



Robust nonlinear control: from continuous time to sampled-data with aerospace applications.

Giovanni Giovanni Mattei

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par

Giovanni MATTEI

Robust Nonlinear Control

from continuous time to sampled-data
with aerospace applications

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"I? What am I?" roared the President, and he rose slowly to an incredible height, like some enormous wave about to arch above them and break. "You want to know what I am, do you? Bull, you are a man of science. Grub in the roots of those trees and find out the truth about them. Syme, you are a poet. Stare at those morning clouds. But I tell you this, that you will have found out the truth of the last tree and the top-most cloud before the truth about me. You will understand the sea, and I shall be still a riddle; you shall know what the stars are, and not know what I am. Since the beginning of the world all men have hunted me like a wolf - kings and sages, and poets and lawgivers, all the churches, and all the philosophies. But I have never been caught yet, and the skies will fall in the time I turn to bay. I have given them a good run for their money, and I will now." (Sunday, ch.13)

G. K. Chesterton - *The Man Who Was Thursday.*

Science furthers ability, not knowledge. The value of having for a time rigorously pursued a rigorous science does not derive precisely from the results obtained from it: for in relation to the ocean of things worth knowing these will be a mere vanishing droplet. But there will eventuate an increase in energy, in reasoning capacity, in toughness of endurance; one will have learned how to achieve an objective by the appropriate means. To this extent it is invaluable, with regard to everything one will afterwards do, once to have been a man of science.

F. Nietzsche - *Human, All Too Human.*

Abstract

The dissertation deals with the problems of stabilization and control of nonlinear systems with deterministic model uncertainties. First, in the context of uncertain systems analysis, we introduce and explain the basic concepts of robust stability and stabilizability. Then, we propose a method of stabilization via state-feedback in presence of unmatched uncertainties in the dynamics. The recursive backstepping approach allows to compensate the uncertain terms acting outside the control span and to construct a robust control Lyapunov function, which is exploited in the subsequent design of a compensator for the matched uncertainties. The obtained controller is called *recursive Lyapunov redesign*. Next, we introduce the stabilization technique through “Immersion & Invariance” (I&I) as a tool to improve the robustness of a given nonlinear controller with respect to unmodeled dynamics. The recursive Lyapunov redesign is then applied to the attitude stabilization of a spacecraft with flexible appendages and to the autopilot design of an asymmetric air-to-air missile. Contextually, we develop a systematic method to rapidly evaluate the aerodynamic perturbation terms exploiting the deterministic model of the uncertainty. The effectiveness of the proposed controller is highlighted through several simulations in the second case-study considered. In the final part of the work, the technique of I&I is reformulated in the digital setting in the case of a special class of systems in feedback form, for which constructive continuous-time solutions exist, by means of backstepping and nonlinear domination arguments. The sampled-data implementation is based on a multi-rate control solution, whose existence is guaranteed for the class of systems considered. The digital controller guarantees, under sampling, the properties of manifold attractivity and trajectory boundedness. The control law, computed by finite approximation of a series expansion, is finally validated through numerical simulations in two academic examples and in two case-studies, namely the cart-pendulum system and the rigid spacecraft.

Keywords. *Nonlinear control, robust control, robust control Lyapunov function, uncertainty modeling, robust backstepping, Lyapunov redesign, recursive Lyapunov redesign, Immersion and Invariance, nonlinear sampled data control, multi-rate control, missile autopilot design, attitude stabilization, flexible spacecraft*

List of the publications

1. Journals

- G. Mattei and S. Monaco - “Nonlinear Autopilot Design for an Asymmetric Missile Using Robust Backstepping Control” - Journal of Guidance, Control, and Dynamics, (2014), doi: <http://arc.aiaa.org/doi/abs/10.2514/1.G000434>.

2. Conferences

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- G. Mattei and S. Monaco - “Robust Backstepping Control of Missile Lateral and Rolling Motions in the Presence of Unmatched Uncertainties” - IEEE CDC 2012 - p. 2878-2883.
- G. Mattei, P. Di Giamberardino, S. Monaco, M.D. Normand-Cyrot - “Lyapunov Based Attitude Stabilization of an Underactuated Spacecraft with Flexibilities” - IAA-AAS DyCoSS2 2014 - 140703.
- G. Mattei, A. Carletti, P. Di Giamberardino, S. Monaco, M.D. Normand-Cyrot - “Adaptive Robust Redesign of Feedback Linearization for a Satellite with Flexible Appendages” - IAA-AAS DyCoSS2 2014 - 141007.
- G. Mattei, S. Monaco, M.D. Normand-Cyrot - “Multi-rate sampled-data I&I stabilization of a class of nonlinear systems” - IEEE ECC 2015 proceedings.
- G. Mattei, S. Monaco, M.D. Normand-Cyrot - “Robust Nonlinear Attitude Stabilization of a Spacecraft through Digital Implementation of an I&I Stabilizer” - IFAC MICNON 2015 proceedings.

3. Technical reports

- G. Mattei and F. Liberati - “Modelli di option pricing: l’equazione di Black & Scholes” - Department of Computer, Control, and Management Engineering “Antonio Ruberti” Technical Reports, Report n. 11, 2013.

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*But the little things they make me so happy
All I want to do is live by the sea
Little things they make me so happy
But it's good, yes it's good . . . it's good to be free.*

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Symbols

Asymmetric missile parameters

m	mass of the missile	kg
I_x, I_y, I_z, I_{xz}	moments of inertia	kg·m ²
d	diameter of the fuselage	m
S	cross-sectional area	m ²
$\hat{x}_b, \hat{y}_b, \hat{z}_b$	unit-vectors of body-axes	m
φ, α, β	bank, attack and sideslip angles	rad
p, q, r	angular velocities	rad·s ⁻¹
$\delta_R, \delta_P, \delta_Y$	overall tail-fins angular deflections	rad
F_y, F_z	aerodynamic forces	N
M_x, M_y, M_z	aerodynamic moments	N·m
ρ	air density	kg·m ⁻³
M	Mach number	-
V_m, V_s	missile velocity and speed of sound	m/s
$\frac{1}{2}\rho V_m^2$	dynamic pressure Q_d	kg·m ⁻¹ s ⁻²
C_{Ny}, C_{Nz}	aerodynamic coefficients (forces)	-
C_l, C_m, C_n	aerodynamic coefficients (moments)	-
G_p, G_q, G_r	aerodynamic coefficients (control deflections)	-
$a_{yi}, b_{yi} \ i = 1, \dots, 4$	parameters defining F_y	-
$a_{zi}, b_{zi} \ i = 1, \dots, 4$	parameters defining F_z	-
$d_i \ i = 1, \dots, 5$	parameters defining M_x	-
$m_i \ i = 1, \dots, 4$	parameters defining M_y	-
$n_i \ i = 1, \dots, 4$	parameters defining M_z	-
C_{l_p}	stability derivative of the rolling moment	-
$C_{m_\alpha}, C_{m_{\alpha\beta}}, C_{m_q}, C_{n_{p\beta}}$	stability derivatives of the pitching moment	-
$C_{n_\beta}, C_{n_{\alpha\beta}}, C_{n_r}, C_{n_{p\alpha}}, C_{n_{\delta_Y}}$	stability derivatives of the yawing moment	-
$C_{mi}, i = 1, 2, 3, 4$	parameters defining the Magnus effect	s ⁻¹ /-
τ	time constant of the actuators dynamics	s
I_3	three-dimensional identity matrix	s

Dedicated to my beloved parents, Virginio and Anna Maria.

Introduction

Automatic control design of nonlinear systems affected by disturbances and model uncertainties is a central topic of interest for the whole scientific community involved in feedback control system analysis and synthesis. In the present work, we develop a procedure to achieve practical asymptotic stabilization of nonlinear systems with matched and unmatched uncertainties, mostly representing modeling errors in the parameters of the system and time-varying disturbances. The core idea is to unify the existing results of Lyapunov redesign and robust backstepping into a single stabilizing controller, which we call recursive Lyapunov redesign. This tool merges the multi-step philosophy of robust backstepping with the implicit control definition of Lyapunov redesign to counteract the effects of both matched and unmatched uncertainties in system dynamics. The property of stability which can be achieved is that of practical robust global asymptotic stability, meaning that the closed-loop trajectories converge to an arbitrarily shrinkable compact set surrounding the origin. Such an approach results particularly suited to design control laws for aerospace systems, whose dynamics is usually highly nonlinear, with time-varying uncertain parameters and disturbances. We apply the proposed technique to two case studies, the attitude stabilization of a spacecraft with flexible appendages and the autopilot design for an asymmetric tail-controlled air-to-air missile.

The construction of a nonlinear stabilizing controller working in the presence of system uncertainties of various nature has captured the attention of control scientists since the early Eighties, starting from linear systems with matched uncertainties [21, 41, 70], next relaxing the matching conditions [8, 9] and finally approaching nonlinear systems [22] and the application to chemical processes [64]. However, it was only in the early Nineties that constructive solutions of robust controllers for nonlinear systems started to appear in the literature, beginning with the work of Isidori and Astolfi [54], mainly devoted to the problem of disturbance attenuation in the nonlinear \mathcal{H}_∞ setting. The robust backstepping approach was first introduced in 1993 by Freeman and Kokotović [31] and by Marino and Tomei [81], from different points of view. A similar construction, however, was found one year before, in 1992, by Qu in [103]. In the following years, the techniques became more and more refined and several important results on robust

nonlinear stabilization were obtained [32, 59, 104, 105], also paving the way for the publication of two fundamental books in the field, the first by Freeman and Kokotović [36] and the second by Qu [106]. Depending on applications, design constraints and methodological improvements, the original core ideas have been progressively specialized and detailed into different areas of interest, like robust adaptive nonlinear control [33, 56, 102, 107], output feedback stabilization and observer design [3, 11–13, 79, 80, 122], saturated feedback problems [34, 123], inverse optimality results [35, 100] and so on. New successful concepts were also developed struggling in making a nonlinear control system robustly stable with respect to external disturbances, one and for all Input-to-State Stability (ISS) [117], with all her friends and cousins, for instance integral Input-to-State Stability (iISS) [2], set Input-to-State Stability [39], incremental stability [1], incremental norm [38] and contraction theory [75]. We cannot keep from mentioning sliding modes. The old idea of sliding surface and the corresponding sliding mode control laws represent the simplest tools guaranteeing robust nonlinear stabilization [128]. They have the advantage of ensuring a finite-time convergence on the surface where the desired behavior of the system is enforced. However, the discontinuity of the control laws obtained may cause serious implementation issues, one for all the problem of “chattering”. A solution has been proposed with higher-order sliding modes, which involve a dynamic extension of the system [10, 30]. Sliding mode control theory and its applications have been explored high and low during the years, with modifications of the original to improve robustness and embrace a broader class of uncertain systems. Notable extensions of the theory lead to sliding-mode observer designs via the super-twisting algorithm [29] and other applications to the measurements feedback case [71].

The basic ideas exploited in robust backstepping design to counteract the effect of the unmatched uncertainties are those of *recursion* and *virtual control*. Moreover, this technique provides a systematic way of construction of a control Lyapunov functions for systems in pure-feedback or strict-feedback form. These ideas were already introduced, although in an embryonic form, in the work of Feuer and Morse [28]. Later, they were developed simultaneously by Byrnes and Isidori [15], Kokotović, Sontag and Sussmann [63, 120] and further refined by Tsinias in the early Nineties [125]. In the Nineties, these ideas were also deeply exploited in adaptive controller design and revealed themselves to be particularly suited in several application domains, as widely shown in [60] and in the book by Krstic, Kanellakopoulos and Kokotović [65]. In the following years, the ideas of backstepping were extended to different system structures. The forwarding technique was introduced to construct nonlinear stabilizer for systems in feedforward form [55, 89, 111], while the interlacing technique to stabilize systems in interlaced form [73, 112]. The reader should be warned with the fact that some of this ideas were already introduced, under slightly different forms, in the most cited book in the control

literature by Alberto Isidori [52], and further developed in [53].

The design of robust controllers for nonlinear systems arouses the interest of the aerospace community, since most of the physical systems in the aerospace domain present nonlinear and uncertain dynamics. Let us consider, for instance, the problem of missile autopilot design, which concerns highly nonlinear dynamics with uncertain, rapidly varying, parameters. Starting from the late Seventies the problem has been faced using classical and Linear Quadratic regulators based upon dynamical models linearized around fixed operating points. These methods lead naturally to the design of several linear time-invariant (LTI) point-regulators around equilibria defined on stationary conditions, usually involving Mach number, altitude and weight. The controller, resulting from the interpolation of the fixed-point regulators, guarantees global stability only in the case of slowly varying conditions on both the states and the parameters. This technique, named *gain scheduling*, still represents the state-of-the-art in control system design for missile autopilots; this is mostly due to the simplicity of implementation of gain-scheduled regulators, which also ensure relatively good performance. Classical gain scheduling design are presented, for instance, in [98] and [40]. In the Nineties, extensions of these techniques brought several improvements, like guaranteed stability margins and performance levels ([113], [99], [7], [109] and references therein). Robustness issues have also been addressed using suitable extensions of H_∞ techniques in [37] and [38]. The breakthrough of LPV (linear parameter-varying) and quasi-LPV approaches in the last two decades has guided the research and the designers towards new, more systematic and rigorous methodologies. Main features of these new approaches are the sounder robustness/performance criteria used and the simplified control synthesis process adopted. The price to pay, especially for quasi-LPV controllers, is an increased level of conservatism due to the non-uniqueness of LPV representations of a given nonlinear system. See, for instance, [130], [121] and [23]. Most of these approaches, with the notable exception of quasi-LPV designs, are still based on local linearizations of the dynamics around operating points. Nevertheless, the introduction of new technologies (e.g. high maneuverability and stealth), together with the developments of nonlinear control methods, has pushed towards new control design methods which take into account the intrinsic nonlinearity of the dynamics. This led to a first generation of nonlinear autopilots based on the inversion of the dynamics [110], on feedback linearization techniques [47], [42], [24], and on sliding modes [124]. New solutions have been proposed in the last decades using the more recent control methods: Lyapunov stabilization techniques [129], \mathcal{L}_1 adaptive control [16], Immersion & Invariance control [69] and the state-dependent Riccati equation approach [18]. The solutions so far proposed in the nonlinear and/or adaptive context fail in presence of large unstructured uncertainties in the dynamics; moreover, the strict requirements on the speed of response cannot often be fulfilled in presence of adaptation

laws. Therefore robust nonlinear approaches based on the geometric theory, as in [49], and others which make extensive use of Lyapunov direct criterion, as in [51] and [50], have also been proposed, with good performances at high angle-of-attack. It should also be pointed out that the solutions based on these methods have only been applied in simple single-input/single-output (SISO) cases, disregarding the dangerous couplings and nonlinearities that arise between lateral and longitudinal dynamics when dealing with Bank-To-Turn maneuvers. Furthermore, these controllers do not take into account Magnus effect and cross-couplings between control channels due to the differential of pressure acting on the tail fins. Notable exceptions are in the works of Jim Cloutier [19], [72] and [96]. In this work we design a robust nonlinear autopilot for an asymmetric air-to-air missile in order to counteract matched and unmatched uncertainties in a MIMO context. Such aim is achieved using the recursive Lyapunov redesign approach introduced in this dissertation, exploiting recursion and virtual control to reach the nested part of the dynamics. Since very little is known about the aerodynamic parameters, uncertainties are mostly treated as unstructured, thus reducing the modeling effort and avoiding the need of a linear parametrization as in adaptive approaches. Moreover, a systematic method to evaluate the uncertain aerodynamic terms is proposed for the first time. Performance is radically increased in both Bank-To-Turn and Skid-To-Turn maneuvers with respect to the standard backstepping design, at the expense of an inevitable increase in control effort during the transient response. Embracing the nonlinear behavior of the system, the proposed controller is suitable to guarantee the fulfilment of both maneuvers, with slight changes in its equations.

A similar discussion can be made about the design of robust nonlinear controllers for flexible spacecraft. The fundamental advantage of the robust nonlinear control techniques presented in this work relies in the fact that they exploit as much as possible the knowledge of the model of the system. This modeling effort reflects itself on the construction of a proper model of the uncertain terms, as it will be shown later in the work. The attitude stabilization of spacecraft with flexible appendages has been widely studied, but very few are the results employing robust nonlinear stabilizers [25, 48]. Furthermore, there is no evidence of results obtained employing robust backstepping or Lyapunov redesign approaches, as done in the present work.

Uncertainty modeling represents a crucial point in robust nonlinear control design, so that in the present work we establish a systematic procedure to evaluate complex aggregates of uncertain terms using a deterministic representation of the uncertain parameters through the Δ -operator and the deviation function. Several techniques of uncertainty modeling were developed in different fields, from linear robust control theory [114, 132] to the nonlinear counterparts [66]. Several kinds of models of uncertain environment were developed, for instance, in the robotics field [27]. A wise exploitation of a good

modeling effort in order to robustify preexisting control laws is one of the nice features characterizing the Immersion & Invariance (I&I) stabilizing controllers. Such technique, first introduced in [6], still represents a strong tool to define and solve stabilization problems, opening new horizons in both the physical and mathematical sides of control theory, as extensively shown in [5]. The basic ingredients of I&I stabilization are the concepts of manifold invariance and immersion of a target system, representing a reduced dynamics with desired properties, often a-priori known as a controlled sub-system of the process to be stabilized. The method is particularly effective when applied to systems with triangular structures, whereas Lyapunov techniques provide constructive solutions. Moreover, it is specifically suited for adaptive controller design [74], also in the discrete-time case [61],[131]. This nonlinear stabilization technique is robust in the sense that it exploits an increased modeling effort in order to “add robustness” to a nominal control law developed on the system disregarding higher-order dynamics. The I&I controller can thus be viewed as a robust version of a preexisting control law which neglects such unmodeled dynamics. These higher-order dynamics are usually faster and in physical systems they often correspond to actuator’s first or second-order dynamics, flexible motions and so on. In this work, a continuous-time solution of the I&I stabilization problem is found for a special class of systems in feedback form. Then, we propose its digital implementation, namely when the control input is maintained constant over time intervals of fixed length, called sampling periods. We obtain approximate sampled-data single rate and multi-rate solutions [91] which follow the continuous-time closed-loop trajectories to ensure manifold attractivity. Trajectory boundedness and manifold invariance are preserved under sampling. The digital controllers are applied to two academic examples and two case-studies, the cart-pendulum system and the rigid spacecraft. Their effectiveness is thus shown by several simulations in different scenarios at increasing sampling periods. The performance improvement of the proposed controllers relies in a remarkable increase of the Maximum Allowable Sampling Period (MASP) with respect to the direct implementation with zero-order hold of the continuous-time solution, which is emulated control (see [97]).

Three fundamental ingredients participate in characterizing a good recipe for robust nonlinear control design: the notion of control Lyapunov function, the representation of system uncertainty together with its, well defined, bounds, and the concept of gain of a nonlinear system. The first is set up to find out whether a nonlinear system is stabilizable or not; the second is strictly related to the robustness issue and yields the concept of *robust* control Lyapunov function, while the third is instrumental in ensuring what is commonly called the *performance* of a system. This work is devoted to develop the theoretical instruments which come along with these three ingredients, with the final aim of constructing a general purpose robust nonlinear stabilizer. The main contribution

of the work lies in the construction of a robust nonlinear stabilizer capable of handling matched uncertainties multiplying the control, using the technique called recursive Lyapunov redesign. This feature has a certain impact on aerospace applications, especially in the case of the tail-controlled air-to-air missile used as case study in this work. It is also, at least to our knowledge, the first time that the technique of the recursive Lyapunov redesign is employed to stabilize the attitude of a spacecraft with flexible appendages, exploiting the particularly control-friendly kinematic representation based on the modified Cayley-Rodrigues parameters. Another contribution is the introduction of a systematic method of evaluation of the uncertain terms, which is based on a particular deterministic model of uncertainty. We show how to propagate easily complex groups of uncertainties into system dynamics, reducing the computational effort in the expression of the corresponding bounding functions. Furthermore, a partially novel contribution lies in the construction of an I&I stabilizer for a particular class of feedback-systems, for which constructive solutions exist. The design employs one step of backstepping and nonlinear domination arguments to show the global asymptotic stability of the controlled system.

The work is organized as follows.

Chapter 1 contains some recalls of the notions of stability and stabilizability of uncertain time-varying nonlinear systems. We start with the definitions of semi-global and practical stability, which are of sure interest in the context of applications, since the properties introduced take into account that there are bounded set of feasible initial conditions and that the steady-state error could also be “sufficiently small” and not exactly zero. Then we introduce the important concepts of Input-to-State Stability (ISS) and Control Lyapunov Function (CLF), which are related to the problems of disturbance attenuation and nonlinear systems stabilizability. Next, we introduce the problem of robust nonlinear stabilizability and the corresponding definitions of nonlinear uncertain system and practical robust stabilizability. Finally, we introduce the concept of robust control Lyapunov function (RCLF), which is the core-idea of the robust nonlinear control design techniques presented later in the work.

In **Chapter 2**, the definition of uncertain nonlinear system introduced in Chapter 1 is refined and particularized using a suitable uncertainty representation. Moreover, we specialize the class of systems considered into that of input-affine nonlinear uncertain systems and we give examples of unstabilizable uncertain systems, highlighting the most common sources of instability which uncertainty can bring. Next, we introduce the fundamental notion of matching conditions together with the corresponding relaxed version, that of generalized matching conditions. Both these conditions set some constraints on system structure which are strictly related to robust stabilizability. In the final part

of the chapter, we propose a deterministic model of the uncertain terms exploiting the concepts of Δ -operator and deviation function. Moreover, we establish some rules to evaluate the complex uncertain terms and a systematic method to propagate them into system dynamics.

In **Chapter 3** we establish the main result of the work. We begin by recalling the Lyapunov redesign technique and introducing a revisited version of it, exploiting suitable differentiable sigmoid functions, which we call “robust control functions”, in order to avoid the problem of chattering in the implementation phase. Next, we recall the robust backstepping technique, and finally we present our main result, consisting of the fusion of the two approaches, the technique called *recursive Lyapunov redesign*. This technique allows to compensate also control-dependent uncertainties, which are common in certain nonlinear aerospace dynamics, like missile dynamics. In the last part of the chapter, we recall the stabilization technique of Immersion & Invariance as a tool of robustification of nonlinear control laws with respect to higher-order dynamics. We propose a solution in the case of a particular class of systems in feedback form using one step of backstepping and a nonlinear domination argument.

In **Chapter 4**, the robust nonlinear control techniques introduced are applied to the dynamics of a spacecraft with flexible appendages. First, the dynamic model is made more effective by using the modified Cayley-Rodrigues parameters for a global and non-redundant attitude representation. Then, stability of the kinematic-flexible subsystem under a linear control law is demonstrated to simplify the subsequent recursive Lyapunov redesign. Application of robust nonlinear control results in practical robust uniform global asymptotic stability (P-RUGAS) of the closed loop system.

In **Chapter 5** the recursive Lyapunov redesign technique is employed to design the autopilot of an asymmetric air-to-air missile, in order for it to perform Skid-To-Turn and Bank-To-Turn maneuvers. After a detailed derivation of the dynamical uncertain model, the problem setting and the construction of a nominal backstepping controller, we design compensators for the matched and unmatched uncertainties of the dynamics using the techniques introduced above. P-RUGAS of the closed-loop system is then shown and several simulations demonstrate the effectiveness of the proposed control law in different scenarios.

In **Chapter 6**, we reformulate the I&I stabilization solution found in Chapter 3 into the sampled-data framework, i.e. when the control input is maintained constant over intervals of fixed length, namely the sampling period. For the particular class of systems considered, a digital multi-rate control solution does exist, and we propose an approximate version for application purposes. The approximate sampled-data single rate and multi-rate solutions found directly match the continuous-time closed-loop trajectories

at the sampling instants to ensure manifold attractivity. Trajectory boundedness and manifold invariance are preserved under sampling. The digital controllers are then applied to two academic examples and to the cart-pendulum system, with simulations showing how the presence of the first-order corrector term enhances the performance and ensures stability at sampling times higher than in the case of direct zero-order hold implementation.

In **Chapter 7** the problem of attitude stabilization of the rigid spacecraft, robustly with respect to first-order actuator dynamics is considered. A continuous time control solution is found using a two-step backstepping transformation and, subsequently, the I&I approach is used to handle actuator dynamics as a dynamic extension of the system. Stabilization is achieved via a nonlinear domination argument. The digital single-rate version of the controller, tailored to the multiple input nature of the system, is then proposed. Its effectiveness is shown by several simulations in two scenarios at increasing sampling periods.

Chapter 1

Robust stability and stabilizability of nonlinear systems

In this chapter, we present and discuss some quite basic facts about the notions of stability and stabilizability of uncertain time-varying nonlinear systems. We start with the definitions of semi-global and practical stability, which are of sure interest in the context of applications, since the properties introduced take into account that there are bounded set of feasible initial conditions and that the steady-state error could also be “sufficiently small” and not exactly zero. Then we introduce the important concepts of Input-to-State Stability (ISS) and Control Lyapunov Function (CLF), which are related to the problems of disturbance attenuation and nonlinear systems stabilizability. Next, we introduce the problem of robust nonlinear stabilizability and the corresponding definitions of nonlinear uncertain system and practical robust stabilizability. Finally, we introduce the concept of robust control Lyapunov function (RCLF), which is the core-idea of the robust nonlinear control design techniques presented later in the work.

1.1 Lyapunov stability of uncertain systems

As the technical definitions of system stability according to Lyapunov should be very well known by the reader, we simply state it as the property of an equilibrium point, a motion or a set of a system to react to perturbations in the initial conditions containing or making vanish their effect on the solution with time. Let us just recall the concept of *domain* (or, alternatively, *basin*) of attraction as the subset of the state space containing all the initial states which lead to asymptotic convergence to a certain equilibrium point,

also known as *attractivity*. These definitions [62] can be extended to the context of uncertain nonlinear systems introducing the concept of *practical stability*. For, let us start by introducing some useful stability and boundedness definitions for time-varying nonlinear systems of the form

$$\dot{x} = f(t, x) \quad (1.1)$$

where $t \in \mathbb{R}_0^+$ represents the time, $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}_0^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to satisfy Carathéodory conditions and to be locally Lipschitz in x . More specifically, for each compact set Q of $\mathbb{R}_0^+ \times \mathbb{R}^n$, we assume that there exists an integrable function $l_Q : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that, for all $(t, x) \in Q$ and all $(t, y) \in Q$,

$$\|f(t, x) - f(t, y)\| \leq l_Q(t) \|x - y\|. \quad (1.2)$$

The Carathéodory conditions plus the Lipschitz condition ensure both existence and uniqueness of the solutions (1.1) [46]. Note that a large variety of physical systems can be described by this equation, which is the starting point to understand stability and stabilizability concepts of uncertain dynamical systems as well. Moreover, the study of time-varying, or nonautonomous, nonlinear systems is useful to solve trajectory tracking problems and, also, in the context of the stabilization of systems which do not satisfy Brockett's necessary condition [14]. These kind of systems are, in fact, not stabilizable by means of a continuous time-invariant state feedback, making the class of time-varying and/or discontinuous controllers eligible for feedback design. This fact is certainly relevant in space systems applications, since underactuated spacecraft belong to the class of systems depicted just above.

In the following, we state stability and boundedness definitions with respect to the origin of (1.1), denoting with \mathcal{D} a closed subset of \mathbb{R}^n containing the origin and with \mathcal{B}_r the ball of radius r centered in the origin.

Definition 1.1.0.1 (UB/UGB). The solutions of (1.1) are said to be *uniformly bounded* on \mathcal{D} if, for any non-negative constant r , there exists a non-negative $c(r)$ such that, for all $t_0 \in \mathbb{R}_0^+$, the flow ϕ satisfies

$$x_0 \in \mathcal{D} \cap \mathcal{B}_r \Rightarrow \|\phi(t, t_0, x_0)\| \leq c, \quad \forall t \geq t_0. \quad (1.3)$$

If \mathcal{D} is \mathbb{R}^n , then the solutions are *uniformly globally bounded*. \diamond

The concept of uniform stability is given according to the ϵ - δ definition for time-varying systems.

Definition 1.1.0.2 (US/UGS). The origin of (1.1) is said to be *uniformly stable* on \mathcal{D} if the solutions starting from it are uniformly bounded on \mathcal{D} and, given any positive constant ϵ , there exists a positive $\delta(\epsilon)$ such that, for all $t_0 \in \mathbb{R}_0^+$, the solution of (1.1) satisfies

$$\|x_0\| \leq \delta \Rightarrow \|\phi(t, t_0, x_0)\| \leq \epsilon, \quad \forall t \geq t_0. \quad (1.4)$$

If \mathcal{D} is \mathbb{R}^n , then the origin is *uniformly globally stable*. \diamond

Note that the difference between the two concepts of boundedness and stability mainly relies in the intrinsic “locality” of the ϵ - δ definition 1.1.0.2. On the other hand, (1.1.0.1) is of practical interest because it is not uniquely related to a given equilibrium point like, for instance, the origin, while rather to an entire domain of initial conditions starting from which the trajectories remain bounded.

When the characteristics of the boundedness property are ensured for every positive ϵ , arbitrarily small, irrespective of the initial conditions in a given set surrounding the origin, we have the notion of attractivity of the origin.

Definition 1.1.0.3 (UA/UGA). The origin of (1.1) is said to be *uniformly attractive* on \mathcal{D} if, for all positive numbers r and ϵ , there exists a positive time $T(r, \epsilon)$ such that, for all $x_0 \in \mathcal{D} \cap \mathcal{B}_r$ and all $t_0 \in \mathbb{R}_0^+$, the solutions of (1.1) satisfy

$$\|\phi(t, t_0, x_0)\| \leq \epsilon, \quad \forall t \geq t_0 + T. \quad (1.5)$$

If \mathcal{D} is \mathbb{R}^n , then the origin is *uniformly globally attractive*. \diamond

Uniform stability plus uniform attractivity yields the property of uniform asymptotic stability.

Definition 1.1.0.4 (UAS/UGAS). The origin of (1.1) is said to be *uniformly asymptotically stable* on \mathcal{D} if it is both uniformly stable and uniformly attractive on \mathcal{D} . If \mathcal{D} is \mathbb{R}^n , then the origin is *uniformly globally asymptotically stable*. \diamond

Remark 1.1.0.5 (Uniformity and robustness). As detailed in [68], the property of uniformity guarantees a certain robustness towards external disturbances, since UAS of the origin of $\dot{x} = f(t, x, 0)$ ensures stability with respect to constantly acting disturbances $d(t)$ of $\dot{x} = f(t, x, d)$, under the assumption of f locally Lipschitz in x uniformly in t . This concept, first introduced by Malkin [78], is known as *total stability* and states that the solutions remain arbitrarily small at all time if the initial state and the disturbance signal are sufficiently small. In some sense, total stability is the progenitor of the concept of input-to-state stability which will be given in the following. This robustness property, however, is strictly related to the uniformity assumption: it is in fact possible to design an arbitrarily small input $d(t)$ such that a GAS (not uniformly) system $\dot{x} = f(t, x, 0)$ becomes unstable under it. \diamond

In the presence of “large”, bounded but non-vanishing, disturbances, asymptotic stability cannot be ensured anymore. In this setting, convergence to, eventually large, neighborhoods of the operating points should be established under the name of *uniform ultimate boundedness* [62].

Definition 1.1.0.6 (UUB/GUUB). The solutions of (1.1) are said to be *uniformly ultimately bounded* if there exist positive numbers ϱ_m and c such that, for every $\varrho \in (0, \varrho_m)$, there exists a positive time $T(\varrho)$ such that, for all $x_0 \in \mathcal{B}_\varrho$ and all $t_0 \in \mathbb{R}_0^+$, they satisfy

$$\|\phi(t, t_0, x_0)\| \leq c, \quad \forall t \geq t_0 + T. \quad (1.6)$$

If the result holds for arbitrarily large ϱ , then the solutions satisfy the *global uniform ultimate boundedness* property. \diamond

Further definitions of stability and boundedness can be given by refining the notation and introducing new concepts. In the literature [43, 133], the concept of *stability of a set* is proposed, under the assumption of *forward completeness*, to simplify the analysis of the uncertain systems considered. However, when dealing with the synthesis of a control system, this extension results particularly useful only in adaptive control design, therefore we leave further details to the reader and simply state the following instrumental definition.

Definition 1.1.0.7 (UAS/UGAS of a set). Assume that (1.1) is forward complete on \mathcal{D} . The set \mathcal{P} is said to be *uniformly asymptotically stable* on \mathcal{D} for the system if it is both uniformly stable and uniformly attractive on \mathcal{D} . If \mathcal{D} is \mathbb{R}^n , then the set \mathcal{P} is *uniformly globally asymptotically stable*. \diamond

For brevity we choose not to state the definitions of uniform stability and attractivity of a set, for which the reader is referred to [17], main source of the definitions given in this section.

We should underline that the above definitions can be equivalently expressed in terms of \mathcal{K} and \mathcal{KL} functions (see the Appendix, [45, 119]), paving the way to the Lyapunov constructions instrumental in control design. We reserve the usage of these functions for later considerations about stability and stabilizability of uncertain nonlinear systems.

1.1.1 Semiglobal and practical stability

These results can be extended to specialize the analysis on the transient behavior of the system and to take into account different kinds of sets to which the trajectories may converge. As a matter of fact, non-vanishing disturbances, parametric uncertainties and, generally speaking, perturbations on the system result in steady-state errors, impeding the exact convergence to the desired operating condition. Therefore, it makes sense to consider a set of operating points to which the trajectories may converge. In particular, we should investigate to what extent it is possible to reduce the dimension of the set of convergence using the external action of a control input, irrespective of all the initial conditions in a prescribed set, which should be on the other hand arbitrarily enlargeable. To do this, it is necessary to move from the analysis point of view to that of systems with inputs, and to consider explicitly the uncertainties into the state-space representation. The notions of practical and semi-global stability introduced in the following are instrumental to this aim.

Let us define the parametrized nonlinear non-autonomous system

$$\dot{x} = f(x, t, \mu) \tag{1.7}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_0^+$, $\mu \in \mathbb{R}^m$ is a tuning parameter and $f : \mathbb{R}^n \times \mathbb{R}_0^+ \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz in x and satisfies Carathéodory conditions for all μ in the domain of interest.

Definition 1.1.1.1 (SUAS). Let $\mathcal{M} \subset \mathbb{R}^m$ be a set of tuning parameters. The system (1.7) is said to be *semi-globally uniformly asymptotically stable* on \mathcal{M} if, given any $R > 0$, there exists an allowable tuning parameter $\mu_s(R) \in \mathcal{M}$ such that the origin is uniformly asymptotically stable with \mathcal{B}_R as domain of attraction for the tuned system $\dot{x} = f(x, t, \mu_s)$. \diamond

Definition 1.1.1.2 (PUGAS). Let $\mathcal{M} \subset \mathbb{R}^m$ be a set of tuning parameters. The system (1.7) is said to be *practically uniformly globally asymptotically stable* on \mathcal{M} if, given any $\varrho > 0$, there exists an allowable tuning parameter $\mu_p(\varrho) \in \mathcal{M}$ such that the ball \mathcal{B}_ϱ is uniformly globally asymptotically stable for the tuned system $\dot{x} = f(x, t, \mu_p)$. \diamond

Definition 1.1.1.3 (SPUAS). Let $\mathcal{M} \subset \mathbb{R}^m$ be a set of tuning parameters. The system (1.7) is said to be *semi-globally practically uniformly asymptotically stable* on \mathcal{M} if, given any $R > \varrho > 0$, there exists an allowable tuning parameter $\mu_{sp}(\varrho, R) \in \mathcal{M}$ such that the ball \mathcal{B}_ϱ is uniformly asymptotically stable with \mathcal{B}_R as domain of attraction for the tuned system $\dot{x} = f(x, t, \mu_{sp})$. \diamond

These definitions do have a practical relevance, also in aerospace applications. They take into account the presence of design parameters μ representing gains (or other tunable variables) of a control law designed to ensure such properties to an uncertain nonlinear system, for instance a missile or a spacecraft. Moreover, the allowed design parameters belong to the set \mathcal{M} , which in practical situations is bounded due to actuators limitations. Thus, good performance in a robust nonlinear control system means achieving the smallest ϱ possible, guaranteeing the largest estimate of R through exploitation of the allowable parameter set \mathcal{M} . We stress that practical stability is more useful in applications than uniform ultimate boundedness, since the latter does not allow the possibility of reducing the dimension of the ultimate bound at will and does not guarantee stability. Moreover, the semi-global feature is just in principle less strong than the global

one, since in real applications the set of all the feasible initial conditions is usually a bounded, closed, set surrounding the operating point of interest. In addition, note that also the basin of attraction \mathcal{B}_R can be made large at will using control parameters.

Juxtaposition between “ideal” stability and semi-global/practical stability also finds reasons in the behavior of physical systems, usually perturbed by external disturbances and whose state-space representation suffers from modeling errors, neglected dynamics and other uncertainties. Non-vanishing perturbations usually yield practical stability results, while neglecting high-order nonlinearities result in relaxing into semi-global the properties. As a consequence, these two properties taken alone or combined together give a measure of the robustness of a nonlinear, non-autonomous, system. Finally, we should note that using the concepts of stability of sets, the origin is not required to be an equilibrium point for the system considered anymore, as expressed by Def. 1.1.1.2 and 1.1.1.3. This is an appealing feature, because several practically stable systems do not have an equilibrium at the origin [17].

1.1.2 Input-to-state stability

Another (quite new) relevant idea that fits into the framework of robust nonlinear stabilization is that of *input-to-state stability* (ISS), introduced by Sontag in [117]. The notion is more general than asymptotic stability, since it requires that the norm of the state trajectory be bounded by a function of the norm of the external input plus an asymptotically vanishing term in the initial state. However, because the ISS notion results to be in many cases too strong and restrictive, a weaker version has been introduced in [2], that of integral input-to-state stability (iISS), which is even more general than the classic ISS. Instead of establishing a connection between the state and the supremum of the input, iISS takes into account an estimate of the energy that the input brings into the system. Just like the ISS, iISS entails global asymptotic stability of the system with zero input and guarantees a certain degree of robustness against external disturbances. A big result in this direction is given by Sontag in [118]: if the energy furnished by the input is finite, the asymptotic state behavior is not affected.

Let us consider, this time, autonomous systems with input of the form

$$\dot{x} = f(x, u) \tag{1.8}$$

where $x \in \mathbb{R}^n$ is the state, u is the locally measurable and essentially bounded (or, more simply, piecewise continuous norm-limited) input signal and $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is locally Lipschitz. The definitions of ISS and iISS are given as follows.

Definition 1.1.2.1 (ISS). The nonlinear system (1.8) is said to be *input-to-state stable* if there exists a class \mathcal{KL} function β and a class \mathcal{K}_∞ function γ such that, for all $x_0 \in \mathbb{R}^n$ and any admissible input signal $u(t)$, the solution satisfies

$$\|\phi(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty), \quad \forall t \in \mathbb{R}_0^+. \quad (1.9)$$

where by $\|u\|_\infty$ we identify $\sup_{\tau \in [0, t]} (\|u(\tau)\|)$. The function γ is called *ISS gain* of (1.8). \diamond

Definition 1.1.2.2 (iISS). The nonlinear system (1.8) is said to be *integral input-to-state stable* if there exists a class \mathcal{KL} function β and class \mathcal{K}_∞ functions γ and μ such that, for all $x_0 \in \mathbb{R}^n$ and any admissible input signal $u(t)$, the solution satisfies

$$\|\phi(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma\left(\int_0^t \mu(\|u\|) d\tau\right), \quad \forall t \in \mathbb{R}_0^+. \quad (1.10)$$

The function μ is called *iISS gain* of (1.8). \diamond

It is evident that these concepts are suitable to describe the robustness of a GAS nominal system affected by disturbances or, more in general, by uncertainties. Namely, ISS and iISS guarantee the global asymptotic stability property of a ball whose amplitude depends on the amplitude (respectively, the energy) of the input signal. Nevertheless these properties don't require that, for a fixed input function $u(t)$, the dimension of such a ball can be arbitrarily decreased by a proper choice of a design parameter. Thus, ISS and iISS, still describing a very interesting feature of a nonlinear system, don't own the points of strength of practical and semi-global stability, "tactical" in view of control design. Instead, in view of solving the particular problem of disturbance attenuation (which is a sub-case of the robust stabilization problem in the nonlinear context), these two properties will result powerful since they embed a notion of gain of a nonlinear system.

1.2 Control Lyapunov Function

After having detailed the notions of stability in the nonlinear uncertain systems framework, we need to establish the sufficient conditions under which such properties are

guaranteed. The classic Lyapunov direct criterion [77] establishes that, given a nonlinear system, the existence of a smooth positive definite function with the property of having a non-positive derivative along the solutions implies that the origin of the system is stable. If such derivative happens to be negative, the origin is asymptotically stable. If, in addition, the function, which is called *Lyapunov function* of the system, is radially unbounded, then the origin is globally asymptotically stable. Furthermore, converse Lyapunov theorems ensure that the existence of a Lyapunov function is also a *necessary condition* for stability. The classic Lyapunov result for a nonlinear nonautonomous system [44] is recalled in what follows.

Theorem 1.2.0.3 (Lyapunov direct criterion). *The origin of (1.1) is uniformly globally asymptotically stable if and only if there exists a C^1 class function $V : \mathbb{R}^n \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ and class \mathcal{K}_∞ functions $\underline{\alpha}$, $\bar{\alpha}$ and α such that, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_0^+$*

$$\underline{\alpha}(\|x\|) \leq V(x, t) \leq \bar{\alpha}(\|x\|) \quad (1.11)$$

$$\dot{V} = \frac{\partial V}{\partial x}(x, t)f(x, t) + \frac{\partial V}{\partial t}(x, t) \leq -\alpha(\|x\|) \quad (1.12)$$

◇

We recall that the property of uniform asymptotic stability is desirable in terms of robustness against perturbations and disturbances. Another powerful tool for stability analysis is given by the following theorem due to LaSalle and Yoshizawa [65].

Theorem 1.2.0.4 (LaSalle-Yoshizawa). *Let the origin be an equilibrium point of (1.1) and suppose f is locally Lipschitz in x uniformly in t . Let $V : \mathbb{R}^n \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be a C^1 function such that, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}_0^+$*

$$\gamma_1(\|x\|) \leq V(x, t) \leq \gamma_2(\|x\|) \quad (1.13)$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(x, t)f(x, t) \leq -\gamma(x) \leq 0 \quad (1.14)$$

where γ_1 and γ_2 are class \mathcal{K}_∞ functions and γ is a continuous function. Then, all the solutions of (1.1) are globally uniformly bounded and satisfy

$$\lim_{t \rightarrow \infty} \gamma(x(t)) = 0. \quad (1.15)$$

Moreover, if $\gamma(x)$ is positive definite, then the equilibrium $x = 0$ is globally uniformly asymptotically stable. ◇

This result is very useful when there is the need to establish convergence to a set defined by $\gamma(x) = 0$, as in most nonlinear adaptive control designs and in their robust versions. We introduce the fundamental concept of invariant set for a given time-invariant system

$$\dot{x} = f(x) \quad (1.16)$$

which fulfills the forward completeness assumption.

Definition 1.2.0.5 (Invariance of a set). A set \mathcal{M} is called an *invariant set* of (1.16) if any solution $x(t)$ that belongs to \mathcal{M} at some time instant t_1 must belong to it for all future and past time:

$$x(t_1) \in \mathcal{M} \Rightarrow x(t) \in \mathcal{M}, \quad \forall t \in \mathbb{R}. \quad (1.17)$$

A set Ω is positively invariant if the above fact is only verified for all future time:

$$x(t_1) \in \Omega \Rightarrow x(t) \in \Omega, \quad \forall t \geq t_1. \quad (1.18)$$

◇

Convergence to a desired invariant set can be shown using the well-known results provided by *LaSalle's Invariance Principle* and its corollary.

Theorem 1.2.0.6 (LaSalle). Let Ω be a positively invariant set of (1.16) and $V : \Omega \rightarrow \mathbb{R}_0^+$ be a C^1 function $V(x)$ such that $\dot{V}(x) \leq 0$, $\forall x \in \Omega$. Moreover, let $E = \{x \in \Omega | \dot{V}(x) = 0\}$, and let \mathcal{M} the largest invariant set contained in E . Then, every bounded solution $x(t)$ starting in Ω converges asymptotically to \mathcal{M} . ◇

Corollary 1.2.0.7 (Asymptotic stability). Let $x = 0$ be the only equilibrium point of (1.16) and $V : \Omega \rightarrow \mathbb{R}_0^+$ be a C^1 , positive definite, radially unbounded function $V(x)$ such that $\dot{V}(x) \leq 0 \quad \forall x \in \mathbb{R}^n$. Let $E = \{x \in \mathbb{R}^n | \dot{V}(x) = 0\}$, and suppose that no solution other than the trivial $x(t) \equiv 0$ can stay forever in E . Then the origin is globally asymptotically stable. ◇

It is clear from these results that the convergence properties of a given nonlinear system are stronger if the dimension of \mathcal{M} is lower. In the ideal case of asymptotic stability, \mathcal{M} coincides with the origin of the considered system.

This work is about control *design*, therefore we need to introduce an extension of the Lyapunov function concept, the *control Lyapunov function* (CLF), which helps us establish the stabilizability of a given nonlinear system. We consider again nonlinear systems with input in the form of (1.8), satisfying $f(0, 0) = 0$. Our aim is to design a feedback control law $u = \alpha(x)$ in such a way that the equilibrium $x = 0$ of the closed-loop system

$$\dot{x} = f(x, \alpha(x)) \quad (1.19)$$

is GAS. Exploiting Theorem 1.2.0.3, we may seek for a couple $(V(x), \alpha(x))$ satisfying, in a domain of interest $\mathcal{D} \subset \mathbb{R}^n$, the differential inequality

$$\frac{\partial V}{\partial x}(x)f(x, \alpha(x)) \leq -\gamma(x) \quad (1.20)$$

with $\gamma(x)$ positive definite function. This is in general really hard to achieve. A stabilizing control law for (1.8) may exist but we may be unable to find proper $V(x)$ and $\gamma(x)$ to satisfy (1.20). If a suitable choice for $V(x)$ and $\gamma(x)$ exists, then the system admits a CLF, as detailed in the following definition.

Definition 1.2.0.8 (Control Lyapunov Function). A smooth, positive definite and radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ is a control Lyapunov function (CLF) for (1.8) if, for all $x \neq 0$,

$$\inf_{u \in U} \left\{ \frac{\partial V}{\partial x} f(x, u) \right\} < 0, \quad (1.21)$$

where U is a convex set of admissible values for the control variable u . ◇

As a result, a CLF is simply a candidate Lyapunov function whose derivative can be made negative pointwise by a proper choice of u . It is then clear that, since f is continuous by assumption, if there exists a continuous state feedback $\alpha(x)$ such that the origin of (1.19) is a GAS equilibrium point, then by converse Lyapunov theorems [36] there must exist a CLF for system (1.8). Moreover, if f is affine in the control variable, then the existence of a CLF for (1.8) is also a sufficient condition for stabilizability via continuous state feedback.

If the existence of a CLF implies the existence of an admissible stabilizing u for (1.8) and, vice-versa, the global asymptotic stabilizability of the origin of (1.19), by converse

theorems, implies the existence of a CLF, we can finally state that the existence of a CLF is equivalent to the stabilizability of $\dot{x} = f(x, u)$ [4].

The CLF concept was introduced by Artstein [4] and Sontag [115] as generalization of the Lyapunov design results by Jurdjevic and Quinn [57]. In the case of systems affine in the control

$$\dot{x} = f(x) + g(x)u \quad (1.22)$$

with $f(0) = 0$, the inequality (1.20) takes the form

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) \alpha(x) = L_f V(x) + L_g V(x) \alpha(x) \leq -\gamma(x) \quad (1.23)$$

Suppose we know a $V(x)$ CLF for (1.22), then one stabilizing control law $\alpha(x)$ of class C^∞ for all non-zero x is given by the *universal construction* known as Sontag's formula [116]

$$\alpha_S(x) = \begin{cases} - \left(c_0 + \frac{a(x) + \sqrt{a^2(x) + (b^T(x)b(x))^2}}{b^T(x)b(x)} \right) b(x) & \text{if } b(x) \neq 0 \\ 0 & \text{if } b(x) = 0 \end{cases} \quad (1.24)$$

where $a(x) = L_f V(x)$ and $b(x) = (L_g V(x))^T$. The control law (1.24) renders \dot{V} evaluated along the closed-loop trajectories negative definite. Namely, for $x \neq 0$, we obtain

$$\dot{V} = a(x) - p(x)b^T(x)b(x) = -\sqrt{a^2(x) + (b^T(x)b(x))^2} - c_0 b^T(x)b(x) < 0 \quad (1.25)$$

where

$$p(x) = \begin{cases} c_0 + \frac{a(x) + \sqrt{a^2(x) + (b^T(x)b(x))^2}}{b^T(x)b(x)} & \text{if } b(x) \neq 0 \\ c_0 & \text{if } b(x) = 0 \end{cases} \quad (1.26)$$

We stress that $c_0 > 0$ is not a necessary requirement for a negative definite \dot{V} since, away from $x = 0$, $a(x)$ and $b(x)$ never vanish together due to (1.23) being satisfied only if

$$\frac{\partial V}{\partial x} g(x) = 0 \Rightarrow \frac{\partial V}{\partial x} f(x) < 0, \quad (1.27)$$

This results in the following expression for $\gamma(x)$

$$\gamma(x) = \sqrt{a^2(x) + (b^T(x)b(x))^2} > 0 \quad (1.28)$$

for all non-zero x . Note that the stabilizing feedback $\alpha(x)$, defined on \mathbb{R}^n , is such that $\alpha(0) = 0$ and is smooth on the open (and dense) subset $\mathbb{R}^n \setminus \{0\}$ of \mathbb{R}^n . In addition, $\alpha(x)$ is continuous at $x = 0$, i.e. it has the property of being an *almost smooth* function, if and only if the corresponding CLF $V(x)$ satisfies the so-called *small control property*.

Definition 1.2.0.9 (Small control property). A control Lyapunov function $V(x)$ is said to satisfy the small control property if, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, if $x \neq 0$ satisfies $0 < \|x\| < \delta$, then there is some u with $\|u\| < \epsilon$ such that

$$\frac{\partial V}{\partial x} (f(x) + g(x)u) = L_f V(x) + L_g V(x)u < 0. \quad (1.29)$$

◇

The small control property represents a mild assumption on V . In fact, let us consider the open set

$$\Xi = \{x | b(x) \neq 0 \text{ or } a(x) < 0\} \quad (1.30)$$

complemented by $\Xi^c = \mathbb{R}^n \setminus \Xi$. Note that, inside Ξ , the control law (1.24) is a smooth function of x if a and b are smooth, because

$$\frac{a + \sqrt{a^2 + (b^T b)^2}}{b^T b} b \quad (1.31)$$

seen as a function of $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$ is analytic when $b \neq 0$ or $a < 0$. When V is a CLF, the set Ξ is the whole state space except for the origin, because of the strict inequality (1.23). As a consequence, Ξ^c is just the origin $x = 0$. If Ξ^c were to include points other than the origin, which happens when $\gamma(x)$ is only positive semi-definite, the continuity of the control law (1.24) would require the small control property to hold at every point of Ξ^c . This is a quite restrictive assumption, and also the reason for which the CLF concept is defined only with a strict inequality.

The idea of control Lyapunov function represents in some sense a breakthrough in nonlinear control design, but it still remains very difficult to find a CLF for a given nonlinear system. It is sometimes harder to focus in finding the CLF than to design directly a stabilizing controller. A way to construct *recursively* a control Lyapunov function and a stabilizing control law is provided by the *backstepping* procedure [28, 125]. This methodology represents a powerful tool for the state-feedback stabilization of nonlinear systems in *feedback form*, the following lower-triangular structure

$$\begin{aligned} \dot{z} &= f(z, \xi_1) \\ \dot{\xi}_1 &= a_1(\xi_1, \xi_2) \\ \dot{\xi}_2 &= a_2(\xi_1, \xi_2, \xi_3) \\ &\vdots \\ \dot{\xi}_n &= a_n(\xi_1, \xi_2, \dots, \xi_n, u). \end{aligned} \quad (1.32)$$

The control Lyapunov function obtained is non-quadratic in the state variables and the control law is intrinsically robust, since the approach is based on an energetic description of the system and avoids wasteful cancellations, unlike feedback linearization. The nonlinear backstepping methodology will be deeply analyzed, also in its robust extensions. As a matter of fact, this approach has a remarkable impact on space systems applications of nonlinear control, as will be highlighted in Chapter 3.

Another interesting tool for stabilization is that of *forwarding* [55, 89], which applies to systems in *feedforward form*, the following upper-triangular structure

$$\begin{aligned}
 \dot{\xi}_1 &= f_1(\xi_1) + \psi_1(\xi_1, \xi_2, \dots, \xi_n, z, u) \\
 \dot{\xi}_2 &= f_2(\xi_2) + \psi_2(\xi_2, \dots, \xi_n, z, u) \\
 &\dots \\
 \dot{\xi}_{n-1} &= f_{n-1}(\xi_{n-1}) + \psi_{n-1}(\xi_{n-1}, \xi_n, z, u) \\
 \dot{z} &= a(z) + b(\xi_n, z)\xi_n \\
 \dot{\xi}_n &= u
 \end{aligned} \tag{1.33}$$

where the subsystems $\dot{\xi}_i = f_i(\xi_i)$ are globally stable with a Lyapunov function $W(\xi)$, i.e. $L_f W(\xi) \leq 0$ for all ξ , the interconnection term $\psi(\xi, z)$ satisfy a not-higher-than-linear growth condition in ξ and $\dot{z} = a(z)$ is GAS and locally exponentially stable with a Lyapunov function $U(z)$. A CLF for the following simplified version of the cascade (1.33)

$$\begin{aligned}
 \dot{\xi} &= f(\xi) + \psi(\xi, z) \\
 \dot{z} &= a(z)
 \end{aligned} \tag{1.34}$$

is of the form

$$V(\xi, z) = W(\xi) + U(z) + \Psi(\xi, z) \tag{1.35}$$

where $W(\xi)$ for $\dot{\xi} = f(\xi)$ and $U(z)$ for $\dot{z} = a(z)$ are known and the cross term Ψ must be constructed to satisfy $\dot{\Psi} = -L_\psi W$, so that

$$\dot{V} = L_f W + L_a U \leq 0 \tag{1.36}$$

In this kind of constructive design, however, a severe critical point relies in the evaluation of the line integral

$$\Psi(\xi, z) = \int_0^\infty L_\psi W(\tilde{\xi}(t, \xi, z), \tilde{z}(t, z)) dt, \tag{1.37}$$

along the solutions of $\tilde{\xi}(t, \xi, z)$ and $\tilde{z}(t, z)$ of (1.34) starting from (ξ, z) at $t = 0$. In many cases, numerical integration is required. To obtain closed-form solutions, $f(\cdot)$ and $a(\cdot)$ should be linear in their arguments and the cross-term $\psi(\xi, z) = p(z)$ a polynomial

function of z only. Nonlinear forwarding design usually leads to very complex control laws, often without analytic solutions, which are easier to implement in many control applications. Thus, it is not really suitable for real-time implementation, also due to lack of robustness improvements, unlike backstepping.

Backstepping and forwarding are also useful tools in recursive feedback passivation of nonlinear systems, as extensively shown in [111]. As a matter of fact, while the first overcomes the relative degree zero or one requisite, the second bypasses the weak minimum phase condition. Both the designs fail in the presence of systems which do not exhibit neither the feedback nor the feedforward forms. Nevertheless, their combined application may lead to constructive designs for the extended class of *interlaced systems* [73, 112], which covers a broad variety of physical systems.

1.3 Robust nonlinear stabilization

To formulate the problem of robust nonlinear stabilization, it is mandatory to define properly the concept of nonlinear uncertain system, the notion of practical-robust uniform global asymptotic stability (P-RUGAS) and the core-idea of robust control Lyapunov function (RCLF), first introduced by Freeman and Kokotovic in [36].

1.3.1 Nonlinear uncertain systems

Consider three finite-dimensional Euclidean spaces: the state space \mathcal{X} , the control space \mathcal{U} and the uncertainty space \mathcal{W} . Given a continuous function $f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \times \mathbb{R}_0^+ \rightarrow \mathcal{X}$, the following system of differential equations depicts a very general representation of a nonlinear and non-autonomous uncertain system:

$$\dot{x} = f(x, u, w, t) \quad (1.38)$$

where $x \in \mathcal{X}$ is the state vector, $u \in \mathcal{U}$ is the control inputs vector, $w \in \mathcal{W}$ is the uncertainties vector, and $t \in \mathbb{R}_0^+$ is the independent (time) variable. Together with this state-space representation, we consider the constraints given by the set of admissible uncertainties, W , and that of admissible controls, U . Note that we could take into account also an uncertain output equation and a measurements vector assuming values in a certain set, but in this work we are interested in state-feedback problems, so we will assume that the output coincides with the state: $y = x$. For the output-feedback version of these results, the reader should refer to [36], which also provides part of the

contents of this section. Let us characterize with precision the spaces above defined in terms of their set-valued constraints.

We define *uncertainty* for (1.38) a function $w : \mathcal{X} \times \mathcal{U} \times \mathbb{R}_0^+ \rightarrow \mathcal{W}$ such that $w(\cdot, \cdot, t)$ is continuous for each fixed $t \in \mathbb{R}_0^+$ and $w(x, u, \cdot)$ is locally \mathcal{L}_∞ for each fixed $(x, u) \in \mathcal{X} \times \mathcal{U}$. Given an uncertainty constraint $W : \mathcal{X} \times \mathcal{U} \times \mathbb{R}_0^+ \rightsquigarrow \mathcal{W}$, we define an uncertainty $w(x, u, t)$ *admissible* when $w(x, u, t) \in W(x, u, t)$ for all $(x, u, t) \in \mathcal{X} \times \mathcal{U} \times \mathbb{R}_0^+$. This kind of representation covers a broad class of admissible static uncertainties, including exogenous and feedback *disturbances* and other plant and input perturbations. Moreover, this description is at the foundations of the *guaranteed stability* framework for robust nonlinear control [21, 41, 70].

A *control* for (1.38) is a function $u : \mathcal{X} \times \mathbb{R}_0^+ \rightarrow \mathcal{U}$ such that $u(\cdot, t)$ is continuous for each fixed $t \in \mathbb{R}_0^+$ and $u(x, \cdot)$ is locally \mathcal{L}_∞ for each fixed $x \in \mathcal{X}$. Given a control constraint $U : \mathcal{X} \times \mathbb{R}_0^+ \rightsquigarrow \mathcal{U}$, we define a control $u(x, t)$ *admissible* when $u(x, t) \in U(x, t)$ for all $(x, t) \in \mathcal{X} \times \mathbb{R}_0^+$ and it is jointly continuous in (x, t) .

The vector-field f , together with the two set-valued constraints U and W , constitute the uncertain nonlinear dynamical system $\Sigma = (f, U, W)$. With the expression “solution” to Σ we mean a solution $x(t)$ to the initial value problem:

$$\dot{x} = f(x, u(x, t), w(x, u(x, t), t), t), \quad x(t_0) = x_0 \quad (1.39)$$

given an uncertainty vector $w(x, u, t)$, a control $u(x, t)$ and an initial condition $(x_0, t_0) \in \mathcal{X} \times \mathbb{R}_0^+$. The regularity assumptions guarantee that the right-hand side of (1.39) is continuous in x and locally \mathcal{L}_∞ in t , and it follows from classical existence theorems that solutions of Σ always exist, at least *locally* in t , but need not be unique. Figure 1.1 shows the signal flow diagram of the system.

The system $\Sigma = (f, U, W)$ is time-invariant, or autonomous, when the mappings f , U and W do not explicitly depend on time t . In this case, with slight abuse of notation, we write $f(x, u, w)$, $U(x)$ and $W(x, u)$. In the same way, we say that a control is time-invariant when $u(x, t) = u(x)$. Although we have included only static uncertainties and controls in our formulation, we can accommodate fixed-order dynamics by redefining Σ according to what is done in [36].

1.3.2 Practical-robust stability and stabilizability

Recall that in Section 1.1.1 we have introduced the notion of PUGAS for a nonlinear non-autonomous parametrized system. We want to re-define and extend the notion to

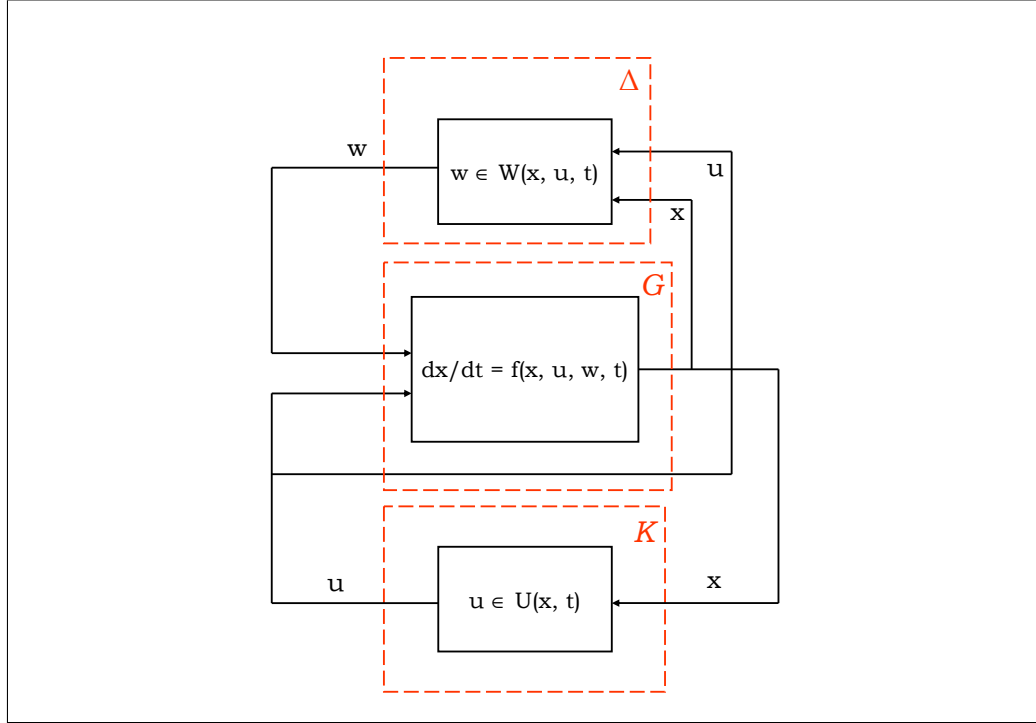


FIGURE 1.1: Nonlinear robust control paradigm.

the context of nonlinear uncertain systems with control input. With this in mind, we examine the behavior of the set of solutions of Σ or, which is the same, of (1.38), for all initial conditions and admissible uncertainties and try to determine which stabilization property can be achieved with a suitable choice of the robust nonlinear control law $u(x)$.

When every solution of (1.38) converges to a compact residual set $\Omega \subset \mathcal{X}$ containing a desired operating point (for convenience, the origin $0 \in \mathcal{X}$), we will say that these solutions (or, equivalently, Ω) are robustly (uniformly globally asymptotically) stable according to the following definition.

Definition 1.3.2.1 (RUGAS- Ω). Given a control law and a compact set $\Omega \subset \mathcal{X}$, containing the origin, the solutions of the nonlinear uncertain system Σ are *robustly uniformly globally asymptotically stable* with respect to Ω (**RUGAS- Ω**) when there exists a class \mathcal{KL} function β such that for all admissible uncertainties and initial conditions $(x_0, t_0) \in \mathcal{X} \times \mathbb{R}_0^+$, all solutions $x(t)$ exist for all $t \geq t_0$ and satisfy

$$|x(t)|_{\Omega} \leq \beta(|x_0|_{\Omega}, t - t_0) \quad (1.40)$$

for all $t \geq t_0$. Moreover, the solutions of Σ are *RUGAS* when they are *RGUAS*- $\{0\}$. \diamond

Note that the RUGAS- Ω implies that the residual set Ω is (robustly) positively invariant. In particular, RUGAS (the particular case with $\Omega = \{0\}$) implies that the origin is a equilibrium point (in forward time) for every admissible uncertainty and initial condition, which is clearly not necessary using the relaxed RUGAS- Ω property. The robust stabilization problem for an uncertain nonlinear system Σ which emerges from Definition (1.3.2.1) is to construct an admissible control law such that the solutions of (1.38) are RUGAS- Ω for some residual set Ω . In particular, we seek for a *practical* version of such property, since we would like to reduce the dimension of Ω at will exploiting control design parameters. With this in mind, and without loss of generality, let us consider the extended system $\Sigma_\mu = (f, U, W, \mathcal{M})$, containing the set $\mathcal{M} \subset \mathbb{R}^m$ of admissible tuning control parameters. The corresponding state-space representation is then

$$\dot{x} = f(x, u(\mu), w, t) \quad (1.41)$$

with $\mu \in \mathcal{M}$ vector of admissible design parameters. It is therefore possible to extend Definition (1.3.2.1) in the following fashion.

Definition 1.3.2.2 (P-RUGAS). Let $\mathcal{M} \subset \mathbb{R}^m$ be a set of tuning parameters. The uncertain system (1.41) is said to be *practically-robustly uniformly globally asymptotically stable* on \mathcal{M} if, given any $\varrho(\Omega) > 0$, with $\mathcal{B}_{\varrho(\Omega)}$ biggest ball contained in the desired convergence set Ω , there exists an allowable vector of tuning parameters $\mu_p(\varrho(\Omega)) \in \mathcal{M}$ such that the solutions of $\dot{x} = f(x, u(\mu_p(\varrho(\Omega))), w, t)$ are robustly uniformly globally asymptotically stable with respect to $\mathcal{B}_{\varrho(\Omega)}$ (RUGAS- $\mathcal{B}_{\varrho(\Omega)}$). \diamond

According to how small we can make the set of convergence of the solutions by a suitable choice of the control law tuning parameters, we define three types of robust stabilizability as follows.

Definition 1.3.2.3 (Robust stabilizability). The system Σ is **robustly uniformly globally asymptotically stabilizable** when there exists an admissible control and a compact set $\Omega \subset \mathcal{X}$, containing the origin, such that the solutions to Σ are RUGAS- Ω . This means that the distance between the trajectories and the origin of the state space depends strictly on the dimension of Ω , and cannot be made arbitrarily small. When Ω collapse to the origin, the system stabilizability property is the simple RUGAS. The system Σ_μ is **practically-robustly uniformly globally asymptotically stabilizable** when for every $\varrho(\Omega) > 0$ there exists an admissible control and a compact set $\Omega \subset \mathcal{X}$, satisfying $0 \in \mathcal{B}_{\varrho(\Omega)} \subset \Omega$, such that the solutions

to Σ_μ are RUGAS- $\mathcal{B}_{\varrho(\Omega)}$. This means that the distance between the trajectories and the origin of the state space can be made arbitrarily small by an opportune choice of the control design parameters μ . \diamond

Clearly RUGAS with respect to the origin implies practical robust (uniform global asymptotic) stabilizability, which in turn implies robust (uniform global asymptotic) stabilizability with respect to a fixed (irreducible in dimension) set Ω .

1.3.3 Robust Control Lyapunov Function

The existence of a Lyapunov function is the most significant necessary and sufficient condition for the stability of a nonlinear system. The sufficiency was proved by Lyapunov [76], and the necessity was established half a century later with the advent of the so-called converse theorems [67]. Since Lyapunov theory deals with dynamical systems without inputs, it has traditionally been applied only to closed-loop control systems, that is, systems for which the input has been replaced by a state-feedback control law. Starting from the early Sixties, some authors began exploiting Lyapunov functions for feedback design by “making the Lyapunov derivative negative” with proper choice of the control law [58, 82]. Such ideas have been progressively activated with the introduction of the concept of control Lyapunov function, of which we have generously discussed in Section 1.2. This idea can be extended to the class of uncertain nonlinear systems, introduced in the previous sections, by defining the *robust control Lyapunov function*, whose existence, as we are going to show, is a necessary and sufficient condition for robust nonlinear stabilizability. Most of the contents of this section are rearranged from [36].

Recall from Section 1.3.1 that an uncertain nonlinear system Σ consists of the triple (f, U, W) with continuous generating function f and sets of admissible controls U and disturbances W . Denote by $\mathcal{V}(\mathcal{X})$ the set of all C^1 functions $V : \mathcal{X} \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ satisfying, for all $(x, t) \in \mathcal{X} \times \mathbb{R}_0^+$, the inequality

$$\underline{\alpha}(\|x\|) \leq V(x, t) \leq \bar{\alpha}(\|x\|) \quad (1.42)$$

with $\underline{\alpha}$ and $\bar{\alpha}$ \mathcal{K}_∞ functions. The set $\mathcal{V}(\mathcal{X})$ is the set of all candidate Lyapunov functions for testing the robust stability of a nonlinear uncertain system Σ . This set $\mathcal{V}(\mathcal{X})$ is contained in the broader set $\mathcal{P}(\mathcal{X})$ of functions which need not to be differentiable or radially unbounded, that is, the set of all continuous functions $\alpha : \mathcal{X} \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such

that there exist $\chi_1, \chi_2 \in \mathcal{K}$ satisfying $\chi_1(\|x\|) \leq \alpha(x, t) \leq \chi_2(\|x\|)$ for all $(x, t) \in \mathcal{X} \times \mathbb{R}_0^+$. Given $V \in \mathcal{V}(\mathcal{X})$ and an uncertain nonlinear system Σ , namely $\dot{x} = f(x, u, w, t)$, we define the Lyapunov derivative along the motion $L_f V: \mathcal{X} \times \mathcal{U} \times \mathcal{W} \times \mathbb{R}_0^+ \rightarrow \mathbb{R}$ by the equation

$$L_f V(x, u, w, t) := \frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t) f(x, w, u, t) \quad (1.43)$$

Clearly $L_f V$ is continuous, and it follows from (1.42) that $L_f V(0, \cdot, \cdot, \cdot) \equiv 0$. Now we are ready to give the definition of robust control Lyapunov function.

Definition 1.3.3.1 (Robust control Lyapunov function). A function $V \in \mathcal{V}(\mathcal{X})$ is called a *robust control Lyapunov function* (RCLF) for an uncertain nonlinear system Σ if there exists $c_v \in \mathbb{R}_0^+$ and $\alpha_v \in \mathcal{P}(\mathcal{X})$ such that

$$\inf_{u \in U(x, t)} \sup_{w \in W(x, u, t)} (L_f V(x, u, w, t) + \alpha_v(x, t)) < 0 \quad (1.44)$$

for all $t \in \mathbb{R}_0^+$ and all $c > c_v$. ◇

The RCLF extends the CLF in different directions. First, the definition takes into account not only the control variable, but also the uncertainties, disturbances or generic perturbations. Second, this notion can be easily extended to measurement feedback as well. Last, but not least, the parameter c_v potentially encompasses all three types of stabilizability introduced in section 1.3.2, not just asymptotic stabilizability.

This definition particularizes to the special case of autonomous (or time-invariant) systems in the following fashion.

Definition 1.3.3.2 (Time-invariant RCLF). A time invariant function $V \in \mathcal{V}(\mathcal{X})$ is called a *robust control Lyapunov function* for an autonomous uncertain nonlinear system Σ if and only if there exist $c_v \in \mathbb{R}_0^+$ and a time invariant function $\alpha_v \in \mathcal{P}(\mathcal{X})$ such that

$$\inf_{u \in U(x)} \sup_{w \in W(x, u)} (L_f V(x, u, w) + \alpha_v(x)) < 0 \quad (1.45)$$

whenever $V(x) > c_v$. ◇

The inequality (1.45) can be interpreted as follows: for every fixed x there exists an admissible value u for the control such that the Lyapunov derivative is negative for any admissible value w of the uncertainty. This is a natural generalization of the control Lyapunov function concept for systems with disturbance inputs.

Example 1.3.3.3 (feedback linearizable systems). Consider a feedback linearizable system $(g_1(\xi))$ nonsingular for all ξ) perturbed by a disturbance input w

$$\dot{\xi} = H\xi + L(g_0(\xi, w) + g_1(\xi)u) \quad (1.46)$$

Suppose that there is no control constraint, i.e. $U(x) \equiv \mathcal{U}$, and suppose that the disturbance constraint is such that $W(x, \mathcal{U})$ is bounded for every $x \in \mathcal{X}$. Let P be the symmetric, positive definite, solution to the matrix Riccati equation

$$H^T P + PH - PLL^T P + I = 0 \quad (1.47)$$

The function $V(x) = \Phi^T(x)P\Phi(x) = \xi^T P\xi$ with the selection $\alpha_v(x) = \frac{1}{2}\xi^T \xi$ is such that

$$\begin{aligned} & \inf_{u \in \mathcal{U}} \sup_{w \in W(x, u)} (L_f V(x, u, w) + \alpha_v(x)) \\ &= \inf_{u \in \mathcal{U}} \sup_{w \in W(x, u)} \left[\xi^T (H^T P + PH)\xi + 2\xi^T PL(g_0(\xi, w) + g_1(\xi)u) + \frac{1}{2}\xi^T \xi \right] \\ &= \inf_{u \in \mathcal{U}} \sup_{w \in W(x, u)} \left[-\frac{1}{2}\xi^T \xi + \xi^T PL(L^T P\xi + 2g_0(\xi, w) + 2g_1(\xi)u) \right] \\ &= \begin{cases} -\frac{1}{2}\xi^T \xi & \text{when } \xi^T PL = 0 \\ -\infty & \text{when } \xi^T PL \neq 0 \end{cases} \end{aligned} \quad (1.48)$$

Thus we deduce from Definition 1.45 that V is RCLF for system (1.46) with $c_v = 0$. With this example, we would like to stress that V is also a CLF for the unperturbed version of system (1.46), the one with $w = 0$. However, note that not every CLF become a RCLF when uncertainties perturb a “nominal” dynamics. Generally speaking, this happens only when the source of uncertainty is *matched* with the control input, i.e. when the disturbance w enters the system through the same channel as the control u . Another way to say this is that the disturbance is *in the span* of the control action. We will return later on this point, crucial for robust stabilizability. \diamond

1.3.3.1 RCLF in absence of disturbance input

The RCLF is a concept more general than that of CLF, since it takes into account uncertainties, measurement feedback and non-constant control constraints. We are interested in determining whether or not the RCLF and the CLF coincide. We consider

an autonomous system Σ with perfect state measurement, constant control constraint ($U(x) \equiv U$), and no uncertainties or disturbance inputs. Recall that a CLF $V \in \mathcal{V}(\mathcal{X})$ for $\dot{x} = f(x, u)$ satisfies

$$x \neq 0 \Rightarrow \inf_{u \in U} L_f V(x, u) < 0 \quad (1.49)$$

Clearly, every RCLF for Σ is also a CLF: we want to find out whether or not every CLF is also a RCLF in this restricted case.

Suppose V is a CLF with the additional property that the left-hand side of the inequality is bounded away from zero for large x , i.e. for all $c > 0$ there exists $\delta > 0$ such that

$$V(x) \geq c \Rightarrow \inf_{u \in U} L_f V(x, u) \leq -\delta \quad (1.50)$$

Every CLF with this additional property is a RCLF according to the definition. As a matter of fact, if (1.50) is true, then we can find a time-invariant function $\alpha_v \in \mathcal{P}(\mathcal{X})$ such that adding $\alpha_v(x)$ to the left-hand side of (1.49) preserves the inequality. On the other hand, if a given CLF does not satisfy such extra property, it is not a RCLF because a suitable function α_v does not exist. However, given a CLF not satisfying (1.50), it is always possible to construct another CLF V satisfying it, and being therefore a RCLF, see [36] for more details.

1.3.3.2 RCLF implies robust stabilizability

In [36] several assumptions under which the existence of a robust control Lyapunov function for a system Σ *implies* its *robust stabilizability*, according to Definition 1.3.2.3, are investigated. There are constraints to be imposed on control and uncertainty spaces U and W , in terms of lower/upper semi-continuity and, in general, regularity. Another set of conditions to be fulfilled concerns the structure of the vector field f , which determines the dynamics of the system Σ , and the structure of W . The first must be a map affine in the control input, while the second should be independent of u , i.e. $W(x, u, t) = W(x, t)$. This second set of assumptions could be relaxed when the map f is jointly affine in u and w : the condition on W changes in that it can be dependent on u , in particular it must be convex in u . Thus we can state the following sufficiency theorem, whose proof is detailed in [36].

Theorem 1.3.3.4 (RCLF \Rightarrow robust stabilizability). *Suppose a nonlinear uncertain system Σ satisfies some regularity assumptions on U and W , detailed in [36], plus the structural conditions on f and W . If there exists a RCLF for Σ , then Σ is robustly stabilizable. If, furthermore, $c_v = 0$, then Σ is practically-robustly stabilizable. Moreover, if Σ and V are*

time-invariant, then the robustly stabilizing admissible control can always be chosen to be time-invariant. \diamond

1.3.3.3 Small control property

Finally, defining a suitable *small control property* for nonlinear uncertain systems like Σ , it is possible to relate the existence of a robust control Lyapunov function to the robust asymptotic stabilizability of the system considered. For more details about the extension of such a property to the framework of nonlinear uncertain systems, the reader is again referred to [36]. Substantially, the existence of a robust control Lyapunov function V which satisfies the small control property, together with all the other conditions briefly presented above, implies *robust asymptotic stabilizability* of Σ . Also, if Σ and V are time-invariant, then the robustly stabilizing admissible control can always be chosen to be time-invariant.

1.3.3.4 Robust stabilizability implies RCLF

The necessity theorem about robust control Lyapunov functions applies to systems of the form

$$\dot{x} = F(x, d) \quad (1.51)$$

where F is locally Lipschitz and $d(t)$ is an \mathcal{L}_∞ exogenous input assuming values in a nonempty compact convex subset D of a finite-dimensional Euclidean space. First, we state the following converse theorem.

Theorem 1.3.3.5. *Suppose there exists a class \mathcal{KL} function β such that, for every initial condition $(x_0, t_0) \in \mathcal{X} \times \mathbb{R}$ and every \mathcal{L}_∞ exogenous input $d(t)$ taking values in D , the solution $x(t)$ of (1.51) exists for all $t \geq t_0$ and satisfies $\|x(t)\| \leq \beta(\|x_0\|, t - t_0)$ for all $t \geq t_0$. Then, there exist time-invariant functions $V \in \mathcal{V}(\mathcal{X})$ and $\alpha_v \in \mathcal{P}(\mathcal{X})$ such that*

$$\sup_{d \in D} (L_F V(x, d) + \alpha_v(x)) \leq 0 \quad (1.52)$$

for all $x \in \mathcal{X}$. \diamond

This converse theorem is useful in showing that, at least in the locally Lipschitz case, the existence of a RCLF is necessary for robust stabilizability. At the same time, we present the result for robust *asymptotic* stabilizability, showing the necessity of the small control property. Again, for the proof the reader is referred to [36].

Theorem 1.3.3.6 (Robust stabilizability \Rightarrow RCLF). *Let Σ be a time-invariant uncertain nonlinear system with generating function f locally Lipschitz. Let the disturbance admissible space W be locally Lipschitz too, with nonempty compact convex values. If Σ is robustly asymptotically stabilizable via locally Lipschitz time-invariant state-feedback, then there exists a time-invariant RCLF for Σ satisfying the small control property.* \diamond

Chapter 2

Structure of an uncertain nonlinear system

Robust nonlinear control, as already stated, takes into account stabilization problems of systems whose dynamics is not completely known. A really important role in this framework is played by the representation used for the uncertain parts of the dynamics, more briefly the *uncertainty*. In this section, we define with more details what does the term “uncertainty” means in the present setting and how uncertainties can be classified. The most important assumption needed in this context is the possibility to define a deterministic model of the uncertainty, exploiting it in the subsequent robust nonlinear design. In particular, only the knowledge of the bounds of these models will be required to obtain the desired control law. Moreover, we refine the representation of nonlinear uncertain system and illustrate some of its structural properties. We specialize the class of systems considered into that of input-affine nonlinear uncertain systems and we give examples of unstabilizable uncertain systems, highlighting the most common sources of instability which uncertainty can bring. Next, we introduce the fundamental notion of matching conditions together with the corresponding relaxed version, that of generalized matching conditions. Both these conditions set some constraints on system structure which are strictly related to robust stabilizability. In the final part of the chapter, we propose a deterministic model of the uncertain terms exploiting the concepts of Δ -operator and deviation function. As a result, we establish some rules to evaluate the complex uncertain terms and a novel systematic method to propagate them into system dynamics.

2.1 Uncertainty representation

Uncertainty represents a part of a physical system that cannot be easily modeled or precisely known. This means that the model of the system is only partially known. In the sequel, we will assume that system dynamics can be divided in a *nominal*, perfectly known, part and in an *uncertain*, only partially known, part. Uncertainty is not completely unknown, because the designer can count on the information about its bounds, assumed to be known. Note that the use of a partially known model to describe a physical system not only reflects the reality, but also makes it possible for us to study the way of designing controls that compensate for the unknowns and achieve better performance in modern sophisticated systems, like those in the aerospace domain. Recall that the approach described in this work is totally deterministic, and so are the models and the uncertainties considered.

Typical uncertainties in deterministic models of physical systems include, but are not limited to, constant or time-varying parameters of the system, unmodeled dynamics, modeling errors, unknown inputs such as disturbances, measurement noise, environmental perturbations of various nature, sampling, quantization noise, etc. In this work we are going to describe uncertainties as a “lumped”, thus extremely synthetic and practical, *description of all the phenomena that are not depicted in the nominal system model*. Below, we follow in part the classification made by Z. Qu in [106].

There are different ways of classifying the uncertainties, for instance according to:

- functional dependence and structure;
- relationship to the control input;
- size of the bounding functions;
- continuity and other regularity assumptions;
- location in system equations.

A first notable distinction to make is between *dynamic* (with memory) uncertainties and *static* (memoryless) uncertainties.

1. **Dynamic uncertainties**, also called *unmodeled dynamics*, are quite difficult to consider in control design, so they are usually neglected due to their high-frequency response. Sometimes, their effect on system dynamics is studied using approximations or asymptotic methods like singular perturbations, center manifold theory or more practical time-scales separation arguments. If neglected, these uncertain

terms may lead to instability of the closed-loop system. They are due to a lack of dynamic description of the model used for control design, often highly simplified to make the design phase easier. A remarkable example of unmodeled dynamics is the high-frequency actuator dynamics of an air-to-air missile, one of the case-studies analyzed in the following. Also the equations of the flexible modes in a spacecraft with appendages can be considered unmodeled dynamics.

2. **Static uncertainties** can be taken into account more easily in control design, nonetheless they still represent a danger for stability. Usually, they are parametric uncertainties, modeling errors or disturbances. Typical examples of static uncertainties, that may be time-varying, state, and control dependent, are the aerodynamic forces and moments in the missile equations of motion. Another notable example is the parametric uncertainty in the inertia moments of a flexible spacecraft, as in one of the application studies made in this work.

Another very important distinction emerges between *unstructured* and *structured* uncertainties.

1. **Unstructured uncertainties** are those that are bounded by a function of the state and time in some norm and can assume any arbitrary value or function within the size of the given bounding function. There is no additional information upon the uncertainty structure, only that of its bounding function.
2. **Structured uncertainties** are those for which some information in addition to their bounding functions is available. This information is related to the functional structure of the uncertainty itself.

Consider a deterministic time-domain uncertainty $\Delta(x, t)$, for instance a scalar function, such that:

$$|\Delta(x, t)| \leq \|x\| \quad (2.1)$$

where $x \in \mathbb{R}^n$. Without any information about $\Delta(x, t)$ other than its bounding function $\|x\|$, the uncertainty is said to be unstructured. This particular uncertainty $\Delta(x, t)$ is also called *sector-bounded* as its bounding function is linear with respect to the norm of the state. Uncertainty $\Delta(x, t)$ is said to be structured if its functional dependence on the state and time can be described to some extent. The most common type of structured uncertainties is the class of so-called *parametrizable uncertainties*, that is, uncertainties that can be characterized through a certain, more or less defined, functional expression. If the function is known except for a finite number of unknown, possibly time-varying, parameters, the uncertainty is called *linearly parametrizable*. An

example of linearly parametrizable uncertainty is $\Delta(x, t) = a^T(t)x$, where $a(t)$, with $\|a(t)\| \leq 1$, is a vector of unknown time-varying or constant parameters. If otherwise, for instance, $\Delta(x, t) = b^T(x, t)x$ where $\|b(x, t)\| \leq 1$, the uncertainty is said to be *non-linearly parametrizable*. It should be noted that several levels of information may be obtained for uncertainties, particularly for structured ones. As an example, consider again the structured uncertainty $\Delta(x, t) = b^T(x, t)x$ with $\|b(x, t)\| \leq 1$. Once the non-linear parametrization is given, we can continue to specify if the uncertain vector $b(x, t)$ is structured. For vector uncertainty $b(x, t)$ to be structured, the expressions of its uncertain elements are important. If

$$b(x, t) = c(x, t) \frac{x}{1 + \|x\|^2} \quad (2.2)$$

with $\|c(x, t)\| \leq 2$, the same question of being structured or not can be raised again for the vector $c(x, t)$. This process of exploring uncertainty information stops when all physical understandings of the system are exhausted. At the end the uncertainties should be either unstructured or linearly parametrized.

Given a bounding function, the unstructured uncertainties entailed by it constitute a set of infinite elements. Obviously, structured uncertainties are special, deterministic, realizations of unstructured uncertainties bounded by the same bounding function. Hence, treating structured uncertainty as unstructured inevitably enlarges the set of admissible elements and in turn makes analysis and control design *conservative*. It will be shown in some examples that the location of the uncertainties in system dynamics and their nature may determine whether the uncertain system is stabilizable, and that exploiting such structural information is a crucial point in control synthesis. Nevertheless, available analysis techniques and control design procedures are not capable of taking into account every piece of information known about uncertainties. As a result, uncertainties being unstructured at a certain level are often much easier to be handled in analysis and control design than structured uncertainties. It is in these cases that a conservative result may have to be concluded. There is clearly a compromise to deal with concerning the complexity of the design and its conservatism.

Once defined in an accurate way the concept of uncertainty, we are now ready to present the general problem of robust nonlinear control.

Definition 2.1.0.7 (Robust nonlinear control problem). The problem of controlling nonlinear uncertain systems, also referred to as the **robust nonlinear control problem**, is to design a fixed (i.e. uncertainty independent) controller which guarantees design requirements in

the presence of significant uncertainties bounded in size by either some constant or some well-defined functions of the state and time. If it exists, such a stabilizing control is called **robust nonlinear control**, and must be suited to achieve **stability** and a desired level of **performance** for the closed-loop system. \diamond

The adjective “robust” is used to reflect the fact that the specific property holds for all possible uncertainties within their bounds (or bounding functions). The other two key phrases in the above definition are “significant” and “bounded in size”. Insignificant uncertainties need not be considered since all stable feedback control systems possess a certain level of robustness against small perturbations. Moreover, the boundedness of uncertainties is a compulsory requirement for the success of nonlinear robust control synthesis. Note that, however, robust nonlinear control laws can be designed merely knowing the *existence* of some bounds or bounding functions of the uncertainties, without requiring the knowledge of the bounds amplitude itself, but with compulsory limitations on the performance of the control system. This happens to be the case, for instance, of the famous redesign technique called *nonlinear damping* presented in [62]. Therefore, the superiority of robust control and its difference from other design methodologies are due to the fact that uncertainties are only required to be bounded. In fact, in comparison, classical adaptive control and learning control can be used to handle only those systems with linearly parametrized uncertainties. For systems with both unstructured and parametrizable uncertainties, a combination of robust and adaptive control can have success.

Basically, the design phase of robust nonlinear control consists of three steps:

- Study the nominal system (in which system uncertainties are assumed to be zero) and design a *stabilizing nominal control*.
- Find a Lyapunov function for the uncertain system.
- Develop a robust controller through a Lyapunov argument in which *enlargement* and *size domination* will be employed.

Robust control design is usually based on a nominal control design for the nominal system, since the presence of uncertainties complicates the synthesis and because robust control reduces to the nominal as the size of uncertainties becomes zero. There are two additional reasons why a nominal design is required. First, stability analysis of uncertain systems is always done with respect to the equilibrium point of the nominal

system. Second, the Lyapunov function for the uncertain system, at least in solving the easiest problems, can be chosen to be that of the nominal system. Nominal control design and robust control design may be combined together to achieve stabilization. In some cases, it is easier to neglect some known dynamics to develop a simpler nominal control and treat them as uncertainties to handle in the following robust control synthesis. The enlargement process consists in exploiting the knowledge of the bounds to develop the Lyapunov derivative inequality, enlarging the uncertainties to their known maximum possible sizes. This implies that a robust nonlinear control design is, in essence, a *worst-case* design. Size domination, finally, means that the control law developed makes the Lyapunov derivative satisfy the dissipation inequality by *dominating in size* all the uncertain terms in it. For further conceptual considerations about the philosophy of nonlinear robust control, the reader could refer to [106].

2.2 Input-affine nonlinear uncertain systems

The generic state-space representation of a nonautonomous uncertain nonlinear system was already given as (1.38). However, it is very difficult to develop a robust nonlinear controller for a system with such a highly-nonlinear structure. Therefore, we are going to take into account for the rest of the work the input-affine version of (1.38), because the control techniques that we will present need this structure to be properly applied. More in detail, we consider systems of the form

$$\dot{x} = a(x, w, t) + B(x, w, t)u \quad (2.3a)$$

$$:= f(x, t) + \Delta \bar{f}(x, w, t) + G(x, t)u + \Delta \bar{G}(x, w, t)u \quad (2.3b)$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} \subset \mathbb{R}^p$ is the control inputs vector, $t \in \mathbb{R}_0^+$ is the time, and $w \in \mathcal{W} \subset \mathbb{R}^s$ is the vector of time-varying uncertain variables. In (2.3a), w is simply a lumped description of all the uncertainty of the system. The second representation (2.3b) is a structured version of (2.3a). As a matter of fact, f and G are the *known*, nominal, parts of a and B respectively, while $\Delta \bar{f}$ and $\Delta \bar{G}$ represent their uncertain components. Thus, the representation (2.3b) emphasizes a structural separation between the nominal dynamics and the uncertain one.

To guarantee existence and uniqueness of the solutions of (2.3), some *regularity assumptions* on system equations are needed. A naïf assumption could be restricting the vector field $a + Bu$ to be *sufficiently smooth*. However, this could be too restrictive and could cause a lack in the representation power of (2.3). As a consequence, we employ the following two assumptions [106], which are indeed less restrictive.

Assumption 2.2.0.8. In the nonlinear uncertain system (2.3), the vector field $a + Bu$ is *Carathéodory*, and the uncertain vector w is *Lebesgue measurable*. \diamond

Assumption 2.2.0.9. In the nonlinear uncertain system (2.3), the vector field $a + Bu$ is locally uniformly bounded. \diamond

If we make the extra assumption of a state-dependent, time-varying, vector of uncertainties $w(x, t)$, we can further simplify (2.3a) and (2.3b) into:

$$\dot{x} = f(x, t) + \Delta f(x, t) + G(x, t)u + \Delta G(x, t)u \quad (2.4)$$

which is the representation used throughout the rest of the work to construct a robust nonlinear controller.

2.2.1 Examples of unstabilizable uncertain systems

Although it would be ideal if robust control could be designed to stabilize all uncertain systems in the fully nonlinear form (1.38), the following examples show that this is too hard to even conceive, since not all the uncertain systems are stabilizable [106].

Example 2.2.1.1 (Structural change). Consider the second-order system:

$$\begin{aligned} \dot{x}_1 &= x_2 + \Delta(x_1, x_2) \\ \dot{x}_2 &= u \end{aligned} \quad (2.5)$$

in which the uncertainty $\Delta(\cdot, \cdot)$ is bounded as follows

$$|\Delta(x_1, x_2)| \leq 2 + x_1^2 + x_2^2 \quad (2.6)$$

One can easily see that the system is not stabilizable since one possibility of additive uncertainty within the given bounding function is $-x_2 + x_1$. Physically, the system becomes unstabilizable since uncertainty can change its structure in a critical manner. Note that, as a matter of fact, part of system dynamics becomes unstable and, also, decoupled from the rest of the system and from the control input. As will be shown later, such an uncertainty causing the loss of controllability belongs to the class of so-called *unmatched uncertainties*. \diamond

Example 2.2.1.2 (Unknown control direction). Consider the scalar system

$$\dot{x} = x + (1 + \Delta(x))u \quad (2.7)$$

where uncertainty is bounded as $|\Delta(x)| \leq C$ for some $C \geq 1$. The system is not stabilizable since $\Delta(x)$ could be -1 , and so the system is not subject to any control and is simply unstable. The uncertain term may also be such that the whole sum $1 + \Delta(x)$ is uncertain because of $C > 1$, and therefore any control introduced may have an adverse effect on stability, since it may cause the state to grow out of bound more quickly. In these cases, keeping the control input at zero is the best choice, and the system would be unstabilizable if any control is needed. \diamond

Example 2.2.1.3 (Unidirectional control). Consider the scalar system

$$\dot{x} = \Delta(x)x + u^2 \quad (2.8)$$

where uncertainty is bounded as $0 < \Delta(x) \leq 1$. The system is not stabilizable since, no matter what choice is made for u , the **control action** in \dot{x} is always *unidirectional* (positive). In fact, any scalar uncertain system is not stabilizable if the designer cannot make \dot{x} be both positive and negative upon his choice through selecting in appropriate way u (specifically, through choosing robust control to dominate all possible uncertainties). \diamond

Although the dynamics of the above examples are quite simple, they furnish intuitive explanations of what may lead uncertain systems to be unstabilizable. In particular, we may distinguish two categories of issues:

1. **Loss of controllability**, due to either the combination of a broken input-output chain and unstable dynamics, as in example 2.2.1.1, or the presence of both feedback and feedforward paths.
2. **Unknown** or **unidirectional** control contribution to the differential equations, as in examples 2.2.1.2 and 2.2.1.3.

The ultimate objective of robust control theory and design of nonlinear uncertain systems is twofold.

1. First, if necessary, determine the least requirements, called *structural conditions*, on the system (2.3) such that it can be stabilized or, in general, controlled. This is done in the two sections below.
2. Second, find procedures under which robust control u can be *systematically designed*. The key issue in the design is the search for robust control Lyapunov functions and their associated robust controllers, which may be different for achieving different types and levels of performances. This is done in the final sections of this part of the work.

2.2.2 Matching conditions

To find a suitable robust control for (2.4), we need to analyze further the structure of the representation in terms of the uncertainties. The main structural property characterizing the uncertain terms in (2.4) is the *matching condition*, under which the dynamics assume the form:

$$\dot{x} = f(x, t) + G(x, t) [u + \Delta(x, u, t)] \quad (2.9)$$

As before, the vector fields f and G represent the known functions of the nominal system model, while $\Delta(x, u, t)$ is an unknown function which lumps together various structured and unstructured uncertainties. Note that the uncertainty $\Delta(x, u, t)$ enters the system in the *same channel of the control input* u . This property is recognized by the fact that $\Delta(x, u, t)$ is multiplied by the same matrix function G as the control input.

Let us define more precisely the matching conditions for systems of the form (2.4) as follows.

Definition 2.2.2.1 (Matching conditions). Consider an input-affine uncertain system of the form

$$\dot{x} = f(x, t) + \Delta f(x, t) + G(x, t)u + \Delta G(x, t)u \quad (2.10)$$

with all the properties of (2.4). The above system is said to satisfy the *matching conditions* if the uncertain terms can be factorized as follows:

$$\begin{aligned} \Delta f(x, t) &= G(x, t) \Delta f'(x, t) \\ \Delta G(x, t) &= G(x, t) \Delta G'(x, t) \end{aligned} \quad (2.11)$$

and if there exists a positive constant ϵ such that

$$\|\Delta G'(x, t)\| \leq 1 - \epsilon \quad (2.12)$$

The uncertainties satisfying the matching conditions are called *matched uncertainties*. ◇

According to Definition 2.2.2.1, the matched uncertainties act “along” the distribution spanned by $G(x, t)$, whereas unmatched uncertainties are off-the-distribution components. Roughly speaking, matched uncertainties represent variations of the intensity in the direction of the control action, while unmatched uncertainties act on directions on which there is no apparent control authority.

To illustrate the rationale for the inequality condition (2.12), let us rewrite system (2.4), assuming satisfied conditions (2.12), as follows

$$\dot{x} = f(x, t) + G(x, t) [\Delta f'(x, t) + (1 + \Delta G'(x, t))u] \quad (2.13)$$

Recalling some basic notions about necessary conditions for output controllability, we observe that if the bounds of the uncertain term multiplying the control are allowed to be greater than unity:

$$\|\Delta G'(x, t)\| > 1 \quad (2.14)$$

then the control direction cannot be determined, thus any control input may cause the system to grow unstable. Furthermore, if the uncertain term multiplying the control input is such that:

$$\|\Delta G'(x, t)\| = 1 \quad (2.15)$$

then a singularity occurs when $\Delta G'(x, t) = -1$. In such a case, with no control authority, stabilization cannot be achieved.

It can be seen for instance from the above example 2.2.1.2, that if the unknown functions were replaced by known bounding functions satisfying the matching conditions, then one could easily choose a control law to cancel their effects and stabilize the system. This is the basic idea of the robust control design methodology called Lyapunov redesign, presented later in Section 3.1.

2.2.3 Generalized matching conditions

A step forward in nonlinear robust control design is made by taking into account systems which do not satisfy the matching conditions, i.e. systems containing the so-called *unmatched uncertainties*, which do not enter the system dynamics in the input channel as the matched ones. For this category of systems, very frequent in applications, recent investigations of the concept of robust control design have been directed toward the discovery of robust stabilizing controllers. Important results can be found in [108]. These results were formulated by considering the idea of a stability margin of the stabilized nominal system. Qu and Dorsey demonstrated that uncertain systems with arbitrarily large unmatched uncertainty can be stabilized if the nominal system can be stabilized with an arbitrarily large convergence rate (i.e. arbitrarily large stability margin). Qu, in [103], proposed a robust control design for a relaxed set of conditions referred to as *equivalently matched uncertainties* (or *EMUs*). The main concept for the EMU based design is to take advantage of the non-uniqueness of the Lyapunov function candidates, and demonstrates a set of conditions on the unmatched uncertainties that allows them

to be handled in the Lyapunov argument as if they were matched. These results were still fairly restrictive in scope, and limited in practical applications. After some results for linear systems, the concept of *generalized matching condition* (or *GMCs*) was finally introduced in an important paper of Qu [104]. In this paper, it was also proposed a systematic robust control design procedure to guarantee global stability under the GMCs.

The definition of generalized matching conditions is somewhat very technical and lies outside the purpose of this work. The only conceptual question is in describing the general structure of an uncertain nonlinear system which meets the GMCs, as follows:

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1, t) + \Delta f_1(x_1, w_1, t) + g_1(x_1, x_2, w_1, t) \\
 &\dots \\
 \dot{x}_i &= f_i(x_1, \dots, x_i, t) + \Delta f_i(x_1, \dots, x_i, w_i, t) + g_i(x_1, \dots, x_{i+1}, w_i, t) \\
 &\dots \\
 \dot{x}_m &= f_m(x_1, \dots, x_m, t) + \Delta f_m(x_1, \dots, x_m, w_m, t) + g_m(x_1, \dots, x_m, u, w_m, t)
 \end{aligned} \tag{2.16}$$

where $\text{col}(x_1 \dots x_m)$ is the state vector of the system, which is decomposed into state sub-vectors $x_i \in \mathbb{R}^{n_i}$, $i = 1, \dots, m$. Clearly, one must have

$$n_1 + n_2 + \dots + n_m = n \tag{2.17}$$

with n dimension of the state space. The vector $u \in \mathbb{R}^{n_{m+1}}$ contains the control inputs of the subsystems. The variables $w_i = 1, \dots, m$, represent the uncertain time-varying parameters, and their domain is a prescribed compact set of \mathbb{R}^{p_i} . The conditions that (2.16) should fulfill to satisfy GMCs are fully presented in [106], to which the reader should refer for a deeper understanding. The reader should only be acknowledged that these conditions impose constraints on the dimensions of the subsystems, in a lower triangular form fashion, on the factorization of the g_i vector fields (that should also be Lipschitz functions) and, of course, on the boundedness of the uncertain terms, namely:

$$\|\Delta f_i(x_1, \dots, x_i, w_i, t)\| \leq \rho_i(x_1, \dots, x_i, t), \quad \forall (x_1, \dots, x_i, w_i, t) \tag{2.18}$$

Note that ρ_i are known continuous functions that are *uniformly bounded* w.r.t. t , and *locally uniformly bounded* w.r.t. x .

Moving ahead to a more general structure of system, we should consider the generic class of uncertain systems which do not satisfy matching conditions, nor the generalized matching conditions. Qu and Kaloust addressed this problem in [59] and developed a

generic procedure for robust nonlinear control design of these systems. In the following, we will consider separately the problem of stabilization of a system which meets the matching conditions, using the approach named *Lyapunov redesign*, and the (more difficult) one of stabilizing a system satisfying just the generalized matching conditions, using the so-called *robust backstepping* approach. The fusion of these two methods results in a novel approach, which we call *recursive Lyapunov redesign*, whose aim is the stabilization of systems with a more general structure, in which the matched uncertainties may be control-dependent.

2.3 Uncertainty modeling

The first step in robust controller design for nonlinear systems is to *model the uncertainty* which affects the nominal equations in terms of deviations from the nominal values of system parameters. In this part of the work, we propose a novel approach to rapidly evaluate the deterministic model of such uncertainty.

First, we need to assume that system (2.4) may be simplified as follows

$$\dot{x} = \bar{f}(x) + \tilde{f}(t) + \Delta f(x, t) + \bar{G}(x)u + \tilde{G}(t)u + \Delta G(x, t)u \quad (2.19)$$

so that we can consider the time-varying terms as part of the uncertain terms, namely

$$\begin{aligned} \Delta \bar{f}(x, t) &= \tilde{f}(t) + \Delta f(x, t) \\ \Delta \bar{G}(x, t) &= \tilde{G}(t) + \Delta G(x, t) \end{aligned} \quad (2.20)$$

Thus, disregarding the $\bar{\cdot}$ for simplicity of notation, system (2.19) can be written in the following form:

$$\dot{x} = f(x) + \Delta f(x, t) + [G(x) + \Delta G(x, t)]u. \quad (2.21)$$

Note that $f(x)$ and $G(x)$ are completely known vector and matrix functions. As a consequence, we can further define

$$\begin{aligned} f_{\text{act}}(x, t) &= f_{\text{nom}}(x) + \Delta f(x, t) \\ G_{\text{act}}(x, t) &= G_{\text{nom}}(x) + \Delta G(x, t) \end{aligned} \quad (2.22)$$

where clearly $f_{\text{nom}}(x) = f(x)$ and $G_{\text{nom}}(x) = G(x)$ are the nominal values of the functions, perfectly known, while with f_{act} and G_{act} we denote the actual, true, values that the vector function f and the matrix function G assume at time t . The gap between the real and the nominal values is embedded into the uncertainty functions Δf and ΔG , which are time-varying, deterministic perturbations of the nominal values of system's

vector and matrix functions. The first thing to do to define these uncertainty functions is to define the scalar Δ operator, which applies point-wisely to the nominal components of system's vectors and matrices giving, one by one, the components of the vector uncertainty functions.

Definition 2.3.0.1. [Δ -operator and deviation function] The Δ -operator, when applied to the nominal value of a function f depending on uncertain parameters, gives a measure of the gap between that value and its actual one. The way the Δ operator perturbs a nominal value really depends on the design purposes and on modeling necessities. The straightforward choice we make is that of using what we call *deviation function* $\sigma(t)$, such that

$$\Delta f = \sigma(t)f_{\text{nom}} \quad (2.23)$$

where $\sigma(t)$ has the form

$$\sigma(t) = k \sin(\omega t)e^{pt} \quad (2.24)$$

Its amplitude k represents the amount of uncertainty which perturbs the nominal value of the function to which $\sigma(t)$ is applied. We let it be a constant value, but in fact it could be allowed to depend on time. The frequency ω describes the speed of variation of the actual function's value around the nominal one. The constant p , less or equal than zero, defines if the perturbation is transient or permanent: throughout the study we will put this equal to zero, considering only persistent uncertainties. \diamond

By definition, the Δ -operator acts on known values like the derivative operator acts on constant values. When a parameter p or a function is considered known, its nominal value coincides with the actual one, and the Δ operator gives zero as result, namely:

$$\Delta(p_{\text{nom}}) = 0 \quad (2.25)$$

Nevertheless, its real purpose is acting as a perturbation operator when applied to an unknown parameter. In this simple case it returns the product of the deviation function with the nominal value of the parameter.

$$\Delta(p) = \sigma_p(t) \cdot p_{\text{nom}} = \Delta p \quad (2.26)$$

When Δ is applied to a generic function f depending on an uncertain vector of parameters p , it takes the general form

$$\Delta(f(\cdot, p)) = \Delta_f(\cdot, p_{\text{nom}}, \sigma_{p_1}, \dots, \sigma_{p_N}) \quad (2.27)$$

with a different deviation function σ_{p_i} for each parameter in p .

In the sequel, with in mind our case study, we characterize the structure of Δf when applied to functions with polynomial dependence on the uncertain parameters. These functions depend also on known parameters a_i and show an affine dependency on them. We start by considering the case of a single uncertain parameter z :

$$f = a_r z^r + a_{r-1} z^{r-1} + \dots + a_1 z + a_0 \quad (2.28)$$

The application of the Δ operator preserves the affinity:

$$\Delta(f) = a_r \Delta(z^r) + a_{r-1} \Delta(z^{r-1}) + \dots + a_1 \Delta(z) \quad (2.29)$$

In a way similar to the derivative operator, Δ acts on the polynomial uncertain term z^r in this way:

$$\begin{aligned} \Delta(z^r) &= \Delta(z \cdot z \cdots z) = (\Delta z)^r + r z^{r-1} \Delta z + \frac{r(r-1)}{2} z^{r-2} (\Delta z)^2 \\ &\quad + \dots + \frac{r!}{i!(r-i)!} z^{r-i} (\Delta z)^i + \dots + r (\Delta z)^{r-1} z \end{aligned} \quad (2.30)$$

that means that it is distributed across the polynomial according to the rule of *simple combinations without repetitions*. At each application of Δ we need to group i distinct elements into a subset, and the number of these subsets is given by the Newton coefficient. This procedure is of direct derivation from the Leibniz rule for the differentiation of a product, with the notable difference that multiple Δ -products are allowed, since we are dealing with finite values and not infinitesimal ones. Keeping in mind that $\Delta z = \Delta(z) = \sigma(t)z^1$ and consequently grouping powers of z , we obtain:

$$\begin{aligned} \Delta(z^r) &= \sigma^r(t) z^r + r \sigma(t) z^r + \frac{r(r-1)}{2} \sigma^2(t) z^r \\ &\quad + \dots + \frac{r!}{i!(r-i)!} \sigma^i(t) z^r + \dots + r \sigma^{r-1}(t) z^r \end{aligned} \quad (2.31)$$

Or in a more compact form, re-ordering and post-multiplying by z^r :

$$\Delta(z^r) = \left(\sum_{i=1}^r \frac{r!}{i!(r-i)!} \sigma^i(t) \right) z^r \quad (2.32)$$

¹For simplicity, in the following we omit the subscript $_{\text{nom}}$, since it is clearly the nominal value of the parameter z the one we refer to when applying the deviation function.

Applying repeatedly the same arguments on every power of z in f we obtain:

$$\begin{aligned} \Delta(f) = & a_r \left(\sum_{i=1}^r \frac{r!}{i!(r-i)!} \sigma^i(t) \right) z^r \\ & + a_{r-1} \left(\sum_{i=1}^{r-1} \frac{(r-1)!}{i!(r-1-i)!} \sigma^i(t) \right) z^{r-1} + \dots + a_1 \sigma(t) z \end{aligned} \quad (2.33)$$

Since we use simple combinations without repetitions, the whole calculus developed is valid also for polynomial functions of r different parameters z_1, z_2, \dots, z_r . Thus the following proposition holds.

Proposition 2.3.0.2. *Let f_{nom} be the nominal value of a scalar function depending on known terms $a_r, a_{r-1}, \dots, a_1, a_0$ and unknown terms whose nominal values are z_1, z_2, \dots, z_r .*

$$f_{\text{nom}} = a_r \cdot (z_1 \cdot z_2 \cdots z_r) + a_{r-1} \cdot (z_1 \cdot z_2 \cdots z_{r-1}) + \dots + a_1 \cdot z_1 + a_0 \quad (2.34)$$

The application of the Δ operator with the associated deviation function $\sigma(t)$ results in:

$$\begin{aligned} \Delta(f_{\text{nom}}) = & a_r \left(\sum_{i=1}^r \frac{r!}{i!(r-i)!} \sigma^i(t) \right) f_r \\ & + a_{r-1} \left(\sum_{i=1}^{r-1} \frac{(r-1)!}{i!(r-1-i)!} \sigma^i(t) \right) f_{r-1} + \dots \\ & + a_{r-j} \left(\sum_{i=1}^{r-j} \frac{(r-j)!}{i!(r-j-i)!} \sigma^i(t) \right) f_{r-j} + \dots + a_1 \sigma(t) f_1 \end{aligned} \quad (2.35)$$

where $f_r = z_1 \cdot z_2 \cdots z_r$, $f_{r-1} = z_1 \cdot z_2 \cdots z_{r-1}$ and so on until $f_1 = z_1$. ◇

Proof. The Δ operator applies to the sets of r parameters z of f_r by constituting subsets using simple combinations without repetitions, more in detail:

- **1-subset** the first subset is only one and given by the simultaneous application of Δ to all the r parameters, i.e. $set_1 = \Delta z_1 \Delta z_2 \cdots \Delta z_r$.
- **r-subsets** There are two subsets composed by r elements, the first given by the application of Δ to the single elements, one-by-one, and the second given by the

application of Δ to $r - 1$ groups of elements, both r times:

first $r - 1$ subset :

$$\Delta z_1(z_2 z_3 \cdots z_r) + \Delta z_2(z_1 z_3 \cdots z_r) + \dots + \Delta z_r(z_1 z_2 \cdots z_{r-1})$$

second $r - 1$ subset :

$$\begin{aligned} & (\Delta z_1 \Delta z_2 \cdots \Delta z_{r-1}) z_r + (\Delta z_1 \cdots \Delta z_{r-2} \Delta z_r) z_{r-1} \\ & + \dots + (\Delta z_2 \Delta z_3 \cdots \Delta z_r) z_1 \end{aligned} \tag{2.36}$$

- Going on with the analysis, we find other two subsets composed by $\frac{r(r-1)}{2}$ elements obtained through the application of Δ to 2 and $r - 2$ elements at once.
- The iterative perturbation process goes on, using always simple combinations to form the remaining couples of subsets, each of them formed by the same number of elements given by the Newton coefficient.

The associative property of the deviation function allow for the simplification, so we obtain:

$$\begin{aligned} \Delta(f_r) &= \sigma^r(t)(z_1 z_2 \cdots z_r) + r \sigma(t)(z_1 z_2 \cdots z_r) \\ &+ \frac{r(r-1)}{2} \sigma^2(t)(z_1 z_2 \cdots z_r) + \dots \\ &+ \frac{r!}{i!(r-i)!} \sigma^i(t)(z_1 z_2 \cdots z_r) + \dots + r \sigma^{r-1}(t)(z_1 z_2 \cdots z_r) \\ &= \left(\sum_{i=1}^r \frac{r!}{i!(r-i)!} \sigma^i(t) \right) f_r \end{aligned} \tag{2.37}$$

This procedure clearly still holds for all the other terms of the polynomial, from f_{r-1} to f_1 . The linearity of the Δ operator then yields (2.35). \square

In the following, we will need to evaluate the uncertain envelopes of quotients between uncertain parameters. With this aim, it is possible to establish the following result.

Proposition 2.3.0.3. *Let f_{nom} be a function expressed by the quotient of two uncertain parameters, whose nominal values are z_1 and z_2 , multiplied by a known value a*

$$f_{\text{nom}} = a \cdot \frac{z_1}{z_2} \tag{2.38}$$

The application of the Δ operator results in the following uncertain envelope:

$$\Delta f_{\text{nom}} = f_{\text{nom}} \frac{\sigma_1(t) - \sigma_2(t)}{1 + \sigma_2(t)} \tag{2.39}$$

where σ_1 and σ_2 are the deviation functions associated to, respectively, z_1 and z_2 . \diamond

Proof. By definition, we have:

$$z_1 = \frac{1}{a} f_{\text{nom}} z_2 \quad (2.40)$$

the linearity-in-the-known-terms of Δ yields:

$$\Delta_1 z_1 = \frac{1}{a} (\Delta f_{\text{nom}} \Delta_2 z_2 + \Delta f_{\text{nom}} z_2 + f_{\text{nom}} \Delta_2 z_2) \quad (2.41)$$

With Δ_i we underline that we are applying to the parameter z_i the deviation function $\sigma_i(t)$. Grouping and rearranging equation (2.41) we obtain:

$$\Delta f_{\text{nom}} = \frac{a \Delta_1 z_1 - f_{\text{nom}} \Delta_2 z_2}{\Delta_2 z_2 + z_2} \quad (2.42)$$

Finally, substituting into the right member the expression of f_{nom} in terms of the other parameters and using the deviation functions σ_1 and σ_2 , we obtain:

$$\Delta f_{\text{nom}} = a \frac{z_1}{z_2} \frac{\sigma_1(t) - \sigma_2(t)}{\sigma_2(t) + 1} = f_{\text{nom}} \frac{\sigma_1(t) - \sigma_2(t)}{\sigma_2(t) + 1} \quad (2.43)$$

which is exactly (2.39), q.e.d. □

We are going to massively use these results in missile autopilot design, where the complex structure of the aerodynamics parameters requires simplified procedures to evaluate the uncertain envelopes.

Chapter 3

Robust nonlinear control design

In this chapter we establish the main results of the work. We begin by recalling the Lyapunov redesign technique and introducing a revisited version of it, exploiting suitable differentiable sigmoid functions, the *robust control functions*, in order to avoid the problem of chattering in the implementation phase. Next, we recall the robust backstepping technique, and finally we present our main result, consisting of the fusion of the two approaches, the technique called *recursive Lyapunov redesign*. This technique allows to compensate also control-dependent uncertainties, which are common in certain nonlinear aerospace dynamics, like tail-controlled missile dynamics. In the last part of the chapter, we recall the stabilization technique of Immersion & Invariance as a tool of robustification of nonlinear control laws with respect to higher-order dynamics. We propose a solution in the case of a particular class of systems in feedback form using one step of backstepping and a nonlinear domination argument.

3.1 Lyapunov redesign revisited

To handle matched uncertainties it is possible to use a classical technique presented in [62], the Lyapunov redesign. This method is of sure effectiveness, but leads to a discontinuous control law. We propose here a revisited version of this method that yields a C^1 (continuous with its first derivative) control law. The procedure applies to systems which satisfy the matching condition, namely:

$$\dot{x} = f(x, t) + G(x, t) [u + \Delta(x, u, t)] \quad (3.1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Note that Δ is an unknown function lumping together various structured and unstructured uncertainties and satisfying the matching condition, since

it lies in the span of the control input. Suppose that the nominal system

$$\dot{x} = f(x, t) + G(x, t)\psi(x, t)$$

is stabilized by a known nominal¹ feedback control law (for instance, feedback linearization or backstepping) $u = \psi(x, t)$. Such state feedback renders the nominal closed-loop system

$$\dot{x} = f(x, t) + G(x, t)\psi(x, t) \quad (3.2)$$

uniformly globally asymptotically stable (UGAS). Suppose further that we know a Lyapunov function for (3.2), i.e. a continuously differentiable function $V(x, t)$ that satisfies the inequalities:

$$\begin{aligned} \alpha_1(\|x\|) &\leq V(x, t) \leq \alpha_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} [f(x, t) + G(x, t)\psi(x, t)] &\leq -\alpha_3(\|x\|) \end{aligned} \quad (3.3)$$

where α_1, α_2 are class \mathcal{K}_∞ functions, while α_3 is a class \mathcal{K} function. Assume that, with $u = \psi(x, t) + v$, the uncertain term satisfies the inequality:

$$\|\Delta(x, \psi(x, t) + v, t)\| \leq \rho(x, t) + k_0 \|v\|, \quad 0 \leq k_0 < 1 \quad (3.4)$$

where ρ is a non-negative continuous function. The above inequality defines the bounding function of the uncertain term Δ , and it is the only information we need to know about it. The bounding function ρ represents a measure of the maximum size the uncertainty may assume: we will not require it to be small, only to be known. We want to find an expression for v such that, under the overall control $u = \psi(x, t) + v$, the uncertain system (3.1) is UGAS. The uncertain closed-loop system under the feedback u is:

$$\dot{x} = f(x, t) + G(x, t)\psi(x, t) + G(x, t)[v + \Delta(x, \psi(x, t) + v, t)]. \quad (3.5)$$

We exploit the same Lyapunov function of the nominal system (from this re-use, the name “re-design”) as a robust control Lyapunov function for the uncertain system. The derivative along the motion takes the form (omitting for brevity the arguments of the various functions):

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} (f + G\psi) + \frac{\partial V}{\partial x} G(v + \Delta) \leq -\alpha_3(\|x\|) + \frac{\partial V}{\partial x} G(v + \Delta). \quad (3.6)$$

¹Here and throughout the work we call “nominal” a control law which is designed using only the perfectly known (nominal) parts of the dynamics.

Set $w^T = \frac{\partial V}{\partial x} G$ and rewrite the last inequality as:

$$\dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T \Delta. \quad (3.7)$$

Note that, due to the matching conditions, the uncertain term Δ appears on the right-hand side exactly at the same point where v appears. Consequently, it is possible to choose v to cancel the (eventually destabilizing) effect of Δ on \dot{V} . Exploiting inequality (3.4), we obtain:

$$w^T v + w^T \Delta \leq w^T v + \|w\| \|\Delta\| \leq w^T v + \|w\| [\rho(x, t) + k_0 \|v\|]. \quad (3.8)$$

Taking

$$v = -\eta(x, t) \cdot \frac{w}{\|w\|} \quad (3.9)$$

with η non-negative function, we obtain

$$w^T v + w^T \Delta \leq -\eta \|w\| + \rho \|w\| + k_0 \eta \|w\| = -\eta(1 - k_0) \|w\| + \rho \|w\|. \quad (3.10)$$

Choosing then

$$\eta(x, t) \geq \frac{\rho(x, t)}{1 - k_0} \quad (3.11)$$

for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ yields

$$w^T v + w^T \Delta \leq -\rho \|w\| + \rho \|w\| = 0. \quad (3.12)$$

Hence under the nonlinear robust control (3.9), the derivative along the trajectories of the uncertain closed-loop system (3.5) is negative definite, and so the feedback system obtained is UGAS.

The control law (3.9) is clearly discontinuous in $w = 0$. This can bring serious implementation issues, one for all the problem of chattering (see [62] for further details). Notice that in the scalar case, (3.9) takes the simple form:

$$v = -\eta \cdot (x, t) \text{sign}(w). \quad (3.13)$$

Our revisited version of this control law is based on its approximation using a C^1 approximation of the signum function, so avoiding the problem of chattering. The function employed is a *sigmoid*, dependent on a tracking error s , which we call *robust control function*, as stated by the following definition.

Definition 3.1.0.4. [Robust (virtual) control function] A *robust control function* is a sigmoid of the form

$$v(s, \sigma_s) = \text{sign}(s)(1 - e^{-\sigma_s|s|}) = \text{sigm}(s, \sigma_s). \quad (3.14)$$

Its structure emulates a discontinuous function, but it can also be used in a two-step recursive procedure as virtual control input, since it has a continuous first derivative:

$$\frac{\partial v}{\partial s} = \sigma_s e^{-\sigma_s|s|} \quad (3.15)$$

◇

Remark 3.1.0.5. The robust control function (3.14), which has the property of size-domination of the uncertain terms, approximates better the signum function by increasing the value of the sigmoid slope σ_s (Fig. 3.1), namely:

$$\lim_{\sigma_s \rightarrow \infty} \text{sigm}(s, \sigma_s) = \text{sign}(s) \quad (3.16)$$

◇

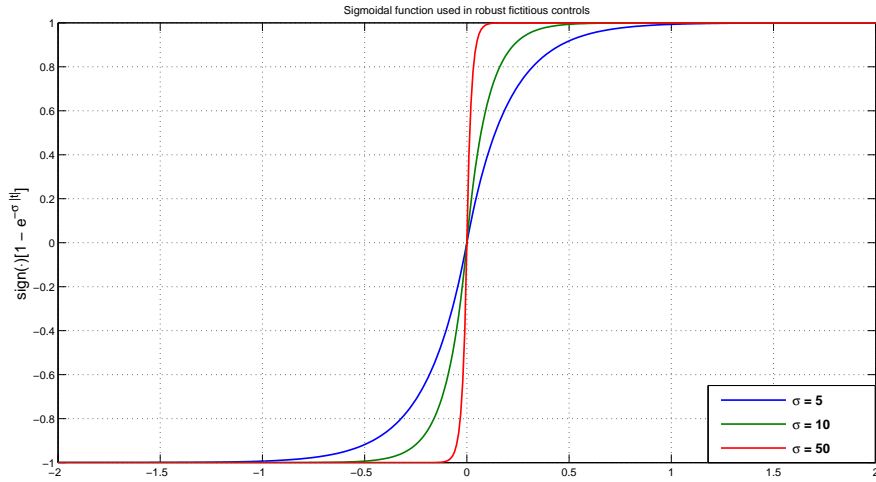


FIGURE 3.1: Sigmoidal function represented for three different values of the slope.

Remark 3.1.0.6. Note that the second derivative of (3.14) with respect to s

$$\frac{\partial^2 v}{\partial s^2} = -\sigma_s^2 \text{sign}(s) e^{-\sigma_s|s|} \quad (3.17)$$

is not continuous anymore. This technical obstruction can be overcome for higher order systems, by raising the sigmoidal function to odd powers. For instance:

$$\text{sigm}_3(s, \sigma_s) = \left[\text{sign}(s)(1 - e^{-\sigma_s |s|}) \right]^3 \quad (3.18)$$

is a continuous approximation of the signum function, which admits continuous first and second derivatives with respect to s , hence it is suitable for a third-order system design. \diamond

With this in mind, we can *revisit* the Lyapunov redesign control law as follows:

$$v = -\eta(x, t) \text{sigm}(w, \sigma) = -\eta(x, t) \text{sign}(w)(1 - e^{-\sigma |w|}). \quad (3.19)$$

It must be underlined that, being (3.19) only an approximation of the discontinuous stabilizing control law (3.13), the UGAS cannot be guaranteed anymore. However, due to the fact that the approximation can be made arbitrarily precise by increasing the value of σ_s , which indeed is a design parameter, we can conclude that a “practical” UGAS can always be obtained using in a smart manner all the degrees of freedom of (3.19). Therefore the residual regulation (or tracking) error in the steady-state can be made arbitrarily small by increasing σ_s , as will be shown in detail in the following sections.

3.2 Robust Backstepping

The recursive approach of backstepping can be used to construct a robust control Lyapunov function for a nonlinear uncertain system which do not satisfy the matching conditions. Such a modification of the nominal control law is called *robust backstepping* and it was first introduced in [31]. The design procedure is based on the idea of exploiting the virtual control input to compensate the unmatched uncertainties. For a detailed discussion and explanation of the procedure in the general case, see [36]. Let us illustrate the core-idea of robust backstepping design in the case of a simple second order uncertain nonlinear system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + \Delta_1(x, t) \\ \dot{x}_2 &= u + \Delta_2(x, t) \end{aligned} \quad (3.20)$$

Remark 3.2.0.7. System (3.20) is the feedback linearized version of the generic second order system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, t) + g_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_1, x_2, t) + g_2(x_1, x_2)u.\end{aligned}\tag{3.21}$$

The uncertainties $\Delta_1(x, t)$ and $\Delta_2(x, t)$ represent the errors committed in the feedback cancellation and inversion of the functions describing the system. Again, as done in 2.3, we assume that the time-varying part of the drift functions can be embedded into the uncertain terms. \diamond

Note that $\Delta f_1(x, t)$ is an unmatched uncertainty. For system (3.20) to fulfill the generalized matching conditions, the uncertain terms should satisfy the following lower-triangular inequalities

$$\begin{aligned}|\Delta_1(x, t)| &\leq \rho_1(x_1) \\ |\Delta_2(x, t)| &\leq \rho_2(x_1, x_2)\end{aligned}\tag{3.22}$$

with ρ_1 and ρ_2 suitable bounding functions, allowed to grow faster than linear. For the sake of simplicity, we assume $\rho_1(0) = 0$ and $\rho_2(0, 0) = 0$.

The procedure starts with the observation that Δ_1 is not matched with u , but in fact it is matched with x_2 . Thus we design a *robust virtual control law* $\phi_r(x_1)$ to achieve P-RUGAS of the x_1 dynamics, using $V = \frac{1}{2}x_1^2$ as RCLF. The derivative along the motion

$$\dot{V} = x_1 [\phi_r(x_1) + \Delta_1(x, t)] \leq x_1 \phi_r(x_1) + |x_1| \rho_1(x_1)\tag{3.23}$$

is made negative definite, for instance, by the choice

$$\phi_r(x_1) = -x_1 - \text{sign}(x_1)\rho_1(x_1)\tag{3.24}$$

Note that this choice is not practical in a recursive control design, since the derivative of ϕ_r would not be defined in the origin. We propose, thus, a modified version using the following choice for the robust virtual control function

$$\phi_r(x_1) = -x_1 - \text{sigm}(x_1, \sigma_1)\rho_1(x_1) = -x_1 - \text{sign}(x_1) \left(1 - e^{-\sigma_1|x_1|}\right) \rho_1(x_1)\tag{3.25}$$

With this choice, if $x_2 = \phi_r(x_1)$ we would obtain P-RUGAS of the x_1 dynamics, in fact

$$\dot{V} \leq -x_1^2 + \rho_1(x_1)|x_1|e^{-\sigma_1|x_1|} \leq -x_1^2 + \rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1}\tag{3.26}$$

is negative whenever $x_1 > \sqrt{\rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1}}$. This means that the x_1 solution converge towards the set of radius $\sqrt{\rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1}}$, which can be made arbitrarily small by increasing σ_1 . In other words, we can conclude the P-RUGAS of the x_1 dynamics.

The backstepping procedure continues by defining the error $\zeta = x_2 - \phi_r(x_1)$ and the corresponding RCLF for the whole system (3.20)

$$W = V + \frac{1}{2}\zeta^2 = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \phi_r(x_1))^2 \quad (3.27)$$

which is non-quadratic in the state x . We compute the derivative along the motion, obtaining

$$\begin{aligned} \dot{W} &= \dot{V} + (x_2 - \phi_r(x_1)) \left(u + \Delta_2 - \frac{\partial \phi_r}{\partial x_1}(x_2 + \Delta_1) \right) \\ &\leq -x_1^2 + \rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1} + (x_2 - \phi_r(x_1)) \left(x_1 + u + \Delta_2 - \frac{\partial \phi_r}{\partial x_1}(x_2 + \Delta_1) \right) \end{aligned} \quad (3.28)$$

where, this time, u matches the composite uncertainty $\Delta_o(x, t) = \Delta_2(x, t) - \frac{\partial \phi_r}{\partial x_1}(x)\Delta_1(x, t)$, for which a bounding function ρ_o can be computed such that $|\Delta_o| \leq \rho_o(x)$. Choosing then

$$u = -(x_2 - \phi_r(x_1)) - x_1 - \frac{\partial \phi_r}{\partial x_1}x_2 - \rho_o(x)\text{sigm}(\zeta, \sigma_2) \quad (3.29)$$

yields

$$\dot{W} \leq -x_1^2 + \rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1} - \zeta^2 + \rho_o^{\max}\sigma_2^{-1}e^{-1} \quad (3.30)$$

with ρ_o^{\max} the value of ρ_o at $\zeta = \sigma_2^{-1}$. Inequality (3.30) is equivalent to P-RUGAS of the origin of (3.20), with residual set Ω defined by:

$$\Omega := \left\{ (x_1, \zeta) \in \mathbb{R}^2 : \sqrt{x_1^2 + \zeta^2} \leq \sqrt{\rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1} + \rho_o^{\max}\sigma_2^{-1}e^{-1}} \right\}. \quad (3.31)$$

The results obtained can be extended to more general system structures, also in the case of more than one control input [36, 86]. We may also assume a more general bounding inequality for the unmatched uncertainty Δ_1 , namely $|\Delta_1(x, t)| \leq \rho_1(x)$ where ρ_1 is allowed to depend also on x_2 . This modification leads to a slightly different Lyapunov inequality in which the control input appears multiplied by the term $\left(1 + \frac{\partial \phi_r}{\partial x_2}\right)$, since the robust virtual control function ϕ_r is now a function of the full state, through the bounding term ρ_1 , namely

$$\phi_r(x_1, x_2) = -x_1 - \text{sigm}(x_1, \sigma_1)\rho_1(x_1, x_2). \quad (3.32)$$

We will discuss this modification with more detail later, when dealing with missile autopilot design.

3.3 Recursive Lyapunov redesign

In this section we introduce a design method which combines robust backstepping and Lyapunov redesign to deal with different kinds of uncertainties in the dynamics. We call it recursive Lyapunov redesign, since it merges the recursion of backstepping with the ability to deal with control-dependent uncertainties of Lyapunov redesign. Let us show the core idea by extending system (3.20) in the following fashion:

$$\begin{aligned}\dot{x}_1 &= x_2 + \Delta_1(x, t) \\ \dot{x}_2 &= (1 + \Delta_3(x, t))u + \Delta_2(x, t)\end{aligned}\tag{3.33}$$

where $\Delta_3(x, t)$ multiplies directly u and it is dangerous because it may alter control direction. We need the following two assumptions before stating our result.

Assumption 3.3.0.8. We know a robust backstepping control law $u_b = \psi(x)$ which renders the system without Δ_3 , namely

$$\begin{aligned}\dot{x}_1 &= x_2 + \Delta_1(x, t) \\ \dot{x}_2 &= \psi(x) + \Delta_2(x, t)\end{aligned}\tag{3.34}$$

practically-robustly uniformly globally asymptotically stable. Such a control law exists, as shown in section 3.2, and depends on the bounding functions of Δ_1 and Δ_2 . Contextually, we know the corresponding robust control Lyapunov function $W(x)$, as determined in (3.27). \diamond

Assumption 3.3.0.9. The uncertain term Δ_3 , with the control law $u = \psi(x) + u_r$, satisfies the inequality:

$$|\Delta_3(x, t)(\psi(x) + u_r)| \leq \rho_3(x, t) + k_0|u_r|, \quad 0 \leq k_0 < 1\tag{3.35}$$

where ρ_3 is a continuous and non-negative bounding function. \diamond

Assumption 3.3.0.8 is essential for the design of the complete control law, while assumption 3.3.0.9 is necessary for Δ_3 not to alter control direction. In fact, inequality (3.35) can be split in two as follows:

$$|\Delta_3(x, t)\psi(x)| \leq \rho_3(x, t)\tag{3.36}$$

$$|\Delta_3(x, t)| \leq k_0,\tag{3.37}$$

where (3.37) ensures that control direction is not modified by the uncertain term. We are now ready to establish our result in the following theorem.

Theorem 3.3.0.10. Consider a system of the form (3.33), for which assumptions 3.3.0.8, 3.3.0.9 are satisfied. The control law

$$u_r(x) = -\eta(x, t) \text{sigm}(w, \sigma_r) \quad (3.38)$$

with $w^T = \frac{\partial W}{\partial x_2}$ and $\eta(x, t)$ non-negative function such that

$$\eta(x, t) \geq \frac{\rho_3(x, t)}{1 - k_0} \quad (3.39)$$

renders the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 + \Delta_1(x, t) \\ \dot{x}_2 &= (1 + \Delta_3(x, t))(\psi(x) + u_r(x)) + \Delta_2(x, t) \end{aligned} \quad (3.40)$$

practically-robustly uniformly globally asymptotically stable. \diamond

Proof. From assumption 3.3.0.8 we know the RCLF $W(x)$, hence we apply the Lyapunov direct criterion on system (3.40), obtaining

$$\dot{W} = \frac{\partial W}{\partial x_1} (x_2 + \Delta_1(x, t)) + \frac{\partial W}{\partial x_2} (\psi(x) + \Delta_2(x, t)) \quad (3.41)$$

$$\frac{\partial W}{\partial x_2} [u_r(x) + \Delta_3(x, t)(\psi(x) + u_r(x))] \quad (3.42)$$

Note that the first part of \dot{W} , namely (3.41), is bounded from above as in (3.30). For the sake of simplicity, let us rename the residual term as follows

$$\rho_1(\sigma_1^{-1})\sigma_1^{-1}e^{-1} + \rho_o^{\max}\sigma_2^{-1}e^{-1} = \text{res}(\sigma_1, \sigma_2) \quad (3.43)$$

and simplify (3.41)-(3.42) into

$$\dot{W} \leq -x_1^2 - \zeta^2 + \text{res}(\sigma_1, \sigma_2) + \frac{\partial W}{\partial x_2} [u_r(x) + \Delta_3(x, t)(\psi(x) + u_r(x))]. \quad (3.44)$$

Using assumption 3.3.0.9, setting $u_r(x)$ as in (3.38) and $w = \frac{\partial W}{\partial x_2}$, inequality (3.44) is developed as follows

$$\begin{aligned} \dot{W} &\leq -x_1^2 - \zeta^2 + \text{res}(\sigma_1, \sigma_2) + wu_r(x) + w\rho_3(x, t) + wk_0u_r(x) \\ &\leq -x_1^2 - \zeta^2 + \text{res}(\sigma_1, \sigma_2) - w\eta(x, t)\text{sigm}(w, \sigma_r) + |w|\rho_3(x, t) + wk_0\eta(x, t)\text{sigm}(w, \sigma_r) \\ &= -x_1^2 - \zeta^2 + \text{res}(\sigma_1, \sigma_2) - \eta|w| + \eta\text{res}_r(\sigma_r) + |w|\rho_3(x, t) + |w|k_0\eta + k_0\eta\text{res}_r(\sigma_r) \end{aligned} \quad (3.45)$$

where $\text{res}_r(\sigma_r) = \sigma_r^{-1}e^{-1}$. Note that choosing η as in (3.39) reduces the expression to

$$\begin{aligned}\dot{W} &\leq -x_1^2 - \zeta^2 + \text{res}(\sigma_1, \sigma_2) + \eta \text{res}_r(\sigma_r) + k_0 \eta \text{res}_r(\sigma_r) \\ &\leq -x_1^2 - \zeta^2 + \text{res}(\sigma_1, \sigma_2) + \eta^{\max} \text{res}_r(\sigma_r) + k_0 \eta^{\max} \text{res}_r(\sigma_r)\end{aligned}\quad (3.46)$$

where η^{\max} is an opportune, constant, upper-bound. Inequality (3.46) is equivalent to P-RUGAS of the origin of (3.40), with residual set Ω_r defined by:

$$\Omega_r := \left\{ (x_1, \zeta) \in \mathbb{R}^2 : \sqrt{x_1^2 + \zeta^2} \leq \sqrt{\text{res}(\sigma_1, \sigma_2) + \sigma_r^{-1}e^{-1}\eta^{\max}(1 + k_0)} \right\}. \quad (3.47)$$

which concludes the proof. \square

In the following sections, we extend the proposed robust nonlinear stabilization procedure of recursive Lyapunov redesign to more general system structures, not in lower-triangular form and with more than one control input. First, we use it to stabilize the attitude of a spacecraft with flexible appendages. Second, we develop a robust nonlinear missile autopilot, simultaneously introducing a novel systematic calculus procedure for the uncertain terms in aerodynamic forces and moments, which can be easily extended to other mechanical systems.

3.4 Stabilization via Immersion & Invariance

The technique of Immersion & Invariance (I&I), first introduced in [6], is a tool for the stabilization of nonlinear systems via state-feedback. The existence of a globally asymptotically stable target dynamics into which the system to be controlled can be “immersed” plus the invariance and attractivity of the corresponding manifold, together with the boundedness of the trajectories of an extended system are sufficient conditions for the GAS of a chosen equilibrium of the controlled system. The method is specifically suitable for systems that admit a fast-slow dynamics decomposition, e.g. singularly perturbed systems, systems in feedback form but also underactuated systems requiring non-standard control solutions. I&I can be regarded as a tool to robustify a given nonlinear controller with respect to higher-order dynamics, exploiting at its best the knowledge of such dynamics during the control design phase. Thus, this approach can be considered, at least in part, “robust” nonlinear control.

Let us recall the continuous-time I&I main result in the general case [5].

Theorem 3.4.0.11. *Consider the nonlinear system*

$$\dot{x} = f(x) + g(x)u \quad (3.48)$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and an equilibrium point $x^* \in \mathbb{R}^n$ to be stabilized. Suppose that (3.48) satisfies the following four conditions.

H1c (Target System) - There exist maps $\alpha(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $\pi(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^n$ such that the subsystem $\dot{\xi} = \alpha(\xi)$ with state $\xi \in \mathbb{R}^p$, $p < n$, has a (globally) asymptotically stable equilibrium at $\xi^ \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$.*

H2c (Immersion condition) - For all $\xi \in \mathbb{R}^p$, there exist maps $c(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^m$ and $\pi(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^n$ such that

$$f(\pi(\xi)) + g(\pi(\xi))c(\xi) = \frac{\partial \pi}{\partial \xi}(\xi)\alpha(\xi) \quad (3.49)$$

H3c (Implicit manifold - \mathcal{M}) - There exists a map $\phi(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n-p}$ such that the identity between sets $\{x \in \mathbb{R}^n | \phi(x) = 0\} = \{x \in \mathbb{R}^n | x = \pi(\xi) \text{ for } \xi \in \mathbb{R}^p\}$ holds.

H4c (Manifold attractivity and trajectory boundedness) - There exists a map $\psi(\cdot, \cdot) : \mathbb{R}^{n \times (n-p)} \rightarrow \mathbb{R}^m$ such that all the trajectories of the system, with initial condition $z_0 = \phi(x_0)$

$$\dot{z} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\psi(x, z)] \quad (3.50a)$$

$$\dot{x} = f(x) + g(x)\psi(x, z) \quad (3.50b)$$

are bounded and satisfy $\lim_{t \rightarrow \infty} z(t) = 0$.

Under these four conditions, x^* is a (globally) asymptotically stable equilibrium of the closed-loop system

$$\dot{x} = f(x) + g(x)\psi(x, \phi(x)) \quad (3.51)$$

◇

The following definition is straightforward.

Definition 3.4.0.12 (I&I Stabilizability). A nonlinear system of the form (3.48) is said to be *I&I stabilizable* with target dynamics $\dot{\xi} = \alpha(\xi)$, if it satisfies conditions H1c to H4c of Theorem 3.4.0.11. ◇

Note that the target dynamics is the restriction of the closed-loop system to the manifold \mathcal{M} , implicitly defined in H3c. The control law $u = \psi(x, z)$ is designed to steer to zero the

off-the-manifold coordinate z and to guarantee the boundedness of system trajectories. On the manifold, the control law is reduced to $\psi(\pi(\xi), 0) = c(\xi)$, and it renders \mathcal{M} invariant according to H2c. The complete control law can thus be decomposed in two parts:

$$u = \psi(x, \phi(x)) = \psi(x, 0) + \tilde{\psi}(x, \phi(x)) \quad (3.52)$$

with $\psi(\pi(\xi), 0) = c(\xi)$ on the manifold and $\tilde{\psi}(x, 0) = 0$. While $\psi(x, 0)$ can be seen as a nominal control law, designed on the model of the dynamics restricted on the manifold to obtain a GAS target dynamics, the term $\tilde{\psi}(x, \phi(x))$ is a robustness-improving addendum which takes into account the off-the-manifold behaviors. The overall control law provides the I&I “robust” nonlinear stabilizer.

3.4.1 The class of systems under study

We consider the following special class of nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2, x_3) \\ \dot{x}_3 &= u \end{aligned} \quad (3.53)$$

where $x_1 \in \mathbb{R}^p$, $x_3, u \in \mathbb{R}$, x_2 is a dynamic extension of the x_1 dynamics, and $x_1 = 0$ is a GAS equilibrium point of $\dot{x}_1 = f_1(x_1, 0)$. For simplicity, in the sequel we will assume $x_2 \in \mathbb{R}$, hence $n - p = 2$. We assume also $f_1(0, 0) = 0$ and f_2 smooth function, $f_2(0, 0) = 0$.

Remark 3.4.1.1. System (3.53) “extends” the class of pure-feedback systems of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= u \end{aligned} \quad (3.54)$$

with the same dimensions and properties, except that $n - p = 1$. As shown in [6], conditions H1c, H2c and H3c are trivially satisfied by (3.54), with target dynamics $\alpha(x_1) = f_1(x_1, 0) : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and reduced control $c(x_1) = 0$. The attractivity and boundedness condition can be verified using a weak control Lyapunov function with a stabilizing feedback of the form

$$u = -k(x_1, x_2)x_2 \quad (3.55)$$

with $k(\cdot, \cdot) > 0$ for any (x_1, x_2) . ◇

Note that system (3.53) satisfies the first three conditions for I&I stabilization. H1c is trivially fulfilled with target dynamics $\dot{\xi} = \alpha(\xi) = f_1(\xi, 0)$. The maps

$$\begin{aligned} x_1 &= \pi_1(\xi) = \xi \\ \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} &= \pi_2(\xi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \quad (3.56)$$

with $\pi_1 : \mathbb{R}^p \rightarrow \mathbb{R}^p$, $\pi_2 : \mathbb{R}^p \rightarrow \mathbb{R}^2$ and so $\pi : \mathbb{R}^p \rightarrow \mathbb{R}^n$, are such that the immersion condition H2c holds with the choice $c(\xi) = 0$. The implicit manifold condition H3c is verified, with the following choice for the off-the-manifold coordinates

$$\phi(x) = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} - \pi_2(\pi_1^{-1}(x_1)) = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}. \quad (3.57)$$

3.4.2 Control design

In the following, we investigate under which conditions (3.53) satisfies the attractivity and boundedness requirements of H4c.

Assumption 3.4.2.1. We assume that system (3.53) can be written in the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, 0) + g_1(x_1, x_2)x_2 \\ \dot{x}_2 &= f_2(x_2, 0) + g_2(x_2, x_3)x_3 \\ \dot{x}_3 &= u \end{aligned} \quad (3.58)$$

with g_1 and g_2 smooth functions of proper dimensions. \diamond

Assumption 3.4.2.2. We assume that, for system (3.58), $g_2(x_2, x_3) = 1$. \diamond

Note that assumption 3.4.2.1 is always verified if the vector-fields of (3.53) are smooth, while assumption 3.4.2.2 is only made to simplify the control design procedure. It can be replaced by the weaker requirement that $g_2(x_2, x_3) \neq 0$ for all (x_2, x_3) in a domain of interest. We are now ready to state the following result.

Theorem 3.4.2.3. Consider system (3.53) and suppose it satisfies assumptions 3.4.2.1 and 3.4.2.2. Then, it fulfills condition H4c of theorem 3.4.0.11, i.e. it is I&I stabilizable, with the

control law

$$\begin{aligned}
 u = & -x_2 - \left(\frac{\partial f_2}{\partial x_2} + k_1(x_1, x_2) + \frac{\partial k_1}{\partial x_2} x_2 \right) (x_3 + f_2(x_2, 0)) \\
 & - \frac{\partial k_1}{\partial x_1} (f_1(x_1, 0) + g_1(x_1, x_2)x_2) x_2 \\
 & - k_2 (x_3 + f_2(x_2, 0) + k_1(x_1, x_2)x_2)
 \end{aligned} \tag{3.59}$$

for suitable choices of $k_1(x_1, x_2), k_2 > 0$. \diamond

Proof. Since assumptions 3.4.2.1 and 3.4.2.2 are satisfied, let the dynamics

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1, 0) + g_1(x_1, x_2)x_2 \\
 \dot{x}_2 &= f_2(x_2, 0) + x_3 \\
 \dot{x}_3 &= u.
 \end{aligned} \tag{3.60}$$

The subsystem (x_2, x_3) is stabilizable by means of backstepping, so it is possible to construct a feedback

$$u_b = b(x_2, x_3) = -x_2 - \left(\frac{\partial f_2}{\partial x_2} + k_1 \right) (\zeta - k_1 x_2) - k_2 \zeta \tag{3.61}$$

with virtual error $\zeta = x_3 - x_{3d} = x_3 + f_2(x_2, 0) + k_1 x_2$, and gains $k_1, k_2 > 0$ such that

$$\begin{aligned}
 \dot{x}_2 &= f_2(x_2, 0) + x_3 \\
 \dot{x}_3 &= b(x_2, x_3)
 \end{aligned} \tag{3.62}$$

has a GAS equilibrium in $\phi(x) = (x_2, x_3)^T = (0, 0)^T$, thus ensuring manifold attractivity. In closed-loop with the backstepping controller and written in ζ coordinates, system (3.60) takes the form

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1, 0) + g_1(x_1, x_2)x_2 \\
 \dot{x}_2 &= -k_1 x_2 + \zeta \\
 \dot{\zeta} &= -x_2 - k_2 \zeta
 \end{aligned} \tag{3.63}$$

Recall that $\dot{x}_1 = f_1(x_1, 0)$ has a GAS equilibrium in the origin, so for the Lyapunov converse theorems there exists a positive definite function $V(x_1)$ such that

$$\frac{\partial V}{\partial x_1} f_1(x_1, 0) < 0 \tag{3.64}$$

for all $\|x_1\| \geq M > 0$. With this in mind, let us up-augment the backstepping Lyapunov candidate, obtaining

$$W = V(x_1) + \frac{1}{2} [x_2^2 + \zeta^2] \quad (3.65)$$

whose derivative along the trajectories

$$\begin{aligned} \dot{W} &= \frac{\partial V}{\partial x_1} f_1(x_1, 0) + \frac{\partial V}{\partial x_1} g_1(x_1, x_2) x_2 - k_1 x_2^2 - k_2 \zeta^2 \\ &\leq \left\| \frac{\partial V}{\partial x_1} \right\|^2 \|g_1(x_1, x_2)\|^2 x_2^2 - k_1 x_2^2 - k_2 \zeta^2 \end{aligned} \quad (3.66)$$

can be made negative definite by allowing k_1 to be a function of x_1 and x_2 , using a nonlinear domination argument. Redesigning the classical backstepping control law u_b to accomplish this goal results in the final expression for the I&I stabilizer

$$\begin{aligned} u &= -x_2 - \left(\frac{\partial f_2}{\partial x_2} + k_1(x_1, x_2) + \frac{\partial k_1}{\partial x_2} x_2 \right) (\zeta - k_1(x_1, x_2) x_2) \\ &\quad - \frac{\partial k_1}{\partial x_1} (f_1(x_1, 0) + g_1(x_1, x_2) x_2) x_2 - k_2 \zeta \end{aligned} \quad (3.67)$$

which, written in terms of state variables, coincides with (3.59). This control law ensures the negativity of (3.66), for all $\|x_1\| \geq M$, with the choice

$$k_1(x_1, x_2) > \left\| \frac{\partial V}{\partial x_1} \right\|^2 \|g_1(x_1, x_2)\|^2 \geq 0. \quad (3.68)$$

Thus (3.59) ensures manifold attractivity and boundedness of the trajectories of (3.53), i.e. condition H4c, which concludes the proof. \square

Note that the continuous-time controller designed is zero when $\phi(x) = 0$. The same result holds in the case of x_2 having more than one dimension. Moreover, we focus on the single-input case, but extensions to vector inputs are straightforward and based on the multi-input version of backstepping design.

Chapter 4

Attitude stabilization of a flexible spacecraft

In this section we study the problem of robust attitude stabilization of a spacecraft with flexible appendages. The kinematic model is made more effective by using the modified Cayley-Rodrigues parameters for a global and non-redundant attitude representation. The parametric uncertainty in the coupling matrix of the flexible dynamics propagates unto the rigid dynamics and its destabilizing effects are counteracted using recursive Lyapunov redesign. Stability of the kinematic-flexible sub-system under a linear control law is demonstrated to simplify the subsequent recursive Lyapunov redesign. Application of robust nonlinear control results in practical robust uniform global asymptotic stability (P-RUGAS) of the closed loop system.

4.1 Dynamic model

The kinematic model used is based on the *modified Cayley-Rodrigues parameters*, which provide a global and non-redundant parametrization of the attitude of a rigid body [126]. In the following, we denote by $S(\cdot)$ the three-dimensional skew-symmetric matrix, which for a generic vector $r \in \mathbb{R}^3$ takes the form

$$S(r) = \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} \quad (4.1)$$

Defining $\rho \in \mathbb{R}^3$ the modified Cayley-Rodrigues parameters vector and $\omega \in \mathbb{R}^3$ the angular velocity in a body-fixed frame, the kinematic equations take the form

$$\dot{\rho} = H(\rho)\omega \quad (4.2)$$

where the matrix-valued function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ denotes the kinematic jacobian matrix of the modified Cayley-Rodrigues parameters, given by

$$H(\rho) = \frac{1}{2} \left(I - S(\rho) + \rho\rho^T - \frac{1 + \rho^T\rho}{2} I \right) \quad (4.3)$$

where I denotes the 3×3 identity matrix. The matrix $H(\rho)$ satisfies the following identity [127]

$$\rho^T H(\rho)\omega = \left(\frac{1 + \rho^T\rho}{4} \right) \rho^T \omega \quad (4.4)$$

for all $\rho, \omega \in \mathbb{R}^3$.

The flexible appendages of the satellite may be, for instance, solar arrays or antennas, as in the case of the *Mars Express Orbiter*, depicted in Fig. 4.1.

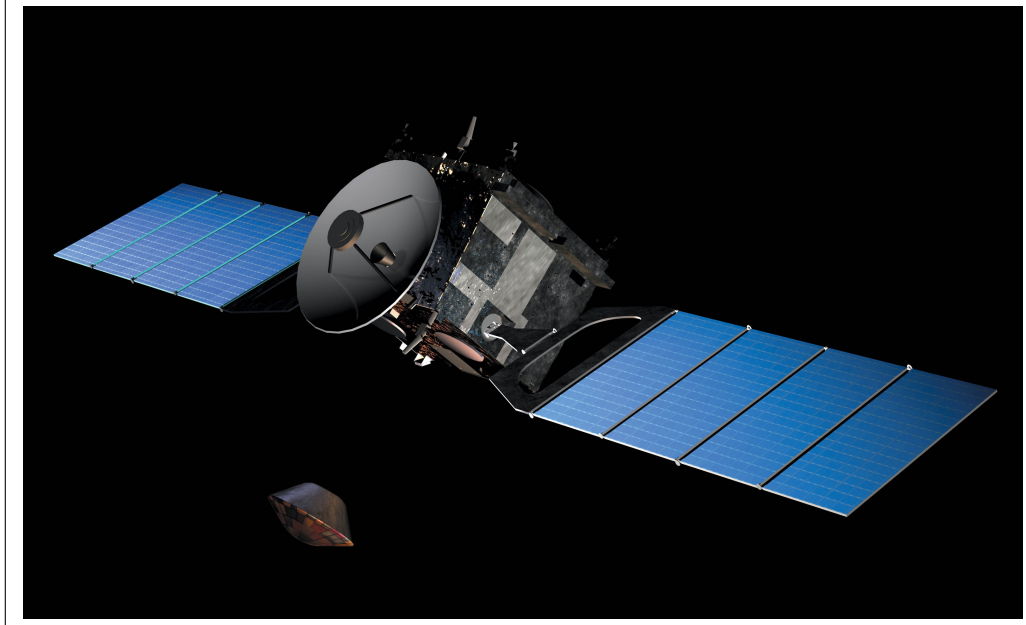


FIGURE 4.1: Artist's impression of the lander Beagle2 leaving the orbiter Mars Express (courtesy of ESA) ©.

The dynamical model of the spacecraft with flexible appendages is built employing the extended Euler laws, as done in [25]. Using the conservation of angular momentum, it is possible to write the coupled rigid and flexible equations as follows:

$$\begin{aligned} J\dot{\omega} + N^T\ddot{\eta} &= S(\omega) [J\omega + N^T\psi] + u \\ \ddot{\eta} + C\dot{\eta} + K\eta &= N\dot{\omega} \end{aligned} \quad (4.5)$$

where $u \in \mathbb{R}^3$ is the control torques vector and $\eta, \psi \in \mathbb{R}^M$ are the elastic coordinates, with $\psi = \dot{\eta}$, in the case of M flexible modes. The matrix $N \in \mathbb{R}^{M \times 3}$ is the coupling matrix between the attitude and the flexible motions, $C \in \mathbb{R}^{M \times M}$ is the damping matrix and $K \in \mathbb{R}^{M \times M}$ is the stiffness matrix. The inertia matrix, evaluated along the principal axes, is diagonal:

$$J = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix} \quad (4.6)$$

The angular velocity equation can be written as

$$\dot{\omega} = J_{eq}^{-1} [G + N^T (C\psi + K\eta - CN\omega) + u] \quad (4.7)$$

where $J_{eq} = J + N^T N$ is the equivalent inertia matrix and $G = S(\omega) [J_{eq}\omega + N^T \psi]$ is the gyroscopic term. Hence we obtain the overall dynamical model as follows:

$$\begin{aligned} \dot{\omega} &= J_{eq}^{-1} [S(\omega) (J_{eq}\omega + N^T \psi) + N^T (C\psi + K\eta - CN\omega) + u] \\ \dot{\eta} &= \psi - N\omega \\ \dot{\psi} &= -(C\psi + K\eta) + CN\omega \end{aligned} \quad (4.8)$$

The coupling matrix is assumed to be unknown, while we assume to have good estimates of the values of C and K . Following the results in [83], we decompose N into its nominal and its deterministic, uncertain, part, as follows:

$$N = N_n + \Delta N \quad (4.9)$$

The uncertain matrix ΔN is given by the element-wise Hadamard product, identified by the symbol \circ , of a matrix νN , whose elements are measures of the percentage of uncertainty $\nu \in [0, \nu_{\max}]$, with a matrix W , whose elements are sinusoidal functions of time at the different frequencies w_{ij} , representing the speed of variation of the actual values of the elements of N , n_{ij} , around their nominal values.

$$\Delta N = \begin{pmatrix} \nu n_{11} & \nu n_{12} & \nu n_{13} \\ \vdots & \vdots & \vdots \\ \nu n_{M1} & \nu n_{M2} & \nu n_{M3} \end{pmatrix} \circ \begin{pmatrix} \sin w_{11}t & \sin w_{12}t & \sin w_{13}t \\ \vdots & \vdots & \vdots \\ \sin w_{M1}t & \sin w_{M2}t & \sin w_{M3}t \end{pmatrix} = \nu N \circ W \quad (4.10)$$

The elements of the matrix W are indeed the deviation functions introduced in 2.3. For the sake of simplicity, but without any loss of generality, we consider the real number ν fixed for all the elements of N . We assume to know upper and lower bounds for ΔN ,

namely

$$\|\Delta N\|_2 = \sqrt{\lambda_{\max}(\Delta N^T \Delta N)} \leq \rho_N \quad (4.11)$$

$$\|\Delta N\|_{\circ} \leq \begin{pmatrix} \rho n_{11} & \rho n_{12} & \rho n_{13} \\ \vdots & \vdots & \vdots \\ \rho n_{M1} & \rho n_{M2} & \rho n_{M3} \end{pmatrix} = R_N \quad (4.12)$$

where $\rho n_{ij} \geq \nu n_{ij}$. For consistency of notation in the following calculations, we use two different kind of norms: the first is the classical induced \mathcal{L}_2 norm and the second is an element-wise norm suitable when dealing with Hadamard products.

4.2 Control design

System dynamics can be put in strict-feedback form in order to apply the backstepping stabilizing procedure. First, let us define the nominal system matrices as follows:

$$F = \begin{pmatrix} 0 & I_M \\ -K & -C \end{pmatrix} \quad B = \begin{pmatrix} -N \\ CN \end{pmatrix} \quad (4.13)$$

with I_M identity matrix of dimension M , and

$$f_n(\omega, \eta, \psi) = (J_{eq}^n)^{-1} [S(\omega) (J_{eq}^n \omega + N_n^T \psi) + N_n^T (C\psi + K\eta - CN_n \omega)] \quad (4.14)$$

where N_n is the nominal value of the decoupling matrix and J_{eq}^n the nominal value of the equivalent inertia matrix, namely $J_{eq}^n = J + N_n^T N_n$. Next, we define the uncertain terms in the dynamics propagating the uncertainty of the coupling matrix ΔN , obtaining:

$$\Delta f(\omega, \eta, \psi) = \tilde{\Delta} J_{eq}^{-1} [S(\omega) (\Delta J_{eq} \omega + \Delta N^T \psi) + \Delta N^T (C\psi + K\eta - C\Delta N \omega)] \quad (4.15)$$

where the uncertain term

$$\Delta J_{eq} = \Delta N^T N_n + N_n^T \Delta N + \Delta N^T \Delta N \quad (4.16)$$

is such that $J_{eq} = J_{eq}^n + \Delta J_{eq}$. Note that the term

$$\tilde{\Delta} J_{eq} = - (J_{eq}^n)^{-1} \left[(J_{eq}^n)^{-1} + \Delta J_{eq} \right]^{-1} (J_{eq}^n)^{-1} \quad (4.17)$$

comes from the direct application of the matrix inversion lemma to $(J_{eq}^n + \Delta J_{eq})^{-1}$. The kinematic and the dynamics of the flexible spacecraft can thus be written as follows:

$$\dot{\rho} = H(\rho)\omega \quad (4.18)$$

$$\begin{pmatrix} \dot{\eta} \\ \dot{\psi} \end{pmatrix} = F \begin{pmatrix} \eta \\ \psi \end{pmatrix} + B\omega \quad (4.19)$$

$$\dot{\omega} = f_n(\omega, \eta, \psi) + \Delta f(\omega, \eta, \psi) + \left[(J_{eq}^n)^{-1} + \tilde{\Delta}J_{eq} \right] u \quad (4.20)$$

Note that the matrix B still contains unmatched model uncertainties, but we will highlight them later in the design. System (4.18)-(4.19)-(4.20) satisfies the generalized matching conditions, thus it is possible to apply the recursive Lyapunov redesign procedure as one step of robust multi-input backstepping to compensate the unmatched uncertainties ΔN and Δf plus compensation of the matched uncertainty $\tilde{\Delta}J_{eq}$. We assume to know suitable bounding functions for the uncertain terms, namely

$$\|\Delta f(\omega, \eta, \psi)\| \leq \rho_f(\omega, \eta, \psi) \quad (4.21)$$

$$\|\tilde{\Delta}J_{eq}\|_1 \leq \frac{1}{3}\tilde{\rho}_J \quad 0 \leq \tilde{\rho}_J \leq 3 \quad (4.22)$$

where $\|\cdot\|_1$ stands for the matrix induced \mathcal{L}_1 norm.

4.2.1 Robust backstepping

Backstepping design is based on the observation that the kinematics and the flexible dynamics (4.18)-(4.19) can be stabilized by the virtual control law

$$\omega_d = -k\rho \quad (4.23)$$

with $k \in \mathbb{R}^+$. First note that, as a matter of fact, the drift term in equation (4.19) is globally exponentially stable, since the flexible dynamics is inherently damped. Thus, there exists a unique, symmetric and positive definite, solution P of the equation

$$P \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix} + \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix}^T P = -2Q \quad (4.24)$$

for each fixed matrix Q , symmetric and positive definite. Suppose we know a solution of the Lyapunov equation (4.24) and let us exploit identity (4.4) to show that, under the virtual control input (4.23), the kinematic equations are also GES. The closed-loop kinematic sub-system under virtual control becomes

$$\dot{\rho} = -kH(\rho)\rho \quad (4.25)$$

Consider now the Lyapunov function

$$V(\rho) = \rho^T \rho \quad (4.26)$$

Using (4.4), the derivative of V along the trajectories of (4.25) is given by

$$\dot{V} = -2k\rho^T H(\rho)\rho = -2k \left(\frac{1 + \rho^T \rho}{4} \right) \rho^T \rho \leq -\frac{k}{2} V \quad (4.27)$$

which yields global exponential stability with rate of decay $k/2$. It is then straightforward to show that, under the same virtual control law (4.23), the whole kinematic plus flexible sub-system is GES. The “virtual” closed-loop system takes the form

$$\dot{\rho} = -kH(\rho)\rho \quad (4.28)$$

$$\begin{pmatrix} \dot{\eta} \\ \dot{\psi} \end{pmatrix} = F \begin{pmatrix} \eta \\ \psi \end{pmatrix} + \begin{pmatrix} kN \\ -kCN \end{pmatrix} \rho. \quad (4.29)$$

Define $\xi = \text{col}(\eta, \psi)$ and consider the Lyapunov function

$$W = V(\rho) + \xi^T P \xi. \quad (4.30)$$

Evaluating the derivative of W along the trajectories of (4.28)-(4.29), we obtain

$$\begin{aligned} \dot{W} &= -k\rho^T H(\rho)\rho + 2\xi^T P \left[F\xi + \begin{pmatrix} kN \\ -kCN \end{pmatrix} \rho \right] \\ &\leq -\frac{k}{2}\rho^T \rho + 2\xi^T P F \xi + 2\xi^T P \begin{pmatrix} kN \\ -kCN \end{pmatrix} \rho \\ &\leq - \underbrace{\begin{pmatrix} \rho^T & \xi^T \end{pmatrix} \begin{bmatrix} \frac{k}{2}I & 0_{3 \times 2M} \\ 2kPB & 2Q \end{bmatrix} \begin{pmatrix} \rho \\ \xi \end{pmatrix}}_R \leq -\lambda_{\min}(R) \|\text{col}(\rho, \xi)\|^2 \end{aligned} \quad (4.31)$$

where $0_{3 \times 2M}$ is a block of zeros of dimension $3 \times 2M$, with M number of flexible modes. Clearly, the matrix $R \in \mathbb{R}^{(3+2M) \times (3+2M)}$ has all positive eigenvalues, hence global exponential stability of the kinematic plus flexible dynamics is established.

Remark 4.2.1.1. Note that the linear control law (4.23) achieves GES of (4.18)-(4.19) irrespective of the uncertain decoupling matrix N , thus for all the possible unmatched uncertainties acting on the kinematic-flexible subsystem. However, since (4.23) is just a virtual control law, we will have to deal with such unmatched terms in the actual robust backstepping design. \diamond

A graphical interpretation of this result can be given by looking at (4.28)-(4.29) as the cascade of two GES systems, the first nonlinear and the second linear, as shown in Fig. 4.2.

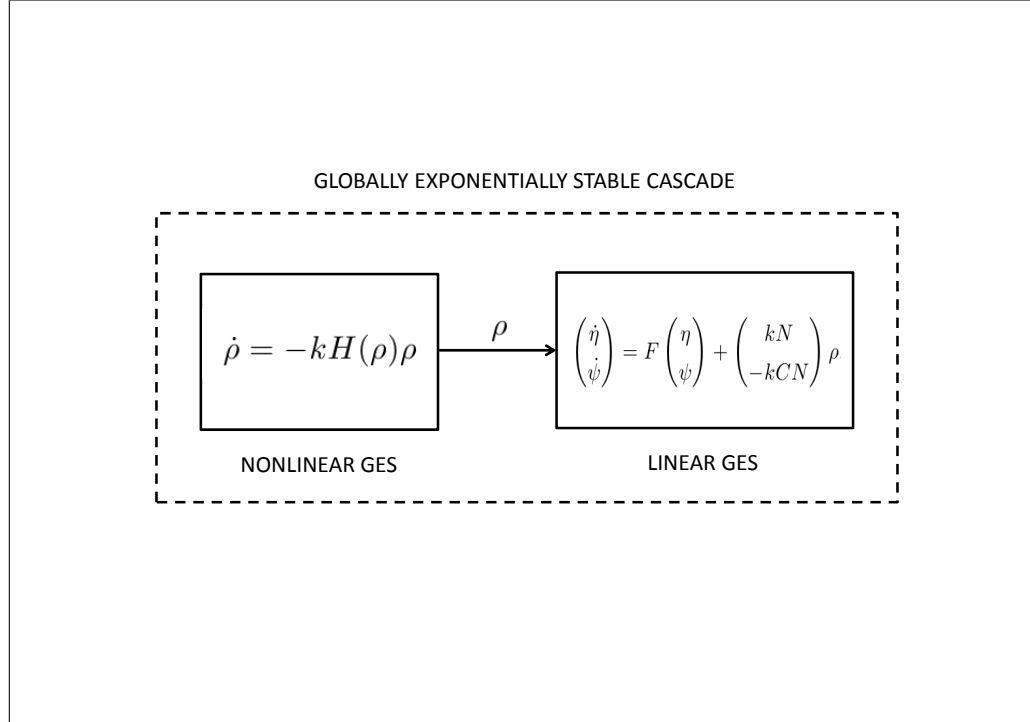


FIGURE 4.2: The nonlinear GES kinematic sub-system cascaded with the linear GES flexible sub-system yields a nonlinear GES cascade.

Linearity of the “sink” flexible sub-system implies the absence of finite-escape times, thus avoiding peaking during the transient and ensuring the GES of the cascade. Note that, if the second subsystem was nonlinear, we should have invoked the stronger property of input-to-state stability to ensure at least the GAS of the cascade.

The backstepping approach takes advantage of this result by introducing the error coordinate

$$\zeta = \omega - \omega_d = \omega + k\rho \quad (4.32)$$

and defining accordingly the mixed state-error system

$$\begin{aligned} \dot{\rho} &= H(\rho) [\zeta - k\rho] \\ \begin{pmatrix} \dot{\eta} \\ \dot{\psi} \end{pmatrix} &= F \begin{pmatrix} \eta \\ \psi \end{pmatrix} + B [\zeta - k\rho] \\ \dot{\zeta} &= f_n(\omega, \eta, \psi) + \Delta f(\omega, \eta, \psi) + (J_{eq}^n)^{-1} u + kH(\rho) [\zeta - k\rho] \end{aligned} \quad (4.33)$$

At this first step of the overall design, we disregard the matched uncertainty $\tilde{\Delta}J_{eq}$. Since we already know a Lyapunov function W for the (ρ, η, ψ) subsystem, we can build

a robust control Lyapunov function for (4.33) as follows

$$U = W(\rho, \xi) + \frac{1}{2} \|\zeta\|^2. \quad (4.34)$$

Evaluating the derivative of (4.34) along the trajectories of (4.33) we can find the robust backstepping control law which renders \dot{U} negative definite outside a compact set, which can be made arbitrarily small by properly choosing some control parameters (PRUGAS). The derivative of U takes the form:

$$\begin{aligned} \dot{U} &= \dot{W}(\rho, \xi) + \zeta^T \left[f_n(\omega, \eta, \psi) + \Delta f(\omega, \eta, \psi) + (J_{eq}^n)^{-1} u + kH(\rho) (\zeta - k\rho) \right] \\ &= \rho^T H(\rho) \zeta - k\rho^T H(\rho) \rho + 2\xi^T P F \xi + 2\xi^T P B \zeta - 2k\xi^T P B \rho \\ &\quad + \zeta^T \left[f_n(\omega, \eta, \psi) + \Delta f(\omega, \eta, \psi) + (J_{eq}^n)^{-1} u + kH(\rho) (\zeta - k\rho) \right] \\ &\leq - \begin{pmatrix} \rho^T & \xi^T \end{pmatrix} \begin{bmatrix} kH(\rho) & 0_{3 \times 2M} \\ 2kPB & 2Q \end{bmatrix} \begin{pmatrix} \rho \\ \xi \end{pmatrix} + \rho^T H(\rho) \zeta + 2\xi^T P B_n \zeta + 2\xi^T P \Delta B \zeta \\ &\quad + \zeta^T \left[f_n(\omega, \eta, \psi) + \Delta f(\omega, \eta, \psi) + (J_{eq}^n)^{-1} u + kH(\rho) (\zeta - k\rho) \right]. \end{aligned} \quad (4.35)$$

where the local variable ΔB depends solely on ΔN , namely

$$B = B_n + \Delta B = \begin{pmatrix} -N_n \\ C N_n \end{pmatrix} + \begin{pmatrix} -\Delta N \\ C \Delta N \end{pmatrix}. \quad (4.36)$$

The presence of this unmatched uncertainty in inequality (4.35) has to be dealt with robust backstepping design. We can define a bounding function for ΔB using an element-wise upper-bound estimate, as done previously for ΔN in (4.12), as follows

$$\|\Delta B\|_o \leq \rho_B \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & \vdots & \vdots \\ b_{2M1} & b_{2M2} & b_{2M3} \end{pmatrix} = R_B \quad (4.37)$$

The control law which solves this first stabilization step is now of straightforward evaluation:

$$\begin{aligned} u_u &= -J_{eq}^n [f_n(\omega, \eta, \psi) + \rho_f(\omega, \eta, \psi) \mathbf{sigm}(\zeta, \sigma_f) + kH(\rho) (\zeta - k\rho) \\ &\quad + H^T(\rho) \rho + 2B_n^T P \xi + 2R_B^T \circ \Sigma(\zeta, \sigma_B) P \xi + k_\zeta \zeta] \end{aligned} \quad (4.38)$$

where $k_\zeta \in \mathbb{R}^+$ is a design gain introduced to force the negativity of \dot{U} , $\sigma_B, \sigma_f \in \mathbb{R}^+$ are the sigmoid slopes and the vector sigmoid functions take the form

$$\mathbf{sigm}(\zeta, \sigma_f) = \begin{pmatrix} \text{sign}(\zeta_1)(1 - e^{-\sigma_f|\zeta_1|}) \\ \text{sign}(\zeta_2)(1 - e^{-\sigma_f|\zeta_2|}) \\ \text{sign}(\zeta_3)(1 - e^{-\sigma_f|\zeta_3|}) \end{pmatrix} \quad (4.39)$$

$$\mathbf{\Sigma}(\zeta, \sigma_B) = \begin{pmatrix} \text{sign}(\zeta_1)(1 - e^{-\sigma_B|\zeta_1|}) & \dots & \text{sign}(\zeta_1)(1 - e^{-\sigma_B|\zeta_1|}) \\ \text{sign}(\zeta_2)(1 - e^{-\sigma_B|\zeta_2|}) & \dots & \text{sign}(\zeta_2)(1 - e^{-\sigma_B|\zeta_2|}) \\ \text{sign}(\zeta_3)(1 - e^{-\sigma_B|\zeta_3|}) & \dots & \text{sign}(\zeta_3)(1 - e^{-\sigma_B|\zeta_3|}) \end{pmatrix} \quad (4.40)$$

Note that $\mathbf{\Sigma}(\zeta, \sigma_B) \in \mathbb{R}^{3 \times 2M}$ is introduced for consistency purposes to compensate the unmatched uncertainty ΔB . For this reason, we need to use the element-wise (Hadamard) matrix product, identified by the symbol \circ , to bound from above the derivative of U using the control.

Under (4.38), (4.35) can be reduced to

$$\begin{aligned} \dot{U} \leq & - \begin{pmatrix} \rho^T & \xi^T \end{pmatrix} R \begin{pmatrix} \rho \\ \xi \end{pmatrix} - k_\zeta \|\zeta\|^2 \\ & + 2\rho_B \zeta^T B_n^T \circ \mathbf{RES}_B(|\zeta_i|, \sigma_B) P \xi + \rho_f(\omega, \eta, \psi) \sum_{i=1}^3 |\zeta_i| e^{-\sigma_f|\zeta_i|} \end{aligned} \quad (4.41)$$

where the matrix $\mathbf{RES}_B(|\zeta_i|, \sigma_B) \in \mathbb{R}^{3 \times 2M}$ has for elements the approximation errors made in using sigmoid functions instead of discontinuities, namely

$$\mathbf{RES}_B(|\zeta_i|, \sigma_B) = \begin{pmatrix} \text{sign}(\zeta_1)e^{-\sigma_B|\zeta_1|} & \dots & \text{sign}(\zeta_1)e^{-\sigma_B|\zeta_1|} \\ \text{sign}(\zeta_2)e^{-\sigma_B|\zeta_2|} & \dots & \text{sign}(\zeta_2)e^{-\sigma_B|\zeta_2|} \\ \text{sign}(\zeta_3)e^{-\sigma_B|\zeta_3|} & \dots & \text{sign}(\zeta_3)e^{-\sigma_B|\zeta_3|} \end{pmatrix}. \quad (4.42)$$

A further simplification of (4.41) leads to

$$\begin{aligned} \dot{U} \leq & - \begin{pmatrix} \rho^T & \xi^T \end{pmatrix} R \begin{pmatrix} \rho \\ \xi \end{pmatrix} - k_\zeta \|\zeta\|^2 \\ & + 2\rho_B \mathbf{res}_B(|\zeta_i|, \sigma_B) P \xi + \rho_f^{\max} \sum_{i=1}^3 |\zeta_i| e^{-\sigma_f|\zeta_i|} \end{aligned} \quad (4.43)$$

where ρ_f^{\max} is a upper-bound on ρ_f for all $\omega \in \mathbb{R}^3$, $(\eta, \psi) \in \mathbb{R}^{2M}$ and the row vector $\mathbf{res}_B(|\zeta_i|, \sigma_B) \in \mathbb{R}^{2M}$ can be expressed as

$$\mathbf{res}_B(|\zeta_i|, \sigma_B) = \left(\sum_{i=1}^3 b_{1i} |\zeta_i| e^{-\sigma_B|\zeta_i|} \quad \sum_{i=1}^3 b_{2i} |\zeta_i| e^{-\sigma_B|\zeta_i|} \quad \dots \quad \sum_{i=1}^3 b_{2Mi} |\zeta_i| e^{-\sigma_B|\zeta_i|} \right) \quad (4.44)$$

It is easy to show that the terms $|\zeta_i|e^{-\sigma_B|\zeta_i|}$ and $|\zeta_i|e^{-\sigma_f|\zeta_i|}$ attain a maximum at $|\zeta_i| = \sigma_B^{-1}$ and $|\zeta_i| = \sigma_f^{-1}$, respectively. Moreover, since P is known and constant and ξ are the flexible variables, bounded solutions of a linear damped system, it is again possible to bound from above \dot{U} as follows

$$\begin{aligned} \dot{U} \leq & - \begin{pmatrix} \rho^T & \xi^T \end{pmatrix} R \begin{pmatrix} \rho \\ \xi \end{pmatrix} - k_\zeta \|\zeta\|^2 \\ & + 2\rho_B \mathbf{res}^{\max}(\sigma_B) P \xi^{\max} + \rho_f^{\max} 3\sigma_f^{-1} e^{-1} \end{aligned} \quad (4.45)$$

where

$$\mathbf{res}^{\max}(\sigma_B) = \begin{pmatrix} \sigma_B^{-1} e^{-1} \sum_{i=1}^3 b_{1i} & \sigma_B^{-1} e^{-1} \sum_{i=1}^3 b_{2i} & \cdots & \sigma_B^{-1} e^{-1} \sum_{i=1}^3 b_{2Mi} \end{pmatrix} \quad (4.46)$$

Note that the non-negative terms in (4.45) can be shrunk by increasing the sigmoid slopes σ_B and σ_f . Let us rearrange the negative terms defining $\tilde{x} = \text{col}(\rho, \xi, \zeta)$, obtaining

$$\begin{aligned} \dot{U} & \leq -\tilde{x}^T \begin{bmatrix} R & 0 \\ 0 & k_\zeta I \end{bmatrix} \tilde{x} + 2\rho_B \mathbf{res}^{\max}(\sigma_B) P \xi^{\max} + \rho_f^{\max} 3\sigma_f^{-1} e^{-1} \\ & \leq -\min\{\lambda_{\min}(R), k_\zeta\} \|\tilde{x}\|^2 + 2\rho_B \mathbf{res}^{\max}(\sigma_B) P \xi^{\max} + \rho_f^{\max} 3\sigma_f^{-1} e^{-1}. \end{aligned} \quad (4.47)$$

As a result, \dot{U} is negative definite provided that

$$\|\tilde{x}\| > \sqrt{\frac{2\rho_B \mathbf{res}^{\max}(\sigma_B) P \xi^{\max} + \rho_f^{\max} 3\sigma_f^{-1} e^{-1}}{\min\{\lambda_{\min}(R), k_\zeta\}}} := \Omega(\sigma_B, \sigma_f) \quad (4.48)$$

thus P-RUGAS of the full-state trajectories is established, since they converge to an “arbitrarily shrinkable” compact set containing the origin.

4.2.2 Recursive Lyapunov redesign

As we have shown in the previous section, the robust backstepping design stabilizes the flexible spacecraft dynamics by compensating the unmatched uncertainties. To deal with the matched uncertainty $\tilde{\Delta}J_{eq}$, of which we know the bounding term $\tilde{\rho}_J$, it is straightforward to use the recursive Lyapunov redesign introduced in section 3.3. The robust backstepping control law (4.38) can be expressed as a function of the full-state $x = \text{col}(\rho, \xi, \omega)$, namely

$$u_u = \psi(\rho, \xi, \omega) = \psi(x). \quad (4.49)$$

Moreover, we group system equations and matrices defining

$$F(x) = \begin{pmatrix} H(\rho)\rho \\ F\xi + B\omega \\ f(\xi, \omega) \end{pmatrix} \quad G = \begin{pmatrix} 0_{3 \times 3} \\ 0_{2M \times 3} \\ (J_{eq}^n)^{-1} \end{pmatrix} \quad (4.50)$$

so that the full flexible-spacecraft dynamics can be expressed as

$$\dot{x} = F(x) + G [\tilde{\Delta} J_{eq} u + u] \quad (4.51)$$

Recall that we know a robust control Lyapunov function for system (4.51) in the case $\tilde{\Delta} J_{eq} = 0$, namely

$$U(\rho, \xi, \omega) = \rho^T \rho + \xi^T P \xi + \|\omega + k\rho\|^2. \quad (4.52)$$

Under the robust backstepping control law (4.49), $U(x)$ satisfies inequality (4.45), so that (4.51) is P-RUGAS with $\tilde{\Delta} J_{eq} = 0$. Moreover, the matched uncertainty $\tilde{\Delta} J_{eq}$ is such that the inequality

$$\left\| \tilde{\Delta} J_{eq}(\psi(x) + u_m) \right\|_1 \leq \rho_r(x) + \frac{1}{3} \tilde{\rho}_J \|u_m\|_1 \quad 0 \leq \tilde{\rho}_J < 3 \quad (4.53)$$

holds, with u_m recursive Lyapunov redesign control law to be determined and $\rho_r(x) \geq \left\| \tilde{\Delta} J_{eq} \psi(x) \right\|_1$ is a suitable bounding function. Since assumptions 3.3.0.8 and 3.3.0.9 are fulfilled, we can apply the recursive Lyapunov redesign procedure to system (4.51), which in closed loop with $u = \psi(x) + u_m$ takes the form

$$\dot{x} = f(x) + G\psi(x) + G [\tilde{\Delta} J_{eq}(\psi(x) + u_m) + u_m]. \quad (4.54)$$

Set $\left(\frac{\partial U}{\partial x} \right)^T = w$. The derivative of U along the trajectories of (4.54) takes the form

$$\begin{aligned} \dot{U} &= \left(\frac{\partial U}{\partial x} \right)^T (f(x) + G\psi(x)) + w \tilde{\Delta} J_{eq}(\psi(x) + u_m) + w u_m \\ &\leq \left(\frac{\partial U}{\partial x} \right)^T (f(x) + G\psi(x)) + \|w\|_1 \left\| \tilde{\Delta} J_{eq}(\psi(x) + u_m) \right\|_1 + w u_m \\ &\leq -\min \{ \lambda_{\min}(R), k_\zeta \} \|\tilde{x}\|^2 + \text{res}_u(\sigma_B, \sigma_f) + \|w\|_1 (\rho_r(x) + \tilde{\rho}_J \|u_m\|_1) + w u_m \end{aligned} \quad (4.55)$$

where $\text{res}_u(\sigma_B, \sigma_f)$ is the residual term in (4.45), namely

$$\text{res}_u(\sigma_B, \sigma_f) = 2\rho_B \mathbf{res}^{\max}(\sigma_B) P \xi^{\max} + \rho_f^{\max} 3\sigma_f^{-1} e^{-1}. \quad (4.56)$$

Setting now

$$u_m = \frac{\rho_r(x)}{1 - \tilde{\rho}_J} \mathbf{sigm}(w, \sigma_J) \quad (4.57)$$

and accordingly evaluating the induced 1-norms in (4.55), yields

$$\begin{aligned}
\dot{U} \leq & -\min\{\lambda_{\min}(R), k_{\zeta}\} \|\tilde{x}\|^2 + \text{res}_u(\sigma_B, \sigma_f) \\
& + \|w\|_1 \left[\rho_r(x) + \frac{1}{3} \tilde{\rho}_J \left(\frac{3\rho_r(x)}{1 - \tilde{\rho}_J} + \frac{3\rho_r(x)}{1 - \tilde{\rho}_J} \sum_{i=1}^3 e^{-\sigma_J |w_i|} \right) \right] \\
& + w \left[\frac{\rho_r(x)}{1 - \tilde{\rho}_J} \begin{pmatrix} \text{sign}(w_1) e^{-\sigma_J |w_1|} \\ \text{sign}(w_2) e^{-\sigma_J |w_2|} \\ \text{sign}(w_3) e^{-\sigma_J |w_3|} \end{pmatrix} - \frac{\rho_r(x)}{1 - \tilde{\rho}_J} \begin{pmatrix} \text{sign}(w_1) \\ \text{sign}(w_2) \\ \text{sign}(w_3) \end{pmatrix} \right]
\end{aligned} \tag{4.58}$$

Simplifying and taking the maximum values of the residual terms, we obtain

$$\dot{U} \leq -\min\{\lambda_{\min}(R), k_{\zeta}\} \|\tilde{x}\|^2 + \text{res}_u(\sigma_B, \sigma_f) + 9 \frac{\tilde{\rho}_J \rho_r^{\max}}{1 - \tilde{\rho}_J} \sigma_J^{-1} e^{-1} + 3 \frac{\rho_r^{\max}}{1 - \tilde{\rho}_J} \sigma_J^{-1} e^{-1} \tag{4.59}$$

so that the trajectories of the closed-loop system (4.54) converge to the compact set

$$\tilde{\Omega}(\sigma_B, \sigma_f, \sigma_J) := \sqrt{\frac{\text{res}_u(\sigma_B, \sigma_f) + 9 \frac{\tilde{\rho}_J \rho_r^{\max}}{1 - \tilde{\rho}_J} \sigma_J^{-1} e^{-1} + 3 \frac{\rho_r^{\max}}{1 - \tilde{\rho}_J} \sigma_J^{-1} e^{-1}}{\min\{\lambda_{\min}(R), k_{\zeta}\}}}. \tag{4.60}$$

P-RUGAS of the origin of (4.54) can thus be concluded. As a result, the flexible spacecraft described by the equations (4.18)-(4.19)-(4.20) is robustly stabilized by the recursive Lyapunov redesign control law $u = u_u(x) + u_m(x)$, with $u_u(x)$ defined in (4.38) and $u_m(x)$ in (4.57).

Chapter 5

Missile autopilot design

In this chapter, the recursive Lyapunov redesign technique is employed to design the autopilot of an asymmetric tail-controlled air-to-air missile, in order for it to perform Skid-To-Turn and Bank-To-Turn maneuvers. After a detailed derivation of the dynamical uncertain model, the problem setting and the construction of a nominal backstepping controller, we design compensators for the matched and unmatched uncertainties of the dynamics using the techniques introduced above. The recursive Lyapunov redesign can handle the uncertainties in the dynamics of a non-minimum phase airframe, thus it is particularly suited for missile autopilot design. P-RUGAS of the closed-loop system is shown and several simulations demonstrate the effectiveness of the proposed control law in different scenarios. The chapter follows the work presented in [86].

5.1 Dynamic model

The missile considered is a very generic, tail-controlled, air-to-air aerodynamic missile in a non axially-symmetric configuration which allows both Skid-To-Turn and Bank-To-Turn maneuvers. An example of such an airframe may be *Meteor* (Fig. 5.1), the innovative active radar guided beyond-visual-range air-to-air missile (BVRAAM) produced by MBDA. While wings act as lifting surfaces, the four tail-fins are the control surfaces which twist and steer the vehicle. The typical use of this kind of missile is that of intercepting aircraft, anti-aircraft missiles and high-speed ballistic missiles in flight. The hybrid maneuvering capabilities of the airframe do require an autopilot able to ensure tracking of different kind of signals on different dynamic behaviors, with slight or even none change of the control laws. These requirements can be fulfilled by using the recursive Lyapunov redesign approach.

The modeling process is based on some quite common assumptions.

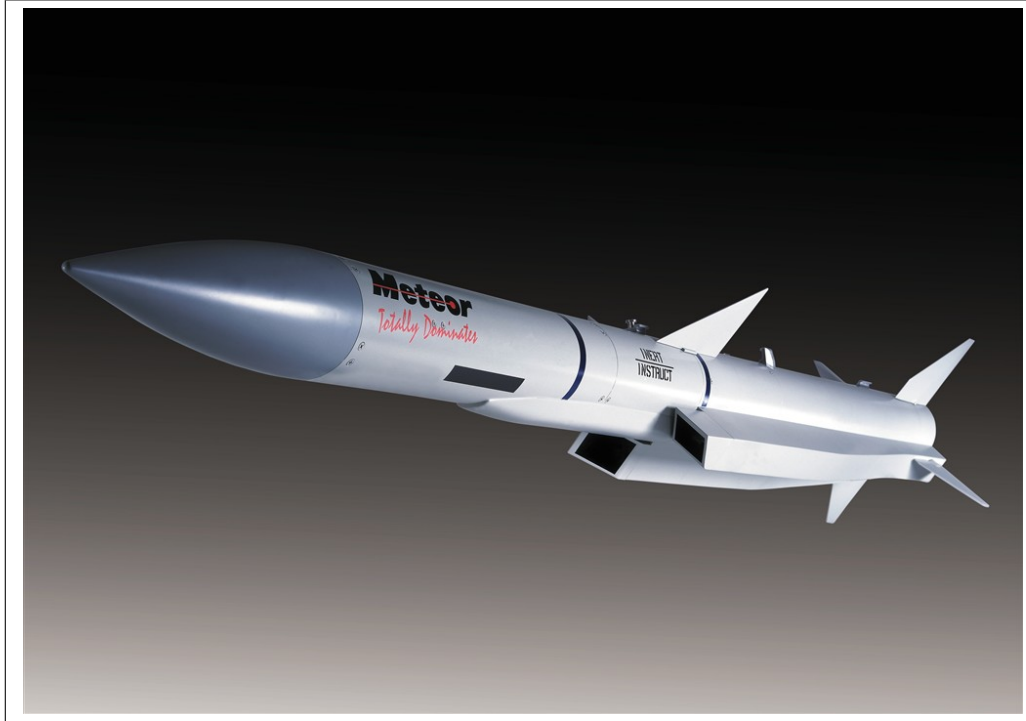


FIGURE 5.1: *Meteor* is an active radar guided beyond-visual-range air-to-air missile (BVRAAM) being developed by MBDA missile systems. *Meteor* will offer a multi-shot capability against long range manoeuvring targets in a heavy electronic countermeasures (ECM) environment with range in excess of 100km. The picture is taken from defenseindustrydaily.com, all rights reserved ©.

- The considered dynamics is governed by rigid body mechanics, that means flexible modes due to aeroelasticity are neglected.
- Reference position for aerodynamic data coincides with the center of mass (abbr. c.o.m.), where all the sensors are mounted.
- Trustworthy measurements or estimates for the angle-of-attack, sideslip angle, angular velocities and bank angle are available through high quality inertial measurement units.
- Thrust force is assumed to be zero, or equivalently fully compensated by drag force.
- As a consequence of the previous assumption, velocity is assumed to be slowly varying (increasing) for the first half-window of simulation and constant during the terminal phase. Wind velocity will be treated as a disturbance.
- Gravity is neglected since it acts on time-scales much longer than those of the autopilot.

- We also assume constant height of operation h , which determines certain known thermodynamic characteristics of the local atmosphere, such as constant air density $\bar{\rho}$ and speed of sound \bar{V}_s .

Note that the presence of aeroelastic effects can be treated, as usual, in the frequency-domain framework, taking care to give the appropriate open-loop attenuation near the flex-mode frequencies (see, for instance, [94] and [95]). On the other hand, a reference position shifted from the c.o.m. does not pose a problem, since it can be compensated by introducing simple correction terms. Furthermore, while the angular velocities are usually available for feedback, α and β angles need to be estimated somehow, since the nonlinear control law developed is in the form of full state-feedback. Finally, last four assumptions are quite usual in autopilot design. In supersonic and hypersonic regimes, particularly in the terminal phases of the flight, it is not such an issue to consider zero-gravity conditions and the missile operating in the so-called *glide-phase*, at zero thrust.

Starting from the Newton linear momentum and the Euler angular momentum laws evaluated along the missile body unit-axes $(\hat{x}_b, \hat{y}_b, \hat{z}_b)^T$ (body frame fixed on the c.o.m.), introducing the inertia tensor and differentiating the equations using the Coriolis theorem, the 6-DOF equations of motion of a rigid body with slowly varying mass and inertia are obtained. These six equations are used to represent missile dynamics in the wind-axes reference frame, after a coordinate transformation [20]. Velocity $V_m(t)$ is assumed to be slowly and linearly increasing in the first part of the simulation window, according to the following Mach number variation:

$$M(t) \in [2.5, 3.2] \quad (5.1)$$

in such a way that $V_0 = 2.5\bar{V}_s$ and $V_f = 3.2\bar{V}_s$. The second half of the flight envelope is therefore with constant velocity V_f .

A non axially-symmetric airframe is considered, with inertia matrix (determined considering the principal inertia axes coincident with the body axes):

$$I(t) = \begin{pmatrix} I_x(t) & 0 & -I_{xz}(t) \\ 0 & I_y(t) & 0 \\ -I_{xz}(t) & 0 & I_z(t) \end{pmatrix} \quad (5.2)$$

also slowly time-varying, in particular linearly decreasing in the first part of the flight envelope due to propellant consumption. For instance, we consider this time profile for

the longitudinal inertia moment:

$$I_x(t) = \begin{cases} -\left(\frac{I_{x0}-I_{xf}}{t_f}\right)t + I_{x0} & \text{if } t \leq t_f \\ I_{xf} & \text{if } t > t_f \end{cases} \quad (5.3)$$

A similar time evolution has been assigned to the slowly time-varying mass $m(t)$. At $t = t_f$, when all the propellant has been consumed, the so-called glide-phase starts: the velocity becomes constant and so the inertia matrix and the mass of the system.

The complete 6-DOF equations of motion are therefore:

$$\begin{aligned} \dot{\phi} &= p \\ \dot{p} &= L_{pq}pq + L_{qr}qr + \frac{I_z}{I_x I_z - I_{xz}^2} M_x + \frac{I_{xz}}{I_x I_z - I_{xz}^2} M_z \\ \dot{q} &= M_{pr}pr + M_{p^2}p^2 + M_{r^2}r^2 + \frac{1}{I_y} M_y + C_{m1}p\alpha \\ \dot{r} &= N_{pq}pq + N_{qr}qr + \frac{I_{xz}}{I_x I_z - I_{xz}^2} M_x + \frac{I_x}{I_x I_z - I_{xz}^2} M_z + C_{m2}p\beta \\ \dot{\alpha} &= -\cos^2(\alpha) \tan(\beta)p + q - \sin(\alpha) \cos(\alpha) \tan(\beta)r + C_{m3}p\alpha + \\ &\quad \frac{Q_{\alpha\beta}}{mV_m} F_z + \sin(\alpha) \cos(\alpha) \tan(\beta) \frac{Q_{\alpha\beta}}{mV_m} F_y - \sin(\alpha) \cos(\alpha) Q_{\alpha\beta}^2 \frac{\dot{V}_m}{V_m} \\ \dot{\beta} &= \cos^2(\beta) \tan(\alpha)p + \sin(\beta) \cos(\beta) \tan(\alpha)q - r + C_{m4}p\beta + \\ &\quad \frac{Q_{\alpha\beta}}{mV_m} F_y + \sin(\beta) \cos(\beta) \tan(\alpha) \frac{Q_{\alpha\beta}}{mV_m} F_z - \sin(\beta) \cos(\beta) Q_{\alpha\beta}^2 \frac{\dot{V}_m}{V_m} \end{aligned} \quad (5.4)$$

with

$$L_{pq} = \frac{I_{xz}(I_x - I_y + I_z)}{I_x I_z - I_{xz}^2} \quad N_{pq} = \frac{I_x(I_x - I_y) + I_{xz}^2}{I_x I_z - I_{xz}^2} \quad (5.5)$$

$$L_{qr} = \frac{I_z(I_y - I_z) - I_{xz}^2}{I_x I_z - I_{xz}^2} \quad N_{qr} = \frac{I_{xz}(I_y - I_z - I_x)}{I_x I_z - I_{xz}^2} \quad (5.6)$$

$$M_{pr} = \frac{I_z - I_x}{I_y} \quad M_{p^2} = -\frac{I_{xz}}{I_y} \quad M_{r^2} = \frac{I_{xz}}{I_y} \quad (5.7)$$

and

$$Q_{\alpha\beta} = \sqrt{1 + \tan^2(\alpha) + \tan^2(\beta)} \quad (5.8)$$

The overall velocity derivative w.r.t. time \dot{V}_m is thought as an additional input, because a fixed velocity profile has been imposed. Actually, since it is not going to be taken into account in control design, it will act as a disturbance. Aerodynamic forces and moments

have the following expressions:

$$\begin{aligned}
F_y &= \frac{1}{2}\rho V_m^2 S C_{Ny} \\
F_z &= \frac{1}{2}\rho V_m^2 S C_{Nz} \\
M_x &= \frac{1}{2}\rho V_m^2 S d (C_l + G_p) \\
M_y &= \frac{1}{2}\rho V_m^2 S d (C_m + G_q) \\
M_z &= \frac{1}{2}\rho V_m^2 S d (C_n + G_r)
\end{aligned} \tag{5.9}$$

They depend on the aerodynamic coefficients, which can be written this way:

$$\begin{aligned}
C_{Ny} &= a_{y1}\beta + a_{y2}\beta|\beta| + a_{y3}\beta^2 + a_{y4}\beta^3 + (b_{y1}\beta + b_{y2}\beta^2)\delta_P + \\
&\quad (b_{y3}\beta + b_{y4}\beta^2)\delta_Y \\
C_{Nz} &= a_{z1}\alpha + a_{z2}\alpha|\alpha| + a_{z3}\alpha^2 + a_{z4}\alpha^3 + (b_{z1}\alpha + b_{z2}\alpha^2)\delta_P + \\
&\quad (b_{z3}\alpha + b_{z4}\alpha^2)\delta_Y \\
C_l &= \frac{d}{2V_m} C_{l_p} p \\
G_p &= L_{\delta_R}(\alpha, \beta)\delta_R + L_{\delta_P}(\alpha)\delta_P + L_{\delta_Y}(\beta)\delta_Y \\
C_m &= C_{m_\alpha}\phi_m(\alpha) + C_{m_{\alpha\beta}}\alpha\beta^2 + \frac{d}{2V_m} C_{m_q} q + \frac{d}{2V_m} C_{n_{p\beta}}\beta p \\
G_q &= C_{m_{\delta_P}}\delta_P \\
C_n &= C_{n_\beta}\phi_n(\beta) + C_{n_{\alpha\beta}}\alpha^2\beta + \frac{d}{2V_m} C_{n_r} r + \frac{d}{2V_m} C_{n_{p\alpha}}\alpha p \\
G_r &= N_{\delta_Y}\delta_Y
\end{aligned} \tag{5.10}$$

and:

$$\begin{aligned}
L_{\delta_R}(\alpha, \beta) &= d_1 + d_2\alpha^2 + d_3\beta^2 \\
L_{\delta_P}(\alpha) &= d_4\alpha \\
L_{\delta_Y}(\beta) &= d_5\beta \\
N_{\delta_Y} &= C_{n_{\delta_Y}} \\
\phi_n(\beta) &= n_1\beta + n_2\beta|\beta| + n_3\beta^2 + n_4\beta^3 \\
\phi_m(\alpha) &= m_1\alpha + m_2\alpha|\alpha| + m_3\alpha^2 + m_4\alpha^3
\end{aligned} \tag{5.11}$$

We would like to stress the presence, in the equations of motion, of the *Magnus effect aerodynamic terms*: C_{m1} , C_{m2} , C_{m3} and C_{m4} . These terms describe the fact that spinning missiles deviate from a rectilinear trajectory in the direction given by the spin. This is clearly going to be considered during control system design, since it is a nonlinear coupling dangerous for stability which can be encountered during bank maneuvers.

The same can be said about the nonlinear cross-couplings which arise between the three control inputs in the expressions of (5.10). These couplings are produced by the differential of pressure acting on the wings and tail-fins, a phenomenon particularly relevant in asymmetric airframes.

Finally, control allocation and actuator dynamics are introduced to make the dynamic model even more realistic. Control inputs produced by the autopilot have to be transformed in physical reference signals to the four actuated control surfaces placed on the tail-fins of the missile. In missile dynamics, only the overall deflections produced by the simultaneous combined deflection of the surfaces are considered, and the same is made in autopilot design. These three overall deflections are the classical roll, pitch and yaw ones around the three body axes, namely:

$$\begin{aligned}\delta_R &= u_1 \\ \delta_P &= u_2 \\ \delta_Y &= u_3\end{aligned}\tag{5.12}$$

These deflections can be thought as commands to tail-fin dynamics, conceptually separated in an *aileron* dynamics, which produces as a final result the roll control torque, an *elevator* dynamics, which is the cause of the pitch control torque, and a *rudder* dynamics, producing a yaw control torque. Due to this fact, control inputs generated by the autopilot are often renamed:

$$\begin{aligned}\delta_R &= \delta_a \quad (\text{command to aileron dynamics}) \\ \delta_P &= \delta_e \quad (\text{command to elevator dynamics}) \\ \delta_Y &= \delta_r \quad (\text{command to rudder dynamics})\end{aligned}\tag{5.13}$$

The transformation between these overall control deflections elaborated by the autopilot and the actual signals to the four wings $(\delta_{w1}, \delta_{w2}, \delta_{w3}, \delta_{w4})^T$ is represented by the *control allocation matrix*:

$$\begin{bmatrix} \delta_{w1} \\ \delta_{w2} \\ \delta_{w3} \\ \delta_{w4} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\tag{5.14}$$

Actuator dynamics on the four fins is an approximation of the fast dynamics of low level control loops, whose aim is to produce torques on the actuators moving the fins in order to follow the reference time-profile associated to δ_{wi} . After a small time, these low level control loops achieve their aim and produce the desired torques, giving as output the

desired δ_i :

$$\delta_i = \frac{1}{1 + \tau_s} \delta_{wi} \quad (5.15)$$

The *time constant* τ is a measure of the speed of response of actuators. Between the commands δ_{wi} and the inputs δ_i there are also a *saturation limit* of ± 45 deg and a *rate limit* of 500 deg per second. These inputs are therefore anti-transformed in overall input torques to airframe dynamics by pseudo-inverting the control allocation matrix

$$\begin{bmatrix} \delta_R \\ \delta_P \\ \delta_Y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \quad (5.16)$$

Actuator dynamics and control transformation are not considered as part of the control design process, therefore they are simply part of the overall simulation model, acting as unmodeled dynamics of the system.

5.1.1 Modeling for nominal control

Equations of motion (5.4) can be rearranged to form the following nonlinear dynamical system

$$\begin{aligned} \dot{x} &= f_x(x, t) + g_x^1(x)z + g_x^2(x, z, t)u \\ \dot{z} &= f_z(x, z, t) + g_z(x, t)u \end{aligned} \quad (5.17)$$

The state variables are $x = (\varphi, \alpha, \beta)^T$ and $z = (p, q, r)^T$. Control inputs, represented by the tail fins' deflection angles, are $u = (\delta_R, \delta_P, \delta_Y)^T$. The vector-fields and matrices

of the system have this structure:

$$\begin{aligned}
 f_x(x, t) &= \begin{pmatrix} 0 \\ f_\alpha(x, t) \\ f_\beta(x, t) \end{pmatrix} & g_x^1(x) &= \begin{pmatrix} 1 & 0 & 0 \\ \xi_\alpha(x) & 1 & \chi_\alpha(x) \\ \xi_\beta(x) & \chi_\beta(x) & -1 \end{pmatrix} \\
 g_x^2(x, t) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{\alpha 1}(x, z, t) & g_{\alpha 2}(x, z, t) \\ 0 & g_{\beta 1}(x, z, t) & g_{\beta 2}(x, z, t) \end{pmatrix} \\
 f_z(x, z, t) &= \begin{pmatrix} f_p(x, z, t) \\ f_q(x, z, t) \\ f_r(x, z, t) \end{pmatrix} \\
 g_z(x, t) &= \begin{pmatrix} g_{p1}(x, t) & g_{p2}(x, t) & g_{p3}(x, t) \\ 0 & g_q(x, t) & 0 \\ g_{r1}(x, t) & g_{r2}(x, t) & g_{r3}(x, t) \end{pmatrix}
 \end{aligned} \tag{5.18}$$

The functions f_i , g_i , ξ_i and χ_i are easily constructed by looking at the dynamics of (5.4). System (5.17) well represents the complex nonlinear and time-varying behavior of the missile in the simulation window considered. It involves highly nonlinear non-stationary aerodynamic terms, mainly in the f_i drift terms, highly coupled and nonlinear differential kinematic terms as $g_x^1(x)z$ and, furthermore, the cross-couplings between the control inputs, due to the different pressure acting on the wings and the tail-fins during maneuvers, especially at high α and β angles. These couplings are going to be taken into account directly in control system design, using a nonlinear MIMO approach.

For the nominal controller design, system equations are going to be simplified into

$$\begin{aligned}
 \dot{x} &= f_x(x) + g_x^1(x)z \\
 \dot{z} &= f_z(x, z) + g_z(x)u
 \end{aligned} \tag{5.19}$$

in which we have dropped the input influence on the second and third equations and we have made the assumption of constant velocity, mass and inertia during the portion of flight considered.

5.1.2 Modeling for robust control

The nonlinear dynamical model developed in the previous section is particularly suited to design nonlinear control laws which are based on *cancellations* of nominal terms in system equations. However, it is really difficult to obtain good estimates of aerodynamic coefficients and Magnus effect terms, which are barely known. Even in the luckier

situations the designer only knows their nominal, often incorrect, value. As a consequence, nonlinear control laws are frequently failing in real applications. This issue led the control systems research community to think about robust versions of the nonlinear controllers already available for stabilization and tracking.

To model the very complex aerodynamic uncertain terms of missile dynamics, we need to exploit the results introduced in Section 2.3 to rapidly evaluate the uncertain envelopes. As an example of application of proposition 2.3.0.2, let us consider the calculus of the uncertainty associated to the aerodynamic force F_y . We assume that the air density ρ and the cross-sectional area S are perfectly known parameters. On the other hand we group the perturbations deriving from time-variance and uncertainty into the velocity V_m and the aero-coefficient C_{Ny} , of which we only know the nominal values and the maximum gap k between them and their actual values, according to the specific definition of deviation function.

Example 5.1.2.1. Consider the nominal expression of the force along \hat{y}_b

$$F_y = \frac{1}{2}\rho V_m^2 S C_{Ny}$$

and separate the known part $\frac{1}{2}\rho S = c$ from the uncertain part $V_m^2 C_{Ny}$, which can be seen as a cubic function of three parameters when written as $V_m V_m C_{Ny}$. To calculate the uncertain envelope of F_y we use the Δ operator in the following fashion

$$\begin{aligned} \Delta(F_y) &= \Delta(c V_m V_m C_{Ny}) = c \Delta(V_m V_m C_{Ny}) \\ &= c(\Delta V_m \Delta V_m \Delta C_{Ny} + V_m \Delta V_m \Delta C_{Ny} + \Delta V_m V_m \Delta C_{Ny} + \\ &\quad V_m^2 \Delta C_{Ny} + \Delta V_m \Delta V_m C_{Ny} + \Delta V_m V_m C_{Ny} + V_m \Delta V_m C_{Ny}) \end{aligned} \quad (5.20)$$

Keeping in mind that it is possible to decompose the single uncertainties in terms of the deviation function σ and assuming that it is the same for every function, we obtain:

$$\begin{aligned} \Delta(F_y) &= \sigma^3 F_y + 2\sigma^2 F_y + \sigma F_y + \sigma^2 F_y + 2\sigma F_y \\ &= (\sigma^3 + 3\sigma^2 + 3\sigma) F_y \end{aligned} \quad (5.21)$$

which can be generalized for a generic polynomial nominal function as shown in 2.3.0.2. Clearly, in the case of a quadratic nominal function, we obtain:

$$\begin{aligned} \Delta(V_m^2) &= \Delta(V_m V_m) = \Delta V_m \Delta V_m + V_m \Delta V_m + \Delta V_m V_m \\ &= \sigma^2 V_m + \sigma V_m + \sigma V_m = (\sigma^2 + 2\sigma) V_m \end{aligned} \quad (5.22)$$

which also fits into 2.3.0.2. ◇

Remark 5.1.2.2. The procedure can be extended to the case of multiple deviation functions. Since we are limiting ourselves to the case of two different deviation functions for at most three uncertain parameters in the same uncertain envelope, we show the results in this case, omitting for brevity the time-dependence of σ_i 's.

- **Two uncertain parameters** as in the case of M_x , in which V_m is not squared and C_{l_p} is the other one. The computation of the uncertain envelope gives:

$$\begin{aligned}\Delta(V_m C_{l_p}) &= \Delta_1 V_m \Delta_2 C_{l_p} + C_{l_p} \Delta_1 V_m + V_m \Delta_2 C_{l_p} \\ &= (\sigma_1 \sigma_2 + \sigma_1 + \sigma_2) V_m C_{l_p}\end{aligned}\tag{5.23}$$

where σ_1 embeds the variations of V_m and σ_2 those of C_{l_p} .

- **Three uncertain parameters** as in the previous case of F_y . Easy calculations result in:

$$\begin{aligned}\Delta(V_m^2 C_{Ny}) &= \Delta_1 V_m \Delta_1 V_m \Delta_2 C_{Ny} + V_m \Delta_1 V_m \Delta_2 C_{Ny} \\ &+ \Delta_1 V_m V_m \Delta_2 C_{Ny} + V_m^2 \Delta_2 C_{Ny} + \Delta_1 V_m \Delta_1 V_m C_{Ny} \\ &+ \Delta_1 V_m V_m C_{Ny} + V_m \Delta_1 V_m C_{Ny} \\ &= (\sigma_1^2 \sigma_2 + 2\sigma_1 \sigma_2 + 2\sigma_1 + \sigma_2 + \sigma_1^2)(V_m^2 C_{Ny})\end{aligned}\tag{5.24}$$

Looking at the equations of motions in (5.4) we find out that there are not only products between uncertain parameters, but also quotients, as in the terms with V_m at denominator in $\dot{\alpha}$ and $\dot{\beta}$ right members. Therefore, to compute the uncertain envelopes of the complete dynamics, we also need to use Proposition 2.3.0.3.

The results established in Section 2.3 yield a simple procedure to construct the uncertain envelopes of system's vectors and matrices by using the Δ -operator in an element-wise fashion. Aerodynamic forces and moments are in fact given by products between some parameters considered known (i.e. scarcely varying around their nominal values), as S , d , ρ and the inertia moments, and others considered uncertain (i.e. with significant variations around their nominal values), as V_m (since it embeds all the flight conditions between 2.5 and 3.2 Mach in a single value), \dot{V}_m and all the aero-coefficients together with the Magnus effect terms. The polynomial-in-the-state structure of most aerodynamic coefficients and their consequent linearity-in-the-parameters implies that the application of the Δ operator to each of them results simply in the product of the corresponding deviation function with the nominal value of the coefficient, as implicitly taken for granted in the calculations above:

$$\Delta(C_{Ny}) = \sigma_y(t) C_{Ny} \quad \Delta(C_{Nz}) = \sigma_z(t) C_{Nz}\tag{5.25}$$

therefore, being σ_V the perturbation associated to V_m , we have:

$$\begin{aligned}\Delta(F_y) &= (\sigma_V^2 \sigma_y + 2\sigma_V \sigma_y + 2\sigma_V + \sigma_y + \sigma_V^2) F_y \\ \Delta(F_z) &= (\sigma_V^2 \sigma_z + 2\sigma_V \sigma_z + 2\sigma_V + \sigma_z + \sigma_V^2) F_z\end{aligned}\tag{5.26}$$

furthermore, paying attention to the power of V_m

$$\begin{aligned}\Delta(M_x) &= \frac{1}{4} \rho S d^2 p \Delta(V_m C_{l_p}) + \frac{1}{2} \rho S d \Delta(V_m^2 G_p) \\ &= c_1 (\sigma_V \sigma_{l_p} + \sigma_V + \sigma_{l_p}) V_m C_{l_p} p \\ &\quad + c_2 (\sigma_V^2 \sigma_{G_p} + 2\sigma_V \sigma_{G_p} + 2\sigma_V + \sigma_{G_p} + \sigma_V^2) (V_m^2 G_p) \\ \Delta(M_y) &= \frac{1}{4} \rho S d^2 (q \Delta(V_m C_{m_q}) + \beta p \Delta(V_m C_{n_{p\beta}})) \\ &\quad + \frac{1}{2} \rho S d (\phi_m(\alpha) \Delta(V_m^2 C_{m_\alpha}) + \alpha \beta^2 \Delta(V_m^2 C_{m_{\alpha\beta}}) + \delta_P \Delta(V_m^2 C_{m_{\delta_P}})) \\ &= c_1 ((\sigma_V \sigma_{m_q} + \sigma_V + \sigma_{m_q}) V_m C_{m_q} q + (\sigma_V \sigma_{n_{p\beta}} + \sigma_V + \sigma_{n_{p\beta}}) V_m C_{n_{p\beta}} \beta p) \\ &\quad + c_2 (\phi_m(\alpha) (\sigma_V^2 \sigma_{m_\alpha} + 2\sigma_V \sigma_{m_\alpha} + 2\sigma_V + \sigma_{m_\alpha} + \sigma_V^2) V_m^2 C_{m_\alpha} \\ &\quad + \alpha \beta^2 (\sigma_V^2 \sigma_{m_{\alpha\beta}} + 2\sigma_V \sigma_{m_{\alpha\beta}} + 2\sigma_V + \sigma_{m_{\alpha\beta}} + \sigma_V^2) V_m^2 C_{m_{\alpha\beta}} \\ &\quad + \delta_P (\sigma_V^2 \sigma_P + 2\sigma_V \sigma_P + 2\sigma_V + \sigma_P + \sigma_V^2) V_m^2 C_{m_{\delta_P}}) \\ \Delta(M_z) &= \frac{1}{4} \rho S d^2 (r \Delta(V_m C_{n_r}) + \alpha p \Delta(V_m C_{n_{p\alpha}})) \\ &\quad + \frac{1}{2} \rho S d (\phi_n(\beta) \Delta(V_m^2 C_{n_\beta}) + \alpha^2 \beta \Delta(V_m^2 C_{n_{\alpha\beta}}) + \delta_Y \Delta(V_m^2 C_{n_{\delta_Y}})) \\ &= c_1 ((\sigma_V \sigma_{n_r} + \sigma_V + \sigma_{n_r}) V_m C_{n_r} r + (\sigma_V \sigma_{n_{p\alpha}} + \sigma_V + \sigma_{n_{p\alpha}}) V_m C_{n_{p\alpha}} \alpha p) \\ &\quad + c_2 (\phi_n(\beta) (\sigma_V^2 \sigma_{n_\beta} + 2\sigma_V \sigma_{n_\beta} + 2\sigma_V + \sigma_{n_\beta} + \sigma_V^2) V_m^2 C_{n_\beta} \\ &\quad + \alpha^2 \beta (\sigma_V^2 \sigma_{n_{\alpha\beta}} + 2\sigma_V \sigma_{n_{\alpha\beta}} + 2\sigma_V + \sigma_{n_{\alpha\beta}} + \sigma_V^2) V_m^2 C_{n_{\alpha\beta}} \\ &\quad + \delta_Y (\sigma_V^2 \sigma_Y + 2\sigma_V \sigma_Y + 2\sigma_V + \sigma_Y + \sigma_V^2) V_m^2 C_{n_{\delta_Y}})\end{aligned}\tag{5.27}$$

where $c_1 = \frac{1}{4} \rho S d^2$, $c_2 = \frac{1}{2} \rho S d$ and the various subscripts for σ are needed to distinguish between different deviation functions. Applying the results of Proposition 2.3.0.3 it is possible to combine the envelopes found for F_y and F_z with those of V_m and \dot{V}_m , whose deviation function is $\sigma_{\dot{V}}$. We obtain, $i = y, z$:

$$\begin{aligned}\Delta\left(\frac{F_i}{V_m}\right) &= \frac{V_m \Delta(F_i) - F_i \Delta(V_m)}{V_m (V_m + \Delta(V_m))} \\ &= \frac{V_m (\sigma_V^2 \sigma_i + 2\sigma_V \sigma_i + 2\sigma_V + \sigma_i + \sigma_V^2) F_i - F_i \sigma_V V_m}{V_m (V_m + \sigma_V V_m)} \\ \Delta\left(\frac{\dot{V}_m}{V_m}\right) &= \frac{\sigma_{\dot{V}} - \sigma_V}{1 + \sigma_V} \frac{\dot{V}_m}{V_m}\end{aligned}\tag{5.28}$$

Finally, the uncertainties of the Magnus terms are simply $\Delta(C_{mi}) = \sigma_{Mi}C_{mi}$, for $i = 1, \dots, 4$.

Organizing the uncertain envelopes found by splitting them into drift parts, identified by the superscript D , depending only on state variables and parameters, and control parts linear in the control inputs it is straightforward to define the uncertain terms affecting system's vector and matrices in the following way:

$$\begin{aligned}
\Delta f_\varphi &= 0 \\
\Delta f_\alpha &= p\alpha\sigma_{M3}C_{m3} + \frac{Q_{\alpha\beta}}{m}\Delta\left(\frac{F_z^D}{V_m}\right) \\
&\quad + s_\alpha c_\alpha t_\beta \frac{Q_{\alpha\beta}}{m}\Delta\left(\frac{F_y^D}{V_m}\right) - s_\alpha c_\alpha Q_{\alpha\beta}^2 \Delta\left(\frac{\dot{V}_m}{V_m}\right) \\
\Delta f_\beta &= p\beta\sigma_{M4}C_{m4} + \frac{Q_{\alpha\beta}}{m}\Delta\left(\frac{F_y^D}{V_m}\right) \\
&\quad + s_\beta c_\beta t_\alpha \frac{Q_{\alpha\beta}}{m}\Delta\left(\frac{F_z^D}{V_m}\right) - s_\beta c_\beta Q_{\alpha\beta}^2 \Delta\left(\frac{\dot{V}_m}{V_m}\right) \\
\Delta f_p &= \frac{I_z}{I_x I_z - I_{xz}^2} \Delta M_x^D + \frac{I_{xz}}{I_x I_z - I_{xz}^2} \Delta M_z^D \\
\Delta f_q &= \frac{1}{I_y} \Delta M_y^D + p\alpha\sigma_{M1}C_{m1} \\
\Delta f_r &= \frac{I_{xz}}{I_x I_z - I_{xz}^2} \Delta M_x^D + \frac{I_x}{I_x I_z - I_{xz}^2} \Delta M_z^D + p\beta\sigma_{M2}C_{m2}
\end{aligned} \tag{5.29}$$

moreover, for $i = 1, 2$

$$\begin{aligned}
\Delta g_{\alpha i} &= (\sigma_V \sigma_{\alpha i} + \sigma_V + \sigma_{\alpha i}) g_{\alpha i} \\
\Delta g_{\beta i} &= (\sigma_V \sigma_{\beta i} + \sigma_V + \sigma_{\beta i}) g_{\beta i}
\end{aligned} \tag{5.30}$$

and for $j = 1, 2, 3$

$$\begin{aligned}
\Delta g_{pj} &= (\sigma_V^2 \sigma_{pj} + 2\sigma_V \sigma_{pj} + 2\sigma_V + \sigma_{pj} + \sigma_V^2) g_{pj} \\
\Delta g_q &= (\sigma_V^2 \sigma_P + 2\sigma_V \sigma_P + 2\sigma_V + \sigma_P + \sigma_V^2) g_q \\
\Delta g_{rj} &= (\sigma_V^2 \sigma_{rj} + 2\sigma_V \sigma_{rj} + 2\sigma_V + \sigma_{rj} + \sigma_V^2) g_{rj}
\end{aligned} \tag{5.31}$$

Such uncertain envelopes can be divided in two groups: matched and unmatched uncertainties. The distinction follows the core Definition 2.2.2.1 given in Section 2.2.

Once defined the uncertain terms it is possible to rearrange them into vectors and matrices to compose a model suited for robust nonlinear control design. Splitting into the two sub-systems and highlighting the Magnus effect's terms in Δf_α and Δf_β with

the purpose of gain more structure, we obtain:

$$\begin{aligned}
\Delta f_{xu} &= \begin{pmatrix} 0 \\ \Delta f'_\alpha \\ \Delta f'_\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \Delta C_{m3}\alpha & 0 & 0 \\ \Delta C_{m4}\beta & 0 & 0 \end{pmatrix} z \\
&\quad + \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{\alpha 1} + \Delta g_{\alpha 1} & g_{\alpha 2} + \Delta g_{\alpha 2} \\ 0 & g_{\beta 1} + \Delta g_{\beta 1} & g_{\beta 2} + \Delta g_{\beta 2} \end{pmatrix} u_M \\
\Delta f_z &= \begin{pmatrix} \Delta f_p \\ \Delta f_q \\ \Delta f_r \end{pmatrix} \quad \Delta g_z = \begin{pmatrix} \Delta g_{p1} & \Delta g_{p2} & \Delta g_{p3} \\ 0 & \Delta g_q & 0 \\ \Delta g_{r1} & \Delta g_{r2} & \Delta g_{r3} \end{pmatrix}
\end{aligned} \tag{5.32}$$

Clearly $\Delta f_{xu} = \Delta f_{xu}(x, z, u_M, t)$, $\Delta f_z = \Delta f_z(x, z, t)$ and $\Delta g_z = \Delta g_z(x, t)$. We denote with u_M an estimate for the upper-bound of control inputs vector, considered as structured uncertainty when it acts on the first part of the dynamics causing the non-minimum phase behavior of the airframe. The uncertain system then can be expressed as

$$\begin{aligned}
\dot{x} &= f_x(x, t) + \Delta f_{xu}(x, z, u_M, t) + g_x^1(x)z \\
\dot{z} &= f_z(x, z, t) + \Delta f_z(x, z, t) + [g_z(x, t) + \Delta g_z(x, t)]u
\end{aligned} \tag{5.33}$$

Note that, disregarding the uncertain terms and the explicit time-dependence of system's equations, this model reduces to the nominal one (5.19).

The *unmatched uncertainties* Δf_u are those acting directly on f :

$$\Delta f_u(x, z, u_M, t) = \begin{pmatrix} \Delta f_{xu}(x, z, u_M, t) \\ \Delta f_z(x, z, t) \end{pmatrix} \tag{5.34}$$

while the *matched uncertainties* Δg_m appear in the same channel of the control inputs

$$\Delta g_m(x, t) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta g_z(x, t) \end{pmatrix} \tag{5.35}$$

This system representation will be used to design robust control once defined the *bounding functions* of the uncertain terms, which are of straightforward calculation using the properties of the deviation functions introduced in Section 2.2. For instance, we can evaluate the bounds of the uncertainties on F_y^D and F_z^D by simply replacing the deviation functions with the corresponding gains k_V , k_y and k_z

$$\begin{aligned}
\rho_{F_y^D} &= (k_V^2 k_y + 2k_V k_y + 2k_V + k_y + k_V^2) |F_y^D| \\
\rho_{F_z^D} &= (k_V^2 k_z + 2k_V k_z + 2k_V + k_z + k_V^2) |F_z^D|
\end{aligned} \tag{5.36}$$

The same can be done for the Magnus effect terms, for which we obtain, $i = 1, \dots, 4$:

$$\rho_{C_{mi}} = k_{Mi}|C_{mi}| \quad (5.37)$$

Similar but much more complicated expressions, which we omit for brevity since their calculation is also straightforward, can be found for ΔM_x^D , ΔM_y^D and ΔM_z^D , resulting in bounding functions $\rho_{M_x^D}$, $\rho_{M_y^D}$ and $\rho_{M_z^D}$. Furthermore it is possible to define the bounding functions for the terms $\Delta \left(\frac{F_i^D}{V_m} \right)$ and $\Delta \left(\frac{\dot{V}_m}{V_m} \right)$, for $i = y, z$, as:

$$\begin{aligned} \rho_{F_i^D/V_m} &= \left| \frac{V_m \rho_{F_i^D} - F_i k_V |V_m|}{V_m (V_m + k_V |V_m|)} \right| \\ \rho_{\dot{V}_m/V_m} &= \frac{|k_{\dot{V}} - k_V|}{1 + k_V} \left| \frac{\dot{V}_m}{V_m} \right| \end{aligned} \quad (5.38)$$

After the derivation of these critical bounds, it is pretty simple to calculate the bounding functions ρ_{f_i} of all the uncertain envelopes introduced in (5.29), $i = \alpha, \beta, p, q, r$. We give just the example of calculus of ρ_{f_α} , because the other ones are very similar to this.

$$\begin{aligned} |\Delta f_\alpha| \leq \rho_{f_\alpha} &= \left| p \alpha k_{M3} |C_{m3}| + \frac{Q_{\alpha\beta}}{m} \rho_{F_z^D/V_m} \right. \\ &\quad \left. + s_\alpha c_\alpha t_\beta \frac{Q_{\alpha\beta}}{m} \rho_{F_y^D/V_m} - s_\alpha c_\alpha Q_{\alpha\beta}^2 \rho_{\dot{V}_m/V_m} \right| \end{aligned} \quad (5.39)$$

Finally, we can evaluate bounding functions for (5.30), $i = 1, 2$, and (5.31), $j = 1, 2, 3$:

$$\begin{aligned} |\Delta g_{\alpha i}| &\leq \rho_{g_{\alpha i}} = (k_V k_{\alpha i} + k_V + k_{\alpha i}) |g_{\alpha i}| \\ |\Delta g_{\beta i}| &\leq \rho_{g_{\beta i}} = (k_V k_{\beta i} + k_V + k_{\beta i}) |g_{\beta i}| \\ |\Delta g_{pj}| &\leq \rho_{g_{pj}} = (k_V^2 k_{pj} + 2k_V k_{pj} + 2k_V + k_{pj} + k_V^2) |g_{pj}| \\ |\Delta g_q| &\leq \rho_{g_q} = (k_V^2 k_P + 2k_V k_P + 2k_V + k_P + k_V^2) |g_q| \\ |\Delta g_{rj}| &\leq \rho_{g_{rj}} = (k_V^2 k_{rj} + 2k_V k_{rj} + 2k_V + k_{rj} + k_V^2) |g_{rj}| \end{aligned} \quad (5.40)$$

Remark 5.1.2.3. The values of the gains k_i are chosen to the extent of size-domination of the particular uncertain terms considered. Together with the choice of the overall ρ_i bounding functions, they determine the level of robustness and associated *conservatism* of the closed-loop system under the robust nonlinear control. For instance, choosing the k_i gains exactly equal to the deviation functions amplitudes guarantees the right compromise between robustness and performance, since no extra-effort than that closely needed is put into uncertainty compensation. \diamond

Grouping the bounding functions into vectors and matrices we are finally able to write:

$$\begin{aligned}
 \rho_{f_{xu}}(x, z) &= \begin{pmatrix} 0 \\ \rho_{f_\alpha} \\ \rho_{f_\beta} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \rho_{C_{m1}}\alpha & 0 & 0 \\ \rho_{C_{m2}}\beta & 0 & 0 \end{pmatrix} z \\
 &\quad + \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{\alpha1} + \rho_{g_{\alpha1}} & g_{\alpha2} + \rho_{g_{\alpha2}} \\ 0 & g_{\beta1} + \rho_{g_{\beta1}} & g_{\beta2} + \rho_{g_{\beta2}} \end{pmatrix} u_M \\
 \rho_{f_z}(x, z) &= \begin{pmatrix} \rho_{f_p} \\ \rho_{f_q} \\ \rho_{f_r} \end{pmatrix} \quad \rho_{g_z}(x, z) = \begin{pmatrix} \rho_{g_{p1}} & \rho_{g_{p2}} & \rho_{g_{p3}} \\ 0 & \rho_{g_q} & 0 \\ \rho_{g_{r1}} & \rho_{g_{r2}} & \rho_{g_{r3}} \end{pmatrix}
 \end{aligned} \tag{5.41}$$

Remark 5.1.2.4. The estimate u_M is a vector formed by the absolute values of the nominal control inputs that are evaluated before designing the robust control law. This choice results in underestimating the effects of u on the x sub-dynamics and it is made exclusively to reduce conservatism and complexity of the design. \diamond

5.2 Control design

After having defined the problem and the control objectives, as first step, a nominal control law disregarding the uncertainties is designed using a MIMO backstepping approach. The second step is the construction, on the shoulders of the nominal design, of a robust control law which is able to counteract the effect of the unmatched uncertainties. The third and final step consists in using a C^1 , revisited version of Lyapunov redesign to compensate for the matched uncertainties.

5.2.1 Control objectives

Control objectives are twofold, since we would like to design a very general autopilot capable of achieving both Skid-To-Turn and Bank-To-Turn maneuvers, independently of the particular airframe on which it is applied and with slight changes in the equations when necessary. This can be done since the equations of motion introduced in Chapter 5.1 and used for control purposes represent an hybrid airframe capable of both movements, as already stressed.

The problem of controlling the motion of a missile achieving maneuvers can be interpreted as the problem of achieving P-RUGAS of a nonlinear uncertain system using an opportune control law. As a consequence, we are able now to formulate the two control problems to be solved in terms of the two main maneuvers.

Skid-To-Turn maneuver

The autopilot has to rapidly steer the bank angle φ to a constant desired value (usually zero or near zero) and so to bring to zero the roll-rate p for the purpose of inertially decoupling the longitudinal from the lateral motion. This is not necessary since the approach is nonlinear, but it is a common practice. We could also track a time-varying motion in φ , but this would not correspond to the typical STT maneuver we want to make. Once the bank angle is regulated, rapidly changing time-profiles in α and β have to be tracked to guarantee the pursuit of the desired vertical and lateral commanded accelerations, respectively a_z and a_y , produced by the guidance system. Viewed in terms of P-RUGAS, control objectives for the STT maneuver are:

$$\begin{aligned}\varphi(t) &\rightarrow \omega_\varphi(\varphi_d) = \omega_\varphi(0) \\ \alpha(t) &\rightarrow \omega_\alpha(\alpha_d(t)) \\ \beta(t) &\rightarrow \omega_\beta(\beta_d(t))\end{aligned}\tag{5.42}$$

ω_i represent the limit sets of convergence of the state trajectories: according to Def. 1.3.2.2 they can be made arbitrarily small through an opportune choice of control design parameters; we will show this peculiarity in the next sections.

Bank-To-Turn maneuver

The autopilot has to rapidly steer the sideslip angle β to a constant desired value (usually zero or near zero) and so to bring to zero the yaw-rate r to prevent the air-breathing engine typical of BTT missiles from flaming out. The first part of the maneuver, the *twisting*, consists in reaching the desired value of φ corresponding to the direction of the commanded acceleration in terms of a_y and a_z . Secondly, the missile has to gain the desired angle of attack, given by the desired acceleration magnitude in that direction: this is called the *steering* part of the maneuver. Viewed in terms of P-RUGAS, control objectives for the STT maneuver are:

$$\begin{aligned}\beta(t) &\rightarrow \omega_\beta(\beta_d) = \omega_\beta(0) \\ \varphi(t) &\rightarrow \omega_\varphi(\varphi_d(t)) \\ \alpha(t) &\rightarrow \omega_\alpha(\alpha_d(t))\end{aligned}\tag{5.43}$$

5.2.2 Nominal control law

To apply backstepping we start from (5.19), which has a suitable structure, and define the error variable $e_x = x_d - x$, that incorporates the desired values of α , β and φ according to the selected maneuver. Suppose there exists a function $\phi(x)$ such that the error-subsystem

$$\dot{e}_x = \dot{x}_d - \dot{x} = \dot{x}_d - f_x(x) - g_x^1(x)\phi(x) \quad (5.44)$$

is *globally asymptotically stable* (abbr. *GAS*) with a Lyapunov function $V(e_x)$. This means that

$$\dot{V}(e_x) = \frac{\partial V}{\partial e_x} (\dot{x}_d - f_x(x) - g_x^1(x)\phi(x)) \leq -w(\|e_x\|) \quad (5.45)$$

where $w(\cdot)$ is a class \mathcal{K}_∞ function (see [62]). Define the following change of coordinates

$$\zeta = z - \phi(x) \quad (5.46)$$

under which the system can be written in terms of the error ζ between the virtual control input z and its desired value $z_d = \phi(x)$

$$\begin{aligned} \dot{e}_x &= \dot{x}_d - f_x(x) - g_x^1(x) (\zeta + \phi(x)) \\ \dot{\zeta} &= f_z(x, z) + g_z(x)u - \frac{\partial \phi}{\partial x} [f_x(x) + g_x^1(x) (\zeta + \phi(x))] \end{aligned} \quad (5.47)$$

The obtained representation is mixed in e_x , x , z and ζ . Define now the augmented Lyapunov function

$$W(e_x, \zeta) = V(e_x) + \frac{1}{2}\zeta^T \zeta \quad (5.48)$$

which is a candidate Lyapunov function for the transformed system (5.47). The derivative along the motion

$$\begin{aligned} \dot{W}(x, \zeta) &= \frac{\partial V}{\partial e_x} [\dot{x}_d - f_x(x) - g_x^1(x) (\zeta + \phi(x))] \\ &\quad + \zeta^T \left[f_z(x, z) + g_z(x)u - \frac{\partial \phi}{\partial x} (f_x(x) + g_x^1(x) (\zeta + \phi(x))) \right] \\ &\leq -w(\|e_x\|) - \frac{\partial V}{\partial e_x} g_x^1(x) \zeta \\ &\quad + \zeta^T \left[f_z(x, z) + g_z(x)u - \frac{\partial \phi}{\partial x} (f_x(x) + g_x^1(x) (\zeta + \phi(x))) \right] \end{aligned} \quad (5.49)$$

is made negative definite by the following choice of u :

$$\begin{aligned} u &= g_z^{-1}(x) \left[-f_z(x, z) + \frac{\partial \phi}{\partial x} (f_x(x) + g_x^1(x) (\zeta + \phi(x))) \right. \\ &\quad \left. + \left(\frac{\partial V}{\partial e_x} g_x^1(x) \right)^T - K_\zeta \zeta \right] \end{aligned} \quad (5.50)$$

which yields:

$$\dot{W}(x, \zeta) \leq -w(\|e_x\|) - \zeta^T K_\zeta \zeta \quad (5.51)$$

A possible choice for $\phi(x)$ is that of a dynamic cancellation with eigenvalue assignment to the error dynamics:

$$\phi(x) = [g_x^1(x)]^{-1} (-f_x(x) + \dot{x}_d + K_x e_x) \quad (5.52)$$

With this choice, the simple quadratic-in-the-error Lyapunov function

$$V(e_x) = \frac{1}{2} e_x^T e_x \quad (5.53)$$

ensures GAS for the error dynamics. The closed loop system

$$\begin{pmatrix} \dot{e}_x \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} -K_x & -g_x^1(x) \\ g_x^1(x)^T & -K_\zeta \end{pmatrix} \begin{pmatrix} e_x \\ \zeta \end{pmatrix} \quad (5.54)$$

is then GAS with Lyapunov function $W(e_x, \zeta) = \frac{1}{2} e_x^T e_x + \frac{1}{2} \zeta^T \zeta$, such that $\dot{W}(e_x, \zeta) \leq -e_x^T K_x e_x - \zeta^T K_\zeta \zeta$ with the (nominal) control law (5.50).

Remark 5.2.2.1. The approach used is called backstepping since it is *recursive* and exploits the concept of *virtual control* ($\phi(x)$ in our case) to stabilize the inner part of the dynamics, in some sense stepping back from the last subsystem up to the first. The recursion is given by the change of coordinates, which can be extended to more complex systems that still have a lower-triangular structure, called *strict-feedback form*. This change of coordinates allows for the construction of a non-quadratic (in the original coordinates) Lyapunov function, making the problem of finding a globally stabilizing control law for the whole system much easier. \diamond

This first result only implies that the nominal system, with the uncertain terms kept to zero, is GAS. To ensure P-RUGAS of the uncertain system we need to make the backstepping *robust* against both matched and unmatched uncertainties.

5.2.3 Robust nonlinear control

The robustification of (5.50) is split up into two parts: the first is the construction of a control law u_u to counteract unmatched uncertainties and the second is the determination of another term u_m which can face the effects of matched uncertainties. Renaming

(5.50) as u_n , the overall robust nonlinear control law will have the form:

$$u = u_n + u_u + u_m \quad (5.55)$$

Unmatched control

Unmatched control, otherwise known as robust backstepping, is built “on the shoulder” of the nominal design, from which inherits recursion and virtual control concepts. In order to compensate for the unmatched uncertain terms, we exploit the concept of robust control function, introduced in Definition 3.1.0.4, which we adapt to the present context with the following straightforward element-wise extension

Definition 5.2.3.1 (Element-wise robust control function). The robust control function can be extended to a vector context by smartly using both matrix product and the element-wise vector and matrix Hadamard product, denoted by the symbol “ \circ ”. For instance, if ρ_u is a three-dimensional vector of uncertain terms and so are the independent variables s and σ_s , we define the *element-wise robust control function* as follows:

$$\rho_u \circ \mathbf{v}(s, \sigma_s) = \rho_u \circ \mathbf{sign}(s) \circ (\mathbf{1} - \mathbf{e}^{-\sigma_s \circ |\mathbf{s}|}) = \rho_u \circ \mathbf{sigm}(s, \sigma_s) \quad (5.56)$$

where

$$\begin{aligned} \mathbf{1} &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \\ \mathbf{sign}(s) &= \begin{pmatrix} \text{sign}(s_1) & \text{sign}(s_2) & \text{sign}(s_3) \end{pmatrix}^T \\ \mathbf{e}^{-\sigma_s \circ |\mathbf{s}|} &= \begin{pmatrix} e^{-\sigma_{s_1}|s_1|} & e^{-\sigma_{s_2}|s_2|} & e^{-\sigma_{s_3}|s_3|} \end{pmatrix}^T \end{aligned} \quad (5.57)$$

and, as a consequence:

$$\mathbf{sigm}(s, \sigma_s) = \begin{pmatrix} \text{sigm}(s_1) \\ \text{sigm}(s_2) \\ \text{sigm}(s_3) \end{pmatrix} = \begin{pmatrix} \text{sign}(s_1) \\ \text{sign}(s_2) \\ \text{sign}(s_3) \end{pmatrix} \circ \begin{pmatrix} 1 - e^{-\sigma_{s_1}|s_1|} \\ 1 - e^{-\sigma_{s_2}|s_2|} \\ 1 - e^{-\sigma_{s_3}|s_3|} \end{pmatrix} \quad (5.58) \quad \diamond$$

We consider this model for unmatched control design:

$$\begin{aligned} \dot{x} &= f_x(x) + \Delta f_{xu}(x, z, u, t) + g_x^1(x)z \\ \dot{z} &= f_z(x, z) + \Delta f_z(x, z, t) + g_z(x)u \end{aligned} \quad (5.59)$$

and prove the following result on trajectory tracking using robust backstepping in the presence of unmatched uncertainties.

Theorem 5.2.3.2. *Given the system with unmatched uncertainties (5.59), the robust backstepping control law :*

$$\begin{aligned} u' &= u_n + u_u \\ &= g_z^{eq}(x, z)^{-1} \left[(e_x^T g_x^1(x))^T - f_z^{eq}(x, z) + \frac{\partial \phi_r}{\partial x} (\dot{x}_d + K_x e_x + g_x^1(x) \zeta) \right. \\ &\quad \left. - K_\zeta \zeta - \rho_{f_z}^{eq} \circ \mathbf{sigm}(\zeta) + \frac{\partial \phi_r(x)}{\partial x} \rho_{f_{xu}} \circ (\mathbf{1} - \mathbf{sigm}(e_x)) \right] \end{aligned} \quad (5.60)$$

where

$$g_z^{eq}(x, z) = g_z(x) - \frac{\partial \phi_r}{\partial z} g_z(x) \quad f_z^{eq} = f_z - \frac{\partial \phi_r}{\partial z} f_z(x) \quad (5.61)$$

and $\rho_{f_z}^{eq}$ is the bounding function of the equivalent uncertainty $\Delta f_z^{eq} = \Delta f_z - \frac{\partial \phi_r}{\partial z} \Delta f_z(x)$, together with the robust virtual control input:

$$\phi_r(x, z) = [g_x^1(x)]^{-1} (-f_x(x) + \dot{x}_d + K_x e_x - \rho_{f_{xu}}(x, z) \circ \mathbf{sigm}(e_x)) \quad (5.62)$$

and robust control Lyapunov function (depending on $\zeta = z - \phi_r(x, z)$):

$$W_r(e_x, \zeta) = \frac{1}{2} e_x^T e_x + \frac{1}{2} \zeta^T \zeta \quad (5.63)$$

renders the closed-loop system (5.59)-(5.60) P-RUGAS about the desired trajectory $x_d(t) = (\varphi_d(t), \alpha_d(t), \beta_d(t))$. \diamond

Proof. The partial change of coordinates:

$$\begin{pmatrix} x \\ \zeta \end{pmatrix} = \begin{pmatrix} x \\ z - \phi_r(x, z) \end{pmatrix} \quad (5.64)$$

transforms system (5.59) into the following mixed representation:

$$\begin{aligned} \dot{x} &= f_x(x) + \Delta f_{xu} + g_x^1(x) (\zeta + \phi_r(x, z)) \\ \dot{\zeta} &= f_z(x, z) + \Delta f_z + g_z(x) u \\ &\quad - \frac{\partial \phi_r}{\partial x} [f_x(x) + \Delta f_{xu} + g_x^1(x) (\zeta + \phi_r(x, z))] \\ &\quad - \frac{\partial \phi_r}{\partial z} [f_z(x, z) + \Delta f_z + g_z(x) u] \end{aligned} \quad (5.65)$$

which, by substituting the expression of the robust fictitious control $\phi_r(x, z)$ (first step of the procedure), can be simplified into:

$$\begin{aligned}\dot{x} &= \dot{x}_d + K_x e_x + g_x^1(x)\zeta + \Delta f_{xu} - \rho_{f_{xu}}(x, z) \circ \mathbf{sign}(e_x) \\ \dot{\zeta} &= f_z^{eq}(x, z) + \Delta f_z^{eq} + g_z^{eq}(x, z)u \\ &\quad - \frac{\partial \phi_r}{\partial x} [\dot{x}_d + K_x e_x + g_x^1(x)\zeta + \Delta f_{xu} - \rho_{f_{xu}}(x, z) \circ \mathbf{sign}(e_x)]\end{aligned}\quad (5.66)$$

In the second step, we close the loop by applying the actual control law (5.60) which, keeping in mind that $e_x = x_d - x$, yields:

$$\begin{aligned}\dot{e}_x &= -K_x e_x - g_x^1(x)\zeta + \Delta f_{xu} - \rho_{f_{xu}} \circ \mathbf{sign}(e_x) \\ \dot{\zeta} &= (e_x^T g_x^1(x))^T + \Delta f_z^{eq} - \rho_{f_z}^{eq} \circ \mathbf{sign}(\zeta) \\ &\quad - \frac{\partial \phi_r}{\partial x} (\Delta f_{xu} - \rho_{f_{xu}} \circ \mathbf{sign}(e_x)) \\ &\quad + \frac{\partial \phi_r}{\partial x} \rho_{f_{xu}} \circ (\mathbf{1} - \mathbf{sign}(e_x)) - K_\zeta \zeta\end{aligned}\quad (5.67)$$

We now apply Lyapunov direct criterion to show that (5.67) is P-RUGAS. The augmented candidate Lyapunov function we choose is $W_r(e_x, \zeta) = \frac{1}{2}e_x^T e_x + \frac{1}{2}\zeta^T \zeta$. Its derivative along the trajectories of (5.67) is:

$$\begin{aligned}\dot{W}_r(e_x, \zeta) &= e_x^T \dot{e}_x + \zeta^T \dot{\zeta} \\ &= -e_x^T K_x e_x - e_x^T g_x^1(x)\zeta + e_x^T (\Delta f_{xu} - \rho_{f_{xu}} \circ \mathbf{sign}(e_x)) \\ &\quad + \zeta^T (e_x^T g_x^1(x))^T + \zeta^T (\Delta f_z^{eq} - \rho_{f_z}^{eq} \circ \mathbf{sign}(\zeta)) \\ &\quad - \zeta^T \frac{\partial \phi_r}{\partial x} (\Delta f_{xu} - \rho_{f_{xu}} \circ \mathbf{sign}(e_x)) \\ &\quad + \zeta^T \frac{\partial \phi_r}{\partial x} \rho_{f_{xu}} \circ (\mathbf{1} - \mathbf{sign}(e_x)) - \zeta^T K_\zeta \zeta \\ &= -e_x^T K_x e_x - \zeta^T K_\zeta \zeta + e_x^T \Delta f_{xu} - (e_x \circ \mathbf{sign}(e_x))^T \rho_{f_{xu}} \\ &\quad + (e_x \circ \mathbf{sign}(e_x))^T \rho_{f_{xu}} \circ \mathbf{e}^{-\sigma_x \circ |e_x|} + \zeta^T \Delta f_z^{eq} - (\zeta \circ \mathbf{sign}(\zeta))^T \rho_{f_z}^{eq} \\ &\quad + (\zeta \circ \mathbf{sign}(\zeta))^T \rho_{f_z}^{eq} \circ \mathbf{e}^{-\sigma_\zeta \circ |\zeta|} - \zeta^T \frac{\partial \phi_r}{\partial x} \Delta f_{xu} + \zeta^T \frac{\partial \phi_r}{\partial x} \rho_{f_{xu}}\end{aligned}\quad (5.68)$$

in which we have used commutativity and associativity of the Hadamard product. In the following, we define in tensorial notation the vector given by the absolute values of e_x and ζ components, which is clearly related to the \mathcal{L}_1 norm structure:

$$\begin{aligned}(e_x \circ \mathbf{sign}(e_x))^T &= |e_x|^T = (|e_{x1}| \quad |e_{x2}| \quad |e_{x3}|)^T \\ (\zeta \circ \mathbf{sign}(\zeta))^T &= |\zeta|^T = (|\zeta_1| \quad |\zeta_2| \quad |\zeta_3|)^T\end{aligned}\quad (5.69)$$

Using this notation, the expression of the derivative along the motion is bounded from above and therefore simplified to:

$$\dot{W}_r(e_x, \zeta) \leq -e_x^T K_x e_x - \zeta^T K_\zeta \zeta \quad (5.70a)$$

$$\begin{aligned} & + |e_x|^T \rho_{f_{xu}} - |e_x|^T \rho_{f_{xu}} + |\zeta|^T \rho_{f_z}^{eq} - |\zeta|^T \rho_{f_z}^{eq} \\ & + |e_x|^T \rho_{f_{xu}} \circ \mathbf{e}^{-\sigma_{\mathbf{x}} \circ |\mathbf{e}_x|} + |\zeta|^T \rho_{f_z}^{eq} \circ \mathbf{e}^{-\sigma_\zeta \circ |\zeta|} \\ & - \zeta^T \frac{\partial \phi_r}{\partial x} \rho_{f_{xu}} + \zeta^T \frac{\partial \phi_r}{\partial x} \rho_{f_{xu}} \\ & = -e_x^T K_x e_x - \zeta^T K_\zeta \zeta \end{aligned} \quad (5.70b)$$

$$+ |e_x|^T \rho_{f_{xu}} \circ \mathbf{e}^{-\sigma_{\mathbf{x}} \circ |\mathbf{e}_x|} + |\zeta|^T \rho_{f_z}^{eq} \circ \mathbf{e}^{-\sigma_\zeta \circ |\zeta|} \quad (5.70c)$$

The differential inequality (5.70) obtained implies that P-RUGAS is ensured for the closed-loop system, as stated in a simpler case in [85] and [84], since \dot{W}_r is negative outside a compact set Ω . In fact, each of the indefinite in sign terms in (5.70c) is bounded and attains a maximum. Consider for instance the residual scalar term $|e_x|^T (\rho_{f_{xu}} \circ \mathbf{e}^{-\sigma_{\mathbf{x}} \circ |\mathbf{e}_x|})$, which is the following three-variables function

$$\text{res}(|e_x|) = |e_x|^1 \rho_{f_{xu}}^1 e^{-\sigma_{x_1} |e_x^1|} + |e_x|^2 \rho_{f_{xu}}^2 e^{-\sigma_{x_2} |e_x^2|} + |e_x|^3 \rho_{f_{xu}}^3 e^{-\sigma_{x_3} |e_x^3|} \quad (5.71)$$

The critical point for the function is found by zeroing the gradient

$$\frac{\partial \text{res}}{\partial |e_x|} = \begin{pmatrix} \rho_{f_{xu}}^1 e^{-\sigma_{x_1} |e_x^1|} (1 - \sigma_{x_1} |e_x^1|) \\ \rho_{f_{xu}}^2 e^{-\sigma_{x_2} |e_x^2|} (1 - \sigma_{x_2} |e_x^2|) \\ \rho_{f_{xu}}^3 e^{-\sigma_{x_3} |e_x^3|} (1 - \sigma_{x_3} |e_x^3|) \end{pmatrix} \quad (5.72)$$

obtaining:

$$|e_x|^* = \begin{pmatrix} \frac{1}{\sigma_{x_1}} & \frac{1}{\sigma_{x_2}} & \frac{1}{\sigma_{x_3}} \end{pmatrix}^T \quad (5.73)$$

Therefore the diagonal Hessian matrix $\text{diag}(|e_x^i| \rho_{f_{xu}}^i \sigma_{x_i}^2 e^{-\sigma_{x_i} |e_x^i|} - 2 \rho_{f_{xu}}^i \sigma_{x_i} e^{-\sigma_{x_i} |e_x^i|})$ evaluated at $|e_x|^*$

$$H_{\text{res}}|_{|e_x|^*} = \begin{pmatrix} -\rho_{f_{xu}}^1 \sigma_{x_1} e^{-1} & 0 & 0 \\ 0 & -\rho_{f_{xu}}^2 \sigma_{x_2} e^{-1} & 0 \\ 0 & 0 & -\rho_{f_{xu}}^3 \sigma_{x_3} e^{-1} \end{pmatrix} \quad (5.74)$$

is negative definite for all (x, z) in the domain of interest being the bounding functions and the sigmoid slopes always greater than zero. As a consequence, $|e_x|^*$ is a global maximum point for the term $\text{res}(|e_x|)$. An identical result can be stated for the term

$\text{res}(|\zeta|) = \sum_{i=1}^3 |\zeta^i| \rho_{f_z}^{eq_i} e^{-\sigma_{\zeta_i} |\zeta^i|}$, whose maximum is attained at $|\zeta|^* = \left(\frac{1}{\sigma_{\zeta_1}} \quad \frac{1}{\sigma_{\zeta_2}} \quad \frac{1}{\sigma_{\zeta_3}} \right)^T$. The maximum values are easily calculated:

$$\max_{|e_x|} \text{res}(|e_x|) = e^{-1} \sum_{i=1}^3 \frac{1}{\sigma_{x_i}} \rho_{f_{xu}}^i \quad (5.75)$$

$$\max_{|e_x|} \text{res}(|\zeta|) = e^{-1} \sum_{i=1}^3 \frac{1}{\sigma_{\zeta_i}} \rho_{f_z}^{eq_i} \quad (5.76)$$

As a result, inequality (5.70b)-(5.70c) can be further simplified to

$$\begin{aligned} \dot{W}_r(e_x, \zeta) &\leq -e_x^T K_x e_x - \zeta^T K_\zeta \zeta \\ &\quad + |e_x|^T \rho_{f_{xu}} \circ \mathbf{e}^{-\sigma_{\mathbf{x}} \circ |\mathbf{e}_x|} + |\zeta|^T \rho_{f_z}^{eq} \circ \mathbf{e}^{-\sigma_{\zeta} \circ |\zeta|} \\ &\leq -\begin{pmatrix} e_x & \zeta \end{pmatrix}^T \begin{pmatrix} K_x & 0 \\ 0 & K_\zeta \end{pmatrix} \begin{pmatrix} e_x \\ \zeta \end{pmatrix} + e^{-1} \left(\sum_{i=1}^3 \frac{1}{\sigma_{x_i}} \rho_{f_{xu}}^i + \sum_{i=1}^3 \frac{1}{\sigma_{\zeta_i}} \rho_{f_z}^{eq_i} \right) \end{aligned} \quad (5.77)$$

Grouping control gains in the diagonal matrix K_b , errors in the vector e_b , sigmoid slopes in the vector σ_b^{-1} and unmatched bounding functions into ρ_u we obtain:

$$\begin{aligned} \dot{W}_r(e_x, \zeta) &= \dot{W}_r(e_b) \leq -e_b^T K_b e_b + e^{-1} \sigma_b^{-1} \cdot \rho_u \\ &\leq -\lambda_{\min}(K_b) \|e_b\|^2 + e^{-1} \sigma_b^{-1} \cdot \rho_u \end{aligned} \quad (5.78)$$

which is negative definite provided that

$$\|e_b\| > \sqrt{\frac{\sigma_b^{-1} \cdot \rho_u}{e \lambda_{\min}(K_b)}} := \omega \quad (5.79)$$

Where the radius ω defines the set (an hyper-sphere) in the error space

$$\Omega = \{\|e_b\| \leq \omega(\sigma_b^{-1}, \lambda_{\min}(K_b))\} \quad (5.80)$$

Let now S be the smallest level set of W containing the hyper-sphere of radius ω . The derivative of W along the trajectories is negative definite as long as the error does not enter S , in symbols:

$$e_b \notin S \Rightarrow \dot{W}(e_b) < 0 \quad (5.81)$$

which is P-RUGAS or global uniform ultimate boundedness of the closed-loop trajectories. \square

Remark 5.2.3.3. The dimension of the set Ω , and so the dimension of S , can be reduced by acting on the gains in K_b and on the sigmoid slopes, as shown by the expression of the radius (5.79). Otherwise, we can use the Hadamard product to derive an alternative version of (5.78) in terms of the following elementwise inequalities, $i = 1, 2, 3$

$$|e_{x_i}| > \sqrt{\frac{\rho_{f_{xu}}^i}{k_{x_i} \sigma_{x_i} e}} \quad (5.82)$$

$$|\zeta_i| > \sqrt{\frac{\rho_{f_z}^{eq_i}}{k_{\zeta_i} \sigma_{\zeta_i} e}} \quad (5.83)$$

which are chosen using a more conservative upper estimate than that of (5.79), but give more degrees of freedom and flexibility to the design, since it is possible to individually set control gains and sigmoid slopes for each error component. \diamond

Matched control

Matched uncertainties are finally compensated using a revisited version of a classic Lyapunov redesign technique (see [62] for the details). The model used for matched control design is simply derived from (5.33), disregarding the time dependence in the nominal part:

$$\begin{aligned} \dot{x} &= f_x(x) + \Delta f_{xu}(x, z, u, t) + g_x^1(x)z \\ \dot{z} &= f_z(x, z) + \Delta f_z(x, z) + g_z(x) [I_3 + \Delta g'_z(x, t)] u \end{aligned} \quad (5.84)$$

whereas, clearly by definition of matched uncertainty:

$$\Delta g'_z(x, t) = g_z^{-1}(x) \Delta g_z(x, t) \quad (5.85)$$

Lyapunov redesign is usually built on the shoulders of a control law and a Lyapunov function which work for a system without uncertainties. Our extension is instead based on the unmatched control law (5.60), u' , which is already a robust control law, and on the associated robust control Lyapunov function W_r . We define the final control law as the sum of the nominal plus unmatched law u' with the matched law u_m

$$u = u' + u_m \quad (5.86)$$

In addition to the assumption of Δg_z being a matched uncertainty, we need a further hypothesis which is strictly related to the already stated matching condition, in order to avoid problems given by a possibly unknown control direction.

Assumption 5.2.3.4. The following inequality holds for the matched uncertainty:

$$\begin{aligned} \|\Delta g'_z(x, t)u\| &= \|\Delta g'_z(x, t)(u'(x, z) + u_m)\| \\ &\leq \|\Delta g'_z(x, t)u'(x, z)\| + \|\Delta g'_z(x, t)\| \|u_m\| \leq \rho_{LR}(x, z) + k_0 \|u_m\| \end{aligned} \quad (5.87)$$

where, as intuition suggests, $\rho_{LR}(x, z, t) \geq \|\Delta g'_z(x, t)u'(x, z)\|$ is the Lyapunov redesign bounding function, and $k_0 \geq \|\Delta g'_z(x, t)\|$ is a scalar design parameter which takes values in the interval $[0, 1)$. \diamond

Clearly assumption 5.2.3.4 can be satisfied only if $\|\Delta g'_z(x, t)\| < 1$ and easy calculations show that this is the case if the bounding functions in ρ_{g_z} , taken for simplicity with a single deviation function, are such that $k < -1 + \sqrt{2}$. If this is true, the following inequalities hold:

$$\|\Delta g'_z(x, t)\| \leq \|g_z^{-1}(x)\| \|\Delta g_z(x, t)\| \leq \|g_z^{-1}(x)\| \|\rho_{g_z}(x)\| \quad (5.88)$$

and assumption 5.2.3.4 is fulfilled. The condition on K clearly limits the maximum amount of matched uncertainty that can be compensated by u_m , whose standard expression is the following:

$$u_m = -\frac{\eta(x, z, t)}{1 - k_0} \cdot \frac{w}{\|w\|} \quad (5.89)$$

where $\eta(x, z, t) \geq \rho_{LR}(x, z, t)$ in the domain of interest and

$$w^T = \frac{\partial W_r}{\partial(e_x, \zeta)} \begin{pmatrix} g_x^2(x) \\ g_z(x) \end{pmatrix} = e_x^T g_x^2(x) + \zeta^T g_z(x)$$

This control law is discontinuous for $w = 0$, thus potentially bringing serious implementation problems like *chattering*. This issue is bypassed using once again the elementwise sigmoid function, which has continuous first derivative. As a result, the complete robust control law is

$$\begin{aligned} u &= u' + u_m \\ &= g_z^{eq}(x, z)^{-1} \left[(e_x^T g_x^1(x))^T - f_z^{eq}(x, z) + \frac{\partial \phi_r}{\partial x} (\dot{x}_d + K_x e_x + g_x^1(x) \zeta) \right. \\ &\quad \left. - K_\zeta \zeta - \rho_{f_z}^{eq} \circ \mathbf{sigm}(\zeta) + \frac{\partial \phi_r}{\partial x} \rho_{f_{xu}} \circ (\mathbf{1} - \mathbf{sigm}(e_x)) \right] \\ &\quad - \frac{\eta(x, z, t)}{1 - k_0} \mathbf{sigm}(w) \end{aligned} \quad (5.90)$$

5.3 Simulation and results

Simulations are made in MATLAB-Simulink ©environment. Two different maneuvers have to be pursued by the missile in order to compare the two controllers (simple non-linear backstepping and robust backstepping) performances in several scenarios with different amounts of uncertainty.

The values of the parameters describing the airframe are listed in tab. 5.1

TABLE 5.1: Missile airframe parameters

Symbol	Definition	Value	Unit
m_0	mass at t_0	185	Kg
I_{x0}	inertia moment at t_0	0.73	$Kg \cdot m^2$
I_{y0}, I_{z0}	inertia moments at t_0	205.7	$Kg \cdot m^2$
I_{xz0}	inertia moment at t_0	0.42	$Kg \cdot m^2$
I_{xf}	inertia moment at t_f	0.64	$Kg \cdot m^2$
I_{yf}, I_{zf}	inertia moments at t_f	180.73	$Kg \cdot m^2$
I_{xxf}	inertia moment at t_f	0.38	$Kg \cdot m^2$
m_f	mass at t_f	137.5	Kg
l	fuselage length	3.65	m
d	fuselage diameter	0.178	m
S	reference area	0.0249	m^2
M	Mach number	[2.5, 3.2]	-
\bar{V}_s	nominal speed of sound	285.4	m/s
$\bar{\rho}$	nominal air density	0.542	Kg/m^3
τ	actuator dynamics time-constant	5	ms

5.3.1 Skid-To-Turn maneuver

The first maneuver performed is the Skid-To-Turn, which requires less effort from the controller, since it does not enhance the couplings and nonlinearities of the airframe dynamics. Both controllers are tested in two scenarios, the first with the 30 % of uncertainty and the second, harder to be dealt with, with the 60 % of deviation from the nominal parameters' values.

Nominal controller The nominal controller behaves good when dealing with low uncertainty, ensuring a rapid convergence to the desired trajectories with small overshoots and control effort. However, increasing the level of the perturbation results in too big deviations from the reference signals and so in a dangerously peaking control signal. In this second scenario the level of performance is clearly unsatisfactory and the nominal autopilot is not able anymore to follow the acceleration commands produced by the guidance system (Fig. 5.2, 5.3, 5.4, 5.5).

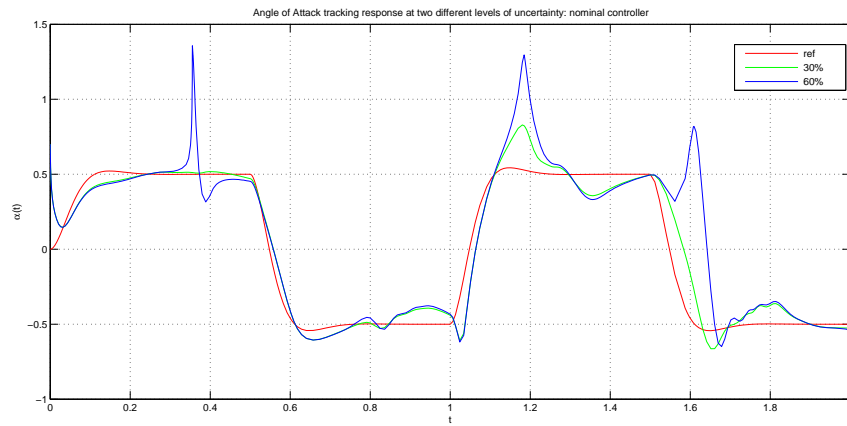


FIGURE 5.2: STT maneuver - Angle of Attack tracking response at two different levels of uncertainty: nominal controller.

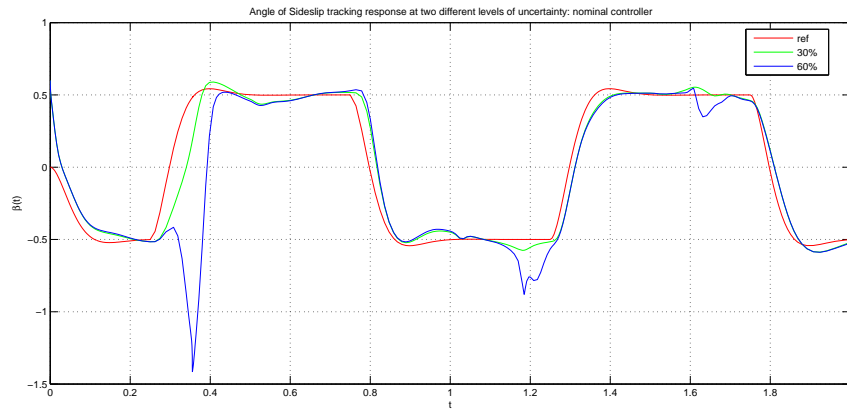


FIGURE 5.3: STT maneuver - Angle of Sideslip tracking response at two different levels of uncertainty: nominal controller.

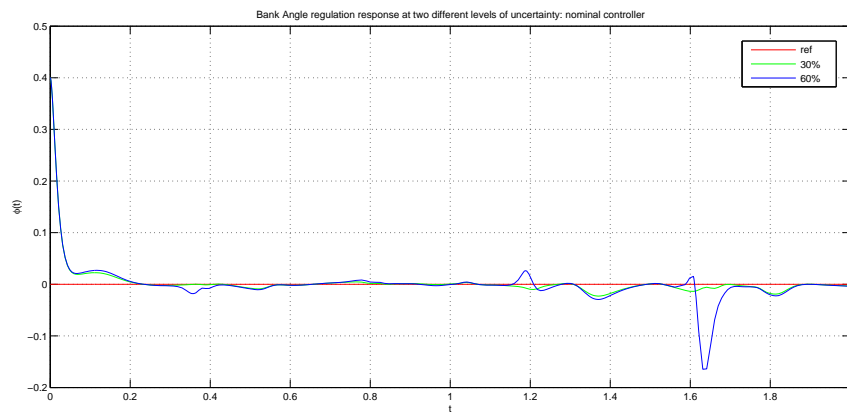


FIGURE 5.4: STT maneuver - Bank Angle regulation response at two different levels of uncertainty: nominal controller.

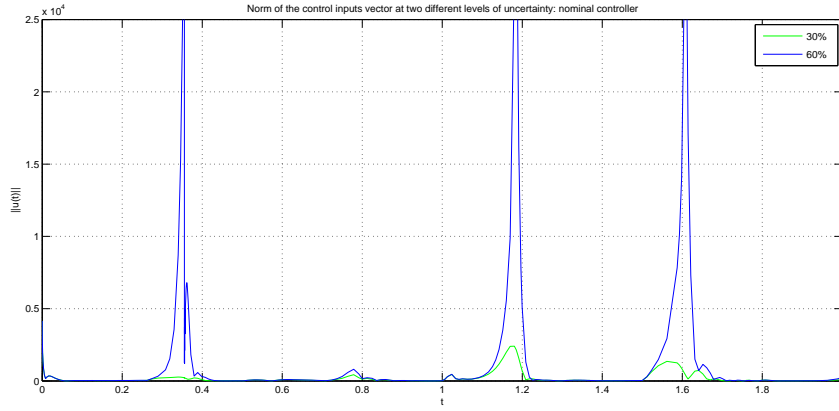


FIGURE 5.5: STT maneuver - Norm of the control inputs vector at two different levels of uncertainty: nominal controller.

Robust controller The robust controller behaves good in both scenarios, with a level of performance a bit poorer when uncertainty is at 60 %, nevertheless maintaining precision and speed of response sufficient to achieve guidance objectives. In the 60 % scenario it should be highlighted the very nervous transient in the control inputs, which is a somewhat structural feature of the robust nonlinear approaches [32] (Fig. 5.6, 5.7, 5.8, 5.9).

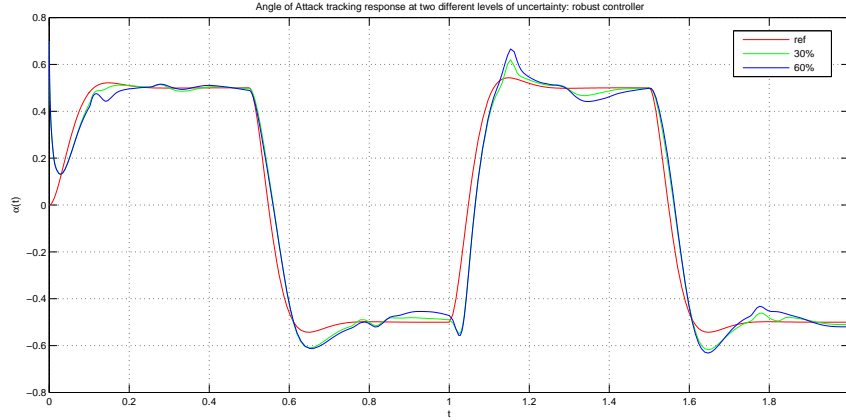


FIGURE 5.6: STT maneuver - Angle of Attack tracking response at two different levels of uncertainty: robust controller.

Comparison A comparison between the two closed-loop behaviors shows the superiority of the robust nonlinear approach in both scenarios, since it ensures higher tracking precision at lower energy (propellant) consumption (Fig. 5.10, 5.11).

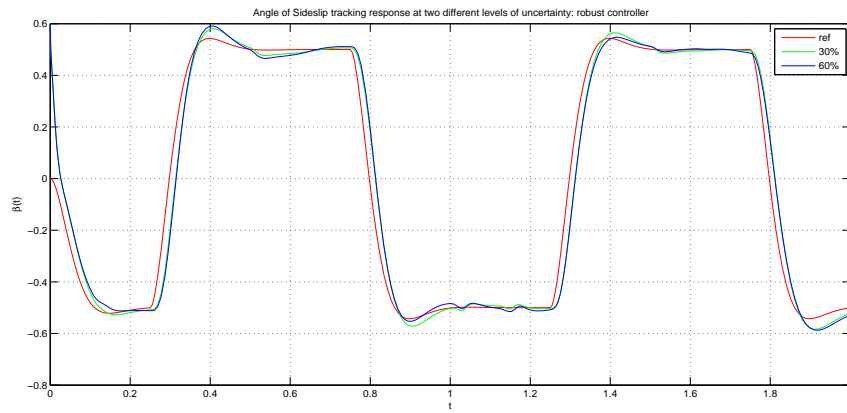


FIGURE 5.7: STT maneuver - Angle of Sideslip tracking response at two different levels of uncertainty: robust controller.

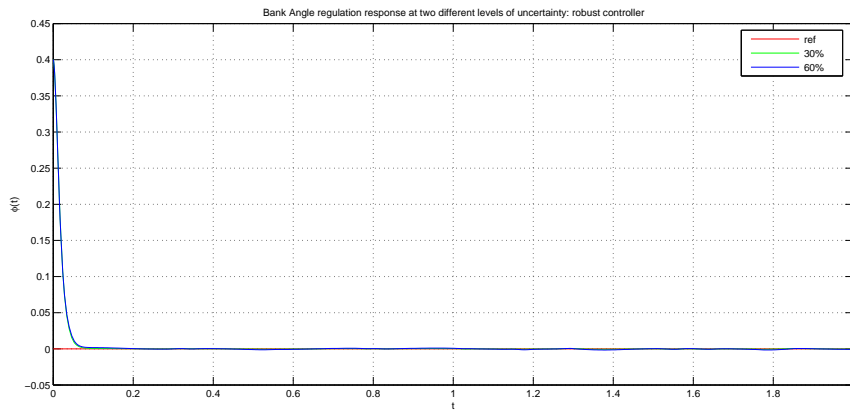


FIGURE 5.8: STT maneuver - Bank Angle regulation response at two different levels of uncertainty: robust controller.

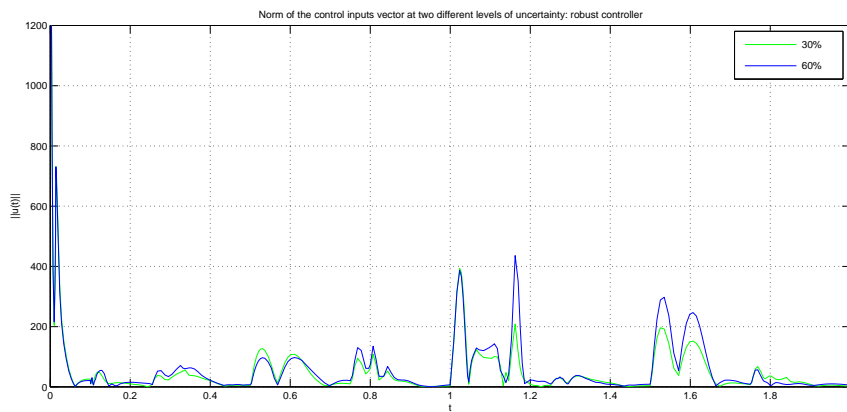


FIGURE 5.9: STT maneuver - Norm of the control inputs vector at two different levels of uncertainty: robust controller.

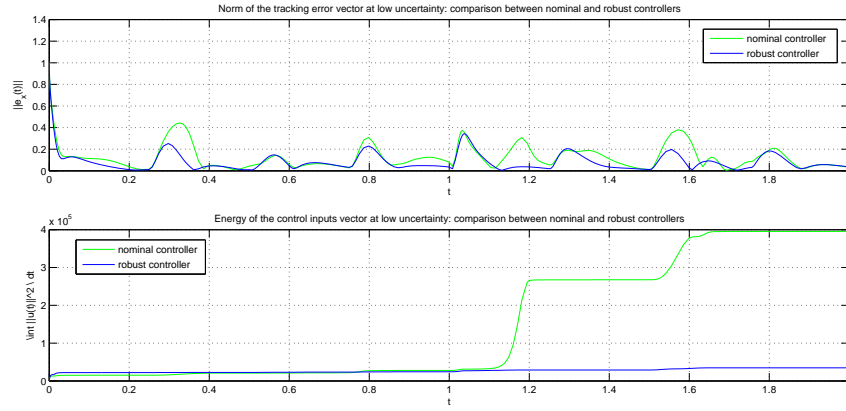


FIGURE 5.10: STT maneuver - Norm of the tracking error vector and energy of the control inputs vector at low uncertainty: comparison between nominal and robust controllers

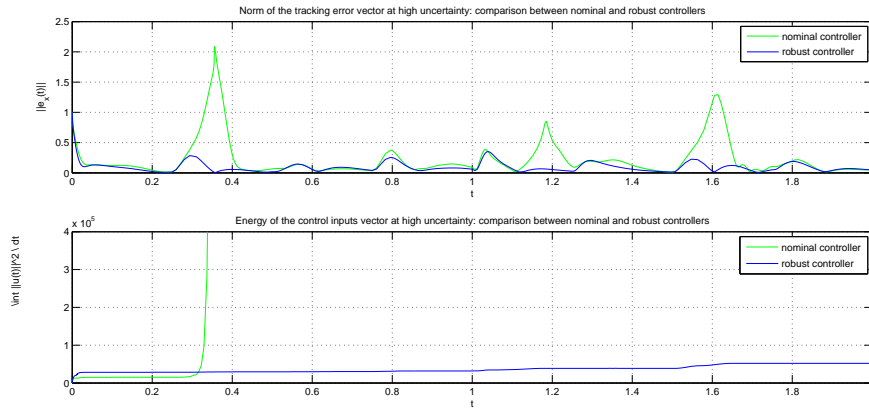


FIGURE 5.11: STT maneuver - Norm of the tracking error vector and energy of the control inputs vector at high uncertainty: comparison between nominal and robust controllers

5.3.2 Bank-To-Turn maneuver

The second maneuver performed is the Bank-To-Turn, which is the more stressful because it does increase the couplings and nonlinearities of the airframe dynamics. Again, the two controllers are tested in two different scenarios, at low (30 %) and high (60 %) levels of uncertainty.

Nominal controller During the Bank-To-Turn maneuver the system exhibits an even more nonlinear and coupled behavior, thus increasing the destabilizing effect of uncertain terms. The nominal controller cannot handle the guidance system requests in both scenarios, because tracking of α is not possible anymore. Tracking of φ and regulation of β can still be achieved, but this is clearly not sufficient for a successful implementation

of the autopilot: the need for a robust version is therefore evident (Fig. 5.12, 5.13, 5.14, 5.15).

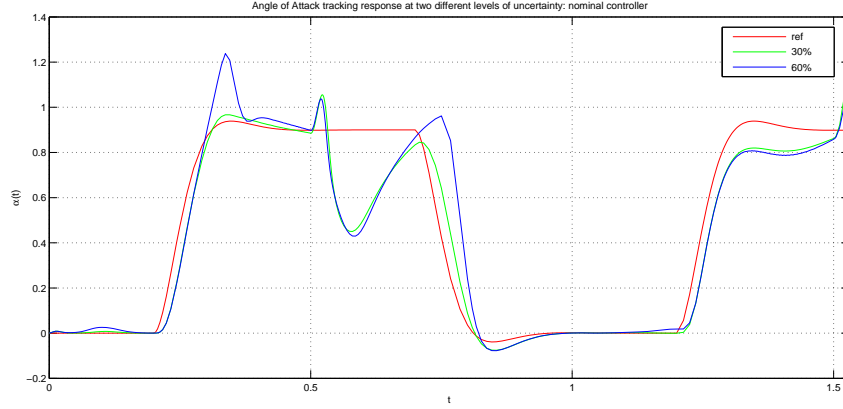


FIGURE 5.12: BTT maneuver - Angle of Attack tracking response at two different levels of uncertainty: nominal controller.

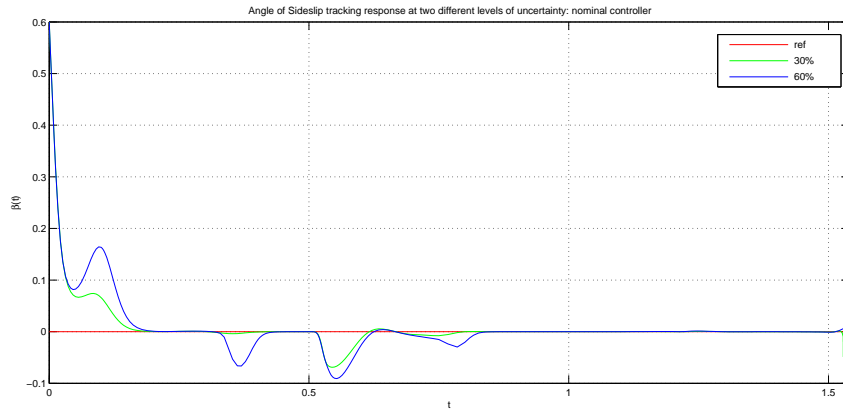


FIGURE 5.13: BTT maneuver - Angle of Sideslip regulation response at two different levels of uncertainty: nominal controller.

Robust controller Robust controller is a bit imprecise in the AoA tracking, while its behavior is very good in both φ tracking and β regulation. Furthermore, the control inputs vector norm is of acceptable amplitude. As a consequence, such controller should be able to follow any acceleration commands given by an external guidance loop, overcoming in performance the nominal one (Fig. 5.16, 5.17, 5.18, 5.19).

Comparison As already stressed before, also in the case of a Bank-To-Turn maneuver the robust version of backstepping is superior to the classic implementation, both in terms of speed and accuracy of the response and even considering the energy exploited. However, the overall performance is definitely worse than in the case of the STT maneuver, especially in tracking of the AoA (Fig. 5.20, 5.21). This fact opens new questions about the need of adaptive versions of the robust backstepping controller, in order to

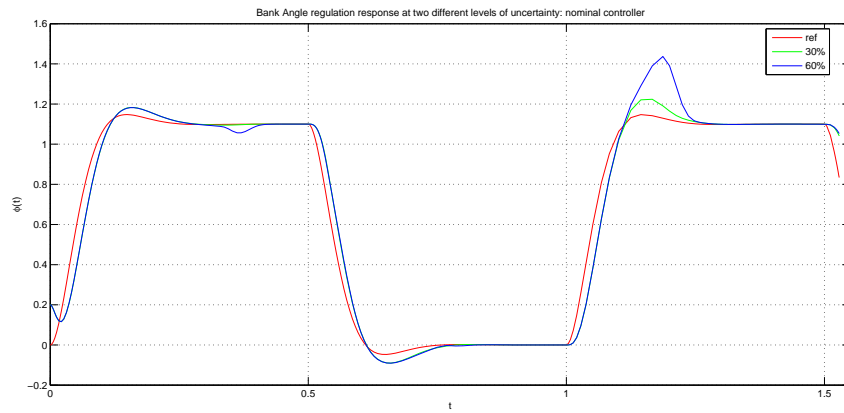


FIGURE 5.14: BTT maneuver - Bank Angle tracking response at two different levels of uncertainty: nominal controller.

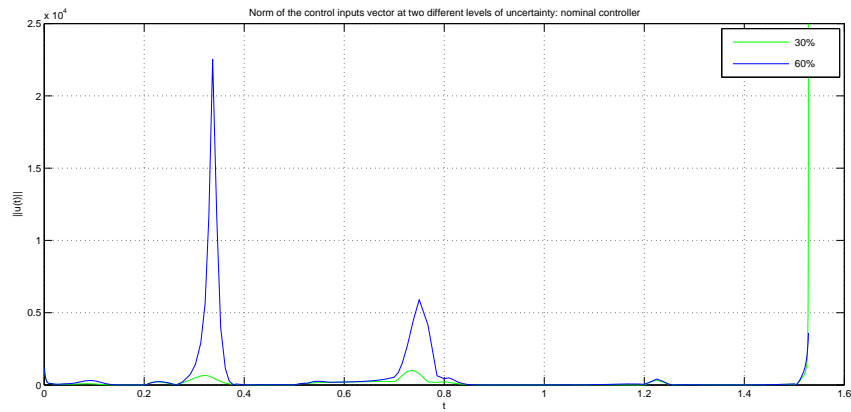


FIGURE 5.15: BTT maneuver - Norm of the control inputs vector at two different levels of uncertainty: nominal controller.

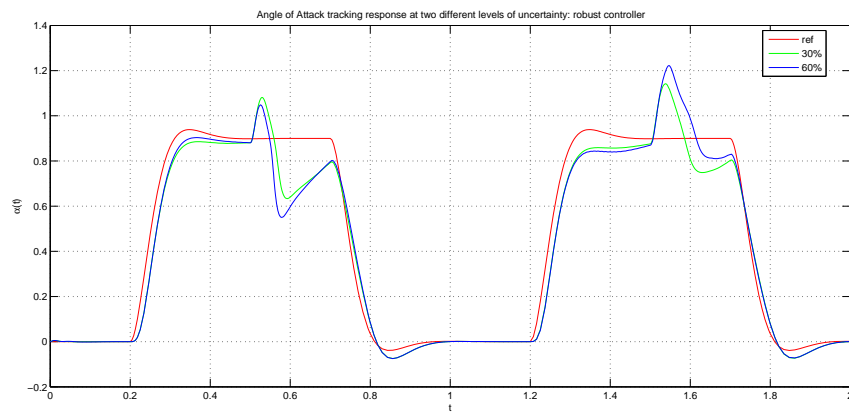


FIGURE 5.16: BTT maneuver - Angle of Attack tracking response at two different levels of uncertainty: robust controller.

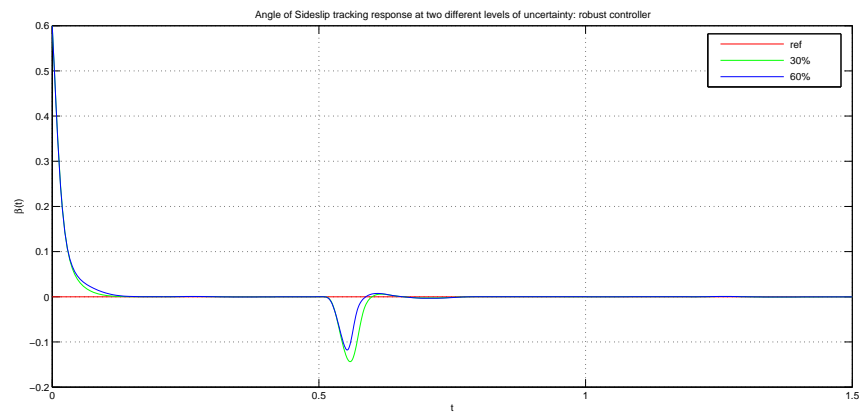


FIGURE 5.17: BTT maneuver - Angle of Sideslip regulation response at two different levels of uncertainty: robust controller.

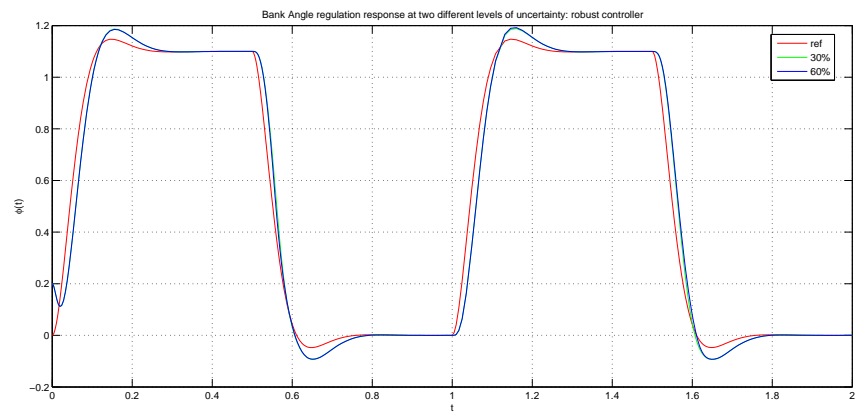


FIGURE 5.18: BTT maneuver - Bank Angle tracking response at two different levels of uncertainty: robust controller.

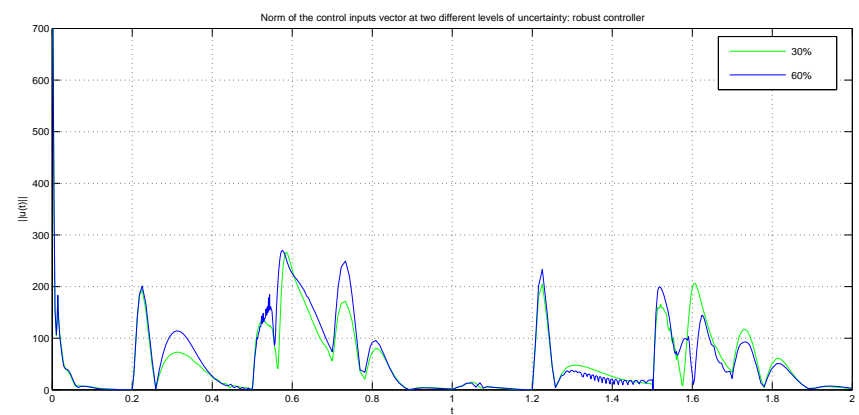


FIGURE 5.19: BTT maneuver - Norm of the control inputs vector at two different levels of uncertainty: robust controller.

maintain a better tracking along the whole reference trajectory even when uncertain terms act in a more aggressive way (i.e. during agile BTT maneuvers).

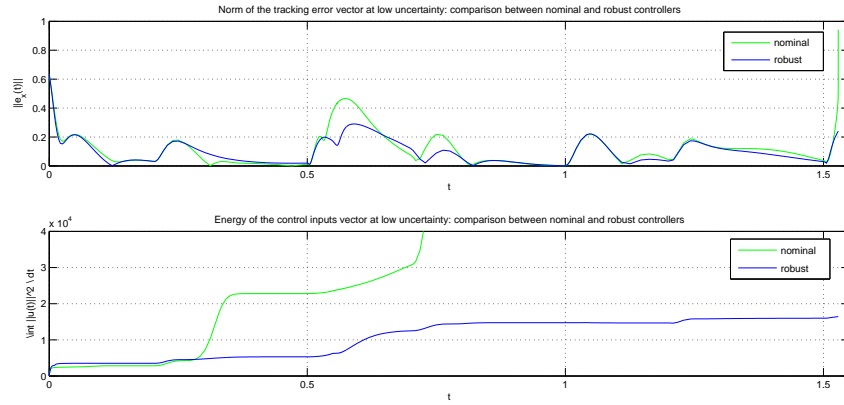


FIGURE 5.20: BTT maneuver - Norm of the tracking error vector and energy of the control inputs vector at low uncertainty: comparison between nominal and robust controllers

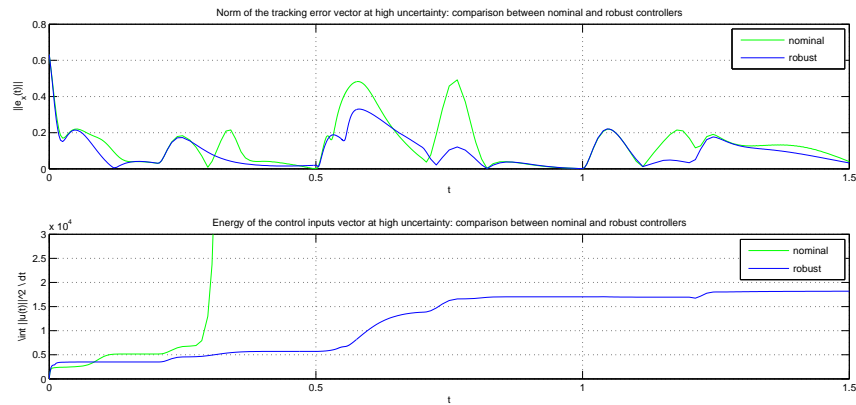


FIGURE 5.21: BTT maneuver - Norm of the tracking error vector and energy of the control inputs vector at high uncertainty: comparison between nominal and robust controllers

Chapter 6

Immersion and Invariance under sampling

The chapter is devoted to the implementation of the Immersion & Invariance stabilization technique under sampling, when, i.e., the control input is maintained constant over intervals of fixed length, namely the sampling period. This approach is not robust in the sense that it counteracts the effects of model uncertainties and disturbances, as the recursive Lyapunov redesign does. More precisely, the I&I technique *adds robustness* to existing nonlinear controllers with respect to neglected dynamics, usually disregarded because faster than the dominant dynamics or, in general, because considered less important in control design. In several cases, such dynamics can critically affect the stability of the closed-loop system, whereas the I&I approach is meant to preserve the desired properties exploiting, at least in part, the knowledge of the model of such dynamics. Often, the presence of higher-order dynamics leads to time-scale separation designs, singular perturbation and center manifold techniques to simplify the design. Such dynamics, for instance associated to fast flexible motions or to actuator dynamics, usually neglected at a first stage of the design, can be suitably handled in the I&I setting.

In this chapter, we reformulate in the sampled-data context the I&I stabilization paradigm for a class of nonlinear systems in feedback form and we propose a constructive solution based on the multi-rate control design philosophy. The main idea is to ensure, via a digital multi-rate construction, the matching of the closed-loop continuous-time evolution of the function $\phi(x)$, describing implicitly the manifold, with the same trajectory under digital control, at the sampling instants. The asymptotic dependency from the sampling period of the multi-rate equivalent model allows for computable approximate solutions. The sampled data design, by reproducing the behavior of ϕ at the sampling instants,

directly guarantees manifold attractivity. Boundedness of the trajectories and the properties of system immersion and manifold invariance are preserved under sampling [93] [90] [87]. The performance improvement of the proposed controllers relies in a remarkable increase of the Maximum Allowable Sampling Period (MASP) with respect to the direct implementation through zero-order hold of the continuous-time solution, which is emulated control. The effectiveness of the proposed control strategy is evaluated on two academic examples and a case study, the cart-pendulum system.

6.1 Sampled-data I&I control design

We consider the problem of I&I stabilization of systems of the form (3.54)-(3.53) and we propose sampled-data solutions in terms of single-rate and multi-rate controllers, which we apply to two academic examples. Next, we apply the digital single-rate solution to the cart-pendulum system.

Problem setting Assume that the control input u is maintained piecewise constant on intervals of fixed length, which is the sampling period. We look for a possibly sampling-dependent controller, which maintains the stability properties achieved by the continuous-time control law under sampling. In particular, the digital controller should verify the sampled-data versions of conditions H1c-H4c of Theorem 3.4.0.11, achieving manifold attractivity and keeping the boundedness of the trajectories under sampling. We seek a sampled-data controller which, following the closed-loop continuous-time evolution of the function $\phi(x)$, ensures attractivity of \mathcal{M} and boundedness under sampling. The closed-loop system under digital control should be globally asymptotically stable. To begin with, let us briefly recall the sampled-data equivalent dynamics under single and multi-rate sampling.

6.1.1 Sampled-data equivalent models

The single-rate *sampled-data equivalent model* is the discrete-time dynamics reproducing, at the sampling instants, the solution of (3.54) when the control variable $u(t)$ is kept constant over time periods of length δ , namely $u(t + \tau) = u(t) = u_k$ for $0 \leq \tau < \delta$, $t = k\delta$, $k \geq 0$. It is described by the δ -parametrized map $F^\delta(\cdot, u_k)$ (the pair u, F^δ) admitting the Lie exponential series expansion:

$$x_{k+1} = F^\delta(x_k, u_k) = e^{\delta(f + u_k g)} x_k \quad (6.1)$$

On the other hand, the *multi-rate* sampled-data equivalent model of order m - MR^m - reproduces, at the sampling instants, the continuous-time behavior of (3.53) when $u(t)$ is maintained constant at values u_{ik} , over intervals of length $\bar{\delta} = \frac{\delta}{m}$ for $\tau \in ((i-1)\bar{\delta}, i\bar{\delta})$. Over each $\delta = m\bar{\delta}$, it is described by the $\bar{\delta}$ -parametrized map (the pair $u_i, F^{m\bar{\delta}}$ for $i = 1, \dots, m$)

$$\begin{aligned} x_{k+1} &= F^{m\bar{\delta}}(x_k, u_{1k}, \dots, u_{mk}) \\ &= e^{\bar{\delta}(f+u_{1k}g)} \circ \dots \circ e^{\bar{\delta}(f+u_{mk}g)} x_k \end{aligned} \quad (6.2)$$

From the computational point of view, it is convenient to define the approximate versions of these representations, namely the *approximate single-rate* (resp. *multi-rate*) *sampled-data model of degree ν* , with reference to truncations of the expansions (6.1) (resp. (6.2)) at finite order ν in δ , i.e.

$$F^\delta(x, u) = F^{\delta[\nu]}(x, u) + O(\delta^\nu) \quad (6.3)$$

$$F^{m\bar{\delta}}(x, u_1, \dots, u_m) = F^{m\bar{\delta}[\nu]}(x, u_1, \dots, u_m) + O(\delta^\nu) \quad (6.4)$$

Remark 6.1.1.1. With $\nu = 1$ (approximation in $O(\delta^2)$), one recovers the well known Euler sampled-data dynamics $x_{k+1} \cong x_k + \delta f(x_k) + \delta u_k g(x_k)$, preserving the structure of the continuous-time system. \diamond

6.1.2 Main result

The sampled-data control is designed employing a multi-rate strategy (see [92]) of order $n - p$, which achieves manifold attractivity under sampling by matching the evolution of $\phi(x)$, under continuous-time control (3.59), at the sampling instants. Since the continuous-time assumptions are fulfilled, we expect ϕ to vanish asymptotically. Boundedness under sampling is ensured by matching between the sampled-data trajectory and the continuous-time one. The behavior on the manifold is reduced to that of the target system under sampling, yielding the global asymptotic stability of the digital control system.

Theorem 6.1.2.1. *Consider the class of feedback systems (3.53) satisfying assumptions 3.4.2.1 and 3.4.2.2, with $n - p = 2$, with an equilibrium x^* to be stabilized. There exists an order $m = 2$ multi-rate sampled-data control law of the form*

$$\begin{aligned} u_{d1} &= \psi_1^\delta(x_k, \phi(x_k)) \\ u_{d2} &= \psi_2^\delta(x_k, \phi(x_k)) \end{aligned} \quad (6.5)$$

such that x^* is a globally asymptotically stable equilibrium of the closed-loop dynamics

$$x_{k+1} = F^{2\bar{\delta}}(x_k, \psi_1^\delta(x_k, \phi(x_k)), \psi_2^\delta(x_k, \phi(x_k))) \quad (6.6)$$

◇

Proof. Following assumptions H1c to H4c of the continuous-time result, we can define for (3.53) a sampled-data equivalent target system, with state $\xi \in \mathbb{R}^p$

$$\xi_{k+1} = \alpha_d(\xi_k) \quad (6.7)$$

where $\alpha_d(\xi) = e^{\delta f_1(\xi, 0)}\xi$ has a globally asymptotically stable equilibrium at $\xi^* \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$. Condition H2c can be reformulated as follows

$$F^{2\bar{\delta}}(\pi(\xi_k), 0) = \pi(\alpha_d(\xi_k)) \quad (6.8)$$

which, exploiting the properties of the exponential representation, yields that π is equal to the identity function for the first p components, while it is necessarily equal to zero for the remaining $n - p$ components. Thus invariance under sampling is ensured. Keep in mind that, for the class of systems at study, $c(\xi) = 0$. As a consequence, the implicit manifold condition H3c is verified under sampling with the choice $\phi_1(x) = x_2, \phi_2(x) = x_3$. The existence of a multi-rate controller of the form (6.5) for (3.53) is ensured provided the following rank condition [93] is fulfilled

$$\text{rank}[g, \text{ad}_f g, \text{ad}_f^2 g, \dots, \text{ad}_f^p g, \dots] \geq n - p. \quad (6.9)$$

When applied to (3.53), condition (6.9) results verified if $\frac{\partial f_2}{\partial x_3} \neq 0$, which is true thanks to the assumptions 3.4.2.1 and 3.4.2.2. Consider now the extended system under sampling ($i = 1, 2$)

$$z_{k+1} = \phi(\tilde{F}^{(n-p)\bar{\delta}}(x_k, \psi_i^\delta(x_k, z_k))) \quad (6.10)$$

$$x_{k+1} = \tilde{F}^{(n-p)\bar{\delta}}(x_k, \psi_i^\delta(x_k, z_k)) \quad (6.11)$$

where $z = \phi(x) = (x_{n-1}, x_n)^T$. Attractivity of \mathcal{M} is ensured by matching the continuous-time trajectories of $\phi(x)$, which for the systems under study reduces to a simple input to partial-state matching, achieved by the existing multi-rate solution according to the results in [93]. As a consequence, $\lim_{k \rightarrow \infty} z_k = 0$. Boundedness of the trajectories of (6.10) is guaranteed by one-step consistency property plus forward completeness of the vector fields of (3.53), hence the thesis. \square

Remark 6.1.2.2. In general, with x_2 of dimension q , the sampled-data I&I solution is given by a multi-rate control law of order $m = q + 1$. The multi-rate controller is built to follow the closed-loop continuous-time evolution of the function $\phi(x)$. \diamond

In the following, we are going to develop an approximate version of the proposed digital controller (6.5), leading to a *computable* control law.

6.1.3 Sampled-data approximate design

We derived the continuous-time I&I stabilizing control law (3.59) for a special class of systems in the case $n - p > 1$, establishing that a sampled-data multi-rate control logic is well suited to solve the problem in a digital setting. Now we look for constructive, approximate, solutions according to (6.3).

6.1.3.1 Single-rate solution

For the class of systems (3.54), with $n - p = 1$, the continuous-time solution (3.55), namely $\gamma(x)$, yields a single-rate sampled-data controller u_d designed to match at the sampling instants the controlled x_2 dynamics. The matching condition is

$$e^{\delta f(\cdot, u_d)} x_2 = e^{\delta f(\cdot, \gamma)} \phi|_{x_1, x_2} \quad (6.12)$$

with $f(x_1, x_2, u) = f_1(x_1, x_2) \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial x_2}$. In this case, since $\dot{x}_2 = u$, (6.12) is satisfied by the single-rate sampled-data control law

$$u_d = \gamma(x) + \frac{\delta}{2!} \dot{\gamma}(x) + \frac{\delta^2}{3!} \ddot{\gamma}(x) + \sum_{i \geq 3} \frac{\delta^i}{(i+1)!} u_c^{(i)}(x) \quad (6.13)$$

According to (6.3), an approximate version of (6.13), truncated at the first term in $O(\delta^2)$, can be given as follows:

$$u_d = u_{d0} + \frac{\delta}{2} u_{d1}. \quad (6.14)$$

Note that the term $u_{d0} = \gamma(x)|_{x_k}$ is the emulated controller, with $x_k = x(t = k\delta)$ and $u_{d1} = \dot{u}_c(x)|_{x_k} = \frac{\partial \gamma}{\partial x}(f + g\gamma(x))|_{x_k}$.

In this particular case, the digital control can be directly expressed in terms of the continuous-time control input as follows:

$$u_d = \frac{1}{\delta} \int_{k\delta}^{(k+1)\delta} u_c(\tau) d\tau, \quad (6.15)$$

which is exactly the average of the continuous control signal evaluated on the sampling period.

6.1.3.2 Multi-rate solution

For systems in the form (3.53), the multi-rate solution is better suited as well as of straightforward derivation. In the case $n - p = 2$, a multi-rate controller of order $m = 2$ can be designed to match the continuous-time controlled trajectories of $\phi(x) = (x_2, x_3)^T$ at the sampling instants. Recall that the continuous-time solution (3.59) is of the form $\psi(x_1, \phi(x))$. The input to partial-state matching conditions in this case are given by

$$\begin{aligned} e^{\bar{\delta}(\bar{f} + u_{d1}\bar{g})} \circ e^{\bar{\delta}(\bar{f} + u_{d2}\bar{g})} x_2 &= e^{\delta(\bar{f} + \psi(\cdot, \cdot)\bar{g})} \phi_1|_{x_1, z} \\ e^{\bar{\delta}(\bar{f} + u_{d1}\bar{g})} \circ e^{\bar{\delta}(\bar{f} + u_{d2}\bar{g})} x_3 &= e^{\delta(\bar{f} + \psi(\cdot, \cdot)\bar{g})} \phi_2|_{x_1, z} \end{aligned} \quad (6.16)$$

where $\bar{f} = (f_2(x_2, x_3), 0)^T$, $\bar{g} = (0, 1)^T$. As stated by Theorem 6.1.2.1, there exists a multi-rate sampled-data control law of order 2, satisfying (6.16), of the form

$$\begin{aligned} u_{d1} &= u_{d10} + \sum_{i \geq 1} \frac{\bar{\delta}^i}{(i+1)!} u_{d1i} \quad \text{over } [0, \bar{\delta}) \\ u_{d2} &= u_{d20} + \sum_{i \geq 1} \frac{\bar{\delta}^i}{(i+1)!} u_{d2i} \quad \text{over } (\bar{\delta}, \delta] \end{aligned} \quad (6.17)$$

The approximate control laws which solve the problem up to an error in $O(\delta^3)$ (in x_3), arresting the expansions (6.17) at $i = 2$, are given by the following expressions for the u_{dji} terms

$$\begin{aligned} u_{d10} &= u_{d20} = \psi(x_1, z) \\ u_{d11} &= \frac{2}{3} u_{c1} \quad u_{d21} = \frac{10}{3} u_{c1} \end{aligned} \quad (6.18)$$

where $u_{c1} = \dot{\psi}(x)$. Substituting (6.18) into (6.17) yields

$$u_{d1} = \psi(x) + \frac{\bar{\delta}}{3}\dot{\psi}(x) \quad u_{d2} = \psi(x) + \frac{5\bar{\delta}}{3}\dot{\psi}(x) \quad (6.19)$$

with $\delta = 2\bar{\delta}$. On the first sub-interval $[0, \bar{\delta})$ the control applied is u_{d1} , which switches to u_{d2} in $[\bar{\delta}, \delta)$. The obtained multi-rate control law (6.19) renders the origin of (3.59) GAS under sampling, if δ is chosen small enough. The approximate single and multi-rate controllers (6.14)-(6.19) overcome in performance the emulated control laws $u_{d0} = u_{d10} = u_{d20}$, preserving stability for increasing values of δ as shown in the two examples below.

6.2 Example with single-rate control

The results can be illustrated through an academic example discussed in [6]

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^3 x_2 \\ \dot{x}_2 &= u \end{aligned} \quad (6.20)$$

with x_1 and $x_2 \in \mathbb{R}$. In this particular case, $z = \phi(x) = x_2$ and, on the manifold, $u_c = -(2 + x_1^8) x_2|_{x_2=0} = 0$. Exploiting the arguments presented in [6], the continuous-time stabilizing I&I control law is found to be

$$u_c = -(2 + x_1^8) x_2. \quad (6.21)$$

The computation of the sampled feedback is made according to the approximate sampled-data controller (6.14) which, truncated at the first term in $O(\delta)$, gives

$$u_d = -(2 + x_1^8) x_2 + \frac{\delta}{2} x_2 (x_1^6 - 8x_2 x_1^4 + 12x_1^8 + 4) \quad (6.22)$$

We compare the target system and the continuous-time trajectories with those under emulated control u_{d0} and those under u_d , for increasing values of the sampling period δ . In the first scenario, with $\delta = 0.01$ s, both controllers behaves very well and the difference in performance is imperceptible. In the second scenario, with $\delta = 0.2$ s, the state trajectories under u_{d0} and u_d track the target dynamics and the continuous time evolution very well, with a slightly increased performance under u_d (Fig. 6.1, 6.2, 6.3).

In the third scenario, with $\delta = 0.8$ s, the state trajectories convergence under u_d is still more than acceptable: boundedness and attractivity are preserved. However, as expected, the behavior under u_{d0} is unstable as shown in Fig. 6.4, 6.5 and 6.6.

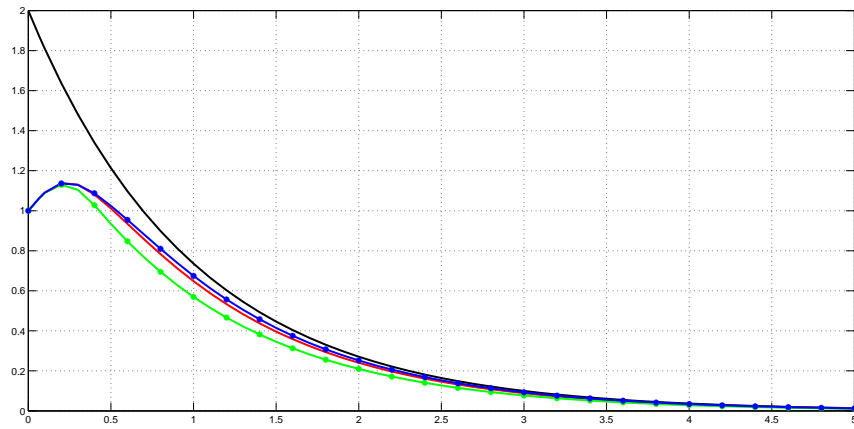


FIGURE 6.1: x_1 evolutions and comparison with the target system when $\delta = 0.2$ s. In black the target system, in red the continuous-time evolution, in blue the evolution under u_d and in green that under u_{d0} .

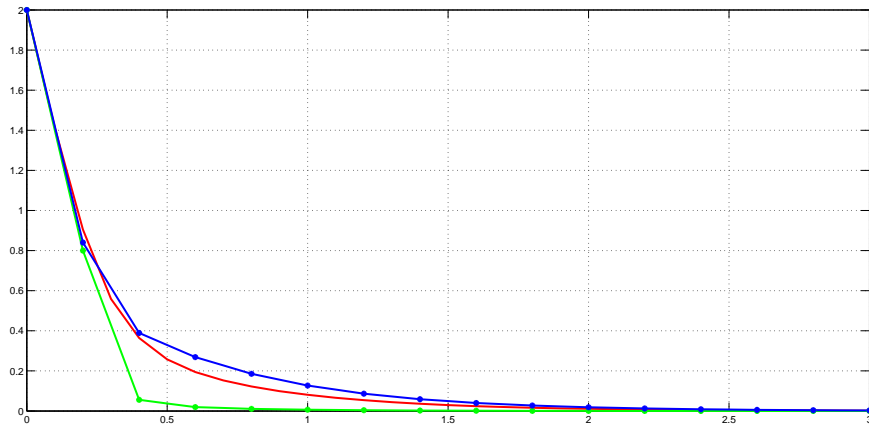


FIGURE 6.2: x_2 evolutions when $\delta = 0.2$ s. In red the continuous-time evolution, in blue the evolution under u_d and in green that under u_{d0} .

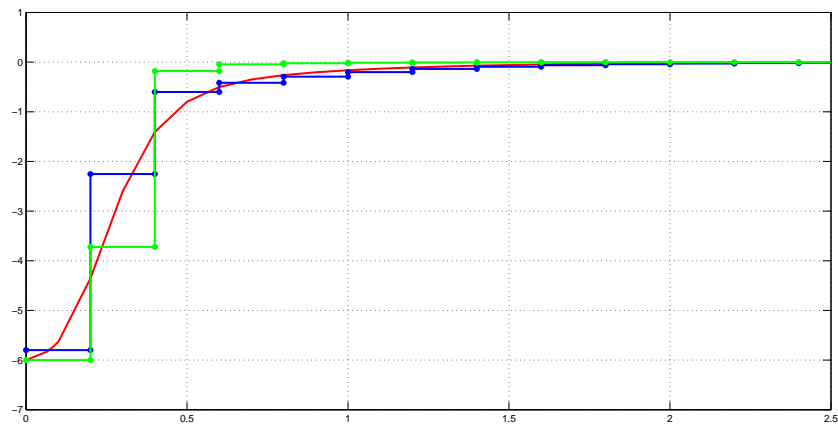


FIGURE 6.3: u when $\delta = 0.2$ s. In red the continuous-time input, in blue u_d and in green u_{d0} .

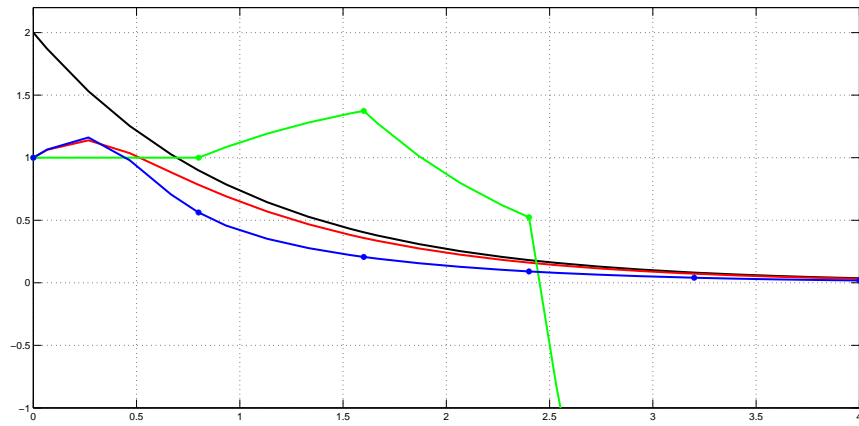


FIGURE 6.4: x_1 evolutions and comparison with the target system when $\delta = 0.8$ s. In black the target system, in red the continuous-time evolution, in blue the evolution under u_d and in green that under u_{d0} .

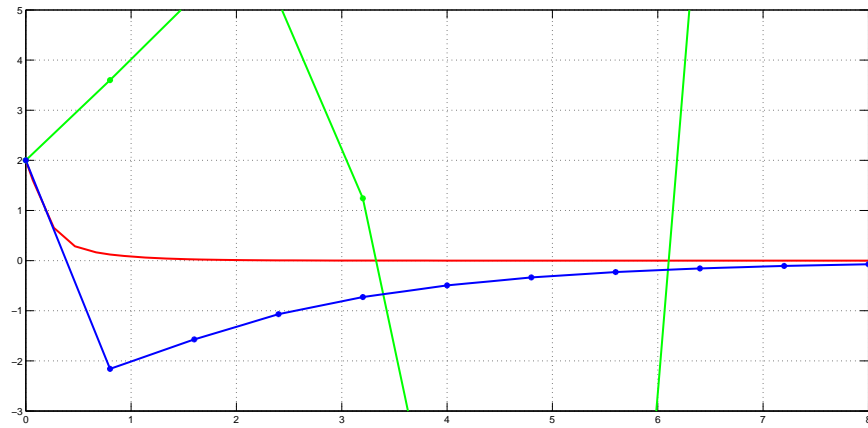


FIGURE 6.5: x_2 evolutions when $\delta = 0.8$ s. In red the continuous-time evolution, in blue the evolution under u_d and in green that under u_{d0} .

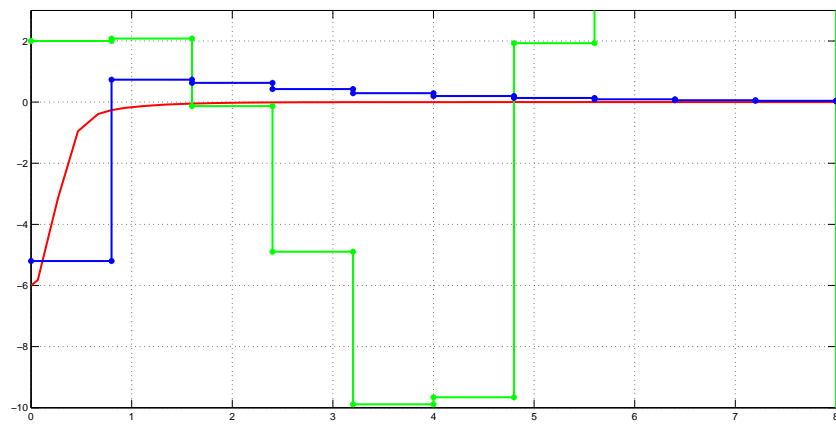


FIGURE 6.6: u when $\delta = 0.8$ s. In red the continuous-time input, in blue u_d and in green u_{d0} .

6.3 Example with multi-rate control

Let us consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2 x_2^2 \\ \dot{x}_2 &= x_2^2 + x_3 \\ \dot{x}_3 &= u\end{aligned}\tag{6.23}$$

which is a system of the form (3.58) with target dynamics $\dot{\xi} = -\xi$, $\alpha(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$. In this case $p = 1$, hence $n - p = 2$. Due to the lower-triangular structure of the system, conditions H2c and H3c hold with $c(\xi) = 0$ and $z = \phi(x) = (x_2, x_3)^T$. Manifold attractivity and boundedness of the trajectories are achieved using simple nonlinear domination arguments, thus applying (3.59) with the choice $k_1 = 2 + x_1^4$, $k_2 > 0$, results in the control law:

$$\begin{aligned}u &= -x_2 - (2x_2 + 2 + x_1^4)(x_3 + x_2^2) \\ &\quad - 4x_2x_1^3(-x_1 + x_1^2x_2^2) - k_2(x_3 + x_2^2 + 2x_2 + x_2x_1^4)\end{aligned}\tag{6.24}$$

which ensures $\lim_{z \rightarrow \infty} = 0$ and boundedness of the trajectories choosing

$$\tilde{W} = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2^2 + (x_3 + x_2^2 + 2x_2 + x_2x_1^4)^2\right)\tag{6.25}$$

Remark 6.3.0.1. The Lyapunov function (6.25) represents a quite strong choice. In fact, it is possible to relax the choice of V , using a weak Lyapunov function, with derivative negative outside a compact set containing the origin, or even a semidefinite Lyapunov function: the boundedness property required for x_1 is still maintained. Other issues, as reducing the dimensions of the (ultimate) bound or the time of convergence, can be taken into account exploiting further design parameters. \diamond

The sampled-data design aims at following the closed-loop behavior of the two-dimensional function $\phi = (x_2, x_3)^T$ with a multi-rate controller of order two. The classical solutions with $m = n - p = 2$ are truncated at the first order in δ , so we apply (6.19) directly.

Setting $k_2 = 4$, the expression of u_{c1} takes the form

$$\begin{aligned}
 u_{c1} = & 4x_1^9x_2^3 + x_1^8x_2^2 + x_1^8x_3 - 20x_1^6x_2^5 - 8x_1^5x_2^4 + 44x_1^5x_2^3 \\
 & - 16x_1^5x_2^2x_3 + 2x_1^4x_2^3 + 16x_1^4x_2^2 + 2x_1^4x_2x_3 \\
 & + 9x_1^4x_2 + 16x_1^4x_3 - 2x_2^4 + 12x_2^3 - 4x_2^2x_3 \\
 & + 45x_2^2 + 12x_2x_3 + 54x_2 - 2x_3^2 + 27x_3.
 \end{aligned} \tag{6.26}$$

Note that the multi-rate solution combines the emulation of the continuous controller with its weighted derivative in closed-loop. Again, we compare the target system and the continuous-time trajectories with those under emulated control $u_{d10} - u_{d20}$ and those under $u_{d1} - u_{d2}$, for increasing values of the sampling period δ .

In the first scenario, with $\delta = 0.05$ s, both the emulated and the multi-rate solutions behave well in steering to zero system trajectories (Fig. 6.7, 6.8). However, a nervous transient behavior can be observed in the emulated case, especially in the control input (Fig. 6.9). In the second scenario, with $\delta = 0.1$ s, it is clear from Fig. 6.10, 6.11 and

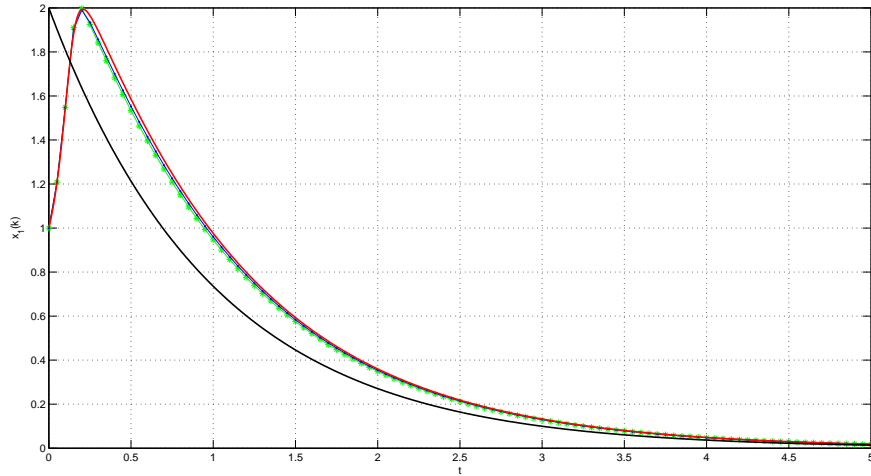


FIGURE 6.7: x_1 evolutions and comparison with the target system when $\delta = 0.05$ s. In black the target system, in red the continuous-time evolution, in blue the evolution under multi-rate and in green that under $u_{d10} - u_{d20}$.

6.12 that the emulated controller cannot guarantee stability anymore, while the behavior under multi-rate control is still more than acceptable.

Next, we propose the application of the single-rate digital controller to an underactuated system with relative degree one, the cart-pendulum. Also in this case, the continuous-time solution has been studied and proposed in [5].

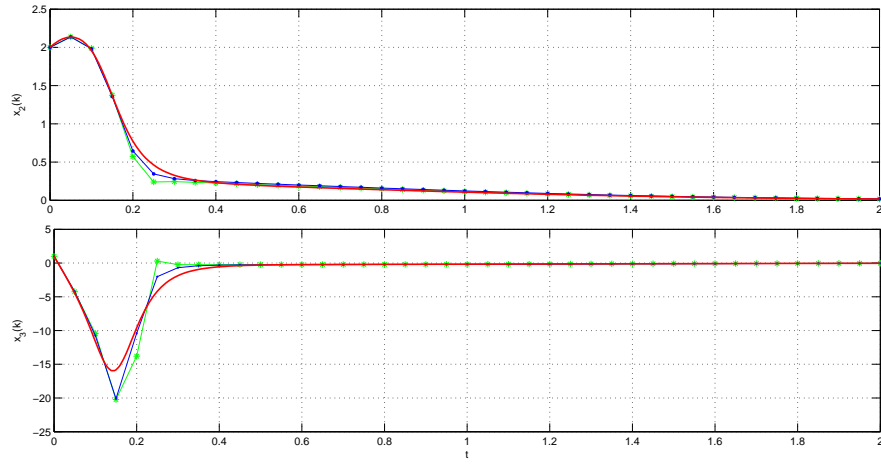


FIGURE 6.8: x_2 and x_3 evolutions when $\delta = 0.05$ s. In red the continuous-time evolution, in blue the evolution under multi-rate and in green that under $u_{d10} - u_{d20}$.

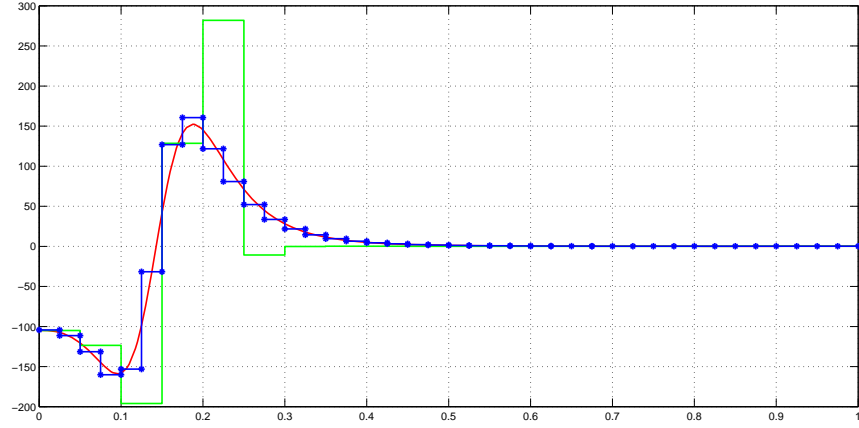


FIGURE 6.9: u when $\delta = 0.05$ s. In red the continuous-time input, in blue the multi-rate and in green the emulated.

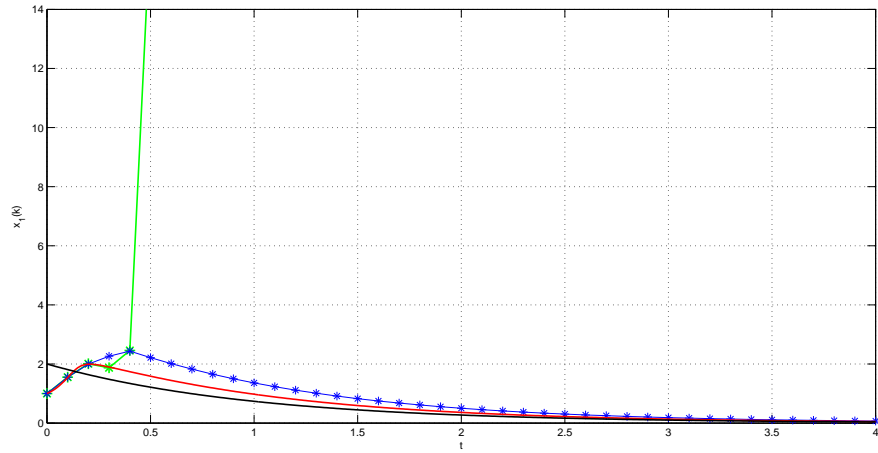


FIGURE 6.10: x_1 evolutions and comparison with the target system when $\delta = 0.1$ s. In black the target system, in red the continuous-time evolution, in blue the evolution under multi-rate and in green that under $u_{d10} - u_{d20}$.

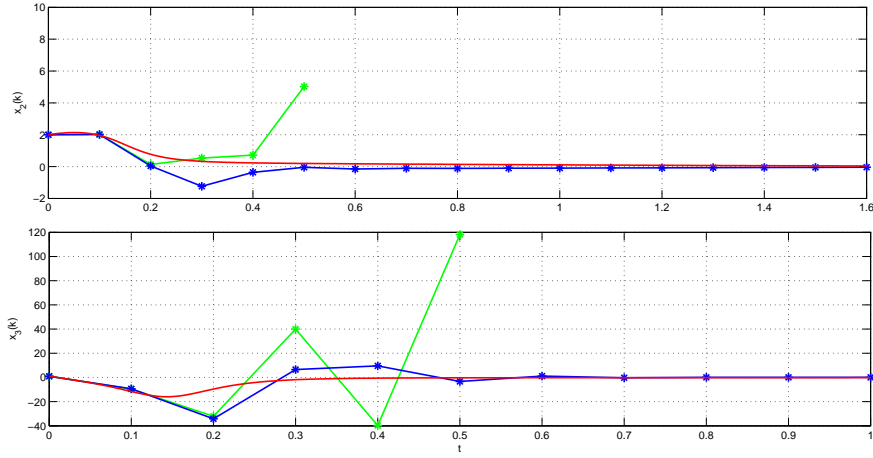


FIGURE 6.11: x_2 and x_3 evolutions when $\delta = 0.1$ s. In red the continuous-time evolution, in blue the evolution under multi-rate and in green that under $u_{d10} - u_{d20}$.

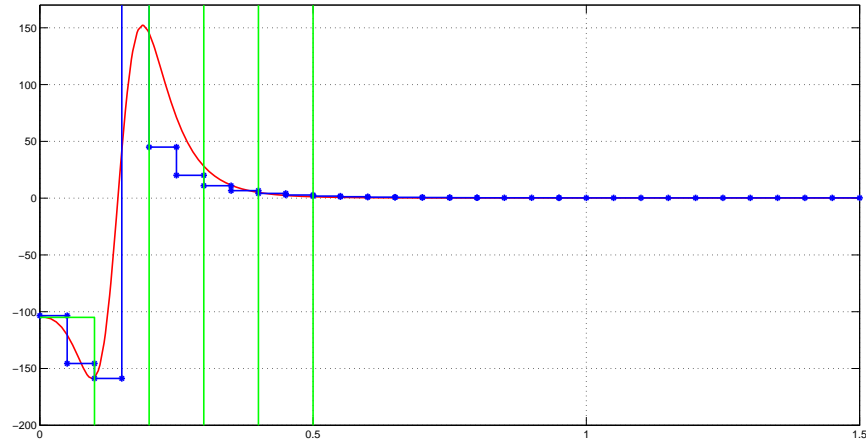


FIGURE 6.12: u when $\delta = 0.1$ s. In red the continuous-time input, in blue the multi-rate and in green the emulated.

6.4 The cart-pendulum system

Following [6], we develop a sampled-data controller for a classical example based on an underactuated system with three state variables, the cart-pendulum. The partially-linearized and normalized cart-pendulum system equations of motion are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) - u \cos(x_1) \\ \dot{x}_3 &= u \end{aligned} \tag{6.27}$$

where $(x_1, x_2) \in \mathcal{S}^1 \times \mathbb{R}$ are the pendulum angle (w.r.t. the upright vertical) and its velocity, $x_3 \in \mathbb{R}$ is the velocity of the cart and $u \in \mathbb{R}$ is the control input. We want to stabilize the pendulum in its upward position, with the cart stopped, corresponding to

the equilibrium $x^* = 0$. Usually, a smart target-system choice for this kind of systems is the unactuated part of the mechanism to which has been assigned a desired dynamics, in this case that of a fully actuated pendulum

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -V'(\xi_1) - R(\xi_1, \xi_2)\xi_2\end{aligned}\tag{6.28}$$

with $R(\cdot)$ damping function, $R(0,0) > 0$, $V(\xi_1)$ potential energy function, $V'(0) = 0$ and $V''(0) > 0$, and $H(\xi_1, \xi_2) = \frac{1}{2}\xi_2^2 + V(\xi_1)$ total energy function. A natural selection of the mapping $\pi(\cdot)$ is

$$\pi(\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \pi_3(\xi_1, \xi_2) \end{pmatrix}\tag{6.29}$$

The choice $\pi_3(x_1, x_2) = -k_1x_1 - \frac{k_2}{\cos(x_1)}x_2$ with $k_1 > 0$ and $k_2 > 1$ yields the following continuous-time control law

$$\begin{aligned}u(x) &= \frac{1}{k_2 - 1} \left[\gamma \left(x_3 + k_1x_1 + \frac{k_2}{\cos(x_1)}x_2 \right) \right. \\ &\quad \left. + k_1x_2 + k_2 \tan(x_1) \left(\frac{x_2^2}{\cos(x_1)} + 1 \right) \right]\end{aligned}\tag{6.30}$$

which, with $\gamma > 0$, solves the problem of I&I stabilization of the origin of the cart-pendulum system. For a detailed discussion of the continuous-time control design, see [6] and the references therein.

Since the relative degree of (6.27), considering x_1 as output function, is one, the computation of the sampled feedback can be done according to (6.13), which for the first term in $O(\delta^2)$ gives

$$\begin{aligned}\dot{u}(x) &= \left(g \left(\frac{k_1 + k_2x_2 \sin(x_1)}{\cos(x_1)^2} + k_2 \tan(x_1)^2 + 1 \right) \left(\frac{x_2^2}{\cos(x_1)} + 1 \right) \right. \\ &\quad \left. + \frac{k_2x_2^2 \sin(x_1) \tan(x_1)}{\cos(x_1)^2(k_2 - 1)} \right) x_2 + \left(k_1 + \frac{gk_2}{\cos(x_1)} + \frac{2k_2x_2}{\tan(x_1) \cos(x_1)(k_2 - 1)} \right) \\ &\quad \times (\sin(x_1) - \cos(x_1)u(x)) + \frac{g}{(k_2 - 1)}u(x)\end{aligned}\tag{6.31}$$

Simulations are carried out to highlight the difference in performance between the emulated controller, which is (6.30) evaluated at the sampling instants $t = k\delta$, and the digital controller with the first order corrector term defined in (6.31). Two scenarios are considered: the first with sampling time $T = 0.1$ s and the second with $T = 0.3$ s.

When the sampling time is sufficiently small, as in the case of $T = 0.1$ s, both the emulated and the first-order controllers behave well in reaching asymptotically the target system evolution, as shown in Fig. 6.13, 6.14, 6.15. The continuous-time trajectories are

well followed by both the controllers. However, note that the transient response shown by the system controlled using the first-order correction term is smoother and shows less overshoot than the emulated response. Similar observations can be made for the control inputs transient responses in Fig. 6.16. The superiority of the controller with correction term is mostly due to the *predictive* action of \dot{u} , which increases damping. In the following figures we show in black the target system evolution, in red the continuous-time one, in green the evolution under emulation of the continuous-time controller and in blue the state trajectories of the system in closed-loop with the first order correction controller.

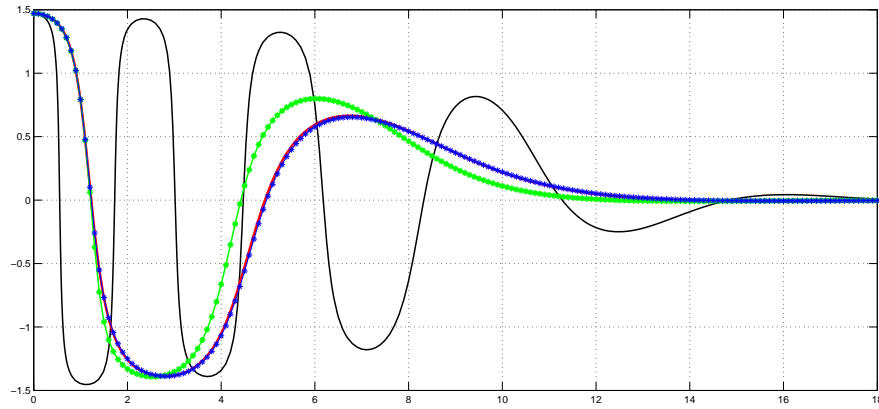


FIGURE 6.13: Pendulum heading angle w.r.t. the vertical, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.1$ s

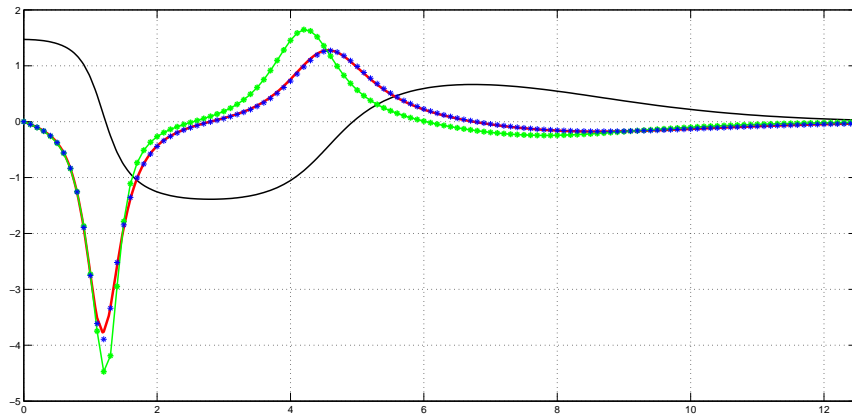


FIGURE 6.14: Pendulum angular velocity, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.1$ s

In the second scenario, the sampling time is increased at the value of $T = 0.3$ s. This seems to be a critical value for the system, since the emulated controller doesn't work anymore, while the first-order corrected controller is still able to follow the target system behavior, as shown in Fig. 6.17, 6.18, 6.19, 6.20.

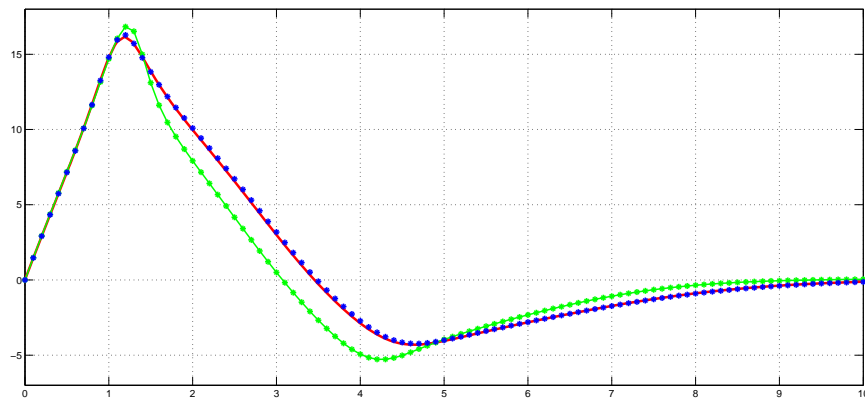


FIGURE 6.15: Cart velocity trajectory, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.1$ s

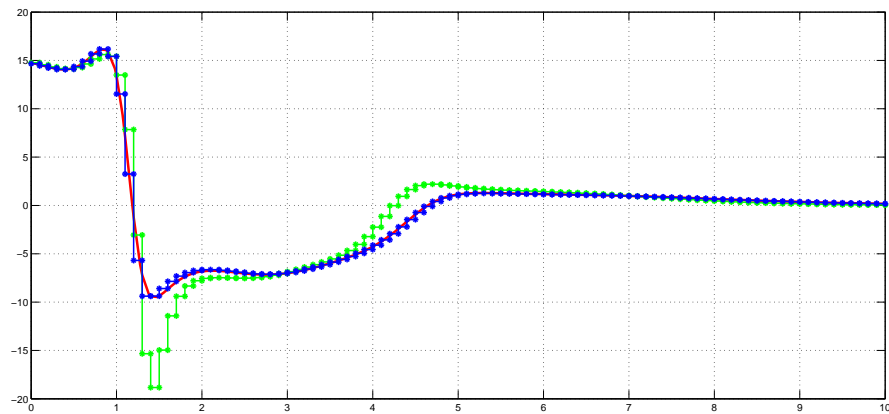


FIGURE 6.16: Control input, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.1$ s

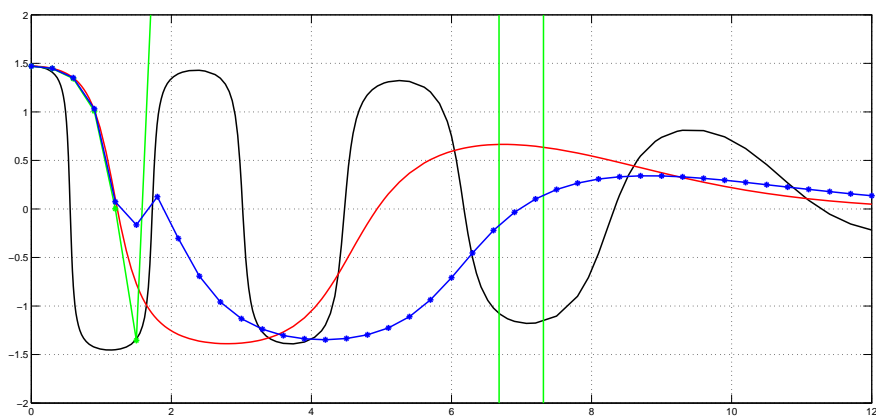


FIGURE 6.17: Pendulum heading angle w.r.t. the vertical, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.3$ s

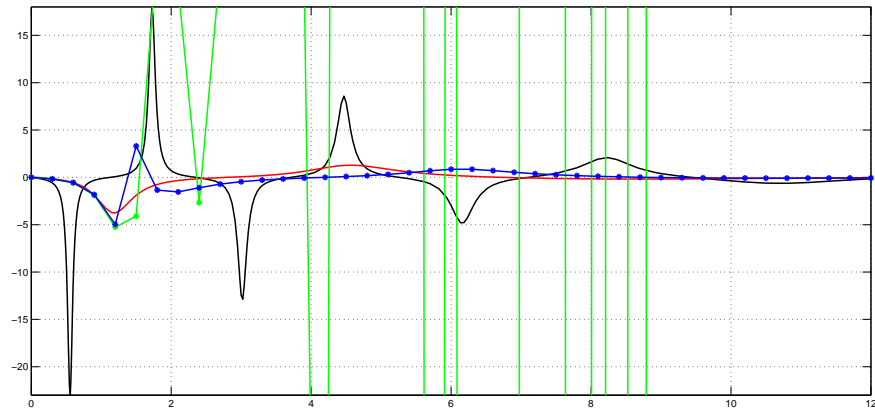


FIGURE 6.18: Pendulum angular velocity, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.3$ s

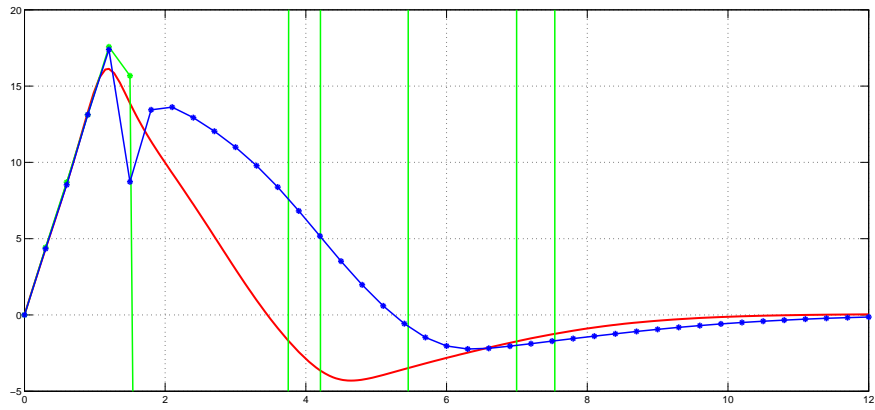


FIGURE 6.19: Cart velocity trajectory, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.3$ s

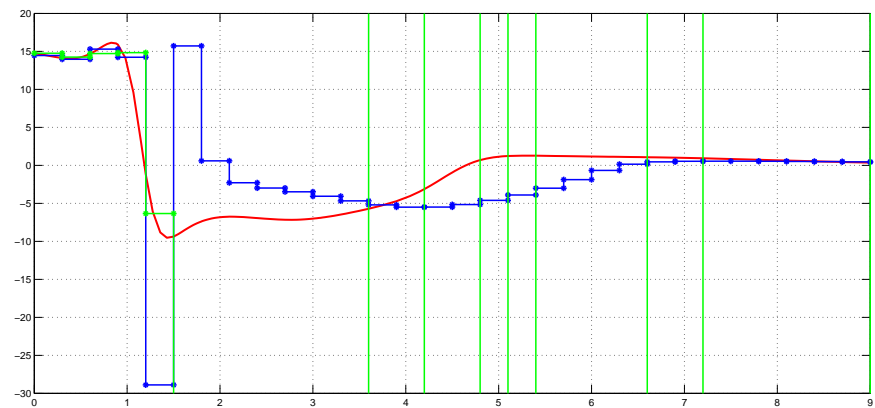


FIGURE 6.20: Control input, comparison between continuous (red) and sampled-data controllers (emulated in green and first order in blue) at $T = 0.3$ s

Chapter 7

Robust attitude stabilization via digital I&I

In this chapter, we face the problem of robust nonlinear attitude stabilization of a rigid spacecraft. An Immersion and Invariance robust attitude stabilizer is proposed, taking into account actuator dynamics in control design. The proposed continuous-time controller is then implemented under sampling using an approximated single-rate strategy to match, at the sampling instants, the zero-going evolution of the off-the-manifold coordinates. The results illustrated in section 6.1 for the single-rate solution are easily extended to the case of more than one control input. Simulations show the effectiveness of the proposed controller. The chapter follows the paper [88] submitted to the first MICNON conference and waiting for review.

7.1 Introduction

A nonlinear control strategy to stabilize the attitude of a rigid spacecraft robustly with respect to actuators dynamics is proposed. The control law is applied under sampling using a single-rate digital approach with first order corrector term, which overcomes in performance the direct implementation through zero-order hold, as shown by simulations. A robust attitude stabilizer is necessary for long range communications satellites, especially when a high throughput is involved. The capability of the spacecraft to maintain a fixed orientation despite external disturbances and modeling uncertainties is crucial when dealing with satellite internet access at high data speeds [101]. In this work we propose an I&I solution for systems in strict-feedback form which is particularly suited to counteract the degrading effect of unmodeled actuator dynamics on the overall control systems. In fact, I&I can be regarded as a tool to robustify a given

nonlinear controller with respect to higher-order dynamics, exploiting at its best the knowledge of such dynamics during the control design phase. Thus, this approach can be considered “robust” nonlinear control. The obtained continuous-time controller is then implemented under sampling using a single-rate control strategy with truncation of series expansions at the second order in the sampling period. Simulations at increasing sampling times show the effectiveness of using a first order corrector term with respect to the simpler implementation through zero-order hold device (emulated control [97]). In particular, the maximum allowable sampling period (MASP) is increased, thus the sampled-data controller shows robustness w.r.t. δ .

7.2 Spacecraft dynamic modeling

Consider a symmetric rigid spacecraft characterized by a diagonal inertia matrix J , namely

$$J = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}.$$

The kinematic model used is based on the modified Cayley-Rodrigues parameters, which provide a global and non-redundant parametrization of the attitude of a rigid body ([26]). In the following, $S(\cdot)$ denotes the three-dimensional skew-symmetric matrix, which for a generic vector $r \in \mathbb{R}^3$ takes the form

$$S(r) = \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} \quad (7.1)$$

Defining $\rho \in \mathbb{R}^3$ the modified Cayley-Rodrigues parameters vector and $\omega \in \mathbb{R}^3$ the angular velocity in a body-fixed frame, the kinematic equations take the form

$$\dot{\rho} = H(\rho)\omega. \quad (7.2)$$

The matrix-valued function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ denotes the kinematic jacobian matrix of the modified Cayley-Rodrigues parameters, given by

$$H(\rho) = \frac{1}{2} \left(I - S(\rho) + \rho \rho^T - \frac{1 + \rho^T \rho}{2} I \right) \quad (7.3)$$

where I denotes the 3×3 identity matrix. The matrix $H(\rho)$ satisfies the following identity ([127])

$$\rho^T H(\rho) \omega = \left(\frac{1 + \rho^T \rho}{4} \right) \rho^T \omega \quad (7.4)$$

for all $\rho, \omega \in \mathbb{R}^3$.

According to Euler's law, the kinematic and dynamic equations can be written as

$$\dot{\rho} = H(\rho) \omega \quad (7.5)$$

$$\dot{\omega} = J^{-1} S(\omega) J \omega + J^{-1} u. \quad (7.6)$$

If first-order actuator dynamics with time-constants T_i ($i = 1, 2, 3$) are considered, equations (7.5)-(7.6) are dynamically extended as follows

$$\dot{\rho} = H(\rho) \omega \quad (7.7)$$

$$\dot{\omega} = J^{-1} S(\omega) J \omega + J^{-1} \tau \quad (7.8)$$

$$\dot{\tau} = A \tau + u. \quad (7.9)$$

where $\tau \in \mathbb{R}^3$ represents the torque generated by the actuators according to the reference torque $u \in \mathbb{R}^3$ and $A = \text{diag}(-\frac{1}{T_1}, -\frac{1}{T_2}, -\frac{1}{T_3})$ is a Hurwitz diagonal matrix whose eigenvalues, all negative real, depend on the time constants of the actuators.

7.3 Immersion and Invariance stabilization

7.3.1 Recalls

Let us recall the continuous-time I&I main result in the general case (the proof is detailed in [5]).

Theorem 7.3.1.1. *Consider the nonlinear system*

$$\dot{x} = f(x) + g(x)u \quad (7.10)$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and an equilibrium point $x^ \in \mathbb{R}^n$ to be stabilized. Suppose that (7.10) satisfies the following four conditions.*

H1c (Target System) - There exist maps $\alpha(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $\pi(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^n$ such that the sub-system $\dot{\xi} = \alpha(\xi)$ with state $\xi \in \mathbb{R}^p$, $p < n$, has a (globally) asymptotically stable equilibrium at $\xi^ \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$.*

H2c (Immersion condition) - For all $\xi \in \mathbb{R}^p$, there exists a map $c(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^m$ such that

$$f(\pi(\xi)) + g(\pi(\xi))c(\xi) = \frac{\partial \pi}{\partial \xi}(\xi)\alpha(\xi) \quad (7.11)$$

H3c (Implicit manifold - \mathcal{M}) - There exists a map $\phi(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n-p}$ such that the identity between sets $\{x \in \mathbb{R}^n | \phi(x) = 0\} = \{x \in \mathbb{R}^n | x = \pi(\xi) \text{ for } \xi \in \mathbb{R}^p\}$ holds.

H4c (Manifold attractivity and trajectory boundedness) - There exists a map $\psi(\cdot, \cdot) : \mathbb{R}^{n \times (n-p)} \rightarrow \mathbb{R}^m$ such that all the trajectories of the system, with initial condition $z_0 = \phi(x_0)$

$$\dot{z} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\psi(x, z)] \quad (7.12a)$$

$$\dot{x} = f(x) + g(x)\psi(x, z) \quad (7.12b)$$

are bounded and satisfy $\lim_{t \rightarrow \infty} z(t) = 0$.

Under these four conditions, x^* is a globally asymptotically stable equilibrium of the closed-loop system

$$\dot{x} = f(x) + g(x)\psi(x, \phi(x)) \quad (7.13)$$

◇

The following definition is straightforward.

Definition 7.3.1.2 (I&I Stabilizability). A nonlinear system of the form (7.10) is said to be I&I stabilizable with target dynamics $\dot{\xi} = \alpha(\xi)$, if it satisfies conditions H1c to H4c of Theorem 7.3.1.1. ◇

Note that the target dynamics is the restriction of the closed-loop system to the manifold \mathcal{M} , implicitly defined in H3c. The control law $u = \psi(x, z)$ is designed to steer to zero the off-the-manifold coordinate z and to guarantee the boundedness of system trajectories. On the manifold, the control law is reduced to $\psi(\pi(\xi), 0) = c(\xi)$, and it renders \mathcal{M} invariant according to H2c. The complete control law can thus be decomposed in two parts:

$$u = \psi(x, \phi(x)) = \psi(x, 0) + \tilde{\psi}(x, \phi(x)) \quad (7.14)$$

with $\psi(\pi(\xi), 0) = c(\xi)$ on the manifold and $\tilde{\psi}(x, 0) = 0$. Note that $\psi(x, 0)$ can be seen as a nominal control law, designed on the model of the dynamics restricted on the manifold to obtain a GAS target dynamics. In this sense, the term $\tilde{\psi}(x, \phi(x))$ is a robustness-improving addendum which takes into account the off-the-manifold behaviors generated, for instance, by higher-order actuator dynamics. The overall control law provides the I&I “robust” nonlinear stabilizer.

7.3.2 The class of systems under study

In this work, we consider the problem of state-feedback stabilization of the following class of systems in feedback form

$$\begin{aligned}\dot{\xi} &= f(\xi) + g(\xi)\eta \\ \dot{\eta} &= u\end{aligned}\tag{7.15}$$

where $\xi \in \mathbb{R}^p$, $\eta \in \mathbb{R}^{n-p}$, $u \in \mathbb{R}^m$ (with $m = n - p$), $x = \text{col}(\xi, \eta)$ and $\xi = 0$ is a globally asymptotically stable equilibrium of $\dot{\xi} = f(\xi)$. It is also assumed that we know a radially unbounded Lyapunov function $V(\xi)$ such that

$$\frac{\partial V}{\partial \xi} f(\xi) < -w(\|\xi\|)\tag{7.16}$$

with $w(\cdot)$ \mathcal{K}_∞ function. Note that the existence of V is guaranteed by the converse Lyapunov theorems, although in many cases its knowledge or construction could be difficult to achieve. For systems like (7.15) constructive solutions of the I&I stabilization problem do exist, as shown with more detail in [5]. In fact, the target dynamics condition H1c is trivially satisfied by $\dot{\xi} = f(\xi)$. Moreover, the mappings

$$x = \begin{bmatrix} \pi_1(\xi) \\ \pi_2(\xi) \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix} \quad u = c(\xi) = 0 \quad \phi(\xi, \eta) = \eta\tag{7.17}$$

are such that conditions H2c and H3c hold. Since the off-the-manifold component $z = \eta$ is a partial-coordinate, condition H4c is verified if it is possible to find a control law $u = \psi(\xi, \eta)$ such that the trajectories of the closed-loop system

$$\begin{aligned}\dot{\xi} &= f(\xi) + g(\xi)\eta \\ \dot{\eta} &= \psi(\xi, \eta)\end{aligned}\tag{7.18}$$

are bounded and $\lim_{t \rightarrow \infty} \eta(t) = 0$ (manifold attractivity). To this end, it is possible to relax the assumption on V to be a weak Lyapunov function, namely such that (7.16) holds for all $\|\xi\| > M > 0$, for a proper, problem-dependent, choice of M (see [5] for more details). With this in mind, we can state the following result.

Theorem 7.3.2.1. *Consider system (7.18) with all the related properties and assumptions. The system satisfies condition H4c, i.e. manifold attractivity and trajectory boundedness, with the*

following choice for the control law:

$$\begin{aligned}\psi(\xi, \eta) &= -(\mu(\xi) + k_\eta)\eta \\ \mu(\xi) &> \left\| \frac{\partial V}{\partial \xi} g(\xi) \right\| \quad k_\eta > 0.\end{aligned}\tag{7.19}$$

◇

Proof. Substituting (7.19) into (7.18) yields

$$\begin{aligned}\dot{\xi} &= f(\xi) + g(\xi)\eta \\ \dot{\eta} &= -(\mu(\xi) + k_\eta)\eta.\end{aligned}\tag{7.20}$$

Consider now the Lyapunov function

$$W(\xi, \eta) = V(\xi) + \frac{1}{2}\eta^T \eta.\tag{7.21}$$

The derivative of (7.21) along the trajectories of (7.20) takes the form

$$\begin{aligned}\dot{W} &= \frac{\partial V}{\partial \xi} f(\xi) + \frac{\partial V}{\partial \xi} g(\xi)\eta - \mu(\xi)\eta^T \eta - k_\eta \eta^T \eta \\ &\leq -w(\|\xi\|) + \left\| \frac{\partial V}{\partial \xi} g(\xi) \right\| \|\eta\|^2 - \mu(\xi) \|\eta\|^2 - k_\eta \|\eta\|^2 \\ &\leq -w(\|\xi\|) + \left(\left\| \frac{\partial V}{\partial \xi} g(\xi) \right\| - \mu(\xi) \right) \|\eta\|^2 - k_\eta \|\eta\|^2 \\ &\leq -w(\|\xi\|) - k_\eta \|\eta\|^2\end{aligned}\tag{7.22}$$

which is negative definite for all $\|\xi\| > M > 0$ thanks to the proper choice of $\mu(\xi)$ and k_η , thus condition H4c is verified, which concludes the proof. \square

7.3.3 Problem setting

Assume now that the control input u is maintained piecewise constant on intervals of fixed length, namely the sampling period. We seek a possibly sampling-dependent controller, which maintains the stability properties achieved by the continuous-time control law under sampling. In particular, the digital controller should verify the sampled-data versions of conditions H1c-H4c of Theorem 3.4.0.11, achieving manifold attractivity and keeping the boundedness of the trajectories under sampling.

7.4 Sampled-data control design

We seek a sampled-data controller which, following the closed-loop continuous-time evolution of the function $\phi(\xi, \eta) = \eta$, ensures attractivity of the manifold \mathcal{M} and boundedness of the closed-loop trajectories under sampling. The closed-loop system under digital control should be globally asymptotically stable. To begin with, let us briefly recall the sampled-data equivalent dynamics under single-rate sampling. In order to simplify the problem and to fit it to that of attitude stabilization of the rigid spacecraft, the following two assumptions are straightforward.

Assumption 7.4.0.1. Consider system (7.15). With the aim of designing the sampled-data controller, it is assumed that the dimension of the vector ξ is equal to that of η and u , namely $p = m$, thus $n = 2m$. As a consequence, the matrix $g(\xi)$ is squared, $g(x) \in \mathbb{R}^{m \times m}$. \diamond

Assumption 7.4.0.2. Consider system (7.15). For the sake of simplicity of the sampled-data control design it is assumed that the matrix $g(\xi)$ is diagonal and does not depend on ξ , namely $g(\xi) = G$. Moreover, it is assumed that it is non-singular, i.e. $\det G \neq 0$. \diamond

Next, let us rewrite system (7.15) in the following form ($i = 1, \dots, m$):

$$\dot{x}_i = \bar{f}_i(x) + u_i \bar{g} \quad (7.23)$$

where $x_i = \text{col}(\xi_i, \eta_i)$, u_i is the i^{th} component of the control input, $\bar{f}_i(x) = \text{col}(f_i(\xi) + G_i \eta_i, 0)$, G_i is the i^{th} row- i^{th} column element of G and $\bar{g} = \text{col}(0, 1)$. In this way, the n -dimensional multi-input system is decomposed into m single-input systems, each of dimension two.

The single-rate *sampled-data equivalent model* is the discrete-time dynamics reproducing, at the sampling instants, the solution of (7.23) when the control variable $u_i(t)$ is kept constant over time periods of length δ , namely $u_i(t + \tau) = u_i(t) = u_{ik}$ for $0 \leq \tau < \delta$, $t = k\delta$, $k \geq 0$. It is described by the δ -parametrized map $F_i^\delta(\cdot, u_{ik})$ (the pair u_i, F_i^δ) admitting the Lie exponential series expansion:

$$x_{i(k+1)} = F_i^\delta(x_{ik}, u_{ik}) = e^{\delta(\bar{f}_i + u_{ik}\bar{g})} x_{ik} \quad (7.24)$$

The sampled-data control is designed employing a single-rate strategy (see [92]), which achieves manifold attractivity under sampling by matching the controlled continuous-time evolution of $\phi = \eta$ at the sampling instants. Since the continuous-time conditions are fulfilled by (7.19), ϕ will vanish asymptotically. A digital control law reproducing the ϕ behavior is proposed to steer system trajectories on the manifold. The behavior on

the manifold is reduced, by construction, to that of the target system under sampling, yielding global asymptotic stability of the digital control system.

Theorem 7.4.0.3. *Consider the class of feedback systems (7.15) satisfying assumptions 7.4.0.1 and 7.4.0.2 with an equilibrium x^* to be stabilized. Moreover, consider the stabilizing continuous-time control law (7.19), $\psi(\xi, \eta) \in \mathbb{R}^m$. There exist sampled-data control laws of the form ($i = 1, \dots, m$)*

$$u_{di} = \psi_i^\delta(\xi_k, \eta_k) \quad (7.25)$$

such that x^ is a globally asymptotically stable equilibrium of the closed-loop dynamics ($i = 1, \dots, m$)*

$$x_{i(k+1)} = F_i^\delta(x_{ik}, \psi_i^\delta(\xi_k, \eta_k)) \quad (7.26)$$

◇

Proof. Following assumptions H1c to H4c of the continuous-time result, we can define for (7.15) a sampled-data equivalent target system, with state $\xi \in \mathbb{R}^p$

$$\xi_{k+1} = f_d(\xi_k) \quad (7.27)$$

where $f_d(\xi) = e^{\delta f(\xi)}\xi$ has a globally asymptotically stable equilibrium at $\xi^* \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$. Condition H2c can be reformulated as follows, for $i = 1, \dots, m$,

$$F^\delta(\pi(\xi_k), 0) = \pi(f_d(\xi_k)) \quad (7.28)$$

which, exploiting the properties of the exponential representation, yields that π is equal to the identity function for the first p components, while it is necessarily equal to zero for the remaining $n - p = m$ components. Thus invariance under sampling is ensured. Keep in mind that, for the class of systems at study, $c(\xi) = 0$. As a consequence, the implicit manifold condition H3c is verified under sampling with the choice $\phi_1(x) = x_{p+1}$, $\phi_2(x) = x_{p+2}, \dots, \phi_{n-p}(x) = x_n$ or, alternatively, $\phi(x) = \eta$. The existence of a single-rate controller of the form (7.25) for (7.15) is ensured provided a controllability-like condition [93] is fulfilled. For systems in strict-feedback form like (7.15) satisfying assumption 7.4.0.1, such condition translates in requiring that $\det g(\xi) \neq 0$, which is automatically ensured by assumption 7.4.0.2, thus the sampled-data control law exists.

Consider now the extended system under sampling ($i = 1, \dots, m$)

$$z_{i(k+1)} = \phi(F_i^\delta(x_{ik}, \psi_i^\delta(\xi_k, \eta_k))) \quad (7.29)$$

$$x_{i(k+1)} = F_i^\delta(x_{ik}, \psi_i^\delta(\xi_k, \eta_k)) \quad (7.30)$$

Attractivity of \mathcal{M} is ensured by matching the continuous-time trajectories of $\phi(x)$, which for the systems at study reduces to a simple input-state matching, guaranteed by the existing single-rate solution according to the results in [93]. As a consequence, $\lim_{k \rightarrow \infty} z_k = 0$. Boundedness of the trajectories of (7.29) is guaranteed by one-step consistency property plus forward completeness of the vector fields of (7.15), hence the thesis. \square

7.5 Attitude stabilization of the rigid spacecraft

We consider the dynamical model introduced in section 7.2 to design a robust nonlinear control law in continuous-time using the result introduced in 7.3 and implementing it under sampling using the result provided in 7.4.

7.5.1 Continuous-time control design

The basic idea of continuous-time control design lies in the exploitation of identity (7.4) to show global exponential stability of the kinematic subsystem with a virtual feedback linear in ρ , namely $\omega = -k\rho$, yielding the closed-loop kinematic subsystem

$$\dot{\rho} = -kH(\rho)\rho \quad (7.31)$$

With this aim, consider the Lyapunov function

$$V(\rho) = \rho^T \rho. \quad (7.32)$$

Using (7.4), the derivative of V along the trajectories of (7.31) is given by

$$\dot{V} = -2k\rho^T H(\rho)\rho = -2k \left(\frac{1 + \rho^T \rho}{4} \right) \rho^T \rho \leq -\frac{k}{2} V \quad (7.33)$$

which yields global exponential stability with rate of decay $k/2$. With this in mind, in the following it is shown how the dynamical model of the spacecraft (7.5)-(7.6)-(7.9) can be cast into the form (7.15) using simple transformations. First, we perform the classical backstepping transformation $\zeta = \omega + k\rho$, which transforms the dynamics into

$$\dot{\rho} = -kH(\rho)\rho + H(\rho)\zeta \quad (7.34)$$

$$\dot{\zeta} = J^{-1}S(\omega)J\omega + J^{-1}\tau - k^2H(\rho)\rho + kH(\rho)\zeta \quad (7.35)$$

$$\dot{\tau} = A\tau + u. \quad (7.36)$$

Now, it is possible to “virtually” stabilize sub-system (7.34)-(7.35) using the backstepping control law

$$\begin{aligned}\tau_b(\rho, \omega) &= \tau_1(\rho, \omega) + \tau_2(\rho, \omega) \\ \tau_1(\rho, \omega) &= -J [S(\omega)J\omega - k^2 H(\rho)\rho + kH(\rho)\zeta] \\ \tau_2(\rho, \omega) &= -k_\zeta \zeta - \frac{1}{2} (1 + \rho^T \rho) \rho.\end{aligned}\tag{7.37}$$

As a consequence, the resulting target-dynamics when $\tau = \tau_b$ in (ρ, ζ) coordinates takes the following form

$$\begin{aligned}\dot{\rho} &= -kH(\rho)\rho + H(\rho)\zeta \\ \dot{\zeta} &= -k_\zeta \zeta - \frac{1}{2} (1 + \rho^T \rho) \rho.\end{aligned}\tag{7.38}$$

Since τ_b is not the actual control law, because non-negligible actuator dynamics are considered, we need to introduce the further change of coordinates

$$z = \tau - \tau_b\tag{7.39}$$

under which the dynamics in (ρ, ω) coordinates takes the form

$$\dot{\rho} = H(\rho)\omega\tag{7.40}$$

$$\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}\tau_b + J^{-1}z\tag{7.41}$$

$$\dot{z} = A\tau + u - \dot{\tau}_b.\tag{7.42}$$

Setting now $u = -A\tau + \dot{\tau}_b + v$, the representation reduces to a strict-feedback form like (7.15), namely

$$\begin{pmatrix} \dot{\rho} \\ \dot{\omega} \end{pmatrix} = \begin{bmatrix} H(\rho)\omega \\ J^{-1}S(\omega)J\omega + J^{-1}\tau_b(\rho, \omega) \end{bmatrix} + \begin{bmatrix} 0_{3 \times 3} \\ J^{-1} \end{bmatrix} z\tag{7.43}$$

$$\dot{z} = v\tag{7.44}$$

where the drift term in (7.43) is exactly the GAS target dynamics.

Keeping in mind that in our case a proper Lyapunov function for the target dynamics is known from backstepping design, namely

$$W(\rho, \zeta) = \rho^T \rho + \frac{1}{2} \zeta^T \zeta\tag{7.45}$$

we can directly apply the result of theorem (7.3.2.1), obtaining the continuous-time control law:

$$\begin{aligned} v &= -(\mu(\rho, \zeta) + k_z)z \\ \mu(\rho, \zeta) &> \|\text{col}(\zeta_1 J_x^{-1}, \zeta_2 J_y^{-1}, \zeta_3 J_z^{-1})^T\| \quad k_z > 0 \end{aligned} \quad (7.46)$$

which ensure global asymptotic stabilization of the attitude of the rigid spacecraft.

7.5.2 Sampled-data control design

Taking as output z , the vector relative degree of system (7.43)-(7.44) is equal to 3, thus it is possible to design three digital single-rate control laws to bring z to zero, by matching the continuous-time control law at the sampling instants. In particular, let us consider the controlled sub-system:

$$\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}\tau_b(\rho, \omega) + J^{-1}z \quad (7.47)$$

$$\dot{z} = -\mu(\rho, \zeta)z - k_z z \quad (7.48)$$

for which assumption 7.4.0.1 and 7.4.0.2 are satisfied, since $\dim(\omega) = \dim(z)$ and $J^{-1} \in \mathbb{R}^{3 \times 3}$ is squared, diagonal, constant and non-singular. We want to construct a digital version of the linear part of the continuous-time controller, i.e. $v_l = -k_z z$, which is exactly the component bringing to zero the off-the-manifold coordinate $z = \phi$. With this aim, let us express (7.47)-(7.48) in the following way, for $i = 1, 2, 3$:

$$\begin{pmatrix} \dot{\omega}_i \\ \dot{z}_i \end{pmatrix} = \begin{pmatrix} \bar{f}_i(\rho, \omega, z_i) \\ -\mu(\rho, \zeta)z_i \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_{l_i} = f_i(\rho, \omega, z_i) + g_i v_{l_i} \quad (7.49)$$

where

$$\begin{aligned} \bar{f}_i(\rho, \omega, z_i) &= [J^{-1}S(\omega)J\omega + J^{-1}\tau_b(\rho, \omega) + J^{-1}z]_i \\ v_{l_i}(z_i) &= -k_z z_i \end{aligned} \quad (7.50)$$

The matching equations for the i -th component of the digital control law take the form

$$e^{\delta(f_i(\cdot, \cdot, \cdot) + v_{di}g_i)} z_i|_{(\rho, \omega, z_i)} = e^{\delta(f_i(\cdot, \cdot, \cdot) + v_{l_i}(z_i)g_i)} \phi_i|_{(\rho, \omega, z_i)}. \quad (7.51)$$

The corresponding digital control laws, truncated at the first term in $O(\delta^2)$, are, for $i = 1, 2, 3$,

$$v_{di} = v_{di0} + \frac{\delta}{2} v_{di1} \quad (7.52)$$

or, in more compact form, $v_d = v_{d0} + (\delta/2) v_{d1}$, with $v_{d0} = v_l(z)|_{z_k}$ emulated controller, and $v_{d1} = \dot{v}_l(z)|_{z_k}$ first order corrector term, for which we simply obtain:

$$v_{d1} = (k_z^2 z + k_z \mu(\rho, \zeta) z) |_{(\rho_k, \zeta_k, z_k)} \quad (7.53)$$

7.5.3 Simulations

The performance of the proposed sampled-data controller (7.52)-(7.53) is tested through numerical simulations in comparison with the emulated controller v_{d0} at different sampling periods. The rigid spacecraft introduced in section 7.2 is characterized by the inertia moments listed below in Tab. 7.1 The following initial conditions in modified

TABLE 7.1: Inertia moments

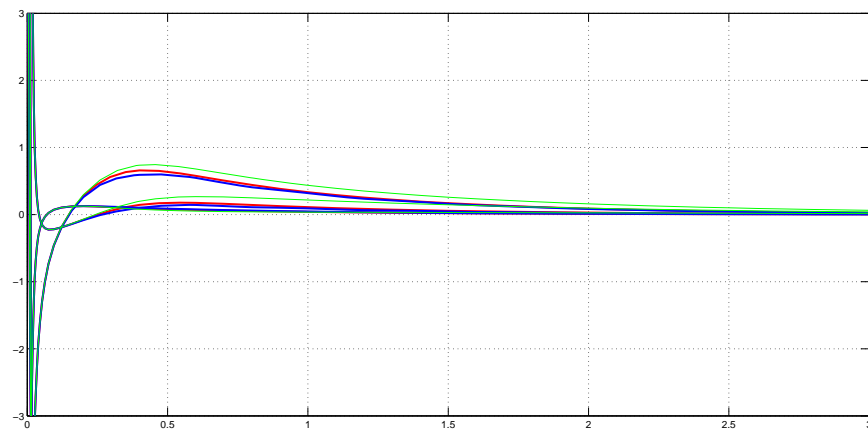
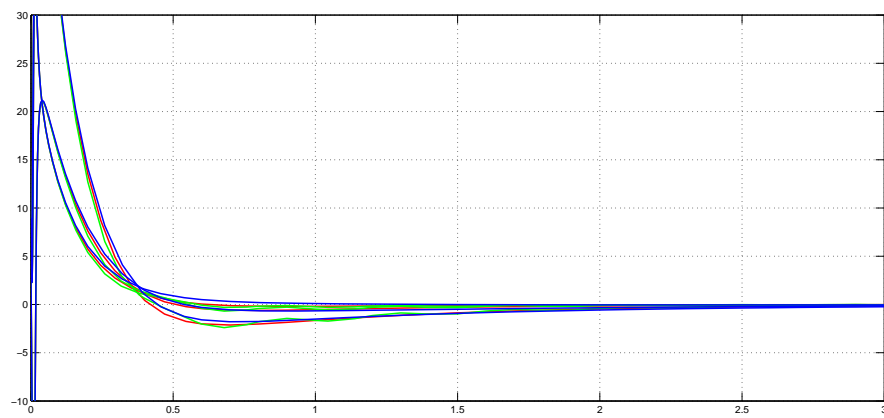
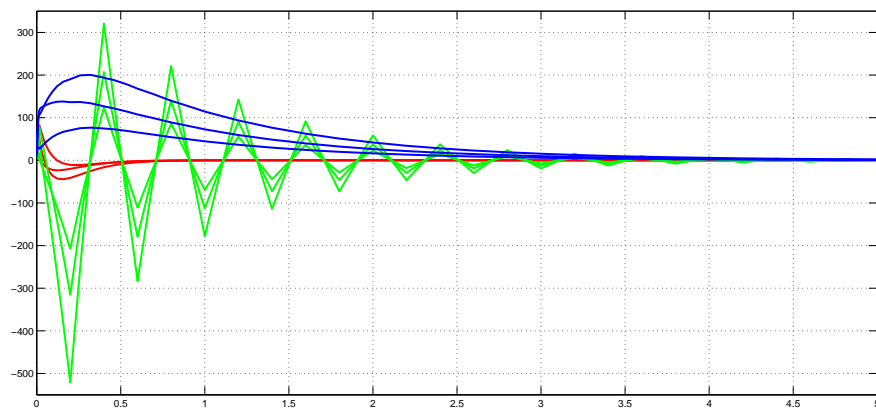
J_x	30.08	$kg \cdot m^2$
J_y	30.12	$kg \cdot m^2$
J_z	29.89	$kg \cdot m^2$

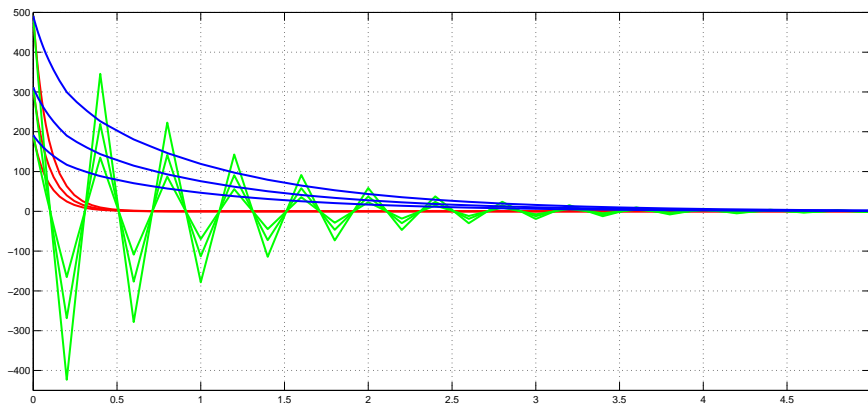
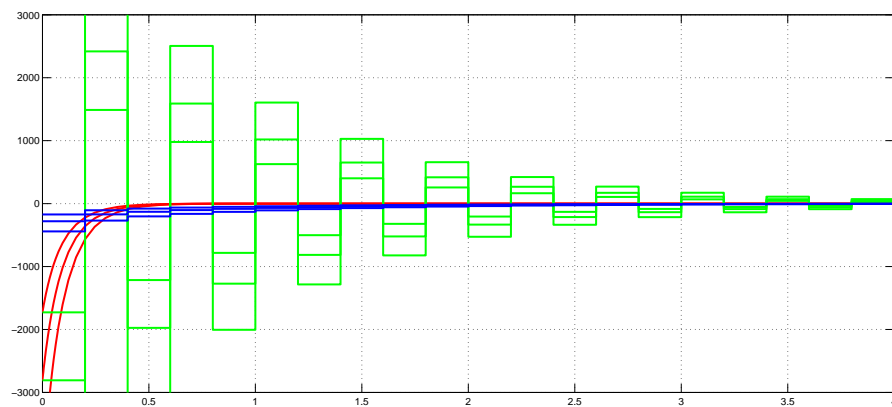
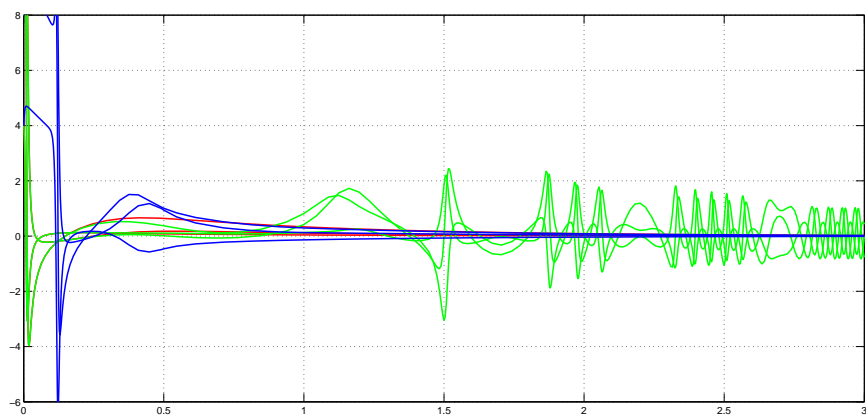
Cayley-Rodrigues (CR) parameters, angular velocities and torques are considered:

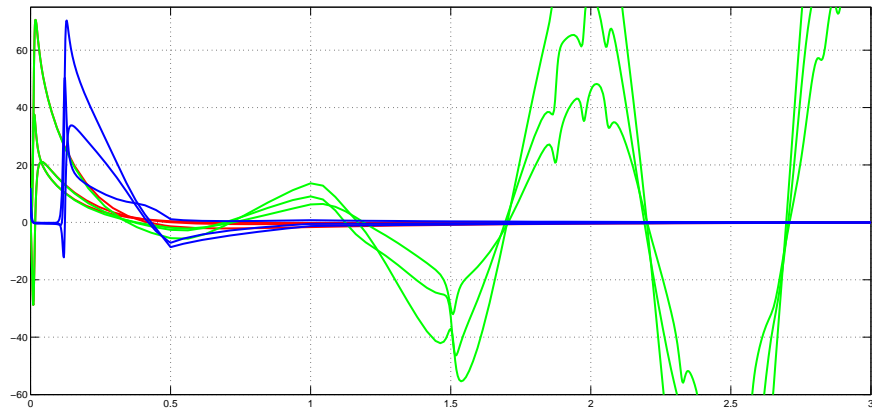
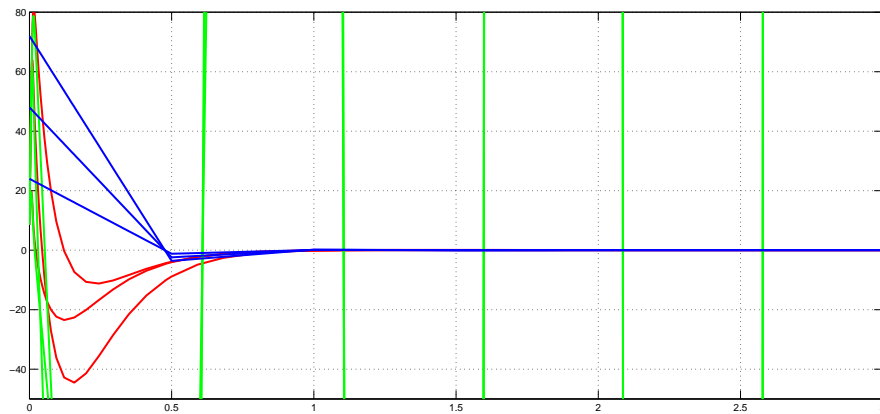
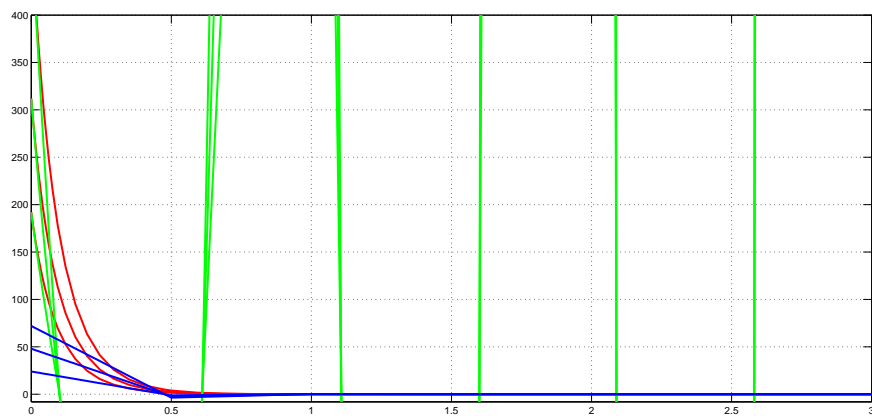
$$\begin{aligned} (\rho_1(0), \rho_2(0), \rho_3(0)) &= (4, 8, 12) \\ (\omega_1(0), \omega_2(0), \omega_3(0)) &= (6, 3, 9) \text{ [rad/s]} \\ (\tau_1(0), \tau_2(0), \tau_3(0)) &= (10, 11, 8.5) \text{ [N} \cdot \text{m]}. \end{aligned}$$

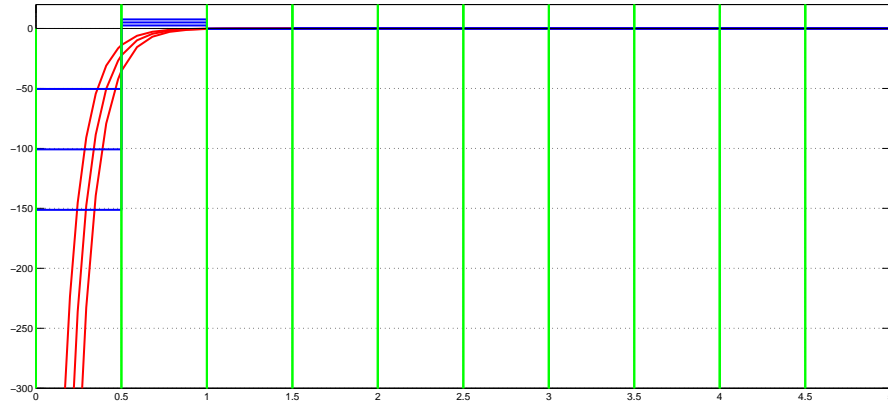
In the first scenario, the emulated and first-order corrector controllers are tested when the sampling period is $T = 0.2$ s. Figures from 7.1 to 7.4 illustrate the results: even if both controllers achieve asymptotic stabilization, the improved performances of the proposed controller in the transients of $\tau(t)$ and $z(t)$ are evident. Moreover, the result is obtained with a control effort which is smaller even in comparison with the continuous-time controller (Fig. 7.5). In all the figures below, the closed-loop continuous-time trajectories are depicted in red, those under emulated control in green and those under the improved controller in blue. In the second scenario, the emulated and first-order corrector controllers are tested when the sampling period is $T = 0.5$ s. As shown in Fig. 7.6, 7.7, 7.8, 7.9 and 7.10 the emulated controller cannot achieve stabilization, while the proposed controller still behaves well, also in terms of control effort.

Note that the maximum allowable sampling time for the emulated controller is about $T_{\text{MASP}} = 0.4$ s, a sampling period at which the improved controller still works.

FIGURE 7.1: $\mathbf{T} = 0.2$ s Modified CR parameters trajectories $\rho(t)$.FIGURE 7.2: $\mathbf{T} = 0.2$ s Angular velocities trajectories $\omega(t)$.FIGURE 7.3: $\mathbf{T} = 0.2$ s Torques trajectories $\tau(t)$.

FIGURE 7.4: $\mathbf{T} = 0.2$ s $z(t)$ trajectories.FIGURE 7.5: $\mathbf{T} = 0.2$ s Control inputs trajectories $u(t)$.FIGURE 7.6: $\mathbf{T} = 0.5$ s Modified CR parameters trajectories $\rho(t)$.

FIGURE 7.7: $\mathbf{T} = 0.5$ s Angular velocities trajectories $\omega(t)$.FIGURE 7.8: $\mathbf{T} = 0.5$ s Torques trajectories $\tau(t)$.FIGURE 7.9: $\mathbf{T} = 0.5$ s $z(t)$ trajectories.

FIGURE 7.10: $T = 0.5$ s Control inputs trajectories $u(t)$.

7.6 Concluding remarks

A robust nonlinear stabilizer for a rigid spacecraft has been developed taking into account actuator dynamics in control design. The proposed control law is a special case of I&I stabilizer and it has been implemented under sampling using a single-rate control strategy with first-order corrector term. The effectiveness of the proposed control solution is shown in several simulations at two different sampling periods.

Conclusions and perspectives

In this work, we have discussed the problem of robust stabilization of nonlinear systems affected by uncertainties in the dynamics. After some recalls on stability and stabilizability of nonlinear systems in the control Lyapunov function framework, we have introduced the reader to the corresponding activated concepts in the context of *uncertain* nonlinear systems, namely subject to modeling uncertainties and disturbances. The digression culminates with the definition of robust nonlinear stabilizability via the concept of robust control Lyapunov function and with the corresponding property of practical-robust global asymptotic stability (P-RUGAS). The trajectories of the system converge to a compact set containing the origin and such set can be made arbitrarily small by properly setting some tuning parameters, an idea particularly useful and exploited in the related robust nonlinear controller design phase. The representation of uncertain nonlinear system gains structure throughout the first two chapters: we start with a generic nonlinear dynamics depending on a vector of uncertain inputs to arrive at an input-affine representation in which the time-varying parts of the system are enclosed in additive uncertain terms, describing modeling errors, disturbances and uncertain parameters. Next, we have classified the uncertainties according to their structure and their influence on the dynamics and given examples of unstabilizable nonlinear uncertain systems, introducing the critical concepts of matching conditions and generalized matching conditions. Uncertainty is modeled using the concepts of Δ -operator and deviation function, defined in the second chapter together with some novel systematic rules of evaluation of complex aggregates of uncertain parameters, often emerging in missiles and launchers aerodynamics. These rules are also massively helpful in shaping the so-called uncertain envelopes, in order to define proper bounding functions for the uncertain terms. Such functions overestimate, with as less conservatism as possible, the size of the uncertain terms taking into account physical limits of variation of system parameters, maximum size of modeling errors and external disturbances. With all this in mind, we have proceeded to the construction of a differentiable version of the classic Lyapunov redesign controller, to counteract the unmatched uncertainties in system dynamics, and then to the development of a robust backstepping control law. Unifying

these two controllers, we have obtained the recursive Lyapunov redesign, a two-step procedure which can handle both matched and unmatched uncertainties, ensuring P-RUGAS of the closed-loop system. The core idea of the design are the robust (virtual) control functions, i.e. sigmoid-like functions emulating a discontinuity, with the aim of dominating in size the uncertain terms when bounding from above the Lyapunov derivative along the trajectories of the closed-loop system. Adjusting the slopes of these sigmoids, it is possible to reduce the size of set of convergence of the closed-loop trajectories, recovering indeed the property of P-RUGAS. The proposed controller is applied to the problem of attitude stabilization of a spacecraft with flexible appendages and to design an autopilot for an air-to-air aerodynamic missile with non-minimum phase characteristics. In the final part of the work we have translated in the digital context the Immersion & Invariance stabilization technique recalled in Chapter 3. In particular, we have developed a controller employing one step of backstepping plus a nonlinear domination argument for a special class of systems in feedback form, for which solutions of the I&I stabilization problem do exist and are constructive. The I&I approach can be viewed as a tool of robustification of a given nonlinear controller with respect to certain unmodeled dynamics, e.g. higher-order dynamics which entails a fast-slow decomposition of the system. Approximate single and multi-rate control strategies are then proposed, assuming that the continuous-time control input is hold constant over intervals of fixed length, the sampling periods. The sampled-data controllers aim at zeroing the off-the-manifold coordinates, so ensuring manifold attractivity, while trajectory boundedness and manifold invariance are preserved under sampling. Two academic examples show how the sampled-data multi-rate controller with first order correction term overcomes in performance the emulated continuous-time control law, allowing the implementation at higher sampling periods. The single-rate controller shows its effectiveness in the cart-pendulum system stabilization. In the final part of the work, its multi-input version has been applied to the attitude stabilization of a rigid spacecraft, robustly with respect to actuator dynamics.

Future works should take into account explicitly in control design the presence of actuator bandwidth and saturation limits, to understand to what extent it is possible to robustly stabilize a given nonlinear system under such constraints. One possibility is to search for “softer” solutions by shaping suitably the robust control Lyapunov function, in order to obtain control laws with less overshoot in the transient. Hard constraints may also be included in control design exploiting the nonlinear optimal control setting, which however comes along with the Hamilton-Jacobi-Bellmann equation, difficult to solve in the unconstrained disturbance attenuation framework of nonlinear \mathcal{H}_∞ control. To bypass this obstruction, it is convenient to find out whether a given robustly stabilizing controller, with the associated RCLF, is inverse optimal with respect to a

meaningful cost functional, which exists in general and whose construction depends on the particular problem to be solved. Input-to-State Stability interpretations of this class of problems have also to be put in light. The links between robust backstepping design satisfying hard constraints, optimality and inverse optimality of such controller, passivity and stability margins should also be investigated and contextualized. In the sampled-data context, we should encompass the problem of a variable sampling time. This can be viewed as an event-triggered system stabilization problem, in which state-dependent triggering conditions yield a maximum allowable inter-event time, critical for the performance of the whole control system. In fact, suitable choices for the triggering criteria could reduce the control effort, avoiding wasteful interventions of the control action, and even satisfy optimality conditions. Hybrid systems analysis tools could also be helpful in this setting. Applications rely mostly in the domain of cyber-physical and networked control systems, but the approach can be helpful, for instance, in shaping shooting-time conditions for micro/nano satellites orbital controllers, whereas limited actuator resources constitute a rigid constraint of the design. Other actuator-level control problems can be addressed, such as jet-engine compressor, propulsion engine and thrust-vectoring for space launchers and power converters control. Robustness issues may also be addressed, since stabilizability and stability margins depend upon the triggering conditions and the related inter-event times through suitable thresholds on the Lyapunov function.

Resumé

La thèse porte sur le développement des techniques non linéaires robustes de stabilisation et commande des systèmes avec perturbations de model. D’abord, on introduit les concepts de base de stabilité et stabilisabilité robuste dans le contexte des systèmes non linéaires. Ensuite, on présente une méthodologie de stabilisation par retour d’état en présence d’incertitudes qui ne sont pas dans l’image de la commande (“unmatched”). L’approche réursive du “backstepping” permet de compenser les perturbations “unmatched” et de construire une fonction de Lyapunov contrôlée robuste, utilisable pour le calcul ultérieur d’un compensateur des incertitudes dans l’image de la commande (“matched”). Le contrôleur obtenu est appelé “recursive Lyapunov redesign”. Ensuite, on introduit la technique de stabilisation par “Immersion & Invariance” (I&I) comme outil pour rendre un donné contrôleur non linéaire, robuste par rapport à dynamiques non modelées. La première technique de contrôle non linéaire robuste proposée est appliquée au projet d’un autopilote pour un missile air-air et au développement d’une loi de commande d’attitude pour un satellite avec appendices flexibles. L’efficacité du “recursive Lyapunov redesign” est mis en évidence dans le deux cas d’étude considérés. En parallèle, on propose une méthode systématique de calcul des termes incertains basée sur un modèle déterministe d’incertitude. La partie finale du travail de thèse est relative à la stabilisation des systèmes sous échantillonnage. En particulier, on reformule, dans le contexte digital, la technique d’Immersion et Invariance. En premier lieu, on propose des solutions constructives en temps continu dans le cas d’une classe spéciale des systèmes en forme triangulaire “feedback form”, au moyen de backstepping et d’arguments de domination non linéaire. L’implantation numérique est basée sur une loi multi-échelles, dont l’existence est garantie pour la classe des systèmes considérée. Le contrôleur digital assure la propriété d’attractivité et des trajectoires bornées. La loi de commande, calculée par approximation finie d’un développement asymptotique, est validée en simulation de deux exemples académiques et deux systèmes physiques, le pendule inversé sur un chariot et le satellite rigide.

Mots-clés. *Contrôle non linéaire, contrôle robuste, fonction de Lyapunov contrôlée robuste, modelisation de l’incertitude, backstepping robuste, “Lyapunov redesign”, “recursive Lyapunov redesign”, “Immersion and Invariance”, contrôle non linéaire sous échantillonnage, contrôle multi-échelles, projet d’autopilote pour missile, stabilisation d’attitude, satellite flexible*

Appendix

Notation for the continuous-time control part.

Throughout the work, we employ a quite common notation, widespread in control theory and applied mathematics literature. Here we introduce and explain the most commonly used terms and symbols.

We denote the non-negative real numbers with the notation \mathbb{R}_0^+ and the positive real numbers with \mathbb{R}^+ . We denote by \mathcal{B}_r the closed ball in \mathbb{R}^n of radius r centered at the origin, i.e. $\mathcal{B}_r := \{x \in \mathbb{R}^n : |x| \leq r\}$.

A class \mathcal{K} function is a continuous function $\alpha(\cdot) : [0, a) \rightarrow [0, \infty)$, which is strictly increasing and such that $\alpha(0) = 0$. A class \mathcal{K}_∞ function is a class \mathcal{K} function with $a = \infty$ and such that $\lim_{r \rightarrow \infty} \alpha(r) = \infty$. A class \mathcal{KL} function is a continuous function $\beta(\cdot) : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ such that:

- for each fixed s , the function $\beta(r, s)$ belongs to class \mathcal{K} ;
- for each fixed r , the function $\beta(r, s)$ is decreasing with respect to s and it is such that $\beta(r, s) \rightarrow 0$ for $s \rightarrow \infty$.

A function $f : \mathbb{R}_0^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the Carathéodory conditions if $f(\cdot, x)$ is measurable for each fixed $x \in \mathbb{R}^n$, $f(t, \cdot)$ is continuous for each fixed $t \in \mathbb{R}_0^+$ and, for each compact U of $\mathbb{R}_0^+ \times \mathbb{R}^n$, there exists an integrable function $m_U : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that $|f(t, x)| \leq m_U(t)$ for all $(t, x) \in U$.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be locally Lipschitz if, for any compact set $Q \in \mathbb{R}^n$, there exists a constant $l_Q \in \mathbb{R}_0^+$ such that $|f(x) - f(y)| \leq l_Q|x - y|$ for all $(x, y) \in U$. We say a function is locally L_∞ when it is (essentially) bounded on a neighborhood of every point.

The notation $|\cdot|_\Omega$ represents the euclidean point-to-set distance function, that is, $|\cdot|_\Omega := d(\cdot, \Omega)$.

A given dynamical system $\dot{x} = f(x) + g(x)u$, with f and g locally Lipschitz functions, is said to be forward complete if for every initial condition x_0 and every input signal u , the corresponding solution is defined for all $t \geq 0$.

Given a smooth vector field f and a scalar function V , $L_f V = \frac{\partial V}{\partial x} f(x)$ is the Lie derivative of the function V along f .

With the notation $\text{col}(x_1, x_2, \dots, x_n)$ we denote the column vector whose elements are the column vectors x_1, x_2, \dots, x_n .

With the expressions “GAS system” or “GES system” we mean that the origin of that system is GAS or GES.

With $\|\cdot\|_2$ we denote the induced \mathcal{L}_2 matrix norm, also called spectral norm, while when not specified in the text the symbol $\|\cdot\|$ denotes the vector Euclidean norm, namely for $v \in \mathbb{R}^n$, $\|v\| := \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$. The symbol $\|\cdot\|_1$ denotes the vector \mathcal{L}_1 norm, namely for $v \in \mathbb{R}^n$, $\|v\| := \sum_{i=1}^n v_i$.

Notation for the sampled-data control part.

All the functions, maps and vector fields are assumed smooth and the associated dynamics forward complete.

Given a vector field f , L_f denotes the associated Lie derivative operator, $L_f = \sum_{i=1}^n f_i(\cdot) \frac{\partial}{\partial x_i}$, e^{L_f} denotes the associated Lie series operator, $e^f := 1 + \sum_{i \geq 1} \frac{L_f^i}{i!}$.

For any smooth real valued function h , the following result holds $e^f h(x) = e^f h|_x = h(e^f x)$ where $e^f x$ stands for $e^{L_f} I_d |x$ with I_d the identity function on \mathbb{R}^n and (x) (or equivalently $|_x$) denotes the evaluation at a point x of a generic map.

The evaluation of a function at time $t = k\delta$ indicated by “ $|_{t=k\delta}$ ” is omitted, when it is obvious from the context.

The notation $O(\delta^p)$ indicates that the absolute value of the approximation error in the series expansions is bounded from above by a linear function of $|\delta|^p$, for δ small enough.

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*And then dance if you wanna dance
Please brother take a chance
You know they're gonna go
Which way they wanna go
All we know is that we don't know
How it's gonna be
Please brother let it be
Life on the other hand
Won't make us understand
We're all part of the Masterplan.*

*Maybe I just want to fly
I want to live, I don't want to die
Maybe I just want to breath
Maybe I just don't believe
Maybe you're the same as me
We see things they'll never see
You and I are gonna live forever.*