \in H_\infty$ are two free parameters. This gives (7.9). \square

However, the Youla-Kučera parametrization (with one parameter) may be more favourable for controller design and as G is H_∞ -stabilizable, we know that G necessarily admits a coprime factorization over H_∞ (Smith, 1989). The next proposition, which characterizes quasi-polynomials with all roots in the open left half-plane, will be useful to find a

coprime factorization of the transfer function of the system, which is required to derive a parametrization of stabilizing controllers using Youla-Kučera formula.

Proposition 7.8. *The equation*

$$(as^\mu + b) + (cs^\mu + d)e^{-s\tau} = 0 \quad (7.10)$$

with $a > 0$, $b, c, d, h \in \mathbb{R}$, $|a| = |c|$, and $\mu \in (0, 1)$ has no roots in the closed right half-plane for all $\tau \geq 0$ if and only if

- $b + d > 0$ and $b - d > 0$ if $a = c$,
- $b + d > 0$ and $b - d \geq 0$ if $a = -c$.

Proof. From (Marshall et al., 1992), we have that the equation has no roots in the closed right half-plane for all τ if and only if

- the equation has no roots in the closed right half-plane for $\tau = 0$,
- infinitely many poles approach the imaginary axis from the left side for τ sufficiently small,
- there is no roots crossing the imaginary axis for $\tau > 0$.

First, we consider the case $a = c$.

When $\tau = 0$, (7.10) becomes

$$2as^\mu + b + d = 0,$$

which has no root in the closed right half-plane if and only if $b + d > 0$.

For $\tau > 0$, no crossings means $W(\omega) := |p(i\omega)|^2 - |q(i\omega)|^2 \neq 0 \forall \omega > 0$ (Bonnet and Partington, 2007). We have

$$W(\omega) = (a^2 - c^2)\omega^{2\mu} + b^2 - d^2 + 2a\omega^\mu \cos\left(\frac{\mu\pi}{2}\right)(ab - cd). \quad (7.11)$$

For $a = c$, the above expression becomes

$$W(\omega) = (b - d) \left[(b + d) + 2a\omega^\mu \cos\left(\frac{\mu\pi}{2}\right) \right].$$

Then $W(\omega) \neq 0 \forall \omega > 0$ if and only if $b \neq d$ and $b + d \geq 0$.

Next, we have

$$\frac{as^\mu + b}{cs^\mu + d} = \frac{a}{c} + \frac{bc - ad}{c^2} \frac{1}{s^\mu} + \mathcal{O}(s^{-2\mu}).$$

Since $bc - ad \neq 0$ from the previous argument, Theorem 3.1 in (Bonnet and Partington, 2007) shows that large roots of (7.10) are stable if and only if

$$\begin{aligned} \frac{bc - ad}{c^2} \frac{c}{a} &> 0 \\ \Leftrightarrow b - d &> 0 \end{aligned} \quad (7.12)$$

From the three conditions, we derive that $b + d > 0$ and $b - d > 0$.

Similarly, we consider the case $a = -c$.

For $\tau = 0$, (7.10) becomes

$$b + d = 0,$$

which has no root in the closed right half-plane if and only if $b + d \neq 0$.

For $a = -c$, (7.11) becomes

$$W(\omega) = (b + d) \left[(b - d) + 2a\omega^\mu \cos\left(\frac{\mu\pi}{2}\right) \right].$$

Then $W(\omega) \neq 0 \forall \omega > 0$ if and only if $b + d \neq 0$ and $b - d \geq 0$.

With $a = -c > 0$, the condition (7.12) for stable chain of poles becomes $b + d > 0$.

From the three conditions, we derive that $b + d > 0$ and $b - d \geq 0$. \square

Now we give a parametrization of all stabilizing controllers of systems given by (7.7).

Theorem 7.9. *Let*

$$G(s) = \frac{1}{(as^\mu + b) + (cs^\mu + d)e^{-s\tau}}$$

with $a, b, c, d \in \mathbb{R}$, $a > 0$, $|a| = |c|$, and $\mu \in (0, 1)$. The set of all H_∞ -stabilizing controllers is given by

$$\frac{V + MQ}{U - NQ} \tag{7.13}$$

where

$$\begin{aligned} N(s) &= \frac{1}{(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}}, \\ M(s) &= \frac{(as^\mu + b) + (cs^\mu + d)e^{-s\tau}}{(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}}, \\ U(s) &= \frac{s^\mu[(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}]}{s^\mu(as^\mu + b + k_p) + k_i + s^\mu(cs^\mu + d)e^{-s\tau}}, \end{aligned} \tag{7.14}$$

$$V(s) = \frac{(k_p s^\mu + k_i)[(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}]}{s^\mu(as^\mu + b + k_p) + k_i + s^\mu(cs^\mu + d)e^{-s\tau}}, \tag{7.15}$$

Q is a free parameter in H_∞ , $k_i > 0$ and k_p satisfy

$$\begin{aligned} \frac{b + d + k_p}{a + c} &> -2\sqrt{\frac{k_i}{a + c}} \cos\left(\frac{\mu\pi}{2}\right) \quad \text{for } a = c, \\ k_i(b + d + k_p) &> 0 \quad \text{for } a = -c, \\ (a(b + k_p) - cd) \cos\left(\frac{\mu\pi}{2}\right) &> 0, \\ (b + k_p)^2 + 2ak_i \cos(\mu\pi) - d^2 &> 0, \\ k_i(b + k_p) \cos\left(\frac{\mu\pi}{2}\right) &> 0, \end{aligned}$$

and $a', b', c', d' \in \mathbb{R}$ satisfy

$$\begin{aligned} a' &> 0, \\ \frac{a'}{c'} &= \frac{a}{c}, \\ b' + d' &> 0, \end{aligned} \tag{7.16}$$

$$b' - d' > 0. \tag{7.17}$$

Proof. Under the above conditions, $(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}$ has no poles in the closed right half-plane.

Now, $M(s)$ can be decomposed as follows

$$M(s) = \frac{a}{a'} + \frac{b - \frac{a}{a'}b'}{(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}} + \frac{d - \frac{a}{a'}d'}{(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}}e^{-s\tau}.$$

Under the conditions (7.16) and (7.17), Corollary 3.2 in (Bonnet and Partington, 2007) shows that $N(s)$ and $M(s)$ belong to H_∞ .

It is also easy to see that $\inf_{\Re(s) > 0} (|N(s)| + |M(s)|) > 0$ so that (N, M) is a coprime factorization of G over H_∞ .

By the same arguments as in (Bonnet and Partington, 2007), knowing a stabilizing controller K_0 , one can derive the pair of Bézout factors U, V from the following expressions

$$\begin{aligned} \frac{1}{1 + GK_0} &= MU, \\ \frac{K_0}{1 + GK_0} &= MV. \end{aligned}$$

Now, U and V in (7.14), (7.15) are obtained by using a PI controller proposed in Proposition 7.5. By decomposing U and V as in (Bonnet and Partington, 2007), we conclude that $U, V \in H_\infty$. \square

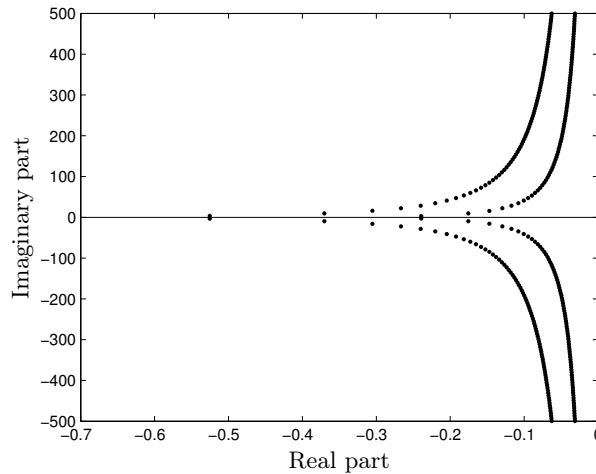
The following example shows another stabilizing controller of the system G_1 in Example 7.1 obtained from the above parametrization.

Example 7.3. A coprime factorization of G_1 is

$$\begin{aligned} N(s) &= \frac{1}{(s^{1/2} + 3) + (s^{1/2} + 2)e^{-s}}, \\ M(s) &= \frac{(s^{1/2} + 1) + (s^{1/2} + 2)e^{-s}}{(s^{1/2} + 3) + (s^{1/2} + 2)e^{-s}}. \end{aligned}$$

Besides, $U(s)$ and $V(s)$ are obtained based on the PI controller as in (7.8).

Now, in order to have another controller we choose $Q \in H_\infty$. The simplest case is a

Figure 7.3 – Poles of the transfer functions of the closed-loop system $[G_1(s), \bar{K}_1(s)]$

constant, e.g. $Q = 1$. The corresponding controller is

$$\begin{aligned} \bar{K}_1(s) &= \frac{V(s) + M(s)}{U(s) - N(s)} \\ &= [4s^{3/2} + 25s + 45s^{1/2} + 20 + (8s^{3/2} + 43s + 68s^{1/2} + 28)e^{-s} \\ &\quad + (4s^{3/2} + 18s + 24s^{1/2} + 8)e^{-2s}] / [s^{3/2} + 5s + 5s^{1/2} - 2 \\ &\quad + (2s^{3/2} + 9s + 10s^{1/2})e^{-s} + (s^{3/2} + 4s + 4s^{1/2})e^{-2s}] \end{aligned}$$

which involves commensurate delays.

7.4 H_∞ -stabilization

Let us denote

$$\begin{aligned} A(s) &= (as^\mu + b) + (cs^\mu + d)e^{-s\tau}, \\ A'(s) &= (a's^\mu + b') + (c's^\mu + d')e^{-s\tau}, \\ B(s) &= k_p s^\mu + k_i, \end{aligned}$$

then the transfer functions of the closed-loop system can be written as

$$\frac{1}{1 + GK} = \frac{A(s^\mu A'^2 - Q(s^\mu A + B))}{A'^2(s^\mu A + B)}, \quad (7.18)$$

$$\frac{G}{1 + GK} = \frac{s^\mu A'^2 - Q(s^\mu A + B)}{A'^2(s^\mu A + B)}, \quad (7.19)$$

$$\frac{K}{1 + GK} = \frac{A(BA'^2 + AQ(s^\mu A + B))}{A'^2(s^\mu A + B)}. \quad (7.20)$$

The transfer functions can have the terms A' or $(s^\mu A + B)$ in the denominator with appropriate values of Q . These terms have all roots in the open left half-plane but have

roots of large modulus approaching the imaginary axis. Small changes in their coefficients may move the asymptotic axis to the right and thus the closed-loop system becomes unstable.

In applications where robust stabilization is required, one may wish to eliminate chains of poles asymptotic to the imaginary axis. However, in the next propositions, we demonstrate that for a large class of controllers, this cannot be achieved. First we consider the case of rational μ and then irrational μ .

Proposition 7.10. *Let G be given as in (7.7) with μ be rational. If a controller that guarantees the internal stability of the closed-loop system has the form $K(s) = N_K(s)/D_K(s)$ where $N_K(s)$ and $D_K(s)$ are quasi-polynomials with real coefficients in $e^{-s\tau}$ and s^ν , $\nu \in (0, 1)$ and is rational, then the closed-loop system necessarily has chains of poles asymptotic to the imaginary axis.*

Proof. Due to (7.13), the controllers of interest can be written as

$$K = \frac{N_K}{D_K} = \frac{BA'^2 + AQ(s^\mu A + B)}{s^\mu A'^2 - Q(s^\mu A + B)}.$$

Then

$$Q = \frac{A'^2(s^\mu N_K - BD_K)}{(s^\mu A + B)(N_K + AD_K)}.$$

If Q is written as $Q = N_Q/D_Q$, then $N_Q(s)$ and $D_Q(s)$ are quasi-polynomials in $e^{-s\tau}$ and s^δ where $\delta \in (0, 1)$ such that $\mu = m\delta$ and $\nu = m'\delta$ with $m, m' \in \mathbb{N}$.

Now, let us consider the following transfer function of the closed-loop system

$$\frac{G}{1 + GK} = \frac{s^\mu A'^2 D_Q - N_Q(s^\mu A + B)}{A'^2 (s^\mu A + B) D_Q}.$$

The denominator of the closed-loop transfer function involves $A'^2(s^\mu A + B)$ which corresponds to chains of poles approaching the imaginary axis.

To eliminate all the chains of poles asymptotic to the imaginary axis of the transfer function, a necessary condition is that all the roots of modulus one of the formal polynomial corresponding to the denominator are roots of the formal polynomial corresponding to the numerator.

Recall that the corresponding formal polynomial is deduced from a quasi-polynomial by picking up highest degree terms. In the numerator of the transfer function, $\deg(s^\mu A'^2 D_Q) > \deg(N_Q(s^\mu A + B))$. Indeed $\deg(s^\mu A'^2) > \deg(s^\mu A + B)$ and $\deg D_Q \geq \deg N_Q$ since $Q = N_Q/D_Q \in H_\infty$. Therefore, the highest degree term of the numerator is a multiple of $s^\mu (a' s^\mu + c' s^\mu e^{-s\tau})^2 s^{d_{dQ}\delta} \tilde{c}_{dQ}(e^{-s\tau})$ where $d_{dQ} = \deg D_Q$ and \tilde{c}_{dQ} is the formal polynomial corresponding to D_Q . The formal polynomial associated to the numerator is $(1 + (c'/a')z)^2 \tilde{c}_{dQ}(z)$ with $z = e^{-s\tau}$.

By similar arguments, we derive that the formal polynomial associated to the denominator is $(1 + (c'/a')z)^2 (1 + (c/a)z) \tilde{c}_{dQ}(z)$. It then has one root of modulus one more than the

formal polynomial associated to the numerator since $|c/a| = 1$ and does not satisfy the necessary condition being that all the roots of modulus one of the formal polynomial corresponding to the denominator are roots of the formal polynomial corresponding to the numerator. Hence, the closed-loop system has at least one chain of poles approaching the imaginary axis for all controllers of the prescribed form. \square

Remark 7.11. Controllers of the form $K(s) = N_K(s)/D_K(s)$ where $N_K(s)$ and $D_K(s)$ are quasi-polynomials in $e^{-s\tau}$, s^μ , and s are a particular case of the controllers considered in Proposition 7.10. Indeed, if $\mu = m/n$ with $m, n \in \mathbb{N}$, then $N_K(s)$ and $D_K(s)$ can be seen as quasi-polynomials in $e^{-s\tau}$ and $s^{1/n}$.

We now give an example to illustrate Proposition 7.10 as well as Remark 7.11.

Example 7.4.

$$G_2(s) = \frac{1}{s^{1/2} + (s^{1/2} + 2)e^{-s\tau}}$$

This system has one chain of poles approaching the imaginary axis from the right.

For $Q = 1/(s + 1)$ and A', B chosen as follows

$$\begin{aligned} A' &= (s^{1/2} + 3) + (s^{1/2} - 1)e^{-s\tau}, \\ B &= 3s^{1/2} + 2, \end{aligned}$$

which satisfy the conditions in Theorem 7.9, the controller is

$$K(s) = \frac{N_K(s)}{D_K(s)}$$

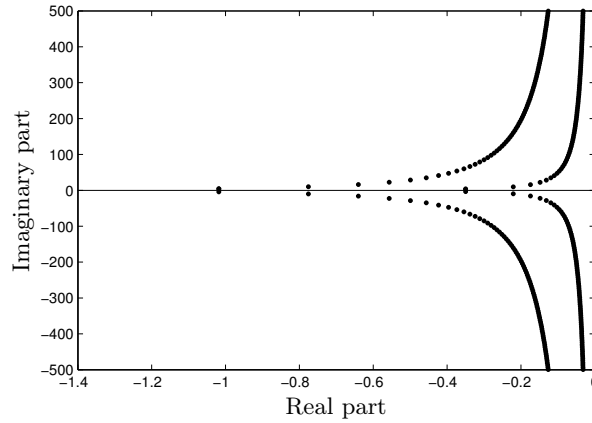
where

$$\begin{aligned} N_K(s) &= (3s^{5/2} - 4s^2 + 3s^{3/2} + 2s + 3s^{1/2} + 2)e^{-2s\tau} \\ &\quad + (6s^{5/2} + 16s^2 - 2s^{3/2} + 11s - 2s^{1/2} - 8)e^{-s\tau} \\ &\quad + 3s^{5/2} + 20s^2 + 43s^{3/2} + 41s + 41s^{1/2} + 18, \\ D_K(s) &= (s^{5/2} - 2s^2 + 2s^{3/2} - 2s + s^{1/2})e^{-2s\tau} \\ &\quad + (2s^{5/2} + 4s^2 - 4s^{3/2} + 3s - 8s^{1/2})e^{-s\tau} \\ &\quad + s^{5/2} + 6s^2 + 10s^{3/2} + 5s + 6s^{1/2} - 2, \end{aligned}$$

which are quasi-polynomials in $e^{-s\tau}$, $s^{1/2}$, and s .

The denominator of the transfer functions of the closed-loop system is $[(s^{1/2} + 3) + (s^{1/2} - 1)e^{-s\tau}]^2[(s - 2s^{1/2})e^{-s\tau} + s + 3s^{1/2} + 2](s + 1)$. It has three chains of poles asymptotic to the imaginary axis. Among them, two chains are identical. The poles of the closed-loop system are showed in Figure 7.4.

We now consider the case of irrational μ . Here the result is restricted to a class of controllers involving s^μ .

Figure 7.4 – Poles of the closed-loop system $[G_2(s), K(s)]$

Proposition 7.12. *Let G be given as in (7.7) with μ be irrational. If a stabilizing controller has the form $K(s) = N_K(s)/D_K(s)$ where $N_K(s)$ and $D_K(s)$ are quasi-polynomials with real coefficients in s^μ and $e^{-s\tau}$, then the closed-loop system necessarily has chains of poles asymptotic to the imaginary axis.*

Proof. Here, we deduce that $N_Q(s)$ and $D_Q(s)$ are quasi-polynomials in $e^{-s\tau}$ and s^μ . Then similar arguments on formal polynomials lead to the conclusion. \square

7.5 Conclusion

In this chapter, we have considered first the stabilization of general fractional delay systems of the neutral type by rational fractional controllers of commensurate order.

Then, for the special class of fractional delay systems with one delay we have derived a two-degree-of-freedom parametrization of the set of all H_∞ -stabilizing controllers and given an explicit expression of coprime and Bézout factors allowing to obtain the Youla-Kučera parametrization of all the H_∞ -stabilizing controllers. However, we have then proved that a large class of stabilizing controllers is unable to put the infinite number of poles far away from the imaginary axis in the left half-plane.

Future work could consider a larger class of stabilizing controllers, namely those which might contain non-commensurate delays or terms in e^{s^ν} as they still remain a simple class of controllers. But, non-commensurate delays increase a lot the difficulty of the analysis and make the implementation of such controllers non trivial. Of course the same question addressed in full generality to the whole class of stabilizing controllers is a theoretically challenging one.

After that, future work will be devoted to the stabilization of a larger class of systems with several delays.

Chapter 8

Conclusions

In this thesis, we have considered the stability analysis and stabilization problems, both in the sense of H_∞ , of linear fractional systems with delays. Concretely, we have dealt with systems with commensurate fractional orders and with commensurate/non-commensurate delays. All the results have been established exclusively in the frequency domain using analytical techniques.

Two classes of systems have been considered.

The first class consists of MISO fractional systems with multiple I/O delays (which are not necessarily commensurate). For this class of systems, the stabilization problem has been addressed in Chapter 3. In the framework of the factorization approach to analysis and synthesis problems, explicit expressions of left and right coprime factorizations and Bézout factors were derived.

The second class involves SISO fractional neutral systems with commensurate delays. In particular, we have been interested in the delicate case where poles approach the imaginary axis. For this class of systems, we have considered both problems of stability analysis and stabilization.

Chapter 4 has been dedicated to answer the stability question for a large class of systems, in particular systems with multiple chains of poles asymptotic to the imaginary axis. The location of neutral chains of poles w.r.t. the imaginary axis has been determined and the necessary and sufficient conditions obtained are related not only to the location of poles but also the relative order between the numerator and the denominator of the transfer function.

The results on approximating poles of neutral chains for fractional delay systems have been carried over to classical delay systems by simply replacing the fractional order $\mu \in (0, 1)$ with the integer order $\mu = 1$ in Chapter 5. Analyzing these approximations in order to determine the pole location about the asymptotic axis leads to different results in certain cases. The phenomena observed in these cases is that while for fractional systems stable chains of poles may be indicated by the first approximation, for classical systems we need higher approximations to detect such chains. Nevertheless, this phenomena is not general.

In Chapter 6, a unified approach to stability analysis has been proposed. This new method addresses both fractional and classical neutral systems and covers not only the cases studied in Chapters 4 and 5 but also all other unsolved cases. The method has been carefully described with the intention of implementation in computation software.

The stabilization problem has been considered in Chapter 7. First, we have studied the stabilization of general fractional delay systems of neutral type by rational fractional controllers of commensurate order. Then, for the special class of fractional delay systems with one delay we have derived two parametrizations of the set of all H_∞ -stabilizing controllers. The first parametrization has two degrees of freedom and has been obtained immediately with a particular stabilizing controller. The second is the usual Youla-Kučera parametrization constructed from the coprime factorizations and Bézout factors derived in explicit forms. However, we have then proved that a large class of stabilizing controllers is unable to put the infinite number of poles far away from the imaginary axis in the left half-plane.

Future work could consist of the following directions.

For MISO systems with I/O delays, in Chapter 3, doubly coprime factorizations have not been obtained in the general case where elements of the transfer matrix may have identical poles. In order to use the Youla-Kučera parametrization, we need to determine the right factors which are now still missing.

In the set of stabilizing controllers constructed from the obtained coprime and Bézout factors, we should investigate methods to choose controllers to be implemented that are not sensitive to parameter uncertainties. This problem was reported for classical systems in (Gumussoy, 2012).

For SISO fractional neutral systems, the stability analysis results presented in Chapters 4, 5 and 6 could be used to decide on H_∞ -stabilizability of several classes of fractional delay systems by rational or fractional controllers (with delays).

The unified method in Chapter 6 allows one to reach stability conclusions in all cases except when the location of poles about the axis cannot be determined from the approximation provided and further analyses may be then needed. Although the next approximation terms can be determined using the same procedure as presented for some cases in the chapter, one has to repeat this procedure for each particular case. Hence, future work could consist of investigating methods to determine approximation terms with less effort required.

For the stabilization problem of fractional neutral systems in the critical case of poles asymptotic to the imaginary axis, future work could consider a larger class of stabilizing controllers, namely those which might contain non-commensurate delays or terms in e^{s^ν} as they still remain a simple class of controllers. Although non-commensurate delays increase a lot the difficulty of the analysis, studying these systems could provide a better understanding of the behaviors of real systems where variation of delays usually occurs. Of course the same question addressed in full generality to the whole class of stabilizing controllers is a theoretically challenging one. After that, future work could be devoted to the stabilization of a larger class of systems with several delays.

We are integrating the results obtained in Chapters 6 and 7 in the Matlab toolbox YALTA which can be downloaded at <http://team.inria.fr/disco/software/>.

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Appendix A

List of publications

Conference papers

- [1] L. H. V. Nguyen, A. R. Fioravanti, and C. Bonnet. Analysis of neutral systems with commensurate delays and many chains of poles asymptotic to same points on the imaginary axis. In *10th IFAC Workshop on Time Delay Systems*, Boston, MA, June 2012.
- [2] L. H. V. Nguyen and C. Bonnet. Coprime factorizations of MISO fractional time-delay systems. In *20th International Symposium on Mathematical Theory of Networks and Systems*, Melbourne, Australia, July 2012.
- [3] L. H. V. Nguyen and C. Bonnet. Stability analysis of fractional neutral time-delay systems with multiple chains of poles asymptotic to same points in the imaginary axis. In *51st IEEE Conference on Decision and Control*, Maui, Hawaii, December 2012.
- [4] L. H. V. Nguyen and C. Bonnet. Right coprime factorizations of MISO fractional time-delay systems. In *1st IFAC Workshop on Control of Systems Modeled by Partial Differential Equations*, Paris, France, September 2013.
- [5] L. H. V. Nguyen and C. Bonnet. Stabilization of fractional neutral systems with one delay and a chain of poles asymptotic to the imaginary axis. In *International Conference on Fractional Differentiation and its Applications*, Catania, Italy, June 2014.

Book chapters

- [1] D. Avanesoff, A. R. Fioravanti, C. Bonnet, and L. H. V. Nguyen. H_∞ -stability analysis of (fractional) delay systems of retarded and neutral type with the Matlab toolbox YALTA. In T. Vyhlídal, J. F. Lafay, and R. Sipahi, editors, *Delay systems: From Theory to Numerics and Applications*, volume 1 of *Advances in Delays and Dynamics*. Springer, 2014.

- [2] L. H. V. Nguyen and C. Bonnet. Stabilization of some fractional neutral delay systems which possibly possess an infinite number of unstable poles. In C. Bonnet, H. Mounier, H. Özbay, and A. Seuret, editors, *Low complexity controllers for time-delay systems*, volume 2 of *Advances in Delays and Dynamics*. Springer, 2014.

Preprints

- [1] L. H. V. Nguyen, C. Bonnet, and A. R. Fioravanti. H_∞ -stability analysis of fractional delay systems of neutral type. Submitted to *SIAM Journal on Control and Optimization*.

Appendix B

Résumé

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B.1 Introduction

Dans ce travail, nous abordons les problèmes d'analyse de stabilité et de stabilisation de plusieurs classes de systèmes SISO et MIMO. Nous travaillons dans le domaine fréquentiel et notre objectif est de trouver des conditions de stabilité faciles à vérifier ainsi que des expressions explicites de contrôleurs ayant à l'esprit une intégration de nos résultats dans un logiciel.

Nous sommes intéressés par deux grandes classes de systèmes : les systèmes à retard et les systèmes fractionnaires. Les deux ont besoin d'outils de la théorie du contrôle de dimension infinie.

Dans le domaine temporel, les modèles comprennent des dérivées et/ou des intégrales d'ordre arbitraire. De même, ils contiennent dans le domaine fréquentiel des puissances d'ordre arbitraire de la variable de Laplace s . Pour plus de détails sur l'analyse fractionnaire et des exemples, voir (Oldham and Spanier, 1974; Podlubny, 1998) et leurs références.

Ces modèles se retrouvent dans de nombreux domaines d'applications, voir par exemple (Westerlund and Ekstam, 1994; Knospe and Zhu, 2011; Vinagre et al., 1998; Grahovac and Zigic, 2010), puisque les lois fractionnaires ont été de plus en plus utilisées pour décrire des phénomènes physiques et les modèles fractionnaires collent mieux aux données recueillies que les modèles d'ordre entier et avec moins de paramètres.

Dans le domaine de la commande, de nombreux résultats sont disponibles sur les contrôleurs fractionnaires et leur mise en œuvre. Voir, par exemple (Oustaloup et al., 1995; Podlubny, 1999; Chen et al., 2009; Magin et al., 2011) et leurs références.

Lorsqu'on considère des schémas de commande avec des systèmes fractionnaires, il est naturel de penser à des systèmes fractionnaires à retards car les retards sont couramment rencontrés dans les systèmes réels en raison de la communication ou des distances de transport et leurs effets sur la stabilité ne peuvent être négligés.

Il y a eu un intérêt croissant pour l'étude des systèmes fractionnaires à retards. La question de la stabilité des systèmes linéaires fractionnaires à retards a été étudiée par de nombreux auteurs (Hotzel, 1998a; Bonnet and Partington, 2002, 2001; Chen and Moore, 2002; Deng et al., 2007; Akbari Moornani and Haeri, 2010, 2011). Toutes les conditions de stabilité obtenues dans ces articles concernent la localisation des pôles dans le plan complexe. Pour les systèmes fractionnaires de type retardé, la condition nécessaire et suffisante pour la stabilité est "pas de pôle dans le demi-plan droite fermé" qui est classique. Afin de vérifier cette condition, plusieurs méthodes numériques ont été proposées (Hwang and Cheng, 2005, 2006; Ozturk and Uraz, 1985; Fioravanti et al., 2012; Mesbahi and Haeri, 2013).

Pour les systèmes de type neutre, la condition "pas de pôle dans le demi-plan droite fermé" n'est que nécessaire. Ceci peut être expliqué par les localisations compliquées des pôles des systèmes neutres : une infinité de pôles isolés se rassemblent dans des bandes verticales dans le plan complexe (Bellman and Cooke, 1963; Hotzel, 1998a; Bonnet and Partington, 2002).

Dans le cas le plus simple des systèmes d'ordres fractionnaires commensurables et à retards commensurables où le phénomène ci-dessus se réduit à des pôles asymptotiques à des axes verticaux, des tentatives ont été faites dans (Bonnet and Partington, 2007; Fioravanti et al., 2010) pour obtenir des conditions nécessaires et suffisantes pour la stabilité H_∞ (qui est une notion plus faible que la stabilité BIBO) pour une sous-classe de ces systèmes.

Certains travaux ayant le même objectif sont également disponibles pour les systèmes classiques (d'ordres entiers) de type neutre pour lesquels la même difficulté est rencontrée. Dans le domaine fréquentiel, nous avons (Bonnet et al., 2011) pour la stabilité H_∞ et (Abusaksaka and Partington, 2014) pour la stabilité BIBO. Et dans le domaine temporel, (Rabah et al., 2012) a considéré la stabilité asymptotique.

Bien qu'il y ait eu de nombreux résultats concernant l'analyse de stabilité, le problème de stabilisation des systèmes fractionnaires à retards a très peu été traité (Hotzel, 1998b; Bonnet and Partington, 2001, 2007; Özbay et al., 2012).

Pour étudier ce problème ouvert de stabilisation des systèmes à retards fractionnaires, nous choisissons l'approche de factorisation (Vidyasagar, 1985). Avec sa nature algébrique, cette approche puissante permet de dériver l'ensemble des contrôleurs stabilisants qui peuvent être utilisés pour étudier divers problèmes de contrôle et en particulier la commande robuste.

Pour les systèmes MIMO, la question de paramétrage de tous les contrôleurs stabilisants a été étudiée dans (Mori, 2002; Quadrat, 2006b; Mirkin and Raskin, 1999; Moelja and Meinsma, 2003). Notre objectif dans ce travail est d'obtenir des expressions explicites de factorisations copremières et de facteurs de Bézout des systèmes fractionnaires MIMO avec retards en entrées et/ou sortie.

Nous considérons deux classes de systèmes fractionnaires linéaires invariants dans le temps avec retards discrets. Le premier se compose de systèmes fractionnaires MISO avec retards quelconques en entrées et/ou sortie. La seconde se compose de systèmes fractionnaires neutres SISO avec retards commensurables.

Le résumé est organisé comme suit. Tout d'abord, quelques préliminaires sont donnés dans la Section B.2. Nous étudions la stabilisation de la première classe de systèmes dans la Section B.3 en utilisant l'approche de factorisation. Nous obtenons des expressions explicites de factorisations copremières à gauche et à droite et les facteurs de Bézout associés, qui sont les éléments pour constituer l'ensemble des contrôleurs stabilisants. Pour la deuxième classe de systèmes, nous sommes intéressés au cas critique où ces systèmes ont des pôles asymptotiques à l'axe imaginaire. Tout d'abord, l'analyse de stabilité est réalisée dans la Section B.4. Cette analyse est applicable pour les systèmes classiques de la même forme. En outre, elle peut facilement être programmée dans des logiciels de calcul. Ensuite, la question de stabilisation est étudiée dans la Section B.5 pour une sous-classe de systèmes fractionnaires, en utilisant les résultats de l'analyse de stabilité et l'approche de factorisation. Enfin, nous donnons les conclusions et les perspectives dans la Section B.6.

Notons que comme il s'agit d'un résumé, les résultats sont tous donnés sans preuves.

B.2 Préliminaires

B.2.1 Notations

\mathbb{C}_+	ensemble de nombres complexes à partie réelle positive
$\overline{\mathbb{C}_+}$	ensemble de nombres complexes à partie réelle non-négative
$\text{card}(\mathcal{L})$	nombre d'éléments de l'ensemble \mathcal{L}
\mathbb{N}	ensemble de nombres naturels (non compris zéro)
\mathbb{N}_N	ensemble des N premiers nombres naturels
\mathbb{R}_+	ensemble de nombres réels positifs
$[x]$	partie entière de $x \in \mathbb{R}$.
\mathbb{Z}_+	ensemble des nombres entiers non-négatifs
\mathbb{Z}_+^*	ensemble des nombres entiers positifs

Les pôles (resp. racines) dans le demi-plan droit fermé $\overline{\mathbb{C}_+}$ sont appelés les pôles (resp. racines) instables.

B.2.2 Stabilité H_∞

Les références utilisées pour cette sous-section sont (Curtain and Zwart, 1995; Zhou et al., 1995).

Définition B.1.

$$L_2[0, \infty) := \{f : [0, \infty) \mapsto \mathbb{C} \mid f \text{ est Lebesgue-mesurable et } \int_0^\infty |f(t)|^2 dt < \infty\}.$$

Définition B.2. Un système linéaire continu défini par un opérateur linéaire

$$\Sigma : L_2[0, \infty) \mapsto L_2[0, \infty)$$

est L_2 -stable si

$$\|\Sigma\|_2 < \infty,$$

où $\|\Sigma\|_2$ est la norme de l'opérateur et est défini par

$$\|\Sigma\|_2 := \sup\{\|\Sigma f\|_2 \mid f \in L_2[0, \infty), \|f\|_2 = 1\} = \sup_{0 \neq f \in L_2[0, \infty)} \frac{\|\Sigma f\|_2}{\|f\|_2}.$$

Autrement dit, un système L_2 -stable produit un signal de sortie à énergie bornée pour un signal d'entrée à énergie bornée.

Définition B.3 (Espaces de Hardy).

$$H_2(\mathbb{C}_+) := \{f : \mathbb{C}_+ \mapsto \mathbb{C} \mid f \text{ est analytique dans } \mathbb{C}_+ \text{ et } \sup_{\sigma > 0} \int_{-\infty}^\infty |f(\sigma + j\omega)|^2 d\omega < \infty\},$$

$$H_\infty(\mathbb{C}_+) := \{f : \mathbb{C}_+ \mapsto \mathbb{C} \mid f \text{ est analytique dans } \mathbb{C}_+ \text{ et } \sup_{s \in \mathbb{C}_+} |f| < \infty\}.$$

Théorème B.4 (Théorème de Paley-Wiener). $L_2[0, \infty)$ est isomorphe à $H_2(\mathbb{C}_+)$ par la transformée de Laplace.

Définition B.5.

$$L_\infty(j\mathbb{R}) := \{f : j\mathbb{R} \mapsto \mathbb{C} \mid \text{ess sup}_{\omega \in \mathbb{R}} |f(j\omega)| < \infty\}.$$

Théorème B.6. Si $G \in H_\infty(\mathbb{C}_+)$ et $\hat{u} \in H_2(\mathbb{C}_+)$, alors $G\hat{u} \in H_2(\mathbb{C}_+)$. De plus, la norme de l'opérateur de multiplication $\hat{\Sigma} : \hat{u} \mapsto G\hat{u}$ définie par

$$\|\hat{\Sigma}\| := \sup_{0 \neq \hat{u} \in H_2(\mathbb{C}_+)} \frac{\|G\hat{u}\|_2}{\|\hat{u}\|_2},$$

satisfait

$$\|\hat{\Sigma}\| = \|G\|_\infty.$$

Lemme B.7. $H_\infty(\mathbb{C}_+)$ est une sous-espace de $L_\infty(j\mathbb{R})$.

Théorème B.8. Si $G \in L_\infty(j\mathbb{R})$, alors $G \in H_\infty(\mathbb{C}_+)$ si et seulement si $G\hat{u} \in H_2(\mathbb{C}_+)$ pour tous $\hat{u} \in H_2(\mathbb{C}_+)$.

Ainsi, d'après les Théorèmes B.4 et B.8, si l'on se restreint aux systèmes linéaires invariants dans le temps dont la fonction de transfert appartient à $L_\infty(j\mathbb{R})$, alors un système linéaire invariant dans le temps est L_2 -stable si et seulement si sa fonction de transfert appartient à H_∞ . Pour cette raison, la stabilité $L_2 - L_2$ est appelée la stabilité H_∞ .

B.2.3 Stabilisation interne

Les références pour cette sous-section sont (Desoer et al., 1980; Vidyasagar et al., 1982; Vidyasagar, 1985).

On note \mathcal{S} un anneau commutatif unitaire intègre et \mathcal{F} le corps de fractions de \mathcal{S} , c'est-à-dire,

$$\mathcal{F} := \{a/b \mid a, b \in \mathcal{S}, b \neq 0\}.$$

Remarque B.9. Un ensemble de systèmes linéaire stable SISO est un anneau commutatif unitaire intègre. En particulier, les connexions parallèles et cascades des systèmes stables sont aussi stables.

Dans la suite, on considère que \mathcal{S} est un ensemble de systèmes linéaire stable SISO. Alors \mathcal{F} comprend des systèmes stables et instables.

Pourtant les résultats basiques suivants sont aussi utiles pour d'autres buts que la stabilisation à condition que l'ensemble de systèmes désirés soit un anneau commutatif unitaire intègre.

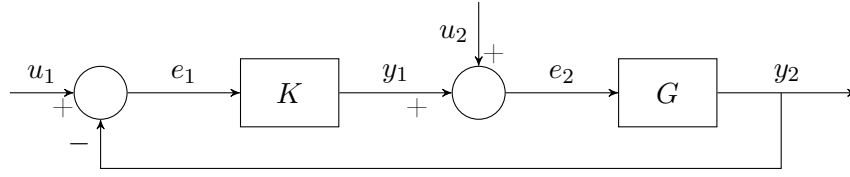


FIGURE B.1 – La boucle fermée

On considère le système bouclé présenté dans la Figure B.1 où G de dimension $n \times m$ est la matrice de transfert du système à contrôler et K de dimension $m \times n$ la matrice de transfert du contrôleur.

La fonction de transfert entre $[u_1, u_2]^T$ et $[e_1, e_2]^T$ est $H(G, K)$, c'est-à-dire,

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = H(G, K) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

avec

$$\begin{aligned} H(G, K) &= \begin{bmatrix} I_n - G(I_m + KG)^{-1}K & -G(I_m + KG)^{-1} \\ (I_m + KG)^{-1}K & (I_m + KG)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I_n + GK)^{-1} & -(I_n + GK)^{-1}G \\ K(I_n + GK)^{-1} & I_m - C(I_n + GK)^{-1}G \end{bmatrix} \end{aligned}$$

car avec quelques manipulations matricielles basiques nous obtenons $G(I_m + KG)^{-1} = (I_n + GK)^{-1}G$.

La fonction de transfert entre $[u_1, u_2]^T$ et $[y_1, y_2]^T$ est $W(G, K)$, c'est-à-dire,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = W(G, K) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

avec

$$\begin{aligned} W(G, K) &= \begin{bmatrix} 0 & I_n \\ -I_m & 0 \end{bmatrix} (H(G, K) - I_{m+n}) \\ &= \begin{bmatrix} K(I_n + GK)^{-1} & -KG(I_m + KG)^{-1} \\ GK(I_n + GK)^{-1} & G(I_m + KG)^{-1} \end{bmatrix}. \end{aligned} \tag{B.1}$$

Définition B.10. Le système bouclé dans la Figure B.1 est stable de manière interne si $H(G, K) \in \mathcal{S}^{(m+n) \times (m+n)}$.

Remarque B.11. Dû à (B.1), la boucle fermée est stable de manière interne si et seulement si $W(G, K) \in \mathcal{S}^{(m+n) \times (m+n)}$. Autrement dit, la boucle fermée est stable de manière interne si et seulement si toutes les relations entrée-sortie du système bouclé sont bornées.

Lemme B.12 (Vidyasagar, 1985). Si $W(G, K) \in \mathcal{S}^{(m+n) \times (m+n)}$, alors $G \in \mathcal{F}^{n \times m}$, $K \in \mathcal{F}^{m \times n}$.

Le lemme précédent montre que seulement les systèmes dont les éléments de la fonction de transfert sont dans \mathcal{F} peuvent être stabilisés avec le schéma de retour de la Figure B.1. Alors, dans la suite, nous considérons les systèmes de ce type.

B.2.4 Factorisations copremières

Définition B.13. $N \in \mathcal{S}^{n \times m}, D \in \mathcal{S}^{m \times m}$ sont copremiers à droite s'il existe $X \in \mathcal{S}^{m \times n}, Y \in \mathcal{S}^{m \times m}$ tels que

$$XN + YD = I_m.$$

Définition B.14. (N, D) avec $N \in \mathcal{S}^{n \times m}, D \in \mathcal{S}^{m \times m}$ est une factorisation à droite de $G \in \mathcal{F}^{n \times m}$ si $\det D \neq 0$ et $G = ND^{-1}$.

Remarque B.15. Comme \mathcal{S} est commutatif, chaque $G \in \mathcal{F}^{n \times m}$ admet des factorisations copremières. L'élément (i, j) de G peut être écrit comme $g_{ij} = p_{ij}/q_{ij}$ où $p_{ij}, q_{ij} \in \mathcal{S}$. En notant $b = \prod_i \prod_j q_{ij} \neq 0$ et A la matrice dont les éléments sont $a_{ij} = bp_{ij}/q_{ij} \in \mathcal{S}$, on obtient $G = A(bI_m)^{-1}$.

Définition B.16. (N, D) avec $N \in \mathcal{S}^{n \times m}, D \in \mathcal{S}^{m \times m}$ est une factorisation copremière à droite de $G \in \mathcal{F}^{n \times m}$ si (N, D) est une factorisation à droite de G et N, D sont copremiers à droite.

Définition B.17. $\tilde{N} \in \mathcal{S}^{n \times m}, \tilde{D} \in \mathcal{S}^{n \times n}$ sont copremiers à gauche s'il existe $\tilde{X} \in \mathcal{S}^{m \times n}, \tilde{Y} \in \mathcal{S}^{n \times n}$ tels que

$$\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_n.$$

Remarque B.18. De manière similaire, on peut construire des factorisations à gauche de tous $G \in \mathcal{F}^{n \times m}$. En fait, $G = (bI_n)^{-1}A$.

Définition B.19. (\tilde{N}, \tilde{D}) avec $\tilde{N} \in \mathcal{S}^{n \times m}, \tilde{D} \in \mathcal{S}^{n \times n}$ est une factorisation à gauche de $G \in \mathcal{F}^{n \times m}$ si $\det \tilde{D} \neq 0$ et $G = \tilde{D}^{-1}\tilde{N}$.

Définition B.20. (\tilde{N}, \tilde{D}) avec $\tilde{N} \in \mathcal{S}^{n \times m}, \tilde{D} \in \mathcal{S}^{n \times n}$ est une factorisation copremière à gauche de $G \in \mathcal{F}^{n \times m}$ si (\tilde{N}, \tilde{D}) est une factorisation à gauche de G et \tilde{N}, \tilde{D} sont copremiers à gauche.

Lemme B.21 (Théorème de la couronne, (Vidyasagar, 1985, Lemme 8.1.12)). *Soit \mathcal{S} une algèbre de Banach sur \mathbb{C} avec l'idéal maximal Ω . Supposons que Γ est un sous-ensemble dense de Ω et que $a_1, \dots, a_n \in \mathcal{S}$. Alors il existe $x_1, \dots, x_n \in \mathcal{S}$ tels que*

$$\sum_{i=1}^n x_i a_i = 1$$

si et seulement si

$$\inf_{\omega \in \Gamma} \sum_{i=1}^n |\hat{a}_i(\omega)| > 0$$

où \hat{a}_i est la transformée de Gelfand de a_i .

Corollaire B.22. *Si \mathcal{S} est un anneau de Bézout, alors tous $G \in \mathcal{F}^{n \times m}$ possèdent des factorisations copremières à gauche et à droite.*

Lemme B.23. H_∞ n'est pas un anneau de Bézout.

Remarque B.24. Le lemme précédent montre qu'il existe $G \in H_\infty^{n \times m}$ qui n'a pas de factorisations à gauche ou/et à droite.

Lemme B.25 (Vidyasagar, 1985, Théorème 8.1.23). *Les trois assertions suivantes sont équivalentes :*

1. \mathcal{S} est un anneau d'Hermite.
2. Si $G \in \mathcal{F}^{n \times m}$ a une factorisation copremière à droite, alors il a une factorisation copremière à gauche.
3. Si $G \in \mathcal{F}^{n \times m}$ a une factorisation copremière à gauche, alors il a une factorisation copremière à droite.

Lemme B.26. *Tous les anneaux de Bézout sont des anneaux d'Hermite.*

Lemme B.27. H_∞ est un anneau d'Hermite.

B.2.5 Paramétrisation de contrôleurs stabilisants

Lemme B.28 (Vidyasagar et al., 1982, Lemme 3.1). *Soient $G \in \mathcal{C}_r^{n \times m}$, $K \in \mathcal{C}_l^{m \times n}$, où $\mathcal{C}_r^{n \times m}$ et $\mathcal{C}_l^{n \times m}$ désignent les ensembles de tous $G \in \mathcal{F}^{n \times m}$ ayant une factorisation copremière à droite et une factorisation copremière à gauche respectivement. Supposons que (N_p, D_p) est une factorisation copremière à droite de G et que $(\tilde{N}_k, \tilde{D}_k)$ est une factorisation copremière à gauche de K . Sous ces conditions la paire (G, K) est stable si et seulement si*

$$\Delta := \tilde{D}_k D_p + \tilde{N}_k N_p$$

est une unité dans $\mathcal{S}^{m \times m}$.

Théorème B.29 (Vidyasagar, 1985, Théorème 8.3.5). *Soit $G \in \mathcal{F}^{n \times m}$ ayant une factorisation copremière à droite (N, D) et une factorisation copremière à gauche (\tilde{N}, \tilde{D}) . Supposons que $X \in \mathcal{S}^{m \times n}$, $Y \in \mathcal{S}^{m \times m}$, $\tilde{X} \in \mathcal{S}^{m \times n}$, $\tilde{Y} \in \mathcal{S}^{n \times n}$ tels que $XN + YD = I_m$, $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_n$. Alors*

$$\begin{aligned} S(G) &= \{(Y - R\tilde{N})^{-1}(X + R\tilde{D}) : R \in \mathcal{S}^{m \times n} \text{ et } \det(Y - R\tilde{N}) \neq 0\} \\ &= \{(\tilde{X} + DR)(\tilde{Y} - NR)^{-1} : R \in \mathcal{S}^{m \times n} \text{ et } \det(\tilde{Y} - NR) \neq 0\}. \end{aligned}$$

Remarque B.30. • $(Y - R\tilde{N})$ et $(X + R\tilde{D})$ sont copremiers à gauche. En fait, $(Y - R\tilde{N})D + (X + R\tilde{D})N = I_m$ car $XN + YD = I_m$ et $\tilde{D}N = \tilde{N}D$.

- $(\tilde{X} + DR)$ et $(\tilde{Y} - NR)$ sont copremiers à droite. En fait, $\tilde{N}(\tilde{X} + DR) + \tilde{D}(\tilde{Y} - NR) = I_n$ car $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_n$ et $\tilde{D}N = \tilde{N}D$.
- Si $\det Y \neq 0$, alors un contrôleur stabilisant est donné par $K = Y^{-1}X$ qui correspond à $R = 0$.
- Si $\det \tilde{Y} \neq 0$, alors un contrôleur stabilisant est donné par $K = \tilde{X}\tilde{Y}^{-1}$ qui correspond à $\tilde{R} = 0$.

B.3 Stabilisation des systèmes fractionnaires MISO à retards

B.3.1 Une classe de systèmes fractionnaires MISO à retards

On considère les systèmes décrits par les matrices de transfert de la forme

$$G(s) = \left[e^{-sh_1} R_1(s^\alpha), \dots, e^{-sh_n} R_n(s^\alpha) \right], \quad (\text{B.2})$$

où

- $0 \leq h_k \in \mathbb{R}$ pour $k = 1, \dots, n$ sont les retards ;
- $\alpha \in \mathbb{R}$, $0 < \alpha < 1$;
- $R_k(s^\alpha) = \tilde{q}_k(s^\alpha)/\tilde{p}_k(s^\alpha)$, où $\tilde{p}_k(s^\alpha)$ et $\tilde{q}_k(s^\alpha)$ sont des polynômes de degré entier en s^α , $\tilde{p}_k(s^\alpha)$ et $\tilde{q}_k(s^\alpha)$ n'ont pas de racines communes, et $\deg \tilde{p}_k(s^\alpha) \geq \deg \tilde{q}_k(s^\alpha)$ pour $k = 1, \dots, n$;
- d_k est le degré en s^α de $\tilde{p}_k(s^\alpha)$;
- s est dans la branche principale $\mathbb{C} \setminus \mathbb{R}_-$, c'est-à-dire $\arg(s) \in (-\pi, \pi)$, afin d'assurer une valeur unique pour la fonction de transfert qui contient des termes en s^α avec $\alpha \in (0, 1)$.

On étudie le problème de stabilisation du système dans le cadre de l'approche de factorisation. Plus précisément, on souhaite chercher des factorisations copremières à gauche et à droite de la matrice de transfert du système ainsi que les facteurs de Bézout associés afin d'obtenir l'ensemble des contrôleurs stabilisants.

Les notations suivantes seront utiles dans la suite.

Notons

- $p(s^\alpha)$ le plus petit commun multiple de tous les dénominateurs des $R_k(s^\alpha)$ pour $k = 1, \dots, n$;
- d le degré en s^α de $p(s^\alpha)$.

Alors, les fonctions de transfert rationnelles $R_k(s^\alpha)$ peuvent se réécrire comme suit

$$R_k(s^\alpha) = \frac{q_k(s^\alpha)}{p(s^\alpha)},$$

où $q_k(s^\alpha)$ sont des polynômes en s^α .

On peut décomposer

$$p(s^\alpha) = (s^\alpha)^{m_0} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_i} \right) \left(\prod_{j=1}^{N'} (s^\alpha - c_j)^{m'_j} \right),$$

où

- $b_i \in \mathcal{D} := \{\sigma \in \mathbb{C} \setminus \{0\} \mid -\pi\alpha/2 \leq \text{Arg}(\sigma) \leq \pi\alpha/2\}$,
- $c_j \in \mathbb{C} \setminus \{\mathcal{D} \cup \{0\}\}$,
- $m_0, m_i, m'_j \in \mathbb{Z}_+$ pour $i = 1, \dots, N$ et $j = 1, \dots, N'$.

Ainsi $s_i = b_i^{1/\alpha}$ sont les racines instables non-nulles en s de $p(s^\alpha)$.

De manière similaire, on écrit

$$\tilde{p}_k(s^\alpha) = (s^\alpha)^{m_{0k}} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ik}} \right) \left(\prod_{j=1}^{N'} (s^\alpha - c_j)^{m'_{jk}} \right),$$

où $m_{0k}, m_{ik}, m'_{jk} \in \mathbb{Z}_+$ pour $i = 1, \dots, N, j = 1, \dots, N'$ et $k = 1, \dots, n$. Il est évident que $m_{0k} \leq m_0, m_{ik} \leq m_i$, et $m'_{jk} \leq m'_j$.

B.3.2 Factorisations copremières à gauche et facteurs de Bézout associés

Dû à la dimension de la matrice de transfert, une factorisation copremière à gauche est facile à trouver.

Proposition B.31. *Soit G décrit par (B.2). Alors*

$$\tilde{M}(s) = \frac{p(s^\alpha)}{(s^\alpha + 1)^d} \text{ et } \tilde{N}(s) = \frac{1}{(s^\alpha + 1)^d} \left[e^{-sh_1} q_1(s^\alpha), \dots, e^{-sh_n} q_n(s^\alpha) \right] \quad (\text{B.3})$$

est une factorisation copremière à gauche de G sur H_∞ .

Avant de donner les facteurs de Bézout associés, notons

$$k_i := \min\{k \mid k \in \{1, \dots, n\}, m_{ik} = m_i\} \quad \text{for } i = 0, \dots, N, \quad (\text{B.4})$$

$$f_k := \sum_{i \in \{1, \dots, N\}, k_i = k} m_i \quad \text{for } k = 1, \dots, n,$$

$$\mathcal{L}(m_{0\alpha}) := \{x \in \mathbb{R} \mid x = a + b\alpha < m_0\alpha, a, b \in \mathbb{Z}_+\}. \quad (\text{B.5})$$

Proposition B.32. *Soit $G(s)$ décrit par (B.2). Alors les facteurs de Bézout associés à la factorisation copremière à gauche obtenue dans (B.3) sont donnés par*

$$\tilde{X}(s) = \frac{(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s)}{p(s^\alpha) u(s^\alpha)},$$

$$\tilde{Y}(s) = \left[\frac{\mu_1(s)}{u(s^\alpha)}, \dots, \frac{\mu_n(s)}{u(s^\alpha)} \right]^T,$$

où $u(s^\alpha)$ est un polynôme de degré supérieur ou égal à d en s^α dont les racines sont stables, et les polynômes fractionnaires (d'ordre non-commensurable) $\mu_k(s)$ pour $k = 1, \dots, n$ ont

la forme suivante

$$\mu_k(s) = \begin{cases} \sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k} s^\lambda + \sum_{j=m_0}^{m_0+f_k-1} \beta_{(j\alpha)k} (s^\alpha)^j & \text{si } k = k_0, \\ \sum_{j=0}^{f_k-1} \beta_{(j\alpha)k} (s^\alpha)^j & \text{si } k \neq k_0, \end{cases}$$

et vérifient

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \right] = O(s^{m_0\alpha}) \quad (\text{B.6})$$

lorsque $s \rightarrow 0$ et

$$\left[(s^\alpha + 1)^d u(s^\alpha) - \sum_{k=1}^n e^{-sh_k} q_k(s^\alpha) \mu_k(s) \right]^{(l)} = 0, \quad (\text{B.7})$$

pour chaque racine instable non-nulle $s = b_i^{1/\alpha}$, $i = 1, \dots, N$, de $p(s^\alpha)$ et pour $0 \leq l \leq m_i - 1$.

Remarque B.33. Si $f_k = 0$, alors

$$\mu_k(s) = \begin{cases} \sum_{\lambda \in \mathcal{L}(m_0\alpha)} \beta_{\lambda k} s^\lambda & \text{si } k = k_0, \\ 0 & \text{si } k \neq k_0. \end{cases}$$

Remarque B.34. Si $m_0\alpha \leq 1$ ou $\alpha = 1/m$ avec $m \in \mathbb{Z}_+ \setminus \{0, 1\}$, alors λ sont des multiples de α et nous obtenons une expression élégante pour μ_{k_0} qui ne contient que des termes en s^α . Plus généralement, si α est rationnel, alors μ_{k_0} contient des puissances de s à exposants commensurables.

Cela est aussi obtenu si nous introduisons plus de coefficients dans $\mu_k(s)$, $k = 1, \dots, n$, $k \neq k_0$ que dans les formes données dans la proposition. Plus précisément, si nous notons x le nombre de valeurs de $\lambda \in \mathcal{L}(m_0\alpha)$ telles que $\lambda \neq b\alpha$, $b \in \mathbb{Z}_+$, alors nous devons ajouter au moins x termes en s^α d'ordres plus élevés. Alors il est possible de choisir $\beta_{\lambda k_0} = 0$ pour $\lambda \in \mathcal{L}(m_0\alpha)$, $\lambda \neq b\alpha$, $b \in \mathbb{Z}_+$ et de résoudre le système d'équations pour les autres coefficients car ce système d'équations admet une solution unique ou une infinité de solutions.

Remarque B.35. Il suffit de choisir $u(s^\alpha)$ de degré en s supérieur ou égal au degré en s de $\mu_k(s)$ pour $k = 1, \dots, n$ afin d'assurer que $\tilde{Y} \in \mathbf{M}(H_\infty)$.

B.3.3 Factorisations copremières à droite et facteurs de Bézout associés

La section précédente a montré que le système $G(s)$ admettait des factorisations copremières à gauche sur H_∞ , et l'une d'entre elles est donnée par (B.3). Comme H_∞ est un

anneau d'Hermité, alors à partir de (Quadrat, 2003a, Corollaire 4.14), nous déduisons qu'il existe des factorisations copremières à droite pour $G(s)$.

Pour nos matrices de transfert, les factorisations copremières à droite et les facteurs de Bézout associés sont des matrices contenant plus d'éléments que celles intervenant dans les factorisations à gauche. Nous considérerons deux larges classes de systèmes. La première classe comprend des systèmes avec des pôles distincts, c'est-à-dire $\tilde{p}_k(s^\alpha)$ et $\tilde{p}_{k'}(s^\alpha)$ n'ont pas de racines communes si $k \neq k'$. Dans ce cas, la matrice $M(s)$ peut être de forme diagonale, ce qui réduit la complexité des calculs car il est facile d'obtenir la matrice inverse. La seconde classe comprend des systèmes à pôles identiques et la matrice $M(s)$ a donc une forme plus compliquée. Pour cette classe, on ne considère qu'une sous-classe assez simple de systèmes.

B.3.3.1 Pôles distincts

Proposition B.36. *Soit $G(s)$ décrit par (B.2). Supposons que toutes les racines instables (nulles ou non-nulles) de $\tilde{p}_k(s^\alpha)$ pour $k = 1, \dots, n$ sont distinctes. Alors une factorisation copremière à droite et les facteurs de Bézout associés sont donnés par*

$$\begin{aligned} N(s) &= [N_1(s), \dots, N_n(s)], \\ M(s) &= \begin{bmatrix} M_{11}(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \\ X(s) &= \begin{bmatrix} X_{11}(s) & \cdots & X_{1n}(s) \\ \vdots & \ddots & \vdots \\ X_{n1}(s) & \cdots & X_{nn}(s) \end{bmatrix}, \\ Y(s) &= [Y_1(s), \dots, Y_n(s)]^T, \end{aligned}$$

où pour $k, k' \in \{1, \dots, n\}$ et $k \neq k'$

$$N_k(s) = \frac{e^{-sh_k} \tilde{q}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (\text{B.8})$$

$$M_{kk}(s) = \frac{\tilde{p}_k(s^\alpha)}{(s^\alpha + 1)^{d_k}}, \quad (\text{B.9})$$

$$Y_k(s) = \frac{\mu_k(s)}{u(s^\alpha)} \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right),$$

$$X_{kk}(s) = \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)},$$

$$X_{kk'}(s) = -Y_k(s) e^{-sh_{k'}} \frac{\tilde{q}_{k'}(s^\alpha)}{\tilde{p}_{k'}(s^\alpha)},$$

où d_k est le degré en s^α de \tilde{p}_k ; $u(s^\alpha)$ est un polynôme de degré d en s^α qui n'a pas de racines instables ; et $\mu_k(s)$ ont la forme suivante

$$\mu_k(s) = \sum_{\lambda \in \mathcal{L}(m_{0k}\alpha)} \beta_{\lambda k} s^\lambda + \sum_{j=m_{0k}}^{m_{0k} + \sum_{i=1}^N m_{ik} - 1} \beta_{(j\alpha)k} (s^\alpha)^j,$$

et vérifient

$$u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left(\prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) = O(s^{m_{0k}\alpha}) \quad (\text{B.10})$$

lorsque $s \rightarrow 0$ si $\tilde{p}_k(s^\alpha)$ a une racine nulle, et pour chaque racine instable non-nulle de $\tilde{p}_k(s^\alpha)$, c'est-à-dire $s = b_i^{1/\alpha}$ avec $m_{ik} \neq 0$ pour $i = 1, \dots, N$,

$$\left[u(s^\alpha)(s^\alpha + 1)^{d_k} - e^{-sh_k} \mu_k(s) \tilde{q}_k(s^\alpha) \prod_{1 \leq j \leq n, j \neq k} \left((s^\alpha)^{m_{0j}} \prod_{i=1}^N (s^\alpha - b_i)^{m_{ij}} \right) \right]^{(l)} = 0 \quad (\text{B.11})$$

où $l = 0, \dots, m_{ik} - 1$.

Remarque B.37. Nous ne pouvons pas éliminer les puissances de s d'ordre non-commensurable dans $\mu_k(s)$ telles que $m_{0k} > 0$ en ajoutant plus de coefficients comme nous le pouvons dans le cas des facteurs de Bézout à gauche.

B.3.3.2 Pôles identiques

Tandis que nous obtenons des expressions simples pour les systèmes avec pôles distincts, le cas de pôles identiques demande plus d'attention. Pour ce dernier, la matrice $M(s)$ sous forme diagonale et $N_k(s)$, $M_{kk}(s)$ sous les formes (B.9), (B.10) ne sont plus possible. Nous déduisons de l'identité de Bézout à droite $X(s)M(s) + Y(s)N(s) = I$ que pour $k, k' \in \{1, \dots, n\}$ et $k \neq k'$

$$X_{kk}(s) = \frac{1 - Y_k(s)N_k(s)}{M_{kk}(s)},$$

$$X_{kk'}(s) = -Y_k(s) \frac{N_{k'}(s)}{M_{k'k'}(s)}.$$

Pour que $X_{kk'}(s)$ appartienne à H_∞ , toutes les racines instables de $M_{k'k'}(s)$ doivent être racines de $Y_k(s)$. Par conséquent, si $M_{kk}(s)$ et $M_{k'k'}(s)$ ont une racine commune, alors $X_{kk}(s)$ est infini en cette racine, et alors $X_{kk}(s) \notin H_\infty$.

Ici, nous considérons les transferts $G(s)$ pour lesquels chaque élément contient un pôle et certains éléments pouvant avoir des pôles communs. Pour la clarté de la présentation de ces résultats, nous commencerons par un lemme concernant l'inverse d'une matrice triangulaire supérieure.

Nous considérons des matrices creuses dont les éléments au-dessus de la diagonale principale satisfont les conditions suivantes : si un élément de la ligne k -ème est non-nul, alors

tous les éléments de la colonne k -ème doivent être nuls ; si un élément de la colonne k -ème est non-nul, alors tous les autres éléments de la même colonne ainsi que ceux de la ligne k -ème doivent être nuls.

Lemme B.38. Soit $M \in \mathbb{R}^{n \times n}$ une matrice triangulaire supérieure donnée par

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn} \end{bmatrix},$$

où les éléments sur la diagonale principale sont non-nuls et les éléments au-dessus de la diagonale principale vérifient les conditions suivantes

- (i) pour $k = 1, \dots, n$, s'il existe $l' \in \mathbb{Z}$, $l' \in (k, n]$ tel que $M_{kl'} \neq 0$ alors $M_{lk} = 0$ pour $l \in \{1, \dots, k-1\}$,
- (ii) pour $k = 1, \dots, n$, s'il existe $l'' \in \mathbb{Z}$, $l'' \in [1, k)$ tel que $M_{l''k} \neq 0$ alors $M_{lk} = 0$ pour $l \in \{1, \dots, k-1\} \setminus \{l''\}$ et $M_{kl'} = 0$ pour $l' \in \{k+1, \dots, n\}$.

Alors, l'inverse de M est donné par

$$M^{-1} = M^{inv}$$

avec

$$M^{inv} := \begin{bmatrix} M_{11}^{inv} & \cdots & M_{1n}^{inv} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}^{inv} \end{bmatrix},$$

où les éléments sur ou au-dessus de la diagonale principale satisfont

$$M_{kk}^{inv} = \frac{1}{M_{kk}}, \tag{B.12}$$

$$M_{kk'}^{inv} = -\frac{M_{kk'}}{M_{kk}M_{k'k'}} \tag{B.13}$$

pour $k, k' \in \{1, \dots, n\}$ et $k < k'$.

Dans la proposition suivante, nous proposons des factorisations copremières et des facteurs de Bézout correspondants des systèmes $G(s)$ pour lesquels chaque élément contient un pôle et certains éléments pouvant avoir des pôles communs. Afin de simplifier la présentation, nous supposons que les retards sont rangés en ordre. L'extension du résultat au cas des retards arbitraires sera discutée après la proposition.

Proposition B.39. Soit $G(s)$ décrit par (B.2) avec

$$h_1 \leq \dots \leq h_n,$$

$$R_k(s^\alpha) = \frac{a_k}{s^\alpha - \sigma_k}$$

avec $a_k, \sigma_k \in \mathbb{R}$ pour $k = 1, \dots, n$. Nous notons $\mathcal{I}_1 := \emptyset$ et $\mathcal{I}_k := \{j \mid j \in \{1, \dots, k-1\}, \sigma_j = \sigma_k\}$ pour $k = 2, \dots, n$. Une factorisation copremière à droite et les facteurs de Bézout associés sont donnés par

$$N(s) = [N_1(s), \dots, N_n(s)], \quad (\text{B.14})$$

$$M(s) = \begin{bmatrix} M_{11}(s) & \cdots & M_{1n}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}(s) \end{bmatrix}, \quad (\text{B.15})$$

$$Y(s) = [Y_1(s), \dots, Y_n(s)]^T,$$

$$X(s) = M^{-1}(s) - Y(s)G(s),$$

où pour $k, k' \in \{1, \dots, n\}$ et $k' \neq k$

$$N_k(s) = \begin{cases} 0 & \text{si } \mathcal{I}_k \neq \emptyset, \\ \frac{a_k e^{-s h_k}}{s^\alpha + 1} & \text{sinon,} \end{cases} \quad (\text{B.16})$$

$$M_{kk}(s) = \begin{cases} 1 & \text{si } \mathcal{I}_k \neq \emptyset \\ \frac{s^\alpha - \sigma_k}{s^\alpha + 1} & \text{sinon,} \end{cases} \quad (\text{B.17})$$

$$M_{k'k}(s) = \begin{cases} -\frac{a_k e^{-s(h_k - h_{k'})}}{a_{k'}} & \text{si } k' = \min \mathcal{I}_k, \\ 0 & \text{sinon,} \end{cases} \quad (\text{B.18})$$

$$Y_k(s) = \begin{cases} 0 & \text{si } \mathcal{I}_k \neq \emptyset, \\ \frac{\beta_k \hat{p}_k(s^\alpha)}{u(s^\alpha)} & \text{sinon,} \end{cases}$$

où $u(s^\alpha)$ est un polynôme de degré d en s^α qui n'a pas de racines instables ; $\hat{p}_k(s^\alpha) = p(s^\alpha)/(s^\alpha - \sigma_k)$; β_k (pour les valeurs de k telles que $\mathcal{I}_k = \emptyset$ et $\sigma_k \geq 0$) sont donnés par

$$\beta_k = \frac{u(\sigma_k)(\sigma_k + 1)e^{\sigma_k^{1/\alpha} h_k}}{a_k p'_k(\sigma_k)}, \quad (\text{B.19})$$

β_k pour les autres valeurs de k peuvent être choisis de façon arbitraire, et $M^{-1}(s)$ sont donnés par

$$M^{-1}(s) = \begin{bmatrix} M_{11}^{inv}(s) & \cdots & M_{1n}^{inv}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn}^{inv}(s) \end{bmatrix} \quad (\text{B.20})$$

où les éléments sur et au-dessus de la diagonale principale satisfont

$$M_{kk}^{inv} = \frac{1}{M_{kk}},$$

$$M_{kk'}^{inv} = -\frac{M_{kk'}}{M_{kk} M_{k'k'}}$$

pour $k, k' \in \{1, \dots, n\}$ et $k < k'$.

Remarque B.40. Une matrice de transfert G donnée par (B.2) dont les éléments ont des retards quelconques peut être transformée en une matrice de transfert G_0 avec des retards

rangés par ordre croissant en multipliant G par une matrice de permutation appropriée P . Il est bien connu que cette matrice P est orthogonale et son inverse est P^T . Supposons que (M_0, N_0) est une factorisation copremière à droite sur H_∞ de G_0 et X_0, Y_0 sont les facteurs de Bézout associés. Nous avons donc $G = G_0 P^{-1} = N_0 M_0^{-1} P^{-1} = N_0 (PM_0)^{-1}$. Il est évident que $PM_0 \in \mathbf{M}(H_\infty)$. De plus, $X_0 P^{-1} PM_0 + Y_0 N_0 = I$ et $X_0 P^{-1} \in \mathbf{M}(H_\infty)$. Alors, (PM_0, N_0) est une factorisation copremière à droite de G et $X_0 P^{-1}, Y_0$ sont les facteurs de Bézout associés.

B.4 Analyse de stabilité des systèmes classiques et fractionnaires SISO à retards commensurables

B.4.1 Une classe de systèmes classiques et fractionnaires de type neutre

Nous considérons les systèmes (fractionnaires) à retards de type neutre dont la fonction de transfert est de la forme

$$G(s) = \frac{t(s)}{p(s) + \sum_{k=1}^N q_k(s) e^{-ks\tau}}, \quad (\text{B.21})$$

où

- $\tau > 0$ est le retard,
- t, p , et q_k pour tous $k \in \mathbb{N}_N$ sont des polynômes réels en s^μ ,
- $0 < \mu \leq 1$,
- $-\pi < \arg(s) < \pi$ dans le cas où $0 < \mu < 1$ afin d'avoir une valeur unique de s^μ ,
- $\deg p \geq \deg t$, $\deg p \geq \deg q_k$ pour tous $k \in \mathbb{N}_N$, et $\deg p = \deg q_k$ pour au moins une valeur de $k \in \mathbb{N}_N$ pour que le système soit propre et de type neutre.

Notons que le degré d'un (quasi-)polynôme signifie le degré en s^μ .

Comme $\deg p \geq \deg q_k$ pour tous $k \in \mathbb{N}_N$, alors pour chaque k nous obtenons

$$\frac{q_k(s)}{p(s)} = \alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}}{s^{l\mu}} + \mathcal{O}(s^{-(M'+1)\mu}) \text{ lorsque } |s| \rightarrow \infty, \quad (\text{B.22})$$

où $M' \in \mathbb{Z}_+$ et peut être arbitrairement large.

Le coefficient de la terme de degré le plus élevé du dénominateur de la fonction de transfert (B.21) peut s'écrire comme un multiple du polynôme en z ci-dessous

$$\tilde{c}_d(z) = 1 + \sum_{k=1}^N \alpha_{0,k} z^k, \quad (\text{B.23})$$

où $z = e^{-s\tau}$. Il est appelé le polynôme formel.

A chaque racine r de (B.23) est associée une chaîne de pôles de type neutre de G et les pôles à grand module de cette chaîne sont approximés par

$$s_n \tau = \lambda_n + o(1), \quad (\text{B.24})$$

où

$$\lambda_n = -\ln(r) + j2\pi n, \quad n \in \mathbb{Z}, \quad (\text{B.25})$$

lorsque $n \rightarrow \infty$ (Bellman and Cooke, 1963; Hotzel, 1998a; Fioravanti et al., 2010).

Par conséquent, la chaîne de pôles approche l'axe vertical

$$\Re(s) = -\frac{\ln(|r|)}{\tau}. \quad (\text{B.26})$$

Si l'axe vertical est strictement à droite ou à gauche de l'axe imaginaire, ce qui est le cas lorsque $|r| < 1$ ou $|r| > 1$, alors les pôles asymptotiques à cet axe vertical sont respectivement à droite ou à gauche de l'axe imaginaire, et alors leur effets sur la stabilité H_∞ sont facilement déduits car ils ne dépendent que leur localisation par rapport à l'axe imaginaire (Bonnet and Partington, 2007; Bonnet et al., 2011).

L'analyse de stabilité est plus délicate dans le cas où des chaînes de pôles sont asymptotiques à l'axe imaginaire. Pour cette analyse, nous serons intéressés dans un premier temps à la localisation des pôles des chaînes par rapport à l'axe. Ensuite, le cas échéant, nous déduirons des conditions nécessaires et suffisantes pour la stabilité H_∞ . Enfin, nous illustrerons les résultats par un exemple d'une sous-classe de systèmes.

Nous présentons dans le lemme suivant quelques propriétés du polynôme formel quand ce dernier a des racines multiples.

Lemme B.41. *Soit r une racine de multiplicité $m > 1$ de $f(z) = 1 + \sum_{k=1}^N \alpha_k z^k$, où $\alpha_k \in \mathbb{C}$. Alors $\sum_{k=1}^N k^l \alpha_k r^k = 0$ pour $l = 1, \dots, m-1$ et $\sum_{k=1}^N k^m \alpha_k r^k \neq 0$.*

B.4.2 Localisation des pôles neutres

Comme nous avons vu précédemment, à chaque racine r du polynôme formel $\tilde{c}_d(z)$ correspond une chaîne de pôles de type neutre. L'approximation de ces pôles donnée dans (B.24) n'indique que l'axe vertical vers lequel la chaîne de pôles s'approche. Afin de déterminer la localisation de la chaîne par rapport à l'axe asymptotique, nous examinerons dans cette section une approximation plus précise des pôles de type neutre de la forme

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}) \quad (\text{B.27})$$

avec

$$\nu_{n,1} = \frac{\nu_1}{n^{y_1}}, \quad \nu_1 \neq 0, y_1 > 0, n \in \mathbb{Z}, n \rightarrow \infty.$$

Autrement dit, le cas échéant, nous déterminerons le terme d'approximation *non-nul* suivant. Un tel terme n'existe pas si les pôles neutres sont précisément $s_n = \lambda_n / \tau$.

A l'exception de ce cas spécial, $\nu_{n,1}$ existe et le signe de $\Re(\nu_1/n^{y_1})$ montre de quel côté de l'axe asymptotique se situent les pôles. Notons que le signe pourrait changer pour n positif et négatif. Ainsi, les parties en haut et en bas de la chaîne de pôles pourraient se situer de différents côtés de l'axe asymptotique.

Notons que nous ne fixons pas une valeur de y_1 à priori mais nous la cherchons pour que $\nu_1 \neq 0$. Cela assure que l'approximation donne des nouvelles informations sur la localisation des pôles. Le seul cas où l'information donnée n'est pas utile est lorsque $\Re(\nu_1/n^{y_1}) = 0$ et nous devons chercher d'autres termes d'approximation pour déterminer la localisation des pôles relatif à l'axe asymptotique.

Avant de présenter les résultats principaux sur la localisation des pôles par rapport à l'axe asymptotique, nous définissons des notions utiles pour la suite.

- Pour une racine r de $\tilde{c}(z)$,

$$AB(r) = \{(a, b) \in \mathbb{Z}_+^2 : a + b \neq 0, \sum_{k=1}^N \alpha_{a,k} k^b r^k \neq 0\}. \quad (\text{B.28})$$

- \mathcal{S} désigne un sous-ensemble de $AB(r)$ tel que $n(\mathcal{S}) \geq 2$ et il existe $\mathbf{m} > 0$ tel que $a + b\mathbf{m} = a' + b'\mathbf{m} \forall (a, b), (a', b') \in \mathcal{S}$ et $a + b\mathbf{m} < a'' + b''\mathbf{m} \forall (a'', b'') \in AB(r) \setminus \mathcal{S}$. Nous appelons \mathcal{S} un *segment de frontière en bas à gauche* de $AB(r)$.
- \mathbf{m} défini précédemment pour chaque \mathcal{S} est évidemment unique et nous l'appelons la *penne* du segment.
- $\mathfrak{S}(AB(r))$ désigne l'ensemble de tous les segments de frontière en bas à gauche de $AB(r)$.

Un segment de frontière en bas à gauche est illustré sur la Figure B.2. Notons que si nous notons \mathbf{m}_2 la penne du segment alors $\mathbf{m}_2 = \tan \gamma_2$ avec γ_2 présenté sur la figure.

L'approximation des chaînes de pôles de type neutre est l'objectif du théorème suivant.

Théorème B.42. *Soit $G(s)$ un système à retards de type neutre décrit par (B.21) et r une racine de multiplicité m du polynôme formel $\tilde{c}_d(s)$ donné par (B.23). Avec $\alpha_{a,k}$ donné comme dans (B.22), nous définissons*

$$C(a, b, \nu) := \frac{\tau^{a\mu}}{(j2\pi)^{a\mu}} \frac{(-1)^b \nu^b}{b!} \sum_{k=1}^N \alpha_{a,k} k^b r^k, \quad (\text{B.29})$$

$$B(\mathcal{S}) := \left\{ (\nu, y) : \nu \text{ est une racine non-nulle de } \sum_{(a,b) \in \mathcal{S}} C(a, b, \nu) = 0, y = \mathbf{m}\mu \right\}. \quad (\text{B.30})$$

Notons n_1 le nombre de chaînes de pôles relatives à r dont les pôles sont donnés par $s_n = \lambda_n/\tau$ où $n \in \mathbb{Z}$, $n \rightarrow \infty$ et λ_n est donné par (B.25). Alors, les pôles des autres chaînes neutres correspondant à r sont approximés par

$$s_n = \frac{1}{\tau} \left(\lambda_n + \frac{\nu_1}{n^{y_1}} \right) + o(n^{-y_1}) \quad (\text{B.31})$$

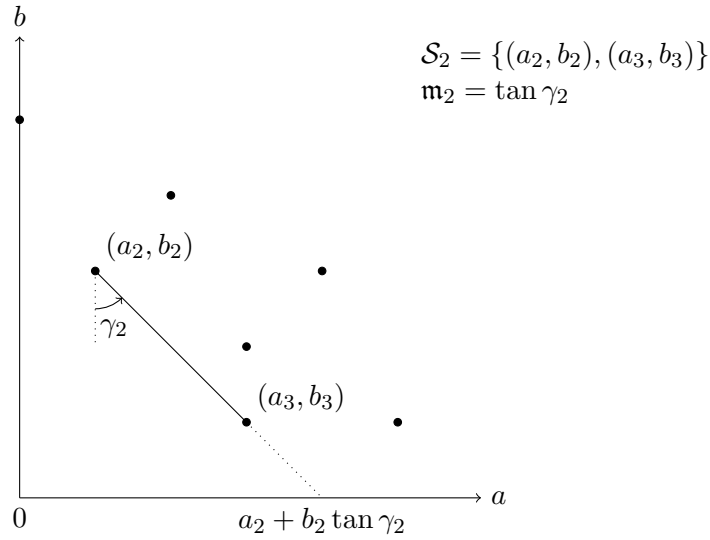


FIGURE B.2 – Un segment de frontière en bas à gauche d’un ensemble de points dans le plan

où pour chaque chaîne de pôles (ν_1, y_1) prend une des $m - n_1$ valeurs (en tenant compte des multiplicités) données par

$$(\nu_1, y_1) \in \bigcup_{\mathcal{S} \in \mathfrak{S}(AB(r))} B(\mathcal{S}).$$

Maintenant nous allons discuter comment construire tous les segments de frontière en bas à gauche de l’ensemble $AB(r)$.

Dans un premier temps, nous mentionnons deux points importants de $AB(r)$ qui limitent un sous-ensemble de $AB(r)$ contenant les segments de frontière en bas à gauche. Le premier point est $(0, m)$. Ce point appartient à $AB(r)$ car $\sum_{k=1}^N \alpha_{0,k} k^m r^k \neq 0$ (voir le Lemme B.41). Le second point, noté par (a_L, b_L) , est le point le plus à gauche parmi ceux les plus bas de $AB(r)$, c’est-à-dire

$$\begin{aligned} b_L &= \min\{b \mid (a, b) \in AB(r)\} \\ a_L &= \min\{a \mid (a, b_L) \in AB(r)\}. \end{aligned} \tag{B.32}$$

Les segments de frontière en bas à gauche de $AB(r)$ appartiennent donc au sous-ensemble $A_L^m = \{(a, b) \in AB(r) \mid a \leq a_L, b \leq m\}$ (voir la Figure B.3). En fait, si $(a, b) \in AB(r)$ et $a > a_L$, alors $a + bm > a_L + b_L m$ pour tous $m > 0$ car $b \geq b_L$ par définition. Si $(a, b) \in AB(r)$ et $b > m$, alors $a + bm > mm$ pour tous $m > 0$ car $a \geq 0$ par définition.

Le sous-ensemble A_L^m a un nombre fini de points et son enveloppe convexe est donc un polygone convexe (De Berg et al., 2008). Les sommets de ce polygone sont dans A_L^m et la ligne contenant chacun de ses côtés définit un demi-plan fermé contenant tous les points de A_L^m . Il n’existe pas d’autres lignes contenant deux points de A_L^m avec une telle caractéristique.

Par conséquent, par définition, les points d'un segment de frontière en bas à gauche de $AB(r)$ appartiennent à un côté de l'enveloppe convexe de A_L^m et deux d'entre eux sont sommets de l'enveloppe.

Il existe de nombreux algorithmes pour déterminer dans un ensemble fini de points dans \mathbb{R}^2 les points qui appartiennent à l'enveloppe convexe (De Berg et al., 2008). Parmi ceux-ci, nous pouvons collecter les points appartenant aux segments de frontière en bas à gauche.

La discussion ci-dessus indique que nous devons connaître les points $(0, m)$ et (a_L, b_L) avant d'utiliser des algorithmes pour déterminer l'enveloppe convexe. Dans le reste de cette section, nous présentons une méthode pour chercher (a_L, b_L) de façon numérique.

D'abord, notons que $b_L = n_1$ où n_1 est le nombre de chaînes de pôles avec $s_n = \lambda_n/\tau$. En fait, $m - n_1$ est le nombre total de valeurs non-nulles de ν_1 . Ce nombre est également égal à $(\max\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{G}(AB(r))} \mathcal{S}\} - \min\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{G}(AB(r))} \mathcal{S}\})$ car le nombre de valeurs non-nulles de ν_1 pour chaque $\mathcal{S} \in \mathfrak{G}(AB(r))$ est $(\max\{b \mid (a, b) \in \mathcal{S}\} - \min\{b \mid (a, b) \in \mathcal{S}\})$ et les segments dans $\mathfrak{G}(AB(r))$ sont interconnectés. De plus, notons que $\max\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{G}(AB(r))} \mathcal{S}\} = m$ et $\min\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{G}(AB(r))} \mathcal{S}\} = b_L$.

Le lemme suivant fournit un outil pour déduire le nombre de chaînes de pôles avec $s_n = \lambda_n/\tau$.

Lemme B.43. *Soit $G(s)$ un système à retards de type neutre défini par (B.21). Son dénominateur peut s'écrire comme suit*

$$D(s, z) = p(s) + \sum_{k=1}^N q_k(s) z^k, \quad z = e^{-ks\tau}.$$

Notons r une racine de multiplicité m de $\tilde{c}_d(z)$ défini par (B.23). Les assertions suivantes sont équivalentes :

- (i) $D(s, z)$ possède n_1 chaînes de pôles identiques qui se situent sur l'axe imaginaire, sont associées à la racine r de $\tilde{c}_d(z)$ et ont les pôles donnés par $s_n = \lambda_n/\tau$ où $n \in \mathbb{Z}$ et λ_n est donné par (B.25).

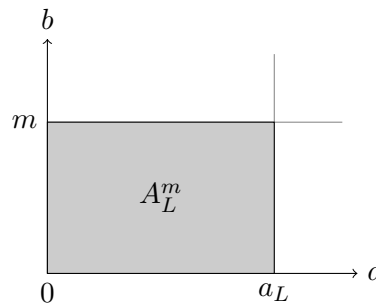


FIGURE B.3 – Le sous-ensemble A_L^m de $AB(r)$ qui contient tous les segments de frontière en bas à gauche de $AB(r)$

(ii)

$$\left. \frac{d^b D(s, z)}{ds^b} \right|_{z=r} \equiv 0, \quad b = 0, \dots, n_1 - 1, \quad (\text{B.33})$$

$$\left. \frac{d^{n_1} D(s, z)}{ds^{n_1}} \right|_{z=r} \neq 0, \quad (\text{B.34})$$

où $d^0 D(s, z)/ds^0 = D(s, z)$.

Après avoir déterminé b_L en appliquant le lemme précédent, nous pouvons déterminer a_L en lançant une boucle pour chercher le plus petit a telle que $\sum_{k=1}^N \alpha_{a,k} k^{b_L} r^k \neq 0$.

B.4.3 Stabilité

Dans cette sous-section, nous étudions la stabilité de type H_∞ des systèmes d'intérêt en utilisant l'approximation de pôles obtenue dans la sous-section précédente. Ici, nous sommes intéressés uniquement par les systèmes avec des chaînes de pôles de type neutre asymptotiques à l'axe imaginaire.

Le théorème suivant donne des critères pour vérifier rapidement si un système est instable. Nous n'avons pas besoin de connaître $\nu_{n,1}$ pour appliquer ces critères.

Théorème B.44. *Soit $G(s)$ un système à retard de type neutre défini par (B.21), et supposons que le polynôme formel $\tilde{c}_d(z)$ défini par (B.23) a des racines de module un. Si pour une racine r , il existe $\mathcal{S} \in \mathfrak{S}(AB(r))$ avec $AB(r)$ défini par (B.28) tel que $n(\mathcal{S}) = 2$ et une des deux conditions suivantes est satisfaites pour $(a_1, b_1), (a_2, b_2) \in \mathcal{S}, b_1 > b_2$*

- $b_1 - b_2 \geq 3$,
- $b_1 - b_2 = 2$, et $(a_2 - a_1)\mu \neq 2k, k \in \mathbb{Z}_+ \setminus \{0\}$,

alors le système est instable.

Plusieurs systèmes instables qui ne satisfont pas les conditions du théorème précédent se trouvent dans la sous-section suivante. Pour ces systèmes, nous serons capable de conclure sur l'instabilité en utilisant d'autres analyses.

Dans le cas favorable où les chaînes neutres approchant l'axe imaginaire par la gauche, le théorème suivant présente d'autres conditions pour que le système soit H_∞ -stable.

Théorème B.45. *Soit $G(s)$ un système à retards de type neutre défini par (B.21), et supposons que G n'a pas de pôles instables de petit module et de chaînes de pôles sur l'axe imaginaire. Supposons aussi que le polynôme formel $\tilde{c}_d(z)$ défini par (B.23) a des racines r de module un et que toutes les valeurs de ν_1 relatives à chaque r vérifient $\Re(\nu_1) < 0$ où ν_1 est défini par (B.31). Alors, G est H_∞ -stable si et seulement si $\deg p \geq \deg t + x$ où $x = \max_r \{a_L\}$ avec (a_L, b_L) défini par (B.32) (il s'agit du point le plus à gauche parmi ceux les plus bas de $AB(r)$).*

B.4.4 Un exemple

On considère ici les systèmes $G(s)$ où $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$, et $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$.

Les conditions ci-dessus impliquent que $(1, 0), (1, 1) \notin AB(r)$ et $(1, 2), (2, 0) \in AB(r)$.

- Si $2 \leq m \leq 3$, alors $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ avec $\mathcal{S}_1 = \{(0, m), (2, 0)\}$ (voir la Figure B.4). Grâce au Théorème B.42, nous obtenons

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{et} \quad y_1 = \mathbf{m}_1 \mu,$$

ce qui donne

$$\nu_1^m = \frac{(-1)^{m+1} m! \tau^{2\mu} \sum_{k=1}^N \alpha_{2,k} r^k}{(j2\pi)^{2\mu} \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{et} \quad y_1 = \frac{2\mu}{m}.$$

Si $m = 2$, nous avons $n(\mathcal{S}_1) = 2$, $b_1 - b_2 = m - 0 = 2$, et $(a_2 - a_1)\mu = (2 - 0)\mu = 2\mu$. Si $\mu \in (0, 1)$ alors $2\mu \neq 2k$, $k \in \mathbb{Z}_+ \setminus \{0\}$, et donc le Théorème B.45 montre que le système est instable. Si $\mu = 1$, d'autres analyses sont nécessaires.

Si $m = 3$, nous avons $n(\mathcal{S}_1) = 2$ et $b_1 - b_2 = m - 0 = 3$, et donc le système est instable.

- Si $m = 4$, alors $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ avec $\mathcal{S}_1 = \{(0, 4), (1, 2), (2, 0)\}$ (voir la Figure B.4). Grâce au Théorème B.42, nous obtenons

$$\sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) = 0 \quad \text{et} \quad y_1 = \mathbf{m}_1 \mu,$$

alors

$$\frac{\nu_1^4}{4!} \sum_{k=1}^N \alpha_{0,k} k^4 r^k + \frac{\nu_1^2 \tau^\mu}{2(j2\pi)^\mu} \sum_{k=1}^N \alpha_{1,k} k^2 r^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}} \sum_{k=1}^N \alpha_{2,k} r^k = 0 \quad \text{et} \quad y_1 = \frac{\mu}{2}.$$

Le Théorème B.44 ne peut pas être appliqué dans ce cas. Pour étudier la stabilité, nous pouvons calculer ν_1 pour chaque système particulier et puis étudier le signe de $\Re(\nu_{n,1})$. D'ailleurs, nous pouvons faire l'analyse pour des classes de systèmes comme cela a été fait dans les Corollaires 4.16 et 5.18 dans lesquels nous avons prouvé que le système était instable.

- Si $m \geq 5$, alors $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1, \mathcal{S}_2\}$ avec $\mathcal{S}_1 = \{(0, m), (1, 2)\}$ et $\mathcal{S}_2 = \{(1, 2), (2, 0)\}$ (voir la Figure B.4). Grâce au Théorème B.42, nous obtenons

$$\begin{aligned} \sum_{(a,b) \in \mathcal{S}_1} C(a, b, \nu_1) &= 0 \quad \text{et} \quad y_1 = \mathbf{m}_1 \mu, \\ \sum_{(a,b) \in \mathcal{S}_2} C(a, b, \nu_1) &= 0 \quad \text{et} \quad y_1 = \mathbf{m}_2 \mu, \end{aligned}$$

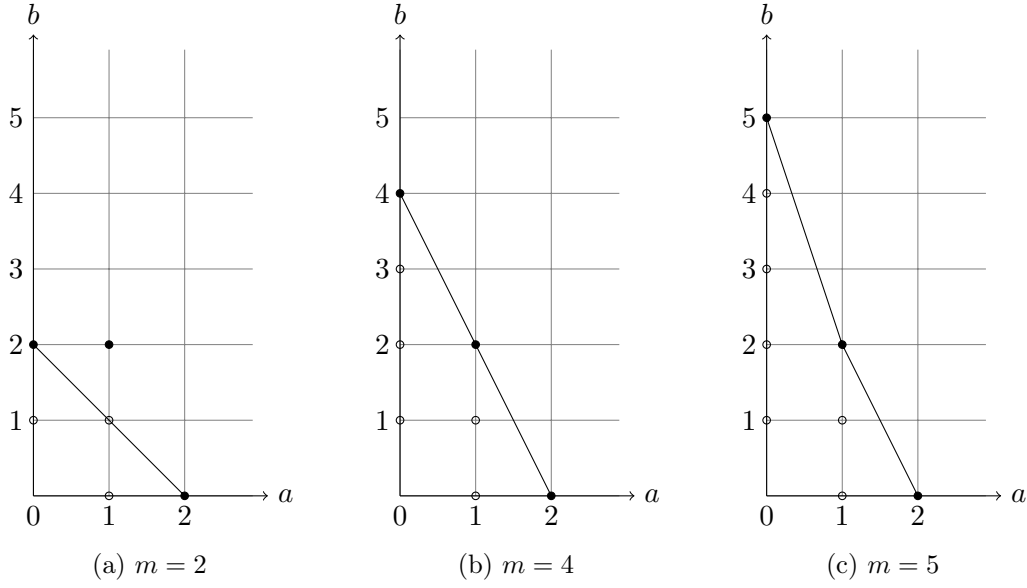


FIGURE B.4 – Les segments de frontière en bas à gauche de $AB(r)$ dans le cas où $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k \alpha_{1,k} r^k = 0$, $\sum_{k=1}^N k^2 \alpha_{1,k} r^k \neq 0$, et $\sum_{k=1}^N \alpha_{2,k} r^k \neq 0$

qui sont respectivement équivalents à

$$\nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau^\mu \sum_{k=1}^N \alpha_{1,k} k^2 r^k}{2(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k} k^m r^k} \quad \text{et} \quad y_1 = \frac{\mu}{m-2},$$

$$\nu_1^2 = -\frac{2\tau^\mu \sum_{k=1}^N \alpha_{2,k} r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{1,k} k^2 r^k} \quad \text{et} \quad y_1 = \frac{\mu}{2}.$$

Nous avons $n(\mathcal{S}_1) = 2$ et $b_1 - b_2 = m - 2 \geq 3$, alors le Théorème B.45 montre que le système est instable.

B.5 Stabilisation des systèmes fractionnaires SISO à retards commensurables

B.5.1 Propriétés de stabilisabilité

Nous étudions la stabilisation H_∞ des systèmes à retards commensurables de type neutre définis comme suit

$$G(s) = \frac{1}{p(s) + \sum_{k=1}^N q_k(s) e^{-sk\tau}} \quad (\text{B.35})$$

où $\tau > 0$ est le retard, p et q_k , $k \in \mathbb{N}_N$ sont des polynômes réels en s^μ , $\mu \in (0, 1)$, $\deg p \geq 1$, $\deg p \geq \deg q_k$, et il existe au moins un q_k , $k \in \mathbb{N}_N$ tel que $\deg p = \deg q_k$. Ici, le degré est interprété comme le degré en s^μ et est donc un entier. Afin d'éviter des fonctions de valeurs multiples, nous considérons s dans un feuillet de Riemann telle que $-\pi < \arg s < \pi$.

Cette classe de systèmes appartient évidemment à celles décrites par (B.21).

Considérons les contrôleurs de la forme

$$K(s) = \frac{N(s)}{D(s)} \quad (\text{B.36})$$

où N et D sont des polynômes réels en s^μ , $\mu \in (0, 1)$. Ces contrôleurs sont appelés les contrôleurs fractionnaires rationnels d'ordre commensurable. A partir de (Partington and Bonnet, 2004, Lemme 4.1), nous savons que $\deg N \leq \deg D$ si $K(s)$ stabilise $G(s)$ au sens H_∞ . Supposons que $N(s)$ et $D(s)$ n'ont pas de racines en commun, et $N(s)$ n'a pas de racines en commun avec le dénominateur de $G(s)$.

Nous considérons la stabilité interne.

Si le polynôme formel $\tilde{c}_d(z)$ défini par (B.23) a une racine r vérifiant $|r| < 1$, alors grâce à (B.26), la chaîne de pôles relative à r est asymptotique à un axe vertical qui se situe dans le demi-plan droit ouvert. Alors cette chaîne a une infinité de pôles dans le demi-plan droit ouvert et le système est instable. La stabilisation de tels systèmes par les contrôleurs de la forme (B.36) est examinée dans la proposition suivante.

Proposition B.46. *Soit G décrit par (B.35). Si $\tilde{c}_d(z)$ a des racines de module strictement inférieur à un, alors G ne peut pas être stabilisé par un contrôleur de la forme (B.36).*

Une autre situation où G a une infinité de pôles dans le demi-plan droit ouvert est lorsque G a des chaînes de pôle de type neutre qui approchent l'axe imaginaire à droite avec des autres chaînes neutres asymptotiques à des axes verticaux dans le demi-plan gauche ouvert. La proposition suivante considère cette classe de systèmes.

Proposition B.47. *Soit G décrit par (B.35). Supposons que le polynôme $\tilde{c}_d(z)$ a des racines de module un et de multiplicité une et que les autres racines sont de module supérieur à un. Supposons aussi qu'au moins une racine de module un de $\tilde{c}_d(z)$, notée r , vérifie*

$$\Re(\nu_{n,1}) > 0 \quad (\text{B.37})$$

où

$$\nu_{n,1} = \frac{\tau^\mu \sum_{k=1}^N \beta_k r^k}{(2jn\pi)^\mu \sum_{k=1}^N k \alpha_k r^k}.$$

Alors G peut être stabilisé par les contrôleurs de la forme (B.36) si et seulement si $\deg p = 1$.

Remarque B.48. Les systèmes considérés dans la Proposition B.47 ne sont pas les seuls à avoir des chaînes de pôles approchant l'axe imaginaire à droite. Cela pourrait arriver pour les systèmes avec $\Re(\nu_{n,1}) = 0$ mais dans ce cas d'autres analyses sont nécessaires comme décrites dans (Bonnet et al., 2011).

B.5.2 Paramétrisation des contrôleurs stabilisants

Les systèmes les plus simples parmi ceux décrits par (B.35) et avec $\deg p = 1$ sont les systèmes à un retard. Ils ont été étudiés dans (Bonnet and Partington, 2007). Pour de tels systèmes dont la fonction de transfert est donnée par

$$G(s) = \frac{1}{(as^\mu + b) + (cs^\mu + d)e^{-s\tau}}, \quad (\text{B.38})$$

où $a, b, c, d \in \mathbb{R}$, $a > 0$, $|a| = |c|$, et $\mu \in (0, 1)$, des contrôleurs PI fractionnaires ont été obtenus.

Ces contrôleurs sont le point de départ pour obtenir des paramétrisations de tous les contrôleurs stabilisants.

Remarque B.49. Les systèmes G définis par (B.38) peuvent avoir une infinité de pôles dans les demi-plans gauche ou droit ouverts.

Avec la connaissance d'un contrôleur stabilisant $K_0(s)$ du système (B.38), nous pouvons obtenir directement une paramétrisation de tous les contrôleurs stabilisants sans la connaissance des factorisations copremières en utilisant (Quadrat, 2003b, Théorème 2).

Proposition B.50. *Soit $G(s)$ décrit par (B.38). Une paramétrisation à deux degrés de liberté de tous les contrôleurs stabilisants H_∞ de $G(s)$ est donnée par*

$$\frac{-T(s^\mu R + T) + (s^{2\mu}Q_1 + Q_2T^2)R}{s^\mu(s^\mu R + T) + (s^{2\mu}Q_1 + Q_2T^2)} \quad (\text{B.39})$$

où $Q_1, Q_2 \in H_\infty$ sont les deux paramètres libres,

$$\begin{aligned} R(s) &= (as^\mu + b) + (cs^\mu + d)e^{-s\tau}, \\ T(s) &= k_p s^\mu + k_i, \end{aligned}$$

$k_i > 0$ et k_p vérifient

$$\begin{aligned} \frac{b + d + k_p}{a + c} &> -2\sqrt{\frac{k_i}{a + c}} \cos\left(\frac{\mu\pi}{2}\right) \quad \text{pour } a = c, \\ k_i(b + d + k_p) &> 0 \quad \text{pour } a = -c, \\ (a(b + k_p) - cd) \cos\left(\frac{\mu\pi}{2}\right) &> 0, \\ (b + k_p)^2 + 2ak_i \cos(\mu\pi) - d^2 &> 0, \\ k_i(b + k_p) \cos\left(\frac{\mu\pi}{2}\right) &> 0. \end{aligned}$$

Cependant, la paramétrisation de Youla-Kučera (avec un paramètre) pourrait être plus favorable pour la synthèse de contrôleurs et comme G est H_∞ -stabilisable, nous savons que G admet nécessairement une factorisation copremière sur H_∞ (Smith, 1989). La proposition suivante qui caractérise les quasi-polynômes dont toutes les racines sont dans le demi-plan gauche ouvert sera utile dans la recherche d'une factorisation copremière de la fonction de transfert du système. Cette dernière est nécessaire à l'obtention d'une paramétrisation des contrôleurs stabilisants en utilisant le formulaire de Youla-Kučera.

Proposition B.51. *L'équation*

$$(as^\mu + b) + (cs^\mu + d)e^{-s\tau} = 0 \quad (\text{B.40})$$

avec $a > 0$, $b, c, d, h \in \mathbb{R}$, $|a| = |c|$, et $\mu \in (0, 1)$ n'a pas de racines dans le demi-plan droit fermé pour tous $\tau \geq 0$ si et seulement si

- $b + d > 0$ et $b - d > 0$ si $a = c$,
- $b + d > 0$ et $b - d \geq 0$ si $a = -c$.

Nous donnons à présent une paramétrisation de tous les contrôleurs stabilisants des systèmes donnés par (B.38).

Théorème B.52. *Soit*

$$G(s) = \frac{1}{(as^\mu + b) + (cs^\mu + d)e^{-s\tau}}$$

avec $a, b, c, d \in \mathbb{R}$, $a > 0$, $|a| = |c|$, et $\mu \in (0, 1)$. L'ensemble de tous les contrôleurs stabilisants H_∞ est donné par

$$\frac{V + MQ}{U - NQ} \quad (\text{B.41})$$

où

$$\begin{aligned} N(s) &= \frac{1}{(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}}, \\ M(s) &= \frac{(as^\mu + b) + (cs^\mu + d)e^{-s\tau}}{(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}}, \\ U(s) &= \frac{s^\mu [(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}]}{s^\mu (as^\mu + b + k_p) + k_i + s^\mu (cs^\mu + d)e^{-s\tau}}, \end{aligned} \quad (\text{B.42})$$

$$V(s) = \frac{(k_p s^\mu + k_i) [(a's^\mu + b') + (c's^\mu + d')e^{-s\tau}]}{s^\mu (as^\mu + b + k_p) + k_i + s^\mu (cs^\mu + d)e^{-s\tau}}, \quad (\text{B.43})$$

Q est le paramètre libre qui appartient à H_∞ , $k_i > 0$ et k_p vérifient

$$\begin{aligned} \frac{b + d + k_p}{a + c} &> -2\sqrt{\frac{k_i}{a + c}} \cos\left(\frac{\mu\pi}{2}\right) \quad \text{pour } a = c, \\ k_i(b + d + k_p) &> 0 \quad \text{pour } a = -c, \\ (a(b + k_p) - cd) \cos\left(\frac{\mu\pi}{2}\right) &> 0, \\ (b + k_p)^2 + 2ak_i \cos(\mu\pi) - d^2 &> 0, \\ k_i(b + k_p) \cos\left(\frac{\mu\pi}{2}\right) &> 0, \end{aligned}$$

et $a', b', c', d' \in \mathbb{R}$ vérifient

$$\begin{aligned} a' &> 0, \\ \frac{a'}{c'} &= \frac{a}{c}, \\ b' + d' &> 0, \end{aligned} \quad (\text{B.44})$$

$$b' - d' > 0. \quad (\text{B.45})$$

B.5.3 Stabilisation H_∞

Notons

$$\begin{aligned} A(s) &= (as^\mu + b) + (cs^\mu + d)e^{-s\tau}, \\ A'(s) &= (a's^\mu + b') + (c's^\mu + d')e^{-s\tau}, \\ B(s) &= k_p s^\mu + k_i, \end{aligned}$$

alors les fonctions de transfert de la boucle fermée peuvent être écrites comme suit :

$$\frac{1}{1 + GK} = \frac{A(s^\mu A'^2 - Q(s^\mu A + B))}{A'^2(s^\mu A + B)}, \quad (\text{B.46})$$

$$\frac{G}{1 + GK} = \frac{s^\mu A'^2 - Q(s^\mu A + B)}{A'^2(s^\mu A + B)}, \quad (\text{B.47})$$

$$\frac{K}{1 + GK} = \frac{A(BA'^2 + AQ(s^\mu A + B))}{A'^2(s^\mu A + B)}. \quad (\text{B.48})$$

Les fonctions de transfert ci-dessus contiennent les termes A' ou $(s^\mu A + B)$ en dénominateur (sauf pour certaines valeurs de Q aboutissant à des simplifications entre numérateur et dénominateur). Ces termes ont toutes les racines dans le demi-plan gauche ouvert mais elles ont des racines à large module approchant l'axe imaginaire. Cependant, de petites variations de leurs coefficients pourraient décaler l'axe asymptotique à droite et la boucle fermée deviendrait donc instable.

Dans des applications où la stabilisation robuste est demandée, nous souhaitons éliminer les chaînes de pôles asymptotiques à l'axe imaginaire. Cependant, dans les propositions suivantes, nous montrons que pour une large classe de contrôleurs, il est impossible d'atteindre cela. Dans un premier temps, nous considérons le cas de μ rationnel et puis celui de μ irrationnel.

Proposition B.53. *Soit G donné par (B.38) avec μ rationnel. Si un contrôleur qui assure la stabilité interne de la boucle fermée est de la forme $K(s) = N_K(s)/D_K(s)$ où $N_K(s)$ et $D_K(s)$ sont des quasi-polynômes à coefficients réels en $e^{-s\tau}$ et s^ν , $\nu \in (0, 1)$ et est rationnel, alors la boucle fermée a nécessairement des chaînes de pôles asymptotiques à l'axe imaginaire.*

Remarque B.54. Les contrôleurs de la forme $K(s) = N_K(s)/D_K(s)$ où $N_K(s)$ et $D_K(s)$ sont des quasi-polynômes en $e^{-s\tau}$, s^μ , et s sont un cas particulier des contrôleurs considérés dans la Proposition B.53. En fait, si $\mu = m/n$ avec $m, n \in \mathbb{N}$, alors $N_K(s)$ et $D_K(s)$ peuvent être vus comme les quasi-polynômes en $e^{-s\tau}$ et $s^{1/n}$.

Proposition B.55. *Soit G donné par (B.38) avec μ irrationnel. Si un contrôleur stabilisant est de la forme $K(s) = N_K(s)/D_K(s)$ où $N_K(s)$ et $D_K(s)$ sont des quasi-polynômes à coefficients réels en s^μ et $e^{-s\tau}$, alors la boucle fermée a nécessairement des chaînes de pôles asymptotiques à l'axe imaginaire.*

B.6 Perspectives

Le travail à venir suivra les directions suivantes.

Pour les systèmes MISO avec retards en entrées/sortie, dans la Section B.3, des factorisations copremières doubles n'ont pas encore été obtenues dans la cas général où des éléments de la matrice de transfert auraient des pôles identiques. Nous allons nous y atteler à l'avenir.

Pour l'ensemble de contrôleurs stabilisants construits à partir des facteurs copremiers et de Bézout obtenus, nous souhaitons chercher des méthodes pour choisir des contrôleurs à implémenter qui ne sont pas sensibles aux incertitudes des paramètres. Ce problème a été étudié pour les systèmes classiques dans (Gumussoy, 2012).

Pour les systèmes fractionnaires SISO de type neutre, les résultats de l'analyse de stabilité obtenus dans la Section B.4 pourraient être utiles pour décider sur la stabilisabilité de plusieurs classes de systèmes fractionnaires à retards par des contrôleurs rationnels ou fractionnaires sans ou avec retards.

La méthode dans la Section B.4 permet de conclure sur la stabilité du système dans tous les cas à l'exception du cas où la localisation des pôles relatifs à l'axe imaginaire ne peut être déterminé à l'aide de l'approximation obtenue et d'autres analyses sont nécessaires. Bien que le terme d'approximation suivant puisse être déterminé en suivant la même procédure présentée pour certains cas dans la section, on doit répéter la procédure pour chaque cas particulier. Alors une piste intéressante serait de déterminer une méthode plus systématique dans l'esprit de ce qui a été réalisé dans la Section B.4 (la même approche ne paraissait toutefois pas pouvoir être étendue à ce cas).

Pour le problème de stabilisation des systèmes fractionnaires de type neutre dans le cas critique où des pôles sont asymptotiques à l'axe imaginaire, le travail à venir pourrait considérer une plus large classe de contrôleurs stabilisants, par exemple ceux qui contiennent des termes en e^{-s^p} ou des retards non-commensurables car il s'agit d'une classe de contrôleurs simple. Bien que les retards non-commensurables augmentent la difficulté de l'analyse, l'étude de ces systèmes pourrait fournir une meilleure compréhension des comportements des systèmes réels où la variation des retards est courante (transformant donc un système à retards commensurables en un système à retards non-commensurables). La même question adressée à la classe plus générale de contrôleurs stabilisants est certainement difficile. Ensuite, le travail à venir pourrait être consacré au problème de stabilisation d'une classe plus large de systèmes à plusieurs retards.

Nous sommes entrain d'intégrer les résultats obtenus dans les Sections B.4 et B.5 dans le toolbox Matlab YALTA qui peut être téléchargé à l'adresse <http://team.inria.fr/disco/software/>.