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École doctorale InfoMaths

Géometrie du champ libre Gaussien en relation avec les processus SLE et la formule KPZ

THÈSE

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Discipline : Mathématiques

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Une introduction en franglais

Merci d'avoir commencé la lecture (de l'introduction) de cette thèse, dans laquelle on essaie de mieux comprendre la géométrie du champ libre Gaussien. Le champ libre Gaussien est un objet central dans la théorie quantique des champs et représente par exemple les fluctuations d'un potentiel électrique d'équilibre dans un domaine, si on fixe une charge sur le bord. Il joue aussi un rôle clé dans la théorie de la gravité quantique – une théorie visant à lier la géométrie de l'espace-temps induite par la gravité avec l'aspect aléatoire de la théorie quantique.

Du coté mathématique, il faut commencer par mentionner qu'en dimension 1 le champ libre est l'objet le plus célèbre de la théorie des probabilités – le mouvement Brownien. Il s'agit donc d'un objet naturel, au moins pour les probabilistes. Comme le mouvement Brownien, il est aussi universel – plusieurs modèles discrets convergent vers le champ libre. Cette partie Gaussienne est la partie « agréable » du champ libre.

Mais cet objet a aussi un coté désagréable. On aimerait le voir comme une surface aléatoire de dimension 2, et comme un variété aléatoire dans les dimensions plus grandes, mais cela n'a pas vraiment de sens mathématique. Notamment, le champ libre en dimension plus grande que 1 n'est plus une fonction au sens usuel – il est trop irrégulier et demander la valeur, la hauteur du champ en un point n'a plus de sens. Alors comment peut-on dans ce cas parler de la géométrie du champ libre, le sujet de cette thèse ?

Il faut de l'imagination ! Et heureusement, il y a des mathématiciens avec beaucoup d'imagination. Ce sont les travaux de Laurent Schwartz qui ont montré comment donner un sens global aux objets aussi irréguliers. L'idée est assez simple et jolie. Il faut juste repenser ce que signifie la valeur d'une fonction en un point donné : regarder la valeur en un point, cela veut dire que l'on moyenne la fonction par rapport à un masse de Dirac en ce point. Si l'objet est trop irrégulier, ce n'est plus possible, mais dès que l'on regarde cette operation comme une moyennisation, on voit que l'on pourrait espérer lui donner un sens si on se permet de moyenner seulement contre des fonctions dont la masse est distribuée d'une manière lisse. La jolie idée de Laurent Schwartz était que l'on peut en fait donner un sens à cet opération de moyennisation, et par cette opération définir notre objet global.

Nous avons donc bien un objet global aléatoire. Mais peut-on toujours parler de sa géométrie ? De nouveau, il faut d'imagination pour répondre positivement à cette question. Notamment, il se trouve que l'on peut parler des lignes de niveau de ce champ libre, et des lignes géométriques qui s'appellent lignes de flot – qui sont effectivement des lignes de flot pour une fonctionnelle du champ libre. L'intuition permettant de comprendre pourquoi cela devrait être possible vient du modèle discret : on sait que le champ libre continu peut-être vu comme la limite d'échelle de champs libres discrets. Mais alors, ces champs libres discrets

sont bien des objets géométriques et on peut bien parler de leurs lignes de niveau. On peut donc espérer que ces lignes de niveau convergent vers des lignes de niveau de GFF continue, et c'est effectivement ce qui est démontré dans [50]. Même plus, il se donne que ces lignes de niveau sont eux mêmes des objets naturels qui s'appellent des processus SLE.

Chapitre 1

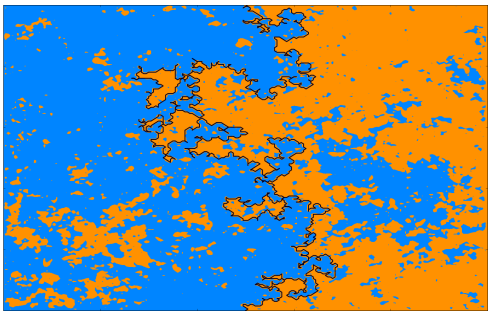
On commence cette thèse aussi en géométrie discrète. On discute une nouvelle façon de voir le champ libre : on considère son gradient comme la partie gradient d'un champ de bruit blanc. Cette façon de penser donne lieu au principe de Donsker dans les dimensions plus grandes que 1. En dimension 1, le principe de Donsker est bien connu et stipule que toutes les marches aléatoires uni-dimensionnelles convergent faiblement vers le champ libre. On peut montrer un énoncé similaire en dimension plus grande que 1.

Après cette observation, on a bien moins peur de commencer à discuter les processus d'exploration du champ libre – ainsi sont nommés les ensembles locaux introduit en [51]. Il s'agit de la bonne analogie aux temps d'arrêt pour les dimensions plus grandes. Un ensemble local, c'est un ensemble aléatoire couplé avec le champ tel qu'après avoir découvert le champ libre sur cet ensemble, il nous reste à découvrir un champ libre indépendant hors de cet ensemble. De plus, pour le cas du champ libre, la situation est particulièrement agréable et on peut même comprendre les conditions au bord de ce nouveau champ libre. Une question assez intéressante surgit : si on découvre le champ libre avec deux ensembles locaux en même temps, quelles sont les conditions au bord? On donne une réponse au cas ou les deux ensembles s'intersectent d'une manière gentille, mais la question reste sans réponse si les ensembles ont une intersection des bords compliquée...

Motivés par cette question, on essaie aussi de mieux comprendre le comportement du champ libre près du bord et on observe de jolies oscillations. Notamment, on observe que près du bord le processus des régularisations circulaires du champ libre se comporte comme un processus de Ornstein-Uhlenbeck. Alors un calcul probabiliste peut nous aider à quantifier la taille et la fréquence de ces oscillations.

Chapitre 2

Dans le Chapitre 2, on commence à étudier des processus d'exploration particuliers – les processus SLE couplés avec le champ libre. Les processus SLE sont aussi des objets naturels en isolation - ils décrivent les interfaces des modèles de physique statistique [49]. Néanmoins, couplés avec champ libre, l'image est même plus joli:



Si on régarde de plus près, on voit aussi un contenu mathématique assez profond : premièrement, on peut utiliser ce couplage pour mieux comprendre les objets séparés – le champ libre et les processus SLE [38]. Deuxièmement, les processus SLE décrivent vraiment des lignes géométriques de ce champ irrégulier. Par exemple, on peut donner un sens aux lignes de niveau en utilisant le processus SLE_4 [51]. Une motivation pour avoir construit le bon formalisme pour les ensembles locaux introduits dans le paragraphe précédent était de prouver que ces lignes de niveau sont intrinsèques au champ libre – qu'elles sont mesurables. Nous n'y sommes malheureusement pas parvenus, mais une partie de thèse décrit ces efforts.

Pour prendre une petite pause du champ libre, on étudie dans ce chapitre aussi l'enroulement des SLEs – en utilisant un peu du calcul stochastique et on peu d'analyse à la main, on obtient des résultats précises pour les moments exponentielles d'enourelement des SLEs. C'est la partie la plus technique dans cette thèse.

Chapitre 3

Finalement, on utilise le couplage entre le GFF et les SLEs pour mieux comprendre un autre objet et une relation: à savoir la mesure de Liouville et la relation KPZ. La mesure de Liouville devrait être la bonne mesure de volume pour une surface aléatoire. La relation KPZ, qui a été introduite dans les années 80 [28] et qui reste un peu un mystère encore aujourd'hui, dit que l'on peut mieux comprendre certains modèles de physique statistique dans une géométrie euclidienne en les considérant dans une géométrie aléatoire bien choisie. Plus exactement, la relation explique comment relier les exposants critiques qui décrivent le modèle dans ces deux géométries différentes. Les questions de savoir comment bien définir cette géométrie aléatoire et comment prouver qu'une telle relation KPZ restent ouvertes, mais on dispose d'ores et déjà d'un cadre mathématique pour jouer avec une version faible de cette relation [19].

Il se trouve que la mesure de Liouville dans le cas critique est donnée par l'exponentielle du champ libre. Alors, on voit que les processus SLE qui sont eux aussi des objets naturels en physique statistique, ont un couplage naturel avec la mesure de Liouville. On peut donc se demander - est-ce que les dimensions fractales des courbes SLE couplées avec le champ libre satisfont une relation KPZ? Comme réponse, on a obtenu qu'elles ne satisfont pas la relation de KPZ canonique, mais qu'elles donnent lieu à des relations très similaires. Les résultats obtenus donnent aussi de nouvelles informations sur la géométrie du champ libre – on décrit les ensembles spéciaux, ou l'on trouve bien moins de fluctuations qu'à la normale.

Conclusions?

En conclusion, dans cette thèse on a essayé de mieux comprendre la géométrie du champ libre. Il y a des petites réussites – le travail sur la relation KPZ a été accepté pour publication dans la revue « Probability Theory and Related Fields », comme de petits échecs – on n'a pas pu achever le travail sur la mesurabilité des lignes de niveau. Dans l'ensemble, on pense quand-même être devenus amis avec le champ libre et on est content d'avoir fini la thèse avec plus des questions que des réponses.

Another introduction

In statistical physics one often observes the following phenomena, called universality: microscopic details of a model do not influence the macroscopic behaviour.

1 A micro-introduction in prose

Mathematics can be seen as a universal language, a language that can describe and explain different phenomena of nature. These descriptions help to simplify how we see the world, to disentangle the important from the unimportant and to show how the seemingly different is in the end all the same.

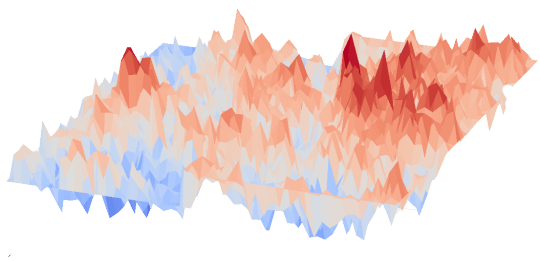
But mathematics itself is not all the same. There are mathematical objects which are more universal and more natural than the others. There are questions which offer more insight, and there are answers that are more beautiful.

In this thesis, we would of course have liked to ask insightful questions and to have obtained beautiful answers. However, we are also happy only with the fact that we have worked on natural and universal mathematical objects.

Indeed all of the mathematical objects of this thesis can be described as a natural model for something:

- The Gaussian free field can be seen as the natural fluctuation of, say, an electric potential.
- The Schramm-Loewner evolution can be seen as the natural random curve joining two points on the boundary of a region.
- The Liouville measure can be seen as the natural measure of mass on a randomly chosen spherical object.

Here is an illustration of the Gaussian free field, our main character:

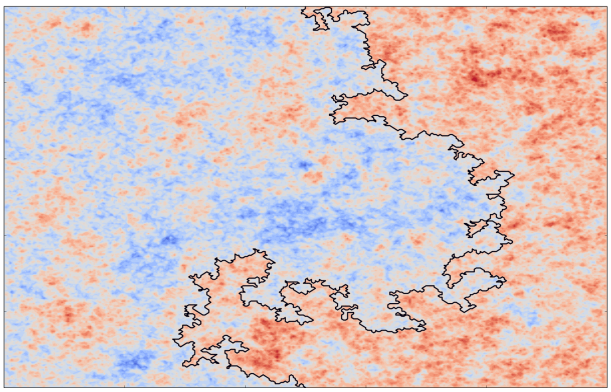


They are also universal in a mathematical sense: they are all limiting objects to a whole range of different mathematical models. Moreover, they can all be seen to be universal in the sense that they bring together different areas of science:

- The Gaussian free field is also the model for the massless bosonic field in quantum field theory, it also describes harmonic crystals.
- The Schramm-Loewner evolution describes the interfaces of different statistical physics models related to polymers, magnets and porous media.
- The Liouville measure enters into the modelling of turbulence and financial markets.

And if this is not enough, they are all related each to the other! Indeed, the Schramm-Loewner evolution gives the contour lines of the Gaussian free field. The Liouville measure, on the other hand, is a multiplicative model of the Gaussian free field. Moreover, whereas the Liouville measure should describe the mass of a randomly chosen spherical object, the GFF should describe its curvature. Finally, the Schramm-Loewner evolution is the natural way to cut this random spherical object into two.

Here is an illustration of the Schramm-Loewner evolution as the contour line of the Gaussian free field:



I hope by now the reader is well motivated to have a light start with mathematics.

2 A macro-introduction

What we mean by universality in the context of probability theory is best described using a few examples:

Theorem 0.2.1 (The strong law of large numbers)**.** *Take a sequence of integrable independent random variables of mean μ. Then their average converges a.s. to μ.*

Here we see that the macroscopic observable - the average of the sequence - does not care about the microscopic details - the exact laws of the random variables. An even more striking example is the following:

Theorem 0.2.2 (The central limit theorem)**.** *Take a sequence of independent, identically distributed random variables of mean μ, variance σ^2. Then the normalized sum*

$$\frac{\sum_i^n X_i - n\mu}{\sqrt{n}}$$

converges in law to a Gaussian random variable of mean 0 *and variance σ^2.*

This can be thought of as describing the fluctuations around the limiting average of the previous example. Here, we do not only observe the universality of these fluctuations, but also a non-trivial universal object - a Gaussian random variable. But let us go further (see e.g. [41]):

Theorem 0.2.3 (1D Donsker for Brownian motion)**.** *Consider the unit interval as a time parameter. Discretize it into n time-segments of equal length. Pick some random variable X of zero mean and unit variance. Start a discrete-time random S_n walk from* 0 *and on each step use an independent copy of X as the jump distribution. If we now interpolate S_n between its points linearly and consider it as a continuous function, then the renormalized function $\frac{S_n}{\sqrt{n}}$ converges to the standard 1D Brownian motion.*

Here our perspective widens even a bit more. Namely, the object itself is more interesting and we obtain different equally fruitful ways of thinking about this limiting object. We could see it as a continuous version of a simple random walk, as a natural model of a random continuous function on $[0, 1]$ pinned at zero, or even more geometrically, we could even say that the trace is a natural fluctuation of a linear function.

The interesting thing is that these different viewpoints give rise to quite different questions in 2 dimensions. In the case of the simple random walk the most straight-forward generalization is maybe the following, see e.g. [33]:

Theorem 0.2.4 (2D Donsker for Brownian motion)**.** *Tile $[-1, 1]^2$ into a square grid of size $2n + 1 \times 2n + 1$. Start an unbiased random walk from zero: move along the grid and on each vertex toss two coins to decide the vertical and horizontal directions. Then this walk, seen as a continuous function on the lattice converges to the planar Brownian motion.*

When we interpret the Brownian motion as a random continuous function, the natural question for 2 dimensions is to ask how a random sphere looks, or slightly more precisely:

Question 0.2.5. What is the natural measure on metrics on the Riemmanian sphere?

This question has motivated quite a lot of recent work in the domain, and in fact we are getting rather close to an answer. Interestingly, this answer is related to the third viewpoint of the Brownian motion: that of seeing it as the natural fluctuation of linear functions. In 2D the equivalent of a linear function is a harmonic function. Thus the question is what is the natural fluctuation of harmonic functions? As we will explain more in chapter 1 this object is the Gaussian free field, one of the central objects of this thesis.

Now we have found these objects through analogy, but are they still universal in the sense above? Are they still universal in the sense of being the limiting objects for a variety of discrete models? In the case of viewpoint 1, the cited theorem itself proves it to be so - planar

Brownian motion is the limit of random walks, however they might differ in detail. In the case of question 3, we refer to section 3 of chapter 1 for a simplistic version of universality. For the case of question 2 we want however to cite the following [37, 35] deep theorem:

Theorem 0.2.6. *Take a uniform rooted 2k-angulation Q_n on the topological sphere of n faces. Consider it as an abstract metric space (Q_n, d_n), with points being the vertex set of the quadrangulations and distances given by the graph distances. Then as $n \to \infty$, in the Gromov-Hausdorff topology of metric spaces, $(Q_n, \frac{cd_n}{n^{1/4}})$ converge to a random metric space with the topology of the sphere. This limiting space is called the Brownian map.*

Comparing this to the Question 0.2.5 above, we see that we really are missing only a bit - an underlying embedding on the sphere.

Yet, as this aspect was too complicated to consider for this thesis, let us leave it aside and continue our taxonomy of universal objects. Consider now the following classical models of statistical physics [57]:

- Self-avoiding walk: take the square lattice, and all walks of fixed length that do not touch themselves.
- Percolation: take the square lattice, colour each edge black or white by tossing an independent fair coin on each square.
- Uniform spanning tree: consider a uniform spanning tree on a very large square grid.

And here is a question:

Question 0.2.7. What links the continuum limits of these objects?

The answer is - a one-parameter family of random curves called the Schramm-Loewner evolution. These random curves should describe the continuum limits of (the interfaces) in the above-mentioned and many other planar classical physics models. This is the second object of our thesis and described in chapter 2.

So it leaves us with the third object. Why do we mix in this Liouville measure? Does it rank as highly on the level of universality? And what is this KPZ relation?

It comes out that a version of the Liouville measure should give the random volume form of the random metric we are looking for in question 0.2.5. So, in particular, it is also related to the Brownian map. Moreover, the KPZ relation is the natural way of relating the masses of sets on the sphere in the Euclidean sense and as seen under the random volume form of the random metric.

Thus we have a nice connection between the GFF and the Liouville measure. But this is not the only connection between these random objects. Over the last decade or so, we have discovered many more connections:

- from [51] we know that SLE_4 curves are the level lines of the GFF;
- in [19] we see that the Liouville measure is "the exponential of the GFF";
- due to [54], we know that SLE curves are the natural way to cut the random surfaces related to the Liouville measure.

In this thesis we look at how the SLE lines coupled with the GFF appear under the Liouville measure. Thus we hope to shed some light on these connections, without trying to answer the third remaining mystery of universality: why are universal objects so closely related?

Finally we remark that this thesis has been written in the hope, quite in defiance of the universality phenomenon: in the hope that even microscopic contributions will affect our understanding.

3 Results presented in this thesis and outline

The underlying theme of this thesis is very simple: trying to understand better the underlying objects and their connections.

One can look at the Gaussian free field from different viewpoints. Even if it is not mathematically a surface, we have tried to interpret it as a random geometric object.

3.1 Overview: chapter 1

Chapter 1 contains a mixture of results on the Gaussian free field.

In section 1 we start from the basics of the zero boundary Gaussian free field. Most of the material in this section has been presented elsewhere, e.g. in [53]. Yet we have reworked and reinterpreted the material slightly. We also obtain a tight continuity result 1.1.3:

Proposition 0.3.1. *let $D_n \subseteq D_{n+1}$ be open domains s.t. $\bigcup D_n = D$ and D bounded. Consider the zero boundary GFF h on D and let $h_n = h^{D_n}$ be its orthogonal projections to D_n. We can extend the h_n by zero outside of D_n and thus make sense of them as elements in $\mathcal{H}^{-\varepsilon}(D)$ for all $\varepsilon > 0$. Then almost surely the restrictions h_n defined above converge to h in the space $\mathcal{H}^{-\varepsilon}(D)$ for any $\varepsilon > 0$.*

In section 2 we discuss the relations between the Gaussian free field and the white noise. We obtain what we fancily call the Hodge decomposition of the white noise 1.2.5:

Theorem 0.3.2 (Hodge decomposition of the white noise)**.** *In the space and sense of distributions, the 2D anharmonic white-noise vector field can be a.s. written as $W_0 = \nabla h_1 + \nabla \times h_2$, where h_1 and h_2 are W_0 measurable and have the law of a pair of independent zero boundary GFFs.*

From the point of mathematical content, this should not earn the title of a theorem, it should be also well-known for experts. Yet the result is well-sounding and has well-sounding corollaries, thus - a theorem. Similar choice of naming holds for the Donsker invariance principle for the GFF 1.2.14

Theorem 0.3.3 (Donsker invariance for the GFF)**.** *Consider the square $[0,1]^2$ and split it into n^2 squares of side-length $1/n$. On each inner edge put an i.i.d random variable of zero mean, unit variance and uniformly bounded $2+\delta$-th moment for some $\delta > 0$. Call this vector field W_n. Solve the discrete Poisson problem $\Delta h_n = \nabla \cdot W_n$ on the vertices with Dirichlet boundary conditions to obtain a random potential on the vertices. Interpolate it linearly to whole of $[0,1]^2$ (i.e. draw diagonals and interpolate in these triangles). Then as $n \to \infty$, the random potentials h_n converge in law in the strong topology of $\mathcal{H}^{-\varepsilon}([0,1]^2)$ to the zero boundary GFF on $[0,1]^2$.*

Personally, we quite like this theorem as for a while we did not see the exact analogue of Donsker's principle for the GFF: whereas the Brownian motion is often introduced via Donsker's principle, the GFF is usually introduced by seeing it solely as a scaling limit of the discrete GFF.

In section 3 we discuss how to explore the GFF using so called local sets. Local sets are random sets coupled with the GFF such that when we sample the random set, we are left with an independent GFF outside of the set, with some boundary conditions determining the expected height of the field outside. The underlying ideas and principles stem from [51]. In this section, we offer a different formalization of these concepts, trying more closely to adhere to the analogy with the Brownian motion. For example, we discuss how to differentiate between the notions of Markov and strong Markov property for the GFF. We believe that as a result both the statements about local sets and their proofs become conceptually slightly clearer.

Finally, we go a small step further to discuss in some more detail the behaviour of the expected height of the field near the intersections of two local sets. We show that if the boundary intersection of two local sets is small enough, then all the necessary information can be read off from separately from the two local sets.

This section was developed in relation with the hope of finding new proofs of measurability of SLE_4, as explained in chapter 2.

In section 4 we discuss the boundary oscillations of the GFF. This was also motivated by trying to understand the possible behaviour of the harmonic extension near boundary intersection of local sets. The material in this section is also related, but not completely analogous to so-called thick points of the GFF. First, we observe that non-trivial oscillations exist 1.4.4:

Proposition 0.3.4. *Consider the zero boundary GFF on the upper half plane. Look at the sets* $T_a^B = \{z \in [-1,1] : \limsup \frac{h_r(z)}{\sigma_0\sqrt{-\log r}} \geq a\}$*, where* $h_r(z)$ *is the semi-circle average around* z *and* σ_0^2 *its variance. Then for* $a \leq \sqrt{2}$*, the set* T_a^B *has Hausdorff dimension* $1 - a^2/2$ *and is empty for* $a > \sqrt{2}$*.*

And then we further make sure that these really are oscillations and not a drift 1.4.5:

Proposition 0.3.5. *For any fixed* a*, the set* $T_a^i = \{z \in [-1,1] : \liminf h_r(z) > a\}$ *is empty.*

3.2 Overview: chapter 2

Chapter 2 deals with the SLE processes and their coupling with the Gaussian free field.

In section 1 we give a short overview of the level and flow line couplings of the GFF with SLE curves. We recap the proofs of the measurability of SLE_4 and describe a failed proof-attempt of the same claim. In this non-proof we tried to circumvent one difficult step - that of identifying the SLE_4 using its boundary condition in the coupling. On paper this section seems short and contains only a few observations. Yet in real life it stole a considerable amount of time and also motivated the work in sections 3 and 4 of chapter 1.

In section 2 we discuss the winding of chordal SLE curves, conditioned to pass near a point. This winding plays a role in the flow line coupling of the GFF and SLE_κ for $\kappa \neq 4$. Thus in some sense we were studying the behaviour of the GFF near its flow lines. We do believe that this result is worthy of the title of a theorem 2.2.3:

Theorem 0.3.6. *Let* CR_0 *be the conformal radius of a fixed point* z_0 *in the upper half plane. Fix* $0 < \kappa < 8$*. Denote by* H_τ *the SLE slit domain component containing* z_0*. Then, for* $\epsilon > 0$ *sufficiently small, conditioned on* $\mathrm{CR}(z_0, H_\tau) \in [\epsilon, C\epsilon]$ *with* $C > 1$*, the exponential moments of the winding* $w(z_0)$ *around the point* z_0 *are given by*

$$\mathbb{E}\left(e^{\lambda w(z_0)} | CR(z_0, H_\tau) \in [\epsilon, C\epsilon]\right) \asymp \epsilon^{-\lambda^2\kappa/8}$$

where the implied constants depend on κ, λ *and for fixed* κ *can be chosen uniform for* $|\lambda| < \lambda_0$ *for any choice of* $\lambda_0 > 0$*.*

3.3 Overview: chapter 3

In chapter 3 we study the Liouville measure, the so-called KPZ relation and the behaviour of the Liouville measure on the flow lines of the GFF.

In section 1 we discuss the Liouville measure. We overview the different KPZ relations - quadratic equations that relate the fractal dimensions of sets as measured in Euclidean geometry, and under the Liouville measure. We also introduce a new version of the quantum dimension: an expected quantum Minkowski dimension that is the direct analogue of the Euclidean Minkowski dimension for random measures.

In section 2 we prove the KPZ relation for the expected Minkowski version. More precisely we show that 3.2.1:

Proposition 0.3.7 (KPZ formula for expected Minkowski dimension)**.** *Let* A *be a fixed (or field-independent) compact subset in the interior of some domain. Let* μ_γ *be the Liouville measure on this domain with* $0 \leq \gamma < 2$*. Then we have the following KPZ formula:*

$$d_M = (2 + \gamma^2/2)q_{M,E} - \gamma^2 q_{M,E}^2/2$$

where by d_M *and* $q_{M,E}$ *we denote respectively the usual (upper) and the expected quantum Minkowski dimensions.*

Moreover, we show that it upper bounds the almost sure Hausdorff dimension given in [45].

In section 3 we do our first step in combining the GFF, SLE and KPZ. We show that the usual KPZ relation is not satisfied for the level lines 3.3.1:

Proposition 0.3.8. *Consider the Liouville measure* μ_γ *with* $0 \leq \gamma < 2$ *in the upper half plane. The expected quantum Minkowski dimension of the zero level line drawn up to some finite stopping time satisfies* $q_{M,E} \leq \frac{3}{4+\gamma^2}$*. Hence the usual KPZ relation does not hold.*

This can be seen as a preliminary step preparing for the final section, where we in fact deduce the exact expected quantum Minkowski dimension also for the level lines.

In section 4 the previous material of this thesis converges when we study the behaviour of the GFF and the Liouville measure near the flow lines. This can also be interpreted as proving a KPZ relation for the flow and level lines of the GFF, i.e. as proving a quadratic relation between the Euclidean and quantum dimensions of the flow and level lines 3.4.1:

Theorem 0.3.9. *Consider the Liouville measure with* $0 \leq \gamma < 2$ *in the unit disc and let* $0 < \kappa < 8$. *Then the expected quantum Minkowski dimension of the* SLE_κ *flow lines is given by* $q_{M,E} < 1$ *satisfying*

$$d_M = (2 + \gamma^2/2)q_{M,E} - \gamma^2(1 - \kappa/4)^2 q_{M,E}^2/2$$

where d_M *is the Minkowski dimension of the respective SLE curve.*

The result relies heavily on the work of chapter 2.

3.4 A note on publications

Chapter 3, and section 2 of chapter 2, resulted in the following publication:

- Aru J. (2014) KPZ relation does not hold for the level lines and the SLE_κ flow lines of the Gaussian free field, 62p. To appear in Probability Theory and Related Fields, available online http://link.springer.com/article/10.1007/s00440-014-0597-1

It also seems that three more minimal publishable units may be extracted from chapter 1: local sets; Hodge decomposition and Donsker invariance principle; boundary oscillations. We are pondering upon whether to publish or not to publish.

4 Outlook

There are several natural questions that we would have liked to have answered during this thesis, that we did not manage to answer and that still ask for an explanation. These questions are also mentioned in the body of the text, but we gather most of them here for a better overview. We hope to answer at least some of them some day!

Already the GFF on its own continues to offer problems to solve. For example, in section 1.2 we discuss how the GFF can be seen to give rise both to the divergence part and the solenoidal part of the white noise vector field. The solenoidal part is naturally linked to loops. Thus one might ask:

Question 0.4.1. Is there a natural way to construct loops using the solenoidal part of the white noise vector field, so that (in the limit) they would be related to the CLEs?

In fact the solenoidal part and the related two-form remain a bit unclear in general:

Question 0.4.2. Can one better understand the Gaussian two-form S coming from the Hodge decomposition of white noise in higher dimensions? Is there anything to understand at all?

In section 1.3 we discuss the local sets. Roughly these are random sets coupled with the GFF such that conditioned on A, the GFF in A^c can be written as a sum of an independent zero boundary GFF and a harmonic function. In some cases like in the local set coupling of

the GFF and SLE-s, this harmonic function can be read off from the curve. But what happens if we couple several local sets with the same GFF? Can we read off this harmonic function by just knowing how it behaves for both of the individual local sets? It comes out that the only worry is what happens at the boundary intersection of two local sets, i.e. on the boundary of their intersection. Thus we ask the following question:

Question 0.4.3. Is it true that the boundary intersection points of local sets never contribute to the expected height?

As discussed in sections 1.3 and 2.1 it comes out that a positive answer to this question would simplify some proofs for SLE and GFF couplings. It also relates to the following at least seemingly non-trivial question of fractal geometry:

Question 0.4.4. Given a continuous function $f : [0,1] \to \mathbb{R}$, define for each $\delta > 0$ the level sets $A_\delta = \{z : f(z) \leq \delta$. Is there a way to uniformly control the fractional Sobolev norms of $\chi(A_\delta)$ for all $\delta > 0$ in terms of say the Minkowski dimension of the zero set of f?

In section 1.4 we study the boundary oscillations, which are reminiscent of thick points of the GFF. There is an interesting question that one has not yet fully understood in this context as well:

Question 0.4.5. Can one give an intrinsic definition of the thick points that does not refer to a regularization process?

Chapter 2 has tormented us for a while with the following questions:

Question 0.4.6. Can one find an alternative proof of the measurability of the level and flow lines of the GFF in the continuum? In particular, could one find a constructive proof? Can one find a proof that does not avoid intersections of the SLE curves?

A related question is a more concrete version of the Question 0.4.3 above. Answering it positively would provide us with a new proof straight away:

Question 0.4.7 (Intersection points cause no harm). Consider two SLE_κ curves γ_1, γ_2 coupled with the GFF as local sets, conditionally independently of the GFF. Is it true that the conditional expectation of the GFF, given their conditionally independent union, can be found by just taking the harmonic extension that has boundary values $-\lambda - \chi\,\mathrm{wind}_i$ and $+\lambda - \chi\,\mathrm{wind}_i$ respectively on the left and right side of both curves?

Later in this chapter we discuss the winding of SLE curves, that exactly gives this harmonic extension in their coupling with the GFF as local sets. The estimates we find are sufficient to prove first moment estimates for the winding spectrum of the SLE.

Question 0.4.8. Can one prove an a.s. version of winding spectrum?

It is well possible that this can be handled by borrowing ideas from the recent proof of the almost sure multifractal spectrum of the SLE [23]. This is linked to the following:

Question 0.4.9. Can one describe the thick-point decomposition of the flow lines?

One can certainly give first moment bounds, at least in the style of section 1.4. For a.s. results, one would first need to answer Question 0.4.8.

Finally, in chapter 3 we observe that there is natural deviation from the KPZ relation for the level and flow lines of the GFF. But by how much can one deviate?

Question 0.4.10. Can one determine non-trivial bounds $ub(\gamma, q)$ and $lb(\gamma, q)$ such that for any $0 \leq \gamma < 2$ and $0 \leq q \leq 2$ and any (possibly field-dependent) set A of Hausdorff dimension $q_H(A) = q$, we have $lb(\gamma, q) \leq q_H^Q(A) \leq ub(\gamma, q)$ where the quantum dimensions are defined with respect to the μ_γ Liouville measure?

As a general question, it feels that there are still things to be discovered about the Gaussian free field, but of course, we also look forward to thinking about questions that are unrelated to this thesis...

1

GFF

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The Gaussian free field (GFF) is a model for random surfaces, even though it is not mathematically a surface itself. On a domain D it may also be motivated as the natural fluctuation of harmonic functions. Indeed, it is known that the harmonic solution to the Dirichlet problem - i.e. the solution to

$$\Delta\phi = 0$$

with boundary condition $\phi = g$, can be obtained by minimizing the Dirichlet energy:

$$\|f\|_{\nabla} = \|\nabla f\|_{L^2}$$

Moreover, any sufficiently nice harmonic function is a solution for the Dirichlet problem for some boundary function.

Thus to obtain natural fluctuations h of the harmonic function, one can consider a Gibbs measure such that, say, for any smooth function D, we have:

$$\mathbb{P}(h = f) \asymp e^{-\frac{\|f\|_{\nabla}^2}{2}}$$

But what is the underlying density? In the discrete case, one can take the underlying density to be the product of Lebesgue measures on each coordinate, and then this definition gives precisely the discrete GFF [53].

In the continuum, there is no "Lebesgue" density and the rigorous definition is slightly less obvious, but the heuristics remain. Thus, as the Dirichlet problem physically solves for the equilibrium potential given a charge density on the boundary, we see that the GFF gives the fluctuations of this potential.

The most thorough review up to now can be found in [53].

1.1 Zero boundary GFF

In this section we define and study the zero boundary GFF. In fact this is the version of the GFF that is most naturally linked to the Dirichlet problem: fixed boundary values should allow for no fluctuations on the boundary.

Consider a bounded open domain D. To define the z.b. GFF, consider an orthonormal basis $(h_i)_{i\geq 1}$ of $\mathcal{H}_0^1(D)$ and let $(X_i)_{i\geq 1}$ be standard normal variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we can formally write:

$$h = \sum X_i h_i \tag{1.1}$$

We have that a.s. $\|h\|_{\nabla} = \infty$ and unfortunately also $\|h\|_{L^2} = \infty$. However, we can make sense of the GFF in the sense of generalized functions as shown in the lemma disguised in the following definition:

Definition 1.1.1. *The zero boundary GFF on an open bounded planar domain D, denoted by $h = h^D$ is given by the formal sum (1.1), where the convergence takes place in any $\mathcal{H}^{-\varepsilon}$ for $\varepsilon > 0$.*

Notice that in particular the GFF can be seen as a random distribution. Moreover, the following characterization shows that the choice of the orthonormal basis does not matter:

Lemma 1.1.2. *Let D be a bounded planar domain. Suppose we have some probability space with a Gaussian process h such that for any smooth $\rho \in \mathcal{C}_0^\infty(D)$, $h(\rho)$ is a centred Gaussian and the covariance structure is given by*

$$\mathbb{E}h(\rho_1)h(\rho_2) = (\Delta^{-1}\rho_1, \Delta^{-1}\rho_2)_\nabla$$

Then this process has a modification that lives in $\mathcal{H}^{-\varepsilon}$ for any $\varepsilon > 0$ and has the law of the GFF given by Definition 1.1.1.

There are several ways to argue for this. For example, one could argue directly using a continuity criteria, as is often done for the Brownian motion. One just needs to replace the usual Kolmogorov's continuity criterion with its more general counterpart: Dudley's entropy criterion. However, given that we already have a regular version at hand, one should be also able to argue as follows:

Proof. By Kolmgorov's extension theorem we can see this Gaussian process as a probability measure on $\mathbb{R}^{\mathcal{C}_0^\infty(D)}$. Similarly we can consider the process given by Definition 1.1.1 as a probability measure on the same space. As the finite-dimensional distributions agree, they give the same measure. Thus we can couple the processes such that one is a modification of the other. □

Notice that for the moment we have only defined the GFF on bounded domains. This is just because this simple construction of $\mathcal{H}_0^\varepsilon(D)$-spaces works only in that setting. The GFF on unbounded domains can be just defined using conformal invariance of the Dirichlet norm, as observed in one of the sections below.

Also, let us comment here on the a multiplicative factor that travels along with the GFF: in the literature one either finds the GFF defined either w.r.t an orthonormal basis of the Dirichlet norm $\|\cdot\|_\nabla$ ([53, 25]) or w.r.t to the weighed norm $\frac{1}{2\pi}\|\cdot\|_\nabla$ in the later literature ([19, 38]).

This $\frac{1}{2\pi}$ factor comes from the Green's function for the Laplacian: using the standard L^2 inner product, the Green's function behaves locally like $\frac{1}{2\pi}\log\frac{1}{|x-y|}$, however when one starts talking about say the exponential of the GFF, it is much more convenient to just move this $\frac{1}{2\pi}$ factor into reweighing the integrals, or thus in some sense into reweighing the Lebesgue measure by $\frac{1}{2\pi}$. In chapters 2,3 it is important for use the latter version to in synchrony with the literature. In chapter 1 it does not really matter.

We will finish this subsection by setting straight the analytical background and in particular giving a precise definition of the $\mathcal{H}^{-\varepsilon}$-spaces.

Some analytical preliminaries For open and bounded D we can find an orthonormal basis $(\phi_i)_{i\geq 1}$ of $L^2(D)$ consisting of the eigenfunctions of the Dirichlet laplacian $-\Delta$. By Weyl's law, for the 2D Dirichlet Laplacian the number of eigenvalues less than n grows like $O(n)$ and thus if we arrange the eigenvalues in increasing order, the n-th eigenvalue satisfies $\lambda_n \asymp n$. Now let $\phi \in \mathcal{C}_0^\infty(D)$ and write

$$\phi = \sum_i (\phi, \phi_i)\phi_i$$

where by (f, g) we denote the L^2 inner product. Then we have that

$$\|\phi\|_{L^2}^2 = \sum_i (\phi, \phi_i)^2$$

And in the space $\mathcal{H}_0^1(D)$,

$$\|\phi\|_\nabla^2 = \|\phi\|_{\mathcal{H}^1}^2 = \sum_i \lambda_i (\phi, \phi_i)^2$$

By analogy, we denote by $\|\phi\|_{\mathcal{H}^\varepsilon}$ the norm obtained by writing

$$\|\phi\|_{\mathcal{H}^\varepsilon}^2 = \sum_i \lambda_n^\varepsilon (\phi, \phi_i)^2$$

One can check that this is indeed a norm on $\mathcal{C}_0^\infty(D)$ and stems in fact from an inner product. In this setup one can explicitly check the following lemma-definition [11]:

Lemma 1.1.3. *For D an open bounded domain and $\varepsilon > 0$, let $\mathcal{H}_0^\varepsilon(D)$ be the closure of $\mathcal{C}_0^\infty(D)$ under the norm $\|\phi\|_{\mathcal{H}^\varepsilon}$. It is a Hilbert space and its Hilbert space dual is denoted by $\mathcal{H}^{-\varepsilon}$. Moreover, one can observe that $\mathcal{H}_0^\varepsilon(D)$ is compactly embedded in $\mathcal{H}_0^{\varepsilon/2}(D)$.*

We also record the following consequence, which basically says that in the context of spaces $\mathcal{H}^{-a}$ we can always get strong convergence out of weak convergence, by just paying a tiny price in the exponent. Or in other words, the weak topology of $\mathcal{H}^{-a}$ controls the strong topology of $\mathcal{H}^{-a-\varepsilon}$ for any positive ε:

Corollary 1.1.4. *If g_n and g in $\mathcal{H}_0^{-a}(D)$ are such that g_n converge weakly to g in $\mathcal{H}_0^{-a}(D)$, then g_n converge to g strongly in $\mathcal{H}_0^{-a-\varepsilon}(D)$ for any $\varepsilon > 0$.*

This just follows from the compact embedding theorem (Rellich theorem) above:

Proof. First notice that g_n, g are bounded in $\mathcal{H}_0^{-a}(D)$ by uniform boundedness principle. Now as $\mathcal{H}_0^{-a}(D)$ is compactly embedded in $\mathcal{H}_0^{-a-\varepsilon}(D)$, it has a convergent subsequence. But the limit of this subsequence can be characterized using the weak topology.

□

1.1.1 Field decompositions and spatial Markov property

Given an open subset $B \subseteq D$ of the domain, we can decompose our Hilbert space as a direct sum:

$$\mathcal{H}_0^1(B) \bigoplus \mathcal{H}_0^1(B)^\perp$$

The space $\mathcal{H}_0^1(B)^\perp$ consists of functions that are harmonic inside B.

This gives a natural way to decompose the GFF: we take the orthonormal basis of $\mathcal{H}_0^1(B)$ and of $\mathcal{H}_0^1(B)^\perp$, associate standard Gaussians to each basis element and write formal sums as in (1.1), and look at them by zero extension as distributions on the whole of D. We obtain a decomposition

$$h = h^B + h^{B^\perp}$$

where by h^B we denote the random distribution coming from $\mathcal{H}_0^1(B)$ and by $h^{B^\perp}$ the rest. We know that h^B restricted to B is the zero boundary GFF in B. Thus we obtain a first version of the spatial Markov property: having observed $h^{B^\perp}$, we have an independent GFF inside B.

Now, one can say a bit more about $h^{B^\perp}$. Using the orthogonal decomposition, one can see that it is weakly harmonic. However, any weakly harmonic distribution is in fact harmonic by Weyl's lemma. Thus the projection of the GFF onto $\mathcal{H}_0^1(B)^\perp$ is a random harmonic function on B. We refer the reader to [53] for a more thorough discussion.

One can also notice that this spatial Markov property is a specific case of a more general Markov property: if we take any orthogonal decomposition of $\mathcal{H}_0^1(D)$, we can consider the corresponding Gaussian sums converging in, say, $\mathcal{H}^{-1}(D)$. These are always independent.

For example, when working in a disc, it is sometimes useful to decompose $\mathcal{H}_0^1(D)$ into radial and angular parts. In this way, we can write the GFF as a sum of a Brownian motion (the radial part parametrized using $t = -\log r$) and log-correlated fields on circles (the angular parts).

Or, instead of exploring the GFF spatially, we could explore it also in the Fourier space.

Side-amusement In the case of the Brownian bridge, one could first explore all the Fourier components with odd frequency components, giving in some sense the symmetric part of the Brownian bridge. Then, after rescaling by $1/2$, what remains to be explored is an independent Brownian bridge inside half of the domain.

This is of course just the even-odd decomposition with respect to the midpoint of the interval. It provides a fun way of decomposing the Brownian bridge. First sample an independent Gaussian W of variance $1/2$, and two Brownian bridges B_1, B_2. Then the following has the distribution of a Brownian bridge on $[0,1]$: put the Gaussian W at the centre of the interval and interpolate linearly to endpoints. Now just add to one half-interval $\frac{B_1+B_2}{2}$ and to the other $\frac{B_1-B_2}{2}$.

This procedure could also be re-iterated in order to write the Brownian bridge as independent components that are just rescalings of the same field.

Sigma-algebras and Blumenthal's 0-1 law

The basic sigma-algebra of the GFF is just the sigma-algebra generated by the associated Gaussians. It is desirable to work with what are called the usual conditions: we consider the completion of the sigma-algebra and augment any filtration by all zero probability events. We now consider the sigma-algebras on the projections of the Gaussian free field.

- for B open, let $\mathcal{F}_B^0$ be the sigma-algebra generated by h^B. Intuitively, this is the sigma-algebra containing information just inside B.
- for B open again, let $\mathcal{F}_B$ be the sigma-algebra generated by the (h, ϕ) for $\phi \in \mathcal{C}_0^\infty(B)$. Intuitively, this is the sigma algebra containing information inside B and on its boundary.
- for C closed, $\mathcal{F}_C$ is the sigma-algebra $\bigcap_{C \subseteq B, B \text{ open}} \mathcal{F}_B$ - intuitively this is again the sigma algebra containing information inside and on the boundary of C.

Notice the filtration on closed subsets is right-continuous w.r.t. set inclusion. Moreover, we have the following strong independence between the sigma-algebra on a closed set and its complement:

Lemma 1.1.5 (Extended Markov property)**.** *Let $C \subseteq D$ be closed and consider $B = C^c$. Then the sigma-algebras $\mathcal{F}_C$ and $\mathcal{F}_B^0$ are independent.*

Or in other words, conditioned on $\mathcal{F}_C$ we are left to sample an independent zero boundary GFF inside its complement. Notice that here we also take infinitesimal information around C.

Proof. Denote by $B = C^c$, which is an open subset. Pick an orthonormal basis of $\mathcal{H}_0^1(B)$ consisting of compactly supported smooth functions. It suffices to show that the independence holds for any finite Gaussian vector associated to basis elements of $\mathcal{H}_0^1(B)$. Thus pick such a finite vector.

As all of our basis elements are compactly supported in B, those in the fixed vector are supported in some open set $B' \subseteq B$. But then $C' = B'^c$ is closed and strictly contains C. However, from the orthogonal decomposition of the GFF, one can see that $\mathcal{F}_{B'}^0$ and $\mathcal{F}_{C'}$ are independent. As now $\mathcal{F}_C \subseteq \mathcal{F}_{C'}$, the lemma follows.

□

Thus we have the following zero-one law, replacing Blumenthal's zero-one law for the Brownian motion:

Corollary 1.1.6 (Blumenthal 0-1)**.** *Let C be a closed subset of D and set $B = C^c$. Then any event contained in both of the sigma-algebras $\mathcal{F}_B^0$ and $\mathcal{F}_C$ is trivial.*

1.1.2 General properties

As the first property, we mention that the GFF is conformally invariant. Indeed, due to the conformal invariance of the Dirichlet norm a conformal map $\phi : D \to D'$ provides a correspondence between the orthonormal basis of D' and the orthonormal basis of D. Thus a GFF on D' provides a GFF on D just by using the orthonormal basis $h_i' \circ \phi$. Often we make use of this property to work on a more convenient domain. Moreover, we can use this property to define the GFF on unbounded domains.

Second, let us see that we can actually consider the GFF as a Gaussian process on a wider space of functions. Fix an ordered orthonormal basis of $\mathcal{H}_0^1(D)$ and let h_n be the sum (1.1) restricted to first n elements. In particular then $h_n \in \mathcal{H}_0^1(D)$. For any fixed $f \in \mathcal{H}_0^1(D)$, one can verify that $(h_n, f)_\nabla$ converges to a zero mean Gaussian of variance $\|f\|_\nabla$.

Thus one can make sense of (h, ρ) for any fixed ρ in $\mathcal{H}^{-1}$ by using the dual pairing

$$\mathcal{H}_0^1(D) \xrightarrow{\Delta} \mathcal{H}^{-1}(D)$$

Here the inverse is given by solving the corresponding Dirichlet problem, i.e. using the Dirichlet Green's kernel:

$$G(x, y) = -\log|x - y| - g(x, y) \tag{1.2}$$

where for fixed x in the interior of D, the function $g(x,y)$ is the solution of the Dirichlet problem with the boundary data given by $-\log|x-y|$.

Thus one can look at the GFF as a Gaussian process on $\mathcal{H}^{-1}(D)$ with the covariance structure given by:

$$Cov((h,\rho_1),(h,\rho_2)) = \int_D \int_D \rho_1(x)\rho_2(y)G(x,y)dxdy \tag{1.3}$$

Notice however that as almost surely the GFF does not belong to $\mathcal{H}_0^1$, this Gaussian process is unbounded. Yet, one can often find subsets on which the GFF can be seen as a bounded and even as a continuous process. For example, using Kolmogorov's continuity theorem, one can observe the following:

Lemma 1.1.7. *Denote by ρ_ϵ^z the measure that distributes unit mass uniformly over the circle of radius ϵ around z. Then the circle average process $h_\epsilon(z) := (h,\rho_\epsilon^z)$ is locally α-Holder jointly in (z,ϵ) for any $\alpha < 1/2$.*

One could also ask whether there is some hope that the GFF would be continuous on the closed subsets A of the domain, i.e. on functions $\mathbb{I}(A)$ with respect to some reasonable topology. That this is cannot be the case can be seen by the following consideration.

Take the domain $D = [0,1]^2$ and define A_n as follows: divide the square into 2^{2n} small squares and use the chess-board colouring of these squares with black in the upper left-hand corner. Now let A_n be the union of all black squares on level n. In any reasonable topology on closed subsets, this sequence of closed subsets remains inside a bounded ball (for example in the Hausdorff topology it just converges to the unit square). Consider now $X_n = (h,\mathbb{I}(A_n))$. This is a Gaussian process of mean zero. Denote by $\mathcal{F}_n$ the sigma-algebra generated by X_m for $m \leq n$. Then we can write X_{n+1} as as sum of a $\mathcal{F}_n$-measurable Gaussian and one of of uniformly positive variance. Thus this process is not bounded and we cannot have continuity.

Nevertheless, as we will see in the next section, the GFF changes continuously in the space of distributions, if we perturb the underlying domain.

1.1.3 Continuity of the GFF

Next, we want to consider the following situation: let $D_n \subseteq D_{n+1}$ be open domains s.t. $\bigcup D_n = D$ and D is bounded. Consider the zero boundary GFF h on D and let $h_n = h^{D_n}$ be its orthogonal projections to D_n. We can extend the h_n by zero outside of D_n and thus make sense of them as elements in $\mathcal{H}^{-\varepsilon}(D)$ for all $\varepsilon > 0$. We claim that as such they a.s. converge to the initial GFF:

Proposition 1.1.8. *The restrictions h_n defined above converge a.s. to h in the space $\mathcal{H}^{-\varepsilon}(D)$ for any $\varepsilon > 0$.*

This proposition has a few nice and useful corollaries. On the one hand, it gives us a way to handle the convergence of GFFs in general:

Corollary 1.1.9. *Let $D_n \subseteq D_{n+1}$ be open domains s.t. $\bigcup D_n = D$. Then the zero boundary GFFs h_n converge in law to h as random elements in the space $\mathcal{H}^{-\varepsilon}(D)$ for any $\varepsilon > 0$. In particular they converge in law as random distributions.*

Second it provides us with (weak) control over what happens near the boundary:

Corollary 1.1.10. *Let ρ_n be elements in $\mathcal{H}_0^\varepsilon(D)$ that are supported in the complements of increasing domains D_n with $\bigcup D_n = D$. Suppose that the ρ_n satisfy $\|\rho_n\|_{\mathcal{H}_0^\varepsilon} < C$. Then a.s. for any choice of ρ_n, we have that (h, ρ_n) converges to zero.*

Proof. Indeed, this follows as we can decompose the GFF $h = h^{D_n} + h^{D_n^\perp}$ as the zero boundary GFF inside D_n plus the rest. We have that $(\rho_n, h) = (\rho_n, h^{D_n^\perp})$ and we know that $h^{D_n^\perp}$ converges to zero in $\mathcal{H}^{-\varepsilon}(D)$. □

It comes out that in the proof of Proposition 1.1.3 it is more convenient to work with the Dirichlet inner product and thus we recall that for compactly supported ϕ we have a.s. $(h, \phi)_{L^2} = (h, -\Delta^{-1}\phi)_\nabla$. In particular, due to 1.1.1 it means that h is a bounded linear operator on $\mathcal{H}_0^{2+\varepsilon}(D)$ when acting via the Dirichlet inner product.

Proof. Let $\phi \in \mathcal{C}_0^\infty(D)$. Then for some N we know that $\mathrm{supp}(\phi) \subseteq D_n$ for all $n \geq N$. By the orthogonal decomposition of the GFF, we can write as above $h = h_n + h^{D_n^\perp}$.

Then, we have that $(h, \phi)_\nabla = (h_n, \phi)_\nabla$ for $n > N$ and thus it follows that a.s. for all $\phi \in \mathcal{C}_0^\infty(D)$, we have that $(h_n, \phi)_\nabla \to (h, \phi)_\nabla$.

We know by 1.1.1 that a.s.

$$\|h_n\|_{\mathcal{H}^{-\varepsilon}(D)} \leq \|h\|_{\mathcal{H}^{-\varepsilon}(D)} < \infty$$

and thus all h_n and h are bounded linear operators on $\mathcal{H}_0^{2+\varepsilon}(D)$ when acting with the Dirichlet inner product.

Consider now $\xi \in \mathcal{H}_0^{2+\epsilon}(D)$. As $\mathcal{C}_0^\infty(D) \cap \mathcal{H}_0^{2+\varepsilon}(D)$ is dense in $\mathcal{H}_0^{2+\varepsilon}(D)$, we can choose $\phi_i \in \mathcal{C}_0^\infty(D) \cap \mathcal{H}_0^{2+\varepsilon}(D)$ such that $\phi_i \to \xi$ in the space $\mathcal{H}_0^{2+\varepsilon}(D)$.

Next write

$$(h - h_n, \xi)_\nabla = (h - h_n, \xi - \phi_i)_\nabla + (h - h_n, \phi_i)_\nabla$$

We can first choose i sufficiently large such that $|(h - h_n, \xi - \phi_i)_\nabla| < \delta$. On the other hand we know that $(h - h_n, \phi_i)_\nabla \to 0$ for any fixed i. Thus by choosing n large enough we can see that $|(h - h_n, \xi)_\nabla| < 2\delta$ and hence a.s. $(h_n, \xi)_\nabla \to (h, \xi)_\nabla$ for any $\xi \in \mathcal{H}_0^{2+\varepsilon}(\mathbb{D})$.

But this means that in particular a.s. for any $\chi \in \mathcal{H}_0^\varepsilon(D)$ we have that

$$(h_n, \chi)_{L^2} \to (h, \chi)_{L^2}$$

In other words we have a.s. weak convergence in $\mathcal{H}_0^{-\varepsilon}(D)$ for any $\epsilon > 0$. But now we know from the analytic corollary 1.1.4, that we can always improve weak topology to the strong topology by paying as little as we wish in terms of Sobolev exponents. Thus the proposition follows.

□

Finally, we describe in slightly more detail the behaviour of the GFF near the boundary using the so-called trace of the Gaussian free field [13]. It is classically known that for $s \geq 1/2$ the trace of a function $f \in \mathcal{H}_0^s(\mathbb{D})$ on a circle $\mathcal{C}_r$ is well-defined by first using the restriction

operator for functions that extend continuously to $\mathcal{C}_r$ [11]. If we denote the trace operator by T, then it maps $\mathcal{H}_0^s(\mathbb{D}) \to \mathcal{H}(\mathcal{C}_r)^{s-1/2}$. In the case of the GFF, one needs a bit more care.

In [13], section 4.3 the author defines the trace of the GFF on the circle $\mathcal{C}_r$ as follows. Pick an orthonormal basis $(\xi_i)_{i\geq 1}$ of the subspace of $\mathcal{H}_0^1(\mathbb{D})$ containing functions harmonic off $\mathcal{C}_r$. Then we define

$$T_{\mathcal{C}_r} h := \sum (h, \xi_i)_\nabla \xi_i$$

The Dirichlet inner product on $\mathcal{H}_0^1(\mathbb{D})$ induces an inner product on $\mathcal{H}_0^{1/2}(\mathcal{C}_r)$ via harmonic extensions and thus this trace can be considered as an element defined on a function space on $\mathcal{C}_r$. We have the following lemma describing this function space in the case of the GFF [13]:

Lemma 1.1.11 (Traces of the GFF)**.** *For any $r \in (0,1)$ one the trace $T_{\mathcal{C}_r} h$ of the GFF on $\mathbb{D}$ is (via the identification above) an element of $\mathcal{H}^{-\varepsilon}(\mathcal{C}_r)$ for any $\varepsilon > 0$.*

The second lemma in [13] shows that these traces a.s. converge to zero, if parametrized such that they could be seen as elements of $\mathcal{H}^{-\varepsilon}(\mathcal{C}_1)$:

Lemma 1.1.12 (Traces converge to zero)**.** *As $r \uparrow 1$, a.s. the traces $T_{\mathcal{C}_r} h$ converge to zero in the strong topology of $\mathcal{H}^{-\varepsilon}(\mathcal{C}_1)$ for any $\varepsilon \downarrow 0$.*

This lemma is proved in [13] for $\varepsilon = 1$, however using a Rellich argument (see Corollary 1.1.4) the convergence can be enhanced to all $\varepsilon > 0$. It also follows from Proposition 1.1.3 by using the equivalence of the $\|\cdot\|_{\mathcal{H}^{1/2}(\mathcal{C}_r)}$ norm and the $\|\cdot\|_{\mathcal{H}^1(\mathbb{D})}$ norm of its harmonic extension.

1.2 GFF and the white noise

In this section we first describe a relation between the GFF and the 2− dimensional white noise vector field. We then obtain a version of the Donsker principle for the 2D GFF. Finally, we discuss what happens in higher dimensions.

The observations below stem from nice discussions with Jean-Christophe Mourrat and Scott Armstrong.

1.2.1 2-dimensional white noise vector field

We first make sense of the 2D white noise process on a domain D. The white noise process is the natural fluctuation with respect to the L^2 norm of the domain. Similarly to the GFF, the white-noise process can be given by a formal sum:

$$w = \sum_i X_i \phi_i \tag{1.4}$$

with X_i i.i.d standard Gaussians and ϕ_i an orthonormal basis of $L^2(D)$. Again, for convergence we have to look for spaces with a weaker norm than that of $L^2(D)$:

Definition 1.2.1. *The white-noise on a planar domain D, denoted by $w = w^D$ is given by the formal sum (1.4), where the convergence takes place in any $\mathcal{H}^{-1-\varepsilon}$ for $\varepsilon > 0$.*

As in the case of the GFF, the uniqueness is then guaranteed by the following proposition that characterizes white noise:

Proposition 1.2.2. *Let D be a bounded planar domain. Suppose we have some probability space with a Gaussian process h such that for any smooth $\rho \in \mathcal{C}_0^\infty(D)$, (h, ρ) is a centred Gaussian of variance $\|\rho\|_{L^2}^2$. Then this process has a modification that lives in $\mathcal{H}^{-1-\varepsilon}$ for any $\varepsilon > 0$. This modification is given by (1.4).*

Again it just follows from the fact that we already have an explicit construction that agrees in law as a Gaussian process. Finally, we define the 2D white noise vector field as follows:

Definition 1.2.3. *The 2D white-noise vector field on a planar domain D, denoted by W, is given by $W = (w_1, w_2)$ where w_1, w_2 are two independent white-noise processes on D.*

1.2.2 Hodge decomposition of the white noise vector field

Now consider a bounded domain with a Lipschitz boundary. The standard Hodge decomposition (in our restricted case also known as the Helmholtz decomposition) expresses a smooth vector field V as a sum of a curl-free and a divergence-free component.

Let us concentrate on 2D. Then we can define the curl or rotation of a vector field as $\nabla \times V = \partial_x V_y - \partial_y V_x$ and of a scalar by $\nabla \times S = (-\partial_y S, \partial_x S)$.

It is known that in 2D any curl-free vector field can be expressed as the gradient of a scalar potential and every divergence-free vector field can be seen as a rotation of another scalar field. Thus we can write:

$$V = \nabla\phi + \nabla \times S + H$$

where ϕ is a zero boundary scalar potential, S a zero boundary vector potential and H a harmonic component satisfying $\nabla \cdot H = 0$ and $\nabla \times H = 0$, which is there to guarantee the zero boundary conditions for ϕ and S. This decomposition gives a orthogonal decomposition of $(L^2(D))^2$.

We will obtain a similar decomposition for the 2D white noise vector field. However, it is more convenient to work with an anharmonic white noise vector field, i.e. to remove the H component.

Definition 1.2.4. *Decompose $(L^2(D))^2$ into an orthogonal sum $H(D) \bigoplus H(D)^\perp$ with $H(D)$ consisting of harmonic vector fields satisfying $\nabla \cdot H = \nabla \times H = 0$. The anharmonic white noise vector W_0 in a domain D is given by taking the white noise-sum as in 1.4 only for the subspace $H(D)^\perp$. Component-wise the convergence for W_0 takes place in any $\mathcal{H}^{-1-\varepsilon}(D)$.*

We will also sometimes need to use the harmonic component W_H corresponding to the other part of the orthogonal complement.

Theorem 1.2.5 (Hodge decomposition of the white noise)**.** *Consider some open planar domain D. Then in the space and sense of distributions, the 2d anharmonic white-noise vector field can be a.s. written as $W_0 = \nabla h_1 + \nabla \times h_2$, where h_1 and h_2 are W_0 measurable and have the law of a pair of independent zero boundary GFFs.*

The statement makes sense in any space of the lowest regularity of all these terms, i.e. in any $\mathcal{H}^{-1-\varepsilon}$.

Remark 1.2.6. Also, if we did not require the white noise to be anharmonic, we could include the harmonic part in one of of our GFF-s, say in h_1 and it would become a free boundary GFF, still independent of h_2. Indeed, consider the one dimensional case:

The Hodge decomposition of a smooth function on $[0,1]$ is simple. There is no rotational component and if we insist on zero boundary conditions, we can just write

$$f = \nabla g_0 + c$$

where c is just a constant - $c = \int_0^1 f dt$ and

$$g_0(t) = \int_0^t f(t)dt - t\int_0^1 f(t)dt$$

If we did not insist that g_0 had zero boundary, then we could of course write $f = \nabla g$, where now

$$g = \int_0^t f(t)dt$$

This gives a good hint of what is what in the case of the 1D white noise on $[0,1]$: if f is the standard white noise, then g is the standard Brownian motion. If furthermore, we ask for zero boundary conditions, then g_0 is the Brownian bridge (thus the zero boundary GFF in 1D) and h is just the integral of the white noise over the interval. It is the terminal value of the related Brownian motion, and thus just a Gaussian of unit variance.

Thus we could actually use the decomposition of the 2D white noise vector field to define the free boundary GFF, and in fact to define a zero boundary and a free boundary GFF from the same 2D white noise vector field. In this interpretation, one would be the harmonic conjugate of the other.

There are also several well-sounding but immediate corollaries to this result, justifying a little why we call it a theorem. We state two of of them:

Corollary 1.2.7. *The anharmonic 2D white noise conditioned to be curl-free is given by the gradient of a zero boundary GFF. The anharmonic 2D white noise conditioned to be divergence-free is given by the curl of a zero boundary GFF.*

Remark 1.2.8. Thus geometrically one can say that the GFF is a natural height function related to a white noise vector field. One can also interpret the GFF as the natural measure on scalar potentials and in the case of 2D also on vector potentials - i.e. potentials coming from divergence-free vector fields.

Now, it is known that the divergence-free vector fields in 2D have a loop representation [55]. This again makes one wonder whether this is a good way to interpret the CLE_4 construction of the GFF, which decomposes the GFF into a sum of nested loops, each contributing value $\pm\lambda$ to their interior.

The latter also motivates the following question:

Question 1.2.9. Is there a natural way to construct loops using the divergence free part of the white noise vector field, so that (in the limit) they would be related to the CLEs?

The second one builds the basis to formulate the Donsker invariance principle for the GFF later on:

Corollary 1.2.10. *In terms of distributions we have $\nabla \cdot W = \Delta h_1$, where W is the white-noise vector field and h_1 is a zero boundary GFF.*

Remark 1.2.11. Moreover, one may notice an interesting interpretation in the context of Liouville quantum gravity. Namely, in Liouville quantum gravity one aims to define a random metric on 2D surfaces (see chapter 3 for a more thorough discussion). Writing it in isothermal coordinates, it should be of the form "$e^h dz^2$", where h is an instance of the GFF. Now the curvature of such a metric is given just by Δh and thus we see that the curvature should match the divergence of a white-noise vector field.

The proof of the theorem itself is not very demanding:

Proof. We will argue in two steps:

First, we show that the equality holds in law and second, we argue that in fact these components are measurable.

Denote by $V = (\rho_1, \rho_2)$ with $\rho_i \in C_0^\infty(D)$ for both $i = 1, 2$. For the first step, it suffices to prove that we have in law:

$$\int \langle W_0, V \rangle = \int \langle \nabla h_1, V \rangle + \int \langle \nabla \times h_2, V \rangle \tag{1.5}$$

with h_1, h_2 independent zero boundary GFF-s, W_0 the an-harmonic 2D white-noise vector field and $\langle \cdot, \cdot \rangle$ denoting the standard inner product on $\mathbb{R}^2$.

Now use the Hodge decomposition for ρ to write $V = \nabla\phi + \nabla \times S + H$, where both ϕ and S satisfy zero boundary condition. Notice that from our definition of W_0, we have that $\int \langle W_0, H \rangle = 0$.

As the three terms in (1.5) are centred Gaussians, we just need to show that the variances match up.

$$\int \langle \nabla h_1, V \rangle = \int \langle \nabla h_1, \nabla\phi \rangle + \int \langle \nabla h_1, \nabla \times S \rangle + \int \langle \nabla h_1, H \rangle$$

The first of those is just $(h, \phi)_\nabla$ and thus a centred Gaussian of variance $\|\nabla\phi\|_{L^2}$. We claim that the second and the third vanish: indeed, it follows by integration by parts and the fact that $\nabla \cdot (\nabla \times S)$ and $\nabla \cdot H$ are both zero.

For h_2 the only contribution comes from $\int \langle \nabla \times h_2, \nabla \times S \rangle$. Using the fact that in two dimensions $\langle \nabla \times f, \nabla \times g \rangle = \langle \nabla f, \nabla g \rangle$ we see that we get a centred Gaussian of variance $\|\nabla S\|_{L^2} = \|\nabla \times S\|_{L^2}$. Again the other two terms vanish. Thus the first step follows.

For the second step consider $\nabla \cdot W$. From above we can see that in the sense of distributions, it has the law of the Laplacian of the GFF. It just remains to argue that h_1 is ∇W-measurable: for any smooth ρ we know $(h_1, \Delta\rho) = (\Delta h_1, \rho) = (\nabla \cdot W, \rho)$ and we know from 1.1.2 that this fully characterizes the zero boundary GFF. A similar argument holds for h_2 and thus we can indeed write the a.s. equation given in the proposition.

□

Remark 1.2.12. When working on the square, one can also obtain the decomposition by working in the Fourier space. Moreover, in this setting an additional nice property reveals itself: for any Fourier coefficient, the vectors associated to h_1, h_2 are in addition orthogonal - one of them points in the direction of the Fourier vector, and the other one is orthogonal to it.

Discrete Hodge decomposition

Let us now discuss the Hodge decomposition of white noise on a finite graph. This will lead us towards a Donsker principle for the GFF.

Consider a finite simple planar graph $G = (V, \partial V, E)$ with boundary vertices ∂V. We denote the set of faces by F (without including the outer face) and let the face boundary to consist of all faces that contain a vertex on the boundary. Every edge has two opposite directions: we denote the edge from x to y by xy and from y to x by yx.

The hodge decomposition also generalizes nicely to this discrete setting, one just has to be a bit more careful about what lives where. So let us briefly specify the setting:

- The scalar potentials ϕ live on vertices of the graph and their gradient lives on edges and for an edge from x to y is given by $(\nabla\phi)(xy) = \phi(y) - \phi(x)$.
- Rotational potentials S live on faces. Their curl lives on edges and for a directed edge e between two faces e_R and e_L, we set $(\nabla \times S)(e) = S(e_R) - S(e_L)$, where e_R is to the right and e_L to the left of the oriented edge e.
- Vector fields live on direct edges and satisfies for any edge $V(yx) = -V(xy)$. Their divergence lives on vertices and is given by $(\nabla \cdot V)(x) = \sum_{yx \in E} V(yx)$. Their curl lives on faces and is given by: $(\nabla \times V)(f) = \sum V(yx)$ where the sum is over the edges around the face in clockwise direction.

The laplacian Δf of a function on the vertices is given by $\Delta f = \nabla \cdot \nabla f$.
The l^2 inner product on vertices, written $(f, g)^v$ is just given by:

$$(f, g)^v = \sum_v f(v)g(v)$$

The l^2 inner product on vertices, written $(f, g)^e$ is just given by:

$$(U, V)^e = \sum_e U(e)V(e)$$

where one can see that the choice in which direction we consider an edge, does not matter.
The Dirichlet inner product $(f, v)^e_\nabla$ is the l^2 inner product on undirected edges:

$$(f, g)^e_\nabla = (\nabla f, \nabla g)^e = \sum_e (f(e_x) - f(e_y))(g(e_x) - g(e_y))$$

Here we use the upper-scripts to remind ourselves if whether we work on edges or vertices. One can verify that if f is a scalar potential with zero boundary condition and V a vector field then

$$(\nabla f, V)^e = -(f, \nabla \cdot V)^v$$

and that in particular for f with zero boundary condition

$$(f, g)_\nabla = -(f, \Delta g)$$

If we want to adhere to the Hodge decomposition above, we should also make clear what it means to be zero boundary. For a scalar potential it just means to be zero on the boundary vertices. For a rotational potential we fix zero rotation on the face boundary, which we defined above. In this setting, the Hodge decomposition of vector fields says that any vector field V can be written as the following sum:

$$V = \nabla\phi + \nabla \times S + H$$

with again ϕ being a zero-boundary scalar potential, S being a zero boundary rotational potential and H a harmonic component, where harmonic here really means that it is harmonic away from the vertex and face boundaries. In this way, we again obtain a orthogonal decomposition of $l_2(E)$.

Let us now introduce the probabilistic setting. First, the white noise vector field on G is defined by just putting i.i.d standard Gaussians on each edge in E. We will relate it to two zero boundary GFF-s related to the graph: one on the vertices and one on the faces.

To define the zero boundary GFF-s, consider the Dirichlet Green's function G_0 on a graph: $G_0(x, y)$ gives the expected occupation time at site y for a simple random walk starting at x and killed when it first hits a boundary vertex. It satisfies $-\Delta G_0(x, y) = 1_{x=y}$.

Then the zero boundary GFF h_G living on V is then defined as a zero mean Gaussian process on vertices V with the covariance given by the Green's function on G:

$$\mathbb{E}h_G(v_1)h_G(v_2) = G_0^G(v_1, v_2)$$

The zero boundary GFF h_G^f living on F is defined as a zero mean Gaussian process on faces F with the covariance given by the Green's function on the dual graph - i.e. on the graph where faces are the vertices.

Thus similarly to above, we can also define the anharmonic white noise vector field and state the discrete white noise decomposition:

Proposition 1.2.13 (Discrete decomposition of the white noise)**.** *Consider a finite simple embedded planar graph $G = (V, \partial V, E)$ with boundary. Then the 2D anharmonic white-noise vector field can be written as $W_0 = \nabla h_1 + \nabla \times h_2$, where h_1 and h_2 are W_0 measurable and have the law of a pair of independent discrete zero boundary GFFs, h_1 living on the vertices and h_2 on the faces of G.*

Moreover, we again have the corollary saying that $\nabla \cdot W_0 = \Delta h_1$. This inspires the following question: what if we put just i.i.d. not necessarily Gaussian noise on the edges of a regular graph? We can then still decompose the noise and define the fields h_1, h_2 by solving the discrete Poisson problems. Of course h_1, h_2 are not any longer Gaussian and moreover, they are not any longer necessarily independent. However they will become Gaussian and independent in the limit. This is the content of the following Donsker principle for the GFF:

1.2.3 Donsker principle for the GFF

Recall the Donsker theorem in 1D: consider a 1D random walk S_n of n steps, with independent steps that have unit variance and zero mean. Interpolate the walk linearly between the steps to get a continuous function. Then the law on the renormalized walks $S_n/\sqrt{n}$ seen as continuous functions converges to the 1D standard Brownian motion. What would be a good generalization to higher dimensions?

In the formulation above, it is not straight-forward to see how to generalize Donsker's principle to higher dimensions. Let us however see that the decomposition obtained in the previous section, and in particular the way of building the scalar potential out of the noise field is exactly the content of Donsker's principle.

Indeed, suppose we work on $[0,1]$. Then, we can think of the SRW of length n as follows: discretize $[0,1]$ into n equal pieces. Now we put a discrete vector field on the edges: i.e. every edge gets and i.i.d. unbiased random variable of zero mean and unit variance. Call this field W_n.

Then calculating the renormalized SRW path just amounts to solving the discrete Poisson problem: $\Delta S_n = \nabla W_n$ on vertices with Dirichlet boundary at 0 and Neumann boundary at 1. The Donsker invariance principle claims that the $S_n/\sqrt{n}$, linearly interpolated and seen as a continuous functions, converge to the standard Brownian motion.

Thus we can state Donsker's invariance principle for the GFF as follows:

Theorem 1.2.14 (Donsker invariance for the GFF)**.** *Consider the square $[0,1]^2$ and split it into n^2 squares of side-length $1/n$. On each inner edge put an i.i.d random variable of zero mean, unit variance and uniformly bounded $2+\delta$-th moment for some positive δ. Call this vector field W_n. Solve the discrete Poisson problem $\Delta h_n = \nabla \cdot W_n$ on the vertices with Dirichlet boundary conditions to obtain a random potential on the vertices. Interpolate it linearly to the whole of $[0,1]^2$ (i.e. draw diagonals and interpolate in these triangles). Then as $n \to \infty$, the random potentials h_n converge in law in the strong topology of $\mathcal{H}^{-\varepsilon}([0,1]^2)$ to the zero boundary GFF on $[0,1]^2$.*

Remark 1.2.15. In fact, it should also be possible to show the joint convergence of the whole noise vector field decomposition. In this case the two potentials - one on vertices, on on faces will both in the limit have the law of the GFF. Moreover, they will become independent in the limit. This is just slightly trickier to properly state and prove.

Finally, the same convergence result holds for any reasonable interpolation. It also holds for other regular tilings - e.g. if you regularity triangulate the unit rhombus.

For non-regular tilings we expect a similar result to hold if the Green's functions converge, only the limiting GFF would have a different underlying metric. For example in the GFF sum representation (1.1) we could replace the orthonormal basis consisting of the eigenfunctions of the regular Laplacian by an orthonormal basis of $g^{-1}\Delta$ with some positive g that represents a change of the metric.

This is analogous in the 1D setting: one can observe that the Donsker principle still holds if you choose a different tiling (i.e. size $2/n$ on $[0,1/2]$ and $1/n$ on $[1/2,1]$), but you need to time-change the Brownian motion accordingly.

Also, notice that renormalization is present in the 1D case but not in the 2D case. The reason is the following: in the discrete set-up above, all the vector-calculus was done intrinsically

to the graph structure. If we however embed the discrete graphs on the plane and want to relate them to the Euclidean vector-calculus, we need to start renormalizing.

In the regular embedding considered here, this just means that the derivative operations need to be multiplied by n and the mass of each vertex and edge will be roughly $1/n^d$. One can check that in d-dimensions one obtains

$$(f,g)^{e,E}_{n,\nabla} = n^2 (f,g)^{e,E}_n = n^{2-d}(f,g)^e_n$$

where by E we denote the embedded version and n marks the level of regularization. Thus only in 2 dimensions we have the miracle that

$$(f,g)^{e,E}_{n,\nabla} = n^2 (f,g)^{e,E}_n = (f,g)^e_n$$

and thus we can take the graph-intrinsic operations without any rescaling. This is of course related to the scale invariance of the GFF.

Proof. It is known that eigenfunctions of the discrete Dirichlet Laplacian on the unit square are just given by the discretizations of the eigenfunctions of the continuous Laplacian.

Indeed, the eigenfunctions in the continuum are given by

$$\sin \pi i x \sin \pi j y$$

where i, j range over positive integers. Denote by $a_{i,j}$ the eigenfunctions that are orthonormal in the space $\mathcal{H}^1_0([0,1]^2)$.

In the case of $1/n$ level discretization, the $a_{i,j}$ for $i, j \in \{1, \dots, n\}$ seen as functions on the vertices also form a orthogonal basis for the discrete Dirichlet energy. Moreover, they roughly form an orthonormal basis for the Dirichlet inner product. One can observe that for any fixed $\rho_1, \rho_2 \in \mathcal{H}^1_0([0,1]^2)$ we have

$$(\rho_1, \rho_2)^v_{n,\nabla} \to (\rho_1, \rho_2)_\nabla \tag{1.6}$$

Indeed, as discussed above, in 2D the intrinsic graph Dirichlet product coincides with the embedded one and the latter just approximates the continuous case. In particular, the $a_{i,j}$ considered as functions on the graph, will have unit norm in the limit. We treat the question of uniform convergence over the whole basis later on.

Now, we will first show the statement of the proposition for a different interpolation than in the statement: instead of the linear interpolation of the field defined on the vertices, we use the interpolation that just comes from the identification of the Discrete and continuous eigenfunctions. More precisely, if we denote this field by h'_n, we can write

$$h'_n = \sum_{i,j \in \{1,\dots,n\}} Y^n_{i,j} a_{i,j}$$

where $Y^n_{i,j} = (h'_n, a_{i,j})_\nabla$.

To show that h'_n converge to h in law in the strong topology of $\mathcal{H}_0^{-1-\varepsilon}([0,1]^2)$ for each $\varepsilon > 0$, it suffices to do two things:

- show that $Y^n_\rho = (h'_n, \rho)_\nabla$ converges to a zero mean Gaussian of variance $\|\rho\|_\nabla$ for any $\rho \in \mathcal{H}^1_0([0,1]^2)$: as in the limit we have a Gaussian process, then this characterizes the limiting law
- prove tightness in $\mathcal{H}^{-\varepsilon}([0,1]^2)$

Characterizing the law

Let us first show that we have the right limiting law. Denote by H_{n_0} the subspace of $\mathcal{H}^1_0([0,1]^2)$ spanned by $a_{i,j}$ for $i,j \in \{1,\dots,n_0\}$. Now fix n_0 and consider first $\rho \in H_{n_0}$.

Now, as $\rho \in H_{n_0}$ it can be written as

$$\rho = \sum_{i,j\in\{1,\dots,n_0\}} \rho_{i,j} a_{i,j}$$

This means that its restriction to the vertices for $n \geq n_0$ is just given by the restriction of this sum to the grid.

By definition Y^n_ρ is given by the Dirichlet inner product of h'_n and ρ:

$$Y^n_\rho = (h'_n, \rho)_\nabla = \sum_{i,j\in\{1,\dots,n_0\}} \rho_{i,j} Y^n_{i,j}$$

This can be again approximately determined by only looking at the restriction to the vertices. Write

$$Z^n_\rho = (h'_n, \rho)^v_{n,\nabla}$$

Then using (1.6), as ρ is supported only on a bounded number of basis elements, we have almost surely:

$$Z^n_\rho \to Y^n_\rho$$

But now on the vertices of the grid $h_n = h'_n$ and thus we can write:

$$Z^n_\rho = (\rho, h_n)^v_{n,\nabla} = -(\rho, \Delta h_n)^v_n = -(\rho, \nabla \cdot W_n)^v_n = (\nabla\rho, W_n)^e_n$$

where as the notation hints all the equations are seen as operations on the square grid and the final equation follows as ρ is zero on the vertex boundary. This puts us well for calculating Y^n_ρ:

Indeed on each edge W_n is just an independent random variable with zero mean and unit variance. One can check that the Lyapunov's condition is satisfied, and thus by the central limit theorem Y^n_ρ converges to a Gaussian of zero mean and variance given exactly by the Dirichlet energy of ρ.

Putting all together, we have that if $\rho \in H_{n_0}$ for some n_0, then we have in law:

$$(h'_n, \rho) \to (h, \rho)$$

But these ρ are dense in $\mathcal{H}_0^1([0,1]^2)$ and thus also in $\mathcal{H}_0^\varepsilon([0,1]^2)$. Hence we have characterized the law as soon as we know that the limiting object lives in $\mathcal{H}^{-\varepsilon}$. To show this, it remains to argue for tightness:

Tightness

From Lemma 1.1.3, one sees that to prove tightness in $\mathcal{H}^{-\varepsilon}$ it suffices just to show that for any $\delta > 0$, we can find an $R = R(\delta)$ s.t. $\mathbb{P}(\|h_n'\|_{\mathcal{H}^{-\varepsilon/2}} > R) \leq \delta$.

To do this, we need to make the approximation by of the continuous Dirichlet energy by the discrete Dirichlet energy slightly more quantitative. I.e. we want to see what is the error in approximating $(\rho,\rho)_\nabla$ by its discrete counterpart. The vertical edges approximate the $\partial_y\rho\partial_y\rho$ part of the Dirichlet energy and horizontal edges approximating the $\partial_x\rho\partial_x\rho$ part. As it is basically the next derivative that controls the error, we obtain that the error is of order:

$$O(\frac{1}{n}\left\|\partial_x\rho\partial_{xx}\rho + \partial_y\rho\partial_{yy}\rho\right\|_{L^2})$$

Thus for each of the orthonormal basis elements $a_{i,j}$ of $\mathcal{H}_0^1$, we see that the error is naively bounded by $O(\frac{|i+j|}{n})$.

Hence we can bound $(h_n', h_n')_\nabla$ using $(h_n', h_n')^v_{n,\nabla}$ by losing

$$\sum Z_{n,i,j}^2 O(\frac{|i+j|}{n})$$

where now $Z_{n,i,j}$ are the coefficients found from discrete inner products $(h_n', a_{i,j})^v_{n,\nabla}$. Thus, as these coefficients are uncorrelated we have:

$$(h_n', h_n')_\nabla = \sum Z_{n,i,j}^2 O(1 + \frac{|i+j|}{n})$$

In the limit these are zero mean Gaussians of unit variance. Moreover, by using again comparison of discrete and continuous Dirichlet energy we see that their variance is always bounded by $O(1)$.

Finally, if instead of Dirichlet inner product, we consider the inner product of $\mathcal{H}^{-\varepsilon/2}$, then for each term we win a factor of $O((i^2+j^2)^{1+\varepsilon/2})$. Thus as $|i+j| = O(n)$, we obtain that uniformly in n

$$\mathbb{E}(h_n', h_n')_{\mathcal{H}^{-\varepsilon/2}} < C$$

This proves tightness in $\mathcal{H}^{-\varepsilon}$ by Markov's inequality.

Returning to the linear interpolation

Finally it remains to argue that we can use the slightly better sounding linear interpolation. This is very similar to the tightness estimate. Indeed, first notice that we can think of this linear interpolation as taking each basis element $a_{i,j}$ and interpolating it linearly to say $b_{i,j}$, instead of just extending it.

Thus let us consider some basis element $a_{i,j}$. Then one can calculate that on the $n-$th level discretization we have again

$$\left\|a_{i,j} - b_{i,j}\right\|_{\nabla}^2 = O(\frac{\sqrt{i^2+j^2}}{n})$$

Hence we see that the norm $\|h_n - h_n'\|_{\mathcal{H}^{-\varepsilon}}^2$ can be bounded given by

$$\sum Y_{a_{k,l}}^n Y_{a_{i,j}}^n O(\frac{(i^2+j^2)^{-1/4-\varepsilon/2}(k^2+l^2)^{-1/4-\varepsilon/2}}{n}) \tag{1.7}$$

where the sum is over $i, j \leq n$.

But now from above we know that the variance of $Y_{a_{i,j}}$ is of order $O(1)$. Moreover, we know that $Z_{a_{i,j}}$ are uncorrelated. Thus again by comparing discrete and continuous Dirichlet inner products, we see that $\mathbb{E}Y_{a_{i,j}}Y_{a_{k,l}}$ can be bounded by $C\frac{(i^2+j^2)^{1/4}(k^2+l^2)^{1/4}}{n}$.

Thus in particular using $\sum_{i=1}^n \sum_{j=1}^n \frac{1}{(i^2+j^2)^{1/2+\varepsilon}} = O(1/n^e ps)$ and Cauchy-Schwarz, we get:

$$\mathbb{E}\left\|h_n - h_n'\right\|_{\mathcal{H}^{-\varepsilon}}^2 = O(1/n^{\varepsilon})$$

and the claim follows.

□

1.2.4 Higher dimensions

Most of the discussion of this section generalizes to higher dimensions.

First, white noise vector fields can be generalized directly to higher dimensions. Second, the decomposition theorem 1.2.5 holds in the given form, only that the S is not a scalar any longer. In 3D it would be a vector potential and in higher dimensions a so-called two-form. It will still represent rotation. We can still ask for zero boundary conditions for the scalar potential, but choosing the natural version for S is slightly more tricky.

In any case, the gradient part of the white noise still remains a gradient of the GFF in any dimension. The S component is slightly more mysterious, for example it remains unclear for now whether S or indeed the divergence free vector field induced by $\nabla \times S$ are of any independent interest. However, one can observe that it contains lots of independent GFF-s inside, staying true to the saying that for any regularity there is one natural Gaussian object. In fact, one can actually obtain a decomposition theorem that decomposes the anharmonic white noise vector field in n dimensions as a sum of n independent vectors, that are independent, orthogonal for each Fourier coefficient and have the same distribution - that of a gradient of the GFF.

The Donsker invariance principle generalizes to higher dimensions with minor changes.

1.3 Local sets

Local sets were introduced in [51] in the context of studying the level lines of the Gaussian free field. In that paper, they were seen as couplings of the GFF with possibly random closed

subsets such that the GFF splits into two: a part of the GFF that is covered and discovered by the local set, and an independent GFF outside that remains to be explored.

This is of course reminiscent of stopping times of the Brownian motion: we look at our process up to some stopping time and what remains is an independent copy of the process. Thus one would like to say that in some sense local sets are just stopping times for random fields, the time is just now parametrized by higher-dimensional sets. In this chapter we try to make this analogy a bit more precise.

This analogy is certainly not exact. For example, in higher dimensions this "time" is no longer totally ordered, and the notions instead of being time-related become more geometric. So when we talk about exploring the field with a local set, this has the flavour of a "local sampling": we look at the field on some possibly field-dependent subset, without gaining any information about the field outside. There is no "time-evolution" and thus local sets are in some sense "static explorations". To introduce "dynamics" into the exploration, one somehow chooses a particular totally-ordered set of subsets and explores the field along these subsets. In our language these processes will correspond to local processes.

The rest of this section proceeds as follows. First we develop the notion of local sets and local processes for general Markov fields. Many of the definitions mimic the equivalent ones in the theory of stopping times. We have tried to look for a level of generality that clarifies the extent of all concepts, but does not ask for too much technical detail. This in particular means that at some point we abruptly restrict ourselves only to the GFF. We convince ourselves that in this particular case everything is especially nice and obtain more precise statements. For example, we discuss how to state and prove the strong Markov property for the GFF.

The term "local set" and the underlying results stem from [51]. At the same time in [13], the author implicitly used "local sets" and their properties, without defining them as a separate object of interest. He also considers "stochastic differential equations driven by the free field", which correspond to local processes in our language. We also remark, that an analogy between stopping times and local sets was also developed in [26] and that another treatment of local sets appears in [58].

1.3.1 Local sets for random fields

Let us start by specifying how we restrict the notion of a random field:

Definition 1.3.1. *A random field h in some open domain $D \subseteq \mathbb{R}^n$ is a random variable taking values in $\mathcal{H}^{-n}(D)$ for some $n \in \mathbb{N}$.*

To talk of the field on different subsets, we define the following sigma-algebras:

- for B open again, let $\mathcal{F}_B$ be the sigma-algebra generated by the (h, ϕ) for $\phi \in \mathcal{C}_0^\infty(B)$. Intuitively, this is the sigma algebra containing information inside B and on its boundary
- for C closed, $\mathcal{F}_C$ is the sigma-algebra $\bigcap_{C \subseteq B, B \text{ open}} \mathcal{F}_B$ - intuitively this is the sigma algebra containing information inside, on the boundary and infinitesimally outside of C.

Recall that they match the definitions given for the GFF earlier on. We always work with what are called the usual conditions: we always consider the completion of the sigma-algebra and augment sigma-algebras in question with all zero probability events.

The notion of a Markov field is then exactly what one expects:

Definition 1.3.2 (Markov field)**.** *We say that a random field is a Markov field if for any closed $C \subseteq D$, the sigma-algebras $\mathcal{F}_C$ and $\mathcal{F}_{C^c}$ are independent, conditioned on $\mathcal{F}_{\partial C}$.*

Remark 1.3.3. Notice here that "Markov" really means "domain Markov" - it is a Markov property in the geometric sense. For some fields, like for example fractional Gaussian fields [36], this might not be the best viewpoint. One might want to relax the strict geometric view to allow for some long-range correlation.

In what follows we will however work exclusively with (domain) Markov fields.

Couplings and enlarged filtrations for Markov fields

We are interested in couplings (h, A) where A is a random closed set. Thus let us also define what we mean by a random closed set. Let Λ be the space of closed subsets of $\bar{D}$ equipped with the Hausdorff metric. Λ is a Polish space and we equip it with its Borel sigma algebra σ_Λ. By A we denote a σ_Λ-measurable random variable.

In order to work in the setting where we have information about the field and the random set A, we define enlarged filtrations for the random field:

Definition 1.3.4 (Enlarged filtration)**.** *Let h be a random field defined on some probability space containing $\sigma(h)$. We call a filtration indexed by the closed sets an enlarged filtration of the field if it satisfies the following conditions:*

- *increasing with respect to set inclusion on the closed sets*
- *right-continuous with respect to set inclusion, i.e. we ask that $\mathcal{G}_C = \bigcap_{C \subsetneq C'} \mathcal{G}_{C'}$*
- *contains the field: for any closed C we have $\mathcal{F}_C \subseteq \mathcal{G}_C$*
- *the filtration retains the Markov property: $\mathcal{G}_C$ is independent of $\mathcal{F}_{C^c}$ conditioned on $\mathcal{G}_{\partial C}$*

Notice that it is the last condition that really puts restrictions on possible filtrations. Also, whereas this is the right definition for geometric Markov fields, this might not always be the right way to enlarge the filtration.

Local sets

Now let us define local sets, which correspond to "local samplings" of the random field:

Definition 1.3.5 (Local set)**.** *Let (A, h) be a coupling as above. Suppose we have an enlarged filtration $\mathcal{G}_C$ of the random field h. We say that the set A is a local set with respect to this filtration, if for any closed set C the event $\{A \subseteq C\}$ is measurable with respect to $\mathcal{G}_C$.*

Notice that a deterministic set is local with respect to any enlarged filtration that retains the Markov property of the field: i.e. for any filtration such that for any closed C, we have $\mathcal{G}_C$ is independent of $\mathcal{F}_{C^c}$.

The role of the filtration is crucial. Some coupling might be local with respect to some well-chosen filtration and not local with respect to some badly chosen one. For example, consider a coupling with an independent random set A. If we choose as the filtration the filtration containing only the information of the field, then A is not a local set w.r.t this filtration. It is however a local set with respect to the natural filtration of the field joined with all the information about A.

In analogy with stopping times, we have the following lemma:

Lemma 1.3.6. *Consider a local set A w.r.t. some enlarged filtration $\mathcal{G}_C$. Then, conditioned on $\{A \subseteq C\}$ the random variable A is measurable w.r.t $\mathcal{G}_C$.*

Proof. As our filtration is increasing, then for any set $B_i \subseteq C$ for dyadic boundary, we have that $\{A \subseteq B_i\}$ is $\mathcal{G}_C$ measurable. But such events characterize A completely, conditioned on $\{A \subseteq C\}$. □

Local couplings with more than one set

Given two random sets A_1, A_2 taking values in Λ each coupled with the free field, we can always consider a three-way coupling on the product $\mathcal{H}$ and two copies of Λ with the product sigma-algebra. Suppose for simplicity that they all live on a common measure space. If we have two local sets A_1, A_2 with respect to a common filtration, it is simple to define their local pair:

Definition 1.3.7 (Local pair of sets)**.** *We say that the pair (A_1, A_2) is a local pair of sets with respect to a filtration $(\mathcal{G})$ if for any closed set C the events $\{A_i \subseteq C\}$ are $\mathcal{G}_C$-measurable.*

Remark 1.3.8. Notice that both sets A_i are also local sets with respect to the same filtration. Also, one can similarly construct local triplets, quadruplets and even countable families of subsets coupled to the free field.

The interesting question is how to construct local pairs or triplets of sets out of local sets that are à priori not defined with respect to the same filtration. It comes out that the most fruitful way is to consider "conditionally independent coupling". This is introduced in [51] and heavily used in the so called imaginary geometry papers (e.g. [38]), but also implicitly in [13].

We start with a simple lemma.

Lemma 1.3.9. *Suppose we have two local sets A_1, A_2 w.r.t filtrations $\mathcal{G}$ and $\mathcal{G}'$ and that for any closed C the sigma-algebras $\mathcal{G}_C$ and $\mathcal{G}'_C$ are independent conditionally on $\mathcal{F}_C$. Then the pair (A_1, A_2) is a local pair w.r.t the join of the filtrations.*

Proof. One has to argue two things: first, that the join of the filtrations is still an enlarged filtration of the field; second, that the events $\{A_i \subseteq C\}$ are both measurable w.r.t the join of the sigma-algebras. The latter is clear, the former follows from a calculation with sigma-algebras, done in Lemma 3.5 of [51]. □

This lemma is however unsatisfactory, as it is unclear how to make sure that we are actually working with conditionally independent filtrations. However, it comes out that given two local sets, we can always construct such filtrations:

Lemma 1.3.10 (Conditionally independent pair of local sets)**.** *Suppose we have two local sets of the field A_1, A_2 w.r.t some enlarged filtrations. Then we can define local sets $(B_1, h), (B_2, h)$ w.r.t. conditionally independent filtrations, such that each has the law of (A_i, h). We call $(A_1, A_2)_{CI} := (B_1, B_2)$ the conditionally independent pair of local sets (A_1, A_2).*

Before proving the lemma, we discuss the conditionally independent unions in general: consider two random subsets A_1, A_2 coupled with h. As we are working on Radon spaces, we can use a disintegration theorem [8] to define the conditional measures $B_i \sim \mathcal{L}(A_i|\mathcal{F}_D)$ on Λ. Next look at the following measure on the product space: sample h and then B_i independently according to their law. The random pair $(A_1, A_2)_{CI}$ obtained this way is called the conditionally independent pair and the set union $A_1 \cup_{CI} A_2 := B_1 \cup B_2$ is called the conditionally independent union of A_1, A_2.

Proof. Given two local sets A_1, A_2 w.r.t some filtrations, we consider the conditional random variables $B_i \sim \mathcal{L}(A_i|\mathcal{F}_D)$ sampled independently. Now define filtrations $\mathcal{G}^i_C$ given by the join of $\mathcal{F}_C$ and $\sigma(\{B_i \subseteq C\}$ and let $(A_1, A_2)_{CI} = (B_1, B_2)$. From the definition of a local set, we see that B_i is a local set w.r.t. $\mathcal{G}^i$. So in order to fulfil the conditions of Lemma 1.3.9, we just need to check that these filtrations are conditionally independent given the field. This follows from our construction. □

Finally, notice that here we see another difference with stopping times: if S and T are stopping times, then so are $T \vee S$ and $T \wedge S$. The first of them would intuitively correspond to the union of two local sets, the second to the intersection. We saw that the union of two local sets is still local. However as the next example shows, the intersection certainly does not need to be local:

Consider the 1D GFF - the standard Brownian bridge $B(t)$ on the interval $[0, 1]$. Let $A_1 = [0, \tau]$ where $\tau = \inf_t\{B(t) = 1\}$ and let $A_2 = [1/2, 1]$ and take the usual Brownian filtration. Then (A_1, A_2) is a local pair in the sense above, yet their intersection cannot be a local set - for example the event $A_1 \cap A_2 \subseteq [1/2, 2/3]$ depends on the values of the process in the interval $[0, 1/2]$.

However, we do have the following:

Lemma 1.3.11. *The intersection of nested local sets is again local: i.e. if $A_1 \supset A_2 \supset A_3$... are local sets w.r.t the same filtration, then so is $\bigcap A_i$.*

Proof. This follows from the definition of the local set and the right-continuity of the filtration. □

Local processes

Sometimes we want however to go even further than coupling several local sets at the same time - we want to explore the field in a continuous fashion. This corresponds to coupling a stochastic process, say, a stochastic curve with the free field.

We assume that our process ν_t takes values in Λ for each t and is growing continuously in the Hausdorff topology. Denote by $\mathcal{N}_t = \sigma(\nu_s, s \leq t)$. Moreover, suppose that $\nu_0 = C_0$ is deterministic and for a closed subset $C \supset C_0$ denote by τ^C the first time $\nu_t \cap \overline{C^c} \neq \emptyset$.

Definition 1.3.12 (Local process)**.** *A continuous stochastic process starting from C_0 and taking values in Λ is local w.r.t. an enlarged filtration $(\mathcal{G})$ of the field, if the following holds: for any closed subset $C \supset C_0$, we have that $\mathcal{N}_{\tau_C} \subseteq \mathcal{G}_C$.*

Notice that for any fixed t, the set ν_t is coupled as a local set with respect to the same filtration. Indeed, we have

$$\{\nu_t \subseteq C\} = \{t \leq \tau_C\} \in \mathcal{N}_{\tau_C} \subseteq \mathcal{G}_C$$

By a standard discretization argument it also follows that:

Lemma 1.3.13. *For any finite stopping time τ of the process $\mathcal{N}_t$, the set ν_τ is local.*

Proof. First consider an upper discretized version τ_n say taking values in n-th level dyadics which we denote by t_k^n. Then the claim is clear from the definition of local sets. Indeed, for any closed C we can write

$$\{\nu_{\tau_n} \subseteq C\} = \bigcup_k \{\tau_n = t_k^n\} \cap \{\nu_{t_k^n} \subseteq C\}$$

However, we know that for each t_k^n,

$$\{\tau_n = t_k^n\} \cap \{\nu_{t_k^n} \subseteq C\} = \{\tau_n = t_k^n\} \cap \{t_k^n \leq \tau_C\} \in \mathcal{N}_{\tau_C} \subseteq \mathcal{G}_C$$

Hence ν_{τ_n} is a local set w.r.t $\mathcal{G}_C$.

Finally it just remains to take the limit as $\tau_n \downarrow \tau$ and use Lemma 1.3.11. □

Sometimes however it is also important to know the converse, i.e. that if we can couple a stochastic process with the field such that it is local at fixed times, then we can couple it as a local process:

Lemma 1.3.14. *Suppose that there is a common enlarged filtration $(\mathcal{G})$ of the field such that for any fixed t, the fixed sets ν_t of a continuous stochastic process are coupled as local sets of the field with respect to $(\mathcal{G})$. Then the process itself is local w.r.t. the same filtration.*

Proof. Recall that any local set, conditioned on $\{A \subseteq C\}$ is $\mathcal{G}_C$-measurable. Thus under the conditioning $\{\nu_t \subseteq C\}$, the whole of $\nu([0,t])$ is $\mathcal{G}_C$-measurable, i.e. for any event $E \in \mathcal{N}_t$, we have that $\{\nu_t \subseteq C\} \cap E \in \mathcal{G}_C$. But this implies that $\mathcal{N}_{\tau^C} \subseteq \mathcal{G}_C$. □

Notice that one can also make sense of pairs and conditionally independent pairs of local processes.

1.3.2 Local sets for the GFF

Now, for Markov fields the natural question is to ask whether this Markov property is also kept with respect to local sets. Indeed, in [51] the local sets are given several different characterizations. The one which is most useful for the main example - the SLE curves - says roughly the following:

Suppose we have a coupling (h, A) of the GFF in the domain D and a random closed set $A \subseteq D$. This set A in the coupling is called a local set if the following holds: conditioned on A, the Gaussian free field inside A and its boundary values, the field outside is given as a sum of a measurable harmonic function and an independent zero boundary Gaussian free field inside A^c.

This can be thought of as the strong Markov property. In what follows we discuss this property only in the context of the GFF. Although the notion and its content are intuitively clear for general random fields, making it precise would already involve too much notation!

Local sigma algebra

First, introduce the local sigma algebra - the sigma-algebra that contains the information about the random set and the field on this random set:

Definition 1.3.15 (Local set sigma algebra)**.** *Given a local set A with respect to the filtration $\mathcal{G}_C$, the local set sigma-algebra $\mathcal{G}^L(A)$ is the sigma-algebra generated by events F such that for any closed C we have $F \cap \{A \subseteq C\} \in \mathcal{G}_C$.*

It is analogous to the stopping time sigma-algebra and satisfies the following properties:

- Local set A itself is measurable with respect to $\mathcal{G}^L(A)$, as for all closed B, C, we have $\{A \subseteq B\} \cap \{A \subseteq C\} \in \mathcal{G}_C$ and the events $\{A \subseteq B_i\}$ for the countable collection of sets B_i with dyadic boundary characterize A.
- Given local sets $A_1 \subseteq A_2$ w.r.t same filtration, then we also have that $\mathcal{G}^L(A_1) \subseteq \mathcal{G}^L(A_2)$.
- The local set sigma-algebras also satisfy a right-continuity property, i.e. for $A = \bigcap A_i$, $\mathcal{G}^L(A) = \bigcap \mathcal{G}^L(A_i)$.

Using the local set sigma algebra one can verify that if two local sets are coupled with the GFF w.r.t. the same filtration, then conditioned on the local sigma algebra of one of them, the other remains a local set in the unexplored part of the GFF:

Lemma 1.3.16. *Conditioned on $\mathcal{G}^L(A_1)$, $A_2 \cap A_1^c$ is a local set w.r.t the random field in A_1^c and the filtration $\mathcal{G}$ restricted to subsets of A_1^c.*

A similar property also holds for local processes :

Lemma 1.3.17. *Consider a pair of local processes ν_t, η_t w.r.t some filtration. Conditioned on $\mathcal{G}^L(\nu_T)$, for some $T > 0$, the process $\eta_t \cap \nu_T^c$ is a local process w.r.t the random field in ν_T^c and the filtration restricted to subsets of ν_T^c.*

In particular this gives an interesting way of generating local sets that goes beyond the restrictions of 1D: we can sample first the process ν_t up to some time T and then sample η_t up to some stopping time that can now depend on the information about ν_t up to time T. This is very useful when one starts studying SLE processes coupled with the GFF, as in chapter 2 or much more thoroughly in the imaginary geometry papers, e.g. [38]. For example, the SLE processes represent flow lines of the GFF and using this property one can study the configuration where you start with one flow line and then sample another flow line up to their first intersection point.

Strong Markov property

A local set that satisfies the Markov property w.r.t its local sigma-algebra is called strongly local. In other words we would like to say the following: for a strongly local set, conditioned on the sigma algebra $\mathcal{G}_A^L$, we can write the random field as a sum of an $\mathcal{G}_A^L$ measurable field plus an independent random field that has the distribution of the "zero boundary field" inside A^c.

Remark 1.3.18. For a general Markov field, it is not entirely clear how one should define a zero boundary field for any open set. This is also one of the reasons why we concentrate on the GFF.

For any open $B \subseteq D$ we denote by $\rho_{|B}$ the projection of ρ on $\mathcal{H}_0^1(B)$.

Proposition 1.3.19 (Strong Markov property for the GFF)**.** *Let A be a local set of the GFF with respect to a filtration $(\mathcal{G})$. Denote by $B = A^c$. Then A is strongly local, or in other words for any $\rho \in C_0^\infty(D)$, the conditional law is given by:*

$$(h, \rho_{|B})_\nabla | \mathcal{G}^L(A) \sim (h_A, \rho_{|B})_\nabla + (h^B, \rho_{|B})_\nabla$$

where by h^B we denote an independent zero boundary GFF in B and h_A is a $\mathcal{G}_A^L$ measurable field, harmonic in A^c.

Remark 1.3.20. Notice that the second term on the RHS really does make sense. We have no difficulty defining an independent zero boundary GFF, when conditioned on the domain.

The proof is similar to proving the strong Markov property for the Brownian Motion via discretizations. See e.g [41].

Proof. We first show the claim for any upper discretized version of the local set A:

For any n we look at the dyadic grid of side-length 2^{-n} and define A_n to be the union of all squares that intersect A. These discretizations are local sets w.r.t the initial filtration.

Notice that A_n can only take finitely many possible values C. Thus for all of them simultaneously the GFF can be a.s. written as an independent sum of the zero boundary GFF in the complement plus a $\mathcal{F}_C \subseteq \mathcal{G}_C$ measurable random field.

So let us argue that conditioned on $\{A_n = C\}$ the following two things are true: first, the field outside remains an independent zero boundary GFF and second, the $\mathcal{G}_C-$ measurable random field is in fact $\mathcal{G}^L(A_n)$-measurable.

The independence of the GFF outside follows from the definition of the local sigma-algebra: indeed, for any event $E \in \mathcal{G}^L(A_n)$ we have that $\{A_n = C\} \cap E \in \mathcal{G}_C$ and thus, conditioned on $\{A_n = C\}$, the event E is independent of $\mathcal{F}_{C^c}$, i.e. the GFF in the complement.

For the second part, consider any event $F \in \mathcal{G}_C$. Then $\{A_n = C\} \cap F \cap \{A_n \subseteq C'\} \in \mathcal{G}_{C'}$ for any closed C'. Thus, for any $F \in \mathcal{G}_C$, $\{A_n = C\} \cap F \in \mathcal{G}^L(A_n)$ and hence, conditioned on $\{A_n = C\}$, any $\mathcal{G}_C$-measurable event is also determined by $\mathcal{G}^L(A_n)$.

Thus the claim holds for discretizations.

It remains to take the discretization to zero and to argue for the convergence of the GFF part and the harmonic part.

Let us start with the second term, i.e. the GFF in the complement. Conditioned on $\mathcal{G}^L(A)$, the set A and its discretizations are determined. By definition, the complements of the A_n grow to the complement of A. Thus we can apply Corollary 1.1.9 to deduce that the zero-boundary GFF-s in A_n converge in law to a zero boundary GFF in A^c.

As the independence is kept in the limiting procedure, we just need to argue for the harmonic part. Denote by h_{A_n} the $\mathcal{G}^L(A_n)$ measurable part for the $n-th$ level discretization. Then for any $\rho \in \mathcal{C}_0^\infty(A^c)$ we have that $(h_{A_n}, \rho)_\nabla$ is a reverse martingale w.r.t. the decreasing filtration $\mathcal{G}^L(A_n)$. But as discussed below definition 1.3.15, these sigma-algebras intersect to $\mathcal{G}_L(A)$ by right-continuity of the filtrations. Thus by the Backwards martingale convergence theorem $(h_{A_n}, \rho)_\nabla$ a.s. converges to a $\mathcal{G}^L(A)$ measurable random variable. As we also have tightness, h_{A_n} a.s. converge as distributions to a $\mathcal{G}^L(A)$-measurable distribution.

Finally, as all h_{A_n} are weakly harmonic in A^c, the weak harmonicity is also satisfied in the limit. Moreover, again by Weyl's lemma strong harmonicity follows from weak harmonicity. □

Thus in fact outside of A, the GFF can be written as a sum of a zero boundary GFF and some harmonic function representing the expected height of the field. We will later study this harmonic function in more detail. First, we make sense of strongly local pairs of sets, and strongly local processes.

The good analogy for strong locality for local pairs uses their union. As the union of a local pair is again a local set, by Proposition 1.3.19 we have the following:

Lemma 1.3.21. *Suppose we have a local pair (A_1, A_2) of the GFF w.r.t. some filtration $\mathcal{G}$. Then their union is a strongly local set.*

Remark 1.3.22. In particular, this means that if we start with two local sets and build their conditionally independent union, then as this can be seen as a union of a local pair, it is also a strongly local set.

We can deduce the similar result for local processes:

Definition 1.3.23 (Strongly local process)**.** *A local process ν_t coupled with the GFF w.r.t some enlarged filtration is strongly local in the following sense: for any fixed time t the set ν_t is strongly local w.r.t the same filtration.*

In fact, as we know that for any finite stopping time ν_τ is also local, it similarly follows that for any finite stopping time ν_τ is strongly local.

The expected value of the field

We start with a convenient definition:

Definition 1.3.24 (GFF with given height)**.** *Given a harmonic function h_D on D, the GFF with height h_D is given by the sum of a zero boundary GFF in D plus h_D.*

It is easy to see that this harmonic function gives the expected height of the field.

The analogy to think of is the Brownian bridge on $[0,1]$ from some starting point a to end-point b. This bridge can then be written as a line from a to b plus a Brownian bridge

from zero to zero. For different choices of a, b we get mutually singular bridges. Indeed, the reason is that the process itself converges a.s. to endpoints a, b and thus their heights are a.s. properties.

Now in the case of the GFF, the 1D harmonic function, the line, is replaced by a more general harmonic function, but everything stays analogous. First, as the harmonic function is determined by its boundary behaviour, the expected height can be read out by nearing the boundary. Second, two GFF-s with different heights are mutually singular:

Lemma 1.3.25. *The expected height of the GFF can be a.s. read off from the field. Thus when we have two GFF-s (h^1, h^2) defined on simply connected D with different heights h_1 and h_2, then they are mutually singular as distributions.*

This is lemma 3.1 in [51] and the basic argument is as follows:

Proof. By uniformization, we can WLOG assume that $D = \mathbb{D}$ and that our point of interest is at 0. However, as h_1 is harmonic, we have $h_1(0) = (h_1, \rho_r^0)$ where ρ_r^z is the distribution that puts unit mass on the circle of radius r around the point z. The variance of (h_1, ρ_r^0) is given by $-\log r$ [19] and thus tends to zero as $r \to 1$. Hence the value of $h_1(0)$ is a.s. determined by h_1. □

If one decomposes the GFF in the unit disc into a radial part and the part that has zero average on each radii, then this lemma uses just the fact that the radial part converge to zero as a function. However, we know from Lemma 1.1.12 that also the angular parts, seen as a family of distributions parametrized by the circle $\mathcal{C}$, converge to zero. This can be restated as follows:

Corollary 1.3.26. *Consider the zero boundary GFF on $\mathbb{D}$. Denote by $\mathcal{C}_n$ the circles at distance 2^{-n} from the boundary of $\mathbb{D}$. Suppose we are given $\rho_n \in \mathcal{H}^\epsilon(\mathcal{C}_n)$ that for some fixed $\varepsilon, C > 0$ satisfy $\|\chi_n(\mathcal{C}_n \backslash \rho_n)\|_{\mathcal{H}^\epsilon} < C$. Then a.s. $(T_{\mathcal{C}_n} h, \rho_n) \to 0$.*

We will now apply this to local sets.

Lemma 1.3.27. *In some open domain D, consider a GFF h with height h_D and a local set (h, A_1) w.r.t some filtration, such that A_1 stays almost surely at distance $\epsilon > 0$ of a boundary component C_0. Then the expected height $h_{A_1}(z)$ agrees with that of h_D on this boundary component in the following sense: a.s. for any sequence of points $z_n \to z$ with $z \in C_0$ that is not a single point inside C_0, we have that $h_{A_1} - h_D$ converges to zero uniformly over C_0.*

Proof. We can separate C_0 from A_1 by a deterministic smooth curve l_ε of finite length and moreover the GFF integrated w.r.t to Lebesgue measure on this line is a.s. finite.

But (l_ε, A_1) is a pair of local sets w.r.t. the filtration of A_1 and thus we can first sample the GFF on l. Then $A_1 \backslash D$ is as a local set strictly inside the component not containing C_0. In particular the height of the field in the component containing C_0 is determined by the boundary of that component only.

Hence it suffices just to show that conditioning on the field values on some line ϵ-far from C_0 does not change the boundary values on C_0. But we can WLOG assume that the component containing C_0 is a disc and then from the Corollary 1.3.26 it follows that its values are a.s. properties of the field. □

This statement can be made stronger. We can say that for almost any instance of (A_1, h), if the local set A_1 does not touch a boundary point z, the expected height near this point does not change:

Lemma 1.3.28. *Consider a GFF h with height h_D and a local set (A_1, h) in a domain D with finite boundary length. Then the expected height $h_{A_1}(z)$ agrees with that of h_D off A_1 in the following sense: a.s. for any sequence of points $z_n \to z$, where z is not a single point inside ∂D and is of positive distance from A_1, we have that $h_{A_1} - h_D$ converges to zero.*

Proof. This is done just by conditioning. Indeed, as in the proof of the previous lemma, for any small region C_0 and some $\varepsilon > 0$, we can consider the local pair (l_ε, A_1) and first sample the GFF on l. On the event that A_1 stays inside the component not containing C_0, we again conclude the expected height after conditioning on A_1 a.s. agrees with the original height when approaching any point on C_0. Finally, we can pick a countable collection of such intervals so that each point is inside intervals of arbitrarily small length. Thus the claim follows. □

Thus we have shown that sampling a local set does in a precise sense not change the boundary values away from it. Now, notice that the closure of $A \backslash \partial D$ is also a local set w.r.t the same filtration. Thus in fact we see that the boundary values do not change also on points $z \in \partial D$ which are at a positive distance from the boundary of $A \cap \partial D$.

The same holds for a pair of local sets:

Proposition 1.3.29. *Consider a pair of local sets (A_1, A_2). Then the expected height $h^{A_1 \cup A_2}(z)$ agrees with that of h^{A_i} on the boundary in the following sense: a.s. for any sequence of points $z_n \to z$ in $D \backslash A_1$, where z is not a single point inside ∂A_1 and of positive distance from A_2, we have that $h^{A_1 \cup A_2} - h^{A_1}$ converges to zero. Similarly when we switch the roles of A_1, A_2. Moreover, this holds for any z in the interior of $\partial A_1 \cap \partial A_2$.*

Proof. From Lemma 1.3.16, we know that conditioned on A_1, A_2 is a local set of the GFF h_1 inside $D \backslash A_1$ w.r.t. the filtration restricted to that region. The same holds when we change the roles of A_1, A_2. Thus we have reduced the claim to the previous lemma. □

Thus, we can treat the harmonic extension everywhere except at the boundary points of $A_1 \cap A_2$. Let's call them boundary intersection points. We can reformulate this question again as a question about a single local set - what happens to the expected height at points on the boundary of $\partial D \cap A_1$?

Dealing with the boundary intersection points in full generality seems quite hard. Roughly one expects that as long as the set of boundary intersection points is small enough, the intersection does not contribute to the expected height of the field. One way to make this precise is to more or less restate Corollary 1.3.26 in this context. The lemma is slightly cumbersome and long to state, thus we just include some extra assumptions just to make the statement more clear:

Lemma 1.3.30. *Consider a local set (h, A_1) in the unit disc $\mathbb{D}$ and with zero boundary GFF such that $\mathbb{D} \backslash A_1$ is simply connected and that h^{A_1} is given by integrating the Poisson kernel of $\mathbb{D} \backslash A_1$ w.r.t*

to some distribution g_∂ on the boundary of $D\backslash A_1$. Denote by $\mathcal{C}_n$ the circles at distance 2^{-n} from the boundary. If for some fixed $\varepsilon > 0$ and for all n large enough, the indicator functions $\chi_n(\mathcal{C}_n\backslash A_1)$ a.s. satisfy $\|\chi_n(\mathcal{C}_n\backslash A_1)\|_{\mathcal{H}^\epsilon} < C$ for some constant C, then the expected height does not see the boundary intersection points in the following sense: for any n we let g_∂^n agree with g_∂ inside $\mathbb{D}_{1-2^{-n}}$ and to be zero on $\mathcal{C}_n\backslash A_1$. Via the Poisson kernel in $\mathbb{D}_{1-2^{-n}}\backslash A_1$ this gives rise to a harmonic function $h_n^{A_1}$ defined inside $\mathbb{D}_{1-2^{-n}}$. We claim that $h_n^{A_1} \to h^{A_1}$.

Remark 1.3.31. It seems that it is not entirely trivial to give a more pleasant description to what "small intersection" means. One can show that uniform control on the Minkowski dimension of the boundaries of $\mathcal{C}_n\backslash A_1$ suffices, but not the Hausdorff dimension. It would be interesting to know whether information only on the boundary would already give a criteria. We have formulated a more precise question in the Outlook as Question 0.4.4.

As above, one then formulates a similar statement for a pair of local sets. It is just slightly more technical to state, as one first wants to map at least locally to a convenient domain.

The following (roughly stated) open question seems to be of interest:

Question 1.3.32. Is it true that the boundary intersection points of local sets never contribute to the expected height?

Let us recapitulate what remains to be shown: we saw above that the answer is yes as long as we have a control on the size of the intersection of local sets. Moreover, we know from Proposition 1.3.29 that if the local sets agree on any small segment, the harmonic extension can be determined over that segment. So the only open case is a very nasty intersection.

Moreover, one may observe that the boundary intersection points cannot contribute, if they correspond to a nasty but deterministic set (in the case of two local sets, it would mean that one of them determines the whole boundary intersection). So to create boundary effects, we would need to have a local set that chooses to hit a large and random set of very specific points on the boundary. It is not even clear whether one can construct such a local set.

If one could answer the question above positively, it would for example imply a very elementary proof on the measurability of the level lines of the Gaussian free field. See the discussion in chapter 2. Moreover, one could also deduce some continuity statements for SLE flow lines of the GFF for different values of κ.

We are currently working on this problem with Avelio Sepuvalda and Wendelin Werner, though there is no great progress to be reported.

One of the reasons why this is difficult is the following: even though the zero boundary GFF has indeed zero boundary, it still exhibits non-trivial behaviour when approaching the boundary. We will exemplify this in the next section.

1.4 Boundary oscillations of the GFF

In this section we study the boundary behaviour of the zero boundary GFF. Although "zero boundary" does really mean that in the sense of distributions the GFF is zero on the boundary, one can still observe an oscillatory type of behaviour when nearing this zero boundary. The statements and proofs in this section are similar to the case of thick points in the bulk of the GFF, thus we will first recall the definition and the main statements in that context.

1.4.1 Thick points

Recall that due to the roughness of the GFF, one cannot evaluate it at any single point. However, one can still discriminate the points based on their asymptotic heights, i.e. based on how the circle-average process $h_r(z)$ behaves as $r \downarrow 0$.

Definition 1.4.1 (Thick points)**.** *Let $z \in D$. We say that z is an $a-$ thick point of the GFF h, if* $\lim_{r\downarrow 0} \frac{h_r(z)}{\log 1/r} \geq a$

In this form, the definition stems from [25]. For orthonormal-basis regularization, thick points were already considered in Kahane's seminal article [27]. From [25], we have the following result determining the Hausdorff dimension of circle-average thick points:

Theorem 1.4.2. *Denote by T_a the set of $a-$thick points. Then for $a \in [0,2]$, the Hausdorff dimension of T_a is given by $2 - a^2/2$. Moreover, for $a > 2$, the set T_a is empty.*

One would expect that the exact way of regularizing the field should not matter too greatly, i.e. that for any reasonable regularization, one could set $T_a = \{z : \lim_n \frac{h_n(z)}{\mathbb{E}h_n(z)^2} \geq a\}$ - and obtain the same sets. Thus being a thick point should be a question of the "height" of the point:

Question 1.4.3. Can one give an intrinsic definition of the thick points that does not refer to a regularization process?

This is supported by [10], where it is shown that the thick points as sets a.s. agree for regularization processes that stay close in a specific sense. Moreover, a possible intrinsic definition could come from considering the Gaussian multiplicative chaos measure. First, we know from [19, 27] that, for example, the Liouville measure of parameter γ - heuristically given by "$e^{\gamma h}$" (see chapter 3) - is supported on γ-thick points [19] and second, [52] gives us an abstract way of defining the multiplicative chaos, without relying on any regularization procedure.

1.4.2 Boundary oscillations: statements

Now, we have seen that in some sense the zero boundary GFF trivialises near the boundary: if one decomposes the GFF on the unit disc into a radial part and a part that has zero average on each radii, then not only does the radial part converge to zero as a function, but also the angular parts seen as a family of distributions parametrized by the circle $\mathbb{S}$, converge to zero in any space $\mathcal{H}_0^{-\varepsilon}(\mathbb{S})$.

This however still allows for non-trivial behaviour when approaching the boundary. This non-trivial behaviour poses problems when we want to understand the behaviour of the field at intersection points of local sets (see e.g. section 3 in chapter 1 or section 1 in chapter 2). In what follows we exemplify the orchestrated boundary behaviour by the following propositions:

Proposition 1.4.4. *Consider the zero boundary GFF on the upper half plane. Look at the sets $T_a^B = \{z \in [-1,1] : \limsup \frac{h_r(z)}{\sigma_0\sqrt{-\log r}} \geq a\}$, where $h_r(z)$ is the semi-circle average around z and σ_0^2 its variance. Then for $a \leq \sqrt{2}$ the set T_a^B has Hausdorff dimension $1 - a^2/2$ and is empty for $a > \sqrt{2}$.*

Now, it is tempting to talk of boundary thick points. There are important differences, however. Firstly, here we cannot see these thick points as intrinsic heights of the field on the boundary - it is really the semi-circle average process that interests us. Second, in contrast to the definition of bulk thick points, it is important to keep the lim sup instead of lim here. The events are not carried by a drift of the process, but rather by oscillations when approaching the boundary. Indeed, one can obtain the following:

Proposition 1.4.5. *For any fixed a the set* $T_a^i = \{z \in [-1,1] : \liminf h_r(z) > a\}$ *is empty.*

Before proving these propositions, we will first study the semi-circle average process around a boundary point z of the zero boundary GFF.

1.4.3 Semi-circle average on the boundary

For simplicity suppose that we work on the upper half plane. We claim that the semi-circle average $h_r(z) = (h, \rho_r^z)$ is a stationary Ornstein-Uhlenbeck process when parametrized by $-\log r$: indeed it is clearly a centred Gaussian process, stationary by the scale-invariance of the GFF and Markovian by the domain Markov property of the GFF. Thus by [12] it is an Ornstein-Uhlenbeck process of mean zero. Its covariance structure for $q \leq r$, is given by $\mathbb{E}h_r h_q = \sigma_0^2 (q/r)^\beta$, for some positive β, σ_0. One can further determine that $\beta = 1$ by letting $q \downarrow 0$. We collect this in a lemma:

Lemma 1.4.6. *Consider the zero boundary GFF on the upper half plane. Then the semi-circle average* $h_r(z) = (h, \rho_r^z)$ *around any boundary point* z*, parametrized by* $-\log r$*, has a law of a stationary Ornstein-Uhlenbeck process of mean zero and covariance structure given by* $\mathbb{E}h_r h_q = \sigma_0^2 (q/r)$ *for some explicit* σ_0*.*

The probability density function of our Ornstein-Uhlbenbeck process is given by:

$$q(t,x,y) = p(\sigma_0^2(1-e^{-2t}), xe^{-t}, y) \tag{1.8}$$

where $p(t,x,y)$ is the transition kernel for the standard 1D Brownian motion. This follows from the fact that in our case the process is already in its stationary regime and has the form

$$X(t) = \sigma_0 e^{-t} B(e^{2t})$$

where $B(t)$ is the standard Brownian motion.

We record the following lemma controlling the regularity of the semi-circle average process. In particular, it says that for a fixed radius our process has the same Holder continuity as the Brownian motion:

Lemma 1.4.7. *Consider a zero boundary GFF on the upper half plane. Then on the real segment* $[-1,1]$ *the semi-circle average process* $h_r(z)$ *possesses a modification* $\tilde{h}_r(z)$ *such that for any* $0 < \beta < 1/2$ *and any* $\delta > 0$*, there exists some* $C = C(\gamma, \delta)$ *such that*

$$|\tilde{h}_r(z) - \tilde{h}_q(s)| \leq C \frac{|(z,r) - (w,s)|^\beta}{r^{\beta+\delta}}$$

holds for all $r, s \in (0,1]$ *satisfying* $1/2 \leq r/s \leq 2$*.*

To prove this lemma, we use the so-called modified Kolmorogov-Centsov theorem given in appendix C of [25]. We state it in a slightly weaker form and will not prove it:

Lemma 1.4.8 (Modified Kolmogorov-Centsov)**.** *Suppose that $U \subseteq \mathbb{R}^d$ is a bounded open set and that $X : U \times (0,1] \to \mathbb{R}$ is a time-varying random field satisfying*

$$\mathbb{E}|X(z,r) - X(w,s)|^\alpha \leq C \left(\frac{|(z,r) - (w,s)}{r \wedge s} \right)^{d+1+\beta} \tag{1.9}$$

for some $\alpha, beta > 0$. Then for each $\delta > 0$, each $\gamma \in (0, \beta/\alpha)$ the field X has a modification $\tilde{X}$, satisfying

$$|\tilde{X}(z,r) - \tilde{X}(w,s)| \leq M(\delta,\gamma) \frac{|(z,r) - (w,s)|^\gamma}{r^{\tilde{\gamma}}}$$

with $z, w \in U$ and $r, s \in (0,1]$ with $1/2 \leq r/s \leq 2$.

We will however indicate a proof of Lemma 1.4.7:

Proof: For a Gaussian process, in order to obtain bounds on $\mathbb{E}(h_r(z) - h_s(w))^\alpha$, it suffices to only obtain bounds for $\mathbb{E}(h_r(z) - h_s(w))^2$.

In [25], it is shown how a small calculation with the Green's kernel implies that for the circle-average process h' in the bulk it holds that

$$E(h'_r(z) - h'_s(w))^2 \leq C \frac{|(z,r) - (w,s)|}{r \wedge s}$$

By Gaussian calculations this implies for all $\alpha > 1$:

$$E(h'_r(z) - h'_s(w))^\alpha \leq C \left(\frac{|(z,r) - (w,s)|}{r \wedge s} \right)^{\alpha/2}$$

Thus, if we knew this bound for our process, the lemma would follow. Now, this bound could be obtained by again doing a direct calculation on the Green's kernel. However, one could also just deduce it from the estimate for the bulk case as follows:

Consider some line $y = i$ on the upper half plane. Take the decomposition of the zero boundary GFF to the zero boundary parts in $\{\Im z > 1\}$ and $\{0 \leq \Im z < 1\}$ plus the part that is harmonic in both of these domains. Call them respectively h, h_0 and h_c.

Now consider the circle average process on this line. We can write it as an independent sum: $h'_r(z) = (h, \rho^z_r) + (h_0, \rho^z_r) + (h_c, \rho^z_r)$, where h' is a zero boundary GFF on the whole upper half-plane. But as h is a zero boundary GFF on $\{\Im z > 1\}$, the first term is, up to a multiplicative factor, exactly our semi-circle average process on the boundary. Thus by independence it follows that

$$\mathbb{E}(h_r(z) - h_q(w))^2 \leq \mathbb{E}(h'_r(z) - h'_q(w))^2$$

Hence we have the same bound on the second moment, as was obtained in the bulk in [25]. Thus the higher moments and the lemma follow. □

1.4.4 Boundary oscillations: proofs

We are now ready to prove the propositions. We start by proving Proposition 1.4.4 and then prove Proposition 1.4.5.

The proof of the first proposition follows the usual line of attack for determining the Hausdorff dimension of a random set: the easier case is the upper bound of the dimension, for which one just proves a first-moment estimate. This then needs to be supplemented with a second-moment estimate to obtain the lower bound. Needless to say, this is also the strategy used in [25] to prove the dimension of $a-$thick points in the bulk case. However, the details differ quite a bit, as here we are dealing more with an oscillation-type of behaviour and moreover, independent increments are not at hand:

First moment estimate

For the first moment argument our aim is to find an efficient covering of T_a. In this respect, consider the events

$$E_a^r(z) = \{h_r(z) \geq a\sigma_0\sqrt{-\log r} + O(1)\}$$

i.e. events such that at time $-\log r$, we have an oscillation of suitable height. We claim that it suffices to consider these events for some discrete sequence of radii r and some discretization of space in order to know whether the $z \in T_a$ for any $z \in [-1,1]$.

Indeed, using Lemma 1.4.7, the continuity in the r parameter for the semi-circle process implies that the event $z \in T_a$ happens iff $E_a^{r_n}(z)$ happens infinitely often for $r_n = n^{-M}$, where M is some large constant. Indeed, the choice of r_n here is such that $r_n - r_{n+1} = O(r_n^{1+1/M})$ and thus we can use the lemma.

Moreover, one can see that the continuity in the space parameter implies that if $E_a^{r_n}(z)$ holds, then it holds inside the disc $B(z, r_n^{1+\epsilon})$ for any $\epsilon > 0$ sufficiently small.

Thus, we can cover the set T_a as follows: for each n, we just cover $[-1,1]$ by balls of radius $r_n^{1+\varepsilon}$. We need around $r_n^{-(1+2\varepsilon)}$ balls for that. Then the expected value of the Hausdorff $d-$content of this covering is given by:

$$\mathbb{E}\mathcal{H}_d(T_a) \lesssim \sum_n \frac{r_n^{(1+\varepsilon)d}}{r_n^{1+2\varepsilon}} \mathbb{P}(E_a^{r_n}(z)) \leq \sum_n r_n^{(1+\varepsilon)(d-1)-\varepsilon+\frac{a^2}{2}}$$

On the other hand, by using the fact that each h_r is just a zero mean Gaussian with fixed variance σ_0^2, the probability of events $E_n(z)$ is given by

$$\mathbb{P}(E_a^r(z)) \asymp r^{\frac{a^2+o(1)}{2}}$$

Thus the expected Hausdorff content is summable for $d = 1 - \frac{a^2}{2} + 2\varepsilon$. But we can let $\varepsilon \downarrow 0$ and thus the upper bound for the Hausdorff dimension follows.

Moreover, for $a^2 > 1$ the expectation tends to zero for $d = 0$ and thus we see that the set T_a is a.s. empty.

Second moment estimate

For the second moment we need to obtain control on $\{z \in T_a\} \cap \{w \in T_a\}$. This is not usually easy to obtain directly and instead one finds sets $\tilde{T}_a \subseteq T_a$ for which the correlation structure is easier to handle, but which are still large enough to provide the correct lower bound.

Indeed, starting from the estimate:

$$\mathbb{P}(\{z \in \tilde{T}_a^n\} \cap \{w \in \tilde{T}_a^n\})|z-w|^{\frac{a^2}{2}} \lesssim \mathbb{P}(\{z \in \tilde{T}_a^n\})\mathbb{P}(\{w \in \tilde{T}_a^n\}) \tag{1.10}$$

for a sequence of finite approximations of sets $\tilde{T}_a$, there is a standard argument to deduce the lower bound $1-\frac{a^2}{2}$ on the Hausdorff dimension. It works by constructing the measures $\frac{1(z\in\tilde{T}_a^n)}{\mathbb{P}(\tilde{T}_a^n)}dz$ and then taking looking at the limiting measure. From 1.10 it then follows that this limiting measure is a finite measure supported on $\tilde{T}_a$ and moreover has finite $1-\frac{a^2}{2}$ -energy. One can conclude that with positive probability the lower bound holds. Then a 0-1 argument can be used that this probability has to be 1. This argument is used in [25], and stated in a general form that applies in our context, for example, in Proposition 4.8 of [40].

Thus in order to prove the lower bound, it suffices to construct the sets $\tilde{T}_a^n$ satisfying the estimate above. The form of estimate 1.10 suggests that one should build the sets $\tilde{T}_a^n$ in a tree-like manner. In other words, we want to construct a family of decreasing subsets of the segment $[-1,1]$ so that the dependency of any two branches comes from their joint path to the root - i.e. only from subsets that contain both. The order of magnitude of the events and their dependency structure determine how one should go about choosing the exact sizes of the intervals. The following set-up seems to work the best in the concrete case. Our only good explanation for it is the proof itself:

Set $r_m = \frac{1}{2^{K^m}}$ for some K to be defined later. Consider for $m = 1,2,3,\dots$ the events:

$$F_m(z) = \{h_{r_m}(z) \in [a\sigma_0\sqrt{-\log r_m}, 2\sigma_0\sqrt{-\log r_m}]\}$$

We call these events oscillations at "level m". Set further

$$E_n(z) = \bigcap_{m\le n} F_m(z)$$

They correspond to events where we have oscillations at each level up to some scale n.

Then we claim the following:

Claim 1.4.9. *For any $\varepsilon > 0$, we can choose $K > 0$ and $C(\varepsilon) > 0$ such that for all z, w with $|z-w| < C(\varepsilon)$ and all $n \ge m(\varepsilon)$, we have*

$$\mathbb{P}(E_n(z) \cap E_n(w))|z-w|^{\frac{a^2+o(1)}{2(1-\varepsilon)}} \lesssim \mathbb{P}(E_n(z))\mathbb{P}(E_n(w)) \tag{1.11}$$

and here the implied constants do not further depend on n, z, w.

Before proving the claim, we show how to conclude. Fix some $\varepsilon > 0$ small. We define the set $\tilde{T}_a = \tilde{T}_a(\varepsilon)$ and the approximating sets $\tilde{T}_a^n$ as follows:

For any n sufficiently large, divide $[-C(\varepsilon), C(\varepsilon)]$ into intervals of size r_n and let $\tilde{T}_a^n$ be the union of these intervals whose centers oscillate up to the scale n. The set $\tilde{T}_a$ is then defined as the intersection of sets $\tilde{T}_a^n$.

From the spatial continuity of the process it follows that $\tilde{T}_a \subseteq T_a$: indeed, by the construction we have guaranteed that any point in $\tilde{T}_a$ has arbitrarily close neighbours that oscillate up to level n.

Finally, the estimate (1.11) implies the desired estimate 1.10 for $\tilde{T}_a^n$ with $\frac{1}{1-\varepsilon}a^2/2$ instead of $a^2/2$ for any $\varepsilon > 0$. Thus we obtain a lower bound of the form $1 - \frac{1}{1-\varepsilon}a^2/2$ on the interval $[-C(\varepsilon), C(\varepsilon)] \subseteq [-1, 1]$ and this implies the lower bound for the whole interval $[-1, 1]$. Now we can make ε as small as we wish and obtain the correct lower bound for the Hausdorff dimension.

Proof of claim: First, we calculate the probability of the events F_m and E_n. For F_m, we just use the fact that h_{r_m} is a zero mean Gaussian with some fixed variance σ_0^2:

$$\mathbb{P}(F_m(z)) \asymp r_m^{a/2}$$

For E_n we use (1.8) and Gaussian estimates to obtain:

$$\Pi_{m \leq n} c_1 (\log \frac{1}{r_m})^{-1} \mathbb{P}(F_m(z)) \leq \mathbb{P}(E_n(z)) \leq \Pi_{m \leq n} c_2 \log \frac{1}{r_m} \mathbb{P}(F_m(z)) \tag{1.12}$$

Now consider $z, w \in [-1, 1]$ and let

$$m_0 = \inf\{m \in \mathbb{N} : 2r_{m_0} < |z - w|\}$$

be the first level where the annuli related to events F_m are disjoint.

Let us first look at the LHS of the equation (1.11). Our choice of r_m and m_0 should imply that the naive bound that counts the large-scale oscillations for only one of the points is rather tight. It gives us:

$$\mathbb{P}(E_n(z) \cap E_n(w)) \leq \mathbb{P}(E_n(z) \bigcap_{m_0 \leq k \leq n} F_k(w))$$

We write this further as

$$\mathbb{P}(\bigcap_{m_0 \leq k \leq n} F_k(w) | E_n(z)) \mathbb{P}(E_n(z)) \tag{1.13}$$

Now from the Markov property of the GFF, we see that conditioned on $F_{m_0}(w) \cap F_{m_0-1}(z)$, the events $E_n^{m_0-1}(z)$ and $\bigcap_{m_0 < k \leq n} F_k(w)$ are independent. Thus we can write the left term of (1.13) as:

$$\mathbb{P}(\bigcap_{m_0 < k \leq n} F_k(w) | F_{m_0}(w) \cap F_{m_0-1}(z)) \mathbb{P}(F_{m_0}(w) | E_n(z))$$

We rewrite this further as

$$\mathbb{P}(\bigcap_{m_0 < k \leq n} F_k(w) \cap F_{m_0-1}(z) | F_{m_0}(w)) \frac{\mathbb{P}(F_{m_0}(w) | E_n(z))}{\mathbb{P}(F_{m_0-1}(z) | F_{m_0}(w))}$$

Now we forget the $F_{m_0-1}(z)$ part in the first term and then it can be bounded by $c\Pi_{m_0 < k \leq n} \mathbb{P}(F_k(w))$. Thus we have reduced (1.11) to proving that for any $\varepsilon > 0$ we can choose the sequence r_m such that in a small interval $[-C(\varepsilon), C(\varepsilon)]$ the following holds:

$$\mathbb{P}(F_{m_0}(w)|E_n(z))\Pi_{m_0<k\leq n}\mathbb{P}(F_k(w))|z-w|^{\frac{a^2+o(1)}{2(1-\varepsilon)}} \lesssim \mathbb{P}(E_n(w))\mathbb{P}(F_{m_0-1}(z)|F_{m_0}(w))$$

By the definition of the events F_k and E_n and (1.12), we can simplify it further. It remains to prove the following:

$$\mathbb{P}(F_{m_0}(w)|E_n(z))|z-w|^{\frac{a^2+o(1)}{2(1-\varepsilon)}} \lesssim \mathbb{P}(F_{m_0-1}(z)|F_{m_0}(w))\Pi_{k\leq m_0}\mathbb{P}(F_k(w))$$

To prove this, we separate two cases with $\delta \ll 1$:

1. $|z-w| < r_{m_0}^{1-\delta}$: this is the case where F_{m_0} is still correlated with E_n
2. $|z-w| \geq r_{m_0}^{1-\delta}$: here we expect independence

In the first case, we upper bound the term $\mathbb{P}(F_{m_0}(w)|E_n(z))$ just by 1. By our choice of m_0, the LHS is bounded by

$$|z-w|^{\frac{a^2+o(1)}{2}} \leq r_{m_0}^{\frac{(1-\delta)a^2+o(1)}{2}}$$

On the other hand, from (1.12) and our choice of r_m we see that

$$r_{m_0}^{\frac{(\frac{K}{K-1})a^2+o(1)}{2}} \lesssim \Pi_{k\leq m_0}\mathbb{P}(F_k(w))$$

Finally, $\mathbb{P}(F_{m_0-1}(z)|F_{m_0}(w)) \gtrsim \mathbb{P}(F_{m_0-1}(z))$ as the corresponding semi-circles can be separated by a semicircle of, say, radius $r_{m_0-1}/2$ around z. Thus in this case 1.11 just follows from choosing K, δ such that $(1-\delta)(\frac{K-1}{K+1}) > 1-\varepsilon$.

Le us now treat the second case. First, notice that in fact in $\mathbb{P}(F_{m_0}(w)|E_n(z))$ the conditioning amounts to conditioning on

$$\bigcap_{k=m_0,m_0-1,m_0-2} F_{m_0}(z)$$

Second, observe that the correlations between the $h_{r_{m_0}}(w)$ and the semi-circle averages $h_{r_k}(z)$ with $k = m_0, m_0-1, m_0-2$ tend to zero $m_0 \to \infty$. Indeed, for δ small, the amount of mass of these three semi-circles around z that stays closer than $r_{m_0}^{1-\delta}$ to w can be bounded by $c' r_{m_0}^{\delta}$ for some absolute constant c'. Calculating then explicitly using the Green's function, one obtains the bound:

$$\mathbb{E}h_{r_{m_0}}(w)(\sum_{k=m_0,m_0-1,m_0-2} h_{r_k}(z)) < c r_{m_0}^{\delta}$$

for some absolute constant c. Thus under this conditioning, the Gaussian $h_{r_{m_0}}$ still contains an independent bit that has at least $1-c' r_{m_0}^{\delta}$ times the original variance. Putting this into Gaussian calculations, it follows that

$$\mathbb{P}(F_{m_0}(w)|E_n(z)) \lesssim P(F_m(w))^{1-c' r_{m_0}^{\delta}}$$

One can similarly handle the $\mathbb{P}(F_{m_0-1}(z)|F_{m_0}(w))$-term: we write $h_{r_{m_0-1}}(z)$ as a sum of an $F_{m_0}(w)$-dependent and an independent term. Again the correlation is smaller than $r_{m_0}^{\delta}$, and we can bound

$$\mathbb{P}(F_{m_0-1}(z)|F_{m_0}(w)) \gtrsim \mathbb{P}(F_{m_0-1}(z))$$

Thus for 1.11 to hold, we need to be able to choose $K, C(\varepsilon)$ (we fix δ by case 1) such that

$$1 > (1-\varepsilon)(1 + cr_{m_0}^{\delta} K + \frac{2}{K-1})$$

Now $r_{m_0} < 2|z-w|$. Hence we can first pick $K = 10\varepsilon^{-1}$ and then choose $C(\varepsilon)$ such that $C(\varepsilon)^{\delta} < 0.01\varepsilon^2$.

□

Proof of proposition 1.4.5

Given the above arguments, the second proposition is not too hard to obtain:

Proof. Recall that we want to show that the lim inf cannot stay above level a. It suffices to prove it for a negative. Consider the radii $r_m = 1/m$ and events

$$F_m^a(z) = \{h_{r_m}(z) \in [a, -a + \log m]\}$$

Thus a priori we are also asking for an upper bound. Yet we will see that by the previous proposition this does no harm.

Pick n_0 very large and set for $n > n_0$

$$E_n^a(z) = \bigcap_{n_0 \leq m \leq n} F_m^a(z)$$

Now, if we condition on F_m^a for some large enough m, then the time interval $r_m - r_{m+1}$ is large enough to decorrelate events F_m and F_{m+1}^a. Indeed, using the transition densities of the Ornstein-Uhlenbeck process (1.8) we can see that for some $0 < c < 1$ and some $d > 0$, we have:

$$\mathbb{P}(E_n^a(z)) \leq c^{n-dn_0}$$

Now, divide the interval $[-1, 1]$ into n^2 equal intervals and consider their midpoints. By the continuity estimates for the semi-circle average process, it follows that if we choose n_0 large enough, then whenever the midpoint of an interval does not satisfy E_n^{2a}, the points inside this interval do not satisfy the event E_n^a.

Now denote by $M_n(z)$ the midpoint of the interval containing z, and consider:

$$T_a^i(n) = \{z \in [-1, 1] : 1(E_n^a(M_n(z))) = 1\}$$

This set can be covered by using sets of which the midpoints satisfy $E_n^{2a}(z)$. From our estimate above we know that there are, in expectation, $n^2 c^{n-n_0}$ such intervals. But this clearly tends to zero as $n \to \infty$ for any fixed n_0. And in particular the set $\bigcap_{n \geq n_0} T_a^i(n)$ is a.s. empty for any fixed n_0.

On the other hand, by Proposition 1.4.4 we know that the set $T_b^B = \{z \in [-1,1] : \limsup \frac{h_r(z)}{\sigma_0\sqrt{-\log r}} \geq b\}$ is a.s. empty for $b > 2$. Thus fixing some N_0 very large and taking, say, $b = 5$, we have

$$T_a^i \subseteq \bigcup_{n_0 \geq N_0} \bigcap_{n \geq n_0} T_a^i(n) \cup T_5^B$$

and the proposition follows. □

2

GFF with SLE

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Schramm-Loewner evolution (SLE) is a one-parameter family of random curves, that was invented to describe the interfaces of models in statistical physics [49]. For a thorough introduction we refer to either [57] or [29].

Whereas one can talk of chordal, radial and whole-plane SLE-s, we here concentrate only on the chordal version. The chordal family in any simply connected domain D can characterized by two properties:

- conformal invariance for the trace of the process
- domain Markov property as a process

We define the chordal SLE in the upper half-plane $\mathbb{H}$, but due to conformal invariance this gives the definition for any simply connected domain.

The idea behind the definition is the following: consider a simple curve $\gamma(t)$ in $\mathbb{H}$ from 0 to ∞. Then the trace of this curve up to some time t can be described by mapping the slit domain $\mathbb{H}\backslash\gamma([0,t])$ back to $\mathbb{H}$ using a conformal map g_t. If we decide to fix the ∞, we are left with two free real parameters for the choice of the conformal map: they correspond to fixing the scaling and horizontal translation of $\mathbb{H}$. We could fix both of them by just using the behaviour of the map near infinity: say, by fixing the translation to zero w.r.t infinity and the derivative to be 1. Thus we have a canonical way of identifying the curve $\gamma([0,t])$ with the conformal maps g_t.

Now, one can observe further that at any time t, the tip of the curve is mapped to some point $\zeta(t)$ on the real line. C. Loewner observed, that we can actually reverse this procedure and construct the curve only from this real-valued process $\zeta(t)$ by using the following differential equation:

Definition 2.0.10 (Loewner differential equation)**.** *Let $\zeta(t)$ be a continuous real-valued function. Then for any $z \in \mathbb{H}$ define $g_0(z) = z$ and*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \zeta(t)}$$

defined up to $\tau(z) = \sup_{t\geq 0} \min|g_t(z) - \zeta(t)| > 0$.

If we write $K_t = \{z : \tau(z) \leq t\}$ then this equation defines a family of conformal maps from the decreasing domains $H_t = \mathbb{H}\backslash K_t$ back to the upper half plane. The family K_t is called the Loewner chain. In the case of simple curves, the hull is equal to the trace of the curve, i.e. $K_t = \gamma([0,t])$.

Loewner differential equation has proved to be a powerful tool in studying univalent maps, and in particular played an important role in de Branges' proof of Bieberbach conjecture, which states bounds on the Taylor coefficients of univalent maps.

O. Schramm realized that one would obtain a very natural family of random curves, if one uses as the driving function $\zeta(t)$ a multiple of the standard Brownian motion. This is in fact the only way to satisfy the two properties above: conformal invariance and domain Markov property.

Definition 2.0.11 (Chordal SLE)**.** *Let B_t denote a standard Brownian motion. Then the Loewner chain given by the driving function $\zeta(t) = \kappa B_t$ with $\kappa \geq 0$ is called an SLE_κ.*

We want to also consider the map that sends the tip of the curve to zero. This can be done by just setting

$$f_t(z) = g_t(z) - \zeta(t)$$

It has been shown that the SLE chains are almost surely generated by a curve [48]. We will be interested in the quantum fractal dimension of these curves, when coupled with the GFF. We use the known fact that the Hausdorff dimension of SLE_κ curve for $\kappa \leq 8$ is $1 + \kappa/8$, first proved in its entirety in [6].

In what follows we first discuss the geometric coupling of the SLE and GFF, introduced in [13, 51, 54] and thoroughly used in the imaginary geometry papers starting with [38]. Thereafter, we study one of the key ingredients of the coupling - the harmonic extension, that is given by the winding of the SLE process.

2.1 Level and flow lines of the GFF

The GFF and SLE processes are coupled in two beautiful ways [13, 54]. One way is to see SLE curves as interfaces for glueing together two quantum surfaces by matching their boundary lengths [54]. Here we work with the other way, which gives SLE curves a geometric meaning, when GFF itself is seen as a random surface [13, 54].

Set

$$\lambda = \frac{\pi}{\sqrt{\kappa}}$$

Then, first the SLE_4 can be seen as, or maybe rather forced to be, the zero level line of the GFF [51]:

Theorem 2.1.1 (Zero level lines of the GFF)**.** *Let η be a chordal SLE_4 curve in $\mathbb{H}$ and h the GFF in $\mathbb{H}$ with boundary conditions $-\lambda$, λ on the negative and positive real axis respectively. Then there is a coupling (h, η) such that*

- *h can be sampled by first sampling the SLE up to some finite stopping time T, and then sampling an independent GFF in the slit domain with boundary conditions set to $-\lambda$ on the negative real axis and to the left of the SLE, and λ to the right of the SLE and on the positive real axis*
- *η is a measurable with respect to h*

One can see that the GFF of the slit domain is in fact well-defined as a distribution on the whole of $\mathbb{H}$. Also notice that the harmonic correction term is uniformly bounded, but not well-defined on the curve. However, it can be set to zero - see the discussion following the statement on theorem 1.1. in [54]. Finally, it is not hard to show that the coupling theorem also holds when we sample the whole SLE curve [51].

The intuitive name "level line" is well justified by the following: consider the discrete GFF with the same boundary conditions as above on a finer and finer regular triangulation. Then the line that starts at the separation between $-\lambda$ and λ boundary components and keeps

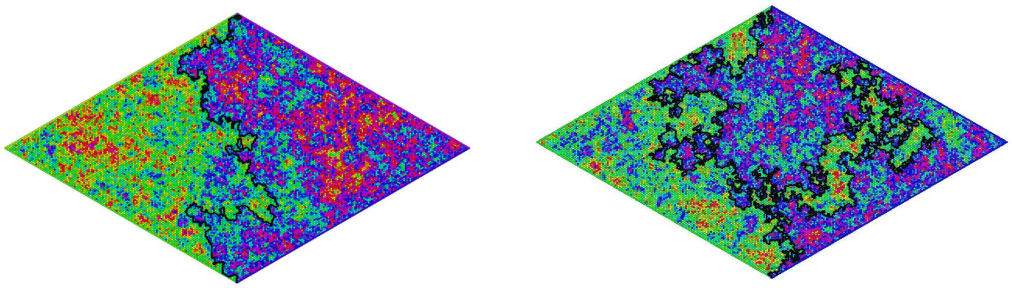


Figure 2.1: On the left, we see the flow line coupling of the $\mathrm{SLE}_{8/3}$ and on the right the zero level line coupling. The colours indicate the height of the GFF. Notice that whereas the zero level line - as by definition it should - really moves along the boundary of positive and negative heights, the $\mathrm{SLE}_{8/3}$ also keeps close to this boundary. We thank Scott Sheffield for allowing us to use these images.

negative field values to the left and positive values to the right converges to the SLE_4 of the above coupling [50].

Secondly, the SLE_κ for $\kappa > 0$ can be seen as flow lines of the GFF [54, 13, 38]. Whereas the intuitive picture of level lines is clear, flow lines are a bit harder to interpret. Nice pictures with nice explanations can be found in [38]. In short and without rigour, for flow lines at any point the angular derivative is given by a multiple of the field height.

Theorem 2.1.2 (Flow lines of the GFF)**.** *For $0 \leq \kappa < 4$, let η_κ be a chordal SLE_κ curve in $\mathbb{H}$ and h the GFF in $\mathbb{H}$ with boundary conditions $-\lambda$, λ on the negative and positive real axis respectively. Then there is a coupling (h, η_κ) such that*

- *the marginal of h can be sampled by*
  - *sampling the SLE η_κ up to some finite stopping time T*
  - *then sampling an independent GFF in the slit domain with boundary conditions as above: $-\lambda$ on the negative real axis and to the left of the SLE, and λ to the right of the SLE and on the positive real axis*
  - *and finally, subtracting $\chi \arg f_T'$ where $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$ and f_T is the normalized SLE map*
- *η is measurable with respect to h*

Notice also that this coupling reduces to the level line coupling for $\kappa = 4$ as then $\chi = 0$.

In this case it is not clear a priori that the GFF in the slit domain summed with the harmonic correction term defines a distribution on the whole of $\mathbb{H}$. However, it either follows a posteriori from the proof in [54] or can be shown by obtaining simple bounds on the winding straight from the Loewner equation.

As above, the harmonic correction term (this time possibly unbounded!) can still be set to zero on the curve. Moreover, this term $\arg f_T'(z) = \operatorname{Im} \log f_T'(z)$ measures the winding of the SLE curve with respect to the point z. We require the argument to be continuous in the slit

domain and tend to 0 at infinity. The winding is discussed in detail in the next section, but we also refer to [38].

Also, in fact $\kappa < 4$ is no real restriction. Everything here can also be stated for $8 > \kappa > 4$. One needs just to take extra care as the SLE curve is no longer simple: first, the winding for any point needs to be calculated just before the point gets separated from infinity by the curve, i.e. as a limit $\lim_{t \uparrow T'} \arg f_t'(z)$ with $T' = T \wedge \tau(z)$, where $\tau(z)$ is the first time z is separated from infinity by the SLE curve. Second, a separate and independent GFF needs to be defined in each isolated domain (they all extend similarly to the whole of $\mathbb{H}$) and for the boundary conditions one needs to take into account in which direction the loops were closed. For details and an extension to $\kappa > 8$ and more generally to $\mathrm{SLE}_{\kappa,\underline{\rho}}$ processes, see [54] or [38].

Finally, it is not hard to show that the coupling theorem also holds when we sample the whole SLE curve [51].

2.1.1 SLE curves as local processes

Let us now verify that in the language of chapter 1, the SLE level and flow lines are coupled with the free field as local processes. Notice that the process η_t indeed takes values in Λ for each t and is continuously growing in the Hausdorff topology. Again we denote by $\mathcal{N}_t = \sigma(\eta_s, s \leq t)$. Moreover we see that $\nu_0 = \{0\}$. Denote by τ^C the first time $\nu_t \cap \overline{C^c} \neq \emptyset$.

Lemma 2.1.3. *In the level and flow line couplings, Theorems 2.1.1 and 2.1.2 respectively, the corresponding SLE processes are coupled as local processes of the GFF with respect to the filtration $(\mathcal{G})$. Here for any fixed C the sigma-algebra $\mathcal{G}_C$ is generated by $\mathcal{F}_C$ and $\mathcal{N}_{\tau^C}$.*

Proof. By definition the filtration $\mathcal{G}$ satisfies $\mathcal{N}_{\tau_C} \subseteq \mathcal{G}_C$ and thus by definition 1.3.12 it is a local process, as soon as the defined filtration is an enlarged filtration of the GFF. The only condition it might not satisfy in the Definition 1.3.4 is the Markov property of the free field. This however follows from Theorems 2.1.1 and 2.1.2. □

Notice that in the case of the level and flow lines of the GFF, the coupling is actually stronger in two ways: first, the harmonic extension itself is measurable w.r.t just the curve itself. In other words the curve contains all the information about the field it has discovered. Second, in fact the whole curve is measurable w.r.t the GFF.

Given that one can couple a single SLE curve as a local process, one can start coupling pairs, triplets etc of local processes, as described in section 1.3.1. This is done in [38] and the subsequent papers, where the authors study in great detail the whole flow line geometry of the GFF.

2.1.2 A side-amusement

It comes out that often intuition about the planar GFF can be obtained by thinking about its one-dimensional counterpart, the Brownian bridge. One could also ask about the converse direction - can thinking about the planar GFF tell us anything interesting about the Brownian bridge?

For example, reflecting on the level lines of the GFF leads naturally to the following question:

Question 2.1.4. Can one choose $\lambda > 0$ and X, a random variable on $[0,1]$, such that we can sample the $-\lambda \to \lambda$ Brownian bridge on $[0,1]$ by first sampling X and then sampling independent $-\lambda \to 0$ and $0 \to \lambda$ bridges on $[0,X]$ and $[X,0]$ respectively? Or in other words, can we find a (random) "zero point" on the $-\lambda \to \lambda$ bridge, cutting it into two bridges?

The answer is no and we leave the pleasure of proving it to the reader.

2.1.3 Thoughts on the measurability of SLE_4

In this section we discuss the measurability part of the coupling theorem 2.1.1, i.e. the result:

Proposition 2.1.5. *The SLE_4 is measurable with respect to the Gaussian free field in the level line coupling.*

We will first discuss the two existing proofs in the literature, one by J. Dubedat [13] and one by O. Schramm and S. Sheffield [51]. As we will see, these two proofs are very similar in spirit and consist of more or less the same steps. Thereafter, we will discuss a different, unfortunately unsuccessful proof strategy.

2.1.4 Recap of known proofs

Suppose we are working in a simply connected domain with two marked points: (D,x,y). Both proofs in [51] and [13] start by considering two independent level lines, running in opposite directions. So let γ_{xy} be a SLE_4 curve from $x \to y$ and γ_{yx} a SLE_4 curve from $y \to x$.

We next couple them as local sets (h,γ_{xy}) and (h,γ_{yx}), and then consider the conditionally independent pair $(\gamma_{xy},\gamma_{yx})_{CI}$. The idea of the proof is now to show that a.s. in this coupling one of the curves determines the other. Then, as the curves were chosen to be independent, conditionally on the field, it follows that the curves are in fact determined by the field.

The key step in both proofs is arguing that if we condition on γ_{xy} up to some stopping time T, the other curve γ_{yx} is distributed as an SLE_4 even in the domain $D\backslash\gamma_{xy}(T)$. Notice that a priori we only know that the second curve is an SLE_4 in the whole domain. By Lemma 1.3.16, the second curve is a local process in the slit domain, but it could well be a different process.

So the key step is showing that the second curve is indeed an SLE_4 process also in the slit domain. This implies straight away that the second curve will exit the domain at the tip of the first curve and thus contains the point $\gamma_{xy}(T)$. Thus, by considering a countable set of times for the first curve one can deduce that a.s. the second curve contains all of these points. Then from the a.s. continuity, it follows that the second curve contains the whole of the first curve. But the curves are simple, and thus their traces agree.

Thus the key is to show this "restriction" or "commutativity" property of the contour lines. This is done in two slightly different ways:

In [51] the authors show that the SLE_4 is the only curve that can be coupled as a local process of the GFF such that the first part of Theorem 2.1.1 is satisfied: i.e. such that the harmonic extension is given by putting boundary values $-\lambda$ and λ respectively to the left and to the right of the curve.

The argument can be retraced to the following: first, by studying the evolution of the harmonic extension at some point, one can see that $\log f_t(z)$ is a martingale. Here f_t is the centred Loewner map. Second, one argues that this property characterizes the SLE_4. This can be done for example using the expansion of $\log f_t$ at infinity and noticing that the coefficients of a few first terms need to be martingales. A good reference is [58], where the very final part is done slightly differently.

In [13] this "commutativity" property is proved using identities for the partition functions of SLE curves and the GFF.

Remark 2.1.6. It is maybe interesting to observe the following. When one considers the discrete GFF on the triangular lattice, then the contour line is measurable w.r.t the GFF for local reasons: at every time there we have only one available step for continuing the contour line. Or in other words, we construct the contour line step by step.

However, in the two continuum proofs above one uses a slightly intertwined argument that firstly is non-constructive and secondly seems to combine a local and global approach.

One might ask why the authors have taken this approach to prove the measurability of the SLE_4 and more generally for other SLE curves, coupled as flow lines. There seem to be two good reasons.

Firstly, arguing uniqueness directly from construction in the continuum seems very tricky. One would like to say that the curve has to go through the place with prescribed "height difference" and that near the boundary the boundary values determine this choice, yet given the roughness of the GFF it is not clear how to do it.

Secondly, one might ask why, even if we prove uniqueness indirectly, we need to consider different seeds for the curves? Why do we not just show that two full SLE curves cannot coexist in the same GFF? Here the reason seems to be that general intersection of local sets is hard to handle. This was already mentioned in the chapter on local sets 1.3.1, and we will see it again in the next subsection, when we discuss a proof attempt, in which we start the two SLE curves from the same point.

2.1.5 A non-proof

In this section we explain a proof strategy for obtaining the measurability of the SLE_4 by looking at the conditional expectation of the GFF to one side of a level line in two ways. For the moment we are not able to push it to work, but the strategy leads us naturally to look at some of the subtleties of the imaginary geometry [38]: the behaviour of the field near intersection points near two curves. The troubles encountered should also indicate why the two proofs cited above carefully avoid any intersections.

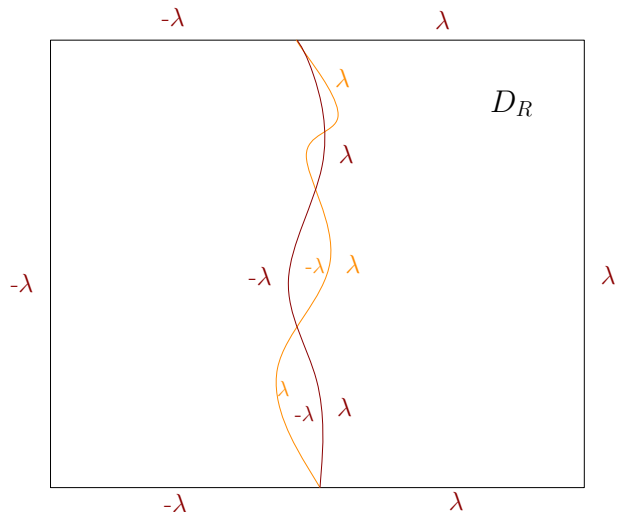
So here is one version of the non-proof:

Non-proof: Let γ_i for $i = 1, 2$ be SLE_4 curves from the level line coupling with the GFF drawn up to infinity. As above, we look at the conditionally independent local pair of $(\gamma_1, \gamma_2)_{CI}$ and notice that for measurability it suffices to show that one curve a.s. determines the other. Moreover, as the curves are continuous, simple and parametrized by half-plane capacity, it suffices to show that their traces agree.

Now, from the coupling theorem 2.1.1 we know that the conditional distribution of the GFF conditioned on a single γ_1 is given by just sampling independent zero boundary GFFs in the left and right components plus adding constant functions $-\lambda, +\lambda$ to left and right respectively.

As explained in section 1.3.1, we can sample this conditional distribution in a fancy manner: after sampling γ_1, we can sample γ_2 from its disintegrated measure, then sample independent GFFs in the remaining connected components and finally again average over the random curve γ_2.

The plan is now to show that the conditional expectation of the field given the curve γ_1 cannot agree if the curve γ_2 does not almost surely coincide with γ_1. The underlying idea is illustrated in the figure below: after also sampling an instance of γ_2, we know its boundary values outside of the intersection as well: it is $-\lambda$ to the left and λ to the right of the curve. Thus the expected height of the field to the left of γ_1 can only go up.



Indeed, if we look at the expected height of the field to the left of the curve γ_1 after sampling γ_1, we see that it is everywhere $-\lambda$. But as soon as γ_2 creates pockets left of the curve γ_1, the expected height in these pockets becomes strictly higher: indeed, a positive proportion (in the sense of harmonic measure) of the boundary has now boundary value λ. One can make this precise by, say, looking at the average of the expected height over the whole domain that remains to the left of γ_1. Thus γ_2 cannot create pockets of positive size to the left of γ_1 and by the same argument it cannot create them to the right. Hence the traces must agree.

□

Why does this argument not work?

In fact, the picture hides the difficulties. Namely, the intersection points of the two curves could affect the harmonic extension. Lemma 1.3.29 indeed tells us that away from what we call the boundary intersection (i.e. the boundary of $\gamma_1 \cap \gamma_2$), the boundary values of the harmonic extension are given by $\pm\lambda$, exactly as in the case of individual couplings. Yet this lemma does not say anything about the behaviour of the harmonic extension at these boundary intersection points.

Now, in a subsequent lemma, Lemma 1.3.30, we were further able to argue that as long as the boundary intersection is small, it cannot contribute to the harmonic extension. Using this lemma, we could state a precise condition for the non-proof to work. We will content ourselves with a slightly rough version for now. Namely, from this lemma it follows that the non-proof above would work, if we could control the boundary intersection:

Question 2.1.7. Can one show that after mapping the first SLE_4 back to the upper half-plane, the boundary intersection of the two curves is small enough in terms of Lemma 1.3.30?

Remark 2.1.8. As mentioned below Lemma 1.3.30 it would suffice to have uniform control on the Minkowski dimension of the intersection of the second curve with horizontal lines at all heights $\varepsilon > 0$.

Slightly surprisingly, we cannot (for the moment) prove that the intersection of two conditionally independent SLE_4 curves does not have such a nasty boundary intersection. The problem is that we only know the marginal laws of the curves and their joint law could be a priori rather complicated. It is true that we have a certain Markovian structure and, say, scale-invariance to use, but this does not seem to suffice to conclude.

There is yet another possibility to make the non-proof work. Instead of trying to obtain information on the joint distribution of two conditionally independently coupled level lines, we could try to treat their boundary intersections in full generality. In other words, the non-proof would also work as sketched above, if we managed an affirmative answer to the following question:

Question 2.1.9 (Intersection points cause no harm). Consider two SLE_κ curves γ_1, γ_2 coupled with the GFF as a conditionally independent local pair. Is it true that the conditional expectation of the GFF, given their conditionally independent union, can be found by just taking the harmonic extension that has boundary values $-\lambda - \chi \operatorname{wind}_i$ and $+\lambda - \chi \operatorname{wind}_i$ respectively on the left and right side of both curves?

This is something we have been working on with Avelio Sepuvalda and Wendelin Werner. For now we cannot confirm it, but we do not have any counterexamples either.

Remark 2.1.10. One could wonder whether it would be more helpful to use higher moments instead of the expectation in the non-proof above. For example, as the conditional variance depends on the size of the domain, one could hope to use this to derive a contradiction. Yet there seems to be no way around treating boundary intersection in this way.

2.2 The harmonic extension of flow lines: winding of SLE_κ

In this section we find the exponential moments for the winding of chordal SLE curves conditioned to pass nearby a fixed point. This winding we study in this section is in exact correspondence with the harmonic correction term in the flow line coupling of theorem 2.1.2. Thus in some sense we try to understand the geometry of the GFF near its flow lines.

2.2.1 Introduction and results

Let us start by defining the winding:

Definition 2.2.1. *Consider a chordal SLE$_\kappa$, $0 < \kappa < 8$ in the upper half plane and fix some point z. Let τ be the disconnection time of z, which is finite for $4 < \kappa < 8$ and a.s. infinite otherwise. We define the winding $w(z)$ around the point z by the following limit $w(z) := \lim_{t\uparrow\tau} \arg f_t'(z)$.*

Remark 2.2.2. It is known that this limits exist, e.g. see [54]. Or indeed, it follows from the proofs below.

Notice that as $\arg f'(z)$ is the imaginary part of an analytic function $\log f'(z)$, it is a harmonic function off the curve itself. We fix the logarithm by requiring it to be continuous in the slit domain and tend to 0 at infinity [54]. The basic intuition behind winding is that whereas $|f'(z)|$ measures the distortion of the length under f, then $\arg(f'(z))$ measures the angular distortion. Very near the curve, this distortion is given by unwinding the SLE curve back to zero. One can also think that this definition of winding gives the amount that a curve from the infinity needs to wind to access the point z. Asymptotically near the curve, this version of winding should coincide with the geometric winding up to some bounded constants [18]. We will henceforth always use the term winding to refer to the definition above and not the usual geometric counterpart.

The coupling of GFF and SLE gives the average winding of SLE over the randomness of the SLE. Here, we prove the following more precise result, calculating the winding around any point depending on its distance to the SLE curve. Recall that we are working with the chordal SLE in the upper half plane.

Theorem 2.2.3. *Let CR_0 be the conformal radius of a fixed point z_0 in the upper half plane. Fix $0 < \kappa < 8$. Denote by H_τ the SLE slit domain component containing z_0. Then, for $\epsilon > 0$ sufficiently small, conditioned on $\mathrm{CR}(z_0, H_\tau) \in [\epsilon, C\epsilon]$ with $C > 1$, the exponential moments of the winding $w(z_0)$ around the point z_0 are given by*

$$\mathbb{E}\left(e^{\lambda w(z_0)} | CR(z_0, H_\tau) \in [\epsilon, C\epsilon]\right) \asymp \epsilon^{-\lambda^2\kappa/8}$$

where the implied constants depend on κ, λ and for fixed κ can be chosen uniform for $|\lambda| < \lambda_0$ for any choice of $\lambda_0 > 0$.

Remark 2.2.4. We have defined the winding in the upper half plane and also stated the theorem in there. However, as defining the chordal SLE in a different nice (for example smooth Jordan boundary) domain would involve conjugations by analytic maps that extend to the boundary and have non-zero derivative on the boundary almost everywhere, the winding in any other such domain will be the same up to a uniformly bounded additive error. Hence, as we determine exponential moments up to multiplicative constants, the theorem 2.2.3 holds also for the chordal SLE in all nice domains and in particular in the unit disc.

Remark 2.2.5. By following the proof carefully, we actually get slightly more: we get that the winding is given by a Gaussian of variance $-\frac{\kappa}{4}\log\epsilon$ plus different error terms. The dependence relations between these error terms are a bit delicate and that is also the reason why we chose the wording above, which, needless to say, is sufficient for our applications.

Comparision to Schramm's study on winding

In this paragraph, we will shortly discuss how this result relates to Schramm's work on winding in his seminal paper [49]. First, Schramm actually studied the geometric winding of radial SLE around its endpoint zero and the variance was approximated by a Gaussian of variance $-\kappa \log \epsilon$, when the tip was ϵ-close to zero. However, in our case we have a Gaussian of variance $-\kappa/4 \log \epsilon$. This seems to be in agreement with predictions by Duplantier (see e.g. [17], ch. 8), where radial SLE ought to correspond to a one-arm event and chordal SLE conditioned to be close to a point - we think - could correspond to a two-arm event. Intuitively for κ small, one could argue that in the chordal case you just pass from one or other side of the point, whereas in the radial case you might still do a turn before finally hitting zero, thus causing a difference in variances.

Also, one needs to remark that notions of winding in [49] and here differ. Schramm is looking at the geometric winding number around zero, which is given by the argument of the tip of the curve, when the argument is chosen to be continuous along this curve. We, however, use the definition of [54] that gives the GFF-SLE couplings above. As explained above and as used in physics literature [18], these two notions should asymptotically agree up to bounded additive errors. In the radial case this is also shown in [39]. Moreover, a few line of calculations show that in the radial case around point zero, the concept used here would give a Gaussian of variance $-\kappa \log \epsilon$, in agreement with Schramm's result.

Finally, there is the question whether Schramm's nice geometric approach could have helped the technical work to follow. It does not seem to be the case, as his method in some sense only helps to relate the winding of the curve to the behaviour of the driving process. Due to conditioning, in our case the work is actually in studying the behaviour of the driving process resulting from conditioning.

2.2.2 Proof of the theorem

To start attacking the theorem, we need a lemma to translate the question to that of diffusion processes and rewrite the geometric conditioning of SLE curves in terms of exit times of a certain diffusion process:

Lemma 2.2.6. *Consider the chordal SLE_κ in the upper half plane with $0 < \kappa < 8$ and set $\mathrm{CR}_0 = CR(z_0, \mathbb{H})$. Parametrize the SLE using "radial parametrization", i.e. so that at any time t we have $CR(z_0, H_t) = \mathrm{CR}_0 e^{-t}$. In this parametrization, the driving function gives rise to a diffusion α_s in $(0, 2\pi)$, satisfying the following equation:*

$$d\alpha_s = \sqrt{\kappa} dB_s + \frac{\kappa - 4}{2} \cot \frac{\alpha_s}{2} ds \tag{2.1}$$

Let τ be the first exit time of a diffusion. Then the winding around z_0 is given by $w(z_0) = \int_0^\tau \cot \frac{\alpha_s}{2} ds$.

Remark 2.2.7. This lemma stems from the first moment argument in [6]. The basic strategy is the following: we transform our chordal SLE in $\mathbb{H}$ to a process in $\mathbb{D}$ for which the image of z_0 is fixed to the origin, then pick a convenient time change, and study the process induced for the driving Brownian motion. We only need slightest adjustments, but for the convenience of the reader, the proof is still provided. Notice that in case of $\kappa > 4$ the exit time of the diffusion corresponds to the first time when the point z_0 is separated by the curve from the infinity, and for $\kappa < 4$ it corresponds to infinity. For more on radial parametrization, see for example [32].

Proof of lemma 2.2.6. The proof is the first moment argument in [6], with two slight differences: 1) we follow the evolution of the conformal radius and not the distance itself 2) we also follow the time evolution of winding. The basic strategy is the following: we transform our chordal SLE in $\mathbb{H}$ to a process in $\mathbb{D}$ for which the image of z_0 is fixed. Then pick a convenient time change, and study the process induced for the driving Brownian motion. As in [6] one works with the map $g_t(z)$ instead of $f_t(z)$ and we want to keep close to his exposition, we first remark that for the question of winding as defined in 2.2.1 this is equivalent - $g_t'(z)$ is equal to $f_t'(z)$.

Fixing the image of z_0

Denote by $H_t = \mathbb{H}\backslash K_t$ the SLE slit domain and consider the map $\tilde{g}_t : H_t \to \mathbb{D}$ from the slit domain to the unit disc, given by

$$\tilde{g}_t : z \to \frac{g_t(z) - g_t(z_0)}{g_t(z) - \overline{g_t(z_0)}}$$

It maps $\infty \to 1$ and $z_0 \to 0$. We have that

$$\log \tilde{g}_t{}'(z) = \log g_t'(z) - \log(g_t(z) - \overline{g_t(z_0)})$$

First of all, one can see that the conformal radius

$$\mathrm{CR}(z_0, H_t) = \frac{1}{|\tilde{g}_t{}'(z_0)|}$$

Second, we have that

$$\arg \tilde{g}_t{}'(z_0) = \arg g_t'(z_0) - \pi/2$$

Hence $\partial_t w(z_0) = \partial_t \arg \tilde{g}_t{}'(z_0)$ and hence we can concentrate on studying $\arg \tilde{g}_t{}'(z_0)$.

The driving function of the Loewner chain maps to a process on the unit circle by:

$$\tilde{\beta}_t = \frac{\beta_t - g_t(z_0)}{\beta_t - \overline{g_t(z_0)}}$$

Defining a time change

$$ds = \frac{(\tilde{\beta}_t - 1)^4}{|g_t(z_0) - \overline{g_t(z_0)}|^2 \tilde{\beta}_t{}^2} dt$$

it is shown in [6] that we can write the time evolution of $\tilde{g}_t$ as $h_s = \tilde{g}_{t(s)}$ where h_s satisfies the following equation:

$$\partial_s h_s(z) = \frac{2\tilde{\beta}_t h_s(z)(h_s(z) - 1)}{(1 - \tilde{\beta}_t)(h_s(z) - \tilde{\beta}_t)}$$

Now differentiating this with respect to s at $z = z_0$, we get

$$\partial_s h_s'(z_0) = \frac{2h_s'(z_0)}{1 - \tilde{\beta}_s}$$

Hence

$$\partial_s \log h_s'(z_0) = \frac{2}{1 - \tilde{\beta}_s}$$

From here two things follow. Firstly, as

$$\mathrm{CR}(z_0, H_t) = \frac{1}{|h_s'(z_0)|}$$

and

$$\partial_s \log|h_s'(z_0)| = 1$$

we can follow the evolution of the conformal radius:

$$\partial_s \log \mathrm{CR}(z_0, H_s) = -1 \tag{2.2}$$

Secondly, after writing $\beta_s = \exp(i\alpha_s)$, a small calculation gives that we can also follow the winding:

$$\partial_s \arg h_s'(z_0) = \cot \frac{\alpha_s}{2} \tag{2.3}$$

Hence, everything is at our hand as soon as we understand the transformed driving process α_s.

The diffusion of the driving process

Indeed, putting faith in [6], Ito's formula gives that α_s defined as above by $\beta_s = \exp(i\alpha_s)$ is a diffusion in $(0, 2\pi)$ starting from $\alpha_0 = 2\arg g_t(z_0)$ and satisfying the following stochastic differential equation:

$$d\alpha_s = \sqrt{\kappa} dB_s + \frac{\kappa - 4}{2} \cot \frac{\alpha_s}{2} ds$$

where B_s is a standard 1D Brownian Motion. This is well-defined & admits a unique strong solution up to the first exit-time.

As for $\kappa < 4$ the drift term is attractive towards the boundary, then comparing to Brownian motion, one can conclude that the exit time τ for the diffusion is almost surely finite. Moreover, looking at 2.2, we can put the hitting time in exact correspondence with the conformal radius. Indeed, we have

$$\mathrm{CR}(z_0, H_\tau) = \mathrm{CR}(z_0, \mathbb{H}) e^{-\tau}$$

Moreover, from (2.3) the claimed form for the winding also follows:

$$w(z_0) = \int_0^\tau \partial_s \arg h_s'(z_0) = \int_0^\tau \cot \frac{\alpha_s}{2} ds$$

□

Now we venture into the proof of the theorem:

Proof of the Theorem 2.2.3. Let τ be the disconnection time of z_0. From lemma 2.2.6, we see that conditioning on

$$CR(z_0, H_\tau) \in [\epsilon, C\epsilon]$$

is equivalent on conditioning the corresponding diffusion to exit $(0, 2\pi)$ during the time interval

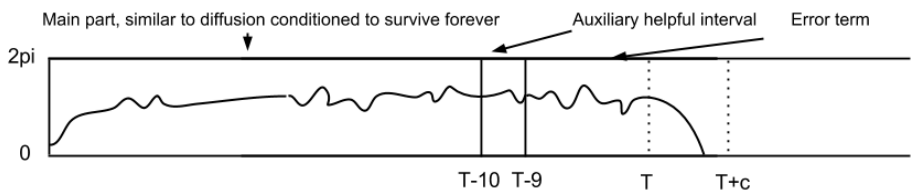
$$(\log \mathrm{CR}_0 + \log \frac{1}{\epsilon} - \log C, \log \mathrm{CR}_0 + \log \frac{1}{\epsilon}]$$

Recall that τ is also the first exit time for the diffusion and set $T = \log \frac{1}{\epsilon} + \log \mathrm{CR}_0 - \log C$. Then it remains to show that conditioned on $\tau \in [T, T+c]$, we have

$$\mathbb{E}\left(\exp(\lambda \int_0^\tau \cot \frac{\alpha_s}{2})|\tau \in [T, T+c]\right) \asymp e^{T\lambda\kappa/8}$$

with uniform constants for $|\lambda| < \lambda_0$ for any choice of of $\lambda_0 > 0$.

We will do this in several steps: first, the main term of the theorem comes from the conditioned diffusion up to time $T - 10$. By gaining control on eigenfunction expansions of survival probability, we show that this part is more or less stationary and absolutely continuous with respect to the process conditioned to everlasting survival. Thereafter, we have to control the rest. As the behaviour of the diffusion starts to change and we need to opt a different strategy. The more dangerous part is the very end and we want to handle it (for $\kappa \neq 4$) independently of the main term, thus we introduce yet another subdivision at time $T - 9$. These error terms are then controlled using probabilistic arguments.



Boundary growth of eigenfunctions for the Green operator

For the main part, the key is obtaining tight estimates of the survival probability of the diffusion. This is also studied in [6], and in several articles by Lawler and co-authors, e.g. in [31], where they study the so called SLE Green's function, which represents the probability that the curve passes ϵ-close to a point. Whereas usually this is done via martingale methods, we do everything analytically by gaining control over the eigenfunction expansion of a related integral operator. This method would apply in quite a larger context of diffusions.

Although inside the interval everything about our diffusion (2.1) is nice and smooth, we have to be cautious because the drift term becomes singular at both ends of the interval. Recall that when one considers one-dimensional diffusions on its natural scale - basically turning it into a martingale - then the speed measure represents the time-change with respect to a standard Brownian motion. In our case this speed measure is seen to be

$$m(dx) = \sin^{2-\frac{8}{\kappa}} \frac{x}{2} dx$$

which is integrable over the interval $[0, 2\pi]$ only for $\kappa > 8/3$.

Given the speed measure of the diffusion, it is known that the Green's function is given by

$$G(x,y) = \begin{cases} \frac{s(x)(s(2\pi)-s(y))}{s(2\pi)} & \text{for} \quad x \leq y \\ \frac{s(y)(s(2\pi)-s(x))}{s(2\pi)} & \text{for} \quad x > y \end{cases} \tag{2.4}$$

where $s(x)$ is a scale function of the diffusion given by

$$s(x) = \int_0^x \sin^{\frac{8}{\kappa}-2} \frac{u}{2} du$$

See for example chapter IV of [4]. Green's function could also be derived purely analytically as in [60].

Now consider the corresponding integral operator on $L^2(I, m(dx))$ with $I = (0, 2\pi)$:

$$Gf(x) = \int G(x,y) f(y) m(dy) \tag{2.5}$$

A direct calculation shows that this satisfies the conditions of a Hilbert-Schmidt integral operator, i.e. its $L^2[(I, m(dx)) \times (I, m(dx))]$ norm of the kernel is finite. Thus from Hilbert-Schmidt expansion theorem and Krein-Rutman theory, it follows that we have a complete orthonormal system of eigenfunctions $\phi_i(x)$ and corresponding eigenvalues λ_i^{-1} such that $0 < \lambda_0 < \lambda_1 \leq \lambda_3 \leq ... < \infty$ (we use the inverses here for the sake of readability later).

In the concrete context this can be shown by hand as in [4] chapter IV, section 5. [1] Also, we remark that all of the claims above would also follow by just considering the corresponding Sturm-Liouville problem: even though the problem is not entirely regular at endpoints, the expansion still applies and we still have eigenvalues as described above.

Now as the corresponding diffusion (or its generator) has C^2 regularity inside any compact interval of $(0, 2\pi)$, the eigenfunctions are also at least C^2 in these respective intervals. Moreover, by writing out the eigenfunction expansion for the Green's function itself and using Bessel inequality, we see that eigenvalues do not grow too hastily:

$$\sum_{i=0} \lambda_i^{-2} < \infty \tag{2.6}$$

We summarize the above conclusions in a lemma:

Lemma 2.2.8. *The integral operator (2.5) has a complete orthonormal system of eigenfunctions $\phi_i(x)$ that are of C^2 regularity inside any compact interval in $(0, 2\pi)$. The corresponding eigenvalues λ_i^{-1} satisfy*

1. $0 < \lambda_0 < \lambda_1 \leq \lambda_3 \leq ... < \infty$
2. $\sum_{i=0} \lambda_i^{-2} < \infty$

Next we would like to get a good control on individual eigenfunctions also near the boundary. An explicit calculation shows that $\lambda_0 = 1 - \frac{\kappa}{8}$ and up to a normalization constant $\phi_0(x)$ is equal to $\sin^{\frac{8}{\kappa}-1} \frac{x}{2}$. This is also well known in the SLE literature, see e.g [31] or [30] where it plays a role in the so called SLE Green's function, which represents the time spent by the SLE curve near a point. In what follows we set $\phi_0(x) = \sin^{\frac{8}{\kappa}-1} \frac{x}{2}$ to ease some subsequent calculations (but keep other eigenfunctions normalized).

[1] Notice that in the reference [4] chapter IV, section 5 the integrability of the speed measure is assumed (which we do not have). Yet one can see that the arguments go through as long as our operator is Hilbert-Schmidt and the Green's function itself is integrable.

For other eigenfunctions, we need some more work. As a first step we can use Cauchy-Schwartz on $\phi_i(x) = \lambda_i G\phi_i(x)$, to obtain

$$\begin{aligned} |\frac{1}{\lambda_i}\phi_i(x)| = |G\phi_i(x)| &= |\int G(x,y)\phi_i(y)m(dy)| \\ &\leq (\int G^2(x,y)m(dy))^{1/2}(\int \phi_i(y)^2 m(dy))^{1/2} \\ &\lesssim 1 \end{aligned} \tag{2.7}$$

or in other words $\phi_i(x) \lesssim \lambda_i$, where the implied constant does not depend on i.

However, this is not yet enough for our purposes. We need to show that the boundary growth of other eigenfunctions is at least of the same order than that of the first eigenfunction ϕ_0. Thus we define for all $i \in \mathbb{N}$

$$g_i(z) = \frac{\phi_i(x)}{\phi_0(x)}$$

and study its behaviour. We prove two lemmas about $g(z)$. First we show that all eigenfunctions scale similarly near the boundary or in other words:

Lemma 2.2.9. *For all $i \in \mathbb{N}$, we have*

$$g_i(x) \lesssim \lambda_i^m$$

for some universal m.

Then we go on to push this control a step further to show that the boundary growth of other eigenfunctions is in fact even nicer:

Lemma 2.2.10. *For all $i \in \mathbb{N}$, we have*

$$g_i'(x) \lesssim \lambda_i^{m+1} \sin\frac{x}{2}$$

where the implied constant does not depend on i.

Proof of lemma 2.2.9. To prove the first lemma, notice that it is enough to show the claim near $x = 0$, as firstly by (2.7) and the fact that ϕ_0 does not vanish inside the interval we know that the claim holds trivially in any compact subinterval of $(0, 2\pi)$ and secondly, our diffusion is symmetric with respect to π and thus boundary behaviour is the same near 0 and 2π.

Now the key is to notice that the Green's function is actually much more regular than needed for being in $L^2(I, m(dx))$. For example from $G_y(x) \lesssim \phi_0(x)$ it already follows that the Green's function lies in $L^1(I, m(dx))$.

Our next aim is to use a bootstrap the scaling of the eigenfunctions $\phi_i(x)$, by improving step by step on the Cauchy-Schwarz in (2.7). In this respect, consider the following expression for x near 0 and for $a \geq 0$:

$$z(x,a) = \int_0^{2\pi} G^2(x,y)\sin^{2a}\frac{y}{2}m(dy)$$

Claim 2.2.11. $z(x,a) \lesssim \max(\sin^{\frac{8}{\kappa}+2a+1}\frac{x}{2}, \sin^{2(\frac{8}{\kappa}-1)}\frac{x}{2})$

Using this claim, it is easy to improve step by step on the regularity of the eigenfunctions and to prove the lemma.

Indeed, notice that in (2.7) the first term on the RHS is given by

$$z(x,0)^{1/2} \lesssim \sin^{\frac{4}{\kappa}+1/2} \frac{x}{2}$$

and thus it follows that $\phi_i(x) \lesssim \lambda_i \sin^{\frac{4}{\kappa}+1/2} \frac{x}{2}$. Notice that for $\kappa \geq 8/3$ we could hence stop here, as $\frac{4}{\kappa} + 1/2 \geq \frac{8}{\kappa} - 1$ and we already have the statement of the lemma. For smaller κ consider the following bootstrap:

Suppose that we already know that $\phi_i(x) \lesssim \lambda_i^k \sin^{\frac{4}{\kappa}-1+a} \frac{x}{2}$. Then using a similar strategy as in (2.7), we could write using claim 2.2.11

$$\begin{aligned}
|\frac{1}{\lambda_i}\phi_i(x)| = |G\phi_i(x)| &= |\int G(x,y)\phi_i(y)m(dy)| \\
&\leq \lambda_i^k (\int G^2(x,y) \sin^{2a} \frac{y}{2} m(dy))^{1/2} (\int \sin^{\frac{8}{\kappa}-2} \frac{y}{2} m(dy))^{1/2} \\
&= O(\lambda_i^k z(x,a)^{1/2})) \\
&\lesssim \lambda_i^k \max(\sin^{\frac{4}{\kappa}+a+\frac{1}{2}} \frac{x}{2}, \sin^{\frac{8}{\kappa}-1} \frac{x}{2})
\end{aligned}$$

and thus $\phi_i(x) \lesssim \lambda_i^{k+1} \max(\sin^{\frac{4}{\kappa}+a+\frac{1}{2}} \frac{x}{2}, \sin^{\frac{8}{\kappa}-1} \frac{x}{2})$. Thereby we can improve on the boundary scaling $m-1$ times until we get $\phi_i(x) \lesssim \lambda_i^m \phi_0(x)$ as needed, whereas the implied constants have been independent of i.

Hence to prove the lemma we just need to prove the claim above.

Proof of claim 2.2.11. Using the form of the Green's function, we can bound

$$z(x,a) = \int_0^{2\pi} G^2(x,y) \sin^{2a} \frac{y}{2} m(dy)$$

by the following:

$$\begin{aligned}
z(x,a) \lesssim \max(&\int_0^x s(y)^2 (s(2\pi) - s(x))^2 \sin^{2a} \frac{y}{2} m(dy), \\
&\int_x^{2\pi} s(x)^2 (s(2\pi) - s(y))^2 \sin^{2a} \frac{y}{2} m(dy))
\end{aligned}$$

This can be further simplified to

$$z(x,a) \lesssim \max(\int_0^x s(y)^2 \sin^{2a} \frac{y}{2} m(dy), s(x)^2)$$

Inserting now the definitions of the scale function and the speed measure, this gives us for x small:

$$z(x,a) \lesssim \max(\sin^{\frac{8}{\kappa}+2a+1} \frac{x}{2}, \sin^{2(\frac{8}{\kappa}-1)} \frac{x}{2})$$

Thus the claim 2.2.11 and the proof of lemma 2.2.9 follow. □

□

The second lemma improves on this multiplicative regularity. And to prove it, we need to go back to the generator of the diffusion and use the fact that any eigenfunction of the Green's operator is also an eigenfunction of the generator [60].

Proof of lemma 2.2.10. From the previous claim, we know than we can write $\phi_i(x) = \phi_0(x)g_i(x)$ for $g_i = O(\lambda_i^m)$. Notice also that g_i has C^2 regularity inside any compact interval of $(0, 2\pi)$ as both ϕ_i, ϕ_0 have this regularity and $\phi_0 = \sin^{\frac{8}{\kappa}-1}\frac{x}{2}$ is non-zero inside the whole interval.

Now every ϕ_i is also an eigenfunction of the generator of the diffusion. This can be stated in the Sturm-Liouville form:

$$\left(\frac{\kappa}{2}\sin^{2-\frac{8}{\kappa}}\frac{x}{2}\phi_i'(x)\right)' = \lambda_i \sin^{2-\frac{8}{\kappa}}\frac{x}{2}\phi_i(x)$$

Replacing now $\phi_i(x) = \phi_0(x)g_i(x)$, using the fact that $\phi_0(x)$ is an eigenfunction, we can calculate inside any compact interval of $(0, 2\pi)$:

$$\frac{\kappa}{2}\sin^{2-\frac{8}{\kappa}}\frac{x}{2}\phi_0'(x)g_i'(x) + \left(\frac{\kappa}{2}\sin^{2-\frac{8}{\kappa}}\frac{x}{2}\phi_0(x)g_i'(x)\right)' = (\lambda_i - \lambda_0)\sin^{2-\frac{8}{\kappa}}\frac{x}{2}\phi_0(x)g_i(x)$$

Plugging in the exact form of $\phi_0(x)$ and a few calculations, we have:

$$2\cos\frac{x}{2}g_i'(x) + \frac{\kappa}{2}\sin\frac{x}{2}g_i''(x) = (\lambda_i - \lambda_0)\sin\frac{x}{2}g_i(x)$$

Thus we obtain the following Sturm-Liouville form for $g_i(x)$, which holds at least inside any compact of $(0, 2\pi)$.

$$\left(\frac{\kappa}{2}\sin^{\frac{8}{\kappa}}\frac{x}{2}g_i'(x)\right)' = (\lambda_i - \lambda_0)\sin^{\frac{8}{\kappa}}\frac{x}{2}g_i(x)$$

But now g is bounded and C^2, the right hand side can be nicely integrated up to any $\epsilon > 0$ and we get

$$\left[\frac{\kappa}{2}\sin^{\frac{8}{\kappa}}\frac{x}{2}g'(x)\right]_{\epsilon}^{x_0} = (\lambda_i - \lambda_0)\int_{\epsilon}^{x_0}\sin^{\frac{8}{\kappa}}\frac{x}{2}g(x)dx \tag{2.8}$$

We first claim the following:

Claim 2.2.12. *As $\epsilon \downarrow 0$ we have*

$$\left[\frac{\kappa}{2}\sin^{\frac{8}{\kappa}}\frac{x}{2}g'(x)\right](\epsilon) = o(1)$$

We know that $\phi_i(x), g_i(x)$ are C^2 inside any compact interval of $(0, 2\pi)$. Thus we can differentiate $\phi_i(x) = \phi_0(x)g(x)$ and using the triangle inequality write

$$|\phi_0(x)g'(x)| \leq |\phi_i'(x)| + |\phi_0'(x)g(x)| \tag{2.9}$$

For the second term of the RHS, we know that $\phi_0(x) = \sin^{\frac{8}{\kappa}-1}\frac{x}{2}$ and from lemma 2.2.9 we know that $g_i = O(\lambda_i^m)$. Hence the second term is of order $O(\lambda_i^m \sin^{\frac{8}{\kappa}-2}\frac{x}{2})$. To get a bound on the first term of the RHS consider again the integral equation satisfied by eigenfunctions:

$$\frac{1}{\lambda_i}\phi_i(x) = \int G(x,y)\phi_i(y)m(dy)$$

Now $\phi_i(x)$ is differentiable inside compacts of $(0, 2\pi)$, and also the Green's function $G(x,y)$ is differentiable unless $x = y$, at which point it is both left and right-differentiable but these derivatives have a finite gap between them. Thus we can differentiate both sides to get:

$$\frac{1}{\lambda_i}\phi_i'(x) = \int \frac{\partial}{\partial x} G(x,y)\phi_i(y)m(dy)$$

Plugging in the form of the Green's function shows that the RHS can be bounded by $O(\lambda_i^m \sin^{\frac{8}{\kappa}-2}\frac{x}{2})$ and thus $|\phi_i'(x)| = O(\lambda_i^{m+1}\sin^{\frac{8}{\kappa}-2}\frac{x}{2})$.

Thus we see that in the triangle inequality (2.9), the whole of RHS is of order

$$O(\lambda_i^{m+1}\sin^{\frac{8}{\kappa}-2}\frac{x}{2})$$

In particular this must hold for the LHS, i.e. we have

$$|\phi_0(x)g'(x)| = O(\lambda_i^{m+1}\sin^{\frac{8}{\kappa}-2}\frac{x}{2})$$

To prove the claim, recall that $\phi_0(x) = \sin^{\frac{8}{\kappa}-1}\frac{x}{2}$. Hence, as $\frac{8}{\kappa} > 1$, it follows that

$$\left[\frac{\kappa}{2}\sin^{\frac{8}{\kappa}}\frac{x}{2}g'(x)\right](\epsilon) = O(\lambda_i^{m+1}\epsilon^{\frac{8}{\kappa}-1}) = o(1)$$

and thus our claim 2.2.12 follows.

Finally return to (2.8). The absolute value of the right hand side can be bounded by $O(\lambda_i^{m+1}\sin^{\frac{8}{\kappa}+1}\frac{x}{2})$ using lemma 2.2.9. From our recent claim we know that by letting $\epsilon \downarrow 0$, only the term $\frac{\kappa}{2}\sin^{\frac{8}{\kappa}}\frac{x}{2}g'(x_0)$ survives. Thus get the claimed derivative bound:

$$\left(\frac{\phi_i(x_0)}{\phi_0(x_0)}\right)' = g'(x_0) \lesssim \lambda_i^{m+1}\sin\frac{x_0}{2}$$

□

Diffusion up to time $s \leq T - 10$

Given a sufficiently regular diffusion of diffusion coefficient $a/2$ and drift term b, one can use either Doob's H-transform [59] or direct calculations as in Pinsky [42] to show that, conditioned on $\tau \in (T, T+c)$, up to time T we have a non-homogeneous diffusion with the following generator

$$L_s^T = 1/2\nabla \cdot a\nabla + b\nabla + a\frac{\nabla \mathbb{P}_x(c+T-s \geq \tau > T-s)}{\mathbb{P}_x(c+T-s \geq \tau > T-s)}\nabla$$

It is also known by same methods that conditioned on everlasting survival, the generator becomes

$$L_s^\infty = 1/2\nabla \cdot a\nabla + b\nabla + a\frac{\nabla\phi_0(x)}{\phi_0(x)}\nabla$$

In our concrete setting this means that conditioned on everlasting survival our diffusion process is given by

$$d\alpha_s^\infty = \sqrt{\kappa}dB_s + 2\cot\frac{\alpha_s^\infty}{2}ds \tag{2.10}$$

Our aim is to then show that for T large at least until some time $T - 10$ the diffusion conditioned to survive up to time T is almost the same. More explicitly, we claim that

Lemma 2.2.13. *The conditioned diffusion can be written as:*

$$d\alpha_s^T = \sqrt{\kappa} dB_s + (2\cot\frac{\alpha_s^T}{2} + E^T(\alpha_s^T, s))ds \tag{2.11}$$

for some independent Brownian motion B_t and the error term

$$E^T(x,s) = \frac{\nabla \mathbb{P}_x(c+T-s \geq \tau > T-s)}{\mathbb{P}_x(c+T-s \geq \tau > T-s)} - \frac{\nabla \phi_0(x)}{\phi_0(x)}$$

satisfies $E^T(x,s) \lesssim e^{-a(T-s)}$ for $s \in [0, T-10]$, for some $a > 0$ and uniformly over the interval $[0, 2\pi]$.

The proof of this lemma just makes use of our control on the eigenfunctions:

Proof. We start by writing out a series representation for $\mathbb{P}_x(c+T-s \geq \tau > T-s)$. To do this, notice first that

$$\mathbb{P}_x(c+T-s \geq \tau > T-s) = \mathbb{P}_x(\tau > T-s) - \mathbb{P}_x(\tau > c+T-s)$$

and so it suffices to find series representation for the similar terms on the RHS.

Now, using lemma 2.2.9 and the condition on the growth of eigenvalues (2.6), it is easy to see [4], that for any $t > 0$ the transition probabilities of the initial process (2.1) can be written as a sum converging absolutely and uniformly over the whole interval $[0, 2\pi]$:

$$\mathbb{P}_x(\alpha_t \in dy) = \sum_{i=0} \phi_i(x) e^{-\lambda_i t} \phi_i(y) m(dy)$$

Thus survival probability can be written as a series

$$\mathbb{P}_x(\tau > T) = \sum_{i=0} c_i \phi_i(x) e^{-\lambda_i T} \tag{2.12}$$

Similarly the convergence of this sum is also absolute and uniform over the interval. Moreover, if we choose some $t_0 > 0$, then for all $T > t_0$ the convergence is uniform in t as well. Any $t_0 > 0$ would do, so we pick $t_0 = 10$.

Notice that then we can in fact write that $\mathbb{P}_x(\tau > T) \asymp e^{-\lambda_0 T}\phi_0$ for all $T > t_0$. This gives us in a slightly more direct manner the conclusion of the first moment argument for the Hausdorff dimension of SLE curves in [5]. More precisely, it replaces the hands-on technical section 1.2 of that paper by the more general setup presented here. It also proves the existence of the conformal radius SLE Green's function [31].

Now plugging in the expansion (2.12) using the remark above, we have

$$e^T(x,s) = \frac{\sum_{i=1} c_i' \left(\phi_i'(x)\phi_0(x) - \phi_0'(x)\phi_i(x)\right) e^{-\lambda_i(T-s)}}{\phi_0(x)\left(\mathbb{P}_x(\tau > T-s) - \mathbb{P}_x(\tau > c+T-s)\right)}$$

with $c_i' = c_i(1 - e^{-\lambda_i c})$.

We start from the denominator. Using the uniform convergence for $T - s > 10$ and $\lambda_1 > \lambda_0$ we have

$$\mathbb{P}_x(\tau > T-s) - \mathbb{P}_x(\tau > c+T-s) = c_0'\phi_0(x)e^{-\lambda_0(T-s)} + O(\phi_0(x))e^{-0.5(\lambda_0+\lambda_1)(T-s)}$$

Thus we have a lower bound:

$$\mathbb{P}_x(\tau > T-s) - \mathbb{P}_x(\tau > c+T-s) \gtrsim \phi_0(x)e^{-\lambda_0(T-s)}$$

For the nominator, write

$$\phi_i'(x)\phi_0(x) - \phi_0'(x)\phi_i(x) = \phi_0^2(x)\left(\frac{\phi_i(x)}{\phi_0(x)}\right)'$$

Plugging in the derivative estimates from lemma 2.2.10 and using the bound on the growth of eigenvalues (2.6), we have for $T-s>10$ uniformly

$$|\sum_{i=1} c_i'\left(\phi_i'(x)\phi_0(x) - \phi_0'(x)\phi_i(x)\right)e^{-\lambda_i(T-s)}| \lesssim e^{-0.5(\lambda_1+\lambda_0)(T-s)}\phi_0^2(x)$$

And thus for $T-s>10$ uniformly over time and space

$$|E^T(x,s)| \lesssim e^{-0.5(\lambda_1-\lambda_0)(T-s)}$$

and the lemma follows. □

Putting things together we find the total winding of this part:

$$\int_0^{T-10} \cot\frac{\alpha_s}{2} \overset{d}{\sim} \frac{\sqrt{\kappa}}{2}B_{T-10} + (\alpha_{T-10}^T - \alpha_0^T) + \int_0^{T-10} E(\alpha_s^T, s)ds$$

Now, α_s^T itself is bounded and due to the exponential decay of the error term, the final term is also uniformly bounded. Finally, from the Brownian part we get a Gaussian of variance $T-10$. This gives us that conditioned on $\tau \in [T, T+c]$ we have

$$\int_0^{T-10} \cot\frac{\alpha_s}{2}ds \overset{d}{\sim} \frac{\sqrt{\kappa}}{2}X + E_B \tag{2.13}$$

with X Gaussian of variance $T-10$ and E_B some uniformly bounded random error (not independent of X). Looking at the exponential moments, we account for the main term of the theorem and a multiplicative error.

Remark 2.2.14. In SLE literature, e.g. in [32], the diffusion conditioned on everlasting survival corresponds to two-sided radial SLE. Hence, one could hope to approach the problem by weighing the everlasting measure by a martingale to obtain the process conditioned on $\{\tau > T\}$. We know this martingale explicitly [30], it is given by the first eigenfunction of the generator - $M_t(x) = e^{\lambda_0 t}\phi_0(x)\mathbb{1}(\tau > t)$. Using this approach one could possibly offer another derivation for the control over the main interval, slightly different in spirit. The error analysis below would be needed in any case.

The remaining part: $T-10 < t \leq \tau$

Now after the time $T-10$, our control on the drift term gets gradually worse and worse and hence our previous strategy doesn't allow the exact estimation of the contribution to winding by relating it to the Brownian motion. This is due to the fact that the initial strong boundary repulsion at time 0 changes gradually to an attraction at time T. Hence we need a different strategy.

We start by reducing our workload considerably:

Claim 2.2.15. *It is sufficient to only deal with the upper bound of the exponential moments for $\lambda > 0$.*

Proof. Indeed, firstly, it is easy to see that uniform upper (lower) bounds on exponential moments for $\lambda > 0$ give also lower (upper) bounds for $\lambda < 0$.

Secondly, notice that the processes starting from a and $2\pi - a$ are symmetric with respect to π, but $\cot \frac{x}{2}$ is antisymmetric. Hence we can couple processes α_1 and α_2 starting from a and $2\pi - a$ by using the Brownian motion B_t and $-B_t$ such that $\cot \frac{\alpha_1(s)}{2} + \cot \frac{\alpha_2(s)}{2} = 0$.

Hence an uniform lower bound on the positive exponential moments of $\int \cot \frac{x}{2}$ starting from 2π, is via Cauchy-Schwarz equivalent to an uniform upper bound on the exponential moments and vice versa. Indeed, we can write

$$1 = \mathbb{E}_c \exp\left(\lambda \int_{T-10}^{\tau} \cot \frac{\alpha_1(s)}{2} + \cot \frac{\alpha_2(s)}{d} s\right)$$

where we write $\mathbb{E}_c$ for the expectation wrt the conditioned measure. We then Cauchy Schwarz to get

$$1 \leq \left[\mathbb{E}_c \exp\left(2\lambda \int_{T-10}^{\tau} \cot \frac{\alpha_1(s)}{2} ds\right)\right]^{1/2} \left[\mathbb{E} \exp\left(2\lambda \int_{T-10}^{\tau} \cot \frac{\alpha_2(s)}{2} ds\right)\right]^{1/2}$$

Thus the claim follows. □

Now we have to treat separately cases $\kappa \neq 4$ and $\kappa = 4$. For the former, we will first discuss how to obtain a bound on the exponential moments from the time $T - 9$ onwards, then deal with the middle part, i.e. the time interval $[T - 10, T - 9]$, and finally put them together to obtain control over the whole remaining part. Thereafter we handle the case $\kappa = 4$ in a more direct manner.

Suppose that at time $T - 9$ the diffusion conditioned to die between T and $T + c$ is at some point $\delta > 0$. Then the process onwards is given by the initial diffusion conditioned to die between $9 < \tau \leq 9 + c$. We claim the following:

Claim 2.2.16. *Suppose we start the diffusion (2.1) conditioned to die between $9 < \tau \leq 9 + c$ from $\delta > 0$. Then we have the following upper bound for the winding over this time interval*

$$\mathbb{E}(\lambda w_\tau) = O(\delta^{1-8/\kappa})$$

Let's first see why this will suffice our needs. The problem is that the estimate blows up as $\delta \downarrow 0$. However, if we were able to well control the probability of being below δ_0 at time $T - 9$ independently of the position at time $T - 10$, we would stand some hope. This is indeed our plan. As is clear from the proof of lemma 2.2.13, absolute continuity with respect to everlasting survival process (2.10) lasts nicely also up to time $T - 9$ (with a slightly worse constant). From say [30] or by following directly [4] and [42], we know that the transition probabilities for this everlasting survival process are given by $\mathbb{P}_x(\alpha_t^\infty \in dy) \lesssim \sin^{\frac{8}{\kappa}} \frac{y}{2} dy$ for any $t > 0$ and thus surely at $t = 1$. Thus our conditioned process will have probability $O(\delta^{8m/\kappa+1})$ to be in the interval $[\delta^{m+1}, \delta^m]$ at time 10. Now taking the above expectation over all possible intervals of this form, we get a geometric sum of terms $O(\delta^{8m/\kappa+1}\delta^{m(1-8/\kappa)}) = O(\delta^{m+1})$ which has a finite value. So everything looks nice. When we put things together in the end of the subsection it is cleaner to condition on the exact position of the diffusion at time $T - 9$, but this just replaces sums by integrals and everything remains nicely bounded.

Proof of claim: Recall that the initial diffusion equation (2.1) has a unique strong solution and so we can work with respect to the filtration of the corresponding Brownian motion B_t. Consider the exponential martingale $\exp(\lambda B_t - \lambda^2 t/2)$ and the bounded stopping time $\tau' = (9+c) \wedge \tau$. We can use the optional stopping theorem to get $\mathbb{E}(\exp(\lambda B_{\tau'} - \lambda^2 \tau'/2)) = 1$. But on the other hand, we know that as α_s remains always bounded, then from the initial diffusion equation 2.1 it follows that we can write the winding as

$$w_{\tau'} = \int_0^{\tau'} \cot \frac{\alpha_s}{2} ds = \frac{2\sqrt{\kappa}}{\kappa - 4} B_{\tau'} + C'$$

with C' random, but in $[0, 2\pi]$. Thus we have

$$\mathbb{E} \exp(\lambda \int_0^{\tau'} \cot \frac{\alpha_s}{2} ds) \lesssim \mathbb{E} \exp(\frac{4\kappa}{(\kappa-4)^2} \lambda^2 \tau'/2) \lesssim \mathbb{E} \exp(\frac{4\kappa}{(\kappa-4)^2} \lambda^2 (9+c)/2)$$

where the implied constants depend on λ, κ. Hence for any event F

$$\mathbb{E} \exp(\lambda \int_0^{\tau'} \cot \frac{\alpha_s}{2} ds) | F) \mathbb{P}(F) \lesssim \mathbb{E} \exp(\frac{4\kappa}{(\kappa-4)^2} \lambda^2 (9+c)/2)$$

In particular, we can choose the event $F = \{9 < \tau \leq 9 + c\}$. Recall from the proof of lemma 2.2.13 that the probability of F is of order $O(\delta^{8/\kappa - 1})$. And thus forgetting the dependence on fixed λ, c, κ we get an upper bound of order $O(\delta^{1-8/\kappa})$ on the exponential moments. □

Control over the interval mid-interval for $\kappa \neq 4$

Now we deal with the small remaining part from $T - 10$ to $T - 9$. Again, as over this time window the process is absolutely continuous with respect to the process conditioned on everlasting survival given by (2.10), it is sufficient to bound exponential moments for the latter.

It might seem that we also have an additional conditioning pushing the endpoints to lie in an interval $[\delta^{m+1}, \delta^m]$. However, in fact when putting the remaining part together in the next paragraph, we will get rid of this dependence. Hence we need to just control the exponential moments independently of the starting point at $T - 10$ for the process that is conditioned on the everlasting survival. Now as $\cot \frac{x}{2}$ is decreasing in $[0, 2\pi]$, then from stochastic coupling of different trajectories using the same Brownian motions, one can see that the exponential moments $\mathbb{E} \exp(\lambda \int_{T-10}^{T-9} \cot \frac{\alpha_s}{2} ds)$ are bounded by those coming from the process that starts at the point 0.

Finally, recall the form of the everlasting survival process (2.10):

$$d\alpha_s^\infty = \sqrt{\kappa} dB_s + 2 \cot \frac{\alpha_s^\infty}{2} ds$$

It follows that we can write the exponential moments of $\int_0^1 1 \cot \frac{\alpha_s^\infty}{2}$ as above using the Brownian part:

$$\int_0^1 \cot \frac{\alpha_s^\infty}{2} ds = \frac{\sqrt{\kappa}}{2} B_1 + C'$$

with C' random, but in $[0, 2\pi]$ and conclude that the exponential moments are finite, independent of where the process is at the time $T - 10$.

Putting the remaining part together for $\kappa \neq 4$
Recall that the main part from the winding came from the time interval $I_1 = [0, T-10]$. Additional error terms come from intervals $I_2 = [T-10, T-9]$ and $I_3 = [T-9, \tau]$. As the winding is given as an integral over time, we can decompose the winding over the remaining part $R = I_2 \cup I_3$ as $w_R = w_{I_2} + w_{I_3}$. Denoting by $\mathcal{F}_{I_1}$ the filtration of the underlying Brownian Motion up to to time $T-10$, we can write the contribution of the remaining part as:

$$\mathbb{E}_c(e^{\lambda(w_{I_2}+w_{I_3})}|\mathcal{F}_{I_1})$$

For now this is a random variable. We Cauchy-Schwarz the expectation to get rid of the dependence at the point $T-9$ and gain an upper bound

$$\mathbb{E}_c(e^{\lambda(w_{I_2}+w_{I_3})}|\mathcal{F}_{I_1}) \leq \mathbb{E}_c(e^{2\lambda w_{I_2}}|\mathcal{F}_{I_1})^{1/2}\mathbb{E}(e^{2\lambda w_{I_3}}|\mathcal{F}_{I_1})^{1/2}$$

Now, start from the first term. As the conditioned process is a nice Markov process, what happens over the time interval $I_2 = [T-10, T-9]$ depends on the filtration $\mathcal{F}_{I_1}$ only through its position at the time $T-10$. But we saw that the positive exponential moments over I_2 have uniform bounds independent of the location of the process at time $T-10$. Thus:

$$\mathbb{E}_c(e^{\lambda(w_{I_2}+w_{I_3})}|\mathcal{F}_{I_1}) \lesssim \mathbb{E}_c(e^{2\lambda w_{I_3}}|\mathcal{F}_{I_1})^{1/2}$$

For the second term, we condition further on the value of α_{T-9}:

$$\mathbb{E}_c(e^{2\lambda w_{I_3}}|\mathcal{F}_{I_1}) = \mathbb{E}_c(\mathbb{E}(e^{2\lambda w_{I_3}}|\alpha_{T-9})|\mathcal{F}_{I_1})$$

In the discussion above we saw that

$$\mathbb{E}_c(e^{2\lambda w_{I_3}}|\alpha_9) \lesssim \alpha_{T-9}^{1-8/\kappa}$$

Thus

$$\mathbb{E}_c(e^{2\lambda w_{I_3}}|\mathcal{F}_{I_1}) \lesssim \mathbb{E}_c(\alpha_{T-9}^{1-8/\kappa}|\mathcal{F}_{I_1})$$

Also, as argued above, the density of α_{T-9} satisfies $\mathbb{P}_x(\alpha_{T-9} \in dy) \lesssim \sin^{\frac{8}{\kappa}} \frac{y}{2} dy$ independently of the starting point at $T-10$. Thus the expectation is nicely finite and indeed, putting everything together

$$\mathbb{E}_c(e^{\lambda w_R}|\mathcal{F}_{I_1}) = O(1)$$

where now the implied constant is deterministic.

Remaining part for $\kappa = 4$
Although the above strategy fails for $\kappa = 4$, the diffusion itself is simpler: the drift term in (2.1) vanishes and the unconditioned process is really just twice a standard Brownian motion. As we are just aiming for bounds of exponential moments, we can well assume that we have the standard Brownian motion, denote it by B_t.

As above we aim to find upper bounds for positive ($\lambda > 0$) exponential moments:

$$\mathbb{E}\left[\exp(\lambda \int_0^\tau \cot \frac{B_s}{2} ds)|\tau \in [10, 10+c)\right]$$

Start by noticing that in the space interval $[0, 2\pi]$ we have $\cot \frac{x}{2} \leq \frac{4}{x}$. Thus it suffices to bound

$$\mathbb{E}\left[\exp(\lambda \int_0^\tau \frac{1}{B_s} ds) | \tau \in [10, 10+c)\right]$$

Next we separate cases $B_\tau = 0$ and $B_\tau = 2\pi$. The latter case is simple, as conditioned on $B_\tau = 2\pi$, we have a Bessel-3 process. With positive probability this process reaches 2π in the time interval $[10, 10+c)$. Thus it suffices to bound just the relevant exponential moments for a Bessel-3 process starting from a point in $[0, 2\pi]$. This we can again do by studying the relevant SDE as above for case $\kappa \neq 4$. The SDE of Bessel-3 is given by

$$d\rho_t = dB_t + \frac{1}{\rho_t} dt$$

Writing $\tau' = \tau \wedge 10 + c$, we have

$$\mathbb{E} \exp(\lambda \int_0^{\tau'} \frac{1}{\rho_s} ds) \lesssim \mathbb{E} e^{\lambda \rho_{\tau'}}$$

Thus, as the exponential moments for Bessel processes on the LHS certainly exist [44], we have the desired upper bound.

For the case $B_\tau = 0$ we need a bit more. Here, the idea is to condition on the exact values of exit times $\tau \in [10, 10+c)$ to obtain a family of Brownian excursions of fixed length and to gain control over these excursions. In other words, we want to write

$$\mathbb{E}\left[\exp(\lambda \int_0^\tau \frac{1}{B_s} ds) | \tau \in [10, 10+c), B_\tau = 0\right] = \mathbb{E}\left[\mathbb{E}(\exp(\lambda \int_0^\tau \frac{1}{B_s} ds) | \tau, B_\tau = 0) | \tau \in [10, 10+c)\right] \tag{2.14}$$

and study $\mathbb{E}(\exp(\lambda \int_0^\tau \frac{1}{B_s} ds) | \tau, B_\tau = 0)$.

First, notice that by stochastic coupling using the same Brownian motion, we can certainly consider the starting point also to be at 0. How to describe this conditioned process? We are conditioning on two events: 1) the process being back at zero at τ and 2) remaining inside the interval for $0 < t < \tau$. Now, as is well known, the probability law of a Brownian excursion can defined as a limit of nicely defined conditional laws. Also, the second event has positive probability in all of the considered measures. Thus we can condition in any order. In particular we can obtain our conditioned process by taking a Brownian excursion and conditioning it to be lower than 2π. Now this latter conditioning has positive probability, and so proving an upper bound on the exponential moments over the usual excursions suffices our needs.

To control the integral over the Brownian excursion over time $[0, 1]$, recall that the scaled Brownian excursion is in fact just a Bessel-3 bridge with the following SDE [44]:

$$d\rho_t = dB_t + \frac{1}{\rho_t} dt - \frac{\rho_t}{1-t} dt$$

Then, as above we can write

$$\int_0^{0.5} \frac{1}{\rho_t} dt = B_t + \int_0^{0.5} \frac{\rho_t}{1-t} dt$$

Thus denoting by M^* the maximum of the Bessel bridge in $[0,1]$, we have for some positive constant c:

$$\mathbb{E}\exp(\lambda \int_0^{0.5} \frac{1}{\rho_s} ds) \leq e^{\lambda^2/8} \mathbb{E} e^{\lambda c M^*}$$

But this maximum of the Bessel 3-bridge is below the maximum of the usual Bessel 3-process in $[0,1]$, and for the latter all exponential moments exist [44]. Thus $\mathbb{E}\exp(\lambda \int_0^{0.5} \frac{1}{\rho_s} ds) = O(1)$. As the bridge is symmetric, it also follows that: $\mathbb{E}\exp(\lambda \int_{0.5}^{1} \frac{1}{\rho_s} ds) = O(1)$

Finally, by Cauchy-Schwarz we have

$$\mathbb{E}\exp(\lambda \int_0^{1} \frac{1}{\rho_s} ds) = O(1)$$

Hence we have showed the existence on the relevant exponential moments over the Bessel-3 bridges of length 1. But by scaling this amounts to the existence of these moments for all bridges of fixed lengths in $[10, 10+c]$. Moreover these bounds are all dominated by those of the longest bridge. Thus we can uniformly upper bound the term $\mathbb{E}(\exp(\lambda \int_0^{\tau} \frac{1}{B_s} ds) | \tau, B_\tau = 0)$ in (2.14) and obtain also $O(1)$ error bound for $\kappa = 4$ uniformly over the starting point of the error interval.

Negative exponential moments and lower bounds

Finally, recall that by claim 2.2.15 in the beginning of this section, the work above for positive exponential moments also implies the upper bound for $\lambda < 0$ and lower bounds for all exponential moments. In other words we have shown that

$$\mathbb{E}_c(e^{\lambda w_R} | \mathcal{F}_{I_1}) \asymp 1 \tag{2.15}$$

with no randomness on the RHS. Here the implied constants depend on λ, κ and can be chosen to be uniform for $|\lambda| < \lambda_0$ for any choice of $\lambda_0 > 0$.

The final result

Now we individually controlled the exponential moments over time intervals $I_1 = [0, T-10]$ and $R = [T-10, \tau]$. There is one moment of dependency between them at time point $T-10$, but this does no harm as our control over the remaining part was uniform. We can write the winding as a sum over the time intervals:

$$w = w_{I_1} + w_R$$

Thus the exponential moments are given by

$$\mathbb{E}_c(e^{\lambda w}) = \mathbb{E}(e^{\lambda(w_{I_1} + w_R)})$$

where again $\mathbb{E}_c$ means that we already consider the expectation with respect the common conditioning of $\tau \in [T, T+c]$. It remains then to condition out the first part:

$$\mathbb{E}_c(e^{\lambda w}) = \mathbb{E}(e^{\lambda w_{I_1}} \mathbb{E}_c(e^{\lambda w_R} | \mathcal{F}_{I_1}))$$

where $\mathcal{F}_{I_1}$ as above denotes the filtration of the underlying Brownian motion up to the end of the first time interval. From (2.15) we know that the second term only can be adds a uniformly bounded by a deterministic constant both from above and below. Thus the proposition follows from plugging in the derived form (2.13) for the first term. □

Remark 2.2.17. Of course this proof method works in a much wider context of conditioned diffusions, hence we hope it could be of some independent interest as well.

2.3 Interlude: topologies on proofs

In writing this thesis, we have found it useful to consider three different types of convergence for an argument.

Definition 2.3.1 (vague convergence of proofs)**.** *A sequence of proofs p_n converges to a proof p vaguely, if the author cannot tell the difference between them.*

Definition 2.3.2 (weak convergence of proofs)**.** *A sequence of proofs p_n converges to a proof p weakly, if no reader can tell the difference between them.*

Definition 2.3.3 (strong convergence of proofs)**.** *A sequence of proofs p_n converges to a proof p strongly, if there is no mathematical difference between them.*

Mostly, one is interested in the special case when p is a correct proof. We have verified that the weak convergence is strictly stronger than the vague convergence. The difference between weak and strong convergence is slightly more delicate.

3

GFF with SLE and KPZ

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The background motivation for this chapter can be traced to statistical physics. Statistical physics models on Euclidean lattices are often difficult to study. Even when for the self-avoiding walk on the hexagonal lattice we know the connective constant [14], we are for example only beginning to gather any rigorous results at all on the square lattices. Also, we still hope for proofs of critical percolation exponents on the same lattice.

However, in the eighties three physicists Knizhnik, Polyakov and Zamolodchikov [28] came up with a far-reaching strategy for studying these models. The proposed plan was to study them in a random environment, or in what they called the Quantum Gravity regime, and then translate the results back to the Euclidean setting. This was a fruitful idea as the study of many models becomes easier in these random environments, and even more - the so called KPZ relation gives an exact translation for critical exponents back to the Euclidean case [15, 16, 1].

Mathematically, however, the understanding of the KPZ relation is still scarce. Mainly, the problem is that in higher than one dimension, we do not yet have a suitable continuum model for the random environment that would allow understanding of the KPZ relation. Even though random planar maps have been shown to converge to a candidate random metric space [37, 34], we are still missing a conformal structure on these spaces, thus making it hard to relate models on these spaces with our usual models on Euclidean lattices.

Still, recently there has been progress in understanding the KPZ relation. In one dimension, we have a quite good understanding [7]. For two dimensions, a more mundane version of the random environment has helped us. Namely, whereas ideally we would like to establish the KPZ relation in a random metric space with a certain topology, we can already give meaning to the KPZ relation when we model the random environment by a random measure on a two-dimensional domain. This measure is called the Liouville measure [19, 21].

In this context of the Liouville measure the KPZ relation can be shown to rigorously relate Euclidean and Quantum fractal dimensions [19, 45]. There is, however, a little catch - all the proofs only work for deterministic sets and sets independent of the random environment. However, in at least a few cases the statistical physics models are coupled with the random environment, as for example in the Ising model. Though expected, it is not a priori clear whether our sets of interest, as for example the interface boundaries, will become independent in the continuum limit. Hence it is also interesting to ask to which extent the KPZ relation holds for sets depending on the measure.

In this chapter, we treat the case of most natural sets coupled with the Liouville measure - the SLE_κ curves corresponding to interface boundaries in statistical physics models. One way of coupling the SLE lines with the GFF and the Liouville measure is using a conformal welding of two quantum surfaces [54, 20]. This ought to correspond to gluing random planar maps in the discrete setting. We already know that in this case one recovers a KPZ relation, if instead of volume measures one considers boundary measures on the SLE [54, 20].

In what follows, we show that on the other hand the usual KPZ relation does not hold for the SLE_κ with $0 < \kappa < 8$ coupled with the GFF as level lines ($\kappa = 4$) or flow lines of the field (see Chapter 2), by determining exactly the quantum fractal behaviour of the SLE curves in this coupling. Notice that this implies that the KPZ relation is of very different character than the Kaufman's theorem on dimension doubling of the Brownian motion. It can also been

seen as evidence that, indeed, in the continuum limit the interface boundaries have to become independent of the random environment.

Finally, we want to also mention that there is another way of interpreting these results: one could just say that we are studying the geometry of the GFF near its level and flow lines. The results presented here are a way of formalizing how exactly the field near flow and level lines differs from its typical behaviour.

The chapter starts by a discussion of the Liouville measure. Then we discuss at more length the different versions of the KPZ relation in the literature [19] [45] and propose yet a third one that we call the expected quantum Minkowski dimension. Next, we study the introduced notion of the expected Minkowski dimension. We prove the relevant KPZ relation and show that, as expected, the expected quantum Minkowski dimension is always larger than the quantum Hausdorff dimension introduced in [45]. Finally, we determine the expected quantum Minkowski for the SLE_κ flow lines of the GFF. A key ingredient is the work on winding of the SLE curves in chapter 2.

3.1 The exponential of the GFF: Liouville measure

Liouville measure should be the right model for a random measure underlying the study of statistical physics models in their "quantum gravity" form. It is one step short of the actual aim - the random metric on a topological sphere.

Mathematically, the Liouville measure ought to be the exponential of the Gaussian free field. However, as GFF is formally a distribution, one needs to define the Liouville measure using some kind of regularization process. There are many ways of achieving this, the roots going back to the beautiful work of Kahane [27] on Gaussian multiplicative chaos and of even more earlier work of Hoegh-Krohn [24] describing quantum fields with exponential interaction. Different ways of defining the Liouville measure and their equivalence are discussed in greater detail in [46].

In this article we use the circle-averaging regularization as used in Duplantier & Sheffield [19]. This suits our needs well as it is local and works well under conditioning as explained below.

3.1.1 Liouville measure

In [19] the following process is used to define the Liouville measure in any sufficiently nice domain D:

- First, regularize the field by taking circle averages around each point, i.e. set

$$h_\delta(z) = h(\rho^z_{\delta_n})$$

  where by $\rho^z_{\delta_n}(z) \in \mathbb{H}^{-1}$ we denote the distribution giving unit mass to the circle of radius δ_n around the point z.

- Now let $0 < \gamma < 2$ and define the $\delta-$approximate Liouville measures as

$$d\mu_\delta(z) = \delta^{\gamma^2/2} e^{\gamma h_\delta(z)} dz$$

Remark 3.1.1. The regularized GFF corresponds to a Gaussian field with the covariance kernel given by

$$G_\delta(x,y) = \log \frac{1}{\delta \vee |x-y|} + \tilde{G}_\delta(x,y)$$

Here $\tilde{G}_\delta(x,y)$ is the harmonic extension to the domain of the function $-\log \frac{1}{\delta \vee |x-y|}$ on boundary. See [19] for details.

Then the following theorem can be then taken as definition of the Liouville measure [19]:

Theorem 3.1.2. *Let D be a domain. For $0 \leq \gamma < 2$, along powers of two in the interior of D, then almost surely $\delta-$approximate Liouville measures weakly converge to a non-degenerate random measure μ_γ, called the Liouville measure. This measure is measurable w.r.t zero-boundary GFF h.*

Remark 3.1.3. Often we denote $\mu = e^{\gamma h}$, as γ can be taken to be a fixed parameter $0 < \gamma < 2$ throughout the rest of this chapter.

Remark 3.1.4. We will later couple the Liouville measure with the SLE in a similar way to the coupling of the GFF and the SLE. Recall that in order to sample a GFF in this coupling, we start by first sampling an SLE, then choosing an independent GFF in the slit domain and adding some harmonic correction terms. In the case of the Liouville measure we would like to obtain the Liouville measure on the whole domain as follows: we sample the SLE, then we define the GFF with correct boundary conditions in the slit domain and construct the Liouville measure in the slit domain.

Thus, in other words, when we depart from the Liouville measure of the whole domain and then condition on say a level line, we would want to see the conditioned measure to be the Liouville measure in the slit domain.

$$"e^{\gamma h_D} | SLE \overset{(d)}{\sim} e^{\gamma h_{D_{SLE}}}"$$

One can see from theorem 3.1.2 that everything works well and we indeed obtain the definition of the Liouville measure in the slit domain.

3.1.2 The 2D KPZ relations for the Liouville measure

We now introduce two canonical versions for the KPZ relation in 2D quantum gravity, and propose yet another one. Then we shortly compare all three. The difference is only in the nature of the fractal dimension used: either using a box-counting, Hausdorff or Minkowski version of the dimension. Throughout we always (more or less silently) assume that we are dealing with sets such that the corresponding fractal dimensions exist. Whereas here all the dimensions are measure-based, we also remark that in [7] a 1D metric version of KPZ relation was proved in the context of dyadic multiplicative cascades.

Expected box-counting version

The first rigorous version of the KPZ relation was given in the work of Duplantier-Sheffield [19], to which an interested reader can find a well-readable introduction in [21]. Here the fractal dimensions for a fixed set A on the Euclidean and on the quantum side are defined as follows (assuming they exist in the first place):

- Euclidean side:
$$x(A) = \lim_{r \downarrow 0} \frac{\log \mathbb{P}(B_r(z) \cap A \neq \emptyset)}{\log r}$$
where we sample according to the uniform measure of the domain.

- Quantum side:
$$\Delta(A) = \lim_{r \downarrow 0} \frac{\log \mathbb{E}\mu_h(B_r^q(z) \cap A \neq \emptyset)}{\log r}$$
Here the quantum ball $B_r^q(z)$ of radius r is defined as the largest Euclidean ball around z for which the Liouville measure is not larger than r.

In other words, to define the euclidean exponent we calculate the probability that a ball of radius r, around a point sampled according to the Lebesgue measure of the domain, touches the set A. Similarly, for the quantum exponent we first calculate the probability that a quantum ball around a point sampled according to an instance of the Liouville measure touches the set A, then average over the random measures and finally calculate the exponent.

With these notions the KPZ relation holds:

Theorem 3.1.5 (Duplantier & Sheffield)**.** *Let A be a deterministic (or field-independent) compact subset in the interior of some domain such that its Euclidean scaling exponent $x(A)$ exists. Let μ_γ be the Liouville measure on this domain with $0 \leq \gamma < 2$. Then we have that:*

- *the quantum scaling exponent $0 \leq \Delta(A) \leq 1$ exists and*
- *satisfies the so called KPZ formula:*
$$x = (2 - \gamma^2/2)\Delta + \gamma^2\Delta^2/2$$

Remark 3.1.6. Here and later, we define the Euclidean using the Euclidean "metric". In particular, this way $x = 2\Delta$ when we let $\gamma \downarrow 0$ as the quantum dimension is measure-based. We opt for this convention, as we will often need to refer to results on Euclidean dimensions of the SLE curves and we feel it would be confusing to translate them into the measure-based context.

Almost sure Hausdorff version

In parallel, Rhodes & Vargas [45] published a version using slightly different notion for the fractal dimension. The proof of the respective KPZ relation can be made quite short [3]. As a basis for their definition of the quantum dimension, they use a measure-based Hausdorff dimension.

- On the Euclidean side we use the usual Hausdorff dimension. I.e. define the Hausdorff content
$$H_\delta(A,r) = \inf\{\sum_{i=1}^{k} r_i^\delta : A \subseteq \cup_1^k B_i(r_i), r_i \leq r\}$$
Then the Hausdorff dimension is defined as
$$d_H(A) = \inf_\delta\{\lim_{r\downarrow 0} H_\delta(A,r) < \infty\}$$

- For the quantum side, we define similarly the quantum Hausdorff content to be
$$H_\delta^Q(A,r) = \inf\{\sum_{i=1}^{k} \mu(B_i(r_i))^\delta : A \subseteq \cup_1^k B_i(r_i), r_i \leq r\}$$
The quantum Hausdorff dimension is then given by
$$q_H(A) = \inf_\delta\{\lim_{r\downarrow 0} H_\delta(A,r) < \infty\}$$

Then the following KPZ relation holds.

Theorem 3.1.7 (Rhodes & Vargas)**.** *Let A be a deterministic (or field-independent) compact subset in the interior of some domain. Let μ_γ be the Liouville measure on this domain with $0 \leq \gamma < 2$. Then, almost surely, the following KPZ formula holds:*
$$d_H = (2+\gamma^2/2)q_H - \gamma^2 q_H^2/2$$
where by d_H and q_H we denote respectively the usual and the quantum Hausdorff dimensions of the set A.

Expected Minkowski version

To make the literature even more colourful, we introduce yet a third version of the dimension which also satisfies the KPZ relation. We use a version of the upper Minkowski dimension, which we will henceforth call just the Minkowski dimension.

There are many ways to define the Minkowski dimension, for us the most convenient version uses only fixed dyadic tiling [9]. We recall that a n-th level dyadic covering of the plane can be defined as the collection of all squares with vertex coordinates of the form $(\frac{k}{2^n}, \frac{l}{2^n}), (\frac{k+1}{2^n}, \frac{l}{2^n}), (\frac{k}{2^n}, \frac{l+1}{2^n}), (\frac{k+1}{2^n}, \frac{l+1}{2^n})$ for $k, l \in \mathbb{Z}$. We restrict this covering to a domain by taking the subset of all these squares intersecting the domain.

Then a dyadic 2^{-n} Minkowski content of A defined by:
$$M_\delta(A, 2^{-n}) = \sum_{S_i \in \mathcal{S}_n} \mathbb{1}(S_i \cap A \neq \emptyset) l(S_i)^\delta$$
where $\mathcal{S}_n$ is the n-th level dyadic covering of the domain and $l(S_i)$ the side-length the square S_i. Then we define the Minkowski dimension as
$$d_M(A) = \inf_\delta\{\limsup_{n\uparrow\infty} M_\delta(A, 2^{-n}) < \infty\}$$

The corresponding quantum version is given by first defining the quantum dyadic 2^{-n} Minkowski content:

$$M_\delta^Q(A, 2^{-n}) = \sum_{S_i \in \mathcal{S}_n} \mathbb{1}(S_i \cap A \neq \emptyset)\mu(S_i)^\delta$$

and then setting

$$q_M(A) = \inf_\delta \{\limsup_{n \uparrow 0} M_\delta^Q(A, 2^{-n}) < \infty\}$$

It is clear that the definitions work nicely also for random sets, in which case the Minkowski contents will just be random variables.

Moreover, it will also make sense to talk about the expected quantum Minkowski dimension, where in the definition of the Minkowski dimension, we just use the expectation of the dyadic Minkowski content w.r.t the measure. So, for deterministic sets we set for example:

$$q_{M,E}(A) = \inf_\delta \{\limsup_{n \uparrow \infty} \mathbb{E}_h \left(M_\delta^Q(A, 2^{-n})\right) < \infty\}$$

Notice that we take the expectation of each dyadic 2^{-n} Minkowski before the lim sup. Whereas this is less natural, it allows us to work only with first moment estimates and nevertheless provide upper bounds for the quantum Hausdorff dimension. Also, it is actually more similar to the order of expectations in the expected box-counting version from [19] introduced above.

In the next section, we will prove the analogous KPZ relation for the expected quantum Minkowski dimension, the proof of which is shorter than for the other two notions:

Proposition 3.1.8. *Let A be a fixed (or field-independent) compact subset in the interior of some domain. Let μ_γ be the Liouville measure on this domain with $0 \leq \gamma < 2$. Then we have the following KPZ formula:*

$$d_M = (2 + \gamma^2/2)q_{M,E} - \gamma^2 q_{M,E}^2/2$$

where by d_M and $q_{M,E}$ we denote respectively the usual (upper) and the expected quantum Minkowski dimensions of the set A.

Relations between the notions

These three different notions of the quantum dimension and hence the KPZ relation all have different benefits:

- Box counting version: it provides a notion of quantum balls having more physical content and is probably easiest to link to discretization of the field, and hence discrete models.

- Almost sure Hausdorff version: whereas the box counting version is averaged over the field, here we have an almost sure relation; it also has the usual advantages and specificities with respect to the Minkowski dimension. However, it proved difficult to use for field-dependent sets.

- Expected Minkowski: this is easiest to work with for both dependent and independent sets; one might say it is less natural, however it certainly has enough substance to give useful bounds on the Hausdorff dimension.

In the next section, we will also prove two relations between the expected quantum Minkowski and quantum Hausdorff dimensions.

Firstly, we show that for deterministic and measure-independent sets we have the following relation: if the Euclidean Minkowski and Hausdorff dimensions of a set agree, then also its expected Minkowski dimension and Hausdorff dimension agree on the quantum side. This shows that we are not losing much in general by using the Minkowski version

Secondly, we show that on the quantum side the quantum Hausdorff dimension is almost surely smaller than the expected Minkowski dimension, even if the measured set depends on the field. This will allow us to prove results about the almost sure Hausdorff version, by fist proving them for the expected Minkowski dimension.

KPZ relation for dependent sets

Notice that in all three theorems we require the sets in question to be either fixed or independent of the underlying measure. Hence it is natural to ask, to what extent the KPZ relation remains true for sets that depend on the measure. It comes out that there is no uniform theorem as for example Kaufman's theorem for dimension doubling in Brownian Motion.

In fact, given that the KPZ relation stems from a multifractal behaviour [46], it is quite intuitive that for example fixed level sets should help us construct already a counterexample. The problem is that the precise counterexamples depend on the "sensitivity" of the definition and the intuitively clearest versions will not always work:

For almost sure Hausdorff dimension finding a counterexample is relatively easy. One just needs to look at $\gamma-$-thick points [27] [25] [2], i.e. points such that $\lim_{r\downarrow 0} \frac{h_\epsilon(z)}{\log 1/r} = \gamma$. Their Hausdorff dimension is smaller than two, but they are of full measure on the quantum side, violating the usual KPZ relation.

For expected box-counting measure and the Minkowski dimension finding a counterexample is somewhat harder, as they are less sensitive. For example $\gamma-$ thick points, being dense, would have trivial dimensions on both sides. To produce a simple counterexample one needs to go one step further. We can still rely on the height of the field to produce a fractal as in [25], but we need to intersect this field-dependent fractal with a deterministic fractal to arrive at the "sensitivity" level of these definitions.

Now these previous examples might look unnatural - in some sense we were really trying to cook up counterexamples. Thus it would be interesting to find counterexamples where the measure-dependent sets are not a priori chosen to violate KPZ. This is exactly the aim of this chapter: we look at the zero level lines and SLE_κ flow lines given by the coupling of the GFF and the SLE and show that the expected Minkowski and almost sure Hausdorff versions of the KPZ relation do not hold for these sets. Thus, even for rather natural couplings the KPZ relation cannot be taken as given.

3.2 Expected Minkowski dimension: KPZ formula and relation to almost sure Hausdorff dimension

In this subsection we will prove the following proposition:

Proposition 3.2.1 (KPZ formula for expected Minkowski dimension)**.** *Let A be a fixed (or field-independent) compact subset in the interior of some domain. Let μ_γ be the Liouville measure on this domain with $0 \leq \gamma < 2$. Then we have the following KPZ formula:*

$$d_M = (2 + \gamma^2/2)q_{M,E} - \gamma^2 q_{M,E}^2/2$$

where by d_M and $q_{M,E}$ we denote respectively the usual (upper) and the expected quantum Minkowski dimensions.

The proof is a simple consequence of the multifractal properties of Euclidean balls under the Liouville measure. We state this as a lemma. For the proof and slightly generalized versions, we refer to one of the many newer works on multiplicative chaos, including [47] [45], but also to [19] where it is approached slightly differently.

Lemma 3.2.2. *Consider the Liouville measure $\mu = \mu_\gamma$ for $0 < \gamma < 2$. Then for any $q \in [0, 1]$ and any fixed ball $B(r) \subseteq D$ of radius r with $0 < r < \epsilon$ at least at distance ϵ from the boundary, we have*

$$\mathbb{E}\mu(B(r))^q \asymp r^{(2+\gamma^2/2)q - \gamma^2 q^2/2}$$

where the implied constant depends on q.

Remark 3.2.3. If the distance of the ball is comparable to the boundary, one needs to be more careful as the exact scaling holds for the covariance kernel given by $\log_+ \frac{1}{|x-y|}$ and the correction term of the Green's function starts playing a greater role near the boundary.

Proof of proposition.

Upper bound

Let $\delta > 0, 1 \geq q > 0$ be such that $d_M + \delta = (2 + \gamma^2/2)q - \gamma^2 q^2/2$. We want to show that $\limsup_n \mathbb{E} M_q^Q(E, 2^{-n}) < \infty$. As the Minkowski dimension of A is d_M, then for sufficiently large n

$$M_{d_M+\delta}(A, 2^{-n}) \lesssim 2^{-n\delta/2}$$

Thus, for the same covering we get using the scaling relation of 3.2.2, that

$$\mathbb{E}(M_q^Q(A, 2^{-n})) \lesssim 2^{-n\delta/2}$$

Thus $q_{M,E} \geq q$. Now letting $\delta \downarrow 0$, we get the upper bound.

Lower bound
The lower bound follows similarly. As d_M is the Minkowski dimension for A, then for any $\delta > 0$, we have infinitely many $n \in \mathbb{N}$ such that $M_{d_M-\delta}(A, 2^{-n}) > R$ for any $R > 0$. Now consider $1 \geq q > 0$ such that $d_M - \delta = (2 + \gamma^2/2)q - \gamma^2 q^2/2$. Then for all the same indexes n, we have $\mathbb{E}M_q^Q(A, 2^{-n}) > R$ and the lower bound follows. □

Remark 3.2.4. Notice that for the upper bound we could use an "almost sure" version of the Minkowski dimension. Indeed, from Markov's inequality

$$\mathbb{P}(M_q^Q(A, 2^{-n}) \geq 2^{-n\delta/4}) \leq 2^{-n\delta/4}$$

Now this sequence of probabilities is summable and thus by Borel-Cantelli the event only happens finitely often.Thus in fact almost surely $\limsup_n M_q^Q(A, 2^{-n}) = 0$.

Remark 3.2.5. Also, it is easy to see that the same result holds for sets that are independent of the field.

3.2.1 Relations between expected Minkowski and almost sure Hausdorff dimension

In this section we bring out two results. First, for fixed (and field-independent) sets we conclude an agreement between the expected Minkowski and almost sure Hausdorff versions of the quantum dimension, given that there is agreement between the dimensions on the Euclidean side. Second, we prove an inequality for the quantum side holding even for dependent sets.

The first relation, as both the Hausdorff and Minkowski dimension satisfy the very same KPZ relation, is a straightforward corollary of the previous proposition:

Corollary 3.2.6. *Consider the Liouville measure for $0 \leq \gamma < 2$ in some domain. Suppose A is deterministic (or field-independent) compact set in the interior of some domain, such that its Euclidean Minkowski and Hausdorff dimensions agree. Then also, its expected quantum Minkowski dimension and quantum Hausdorff dimensions agree.*

The second relation importantly also holds for sets that can depend on the measure:

Proposition 3.2.7. *Consider the Liouville measure with $0 \leq \gamma < 2$ in some domain. For any random set coupled with the field, the quantum Hausdorff dimension is almost surely bounded above by the expected quantum Minkowski dimension.*

To prove this, first notice that in fact we could equally well use squares instead of balls in our definition of the (quantum) Hausdorff dimension.

Proof. Suppose that with positive probability $p > 0$ the quantum Hausdorff dimension of the set A satisfies $q_H(A) > \delta$. Then also

$$\mathbb{P}\left(\lim_{n\uparrow\infty} H_\delta^Q(A, 2^{-n}) = \infty\right) = p$$

where we use squares instead of balls in the covering. But now every covering used in the Minkowski dimension also provides a suitable covering whose content must be larger than $H_\delta^Q(A, 2^{-n})$. Hence it follows that

$$\mathbb{P}\left(\liminf_{n\uparrow\infty} M_\delta^Q(A, 2^{-n}) = \infty\right) \geq p$$

Now fix some $R > 0$ large and define the event

$$E_{N,R} = \{M_\delta^Q(A, 2^{-n}) > R \text{ for all } n \geq N\}$$

The events $E_{N,R}$ are increasing in N and

$$\bigcup_N E_{N,R} \supset \{\liminf_{n\uparrow\infty} M_\delta^Q(A, 2^{-n}) = \infty\}$$

Thus by countable additivity there is some N_R such that $\mathbb{P}(E_{N_R,R}) > p/2$. But then for all $n > N_R$

$$\mathbb{E}(M_\delta^Q(A, 2^{-n})) \geq Rp/2$$

And thus

$$\limsup_{n\uparrow\infty} \mathbb{E}\left(M_\delta^Q(A, 2^{-n})\right) \geq Rp/2$$

But $p > 0$ was fixed and we can pick R arbitrarily large. Therefore

$$\limsup_{n\uparrow\infty} \mathbb{E}\left(M_\delta^Q(A, 2^{-n})\right) = \infty$$

and $q_{M,E}(A) \geq \delta$. As this holds for all δ with $\mathbb{P}(q_H(A) > \delta) > 0$, we have the claim. □

Remark 3.2.8. Notice that we do indeed need a proof. Namely, we have no scaling result similar to lemma 3.2.2 at our disposal. So we do not a priori know that the Hausdorff and Minkowski contents scale well on the quantum side. Secondly, more direct approaches are limited by the fact that our definition of the Minkowski dimension involved an expectation inside the lim sup.

3.3 Almost sure Hausdorff dimension of the zero level line does not satisfy the KPZ relation

In this section we show that the expected Minkowski and almost sure Hausdorff versions of the usual KPZ relation do not hold for zero level lines of the Gaussian free field. Although this follows also from the more general result in the next section, the proof here is shorter, self-contained and partly used in the next section.

Fix the underlying domain to be the upper half plane. Recall from section 2, theorem 2.1.1 the precise meaning of the level line: we couple the GFF with certain boundary conditions with the SLE_4 curve such that the in this coupling the GFF can be sampled by first sampling the SLE, then an independent GFF in the remaining domain + adding a bounded harmonic function.

Proposition 3.3.1. *Consider the Liouville measure μ_γ with $0 \leq \gamma < 2$ in the upper half plane. The expected quantum Minkowski dimension of the zero level line drawn up to some finite stopping time satisfies $q_{M,E} \leq \frac{3}{4+\gamma^2}$. Hence the usual KPZ relation does not hold.*

By using proposition 3.2.7, we have a straightforward corollary:

Corollary 3.3.2. *Almost surely the quantum Hausdorff dimension of the zero level line drawn up to some finite stopping time is bounded from above by $\frac{3}{4+\gamma^2}$ and hence the usual KPZ relation is not satisfied for quantum Hausdorff dimension.*

Remark 3.3.3. In fact, this proposition can also be seen as a straightforward corollary of the later work on flow lines by setting $\kappa = 4$. In fact, we then also confirm that the expected Minkowski dimension of the zero level line is equal to $q = \frac{3}{4+\gamma^2}$. However, the proof here is much shorter and simpler in spirit. The underlying intuition is that near the zero level line the field is lower and this intuition can be nicely expressed with rigour.

We start with a key lemma that replaces the usual scaling lemma 3.2.2 and gives the multifractal behaviour of the balls around points on the zero level line under the Liouville measure:

Lemma 3.3.4. *Sample a zero level line η_τ drawn up to some finite stopping time τ. Let S be a dyadic square of side-length $l(S)$ intersecting this zero level line. Now denote by $h = h_\eta$ the Gaussian free field in this slit domain and by $\mu = \mu_{h_\eta}$ the corresponding Liouville measure with $0 \leq \gamma < 2$. Then we have that $\mathbb{E}_{h|\eta_\tau}(\mu_h(S)) \lesssim l(S)^{2+\gamma^2/2}$*

Here we write $\mathbb{E}_{h|\eta_\tau}$ to recall that we are actually working in the conditioned measure.

Proof. As usual in working with the Liouville measure, it is cleaner to work with a regularized field. From theorem 3.1.2 we know $\delta_n = 2^{-n}$ regularized fields converge to the Liouville measure. Hence, we can write

$$\mu_h(S) = \lim_{\delta_n \downarrow 0} \mu_{h_{\delta_n}}(S) = \lim_{\delta_n \downarrow 0} \int_S \delta^{\gamma^2/2} e^{\gamma h_{\delta_n}(z)} dz$$

Recall from definitions preceding 3.1.2 that the regularized field $h_{\delta_n}(z)$ is a Gaussian field, defined by taking circle averages of the GFF. It is defined nicely point-wise. Its mean is given by the bounded harmonic SLE-measurable correction term described in section 2.1, and the covariance kernel is described by the regularized Green's function of the slit domain:

$$G_{\delta_n}(x,y) = \log \frac{1}{\delta_n \vee |x-y|} + \tilde{G}_{\delta_n}(x,y)$$

Here $\tilde{G}_{\delta_n}(x,y)$ is the harmonic extension of the function equal to $-\log \frac{1}{\delta_n \vee |x-y|}$ when one of the points is on the boundary of the domain. Notice that if at least one of x, y is of distance δ_n from the boundary, then $\tilde{G}_{\delta_n}(x,y) = \tilde{G}(x,y)$ where the latter is the harmonic correction term for the usual Green's function. This is useful, as we know that $\tilde{G}(x,x) = \mathrm{CR}(x, H_t)$ where the latter denotes the conformal radius of the point x for the slit domain.

Now we can write the GFF h as a sum of a zero-boundary GFF h^0 and the bounded harmonic correction term C_h that can be defined to be zero on the SLE (see discussion after the statement on theorem 2.1.1.) Using Fatou's lemma, we can write

$$\mathbb{E}_{h|\eta_\tau}\left(\mu_h(S)\right) \leq \lim_{\delta_n \downarrow 0} \mathbb{E}_{h|\eta_\tau}\left(\int_S \delta_n^{\gamma^2/2} e^{\gamma h^0_{\delta_n}(z)+\gamma C_h} dz\right)$$

Firstly notice that as the harmonic correction is uniformly bounded by a constant, it will only influence the expectation by a bounded constant and thus we can henceforth neglect the term γC_h by absorbing it in some multiplicative constant. Thus we want to bound

$$\mathbb{E}_{h|\eta_\tau}\left(\int_S \delta_n^{\gamma^2/2} e^{\gamma h^0_{\delta_n}(z)} dz\right)$$

We will split the integral into two:

1. the part that is at least of distance δ_n off the curve
2. the curve together with its δ_n neighbourhood

For the first part, start by taking the expectation inside the integral (everything is nicely bounded). Then using exponential moments for Gaussian random variables, we have the following estimate for the integrand:

$$\mathbb{E}_{h|\eta_\tau}\left(\delta_n^{\gamma^2/2} e^{\gamma h^0_{\delta_n}}\right) \leq \mathrm{CR}(z, H_t)^{\gamma^2/2} \tag{3.1}$$

Recall that the conformal radius satisfies $\mathrm{CR}(z, H_t) \asymp d(z, H_t)$ where $d(z, H_t)$ is the distance from the boundary. But $d(z, H_t) \leq l(S)$ and hence we get a bound of $O(l(S))^{\gamma^2/2}$. Thus integrating over the whole square (minus the δ_n neighbourhood) we get a contribution of $O(l(S)^{2+\gamma^2/2})$.

Now we treat the part near the curve. We could use Kahane convexity inequalities [27] or a global argument as in 3.4.9. However, it follows also elementarily by using bare hands. Start again by taking the expectation inside the integral. Then we need to bound the variance of $h^0_{\delta_n}(z)$. By the definition of the GFF in H_t it is given by integrating

$$\int_{H_t \times H_t} G_{\delta_n}(x,y) \rho^z_{\delta_n}(x) \rho^z_{\delta_n}(y) dx dy$$

where by $\rho^z_{\delta_n}$ we denote the distribution giving unit mass to the circle of radius δ_n around the point z.

But $G(x,y) \geq G_{\delta_n}(x,y)$ and hence the variance is bounded by

$$\int_{\mathbb{H} \times \mathbb{H}} G(x,y) \rho^z_{\delta_n}(x) \rho^z_{\delta_n}(y) dx dy$$

i.e. by that of the δ_n regularized GFF in $\mathbb{H}$. But this we can calculate as above to get

$$\mathbb{E}_{h|\eta_\tau}\left(\delta_n^{\gamma^2/2} e^{\gamma h^0_{\delta_n}}\right) \leq \mathrm{CR}(z, \mathbb{H})^{\gamma^2/2}$$

Now we now that the SLE_4 is not space-filling and in fact has Hausdorff dimension of $3/2$ [6]. Thus we may bound this part with $o(\delta_n^{1/3})$

Thus

$$\mathbb{E}_{h|\eta_\tau}\left(\int_S \delta_n^{\gamma^2/2} e^{\gamma h_{\delta_n}(z)} dz\right) \lesssim (l(S))^{2+\gamma^2/2} + o(\delta_n^{1/3})$$

and letting finally $\delta_n \downarrow 0$, we get

$$\mathbb{E}_{h|\eta_\tau}(\mu_h(S)) \lesssim l(S)^{2+\gamma^2/2}$$

□

Now we are ready to attack the proposition:

Proof of proposition. We will sample the GFF as above: we first sample an SLE_4 up to some finite stopping time τ, then the field in the slit domain with its bounded harmonic correction term.

Now, we know that the Minkowski dimension of the SLE_4 curve is $3/2$ [48, 6]. Thus for any $\delta > 0$ we can cover it with $O(r^{-3/2-\delta})$ dyadic squares $S_i \in \mathcal{S}$ of radius $r = 2^{-n}$. Fix $q < 1$ to be defined later.

By linearity of expectation we can write

$$\mathbb{E}_{h|\eta_\tau}\left(M_q^Q(A,r)\right) = \mathbb{E}_{h|\eta_\tau}\left(\sum_{S_i\in\mathcal{S}} \mu_h(S_i)^q\right) = \sum_{S_i\in\mathcal{S}} \mathbb{E}_{h|\eta_\tau}(\mu_h(S_i)^q)$$

Now by lemma 3.3.4, $\mu_h(S_i)$ is an integrable random variable with respect to the randomness of the GFF h. Hence as $q \leq 1$, we can use Jensen's inequality for the concave function x^q to get

$$\mathbb{E}_{h|\eta_\tau}(\mu_h(S_i)^q) \leq \left(\mathbb{E}_{h|\eta_\tau}\mu_h(S_i)\right)^q$$

But using lemma 3.3.4 again, we have for any ball S_i

$$\left(\mathbb{E}_{h|\eta_\tau}\mu_h(S_i)\right)^q \lesssim r^{q(2+\gamma^2/2)}$$

and so

$$\mathbb{E}_{h|\eta_\tau}\left(\sum_{S_i\in\mathcal{S}} \mu_h(S_i)^q\right) \lesssim r^{-3/2-\delta+q(2+\gamma^2/2)}$$

Choosing $q = (1+\delta)\frac{3}{4+\gamma^2}$ and averaging over the curve, we thus have

$$\mathbb{E}M_q^Q(A,r) \lesssim r^{\delta/2}$$

It follows that $q_M \leq q$ and by letting $\delta \downarrow 0$, we see that $q_M \leq \frac{3}{4+\gamma^2}$. □

3.4 Expected quantum Minkowski dimension of the SLE$_\kappa$ flow lines

In this section we aim to find the exact expected quantum Minkowski dimension of the SLE$_\kappa$ flow lines and show that this does not satisfy the KPZ relation and to deduce that the almost sure Hausdorff version of the KPZ relation is not satisfied either. For technical reasons we now consider the unit disc as our underlying domain.

The main result can be then stated as follows:

Theorem 3.4.1. *Consider the Liouville measure with $0 \leq \gamma < 2$ in the unit disc and let $0 < \kappa < 8$. Then the expected quantum Minkowski dimension of the SLE_κ flow lines is given by $q_{M,E} < 1$ satisfying*

$$d_M = (2 + \gamma^2/2)q_{M,E} - \gamma^2(1 - \kappa/4)^2 q_{M,E}^2/2$$

where d_M is the Minkowski dimension of the respective SLE curve.

Hence for $0 < \kappa < 8$ the KPZ relation is not satisfied for the expected Minkowski dimension. And from proposition 3.2.7, we straight away deduce that:

Corollary 3.4.2. *Consider the Liouville measure with $0 \leq \gamma < 2$ in the unit disc and let $0 < \kappa < 8$. Then almost surely the quantum Hausdorff dimension for the flow lines SLE_κ is below the dimension predicted by KPZ relation and hence the KPZ relation is not satisfied in the almost sure Hausdorff version.*

The intuition behind this result can be gained by comparing the two images on figure 2.1 that illustrate the $SLE_{8/3}$ flow line and level line couplings.

Indeed, we see that zero level lines acted like the boundary of the domain and hence the KPZ relation was not satisfied as the field was considerably lower around them. Now looking at figure 2.1 we can also see that at least for κ close to 4, the SLE_κ flow lines still stick close to the level line. Hence similarly to the zero level line case, the corresponding quantum contents of the coverings should be smaller and thus the quantum dimension lower.

For $\kappa = 0, \kappa = 8$ we regain the KPZ relation, which is nice but not surprising as $\kappa = 0$ should correspond to a straight line joining zero and infinity, i.e. become independent of the field, and for $\kappa = 8$ the winding part itself should form the whole field. So in some sense their behaviour is "field-independent". Here we provide two illustrative images that indicate what happens when κ is near 0 or 8:

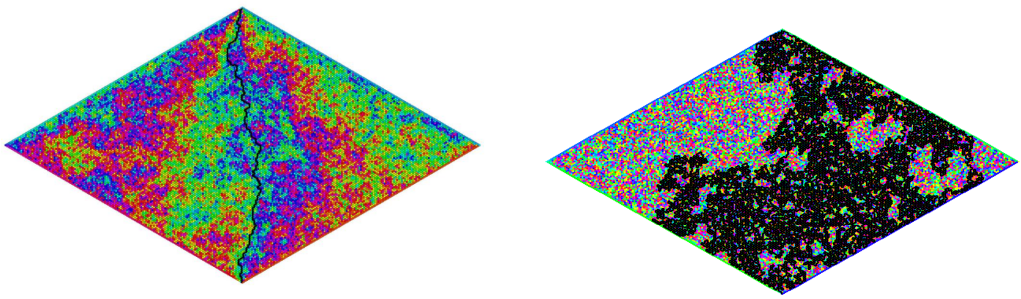


Figure 3.1: On the left the flow line corresponding to $SLE_{0.5}$ is represented. Notice that it does not really hold close to the level line anymore, but shoots quite straight from one end-point to the other. On the right we have the $SLE_{7.5}$ flow line. One can see that it starts filling the space, not being too picky about which points to step on. We thank Scott Sheffield for allowing us to use these images.

Proof strategy

Recall our simple proof strategy for SLE_4: cover the curve with balls, look at their scaling

using Jensen to bring expectation inside integrals, and conclude. This does not seem to work here. Of course already the fact we also want lower bounds asks for some additional ideas. However, main problems are related to the additional winding term in the coupling theorem 2.1.2 for the flow lines:

- First, it is crucial to take averages here over the SLE process to make use of the winding theorem 2.2.3. This requires us to (in some sense) fix the covering balls we are working with. Hence also the usefulness of the Minkowski version of the KPZ relation.
- Second, the fact that winding is not defined on the SLE curve and that we can only calculate it for a specific conditioning poses its constraints.
- Third, as a minor modification we now need to work with the chordal SLE drawn up to the very end. the underlying domain is then cut into two pieces and it needs some extra care.

Our strategy of attack makes use of a variant of the dyadic Whitney decomposition which we call conformal-radius or CR-Whitney decomposition. It allows us at the same time to work off the curve, nicely incorporate the results on winding and still get the necessary information on the fractal geometry of the curve. Whitney decomposition has been also used to study the geometry of the SLE in many of the relevant papers. For example in the beautiful seminal paper by Rohde & Schramm on basic properties of the SLE [48], it was used to provide the correct upper bounds for the Minkowski dimension and thus Hausdorff dimension of traces for the SLE curves.

By using the CR-Whitney decomposition, the proofs of both the upper and lower bounds for the quantum expected Minkowski dimension will follow the same outline. To bound the Minkowski dimension we need to provide bounds for the Liouville measure of a dyadic covering. We will do this in three steps: first, we estimate the expected Liouville measure of a single CR-Whitney square for the SLE slit domain; second, we provide an estimate on the expected Liouville measure over a collection of suitable CR-Whitney squares; and finally, we translate this estimate into an estimate about the combined measure of a dyadic covering.

3.4.1 CR-Whitney decomposition

Recall that dyadic Whitney decomposition of a domain is composed of dyadic squares Q that satisfy: $l(Q) \leq d(Q) \leq 4l(Q)$ where $d(Q)$ is the distance of the square from the boundary of the domain and $l(Q)$ the side-length of the square.One way to achieve a dyadic Whitney decomposition is to just pick all maximal dyadic squares with $d(Q) \geq l(Q)$. The maximality will guarantee the other inequality. See for example [22] or [48] for an usage in context.

It comes however out that it is easier for us not to work with the usual Whitney squares, as this would make incorporating information on winding rather technical. We hence work with a slight modification, where instead of normal distance we use the conformal radius. Thus we define CR-Whitney squares as dyadic squares Q such that they satisfy $4l(Q) \leq \mathrm{CR}(z_0) \leq 12l(Q)$. Notice that here we really condition on the conformal radius of the centre, thus allowing to use the results on winding, i.e. theorem 2.2.3. We have an analogous CR-Whitney decomposition, which we state for clarity as a separate lemma.

Lemma 3.4.3 (CR-Whitney decomposition)**.** *For every Jordain-domain of the complex plane, we can find a decomposition of dyadic squares such that any $Q \in \mathcal{W}$ satisfies $4l(Q) \leq \mathrm{CR}(z_0) \leq 12l(Q)$, where $\mathrm{CR}(z_0)$ is the conformal radius of the centre of z_0 of Q, and that the interiors of the squares do not overlap.*

Proof. Again, pick all maximal dyadic squares satisfying $4l(Q) \leq CR(z_0)$. Then using the triangle inequality and the relation $CR(z_0)/4 \leq d(z_0, \partial D) \leq CR(c_0)$, we arrive that the maximality imposes $CR(z_0) \leq l(8 + 2\sqrt{2}) \leq 12l$. □

It is important for us that we can fully cover the slit domain with CR-Whitney squares. However, we do not actually want to further use the disjointness condition. We would like the event $\{Q$ is a CR-Whitney square$\}$ to be in exact correspondence with conditioning on the conformal radius of its centre and sticking to the disjointness condition would ruin this.

Hence we stress that from now on, being a CR-Whitney square only means conditioning on its centre to satisfy certain inequalities.

An estimate on the Green's function

To estimate the Liouville measure of a CR-Whitney square, we need tight control on the Green's function inside a CR-Whitney square. This is established in the following lemma, which might be well-known, but we could not locate a concrete reference in the literature. It is similar to Harnack type of inequalities, only that we ask for additive bounds. We state and prove it first for typical Whitney squares.

Lemma 3.4.4. *Let D be some bounded simply connected domain. Write the Green's function in D in the form $G_D(x,y) = \log \frac{1}{|x-y|} + \tilde{G}_D(x,y)$. Then if x, y belong to the same Whitney square with $l(Q) < 1$ of D, we have*

$$-\log \frac{1}{d(Q, \partial D)} - C_1 \leq \tilde{G}_D(x,y) \leq -\log \frac{1}{d(Q, \partial D)} + C_2$$

for some universal constants C_1, C_2.

However in fact we make use of the following straightforward corollary:

Corollary 3.4.5. *The same holds for CR-Whitney squares with possibly different constants*

This indeed follows quickly, as one can for example notice that any CR-Whitney square is either contained in a at most M-times bigger Whitney square or is tiled into at most M-times smaller Whitney squares for some absolute constant M. The proof of the lemma itself needs a bit more:

Proof of lemma 3.4.4. The left-hand side is simple. For fixed x, $\tilde{G}_D(x,y)$ is by definition the harmonic extension to D of $-\log \frac{1}{x-y}$ on ∂D. Now we know that a harmonic function inside a bounded domain achieves its minimum on the boundary. Combining this with the fact that the boundary of D is at least at distance $d(Q, \partial D)$ for any $x, y \in Q$, we get the lower bound.

For the upper bound, we argue as follows: we know that the Green's function in the upper half plane is given by

$$G_{\mathbb{H}}(z,w) = \log \frac{1}{|z-w|} - \log \frac{1}{|z-\overline{w}|}$$

Now pick $f : \mathbb{H} \to D$ to be a conformal map and set $x = f(z)$, $y = f(w)$. Then by the conformal invariance of the Green's function, we have

$$\log \frac{1}{|z-w|} - \log \frac{1}{|z-\overline{w}|} = \log \frac{1}{|x-y|} + \tilde{G}_D(x,y)$$

Now using the complex version of the Mean Value Theorem, write $x - y = f(z) - f(w) = A(z-w)$ where $A = \mathrm{Re}(f'(u)) + i\,\mathrm{Im}(f'(v))$ for some u, v on the line between z and w. Plugging this into the previous equation, we get

$$\tilde{G}_D(x,y) = -\log \frac{1}{|z-\overline{w}|} - \log \frac{1}{|A|}$$

Now using triangle inequality, we have $|z-\overline{w}| \leq |z-w| + 2\,\mathrm{Im}(w)$. So using also the definition of A again,

$$\tilde{G}_D(x,y) \leq -\log \frac{|A|}{|x-y| + 2|A|\,\mathrm{Im}(w)} - \log \frac{1}{|A|} = -\log \frac{1}{|x-y| + 2|A|\,\mathrm{Im}(w)}$$

Now we know that $|x-y| \leq \sqrt{2}l(Q)$. Also, we know that for Whitney squares the side-length is up to fixed multiplicative constants equal to the distance of the boundary. Thus $|x-y| \leq cd(Q, \partial D)$.

Recall that from distortion theorems [43] it follows that for f analytic from $D_1 \to D_2$ we have

$$|f'(z_0)| \asymp \frac{d(f(z_0), D_2)}{d(z_0, D_1)} \tag{3.2}$$

where the implied constants are absolute. Thus we get that

$$d(Q) \lesssim \mathrm{Im}(w)|f'(w)| \lesssim d(Q))$$

for some absolute constants and hence

$$\tilde{G}_D(x,y) \leq -\log \frac{1}{cd(Q, \partial D)} - \log \frac{|f'(w)|}{|A|} + C$$

for some absolute constant C. It finally remains to show an absolute bound on $|A|/|f'(w)|$ to conclude the lemma.

Now, we know that Q can be covered by at most M images of Whitney squares in $\mathbb{H}$, where M is a universal constant [22]. Join these M Whitney squares with further Whitney squares in $\mathbb{H}$ to make the region covered convex, i.e. a big rectangle. The number of these additional squares can again be universally bounded.

Then z, w, u, v lie inside this region, and as they are only bounded hyperbolic distance apart, the ratio of their imaginary parts is bounded. On the other hand this bounded number of Whitney squares can be in turn covered by a uniformly bounded number of connected Whitney squares in D. Thus also the ratios of distances of $f(z), f(w), f(u), f(v)$ from the boundary are bounded by constants. It follows again from the distortion theorems (3.2) that also the ratios of the different $f'(\cdot)$ are bounded, giving us the claim. □

Remark 3.4.6. The proof can be done in many different ways. For example, for the right-hand side, i.e. the upper bound, one could also represent $\tilde{G}_D(x,y)$ using the Brownian motion and use Beurling type of estimates. The proof using Whitney decomposition seems to better fit the spirit of the rest of the chapter.

Controlling winding inside a CR-Whitney square

A priori, conditioned on a dyadic square to be a CR-Whitney square we have information on its winding only at the center of the square. This could be a problem, as we have no control on the covariance structure of the winding. However, from the geometric intuition of the winding number, it is clear that inside a CR-Whitney square the winding has to be bounded up to an additive constant. Although the definition of winding in our case is different (see discussion after the statement of theorem 2.2.3), this result also holds in our case. Again we state and prove it for more traditional Whitney squares, but use for CR-Whitney squares and although we cannot find a direct reference, this does follow by a standard argument:

Lemma 3.4.7. *Suppose Q is a Whitney square in the slit domain. Then the winding $w(z)$ satisfies $w(z) - c \leq w(z_0) \leq w(z) + c$, where z_0 is the centre of the square and $c > 0$ is some absolute constants.*

Proof. By distortion theorems, we control well the ratios of absolute values of f_T', we want to translate this control to that of imaginary part of f_T'. To do this, we use the Borel-Carathéodory theorem [56], which is an easy consequence of the Schwarz lemma or Poisson representation. In a slightly constrained form it states that for an analytic function $g(z)$ with $g(0) = 0$ we can control its modulus inside a closed disc of radius $r < R$ by the maximum of its real part on the circle of radius R. More explicitly, we have

$$|g(z)| \leq \frac{2r}{R-r} \sup_{z \in \partial B(0,R)} \Re g(z)$$

We apply this theorem

- $g(z) = \log f_T'(z) - \log f_T'(z_0)$, where f_T is the map from the SLE slit domain back to the upper half plane $\mathbb{H}$ and z_0 is the center of our Whitney square Q
- $r = \frac{l(Q)}{\sqrt{2}}$ and $R = l(Q)$ with $l(Q)$ as before the sidelength of Q

Firstly, as our domain in question is simply connected and $f_T'(z)$ is non-zero everywhere, it follows that $g(z)$ is analytic. Secondly, the whole square Q fits in the closed disc of radius r and the larger disc still fits into the domain as $d(z_0, \partial H_t) \geq \frac{3l(Q)}{2}$.

Next, we need to control the real part of $g(z)$. This real part is given by

$$\Re g(z) = \log \frac{|f_T'(z)|}{|f_T'(z_0)|}$$

Now it can be seen that the disc of radius R centred at z_0 is of bounded hyperbolic diameter that is independent of the sidelength of the square $l(Q)$ and the domain. Hence by conformal invariance of the hyperbolic distance, also the images $f_T(z)$ and $f_T(z_0)$ are only at bounded

hyperbolic distance. It follows from distortion theorems (3.2) that the ratio $\frac{|f_T'(z)|}{|f_T'(z_0)|}$ is bounded by an absolute constant. Thus the same holds for $\Re g(z)$.

Finally, the relative change in winding w.r.t z_0 is given exactly by the imaginary part of $g(z)$ and the lemma follows. □

Corollary 3.4.8. *The same holds for CR-Whitney squares with a slightly different constant.*

3.4.2 Proof of the theorem 3.4.1

Now we are set to prove the theorem 3.4.1. We start with the upper bound and follow the strategy outlined in the beginning of the section. In all sections we start by sampling an SLE_κ and then constructing the Liouville measure in the slit domain, using the coupling results between the GFF and SLE. We make a few remarks that simplify the further work and its write-up

1. We ignore at all phases the bounded harmonic correction term in the coupling, because it only gives a bounded multiplicative constant.

2. As we sample the SLE curve until it cuts the unit disc into two, we are left with two independent GFFs in both subdomains. However we can still consider the Whitney decomposition of the unit disc with the SLE curve, and all estimates for a single Whitney square depend only one one of these GFFs, hence we can also forget about this additional issue. due.

3. For $\kappa = 4$ one needs to forget about winding and everything will go through. For $\kappa > 4$ one needs to notice that χ changes sign and additionally take care of sampling GFF independently in every subdomain as explained in remarks after theorem 2.1.2. Otherwise everything is exactly the same - indeed, even for points cut-off from infinity by the curve, the winding is defined similarly in the coupling theorem 2.1.2 and the theorem on winding 2.2.3.

Upper bound

Upper bound for a CR-Whitney square

Consider a dyadic square Q of side-length $l(Q)$ and denote by $\mathcal{W}$ the collection of all CR-Whitney squares of the unit disc cut by the SLE curve. We will find an upper bound to

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right]$$

where informally $\tilde{\mu}(dz) \asymp \mu(dz)e^{-\gamma\chi w(z)}$ is the Liouville measure now weighted by the winding. This can be given concrete meaning using the circle-average regularization process as in section 4. As winding is harmonic inside the slit domain, then taking the regularization term $\delta_n \leq 0.01 l(Q)$, the circle-averages for winding give its value at the centre. Now, from the corollary to lemma 3.4.7 one can see that inside a CR-Whitney square, the winding is equal up to a constant. So setting z_0 to be the centre of Q we can write

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right] \asymp \mathbb{E}_{SLE}\left[e^{-\gamma\chi q w(z_0)}\mathbb{E}_{h|SLE}\left(\mu(Q)^q\right)|Q \in \mathcal{W}\right]$$

Now, with only minor modifications we can use Lemma 3.3.4, to upper bound the Liouville part without winding and get:

$$\mathbb{E}_{h|SLE}\left(\mu(Q)^q)\right) \leq l(Q)^{(2+\gamma^2/2)q}$$

So we are left with

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right] \lesssim l(Q)^{(2+\gamma^2/2)q}\mathbb{E}_{SLE}\left[e^{-\gamma\chi q w(z_0)}|Q \in \mathcal{W}\right]$$

But as Q has side-length $l(Q)$ and is conditioned to be a CR-Whitney square, we are exactly conditioning the conformal radius $\mathrm{CR}(z_0, SLE) \in [4l(Q), 12l(Q)]$. Hence using the theorem on winding 2.2.3, we have

$$\mathbb{E}_{SLE}\left[e^{-\gamma\chi q w(z_0)}|Q \in \mathcal{W}\right] \lesssim l(Q)^{-\gamma^2(1-\kappa/4)^2q^2/2}$$

Putting everything together, gives us

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right] \lesssim l(Q)^{(2+\gamma^2/2)q-\gamma^2(1-\kappa/4)^2q^2/2}$$

Upper bound for Liouville measure over all CR-Whitney squares

Next, let $\mathcal{W}_{\geq n}$ denote the collection of Whitney squares of side-length at most 2^{-n} we provide an upper bound for the sum

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\sum_{Q\in\mathcal{W}_{\geq n}}\tilde{\mu}(Q)^q\right) = \sum_{Q\in\mathcal{S}_{\geq n}}\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right]\mathbb{P}_{SLE}(Q \in \mathcal{W})$$

where the sum is over the collection $\mathcal{S}_{\geq n}$ of dyadic squares of side-length at most 2^{-n}. Now for Q to be a CR-Whitney square, we certainly need its center $z_0(Q)$ to satisfy $\mathrm{CR}(z_0) \leq 12l(Q)$. However, we know from [6] that the probability of this happening is bounded by $O(1)l(Q)^{1-\kappa/8}$ and so

$$\mathbb{P}_{SLE}(Q \in \mathcal{W}) \leq \mathbb{P}_{SLE}\left[\mathrm{CR}(z_0) \leq 12l(Q)\right] \lesssim l(Q)^{1-\kappa/8}$$

Hence, fixing some $n \in \mathbb{N}$ as the maximal size of the dyadic squares used, and combing this previous estimate with the previous one for CR-Whitney squares, we have that for any $1 > q > 0$, $\delta > 0$ with

$$(2+\gamma^2/2)q - \gamma^2(1-\kappa/4)^2q^2/2 = 1 + \kappa/8 + \delta$$

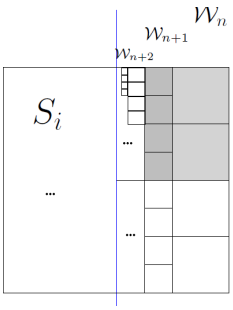
the following upper bound bound holds:

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\sum_{Q\in\mathcal{W}_{\geq n}}\tilde{\mu}(Q)^q\right) \lesssim \sum_{k\geq n}\sum_{l(Q)=2^{-k}}2^{2k}2^{-k(2+\delta)} = \frac{2^{-n\delta}}{1-2^{\delta}}$$

Notice that by making n large enough we can in fact make this sum arbitrarily small.

Almost sure upper bound for the covering

The final step of the proof is inspired by the (not yet published) book of Bishop & Peres [9], where they discuss the notion of dimension related to Whitney decompositions. Suppose we have a covering of the SLE by dyadic squares $S_i \in \mathcal{S}_n$ such that their side-length is 2^{-n}. The idea is to cover each dyadic squares by CR-Whitney squares and obtain an estimation this way for the dyadic covering. See the figure below, where we have illustrated the curve by a blue linear segment, a dyadic square and (usual) Whitney squares covering this dyadic square:



The problem is that with Whitney square we never touch the curve itself, so in order to proceed we need the following claim:

Claim 3.4.9. *For $\kappa < 8$ the Liouville measure of SLE_κ in forward coupling with the GFF is almost surely zero.*

Before proving the claim, let us show it implies the upper bound. Consider again the collection of dyadic CR-Whitney squares of side-length at most 2^{-n}, denoted by $\mathcal{W}_{\geq n}$ and a dyadic square $S_i \in \mathcal{S}_n$ intersecting the SLE curve. Recall that the CR-Whitney squares cover the whole slit domain, also notice that no CR-Whitney square intersecting S_i can be larger than S_i itself. Hence if the Liouville measure of the curve itself is almost surely zero, we a.s. have:

$$\tilde{\mu}(S_i) \leq \sum_{Q \in \mathcal{W}_i} \tilde{\mu}(Q)$$

where $\mathcal{W}_i$ denotes the collection of dyadic CR-Whitney squares intersecting the interior of S_i.

Write

$$\sum_{Q \in \mathcal{W}_i} \tilde{\mu}(Q)^q = \sum_{\mathcal{W}_i} \tilde{\mu}(Q)\tilde{\mu}(Q)^{q-1}$$

Then for $q < 1$, we have $\tilde{\mu}(Q)^{q-1} \geq \tilde{\mu}(S_i)^{q-1}$ and so

$$\sum_{Q \in \mathcal{W}_i} \tilde{\mu}(Q)^q \geq \tilde{\mu}(S_i)^q$$

Now as the collections of CR-Whitney squares $\mathcal{W}_i$ used to cover each dyadic square that intersects the SLE curve are disjoint, we have:

$$\sum_{S_i \in \mathcal{S}_n} \mathbb{1}(S_i \cap SLE \neq \emptyset)\tilde{\mu}(S_i)^q \leq \sum_i \sum_{Q \in \mathcal{W}_i} \tilde{\mu}(Q)^q \leq \sum_{Q \in \mathcal{W}_{\geq n}} \tilde{\mu}(Q)^q$$

We can put everything together in expectation to get:

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\sum_{S_i \in \mathcal{S}_n} \mathbb{1}(S_i \cap SLE \neq \emptyset)\tilde{\mu}(S_i)^q\right) \leq \mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\sum_{Q \in \mathcal{W}_{\geq n}} \tilde{\mu}(Q)^q\right)$$

Plugging in the estimate from the last section, we obtain:

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(M_q^Q(SLE, 2^{-n})\right) \lesssim \frac{2^{-n\delta}}{1-2^\delta}$$

and thus certainly

$$\limsup_{n\uparrow\infty} \mathbb{E}_{SLE,h}\left(M_q^Q(SLE, 2^{-n})\right) < \infty$$

Hence we see that $q_{M,E} < q$ for any q such that there is a $\delta > 0$ with

$$(2+\gamma^2/2)q - \gamma^2(1-\kappa/4)^2q^2/2 = 1+\kappa/8+\delta$$

Now we can just let $\delta \downarrow 0$ to obtain the claimed upper bound.

Proof of claim 3.4.9.
It only remains to prove that the Liouville measure for the SLE$_\kappa$ flow lines with $\kappa < 8$ is zero. We do it using a global "no loss of mass" argument. As this involves several changes of integrals and limits, we have to be careful at all steps.

Denote by D the unit disc. Pick $\delta \to 0$ along powers of two. Recall that the Liouville measure is defined as the limit of the $\delta-$ regularized measures, see theorem 3.1.2. Thus we have that

$$\mathbb{E}\mu(D) = \mathbb{E}\lim_{\delta\to 0}\mu_\delta(D)$$

Now, from the proof of proposition 1.2 in [19] (or indeed, by a small calculation) it follows that the limit can be taken outside the expectation:

$$\mathbb{E}\mu(D) = \lim_{\delta\to 0}\mathbb{E}(\mu_\delta(D)$$

Hence we can write

$$\begin{aligned}\mathbb{E}\mu(D) &= \lim_{\delta\to 0}\mathbb{E}\int_D \mu_\delta(z)dz \\ &= \lim_{\delta\to 0}\int_D \mathbb{E}\mu_\delta(z)dz \\ &= \int_D \lim_{\delta\to 0}\mathbb{E}\mu_\delta(z)dz\end{aligned}$$

Here, the second equality follows from Fubini and the third from dominated convergence. Now fix m large and write A_m for the event that the flow line avoids the δ^m ball around z, i.e. set $A_m = \{SLE \cap B_{\delta^m}(z) = \emptyset\}$. Then we can continue by writing

$$\begin{aligned}\mathbb{E}\mu(D) &= \int_D \lim_{\delta\to 0}\left(\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)\mathbb{1}(A_m)) + \mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)\mathbb{1}(A_m^c))\right)dz \\ &= \int_D \left(\lim_{\delta\to 0}\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)\mathbb{1}(A_m)) + \lim_{\delta\to 0}\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)\mathbb{1}(A_m^c))\right)dz\end{aligned}$$

By boundedness and positivity writing the limit of a sum as sum of limits is fine. We bound the second term using Cauchy-Schwarz:

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\mu_\delta(z)\mathbb{1}(A_m^c)\right) \leq \left(\mathbb{E}\mu_\delta(z)^2\right)^{1/2}\mathbb{P}(A_m^c)^{1/2}$$

But we know that $\mathbb{P}(A_m^c) \asymp \delta^{m(1-\kappa/8)}$. By plugging in $\mu_\delta(z) = \delta^{\gamma^2/2} e^{h_\delta(z)}$ and using the exponential moments of Gaussians, we see that the first term is bounded by $\delta^{-\gamma^2/2}$. Thus the whole term is of order $O(\delta^{-\gamma^2/2+m/2(1-\kappa/8)})$ and by picking m large enough, we can force it to be $o(\delta)$. But then

$$\begin{aligned}\mathbb{E}\mu(D) &= \int_D \left(\lim_{\delta\to 0} \mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)|A_m)\mathbb{P}(A_m) + \lim_{\delta\to 0} \mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)|A_m^c)\mathbb{P}(A_m^c) \right) dz \\ &= o(\delta) + \int_D \lim_{\delta\to 0} \mathbb{E}_{SLE}\mathbb{E}_{h|SLE}(\mu_\delta(z)|A_m)\mathbb{P}(A_m) dz\end{aligned}$$

Here we have also integrated the error term over the domain that has bounded mass.

Now notice that in the second term of the final expression, we never consider the mass on the curve itself. Yet there is no loss of total mass. Thus, in expectation, the mass on the curve is zero. Finally, the mass is clearly non-negative and hence it must be almost surely zero. □

Remark 3.4.10. In fact this is the claim where really the fractal geometry of the SLE, the coupling of GFF & SLE and the construction of Liouville measure are all mixed together.

Lower bound

The strategy is very similar, though small changes are needed at every step:

Lower bound for a CR-Whitney square

Again, to start off consider a dyadic square Q of side-length $l(Q)$ and denote by $\mathcal{W}$ the collection of CR-Whitney squares of the unit disc cut by the SLE curve. We aim to provide a lower bound to

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right]$$

where as before $\tilde{\mu} \asymp \mu(z)e^{-\gamma\chi w(z)}$ is informally the Liouville measure weighed by the winding. From lemma 3.4.7 we see that $w(z) \leq w(z_0) + C'$, where z_0 is the centre of Q. So we can write

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right] \asymp \mathbb{E}_{SLE}\left[e^{-\gamma\chi q w(z_0)}\mathbb{E}_{h|SLE}\left(\mu(Q)^q\right)|Q \in \mathcal{W}\right]$$

We need to lower bound $\mathbb{E}_{h|SLE}\left(\mu(Q)^q\right)$ and this can be done using Kahane convexity inequality [27, 47], that reduces comparing the moments of balls in multiplicative chaos measures to a comparison of covariance kernels. For the convenience of the reader we restate the lemma from [27] in a slightly more convenient manner:

Lemma 3.4.11 (Convexity inequality of GMC)**.** *Let $G_1(x,y) \leq G_2(x,y)$ be two covariance kernels and let μ_1 and μ_2 be the two associated Gaussian multiplicative chaos measures. Let F be a convex function on $\mathbb{R}_+$ and K some compact subset. Then $\mathbb{E}F(\mu_1(K)) \leq \mathbb{E}F(\mu_2(K))$.*

Remark 3.4.12. A priori this is proved in [47] only as long as the multiplicative chaos measure has a density w.r.t the Lebesgue measure. However, it follows from that that as long as two general multiplicative chaos measures can be constructed using the same approximations, the lemma also holds form those measures.

To apply this convexity inequality directly, we need to change the regularization of the Liouville measure to use the exact variance, as used in the literature on the multiplicative chaos. Start by picking $\delta_n = 2^{-n}$ to get the regularization sequence for the construction of Liouville measure in theorem 3.1.2. We have for $\delta_n < 0.01 l(Q)$,

$$\mu_h(Q) = \lim_{\delta_n \downarrow 0} \mu_{h^{\delta_n}}(Q) = \lim_{\delta_n \downarrow 0} \int_Q \delta_n^{\gamma^2/2} e^{\gamma h_{\delta_n}(z)} dz$$

where $h_{\delta_n}(z)$ is a Gaussian field with the kernel

$$G_{\delta_n}(x,y) = \log \frac{1}{\delta_n \vee |x-y|} + \tilde{G}(x,y)$$

Notice that as in the whole square we are at distance at least say $10\delta_n$ from the boundary, we indeed have inside our square $\tilde{G}(x,y) = \tilde{G}_{\delta_n}(x,y)$ where the former is the harmonic correction corresponding to the usual Green's function of the domain, and the latter is the harmonic correction corresponding to regularized Green's function.

Thus $\mu_h(Q)$ can be rewritten in terms of Gaussian multiplicative chaos as

$$\mu_h(Q) \asymp l(Q)^{\gamma^2/2} \lim_{\delta_n \downarrow 0} \int_Q e^{\gamma h_{\delta_n}(z) - \gamma^2/2 \mathbb{E}(h_{\delta_n}(z)^2)} dz \tag{3.3}$$

We now consider two Gaussian fields h_1, h_2, with the covariance kernels respectively denoted by $G_1(x,y)$ and $G_2(x,y)$ and given as follows:

$$G_1(x,y) = G(x,y) + \log \frac{1}{l(Q)} + C$$

for some constant C and the usual Green's function G. Now, we take the constant $C = C_2$ from lemma 3.4.4. Thus when we define

$$G_2(x,y) = \log \frac{1}{|x-y|}$$

we have that $G_2 \geq G_1$. Moreover, we can consider only sufficiently small Whitney squares such that $\log \frac{1}{l(Q)} + C$ is positive and hence h_1 can be written as a sum of the Gaussian free field and an independent Gaussian Y of variance $\log \frac{1}{l(Q)} + C$. Now, by [27, 46] we know that the multiplicative chaos measures for these fields are nicely defined and we will denote them by just "$e^{h_1(z) - \mathbb{E}(h_1(z)^2)}$" etc. Hence as $q < 1$ and thus $x \to x^q$ is concave, we have by Kahane convexity inequality cited above [27, 47]:

$$\mathbb{E}\left(\int_Q e^{\gamma h_1(z) - \gamma^2/2\mathbb{E}(h_1(z)^2)}\right)^q \geq \mathbb{E}\left(\int_Q e^{\gamma h_2(z) - \gamma^2/2\mathbb{E}(h_2(z)^2)}\right)^q$$

Using the fact that $h_1 = h + Y$, that Y is an independent Gaussian and that h, h_2 satisfy the scaling relation 3.2.2 [47], we have

$$\mathbb{E}\left(\int_Q e^{\gamma h(z) - \gamma^2/2\mathbb{E}(h(z)^2)}\right)^q \geq l(Q)^{2q}$$

Finally as $G_{\delta_n}(x,y) \leq G(x,y)$ we can translate this back to the regularized field to get:

$$\mathbb{E}\left(\int_Q e^{\gamma h_{\delta_n}(z)-\gamma^2/2\mathbb{E}(h_{\delta_n}(z)^2)}dz\right)^q \geq l(Q)^{2q}$$

and thus $\mu_h(Q) \gtrsim l(Q)^{(2+\gamma^2/2)q}$ So taking the expectation w.r.t. SLE, we have

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right] \gtrsim l(Q)^{(2+\gamma^2/2)q}\mathbb{E}_{SLE}\left[e^{-\gamma\chi q w(z_0)}|Q \in \mathcal{W}\right]$$

But as Q has side-length $l(Q)$ and is conditioned to be a CR-Whitney square, we are conditioning on

$$\mathrm{CR}(z_0, SLE) \in [4l(Q), 12l(Q)]$$

Hence using the theorem 2.2.3, we have

$$\mathbb{E}_{SLE}\left[e^{-\gamma\chi q w(z_0)}|Q \in \mathcal{W}\right] \gtrsim l(Q)^{-\gamma^2(1-\kappa/4)^2q^2/2}$$

Putting everything together, gives us

$$\mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}\right] \gtrsim l(Q)^{(2+\gamma^2/2)q-\gamma^2(1-\kappa/4)^2q^2/2}$$

Lower bound for Liouville measure over level-n CR-Whitney squares

This time we do not aim to bound the whole CR-Whitney decomposition, but are happy with analysing the collection of level-n CR-Whitney squares $\mathcal{W}_n$. Moreover, we relax the definition of CR-Whitney square and call every dyadic square satisfying $4l(Q) \leq \mathrm{CR}(z_0) \leq 150l(Q)$ a CR-Whitney square, where as before z_0 is the centre of Q. The reason will become clear when we aim for the lower bound of the dyadic covering.

Write as earlier

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\sum_{\mathcal{W}_n}\tilde{\mu}(Q)^q\right) = \sum_Q \mathbb{E}_{SLE}\left[\mathbb{E}_{h|SLE}\left(\tilde{\mu}(Q)^q\right)|Q \in \mathcal{W}_n\right]\mathbb{P}_{SLE}(Q \in \mathcal{W}_n)$$

and pick $1 > q > 0$, $\delta > 0$ with

$$(2+\gamma^2/2)q - \gamma^2(1-\kappa/4)^2q^2/2 = 1 + \kappa/8 - \delta$$

Now the probability of being a CR-Whitney square can be exactly calculated using the SLE Green's function [31], and is still up to some multiplicative constant of order $l(Q)^{1-\kappa/8}$. Thus using this probability and the estimate on the CR-Whitney square itself we finally get

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE}\left(\sum_{\mathcal{W}_n}\tilde{\mu}(Q)^q\right) \gtrsim 2^{2n}2^{-n(2-\delta)} \geq 2^{n\delta}$$

which is arbitrarily large for n large.

Lower bound for the covering

To make the final step from the lower bound on CR-Whitney squares to a lower bound on the covering, our idea is to locate at least one CR-Whitney square inside each dyadic square in the covering of the SLE. At first sight this might seem hard, because we would also need to handle the case when SLE almost fills the square. However, due to estimates of the SLE Green's function, it costs us nothing to require the SLE curve to leave some open space around the centre of the square, just enough to fill in some CR-Whitney squares.

To be more precise, notice first that in order for a dyadic square S of side-length $l(S) = 2^{-n}$ to intersect the SLE curve, it suffices that the centre of this square has conformal radius less than $l(S)/2$. On the other hand we can also require the conformal radius to be more than $l(S)/3$ without changing the order of magnitude of our event [31].

Then a small geometrical calculation shows that all four dyadic squares of side-length $l(S)2^{-6}$ neighbouring the centre of square S will necessarily be CR-Whitney squares. This is of course also the reason for relaxing the definition of CR-squares in the previous section.

The rest now follows easily. Indeed, cut S_i first into four dyadic square $Q'_{i,j}$ with $j = 1, 2, 3, 4$ of sidelength $l(S_i)2^{-1}$. Then from Jensen applied to the concave function x^q:

$$\sum_{S_i \in \mathcal{D}_n} \mathbb{1}(SLE \cap S_i)\tilde{\mu}(S_i)^q \gtrsim \sum_{S_i \in \mathcal{D}_n} \mathbb{1}(SLE \cap S_i) \sum_{j=1,2,3,4} \tilde{\mu}(Q'_{i,j})^q$$

Now denote by $Q_{i,j}$ the corresponding dyadic squares of sidelength $l(S_i)2^{-1}$ that have the centre of S_i as one corner. Thus

$$\sum_{j=1,2,3,4} \tilde{\mu}(Q'_{i,j})^q \geq \sum_{j=1,2,3,4} \tilde{\mu}(Q_{i,j})^q$$

But we saw above $\{SLE \cap S_i\} \supset \cup_{j=1,2,3,4}\{Q_{i,j} \in \mathcal{W}_{n+6}\}$ and so

$$\mathbb{1}(SLE \cap S_i) \geq 1/4 \sum_{j=1,2,3,4} \mathbb{1}(Q_{i,j} \in \mathcal{W}_{n+6})$$

Thus we can further lower bound the RHS by a sum over the CR-Whitney squares on level $n+6$ that are around the centre of a level n dyadic square. When we denote this specific collection by $\mathcal{W}'_{n+6}$, we have:

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE} \sum_{S_i \in \mathcal{D}_n} \mathbb{1}(SLE \cap S_i)\tilde{\mu}(S_i)^q \gtrsim \mathbb{E}_{SLE}\mathbb{E}_{h|SLE} \left(\sum_{\mathcal{W}'_{n+6}} \tilde{\mu}(Q)^q \right)$$

Now, $\mathcal{W}'_{n+6}$ forms a constant proportion of all CR-Whitney squares of size $n+6$, and thus we can use the previous estimate on the sum of n-th level Whitney squares. Thus we get that for n large enough

$$\mathbb{E}_{SLE}\mathbb{E}_{h|SLE} M_q^Q(SLE, 2^{-n}) \gtrsim 2^{n\delta}$$

From this it follows that $q < q_{M,E}$ for any q such that there is a $\delta > 0$ with

$$(2 + \gamma^2/2)q - \gamma^2(1 - \kappa/4)^2 q^2/2 = 1 + \kappa/8 - \delta$$

The lower bound for the expepcted quantum Minkowski dimension follows by taking $\delta \downarrow 0$. This also finishes the proof of the theorem 3.4.1.

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Résumé en français

Cette thèse porte sur la géométrie du champ libre Gaussien. Le champ libre Gaussien est un objet central en théorie quantique des champs et représente entre autre les fluctuations naturelles d'un potentiel électrique ou d'un modèle de dimères.

La thèse commence dans le discret avec la démonstration d'un principe de Donsker en dimension plus grande que 1. Ce résultat est établi grâce à une nouvelle façon de représenter le champ libre en exprimant son gradient comme la partie gradient d'un champ de bruits blancs.

Ensuite, les processus d'exploration du champ libre - ou ensembles locaux - introduits par Schramm-Sheffield sont étudiés en détail. Ces ensembles locaux généralisent de façon naturelle le concept de temps d'arrêt. On formalise cette théorie d'une nouvelle manière en procédant par analogie au cas 1D. Pour mieux comprende le comportement du champs libre près des points d'intersection des ensembes locaux, un étude fine des oscillations du champ libre 2D près du bord s'avère utile.

Enfin, la partie principale de cette thèse étudie des processus d'explorations particuliers – les processus SLE qui sont couplés naturellement avec le champ libre. On peut donner par exemple un sens aux lignes de niveau en utilisant le processus SLE_4 (Schramm-Sheffield).

Nous avons utilisé ce couplage pour mieux comprendre la relation dite de KPZ qui intervient dans la théorie de la gravité quantique de Liouville. A l 'aide de résultats fins sur l'enroulement des SLEs, nous avons montré comment adapter la relation de KPZ à la famille ci-dessus de processus d'explorations du champ libre. On peut interpéter ces résultats aussi comme une déscription de la géometrie du champs libre près des ces lignes d'exploration.