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Université Pierre et Marie Curie

Ecole Doctorale
Institut de Mathématiques de Jussieu

A study of skeleta in non-Archimedean geometry

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Thèse de doctorat de Mathématiques

Dirigée par Prof. François LOESER

Soutenue le 30 juin 2015 devant le jury composé de :

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Abstract

This thesis is a reflection of the interaction between Berkovich geometry and model theory. Using the results of Hrushovski and Loeser [HL], we show that several interesting topological phenomena that concern the analytifications of varieties are governed by certain finite simplicial complexes embedded in them. Our work consists of the following two sets of results.

Let k be an algebraically closed non-Archimedean non trivially real valued field which is complete with respect to its valuation.

1. Let $\phi : C' \to C$ be a finite morphism between smooth projective irreducible k-curves. The morphism ϕ induces a morphism $\phi^{\mathrm{an}} : C'^{\mathrm{an}} \to C^{\mathrm{an}}$ between the Berkovich analytifications of the curves. We construct a pair of deformation retractions of C'^{an} and C^{an} which are compatible with the morphism ϕ^{an} and whose images $\Upsilon_{C'^{\mathrm{an}}}$, $\Upsilon_{C^{\mathrm{an}}}$ are closed subspaces of C'^{an}, C^{an} that are homeomorphic to finite metric graphs. We refer to such closed subspaces as skeleta. In addition, the subspaces $\Upsilon_{C'^{\mathrm{an}}}$ and $\Upsilon_{C^{\mathrm{an}}}$ are such that their complements in their respective analytifications decompose into the disjoint union of isomorphic copies of Berkovich open balls. The skeleta can be seen as the union of vertices and edges, thus allowing us to define their genus. The genus of a skeleton in a curve C is in fact an invariant of the curve which we call $g^{\mathrm{an}}(C)$. The pair of compatible deformation retractions forces the morphism ϕ^{an} to restrict to a map $\Upsilon_{C'^{\mathrm{an}}} \to \Upsilon_{C^{\mathrm{an}}}$. We study how the genus of $\Upsilon_{C'^{\mathrm{an}}}$ can be calculated using the morphism $\phi^{\mathrm{an}}_{|\Upsilon_{C'^{\mathrm{an}}}}$ and invariants defined on $\Upsilon_{C^{\mathrm{an}}}$.

2. Let ϕ be a finite endomorphism of $\mathbb{P}^1_k$. Given a closed point $x \in \mathbb{P}^1_k$, we are interested in the radius $f(x)$ of the largest Berkovich open ball centered at x over which the morphism ϕ^{an} is a topological fibration. Interestingly, the function $f : \mathbb{P}^1_k(k) \to \mathbb{R}_{\geq 0}$ admits a strong tameness property in that it is controlled by a non-empty finite graph contained in $\mathbb{P}^{1,\mathrm{an}}_k$. We show that this result can be generalized to the case of finite morphisms $\phi : V' \to V$ between integral projective k-varieties where V is normal.

Abstract

Cette thèse s'appuie sur et reflète l'interaction entre la théorie des modèles et la géométrie de Berkovich. En utilisant les méthodes de Hrushovski et Loeser [HL], nous montrerons que plusieurs phénomènes topologiques concernant des analytifications de variétés sont contrôlés par certains complexes simpliciaux contenus dans les analytifications. Ce travail comporte les résultats suivants.

Soit k un corps algébriquement clos et complet pour une valuation non-archimédienne non-triviale à valeurs réelles.

1. Soit $\phi : C' \to C$ un morphisme fini entre deux courbes projectives, lisses et irréductibles. Le morphisme ϕ induit un morphisme $\phi^{\mathrm{an}} : C'^{\mathrm{an}} \to C^{\mathrm{an}}$ entre les deux analytifications. Nous construisons une paire de rétractions par déformations qui sont compatible pour le morphisme ϕ^{an}. Les images des déformations $\Upsilon_{C'^{\mathrm{an}}}$, $\Upsilon_{C^{\mathrm{an}}}$ sont des sous-espaces fermés de C'^{an} and C^{an} et homéomorphes à des graphes finis. Ce type de sous-espace est appelé *squelette*. En outre, les espaces analytiques $C'^{\mathrm{an}} \smallsetminus \Upsilon_{C'^{\mathrm{an}}}$ et $C^{\mathrm{an}} \smallsetminus \Upsilon_{C^{\mathrm{an}}}$ se décomposent en une union disjointe de copies de disques unités de Berkovich. Un squelette $\Upsilon \subset C^{\mathrm{an}}$ peut-être décomposé en un ensemble des sommets et un ensemble d'arêtes et on peut définir son genre $g(\Upsilon)$. Nous montrons que $g(\Upsilon)$ est un invariant bien défini de la courbe C. On appelle cet invariant $g^{\mathrm{an}}(C)$. Le morphisme ϕ^{an} induira un morphisme $\Upsilon_{C'^{\mathrm{an}}} \to \Upsilon_{C^{\mathrm{an}}}$ entre les deux squelettes. Nous montrons que le genre du squelette $\Upsilon_{C'^{\mathrm{an}}}$ peut être calculé en utilisant certains invariants associés aux points de $\Upsilon_{C^{\mathrm{an}}}$.

2. Soit ϕ un endomorphisme fini de $\mathbb{P}^1_k$. Soit $x \in \mathbb{P}^1_k(k)$ et $f(x)$ le rayon de la plus grande boule de Berkovich de centre x, sur laquelle le morphisme ϕ^{an} est une fibration topologique. Nous voyons que la fonction $f : \mathbb{P}^1_k(k) \to \mathbb{R}_{\geq 0}$ est contrôlée par un graphe fini et non-vide contenu dans $\mathbb{P}^{1,\mathrm{an}}_k$. Nous montrons que ce résultat peut être généralisé au cas d'un morphisme fini $\phi : V' \to V$ entre deux variétés intégrales, projectives avec V normale.

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These years in Paris during which I made my way fumbling, through the arcane, esoteric world of pure mathematics were always dotted by happy coincidence - itinerant post-docs - Yimu, Michel and Ethan setting up shop down the corridor or a long list of fellow PhD students in neighbouring offices with whom I learnt to speak in french and in whose company, the taste of cold beer after a days work never tasted as sweet. My deepest gratitude to Arthur with whom I shared an office for more than a year and from whom I learnt a great deal of interesting mathematics. I must also mention Hsueh-Yung and Valentin who have always been available for coffee or the quick discussion of a problem I found myself stuck on. And then there is Ildar Gaisin whose presence greatly alleviated the sobering effects that often accompanies the writing of a PhD thesis. It is the afternoon breaks and the relaxed discussions of mathematics amidst our frequent coffees that sustained me through several difficult passages during the last two years. Despite long since having fled these shores, Giovanni Rosso continues to offer his services at a moments notice and though his physical presence and famous risotto are things of the past, his friendship and support are forever present.

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Chapter 1

Introduction

Non-Archimedean fields were discovered at the turn of the twentieth century when K. Hensel defined the field of p-adic numbers $\mathbb{Q}_p$. Ever since, there have been attempts, each with its merits, to develop a theory of geometry over such fields analogous to the theory of complex geometry. However, it was only in the early nineties that Vladmir Berkovich developed a theory of non-Archimedean geometry which provided analytic spaces endowed with reasonable topological properties. In the framework of this geometry, a variety defined over a non-Archimedean field defines an associated Berkovich analytic space called its analytification. Even though such varieties when endowed with the topology induced by the valuation of the ground field are totally disconnected, their analytifications are Hausdorff, locally compact and have a finite number of path connected components. It is hence natural to investigate the homotopy type of the analytification of such algebraic varieties.

In 2010, Hrushovski and Loeser using techniques from Model theory studied the homotopy type of the analytification of an algebraic variety defined over a non trivially valued non-Archimedean real valued field. They showed that these homotopy types are determined completely by finite simplicial complexes embedded in the analytification by constructing deformation retractions of the analytifications onto such complexes. In [HL], Hrushovski and Loeser define a model theoretic analogue of the Berkovich analytification of a variety. One of the advantages of this viewpoint is that it provides a framework within which we can discuss model theoretic notions such as definability and employ powerful methods such as compactness. This thesis is a reflection of this interaction between Berkovich geometry and model theory. We show that several interesting topological phenomena that concern the analytifications of varieties are governed by certain finite simplicial complexes embedded in them.

1.1 A Riemann-Hurwitz formula for the analytic genus

Let k be an algebraically closed, complete non-Archimedean non trivially real valued field. Let C be a k-curve. By k-curve, we mean a one dimensional connected reduced separated scheme of finite type over the field k. It is well

known that there exists a deformation retraction of C^{an} onto a closed subspace Υ which is homeomorphic to a finite metric graph [[B], Chapter 4], [[HL], Section 7]. We call such subspaces *skeleta*. The skeleton Υ can be decomposed into a set of vertices $V(\Upsilon)$ and a set of edges $E(\Upsilon)$. We define the genus of the skeleton Υ as follows.

$$g(\Upsilon) = 1 - V(\Upsilon) + E(\Upsilon).$$

In Proposition 7.1.24, we show that $g(\Upsilon)$ is a well defined invariant of the curve and does not depend on the retract Υ. Let $g^{\mathrm{an}}(C) := g(\Upsilon)$ for any such Υ. We study how g^{an} varies for a finite morphism using a compatible pair of deformation retractions.

Let C', C be smooth projective irreducible k-curves and $\phi : C' \to C$ be a finite morphism. The morphism ϕ induces a morphism between the respective analytifications which we denote ϕ^{an}. Hence we have

$$\phi^{\mathrm{an}} : C'^{\mathrm{an}} \to C^{\mathrm{an}}.$$

In Theorem 7.2.1, we prove that there exists a pair of *compatible* deformation retractions. The exact statement is as follows.

Theorem 7.2.1 *Let C and C' be smooth projective irreducible k-curves and $\phi : C' \to C$ be a finite morphism. There exists a pair of deformation retractions*

$$\psi : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$$

and

$$\psi' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$$

with the following properties.

1. *The images $\Upsilon_{C'^{\mathrm{an}}} := \psi'(1, C'^{\mathrm{an}})$ and $\Upsilon_{C^{\mathrm{an}}} := \psi(1, C^{\mathrm{an}})$ are closed subspaces of C'^{an} and C^{an} respectively, each with the structure of a connected, finite metric graph. Furthermore, we have that $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$.*

2. *The analytic spaces $C'^{\mathrm{an}} \smallsetminus \Upsilon_{C'^{\mathrm{an}}}$ and $C^{\mathrm{an}} \smallsetminus \Upsilon_{C^{\mathrm{an}}}$ decompose into the disjoint union of isomorphic copies of Berkovich open disks i.e. there exist weak semistable vertex sets (cf. Definition 7.1.18) $\mathfrak{A} \subset C^{\mathrm{an}}$ and $\mathfrak{A}' \subset C'^{\mathrm{an}}$ such that $\Upsilon_{C^{\mathrm{an}}} = \Sigma(C^{\mathrm{an}}, \mathfrak{A})$ and $\Upsilon_{C'^{\mathrm{an}}} = \Sigma(C'^{\mathrm{an}}, \mathfrak{A}')$.*

3. *The deformation retractions ψ and ψ' are* ***compatible*** *i.e. the following diagram is commutative.*

$$\begin{tikzcd}
{[0,1] \times C'^{\mathrm{an}}} \arrow[r, "\psi'"'] \arrow[d, "id \times \phi^{\mathrm{an}}"] & C'^{\mathrm{an}} \arrow[d, "\phi^{\mathrm{an}}."] \\
{[0,1] \times C^{\mathrm{an}}} \arrow[r, "\psi"'] & C^{\mathrm{an}}
\end{tikzcd}$$

In Sections 7.3 and 7.4, we study how $g^{\mathrm{an}}(C')$ and $g^{\mathrm{an}}(C)$ relate to each other under the added assumption that $\phi : C' \to C$ is a finite morphism between smooth projective irreducible curves. The necessary notation to make sense of the following result - Corollary 7.3.9 can be found in Section 7.3.1 and Definitions 7.3.6 and 7.3.8.

Corollary 7.3.9 *Let $\phi : C' \to C$ be a finite separable morphism between smooth projective irreducible curves over the field k. Let $g^{\mathrm{an}}(C'), g^{\mathrm{an}}(C)$ be as in Definition 7.1.25. We have the following equation.*

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)(2g^{\mathrm{an}}(C) - 2) + \Sigma_{p \in C^{\mathrm{an}}} 2i(p)g_p + R - \Sigma_{p \in C^{\mathrm{an}}} R^1_p.$$

In Section 7.4, we present another method to calculate the invariant $g^{\mathrm{an}}(C')$ using the existence of a pair of compatible deformation retractions ψ and ψ' on C^{an} and C'^{an} whose images are skeleta $\Upsilon_{C^{\mathrm{an}}}$ and $\Upsilon_{C'^{\mathrm{an}}}$. We assume in addition that the morphism $\phi : C' \to C$ is such that the induced extension of function fields $k(C) \hookrightarrow k(C')$ is Galois. By construction of ψ' and ψ, ϕ^{an} restricts to a morphism between the two skeleta. We show that the genus of the skeleton $\Upsilon_{C'^{\mathrm{an}}}$ can be calculated using invariants associated to the points of $\Upsilon_{C^{\mathrm{an}}}$. In order to do so we define a divisor w on $\Upsilon_{C^{\mathrm{an}}}$ whose degree is $2g(\Upsilon_{C'^{\mathrm{an}}}) - 2$. A divisor on a finite metric graph is an element of the free abelian group generated by the points on the graph.

We define w as follows. For a point $p \in \Upsilon_{C^{\mathrm{an}}}$, let $w(p)$ denote the order of the divisor at p. We set

$$w(p) := (\sum_{e_p \in E_p, p' \in (\phi^{\mathrm{an}})^{-1}(p)} l(e_p, p')) - 2n_p.$$

The terms in this expression are defined as follows. Let T_p denote the tangent space at the point p (cf. 7.1.3, 7.1.6).

1. Let $E_p \subset T_p$ be those elements for which there exists a representative starting from p and contained completely in $\Upsilon_{C^{\mathrm{an}}}$.

2. Let $p' \in C'^{\mathrm{an}}$ such that $\phi^{\mathrm{an}}(p') = p$. The morphism ϕ^{an} induces a map $d\phi_{p'}$ between the tangent spaces $T_{p'}$ and T_p (cf. 7.1.3, 7.1.6). Let $e_p \in E_p$. We define $L(e_p, p') \subset T_{p'}$ to be the preimages of e_p for the map $d\phi_{p'}$. As $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$, any element of $L(e_p, p')$ can be represented by a geodesic segment that is contained completely in $\Upsilon_{C'^{\mathrm{an}}}$. Let $l(e_p, p')$ denote the cardinality of the set $L(e_p, p')$.

3. We define n_p to be the cardinality of the set of preimages of the point p i.e. $n_p := \mathrm{card}\{(\phi^{\mathrm{an}})^{-1}(p)\}$.

In Proposition 7.4.4, we show that w is indeed a well defined divisor whose degree is equal to $2g(\Upsilon_{C'^{\mathrm{an}}}) - 2$. We then study the values n_p and $l(e_p, p')$ described above. These results are sketched below.

We study the value n_p for $p \in \Upsilon_{C^{\mathrm{an}}}$ in terms of two invariants - $\mathrm{ram}(p)$ and $c_1(p)$ which are defined as follows.

Let $p \in \Upsilon_{C^{\mathrm{an}}}$.

1. If p is a point of type I then we set $\mathrm{ram}(p)$ to be the ramification degree $\mathrm{ram}(p'/p)$ for any $p' \in C'^{\mathrm{an}}$ such that $\phi^{\mathrm{an}}(p') = p$. As the morphism ϕ is Galois, $\mathrm{ram}(p)$ is well defined. If p is not of type I then we set $\mathrm{ram}(p) := 1$.

2. In order to define the invariant c_1, we introduce an equivalence relation on $C'(k)$. For $y_1, y_2 \in C'(k)$, we set $y_1 \sim_{c(1)} y_2$ if $\phi(y_1) = \phi(y_2)$ and $\psi'(1, y_1) = \psi'(1, y_2)$. Let $c_1(y)$ denote the cardinality of the equivalence class that contains y. In Lemma 7.4.8, we show that the function $c_1 : C(k) \to \mathbb{Z}_{\geq 0}$ defined by setting $c_1(x) = c_1(y)$ for any $y \in \phi^{-1}(x)$ is well defined. We proceed to show that if $x \in C(k)$ then $c_1(x)$ depends only on the point $\psi(1, x) \in \Upsilon_{C^{\mathrm{an}}}$. This defines $c_1 : \Upsilon_{C^{\mathrm{an}}} \to \mathbb{Z}_{\geq 0}$.

The values $c_1(p)$ and $\mathrm{ram}(p)$ can be used to calculate n_p by the following relation (Proposition 7.4.10).

$$n_p = [k(C') : k(C)]/(c_1(p)\mathrm{ram}(p)).$$

We simplify the term $l(e_p, p')$ which appears in the expression defining w. Let $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$. In Lemma 7.4.12 we show that $l(e_p, p')$ remains constant as p' varies through the set of preimages $p' \in (\phi^{\mathrm{an}})^{-1}(p)$. We set $l(e_p) := l(e_p, p')$. We introduce the invariants - $\widetilde{\mathrm{ram}}(e_p)$ and $\widetilde{\mathrm{ram}}(p)$ to study $l(e_p)$.

1. Let $p \in \Upsilon_{C^{\mathrm{an}}}$. By definition e_p is an element of the tangent space T_p at p (cf. Sections 7.1.3, 7.1.6). As p is of type II, it corresponds to a discrete valuation of the $\tilde{k}$-function field $\widetilde{\mathcal{H}(p)}$. For any $p' \in (\phi^{\mathrm{an}})^{-1}(p)$, the extension of fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ can be decomposed into the composite of a purely inseparable extension and a Galois extension. Hence the ramification degree $\mathrm{ram}(e'/e_p)$ is constant as e' varies through the set of preimages of e_p at $T_{p'}$ for the map $d\phi^{alg}_{p'} : T_{p'} \to T_p$ (cf. 7.1.6). Let $\widetilde{\mathrm{ram}}(e_p)$ be this number. When p is of type I, we set $\widetilde{\mathrm{ram}}(e_p) = \mathrm{ram}(p)$ and when p is of type III, we set $\widetilde{\mathrm{ram}}(e_p) = c_1(p)$.

2. For $p \in \Upsilon_{C^{\mathrm{an}}}$, we define $\widetilde{\mathrm{ram}}(p) := \Sigma_{e_p \in E_p} 1/\widetilde{\mathrm{ram}}(e_p)$.

In Proposition 7.4.15, we show that if $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$ then

$$l(e_p) = [k(C') : k(C)]/(n_p \widetilde{\mathrm{ram}}(e_p)).$$

The results of Section 7.4 are compiled so that the value $2g^{\mathrm{an}}(C') - 2$ can be computed in terms of the invariants $c_1, \widetilde{\mathrm{ram}}$ and ram.

Theorem 7.4.17 *Let $\phi : C' \to C$ be a finite morphism between smooth projective irreducible k-curves such that the extension of function fields $k(C) \hookrightarrow k(C')$ induced by ϕ is Galois. Let $g^{\mathrm{an}}(C')$ be as in Definition 7.1.25. We have that*

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)\Sigma_{p \in \Upsilon_{C^{\mathrm{an}}}}[\widetilde{\mathrm{ram}}(p) - 2/(c_1(p)\mathrm{ram}(p))].$$

The results of chapter 7 form the content of the article "*A Riemann-Hurwitz formula for skeleta in non-Archimedean geometry*". Immediately following this work and related to it, were two papers ([ABBR1], [ABBR2]) by Amini, Baker, Brugallé and Rabinoff wherein the authors study the extent to which morphisms between algebraic k-curves are determined by skeleta. Amongst the striking results of these papers, is the study of obstructions to the lifting of a harmonic morphism between metric graphs to a corresponding morphism of k-curves such that the graphs can be realized as skeleta of these curves. Also, a similar proof of Theorem 7.2.1 can be found in [ABBR1] (cf. Theorem A).

Recently, in [TEM2] and [TEM3], Michael Temkin, Adina Cohen and Dmitri Trushin have obtained results on wild ramification for finite morphisms between quasi-smooth Berkovich curves which bear some resemblance to results considered in this paper. In [TEM3], Temkin considers a morphism $f : Y \to X$ between connected separated quasi-smooth strictly k-analytic curves. The curves Y and X possess a natural metric and the morphism f is piecewise monomial on an interval $I \subset Y$ with respect to this metric. The author is interested in the set $N_{f,\geq d} \subset Y$ of points $y \in Y$ such that the multiplicity $n_f(y)$ of f at y is at least d. The topologically tame case does not present a challenge as the set $N_{f,\geq 2}$ is contained in a finite graph and it is the topologically wild case which is of greater interest. In [TEM2], the authors study the set $N_{f,\geq p}$ by using the *different*. They show that the different function $\delta_f : Y \to [0,1]$ is piece-wise monomial, relates the genus of Y and X and in addition completely controls the set $N_{f,p}$ for morphisms of degree p. In [TEM3], Temkin goes on to prove that the set $N_{f,\geq d}$ is *radial* with respect to a large enough skeleton of f (a compatible choice of skeleta $\Upsilon' \subset Y$ and $\Upsilon \subset X$) and that their Υ'-radii are piece-wise $|k^*|$-monomial functions on Υ'. The author then relates the radii of $N_{f,\geq d}$ to classical ramification invariants.

1.2 Finite morphisms and skeleta

Our second set of results concerns finite surjective morphisms between irreducible projective varieties over non-Archimedean real valued fields. We study such morphisms in terms of the morphisms they induce between the analytifications of the varieties. The theorem we prove implies that the induced morphism when viewed as a continuous map between topological spaces admits a certain uniform behaviour. Before stating the theorem in full generality, we provide its motivation by considering the case of a finite endomorphism of the projective line.

Let k be an algebraically closed, complete non-Archimedean real valued field whose value group $|k^*|$ contains at least two elements and is a sub group of $(\mathbb{R}_{>0}, \times)$. Let $\mathbb{P}^{1,\mathrm{an}}_k$ be the Berkovich analytification of the projective line $\mathbb{P}^1_k$. The analytification $\mathbb{P}^{1,\mathrm{an}}_k$ allows us to use the valuative topology provided by the field to study an algebraic endomorphism. For a point $x \in \mathbb{P}^1_k(k) \subset \mathbb{P}^{1,\mathrm{an}}_k$, we have the notion of a Berkovich closed disk or Berkovich open ball centred at x within the space $\mathbb{P}^{1,\mathrm{an}}_k$ which contains the naive closed or open disk around x. By the naive closed (open) disk around $x \in k$ of radius $r \in \mathbb{R}_{>0}$, we mean the set $\{y \in k | |y-x| \leq r\}$ ($\{y \in k | |y-x| < r\}$). As opposed to their naive counterparts, the Berkovich open and closed disks are locally compact and contractible.

Let $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a finite morphism. For a complete non-Archimedean real valued algebraically closed field extension L/k, let ϕ_L denote the morphism

$$\phi \times \mathrm{id}_L : \mathbb{P}^1_k \times_k \mathrm{Spec}(L) \to \mathbb{P}^1_k \times_k \mathrm{Spec}(L).$$

The analytification of a k-variety of finite type (cf. Section 5.6) is functorial and hence endomorphisms of the projective line will induce endomorphisms of its analytification. That is, the morphism ϕ induces a morphism

$$\phi^{\mathrm{an}} : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k.$$

The morphism ϕ^{an}_L is similarly defined and it is to be noted that $\phi^{\mathrm{an}}_L = \phi^{\mathrm{an}} \times \mathrm{id}^{\mathrm{an}}_L$.

Our reason for introducing the morphism ϕ_L for a complete non-Archimedean real valued algebraically closed field extension L/k is to deal with all points of the analytification of the projective line over k and not just those points for which $\mathcal{H}(x) = k$ (cf. 5.6.2). When discussing the points of the analytification of a k-variety, we make use of the description outlined in Section 5.6. Let $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$ i.e. $x \in \mathbb{P}^{1,\mathrm{an}}$ and there exists an embedding of valued fields $\mathcal{H}(x) \hookrightarrow L$. The image of $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$ for the morphism $\pi : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^1_k$ (cf. 5.6) is an L-point of $\mathbb{P}^1_k$ i.e. an element of the set $\mathbb{P}^1_k(L)$. We abuse notation and refer to this point as x as well. The pair $x : \mathrm{Spec}(L) \to \mathbb{P}^1_k$ and $\mathrm{id}_L : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ defines a closed point of the variety $\mathbb{P}^1_k \times_k \mathrm{Spec}(L)$ which we denote x_L. The following remark generalizes this construction.

Remark 1.2.1. *Let V be a k-variety and let $x \in V^{\mathrm{an}}$. The field $\mathcal{H}(x)$ is the non-Archimedean geometry analogue of the residue field of a point in algebraic geometry and is defined in 5.6. Let L be a complete non-Archimedean real valued algebraically closed field extension of k. The notation $x \in V^{\mathrm{an}}(L)$ will be used to mean that $x \in V^{\mathrm{an}}$ and in addition there exists an embedding of valued fields $\mathcal{H}(x) \hookrightarrow L$. It follows that the image of the point x for the morphism $V^{\mathrm{an}} \to V$ which is defined in Section 5.6 is an L-point i.e. an element of the set $V(L)$. We abuse notation and refer to this point as x as well. The pair $x : \mathrm{Spec}(L) \to V$ and $\mathrm{id}_L : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ defines a closed point of the variety $V \times_k \mathrm{Spec}(L)$ which we denote x_L. This construction will be referred to frequently in what follows.*

We now introduce the theorem concerning finite endomorphisms of the projective line over k.

Theorem 1.2.2. *Let $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a finite morphism. Let $x \in \mathbb{P}^{1,\mathrm{an}}_k$ and L/k be any complete non-Archimedean real valued algebraically closed field extension of k such that $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$. Let $f(x)$ be the minimum of 1 and the radius of the largest Berkovich open ball $B \subset \mathbb{P}^{1,\mathrm{an}}_L$ around $x_L \in \mathbb{P}^1_L(L) \subset \mathbb{P}^{1,\mathrm{an}}_L$ whose preimage under ϕ^{an}_L is the disjoint union of homeomorphic copies of B via ϕ^{an}_L. The function $f : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{R}_{\geq 0}$ is not identically zero and well defined. There exists a finite simplicial complex $\Upsilon \subset \mathbb{P}^{1,\mathrm{an}}_k$, a generalised real interval $I := [i, e]$ and a deformation retraction*

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

such that $\psi(e, \mathbb{P}^{1,\mathrm{an}}_k) = \Upsilon$ and the function f is constant on the fibres of this retraction i.e. for every $x \in \mathbb{P}^{1,\mathrm{an}}_k$ we have that $f(x) = f(\psi(e, x))$. Furthermore, the function $log_c|f|$ is piecewise linear when restricted to Υ where $0 < c < 1$ is a real number.

Remark 1.2.3. *We fix the real number c which appears in the portion of the theorem above concerning piecewise linearity and hence forth write* $\log(|f|)$ *in place of* $\log_c(|f|)$.

The notion of a generalised interval is discussed in Section 3.9 [HL]. We now provide an example which illustrates the behaviour of the function f in Theorem 1.2.2 clearly.

Example 1.2.4. *Let k be an algebraically closed complete non-trivially valued non-Archimedean field which is of characteristic p. Consider the morphism $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $z \mapsto z^p - z$. Let L/k be a non-Archimedean real valued field extension. The morphism ϕ_L is étale over every point other than ∞. Furthermore it can be shown that $f(x) = 1$ if $x \neq \infty$ and 0 at ∞. Let Υ be a finite graph containing the point ∞ and which contains at least one other point. Since there is a deformation retraction of $\mathbb{P}^{1,\mathrm{an}}_k$ onto any non - empty finite sub-graph, it follows that there exists a deformation retraction*

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

such that the function f is constant along the fibres of the retraction.

Our first goal is to generalize Theorem 1.2.2 to the case of finite surjective morphisms between irreducible, projective varieties. A problem standing in the way of any attempt at a generalisation is that there is no intrinsic notion of an open disk in V^{an} if V is a projective k-variety of finite type. However, as V is projective there exists a closed immersion $V \hookrightarrow \mathbb{P}^n_k$ for some $n \in \mathbb{N}$. We identify V with its image under the closed immersion. The space $\mathbb{P}^{n,\mathrm{an}}_k$ can be equipped with a finite formal cover [[B], Section 4.3] such that each element of this cover is isomorphic to the n-dimensional Berkovich closed disk $\mathcal{M}(k\{T_1, \ldots, T_n\})$. Let $\{A_i\}_i$ denote this cover. The intersection of the elements of the formal cover with the image of the immersion $V^{\mathrm{an}} \hookrightarrow \mathbb{P}^{n,\mathrm{an}}_k$ defines a formal cover of the space V^{an}, namely $\{A_i \cap V^{\mathrm{an}}\}_i$. Furthermore, for a non-Archimedean real valued algebraically closed field extension L/k the construction extends to the analytic space $(V_L)^{\mathrm{an}} := (V \times_k L)^{\mathrm{an}}$. The n - dimensional Berkovich open balls contained in $\mathbb{P}^{n,\mathrm{an}}_L$ allow us to generalise Theorem 1.2.2. We now provide a sketch of the details of this construction.

Let L/k be a non-Archimedean real valued algebraically closed field extension and $x \in \mathbb{P}^{n,\mathrm{an}}_L(L)$ (Remark 1.1). Let i be such that $x_L \in A_{i,L}$. Associated to the point x_L is a collection of open neighbourhoods in $\mathbb{P}^{n,\mathrm{an}}_L(L)$, namely the Berkovich open balls around x_L contained in $A_{i,L}$. We denote this family of open neighbourhoods $\mathcal{O}_{x_L}$. Let $G \in \mathcal{O}_{x_L}$. For every j such that x_L belongs to $A_{j,L}$, it can be checked that G is a Berkovich open ball in $A_{j,L}$ as well. This implies that the family $\mathcal{O}_{x_L}$ is independent of the element of the affinoid chart which contains x_L and is hence well defined. We now define a poly radius associated to the elements of $\mathcal{O}_{x_L}$. Let x_L have homogenous coordinates $[x_{1,L} : \ldots : x_{n+1,L}]$ and let $W \in \mathcal{O}_{x_L}$. We can associate an $(n+1)^2$-tuple denoted $h_L(W)$ to the open neighbourhood W as follows. If for an index t, $x_L \in A_{t,L}$ then let $\mathbf{r}_t = (r_1, \ldots, r_{n+1})$ be such that $r_t = 1$ and the Berkovich open ball $B_{W,t}$ is defined by the equations $|(T_j/T_t - x_{j,L}/x_{t,L})(p)| < r_j$ for $j \neq t$. If on the other hand x_L does not belong to $A_{t,L}$ then let $\mathbf{r}_t = (1, \ldots, 1)$. We define $h_L(W) := (\mathbf{r}_t)_t$. Let $\mathcal{O}_L := \bigcup_{x \in \mathbb{P}^{n,\mathrm{an}}_k(L)} \mathcal{O}_{x_L}$. We can extend the above

construction to define a function $h_L : \mathcal{O}_L \to \mathbb{R}_{\geq 0}^{(n+1)^2}$. Note that the sets $\mathcal{O}_{x_L}$ depend on the affine chart chosen for $\mathbb{P}^n$. In the case of $\mathbb{P}^1_k$ with $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$, the set $\mathcal{O}_{x_L}$ associated to the construction above is discussed in Section 8.1. In what follows we explain how the family $\mathcal{O}_L$ can be ordered.

Remark 1.2.5. *We introduce a collection S of functions from $\mathbb{R}_{>0}^{(n+1)^2}$ to $\mathbb{R}_{>0}$. The set S consists of those functions $g : \mathbb{R}_{>0}^{(n+1)^2} \to \mathbb{R}_{>0}$ which satisfy the following properties.*

1. *The function g is continuous.*
2. *If $(r_{i,j})_{i,j}$ and $(s_{i,j})_{i,j}$ are $(n+1)^2$-tuples such that $r_{i,j} \leq s_{i,j}$ for all i, j then $g((r_{i,j})_{i,j}) \leq g((s_{i,j})_{i,j})$.*
3. *g is a definable function (in the model theoretic sense) in the language of Ordered Abelian groups.*

Let $g \in S$. The function g defines a total ordering on the set $\mathbb{R}_{>0}^{(n+1)^2}$ as follows. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}_{>0}^{(n+1)^2}$. We set $\mathbf{r} \leq_g \mathbf{s}$ if $g(\mathbf{r}) \leq g(\mathbf{s})$. Observe that the total ordering $\leq_g$ on $\mathbb{R}_{>0}^{(n+1)^2}$ extends the partial ordering given by : $(r_{i,j})_{i,j} \leq (s_{i,j})_{i,j}$ if $r_{i,j} \leq s_{i,j}$ for all i, j, where $(r_{i,j})_{i,j}$ and $(s_{i,j})_{i,j}$ are $\mathbb{R}_{>0}^{(n+1)^2}$ tuples.

We can extend g to a function $\tilde{g} : \mathbb{R}_{\geq 0}^{(n+1)^2} \to \mathbb{R}_{\geq 0}$ as follows. For $\mathbf{r} \in \mathbb{R}_{>0}^{(n+1)^2}$, we set $\tilde{g}(\mathbf{r}) := g(\mathbf{r})$ and if $\mathbf{r} \in \mathbb{R}_{\geq 0}^{(n+1)^2} \smallsetminus \mathbb{R}_{>0}^{(n+1)^2}$ then $\tilde{g}(\mathbf{r}) := 0$. As for g, the function $\tilde{g}$ defines a total ordering on $\mathbb{R}_{\geq 0}^{(n+1)^2}$ which extends the partial ordering given by : $(r_{i,j})_{i,j} \leq (s_{i,j})_{i,j}$ if $r_{i,j} \leq s_{i,j}$ for all i, j, where $(r_{i,j})_{i,j}$ and $(s_{i,j})_{i,j}$ are $\mathbb{R}_{\geq 0}^{(n+1)^2}$ tuples. Henceforth, we abuse notation and write g for the extension $\tilde{g}$.

Let $g \in S$. By Lemma 8.1.2, the function $g \circ h_L : \mathcal{O}_L \to \mathbb{R}_{\geq 0}$ has the following property. If $O_1, O_2 \in \mathcal{O}_L$ such that $O_1 \subseteq O_2$ then $(g \circ h_L)(O_1) \leq (g \circ h_L)(O_2)$. The functions $g \in S$ hence allow us to quantify the size of elements belonging to $\mathcal{O}_L$.

We now provide an equivalent form of Theorem 1.2.2 which we generalize. We begin by motivating the reformulation. The goal of Theorem 1.2.2 is to prove the existence of a finite simplicial complex Υ contained in $\mathbb{P}^{1,\mathrm{an}}_k$ such that the function f is constant along the fibres of the retraction morphism $\psi(e, _)$. Let us assume that Theorem 1.2.2 is true. We define a function $M : \Upsilon \to \mathbb{R}_{\geq 0}$ as follows. Let $\gamma \in \Upsilon$ and $x \in \mathbb{P}^{1,\mathrm{an}}_k$ be any point for which $\psi(e, x) = \gamma$. We set $M(\gamma) := f(x)$. Let L/k be any complete non-Archimedean real valued algebraically closed field extension such that $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$. By definition $M(\gamma)$ is the minimum of 1 and the radius of the largest Berkovich open ball in $\mathbb{P}^{1,\mathrm{an}}_L$ around x_L whose preimage is the disjoint union of homeomorphic copies of itself for the morphism ϕ^{an}_L. The function M is well defined since we assumed Theorem 1.2.2 is true. Hence a suitable restatement of 1.2.2 is the following theorem.

Theorem 1.2.6. *Let $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a finite morphism. There exists a generalised real interval $I := [i, e]$ and a deformation retraction*

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

which satisfies the following properties.

1. *The image $\psi(e, \mathbb{P}_k^{1,\mathrm{an}})$ of the deformation retraction ψ is a finite simplicial complex. Let Υ denote this finite simplicial complex.*

2. *There exists a well defined function $M : \Upsilon \to [0,1]$ which satisfies the following conditions. The function M is not identically zero and $\log(M)$ is piecewise linear. Let $\gamma \in \Upsilon$ such that $M(\gamma) > 0$ and $x \in \psi(e, _)^{-1}(\gamma)$. Let L/k be any complete non-Archimedean real valued algebraically closed field extension such that $x \in \mathbb{P}_k^{1,\mathrm{an}}(L)$. Then the following are true.*

   (a) *The preimage under the morphism ϕ_L^{an} of the Berkovich open ball $B(x_L, M(\gamma)) \subset \mathbb{P}_L^{1,\mathrm{an}}$ around x_L of radius $M(\gamma)$ decomposes into the disjoint union of Berkovich open balls each homeomorphic to $B(x_L, M(\gamma))$ via the morphism ϕ_L^{an}.*

   (b) *Let O be any other Berkovich open ball around x_L whose radius is less than or equal to 1 such that its preimage under the morphism ϕ_L^{an} decomposes into the disjoint union of Berkovich open balls each homeomorphic to O via ϕ_L^{an}. Then the radius of O must be less than or equal to $M(\gamma)$.*

We will show that Theorems 1.2.2 and 1.2.6 are equivalent in Section 8.2.1. We now state a theorem which in Section 8.5 we will show to be a generalisation of Theorem 1.2.2. Let $\phi : V' \to V$ be a finite surjective morphism between irreducible, projective varieties of finite type over k. For a complete non-Archimedean real valued algebraically closed field extension L/k, let ϕ_L denote the morphism

$$\phi \times \mathrm{id}_L : V \times_k L \to V \times_k L.$$

As in the case of $\mathbb{P}_k^1$, we write $\phi^{\mathrm{an}} : V'^{\mathrm{an}} \to V^{\mathrm{an}}$ for the induced morphism between the respective analytifications. The morphism ϕ_L^{an} is similarly defined and it is to be noted that $\phi_L^{\mathrm{an}} = \phi^{\mathrm{an}} \times \mathrm{id}_L^{\mathrm{an}}$. We fix an embedding $V \hookrightarrow \mathbb{P}_k^n$ and an affine chart of $\mathbb{P}^n$. The theorem will be stated in terms of the sets $\mathcal{O}_{x_L}$ and the functions h_L and g described above.

Theorem 8.4.3. *Let $\phi : V' \to V$ be a finite surjective morphism between irreducible, projective varieties with V normal. Let $g \in S$. There exists a generalized real interval $I := [i, e]$ and a deformation retraction*

$$\psi : I \times V^{\mathrm{an}} \to V^{\mathrm{an}}$$

which satisfies the following properties.

1. *The image $\psi(e, V^{\mathrm{an}})$ of the deformation retraction ψ is a finite simplicial complex which we denote Υ_g.*

2. *There exists a well defined function $M_g : \Upsilon_g \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions. The function M_g is not identically zero. Let $\gamma \in \Upsilon_g$ be a point on the finite simplicial complex for which $M_g(\gamma) \neq 0$ and $x \in \psi(e, _)^{-1}(\gamma)$. Let L/k be any complete non-Archimedean real valued algebraically closed field extension such that $x \in V^{\mathrm{an}}(L)$. There exists*

$W \in (g \circ h_L)^{-1}(M_g(\gamma)) \cap \mathcal{O}_{x_L}$ such that the open set $(\phi_L^{\mathrm{an}})^{-1}(W \cap V_L^{\mathrm{an}}) \subset V_L'^{\mathrm{an}}$ decomposes into the disjoint union of open sets, each homeomorphic to $W \cap V_L^{\mathrm{an}}$ via ϕ_L^{an}. Furthermore, let $O \in \mathcal{O}_{x_L}$ be such that the preimage of $O \cap V_L^{\mathrm{an}}$ under ϕ_L^{an} decomposes into the disjoint union of open sets in $V_L'^{\mathrm{an}}$, each homeomorphic to O via the morphism ϕ_L^{an}. Then $(g \circ h_L)(O) \leq M_g(\gamma)$. Lastly, the function $\log(M_g)$ is piecewise linear on Υ_g.

Remark 1.2.7. *The second property we require the function M_g to satisfy involves choosing a point x_L over x which is an L-point of V_L^{an} and then requiring that M_g fulfill a condition concerning x_L. It may hence seem that the function is dependent on the field L chosen. However, showing that the function M_g is well defined implies in particular that the value $M_g(x)$ for $x \in V^{\mathrm{an}}$ is determined entirely by x.*

It is worth mentioning that the result stated above when applied to smooth Berkovich analytic curves over a field of characteristic zero bears some similarity with theorems proved in [PP] and [Bal]. In [Bal], the author - F. Baldassarri studies a system of differential equations defined over an analytic domain of the affine line over a non-Archimedean real valued field of characteristic zero. More precisely, let k be a non-Archimedean field of characteristic zero and X be a relatively compact analytic domain of the affine line $\mathbb{A}_k^{1,\mathrm{an}}$. Let

$$\Sigma : dy/dT = Gy$$

be a system of linear differential equations where G is a $\mu \times \mu$ matrix of k-analytic functions on X. If $x \in X$ is a k-rational point, let $R(x) = R(x, \Sigma)$ denote the radius of the maximal open disk in X with center at x on which all solutions of Σ converge. The author shows that the function R is continuous. Also, he illustrates how when $X = \mathbb{A}_k^{1,\mathrm{an}}$ there exists a finite graph $\Gamma \subset \mathbb{A}_k^{1,\mathrm{an}}$ which *controls* the behaviour of the function R. Since $\mathbb{A}_k^{1,\mathrm{an}}$ retracts to any of its finite subgraphs, this means that the function R is constant along the fibres of the retraction on Γ. In the paper the control by a finite graph is illustrated by an example. If one preserves the restrictions on the field k and considers the case of a system of differential equations defined instead over a smooth Berkovich curve then a similar result holds true. In [PP], the authors - Poineau and Pulita prove that associated to a system of differential equations over a smooth Berkovich curve, there exists a locally finite graph contained in the curve and a retraction of the curve onto it such that the radius of convergence function is constant along the fibres of the retraction. The results we prove and the results of Poineau-Pulita and Baldassari show that the behaviour of certain functions of interest are controlled by finite simplicial complexes associated to them.

Notation : To prove the main theorems that follow, we require techniques from Model theory where it is standard to write the value group of a non-Archimedean valued field additively. However, when discussing objects from Berkovich geometry such as affinoid algebras and the reduction morphism it is standard to endow the value group with a multiplicative structure as it aids in intuition. Instead of resolving this dichotomy in notation, we preserve both notation and eliminate ambiguity by specifying at every instance the structure of the value group, i.e. whether we look at it additively or multiplicatively.

Chapter 2

Introduction en français

Les corps non-archimédiens ont été découverts au début du vingtième siècle, lorsque K. Hensel a défini le corps des nombres p-adique. Dans les années qui suivirent, on observa plusieurs tentatives, chacune ayant ses mérites, de développement d'une théorie géométrique sur ces corps comparable a celle développée sur le corps des nombres complexes. Cependant, ce n'est qu'au début des années 90 que Vladimir Berkovich introduit une théorie de la géométrie non-archimédienne qui nous fourni des espaces analytiques possédant les bonnes propriétés topologiques. Dans le cadre de cette théorie, on peut associer un espace analytique au sens de Berkovich à une variété définie sur un corps non-archimédien. On appelle cet espace analytique l'analytification de la variété. Même si une variété sur un corps non-archimédien est totalement discontinue pour la topologie induite par la valuation du corps, son analytification est Hausdorff, localement compacte avec un nombre fini de composantes connexes par arcs. Ainsi, il semble naturel d'étudier le type d'homotopie de l'analytification de telles variétés.

En 2010, Hrushovski et Loeser ont utilisé la théorie des modèles pour étudier le type d'homotopie de l'analytification d'une variété sur un corps non-archimédien non-trivialement valué. Ils ont montré que les types d'homotopie sont completement déterminé par certains complexes simpliciaux contenus dans les analytifications. Plus précisément, étant donné une variété sur un corps non-archimédien, ils ont construit une rétraction par déformation de l'analytification de la variété sur un complexe simplicial contenu dans l'analytification. Dans [HL], Hrushovski et Loeser ont introduit l'espace chapeauté associé à une variété qui est la version analogue dans la théorie des modèles de l'analytification. Un des avantages de ce point de vu est le fait qu'on peut parler des notions de définissabilité et également utiliser les méthodes puissantes comme compacité. Cette thèse reflète et s'appuie sur cette interaction entre la théorie des modèles et la géométrie de Berkovich. Nous montrerons que plusieurs phénomènes topologiques concernant des analytifications de variétés sont contrôlés par certains complexes simpliciaux contenus dans les analytifications.

2.1 Une formule de Riemann-Hurwitz pour le genre analytique

Soit k un corps algébriquement clos et complet pour une valuation non-archimédienne à valeurs réelles et non-triviale. Soit C une courbe sur k. Par une courbe sur k, on veut dire un k-schéma réduit, connexe et séparé de dimension 1. Il est bien connu qu'il existe une rétraction par déformation de l'analytification C^{an} de C sur un sous-ensemble fermé Υ qui est homéomorphe à un graphe fini [[B], Chapter 4], [[HL], Section 7]. Ces types des sous-espaces sont appelés *squelettes*. Le squelette Υ peut-être décomposé en un ensemble des sommets $V(\Upsilon)$ et un ensemble d'arêtes $E(\Upsilon)$. On définit le genre du graphe Υ par l'équation suivante.

$$g(\Upsilon) = 1 - V(\Upsilon) + E(\Upsilon).$$

Dans la Proposition 7.1.24, nous montrons que $g(\Upsilon)$ est un invariant bien défini de la courbe. Soit $g^{\mathrm{an}}(C) := g(\Upsilon)$ pour un tel squelette Υ. Nous étudions le comportement de l'invariant g^{an} pour un morphisme fini en utilisant une paire de rétractions compatibles.

Soit $\phi : C' \to C$ un morphisme fini entre deux courbes projectives, lisses et irréductibles. Le morphisme ϕ va induire un morphisme $\phi^{\mathrm{an}} : C'^{\mathrm{an}} \to C^{\mathrm{an}}$ entre les deux analytifications. Nous démontrons qu'il existe une paire *compatible* de rétractions par déformations

$$\psi : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$$

et

$$\psi' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$$

avec les propriétés suivantes:

1. Les ensembles $\Upsilon_{C'^{\mathrm{an}}} := \psi'(1, C'^{\mathrm{an}})$ et $\Upsilon_{C^{\mathrm{an}}} := \psi(1, C^{\mathrm{an}})$ sont des sous-espaces fermés de C'^{an} et C^{an}. Ils sont homéomorphes à des graphes finis. De plus, on a l'égalité $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$.

2. Les espaces analytiques $C'^{\mathrm{an}} \smallsetminus \Upsilon_{C'^{\mathrm{an}}}$ et $C^{\mathrm{an}} \smallsetminus \Upsilon_{C^{\mathrm{an}}}$ se décomposent en une union disjointe de copies de disques unités de Berkovich i.e. il existe deux ensembles de sommets faiblement semi-stables (cf. Definition 7.1.18) $\mathfrak{A} \subset C^{\mathrm{an}}$ et $\mathfrak{A}' \subset C'^{\mathrm{an}}$ tel que $\Upsilon_{C^{\mathrm{an}}} = \Sigma(C^{\mathrm{an}}, \mathfrak{A})$ et $\Upsilon_{C'^{\mathrm{an}}} = \Sigma(C'^{\mathrm{an}}, \mathfrak{A}')$.

3. Les rétractions par déformations ψ and ψ' sont **compatibles**, cela veut dire que le diagramme suivant est commutatif.

$$\begin{tikzcd}
{[0,1] \times C'^{\mathrm{an}}} \arrow[r, "\psi'"'] \arrow[d, "id \times \phi^{\mathrm{an}}"] & C'^{\mathrm{an}} \arrow[d, "\phi^{\mathrm{an}}"] \\
{[0,1] \times C^{\mathrm{an}}} \arrow[r, "\psi"'] & C^{\mathrm{an}}
\end{tikzcd}$$

Les notations utilisées pour le résultat suivant sont explicitées dans la Section 7.3.1 et les définitions 7.3.6 et 7.3.8.

Corollaire 7.3.9 *Soit $\phi : C' \to C$ un morphisms séparable fini entre deux courbes lisses et projectives sur le corps k. Soit $g^{\mathrm{an}}(C'), g^{\mathrm{an}}(C)$ les invariants définis dans la Definition 7.1.25. On a l'équation suivante.*

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)(2g^{\mathrm{an}}(C) - 2) + \Sigma_{p \in C^{\mathrm{an}}} 2i(p)g_p + R - \Sigma_{p \in C^{\mathrm{an}}} R_p^1.$$

Dans la Section 7.4, on propose une autre méthode pour calculer l'invariant $g^{\mathrm{an}}(C')$ en utilisant l'existence de la paire de rétractions par déformations ψ et ψ' sur C^{an} et C'^{an} dont les images sont des squelettes $\Upsilon_{C^{\mathrm{an}}}$ et $\Upsilon_{C'^{\mathrm{an}}}$. On suppose également que le morphisme $\phi : C' \to C$ est tel que l'extension du corps des fonctions $k(C) \hookrightarrow k(C')$ est Galois. Par construction de ψ' et ψ, ϕ^{an} induira un morphisme entre les deux squelettes. Nous montrons que le genre du squelette $\Upsilon_{C'^{\mathrm{an}}}$ peut être calculé en utilisant certains invariants associés aux points de $\Upsilon_{C^{\mathrm{an}}}$. On définit un diviseur w sur $\Upsilon_{C^{\mathrm{an}}}$ de degré $2g(\Upsilon_{C'^{\mathrm{an}}}) - 2$. Un diviseur sur un graphe fini est un élément du groupe abélien libre engendré par les points du graphe.

Le diviseur w est défini comme suit. Pour un point $p \in \Upsilon_{C^{\mathrm{an}}}$, soit $w(p)$ l'ordre du diviseur au point p. Soit

$$w(p) := (\sum_{e_p \in E_p, p' \in (\phi^{\mathrm{an}})^{-1}(p)} l(e_p, p')) - 2n_p.$$

Les termes dans cette expression sont définis comme suit. Soit T_p l'espace tangent au point p (cf. 7.1.3, 7.1.6).

1. Soit $E_p \subset T_p$ l'ensemble des éléments pour lesquels il existe un représentant à partir de p contenu dans $\Upsilon_{C^{\mathrm{an}}}$.

2. Soit $p' \in C'^{\mathrm{an}}$ tel que $\phi^{\mathrm{an}}(p') = p$. Le morphisme ϕ^{an} induit un morphisme $d\phi_{p'}$ entre les espaces tangents $T_{p'}$ et T_p (cf. 7.1.3, 7.1.6). Soit $e_p \in E_p$. On désigne par $L(e_p, p') \subset T_{p'}$ l'ensemble des préimages de e_p pour le morphisme $d\phi_{p'}$. Comme $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$, un élément de $L(e_p, p')$ peut être représenté par un segment géodésique contenu dans $\Upsilon_{C'^{\mathrm{an}}}$. Soit $l(e_p, p')$ la cardinalité de l'ensemble $L(e_p, p')$.

3. On désigne par n_p la cardinalité de l'ensemble des préimages du point p i.e. $n_p := \mathrm{card}\{(\phi^{\mathrm{an}})^{-1}(p)\}$.

Dans la Proposition 7.4.4, nous montrons que w est un diviseur bien défini de degré $2g(\Upsilon_{C'^{\mathrm{an}}}) - 2$. Puis, nous étudions les invariants n_p et $l(e_p, p')$ de la maniére suivante.

On étudie l'invariant n_p pour $p \in \Upsilon_{C^{\mathrm{an}}}$ en termes de deux invariants - $\mathrm{ram}(p)$ et $c_1(p)$ qui sont définis comme suit.
Soit $p \in \Upsilon_{C^{\mathrm{an}}}$.

1. Si p est un point de type I, on désigne par $\mathrm{ram}(p)$ le degré de ramification $\mathrm{ram}(p'/p)$ pour tout $p' \in C'^{\mathrm{an}}$ tel que $\phi^{\mathrm{an}}(p') = p$. Comme le morphisme ϕ est Galois, $\mathrm{ram}(p)$ est bien défini. Si p n'est pas de type I, on définit $\mathrm{ram}(p) := 1$.

2. Afin de définir l'invariant c_1, nous introduisons une relation d'équivalence sur $C'(k)$. Soient $y_1, y_2 \in C'(k)$. Nous définissons $y_1 \sim_{c(1)} y_2$ si $\phi(y_1) = \phi(y_2)$ et $\psi'(1, y_1) = \psi'(1, y_2)$. On désigne par $c_1(y)$ la cardinalité de la classe d'équivalence qui contient y. Dans le Lemme 7.4.8, nous montrons que la fonction $c_1 : C(k) \to \mathbb{Z}_{\geq 0}$ donnée par $c_1(x) = c_1(y)$ pour tout $y \in \phi^{-1}(x)$ est bien définie. Nous montrons que $c_1(x)$ pour $x \in C(k)$ ne dépend que du point $\psi(1, x) \in \Upsilon_{C^{\mathrm{an}}}$. Ceci définit $c_1 : \Upsilon_{C^{\mathrm{an}}} \to \mathbb{Z}_{\geq 0}$.

Les valeurs $c_1(p)$ et (p) peuvent être utilisées pour calculer n_p via l'équation suivante (Proposition 7.4.10).

$$n_p = [k(C') : k(C)]/(c_1(p)\mathrm{ram}(p)).$$

Nous simplifions le terme $l(e_p, p')$. Soit $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$. Dans le Lemme 7.4.12, nous montrons que $l(e_p, p')$ est constant pour tout $p' \in (\phi^{\mathrm{an}})^{-1}(p)$. On définit $l(e_p) := l(e_p, p')$. Nous introduisons les invariants - $\widetilde{\mathrm{ram}(e_p)}$ et $\widetilde{\mathrm{ram}}(p)$ pour étudier $l(e_p)$.

1. Soit $p \in \Upsilon_{C^{\mathrm{an}}}$. Par définition, e_p est un élément de l'éspace tangent T_p à p (cf. Sections 7.1.3, 7.1.6). Comme p est de type II, ça correspond à une valuation discrète du $\tilde{k}$-corps des fonctions $\widetilde{\mathcal{H}(p)}$. Pour tout $p' \in (\phi^{\mathrm{an}})^{-1}(p)$, l'extension de corps $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ peut être décomposée en une extension inséparable et une extension Galoisienne.
   Ainsi, le degré de ramification $\mathrm{ram}(e'/e_p)$ est constant pour tous les préimages de e_p pour le morphisme $d\phi^{alg}_{p'} : T_{p'} \to T_p$ (cf. 7.1.6). On désigne par $\widetilde{\mathrm{ram}}(e_p)$ ce degré de ramification. Si p est un point de type I, on définit $\widetilde{\mathrm{ram}}(e_p) := \mathrm{ram}(p)$ et si p est de type III, on définit $\widetilde{\mathrm{ram}}(e_p) = c_1(p)$.

2. Pour tout $p \in \Upsilon_{C^{\mathrm{an}}}$, on définit $\widetilde{\mathrm{ram}}(p) := \Sigma_{e_p \in E_p} 1/\widetilde{\mathrm{ram}}(e_p)$.

Soit $p \in \Upsilon_{C^{\mathrm{an}}}$ et $e_p \in E_p$. Dans la Proposition 7.4.15, nous montrons que

$$l(e_p) = [k(C') : k(C)]/(n_p\widetilde{\mathrm{ram}}(e_p)).$$

On peut résumer l'ensemble des résultats de la Section 7.4 par le résultat suivant où on calcule la valeur $2g^{\mathrm{an}}(C') - 2$ en termes des invariants $c_1, \widetilde{\mathrm{ram}}$ et ram.

Theoreme 7.4.17 *Soit $\phi : C' \to C$ un morphisme fini entre deux courbes lisses projectives et irréductibles tels que l'extension du corps des fonctions $k(C) \hookrightarrow k(C')$ induite par ϕ est Galoisienne. Soit $g^{\mathrm{an}}(C')$ le genre analytique de la Definition 7.1.25. On a l'équation suivante.*

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)\Sigma_{p\in\Upsilon_{C^{\mathrm{an}}}}[\widetilde{\mathrm{ram}}(p) - 2/(c_1(p)\mathrm{ram}(p))].$$

2.2 Morphismes finis et squelettes

La deuxième série de résultats dans cette thèse concerne les morphismes finis surjectifs entre des variétés projectives irréductibles sur un corps non archimédien. On étudie un tel morphisme en termes du morphisme induit entre les analytifications. Le théorème principal (Theorem 8.4.3) implique que le comportement de ce morphisme entre les analytifications est modéré. Avant de le préciser, nous donnons la motivation en considérant le cas d'un endomorphisme de la droite projective.

Soit k un corps algébriquement clos complet pour une valuation non triviale à valeurs réelles. Soit $\phi : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$ un morphisme fini. Comme l'analytification dans le sens de Berkovich d'un morphisme est une construction fonctorielle, le morphisme ϕ induit un morphisme $\phi^{\mathrm{an}} : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$ entre les deux analytifications. Pour un point $x \in \mathbb{P}^1_k(k) \subset \mathbb{P}^{1,\mathrm{an}}_k$, on a la notion de boule fermée de Berkovich ou de boule ouverte de centre x et contenue dans l'éspace $\mathbb{P}^{1,\mathrm{an}}_k$. Une boule fermé (ouverte) de Berkovich de centre x contient la boule fermée (ouverte) naïve de centre x. On désigne par la boule fermée (ouverte) naïve de centre x et rayon r, l'ensemble des points $\{y \in k||y-x| \leq r\}$ ($\{y \in k||y-x| < r\}$). Les boules de Berkovich sont localement compactes et contractiles.

Pour une extension L/k non archimédienne algébriquement close complète, on désigne par ϕ_L le morphisme

$$\phi \times \mathrm{id}_L : \mathbb{P}^1_k \times_k \mathrm{Spec}(L) \to \mathbb{P}^1_k \times_k \mathrm{Spec}(L).$$

Le morphisme ϕ^{an}_L est défini comme avant et il faut noter que $\phi^{\mathrm{an}}_L = \phi^{\mathrm{an}} \times \mathrm{id}^{\mathrm{an}}_L$. Les morphismes ϕ_L nous permettent de traiter tous les points de l'analytification de la droite projective et pas seulement les points pour lesquels $\mathcal{H}(x) = k$ (cf. 5.6.2). Lorsque l'on parle des points de l'analytification d'une variété sur k, on utilise la description de Section 5.6. Soit $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$ i.e. $x \in \mathbb{P}^{1,\mathrm{an}}$ et il existe une immersion de corps valués $\mathcal{H}(x) \hookrightarrow L$. L'image du point $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$ pour le morphisme $\pi : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^1_k$ (cf. 5.6) est un L-point de $\mathbb{P}^1_k$ i.e. un élément de l'ensemble $\mathbb{P}^1_k(L)$ que l'on désigne aussi par x. La paire $x : \mathrm{Spec}(L) \to \mathbb{P}^1_k$ et $\mathrm{id}_L : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ définit un point fermé de la variété $\mathbb{P}^1_k \times_k \mathrm{Spec}(L)$

désigné par x_L. La remarque suivante généralise cette construction.

Remark 1.1. *Soit V une variété sur k et soit $x \in V^{\mathrm{an}}$. Le corps $\mathcal{H}(x)$ (cf. Section 5.6) est la version non archimédienne du corps résiduel d'un point en géométrie algébrique. Soit L une extension non archimédienne complète de k. La notation $x \in V^{\mathrm{an}}(L)$ veut dire que $x \in V^{\mathrm{an}}$ et qu'il existe une immersion de corps valués $\mathcal{H}(x) \hookrightarrow L$. On peut en déduire que l'image du point x pour le morphisme $V^{\mathrm{an}} \to V$ qui est défini dans Section 5.6 est un L-point i.e. un élément de l'ensemble $V(L)$ que l'on désigne aussi par x. La paire $x : \mathrm{Spec}(L) \to V$ et $\mathrm{id}_L : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ définit un point fermé de la variété $V \times_k \mathrm{Spec}(L)$ qu'on désigne par x_L. On utilisera cette construction dans la Section 8.*

Nous introduisons maintenant le théorème concernant un endomorphisme fini de la droite projective sur k.

Theorem 1.2.2. *Soit $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ un morphisme fini. Soit $x \in \mathbb{P}^{1,\mathrm{an}}_k$ et L/k une extension algébriquement close non archimédienne valuée et complète telle que $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$. Soit $f(x)$ le minimum de 1 et du rayon de la plus grande boule ouverte de Berkovich de centre $x_L \in \mathbb{P}^1_L(L) \subset \mathbb{P}^{1,\mathrm{an}}_L$ dont l'image réciproque pour le morphisme ϕ^{an}_L est union disjointe de copies homéomorphes de B via ϕ^{an}_L. La fonction $f : \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{R}_{\geq 0}$ n'est pas identiquement 0 et elle est bien définie. Il existe un complexe simplicial fini $\Upsilon \subset \mathbb{P}^{1,\mathrm{an}}_k$, un intervalle réel généralisé $I := [i, e]$ et une rétraction par déformation*

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

tels que $\psi(e, \mathbb{P}^{1,\mathrm{an}}_k) = \Upsilon$ et la fonction f soit constante le long des fibres de la rétraction i.e. pour tout $x \in \mathbb{P}^{1,\mathrm{an}}_k$, $f(x) = f(\psi(e, x))$. De plus, la restriction sur Υ de la fonction $\log_c|f|$ est linéaire par morceaux où $0 < c < 1$ est un nombre réel.

Remark 1.3. *Nous fixons le nombre réel c qui apparaît dans le Théorème 1.2.2 et désignons par $\log(|f|)$ la fonction $\log_c(|f|)$.*

La notion d'intervalle réel généralisé est défini dans la Section 3.9 de [HL]. Nous donnons maintenant un exemple pour comprendre le comportement de la fonction f du Théorème 1.2.2.

Example 1.3. *Soit k un corps algébriquement clos complet et non archimédien de caractéristique p. Considèrons le morphisme $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ défini par $z \mapsto z^p - z$. Soit L/k une extension non archimédienne valuée. Le morphisme ϕ_L est étale au-dessus tous les points sauf ∞. On peut vérifier que $f(x) = 1$ si $x \neq \infty$ et $f(\infty) = 0$. Soit Υ un graphe fini qui contient le point ∞ et au moins un autre point de $\mathbb{P}^{1,\mathrm{an}}_k$. Comme il existe une rétraction par déformation de $\mathbb{P}^{1,\mathrm{an}}_k$ sur tout sous-graphe fini et non-vide, on en déduit l'existence d'une rétraction par déformation*

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

telle que la fonction f soit constante le long des fibres de la rétraction.

Notre but premier est de généraliser le Théorème 1.2.2 au cas d'un morphisme fini surjectif entre des variétés irréductibles et projectives. Un des obstacles à une telle généralisation est le fait qu'il n'y a pas de notion intrinsèque de boule ouverte dans V^{an} si V est une variété projective sur k de type fini. Cependant, comme V est projective, il existe une immersion fermée $V \hookrightarrow \mathbb{P}^n_k$ où $n \in \mathbb{N}$. Nous identifions V avec son image sous cette immersion fermée. On peut imposer sur $\mathbb{P}^{n,\mathrm{an}}_k$ un recouvrement formel [[B], Section 4.3] tel que chaque élément dans ce recouvrement soit isomorphe à la boule fermée de Berkovich de dimension n, $\mathcal{M}(k\{T_1, \ldots, T_n\})$. On désigne par $\{A_i\}$ ce recouvrement. L'intersection des éléments de ce recouvrement avec l'image de l'immersion $V^{\mathrm{an}} \hookrightarrow \mathbb{P}^{n,\mathrm{an}}_k$ définit un recouvrement formel de l'espace V^{an}, notamment $\{A_i \cap V^{\mathrm{an}}\}_i$. En outre, pour une extension non archimédienne valuée et algébriquement close L/k, la construction s'étend à l'espace analytique $(V_L)^{\mathrm{an}} := (V \times_k L)^{\mathrm{an}}$. Les boules de Berkovich de dimension n contenues dans $\mathbb{P}^{n,\mathrm{an}}_L$ nous permettent de généraliser le Théorème 1.2.2. Voici les détails de cette construction.

Soit L/k une extension de corps algébriquement close et non archimédienne. Soit $x \in \mathbb{P}^{n,\mathrm{an}}_L(L)$ (Remark 1.1). Soit $i \in \{1, \ldots, n+1\}$ tel que $x_L \in A_{i,L}$. La famille des boules ouvertes de Berkovich de centre x_L et contenues dans $A_{i,L}$, définit une collection de voisinages ouverts de x_L dans $\mathbb{P}^{n,\mathrm{an}}_L(L)$. On désigne par $\mathcal{O}_{x_L}$ cette famille des voisinages. Soit $G \in \mathcal{O}_{x_L}$. Pour chaque j tel que $x_L \in A_{j,L}$, on peut vérifier que G est une boule de Berkovich contenue dans $A_{j,L}$. Cela implique que la famille $\mathcal{O}_{x_L}$ ne dépend pas de l'élément de récouvrement formel qui contient x_L et on déduit que $\mathcal{O}_{x_L}$ est bien défini. Nous définissons le polyrayon d'un élément de la famille $\mathcal{O}_{x_L}$. Supposons que le point x_L a pour cordonnées homogènes $[x_{1,L} : \ldots : x_{n+1,L}]$ et $W \in \mathcal{O}_{x_L}$. On peut lui associer un élément $h_L(W)$ de $\mathbb{R}^{n+1^2}$ à W comme suit. Si $x_L \in A_{t,L}$ pour $t \in \{1, \ldots, n+1\}$, on définit $\mathbf{r}_t := (r_1, \ldots, r_{n+1})$ où $r_t = 1$ et la boule ouverte de Berkovich $B_{W,t}$ est définie par les équations $|(T_j/T_t - x_{j,L}/x_{t,L})(p)| < r_j$ pour $j \neq t$. Si $x_L \notin A_{t,L}$, on définit $\mathbf{r}_t := (1, \ldots, 1)$. Soit $h_L(W) := (\mathbf{r}_t)_t$ et $\mathcal{O}_L := \bigcup_{x \in \mathbb{P}^{n,\mathrm{an}}_k(L)} \mathcal{O}_{x_L}$. On peut étendre cette construction pour obtenir une fonction $h_L : \mathcal{O}_L \to \mathbb{R}^{(n+1)^2}_{\geq 0}$. Il faut noter que la famille $\mathcal{O}_{x_L}$ dépend de la carte affine de $\mathbb{P}^n$ qu'on avait choisi. Dans le cas de $\mathbb{P}^1_k$ avec $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$, la famille $\mathcal{O}_{x_L}$ obtenue par cette construction est analysée dans la Section 8.1. Dans ce qui suit, nous expliquons comment imposer un ordre sur la famille $\mathcal{O}_L$.

Remark 1.5.

Nous introduisons une famille S des fonctions de $\mathbb{R}^{(n+1)^2}_{>0}$ à $\mathbb{R}_{>0}$. L'ensemble S se compose des fonctions $g : \mathbb{R}^{(n+1)^2}_{>0} \to \mathbb{R}_{>0}$ qui satisfont les propriétés suivantes.

1. *La fonction g est continue.*
2. *Si $(r_{i,j})_{i,j}, (s_{i,j})_{i,j} \in \mathbb{R}^{n+1^2}$ sont tels que $r_{i,j} \leq s_{i,j}$ pour tout i, j, on a l'inégalité $g((r_{i,j})_{i,j}) \leq g((s_{i,j})_{i,j})$.*
3. *La fonction g est définissable dans le langage des groupes Abéliennes ordonnés.*

Soit $g \in S$. La fonction définit un ordre total sur l'ensemble $\mathbb{R}^{(n+1)^2}_{>0}$ comme

suit. Soit $\mathbf{r}, \mathbf{s} \in \mathbb{R}_{>0}^{(n+1)^2}$. *Nous définissons* $\mathbf{r} \leq_g \mathbf{s}$ *si* $g(\mathbf{r}) \leq g(\mathbf{s})$. *Il faut noter que l'ordre total* $\leq_g$ *sur* $\mathbb{R}_{>0}^{(n+1)^2}$ *est donné par* $(r_{i,j})_{i,j} \leq (s_{i,j})_{i,j}$ *si* $r_{i,j} \leq s_{i,j}$ *pour tout* i, j *où* $(r_{i,j})_{i,j}, (s_{i,j})_{i,j} \in \mathbb{R}_{>0}^{(n+1)^2}$.

On peut étendre g *pour obtenir une fonction* $\tilde{g} : \mathbb{R}_{\geq 0}^{(n+1)^2} \to \mathbb{R}_{\geq 0}$ *comme suit. Pour tout* $\mathbf{r} \in \mathbb{R}_{>0}^{(n+1)^2}$, *on définit* $\tilde{g}(\mathbf{r}) := g(\mathbf{r})$ *et si* $\mathbf{r} \in \mathbb{R}_{\geq 0}^{(n+1)^2} \smallsetminus \mathbb{R}_{>0}^{(n+1)^2}$ *on suppose* $\tilde{g}(\mathbf{r}) := 0$. *Comme pour la fonction* g, $\tilde{g}$ *définit un ordre total sur* $\mathbb{R}_{\geq 0}^{(n+1)^2}$ *qui étend l'ordre partiel donné par* $(r_{i,j})_{i,j} \leq (s_{i,j})_{i,j}$ *si* $r_{i,j} \leq s_{i,j}$ *pour tout* i, j *où* $(r_{i,j})_{i,j}, (s_{i,j})_{i,j} \in \mathbb{R}_{\geq 0}^{(n+1)^2}$. *Dans ce qui suit, on utilise* g *à la place de l'extension* $\tilde{g}$.

Soit $g \in S$. Par Lemme 8.1.2, la fonction $g \circ h_L : \mathcal{O}_L \to \mathbb{R}_{\geq 0}$ a la propriété suivante. Si $O_1, O_2 \in \mathcal{O}_L$ tel que $O_1 \subseteq O_2$, on a $(g \circ h_L)(O_1) \leq (g \circ h_L)(O_2)$. Les fonctions $g \in S$ nous permettent de quantifier la taille des éléments dans $\mathcal{O}_L$. Nous donnons, maintenant, une version équivalente du Théorème 1.2.2 que nous pouvons généraliser. Le but du Théorème 1.2.2 est de démontrer l'éxistence d'un complexe simplicial fini Υ contenu dans $\mathbb{P}_k^{1,\mathrm{an}}$ tel que la fonction f soit constante le long des fibres de la rétraction $\psi(e, _)$. Supposons que le Théorème 1.2.2 est vrai. Nous définissons une fonction $M : \Upsilon \to \mathbb{R}_{\geq 0}$ comme suit. Soit $\gamma \in \Upsilon$ et un point $x \in \mathbb{P}_k^{1,\mathrm{an}}$ tel que $\psi(e, x) = \gamma$. Nous fixons $M(\gamma) := f(x)$. Soit L/k une extension non archimédienne et algébriquement close telle que $x \in \mathbb{P}_k^{1,\mathrm{an}}(L)$. Par définition, $M(\gamma)$ est le minimum de 1 et du rayon de la plus grande boule de Berkovich contenue dans $\mathbb{P}_L^{1,\mathrm{an}}$ de centre x_L dont l'image réciproque est union disjointe de copies homéomorphes de lui-même pour le morphisme ϕ_L^{an}. Par hypothèse, la fonction M est bien définie. En conséquence, une autre formulation du Théorème 1.2.2 est la suivante.

Theorem 1.2.6. *Soit* $\phi : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ *un morphisme fini. Il existe un intervalle réel généralisé* $I := [i, e]$ *et une rétraction par déformation*

$$\psi : I \times \mathbb{P}_k^{1,\mathrm{an}} \to \mathbb{P}_k^{1,\mathrm{an}}$$

qui satisfasse les propriétés suivantes.

1. *L'image* $\psi(e, \mathbb{P}_k^{1,\mathrm{an}})$ *de la rétraction par déformation* ψ *est un complexe simplicial. Soit* Υ *ce complexe fini.*

2. *Il existe une fonction* $M : \Upsilon \to [0, 1]$ *avec les propriétés suivantes. La fonction* M *n'est pas identiquement zéro et* $\log(M)$ *est linéaire par morceaux. Soit* $\gamma \in \Upsilon$ *tel que* $M(\gamma) > 0$ *et* $x \in \psi(e, _)^{-1}(\gamma)$. *Soit* L/k *une extension non archimédienne et algébriquement close telle que* $x \in \mathbb{P}_k^{1,\mathrm{an}}(L)$.

   (a) *L'image réciproque par le morphisme* ϕ_L^{an} *de la boule ouverte de Berkovich* $B(x_L, M(\gamma)) \subset \mathbb{P}_L^{1,\mathrm{an}}$ *centrée à* x_L *de rayon* $M(\gamma)$ *se decompose comme union disjointe de boules ouvertes de Berkovich, chacune homéomorphe à* $B(x_L, M(\gamma))$ *via le morphisme* ϕ_L^{an}.

   (b) *Soit* O *une boule ouverte de Berkovich de centre* x_L *dont le rayon est inférieur ou égal à* 1, *telle que son image réciproque par le morphisme* ϕ_L^{an} *se decompose comme union disjointe de boules de Berkovich chacune homéomorphe à* O *via* ϕ_L^{an}. *Le rayon* O *doit être inférieur ou égal à* $M(\gamma)$.

Dans la Section 8.5, nous montrons que les Théorèmes 1.2.2 et 1.2.6 sont équivalents. Nous introduisons, maintenant, une généralisation du Théorème 1.2.2. Soit $\phi : V' \to V$ un morphisme fini et surjectif entre deux variétés irréductibles, projectives de type finis sur k. Pour une extension L/k non archimédienne complète et algébriquement close, on désigne par ϕ_L le morphisme

$$\phi \times \mathrm{id}_L : V \times_k L \to V \times_k L.$$

Comme avant, nous écrivons $\phi^{\mathrm{an}} : V'^{\mathrm{an}} \to V^{\mathrm{an}}$ pour le morphisme induit entre les deux analytifications. Le morphisme ϕ_L^{an} est défini de manière similaire et il faut noter que $\phi_L^{\mathrm{an}} = \phi^{\mathrm{an}} \times \mathrm{id}_L^{\mathrm{an}}$. Nous fixons une immersion $V \hookrightarrow \mathbb{P}_k^n$ et une carte affine de $\mathbb{P}^n$. On utilisera les familles $\mathcal{O}_{x_L}$ et les fonctions g et h_L dans l'énoncé suivant.

Theorem 8.4.3. *Soit $\phi : V' \to V$ un morphisme fini surjectif entre deux variétés irréductibles, projectives avec V normale. Soit $g \in S$. Il existe un intervalle réel généralisé $I := [i, e]$ et une rétraction par déformation*

$$\psi : I \times V^{\mathrm{an}} \to V^{\mathrm{an}}$$

qui satisfasse les propriétés suivantes.

1. *L'image $\psi(e, V^{\mathrm{an}})$ de la rétraction par déformation ψ est un complexe simplicial fini qu'on désigne par Υ_g.*

2. *Il existe une fonction $M_g : \Upsilon_g \to \mathbb{R}_{\geq 0}$ qui satisfasse les propriétés suivantes. La fonction M_g n'est pas identiquement zéro. Soit $\gamma \in \Upsilon_g$ un point du complexe fini simplicial pour lequel $M_g(\gamma) \neq 0$ et $x \in \psi(e, _)^{-1}(\gamma)$. Soit L/k une extension non archimédienne et algébriquement close telle que $x \in V^{\mathrm{an}}(L)$. Il existe $W \in (g \circ h_L)^{-1}(M_g(\gamma)) \cap \mathcal{O}_{x_L}$ tel que l'ensemble ouvert $(\phi_L^{\mathrm{an}})^{-1}(W \cap V_L^{\mathrm{an}}) \subset V_L'^{\mathrm{an}}$ se décompose comme union disjointe d'ensembles ouverts, chacun homéomorphe à $W \cap V_L^{\mathrm{an}}$ via ϕ_L^{an}. De plus, soit $O \in \mathcal{O}_{x_L}$ telle que l'image réciproque de $O \cap V_L^{\mathrm{an}}$ pour ϕ_L^{an} se decompose comme union disjointe d'ensembles ouverts contenus dans $V_L'^{\mathrm{an}}$, chacun homéomorphe à O via ϕ_L^{an}. On a l'inégalité $(g \circ h_L)(O) \leq M_g(\gamma)$. Enfin, la fonction $\log(M_g)$ est linéaire par morceaux sur Υ_g.*

Remark 1.7. *Soit x un point défini sur L. Dans la démonstration de 8.4.3, on montrera que la fonction M_g est bien définie. Cela impliquera, en particulier, que la valeur $M_g(x)$ ne dépend que du point x.*

Il faut mentionner qu'il y a une similarité entre ce résultat dans le cas des courbes lisses analytiques de Berkovich sur un corps de caractéristique zéro et les théorèmes de [PP] et [Bal]. Dans [Bal], F. Baldassarri a étudié un système d'équations différentielles défini sur un domaine analytique de la droite affine sur un corps non archimédien de caractéristique zéro. Plus précisément, soit k un corps non archimédien de caractéristique zéro et X un domaine relativement compact de la droite affine $\mathbb{A}_k^{1,\mathrm{an}}$. Soit

$$\Sigma : dy/dT = Gy$$

un système d'équations différentielles où G est une matrice $\mu \times \mu$ de fonctions k-analytiques sur X. Si $x \in X$ est un point rationnel, soit $R(x) = R(x, \Sigma)$ le rayon du disque ouvert dans X de centre x sur lequel toutes les solutions de Σ convergent. L'auteur montre que la fonction R est continue. En outre, il démontre que dans le cas $X = \mathbb{A}_k^{1,\mathrm{an}}$, le comportement de la fonction R est contrôlé par un graphe fini $\Gamma \subset \mathbb{A}_k^{1,\mathrm{an}}$. Comme il existe une rétraction de $\mathbb{A}_k^{1,\mathrm{an}}$ sur tout sous-graphe fini, la fonction R est constante le long des fibres de la rétraction sur Γ. Si on garde les restrictions sur le corps k et que l'on considère le cas d'un système d'équations différentielles défini sur une courbe lisse de Berkovich, on a un résultat similaire. Dans [PP], les auteurs - Poineau et Pulita - démontrent qu'il existe un graphe localement fini contenu dans la courbe qui est l'image d'une rétraction de la courbe telle que la fonction du rayon de convergence soit constante le long des fibres de la rétraction. Nos résultats et les résultats de Poineau-Pulita et Baldassari montrent que le comportement de certaines fonctions intéressantes est contrôlé par les complexes simplicaux associés.

Chapter 3

Model Theory

The goal of Model theory is to study mathematical structures. The field of real numbers is an example of a mathematical structure. Naively speaking, the field $\mathbb{R}$ is a set with distinguished elements - 0 and 1, distinguished functions $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\times : \mathbb{R}^2 \to \mathbb{R}$ and a binary relation $<$. This collection $\mathfrak{L}_{or} := \{0, 1, +, -, \times, <\}$ of constants, functions and relations allows us to describe many properties of the real numbers. For instance, the fact that every real number has a multiplicative inverse can be expressed in the statement $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x \times y = 1$. The collection $\mathfrak{L}_{or}$ is an example of a *language* and such collections are the basic tools with which we describe objects of interest.

Definition 3.0.1. *A language* $\mathfrak{L}$ is given by specifying the following data.

1. A set of function symbols $\mathfrak{F}$ and positive integers n_f for each $f \in \mathfrak{F}$.
2. A set of relation symbols $\mathfrak{R}$ and positive integers n_r for each $r \in \mathfrak{R}$.
3. A set of constant symbols $\mathfrak{C}$.

The numbers n_f and n_r tell us that f is a function of n_f variables and r is an n_r-ary relation.

A language $\mathfrak{L} = (\mathfrak{F}, \mathfrak{R}, \mathfrak{C})$ can be used to describe the properties of sets in which the functions in $\mathfrak{F}$, the relations in $\mathfrak{R}$ and the constants in $\mathfrak{C}$ can be interpreted. Such sets are called the structures associated to $\mathfrak{L}$.

Definition 3.0.2. An $\mathfrak{L}$-*structure* $\mathcal{M}$ consists of a non-empty set M which we call the domain of $\mathcal{M}$, distinguished elements $\{c^{\mathcal{M}}\}$ for every $c \in \mathfrak{C}$, functions $f^{\mathcal{M}} : M^{n_f} \to M$ for every $f \in \mathfrak{F}$ and relations $r^{\mathcal{M}} \subseteq M^{n_r}$ for every $r \in \mathfrak{R}$.

When there is no ambiguity about the given structure, we simplify notation and write f in place of $f^{\mathcal{M}}$ for $f \in \mathfrak{L}$. Likewise, for the constants $c^{\mathcal{M}}$ and the relations $r^{\mathcal{M}}$. We will assume that every language $\mathfrak{L}$ includes a 2-ary relation $=$ which is to be interpreted as equality of elements in any of its structures.

Definition 3.0.3. An *isomorphism of* $\mathfrak{L}$-*structures* $\mathcal{M}$ *and* $\mathcal{N}$ is a bijective map $\psi : M \to N$ such that

1. For every $c \in \mathfrak{C}$, $\psi(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

2. For every $f \in \mathfrak{F}$ and $\bar{a} := (a_1, \ldots, a_{n_f}) \in M^{n_f}$, $\psi(f^{\mathcal{M}}(\bar{a})) = f^{\mathcal{N}}(\psi(\bar{a}))$. where $\psi(\bar{a}) := (\psi(a_1), \ldots, \psi(a_{n_f}))$.

3. For every $r \in \mathfrak{R}$ and $\bar{a} := (a_1, \ldots, a_{n_r}) \in M^{n_r}$, $\bar{a} \in r^{\mathcal{M}}$ if and only if $\psi(\bar{a}) \in r^{\mathcal{N}}$.

In the above definition if the map ψ were only injective then we say that it is an *embedding*.

Example 3.0.4.

1. The language $\mathfrak{L}_{or} = \{0, 1, +, -, \times, <\}$ is the language of ordered rings. The fields $\mathbb{R}$ and $\mathbb{Q}$ are $\mathfrak{L}_{or}$ structures. Clearly, not every $\mathfrak{L}_{or}$-structure need be a field. The integers $\mathbb{Z}$ for instance is an $\mathfrak{L}_{or}$-structure.

2. The language of rings $\mathfrak{L}_r$ is given by $\{0, 1, -, +, \times\}$. The field of complex numbers $\mathbb{C}$ is an $\mathfrak{L}_r$-structure. Any $\mathfrak{L}_{or}$-structure is also an $\mathfrak{L}_r$-structure.

3. Consider the language $\mathfrak{L}_s = \emptyset$. The structures of $\mathfrak{L}_s$ are sets.

Let $\mathfrak{L}$ be a language and $\mathcal{M}$ be an $\mathfrak{L}$-structure. The properties of $\mathcal{M}$ can be described by strings of symbols built using the symbols of the language $\mathfrak{L}$, variable symbols, equality - '=', Boolean connectives $\wedge$, $\vee$ and $\neg$ and the quantifiers $\forall$ and $\exists$. We call such strings formulae. In order to provide a precise definition of a formula in a given language, we introduce the notion of a term.

Definition 3.0.5. Let $\mathfrak{L} = (\mathfrak{F}, \mathfrak{R}, \mathfrak{C})$ be a language. *The set of $\mathfrak{L}$-terms $\mathcal{T}$* is the smallest set such that

1. For every constant symbol $c \in \mathfrak{C}$, $c \in \mathcal{T}$.

2. Each variable symbol belongs to $\mathcal{T}$.

3. If $f \in \mathfrak{F}$ and $t_1, \ldots, t_{n_f} \in \mathcal{T}$ then $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

Consider as an example, the expression $t := \times(+(v_1, v_2), +(v_3, 1))$ in the language $\mathfrak{L}_r$. This is an $\mathfrak{L}_r$ term. When given a ring R, this terms defines a function $t : R^3 \to R$ by sending $(x, y, z) \in R$ to $(x + y)(z + 1)$. This is in fact a specific case of a more general phenomenon.

Lemma 3.0.6. *Let $\mathfrak{L}$ be a language and $\mathcal{M}$ be an $\mathfrak{L}$-structure. Let t be an $\mathfrak{L}_r$-term built using the variables $(v_1, \ldots, v_n)$. Then t can be interpreted as a function $t^{\mathcal{M}} : M^n \to M$.*

Proof. Let s be a sub - term of t. Let $\bar{a} := (a_1, \ldots, a_n) \in M^n$. We define $s^{\mathcal{M}}(\bar{a})$ inductively. This defines the function $t^{\mathcal{M}} : M^n \to M$.

Suppose s is a constant symbol $c \in \mathfrak{C}$. We interpret $s^{\mathcal{M}}$ to be the constant symbol $c^{\mathcal{M}}$. Let s be a variable symbol v_i. We set $s^{\mathcal{M}}(\bar{a})$ to be a_i. Lastly, if $s = f(t_{j_1}, \ldots, t_{j_s})$ where $f \in \mathfrak{F}$ and the t_{j_o} are sub terms of t for which $t_{j_o}^{\mathcal{M}}(\bar{a})$ is well defined then we set $s^{\mathcal{M}}(\bar{a})$ to be $f^{\mathcal{M}}(t_{j_1}^{\mathcal{M}}(\bar{a}), \ldots, t_{j_s}^{\mathcal{M}}(\bar{a}))$. $\square$

The formulae associated to a language $\mathfrak{L}$ can be built up using $\mathfrak{L}$-terms and the relation symbols $r \in \mathfrak{R}$.

Definition 3.0.7. *The set of atomic $\mathfrak{L}$-formulae* is the smallest set such that

1. If t_1, t_2 are $\mathfrak{L}$-terms then the expression $t_1 = t_2$ is an atomic $\mathfrak{L}$-formula.

2. If $r \in \mathfrak{R}$ is an n-ary relation and $t_1, \ldots, t_n$ are $\mathfrak{L}$-terms then $r(t_1, \ldots, t_n)$ is an atomic $\mathfrak{L}$-formula.

The set $\mathcal{W}$ of $\mathfrak{L}$-formulae is the smallest set containing the atomic $\mathfrak{L}$-formulae such that

1. If $\phi \in \mathcal{W}$ then $\neg\phi \in \mathcal{W}$.

2. If $\phi, \psi \in \mathcal{W}$ then $\phi \wedge \psi \in \mathcal{W}$ and $\phi \vee \psi \in \mathcal{W}$.

3. If $\phi \in \mathcal{W}$ and the variable w occurs in ϕ then $\exists w\ \phi \in \mathcal{W}$ and $\forall w\ \phi \in \mathcal{W}$.

Let ϕ be an $\mathfrak{L}$-formula. A variable in ϕ that is not bounded by a quantifier is called a *free variable of* ϕ. For example, consider the formula $\phi := \exists y\ x = y^2$ in the language $\mathfrak{L}_r$. The variable x is not bounded by a quantifier and is hence free. If the free variables occuring in the formula ϕ are $\{v_1, \ldots, v_n\}$ then we write $\phi(v_1, \ldots, v_n)$ in place of ϕ. A formula which does not have any free variables is called a *sentence*. The $\mathfrak{L}_r$-formula given by $\forall x\ \exists y\ y^2 = x$ is an example of a sentence.

An $\mathfrak{L}$-sentence ϕ expresses a property that might or might not be true for a given $\mathfrak{L}$-structure $\mathcal{M}$. When ϕ holds for $\mathcal{M}$, we write $\mathcal{M} \models \phi$. We now provide a precise definition.

Definition 3.0.8. Let $\phi(v_1, \ldots, v_n)$ be a formula and $\mathcal{M}$ be an $\mathfrak{L}$-structure. Let $\bar{a} := (a_1, \ldots, a_n) \in M^n$. *We define inductively what it means to say* $\mathcal{M} \models \phi(a_1, \ldots, a_n)$.

1. Suppose ϕ were of the form $t_1 = t_2$ where the t_i are $\mathfrak{L}$-terms. Then $\mathcal{M} \models \phi$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

2. Suppose $\phi = r(t_1, \ldots, t_n)$ where $r \in \mathfrak{R}$ and the t_i are $\mathfrak{L}$-terms. Then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \in r^{\mathcal{M}}$.

3. We say that $\mathcal{M} \models \neg\phi(\bar{a})$ if $M \not\models \phi(\bar{a})$.

4. If $\phi = \psi \wedge \psi'$ then $\mathcal{M} \models \phi$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \psi'(\bar{a})$. Likewise, if $\phi = \psi \vee \psi'$ then $\mathcal{M} \models \phi$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \psi'(\bar{a})$.

5. If $\phi(\bar{v})$ is of the form $\exists w\ \psi(w, \bar{v})$ then $\mathcal{M} \models \phi(\bar{a})$ if for some $b \in M$, $\mathcal{M} \models \psi(b, \bar{a})$. Similarly, if $\phi(\bar{v})$ is of the form $\forall w\ \psi(w, \bar{v})$ then $\mathcal{M} \models \phi(\bar{a})$ if for all $b \in M$, $\mathcal{M} \models \psi(b, \bar{a})$.

Remark 3.0.9. We make use of certain abbreviations which are commonly used. For example, given sentences ϕ and ψ in some language, we write $\phi \to \psi$ in place of the sentence $\neg\phi \vee \psi$ which means that ψ if ϕ holds. Likewise, $\phi \leftrightarrow \psi$ is a restatement of the sentence $(\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi)$.

3.1 Theories

A given language can be used to study a multitude of mathematical structures. The language of rings $\mathfrak{L}_r$, for example, can be used to describe the structures $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$. However, there exists sentences in $\mathfrak{L}_r$ which hold for the complex numbers but do not hold for the rationals. Similarly, the sentence which asserts the existence of a multiplicative inverse for every element holds in the rationals but fails to do so in the integers. To understand a structure, it is hence natural to understand the sentences in the language which hold true in that structure and the other structures of that language which also satisfy this collection of sentences. In the case of the complex numbers for example, we find that any $\mathfrak{L}_r$-structure which satisfies every $\mathfrak{L}_r$-sentence that holds true in $\mathbb{C}$ must be an algebraically closed field of characteristic zero. Conversely, any algebraically closed field of characteristic zero satisfies every $\mathfrak{L}_r$-sentence that holds true in $\mathbb{C}$.

Definition 3.1.1. Let $\mathfrak{L}$ be a language. *An $\mathfrak{L}$-theory* $\mathfrak{T}$ is a collection of $\mathfrak{L}$-sentences. We say that a theory is *satisfiable* if there exists an $\mathfrak{L}$-structure M such that $M \models \phi$ for every $\phi \in \mathfrak{T}$. The structure $\mathcal{M}$ is then called a *model of* $\mathfrak{T}$ and we write $T \models \mathcal{M}$.

Example 3.1.2. In what follows, we refer to theories of a certain class of mathematical structures. For example, the $\mathfrak{L}_r$-theory ACF of algebraically closed fields. What we mean in such instances is that a structure of the underlying language is a model for that theory if and only if it belongs to that class of mathematical structures. An $\mathfrak{L}_r$-structure is a model for ACF if and only if it is an algebraically closed field.

1. Let $\mathfrak{L}_g := \{0, +\}$ be the language of groups. The theory $\mathfrak{T}_{ag}$ of abelian groups consists of the sentences which form the commutative group axioms. For example, the sentence $\forall x\ \exists y\ x + y = 0$ which asserts the existence of an additive inverse for every element in a model of $\mathfrak{T}_{ag}$.

2. Let $\mathfrak{L}_r = \{0, 1, \times, +, -\}$ denote the language of rings. The theory of rings $\mathfrak{T}_r$ consists of the sentences which form the commutative group axioms with respect to the operation $+$, the axioms which assert that $\times$ is a commutative operation and the fact that $\times$ distributes over $+$ i.e. $\forall x, y, z\ x \times (y + z) = x \times y + x \times z$. The theory $\mathfrak{T}_f$ of fields includes the sentences of the theory $\mathfrak{T}_r$ and in addition contains the axioms which assert that a model of $\mathfrak{T}_r$ must be an abelian group with respect to the operation $\times$. The theory of algebraically closed fields ACF contains the sentences of the theory $\mathfrak{T}_f$ and in addition every sentence of the form $\forall y_1, \ldots, y_n\ \exists x\ x^n + y_n \times x^{n-1} + \ldots + y_0 = 0$ where $n \in \mathbb{N}$. Let p be a prime. By adding the sentence $t_p := p = 0$ to ACF we get ACF_p the theory of algebraically closed fields of characteristic p. By p in the formula t_p, we mean the iterated value of $1 + 1 + \ldots + 1$, p - times. The theory ACF_0 of algebraically closed fields of characteristic zero can be obtained by adding the set of sentences $\{\neg t_p\}_p$ to ACF.

3. Let $\mathfrak{L}_s = \emptyset$ be the theory of sets. If $T_s := \emptyset$ then the models of T_s are sets. If $\mathfrak{T}_{is}$ is the set of sentences of the form $\exists x\ \forall y_1, \ldots, y_n\ x \neq y_1 \wedge \ldots x \neq y_n$ then any model of $\mathfrak{T}_{is}$ is an infinite set.

Definition 3.1.3. Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. An $\mathfrak{L}$-sentence ϕ is said to be *a logical consequence of the theory* $\mathfrak{T}$ if $\mathcal{M} \models \phi$ for every model $\mathcal{M}$ of $\mathfrak{T}$. In this case, we write $\mathfrak{T} \models \phi$.

A great deal of mathematics is concerned with showing that a certain sentence ϕ is a logical consequence of a given theory $\mathfrak{T}$. The method by which this is accomplished is to provide a proof of the sentence. In model theoretic terms, a proof is a sequence of sentences $\psi_1, \ldots, \psi_n$ such that $\psi_n = \phi$ and ψ_i either belongs to $\mathfrak{T}$ or follows from ψ_{i_1} by some simple logical rule. An example of a simple logical rule is "if ϕ and ψ then conclude $\phi \wedge \psi$". If there exists a proof for the sentence ϕ then we write $\mathfrak{T} \vdash \phi$. Although there are several proof systems and we do not concern ourselves with discussing any of them here, they share certain salient features, namely :

1. All proofs are finite.
2. All proofs are sound i.e. if $\mathfrak{T} \vdash \phi$ then $\mathfrak{T} \models \phi$.
3. There exists an algorithm to check whether or not a given sequence of sentences is indeed a proof.

The following result of Godel is one of the most well known theorems in mathematical logic and effectively says that the notions of logical consequence and that of giving a proof are equivalent.

Theorem 3.1.4. *Let $\mathfrak{L}$ be a language, $\mathfrak{T}$ be an $\mathfrak{L}$-theory and ϕ be an $\mathfrak{L}$-sentence. Then $\mathfrak{T} \models \phi$ if and only if $\mathfrak{T} \vdash \phi$.*

Definition 3.1.5. An $\mathfrak{L}$-theory $\mathfrak{T}$ is said to be *inconsistent* if there exists an $\mathfrak{L}$-sentence ϕ for which $\mathfrak{T} \vdash \phi$ and $\mathfrak{T} \vdash \neg\phi$. The theory $\mathfrak{T}$ is *consistent* if and only if it is not inconsistent.

An easy consequence of the completeness theorem is the following.

Corollary 3.1.6. *An $\mathfrak{L}$-theory $\mathfrak{T}$ is consistent if and only if it is satisfiable.*

Proof. Suppose $\mathfrak{T}$ is satisfiable and $\mathcal{M}$ is a model of $\mathfrak{T}$. Let ϕ be a sentence such that $T \vdash \phi$ and $T \vdash \neg\phi$. It follows from the soundness of a proof system, that $\mathcal{M} \models \phi \wedge \neg\phi$. This is not possible and implies that $\mathfrak{T}$ is consistent.

Conversely, suppose that $\mathfrak{T}$ is consistent and unsatisfiable. It follows that for every model $\mathcal{M}$ of $\mathfrak{T}$ and every $\mathfrak{L}$-sentence ϕ, $\mathcal{M} \models \phi \wedge \neg\phi$. By the completeness theorem, this implies that for any $\mathfrak{L}$-sentence ϕ, $\mathfrak{T} \vdash \phi \wedge \neg\phi$ and in turn that $\mathfrak{T}$ is inconsistent. It follows that $\mathfrak{T}$ is indeed satisfiable. $\square$

The completeness theorem can be used to prove the following proposition which we refer to as the *compactness proposition*. It asserts that a theory is satisfiable if and only if it is finitely satisfiable. It will be of great use to us in subsequent chapters.

Proposition 3.1.7. *Let $\mathfrak{T}$ be an $\mathfrak{L}$-theory. Suppose that for every finite set of sentences $\Delta \subset \mathfrak{T}$, there exists an $\mathfrak{L}$-structure $\mathcal{M}$ such that $\Delta \models \mathcal{M}$. Then the theory $\mathfrak{T}$ is satisfiable.*

Proof. Let us suppose that the theory $\mathfrak{T}$ is not satisfiable. It follows that for any $\mathfrak{L}$-sentence ϕ, $\mathfrak{T} \models \phi \wedge \neg\phi$. By the completeness theorem, we must have that $T \vdash \phi \wedge \neg\phi$. As all proofs are finite, there exists a finite set Δ of sentences in $\mathfrak{T}$ such that $\Delta \vdash \phi \wedge \neg\phi$. The hypothesis of the proposition implies that Δ is satisfiable. The corollary to the completeness theorem however implies that Δ cannot be satisfiable. Hence, we must have that $\mathfrak{T}$ is satisfiable. $\square$

Amongst the several applications of the above proposition, we state the following striking consequence.

Theorem 3.1.8. (Lowenheim - Skolem) *Suppose there exists an infinite model $\mathcal{M}$ of a given $\mathfrak{L}$-theory $\mathfrak{T}$. Then there exists models of $\mathfrak{T}$ of every infinite cardinality κ where $\kappa \geq |\mathfrak{L}|$.*

3.1.1 Complete Theories

Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. A way to classify different $\mathfrak{L}$-structures would be to try and understand which $\mathfrak{L}$-sentences hold in different structures i.e. the various theories for which these structures are models. Given an $\mathfrak{L}$-structure $\mathcal{M}$, the theory $\mathrm{Th}(\mathcal{M})$ is the collection of all $\mathfrak{L}$-sentences ϕ such that $\mathcal{M} \models \phi$. This theory has the property that if ϕ is an $\mathfrak{L}$ - sentence then either $\mathrm{Th}(\mathcal{M}) \models \phi$ or $\mathrm{Th}(\mathcal{M}) \models \neg\phi$. In practice however, listing out the sentences which hold in a given structure might be untenable. It is often simpler to find a theory T for which the structure $\mathcal{M}$ is a model and which has the additional property that if ϕ is an $\mathfrak{L}$ - sentence then either $T \models \phi$ or $T \models \neg\phi$. Such a theory would then be equivalent to $\mathrm{Th}(\mathcal{M})$ i.e. every model of $\mathrm{Th}(\mathcal{M})$ is a model of T and conversely, every model of T would be a model of $\mathrm{Th}(\mathcal{M})$. The theory T, in this case, is said to be complete.

Definition 3.1.9. An $\mathfrak{L}$-theory $\mathfrak{T}$ is said to be *complete* if for every $\mathfrak{L}$-sentence ϕ either $\mathfrak{T} \models \phi$ or $\mathfrak{T} \models \neg\phi$.

Observe that if the theory $\mathfrak{T}$ is complete and satisfiable then for any $\mathfrak{L}$-sentence either it or its negation must be a logical consequence of $\mathfrak{T}$ but not both. The property of being complete can be restated using the notion of elementary equivalence.

Definition 3.1.10. Let $\mathcal{M}, \mathcal{N}$ be models of an $\mathfrak{L}$-theory T. We say $\mathcal{M}$ *is elementarily equivalent to* $\mathcal{N}$ if for every $\mathfrak{L}$-sentence ϕ, $\mathcal{M} \models \phi \leftrightarrow \mathcal{N} \models \phi$. We then write $\mathcal{M} \equiv \mathcal{N}$.

Proposition 3.1.11. *Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. The following statements are equivalent.*

1. *The theory $\mathfrak{T}$ is complete.*
2. *If $\mathcal{M}, \mathcal{N}$ are two models of $\mathfrak{T}$ then $\mathcal{M} \equiv \mathcal{N}$.*

Vaught's test provides us with a necessary condition for a theory to be complete. However in order to state it, we must first introduce the notion of categoricity.

Definition 3.1.12. Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. Let κ be a cardinal. The theory $\mathfrak{T}$ is said to be *κ-categorical* if any two models of $\mathfrak{T}$ of cardinality κ are elementarily equivalent.

Example 3.1.13.

1. Let $\mathfrak{L}_s$ be the language of sets and $\mathfrak{T}_s$ be the theory of sets. The theory $\mathfrak{T}_s$ is κ-categorical for every cardinal κ.

2. Let $\mathfrak{L}_g$ be the language of groups and TFDAG be the theory of torsion free divisible abelian groups. Every model of TFDAG can be seen as a $\mathbb{Q}$-vector space. Conversely, every $\mathbb{Q}$-vector space is a model of TFDAG. Furthermore, if V_1 and V_2 are $\mathbb{Q}$-vector spaces and $f : V_1 \to V_2$ is an isomorphism of vector spaces then it can be checked that f is in fact an isomorphism of $\mathfrak{L}_g$-structures, implying that $V_1 \equiv V_2$. We claim that the theory TFDAG is κ-categorical when κ is any uncountable cardinal. Indeed, let V_1 and V_2 be $\mathbb{Q}$-vector spaces of cardinality κ where κ is an uncountable cardinal. If B_1 and B_2 are bases of V_1 and V_2 respectively then we must have that the B_i are of cardinality κ as well. It follows that there exists a set theoretic bijection between B_1 and B_2. But any such bijection implies a linear isomorphism between the vector spaces V_1 and V_2.

3. Let $\mathfrak{L}_r$ denote the language of rings and ACF_p denote the theory of algebraically closed fields of characteristic p where p is either 0 or prime. We claim that ACF_p is κ-categorical when κ is an uncountable cardinal. Indeed, two algebraically closed fields are isomorphic if and only if they have the same characteristic and the same transcendence degree. If F is a model of ACF_p then $|F| = \mathrm{tr.deg}(F) + |\mathbb{N}|$. It follows that if $|F|$ is uncountable of cardinality κ then its transcendence degree must also be κ.

Proposition 3.1.14. *(*Vaught's Test*) Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. Suppose that the theory $\mathfrak{T}$ is κ-categorical where κ is some infinite cardinal and in addition that each of its models is infinite. Then $\mathfrak{T}$ is complete.*

Corollary 3.1.15. *The following theories are complete.*

1. *The theory* ACF_p*, where p is either* 0 *or prime.*

2. *The theory TFDAG of torsion free divisible abelian groups.*

3. *The theory $\mathfrak{T}_{is}$ of infinite sets.*

We end our discussion on completeness with the following application of the corollary and the completeness theorem.

Proposition 3.1.16. *Let $\mathfrak{L}_r$ be the theory of rings and ϕ be an $\mathfrak{L}_r$-sentence. The following statements are equivalent.*

1. *The sentence ϕ is a logical consequence of the theory of algebraically closed fields of characteristic zero i.e.* $\mathrm{ACF}_0 \models \phi$.

2. *There exists a model F of* ACF_0 *such that $F \models \phi$.*

3. *The sentence ϕ holds for the complex numbers i.e.* $\mathbb{C} \models \phi$.

4. *For arbitrary large primes p, we have that* $\mathrm{ACF}_p \models \phi$.

5. *For arbitrary large primes p, there exists a model F_p of* ACF_p *such that* $F_p \models \phi$.

Proof. The equivalence of (1), (2) and (3) follows from the completeness of ACF_0. Likewise, (4) and (5) are equivalent because of the completeness of ACF_p. It suffices hence to show that (1) is equivalent to (4).

Suppose (1) holds. Then by the Completeness theorem, there exists a finite set of sentences $\Delta \subset \mathrm{ACF}_0$ such that $\Delta \vdash \phi$. As the set of sentences $t_p := p \neq 0$ is infinite, we may assume that there exists a p_0 such that for every $p > p_0$, $t_p \notin \Delta$. It follows that for such $p > p_0$, $\Delta \subset \mathrm{ACF}_p$ and hence $\mathrm{ACF}_p \vdash \phi$. By the completeness theorem, this implies that $\mathrm{ACF}_p \models \phi$.

Suppose (4) holds. Let assume that (1) does not hold. As ACF_0 is a complete theory, we must have that $\mathrm{ACF}_0 \models \neg\phi$. It follows from our arguments above that for arbitrary large primes p, $\mathrm{ACF}_p \models \neg\phi$. However, as ACF_p is satisfiable, this is not possible. Hence (1) must hold. □

3.2 Quantifier Elimination

In the previous section, we introduced the notion of a complete theory and stated without proof that a theory is complete if and only if any two of its models are elementarily equivalent. A weaker notion related to completeness is model completeness.

Definition 3.2.1. Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. The theory $\mathfrak{T}$ is said to be *model complete* if whenever $\mathcal{M}, \mathcal{N}$ are two models of $\mathfrak{T}$ such that $\mathcal{M} \subset \mathcal{N}$ then $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ i.e. any $\mathfrak{L}$-sentence holds in $\mathcal{M}$ if and only if it holds in $\mathcal{N}$.

The definition above can be restated as follows. A theory is model complete if and only if every embedding of a model of the theory into another model is elementary.

Example 3.2.2. Consider the following example of a theory that is not model complete. Let $\mathfrak{L}_g$ be the language of groups and T be the theory of 2-torsion abelian groups. It can be checked that every model of T is a vector space over $\mathbb{F}_2$-the field of two elements and conversely, every vector space over $\mathbb{F}_2$ is a model of T. The model $\mathbb{F}_4$-the field of order 4, is one in which the sentence that asserts the existence of at least three different elements holds true. However this sentence is not true in the sub field $\mathbb{F}_2$ which is also a model of T.

Let $\mathfrak{T}$ be an $\mathfrak{L}$-theory that is not necessarily model complete. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models of $\mathfrak{T}$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Let ϕ be an $\mathfrak{L}$-sentence. Our discussion above implies that ϕ might not hold in one of the $\mathcal{M}_i$ even if it holds in the other. However, if ϕ does not have quantifiers then $\mathcal{M}_1 \models \phi$ if and only if $\mathcal{M}_2 \models \phi$ (Proposition 3.28). In general, a formula is said to be *quantifier free* if it does not contain quantifiers.

Proposition 3.2.3. *Let $\mathfrak{L}$ be a language and $\mathcal{M}_1 \subset \mathcal{M}_2$ be two $\mathfrak{L}$-structures. Let $\bar{v} := (v_1, \ldots, v_n)$ and $\phi(\bar{v})$ be a quantifier free formula in the variables v_i. Then for any $\bar{a} := (a_1, \ldots, a_n) \in M_1^n$, $\mathcal{M}_1 \models \phi(\bar{a})$ if and only if $\mathcal{M}_2 \models \phi(\bar{a})$.*

Definition 3.2.4. An $\mathfrak{L}$-theory is said to have *quantifier elimination* if for every formula $\phi(\bar{v})$, there exists a quantifier free formula $\psi(\bar{v})$ such that $\mathfrak{T} \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

We now provide a necessary and sufficient condition for a theory to possess quantifier elimination.

Proposition 3.2.5. *Let $\mathfrak{L}$ be a language with at least one constant and $\mathfrak{T}$ be an $\mathfrak{L}$-theory. Let $\bar{v} = (v_1, \ldots, v_n)$ and $\phi(\bar{v})$ be an $\mathfrak{L}$-formula. The following statements are then equivalent.*

1. *There exists a quantifier free formula $\psi(\bar{v})$ such that $\mathfrak{T} \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.*
2. *If $\mathcal{M}, \mathcal{N}$ are models of $\mathfrak{T}$ and $\mathcal{A}$ is any sub structure contained in $\mathcal{M} \cap \mathcal{N}$ then for any $\bar{a} := (a_1, \ldots, a_n)$, $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{a})$.*

The proposition above can be used to prove the following corollary.

Corollary 3.2.6. *The theory of algebraically closed fields has quantifier elimination.*

It can be shown that the property of having quantifier elimination is stronger than being model complete.

Proposition 3.2.7. *Let $\mathfrak{L}$ be a language and $\mathfrak{T}$ be an $\mathfrak{L}$-theory that has quantifier elimination. The theory $\mathfrak{T}$ is then model complete.*

Proof. Let $\mathcal{M} \subset \mathcal{N}$ and ϕ be an $\mathfrak{L}$ - sentence. As $\mathfrak{T}$ has quantifier elimination, there exists an $\mathfrak{L}$ - sentence ψ such that $\mathfrak{T} \models (\phi \leftrightarrow \psi)$. By Proposition 3.28, $\mathcal{N} \models \psi$ if and only if $\mathcal{M} \models \psi$. It follows that $\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models \phi$. As ϕ was an arbitrarily chosen $\mathfrak{L}$-sentence, we must have that $\mathcal{M} \subset \mathcal{N}$ is an elementary embedding. $\square$

3.3 Definable sets

The notion of a definable set is central to Model theory. In fact, one of the principal results from Section 8 reduces to showing that the graph of a certain function is a definable set. Further on, we provide an equivalent description of definable sets which is the framework within which Hrushovski and Loeser's results from [HL] are presented. The following definition is phrased in a manner concurrent to the material presented up to this point.

Definition 3.3.1. Let $\mathfrak{L}$ be a language and $\mathcal{M}$ be an $\mathfrak{L}$-structure. A set $X \subset \mathcal{M}^n$ is said to be *definable* if there exists an $\mathfrak{L}$-formula $\phi(\bar{x}, \bar{y})$ where $\bar{x} := (x_1, \ldots, x_n)$ and $\bar{y} := (y_1, \ldots, y_m)$ and an element $\bar{b} := (b_1, \ldots, b_m) \in \mathcal{M}^m$ such that $X = \{\bar{a} = (a_1, \ldots, a_n) \in \mathcal{M}^n | \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$.

In the definition above, if $A \subset M$ contains the b_i then we say that X is an A-definable set. The set A is often referred to as the parameters with which X is defined. When the set A is empty, we say that X is $\emptyset$-definable. Equivalently, we can extend the language $\mathfrak{L}$ by adding constant symbols for each $a \in A$ and let $\mathfrak{L}_A$ denote the extended language. The formula $\phi(\bar{x}, \bar{b})$ is an $\mathfrak{L}_A$-formula and for every $\mathfrak{L}_A$-structure $\mathcal{M}$ we can define a set $X(\mathcal{M}) = \{\bar{a} = (a_1, \ldots, a_n) \in \mathcal{M}^n | \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. In this way, we have realized X as a functor from the category whose objects are $\mathfrak{L}_A$-structures and morphisms are elementary embeddings to the category of sets. We will elaborate on this shortly, before which we suggest the following examples.

Example 3.3.2.

1. Let $\mathfrak{L}_r$ denote the language of rings and K be a model of the $\mathfrak{L}_r$-theory ACF. Let $n \in \mathbb{N}_{>0}$. The following lemma provides a complete description of the definable subsets of K^n.

Lemma 3.3.3. *Let $X \subset K^n$ be a K-definable set. Then X is a Boolean combination of Zariski closed subsets of K^n.*

Proof. Let $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$. We claim that X is defined by an atomic $\mathfrak{L}_r$-formula $\phi(\bar{x}, \bar{y})$ if and only if it is the zero set of a polynomial $p \in K[\bar{X}]$.

Let $\phi(\bar{x}, \bar{y})$ be an atomic $\mathfrak{L}_r$-formula and $\bar{a} \in K^m$. There exists a polynomial $q(\bar{x}, \bar{y}) \in \mathbb{Z}[\bar{X}, \bar{Y}]$ such that $\phi(\bar{x}, \bar{a})$ is equivalent to $q(\bar{x}, \bar{a}) = 0$. As $q(\bar{x}, \bar{a}) \in K[\bar{X}]$, we conclude the forward implication of the claim.

Conversely, let $q \in K[\bar{X}]$ and $X = \{x \in K^n | q(x) = 0\}$. There exists $p \in \mathbb{Z}[\bar{X}, \bar{Y}]$ and $\bar{a} \in K^m$ such that $q = p(\bar{X}, \bar{a})$. It follows that X can be defined by an atomic $\mathfrak{L}$-formula.

We deduce from the claim that $X \subset K^n$ is defined using a quantifier free formula if and only if it is the Boolean combination of Zariski closed subsets of K^n. The lemma now follows as the theory ACF has quantifier elimination. □

Corollary 3.3.4. *Let $K \models ACF$ and $X \subset K$ be a definable set. Then either X or $K \setminus X$ is finite.*

The theory ACF is an example of a strongly minimal theory. A theory $\mathfrak{T}$ is *strongly minimal* if for every model $\mathcal{M} \models \mathfrak{T}$ and every definable subset $X \subset \mathcal{M}$, either X or $M \setminus X$ is finite.

2. Let $\mathfrak{L}_{og}$ be the language of ordered groups and DOAG be the $\mathfrak{L}_{og}$-theory of divisible ordered abelian groups. The theory DOAG has quantifier elimination. As a consequence of this fact it can be shown that if $G \models$ DOAG then a subset $D \subset G$ is definable if and only if it is the union of a finite set and finitely many intervals. These intervals may be unbounded on one side in which case we say that their end points are either $+\infty$ or $-\infty$. The theory DOAG is an example of an o-minimal theory.

Definition 3.3.5. Let $\mathcal{M} := (M, <, ..)$ be an infinite structure. We say that $\mathcal{M}$ is *o-minimal* if every definable subset $X \subset M$ is the union of a finite set and a finite set of intervals with end points in $M \cup \{+\infty, -\infty\}$. A theory T is *o-minimal* if all of its models are o-minimal structures.

Definable sets of a given theory are often the primary objects of study in certain branches of mathematics. For instance in the theory of ACF, the definable sets are algebraic varieties. It is hence useful to introduce the notion of a morphism between definable sets.

Definition 3.3.6. Let $\mathfrak{L}$ be a language and $\mathcal{M}$ be an $\mathfrak{L}$-structure. Let $X \subset M^n$ and $Y \subset M^m$ be definable sets. A map $\phi : X \to Y$ is *definable* if the graph of ϕ defined by $\{(x, y)|\phi(x) = y\}$ is a definable subset of M^{n+m}.

A morphism $\phi : X \to Y$ between definable sets is an isomorphism if there exists a morphism $\phi' : Y \to X$ such that $\phi \circ \phi'$ and $\phi' \circ \phi$ are id_Y and id_X respectively.

The following proposition can be used to determine whether or not a given subset of a structure is definable.

Proposition 3.3.7. *Let $\mathfrak{L}$ be a language and $\mathcal{M}$ be an $\mathfrak{L}$-structure. Let $A \subset M$ and $X \subset M^n$ be an A-definable set. If σ is an automorphism of $\mathcal{M}$ that fixes the elements of the set A then σ restricts to an automorphism of X.*

Proof. Let $\bar{x} = (x_1, \ldots, x_n)$. There exists an $\mathfrak{L}$-formula $\phi(\bar{x}, \bar{y})$ where $\bar{y} = (y_1, \ldots, y_m)$ such that for some $(b_1, \ldots, b_m) \in M^m$, $X = \{(a_1, \ldots, a_n)|\mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. It suffices to show that if $\bar{a} \in X$ then $\sigma(\bar{a}) \in X$ to conclude that σ induces an automorphism of X as the same will then be true for σ^{-1}.

Let $\bar{a} \in X$. Then $\mathcal{M} \models \phi(\bar{a}, \bar{b})$. As σ is an autmorphism this is equivalent to saying that $\mathcal{M} \models \phi(\sigma(\bar{a}), \sigma(\bar{b}))$. The hypothesis implies that σ fixes the b_i and hence we must have that $\mathcal{M} \models \phi(\sigma(\bar{a}), \bar{b})$ which is equivalent to saying that $\sigma(\bar{a}) \in X$. □

An application of the proposition is the following corollary which we state without proof.

Corollary 3.3.8. *The set of real numbers $\mathbb{R}$ is not definable in the complex numbers $\mathbb{C}$.*

3.4 Types

Given a certain structure, the space of *types* allows us to understand the first order properties of elements in elementary extensions of that structure. Later in this section we will make this statement more precise. We follow the presentation in David Marker's text [DM]. Let $\mathfrak{L}$ be a language and $\mathcal{M}$ be an $\mathfrak{L}$-structure. Let $A \subset M$. We may extend the language $\mathfrak{L}$ by adding constant symbols for every element in A. Let $\mathfrak{L}_A$ denote this extended language. We can view $\mathcal{M}$ as an $\mathfrak{L}_A$-structure by interpreting the additional constants in an obvious manner. Let $Th_A(\mathcal{M})$ be the set of $\mathfrak{L}_A$ - sentences which hold in $\mathcal{M}$.

Definition 3.4.1. An *n-type* p is a collection of formulae in n-free variables $(v_1, \ldots, v_n)$ such that $p \cup Th_A(\mathcal{M})$ is satisfiable. We say that the type p is *complete* if for every $\mathfrak{L}_A$-formula ϕ in n-free variables either $\phi \in p$ or $\neg\phi \in p$. The space of complete n-types is denoted $S_n^{\mathcal{M}}(A)$.

We often write $p(v_1, \ldots, v_n)$ to emphasize the fact that p is an n-type. By compactness, a collection p of formulae in n-free variables $(v_1, \ldots, v_n)$ is a type if and only if $p \cup Th_A(\mathcal{M})$ is finitely satisfiable.

Example 3.4.2. Consider the structure $(\mathbb{Q}, <)$ and let $\mathbb{Z} \subset \mathbb{Q}$ be the set of parameters.

1. The set of formulae $\{v > n | n \in \mathbb{Z}\}$ defines a 1-type. Indeed, every finite sub collection of formulae in $p \cup Th_{\mathbb{Z}}(\mathbb{Q})$ is satisfiable in $\mathbb{Q}$.

2. Let $c \in \mathbb{Q}$. Then the set of $\mathfrak{L}_{\mathbb{Z}}$-formulae ϕ in one variable for which $\mathbb{Q} \models \phi(c)$ defines a complete 1-type.

The example above can be generalized to generate complete n-types in any structure.

Definition 3.4.3. Let $\mathcal{M}$ be an $\mathfrak{L}_A$-structure. A complete n-type $p \in S_n^{\mathcal{M}}(A)$ is said to be *realized* in $\mathcal{M}$ if there exists $\bar{a} := (a_1, \ldots, a_n) \in M^n$ such that $p := \{\phi(v_1, \ldots, v_n) \in \mathfrak{L}_A | \mathcal{M} \models \phi(\bar{a})\}$. If p is not realized in $\mathcal{M}$ then we say that $\mathcal{M}$ *omits* p.

In general, given an $\mathfrak{L}$-structure $\mathcal{M}$ with $A \subset M$ and $\bar{a} \in M^n$, we use $tp^{\mathcal{M}}(\bar{a}/A)$ to denote the complete n-type defined by $\{\phi(v_1, \ldots, v_n) \in \mathfrak{L}_A | \mathcal{M} \models \phi(\bar{a})\}$.

Proposition 3.4.4 ([DM], Proposition 4.1.3)**.** *Let $p \in S_n^{\mathcal{M}}(A)$. There exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that p is realized in $\mathcal{N}$ i.e. there exists $\bar{a} \in N^n$ such that $\mathcal{N} \models \phi(\bar{a})$ for every $\phi(\bar{v}) \in p$.*

Observe that if $\mathcal{N}$ is an $\mathfrak{L}_A$-structure which is an elementary extension of $\mathcal{M}$ then for every $n \in \mathbb{N}$, $S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$.

The following propositions should clarify the aforementioned fact that given a structure, the space of types allows us to study the first order properties in elementary extensions of that structure.

Proposition 3.4.5 ([DM], Corollary 4.1.4)**.** *Let $\mathcal{M}$ be an $\mathfrak{L}$-structure. The following statements are equivalent*

1. *$p \in S_n^{\mathcal{M}}(A)$.*

2. *There exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ such that $p = tp^{\mathcal{N}}(\bar{a}/A)$.*

Proposition 3.4.6 ([DM], Corollary 4.1.6)**.** *Let $\mathcal{M}$ be an $\mathfrak{L}$-structure. Let $\bar{a}, \bar{b} \in M^n$ for some $n \in \mathbb{N}$. If $tp^{\mathcal{M}}(\bar{a}/A) = tp^{\mathcal{M}}(\bar{b}/A)$ then there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and an automorphism α of $\mathcal{N}$ such that α fixes every element of A and $\alpha(\bar{a}) = \bar{b}$.*

Let $\mathcal{M}$ be an $\mathfrak{L}$ structure and $A \subset M$. The stone space $S_n^{\mathcal{M}}(A)$ of complete n-types can be endowed with a natural topology which we refer to as the *stone topology.*

Let $\phi(v_1, \ldots, v_n)$ be an $\mathfrak{L}_A$ formula in n-free variables. Let $[\phi] := \{p \in S_n^{\mathcal{M}}(A) | \phi \in p\}$. The family $\{[\phi(v_1, \ldots, v_n)] | \phi \in \mathfrak{L}_A\}$ of subsets of $S_n^{\mathcal{M}}(A)$ will be the basic open sets of the stone topology. It can be checked that if ϕ and ψ are $\mathfrak{L}_A$ formulae in n-free variables then $[\phi \wedge \psi] = [\phi] \cap [\psi]$ and $[\phi \vee \psi] = [\phi] \cup [\psi]$. Observe that as the types in $S_n^{\mathcal{M}}(A)$ are satisfiable, we must have that $[\phi] = S_n^{\mathcal{M}}(A) \setminus [\neg\phi]$. It follows that the basic open sets $[\phi]$ are in fact both closed and open for the stone topology.

Proposition 3.4.7 ([DM], Lemma 4.1.8)**.** *The stone space $S_n^{\mathcal{M}}(A)$ is compact and totally disconnected.*

We end our discussion on types by providing a complete description of the space $S_n^{\mathcal{M}}(A)$ when the structure $\mathcal{M}$ is a model of ACF.

Example 3.4.8. Let $\mathfrak{L}_r$ denote the language of rings and let K be an algebraically closed field. Let $A \subset K$ and k denote the sub field of K generated by the set A. Given a complete n-type $p \in S_n^K(k)$, we define its restriction $p_{|A}$ to be the sub collection of $\mathfrak{L}_A$-formulae. This defines an element of $S_n^K(A)$. The following lemma allows us to reduce to studying the space $S_n^K(k)$ in order to describe $S_n^K(A)$.

Lemma 3.4.9 ([DM], Example 4.1.14)**.** *The restriction map described above from $S_n^K(k) \to S_n^K(A)$ is a bijection.*

Let $p \in S_n^K(k)$. Let $I_p := \{\phi \in k[\bar{X}] | (\phi = 0) \in p\}$. We claim that I_p is a prime ideal of $k[\bar{X}]$. Firstly, if $f, g \in I_p$ then as $(f(\bar{v}) = 0 \wedge g(\bar{v}) = 0) \to (f + g)(\bar{v}) = 0$ we must have that $f + g \in I_p$. Likewise, if $f, g \in k[\bar{X}]$ such that $f \in I_p$ then $fg \in I_p$. It follows that I_p is an ideal. To prove the claim we need only show that I_p is a prime ideal. Let $f, g \in k[\bar{X}]$ such that $(fg = 0) \in I_p$. As $K \models \forall \bar{v}(f(\bar{v})g(\bar{v}) = 0 \to f(\bar{v}) = 0 \vee g(\bar{v}) = 0)$, we must have that either $(f = 0) \in I_p$ or $(g = 0) \in I_p$. The following proposition describes the space $S_n^K(k)$.

Proposition 3.4.10 ([DM], Proposition 4.1.16)**.** *The map $p \mapsto I_p$ defines a continuous bijection from $S_n^K(k)$ onto* $\mathrm{Spec}(k[X_1, \ldots, X_n])$ *where* $\mathrm{Spec}(K[X_1, \ldots, X_n])$ *is endowed with the Zariski topology*

3.5 Multi-sorted languages and structures

The focus of this thesis is contained in the study of geometric objects over non-Archimedean valued fields. A non-Archimedean valued field consists of a field K provided with a group homomorphism $v : K^* \to \Gamma(K)$ where $(\Gamma(K), +, <, 0)$ is an ordered abelian group. In addition, we require that if $x, y \in K^*$ and $x+y \neq 0$ then $v(x+y) \geq \min\{v(x), v(y)\}$. The morphism v is called a valuation. The valuation v permits us to define a residue field $k(K) := K^0/K^{00}$ where $K^0 := \{x \in K | x = 0 \vee v(x) \geq 0\}$ and $K^{00} := \{x \in K | x = 0 \vee v(x) > 0\}$. We can study a non-Archimedean field using the language of rings for the fields K and $k(K)$ and the language of ordered abelian groups for $\Gamma(K)$. However,

another view point is to broaden our existing notions of a language so as to permit structures with several domains. A *multi-sorted language* accomplishes this and the different domains of a structure in such a language are referred to as its sorts.

Definition 3.5.1. *(Multi-sorted languages) A multi-sorted language* $\mathfrak{L}$ is given by specifying the following set of data.

1. A set $\mathfrak{S}$ of sorts.

2. A set $\mathfrak{F}$ of function symbols and a function $arity : \mathfrak{F} \to \mathbb{N}$ such that for every $f \in \mathfrak{F}$ there exists a unique $(arity(f) + 1)$-tuple of sorts $< S_1, \ldots, S_{arity(f)}, S_{arity(f)+1} >$. We call this tuple $sort(f)$. The intention here is that $sort(f)$ defines the domain and range of the function f i.e. S_i is the sort in which the i-th entry of the argument takes its values and $S_{arity(f)+1}$ is the sort in which the image of f lies.

3. A set $\mathfrak{R}$ of relation symbols and a function $arity : \mathfrak{R} \to \mathbb{N}$ such that for every $R \in \mathfrak{R}$ there exists a unique $arity(R)$-tuple of sorts $< S_1, \ldots, S_{arity(R)} >$. We call this tuple $sort(R)$. The intention here is that $R(x_1, \ldots, x_{arity(R)})$ can hold only if each of the x_i belong to the sort S_i.

4. A set $\mathfrak{C}$ of constant symbols such that for each $c \in \mathfrak{C}$ there exists a unique sort which we call $sort(c)$.

As before, a structure for a multi-sorted language $\mathfrak{L}$ is a set in which the various elements of the language can be interpreted.

Definition 3.5.2. *(Multi-sorted structures)* Let $\mathfrak{L}$ be a multi-sorted language as in the previous definition. *A structure* $\mathcal{M}$ for the language $\mathfrak{L}$ consists of the following.

1. A non-empty set $|\mathcal{M}|$ which is called the universe of $\mathcal{M}$ and for every sort S_i, we have $S_i^{\mathcal{M}} \subset |\mathcal{M}|$ which is said to be the set of members of $\mathcal{M}$ of sort S_i. The universe of $\mathcal{M}$ is the union of its sorts i.e. $|\mathcal{M}| = \bigcup_{S \in \mathfrak{S}} S^{\mathcal{M}}$.

2. The structure $\mathcal{M}$ interprets the the function, relation and constant symbols appropriately. Precisely,

   (a) If $R \in \mathfrak{R}$ and $sort(R) =< S_1, \ldots, S_{arity(R)} >$ then $R^{\mathcal{M}} \subset S_1^{\mathcal{M}} \times \ldots \times S_{arityR}^{\mathcal{M}}$.

   (b) If $f \in \mathfrak{F}$ and $sort(f) =< S_1, ..., S_{arity(f)}, S_{arity(f)+1} >$ then $f^{\mathcal{M}} : S_1^{\mathcal{M}} \times \ldots \times S_{arity(f)}^{\mathcal{M}} \to S_{arity(f)+1}^{\mathcal{M}}$ is a well defined function.

   (c) If $c \in \mathfrak{C}$ and $sort(c) = S$ then $c^{\mathcal{M}} \in S$.

Definition 3.5.3. *(Isomorphisms)* Let $\mathfrak{L} =< \mathfrak{S}, \mathfrak{F}, \mathfrak{R}, \mathfrak{C}, arity, sort >$ be a multi-sorted language. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\mathfrak{L}$-structures. A function $f : |\mathcal{M}_1| \to |\mathcal{M}_2|$ from the universe of $\mathcal{M}_1$ to the universe of $\mathcal{M}_2$ is an *isomorphism* if the following conditions are satisfied

1. The function f is one-one and onto.

2. If $S \in \mathfrak{S}$ then $m \in S^{\mathcal{M}_1}$ if and only if $f(m) \in S^{\mathcal{M}_2}$.

3. Let $R \in \mathfrak{R}$ be an n-ary relation symbol. Then for any $\bar{m} := (m_1, \ldots, m_n) \in |\mathcal{M}_1|^n$, $\bar{m} \in R^{\mathcal{M}_1}$ if and only if $f(\bar{m}) \in R^{\mathcal{M}_2}$.

4. If $F \in \mathfrak{F}$ and $arity(F) = n$ then for any $\bar{m} := (m_1, \ldots, m_n) \in |\mathcal{M}_1|^n$ we have that $f(F^{\mathcal{M}_1}(\bar{m})) = F^{\mathcal{M}_2}(f(\bar{m}))$.

Example 3.5.4. We introduce a multi-sorted language $\mathfrak{L}_{k\Gamma}$ in which value fields can be realized as structures. The sorts of $\mathfrak{L}_{k\Gamma}$ are the value field sort VF, the value group sort Γ and the residue field sort k. Associated to each sort, we add certain constants, relations and functions. For the value field and residue field sort, we add the constants, functions and relations of the language of rings and for the value group sort, we add the functions, relations and constants of the language of ordered Abelian groups. In addition to these, we add a symbol for the valuation which is a function $v : VF^* \to \Gamma$ and a function $\mathrm{Res} : VF^2 \to k$ sending (x, y) to the residue of xy^{-1} if $v(x) \geq v(y)$ and $y \neq 0$ and to 0 otherwise.

3.5.1 Elimination of Imaginaries

Definition 3.5.5. Let $\mathfrak{L}$ be a multi-sorted language and T be an $\mathfrak{L}$-theory. We say that the theory T has *elimination of imaginaries* if for every model $T \models \mathcal{M}$ and every $\emptyset$-definable equivalence relation E on $S_1^{\mathcal{M}} \times \ldots \times S_n^{\mathcal{M}}$ where the S_i are sorts of $\mathfrak{L}$, there exists a $\emptyset$-definable function $f : S_1^{\mathcal{M}} \times \ldots \times S_n^{\mathcal{M}}$ to a finite product of sorts of $\mathcal{M}$ such that $f(a) = f(b)$ if and only if $(a, b) \in E$.

If T is a complete theory that does not have elimination of imaginaries then one can extend the language $\mathfrak{L}$ to a language $\mathfrak{L}^{eq}$ such that the theory T extends to a complete $\mathfrak{L}^{eq}$ theory T^{eq} which eliminates imaginaries. This can be done as follows. For every $\emptyset$ - equivalence relation E on a finite product of sorts $S_1 \times \ldots \times S_n$, we add a sort $(S_1 \times \ldots \times S_n)/E$ and a function symbol $f : S_1 \times \ldots \times S_n \to (S_1 \times \ldots \times S_n)/E$ to the language $\mathfrak{L}$ to define the language $\mathfrak{L}^{eq}$. The $\mathfrak{L}$-structure $\mathcal{M}$ extends in a natural way to an $\mathfrak{L}^{eq}$ - structure which we denote $\mathcal{M}^{eq}$. Similarly, the theory T can be extended to an $\mathfrak{L}^{eq}$ - theory by setting $T^{eq} := Th(\mathcal{M}^{eq})$. The theory T^{eq} has elimination of imaginaries and is complete. The sorts of T^{eq} are called *imaginary sorts* and the elements of these sorts are referred to as *imaginaries*.

The theory ACF of algebraically closed fields is an example of a theory that eliminates imaginaries.

3.6 Ind and pro-definable sets

We begin with a description of definable sets which is equivalent to the one given earlier. This is the perspective employed by Hrushovski and Loeser in [HL].

Let $\mathfrak{L}$ be a multi-sorted language and T be a complete $\mathfrak{L}$-theory. If S is a sort of $\mathfrak{L}$ and A is an $\mathfrak{L}$-structure then we use $S(A)$ to denote the part of A that belongs to S. Given a set of parameters $C \subset A$, we use $\mathfrak{L}_C$ to denote the the language obtained from $\mathfrak{L}$ by adding constant symbols for every $c \in C$. If x denotes a finite set of sort specific variables $\{x_1, \ldots, x_n\}$ then let $S_x(C)$ denote the stone space of complete types in the variables x defined with parameters in C. Hence, an element $p \in S_x(C)$ is a collection of formulae in the variables x up to equivalence in the theory T such that every finite sub collection of p is satisfiable in some $\mathfrak{L}_C$ model of T.

We work within a large saturated model $\mathbb{U}$ of T. By this we mean the following. Let κ be a large uncountable cardinal and $\mathbb{U}$ be a model of cardinality κ such that if $A \subset \mathbb{U}$ is of cardinality strictly less than κ and $p \in S_x(A)$ then p has a realization in $\mathbb{U}$. Any model of interest will be an elementary sub structure of $\mathbb{U}$ of cardinality $< \kappa$ and and likewise, any set of parameters $A \subset \mathbb{U}$ will be a small subset of $\mathbb{U}$ i.e. a subset of cardinality $< \kappa$.

Let $C \subset \mathbb{U}$ be a parameter set. Let ϕ be an $\mathfrak{L}_C$-formula. The formula ϕ can be used to define a functor Z_ϕ from the category whose objects are $\mathfrak{L}_C$-models of T and morphisms are elementary embeddings to the category of sets. Let $\{x_{S_1,1}, \ldots, x_{S_1,n_{S_1}}, \ldots, x_{S_m,1} \ldots, x_{S_m,n_{S_m}}\}$ be the set of variables which occur in ϕ where the variable $x_{S_i,j}$ takes values in the sort S_i. Given an $\mathfrak{L}_C$-model $\mathcal{M}$ of T, we set

$$Z_\phi(\mathcal{M}) = \{\bar{a} \in S_1(\mathcal{M})^{n_{S_1}} \times \ldots S_m(\mathcal{M})^{n_{S_m}} | \mathcal{M} \models \phi(\bar{a})\}.$$

Clearly, Z_ϕ is well defined.

Definition 3.6.1. A *C-definable set Z* is a functor from the category whose objects are $\mathfrak{L}_C$-models of T and morphisms are elementary embeddings to the category of sets such that there exists an $\mathfrak{L}_C$-formula ϕ and $Z = Z_\phi$.

A definable set X is completely determined by $X(\mathbb{U})$ - its values in the universe $\mathbb{U}$. The set $X(\mathbb{U})$ is definable in the sense of Definition 3.3.1. The C-definable sets form a category which we denote Def_C. A morphism between objects $D_1, D_2 \in \mathrm{Def}_C$ is a map from $f : D_1(\mathbb{U}) \to D_2(\mathbb{U})$ such that its graph is a C-definable subset of $D_1(\mathbb{U}) \times D_2(\mathbb{U})$ in the sense of Definition 3.3.1.

Definable sets satisfy the following version of compactness.

Proposition 3.6.2. *Let X be a C-definable set. Let I be a small index set i.e. of cardinality $< \kappa$ and for every $i \in I$, let X_i be a C-definable subset of X. Suppose $X = \bigcup_{i \in I} X_i$. Then there exists a finite subset A of I such that $X(\mathbb{U}) = \bigcup_{i \in A} X_i(\mathbb{U})$.*

Given a set C contained in a model M of the theory T, its definable and algebraic closure approximate the smallest structure and model that contain C contained in M.

Definition 3.6.3. *(Definable closure and Algebraic closure)* Let $\mathcal{M}$ be a model of the theory T and $C \subseteq |\mathcal{M}|$. The *definable closure of C* denoted $\mathrm{dcl}(C)$ is the set of all $c \in |\mathcal{M}|$ such that there exists an $\mathfrak{L}_C$-formula ϕ in one free variable and $Z_\phi(\mathcal{M}) = \{c\}$. The *algebraic closure of C* denoted $\mathrm{acl}(C)$ is the set of all $c \in |\mathcal{M}|$ such that there exists an $\mathfrak{L}_C$-formula ϕ in one free variable and $Z_\phi(\mathcal{M})$ is a finite set which contains c.

Although the definition of definable sets given above is seemingly more complicated than that of 3.3.1, it is a more natural perspective when applying model theory to study problems in algebraic geometry.

The application of model theory to non-Archimedean geometry will require that we enlarge the category Def_C to the category ProDef_C consisting of pro-objects of the category Def_C indexed by a small directed partially ordered set.

Let I be a small directed partially ordered index set. For every $i \in I$, let $X_i \in \mathrm{Def}_C$ and if $i \leq j$, let $\phi_{ji} : X_j \to X_i$ be a morphism of C-definable sets

such that the family $\{\phi_{ji}\}_{i,j}$ satisfies the obvious compatibility properties. This data defines a functor $\varprojlim_{i\in I} X_i$ from the category of $\mathfrak{L}_C$-models of the theory T whose morphisms are elementary embeddings to the category of sets. Indeed, given an $\mathfrak{L}_C$-model $\mathcal{M}$ of T, we define

$$(\varprojlim_{i\in I} X_i)(\mathcal{M}) := \varprojlim_{i\in I}(X_i(\mathcal{M})).$$

Definition 3.6.4. (The category ProDef_C) An object X of the category ProDef_C is of the form $\varprojlim_{i\in I} X_i$ where I is a small directed partially ordered set and for every $i \in I$, X_i is a C-definable set. Let $X, Y \in \mathrm{ProDef}_C$ such that $X = \varprojlim_{i\in I} X_i$ and $Y = \varprojlim_{j\in J} Y_j$ where I, J are suitable index sets and for every $i \in I$, $j \in J$, X_i and Y_j are C-definable sets. We define

$$\mathrm{Hom}_{\mathrm{ProDef}_C}(X, Y) := \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{Def}_C}(X_i, Y_j).$$

If X is a pro-definable set then it is completely determined by its values in $\mathbb{U}$ i.e. the functor of taking $\mathbb{U}$-points induces an equivalence between the category ProDef_C and the sub category of sets whose objects are inverse limits over a small directed partially ordered index set of $\mathbb{U}$-points of C-definable sets. The object X is *strict pro-definable* if for every $i, j \in I$ such that $i < j$, the transition morphism $X_j \to X_i$ is surjective.

In a similar manner, we can define the category - IndDef_C of ind-definable sets over C whose objects are of the form $\varinjlim_{i\in I} X_i$ where I is a small directed partially ordered index set, if $i \in I$ then X_i is a C-definable set and if $i \leq j$ then $\phi_{ij} : X_i \to X_j$ is a C-definable morphism such that the family $\{\phi_{ij}\}$ satisfies the obvious compatibility properties. The object X is *strict ind-definable* if for every $i, j \in I$ where $i < j$, the induced transition morphisms $X_i(\mathbb{U}) \to X_j(\mathbb{U})$ is injective. Let $X = \varprojlim_{i\in I} X_i$ be an object of ProDef_C and $Y = \varinjlim_{j\in J} Y_j$ an object of IndDef_C. A morphism from Y to X is a compatible family of morphisms $Y_i \to X_j$ for every $i \in I$ and $j \in J$. A morphism from X to Y can be defined by a morphism from $X_i \to Y_j$ for some $i \in I$ and $j \in J$.

There are certain classes of sub sets of a pro-definable set which will be of interest to us later on.

Definition 3.6.5. *([HL], Definition 2.2.2)* Let $Y = \varprojlim_{i\in I} Y_i$ be pro-definable. Assume given, for each i, $X_i \subset Y_i$ such that for $j < i$ the transition maps $Y_i \to Y_j$ restrict to maps $X_i \to X_j$. Let $X := \varprojlim_{i\in I} X_i$.

1. If each X_i is definable and for some i_0, the maps $X_i \to X_{i'}$ for all $i \geq i' \geq i_0$ are bijections, we say that X is *iso-definable*

2. An ∞-definable set over C is of the form $\bigcap_{j\in J} Z_j$ where J is a small index set and the Z_j are C-definable. If each X_i is ∞-definable and for some i_0, the maps $X_i \to X_{i'}$ are bijections for all $i \geq i' \geq i_0$, we say X is *iso-∞-definable.*

3. If there exists a definable set W and a pro-definable morphism $g : W \to Y$ such that for each i, the composition of g and the projection $Y \to Y_i$ has image X_i, we say X is *definably parameterized.*

4. If there exists a strict ind-definable set Z and an injective morphism $g : Z \to Y$ with image X, we say by abuse of language that X is *strict ind-definable.*

5. If there exists $i_0 \in I$ such that X_{i_0} is a definable set, for every $i > i_0$, X_i is the pull back of X_{i_0} by the transition morphism $Y_i \to Y_{i_0}$ and $X = \varprojlim_{i \in I, i \geq i_0} X_i$ then we say that X is *relatively definable.*

It is not clear whether a subset of a pro-definable that is definably parametrized is iso-definable. However, we have the following characterization of iso-definable sets.

Proposition 3.6.6. *([HL], Corollary 2.2.4) Let Y be pro-definable and let $X \subseteq Y$ be a pro-definable subset. Then X is iso-definable if and only if X is in (pro-definable) bijection with a definable set.*

Iso-definable sets are stable when taking quotients for the action of a finite group.

Proposition 3.6.7. *Let Y be pro-definable, X an iso-definable subset. Let G be a finite group acting on Y, and leaving X invariant. Let $f : Y \to Y'$ be a map of pro-definable sets, whose fibers are exactly the orbits of G. Then $f(X)$ is iso-definable.*

3.7 Definable types

We fix a multi-sorted language $\mathfrak{L}$, a complete $\mathfrak{L}$-theory T and a universe $\mathbb{U}$ as in 3.6. Before discussing definable types, we make the following remarks concerning types. We make use of the notation introduced above.

Remark 3.7.1. Let $A \subset \mathbb{U}$ be a set of parameters. Let $p \in S_x(A)$ for some finite set of sort specific variables x. Let $\mathfrak{L}_{Ax}$ denote the set of formulae in the variables x with parameters in A up to equivalence in the theory T. As the type p is complete, given a formula $\phi \in \mathfrak{L}_{Ax}$, either $\phi \in p$ or $\neg\phi \in p$. Hence the type p defines a Boolean retraction from $\mathfrak{L}_{Ax}$ to the two element Boolean algebra. The type p is said to concentrate on an A-definable set V if the formula ϕ which defines V belongs to p. This is equivalent to saying that if $\bar{a}$ is a realization of p in some $\mathfrak{L}_A$-model $\mathcal{M}$ of T then $\bar{a} \in V(\mathcal{M})$. Suppose V is an A-definable set on which p concentrates. Let L_V denote the A-definable subsets of V. The type p can then be realized as a Boolean retraction from the space L_V to the two element Boolean algebra.

Definition 3.7.2. Let $x = (x_1, \ldots, x_n)$ be a set of variables and $A \subset \mathbb{U}$ be a finite set of parameters. A *definable type* $p(x)$ is a Boolean retraction $d_p x : \mathfrak{L}_{Ax,y_1,\ldots,y_r} \to \mathfrak{L}_{Ay_1,..,y_r}$. Here the $(y_i)_i$ are variables running through all finite products of sorts and r varies along $\mathbb{N}$. Equivalently, for an A-definable set V, let L_V denote the Boolean algebra of A-definable subsets of V. A *definable type on V* is a compatible family of elements of $\mathrm{Hom}_W(L_{V \times W}, L_W)$, where Hom_W denotes the set of Boolean homomorphisms h such that $h(V \times X) = X$ for $X \subset W$.

Let p be an A-definable type where $A \subset \mathbb{U}$ is a set of parameters. For every $\mathfrak{L}_A$-model $\mathcal{M}$ of T, we can define a type $p_{|\mathcal{M}}$ as follows. Let $p_{|\mathcal{M}} := \{\phi(x, \bar{b}) : \mathcal{M} \models d_p x(\phi)(\bar{b})\}$. The definable type p is completely determined by its restriction $p_{|\mathbb{U}}$. If X is an A-definable set then we say that p concentrates on X if the type $p_{|\mathbb{U}}$ concentrates on $X(\mathbb{U})$. The space of A-definable types which concentrate on an A-definable set X is denoted $S_{def,X}$.

Let X, Y be A-definable sets and $f : X \to Y$ be a definable morphism from X to Y. We define a morphism $f_* : S_{def,X} \to S_{def,Y}$ as follows. If $p \in S_{def,X}$ then we define $d_{f_*(p)} z(\phi(z, y)) := d_p x(\phi(f(x), y)$ for every $\phi \in \mathfrak{L}_{z,y}$. Here z, x and y denote a finite set of sort specific variables.

It is possible also to construct the product of types. If $p(x)$ and $q(y)$ are C - types then there exists a type $p(x) \otimes q(y)$ such that $(a, b) \models p(x) \otimes q(y)$ if and only if $a \models p$ and $b \models q_{|C(a)}$. In general, the product $\otimes$ is associative but not symmetric.

Chapter 4

The theory ACVF

We recall the definition of a non-Archimedean valued field.

Definition 4.0.3. A *non-Archimedean valued field* consists of a field K and a group homomorphism $v : K^* \to \Gamma(K)$ where $\Gamma(K)$ is an ordered Abelian group. In addition, for every $x, y \in K^*$ such that $x + y \neq 0$, the morphism v satisfies the inequality $v(x + y) \geq \min\{v(x), v(y)\}$. The homomorphism v is called the *valuation* and the group $\Gamma(K)$ is called the *value group.*

We enlarge the group $\Gamma(K)$ so that the valuation v extends to K. More precisely, we add a symbol ∞ to the value group $\Gamma(K)$ with the convention that for every $x \in \Gamma(K)$, $x < \infty$ and $x + \infty = \infty$. Let $v(0) := \infty$. We can associate to the value field K, a field $k(K)$ which we call the *residue field* of K as follows. The *valuation ring* $R(K)$ is defined by $\{x \in K | v(x) \geq 0\}$ and its maximal ideal $\mathcal{M}(K) := \{x \in K | v(x) > 0\}$. Let $k(K) := R(K)/\mathcal{M}(K)$. When the field K is algebraically closed, its value group $\Gamma(K)$ is divisible and its residue field $k(K)$ is algebraically closed as well.

In Example 3.5.4, we introduced the 3-sorted language $\mathfrak{L}_{k\Gamma}$ in which a valued field can be realized as a structure. A classical result of A. Robinson states that the $\mathfrak{L}_{k\Gamma}$-theory of algebraically closed non-Archimedean valued fields of a given characteristic and residue field characteristic is complete. The $\mathfrak{L}_{k\Gamma}$-theory ACVF of algebraically closed valued fields admits quantifier elimination. However, it does not allow for elimination of imaginaries.

In what follows we use the extended language $\mathfrak{L}_{\mathcal{G}}$ to describe ACVF wherein we have elimination of imaginaries. The language $\mathfrak{L}_{\mathcal{G}}$ is defined as follows. Let S_n denote the space of free R-modules in K^n. For each element of S_n, we add an imaginary element or *code* to the language $\mathfrak{L}_{k\Gamma}$. Let $S := \bigcup_{n\in\mathbb{N}} S_n$. If $s \in S_n$ is a code and if $\lambda(s)$ denotes the free R-module associated to s, let $\mathrm{red}(s) := \lambda(s)/\mathcal{M}(\lambda(s))$. This is $\emptyset$-definably isomorphic to a k-vector space of rank n. Let T_n be the set of codes of elements in $\bigcup_{s\in S_n} \mathrm{red}(s)$. Hence, an element of T_n is the code of a coset of the space $\lambda(s)$ modulo $\mathcal{M}(\lambda(s))$. By [HHM], the theory ACVF admits elimination of imaginaries in the extended language $\mathfrak{L}_{\mathcal{G}}$. When discussing the sorts of the language $\mathfrak{L}_{\mathcal{G}}$, we might at times refer to them as $\emptyset$-definable sets.

The fact that the theory ACVF has quantifier elimination can be used to show the following.

Proposition 4.0.4. *([HHM], Proposition 2.1.3)*

1. *The definable set* Γ *is o-minimal in the sense that every definable subset of* Γ *is a finite union of intervals.*

2. *Any* K*-definable subset of* k *is finite or co-finite i.e.* k *is strongly minimal.*

3. *The definable set* Γ *is stably embedded (cf. Definition 4.0.5).*

4. *If* $A \subset K$ *then* $\mathrm{acl}(A) \cap K$ *is equal to the field algebraic closure of* A *in* K*.*

5. *If* $S \subseteq k$ *and* $\alpha \in k$ *belongs to* $\mathrm{acl}(S)$ *in the* K^{eq} *sense, then* α *belongs to the field algebraic closure of* S*.*

6. *The definable set* k *is stably embedded. In fact,* Γ *is endowed with the structure of a pure divisible ordered abelian group and* k *with the structure of a pure algebraically closed field.*

Definition 4.0.5. A C-definable set D is *stably embedded* if for any definable set E and any $r > 0$, the set $E \cap D^r$ is a $C \cup D$ definable subset of D^r.

4.0.1 k-internal sets

We introduce k-internal sets and the lemma that follows states equivalent characterizations of such objects.

Definition 4.0.6. A C-definable set D is called *k-internal* if there exists a finite $F \subset \mathbb{U}$ such that $D \subset \mathrm{dcl}(k \cup F)$.

Lemma 4.0.7. *([HHM], Lemma 2.6.2). Let* D *be a* C*-definable set. The following conditions are equivalent:*

1. D *is* k*-internal.*

2. *For any* $m \geq 1$*, there is no surjective definable map from* D^m *onto an infinite interval in* Γ*.*

3. D *is finite or, up to permutation of coordinates, is contained in a finite union of sets of the form* $\mathrm{red}(s_1) \times \ldots \times \mathrm{red}(s_m) \times F$*, where* $s_1, ..., s_m$ *are* $\mathrm{acl}(C)$*-definable elements of* $S := \bigcup_{n \in \mathbb{N}} S_n$ *and* F *is a* C*-definable finite set of tuples from* $\mathcal{G}$*.*

4.0.2 Γ-internal sets

We introduce the notion of Γ-internal definable sets and present characterizations of such sets. Such objects play a central role in the Hrushovski-Loeser theory of non-Archimedean geometry. We will later dwell briefly on how they occur naturally in the spaces $\widehat{V}$ which are analogues of the Berkovich analytification of quasi-projective varieties and how they completely determine the homotopy type of such spaces.

Definition 4.0.8. Let D be an F-definable set. We say that D is Γ*-internal* if there exists $F' \supseteq F$, a definable subset $D' \subset \Gamma^n$ for some $n \in \mathbb{N}$ and a bijective F'-definable morphism $f : D' \to D$. If $F' = F$ then we say that X is *directly* Γ*-internal.*

The following proposition describes equivalent characterizations of Γ-internal sets.

Proposition 4.0.9. *([HL], Lemma 2.8.1) Let X be an F-definable set. The following conditions are equivalent:*

1. *X is Γ-internal.*
2. *X is internal to some o-minimal definable linearly ordered set.*
3. *X admits a definable linear ordering.*
4. *Every stably dominated type on X (over any base set) is constant (i.e. contains a formula $x = a$).*
5. *There exists an $\mathrm{acl}(F)$ - definable injective $h : X \to \Gamma^*$, where Γ^* means Γ^n for some n.*

It will also be of interest to study a relative version of Γ-internality.

Definition 4.0.10. Let V and U be C-definable sets and $f : U \to V$ a C-definable function. The morphism f is a Γ-*internal cover* if for every $v \in V$, the fibre $f^{-1}(v) \subset U$ is Γ-internal. We say that f is a directly Γ-internal set if the fibres are *directly* Γ-*internal sets.*

Proposition 4.0.11. *([HL], Lemma 2.8.2) Let V be a definable set in* ACVF*. Then any Γ-internal cover $f : U \to V$ is isomorphic over V to a finite disjoint union of sets which are a fiber product over V of a finite cover and a directly Γ-internal cover.*

4.1 The space $\widehat{V}$

4.1.1 Stably dominated types

We begin by presenting the notion of a stably dominated type, using which the spaces $\widehat{V}$ are defined. The definition presented below in terms of orthogonality to the sort Γ is *not* the general definition of a stably dominated type and instead a characterization particular to the theory ACVF.

In the definition that follows and for the rest of this text we use the following convention. If $A \subset \mathbb{U}$ and $a \in \mathbb{U}$ then by $A(a)$ we mean the sub-structure generated by $A \cup a$ i.e. $\mathrm{dcl}(A \cup a)$. By $\Gamma(A)$ we mean the part of A belonging to the sort Γ i.e. $\mathrm{dcl}(A) \cap \Gamma$.

Definition 4.1.1. Let $A \subset \mathbb{U}$ and $p = \mathrm{tp}(a/A)$ be an A-type. We say that p is *almost orthogonal* to Γ if $\Gamma(A(a)) = \Gamma(A)$. An A-definable type q is *stably dominated* if for any sub-structure B containing A the B-type $q_{|B}$ is almost Γ-orthogonal.

Example 4.1.2. Let $a \in VF(\mathbb{U})$ and $\alpha \in \Gamma_\infty(\mathbb{U})$. The *generic type* associated to the definable set $B(a;\alpha) := \{x \in VF | v(x-a) \geq \alpha\}$ is an $\{a, \alpha\}$-stably dominated type.

The following theorem provides us further examples of stably dominated types.

Theorem 4.1.3. *([HHM2], Theorem 12.1.8)*

1. *Suppose that $C \subseteq L$ are non-Archimedean valued fields such that C is maximally complete, $k(L)$ is a regular extension of $k(C)$ and $\Gamma(L)/\Gamma(C)$ is torsion free. Let a be a sequence in $\mathbb{U}$ such that $a \in \mathrm{dcl}(L)$. Then $\mathrm{tp}(a/C \cup \Gamma(C(a)))$ is stably dominated.*

2. *Let C be a maximally complete algebraically closed non-Archimedean valued field and a be a sequence in $\mathbb{U}$. Then $\mathrm{tp}(\mathrm{acl}(Ca)/C \cup \Gamma(Ca))$ is stably dominated.*

Over course of this text, we will treat quasi-projective varieties over a non-Archimedean valued field as definable sets. This can be accomplished by adding sorts corresponding to $\mathbb{P}^n$ for every $n \in \mathbb{N}$.

Example 4.1.4. The space $\mathbb{A}^1$ is $\emptyset$-definable. If $b \subseteq \mathbb{A}^1$ is a closed ball then its generic type denoted p_b is an element of the space $\widehat{\mathbb{A}^1}$ (cf. Example 4.1.2). By [[HHM], 2.3.6, 2.3.8, 2.5.5], $\widehat{\mathbb{A}^1}$ is the set of generic types of closed balls contained in $\mathbb{A}^1$. It can be checked that the space $\widehat{\mathbb{P}^1}$ is the union of $\widehat{\mathbb{A}^1}$ and the point ∞.

Let $f : U \to V$ be a definable map between definable sets. In section 3.7, we defined a map $f_* : S_{def,U} \to S_{def,V}$ from the set of definable types which concentrate on U to the definable types which concentrate on V. It can be checked that the map f_* restricts to a map $\widehat{U} \to \widehat{V}$ which we denote by $\widehat{f}$.

Definition 4.1.5. Let V be a C-definable set. We define $\widehat{V}$ to be a functor from the category whose objects are models of ACVF that contain C and morphisms are elementary embeddings to the category of sets such that if M is a model that contains C then $\widehat{V}(M)$ is the set of M-definable stably dominated types.

Although $\widehat{V}$ is almost always not definable, it is pro-definable.

Theorem 4.1.6. *([HL], Theorem 3.1.1) Let V be a C-definable set. Then there exists a canonical pro-C-definable set E and a canonical identification $\widehat{V}(F) = E(F)$ for any $F \supseteq C$. Moreover, E is strict pro-definable.*

4.1.2 The topology of $\widehat{V}$

We begin by introducing the notion of an A-definable topology where $A \subset \mathbb{U}$ is a set of parameters. In what follows, if X is definable or pro-definable then we mean $X(\mathbb{U})$ when we refer to the *set* X and by a topology on X, we mean a topology on the set $X(\mathbb{U})$. For instance, when we define a topology on $\widehat{V}$, we mean a topology on $\widehat{V}(\mathbb{U})$.

Definition 4.1.7. Let V be an A-pro-definable set. A *topology $\mathcal{J}$ on V is A-definable* if the following conditions are satisfied.

1. Let $\mathcal{J}_d$ denote the sub collection of elements of $\mathcal{J}$ which are relatively definable. The topology $\mathcal{J}$ is generated by $\mathcal{J}_d$.

2. Let $\mathcal{W} := \{W_u | u \in U\}$ be an A-definable family of relatively definable sets. The family $\mathcal{W} \cap \mathcal{J}$ is an ind-definable family over A.

Equivalently, the topology $\mathcal{J}$ is *A-definable* if it is generated by an ind-definable family over A of relatively definable subsets of V.

Let V be an A-definable set endowed with a definable topology. We suppose in addition that there exists an ind-definable sheaf $\mathcal{O}$ of definable functions into Γ_∞. Let $\mathcal{J}_b$ denote the family of subsets of $\widehat{V}$ of the form $\{p \in \widehat{O} | f_*(p) \in U\}$ where O is an open subset of V, $f \in \mathcal{O}(O)$ and U is an open subset of $\Gamma_\infty(\mathbb{U})$ for the order topology. Let $\mathcal{J}$ be the topology generated by the elements of the family $\mathcal{J}_b$. Let M be a model of ACVF. The topology on $\widehat{V}(M)$ is generated by the M-definable open subsets of $\widehat{V}(\mathbb{U})$. Note that we do not define the topology on $\widehat{V}(M)$ by viewing it as a subset of $\widehat{V}(\mathbb{U})$ as doing so would result in the discrete topology.

Remark 4.1.8. When V is contained in an algebraic variety, we use the topology induced by the Zariski topology to define a topology on $\widehat{V}$ and the ind-definable sheaf to be the sheaf of regular functions composed with the valuation $\mathrm{val} : VF \to \Gamma_\infty$.

Example 4.1.9. Let X be a definable subset contained in $\mathbb{A}^n$ for some $n \in \mathbb{N}$ and Y denote its Zariski closure. Let $\mathrm{Fn}_r(X, \Gamma_\infty)$ be the set of functions from X to Γ_∞ of the form $\mathrm{val}(F)$ where F is a regular function on Y. Let $\mathcal{J}'$ be the topology on $\widehat{X}$ whose pre basis consists of sets of the form $\{p \in \widehat{X} | f_*(p) < g_*(p)\}$ where $f, g \in \mathrm{Fn}_r(X, \Gamma_\infty)$. The topology $\mathcal{J}'$ is the same topology defined above using the topology on X induced by the Zariski topology and the ind sheaf of regular functions composed with the valuation. Infact, the topology on $\widehat{X}$ is the weakest topology such that the functions $f_* : \widehat{X} \to \Gamma_\infty$ are continuous when $f \in \mathrm{Fn}_r(X, \Gamma_\infty)$ and Γ_∞ is provided with the order topology. The topology $\mathcal{J}'$ is definable in the sense of Definition 4.16 ([HL], 3.3). The construction above can be generalized to when X is contained in projective variety V by defining a topology on each of the affine pieces and then glueing these together. It can be shown that the topology so obtained is the restriction of the topology defined on $\widehat{V}$ using the Zariski topology on V and the ind-definable sheaf of regular functions composed with val.

Observe that Remark 4.1.8 and Example 4.1.9 define topologies on spaces of the form $\widehat{V}$ where V is a definable subset of an algebraic variety. When defining homotopies and at several other instances, we will work with definable objects of the value group sort and hence we enlarge the class of spaces for which we have introduced an explicit topology. If $X \subset \Gamma_\infty^n$ is a definable set then as every stably dominated type on X concentrates at a point, we have that $\widehat{X} = X$ holds canonically. This is true in slightly greater generality.

Lemma 4.1.10. *If X is a definable subset of Γ_∞^n then $X = \widehat{X}$ canonically. More generally if U is a definable subset of VF^n or a definable subset of an algebraic variety over VF and W is a definable subset of Γ_∞^m, then the canonical map $\widehat{U \times W} \to \widehat{U} \times W$ is a bijection.*

Definition 4.1.11. *(Topology on Γ_∞^n)* The $\emptyset$-definable function $\mathrm{val} : \mathrm{VF} \to \Gamma_\infty$ defines maps $\mathrm{val}^n : \mathbb{A}^n \to \Gamma_\infty^n$ for every $n \in \mathbb{N}$. These in turn induce maps $\widehat{\mathrm{val}^n} : \widehat{\mathbb{A}^n} \to \Gamma_\infty^n$. Let U be a definable subset of an algebraic variety over VF. We endow $\widehat{U} \times \Gamma_\infty^n \simeq \widehat{U \times \Gamma_\infty^n}$ *with the weakest topology such that* $\mathrm{id} \times \widehat{\mathrm{val}^n} : \widehat{U \times \mathbb{A}^n} \to \widehat{U \times \Gamma_\infty^n}$ *is continuous.*

The following lemmas imply in particular that the topology defined above on Γ_∞ coincides with the order topology while the topology on Γ_∞^n is the product topology.

For $\gamma := (\gamma_1, \ldots, \gamma_n) \in \Gamma_\infty^n$, let $b(\gamma) := \{(x_1, \ldots, x_n) \in \mathbb{A}^n | \mathrm{val}(x_i) \geq \gamma_i\}$. Let $p_\gamma = p_{b(\gamma)} \in \widehat{\mathbb{A}^n}$ denote the generic type associated to the n - dimensional closed ball $b(\gamma)$.

Lemma 4.1.12. *([HL], Lemma 3.5.2) The map $j : \mathbb{A}^n \times \Gamma_\infty^n \to \mathbb{A}^{n+1}$ given by $(q, \gamma) \mapsto q \otimes p_\gamma$ is continuous for the product topology of $\widehat{\mathbb{A}}^n$ with the order topology on Γ_∞^n.*

Lemma 4.1.13. *([HL], Lemma 3.5.3) If U is a definable subset of $\mathbb{A}^n \times \Gamma_\infty^n$ and W is a definable subset of Γ_∞^m provided with the order topology, the induced topology on $\widehat{U \times W} = \widehat{U} \times W$ coincides with the product topology.*

4.2 Simple points

Let V be an A-definable set. For $x \in V$, the definable type $\mathrm{tp}(x/\mathbb{U})$ which concentrates on the point x is stably dominated. It follows that $\mathrm{tp}(x/\mathbb{U})$ is an element of $\widehat{V}(\mathbb{U})$. We can thus view V as a subset of $\widehat{V}$. This subset of points in $\widehat{V}$ is called the set of *simple points*.

Lemma 4.2.1. *([HL], Lemma 3.6.1) Let X be a definable subset of VF^n.*

1. *The set of simple points of $\widehat{X}$ (which we identify with X) is an iso-definable and relatively definable dense subset of $\widehat{X}$. If M is a model of* ACVF *then $X(M)$ is dense in $\widehat{X}(M)$.*

2. *The induced topology on X agrees with the valuation topology on X.*

4.3 Canonical Extensions

Let V be an A-definable set where $A \subset \mathbb{U}$ is a small set of parameters. Let W be a definable subset of $\mathbb{P}^m \times \Gamma_\infty^n$ for some $n \in \mathbb{N}$ and $f : V \to \widehat{W}$ be an A-definable map. We define a map $\widehat{f} : \widehat{V} \to \widehat{W}$ which we call the *canonical extension of f*.

Let $p \in \widehat{V}(M)$ and $c \models p_{|M}$. Let $d \models f(c)_{|M(c)}$. By ([HL], Proposition 2.6.5), the type $\mathrm{tp}(cd|M)$ is stably dominated. It follows that $\mathrm{tp}(d|M)$ is stably dominated as well. We set $\widehat{f}(p) := \mathrm{tp}(d|M)$. The map $\widehat{f}$ is well defined and pro-A-definable. We now provide conditions on the map f which imply that its canonical extension is well defined.

Definition 4.3.1. *(v and g - open sets)* Let V be an algebraic variety defined over a non-Archimedean valued field F. A set $U \subset V$ is v-open if it is open for the valuation topology on V. A set G is g-open if it is a positive Boolean combination of Zariski closed and open subsets and sets of the form $\{u : \mathrm{val}(f)(u) < \mathrm{val}(g)(u)\}$ where f, g are regular functions on some Zariski open set. More generally, if U is a definable subset of the variety V then a set

$W \subset U$ is v-open (g-open) if it is of the form $U \cap O$ where O is v-open (g-open). If $X \subset V \times \Gamma_\infty^n$ is a definable set then X is v-open (g-open) if its pull back to $V \times \mathbb{A}^n$ is v-open (g-open).

Remark 4.3.2. The v-topology restricted to Γ is discrete while the neighborhoods of ∞ are the same as those defined by the order topology. The g-topology when restricted to Γ coincides with the order topology while the point ∞ is isolated. It follows that the v + g topology which is the topology generated by the class of sets which are both v and g-open, induces the order topology on Γ_∞. In general the v,g and v + g-open sets are definable. Observe that in the case of a variety V over a non-Archimedean valued field F, for a given model of ACVF, the collection of v-open sets definable in that model generate the valuation topology on V. The g-topology however does not necessarily generate a topology.

Definition 4.3.3. *(v-continuity and g-continuity)* Let V be an algebraic variety over a non-Archimedean valued field F or a definable subset of such a variety. A definable function $h : V \to \Gamma_\infty$ is called v-continuous (resp. g-continuous) if the pullback of any v-open (resp. g-open) set is v-open (resp. g-open). A function $h : V \to W$ with W an affine F-variety is called v-continuous (resp. g-continuous) if, for any regular function $f : W \to \mathbb{A}^1$, $\mathrm{val} \circ f \circ h$ is v-continuous (resp. g-continuous).

The following lemmas provide necessary conditions for a map f in order that its canonical extension be continuous.

Lemma 4.3.4. *([HL], Lemma 3.8.1) Let K be a non-Archimedean valued field and V be an algebraic variety over K. Let X be a K-definable subset of V and let $f : X \to \widehat{W}$ be a pro-K-definable function with W a K-definable subset of $\mathbb{P}^n \times \Gamma_\infty^m$. Assume that f is v + g-continuous i.e. $f^{-1}(G)$ is g-open whenever G is open, and $f^{-1}(G)$ is v-open at x whenever G is open, for any $x \in f^{-1}(G)$. Then f extends uniquely to a continuous pro-K-definable morphism $F : \widehat{X} \to \widehat{W}$.*

Lemma 4.3.5. *([HL], Lemma 3.8.2) Let K be a non-Archimedean valued field and V be an algebraic variety over K. Let $f : I \times V \to \widehat{V}$ be a g-continuous K-definable function where $I = [a, b]$ is a closed interval. Let i_I denote one of a or b and e_I denote the remaining point. Let X be a K-definable subset of V. Assume f restricts to a definable function $g : I \times X \to \widehat{X}$ and that f is v-continuous at every point of $x \in X$. Then g extends uniquely to a continuous pro-K-definable morphism $G : I \times \widehat{X} \to \widehat{X}$. If moreover, for every $v \in X$, $g(i_I, v) = v$ and $g(e_I, v) \in Z$, with Z a Γ-internal subset then $G(i_I, x) = x$ and $G(e_I, x) \in Z$.*

4.4 Paths and homotopies

We introduce conventions and certain fundamental notions which we require in order to state the main results that follow. We begin with the notion of a generalized interval.

Up to this point, when referring to an interval, we meant a sub-interval of Γ_∞. This notion is fairly restrictive as the homotopies we construct later come about by *glueing* other homotopies together. To define what we mean by a

glueing of intervals, we compactify Γ_∞ by adding a point $-\infty$ with the obvious conventions. In practice, any function defined on $-\infty \cup \Gamma_\infty$ will be constant on a sub interval of the form $\{x|x < a\}$ for some $a \in \Gamma_\infty$. If $J = -\infty \cup \Gamma_\infty$ then there exists two possible orders on J, namely the natural order or its reverse. The choice of one of these is called an *orientation* of J.

Definition 4.4.1. A *generalized interval* is a sub-interval of a union of a finite collection of oriented copies $I_1, \ldots, I_n$ of $-\infty \cup \Gamma_\infty$ glued end to end so that the orientation on each copy is respected. If $I := I_1 \cup \ldots \cup I_n$ and $I' \subset I$ is a sub-interval then we use $e_{I'}$ to denote the largest element of the interval I' and $i_{I'}$ to denote its smallest element. A function $f : I' \times V \to W$ can be extended to a function $\tilde{f} : I \times V \to W$ by setting $f(t, x) = f(i_{I'}, x)$ for every $t < i_{I'}$ and $f(t, x) = f(e_{I'}, x)$ for every $t > e_{I'}$. We say that the function $\tilde{f}$ is continuous (v + g-continuous / definable) if the function f is continuous (v + g-continuous / definable). Similarly, if $f : J \times V \to W$ is a function where J is obtained by glueing intervals $J_1, \ldots, J_n$ then we say that f is continuous (v + g-continuous / definable) if the restriction of f to each of the $J_i \times V$ is continuous (v + g continuous / definable).

Definition 4.4.2. Let V be an algebraic variety and $X \subset \widehat{V} \times \Gamma_\infty^n$ be a pro-definable subset. A *homotopy* is a continuous pro-definable map $h : I \times X \to X$ where I is a generalized closed interval. In this case, the maps $h_{|(i_I,\text{-})}$ and $h_{|(e_I,\text{-})}$ are said to be *homotopic.*

If W is a definable subset of $V \times \Gamma_\infty$ then a pro-definable v + g-continuous map $h : I \times W \to \widehat{W}$ is also called a *homotopy.* By Lemma 4.3.5, such a map extends to a continuous pro-definable map $\widehat{h} : I \times \widehat{W} \to \widehat{W}$.

Definition 4.4.3. Let $X \subset \widehat{V} \times \Gamma_\infty^n$ be a pro-definable set and $Z \subset X$. A *deformation retraction of X to Z* is a homotopy $h : I \times X \to X$ such that

1. For every $x \in X$, $h(i_I, x) = x$ and $h(e_I, x) \in Z$.
2. For every $z \in Z$ and $t \in I$, $h(t, z) = z$.

The deformation retraction h is said to have the ** property* if for every $x \in X$ and $t \in I$, $h(e_I, (h(t, x))) = h(e_I, x)$.

Let $h : I \times X \to X$ be a deformation retraction. Let $\rho : X \to X$ be defined by $\rho(x) := h(e_I, x)$. The pair $(\rho, \rho(X))$ is called a *deformation retract.*

Definition 4.4.4. *(Glueing homotopies)* Let $h_1 : I_1 \times X \to X$ and $h_2 : I_2 \times X \to X$ be two homotopies. Suppose that $h_2(i_{I_2}, h_1(e_{I_1}, x)) = h_1(e_{I_1}, x)$ for every $x \in X$. The *glueing of h_1 and h_2* is a homotopy $h_2 \circ h_1 : I_2 + I_1 \times X \to X$ defined by setting $h_2 \circ h_1(t, x) = h_1(t, x)$ for $t \in I_1$ and $h_2 \circ h_1(t, x) := h_2(t, h_1(e_{I_1}, x))$ for $t \in I_2$.

4.5 The space $\widehat{\mathbb{A}^n}$

In what follows, we provide an explicit description of $\widehat{\mathbb{A}^n}$ as a pro-definable space. In order to do so we introduce the space of semi-lattices $L(V)$ associated to a vector space and provide equivalent realizations of it.

Let K be a model of ACVF and V be a K-vector space. Let $\mathcal{O}$ denote the valuation ring of the non-Archimedean valued field K and $\mathcal{M}$ its maximal ideal. A *lattice* in V is a free $\mathcal{O}$-sub module of rank $\dim(V)$. A *semi-lattice* u of V is an $\mathcal{O}$-sub module of V such that there exists a K-sub space U contained in u and u/U is a lattice in V/U. Let $L(V)$ denote the set of semi-lattices. It is a K-definable set.

The set $L(V)$ can be given a topology as follows. A subset of $L(V)$ is a pre-basic open set if it is of the form $\{u \in L(V) | h \notin u\}$ or $\{u \in L(V) | h \in \mathcal{M}u\}$ where $h \in V$. This topology on $L(V)$ is referred to as the linear topology.

An equivalent description of the space $L(V)$ is possible using linear semi-norms. A *linear semi-norm* on V is a definable map $w : V \to \Gamma_\infty$ such that $w(x+y) \geq \min(w(x), w(y))$ and $w(cx) = \mathrm{val}(c) + w(x)$ for every $x, y \in V$ and $c \in K$. We endow the set of linear semi-norms on V with the weakest topology such that for every $f \in V$, the map on the set of semi-norms defined by $w \to w(f)$ is continuous.

Let w be a linear semi-norm. The set $L(w) := \{x \in V | w(x) \geq 0\}$ is a semi-lattice on V. It can be checked that the map which sends a linear semi-norm w to $L(w)$ defines a homeomorphism between the space of linear semi-norms and the space $L(V)$.

We relate the pro-definable space $\widehat{\mathbb{A}^n}$ to the space $L(V)$. Let $H_d(K)$ denote the finite dimensional K-vector space of polynomials of degree at most d and $L(H_d(K))$ the associated space of semi-lattices or linear semi-norms. The sets H_d and $L(H_d)$ are both $\emptyset$-definable. Let $p \in \widehat{V}$ and $J_d(p) := \{f \in H_d | \mathrm{val}(f)_*(p) \geq 0\}$. Since $f \mapsto \mathrm{val}(f)_*(p)$ is a linear seminorm, the set $J_d(p)$ is a semi-lattice in H_d and hence defines an element of $L(H_d)$ i.e. we have a map $J_d : \widehat{\mathbb{A}^n} \to L(H_d)$. The spaces $L(H_d)$ define a projective system indexed by $\mathbb{N}$ in which the transition morphisms are induced by the inclusions $H_{d_1} \subset H_{d_2}$.

Lemma 4.5.1. *([HL], Theorem 5.1.4) The system $(J_d)_{d=1,2,\ldots}$ induces a continuous morphism of pro-definable sets $J : \widehat{\mathbb{A}^n} \to \varprojlim_d L(H_d)$. The morphism J is injective and induces a homeomorphism between $\widehat{\mathbb{A}^n}$ and its image.*

4.6 The space $\widehat{\mathbb{P}^n}$

Analogous to the discussion provided above concerning the space $\widehat{\mathbb{A}^n}$, we obtain in what follows an explicit description of the pro-definable space $\widehat{\mathbb{P}^n}$.

We begin by introducing the *tropical projective space* $\mathrm{Trop}(\mathbb{P}^n)$.

Definition 4.6.1. The *tropical projective space* $\mathrm{Trop}(\mathbb{P}^n)$ is defined to be the quotient space $(\Gamma_\infty^{n+1} \setminus \infty^{n+1})/\Gamma$ where Γ acts by translation.

We have a map $\tau : \mathbb{P}^n \to \mathrm{Trop}(\mathbb{P}^n)$ defined by sending an element $[x_0 : \ldots : x_n] \in \mathbb{P}^n$ to $[\mathrm{val}(x_0) : \ldots : \mathrm{val}(x_n)]$. Observe that the space $\mathrm{Trop}(\mathbb{P}^n)$ embeds into Γ_∞^{n+1} with image $\{(a_0, \ldots, a_n) \in \Gamma_\infty^{n+1} | \min(a_i) = 0\}$.

Let H_d denote the definable set of homogenous polynomials in $n+1$ variables and $H_{d,m} := H_d^{m+1}$. Given $h := (h_0, \ldots, h_m) \in H_{d,m}$ where $h_i \in H_d$ and $x \in \mathbb{P}^n$, we define $c(x, h) := [h_0(x) : \ldots, h_m(x)]$. This defines a morphism $c : \mathbb{P}^n \times H_{d,m} \to \mathbb{P}^m$. Composing the morphism c with the map $\tau : \mathbb{P}^m \to \mathrm{Trop}(\mathbb{P}^m)$ defines a map $\tau : \mathbb{P}^n \times H_{d,m} \to \mathrm{Trop}(\mathbb{P}^m)$. This map factors through

a morphism $\tau : \mathbb{P}^n \times \mathbb{P}(H_{d,m}) \to \mathrm{Trop}(\mathbb{P}^m)$ where $\mathbb{P}(H_{d,m})$ is the projectivization of the finite dimensional vector space $H_{d,m}$. Let $h \in \mathbb{P}(H_{d,m})$ and let τ_h denote the morphism $\mathbb{P}^n \to \mathrm{Trop}(\mathbb{P}^m)$ defined by sending $x \in \mathbb{P}^n$ to $\tau(x,h)$. This induces a map $\widehat{\tau_h} : \widehat{\mathbb{P}^n} \to \mathrm{Trop}(\mathbb{P}^m)$. Let $T_{d,m}$ denote the set of functions $\mathbb{P}(H_{d,m}) \to \mathrm{Trop}(\mathbb{P}^m)$ of the form $h \mapsto \widehat{\tau_h}(x)$ for some $x \in \widehat{\mathbb{P}^n}$. Note that $T_{d,m}$ is a definable set.

Proposition 4.6.2. *([HL],Proposition 5.2.1) The space $\widehat{\mathbb{P}^n}$ may be identified via the canonical mappings $\widehat{\mathbb{P}^n} \to T_{d,m}$ with the projective limit of the spaces $T_{d,m}$. If one endows $T_{d,m}$ with the topology induced from the Tychonoff topology, this identification is a homeomorphism.*

4.7 Γ-internal spaces

In this section we continue our previous discussion on Γ-internal sets in a more general setting. We begin by defining what we mean by a Γ-internal subset of a space $\widehat{V}$ where V is a definable set.

Definition 4.7.1. Let V be a definable set. A subset $X \subset \widehat{V}$ is Γ-*parametrized* if there exists a definable set $Y \subset \Gamma_\infty^n$ and pro-definable map $Y \to \widehat{V}$ with image X. The set X is Γ-*internal* if it is Γ-parametrized and if there exists a canonical projection $\pi : \widehat{V} \to H$ with H definable such that the restriction of π to X is injective.

When discussing Γ-internal subsets, we can suppose that the ambient space is affine.

Lemma 4.7.2. *([HL], Lemma 6.2.1) Let V be a quasi-projective variety over an infinite non-Archimedean valued field F, and let $f : \Gamma^n \to \widehat{V}$ be F-definable. There exists an affine open $V' \subseteq V$ with $f(\Gamma^n) \subseteq \widehat{V'}$. If $V = \mathbb{P}^n$, there exists a linear hyperplane H such that $f(\Gamma_n) \cap \widehat{H} = \emptyset$.*

Over a model F of ACVF, we have a description of the injective map from an F-iso-definable subset of $\widehat{\mathbb{A}^n}$ which is Γ-internal to Γ_∞^m for some $m \in \mathbb{N}$.

Proposition 4.7.3. *([HL], Corollary 6.2.5) Let $X \subseteq \widehat{\mathbb{A}^N}$ be iso-definable and Γ-internal over an algebraically closed non-Archimedean valued field F. Then for some d and finitely many polynomials h_i of degree $\leq d$, the map $p \mapsto (p_*(\mathrm{val}(h_i)))_i$ is injective on X.*

Observe that in the proposition stated above, we required that the set of parameters be a model of ACVF. We discuss certain points of importance if one were to relax this hypothesis.

Let F be a non-Archimedean valued field and ACVF_F be the theory of algebraically closed fields in the language of ACVF with constant symbols corresponding to the elements of F. The sort $(\mathbf{k}, \Gamma_\infty)$, where $\mathbf{k}$ is the residue field sort and Γ the value group sort, eliminates imaginaries. However, it does not eliminate imaginaries topologically. This means for instance that if $X \subset \Gamma_\infty^n$ is a definable set and E is a closed equivalence relation on X then though the quotient space X/E exists as a definable subset of Γ_∞^m for some $m \in \mathbb{N}$, the quotient topology does not necessarily coincide with linear topology. One reason

for this is the fact that there exists non-trivial quotient spaces with a Galois action on cohomology while connected definable spaces contained in Γ^m have a trivial Galois action on cohomology. As a result, the proposition above must be suitably modified when working over a field that is not algebraically closed.

Proposition 4.7.4. *([HL], Proposition 6.2.8) Let A be a base structure consisting of a field F and a set S of elements of Γ. Let V be a projective variety over F, X a Γ-internal, A-definable subset of $\widehat{V}$. Then there exists an A-definable continuous injective map $\phi : X \to [0, \infty]^w$ for some finite A-definable set w. If X is closed then ϕ is a topological embedding.*

4.8 The homotopy type of $\widehat{V}$

We state two of the principal results in Hrushovski and Loeser's paper [HL] which we apply later. These statements can be adapted to the setting of Berkovich spaces and it is the goal of the subsequent sections to introduce these analogues.

Theorem 4.8.1. *([HL], Theorem 11.1.1) Let V be a quasi-projective variety, X a definable subset of $V \times \Gamma_\infty^l$ over some base set $A \subset VF \cup \Gamma$. Then there exists an A-definable deformation retraction $h : I \times \widehat{X} \to \widehat{X}$ to a pro-definable subset Υ definably homeomorphic to a definable subset of Γ_∞^w, for some finite A-definable set w. One can furthermore require the following additional properties for h.*

1. *Given finitely many A-definable functions $\xi_i : V \to \Gamma_\infty$, one can choose h to respect the ξ_i, i.e. $\xi_i(h(t, x)) = \xi_i(x)$ for all t. In particular, finitely many subvarieties or more generally definable subsets U of X can be preserved, in the sense that the homotopy restricts to one of $\widehat{U}$.*

2. *Assume given, in addition, a finite algebraic group action on V. Then the retraction h can be chosen to be equivariant.*

3. *Assume $l = 0$. The homotopy h is Zariski - generalizing, i.e. for any Zariski open subset U of V, $\widehat{U} \cap X$ is invariant under h.*

4. *The homotopy h satisfies condition (*) of 4.4.3, i.e. $h(e_I, h(t, x)) = h(e_I, x)$ for every t and x.*

5. *The homotopy h restricts to $h^\# : I \times X^\# \to X^\#$ [[HL], Definition 2.6.8 and §8.1].*

6. *One has $h(e_I, X) = \Upsilon$, i.e. Υ is the image of the simple points. Hence by (5) it consists of strongly stably dominated points.*

7. *Assume $l = 0$ and $X = V$. Given a finite number of closed irreducible sub varieties W_i of V, one can assume $\Upsilon \cap W_i$ has pure dimension $\dim(W_i)$.*

We state a relative version of Theorem 4.8.1. However, before doing so we introduce the following notion.

Definition 4.8.2. Let T be a definable set and X, Y be pro-definable sets. Let $f : X \to T$ and $g : Y \to T$ be a pair of pro-definable maps. For every $t \in T$, suppose there exists a pro-definable map $\phi_t : X_t \to Y_t$ where $X_t := f^{-1}(t)$ and $Y_t := g^{-1}(t)$. The family of maps $\{\phi_t\}_t$ is said to be *uniformly pro-definable* if there exists a map $\phi : X \to Y$ such that $g \circ \phi = f$ and $\phi_t = \phi_{|X_t}$.

Let T be a definable set. Suppose that $\{(X_t, V_t)\}$, with $X_t \subset V_t$, X_t a definable set and V_t a quasi-projective variety, is a family that *varies uniformly* in the parameter $t \in T$. By this we mean that there exists definable sets X, V with $X \subset V$ and a definable map $\phi : V \to T$ such that for every $t \in T$, $V_t = \phi^{-1}(t)$ and $X_t = X \cap V_t$. For every $t \in T$, Theorem 4.8.1 implies the existence of a deformation retraction $h_t : I_t \times \widehat{X_t} \to \widehat{X_t}$ such that the image $h(e_{I_t}, \widehat{X_t})$ is definably homeomorphic to a definable subset of Γ^n_∞ for some $n \in \mathbb{N}$. This family of deformation retractions indexed by $t \in T$ can be taken *uniformly* in the parameter $t \in T$.

Proposition 4.8.3. *([HL], Proposition 11.7.1) Let V_t be a quasi-projective variety, X_t a definable subset of $V_t \times \Gamma^l_\infty$, definable uniformly in $t \in T$ over some base set A. Then there exists a uniformly pro-definable family $h : I \times \widehat{X} \to \widehat{X}$, a finite set $w(t)$, a definable set $W_t \subseteq \Gamma^{w(t)}$ and $j : W_t \to h(e_I, \widehat{X}_t)$, pro-definable uniformly in $t \in T$, such that for each $t \in T$, h is a deformation retraction, and $j : W \to h(e_I, \widehat{X})$ is a pro-definable homeomorphism. Moreover, (1), (2) of Theorem 11.1.1 can be made to hold if the ξ_i and the group action are given uniformly, as can (4), (5), (6) and (7).*

Chapter 5

Berkovich Spaces

Non-Archimedean valued fields were discovered at the turn of the twentieth century when K. Hensel introduced the field of p-adic numbers $\mathbb{Q}_p$ and ever since, there have been attempts to develop a theory of geometry over such fields analogous to the theory of complex geometry. However, complete algebraically closed non-Archimedean fields display certain anomalies that make them fundamentally different from the complex numbers.

When discussing non-Archimedean fields we will assume that they are complete unless otherwise stated. It is in fact standard to define a non-Archimedean valued field to be a field which is complete with respect to a non-Archimedean valuation. However when studying such fields from a model theoretic perspective, this convention becomes slightly restrictive and it is for this reason that we do not insist on completeness.

The Gelfand-Mazur theorem asserts that every commutative Banach field over $\mathbb{C}$ coincides with $\mathbb{C}$. The theorem does not generalize for algebraically closed non-Archimedean fields. For instance there exists complete models K, k of ACVF such that $k \subset K$ and K strictly contains k. As a result, if $\alpha \in K$ and does not belong to k then the map $x \mapsto \alpha x$ defines a k-linear operator T on the k-vector space K whose spectrum, as defined in the classical situation, is empty. Indeed, there does not exist $\lambda \in k$ such that $(T - \lambda I)$ is not invertible.

The topology induced on a non-Archimedean valued field K by its valuation presents several obstacles towards developing a good theory of analytic functions. The set of open balls contained in K form an open basis for the valuation topology. It can be verified that the non-Archimedean nature of the valuation causes every such open ball to be closed as well. It follows that the field K is totally disconnected. In addition if the field K is algebraically closed, we find that it is not locally compact. If one were to define an analytic function $f : K \to K$ as in the complex case, by insisting that for every $x \in K$, there must exist an $r \in \mathbb{R}$ such that the restriction of f to the closed ball of radius r is a convergent power series of the form $\Sigma_{i \in \mathbb{Z}_{\geq 0}} a_i T^i$ then as the topology on K is totally disconnected, the resulting class of analytic functions would be far too large. The local properties of such analytic functions would not determine their global properties.

Beginning with M. Krasner in the 1940's, the challenge of developing a good theory of analytic functions has been undertaken from several perspectives and resulted in a plethora of famous results. In the 1960's, John Tate

discovered a uniformization theorem for elliptic curves over $\mathbb{Q}_p$ with bad reduction. He showed that for an elliptic curve E over $\mathbb{Q}_p$ there exists an isomorphism $\mathbb{C}_p^*/p^{\mathbb{Z}} \simeq E(\mathbb{C}_p)$ where $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$. It is said that even experts such as Grothendieck believed such an isomorphism was only a brute force manipulation and not evidence of a deeper phenomenon. Tate suspected that the isomorphism must result from a global theory of analytic spaces and developed the theory of rigid analytic spaces in order to make the morphism rigorous. In his theory, Tate introduced algebras of the form $\mathcal{A} := k\{T_1, \ldots, T_n\}$ which model the set of convergent power series on the n-dimensional closed disk defined over a non-Archimedean valued field k and called such objects, Tate algebras. He also introduced a class of open sets on their maximal spectrum $X := \mathrm{Specmax}(\mathcal{A})$ and families of coverings of such open sets, to get a Grothendieck topology. He then defined structure sheaves $\mathcal{O}_X$ on such spaces and general analytic spaces to be spaces equipped with a Grothendieck topology and a structure sheaf that are locally isomorphic to affinoid spaces. The resulting theory of coherent sheaves is well developed and analogous to the theory for complex analytic spaces. However, topological notions such as local compactness and arc wise connectedness cannot be discussed in the rigid analytic framework and it is here that Berkovich's theory of non-Archimedean geometry is particularly satisfying.

We describe briefly the Berkovich theory of non-Archimedean geometry in terms of the most obvious example - the closed ball of unit radius. Let k be a non-Archimedean field and $B \subset k$ denote the closed ball of unit radius. Let us assume that $k\{T\}$ - the commutative k-algebra of power series $\Sigma_{i\in\mathbb{N}} a_i T^i$ such that $a_i \to 0$ as $i \to \infty$ represents the set of analytic functions on B. The algebra $k\{T\}$ is equipped with a norm - the Gauss norm and is complete with respect to it. To overcome the disconnectedness of the valuation topology, we enlarge the space B. Let $\mathcal{M}(k\{T\})$ denote the set of multiplicative semi-norms on the space $k\{T\}$ which are bounded with respect to the Gauss norm. The space B embeds into $\mathcal{M}(k\{T\})$ and furthermore, $\mathcal{M}(k\{T\})$ is Hausdorff, compact and contractible. This construction can be generalized to a class of k-Banach algebras which contain the class of Tate algebras. In this chapter, we describe this construction and highlight the topological properties the resulting spaces enjoy.

5.0.1 Notation

Over the course of this section, we will adopt the convention of writing the group structure of the value group multiplicatively. This is standard practice when discussing results in Berkovich geometry. However, when quoting results from the Hrushovski-Loeser theory of non-Archimedean geometry we adopt their convention and write the value group additively.

5.1 Real valued fields

Definition 5.1.1. A *real valuation* on a field k is a function $|.| : k \to \mathbb{R}_{\geq 0}$ which satisfies the following properties.

1. If $a \in k$ and $|a| = 0$ then $a = 0$.

2. For every $a, b \in k$, $|ab| \leq |a||b|$.

3. For every $a, b \in k$, $|a + b| \leq |a| + |b|$.

We abbreviate notation and use $\mathbb{R}_+$ instead of $\mathbb{R}_{\geq 0}$. A real valuation $|.| : k \to \mathbb{R}_+$ induces a metric on the field k, by sending (x, y) to the value $|x - y|$. A field k equipped with a real valuation is said to be *complete* if it is complete with respect to the metric induced by its valuation. A valuation is *non-Archimedean* if in place of property (3) of the above definition we insist on the stronger condition $|a + b| \leq \max\{|a|, |b|\}$.

Example 5.1.2.

1. The fields $\mathbb{R}$ and $\mathbb{C}$ are complete with respect to the standard Archimedean valuation $|.|_\infty$. More generally, given a real number $0 < \epsilon < 1$, it can be verified that $x \mapsto |x|_\infty^\epsilon$ defines an equivalent Archimedean valuation and that $\mathbb{R}$ and $\mathbb{C}$ are complete with respect to $|.|_\infty^\epsilon$. By equivalent, we mean that the topology generated by the metric induced by $|.|_\infty^\epsilon$ coincides with the topology defined by the valuation $|.|_\infty$.

2. The p-adic integers $\mathbb{Q}_p$ are defined using the p-adic valuation defined on the field $\mathbb{Q}$ as follows. Let p be a prime number. If $x \in \mathbb{Q}$ then x can be written uniquely as $p^r(a/b)$ where $r \in \mathbb{Z}$ and $a, b \in \mathbb{N}$. We define $|x|_p := p^{-r}$. It can be checked that $|.|_p$ is a well defined non-Archimedean valuation on the field $\mathbb{Q}$. The completion of $\mathbb{Q}$ for the valuation $|.|_p$ is a field $\mathbb{Q}_p$ which we call the field of p-adic numbers. Given a real number $0 < \epsilon < 1$, we can define an equivalent p-adic valuation $|.|_p^\epsilon$ as we did in the case of an Archimedean valuation.

3. Let k be a field. The field of Laurent series which is the set of power series of the form $f := \Sigma_{i \in \mathbb{Z}} a_i T^i$ where $a_i \in k$ for all i and there exists $i_0 \in \mathbb{Z}$ such that $a_i = 0$ for $i \leq i_0$. Let $0 < \epsilon < 1$ be a real number. We define a valuation on the field of Laurent series by setting $|f| := \epsilon^n$ where n is smallest integer such that $a_n \neq 0$. It can be checked that the valuation so defined is non-Archimedean.

4. Any field k is complete with respect to the trivial valuation $|.|_0$ defined by $|x|_0 := 1$ if $x \neq 0$ and $|0| = 0$.

As outlined in the introduction, the class of fields which are complete with respect to a non-Archimedean valuation present certain anomalies when compared to their Archimedean counterparts. The following theorems show that there are many more complete fields with a non-Archimedean valuation than there are fields which are complete for an Archimedean valuation.

Theorem 5.1.3. *(Ostrowski's Thorem) Any valuation on the field of rational numbers $\mathbb{Q}$ is either $|.|_\infty^\epsilon$ for $0 < \epsilon \leq 1$ or $|.|_p^\epsilon$ for a prime p and $0 < \epsilon < 1$,or $|.|_0$.*

Theorem 5.1.4. *Any field k which is complete with respect to a valuation is either Archimedean (i.e., it is $\mathbb{R}$ or $\mathbb{C}$ provided with $|.|_\infty^\epsilon$ for $0 < \epsilon \leq 1$), or non-Archimedean (i.e. its valuation is non- Archimedean).*

The next lemma introduces a class of valuations which play an important role in what follows.

Lemma 5.1.5. *(Gauss Lemma). Let k be a non-Archimedean field, i.e. a field which is complete with respect to a non-Archimedean valuation $|.|$. Given a real number $r > 0$, we define a real valued function $|.|_r$ on the field of rational functions $k(T)$ as follows. Let $|\Sigma_{i=0}^n a_i T^i|_r := \max_{0\leq i\leq n}|a_i|r^i$ and $|f/g|_r = |f|_r/|g|_r$ for $f \in k[T]$ and $g \in k[T] \setminus \{0\}$.*

1. *The function $|.|_r$ is a well defined non-Archimedean valuation.*
2. *The field $k(T)$ is complete with respect to $|.|_r$ if and only if the valuation on k is trivial and $r \geq 1$.*

5.2 Commutative Banach rings and their spectrum

Definition 5.2.1. Let M be an Abelian group.

1. A *semi-norm* on M is a function $|.| : M \to \mathbb{R}_+$ which satisfies the following properties.

   (a) $|0| = 0$.

   (b) For every $f, g \in M$, $|f - g| \leq \max\{|f|, |g|\}$.

   The semi-norm $|.|$ is a *norm* if the set $\{m \in M||m| = 0\}$ does not contain an element different from 0.

2. Let $|.|$, $|.|'$ be semi-norms defined on M. We say that $|.|$ and $|.|'$ are *equivalent* if there exists $C, C' > 0$ such that $C|f| < |f|' < C'|f|$.

3. Let $N \subset M$ be a sub-group. The *residue semi-norm* on the quotient M/N is defined as follows. Let $x \in M/N$. We set $|x|_{M/N} := \inf_{y\in\pi^{-1}(x)}\{|y|_M\}$ where $|.|_M$ denotes the semi-norm on M and $\pi : M \to M/N$ is the quotient map.

4. Let N be an Abelian group equipped with a semi-norm $|.|_N$. A group homomorphism $f : M \to N$ is said to be *bounded* if for every $m \in M$, we have that $|f(m)|_N \leq |m|_M$ where $|.|_M$ denotes the semi-norm on M. The morphism f is *admissible* if the residue semi-norm on $M/\mathrm{Ker}(f)$ is equivalent to the restriction of the semi-norm on N to $\mathrm{Im}(f)$.

Remark 5.2.2. *1.* Let M be an Abelian group equipped with a semi-norm $|.|_M$. The semi-norm $|.|_M$ induces a topology on M which is Hausdorff if and only if $|.|_M$ is a norm.

2. Let $N \subset M$ be a sub group and $|.|_M$ a norm on M. The residue semi-norm is a norm if and only if the sub group N is closed in the topology induced by $|.|_M$.

The notions introduced above can be adapted to the case of commutative rings. The rings we discuss henceforth will be commutative with unity.

Definition 5.2.3. Let M be a ring. A *semi-norm* on M is a function $|.| : M \to \mathbb{R}_+$ which satisfies the following properties.

1. $|0| = 0$.
2. For every $a, b \in M$, $|ab| \leq |a||b|$.
3. For every $a, b \in M$, $|a + b| \leq |a| + |b|$.

The semi-norm $|.|$ is a *norm* if the set $\{x \in M||x| = 0\}$ does not contain an element different from 0. The norm induces a metric on M and we say that the ring is a *Banach ring* if it is complete with respect to this metric. A semi-norm is *multiplicative* if instead of property (2), we have that for every $a, b \in M$, $|ab| = |a||b|$. It is *power multiplicative* if for every $m \in M$ and $n \in \mathbb{Z}_+$, we have that $|m^n| = |m|^n$. If the ring M is endowed with a norm $||.||$ then we say that a semi norm $|.|$ is *bounded with respect to* $||.||$ if there exists $C > 0$ such that for every $a \in M$, $|a| \leq C||a||$. A *bounded morphism* $f : X \to Y$ of semi-normed rings is a ring homomorphism between the underlying rings for which there exists $C > 0$ such that for every $x \in X$, $|f(x)|_Y \leq C|x|_X$ where $|.|_X$ and $|.|_Y$ denote the semi-norms on X and Y respectively.

Example 5.2.4.

1. The ring of integers $\mathbb{Z}$ can be equipped with the restriction of the standard Archimedean valuation $|.|_\infty$. The topology induced by this norm is the discrete topology. As a result $(\mathbb{Z}, |.|_\infty)$ is an example of a Banach ring.
2. Let $r := (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. We define the ring $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ (abbreviated $k\{r^{-1}\underline{T}\}$) to be the set of all formal power series $f = \Sigma_{v \in \mathbb{Z}_+^n} a_v T^v$ such that $|a_v|r^v \to 0$ as $|v| := v_1 + ... + v_n \to \infty$. Then $k\{r^{-1}\underline{T}\}$ is a commutative Banach ring with respect to the Banach norm $|f| = \max\{|a_v|r^v\}$.

Definition 5.2.5. *(Normed and semi-normed modules)* Let $(\mathcal{A}, |.|_\mathcal{A})$ be a normed ring. A *semi-normed $\mathcal{A}$-module* is an $\mathcal{A}$-module M equipped with a semi-norm $|.|_M$ for which there exists $C > 0$ such that for every $a \in \mathcal{A}$ and $m \in \mathcal{M}$, we have that $|am|_M \leq C|a|_\mathcal{A}|m|_\mathcal{M}$. A *normed $\mathcal{A}$-module* is a semi-normed module whose semi-norm $|.|_M$ is a norm. A *Banach $\mathcal{A}$-module* is a complete normed $\mathcal{A}$-module.

Remark 5.2.6. Let $(\mathcal{A}, |.|_\mathcal{A})$ be a normed ring and $(M, |.|_M)$ be a semi-normed $\mathcal{A}$-module. There exists a semi-norm $|.|'_M$ on M which is equivalent to $|.|_M$ such that for every $a \in \mathcal{A}$ and $m \in M$, $|am|'_M \leq |a|_\mathcal{A}|m|'_M$.

Given a semi-normed ring $\mathcal{A}$, we will often denote its semi-norm by $|.|_\mathcal{A}$. Likewise, for semi-normed groups or modules.

Definition 5.2.7. *(Complete tensor products)* Let $\mathcal{A}$ be a normed ring. Let M and N be semi-normed $\mathcal{A}$-modules. We define a semi-norm on the tensor product $M \otimes_\mathcal{A} N$ as follows. Let $x \in M \otimes_\mathcal{A} N$. We set $|x|_{M \otimes_\mathcal{A} N} := \inf\{\Sigma_i |m_i|_M |n_i|_N\}$ where the infimum is taken over all representations $x = \Sigma_i m_i \otimes n_i$. The completion of $M \otimes_\mathcal{A} N$ with respect to this semi-norm is denoted $M \widehat{\otimes}_\mathcal{A} N$ and referred to as the *complete tensor product.*

The notions of semi-normed modules over a normed ring and the complete tensor product can be adapted to algebras over a normed ring.

Definition 5.2.8. *(Normed algebras)* Let $\mathcal{A}$ be a normed ring. A *semi-normed $\mathcal{A}$-algebra B* is an $\mathcal{A}$-algebra equipped with a function $|.| : B \to \mathbb{R}_+$ such that $(B, |.|)$ is a semi-normed ring as well as a semi-normed $\mathcal{A}$-module.

Analogous to the construction of the complete tensor product in the case of semi-normed modules, the complete tensor product of normed $\mathcal{A}$-algebras can be realized as a normed $\mathcal{A}$-algebra.

Let B_1, B_2 be normed $\mathcal{A}$-algebras and $B_1 \widehat{\otimes}_{\mathcal{A}} B_2$ denote the complete tensor product defined by regarding the B_i as $\mathcal{A}$-modules. By 3.1.1 in [BGR], $B_1 \widehat{\otimes}_{\mathcal{A}} B_2$ is a normed $\mathcal{A}$-algebra as well with a unique multiplication such that for $b_1, b_1' \in B_1$ and $b_2, b_2' \in B_2$, we have that

$$(b_1 \widehat{\otimes} b_2)(b_1' \widehat{\otimes} b_2') = b_1 b_1' \widehat{\otimes} b_2 b_2'.$$

Definition 5.2.9. *(Spectrum of a Banach ring)* Let $(\mathcal{A}, |.|)$ be a Banach ring. The *spectrum* associated to $\mathcal{A}$ is the set of multiplicative semi-norms bounded with respect to the norm $|.|$. We use $\mathcal{M}(\mathcal{A})$ to denote the spectrum of $(\mathcal{A}, |.|)$. The spectrum $\mathcal{M}(\mathcal{A})$ is given the weakest topology such that, for every $f \in \mathcal{A}$ the function $\mathcal{M}(\mathcal{A}) \to \mathbb{R}_+$ defined by $x \mapsto |f(x)|$ is continuous. Here $|f(x)|$ denotes the image in $\mathbb{R}_+$ of the element f for the semi-norm x.

Definition 5.2.10. *(Spectral radius)* Let $(\mathcal{A}, |.|)$ be a Banach ring and $f \in \mathcal{A}$. The *spectral radius* of f is the real number $\rho(f) := \lim_{n\to\infty}(|f^n|)^{1/n} = \inf_{n\in\mathbb{Z}}\{|f^n|^{1/n}\}$. It can be shown that $\rho(f) = \max_{x\in\mathcal{M}(\mathcal{A})}\{|f(x)|\}$ and that the function $\mathcal{A} \to \mathbb{R}_+$ given by $f \mapsto \rho(f)$ defines a power multiplicative semi-norm bounded with respect to $|.|$.

We now introduce the notion of the residue field of a point in the non-Archimedean sense.

Definition 5.2.11. Let $\mathcal{A}$ be a Banach ring. Let $x \in \mathcal{M}(\mathcal{A})$. Let $|.|_x$ denote the semi-norm defined by the point x. Let $\mathrm{Ker}(|.|_x) := \{f \in \mathcal{A} | |f|_x = 0\}$. It can be verified that $\mathrm{Ker}(|.|_x)$ is a prime ideal of $\mathcal{A}$. It follows that $\mathcal{A}/\mathrm{Ker}(|.|_x)$ is an integral domain and the semi-norm $|.|_x$ defines a norm on the quotient field $Q(\mathcal{A}/\mathrm{Ker}(|.|_x))$. Let $\mathcal{H}(x)$ denote the completion of this field with respect to the norm induced by x. If we abuse notation and use $|.|_x$ to denote the norm on $\mathcal{H}(x)$ then we have a bounded character $\mathcal{A} \to \mathcal{H}(x)$ defined by mapping an element $f \in \mathcal{A}$ to its class in the quotient $\mathcal{A}/\mathrm{Ker}(|.|_x) \subset \mathcal{H}(x)$.

Theorem 5.2.12. *Let $\mathcal{A}$ be a Banach ring. The spectrum $\mathcal{M}(\mathcal{A})$ is a non empty, compact Hausdorff space.*

5.3 Affinoid spaces

Let k be a non-Archimedean field. Let $(X, |.|)$ be a Banach space over k and $Y \subset X$ be a closed sub space. A Banach space over k is a particular instance of Definition 5.2.5. Nonetheless, we provide a definition. A *Banach space over k* is a k-vector space endowed with a function $|.| : X \to \mathbb{R}_+$ called its norm, which satisfies the following properties.

1. $|x| = 0$ if and only if $x = 0$,

2. For $x, y \in X$, $|x+y| \leq |x| + |y|$.

3. For $x \in X$ and $c \in k$, $|cx| = |c|_k |x|$ where $|.|_k$ denotes the valuation on the field k .

Let $x \in X$ and $[x]$ denote its image in X/Y. We set $|[x]| := \inf_{y \in [x]}\{|y|\}$. It can be checked that this defines a norm on the space X/Y which we call the *quotient* norm. A morphism $f : X \to Y$ of k-Banach spaces is *bounded* if there exists a real number $C > 0$ such that for every $x \in X$, $|f(x)|_Y \leq C|x|_X$ where $|.|_X$ and $|.|_Y$ denote the norms on X and Y respectively.

Definition 5.3.1. A bounded k-linear map $f : X \to Y$ between k-Banach spaces is *admissible* if the induced map $X/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is an isomorphism of Banach spaces when $X/\mathrm{Ker}(f)$ is provided with the quotient norm and $\mathrm{Im}(f)$ is endowed with the restriction of the norm on Y.

Definition 5.3.2. A Banach k-algebra $\mathcal{A}$ is an *affinoid algebra* if for some non - negative integer n, there exists an admissible epimorphism $\phi : k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{A}$. If such an epimorphism exists with $r_1 = \ldots = r_n = 1$ then we say that $\mathcal{A}$ is *strictly affinoid.*

Proposition 5.3.3. *1. Any k-affinoid algebra is Noetherian and all of its ideals are closed.*

2. The k-affinoid algebra $k\{r_1^{-1}T, ..., r_n^{-1}T\}$ is strictly k-affinoid if and only if $r_i \in \sqrt{|k^*|} := \{\alpha \in \mathbb{R}_{>0} | \exists n \in \mathbb{Z}_{\geq 1}$ such that $\alpha^n \in |k^*|\}$.

Recall that in Definition 5.2.10, we introduced the notion of the spectral radius of an element of a Banach ring and hence defined the spectral norm. The following proposition implies in particular that for a reduced affinoid algebra, the spectral norm is equivalent to the given norm.

Proposition 5.3.4. *1. Let $\mathcal{A}$ be a k-affinoid algebra and $f \in \mathcal{A}$. There exists $C > 0$ and $N \in \mathbb{N}$ such that $|f^n|_{\mathcal{A}} \leq C\rho(f)^n$ for every $n \in \mathbb{N}$ and $n \geq N$. In particular, if $f \in \mathcal{A}$ is quasi-nilpotent (i.e. $\rho(f) = 0$) then f is nilpotent.*

2. If $\mathcal{A}$ is reduced then there exists $C > 0$ such that $|f|_{\mathcal{A}} \leq C\rho(f)$ for every $f \in \mathcal{A}$.

Definition 5.3.5. *(Affinoid spaces) The category of k-affinoid spaces* denoted k - $\mathcal{A}ff$ is the category dual to the category whose objects are k-affinoid algebras and morphisms are bounded homomorphisms.

5.3.1 Dimension of an affinoid space

Proposition 5.3.6. *Let $\mathcal{A}$ be a k-affinoid algebra and k', k'' be non-Archimedean fields which contain k and such that the algebras $\mathcal{A}\widehat{\otimes}_k k', \mathcal{A}\widehat{\otimes}_k k''$ are strictly k' and k''-affinoid respectively. The Krull dimension of $\mathcal{A}\widehat{\otimes}_k k'$ is the same as the Krull dimension of $\mathcal{A}\widehat{\otimes}_k k''$.*

Definition 5.3.7. The *dimension* $\dim(X)$ of a k-affinoid space $X = \mathcal{M}(A)$ is the Krull dimension of the algebra $\mathcal{A}\widehat{\otimes}_k k'$ for some non-Archimedean field k' over k such that $\mathcal{A}\widehat{\otimes} k'$ is strictly k'-affinoid.

Proposition 5.3.8. *1. For any finite affinoid covering $\{X_i\}_{i\in I}$ of X, one has* $\dim(X) = \max\{\dim(X_i)\}$.

2. For any point $x \in X$, one has $\mathrm{cdl}(\mathcal{H}(x)) \leq \mathrm{cdl}(k) + \dim(X)$. *Here l is a prime integer and* $\mathrm{cdl}(k)$ *is the l-cohomological dimension of k, i.e. the minimal integer n (or ∞) such that $H^i(G_k, A) = 0$ for all $i > n$ and all l-torsion discrete G_k-modules A, where G_k is the absolute Galois group of k.*

Let $\mathcal{A}$ be a k-affinoid algebra. The notion of a k-affinoid algebra can be generalized as follows.

Definition 5.3.9.

1. Let $r := (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. The algebra $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ (abbreviated $\mathcal{A}\{r^{-1}\underline{T}\}$) is the set of formal power series of the form $f := \Sigma_{v\in\mathbb{Z}^n_{\geq 0}} a_v T^v$ such that $a_v r^v \to 0$ as $|v| \to \infty$. It can be verified that we define a norm $|.|$ on $\mathcal{A}\{r^{-1}\underline{T}\}$ by setting $|f|$ to be $\max\{a_v r^v\}$.

2. An $\mathcal{A}$-algebra B is *$\mathcal{A}$-affinoid* if there exists an admissible epimorphism $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to B$ for some $(r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$.

5.3.2 Modules over an affinoid algebra

By a finite $\mathcal{A}$-module M, we mean an $\mathcal{A}$-module M such that for some $n \in \mathbb{N}$ there exists an epimorphism of $\mathcal{A}$-modules $\mathcal{A}^n \to M$ (i.e. the standard notion of a finite module over a ring). Let Mod^h denote the category of finite $\mathcal{A}$-modules.

Definition 5.3.10. A *finite Banach $\mathcal{A}$-module* is a normed $\mathcal{A}$-module B such that for some $n \in \mathbb{N}$ there exists an admissible epimorphism $\mathcal{A}^n \to B$ of normed $\mathcal{A}$-modules. The category of finite Banach $\mathcal{A}$-modules is denoted Mod^h_b.

Proposition 5.3.11. *1. The forgetful functor $\theta : \mathrm{Mod}^h_b \to \mathrm{Mod}^h$ is an equivalence of categories.*

2. Any $\mathcal{A}$-linear map between finite Banach $\mathcal{A}$-modules is admissible.

3. Given $M, N \in \mathrm{Mod}^h_b(\mathcal{A})$ and an $\mathcal{A}$-affinoid algebra B, one has $M \otimes_{\mathcal{A}} N \simeq M\widehat{\otimes}_{\mathcal{A}} N \in \mathrm{Mod}^h_b$ and $M \otimes_{\mathcal{A}} B \simeq M\widehat{\otimes}_{\mathcal{A}} B \in \mathrm{Mod}^h_b(B)$.

5.3.3 Affinoid domains

Let $\mathcal{A}$ be a k-affinoid algebra and $X := \mathcal{M}(\mathcal{A})$.

Definition 5.3.12. *(Affinoid domain)* A subset $V \subset X$ is an *affinoid domain* if there exists an affinoid algebra $\mathcal{A}_V$ and a morphism of k-affinoid spaces $\phi : Y \to X$ where $Y := \mathcal{M}(\mathcal{A}_V)$ such that the following conditions are satisfied.

1. $\mathrm{Im}(\phi) = V$.

2. For every morphism of affinoid spaces $\psi : Z \to X$ with $\mathrm{Im}(\psi) \subset V$, there exists a morphism of affinoid spaces $\phi' : Z \to Y$ such that $\psi = \phi \circ \phi'$.

Example 5.3.13. We introduce three classes of affinoid domains contained in the affinoid space $X = \mathcal{M}(\mathcal{A})$.

1. Let $f := (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_m)$ be tuples of elements of $\mathcal{A}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}, s = (s_1, \ldots, s_m) \in \mathbb{R}^m_{>0}$. Let $X(r^{-1}f, sg^{-1})$ denote the sub space $\{x \in X | \wedge_{i,j} (|f_i(x)| \leq r_i \text{ and } |g_j(x)| \geq s_j)\}$ endowed with the topology induced by the topology on X. The sub space $X(r^{-1}f, sg^{-1})$ is an affinoid domain with respect to the bounded homomorphism

$$\mathcal{A} \to \mathcal{A}_{X(r^{-1}f, sg^{-1})} = \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, s_1S_1, \ldots, s_mS_m\}/(T_i - f_i, g_jS_j - 1)_{i,j}.$$

   The space $X(r^{-1}f, sg^{-1})$ is an affinoid domain and sub spaces of X of this form are called *Laurent domains.* Those affinoid domains of X which are of the form $X(r^{-1}f)$ for some $f \in \mathcal{A}^n$ and $r \in \mathbb{R}^n_+$ are called *Weierstrass domains.*

2. Let $\{f_1, \ldots, f_n, g\}$ be a set of elements in $\mathcal{A}$ which do not have common zeros in X and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. Let $X(r^{-1}f/g)$ denote the sub space $\{x \in X | \wedge_i (f_i(x) \leq r_i g(x))\}$ endowed with the topology induced by the topology on X. The space $X(r^{-1}f/g)$ is an affinoid domain with respect to the following bounded homomorphism of affinoid algebras.

$$\mathcal{A} \to \mathcal{A}_{X(r^{-1}f/g)} = \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}/(gT_i - f_i)_i.$$

   Such an affinoid domain is called a *rational domain.*

Remark 5.3.14. The intersection of two affinoid domains of X of the same type (Laurent, Weierstrass or rational) is again an affinoid domain of that type and every point of X possesses a fundamental system of neighborhoods which are Laurent domains.

Proposition 5.3.15. *Let V be an affinoid domain in X.*

1. $\mathcal{M}(\mathcal{A}_V) \tilde{\to} V$.
2. *$\mathcal{A}_V$ is a flat $\mathcal{A}$-algebra.*
3. *For any point $x \in V$, one has $\mathcal{H}(x) \tilde{\to} \mathcal{H}_V(x)$.*
4. *V is a Weierstrass (resp. rational) domain if and only if the image of $\mathcal{A}$ (resp. $\mathcal{A}_V$) is dense in $\mathcal{A}_V$, where $\mathcal{A}_V$ is the localization of $\mathcal{A}$ with respect to the elements that do not vanish at any point of V.*

Theorem 5.3.16. *(Gerritzen - Grauert) Let $X = \mathcal{M}(\mathcal{A})$ be an affinoid space. Every domain of X is the union of a finite number of rational domains.*

Theorem 5.3.17. *Let $\mathcal{A}$ be an affinoid algebra and $X := \mathcal{M}(\mathcal{A})$. Let $\{V_i\}_{i \in I}$ be a finite covering of X by affinoid domains.*

1. *(Tate's acyclicity theorem) For any finite Banach $\mathcal{A}$-module M, the Čech complex*

$$0 \to M \to \prod_i M_{V_i} \to \prod_{i,j} M_{V_i \cap V_j} \to \ldots$$

   is exact and admissible.

2. *(Kiehl's theorem). Suppose we are given, for each $i \in I$, a finite $\mathcal{A}_{V_i}$-module M_i and for each pair $i, j \in I$, an isomorphism of finite $\mathcal{A}_{V_i \cap V_j}$-modules $\alpha_{i,j} : M_i \otimes_{\mathcal{A}_{V_i}} \mathcal{A}_{V_i \cap V_j} \tilde{\to} M_j \otimes_{\mathcal{A}_{V_j}} \mathcal{A}_{V_i \cap V_j}$ such that $(\alpha_{i,l})_{|W} = (\alpha_{i,j})_{|W} \circ (\alpha_{j,l})_{|W}$, $W = V_i \cap V_j \cap V_l$, for all $i, j, l \in I$. Then there exists a finite $\mathcal{A}$-module M that gives rise to the $\mathcal{A}_{V_i}$-modules M_i and the isomorphisms $\alpha_{i,j}$.*

Recall that we defined the category of k-affinoid spaces k-$\mathcal{A}ff$ to be the category dual to the category whose objects are affinoid algebras and morphisms bounded k-algebra homomorphisms. In the construction of Analytic spaces that follows we will require a sub category of k-$\mathcal{A}ff$.

Definition 5.3.18. Let X, Y be k-affinoid spaces. A morphism $\phi : Y \to X$ is an *affinoid domain embedding* if it induces an isomorphism between Y and an affinoid domain of X i.e. ϕ is a homeomorphism from Y to $\phi(Y)$ and if $\psi : Z \to X$ is a morphism of affinoid spaces for which $\psi(Z) \subset \phi(Y)$ then there exists a morphism $f : Z \to Y$ such that $\psi = \phi \circ f$. The category $k - f\mathcal{A}ff^{ad}$ is the category whose objects are k-affinoid spaces and morphisms are affinoid domain embeddings.

5.4 Analytic spaces

A scheme is a locally ringed space such that every point has an open neighborhood isomorphic to an affine scheme. A similar attempt to define analytic spaces using affinoid spaces meets an immediate obstacle - affinoid spaces are compact. In what follows, we present a brief summary of Berkovich's construction of analytic spaces. The results we state later in this text concern themselves with a smaller class of k-analytic spaces - those arising from algebraic varieties via the *analytification* functor and we discuss such spaces subsequently.

5.4.1 Nets and Quasi-nets

Let X be a topological space. The notions that follow do not require that X be Hausdorff and so we do not insist on it being so.

Definition 5.4.1. *(Quasi-net)* A family τ of subsets of X is a *quasi-net* if for every $x \in X$ there exists $V_1, \ldots, V_n \in \tau$ such that $x \in \cap_i V_i$ and $\cup_i V_i$ is a neighborhood of x.

Given a quasi-net τ on a space X, we regard the elements of τ as sub spaces of X whose topologies are induced by the topology on X.

Remark 5.4.2 ([B2], Lemma 1.1.1)**.** Let τ be a quasi-net on X.

1. A subset $U \subset X$ is open if and only if for every $V \in \tau$, $U \cap V$ is open in V.

2. Suppose that every element of τ is compact and Hausdorff. The space X is Hausdorff if and only if for every $V_1, V_2 \in \tau$, $V_1 \cap V_2$ is compact.

3. If the space X is equipped with a quasi-net composed entirely of compact sub spaces then it is locally compact.

Definition 5.4.3. A family of subsets τ of X is a *net* if it is a quasi-net and in addition, for every $U, V \in \tau$, the restriction $\tau_{|U \cap V}$ is a quasi-net on the space $U \cap V$.

Let τ be a net on X. Then τ defines in a natural way a category whose objects are the elements of τ and whose morphisms are the inclusions between elements of the net. Let $\mathcal{T}op$ denote the category of topological spaces and $\mathcal{T} : \tau \to \mathcal{T}op$ denote the functor which maps an element of τ to its underlying topological space. Recall that we introduced the category $k - \mathcal{A}ff^{ad}$ whose objects are k-affinoid spaces and morphisms are affinoid domain embeddings. Let $\mathcal{T}^a : k - \mathcal{A}ff^{ad} \to \mathcal{T}op$ denote the functor that maps a k-affinoid space to its underlying topological space.

Definition 5.4.4. Let the space X be locally Hausdorff and τ be a net of compact sets on X. A *k-affinoid atlas $\mathcal{A}$ on X with net τ* consists of a functor $A : \tau \to k - \mathcal{A}ff^{ad}$ and an isomorphism of functors $\mathcal{T}^a \circ A \tilde{\to} \mathcal{T}$.

An atlas $\mathcal{A}$ on a space X with net τ implies that for every $V \in \tau$, there exists a k-affinoid algebra $\mathcal{A}_V$ and a homeomorphism $\mathcal{M}(\mathcal{A}_V) \to V$. In addition, if $U \subset V$ are elements of τ then the map $\mathcal{M}(\mathcal{A}_U) \to \mathcal{M}(\mathcal{A}_V)$ obtained by functoriality of $\mathcal{A}$ is an affinoid domain embedding.

5.4.2 k-Analytic spaces

Definition 5.4.5. A *k-analytic space* consists of a triple $(X, \mathcal{A}, \tau)$ where X is a locally Hausdorff topological space, τ is a net of compact sets on X and $\mathcal{A}$ is a k-affinoid atlas on X with net τ.

Proposition 5.4.6. *Let $(X, \mathcal{A}, \tau)$ be a k-analytic space.*

1. *Let $V \in \tau$. If $W \subset V$ is an affinoid domain in V then it is an affinoid domain in any $V' \in \tau$ such that $W \subset V'$.*

2. *Let $\bar{\tau}$ be the collection of subspaces W of X such that there exists $V \in \tau$ which contains W and W is an affinoid domain in V. The family $\bar{\tau}$ is a net on X and there is a unique k-affinoid atlas $\bar{\mathcal{A}}$ which extends $\mathcal{A}$ such that the triple $(X, \bar{\mathcal{A}}, \bar{\tau})$ is a k-analytic space.*

In order to define the category of k-analytic spaces, we must define what we mean by a morphism of k-analytic spaces. We begin by introducing the notion of a strong morphism of k-analytic spaces.

Definition 5.4.7. Let $(X', \mathcal{A}', \tau')$ and $(X, \mathcal{A}, \tau)$ be a pair of k-analytic spaces. A *strong morphism* $\phi : (X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ of k-analytic spaces consists of the following data.

1. A continuous map $\phi : X' \to X$ such that for every $V' \in \tau'$ there exists $V \in \tau$ with $\phi(V') \subseteq V$.

2. A system of compatible morphisms of k-affinoid spaces $\phi_{V'/V} : (V', \mathcal{A}'_{V'}) \to (V, \mathcal{A}_V)$ for every pair $V' \in \tau'$ and $V \in \tau$ such that $\phi(V') \subset V$.

Lemma 5.4.8. *1. A strong morphism $\phi : (X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ of k-analytic spaces extends uniquely to a strong morphism $\bar{\phi} : (X', \overline{\mathcal{A}'}, \bar{\tau}) \to (X, \mathcal{A}, \tau)$.*

2. *Given a pair of strong morphisms* $\phi : (X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ *and* $\psi : (X'', \mathcal{A}'', \tau'') \to (X', \mathcal{A}', \tau')$ *between* k*-analytic spaces, the composition* $\phi \circ \psi : (X'', \mathcal{A}'', \tau'') \to (X, \mathcal{A}, \tau)$ *is a well defined strong morphism of* k*-analytic spaces.*

Let $k - \widetilde{\mathcal{A}n}$ denote the category whose objects are k-analytic spaces and morphisms are strong morphisms of k-analytic spaces.

Definition 5.4.9. A strong morphism $\phi : (X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ of k-analytic spaces is a *quasi-isomorphism* if it induces a homeomorphism $\phi : X' \to X$ and for every $V' \in \tau'$ and $V \in \tau$ such that $\phi(V') \subseteq V$ the induced morphism $\phi_{V'/V} : V' \to V$ is an affinoid domain embedding.

Lemma 5.4.10. *The system of quasi-isomorphisms in* $k - \widetilde{\mathcal{A}n}$ *admits a calculus of right fractions i.e. it satisfies the following properties.*

1. *The composition of two quasi-isomorphisms is again a quasi-isomorphism. The identity map is a quasi-isomorphism.*

2. *Let* $f : (X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ *be a strong morphism of* k*-analytic spaces and* $s : (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau}) \to (X, \mathcal{A}, \tau)$ *be a quasi-isomorphism. There there exists a* k*-analytic space* $(\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ *and strong morphisms* $\tilde{f} : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau})$ *and* $\tilde{s} : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (X', \mathcal{A}', \tau')$ *such that the following diagram commutes.*

$$\begin{tikzcd}
(\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \arrow[r, "\tilde{f}"'] \arrow[d, "\tilde{s}"] & (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau}) \arrow[d, "s"] \\
(X', \mathcal{A}', \tau') \arrow[r, "f"'] & (X, \mathcal{A}, \tau)
\end{tikzcd}$$

.

3. *Let* $f, g : (X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ *be a pair of strong morphisms between* k*-analytic spaces and suppose there exists a quasi-isomorphism* $s : (X, \mathcal{A}, \tau) \to (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau})$ *such* $s \circ f = s \circ g$*. Then there exists a quasi-isomorphism* $\tilde{s} : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (X', \mathcal{A}', \tau')$ *such that* $f \circ \tilde{s} = g \circ \tilde{s}$*.*

Definition 5.4.11. *The category* $k - \mathcal{A}n$ is the localization of the category $k - \widetilde{\mathcal{A}n}$ for the system of quasi-isomorphisms.

We now provide a description of the set of morphisms in the category $k - \mathcal{A}n$ between two k-analytic spaces.

Proposition 5.4.12. *Let* $(X, \mathcal{A}, \tau)$ *and* $(X', \mathcal{A}', \tau')$ *be* k*-analytic spaces. Given a net* σ *on* X*, we write* $\sigma \leq \tau$ *if* $\sigma \subset \bar{\tau}$*. If* $\sigma \leq \tau$ *then let* $\mathcal{A}_\sigma$ *denote the*

restriction of $\bar{\mathcal{A}}$ to σ. The system of nets $\{\sigma|\sigma \leq \tau\}$ is filtered and we have the following equality.

$$\mathrm{Hom}_{k-\mathcal{A}n}((X,\mathcal{A},\tau),(X,\mathcal{A}',\tau')) = \varinjlim_{\sigma\leq\tau} \mathrm{Hom}_{k-\widetilde{\mathcal{A}n}}((X,\mathcal{A}_\sigma,\sigma),(X',\mathcal{A}',\tau')).$$

Example 5.4.13. *1.* Every k-affinoid space can be realized as a k-analytic space. Let $\mathcal{A}$ be a k-affinoid algebra and $X = \mathcal{M}(\mathcal{A})$ be the associated affinoid space. The triple $(X,\mathcal{A},\{X\})$ defines a k-analytic space. Clearly, this defines a functor $k-\mathcal{A}ff \to k-\mathcal{A}n$ and it can be shown that this functor is fully faithful.

2. Let $n \in \mathbb{N}$. We define $\mathbb{A}^{n,\mathrm{an}}$ to be the set of mulitplicative bounded semi-norms on $k[T_1,\ldots,T_n]$ which restrict to the valuation on k. A polynomial $f \in k[T_1,\ldots,T_n]$ defines a function on $\mathbb{A}^{n,\mathrm{an}}$ to $\mathbb{R}_{\geq 0}$ by mapping a semi-norm x to the value $|f(x)|$ of f evaluated at x. We endow $\mathbb{A}^{n,\mathrm{an}}$ with the weakest topology such that for every $f \in k[T_1,\ldots,T_n]$, the function $f : \mathbb{A}^{n,\mathrm{an}} \to \mathbb{R}_{\geq 0}$ is continuous.

We endow the space $\mathbb{A}^{n,\mathrm{an}}$ with the structure of a k-analytic space as follows. Let $r = (r_1,\ldots,r_n) \in \mathbb{R}^n_{>0}$. We define $E(0,r) := \{x \in \mathbb{A}^{n,\mathrm{an}} | \wedge_i (|T_i(x)| \leq r_i)\}$. Clearly, $\mathbb{A}^{n,\mathrm{an}} = \bigcup_{r\in\mathbb{R}^n_{>0}} E(0,r)$ and it can be checked that $\tau := \{E(0,r)\}_r$ defines a net on $\mathbb{A}^{n,\mathrm{an}}$. Let $\mathcal{A}_r := k\{r_1^{-1}T_1,\ldots,r_n^{-1}T_n\}$ where $k\{r^{-1}\underline{T}\}$ was defined in Example 5.2.4 (2). Let $X_r := \mathcal{M}(\mathcal{A}_r)$. Observe that $k[T_1,\ldots,T_n] \subset \mathcal{A}_r$ for any $r \in \mathbb{R}^n_{>0}$ and is in fact a dense subset of $\mathcal{A}_r$. It follows that we have a well defined injective continuous map $\phi_r : X_r \to E(0,r)$ defined by restricting a bounded multiplicative semi-norm on $\mathcal{A}_r$ to the polynomial algebra $k[T_1,\ldots,T_n]$. It can be shown that the map ϕ_r is surjective and bi - continuous which implies that the affinoid space X_r is homeomorphic to $E(0,r)$ via ϕ_r. This defines a k-affinoid atlas $\mathcal{A}$ on $\mathbb{A}^{n,\mathrm{an}}$ with the net τ. Hence $(\mathbb{A}^{n,\mathrm{an}},\mathcal{A},\tau)$ is a k-analytic space.

5.5 Analytic domains

Let $(X,\mathcal{A},\tau)$ be a k-analytic space. Let $U \subset X$ be an open subset. We regard U as a topological space with the topology induced by the topology on X. Let $\bar{\tau}_{|U}$ (cf. Proposition 5.4.6 (2)) denote the collection of those elements of the net $\bar{\tau}$ that are contained in U and $\bar{\mathcal{A}}_{|U}$ be the restriction of the atlas $\bar{\mathcal{A}}$ to the family $\bar{\tau}_{|U}$. The family $\bar{\tau}_{|U}$ of subsets of U is a net and the triple $(U,\bar{\mathcal{A}}_{|U},\bar{\tau}_{|U})$ is a k-analytic space. The open set U is an example of an analytic domain.

Definition 5.5.1. A subspace $Y \subset X$ is an *analytic domain* if for every $y \in Y$, there exists $V_1,\ldots,V_n \in \bar{\tau}$ such that the following hold.

1. For every i, $V_i \subseteq Y$.
2. The set $\bigcup_i V_i$ is an open neighborhood in Y of y.
3. $y \in \bigcap_i V_i$.

The definition above is equivalent to saying that the family $\bar{\tau}_{|Y}$ of subsets of Y is a quasi-net.

Proposition 5.5.2. *Let $Y \subset X$ be an analytic domain.*

1. *The canonical morphism $Y \to X$ possesses the following property. Any morphism of k-analytic spaces $\phi : Z \to X$ with $\phi(Z) \subseteq Y$ factors through a unique morphism $Z \to Y$.*

2. *The intersection of two analytic domains is an analytic domain and the preimage of an analytic domain with respect to a morphism of k-analytic spaces is an analytic domain.*

3. *If $\{X_i\}_{i \in I}$ is a family of analytic domains in X which forms a quasi-net, then for any k-analytic space X' the following sequence of maps is exact.*

$$\mathrm{Hom}(X, X') \to \prod_i \mathrm{Hom}(X_i, X') \to \prod_{i,j} \mathrm{Hom}(X_i \cap X_j, X').$$

Definition 5.5.3. An *analytic domain* in X is an affinoid domain if it is isomorphic to an affinoid space.

The following proposition provides necessary and sufficient conditions for a subset Y of X to be an affinoid domain.

Proposition 5.5.4. *A subset Y of X is an affinoid domain if and only if the following conditions hold.*

1. *There exists $V_1, \ldots, V_n \in \bar{\tau}_{|Y}$ such that $Y = \bigcup_i V_i$ and for every i, j, $V_i \cap V_j$ is an element of $\bar{\tau}$ and the induced morphism $\mathcal{A}_{V_i} \widehat{\otimes} \mathcal{A}_{V_j} \to \mathcal{A}_{V_i \cap V_j}$ is an admissible epimorphism.*

2. *The Banach k-algebra $\mathcal{A}_Y := \mathrm{Ker}(\prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_i \cap V_j})$ is k-affinoid and $V \tilde{\to} \mathcal{M}(\mathcal{A}_Y)$.*

Definition 5.5.5. Let $\hat{\tau}$ denote the family of subsets of X which are affinoid domains. It can be verified that $\hat{\tau}$ is a well defined net on X and that the k-affinoid atlas $\mathcal{A}$ on X with net τ extends to an atlas $\widehat{\mathcal{A}}$ on X with net $\hat{\tau}$. The k-affinoid atlas $\widehat{\mathcal{A}}$ is called the *maximal atlas on X*.

Henceforth, when discussing an analytic space, we will assume that it is endowed with the maximal atlas. Recall that in Definition 5.2.11, we introduced the non-Archimedean version of the residue field $\mathcal{H}(x)$ of a point x of an affinoid space. This notion extends in a natural way to a point x of the analytic space X.

Let $\mathcal{V}$ denote the family of affinoid domains which contain the point x. We set $\mathcal{H}(x) := \varinjlim_{V \in \mathcal{V}} \mathcal{H}_V(x)$ where for $V \in \mathcal{V}$, $\mathcal{H}_V(x)$ is as in Definition 5.2.11. The inductive system $\{\mathcal{H}_V(x)\}_{V \in \mathcal{V}}$ is filtered and the transition morphisms can be checked to be isomorphisms.

5.5.1 The G-topology on an analytic space

In order to discuss a theory of coherent sheaves on the analytic space X, we introduce a Grothendieck topology. The family of analytic domains of X forms a category whose objects are analytic domains and morphisms are inclusions of

analytic domains. This category can be endowed with a Grothendieck topology generated by the pre-topology wherein the set of coverings of an analytic domain $Y \subset X$ is given by a family a of analytic domains of X contained in Y such that a is a quasi-net of Y. We refer to this topology as the G-topology on X.

We define the structure sheaf on X with its G-topology analogous to the construction of the structure sheaf on a scheme. Recall that in Example 5.4.13 (2), we introduced the k-analytic space $\mathbb{A}^{1,\mathrm{an}}$.

Proposition 5.5.6. *1. Let $Z = \mathcal{M}(\mathcal{A})$ be an affinoid space. We have that* $\mathrm{Hom}(Z, \mathbb{A}^{1,\mathrm{an}}) \tilde{\to} \mathcal{A}$.

2. The functor $k - \mathcal{A}n \to Set$ defined by mapping a k-analytic space Y to $\mathrm{Hom}(Y, \mathbb{A}^{1,\mathrm{an}})$ *defines a sheaf of rings on the analytic space X equipped with the G-topology.*

Definition 5.5.7. The sheaf defined on X for the G-topology is denoted $\mathcal{O}_{X_G}$ and referred to as the *structure sheaf on X.*

Let $Z = \mathcal{M}(\mathcal{A})$ be an affinoid space and M be an $\mathcal{A}$-module. Let V be an affinoid domain in Z. It can be shown that the association $V \mapsto \mathcal{A}_V \otimes_{\mathcal{A}} M$ defines a sheaf of $\mathcal{O}_{Z_G}$-modules on Z, which we call $\mathcal{O}_{Z_G}(M)$.

Definition 5.5.8. A sheaf F of $\mathcal{O}_{X_G}$-modules is *coherent* if there exists a family τ' of affinoid domains in X such that τ' is a quasi-net on X and for every $V \in \tau'$, the restriction of F to V with its G-topology is isomorphic to a sheaf of the form $\mathcal{O}_{V_G}(M)$ where M is a finite $\mathcal{A}_V$-module.

The structure sheaf $\mathcal{O}_{X_G}$ restricts to the open sets of X to define a sheaf $\mathcal{O}_X$ on the topological space X and it can be shown that the pair $(X, \mathcal{O}_X)$ is a locally ringed space. We introduce a class of k-analytic spaces whose structures as locally ringed spaces determine their analytic structure.

Definition 5.5.9. A k-analytic space X is *good* if for every $x \in X$ there exists a neighborhood of x that is an affinoid domain.

Proposition 5.5.10. *Suppose that the valuation on k is nontrivial. Then the functor $X \mapsto (X, \mathcal{O}_X)$ from the full subcategory of good k-analytic spaces to that of locally ringed spaces is fully faithful.*

5.6 Analytification of an algebraic variety

Let X be a scheme which is locally of finite type over k. Let $k - \mathcal{A}n$ denote the category of k-analytic spaces, Set denote the category of sets and $Sch_{lft/k}$ denote the category of schemes which are locally of finite type over k. We define a functor

$$\begin{aligned} F : k - \mathcal{A}n &\to Set \\ Y &\mapsto Hom(Y, X) \end{aligned}$$

where $Hom(Y, X)$ is the set of morphisms of k-ringed spaces. The following theorem defines the space X^{an}.

Theorem 5.6.1 ([B], 1.2.4)**.** *The functor F is representable by a k-analytic space X^{an} and a morphism $\pi : X^{\mathrm{an}} \to X$. For any non-Archimedean complete real valued field K extending k, there is a bijection $X^{\mathrm{an}}(K) \to X(K)$. Furthermore, the map π is surjective.*

The associated k-analytic space X^{an} is *good*. Theorem 5.6.1 implies the existence of a well defined functor

$$\begin{aligned} ()^{\mathrm{an}} :& Sch_{lft/k} \to good\ k-an \\ & X \mapsto X^{\mathrm{an}}. \end{aligned}$$

As a set X^{an} is the collection of pairs $\{(x, \eta)\}$ where x is a scheme theoretic point of X and η is a rank one valuation on the residue field $k(x)$ which extends the valuation on the field k. We endow this set with a topology as follows. A pre-basic open set is a set of the form $\{(x, \eta) \in U^{\mathrm{an}} || f(\eta)| \in W\}$, where U is an open subvariety of X with $f \in O_X(U)$, W is an open subspace of $\mathbb{R}_{\geq}0$ and $|f(\eta)|$ is the image of f in the residue field $k(x)$ evaluated at η. A basic open set is any set which is equal to the intersection of a finite number of pre-basic open sets.

Definition 5.6.2. *(The field $\mathcal{H}(\mathbf{x})$ for $\mathbf{x} \in X^{\mathrm{an}}$)* Let $\mathbf{x} := (x, \eta)$ be an element of X^{an}. We define $\mathcal{H}(\mathbf{x})$ to be the completion of the field $k(x)$ with respect the valuation η.

Let L be an algebraically closed non-Archimedean real valued field that contains k. As in Remark 1.1, we say that $\mathbf{x} \in X^{\mathrm{an}}(L)$ if there exists an embedding $\mathcal{H}(\mathbf{x}) \hookrightarrow L$. It follows that we have an embedding $k(x) \hookrightarrow L$ and hence $x \in X(L)$. The mapping $\mathbf{x} = (x, \eta) \mapsto x$ is an explicit description of the map $X^{\mathrm{an}}(L) \to X(L)$ of Theorem 5.6.1.

Example 5.6.3. If $V = \mathbb{A}^n_k$ then V^{an} is the k-analytic space $\mathbb{A}^{n,\mathrm{an}}$ (cf. Example 5.4.13 (2)). Similarly, the analytification of a Zariski closed subset Z of $\mathbb{A}^n_k$ whose ring of regular functions is $k[T_1, \ldots, T_n]/I$, is the collection of multiplicative seminorms on $k[T_1, \ldots, T_n]/I$ which restrict to the given valuation on k and endowed with the weakest topology such that every function $f \in k[T_1, \ldots, T_n]/I$ is continuous on Z^{an}.

We now discuss the notion of the reduction map associated to a k-affinoid space. The reduction map and the $\tilde{k}$-scheme associated to an affinoid space will be of use when discussing formal covers in the following subsection.

5.7 The Reduction Morphism

Let $\mathcal{A}$ be a Banach ring. We recall the definition of the field $\mathcal{H}(x)$ associated to a point $x \in \mathcal{M}(\mathcal{A})$. It can be checked that $\mathrm{Ker}(x) := \{f \in \mathcal{A} || f(x)| = 0\}$ is a prime ideal. Let $\mathcal{H}(x)$ denote the completion of the field of fractions of $\mathcal{A}/(\mathrm{Ker}(x))$ with respect to the norm induced on this quotient by x.

Recall the definition of the spectral norm from Definition 5.2.10. The set $\mathcal{A}^\circ := \{x \in \mathcal{A} | \rho(x) \leq 1\}$ is a complete sub ring of $\mathcal{A}$ in which $\mathcal{A}^{\circ\circ} := \{x \in \mathcal{A} | \rho(x) < 1\}$ is an ideal. We will set $\tilde{\mathcal{A}} := \mathcal{A}^\circ / \mathcal{A}^{\circ\circ}$. A bounded homomorphism of Banach rings $\mathcal{A} \to \mathcal{B}$ induces a morphism $\tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$. Note that if $\mathcal{A}$ is a field then $\tilde{\mathcal{A}}$ is a field as well.

For every $x \in \mathcal{M}(\mathcal{A})$, we have the map $\mathcal{A} \to \mathcal{A}/(\mathrm{Ker}(x))$ which is bounded and hence induces a bounded homomorphism of Banach rings $\mathcal{A} \to \mathcal{H}(x)$. This in turn defines a morphism $\tilde{A} \to \widetilde{\mathcal{H}(x)}$. We have in fact defined a map $\mathcal{M}(\mathcal{A}) \to \mathrm{Spec}(\tilde{\mathcal{A}})$ which we call the reduction map and denote it by π. Explicitly stated,

$$\begin{aligned} \pi : \mathcal{M}(\mathcal{A}) &\to \mathrm{Spec}(\tilde{\mathcal{A}}) \\ x &\mapsto \mathrm{Ker}(\tilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}). \end{aligned}$$

For a k-affinoid space $X := \mathcal{M}(\mathcal{B})$ where $\mathcal{B}$ is a k-affinoid algebra, we will use $\tilde{X}$ to denote the $\tilde{k}$-scheme $\mathrm{Spec}(\tilde{\mathcal{B}})$. Concerning the reduction map in the case of strict k-affinoid algebras [[B], 2.1] we have the following proposition.

Proposition 5.7.1. *[[B], 2.4.4] [[B3], 2.3.6]. Let $\mathcal{A}$ be a strict k-affinoid algebra. Set $X := \mathcal{M}(\mathcal{A})$, $\tilde{X} := \mathrm{Spec}(\tilde{\mathcal{A}})$, and let $\tilde{X}_{gen}$ be the set of generic points of the irreducible components of the scheme $\tilde{X}$. With this notation, the following statements are true.*

1. *The reduction map $\pi : X \to \tilde{X}$ is surjective.*
2. *For any $\tilde{x} \in \tilde{X}_{gen}$, there exists a unique point $x \in X$ with $\pi(x) = \tilde{x}$. If $\rho(\mathcal{A}) = |k|$ then there is an isomorphism $\tilde{k}(\tilde{x}) \simeq \widetilde{\mathcal{H}(x)}$.*
3. *The set $\pi^{-1}(\tilde{X}_{gen})$ is the Shilov boundary of X.*
4. *The pre-image of an open (resp. closed) subset of $\tilde{X}$ is closed (resp. open) under the morphism π.*

5.7.1 Formal Covers

In this section we define for any projective k-variety V, the $\tilde{k}$-scheme $\tilde{V}$ and the reduction map $V^{\mathrm{an}} \to \tilde{V}$. One way of doing so would be to use an affinoid covering of the space V^{an} and glue the $\tilde{k}$-schemes arising from each element of the covering. However in order for the gluing to make sense, we must be restrictive in our choice of covering. It is to this end that we introduce the notion of a formal cover of a separated k-analytic space.

Definition 5.7.2 ([B], Section 4.3)**.** A *formal domain* W in a k-affinoid space X is either the empty set or an affinoid domain such that the induced morphism $\widetilde{W} \to \widetilde{X}$ is an open immersion.

The definition that follows is stated for a separated k-analytic space. The analytification of a k-variety is an example of a separated k-analytic space. We will be interested only in this case.

Definition 5.7.3. Let X be a separated k-analytic space. An affinoid covering $\{W_i\}$ is *formal* if the W_i are strict k-affinoid spaces and for any W_i, W_j belonging to the cover, $W_i \cap W_j$ is a formal domain of both W_i and W_j.

Let X be a separated k-analytic space provided with a formal cover $\mathfrak{W} := \{W_i\}$. Gluing the $\tilde{W}_i$ defines a $\tilde{k}$-scheme $\tilde{X}_{\mathfrak{W}}$ which is reduced and locally of finite type. Furthermore, the reduction map $\pi : W_i \to \tilde{W}_i$ for each element of the covering extends to a map $\pi : X \to \tilde{X}_{\mathfrak{W}}$.

Our notation $\tilde{X}_{\mathfrak{W}}$ was to specify the importance of the choice of cover involved in defining the scheme $\tilde{X}$. In what follows we will suppress the sub script and simply write $\tilde{X}$.

The following proposition ensures that if X is a projective k-variety of finite type then it always admits a formal cover. We first prove the theorem when $X = \mathbb{P}_k^{n,\mathrm{an}}$ for some n. The result in this case is obtained by exploiting the standard chart associated to $\mathbb{P}_k^n$.

Proposition 5.7.4. *Let V be a projective k-variety. For every complete non-Archimedean real valued field extension L/k there exists a finite formal cover $\mathfrak{W}_L$ of V_L^{an} such that the collection $\{\mathfrak{W}_L\}_L$ has the following property. Let L_1/k and L_2/k be complete non-Archimedean real valued field extensions such that L_1 embeds into L_2. If $\mathfrak{W}_{L_1} = \{D_i\}_i$ then $\mathfrak{W}_{L_2} = \{D_i \times_{L_1} L_2\}_i$.*

Proof. We provide a brief sketch of the proof which we then expand upon in detail. We first construct the formal cover $\mathfrak{W}_k$. In the case $V = \mathbb{P}_k^1$, the construction is straightforward. After choosing coordinates, we identify $\mathbb{P}_k^1$ with $\mathrm{Spec}(k[T]) \cup \{\infty\}$. The space $\mathbb{P}_k^{1,\mathrm{an}}$ can be realized as the union of two Berkovich closed disks of unit radius i.e. $\mathbb{P}^{1,\mathrm{an}} = A_1 \cup A_2$ where $A_1 := \{x \in \mathbb{P}^{1,\mathrm{an}} || T(x)| \leq 1\}$ and $A_2 := \{x \in \mathbb{P}^{1,\mathrm{an}} || T(x)| \neq 0 \wedge |(1/T)(x)| \leq 1\}$. The affinoid spaces A_1 and A_2 are glued together along the annulus $A_1 \cap A_2 = \{x \in \mathbb{P}_k^{1,\mathrm{an}} || T(x)| = 1\}$. It can be shown that $\{A_1, A_2\}$ is a formal cover of $\mathbb{P}_k^{1,\mathrm{an}}$. This construction can be generalized to when $V = \mathbb{P}_k^n$. After choosing coordinates, $\mathbb{P}_k^{n,\mathrm{an}}$ can be seen as the union of $n+1$, n - dimensional Berkovich polydisks $\{A_1, \ldots, A_{n+1}\}$ each of poly radius $(1, \ldots, 1)$. We show that $\{A_1, \ldots, A_{n+1}\}$ is a formal cover of $\mathbb{P}_k^{n,\mathrm{an}}$. When V is a general projective variety, after choosing an embedding $V \hookrightarrow \mathbb{P}_k^n$ for some $n \in \mathbb{N}$, we identify V^{an} with a closed subspace of $\mathbb{P}_k^{n,\mathrm{an}}$ and show that $\{A_1 \cap V^{\mathrm{an}}, \ldots, A_{n+1} \cap V^{\mathrm{an}}\}$ is a formal cover of V^{an}. Finally, it can be verified that if L/k is a complete non-Archimedean real valued field extension then extending scalars for each element of the cover $\mathfrak{W}_k$ by L defines a formal cover $\mathfrak{W}_L$.

Let $n \in \mathbb{N}$ and $V = \mathbb{P}_k^n$. The analytification of the projective space $\mathbb{P}_k^{n,\mathrm{an}}$ can be described in a fashion reminiscent of the `Proj´construction in the theory of schemes [[L], 2.3.3].

Consider the k-algebra $k[T_1, \ldots, T_{n+1}]$. Let S denote the set of all multiplicative seminorms on this algebra which restrict to the valuation of the field k such that if $x \in S$ then $|T_i(x)| \neq 0$ for some i. We define an equivalence relation $\sim$ on S as follows.

$$x \sim y \iff \text{There exists } c \in \mathbb{R}_{>0} \text{ such that for any homogenous } f \in k[T_1, ..T_{n+1}], |f(x)| = c^{deg(f)}|f(y)|.$$

The set $S/\sim$ can be endowed with a topology in a natural fashion [[Ba], 2.2] so that it becomes a compact, Hausdorff topological space. We proceed further and give $S/\sim$ the structure of a k-analytic space. Let

$$A_j := \{x \in S || T_r(x)| \leq |T_j(x)| \text{ for every } 1 \leq r \leq n+1\}.$$

Observe that if $a \in A_j$ and $a \sim b$ then $b \in A_j$. We will abuse notation and denote $A_j/\sim$ by A_j as well. It follows that $S/\sim = \cup_j A_j$. Furthermore, for any $1 \leq j \leq n+1$, A_j is in bijection with the set of multiplicative seminorms

on $k[T_1/T_j, ., T_{n+1}/T_j]$ which when evaluated at (T_i/T_j) for any possible choice of i is less than or equal to 1. But this is exactly the set of all multiplicative seminorms on the affinoid algebra $\mathcal{B}_j := k\{T_1/T_j, ., T_{n+1}/T_j\}$ which restrict to the given valuation on k. In fact we have a homeomorphism

$$\delta_j : A_j \to \mathcal{M}(\mathcal{B}_j).$$

Consequently, the collection $\{\cap_{r\in Q} A_r\}_{Q\in\mathfrak{P}}$ where $\mathfrak{P}$ is the set of all subsets of the set $\{1, \ldots, n+1\}$, forms a net [[B2], 1.1] of compact sets on the space $S/\sim$. If $i \neq j$ then δ_j restricts to an isomorphism between $A_j \cap A_i$ and the affinoid domain

$$\mathcal{M}(\mathcal{B}_{ji}) := k\{T_1/T_j, ..T_i/T_j, (T_i/T_j)^{-1}, ., T_{n+1}/T_j\}.$$

We use δ_{ij} to denote this isomorphism. Note that $\mathcal{B}_{ji} = \mathcal{B}_{ij}$ and the isomorphisms δ_{ij} and δ_{ji} are the same.

Similarly let $Q \in \mathfrak{P}$. The space $\cap_{r\in Q\cup\{i\}} A_r$ is homeomorphic to the affinoid space

$$\mathcal{M}(\mathcal{B}_{iQ}) := \mathcal{M}(k\{T_j/T_i, (T_s/T_i)^{-1}\})_{s\in Q, j\in\{1,..,n+1\}}$$

via the restriction of any δ_s where $s \in Q \cup \{i\}$. For any $s \in Q \cup \{i\}$, $\mathcal{B}_{iQ} = \mathcal{B}_{sQ}$ and the restrictions δ_s are the same for all such s.

The triple $(S/\sim, (\mathcal{B}_{iQ})_{i\in 1,..,n+1, Q\in\mathfrak{P}}, \cap_{r\in Q\cup i} A_r)$ is hence a k-analytic space [[B2], pg 17] and is isomorphic as an analytic space to $\mathbb{P}_k^{n,\mathrm{an}}$.

This description of $\mathbb{P}_k^{n,\mathrm{an}}$ enables us to see it as the union of $n+1$ affinoid domains, each isomorphic to the n-dimensional Berkovich closed polydisc of poly radius $(1, \ldots, 1)$. We claim that $\{A_j\}$ is a formal cover of $\mathbb{P}_k^{n,\mathrm{an}}$.

Since

$$\widetilde{\mathcal{B}_{ji}} = \tilde{k}[T_1/T_j, ..T_i/T_j, (T_i/T_j)^{-1}, ..T_{n+1}/T_j],$$

the $\tilde{k}$-algebra $\widetilde{\mathcal{B}_{ji}}$ corresponds to an open affine sub scheme of $\widetilde{\mathcal{M}(\mathcal{B}_j)}$ and $\widetilde{A_j \cap A_i}$ is an open affine subset of $\tilde{A}_j$. We conclude that our claim is verified, thus proving the proposition for the case $V = \mathbb{P}_k^{n,\mathrm{an}}$.

In the general case we make use of the fact that V is projective and hence can be seen as a closed subset of $\mathbb{P}_k^n$ for some n. Furthermore for every j, δ_j restricts to an isomorphism

$$\delta_j : V^{\mathrm{an}} \cap A_j \to \mathcal{M}(k\{T_1/T_j, ..T_i/T_j, ..T_{n+1}/T_j\}/I_j)$$

where I_j is an ideal determined by the embedding of V into $\mathbb{P}_k^n$. The generators of I_j can be chosen to be polynomials $\{f_1, \ldots, f_u\}$ belonging to $k[T_1/T_j, ..T_{n+1}/T_j]$ such that if ρ denotes the Gauss norm (or spectral norm) of the affinoid algebra $k\{T_1/T_j, ..T_i/T_j, ., T_{n+1}/T_j\}$ then $\rho(f_i) \leq 1$. The justification for this follows from [[B], (2) Page 64].

We claim that the cover $\{V^{\mathrm{an}} \cap A_i\}_i$ is a formal cover of V^{an}. Let $1 \leq i, j \leq n+1$ with $i \neq j$. We need only show that the affinoid space $V^{\mathrm{an}} \cap A_i \cap A_j$ is a formal domain in $V^{\mathrm{an}} \cap A_i$. We may assume that $V^{\mathrm{an}} \cap A_i \cap A_j$ is not empty, since otherwise it is trivially a formal domain. Observe that we have the following equality.

$$(V^{\mathrm{an}} \cap A_i) \cap A_j = \{x \in V^{\mathrm{an}} \cap A_i | |T_j/T_i(x)| = 1\}.$$

By assumption, there exists an $x \in V^{\mathrm{an}} \cap A_i$ such that $|T_j/T_i(x)| = 1$. It follows that if ρ denotes the spectral norm of the strict affinoid algebra corresponding to $V^{\mathrm{an}} \cap A_i$ then $\rho(T_j/T_i) = 1$. The above equality and [[B], (ii) pg.28] imply that the affinoid algebra corresponding to the space $(V^{\mathrm{an}} \cap A_i) \cap A_j$ is $(\mathcal{B}_i/I_i)\{T_i/T_j\}$. Applying [[BGR], Proposition 7.2.6/3] will give the result.

Let L be a complete non-Archimedean real valued field extension of k. We check that $\mathfrak{W}_L := \{(V^{\mathrm{an}} \cap A_j) \times_k L)\}_j$ is a formal cover of V_L^{an}. The topological subspace $A_j \subset \mathbb{P}_k^{n,\mathrm{an}}$ is endowed with the structure of a k-affinoid space via the homeomorphism δ_j. Hence we refer to it in future as an affinoid domain in $\mathbb{P}_k^{n,\mathrm{an}}$ and identify it with the strict affinoid space $\mathcal{M}(\mathcal{B}_j)$. Let L/k be a complete non-Archimedean real valued field extension. For every j, we have that $A_j \times_k L = \mathcal{M}(\mathcal{B}_{j,L})$ where $\mathcal{B}_{j,L} := \mathcal{B}_j \hat{\otimes}_k L = L\{T_1/T_j, ., T_{n+1}/T_j\}$. Clearly, $\mathbb{P}_L^{n,\mathrm{an}} = \bigcup_j A_j \times_k L$. Identical arguments as those used above imply that $\{A_j \times_k L\}_j$ is a formal cover of $\mathbb{P}_L^{n,\mathrm{an}}$. Likewise, it can be shown that $\{V_L^{\mathrm{an}} \cap (A_j \times_k L)\}_j$ is a formal cover of V_L^{an}. The equality $V_L^{\mathrm{an}} \cap (A_j \times_k L) = (V^{\mathrm{an}} \cap A_j) \times_k L$ completes the proof. $\square$

Chapter 6

An application of Model theory to Berkovich geometry

The purpose of this chapter is to describe briefly how the results of Hrushovski and Loeser from Chapter 2 imply powerful tameness properties of certain Berkovich spaces. These results are amongst the goals of the paper [HL] and form a part of the motivation behind defining the spaces $\widehat{V}$. Our presentation follows Section 14 of [HL].

6.1 The Berkovich space $B_{\mathbf{F}}(X)$

We begin by providing a model theoretic reinterpretation of the Berkovich space discussed previously, one for which a connection with the space of stably dominated types can be easily made.

Let F be a non-Archimedean real valued field. Let $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. In this chapter, we adopt the convention of writing the group structure on the value group sort additively. Let $\mathbf{F}$ denote the structure defined by the pair $(F, \mathbb{R}_\infty)$. Let V be a quasi-projective F-variety. As a set, the Berkovich space $B_{\mathbf{F}}(V)$ is defined as follows. It can be given a topology similar to the spaces $\widehat{V}$ and V^{an}.

Definition 6.1.1. Let X be an $\mathbf{F}$-definable subset of $V \times \Gamma_\infty^l$ for some $l \in \mathbb{N}$. Let $B_{\mathbf{F}}(X)$ be the set of almost orthogonal $\mathbf{F}$-types which concentrate on X

Observe that if $f : X \to \Gamma_\infty$ is an $\mathbf{F}$-definable function and $p \in B_{\mathbf{F}}(X)$ such that $a \models p$ then $f(a) \in \mathbb{R}_\infty$ depends only on the type p i.e. if $a_1 \models p$ and $a_2 \models p$ then $f(a_1) = f(a_2)$. We set $f(p) := f(a)$. Thus we have a well defined function $f : B_{\mathbf{F}}(X) \to \mathbb{R}_\infty$. The set $B_{\mathbf{F}}(X)$ is endowed with the topology generated by pre - basic open sets of the form $\{q \in B_{\mathbf{F}}(X \cap U) | \mathrm{val}(f)(q) \in W\}$ where $U \subset V$ is an open affine subspace, f is a regular function on U and $W \subset \mathbb{R}_\infty$ is an open interval.

Lemma 6.1.2. *The spaces V^{an} and $B_{\mathbf{F}}(V)$ are canonically homeomorphic.*

Proof. Recall that as a set V^{an} can be realized as the set of pairs $\{(x, v_x)\}$ where x is a scheme theoretic point of the variety V and $v_x : F(x) \to \mathbb{R}_\infty$ is a non-Archimedean valuation of the residue field $F(x)$ which restricts to the valuation on F. The pair $x : F(x) \to V$ and $\text{id} : F(x) \to F(x)$ define a closed point of the variety $V \times_F F(x)$. We can regard the variety V as an F-definable set. The set of $F(x)$-points of $V \times_F F(x)$ in the scheme theoretic sense is the set $V(F(x))$ in the model theoretic sense, where the latter is not to be confused with the scheme theoretic notion $\text{Hom}_F(\text{Spec}(F(x), V)$. Thus the pair (x, v_x) determines an $F(x)$-point $\mathbf{x}$ of the definable set V. Let $p_{(x,v_x)} := \text{tp}(\mathbf{x}/\mathbf{F})$. It can be checked that $p_{(x,v_x)}$ belongs to $B_{\mathbf{F}}(V)$. We have thus defined a function $\phi : V^{\text{an}} \to B_{\mathbf{F}}(V)$ and it can be checked that this function is continuous.

Let $p \in B_{\mathbf{F}}(V)$. Let U be an affine open subset of V on which the type p concentrates. Let I_p denote the ideal $\{f \in \mathcal{O}_V(U) | \text{val}(f)(p) = \infty\}$. It can be checked that I_p defines a prime ideal of the ring of regular functions $\mathcal{O}_V(U)$ on U and consequently a scheme theoretic point x_p of the variety V. The map $\mathcal{O}_V(U) \to \mathbb{R}_\infty$ defined by $f \mapsto \text{val}(f)(p)$ factors through a map $\mathcal{O}_V(U)/I_p \to \mathbb{R}_\infty$ that extends to define a non-Archimedean valuation v_p on the residue field $k(x_p) = Q(\mathcal{O}_V(U)/I_p)$. Here $Q(\mathcal{O}_V(U)/I_p)$ denotes the field of fractions of the domain $\mathcal{O}_V(U)/I_p$. We thus have a map $\psi : B_{\mathbf{F}}(V) \to V^{\text{an}}$ that is well defined and it can be verified that it is continuous. Furthermore, it can be checked that the map ψ is the inverse of the map ϕ. □

We relate the space $B_{\mathbf{F}}(V)$ to the space $\widehat{V}$. Let K be a non-Archimedean algebraically closed, spherically complete real valued field whose residue field is the algebraic closure of the residue field $k(F)$ of F and value group $\Gamma(K)$ is $\mathbb{R}$. Such a field is unique up to isomorphism over the structure $\mathbf{F}$. We fix one such copy and call it F^{max}.

Lemma 6.1.3. *There exists a surjective continuous function $\pi : \widehat{V}(F^{max}) \to B_{\mathbf{F}}(V)$ such that if X is an $\mathbf{F}$-definable subset of V then $\pi^{-1}(B_{\mathbf{F}}(X)) = \widehat{X}(F^{max})$.*

Proof. Let p be a stably dominated type defined over F^{max} that concentrates on V. Then $p_{|F^{max}}$ is an F^{max}-type. Let $\pi(p)$ denote the $\mathbf{F}$-type defined by those formulae with parameters in $\mathbf{F}$ that are contained in p. Let $a \models \pi(p)$. We must have that $\Gamma(\mathbf{F}(a)) \subseteq \Gamma(F^{max}(a)) = \Gamma(F^{max}) = \Gamma(\mathbf{F})$. It follows that π is a well defined function. It can be checked that it is continuous as well.

We show that π is surjective. Let $p \in B_{\mathbf{F}}(V)$ and a be a realization of p. We choose an embedding $F \hookrightarrow F(a)$ which induces an embedding $F^{max} \hookrightarrow F(a)^{max}$. By Theorem 4.1.3, the type $\text{tp}(a|F^{max})$ extends to an F^{max}-stably dominated type which concentrates on V, thus defining an element of $\widehat{V}(F^{max})$. The equality $\widehat{X}(F^{max}) = \pi^{-1}(B_{\mathbf{F}}(X))$ follows from the construction of the function π. □

We make use of the following lemmas in Sections 6 and 8.

Lemma 6.1.4. *Let F be a non-Archimedean complete non trivially real valued field. Let $\phi : V \to W$ be a finite surjective morphism between irreducible F-varieties with W normal. The induced morphism $\phi^{\text{an}} : V^{\text{an}} \to W^{\text{an}}$ is an open morphism.*

Proof. We apply Lemma 3.2.4 in [B] to prove the lemma. Clearly, we need only show that if W is normal then W^{an} is locally irreducible. By 3.4.3 in loc.cit, W^{an} is a normal F-analytic space. Let $x \in W^{\mathrm{an}}$ and $U \subset W^{\mathrm{an}}$ be an F-analytic neighborhood of x. Let $U' \subset U$ be the connected component that contains x. The space U' is a normal F-analytic space. By 3.1.8 in loc.cit, it must be irreducible. This completes the proof. $\square$

Lemma 6.1.5. *Let k be a non-Archimedean real valued algebraically closed complete field. Let V be an irreducible k-variety of finite type. Let $U \subset V$ be a Zariski open subset of V. Let M be a real valued model of* ACVF *which contains k and is complete. The subspace $U(M)$ is dense in $\widehat{V}(M)$. In fact, the stronger statement - the set U is dense in $\widehat{V}$, is also true.*

Proof. Let W be an M-definable basic open set of $\widehat{V}$. By our description of basic open sets, $W = \widehat{O}$ where O is an M-definable subset of V. Recall that we have a map, $\pi : V \widehat{\times_k M^{max}}(M^{max}) \to (V \times_k M^{max})^{\mathrm{an}}$ which is a homeomorphism. As $V \widehat{\times_k M^{max}} = \widehat{V}$, we can write $\pi : \widehat{V}(M^{max}) \to (V \times_k M^{max})^{\mathrm{an}}$. By [[H], 3.15], the scheme $V \times_k L$ is irreducible for any field extension L of k. In particular, $V \times_k M^{max}$ is irreducible. The dimension of the open analytic domain $\pi(W(M^{max})) \subset (V \times_k M^{max})^{\mathrm{an}}$ is $\dim(V \times_k M^{max}) = \dim(V)$ (cf. Lemma 6.1.6).

To conclude a proof of the first part of the lemma, we must show that $U(M) \cap W(M)$ is non-empty. Suppose, $U(M) \cap W(M)$ was empty. In particular, we must have that $U(M) \cap O(M)$ is empty. As M embeds elementarily into any model of ACVF that contains it, we must have that $U \cap O$ is empty and consequently $\widehat{U} \cap \widehat{O}$ is empty. Let $Z := V \setminus U$. By assumption, $W \subset \widehat{Z}$. It follows that $\pi(W(M^{max})) \subset (Z \times M^{max})^{\mathrm{an}}$ and hence the dimension of the open analytic domain $\pi(W(M^{max}))$ must be less than or equal to $\dim(Z)$. As V is irreducible, $\dim(Z) < \dim(V)$. Hence we have a contradiction, thus showing that $W(M) \cap U(M)$ is non-empty.

We now show that U is dense in $\widehat{V}$. Observe that the first part of the lemma implies that $U(M) \subset \widehat{V}(M)$ is dense when M is a model of ACVF containing k that is *real* valued and complete. The topology on $\widehat{V}$ is generated by sets each of which are of the form $\widehat{O}$ where $O \subset V$ is a definable subset with parameters in a model of ACVF that is not necessarily real valued. It is for this reason that an additional argument is required to show part 2 of the lemma.

Let $\{A_i\}_i$ be a finite affine cover of V. Suppose that for every i, $U \cap A_i$ is dense in $\widehat{A_i}$. It can then be verified that U is dense in $\widehat{V}$. Hence, we may suppose that V is affine and V embeds into $\mathbb{A}^n$ for some $n \in \mathbb{N}$. We identify V with its image for this embedding. Let W be an open subset of $\widehat{V}$. By definition of the topology on $\widehat{V}$, there exists a model M of ACVF which contains k such that $W = \widehat{O}$ where $O \subset V$ is M-definable. Furthermore there exists $m \in \mathbb{N}$ such that O must be of the form $\cap_{1 \leq i \leq m} O_i$ where $O_i := \{x \in V | \mathrm{val}(f_i)(x) \in (\alpha_i, \beta_i)\}$ for an M-definable function f_i which is regular on V and $\alpha_i < \beta_i$ elements of the value group $\Gamma_\infty(M)$. Let $r \in \mathbb{N}$ be such that for every $1 \leq i \leq m$, f_i is a polynomial in $M[T_1, \ldots, T_n]$ of degree at most r. Let D denote the 0-definable set of polynomials in n-variables of degree less than or equal to r. Let S denote the sentence : For every $(g_1, \ldots, g_m) \in D^m$ and $(a_1, \ldots, a_m, b_1, \ldots, b_m) \in \Gamma_\infty^{2m}$ with $a_i < b_i$, there exists $x \in U$ such that for every i, $\mathrm{val}(g_i)(x) \in (a_i, b_i)$. The first part of the lemma implies that $k \models S$. As k embeds elementarily into

any model of ACVF that contains it, we must have $M \models S$. It follows that $U(M) \cap O(M)$ is non-empty and hence that $U(M) \cap W(M)$ is non-empty. Thus U is dense in $\widehat{V}$. $\square$

Lemma 6.1.6. *Let F be a real valued complete model of* ACVF*. Let V be an irreducible variety over F. Let $W \subset V^{\mathrm{an}}$ be an open subset. Then* $\dim(W) = \dim(V)$.

Proof. Let $A \subset V$ be an affine open subset of V such that $A^{\mathrm{an}} \cap W$ is non-empty. As V can be covered by affine open subsets, such an affine open subset exists. Since V is irreducible, we must have that $\dim(A) = \dim(V)$ and hence it suffices to prove the lemma for the open set $W \cap A^{\mathrm{an}}$. We assume henceforth that $W \subset A^{\mathrm{an}}$.

By Noetherian normalization, there exists a finite morphism $f : A \to \mathbb{A}^n$ where $n = \dim(A)$. By Lemma 6.1.4, the morphism $f^{\mathrm{an}} : A^{\mathrm{an}} \to \mathbb{A}^{n,\mathrm{an}}$ is open. Let $U' := f^{\mathrm{an}}(W)$. As the morphism f^{an} is finite, $\dim(U') = \dim(W)$. The set of Zariski closed points of $\mathbb{A}^{n,\mathrm{an}}$ are dense. Let x be a Zariski closed point contained in U'. As the Berkovich open balls in $\mathbb{A}^{n,\mathrm{an}}$ and centered at x form a fundamental system of neighborhoods of the point x, there exists a Berkovich open ball B such that $B \subset U'$. As $\dim(B) = n$, it follows that $\dim(U') = n$ which in turn implies that $\dim(W) = \dim(V)$. $\square$

Several topological properties of the Berkovich space can be deduced using the map π defined above. We state the following lemma from [HL] as a preliminary to the deeper results concerning the homotopy type of Berkovich spaces deduced from analogous results in the Hrushovski-Loeser setting.

Proposition 6.1.7. *([HL], Proposition 14.1.2) Let X be an F-definable subset of an algebraic variety V over F. Let $\pi : \widehat{V}(F^{max}) \to B_{\mathbf{F}}(V)$ be the natural map. Then $\pi^{-1}(B_{\mathbf{F}}(X)) = \widehat{X}(F^{max})$ and $\pi : \widehat{X}(F^{max}) \to B_{\mathbf{F}}(X)$ is a closed map. Moreover, the following conditions are equivalent:*

1. *$\widehat{X}$ is definably compact.*
2. *X is bounded and $v + g$-closed.*
3. *$\widehat{X}(F^{max})$ is compact.*
4. *$B_{\mathbf{F}}(X)$ is compact.*
5. *$B_{\mathbf{F}}(X)$ is closed in $B_{\mathbf{F}}(V')$,where V' is any complete F-variety containing V*

The natural map $B_{\mathbf{F}'}(X) \to B_{\mathbf{F}}(X)$ is also closed, if $F \leq F'$ and $\Gamma(F') \leq \mathbb{R}$. In particular, $B_{\mathbf{F}}(X)$ is closed in $B_{\mathbf{F}}(V)$ if and only if $B_{\mathbf{F}'}(X)$ is closed in $B_{\mathbf{F}'}(V)$.

As seen in Chapter 2, the construction of the Hrushovski-Loeser space associated to a variety V is functorial. The Berkovich space associated to a definable subset of a variety possesses similar properties.

Proposition 6.1.8. *([HL], Proposition 14.1.3) Assume X and W are F-definable subsets of some algebraic variety over F.*

1. *Let $h_0 : X \to \widehat{W}$ be an F-definable function. Then h_0 induces functorially a function $h : B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(W)$ such that $\pi_W \circ h_0 = h \circ \pi_X \circ i$ with $i : X \to \widehat{X}$ the canonical inclusion.*

2. *Any continuous F-definable function $h : \widehat{X} \to \widehat{W}$ induces a continuous function $h : B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(W)$ such that $\pi_W \circ h = h \circ \pi_X$.*

3. *The same applies if either X or W is a definable subset of Γ_∞^n and we read $B_{\mathbf{F}}(X) = X(\mathbf{F})$, respectively $B_{\mathbf{F}}(W) = W(\mathbf{F})$.*

The theorems concerning the existence of deformation retractions of the spaces $\widehat{V}$ induce similar results on the associated Berkovich spaces. The precise statements are as follows.

Corollary 6.1.9. *([HL], Corollary 14.1.6)*

1. *Let X be an $\mathbf{F}$-definable subset of some algebraic variety over $\mathbf{F}$. Let $\widehat{h} : I \times \widehat{X} \to \widehat{X}$ be an $\mathbf{F}$-definable deformation retraction with image $h(e_I, X) = Z$. Let $\mathbf{I} = I(\mathbb{R}_\infty)$ and $\mathbf{Z} := \pi(Z(F^{max}))$. Then h induces a deformation retraction $h : \mathbf{I} \times B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)$ with image $\mathbf{Z}$.*

2. *Let $X \to Y$ be an F-definable morphism between F-definable subsets of some algebraic variety over F. Let $h : I \times \widehat{X/Y} \to \widehat{X/Y}$ be an F-definable deformation retraction satisfying (*), with fibers h_y having image Z_y. Let $q \in B_{\mathbf{F}}(Y)$. Then h induces a deformation retraction $h_q : \mathbf{I} \times B_{\mathbf{F}}(X)_q \to B_{\mathbf{F}}(X)_q$ with image $\mathbf{Z}_q$.*

3. *Assume in addition there exists a definable $\Upsilon \subseteq \Gamma_\infty^n$ and definable homeomorphisms $\alpha_y : Z_y \to \Upsilon$, given uniformly in y. Then $\mathbf{Z}_q \simeq \Upsilon(\mathbb{R}_\infty)$. More generally if $\Upsilon \subseteq \Gamma_\infty^w$ with w a finite, Galois invariant subset of a finite field extension F' of F, $\alpha_y : Z_y \to Y$ then $\mathbf{Z}_q \simeq \Upsilon(\mathbb{R}_\infty)/G$ where $G = \mathrm{Gal}(F'/F)$ acting naturally on w.*

6.2 Tame topological properties of Berkovich spaces

Let F be a valued field and V be an F-variety. The homotopy type of the space $\widehat{V}$ is determined by a definable Γ-internal subset of $\widehat{V}$. The corollary above implies that the homotopy type of the Berkovich space is determined by a finite simplicial complex. Before making this statement precise we discuss the role of the parameter space in determining the homotopy type.

Let S be an $\mathbf{F}$-definable Γ-internal subset of the space $\widehat{V}$. By Proposition 6.2.8 in [HL], there exists an $\mathbf{F}$-definable embedding $f : S \to \Gamma_\infty^w$ where w is a finite $\mathbf{F}$-definable set. It follows that there exists a finite Galois extension F' of F such that $\mathrm{Gal}(F^{alg}/F')$ fixes every element of the set w. In this case, we say that S splits over F'. We then have an $\mathbf{F}'$-definable embedding $f' : S \to \Gamma_\infty^n$ where $n = |w|$. It follows that if F'' is any real valued field extension of F' then $S(\mathbf{F}'') = S(\mathbf{F}')$. Hence, $S(F^{max}) = S(\mathbf{F}')$ and if $S_{\mathbf{F}}$ denotes the image of $S(F^{max})$ in $B_{\mathbf{F}}(V)$ then $S_{\mathbf{F}} = S(\mathbf{F}')/\mathrm{Gal}(F'/F)$. Observe that as a result the homotopy type of $B_{\mathbf{F}}(V)$ and $B_{\mathbf{F}'}(V)$ are not necessarily the same (Example 14.2.2, [HL]).

A $\mathbb{Q}$ - tropical structure on a topological space X is a homeomorphism $g : X \to Y$ where Y is a subspace of $[0, \infty]^n$ for some $n \in \mathbb{N}$ and is defined by

a Boolean combination of equalities or inequalities between expressions of the form $\Sigma_i \alpha_i x_i + c$ where $\alpha_i \in \mathbb{Q}_{\geq 0}$ and $c \in \mathbb{R}$. As Y is definable in $(\mathbb{R}, +, .)$, it is a simplicial complex.

Theorem 6.2.1. *([HL], Theorem 14.2.1) Let X be an F-definable subset of a quasi-projective algebraic variety V over a valued field F with $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$. There exists a (strong) deformation retraction $H : I \times B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)$ whose image $\mathbf{Z}$ is of the form $S_{\mathbf{F}}$ with S an F-definable Γ-internal subset of $\widehat{V}$. Thus, $\mathbf{Z}$ has a $\mathbb{Q}$-tropical structure, in particular it is homeomorphic to a finite simplicial complex.*

Given a family of definable subsets of algebraic varieties, Proposition 4.8.3 allows us to study how the homotopy type varies along the family.

Theorem 6.2.2. *([HL], Theorem 14.3.1) Let X and Y be F-definable subsets of algebraic varieties defined over a valued field F . Let $f : X \to Y$ be an F-definable morphism that may be factored through a definable injection of X in $Y \times \mathbb{P}^m$ for some m followed by projection to Y.*

1. *For $b \in Y$, let $X_b = f^{-1}(b)$. Then there are finitely many possibilities for the homotopy type of $B_{\mathbf{F}(b)}(X_b)$, as b runs through Y. More generally, let $U \subseteq X$ be $\mathbf{F}$-definable. Then as b runs through Y there are finitely many possibilities for the homotopy type of the pair $(B_{\mathbf{F}(b)}(X_b), B_{\mathbf{F}(b)}(X_b \cap U))$. Similarly for other data, such as definable functions into Γ_∞.*

2. *For any valued field extension $F \leq F'$ with $\Gamma(F') \subseteq \mathbb{R}$ and $q \in B_{\mathbf{F}'}(Y)$, let $B_{\mathbf{F}'}(X)_q$ denote the fibre over q for the canonical map $B_{\mathbf{F}'}(X) \to B_{\mathbf{F}'}(Y)$. Then there are only finitely many possibilities for the homotopy type of $B_{\mathbf{F}'}(X)_q$ as q runs over $B_{\mathbf{F}'}(Y)$ and F' over extensions of F. More generally, let $U \subset X$ be F-definable. Then as q runs over $B_{\mathbf{F}'}(Y)$ and F' over extensions of F there are finitely many possibilities for the homotopy type of the pair $(B_{\mathbf{F}'}(X)_q, B_{\mathbf{F}'}(X)_q \cap B_{\mathbf{F}'}(U))$. Similarly for other data, such as definable functions into Γ_∞.*

Chapter 7

A Riemann-Hurwitz formula for skeleta

7.1 Semistable vertex sets

Let k be a non-trivially non-Archimedean real valued, algebraically closed complete field. Let $|.|$ denote the valuation on k. By definition, $|.| : k^* \to \mathbb{R}_{>0}$ which we extend to $|.| : k \to \mathbb{R}_{\geq 0}$ by setting $|0| := 0$. Similarly, we define $\mathrm{val} : k^* \to \mathbb{R}$ by setting $\mathrm{val} = -\log(|.|)$ and extend it to $\mathrm{val} \ : k \to \mathbb{R} \cup \infty$ by setting $\mathrm{val}(0) = \infty$. We begin with a brief discussion on the analytification of a k-curve. By curve, we mean a one dimensional connected reduced separated scheme of finite type over k.

Let C be a k-curve. As outlined in Section 5.6, the set C^{an} is the collection of pairs $\{(x, \eta)\}$ where x is a scheme theoretic point of C and η is a rank one valuation on the residue field $k(x)$ which extends the valuation on the field k. We divide the points of C^{an} into four groups using this description. For a point $\mathbf{x} := (x, \mu) \in C^{\mathrm{an}}$, let $\mathcal{H}(x)$ denote the completion of the residue field $k(x)$ for the valuation η. Let $s(\mathbf{x})$ denote the trancendence degree of the residue field $\widetilde{\mathcal{H}(\mathbf{x})}$ over $\tilde{k}$ and $t(\mathbf{x})$ the rank of the group $|\mathcal{H}(\mathbf{x})^*|/|k^*|$. Abhyankar's inequality implies that $s(\mathbf{x}) + t(\mathbf{x}) \leq 1$. This allows us to classify points. We call $\mathbf{x}$ a type I point if it is a k-point of the curve. In which case, both $t(\mathbf{x}) = s(\mathbf{x}) = 0$. If $s(\mathbf{x}) = 1$ then $t(\mathbf{x}) = 0$ and such a point is said to be of type II. If $t(\mathbf{x}) = 1$ then $s(\mathbf{x}) = 0$ and such a point is considered to be of type III. Lastly, if $t(\mathbf{x}) = d(\mathbf{x}) = 0$ and $\mathbf{x}$ is not a k-point of the curve then we call $\mathbf{x}$ a point of type IV.

The fact that C is connected and separated implies that the analytification C^{an} is Hausdorff and pathwise connected. When C is in addition projective, the analytification C^{an} compact. As an example we describe the analytification of the projective line $\mathbb{P}^{1,\mathrm{an}}_k$.

7.1.1 $\mathbb{P}^{1,\mathrm{an}}_k$- The analytification of the projective line over k

The points of $\mathbb{P}^{1,\mathrm{an}}_k$ can be classified as follows. The set of type I points are the k-points $\mathbb{P}^1_k(k)$ of the projective line. The type II, III and IV points are

of the form (ζ, μ) where ζ is the generic point of $\mathbb{P}^1_k$ and μ is a multiplicative norm on the function field $k(\mathbb{P}^1_k)$ which extends the valuation on the field k. The field $k(\mathbb{P}^1_k)$ can be identified with $k(T)$ by choosing coordinates. Hence describing the set of points of $\mathbb{P}^{1,an}_k \setminus \mathbb{P}^1_k(k)$ is equivalent to describing the set of multiplicative norms on the function field $k(T)$ which extend the valuation on k. Let $a \in \mathbb{P}^1_k(k)$ be a k-point and $B(a,r) \subset k$ denote the closed disk around a of radius r contained in $\mathbb{P}^1_k(k)$. We define a multiplicative norm $\eta_{a,r}$ on $k(T)$ as follows. Let $f \in k(T)$. We set $|f(\eta_{a,r})| := \sup_{y \in B(a,r)}\{|f(y)|\}$. It can be checked that this is a multiplicative norm on the function field. If r belongs to $|k^*|$ then $(\zeta, \eta_{a,r})$ is a type II point. Otherwise $(\zeta, \eta_{a,r})$ defines a type III point. It can be shown that every type II and type III point is of this form. A type IV point corresponds to a family of nested closed disks with empty intersection. Let J be a directed index set and for every $j \in J$, $B(a_j, r_j)$ be a closed disk around $a_j \in k$ of radius r_j such that $\bigcap_{j \in J} B(a_j, r_j) = \emptyset$. Let $\mathfrak{E} := \{B(a_j, r_j) | j \in J\}$. We define a multiplicative norm $\eta_{\mathfrak{E}}$ on the function field as follows. For $f \in k(T)$, let $|f(\eta_{\mathfrak{E}})| := \inf_{j \in J}\{\sup_{y \in B(a_j, r_j)} |f(y)|\}$. The set of multiplicative norms on $k(T)$ defined in this manner corresponds to the set of type IV points in $\mathbb{P}^{1,an}_k$.

As in Example 5.4.13 (2), it is standard practice to describe the points of $\mathbb{A}^{1,an}_k$ as the collection $\mathcal{M}(k[T])$ of multiplicative seminorms on the algebra $k[T]$ which extend the valuation of the field k. As a set $\mathbb{P}^{1,an}_k = \mathbb{A}^{1,an}_k \cup \{\infty\}$ where $\infty \in \mathbb{P}^1_k(k)$ is the complement of the affine subspace $\mathrm{Spec}(k[T]) \subset \mathbb{P}^1_k$.

7.1.2 The standard analytic domains in $\mathbb{A}^{1,\mathrm{an}}$.

We follow the treatment in Section 2 of [BPR]. The topological space $\mathbb{P}^{1,an}_k$ is compact, simply connected and Hausdorff. We now describe certain subspaces of $\mathbb{A}^{1,an}_k \subset \mathbb{P}^{1,an}_k$. The tropicalization map, trop : $\mathcal{M}(k[T]) = \mathbb{A}^{1,\mathrm{an}} \to \mathbb{R} \cup \infty$ is defined by $p \mapsto -\log|T(p)|$. Using trop, we define certain analytic domains contained in $\mathbb{A}^{1,\mathrm{an}}$.

1. For $r \in |k^*|$, the standard closed ball of radius r, $\mathbf{B}(r)$ is the set $\mathrm{trop}^{-1}([-\log(r), \infty])$. The space $\mathbf{B}(r)$ is the affinoid space $\mathcal{M}(k\{r^{-1}T\})$.

2. For $r \in |k^*|$, the standard open ball of radius r denoted $\mathbf{O}(r)$ is the set $\mathrm{trop}^{-1}((-\log(r), \infty])$. The space $\mathbf{O}(r)$ is an open analytic domain contained in $\mathbb{A}^{1,\mathrm{an}}$.

3. For $r_1, r_2 \in |k^*|$ with $r_1 \leq r_2$, the standard closed annulus $\mathbf{S}(r_1, r_2)$ of inner radius r_1 and outer radius r_2 is the set $\mathrm{trop}^{-1}([-\log(r_2), -\log(r_1)])$. It is the affinoid space $\mathcal{M}(k\{r_1T^{-1}, r_2^{-1}T\})$. The (logarithmic) modulus of $\mathbf{S}(r_1, r_2)$ is defined to be the value $\log(r_2) - \log(r_1)$.

4. For $r_1, r_2 \in |k^*|$ with $r_1 < r_2$, the standard open annulus of inner radius r_1 and outer radius r_2 denoted $\mathbf{S}(r_1, r_2)_+$ is the set $\mathrm{trop}^{-1}((-\log(r_2), -\log(r_1)))$. The (logarithmic) modulus of $\mathbf{S}(r_1, r_2)_+$ is defined to be the value $\log(r_2) - \log(r_1)$.

5. Let $r \in |k^*|$. The standard punctured Berkovich open disk of radius r is the set $\mathbf{O}(r) \setminus \{0\}$ which we denote $\mathbf{S}(0, r)_+$.

We now highlight certain sub spaces of the analytic domains defined above. The tropicalization map defined above restricts to a map trop : $\mathbf{G}_m^{\mathrm{an}} \to \mathbb{R}$. We define a section $\sigma : \mathbb{R} \to \mathbf{G}_m^{\mathrm{an}}$ of the restriction of the tropicalization map to $\mathbf{G}_m^{\mathrm{an}}$ by mapping $r \in \mathbb{R}$ to the point $\eta_{0,-\exp(r)}$ (cf. 8.1.1).

Definition 7.1.1. Let A be a standard open annulus, a standard closed annulus or a standard punctured open disk. The *skeleton of* A denoted $\Sigma(A)$ is the set $\sigma(\mathbb{R}) \cap A$.

Example 7.1.2. *1.* If $r_1, r_2 \in |k^*|$ with $r_1 < r_2$ then the skeleton of the standard open annulus $\mathbf{S}(r_1, r_2)_+$ is the set $\sigma((-\log(r_2), -\log(r_1)))$.

2. If $r_1, r_2 \in |k^*|$ with $r_1 \leq r_2$ then the skeleton of the standard closed annulus $\mathbf{S}(r_1, r_2)$ is the set $\sigma([-\log(r_2), -\log(r_1)])$.

Following [BPR], we introduce the following definition to distinguish those properties of the standard analytic domains above and their skeleta which are invariant under isomorphism.

Definition 7.1.3. A *general closed disk (resp. general closed annulus, resp. general open annulus, resp. general open disk, resp. general punctured Berkovich open disk)* is an analytic space that is isomorphic to a standard closed disk (resp. standard closed annulus, resp. standard open annulus, resp. standard open disk, resp. standard punctured open disk).

Proposition 7.1.4 ([BPR], 2.8)**.** *Let A, A' be standard closed annuli or open annuli or punctured open disks. Let $\phi : A \to A'$ be an isomorphism. Then $\Sigma(A) = \phi^{-1}(\Sigma(A'))$.*

Definition 7.1.5. Let A be a general open annulus (resp. general closed annulus, resp. general punctured Berkovich open disk). Let A' be a standard open annulus (resp. standard closed annulus, resp. standard punctured Berkovich open disk) such that there exists an isomorphism of analytic spaces $\phi : A \to A'$. The *skeleton* $\Sigma(A)$ *of* A is the set $\phi^{-1}(\Sigma(A'))$. The skeleton of A is well defined by Proposition 7.4.

When A is a general open or closed annulus or general punctured open disk, the skeleton $\Sigma(A)$ can be identified with a real interval upto linear transformations of the form $x \mapsto x + \mathrm{val}(\alpha)$ for some $\alpha \in k^*$. The skeleton $\Sigma(A)$ is endowed with the structure of a metric space.

We introduce the notion of a semistable vertex set of a smooth, projective curve and then generalize this notion to the case of any curve C over k. As above, given a semistable vertex set we associate to it a closed subspace called its skeleton. We then show that the homotopy type of C^{an} is determined by such skeleta. What follows is inspired by the treatment in [[AB], 4.4], [[HL], Section 7] and [[BPR], Section 6].

Definition 7.1.6. Let C be a smooth, projective, irreducible curve defined over the field k and C^{an} be its analytification. *A semistable vertex set* $\mathfrak{V}$ *for* C^{an} is a finite collection of type II points such that if $\mathcal{C}$ denotes the set of connected components of $C^{\mathrm{an}} \smallsetminus \mathfrak{V}$ then there exists a finite subset $S \subset \mathcal{C}$ such that every $A \in S$ is isomorphic to a standard open annulus whose inner and outer radius belong to $|k^*|$ and every $A \in \mathcal{C} \smallsetminus S$ is isomorphic to the standard open disk of unit radius $\mathbf{O}(1)$. Such a decomposition of the space $C^{\mathrm{an}} \smallsetminus \mathfrak{V}$ is called a *semistable decomposition.*

The existence of semistable vertex sets in C^{an} follows from Section 4 in [BPR].

Definition 7.1.7. *An abstract finite metric graph* comprises the following data: A finite set of *vertices* W, a set of *edges* $E \subset W \times W$ which is symmetric and a function $l : E \to \mathbb{R}_{>0} \cup \infty$ such that if $(x, y) \in E$ then $l(x, y) = l(y, x)$.

The function l is called the length function.

Definition 7.1.8. *A finite metric graph* G is the geometric realisation of an abstract finite metric graph (V, E, l) in which every edge e can be identified with a real interval of length $l(e)$. *The genus* $g(G)$ *of the graph* G is defined to be the number $1 - \mathrm{card}(V) + \mathrm{card}(E)$.

It can be verified that if G is a graph which is the geometric realisation of two abstract finite metric graphs (V, E, l) and (V', E', l') then $1 - \mathrm{card}(V) + \mathrm{card}(E) = 1 - \mathrm{card}(V') + \mathrm{card}(E')$.

Definition 7.1.9. Let C be a smooth projective irreducible curve over k. *The skeleton associated to a semistable vertex set* $\mathfrak{V}$ *in* C^{an} is defined to be the union of the skeleta of all open annuli which occur in the semistable decomposition along with the vertex set $\mathfrak{V}$. It is denoted $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$.

Let C be as in the definition above. The skeleton $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ can be seen as a graph whose edges correspond to the closures of the skeleta of the generalized open annuli that occur in the semistable decomposition associated to the set $\mathfrak{V}$. The space C^{an} is pathwise connected. Hence for any two points $v_1, v_2 \in \mathfrak{V}$, there exists a path between them. From the nature of the semistable decomposition, each such path can be taken to be the union of a finite number of edges of the skeleton $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$. It follows that $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ is pathwise connected. By Corollary 2.6 in [BPR], the modulus of the skeleton of every open annulus which occurs in the semistable decomposition defines a length function on the set of edges of $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$. The skeleton $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ is thus a finite, metric graph. Let $x, y \in \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ and P be an injective path from x to y. The path P can be seen as the finite union of injective closed paths $\bigcup_i P_i$ such that for every i, P_i is contained in the closure of the skeleton of a general open annulus which occurs in the semistable decomposition associated to $\mathfrak{V}$. The skeleton of such an open annulus is a metric space, and its metric extends to its closure in C^{an}. It follows that the length $l(P_i)$ of the path P_i is well defined. For instance, if P_i is an injective path from x_i to y_i then $l(P_i) := d(x_i, y_i)$ where d is the metric on the skeleton of the closure of the open annulus that contains x_i and y_i. We set $l(P) := \Sigma_i l(P_i)$. The graph $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ can be given the structure of a metric space by defining the distance between two points x, y in $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ to be $\min_{P \in \mathcal{P}(x,y)}\{l(P)\}$ where $\mathcal{P}(x, y)$ is the set of injective paths between x and y. This defines a metric on $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$.

Let $\mathfrak{V}'$ be a semistable vertex set that contains $\mathfrak{V}$. By Propositions 3.13 and 5.3 in [BPR], $\Sigma(C^{\mathrm{an}}, \mathfrak{V}) \subseteq \Sigma(C^{\mathrm{an}}, \mathfrak{V}')$ and the inclusion is an isometry. Let $\mathbf{H}_0(C^{\mathrm{an}})$ denote the set of points in C^{an} of type II or III. By Corollary 5.1 in loc.cit.,

$$\mathbf{H}_0(C^{\mathrm{an}}) = \varinjlim_{\mathfrak{V}} \Sigma(C^{\mathrm{an}}, \mathfrak{V}).$$

The limit in the above equation is taken over the family of semistable vertex sets $\mathfrak{V}$ in C^{an}. As each of the $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ are metric spaces and the inclusions in the inductive limit are isometries, we have a metric on the space $\mathbf{H}_0(C^{\mathrm{an}})$ which is called its skeletal metric. By Corollary 5.7 in [BPR], this metric extends in a unique way to the space $\mathbf{H}(C^{\mathrm{an}}) := C^{\mathrm{an}} \smallsetminus C(k)$.

7.1.3 The tangent space at a point on C^{an}

Let C be a smooth projective irreducible k-curve. We begin with the notion of a geodesic segment in a metric space T.

Definition 7.1.10. A *geodesic segment from x to y in a metric space T* is the image of an isometric embedding $[a, b] \to T$ with $[a, b] \subset \mathbb{R}$ and $a \mapsto x$, $b \mapsto y$. We often identify a geodesic segment with its image in T and denote it $[x, y]$.

Let $p \in \mathbf{H}(C^{\mathrm{an}})$. A *non trivial geodesic segment starting at p* is a geodesic segment $\alpha : [0, a] \hookrightarrow \mathbf{H}(C^{\mathrm{an}})$ such that $a > 0$ and $\alpha(0) = p$. We say that two non trivial geodesic segments starting from p are *equivalent at p* if they agree in a neighborhood of zero. If α is a geodesic segment starting at the point p then we refer to the equivalence class defined by α as its *germ*. These notions can be adapted to the case $p \in C(k)$ as follows. A *non trivial geodesic segment starting at p* is an embedding $\alpha : [\infty, a] \hookrightarrow C^{\mathrm{an}}$ such that $a < \infty$, $\alpha(\infty) = p$, $\alpha((\infty, a]) \subset \mathbf{H}(C^{\mathrm{an}})$ and the restriction $\alpha_{|(\infty,a]}$ is an isometry. As before, we say that two non trivial geodesic segments starting from p are *equivalent at p* if they agree in a neighborhood of ∞ and if α is a geodesic segment starting at the point p then we refer to the equivalence class defined by α as its *germ*. We now define the tangent space at a point C^{an}.

Definition 7.1.11. Let $x \in C^{\mathrm{an}}$. *The tangent space at x* denoted T_x is the set of non trivial geodesic segments starting from x upto equivalence at x.

Let $x \in C^{\mathrm{an}}$. The tangent space at x depends solely on a neighborhood of x. Following Sections 4 and 5 of [BPR], we introduce the concept of a simple neighborhood U of x and state the result which relates the tangent space T_x to $\pi_0(U \smallsetminus x)$.

Proposition 7.1.12. *([BPR], Corollary 4.27) Let C be a smooth projective irreducible k-curve. Let $x \in C^{\mathrm{an}}$. There is a fundamental system of open neighborhoods $\{U_\alpha\}$ of x of the following form:*

1. *If x is a type-I or a type-IV point then the U_α are open balls.*
2. *If x is a type-III point then the U_α are open annuli with $x \in \Sigma(U_\alpha)$.*
3. *If x is a type-II point then $U_\alpha = \tau^{-1}(W_\alpha)$ where W_α is a simply-connected open neighborhood of x in $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ for some semistable vertex set $\mathfrak{V}$ of C^{an} that contains x and $\tau : C^{\mathrm{an}} \to C^{\mathrm{an}}$ is defined by $x \mapsto \lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(1, x)$ (Proposition 7.1.20). Each $U_\alpha \smallsetminus \{x\}$ is a disjoint union of open balls and open annuli*

Definition 7.1.13 ([BPR], Definition 4.28)**.** Let C be a smooth projective irreducible k-curve. A neighborhood of $x \in C^{\mathrm{an}}$ of the form described in Proposition 7.1.12 is called a *simple neighborhood of x*.

The following proposition is a minor modification of Lemma 5.12 of [BPR] to include points of type I as well.

Proposition 7.1.14. *Let $x \in C^{\mathrm{an}}$ and let U be a simple neighborhood of x in C^{an}. Then $[x, y] \mapsto y$ establishes a bijection $T_x \to \pi_0(U \smallsetminus \{x\})$. Moreover,*

1. *If x is of type I, IV then there is only one tangent direction at x.*
2. *If x has type III then there are two tangent directions at x.*
3. *If x has type II then $U = \mathrm{red}^{-1}(E)$ for a smooth irreducible component E of the special fiber of a semistable formal model $\mathfrak{C}$ of C by (cf. 4.4.1 loc.cit.) and $T_x \tilde{\to} \pi_0(U \smallsetminus \{x\}) \tilde{\to} E(\tilde{k})$.*

Remark 7.1.15. *It should be pointed out that the notation $[x, y]$ was introduced only when $x, y \in \mathbf{H}(C^{\mathrm{an}})$. When $x \in C^{\mathrm{an}}$ is a point of type I and $\alpha : [\infty, a] \hookrightarrow C^{\mathrm{an}}$ is a geodesic segment starting from x, by $[x, \alpha(a)]$ we mean the image of the embedding $\alpha([\infty, a])$.*

Let $\rho : C' \to C$ be a finite morphism between smooth projective k-curves. If $x' \in C'^{\mathrm{an}}$ then the tangent space at x' maps to the tangent space at $\rho(x')$ in an obvious fashion. Suppose x was not of type I. Let $\lambda : [0, 1] \to C'^{\mathrm{an}}$ be a representative of a point on the tangent space at x'. Let U be a simple neighborhood (cf. 7.1.13) of the point $\rho(x')$. We can find $a > 0$ such that $\rho \circ \lambda((0, a])$ lies in a connected component of the space $U \smallsetminus \rho(x')$. This connected component which contains $\rho \circ \lambda((0, a])$ depends only on the equivalence class of λ i.e. on the element of the tangent space that is represented by λ. A similar argument can be used when $x' \in C'(k)$. By 7.1.14, we have thus defined a map

$$d\rho_{x'} : T_{x'} \to T_{\rho(x')}$$

7.1.4 Weak semistable vertex sets

Let C be a smooth projective irreducible k-curve and let $\mathfrak{V}$ be a semistable vertex set in C^{an}. Recall that we defined the skeleton associated to $\mathfrak{V}$ and denoted it $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$. Observe that by construction, the connected components of the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ are isomorphic to Berkovich open balls. If C was not smooth or not complete then there does not exist a finite set of type II points $\mathfrak{V} \subset C^{\mathrm{an}}$ such that C^{an} decomposes into the disjoint union of general open annuli and general open disks. However, we can find a finite set of points $\mathfrak{V}$ in C^{an} and as before define a finite graph $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ such that the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ is the disjoint union of general open disks. It is with this goal in mind that we introduce the notion of weak semistable vertex sets, first for smooth projective irreducible curves and then for any k-curve.

Definition 7.1.16. Let C be a smooth projective irreducible k-curve. *A weak semistable vertex set $\mathfrak{W}$ in C^{an}* is defined to be a finite collection of points of type I or II in C^{an} such that if $\mathcal{C}$ denotes the set of connected components of $C^{\mathrm{an}} \smallsetminus \mathfrak{W}$ then there exists a finite subset $S \subset \mathcal{C}$ such that every $A \in S$ is isomorphic to a standard open annulus or a standard punctured Berkovich open unit disk and every $A \in \mathcal{C} \smallsetminus S$ is isomorphic to a standard Berkovich open unit disk.

As before, we define the skeleton $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ associated to such a set. Let $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ be the union of $\mathfrak{W}$ and the skeleton of every open annulus and punctured open disk in the decomposition of $C^{\mathrm{an}} \smallsetminus \mathfrak{W}$. The closed subspace $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ is homeomorphic to a connected, finite metric graph whose length function is not necessarily finite by which we mean that there could be edges of length ∞.

We generalise this notion of weak semistable vertex sets to the case of curves over k.

Remark 7.1.17. *Let C be a k-curve. Let $j : C \hookrightarrow \bar{C}$ be a dense open immersion where $\bar{C}$ is projective over k. The pair $(j, \bar{C})$ is called a compactification of $C_{/k}$. Let $F := \bar{C} \smallsetminus C$. We know that F is a finite set of points and $C^{\mathrm{an}} = \bar{C}^{\mathrm{an}} \smallsetminus F$. Let $\bar{C}_i$ denote the irreducible components of $\bar{C}$ and $\bar{C'}_i$ denote their respective normalisations. The canonical morphisms $\bar{C'}_i \to \bar{C}$ define a morphism $\rho_{\bar{C}} : \bigcup_i \bar{C}'_i \to \bar{C}$.*

We make use of the notation introduced in Remark 7.1.17 in the definition that follows.

Definition 7.1.18. Let C be a k-curve. Let $\bar{C}$ be a compactification of C. *A weak semistable vertex set $\mathfrak{W}$ for C^{an}* is a finite collection of points of type I or II in $\bar{C}^{\mathrm{an}}$ such that

1. The set $\mathfrak{W}$ contains the set of singular points of $\bar{C}$ and the points $\bar{C} \smallsetminus C$.
2. $\rho_{\bar{C}}^{-1}(\mathfrak{W}) \cap \bar{C}'_i$ is a weak semistable vertex set of the irreducible smooth projective curve $\bar{C}'_i$.

As the above definition requires a compactification $j : C \hookrightarrow \bar{C}$, we should have said a weak semistable set for the pair $(C^{\mathrm{an}}, j : C \hookrightarrow \bar{C})$. However, we abbreviate notation and refer to a set $\mathfrak{W}$ which satisfies the conditions of the above definition, simply as a weak semistable vertex set for C^{an}.

As before, we define the skeleton associated to a weak semistable vertex set for C^{an} as follows.

Definition 7.1.19. Let C be a k-curve and $\bar{C}$ be a compactification of $C_{/k}$. Let $\mathfrak{W}$ be a weak semistable vertex set for C^{an}. Let $\mathfrak{W}'_i := \rho_{\bar{C}}^{-1}(\mathfrak{W}) \cap \bar{C}'^{\mathrm{an}}_i$. We define *the skeleton associated to $\mathfrak{W}$ to be* $\Sigma(C^{\mathrm{an}}, \mathfrak{W}) := [\bigcup_i \rho_{\bar{C}}(\Sigma(\mathfrak{W}'_i, \bar{C}'^{\mathrm{an}}_i))] \cap C^{\mathrm{an}}$.

It can be verified directly from the definition of the skeleta $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ associated to $\mathfrak{W}$ that the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{W})$ decomposes into the disjoint union of sets each of which are isomorphic as analytic spaces to the Berkovich open disk $\mathbf{O}(0, 1)$.

Proposition 7.1.20. *Let C be a k-curve. Let $\mathfrak{V} \subset \mathfrak{W}$ be weak semistable vertex sets of C^{an}. There exists a deformation retraction $\lambda_{\Sigma(C^{\mathrm{an}}, \mathfrak{V})} : [0, 1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ whose image is the skeleton $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ and a deformation retraction $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}}, \mathfrak{V})}$ with image $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$.* (The image of a deformation retraction $\lambda : [0, 1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ is the set $\lambda(1, C^{\mathrm{an}}) := \{\lambda(1, p) | p \in C^{\mathrm{an}}\}$).

Proof. We begin by constructing a deformation retraction $\lambda_{\Sigma(C^{\mathrm{an}}, \mathfrak{V})} : [0, 1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ with image $\lambda_{\Sigma(C^{\mathrm{an}}, \mathfrak{V})}(1, C^{\mathrm{an}}) = \Sigma(C^{\mathrm{an}}, \mathfrak{V})$.

Let $\mathcal{D}$ denote the set of connected components of the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$. By definition, each element $D \in \mathcal{D}$ is isomorphic to the Berkovich open ball $\mathbf{O}(0,1)$. We fix isomorphisms $\rho_D : D \to \mathbf{O}(0,1)$ for every $D \in \mathcal{D}$.

We define $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})} : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ as follows. For $p \in \Sigma(C^{\mathrm{an}}, \mathfrak{V})$, we set $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := p$ for every $t \in [0,1]$. Let $p \in C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$. There exists $D \in \mathcal{D}$ such that $p \in D$. Suppose, p was not a type IV point. By 7.1.1, there exists $a \in k$ and $r \in [0,1)$ such that $\rho_D(p) = \eta_{a,r}$. When $r = 0$, we maintain that $\eta_{a,r}$ is the point a. For $t \in [0,r]$, we set $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := p$ and when $t \in (r,1)$, let $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := \rho_D^{-1}(\eta_{a,t})$. Lastly, let $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(1,p) := \bar{D} \cap \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ where $\bar{D}$ is the closure of D in C^{an}. When p is of type IV, $\rho_D(p)$ corresponds to the semi-norm associated to a nested sequence of closed disks $\{B(x_i, u_i) \subset k\}_i$ whose intersection is empty. Let $u := \lim_i(u_i)$. For any $t > u$, there exists a unique closed disk $B(x,t)$ such that its analytification $\mathbf{B}(x,t)$ contains the point $\rho_D(p)$. We set $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := p$ when $t \in [0,u]$ and $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := \rho_D^{-1}(\eta_{x,t})$ for $t \in (u,1)$ where $\mathbf{B}(x,t) \subset \mathbf{O}(0,1)$ is the unique Berkovich closed disk of radius t that contains the point $\rho_D(p)$. As before, let $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(1,p) := \bar{D} \cap \Sigma(C^{\mathrm{an}}, \mathfrak{V})$.

The function $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})} : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ is well defined and the only points p in C^{an} such that $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) = p$ for every $t \in [0,1]$ are those points which belong to $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$. Furthermore, $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(1, C^{\mathrm{an}}) = \Sigma(C^{\mathrm{an}}, \mathfrak{V})$. We show that $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ is continuous when $[0,1] \times C^{\mathrm{an}}$ is endowed with the product topology. Let $W \subset C^{\mathrm{an}}$ be a connected open set. If W is disjoint from $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ then it must be contained in some $D \in \mathcal{D}$ and $\lambda^{-1}(W)$ does not intersect $\{1\} \times C^{\mathrm{an}} \subset [0,1] \times C^{\mathrm{an}}$. We show that $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$ is open in $[0,1] \times C^{\mathrm{an}}$. As $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}([0,1) \times D) \subseteq D$, the map $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})_{|[0,1)\times D}}$ defines a map $\lambda' : [0,1) \times \mathbf{O}(0,1) \to \mathbf{O}(0,1)$ given by $\lambda'(t,x) := \rho_D(\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t, \rho_D^{-1}(x)))$ for $t \in [0,1)$ and $x \in \mathbf{O}(0,1)$. We need only check that $(\lambda'_D)^{-1}(\rho_D(W))$ is open in $\mathbf{O}(0,1)$. This can be verified using the explicit description of connected open sets in $\mathbf{O}(0,1)$ provided by Lemma 7.1.31. Let W be a connected open set which intersects $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ in an open set W'. Let $\mathcal{D}' := \{D \in \mathcal{D} | \bar{D} \cap \Sigma(C^{\mathrm{an}}, \mathfrak{V}) \subset W'\}$. The semistable decomposition of C^{an} and the connectedness of W imply that it must be contained in $\bigcup_{D \in \mathcal{D}'} D \cup W'$. We can decompose W as the disjoint union $\bigcup_{D \in \mathcal{D}'} (W \cap D) \bigcup W'$. The set $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W \cap D)$ is open in $[0,1] \times C^{\mathrm{an}}$ for every $D \in \mathcal{D}'$. By construction $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W') = ([0,1] \times W') \cup (\{1\} \times (\bigcup_{D \in \mathcal{D}'} D))$. We show that every point in $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W')$ has an open neighbourhood contained in $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$. It can be verified that $[0,1] \times W \subset \lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$. Observe that the set $[0,1] \times W \subset \lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$ forms an open neighborhood of every point in $[0,1] \times W'$. It remains to show that every point in $\{1\} \times (\bigcup_{D \in \mathcal{D}'} D)$ has an open neighbourhood contained in $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$. Let $x \in D$ for some $D \in \mathcal{D}'$. As W is connected and W is an open neighborhood of $\overline{D} \smallsetminus D$, we must have that $W \cap D$ is connected as well. By Remark 7.30, we can reduce to the case when $W \cap D$ is the complement in D of the union of a finite number of Berkovich closed disks and points of types I and IV. It follows that there exists r_D such that $(r_D, 1] \times D \subset \lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$. The set $(r_D, 1] \times D$ is an open neighborhood of $(1,x)$ contained in $\lambda^{-1}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(W)$. We have thus proved that $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})} : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ is continuous and hence a deformation retraction.

We now prove the second part of the proposition. We define a deformation

retraction $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})} : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ with image $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ as follows. For $p \in C^{\mathrm{an}}$, let $s_p \in [0,1]$ be the smallest real number such that $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(s_p, p) \in \Sigma(C^{\mathrm{an}}, \mathfrak{W})$. We define $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ as follows. For $p \in C^{\mathrm{an}}$, $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := \lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p)$ when $t \in [0, s_p]$ and $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p) := \lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(s_p, p)$ when $t \in (s_p, 1]$. Using arguments as before, it can be checked that $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ is indeed a deformation retraction with image $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$. □

Remark 7.1.21. Recall that if C is a smooth projective k-curve then the space $\mathbf{H}(C) := C^{\mathrm{an}} \smallsetminus C(k)$ is a metric space. We can hence define isometries $\alpha : [a,b] \hookrightarrow \mathbf{H}(C)$ where $[a,b] \subset \mathbb{R}$. This fact can be generalized to any k-curve. Let C be a k-curve. By Remark 7.1.17, there exists a finite set of smooth projective curves $\{\bar{C}'_i\}$ such that $\bigcup_i \mathbf{H}((\bar{C}'_i)^{\mathrm{an}}) = \mathbf{H}(C^{\mathrm{an}}) := C^{\mathrm{an}} \smallsetminus C(k)$. We say that a continuous function $\alpha : [a,b] \hookrightarrow \mathbf{H}(C^{\mathrm{an}})$ is an *isometry* if $\alpha([a,b]) \subset \mathbf{H}((\bar{C}'_i)^{\mathrm{an}})$ for some i and $\alpha : [a,b] \hookrightarrow \mathbf{H}((\bar{C}'_i)^{\mathrm{an}})$ is an isometry.

Let the notation be as in Proposition 7.1.20. For $p \in C^{\mathrm{an}}$, let $(\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})})^p : [0,1] \to C^{\mathrm{an}}$ be the path defined by $t \mapsto \lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}(t,p)$. Observe that the deformation retraction $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ is such that if $a, b \in [0,1]$ and $p \in \mathbf{H}(C^{\mathrm{an}})$ then there exists $a_1 < b_1 \in [a,b]$ such that $(\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})})^p$ is constant on the segments $[a, a_1]$ and $[b_1, b]$ and $(\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})})^p \circ -\exp : [-\log(a_1), -\log(b_1)] \to \mathbf{H}(C^{\mathrm{an}})$ is an isometry.

Definition 7.1.22. Let C be a k-curve. Let $\mathfrak{V} \subset \mathfrak{W}$ be weak semistable vertex sets. Let $\lambda : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ be a deformation retraction with image $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$. For $p \in C^{\mathrm{an}}$, let $s_p \in [0,1]$ be the smallest real number such that $\lambda(s_p, p) \in \Sigma(C^{\mathrm{an}}, \mathfrak{W})$. A deformation retraction $\lambda' : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ is said to *extend the deformation retraction* λ if for $p \in C^{\mathrm{an}}$, $\lambda'(t,p) = \lambda(t,p)$ when $t \in [0, s_p]$ and $\lambda'(t,p) = \lambda(s_p, p)$ when $t \in (s_p, 1]$.

Remark 7.1.23. In the proof of Proposition 7.1.20, we constructed deformation retractions $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ and $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$. Observe that $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ is an extension of $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$ by $\mathfrak{W}$.

As outlined in the introduction, we show that given a weak semistable vertex set $\mathfrak{W}$ of a complete curve C, the genus of the finite graph $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ is an invariant of the curve.

Proposition 7.1.24. *Let C be a complete k-curve and $\mathfrak{W}$ be a weak semistable vertex set in C^{an}. Let $\Upsilon \subset C^{\mathrm{an}}$ be a closed subset that does not contain any points of type IV and is a finite graph. Suppose that there exists a deformation retraction $\lambda : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ with image $\lambda(1, C^{\mathrm{an}}) = \Upsilon$. We have that*

$$g(\Sigma(C^{\mathrm{an}}, \mathfrak{W})) = g(\Upsilon).$$

Proof. Let $\psi : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ be the deformation retraction associated to the set $\mathfrak{W}$ with image $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ as constructed in Proposition 7.1.20. As the graph Υ is finite and does not contain any points of type IV, we can find a weak semistable vertex set $\mathfrak{W}'$ such that $\Sigma(C^{\mathrm{an}}, \mathfrak{W}')$ contains Υ. We can choose $\mathfrak{W}'$ so that $\mathfrak{W} \subset \mathfrak{W}'$. The restriction $\lambda : [0,1] \times \Sigma(C^{\mathrm{an}}, \mathfrak{W}') \to C^{\mathrm{an}}$ implies that $\Sigma(C^{\mathrm{an}}, \mathfrak{W}')$ and Υ are homotopy equivalent. It follows that $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ is homotopic to Υ as $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{W})} : [0,1] \times \Sigma(C^{\mathrm{an}}, \mathfrak{W}') \to \Sigma(C^{\mathrm{an}}, \mathfrak{W}')$ is a deformation retraction onto $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$. Hence $g(\Sigma(C^{\mathrm{an}}, \mathfrak{W})) = g(\Upsilon)$. □

Definition 7.1.25. Let C be a k-curve and $\bar{C}$ be a compactification of C. Let f denote the cardinality of the finite set of points $\bar{C}(k) \smallsetminus C(k)$. We define $g^{\mathrm{an}}(C)$ to be $g(\Sigma(\bar{C}^{\mathrm{an}}, \mathfrak{W})) + f$ where $\Sigma(\bar{C}^{\mathrm{an}}, \mathfrak{W})$ is the skeleton associated to a weak semistable vertex set $\mathfrak{W}$ for C^{an}.

It can be checked easily that this definition does not depend on the compactification of C chosen. Proposition 7.1.24 implies that $g^{\mathrm{an}}(C)$ is a well defined invariant of the k-curve C. We end this section with the following proposition concerning finite graphs.

Proposition 7.1.26. *Let $\phi : C' \to C$ be a finite morphism of smooth projective irreducible curves. Let $H \subset C^{\mathrm{an}}$ be a finite graph which does not contain any points of type IV. We then have that $(\phi^{\mathrm{an}})^{-1}(H)$ is a finite graph.*

Proof. We may suppose at the outset that the graph H is connected. We show that we may reduce to the case when the extension of function fields $k(C) \hookrightarrow k(C')$ induced by ϕ is Galois. The extension of function fields $k(C) \hookrightarrow k(C')$ decomposes into a pair of extensions $k(C) \hookrightarrow L$ which is separable and $L \hookrightarrow k(C')$ which is purely inseparable. Let C_1 denote the smooth projective irreducible curve that corresponds to the function field L. The morphisms $C' \to C_1$ and $C'^{\mathrm{an}} \to C_1^{\mathrm{an}}$ are homeomorphisms. It follows that if the preimage of H for the morphism $C_1^{\mathrm{an}} \to C^{\mathrm{an}}$ is a finite graph then $(\phi^{\mathrm{an}})^{-1}(H)$ is a finite graph as well. We may hence suppose that $k(C) \hookrightarrow k(C')$ is separable. Let L' denote the Galois closure of the extension $k(C) \hookrightarrow k(C')$ and C'' be the smooth projective irreducible curve that corresponds to the function field L'. We have a sequence of morphisms $C'' \to C' \to C$. Let $\psi : C'' \to C'$. If the preimage H'' of H for the morphism $C''^{\mathrm{an}} \to C^{\mathrm{an}}$ is a finite graph then its image $\psi^{\mathrm{an}}(H'') = (\phi^{\mathrm{an}})^{-1}(H)$ for the morphism $\psi^{\mathrm{an}} : C''^{\mathrm{an}} \to C'^{\mathrm{an}}$ is a finite graph. Indeed, the group $G' := \mathrm{Gal}(k(C'')/k(C'))$ acts on H''. The graph H'' is defined by combinatorial data i.e. a finite set of vertices $W \subset C''^{\mathrm{an}}$ and a set of edges $E \subset W \times W$ which can be realized as subspaces of C''^{an}. The group G' must act on the sets W and E. It follows that the quotient of H'' for the action of the group G' can be described in terms of the G'-orbits in W and E. Hence $\psi^{\mathrm{an}}(H'')$ is a finite graph. We have thus reduced to the case when the morphism $\phi : C' \to C$ is Galois.

Let $\mathfrak{W}$ be a semistable vertex set in C'^{an}. Let $\mathfrak{V}$ be a weak semistable vertex set in C^{an} that contains $\phi^{\mathrm{an}}(\mathfrak{W})$ and is such that $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ contains the finite graph H. We may suppose in addition that $\mathfrak{V}$ was chosen so that the graph $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ contains no loop edges. It suffices to prove the lemma for $H = \Sigma(C^{\mathrm{an}}, \mathfrak{V})$. We show that there exists a finite graph $H' \subset C'^{\mathrm{an}}$ such that $\phi^{\mathrm{an}}(H') = H$. Let $\mathcal{C}$ denote the connected components of the space $C^{\mathrm{an}} \smallsetminus \mathfrak{V}$. As $\mathfrak{V}$ is a weak semistable vertex set, there exists a finite set $S \subset \mathcal{C}$ such that every $A \in S$ is isomorphic to a standard Berkovich punctured open disk of unit radius or a standard open annulus and if $A \in \mathcal{C} \smallsetminus S$ then A is isomorphic to a standard Berkovich open disk. Likewise, let $\mathcal{C}'$ denote the set of connected components of the space $C'^{\mathrm{an}} \smallsetminus \mathfrak{W}$. As $\mathfrak{W}$ is a semistable vertex set, there exists a finite set $S' \subset \mathcal{C}'$ such that every $A \in S'$ is isomorphic to a standard open annulus and if $A \in \mathcal{C}' \smallsetminus S'$ then A is isomorphic to a standard Berkovich open disk.

Let $\mathfrak{V}' := (\phi^{\mathrm{an}})^{-1}(\mathfrak{V})$. The morphism ϕ^{an} is surjective, open and closed (cf. 6.1.4). It follows that the restriction of ϕ^{an} to $C'^{\mathrm{an}} \smallsetminus \mathfrak{V}'$ is a surjective

clopen morphism onto $C^{\mathrm{an}} \smallsetminus \mathfrak{V}$. Hence if D' is a connected component of the space $C'^{\mathrm{an}} \smallsetminus \mathfrak{V}'$ then there exists a connected component D of the space $C^{\mathrm{an}} \smallsetminus \mathfrak{V}$ such that ϕ^{an} restricts to a surjective morphism from D' onto D. Let D be a connected component of the space $C^{\mathrm{an}} \smallsetminus \mathfrak{V}$ which is not a general Berkovich open ball. We show that if D' is a connected component of the space $C'^{\mathrm{an}} \smallsetminus \mathfrak{V}'$ such that $\phi^{\mathrm{an}}(D') = D$ then D' cannot be a general Berkovich open ball. Suppose that D' was a general Berkovich open ball. There exists a point $q \in C'^{\mathrm{an}}$ such that $D' \cup \{q\}$ is compact. It follows that $D \cup \phi^{\mathrm{an}}(q)$ is compact. The only elements in $\mathcal{C}$ for which this is possible are general Berkovich open disks which contradicts our assumption.

Let $D \in \mathcal{C}$ be a punctured open disk or open annulus. Let D' be a connected component in $C'^{\mathrm{an}} \smallsetminus \mathfrak{V}'$ such that $\phi^{\mathrm{an}}(D') = D$. There exists a finite set of points $P_{D'}$ such that $D' \smallsetminus P_{D'}$ is the disjoint union of general Berkovich open disks and finitely many general open annuli or punctured Berkovich disks. Let $\mathcal{C}'_{D'}$ denote the connected components of $D' \smallsetminus P_{D'}$. Let O be a Berkovich open disk in $D' \smallsetminus P_{D'}$. The image $\phi^{\mathrm{an}}(O)$ is a connected open subset of D for which there exists $p \in D$ such that $\phi^{\mathrm{an}}(O) \cup \{p\}$ is compact. It follows from Lemma 7.1.31 that $\phi^{\mathrm{an}}(O)$ must be a Berkovich open disk D and hence lies in the complement of the skeleton $\Sigma(D)$. Let $S_{D'}$ be the set of open annuli or punctured open disks in $\mathcal{C}'_{D'}$. The set $S_{D'}$ is finite. If $A \in S_{D'}$ then by Proposition 2.5 in [BPR], we must have that $\phi^{\mathrm{an}}(\Sigma(A)) \subset \Sigma(D)$. We showed that if $O \in \mathcal{C}'_{D'} \smallsetminus S_{D'}$ then $\phi^{\mathrm{an}}(O) \subset D \smallsetminus \Sigma(D)$. It follows that $\Sigma(D) \smallsetminus \bigcup_{A \in S_{D'}} \phi^{\mathrm{an}}(\Sigma(A))$ is at most a finite set of points. Let $\Sigma(D') := \bigcup_{A \in S_{D'}} \Sigma(A) \cup P_{D'}$. The set $\Sigma(D')$ is a closed connected subset of D'. As ϕ^{an} restricted to D' is closed, its image $\phi^{\mathrm{an}}(\Sigma(D')) = \{\phi^{\mathrm{an}}(p) | p \in P_{D'}\} \cup \bigcup_{A \in S_{D'}} \phi^{\mathrm{an}}(\Sigma(A))$ is a closed connected subset of D. Hence $\{\phi^{\mathrm{an}}(p) | p \in P_{D'}\} \subset \Sigma(D)$ and $\phi^{\mathrm{an}}(\Sigma(D')) = \Sigma(D)$.

For every $D \in S$, let D'_D be a connected component of $C'^{\mathrm{an}} \smallsetminus \mathfrak{V}'$ such that $\phi^{\mathrm{an}}(D'_D) = D$. We showed that there exists $\Sigma(D'_D) \subset D'_D$ such that $\phi^{\mathrm{an}}(\Sigma(D'_D)) = \Sigma(D)$. If $H'_0 := \bigcup_{D \in S} \Sigma(D'_D)$ then $H' := H'_0 \cup \mathfrak{V}'$ is a finite graph. We must have that $\phi^{\mathrm{an}}(H') = H$. Let $G := \mathrm{Gal}(k(C')/k(C))$. The set $\bigcup_{g \in G} g(H')$ is a finite graph and since the morphism $C' \to C$ is Galois, we must have that $(\phi^{\mathrm{an}})^{-1}(H) = \bigcup_{g \in G} g(H')$.

□

7.1.5 The Non-Archimedean Poincaré-Lelong Theorem

The non-Archimedean
Poincaré-Lelong theorem is used in Sections 7.3 and 7.4. Our treatment follows that of [BPR].

Let C be a smooth projective irreducible k-curve. Let $x \in C^{\mathrm{an}}$ be a point of type II. The field $\widetilde{\mathcal{H}(x)}$ (cf. 5.7) is an algebraic function field over $\tilde{k}$. Let $\tilde{C}_x$ denote the smooth projective $\tilde{k}$-curve that corresponds to the field $\widetilde{\mathcal{H}(x)}$. Let $\mathrm{Prin}(C)$ and $\mathrm{Prin}(\tilde{C}_x)$ denote the group of principal divisors on the curves $C_{/k}$ and $\tilde{C}_{x/\tilde{k}}$ respectively. We define a map $\mathrm{Prin}(C) \to \mathrm{Prin}(\tilde{C}_x)$ as follows. Let $f \in k(C)$ be a rational function on C and c be any element in k such that $|f(x)| = |c|$. This implies that $(c^{-1}f) \in \mathcal{H}(x)^0$. Let f_x denote the image of $c^{-1}f$ in $\widetilde{\mathcal{H}(x)}$. Although $f_x \in \widetilde{\mathcal{H}(x)}$ depends on the choice of $c \in k$, the divisor that f_x defines on $\tilde{C}_x$ is independent of c. Hence we have a well defined map

$\mathrm{Prin}(C) \to \mathrm{Prin}(\tilde{C}_x)$. It can be shown that this map is a homomorphism of groups.

A function $F : C^{\mathrm{an}} \to \mathbb{R}$ is piecewise affine if for any geodesic segment $\lambda : [a, b] \to \mathbf{H}(C^{\mathrm{an}})$, the composition $F \circ \lambda : [a, b] \to \mathbb{R}$ is piecewise affine. The outgoing slope of a piecewise affine function F at a point $x \in \mathbf{H}(C^{\mathrm{an}})$ along a tangent direction $v \in T_x$ is defined to be

$$\delta_v F(x) := \mathrm{Lim}_{\epsilon \to 0}(F \circ \lambda)'(\epsilon).$$

where $\lambda : [0, a] \hookrightarrow \mathbf{H}(C^{\mathrm{an}})$ is a representative of the element v. It is evident from the definition that $\delta_v F(x)$ depends only on the equivalence class of λ i.e. it depends only on the element $v \in T_x$.

Theorem 7.1.27. (Non-Archimedean Poincaré-Lelong Theorem) *Let $f \in k(C)$ be a non-zero rational function on the curve C and S denote the set of zeros and poles of f. Let $\mathfrak{W}$ be a weak semistable vertex set whose set of k-points is the set S. Let $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ be the skeleton associated to $\mathfrak{W}$ and $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{W})} : [0, 1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ be the deformation retraction with image $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$. We will use λ_e to denote the morphism $\lambda_{\Sigma(C^{\mathrm{an}},\mathfrak{W})}(1, _) : C^{\mathrm{an}} \to C^{\mathrm{an}}$ (cf. Proposition 7.1.20). If $F := -\log|f| : C^{\mathrm{an}} \smallsetminus S \to \mathbb{R}$. Then we have that*

1. *$F = F \circ \lambda_e$.*
2. *F is piecewise affine with integer slopes and F is affine on each edge of $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$.*
3. *If x is a type II point of C^{an} and v is an element of the tangent space T_x, then $\mathrm{ord}_{\tilde{v}}(f_x) := \delta_v F(x)$ defines a discrete valuation $\mathrm{ord}_{\tilde{v}}$ on the $\tilde{k}$-function field $\tilde{k}(\tilde{C}_x)$.*
4. *If $x \in C^{\mathrm{an}}$ is of type II or III then $\sum_{v \in T_x} \delta_v F(x) = 0$.*
5. *Let $x \in S$, c be the ray in $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ whose closure in C^{an} contains x and $y \in \mathfrak{W}$ the other end point of e. If $v \in T_y$ is that element of the tangent space T_y for which c is a representative then $\delta_v F(y) = ord_x(f)$.*

7.1.6 An alternate description of the tangent space at a point x of type II

Let $x \in C^{\mathrm{an}}$ be a point of type II. We define the algebraic tangent space at a point of type II and show how this notion reconciles nicely with the definition we introduced above. Recall that the field $\widetilde{\mathcal{H}(x)}$ is of transcendence degree 1 over $\tilde{k}$ and uniquely associated to this $\tilde{k}$-function field is a smooth, projective $\tilde{k}$-curve which is denoted $\tilde{C}_x$.

Definition 7.1.28. *The algebraic tangent space at x* denoted T_x^{alg} is the set of closed points of the curve $\tilde{C}_x$.

We now write out a map $B : T_x \to T_x^{\mathrm{alg}}$. The closed points of the $\tilde{k}$-curve $\tilde{C}_x$ correspond to discrete valuations on the field $\widetilde{\mathcal{H}(x)}$. Given a germ $e_x \in T_x$ and $f \in \widetilde{\mathcal{H}(x)}$ there exists $g \in \mathcal{H}(x)$ such that $|g(x)| = 1$ and $\tilde{g} = f$. Let $B(e_x)(f)$ be the slope of the function $-\log|g|$ along the germ e_x directed outwards. By the

Non-Archimedean Poincaré-Lelong Theorem, $B(e_x)$ defines a discrete valuation on the function field $\widetilde{\mathcal{H}(x)}$ i.e. a closed point of the curve $\tilde{C}_x$. The map B is a well defined bijection.

Let C' be a smooth, projective, irreducible curve over the field k and $\rho : C' \to C$ a finite morphism. If x' is a preimage of the point x then it must be of type II as well. The inclusion of non-Archimedean valued complete fields $\mathcal{H}(x) \hookrightarrow \mathcal{H}(x')$ induces an extension of $\tilde{k}$-function fields $\widetilde{\mathcal{H}(x)} \hookrightarrow \widetilde{\mathcal{H}(x')}$. This defines a morphism $d\rho^{\mathrm{alg}}_{x'} : T^{\mathrm{alg}}_{x'} \to T^{\mathrm{alg}}_{x}$ between the algebraic tangent space at x' and the algebraic tangent space at x. Recall that we have in addition a map $d\rho_{x'} : T_{x'} \to T_x$. These maps are compatible in the sense that the following diagram is commutative.

$$\begin{tikzcd}
T_{x'} \arrow[r, "d\rho_{x'}"'] \arrow[d, "B"] & T_x \arrow[d, "B"] \\
T^{\mathrm{alg}}_{x'} \arrow[r, "d\rho^{\mathrm{alg}}_{x'}"'] & T^{\mathrm{alg}}_{x}
\end{tikzcd}$$

7.1.7 Continuity of lifts

Let $\phi : C' \to C$ be a finite morphism between irreducible smooth projective curves. In Section 7.2, we construct a pair of deformation retractions $\lambda' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$ and $\lambda : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ which are compatible for the morphism ϕ^{an}. Our method of proof is to first construct a suitable deformation retraction λ on C^{an} and then lift it to a function $\lambda' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$ such that for every $q \in C'^{\mathrm{an}}$, the map $\lambda'^q : [0,1] \to C'^{\mathrm{an}}$ defined by setting $\lambda'^q(t) = \lambda'(t,q)$ is continuous. Our goal in this section is to show that given a deformation retraction λ and a lift λ' as above, the function λ' is continuous.

Lemma 7.1.29. *Let $\phi : C' \to C$ be a finite morphism between k-curves and suppose in addition that C is normal. Let $\mathfrak{V} \subset C^{\mathrm{an}}$ be a weak semistable vertex set and suppose $\mathfrak{V}'$ is a weak semistable vertex set for C'^{an} such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}') = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{V}))$. Let $\mathcal{D}$ denote the set of connected components of the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ and likewise, $\mathcal{D}'$ denote the set of connected components of the space $C'^{\mathrm{an}} \smallsetminus \Sigma(C'^{\mathrm{an}}, \mathfrak{V}')$. If $D' \in \mathcal{D}'$ then there exists $D \in \mathcal{D}$ such that $\phi^{\mathrm{an}}(D') = D$. Furthermore, the restriction $\phi^{\mathrm{an}}_{|D'} : D' \to D$ is both closed and open.*

Proof. Let $D' \in \mathcal{D}'$. As D' is connected and ϕ^{an} is continuous, we must have that $\phi^{\mathrm{an}}(D')$ is connected. Furthermore, $\phi^{\mathrm{an}}(D') \subset C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ because $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}') = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{V}))$. The open subspace $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{V})$ decomposes into the disjoint union $\bigcup_{A \in \mathcal{D}} A$. It follows that there exists $D \in \mathcal{D}$ such that $\phi^{\mathrm{an}}(D') \subset D$.

Let A' be a connected component of the space $(\phi^{\mathrm{an}})^{-1}(D)$ that contains D'. As $A' \subset C'^{\mathrm{an}} \smallsetminus \Sigma(\mathfrak{W}, C'^{\mathrm{an}})$ and $C'^{\mathrm{an}} \smallsetminus \Sigma(\mathfrak{W}, C'^{\mathrm{an}})$ decomposes into the disjoint union $\bigcup_{U \in \mathcal{D}'} U$, we must have that $D' = A'$. The morphism ϕ^{an} is a finite morphism and hence closed. By Lemma 6.1.4, it is open as well. It follows that ϕ^{an} restricts to a morphism $D' \to D$ which is both open and closed. As D is connected, we must have that $\phi^{\mathrm{an}}(D') = D$. □

Remark 7.1.30. Recall that for $a \in k$ such that $|a| < 1$ and $r < 1$, we used $\mathbf{O}(a, r)$ to denote the Berkovich open disk around a of radius r and $\mathbf{B}(a, r)$ to denote the Berkovich closed disk around a of radius r. By Proposition 1.6 in [Ba], a basis $\mathcal{B}$ for the open sets of $\mathbf{O}(0, 1)$ is given by the sets

$$\mathbf{O}(a, r),\ \mathbf{O}(a, r) \smallsetminus \bigcup_{i \in I} X_i,\ \mathbf{O}(0, 1) \smallsetminus \bigcup_{i \in I} X_i$$

where I ranges over finite index sets, a ranges over $O(0, 1)$, where $r \in (0, 1)$ and X_i is either a Berkovich closed sub disk of the form $\mathbf{B}(a_i, r_i)$ with $a_i \in O(0, 1)$ and $r_i \in [0, 1)$ or a point of type I or IV. We classify the elements of this basis by referring to Berkovich open sub disks as sets of form 1, Berkovich open disks from which a finite number of closed disks have been removed as sets of form 2 and the complement of the union of a finite number of closed sub disks as sets of form 3.

Lemma 7.1.31. *Let $U \subset \mathbf{O}(0, 1)$ be a connected open set. Then U is of the form $\mathbf{O}(a, r) \smallsetminus \bigcup_{j \in J} X_j$ where $a \in O(0, 1)$, $r \in (0, 1]$, J is an index set and the X_j are Berkovich closed disks or points of type I or IV. In addition, the X_j can be taken to be disjoint from each other.*

Note that we do not claim every set of the form $\mathbf{O}(a, r) \smallsetminus \bigcup_{j \in J} X_j$ is open, as this is false. For instance the set $\mathbf{H}(\mathbf{O}(0, 1)) := \mathbf{O}(0, 1) \smallsetminus O(0, 1)$ is not open in $\mathbf{O}(0, 1)$ as $O(0, 1)$ is dense in $\mathbf{O}(0, 1)$.

Lemma 7.1.32. *Let $f : \mathbf{O}(0, 1) \to \mathbf{O}(0, 1)$ be a surjective, open and closed continuous function.*

1. *If $U \in \mathcal{B}$ is an open set of form i where i is 1 or 3 then $f(U)$ is of form i as well. If $U \in \mathcal{B}$ is of form 2 then $f(U)$ is of form 1 or 2. If we suppose in addition that f is bijective and U is of form 2 then $f(U)$ is also of form 2.*

2. *If $Y \subset \mathbf{O}(0, 1)$ is a Berkovich closed disk then $f(Y)$ is a Berkovich closed disk.*

Proof. 1. Let U be a Berkovich open sub ball. We show that $f(U)$ is a Berkovich open ball. The closure $\overline{U}$ is a compact subspace of $\mathbf{O}(0, 1)$. Observe that $\overline{U} \smallsetminus U$ is a single point which we denote p. As f is continuous, $f(\overline{U}) = f(U) \cup \{f(p)\}$ must be compact as well. As U is connected, $f(U)$ must be a connected open set as well. By Lemma 7.1.31, it suffices to verify which connected open sets in $\mathbf{O}(0, 1)$ are such that they can be compactified by adding a single point of $\mathbf{O}(0, 1)$. It can be checked by hand that the only possibility for $f(U)$ is a Berkovich open ball contained in $\mathbf{O}(0, 1)$.

Let $\{D_1, \ldots, D_m\}$ be a finite number of Berkovich closed disks or points of types I or IV in $\mathbf{O}(0,1)$ and $U := \mathbf{O}(0,1) \smallsetminus (\bigcup_i D_i)$. The set U is of form 3 and we show that $f(U)$ is also of form 3. As U is connected, the image $f(U)$ is a connected open set as well. By Lemma 7.1.31, $f(U)$ must be of the form $\mathbf{O}(a,r) \smallsetminus \bigcup_{j \in J} X_j$ where $a \in O(0,1)$, $r \in (0,1]$, J is an index set and the X_j are Berkovich closed disks or points of type I or IV. In addition, the X_j can be taken to be disjoint. We claim that $r = 1$. Suppose $r < 1$. Then the closure of $f(U)$ in $\mathbf{O}(0,1)$ denoted $\overline{f(U)}$ is compact. The D_i are Berkovich closed disks or points of types I or IV and hence compact. As a result, the $f(D_i)$ are compact subsets of $\mathbf{O}(0,1)$. The surjectivity of f implies that $\mathbf{O}(0,1) = \bigcup_i f(D_i) \cup (\overline{f(U)})$ is compact. This is a contradiction and we must hence have that $r = 1$. We claim that the index set J is finite. There exists a finite set of points $S := \{p_1, \ldots, p_r\}$ in $\mathbf{O}(0,1)$ such that $U \cup S$ is closed in $\mathbf{O}(0,1)$. As the map f is closed, $f(U \cup S)$ is closed in $\mathbf{O}(0,1)$. Uniquely associated to each $j \in J$ is an element $x_j \in \mathbf{O}(0,1)$ that lies in the closure of $f(U)$. Hence we must have that the index set J is finite. This implies that $f(U) \in \mathcal{B}$ and is of form 3.

Let $U \in \mathcal{B}$ be an open set of form 2. As U is contained in a Berkovich open disk U', we must have that its image is a connected open set which is contained in the Berkovich open disk $f(U')$ strictly contained in $\mathbf{O}(0,1)$. Repeating the arguments above, it can be shown that $f(U) \in \mathcal{B}$ is of form either 1 or 2. Suppose that f is bijective and let $U = U' \smallsetminus (\bigcup_i D_i)$ where the $D_i \subset U'$ are Berkovich closed sub disks or points of type I or IV. It follows that $f(U) = f(U') \smallsetminus \bigcup_i f(D_i)$. As the only connected open subsets of $\mathbf{O}(0,1)$ which are open balls from which a finite number of closed subspaces have been removed are of form 2, we conclude that $f(U)$ is of form 2.

2. Let Y be a closed disk of radius r in $O(0,1)$. The closed disk Y can be seen as the union of a family of Berkovich open sub disks in $O(0,1)$ of radius r and a point. Hence we can write $Y = (\bigcup_{i \in I} V_i) \cup \{q\}$ where I is an index set, the V_i are Berkovich open disks of radius r and q is the unique point such that for every i, $V_i \cup \{q\}$ is compact. It follows that $f(V_i \cup \{q\}) = f(V_i) \cup f(q)$ is a compact set. Let $U_i := f(V_i)$ and $p = f(q)$. By part (1), the U_i are Berkovich open balls contained in $\mathbf{O}(0,1)$. The point p of $\mathbf{O}(0,1)$ such that $U_i \cup \{p\}$ is compact is uniquely determined by U_i. Furthermore, this point p determines the radius of the Berkovich open ball U_i. It follows that the radii of the Berkovich open balls U_i are the same. Let t be the radius of Berkovich open balls U_i. Let X denote the Berkovich closed ball corresponding to the point p. The radius of X is t. The tangent space at p is in bijection with the set of Berkovich open balls of radius t contained in X and the open annulus $O(0,1) \smallsetminus X$. Likewise, the tangent space at q is the set of Berkovich open balls V_i of radius r contained in Y and the open annulus $O(0,1) \smallsetminus Y$. The tangent space at q surjects onto the tangent space at p. Furthermore, for every $i \in I$, the image U_i of the Berkovich open ball V_i is a Berkovich open ball of radius t contained in X. Hence if $U \subset X$ is a Berkovich open disk of radius t then there exists $j \in I$ such that $f(V_j) = U$. It follows that $f(Y) = X$.

□

Lemma 7.1.33. *Let $a \in k$ and r be a positive real number belonging to $|k^*|$. Let $\mathbf{B}(a,r)$ denote the Berkovich closed disk around a of radius r and let $\sigma : \mathbf{B}(a,r) \to \mathbf{B}(a,r)$ be an automorphism of analytic spaces. Suppose $W \subset \mathbf{B}(a,r)$ is a Berkovich open disk of radius $0 < s < r$ then $\sigma(W)$ is a Berkovich open disk of radius s. Likewise, if W is a Berkovich closed disk of radius $0 \leq s < r$ then $\sigma(W)$ is a Berkovich closed disk of radius s.*

Proof. As $r \in |k^*|$, we can suppose that $r = 1$, $a = 0$. We choose coordinates and write $\mathbf{B}(0,1) = \mathcal{M}(k\{T\})$. The automorphism σ induces an automorphism $\sigma' : k\{T\} \to k\{T\}$ of affinoid algebras. By the Weierstrass preparation theorem, we must have that $\sigma'(T) = f(T)u$ where $f(T) = c(T-a_1)(T-a_r)$ is a polynomial in T with $c \in k^*$ and u is an invertible element in $k\{T\}$. As σ' is an automorphism, we must have that $|c| = 1$ and that $f(T)$ is of degree 1. It follows that $f(T) = c(T-b)$ for some $b \in B(0,1)$. We now show that if W is a Berkovich open sub ball around a point $x \in B(0,1)$ of radius s then $\sigma(W)$ is a Berkovich open ball around $\sigma(x)$ of radius s. By definition, $W = \{p \in \mathbf{B}(0,1) || (T-x)(p)| < s\}$. It follows that $\sigma(W) = \{q \in \mathbf{B}(0,1) || (1/(cu)T) - (x-b)| < s\}$. As $|cu| = 1$, it can be checked that the claim has been verified. The proof can be repeated when W is the closed disk $\{p \in \mathbf{B}(0,1) || (T-x)(p)| \leq s\}$. $\square$

We make use of the following notation in the statements that follow. Let $\phi : C' \to C$ be a finite morphism between smooth projective curves. Let $\mathfrak{W}', \mathfrak{W}$ be weak semistable vertex sets for C'^{an} and C^{an} respectively such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}))$. Let $\mathcal{D}'$ denote the set of connected components of the space $C'^{\mathrm{an}} \smallsetminus \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$. If $D' \in \mathcal{D}'$ then D' is isomorphic to the Berkovich unit ball $\mathbf{O}(0,1)$ and we identify D' via this isomorphism. Likewise, let $\mathcal{D}$ denote the set of connected components of the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{W})$.

Lemma 7.1.34. *Let $\phi : C' \to C$ be a finite morphism between irreducible projective smooth k-curves. Let $\mathfrak{V} \subset \mathfrak{W}$ be weak semistable vertex sets of C^{an}. Let $\mathfrak{W}' \subset C'^{\mathrm{an}}$ be a weak semistable vertex set such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}))$. Let $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})} : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ be the deformation retraction constructed in Proposition 7.1.20 whose image is $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$. Let $\lambda' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$ be a function such that for every $q \in C'^{\mathrm{an}}$, the path $\lambda'^q : [0,1] \to C'^{\mathrm{an}}$ defined by $t \mapsto \lambda'(t,q)$ is continuous and $\lambda'^q(1) \in \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$. Furthermore, if $\phi^{\mathrm{an}}(q) = p$ then λ'^q is the unique path starting from q such that $\phi^{\mathrm{an}} \circ \lambda'^q = (\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})})^p$. Let $D' \in \mathcal{D}'$ and $x_1, x_2 \in D'$. There exists $r \in [0,1]$ such that $\lambda'^{x_1}(r), \lambda'^{x_2}(r) \in D'$ and $\lambda'^{x_1}_{|[r,1]} = \lambda'^{x_2}_{|[r,1]}$.*

We simplify notation and write λ in place of $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})}$.

Proof. Recall that when constructing the deformation retraction λ, we identified every $D \in \mathcal{D}$ with the standard Berkovich open unit disk. Let D' be as in the statement of the lemma. By Lemma 7.1.29, there exists $D \in \mathcal{D}$ such that $\phi^{\mathrm{an}}(D') = D$. If $\overline{D}$ is the closure of D in C^{an} then $\overline{D} \smallsetminus D$ is a single point η. By construction of the deformation retraction λ there exists $s \in [0,1]$ such that for every $y \in D$, $\lambda(s,y) = \eta$ and the restriction $[0,s) \times D \to D$ given by $(t,x) \mapsto \lambda(t,x)$ is well defined. We must hence have that the restriction $\lambda' : [0,s) \times D' \to D'$ is well defined. If $\overline{D'}$ is the closure of D' in C'^{an} then $\overline{D'} \smallsetminus D'$ is a single point η'. Furthermore, for every $x \in D'$, $\lambda'(s,x) = \eta'$.

If U' is a simple neighborhood of the point η' then the tangent space $T_{\eta'}$ is in bijection with the connected components of the space $U' \smallsetminus \eta'$. The set D' corresponds to a single element of the tangent space at η'. It follows that for some $r \in [0,s)$, $\lambda'^{x_1}(r) \in D' \cap \lambda'^{x_2}([0,s])$. Let $q := \lambda'^{x_1}(r)$ and $p := \phi^{\mathrm{an}}(q)$. By construction of the deformation retraction λ, $\lambda(t,p) = p$ for every $t \in [0,r]$. Also, the deformation λ satisfies the following property. For every $a < b \in [0,1]$ and $y \in C^{\mathrm{an}}$, $\lambda(b,(\lambda(a,y)) = \lambda(b,y)$. It follows that $\lambda'^{q}_{|[r,1]}$, $\lambda'^{x_1}_{|[r,1]}$ and $\lambda'^{x_2}_{|[r,1]}$ are all lifts of the path $\lambda^{p}_{|[r,1]}$. As the lift of the path starting from p is unique, we must have that $\lambda'^{q}_{[r,1]} = \lambda'^{x_1}_{[r,1]} = \lambda'^{x_2}_{[r,1]}$. $\square$

Proposition 7.1.35. *Let $\phi : C' \to C$ be a finite morphism between irreducible projective smooth k-curves. Assume in addition that the extension of function fields $k(C) \hookrightarrow k(C')$ is a finite Galois extension and let $G := \mathrm{Gal}(k(C')/k(C))$. Let $\mathfrak{V} \subset \mathfrak{W}$ be weak semistable vertex sets of C^{an}. Let $\mathfrak{W}' \subset C'^{\mathrm{an}}$ be a weak semistable vertex set such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}))$. Let $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{V})} : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ be the deformation retraction as constructed in Proposition 7.1.20 whose image is $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$. Let $\lambda' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$ be a function such that for every $q \in C'^{\mathrm{an}}$, the path $\lambda'^{q} : [0,1] \to C'^{\mathrm{an}}$ defined by $t \mapsto \lambda'(t,q)$ is continuous and also that the following diagram commutes.*

$$\begin{tikzcd}
{[0,1] \times C'^{\mathrm{an}}} \arrow[r, "\lambda'"'] \arrow[d, "id \times \phi^{\mathrm{an}}"] & C'^{\mathrm{an}} \arrow[d, "\phi^{\mathrm{an}}"] \\
{[0,1] \times C^{\mathrm{an}}} \arrow[r, "\lambda^{\mathfrak{W}}_{\Sigma(\mathfrak{V},C^{\mathrm{an}})}"'] & C^{\mathrm{an}}
\end{tikzcd}$$

.

We suppose in addition that for every $q \in C'^{\mathrm{an}}$, the path λ'^{q} is the unique lift starting from q of the path $(\lambda^{\mathfrak{W}}_{\Sigma(\mathfrak{V},C^{\mathrm{an}})})^{\phi^{\mathrm{an}}(q)}$ and also that λ' is G-invariant i.e. for every $g \in G$, $t \in [0,1]$ and $x \in C'^{\mathrm{an}}$, $g(\lambda'(t,x)) = \lambda'(t,g(x))$. The following statements are then true.

1. *Let $\mathcal{D}'$ denote the set of connected components of the space $C'^{\mathrm{an}} \smallsetminus \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$. If $D' \in \mathcal{D}'$ then D' is isomorphic to the Berkovich unit ball $\mathbf{O}(0,1)$ and we identify D' via this isomorphism. By Lemma 7.1.29, the group G has a well defined action on the set $\mathcal{D}'$. Let $H \subset G$ be the sub group which fixes D'. Let W be a Berkovich closed or open ball strictly contained in D'. There exists a Berkovich closed sub ball $\mathbf{B}(0,r) \subset D'$ with $r \in |k^*|$ such that H stabilizes $\mathbf{B}(0,r)$ and $W \subset \mathbf{B}(0,r)$.*

2. *The map $\lambda' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$ is continuous.*

Over the course of the proof, we simplify notation and write λ in place of $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}},\mathfrak{W})}$. The hypothesis that λ' is G-invariant is redundant as it can be deduced from the uniqueness of the lifts.

Proof. 1. Let $\mathcal{D}$ denote the set of connected components of the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{W})$. In the proof of Proposition 7.1.20, we constructed the deformation retraction λ by identifying every $D \in \mathcal{D}$ with Berkovich open unit disks centered at 0. By Lemma 6.1.4, the morphism ϕ^{an} is open. As ϕ is a finite morphism, the morphism ϕ^{an} is closed as well. Let $D' \in \mathcal{D}'$. By Lemma 7.1.29, there exists $D \in \mathcal{D}$ such that $\phi^{\mathrm{an}}(D') = D$. We also showed in 7.1.29, that D' is a connected component of the space $(\phi^{\mathrm{an}})^{-1}(D)$ and hence the morphism ϕ^{an} restricts to a closed and open morphism from D' onto D. We use $\phi^{\mathrm{an}}_{D'}$ to denote the restriction of ϕ^{an} to D'. Let $x \in D$ and $R := (\phi^{\mathrm{an}}_{D'})^{-1}(x) = \{y_1, \ldots, y_m\}$. Recall that H is the sub group of G which stabilizes D'. It follows that $R = \{h(y_1)\}_{h \in H}$. There exists $s \in [0,1]$ such that for every $z \in D$, $\lambda(t,z) \in D$ for $t \in [0,s)$ and $\lambda(s,z) \in \overline{D} \smallsetminus D$. For every i, let p_{y_i} denote the path $[0,s] \to \overline{D'}$ defined by $t \mapsto \lambda'(t, y_i)$. The paths p_{y_i} are each lifts of the path $\lambda^x : [0,s] \to \overline{D}$ defined by $t \mapsto \lambda(t,x)$. Observe that $p_{y_i}(s) = \overline{D'} \smallsetminus D'$. By Lemma 7.1.34, there exists $r \in [0,s)$ such that for every $y, y' \in R$, $p_{y'|[r,s)} = p_{y|[r,s)}$. Let $r' \in [r,s)$ be such that $\lambda^x(r') = \eta_{0,u}$ (cf. 7.1.1) for some $u \in |k^*|$ and $\mathbf{B}(0,u)$ contains $\phi^{\mathrm{an}}_{D'}(W)$. Since $p_{y_i}(r') = p_{y_j}(r')$ for every $y_i, y_j \in R$ and the paths p_y for $y \in R$ are Galois conjugates of each other, we must have that $q := p_y(r')$ is fixed by H and hence $q = (\phi^{\mathrm{an}}_{D'})^{-1}(\lambda(r', x))$. We simplify notation and use X to denote the closed disk $\mathbf{B}(0,u)$. The group H restricts to an action of the space $Y := (\phi^{\mathrm{an}}_{D'})^{-1}(X)$. We claim that Y is a Berkovich closed disk in D'. Let Y' denote the complement of Y in D'. The image $\phi^{\mathrm{an}}(Y')$ is the complement of X in D as $\phi^{\mathrm{an}}_{D'}$ is surjective. Let $X' := D \smallsetminus X$. We claim that the space Y' is a connected open set. Suppose that Y' is not connected. The morphism $\phi^{\mathrm{an}}_{D'}$ is clopen and hence maps each connected component of Y' onto the complement of X in D. Let Z be a connected component of Y'. Lemmas 7.1.31 and 7.1.32 can be used to show that Z must be of the form $\mathbf{O}(0,1) \smallsetminus \bigcup_{j \in J} X_j$ where the X_j are Berkovich closed disks or points of type I or IV. As the morphism ϕ^{an} restricts to a finite morphism from Z onto X', it can be deduced that there can be only a finite number of points in $\overline{Y'} \smallsetminus Y'$ where $\overline{Y'}$ denotes the closure of Y' in D'. We must hence have that $Y' \in \mathcal{B}$ and is of form 3. As the union of open sets in $\mathcal{B}$ of form 3 is connected, we conclude that Y' is a connected open set in $\mathcal{B}$ of form 3. It must be the complement of the union of a finite number of Berkovich closed disks or points of type I or IV. The complement of Y' is the space Y and $Y = (\phi^{\mathrm{an}}_{D'})^{-1}(X)$. We showed that there exists a point $q \in Y$ which is H-invariant. As $\phi^{\mathrm{an}}_{D'}$ is clopen, every connected component of Y must contain the point q and hence Y is connected. If the union of a finite number of Berkovich closed disks and points of types I or IV in D' is connected then that union must be a Berkovich closed disk as well. Hence Y is a Berkovich closed disk. The morphism $\phi^{\mathrm{an}}_{D'}$ restricts to a finite morphism from Y onto X. As k is algebraically closed, the radius of Y belongs to the group $|k^*|$. This proves the first part of the proposition

2. We make use of the notation introduced in the proof of part 1 of the proposition. Let $W \subset C'^{\mathrm{an}}$ be a connected open set. We must show that $\lambda'^{-1}(W)$ is an open subset of $[0,1] \times C'^{\mathrm{an}}$. We divide the proof into two cases - when $W \cap \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$ is empty and when $W \cap \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$ is

non-empty.

We treat the first case - $W \cap \Sigma(C'^{\mathrm{an}}, \mathfrak{W}') = \emptyset$. As W is connected, there exists $D' \in \mathcal{D}'$ such that $W \subset D'$. By 7.1.29, there exists $D \in \mathcal{D}$ such that $\phi^{\mathrm{an}}(D') = D$. We may suppose further that W belongs to $\mathcal{B}$. It must be of form 1, 2 or 3. Suppose that W is a Berkovich open disk contained in D'. Let $V := \phi^{\mathrm{an}}(W) \subset D$. By Lemma 7.1.32, V is a Berkovich open disk in D. By construction of λ, we must have that there exists $s \in [0,1]$ such that $\lambda^{-1}(V) = [0,s) \times V$. By assumption, λ' is compatible with λ in that the following diagram is commutative.

$$\begin{tikzcd}
{[0,1] \times C'^{\mathrm{an}}} \arrow[r, "\lambda'"'] \arrow[d, "id \times \phi^{\mathrm{an}}"] & C'^{\mathrm{an}} \arrow[d, "\phi^{\mathrm{an}}"] \\
{[0,1] \times C^{\mathrm{an}}} \arrow[r, "\lambda"'] & C^{\mathrm{an}}
\end{tikzcd}$$

.

It follows that $\lambda'^{-1}(W) \subset [0,s) \times (\phi^{\mathrm{an}})^{-1}(V)$. Let $A_1, \ldots, A_m$ denote the connected components of $(\phi^{\mathrm{an}})^{-1}(V)$. Observe that $(\phi^{\mathrm{an}})^{-1}(V) = \bigcup_{\sigma \in G} \sigma(W)$. We suppose without loss of generality that $W \subset A_1$. As A_1 is connected, we must have that $A_1 \subset D'$. We claim that $W = A_1$. By 7.1.29, if $\sigma \in G$ then $\sigma(D') \in \mathcal{D}'$. Let $H := \{\sigma \in G | \sigma(D') = D'\}$. We must have that $A_1 \subset \bigcup_{\sigma \in H} \sigma(W)$ and $A_1 \cap \sigma(W) = \emptyset$ if $\sigma \notin H$. By part (1) of the proposition and Lemma 7.1.33, if $\sigma \in H$ then $\sigma(W)$ is a Berkovich open ball whose radius is equal to that of W. It follows that one of the connected components of $\bigcup_{\sigma \in H} \sigma(W)$ is the ball W. Hence $W = A_1$. Observe that if A_i is a connected component and $x \in A_i$ then for every $t \in [0,s)$, we must have that $\lambda'(t,x) \in A_i$. Indeed, the path $\lambda'^x : [0,s) \to C'^{\mathrm{an}}$ defined by $t \mapsto \lambda'(t,x)$ is contained in $(\phi^{\mathrm{an}})^{-1}(V)$. As $\{A_j\}$ is the set of connected components of $(\phi^{\mathrm{an}})^{-1}(V)$ we must have that $\lambda'(t,x) \in A_i$ for every $t \in [0,s)$. It follows from this observation that $\lambda'^{-1}(W) = [0,s) \times W$.

Let $W \subseteq D$ be a Berkovich open ball and $Y_1, \ldots, Y_m$ be disjoint Berkovich closed sub disks of W or points of type I or IV. Let $Z := \bigcup_{1 \le i \le n} Y_i$. We show that $\lambda'^{-1}(W \smallsetminus Z)$ is an open subset of $[0,1] \times C'^{\mathrm{an}}$. We have already shown that there exists $s \in [0,1]$ such that $\lambda'^{-1}(W) = [0,s) \times W$. Hence $\lambda'^{-1}(W \smallsetminus Z) = ([0,s) \times W) \smallsetminus \lambda'^{-1}(Z)$. As Z is the disjoint union of the Y_i, we must have that $\lambda'^{-1}(Z) = \bigcup_i \lambda'^{-1}(Y_i)$. It suffices hence to show that if Y is a Berkovich closed disk contained in D' then there exists $t \in [0,1]$ such that $\lambda'^{-1}(Y) = [0,t] \times Y$. By Lemma 7.1.32, the image of Y for the morphism $\phi^{\mathrm{an}}_{D'}$ is a Berkovich closed disk or a point of type I or IV. We can use essentially the same argument above wherein we showed that the preimage of a Berkovich open disk O in D' for the function λ' is an open subset of $[0,1] \times C'^{\mathrm{an}}$ of the form $[0,s') \times O$ to show that

$\lambda'^{-1}(Y) = [0,t] \times Y$.

To conclude the proof, we treat the case when $W \subset C'^{\mathrm{an}}$ is a connected open set that intersects $\Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$. Let $S := W \cap \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$ and let $\mathcal{D}'_S := \{D' \in \mathcal{D} | \overline{D'} \cap S \neq \emptyset\}$. As W is connected, for every $D' \in \mathcal{D}'_S$, $W \cap D'$ is a non empty connected open neighborhood which is the union of open sets in $\mathcal{B}$ of form 3. It suffices to prove $\lambda'^{-1}(W)$ is open under the assumption that $W \cap D'$ for $D' \in \mathcal{D}'_S$ belongs to $\mathcal{B}$ and is of form 3. We have the equality $W = \bigcup_{D' \in \mathcal{D}'_S} (W \cap D') \cup (W \cap \Sigma(C'^{\mathrm{an}}, \mathfrak{W}'))$. It follows that $\lambda'^{-1}(W) = \bigcup_{D' \in \mathcal{D}'_S} \lambda'^{-1}(W \cap D') \cup \lambda'^{-1}(\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') \cap W)$. We showed that for every $D' \in \mathcal{D}'_S$, $\lambda'^{-1}(W \cap D')$ is an open set of $[0,1] \times D'$. Furthermore, it can be checked that $[0,1] \times W \subset \lambda'^{-1}(W)$. By construction, $\lambda'^{-1}(\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') \cap W) = ([0,1] \times (\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') \cap W)) \bigcup (\{1\} \times \bigcup_{D' \in \mathcal{D}'_S} D')$. The set $[0,1] \times W$ is an open subset of $[0,1] \times C'^{\mathrm{an}}$ that is contained in $\lambda'^{-1}(W)$ and is a neighborhood of every point of $[0,1] \times (\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') \cap W)$. It remains to show that if $x \in D'$ for some $D' \in \mathcal{D}'_S$ then there exists an open subset of $[0,1] \times C'^{\mathrm{an}}$ contained in $\lambda'^{-1}(W)$ that is a neighborhood of $(1,x)$. We proceed as follows. As $W \cap D'$ is a connected open subset in $\mathcal{B}$ of form 3, it can be verified using the arguments above (case 1) that there exists $r_{W \cap D'} \in [0,1)$ such that $(r_{W \cap D'}, 1] \times D' \subset \lambda'^{-1}(W)$. As $(r_{W \cap D'}, 1] \times D'$ is an open subset of $[0,1] \times C'^{\mathrm{an}}$ that contains $1 \times D'$ we conclude the claim and the proof.

□

7.2 Compatible deformation retractions

Our goal in this section is to prove the existence of a pair of compatible deformation retractions in the case of a finite morphism between curves. The precise statement is the following.

Theorem 7.2.1. *Let C and C' be smooth projective irreducible k-curves and $\phi : C' \to C$ be a finite morphism. There exists a pair of deformation retractions*

$$\psi : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$$

and

$$\psi' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$$

with the following properties.

1. *The images $\Upsilon_{C'^{\mathrm{an}}} := \psi'(1, C'^{\mathrm{an}})$ and $\Upsilon_{C^{\mathrm{an}}} := \psi(1, C^{\mathrm{an}})$ are closed subspaces of C'^{an} and C^{an} respectively, each with the structure of a connected, finite metric graph. Furthermore, we have that $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$.*

2. *There exists weak semistable vertex sets $\mathfrak{A}' \subset C'^{\mathrm{an}}$ and $\mathfrak{A} \subset C^{\mathrm{an}}$ such that $\Upsilon_{C'^{\mathrm{an}}} = \Sigma(C'^{\mathrm{an}}, \mathfrak{A}')$ and $\Upsilon_{C^{\mathrm{an}}} = \Sigma(C^{\mathrm{an}}, \mathfrak{A})$.*

3. *The deformation retractions ψ and ψ' are **compatible** i.e. the following diagram is commutative.*

$$\begin{tikzcd}
{[0,1] \times C'^{\mathrm{an}}} \arrow[r, "\psi'"'] \arrow[d, "id \times \phi^{\mathrm{an}}"] & C'^{\mathrm{an}} \arrow[d, "\phi^{\mathrm{an}}."] \\
{[0,1] \times C^{\mathrm{an}}} \arrow[r, "\psi"'] & C^{\mathrm{an}}
\end{tikzcd}$$

Remark 7.2.2. *In [HL], Hrushovski and Loeser construct compatible deformation retractions in greater generality. Let $\pi : V' \to V$ be a finite morphism between quasi-projective varieties V' and V over a non-Archimedean non-trivially real valued field F. There exists a generalised real interval [[HL], Section 3.9] I and a pair of deformation retractions $H : I \times V^{\mathrm{an}} \to V^{\mathrm{an}}$ and $H' : I \times V'^{\mathrm{an}} \to V'^{\mathrm{an}}$ which are compatible in the sense defined above. This result follows from Remark 11.1.3 (2) and Corollary 14.1.6 in loc.cit.*

The proof of 7.2.1, though involved, is elementary in that our techniques are essentially those developed in [B]. We do not make use of the results of Hrushovski-Loeser in this section. Even though verifying the continuity of the homotopies defined is a painstaking process, it is advantageous that the construction of the compatible pair is explicit. Our strategy is inspired by the treatment of deformation retractions for curves in [B] and also Chapter 7 of [HL].

To prove Theorem 7.2.1, we adapt the strategy employed in Section 7 of [HL]. Our method of proof is as follows. We begin by proving the theorem for a finite morphism $\phi : C \to \mathbb{P}^1_k$ where C is a smooth, projective curve and the extension of function fields $k(\mathbb{P}^1_k) \hookrightarrow k(C)$ induced by the morphism ϕ is Galois. We then use this result to prove the theorem for a finite morphism $\phi : C' \to C$ between smooth projective curves. We begin with the following lemma which provides us compatible weak semistable vertex sets for a finite morphism between smooth projective irreducible curves.

Lemma 7.2.3. *Let $\phi : C' \to C$ be a finite morphism between smooth projective irreducible curves. Let $S \subset C^{\mathrm{an}}$ be a finite set of points none of which are of type IV. There exists a weak semistable vertex set $\mathfrak{W} \subset C^{\mathrm{an}}$ such that $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$ contains S, $\mathfrak{V} := (\phi^{\mathrm{an}})^{-1}(\mathfrak{W})$ is a weak semistable vertex set of C'^{an} and $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}) = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}))$.*

Proof. Let $\mathfrak{W}_0$ be a weak semistable vertex set for C^{an} such that $\Sigma(C^{\mathrm{an}}, \mathfrak{W}_0)$ contains S. As the morphism ϕ^{an} is finite, the preimage $(\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}_0))$ is a finite graph which does not contain a point of type IV (Proposition 7.1.26). Let $\mathfrak{V}_0$ be a weak semistable vertex set such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}_0)$ contains $(\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}_0))$. Let $\mathfrak{W}_1$ be a weak semistable vertex set such that the skeleton $\Sigma(C^{\mathrm{an}}, \mathfrak{W}_1)$ contains $\phi^{\mathrm{an}}(\Sigma(C'^{\mathrm{an}}, \mathfrak{V}_0))$. We claim that the preimage $A := (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}_1))$ is connected. Let $A_1, \ldots, A_m$ denote the connected components of A such that A_1 contains $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}_0)$. The morphism ϕ^{an} is an open and closed morphism (cf. Lemma 6.1.4). It follows that ϕ^{an} restricts to a surjective map from each of the

A_i onto $\Sigma(C^{\mathrm{an}}, \mathfrak{W}_1)$. However since A_1 contains the set $(\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}_0))$ we must have that $A = A_1$. It follows that A is a connected graph that contains the skeleton $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}_0)$, using which it can be checked that $C'^{\mathrm{an}} \smallsetminus A$ is the disjoint union of sets each of which are isomorphic to Berkovich open balls. We claim that these open balls have radii belonging to $|k^*|$. Let D' be a connected component of $C'^{\mathrm{an}} \smallsetminus A$. As the morphism ϕ^{an} is clopen and $A = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}_1))$, there exists a connected component D of $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{W}_1)$ such that the morphism ϕ^{an} restricts to a finite morphism from D' onto D. As D is isomorphic to a Berkovich open ball of radius belonging to $|k^*|$, we must have that D' is a Berkovich open ball whose radius belongs to $|k^*|$. It follows that there exists a weak semistable vertex set $\mathfrak{V}_1$ in C'^{an} such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}_1) = A$. The set $\mathfrak{W} := \mathfrak{W}_1 \cup \phi^{\mathrm{an}}(\mathfrak{V}_1)$ is a weak semistable vertex set for C^{an} and $\Sigma(C^{\mathrm{an}}, \mathfrak{W}) = \Sigma(C^{\mathrm{an}}, \mathfrak{W}_1)$. Let $\mathfrak{V} := (\phi^{\mathrm{an}})^{-1}(\mathfrak{W})$. As $\mathfrak{V}$ contains $\mathfrak{V}_1$ and is contained in $\Sigma(C'^{\mathrm{an}}, \mathfrak{V})$, we must have that $\mathfrak{V}$ is a weak semistable vertex set and $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}) = \Sigma(C'^{\mathrm{an}}, \mathfrak{V}_1)$. The pair $\mathfrak{V}, \mathfrak{W}$ satisfy the claims made in the lemma.

□

7.2.1 Lifting paths

Let $\phi : C' \to C$ be a finite morphism between k-curves. A path in C^{an} is a continuous function $u : [a, b] \to C^{\mathrm{an}}$ where $[a, b]$ is a real interval. To construct deformation retractions which are compatible for the morphism ϕ, we must understand to what extent certain paths on C^{an} can be lifted. By a lift of a path, we mean the following.

Definition 7.2.4. Let $a < b$ be real numbers and $u : [a, b] \to C^{\mathrm{an}}$ be a continuous function. *A lift of the path* u is a path $u' : [a, b] \to C'^{\mathrm{an}}$ such that $u = \phi^{\mathrm{an}} \circ u'$.

Lemma 7.2.5. *Let $\phi : C' \to C$ be a finite separable morphism between irreducible projective smooth k-curves such that the extension of function fields induced by ϕ is separable. Let $\mathfrak{V} \subset \mathfrak{W}$ be weak semistable vertex sets of C^{an}. Assume that $\mathfrak{W}$ contains the set of k-points of C over which the morphism ϕ is ramified. Let $\mathfrak{W}' \subset C'^{\mathrm{an}}$ be a weak semistable vertex set such that $\Sigma(C'^{\mathrm{an}}, \mathfrak{W}') = (\phi^{\mathrm{an}})^{-1}(\Sigma(C^{\mathrm{an}}, \mathfrak{W}))$. Let $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}}, \mathfrak{V})} : [0, 1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ be the deformation retraction constructed in Proposition 7.1.20 whose image is $\Sigma(C^{\mathrm{an}}, \mathfrak{W})$. We simplify notation and write λ in place of $\lambda^{\mathfrak{W}}_{\Sigma(C^{\mathrm{an}}, \mathfrak{V})}$. Let $p \in C^{\mathrm{an}}$. Let $\lambda^p : [0, 1] \to C^{\mathrm{an}}$ be the path defined by $t \mapsto \lambda(t, p)$. Let $q \in (\phi^{\mathrm{an}})^{-1}(p)$. There exists a unique path $u : [0, 1] \to C'^{\mathrm{an}}$ such that $u(0) = q$ and $\phi^{\mathrm{an}} \circ u = \lambda^p$.*

Proof. We split the proof into two cases.

1. *Let $p \in C^{\mathrm{an}}$ be a point which is not of type IV.* We can suppose that $p \notin \Sigma(C^{\mathrm{an}}, \mathfrak{W})$ since when $p \in \Sigma(C^{\mathrm{an}}, \mathfrak{W})$, the path λ^p is constant and hence can always be lifted. We show firstly that for every $t \in [0, 1)$, there exists $\epsilon > 0$ such that if $z' \in (\phi^{\mathrm{an}})^{-1}(\lambda^p(t))$ then $\lambda^p_{|[t,t+\epsilon]}$ lifts *uniquely* to a path starting from z'.

   Suppose $z := \lambda^p(t)$ was a point of type I. By construction of λ, we must have that $z = p$ and $t = 0$. Our choice of weak semistable vertex sets

implies that ϕ is étale over $\lambda^p(t)$. It follows from Hensel's lemma that there exists neighborhoods $V_{z'}$ in C'^{an} around z' and V_p around p such that ϕ^{an} restricts to a homeomorphism from $V_{z'}$ onto V_p. We conclude from this fact that there does indeed exist $\epsilon > 0$ such that $\lambda^p_{|[0,\epsilon]}$ lifts uniquely to a path starting from z'.

Let $z := \lambda^p(t)$ be a point of type II or III. By construction, for every $s \in [t,1]$, $\lambda^p(s) = \lambda^z(s)$ where $\lambda^z : [0,1] \to C^{\mathrm{an}}$ is the path defined by $s \mapsto \lambda(s,z)$. Furthermore, for every $s \in [0,t]$, $\lambda^z(s) = z$. It suffices to show that there exists $\epsilon > 0$ such that the path $\lambda^z_{|[t,t+\epsilon]}$ lifts uniquely to a path starting from z'. If there exists $\epsilon > 0$ such that $\lambda^z_{|[t,t+\epsilon]}$ is constant then our claim is obviously true. Let us hence suppose no such ϵ exists.

By assumption, $z \notin \Sigma(C^{\mathrm{an}}, \mathfrak{W})$. It follows that $z' \in C'^{\mathrm{an}} \smallsetminus \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$. Recall that we used $\mathcal{D}$ to denote the set of connected components of the space $C^{\mathrm{an}} \smallsetminus \Sigma(C^{\mathrm{an}}, \mathfrak{W})$ and when constructing the deformation λ we identified each $D \in \mathcal{D}$ with a Berkovich open ball whose radius belongs to the value group $|k^*|$. Likewise, let $\mathcal{D}'$ denote the set of connected components of the space $C'^{\mathrm{an}} \smallsetminus \Sigma(C'^{\mathrm{an}}, \mathfrak{W}')$. We identify each $D' \in \mathcal{D}'$ with a Berkovich open ball of unit radius. Let $D \in \mathcal{D}$ be such that $z \in D$ and $D' \in \mathcal{D}'$ such that $z' \in D'$. By Lemma 7.1.29, we have that $\phi^{\mathrm{an}}(D') = D$ and in addition ϕ^{an} restricts to an open and closed map on D'. By construction of the deformation retraction λ, there exists $\beta \in [0,1]$ such that for every $x \in D$, $\lambda(s,x) \in D$ when $s \in [0,\beta)$ i.e. $\lambda : [0,\beta) \times D \to D$ is well defined and $\lambda(s,x) = \lambda(\beta,x)$ when $s \in [\beta,1]$. Our assumption that there does not exist $\epsilon > 0$ such that $\lambda^z_{|[t,t+\epsilon]}$ is constant and the construction of λ imply that the path $\lambda^z_{|[t,\beta]}$ is injective and that $\lambda^z([t,\beta]) \subset \mathbf{H}(C^{\mathrm{an}})$. Furthermore, the composition $\lambda^z \circ (-\exp) : [-\log(\beta), -\log(t)] \to C^{\mathrm{an}}$ is an isometry (cf. Remark 7.1.21).

Let $r \in |k^*|$ denote the radius of the ball D. The point z must be of the form $\eta_{a,t}$ for some $a \in D$. Likewise, the point $z' \in D'$ must be of the form $\eta_{b,t'}$ for some $b \in D'$ and $t' \in (0,1)$. We may choose b so that $\phi^{\mathrm{an}}(b) = a$. We show now that we can reduce to the case when $a = b = 0$. The translation automorphism $t_{-a} : O(0,r) \to O(0,r)$ defined by $x \mapsto x - a$ induces an automorphism $t^{\mathrm{an}}_{-a} : D \to D$ that maps the point z to $\eta_{0,t}$. We have that $\lambda : [0,\beta) \times D \to D$ is well defined and it can be checked that λ is t^{an}_{-a} invariant i.e. for every $s \in [0,\beta)$ and $x \in D$, $t^{\mathrm{an}}_{-a}(\lambda(s,x)) = \lambda(s, t^{\mathrm{an}}_{-a}(x))$. Similarly, let $t_{-b} : O(0,1) \to O(0,1)$ denote the translation morphism $y \mapsto y - b$. The map t_{-b} induces an automorphism $t^{\mathrm{an}}_{-b} : D' \to D'$ that maps $z' = \eta_{b,t'}$ to $\eta_{0,t'}$. Let $\phi^{\mathrm{an}}_{D'}$ denote the restriction of ϕ^{an} to D'. As t^{an}_{-a} and t^{an}_{-b} are automorphisms, there exists a morphism $f : D' \to D$ such that the following diagram commutes.

$$\begin{tikzcd} D' \arrow[r, "t^{\mathrm{an}}_{-b}"'] \arrow[d, "\phi^{\mathrm{an}}_{D'}"'] & D' \arrow[d, "f"] \\ D \arrow[r, "t^{\mathrm{an}}_{-a}"'] & D \end{tikzcd}$$

Suppose there exists $\epsilon > 0$ and a unique path $u' : [t, t+\epsilon]$ starting from $\eta_{0,t'}$ such that $f \circ u' = \lambda^{\eta_{0,t}}_{|[t,t+\epsilon]}$. The commutativity of the above diagram and the fact that t^{an}_{-a} and t^{an}_{-b} are automorphisms imply that there exists a unique path $u : [t, t+\epsilon]$ starting at z' such that $\phi^{\mathrm{an}} \circ u = \lambda^{z}_{|[t,t+\epsilon]}$. We may hence assume that $z = \eta_{0,t}$ and $z' = \eta_{0,t'}$ and that $\phi^{\mathrm{an}}(0) = 0$.

Let $F \subset D'$ denote the set $(\phi^{\mathrm{an}})^{-1}(0)$. Let $\mathfrak{A} \subset D'$ be a finite set of points of type II with the following property. Let $\mathcal{C}$ denote the set of connected components of $D' \smallsetminus (\mathfrak{A} \cup F)$. There exists a finite set $\Upsilon \subset \mathcal{C}$ such that if $A \in \Upsilon$ then A is isomorphic to a standard open annulus or a standard punctured Berkovich open disk and if $A \in \mathcal{C} \smallsetminus \Upsilon$ then A is isomorphic to Berkovich open ball with radius in $|k^*|$. Let $\Sigma(D')$ be the union of $\mathfrak{A} \cup F$ and the skeleton of every element $A \in \Upsilon$. By assumption, we must have that $z' \in \Sigma(D')$. Suppose, z' is not a vertex of the finite graph $\Sigma(D')$. It follows that there exists a standard open annulus $A' \in \Upsilon$ that contains z and in particular does not intersect F. The map ϕ^{an} restricts to a morphism $A' \to D \smallsetminus \{0\}$. By [[BPR], Proposition 2.5], $\phi^{\mathrm{an}}(A') \subset D \smallsetminus \{0\}$ must be a standard open annulus A as well and $\phi^{\mathrm{an}}(\Sigma(A')) = \Sigma(A)$. By assumption, we have that $z \in \Sigma(A)$ and for small enough $\epsilon > 0$, the path $\lambda^{z}_{|[t,t+\epsilon]}$ is contained in $\Sigma(A)$. Recall that we defined sections $\sigma : \mathrm{trop}(A) \to \Sigma(A)$ and $\sigma : \mathrm{trop}(A') \to \Sigma(A')$ of the tropicalization maps $\mathrm{trop} : \Sigma(A) \to \mathrm{trop}(A)$ and $\mathrm{trop} : \Sigma(A') \to \mathrm{trop}(A')$ (cf. 7.1.2). By definition of the tropicalization map, we must have that $[-\log(t), -\log(t+\epsilon)] \subset \mathrm{trop}(A)$. By construction, $\lambda^{z} \circ -\exp : [-\log(t), -\log(t+\epsilon)] \to \Sigma(A)$ coincides with $\sigma_{|[-\log(t),-\log(t+\epsilon)]}$. As σ is a homeomorphism, the morphism $\phi^{\mathrm{an}}_{|\Sigma(A')} : \Sigma(A') \to \Sigma(A)$ induces a map $\phi^{\mathrm{trop}} : \mathrm{trop}(A') \to \mathrm{trop}(A)$. By loc.cit, there exists a non zero $d \in \mathbb{Z}$ such that ϕ^{trop} is of the form $d(.) + -\log(|\delta|)$ for some $\delta \in k^*$. Let $(\phi^{\mathrm{trop}})^{-1}$ denote the inverse of ϕ^{trop}. It can be verified that $u := \sigma \circ (\phi^{\mathrm{trop}})^{-1} \circ -\log : [t, t+\epsilon] \to D'$ is a lift of $\lambda^{z}_{|[t,t+\epsilon]}$ starting from z'. In fact, u is the unique lift of $\lambda^{z}_{|[t,t+\epsilon]}$ starting from z'. Indeed, by Proposition 2.5 in loc.cit., it can be deduced that $\phi^{\mathrm{an}}(A' \smallsetminus \Sigma(A')) \subset A \smallsetminus \Sigma(A)$ and furthermore, the map ϕ^{an} restricted to $\Sigma(A')$ is a bijection onto $\Sigma(A)$. As $\lambda^{z}_{|[t,t+\epsilon]}$ is a path along $\Sigma(A)$, u must be a lift along $\Sigma(A')$ and hence unique.

Suppose $z' = \eta_{0,t'} \in \Sigma(D')$ is a vertex. We must have that z' is a type II point. It follows that there exists a standard open annulus $A' \in \Upsilon$ with inner radius t' and outer radius belonging to $|k^*|$. By [BPR], the image

$\phi^{\mathrm{an}}(A')$ is a standard open annulus $A \subset D$ and we choose $\epsilon > 0$ small enough so that $\lambda^z_{|(t,t+\epsilon]}$ is a path contained in $\Sigma(A)$. We now proceed as above to obtain a lift u of $\lambda^z_{|(t,t+\epsilon]}$ starting from z' and for reasons mentioned above it is the unique lift contained in $\Sigma(A)$. It remains to show that u is the unique lift in C'^{an} and this requires an additional argument for the following reason. Observe that the annulus A contains exactly one element of the tangent space $T_{z'}$, whereas previously when z' was in the interior of A, every tangent direction was contained in A. It follows that although u might be the unique lift in A, it might not be the only lift in C'^{an}. However, it is the only possible lift since by Lemma 7.1.32, the Berkovich closed ball $\mathbf{B}(0,t')$ maps onto the closed ball $\mathbf{B}(0,t)$ and the path $\lambda^z_{|(t,t+\epsilon]}$ is contained in the complement of the $\mathbf{B}(0,t)$. Hence any lift of the $\lambda^z_{|(t,t+\epsilon]}$ must be contained entirely in A. This proves that the lift u is unique.

Let L denote the set of $s \in [0,1]$ for which there exists a lift of the path $\lambda^p_{|[0,s]}$ starting from q. Let $s \in L$. We claim that there exists a unique lift of the path $\lambda^p_{|[0,s]}$ starting from q. Let u', u be two lifts of the path $\lambda^p_{|[0,s]}$ starting from q. Let $s_0 \in [0,s]$ be the largest real number such that $u'_{|[0,s_0]} = u_{|[0,s_0]}$. Let $q' := u'(s_0) = u(s_0)$ and $p' := \phi^{\mathrm{an}}$. By the first part of the proof, there exists $\epsilon > 0$ and a unique lift v of the path $\lambda^p_{|[s_0,s_0+\epsilon]}$. As $u'_{|[s_0,s_0+\epsilon]}$ and $u_{|[s_0,s_0+\epsilon]}$ are both lifts of $\lambda^p_{|[s_0,s_0+\epsilon]}$, we must have that $v = u_{|[s_0,s_0+\epsilon]} = u'_{|[s_0,s_0+\epsilon]}$. This implies that $u_{|[0,s_0+\epsilon]} = u'_{|[0,s_0+\epsilon]}$ which contradicts our assumption on s_0. We have thus verified our claim. It can be deduced from this that the set L contains its supremum. Let t be the largest real number in $[0,1]$ such that the path $\lambda^p_{|[0,t]}$ lifts to a path $u : [0,t] \to C'^{\mathrm{an}}$ starting from q. Suppose $t < 1$. By the first half of the proof, there exists a small real number ϵ such that $\lambda^p_{|[t,t+\epsilon]}$ lifts to a path starting from $u' : [t, t+\epsilon] \to C'^{\mathrm{an}}$ starting from $u(t)$. Glueing u and u' gives a path $u'' : [0, t+\epsilon] \to C'^{\mathrm{an}}$ which lifts $\lambda^p_{|[0,t+\epsilon]}$. Hence we have a contradiction to our assumption that $t < 1$. Therefore there exists a unique lift of the path λ^p.

2. *Let p be a point of type IV.* We must have that $p \notin \Sigma(C^{\mathrm{an}}, \mathfrak{W})$. We make use of the notation introduced in part (1) of the proof. There exists $D \in \mathcal{D}$ such that $p \in D$. The path λ^p is injective on $[a,b]$. Let U be a connected neighborhood in D of the point p such that $(\phi^{\mathrm{an}})^{-1}(U)$ decomposes into the disjoint union of connected open sets $\{U_1, \ldots, U_m\}$ and each U_i contains exactly one preimage of the point p. We claim that we can shrink U and choose $a < t'_1 < b$ such that $\lambda^p([0,t'_1]) \subset U$ and for every $x \in \lambda^p([0,t'_1])$ there exists exactly one preimage of x in each of the U_i. This can be accomplished as follows. We show that there exists $t' \in (a,b)$ such that for every element $x \in \lambda^p([a,t'))$ the cardinality of the set $(\phi^{\mathrm{an}})^{-1}(x)$ is constant. We can then shrink U so that it does not intersect $\lambda^p([t',b])$ and choose $t'_1 \in (a,t')$ suitably small. Such a U must satisfy the claim since the morphism ϕ^{an} being closed and open (cf. Lemma 6.1.4) is surjective from each of the U_i onto U. Observe that if $t_1, t_2 \in (a,s]$ are such that $t_1 < t_2$ then the number of preimages of $\lambda^p(t_1)$ is greater than or equal to the number of preimages of $\lambda^p(t_2)$. This follows

from the uniqueness of lifts of part (1) and that if P is a lift of the path $\lambda^p_{|[t_1,s]}$ then $P_{|[t_2,s]}$ is a lift of the path $\lambda^p_{|[t_2,s]}$. As the morphism ϕ^{an} is finite, there exists $t' \in (a,b)$ such that the number of preimages of every point $x \in \lambda^p((a,t'))$ is constant. The preimages in C'^{an} of the point p are of type IV and the tangent space at any such point is a single element. It follows that the number of preimages of every point in $\lambda^p([a,t'))$ is a constant. Let $t'_1 \in (a,t')$ be such that $\lambda^p(t') \in U$. This verifies the claim. We suppose without loss of generality that $q \in U_1$. We show firstly that the path $\lambda^p_{[0,t'_1]}$ can be lifted to a path in C^{an} starting from q. It suffices to show that the path $\lambda^p_{[a,t'_1]}$ can be lifted to a path in C^{an} starting from q. Let I denote the set of real numbers $r \in [a,t'_1]$ for which there exists a lift P_r of the path $\lambda^p_{|[r,t'_1]}$ contained in U_1. As U_1 contains exactly one preimage of the point $\lambda^p(r)$, the uniqueness of lifts from part (1) of the proposition implies that the set I is closed. Let t_0 denote the smallest element of the set I. Suppose $t_0 > a$. Let $a' \in (a,t_0)$ and $p' := \lambda^p(a')$. By construction, p' is not a point of type IV. Let q' be the unique preimage of p' in U_1. By part (1), there exists a lift P' of the path $\lambda^{p'}_{|[a',t'_1]}$ starting from q'. By construction, $\lambda^{p'}_{|[t_0,t'_1]}$ coincides with $\lambda^p_{|[t_0,t'_1]}$. As the lifts are unique, we must have that $P'_{|[t_0,t'_1]} = P$. This is a contradiction to our assumption that $t_0 > a$. It follows that there exists a lift of the path $\lambda^p_{|[a,t'_1]}$ in U_1 which in turn implies that there exists a lift of the path $\lambda^p_{|[0,t'_1]}$ in U_1 as $\lambda^p_{|[0,a]}$ is constant. We abuse notation and refer to this lift as P' as well. Let $p'' := \lambda^p(t'_1)$. By construction, $\lambda^{p''}_{|[t'_1,1]}$ coincides with the path $\lambda^p_{|[t'_1,1]}$. Let $q'' := P'(t'_1)$. By part (1), there exists a lift P'' of the path $\lambda^{p''}_{|[t'_1,1]}$ starting from q''. Glueing the paths P' and P'' results in a lift of the path λ^p starting from q.

□

7.2.2 Finite morphisms to $\mathbb{P}^1_k$

Let C be a smooth projective irreducible k-curve. Let $\phi : C \to \mathbb{P}^1_k$ be a finite morphism such that the extension of function fields $k(\mathbb{P}^1_k) \hookrightarrow k(C)$ induced by ϕ is separable. Let R be the finite set of k-points of $\mathbb{P}^1_k$ over which the morphism ϕ is ramified. Let $\mathfrak{W}$ be a weak semistable vertex set that contains R such that $\mathfrak{V} := (\phi^{\mathrm{an}})^{-1}(\mathfrak{W})$ is a weak semistable vertex set for C^{an} and $\Sigma(C^{\mathrm{an}}, \mathfrak{V}) = (\phi^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{W}))$. By Proposition 7.1.20, there exists a deformation retraction

$$\lambda_{\Sigma(\mathbb{P}^{1,\mathrm{an}}_k,\mathfrak{W})} : [0,1] \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

whose image is the skeleton $\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{W})$. Recall that for a point $p \in \mathbb{P}^{1,\mathrm{an}}_k$, the deformation retraction $\lambda_{\Sigma(\mathbb{P}^{1,\mathrm{an}}_k,\mathfrak{W})}$ defines a path $\lambda^p_{\Sigma(\mathbb{P}^{1,\mathrm{an}}_k,\mathfrak{W})} : [0,1] \to \mathbb{P}^{1,\mathrm{an}}_k$ by $t \mapsto \lambda_{\Sigma(\mathbb{P}^{1,\mathrm{an}}_k,\mathfrak{W})}(t,p)$. We are now in a position to prove Theorem 7.2.1 for the morphism $\phi : C \to \mathbb{P}^1_k$. We suppose in addition that the extension of function fields $k(\mathbb{P}^1_k) \hookrightarrow k(C)$ induced by ϕ is Galois.

Proposition 7.2.6. *Let C be a smooth projective irreducible curve and let $\phi : C \to \mathbb{P}^1_k$ be a finite morphism such that the extension of function fields $k(\mathbb{P}^1_k) \hookrightarrow k(C)$ is Galois. Let $\mathfrak{W}$ be a weak semistable vertex set for $\mathbb{P}^{1,\mathrm{an}}_k$ that contains the closed points over which the morphism is ramified such that $\mathfrak{V} := (\phi^{\mathrm{an}})^{-1}(\mathfrak{W})$ is a weak semistable vertex set for C^{an} and $\Sigma(C^{\mathrm{an}}, \mathfrak{V}) = (\phi^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{W}))$. There exists a pair of compatible deformation retractions $\psi' : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ and $\psi : [0,1] \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$ whose images are the connected finite graphs $\Sigma(C^{\mathrm{an}}, \mathfrak{V})$ and $\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{W})$ respectively.*

Proof. Let $\psi := \lambda_{\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{W})}$. We define the deformation retraction $\psi' : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ as follows. Let $q' \in C^{\mathrm{an}}$ and $q := \phi^{\mathrm{an}}(q')$. By Lemma 7.2.5, there exists a unique lift $\psi'^{q'}$ of the path ψ^q starting at q'. For $t \in [0,1]$ and $q' \in C'^{\mathrm{an}}$, we set $\psi'(t, q') := \psi'^{q'}(t)$. The uniqueness of the lifts $\psi'^{q'}$ imply that ψ' is well defined. Let $G = \mathrm{Gal}(k(C)/k(\mathbb{P}^1_k))$. The uniqueness of the lift implies that for every $g \in G$, $g \circ \psi'^{q'} = \psi'^{g(q')}$. It follows that for every $t \in [0,1]$, $g(\psi'(t, q')) = \psi'(t, g(q'))$. The compatibility of ψ' and ψ implies that $\psi'(1, C^{\mathrm{an}})$ is equal to $(\phi^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{W})) = \Sigma(C^{\mathrm{an}}, \mathfrak{V})$. The continuity of ψ' follows from 7.1.35. □

We show that Theorem 7.2.1 can be deduced from Proposition 7.2.6.

Proof. Let $\phi : C' \to C$ be a finite morphism between smooth projective irreducible k-curves. It suffices to prove the theorem when the extension of function fields $k(C) \hookrightarrow k(C')$ induced by the morphism ϕ is separable. Indeed, the extension $k(C) \hookrightarrow k(C')$ can be decomposed into a separable field extension $k(C) \hookrightarrow L$ and a purely inseparable extension $L \hookrightarrow k(C')$. Let C'' denote the smooth projective irreducible k-curve that corresponds to the function field L. The corresponding morphism of curves $C' \to C''$ and its analytification $C'^{\mathrm{an}} \to C''^{\mathrm{an}}$ are homeomorphisms. If $\mathfrak{V}''$ is a weak semistable vertex set for C''^{an} then its preimage $\mathfrak{V}'$ in C'^{an} is a weak semistable vertex set as well and a deformation retraction of C''^{an} with image $\Sigma(C''^{\mathrm{an}}, \mathfrak{V}'')$ lifts to a deformation retraction on C'^{an} with image $\Sigma(C'^{\mathrm{an}}, \mathfrak{V}')$.

Let $a : C \to \mathbb{P}^1_k$ be a finite separable morphism and let K be a finite Galois extension of $k(\mathbb{P}^1_k)$ that contains $k(C')$. Let C'' denote the smooth projective irreducible curve corresponding to the function field K. By construction we have the following sequence of morphisms : $C'' \to C' \to C \to \mathbb{P}^1_k$. Let $c : C'' \to \mathbb{P}^1_k$ denote this composition. Using Lemma 7.38, it can be checked that there exists a weak semistable vertex set $\mathfrak{A}$ for $\mathbb{P}^{1,\mathrm{an}}_k$ that contains the points over which the morphism $c : C'' \to \mathbb{P}^1_k$ is ramified and in addition that $(a^{\mathrm{an}})^{-1}(\mathfrak{A})$ is a weak semistable vertex set of C^{an}, $(a \circ \phi)^{\mathrm{an}-1}(\mathfrak{A})$ is a weak semistable vertex set of C'^{an} and $(c^{\mathrm{an}})^{-1}(\mathfrak{A}) = \mathfrak{A}''$ is a weak semistable vertex set for C''^{an}. Furthermore, $\Sigma(C^{\mathrm{an}}, (a^{\mathrm{an}})^{-1}(\mathfrak{A})) = (a^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A}))$, $\Sigma(C'^{\mathrm{an}}, (a \circ \phi)^{\mathrm{an}-1}(\mathfrak{A})) = (a \circ \phi)^{\mathrm{an}-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A}))$ and $\Sigma(C''^{\mathrm{an}}, \mathfrak{A}'') = (c^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A}))$.

The deformation retraction $\lambda_{\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A})}$ has image $\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A})$ and lifts to a deformation retraction $\lambda''^{\mathfrak{A}''}$ with image $\Sigma(C''^{\mathrm{an}}, \mathfrak{A}'')$. Let $G := \mathrm{Gal}(k(C'')/k(\mathbb{P}^1_k))$. The deformation retraction $\lambda''^{\mathfrak{A}''}$ is G-invariant. There exists sub groups $H_{C'} \subset G$ and $H_C \subset G$ such that $C'' \to C'$ and $C'' \to C$ are the quotient morphisms $C'' \to C''/H_{C'}$ and $C'' \to C''/H_C$. As $\lambda''^{\mathfrak{A}''}$ is H_C and $H_{C'}$ invariant, it must induce deformations λ_C and $\lambda_{C'}$ whose images are $(a^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A}))$ and $((a \circ \phi)^{\mathrm{an}})^{-1}(\Sigma(\mathbb{P}^{1,\mathrm{an}}_k, \mathfrak{A}))$ respectively. This proves the theorem. □

7.3 Calculating the genera $g^{\mathrm{an}}(C')$ and $g^{\mathrm{an}}(C)$

In the previous sections we showed that given a k-curve, there exists a deformation retraction of the curve onto a closed subspace which is a finite metric graph. We called such subspaces skeleta. In Definition 7.1.25, we introduced the genus of a skeleton and by Proposition 7.1.24 it is independent of the weak semistable vertex sets that define it, implying that it is in fact an invariant of the curve. In what follows we study how these invariants relate to each other given a finite morphism between the spaces they are associated to.

The theorem that follows is analogous to the Riemann-Hurwitz formula in algebraic geometry. We introduce the notation involved in the statement of 7.3.1. Let $\phi : C' \to C$ be a finite separable morphism between smooth projective curves over the field k.

7.3.1 Notation

We define the genus of a point $p \in C^{\mathrm{an}}$ as follows. If $p \in C^{\mathrm{an}}$ is of type II then let g_p denote the genus of the smooth projective curve $\tilde{C}_p$ which corresponds to the $\tilde{k}$-function field $\widetilde{\mathcal{H}(p)}$ and if $p \in C^{\mathrm{an}}$ is not of type II then we set $g_p = 0$.

Let $p' \in C'^{\mathrm{an}}$ which is of type II and let $p := \phi^{\mathrm{an}}(p')$. The $\tilde{k}$-function fields $\widetilde{\mathcal{H}(p')}$, $\widetilde{\mathcal{H}(p)}$ define smooth projective $\tilde{k}$-curves $\tilde{C}'_{p'}$, $\tilde{C}_p$ respectively. The morphism ϕ^{an} induces an injection $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ which implies a morphism $\tilde{C'}_{p'} \to \tilde{C}_p$. This morphism is not necessarily separable. The extension $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ can be decomposed so that there exists an intermediate $\tilde{k}$-function field $\widetilde{I(p',p)}$ and the extension $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{I(p',p)}$ is purely inseparable while $\widetilde{I(p',p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ is separable of degree $s(p',p)$. Such a decomposition exists by [MOU]. Let $\tilde{C}_{p',p}$ denote the smooth projective $\tilde{k}$-curve which corresponds to the field $\widetilde{I(p',p)}$. By construction, the genus of the curve $\tilde{C}_{p',p}$ is equal to g_p.

The finite separable morphism $\tilde{C'}_{p'} \to \tilde{C}_{p',p}$ can be used to relate the genera of the two curves via the Riemann-Hurwitz formula. As in [[H], IV.2], let

$$R_{p',p} := \Sigma_{P \in \tilde{C'}_{p'}} \mathrm{length}(\Omega_{\tilde{C'}_{p'}/\tilde{C}_{p',p}})_P.P$$

and

$$R := \Sigma_{P \in C'} \mathrm{length}(\Omega_{C'/C})_P.P.$$

We define invariants on the points of C^{an} which relate the values $g^{\mathrm{an}}(C'), g^{\mathrm{an}}(C)$ from Definition 7.1.25. For $p \in C^{\mathrm{an}}$ of type II, let

$$\begin{aligned} s(p) &:= \Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} s(p',p), \\ R^1_{p',p} &:= \deg(R_{p',p}) - (2s(p',p) - 2), \\ R^1_p &:= \Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} R^1_{p',p}. \end{aligned}$$

When p is not of type II, let $s(p)$ be the cardinality of the fibre $(\phi^{\mathrm{an}})^{-1}(p)$ and $R^1_p := 0$.

7.3.2 A Riemann-Hurwitz formula for the analytic genus

Theorem 7.3.1. *Let $\phi : C' \to C$ be a finite separable morphism between smooth projective curves over the field k. Let $g^{\mathrm{an}}(C'), g^{\mathrm{an}}(C)$ be as in Definition 7.1.25. We have the following equation.*

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)(2g^{\mathrm{an}}(C) - 2) + \Sigma_{p \in C^{\mathrm{an}}} 2s(p)g_p + \deg(R) - \Sigma_{p \in C^{\mathrm{an}}} R^1_p.$$

Proof. In order to prove Theorem 7.3.1, we make use of the fact that there exists a pair of deformation retractions $\psi' : [0,1] \times C'^{\mathrm{an}} \to C'^{\mathrm{an}}$ and $\psi : [0,1] \times C^{\mathrm{an}} \to C^{\mathrm{an}}$ which are compatible with the morphism ϕ^{an} (Theorem 7.2.1). Let $\Upsilon_{C'^{\mathrm{an}}}$ and $\Upsilon_{C^{\mathrm{an}}}$ denote the images of the deformation retractions ψ' and ψ respectively. We can assume that $\Upsilon_{C^{\mathrm{an}}}$ contains the ramification locus of the morphism ϕ. Furthermore, there exists weak semistable vertex sets $\mathfrak{A} \subset C^{\mathrm{an}}$ and $\mathfrak{A}' \subset C'^{\mathrm{an}}$ such that $\Upsilon_{C^{\mathrm{an}}} = \Sigma(C^{\mathrm{an}}, \mathfrak{A})$ and $\Upsilon_{C'^{\mathrm{an}}} = \Sigma(C'^{\mathrm{an}}, \mathfrak{A}')$.

We identify a set of vertices $V(\Upsilon_{C^{\mathrm{an}}}), V(\Upsilon_{C'^{\mathrm{an}}})$ for the skeleta $\Upsilon_{C^{\mathrm{an}}}$ and $\Upsilon_{C'^{\mathrm{an}}}$ which satisfy the following conditions.

1. $V(\Upsilon_{C'^{\mathrm{an}}}) = (\phi^{\mathrm{an}})^{-1}(V(\Upsilon_{C^{\mathrm{an}}}))$.
2. $\mathfrak{A} \subset V(\Upsilon_{C^{\mathrm{an}}})$ and $\mathfrak{A}' \subset V(\Upsilon_{C'^{\mathrm{an}}})$.
3. If p (resp. p') is a point on the skeleton $\Upsilon_{C^{\mathrm{an}}}$ (resp. $\Upsilon_{C'^{\mathrm{an}}}$) for which there exists a sufficiently small open neighbourhood $U \subset \Upsilon_{C^{\mathrm{an}}}$ (resp. $U' \subset \Upsilon_{C'^{\mathrm{an}}}$) such that $U \smallsetminus \{p\}$ (resp. $U' \smallsetminus \{p'\}$) has atleast three connected components then $p \in V(\Upsilon_{C^{\mathrm{an}}})$ (resp. $p' \in V(\Upsilon_{C'^{\mathrm{an}}})$).

It can be verified that a pair $(V(\Upsilon_{C^{\mathrm{an}}}), V(\Upsilon_{C'^{\mathrm{an}}}))$ satisfying these properties does indeed exist. We define the set of edges $E(\Upsilon_{C^{\mathrm{an}}})$ (resp. $E(\Upsilon_{C'^{\mathrm{an}}})$) for the skeleton $\Upsilon_{C^{\mathrm{an}}}$ ($\Upsilon_{C'^{\mathrm{an}}}$) to be the collection of all paths contained in $\Upsilon_{C^{\mathrm{an}}}$ (resp. $\Upsilon_{C'^{\mathrm{an}}}$) connecting any two vertices. Since $\Upsilon_{C^{\mathrm{an}}}$ (resp. $\Upsilon_{C'^{\mathrm{an}}}$) is the skeleton associated to a weak semistable vertex set, the edges of the skeleton are identified with real intervals. This defines a length function on the set of edges.

By definition, $g^{\mathrm{an}}(C) = g(\Upsilon_{C^{\mathrm{an}}})$ and $g^{\mathrm{an}}(C') = g(\Upsilon_{C'^{\mathrm{an}}})$ The genus formula [[AB], 4.5] implies that

$$g(C) = g(\Upsilon_{C^{\mathrm{an}}}) + \Sigma_{p \in V(\Upsilon_{C^{\mathrm{an}}})} g_p$$

and

$$g(C') = g(\Upsilon_{C'^{\mathrm{an}}}) + \Sigma_{p' \in V(\Upsilon_{C'^{\mathrm{an}}})} g_{p'}.$$

By definition, the spaces $C'^{\mathrm{an}} \smallsetminus \mathfrak{A}$ and $C'^{\mathrm{an}} \smallsetminus \mathfrak{A}'$ decompose into the disjoint union of Berkovich open balls and open annuli. It follows that if $p \notin \mathfrak{A}$ or $p' \notin \mathfrak{A}'$ then $g_p = 0$ and $g_{p'} = 0$. As $\mathfrak{A} \subset V(\Upsilon_{C^{\mathrm{an}}})$ and $\mathfrak{A}' \subset V(\Upsilon_{C'^{\mathrm{an}}})$, the equations above can be rewritten as

$$g(C) = g(\Upsilon_{C^{\mathrm{an}}}) + \Sigma_{p \in C^{\mathrm{an}}} g_p \tag{7.1}$$

and

$$g(C') = g(\Upsilon_{C'^{\mathrm{an}}}) + \Sigma_{p' \in C'^{\mathrm{an}}} g_{p'}. \tag{7.2}$$

The morphism $\phi : C' \to C$ is a finite separable morphism between smooth, projective curves. The Riemann-Hurwitz formula [[H], Corollary IV.2.4] enables us to relate the genera of the curves C' and C. Precisely,

$$2g(C') - 2 = \deg(\phi)(2g(C) - 2) + \deg(R) \tag{7.3}$$

where R is a divisor on the curve C' such that if ϕ is tamely ramified at $x' \in C'$ then $\mathrm{ord}_{x'}(R) = \mathrm{ram}(x', x) - 1$. Using the above, we obtain the following equation relating $g^{\mathrm{an}}(C')$ and $g^{\mathrm{an}}(C)$.

$$2g^{\mathrm{an}}(C') - 2 + 2(\Sigma_{p' \in C'^{\mathrm{an}}} g_{p'}) = \deg(\phi)(2g^{\mathrm{an}}(C) - 2) + \\ \deg(\phi)(2\Sigma_{p \in C^{\mathrm{an}}} g_p) + \deg(R).$$

The only points $p' \in C'^{\mathrm{an}}$ for which $g_{p'} \neq 0$ belong to $\mathfrak{A}'$ and are of type II. Let p' be such a point and $p := \phi^{\mathrm{an}}(p')$. Applying the Riemann-Hurwitz formula to the extension $\widetilde{I(p', p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ relates $g_{p'}$ and g_p by the following equation.

$$2g_{p'} - 2 = s(p', p)(2g_p - 2) + \deg(R_{p',p}).$$

This equation holds for all points of type II. When p' and p are not of type II, we set $s(p', p) := 1$ and $R_{p',p} := 0$. These invariants imply the following equation.

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)(2g^{\mathrm{an}}(C) - 2) - \Sigma_{p' \in C'^{\mathrm{an}}}[s(p', p)(2g_p - 2) + \deg(R_{p',p}) + 2] + \\ \deg(\phi)\Sigma_{p \in C^{\mathrm{an}}}(2g_p) + \deg(R).$$

Let $s(p) := \Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} s(p', p)$, $R^1_{p',p} := \deg(R_{p',p}) - (2s(p', p) - 2)$ and $R^1_p := \Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} R^1_{p',p}$. These invariants further simplify the equation above to the following form.

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)(2g^{\mathrm{an}}(C) - 2) - \Sigma_{p \in C^{\mathrm{an}}} 2s(p) g_p - \Sigma_{p \in C^{\mathrm{an}}} R^1_p + \deg(R).$$

□

The rest of this section is dedicated to studying the invariant $s(p)$ arising in the equation above.

7.3.3 Calculating $i(p')$ and the defect

Let M be a non-Archimedean valued field with valuation v. Let $|M^*|$ denote the value group and $\tilde{M}$ denote the residue field. Let M' be a finite extension of the field M such that the valuation v extends uniquely to M'. By Ostrowski's lemma, we have the following equality.

$$[M' : M] = (|M'^*| : |M^*|)[\tilde{M}' : \tilde{M}]c^r.$$

Here c is the characteristic of the residue field if it is positive and one otherwise. The value $d(M', M) := c^r$ is called the *defect* of the extension. If $r = 0$ then we call the extension M'/M *defectless*.

We now relate this definition to the situation we are dealing with. Let p be a point of type II belonging to C^{an} and $p' \in (\phi^{\mathrm{an}})^{-1}(p)$. Since the field k is algebraically closed non-Archimedean valued and the points p, p' are of type II, the value groups of the fields $\mathcal{H}(p)$ and $\mathcal{H}(p')$ remain the same. We have the following equality

$$[\mathcal{H}(p') : \mathcal{H}(p)] = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}]d(p', p)$$

where $d(p', p)$ is the defect of the extension $\mathcal{H}(p')/\mathcal{H}(p)$.

Lemma 7.3.2. *Let $p \in C^{\mathrm{an}}$ and $p' \in (\phi^{\mathrm{an}})^{-1}(p)$. The extension $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ is defectless i.e. $d(p', p) = 1$.*

Proof. We make use of the Poincaré-Lelong theorem and our construction in Section 7.3 of the pair of compatible deformation retractions ψ and ψ'. Let $r \in [0, 1]$ be the smallest real number such that $p \in \psi(r, C(k)) = \{\psi(r, x) | x \in C(k)\}$. Since the deformation retractions are compatible it follows that if $p' \in (\phi^{\mathrm{an}})^{-1}(p)$ then $p' \in \psi'(r, C'(k))$.

Let $x \in C(k)$ be such that $\psi(r, x) = p$. Observe that our choice of $\Upsilon_{C^{\mathrm{an}}}$ implies that the morphism ϕ is unramified over x. Let P_x denote the path $\psi(_, x) : [0, r] \to C^{\mathrm{an}}$. Given a simple neighborhood [[BPR], Definition 4.28], U of p, the germ of the path P_x at p lies in a connected component of $U \smallsetminus \{p\}$ and hence defines an element of the tangent space which we refer to as e_x. Equivalently, for some $a > 0$, the path $(P_x)_{|[a,r]} \circ -\exp : [-\log(a), -\log(r)] \to C^{\mathrm{an}}$ is a geodesic segment and its germ defines the element e_x of the tangent space T_p (cf. Remark 7.1.21). Let t_x be a uniformisant of the local ring $O_{C,x}$ such that $|t_x(p)| = 1$ and $\tilde{t}_x$- the image of t_x in the residue field $\widetilde{\mathcal{H}(p)}$ is a uniformisant at the point e_x in $\widetilde{\mathcal{H}(p)}$. This can be accomplished by choosing t_x so that it has no zeros or poles at any k-point y for which the path $\psi(_, y) : [0, r] \to C^{\mathrm{an}}$ coincides with e_x in the tangent space and using the Poincaré-Lelong theorem. Such a choice is possible by the semistable decomposition associated to the skeleton $\Upsilon_{C^{\mathrm{an}}}$. Let t'_x denote the image of t_x in the function field $k(C')$. Our choice of t_x implies that for every $p' \in (\phi^{\mathrm{an}})^{-1}(p)$, $|t'_x(p')| = 1$.

The inclusion $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ induces an inclusion of $\tilde{k}$-function fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. As before, let $\tilde{C}_p$ and $\tilde{C}'_{p'}$ denote the smooth projective curves associated to these function fields. As explained above, the path P_x defines a $\tilde{k}$-point e_x of the curve $\tilde{C}_p$. Let $E(e_x, p')$ denote the set of preimages of this point on the curve $\tilde{C}'_{p'}$.

Let $S := \{x'_1, ...x'_k\}$ denote the preimages of the point x and $\mathrm{ram}(x'_i, x)$ denote the ramification index of the morphism ϕ at the point x'_i. Since the skeleton $\Upsilon_{C^{\mathrm{an}}}$ contains the set of k-points over which the morphism is ramified, we have that $\mathrm{ram}(x'_i, x) = 1$ for all i. If $x' \in S$ then the path $P_{x'} := \psi'(_, x') : [0, r] \to C^{\mathrm{an}}$ defines an element of the tangent space at $\psi'(r, x')$. Indeed, if U' is a simple neighborhood of the point $\psi'(r, x')$ then there exists $a \in [0, r)$ such that $P_{x'|[a,r)}$ is contained in exactly one connected component of the space $U' \smallsetminus \psi'(r, x')$.

The set of elements of the tangent spaces $T_{p'}$ for $p' \in (\phi^{\mathrm{an}})^{-1}(p)$ that are defined by the paths $\{\psi'(_, x'_i) : [0, r] \to C^{\mathrm{an}}\}$ coincides with the set $E(e_x) := \cup_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} E(e_x, p')$. Our choice of t_x implies that if $y' \in C'(k) \smallsetminus \phi^{-1}(x)$ and $\psi'(_, y') : [0, r] \to C'^{\mathrm{an}} \in E(e_x)$ then t'_x cannot have a zero or pole at y'.

For $p' \in (\phi^{\mathrm{an}})^{-1}(p)$ and $e' \in E(e_x, p')$, let $S_{e',p'}$ be the collection of those $x' \in S$ such that $\psi'(r, x') = p'$ and $\psi'(_, x') : [0, r] \to C^{\mathrm{an}} = e'$. The non-Archimedean Poincaré-Lelong theorem implies that

$$\delta_{e'}(-|\log(t'_x)|)(p') = \Sigma_{x' \in S_{e',p'}} \mathrm{ram}(x', x) = \mathrm{card}(S_{e',p'}).$$

The second equality follows from the fact that $\mathrm{ram}(x', x) = 1$. Furthermore,

$$\delta_{e'}(-|\log(t'_x)|)(p') = \mathrm{ord}_{e'}(\tilde{t}'_x).$$

Since $\Sigma_{e' \in E(e_x,p')} \mathrm{ord}_{e'}(\tilde{t'}_x) = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}]$, we must have that

$$\Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \Sigma_{x' \in S} \mathrm{ram}(x', x) = \Sigma_{e',p'} \mathrm{card}(S_{e',p'}).$$

Hence

$$\Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \mathrm{card}(S).$$

As the field k is algebraically closed, the expression on the right is equal to the degree of the morphism ϕ and we have that

$$\Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} [\mathcal{H}(p') : \mathcal{H}(p)].$$

This implies that for every $p' \in (\phi^{\mathrm{an}})^{-1}(p)$ the extension $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ is defectless. □

The result above follows from the more general fact that the residue field $\mathcal{H}(p)$ associated to a point p of type II on the analytification C^{an} of a k-curve C is stable [[TEM], Corollary 6.3.6], [DUC]. Lemma 7.3.2 can in fact be used to prove this result. Propositions 2 and 4 of Section 3.6 in [BGR] allow us to give the following definition of a stable field which is complete.

Definition 7.3.3. A complete field K is *stable* if and only if for every finite separable field extension L/K the following equality holds

$$[L : K] = [|L^*| : |K^*|][\widetilde{L} : \widetilde{K}].$$

Proposition 7.3.4. *Let S be a k-curve. Let $p \in S^{\mathrm{an}}$ be a point of type II. The complete field $\mathcal{H}(p)$ is stable.*

Proof. Let S_i be an irreducible component of S such that $p \in S_i^{\mathrm{an}}$. Let S'_i denote the normalisation of S_i. There exists a finite set of k-points F' and F in S'_i and S_i respectively such that ${S'}_i^{\mathrm{an}} \smallsetminus F' = S_i^{\mathrm{an}} \smallsetminus F$. It follows that we may reduce to the case when S is smooth, projective and integral.

Let L be a finite separable extension of $\mathcal{H}(p)$. By definition, $\mathcal{H}(p)$ is the completion of the function field of the curve S with respect to the valuation associated to p. Let L_0 denote the integral closure of $k(S)$ in L. By construction, L_0 is a finite separable field extension of $k(S)$. Hence there exists a smooth projective k-curve S' such that $k(S') = L_0$. Let $\widehat{L_0}$ denote the completion of L_0 induced by the restriction of the valuation of L. We claim that $\widehat{L_0} = L$. Let $\alpha \in L$. By construction α is algebraic over $\widehat{L_0}$. Let $g \in \widehat{L_0}[X]$ be the minimal polynomial of α. Suppose that g is not a monomial. As $\widehat{L_0}$ is the completion

of L_0, there exists a sequence $(f_i)_i$ of polynomials of $\deg(g)$ in $L_0[X]$ which converge to g with respect to the Gauss norm i.e. the coefficients of the $(f_i)_i$ converge to the coefficients of g. Let α_i denote a root of f_i for each i. By Corollary 2 of [[BGR], Section 3.4], there exists a sub sequence of $(\alpha_i)_i$ which converges to a root α' of g. As the α_i are algebraic over L_0 they must be algebraic over $k(S)$. Furthemore, by Proposition 3 in [[BGR], Section 3.4], for large enough i we must have that $\alpha_i \in L$. Hence by definition of L_0, $\alpha_i \in L_0$. Hence $\alpha' \in \widehat{L_0}$. This implies a contradiction to our assumption that g is irreducible and of degree greater than or equal to 2. Consequently, $\alpha \in \widehat{L_0}$ and $\widehat{L_0} = L$. Hence there exists a point p' of type II on S'^{an} such that $\mathcal{H}(p') = L$. The proposition follows from Lemma 7.3.2.

□

We study the invariants $s(p)$ and $s(p', p)$ of Theorem 7.3.1 using the deformation retractions ψ and ψ'.

Definition 7.3.5. *(The equivalence relation $\sim_{i(r)}$)* Let $r \in [0, 1]$. We define an equivalence relation $\sim_{i(r)}$ on $C'(k)$ as follows. We set $x'_1 \sim_{i(r)} x'_2$ if and only if $\phi(x'_1) = \phi(x'_2)$, $\psi'(r, x'_1) = \psi'(r, x'_2)$ and the elements of the tangent space $T_{\psi'(r,x'_1)} = T_{\psi'(r,x'_2)}$ defined by the paths $\psi'(_, x'_1) : [0, r] \to C'^{\mathrm{an}}$, $\psi'(_, x'_2) : [0, r] \to C'^{\mathrm{an}}$ coincide. For $x' \in C'(k)$, let $\mathrm{card}[x']_{i(r)}$ be the cardinality of the equivalence class which contains x'.

Definition 7.3.6. *(The real number r_p, the set $Q_{p',p}$ and the invariant $i(p', p)$)* Let $p \in C^{\mathrm{an}}$ be a point which is not of type IV and $p' \in (\phi^{\mathrm{an}})^{-1}(p)$.

1. We define $r_p \in [0, 1]$ to be the smallest real number for which $p \in \psi(r_p, C(k))$ where $\psi(r_p, C(k)) := \{\psi(r_p, x) | x \in C(k)\}$.
2. We define $Q_{p',p} := \{x' \in C'(k) | \psi'(r_p, x') = p'\}$.
3. Let $i(p', p) := \min_{x' \in Q_{p',p}} \{\mathrm{card}[x']_{i(r_p)}\}$.

Recall that if p is a point of type II then we used $s(p', p)$ to denote the separable degree of the field extension $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ and we set $s(p', p) = 1$ otherwise.

Proposition 7.3.7. *Let $p' \in C'^{\mathrm{an}}, p := \phi^{\mathrm{an}}(p')$.*

1. *When p is of type II, the number $i(p', p)$ (cf. Definition 7.3.6) is the degree of inseparability of the extension $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. Hence $s(p', p) = [\mathcal{H}(p') : \mathcal{H}(p)]/i(p', p)$.*
2. *When p is not of type II or IV, $s(p) := \Sigma_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} s(p', p)$ is the number of $\sim_{i(r(p))}$ equivalence classes in $\phi^{-1}(x)$ for any $x \in C(k)$ such that $\psi(r_p, x) = p$.*

Proof. The second assertion can be verified directly and we restrict to proving the proposition for points of type II. Let $\tilde{C}_p$ and $\tilde{C}'_{p'}$ denote the smooth projective curves corresponding to the function fields $\widetilde{\mathcal{H}(p)}$ and $\widetilde{\mathcal{H}(p')}$ respectively. For a point $e \in \tilde{C}_p$, let s_e denote the uniformisant of the local ring $O_{\tilde{C}_p,e}$.

Let $x \in C(k)$ be such that $\psi(r_p, x) = p$. Since $\Upsilon_{C^{\mathrm{an}}}$ contains every k-point over which the morphism ϕ is ramified, ϕ is unramified over x. Let $e_x \in \tilde{C}_p$

correspond to the path $\psi(_, x) : [0, r_p] \to C^{\text{an}}$. Let t_x be a uniformisant of x such that $|t_x(p)| = 1$ and it does not have any zeros or poles at any y for which the element of the tangent space T_p defined by $\psi(_, y) : [0, r_p] \to C^{\text{an}}$ coincides with e_x.

It follows that the image of t_x in the field $\widetilde{\mathcal{H}(p)}$ is a uniformisant at the point e_x. We can hence assume $\tilde{t}_x = s_e$. Let $e' \in \tilde{C}'_{p'}$ map to e_x and $y' \in C'(k)$ be such that the element of $T_{p'}$ defined by the path $\psi'(_, y') : [0, r_p] \to C'^{\text{an}}$ coincides with e'.

By the Non-Archimedean Poincaré-Lelong Theorem, the order of vanishing of the uniformisant $\tilde{t}_x$ at e' is equal to the cardinality of the equivalence class $[y']_{i(r_p)}$. The inseparable degree of $\widetilde{\mathcal{H}(p')}/\widetilde{\mathcal{H}(p)}$ is equal to $\min_{\{(e',e)|e' \in \tilde{C}'_{p'}, e' \mapsto e\}}\{\text{ord}_{e'}(s_e)\}$ i.e. $\min_{\{(e',e)|e' \in \tilde{C}'_{p'}, e' \mapsto e\}}\{\text{ord}_{e'}(\tilde{t}_x)\}$. Hence $i(p', p) = \min_{x' \in Q_{p',p}}\{\text{card}[x']_{i(r(p))}\}$ is the degree of inseparability of the extension $\widetilde{\mathcal{H}(p')}/\widetilde{\mathcal{H}(p)}$. The equality $s(p', p) = [\mathcal{H}(p') : \mathcal{H}(p)]/i(p', p)$ follows from Lemma 7.3.2. □

Definition 7.3.8. Let $p \in C^{\text{an}}$. We define $i(p) := \Sigma_{p' \in (\phi^{\text{an}})^{-1}(p)}[\mathcal{H}(p') : \mathcal{H}(p)]/i(p', p)$ where $i(p', p)$ is as in Definition 7.3.6.

Proposition 7.3.7, Theorem 7.3.1 and the fact that $g_p = 0$ when p is not of type II imply the following corollary.

Corollary 7.3.9. *Let $\phi : C' \to C$ be a finite separable morphism between smooth projective curves over the field k. Let $g^{\text{an}}(C'), g^{\text{an}}(C)$ be as in Definition 7.1.25. We have the following equation.*

$$2g^{\text{an}}(C') - 2 = \deg(\phi)(2g^{\text{an}}(C) - 2) + \Sigma_{p \in C^{\text{an}}} 2i(p)g_p + \deg R - \Sigma_{p \in C^{\text{an}}} R^1_p.$$

7.4 A second calculation of $g^{\text{an}}(C')$

Let $\phi : C' \to C$ be a finite morphism between smooth projective curves over the field k. Our results in Section 7.4 imply the existence of a pair of deformation retractions ψ', ψ on C'^{an} and C^{an} which are compatible with the morphism ϕ^{an}. We choose ψ and ψ' as in the proof of Theorem 7.3.1. Let $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$ denote the images of the retractions ψ' and ψ respectively. The deformation retractions ψ, ψ' can be constructed so that $\Upsilon_{C^{\text{an}}}$ contains those points of $C(k)$ over which the morphism ϕ is ramified and does not contain any point of type IV. We have that $g^{\text{an}}(C') = g(\Upsilon_{C'^{\text{an}}})$ and $g^{\text{an}}(C) = g(\Upsilon_{C^{\text{an}}})$.

Definition 7.4.1. A *divisor* on a finite metric graph is an element of the free abelian group generated by the points of the graph.

As outlined in the introduction, in this section we introduce a divisor w on the skeleton $\Upsilon_{C^{\text{an}}}$ and relate the degree of this divisor to the genus of the skeleton $\Upsilon_{C'^{\text{an}}}$. The point of doing so is to study how $g(\Upsilon_{C'^{\text{an}}})$ can be calculated in terms of $g(\Upsilon_{C^{\text{an}}})$ and the behaviour of the morphism between the sets of vertices.

We preserve our choices of vertex sets and edge sets for the two skeleta from the proof of Theorem 7.3.1.

Definition 7.4.2. *(The invariant n_p and the sets of tangent directions E_p, $L(e_p, p')$.)* Let $p \in \Upsilon_{C^{\text{an}}}$.

1. Let n_p denote the number of preimages of p for the morphism ϕ^{an}.

2. Let T_p denote the tangent space at the point p (cf. 7.1.3, 7.1.6). We define $E_{p,\Upsilon_{C^{\mathrm{an}}}} \subset T_p$ to be those elements for which there exists a representative starting from p and contained completely in $\Upsilon_{C^{\mathrm{an}}}$. When there is no ambiguity concerning the graph $\Upsilon_{C^{\mathrm{an}}}$, we simplify notation and write E_p.

3. For any $p' \in C'^{\mathrm{an}}$ such that $\phi^{\mathrm{an}}(p') = p$, the morphism ϕ^{an} induces a map $d\phi_{p'}$ between the tangent spaces $T_{p'}$ and T_p (cf. 7.1.3, 7.1.6). For $p' \in (\phi^{\mathrm{an}})^{-1}(p)$ and $e_p \in E_p$, we define $L(e_p, p') \subset T_{p'}$ to be the set of preimages of e_p for the map $d\phi_{p'}$ and $l(e_p, p')$ to be the cardinality of the set $L(e_p, p')$.

Observe that as $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$, any element of $L(e_p, p')$ can be represented by a geodesic segment that is contained completely in $\Upsilon_{C'^{\mathrm{an}}}$.

Definition 7.4.3. *(The divisor w of $\Upsilon_{C^{\mathrm{an}}}$)* Let the notation be as in Definition 7.4.2. For a point $p \in \Upsilon_{C^{\mathrm{an}}}$, let $w(p) := (\sum_{e_p \in E_p, p' \in (\phi^{\mathrm{an}})^{-1}(p)} l(e_p, p')) - 2n_p$. We define w to be the divisor $\Sigma_{p \in \Upsilon_{C^{\mathrm{an}}}} w(p)p$.

Proposition 7.4.4. *The degree of the divisor w is equal to $2g(\Upsilon_{C'^{\mathrm{an}}}) - 2$.*

Proof. We begin by stating the following fact concerning connected, finite metric graphs. Let Σ be a connected, finite metric graph. Let $p \in \Sigma$. Let U be a simply connected neighborhood of p in Σ. We define t_p to be the cardinality of the set of connected components of the space $U \smallsetminus \{p\}$ and $D_\Sigma := \sum_{p \in \Sigma}(t_p - 2)p$. It can be verified that D_Σ is a divisor on the finite graph Σ whose degree is equal to $2g(\Sigma) - 2$.

The connected, finite graphs $\Upsilon_{C'^{\mathrm{an}}}$ and $\Upsilon_{C^{\mathrm{an}}}$ are the images of a pair of compatible deformation retractions. Hence the morphism ϕ^{an} restricts to a continuous map $\Upsilon_{C'^{\mathrm{an}}} \to \Upsilon_{C^{\mathrm{an}}}$. This map induces a homomorphism $\phi_* : \mathrm{Div}(\Upsilon_{C'^{\mathrm{an}}}) \to \mathrm{Div}(\Upsilon_{C^{\mathrm{an}}})$ defined as follows. We define ϕ_* only on the generators of the group $\mathrm{Div}(\Upsilon_{C'^{\mathrm{an}}})$. If $1.p' \in \mathrm{Div}(\Upsilon_{C'^{\mathrm{an}}})$ then we set $\phi_*(1.p') = 1.\phi(p')$. Note that for any divisor $D' \in \mathrm{Div}(\Upsilon_{C'^{\mathrm{an}}})$, $\deg(\phi_*(D')) = \deg(D')$.

We will show that $w = \phi_*(D_{\Upsilon_{C'^{\mathrm{an}}}})$. By definition,

$$\phi_*(D_{\Upsilon_{C'^{\mathrm{an}}}})(p) = (\sum_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} t_{p'}) - 2n_p.$$

Let $p' \in \Upsilon_{C'^{\mathrm{an}}}$ and $p = \phi^{\mathrm{an}}(p')$. We must have that the number of distinct germs of geodesic segments starting from p' and contained in $\Upsilon_{C'^{\mathrm{an}}}$ is $t_{p'}$. We have a map $d\phi_{p'} : T_{p'} \to T_p$ which maps germs of geodesic segments starting at p' to germs of geodesic segments starting at p. As $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$, we must have that the image via $d\phi_{p'}$ of a germ for which there exists a representative contained in $\Upsilon_{C'^{\mathrm{an}}}$ and starting from p' must be a germ starting at p for which there exists a representative contained in $\Upsilon_{C^{\mathrm{an}}}$. Likewise, if e_p is a germ starting at p which has a representative contained in $\Upsilon_{C^{\mathrm{an}}}$ then its preimage for the map $d\phi_{p'}$ is a germ starting at p' for which there exists a representative contained in $\Upsilon_{C'^{\mathrm{an}}}$. It follows that $\sum_{p' \in (\phi^{\mathrm{an}})^{-1}(p)} t_{p'} = \sum_{e_p \in E_p, p' \in \phi^{-1}(p)} l(e_p, p')$. Hence $\phi_*(D_{\Upsilon_{C^{\mathrm{an}}}}) = w$. $\square$

7.4.1 Calculating n_p

We extend the invariant n_p of Definition 7.4.2 to all points of C^{an}.

Definition 7.4.5. *(The invariant n_p)* Let $p \in C^{\mathrm{an}}$. Let n_p denote the number of preimages of p for the morphism ϕ^{an}.

In this section we study n_p for $p \in C^{\mathrm{an}}$ with the added restriction that the extension of function fields $k(C) \hookrightarrow k(C')$ associated to the morphism ϕ is Galois.

Definition 7.4.6. *(The invariant* $\mathrm{ram}(p)$ *for $p \in C^{\mathrm{an}}$)* Let $p \in C^{\mathrm{an}}$.

1. Let p be a point of type I i.e. $p \in C(k)$. Let $p' \in C'(k)$ such that $\phi(p') = p$. Let $\mathrm{ram}(p', p)$ denote the ramification degree associated to the extension of the discrete valuation rings $O_{C,p} \hookrightarrow O_{C',p'}$. Since the morphism ϕ is Galois, for $p \in C(k)$, the ramification degree $\mathrm{ram}(p', p)$ is a constant as p' varies along the set of preimages of the point p. The ramification degree depends only on the point $p \in C(k)$ and we denote it $\mathrm{ram}(p)$. As k is algebraically closed we have that
$$[k(C') : k(C)] = n_p \mathrm{ram}(p).$$
2. When p is not of type I, we define $\mathrm{ram}(p) := 1$.

Let p be a point of C^{an} which is not of type IV. Recall that r_p is the smallest real number in the real interval $[0, 1]$ such that p belongs to $\psi(r_p, C(k)) = \{\psi(r_p, x) | x \in C(k)\}$. Since the pair of deformation retractions ψ' and ψ are compatible with the morphism ϕ^{an}, we must have that $(\phi^{\mathrm{an}})^{-1}(p) \subset \psi'(r_p, C'(k))$.

Definition 7.4.7. *(The equivalence relation $\sim_{c(r)}$ on $C(k)$)* For $r \in [0, 1]$, we define an equivalence relation $\sim_{c(r)}$ on the set of k-points of the curve C'. Let $x'_1, x'_2 \in C'(k)$. We set $x'_1 \sim_{c(r)} x'_2$ if $\phi(x'_1) = \phi(x'_2)$ and $\psi'(r, x'_1) = \psi'(r, x'_2)$. Observe that each equivalence class is finite. For $x' \in C'(k)$, let $[x']_{c(r)}$ denote that equivalence class containing the point x'.

Lemma 7.4.8. *If $x'_1, x'_2 \in C'(k)$ such that $\phi(x'_1) = \phi(x'_2)$ then*
$$\mathrm{card}[x'_1]_{c(r)} = \mathrm{card}[x'_2]_{c(r)}$$
for all $r \in [0, 1]$.

Proof. The lemma is tautological when $x'_1 \sim_{c(r)} x'_2$. Let us hence assume that $\psi'(r, x'_1) = p'_1$ and $\psi'(r, x'_2) = p'_2$ where p'_1 and p'_2 are two points on C'^{an}. Observe that since $\Upsilon_{C'^{\mathrm{an}}}$ and $\Upsilon_{C^{\mathrm{an}}}$ are the images of a pair of compatible deformation retractions, $\phi^{\mathrm{an}}(p'_1) = \phi^{\mathrm{an}}(p'_2)$. Let $p := \phi^{\mathrm{an}}(p'_1)$. The Galois group $G := \mathrm{Gal}(k(C')/k(C))$ acts trasitively on the set of preimages $\phi^{-1}(p)$. Let $\sigma \in G$ be an element of the Galois group such that $\sigma(p'_1) = p'_2$. By construction, the deformation retraction ψ' is Galois invariant i.e. if $t \in [0, 1], q \in C'^{\mathrm{an}}$ and $g \in \mathrm{Gal}(k(C')/k(C))$ then $\psi'(t, g(q)) = g(\psi'(t, q))$. It follows that if $a \sim_{c(r)} x'_1$ then $\sigma(a) \sim_{c(r)} x'_2$. As σ is bijective, $\mathrm{card}[x'_1]_{c(r)} \leq \mathrm{card}[x'_2]_{c(r)}$. By symmetry we conclude that the lemma is true. $\square$

Definition 7.4.9. *(The invariant $c_r(x)$ for $x \in C(k)$)* Let $x \in C(k)$ and $x' \in C'(k)$ such that $\phi(x') = x$. We define

$$c_r(x) := \mathrm{card}[x']_{c(r)}.$$

Lemma 7.4.8 implies that $c_r(x)$ is well defined.

Proposition 7.4.10. *Let $p \in C^{\mathrm{an}}$ be a point which is not of type IV. We have the following equality.*

$$n_p = [k(C') : k(C)]/(c_{r_p}(x)\mathrm{ram}(x))$$

for any $x \in C(k)$ such that $\psi(r_p, x) = p$.

Proof. When $p \in C(k)$, we must have that if $x \in C(k)$ is such that $\psi(r_p, x) = p$ then $x = p$ and $r_p = 0$. Hence $c_{r_p}(p) = 1$ and the proposition amounts to showing that $[k(C') : k(C)] = n_p\mathrm{ram}(p)$ which is a well known calculation.

Suppose $p \in C^{\mathrm{an}} \smallsetminus C(k)$. Let $x \in C(k)$ be such that $\psi(r_p, x) = p$. As the deformation retractions ψ and ψ' are compatible we must have that $\psi'(r_p, y) \in (\phi^{\mathrm{an}})^{-1}(p)$ for every $y \in \phi^{-1}(x)$. Furthermore, given $q \in C'^{\mathrm{an}}$ which maps to p via ϕ^{an}, there exists a $y \in \phi^{-1}(x)$ such that $\psi'(r_p, y) = q$. This can be deduced from the Galois invariance of the deformation retraction ψ'. As ψ fixes the points of $C(k)$ which are ramified, we must have that if $x \in C(k)$ such that $\psi(r_p, x) = p$ then $n_x = [k(C') : k(C)]$ and $\mathrm{ram}(x) = 1$. The proposition can be deduced from these observations. □

Observe that if $x \in C(k)$ is such that $\psi(r_p, x) = p$ then $c_{r_p}(x) = c_s(x)$ for any $s \in [r_p, 1]$. This observation and Proposition 7.4.10 allow us to define the following invariant - $c_1(p)$ for $p \in \Upsilon_{C^{\mathrm{an}}}$.

Definition 7.4.11. *(The invariant $c_1(p)$ for $p \in \Upsilon_{C^{\mathrm{an}}}$)* Let $p \in \Upsilon_{C^{\mathrm{an}}}$. The function $c_1 : C(k) \to \mathbb{Z}_{\geq 0}$ factors through $\Upsilon_{C^{\mathrm{an}}}$ via the retraction $\psi(1, _)$. Hence we have $c_1 : \Upsilon_{C^{\mathrm{an}}} \to \mathbb{Z}_{\geq 0}$. By Proposition 7.4.10,

$$n_p = [k(C') : k(C)]/(c_1(p)\mathrm{ram}(p))$$

where $\mathrm{ram}(p)$ is as in Definition 7.4.6.

7.4.2 Calculating $l(e_p, p')$

Lemma 7.4.12. *Let $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$. Then $l(e_p, p')$ is a constant as p' varies through the set of preimages of p for the morphism ϕ^{an}.*

Proof. Let $p'_1, p'_2 \in (\phi^{\mathrm{an}})^{-1}(p)$. The Galois group $\mathrm{Gal}(k(C')/k(C))$ acts transitively on the set of preimages of the point p. As $\Upsilon_{C'^{\mathrm{an}}} = (\phi^{\mathrm{an}})^{-1}(\Upsilon_{C^{\mathrm{an}}})$, the elements of the Galois group are homeomorphisms on C'^{an} which restrict to homeomorphisms on $\Upsilon_{C'^{\mathrm{an}}}$. It follows that if $\sigma \in \mathrm{Gal}(k(C')/k(C))$ is such that $\sigma(p'_1) = p'_2$ then σ maps the set of germs $L(e_p, p'_1)$ injectively to the set $L(e_p, p'_2)$. By symmetry, we conclude that our proof is complete. □

Definition 7.4.13. *(The invariants $l(e_p)$ and $\widetilde{\mathrm{ram}}(e_p)$ for $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$)* Let $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$ (Definition 7.4.2).

1. We define $l(e_p) := l(e_p, p')$ for any $p' \in (\phi^{\mathrm{an}})^{-1}(p)$. Lemma 7.4.12 implies that $l(e_p)$ is well defined.

2. (a) By Section 7.1.6, when p is a point of type II, e_p corresponds to a discrete valuation of the $\tilde{k}$-function field $\widetilde{\mathcal{H}(p)}$. For any $p' \in (\phi^{\mathrm{an}})^{-1}(p)$, the extension of fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ can be decomposed into the composite of a purely inseparable extension and a Galois extension. Hence the ramification degree $\mathrm{ram}(e'/e_p)$ is constant as e' varies through the set of preimages of e_p for the morphism $d\phi^{alg}_{p'} : T_{p'} \to T_p$ (cf. 7.1.6). Let $\widetilde{\mathrm{ram}}(e_p)$ be this number.

   (b) When p is of type I, the set E_p contains only one element and we set $\widetilde{\mathrm{ram}}(e_p) := \mathrm{ram}(p)$.

   (c) When p is of type III, let $\widetilde{\mathrm{ram}}(e_p) := c_1(p)$.

Applying Propositions 7.4.4 and 7.4.12, the value $2g^{\mathrm{an}}(C') - 2$ can be calculated in terms of $l(e_p)$ as follows.

Proposition 7.4.14. *Let the notation be as in Definition 7.4.13. We have that*

$$2g^{\mathrm{an}}(C') - 2 = \Sigma_{p \in \Upsilon_{C^{\mathrm{an}}}} n_p((\Sigma_{e_p \in E_p} l(e_p)) - 2).$$

Proposition 7.4.15. *Let $p \in \Upsilon_{C^{\mathrm{an}}}$ and $e_p \in E_p$. The following equality holds.*

$$l(e_p) = [k(C') : k(C)]/(n_p \widetilde{\mathrm{ram}}(e_p)).$$

Proof. When p is a point of type I or III, we must have that $l(e_p)$ is 1 and hence the proposition can be easily verified by applying Proposition 7.4.10. Let us suppose that p is a point of type II. The morphism $\phi : C' \to C$ corresponds to an extension of function fields $k(C) \hookrightarrow k(C')$ which is Galois. As $p \in C^{\mathrm{an}}$ is of type II, it corresponds to a multiplicative norm on the function field $k(C)$. The set of preimages $\phi^{-1}(p)$ corresponds to those multiplicative norms on $k(C')$ which extend the multiplicative norm p on $k(C)$. For every $p' \in (\phi^{\mathrm{an}})^{-1}(p)$, $\mathcal{H}(p')$ is the completion of $k(C')$ for p' and is a finite extension of the non-Archimedean valued complete field $\mathcal{H}(p)$. The Galois group $\mathrm{Gal}(k(C')/k(C))$ acts transitively on the set $(\phi^{\mathrm{an}})^{-1}(p)$. It follows that degree of the extension $[\mathcal{H}(p') : \mathcal{H}(p)]$ is a constant as p' varies through the set $(\phi^{\mathrm{an}})^{-1}(p)$. We denote this number $f(p)$. Hence we have that

$$[k(C') : k(C)] = n_p f(p).$$

By Lemma 7.3.2, $f(p) = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}]$. Uniquely associated to the $\tilde{k}$-function fields $\mathcal{H}(p)$ and $\mathcal{H}(p')$ are smooth, projective $\tilde{k}$-curves denoted $\tilde{C}_p$ and $\tilde{C}'_{p'}$. The germ e_p corresponds to a closed point on the former of these curves. The number $l(e_p)$ is the cardinality of the set of preimages of the closed point e_p for the morphism $\tilde{C}_{p'} \to \tilde{C}_p$ induced by ϕ^{an}. The result now follows from [[L], Theorem 7.2.18] applied to the $\tilde{k}$-function fields $\widetilde{\mathcal{H}(p)}, \widetilde{\mathcal{H}(p')}$ and the divisor e_p. □

The results of this section can be compiled so that the value $2g^{\mathrm{an}}(C') - 2$ can be computed in terms of the invariant $\widetilde{\mathrm{ram}}$ introduced below and the invariants ram and c_1 from Definition 7.4.11.

Definition 7.4.16. *(The invariant $\widetilde{\mathrm{ram}}(p)$ for $p \in \Upsilon_{C^{\mathrm{an}}}$)* Let $p \in \Upsilon_{C^{\mathrm{an}}}$. We define $\widetilde{\mathrm{ram}}(p) := \Sigma_{e_p \in E_p}(1/\widetilde{\mathrm{ram}}(e_p))$.

The following theorem can be verified using 7.4.14 and 7.4.10.

Theorem 7.4.17. *Let $\phi : C' \to C$ be a finite morphism between smooth projective irreducible k-curves such that the extension of function fields $k(C) \hookrightarrow k(C')$ induced by ϕ is Galois. Let $g^{\mathrm{an}}(C')$ be as in Definition 7.1.25. For $p \in \Upsilon_{C^{\mathrm{an}}}$, let $\widetilde{\mathrm{ram}}(p), c_1(p)$ and $\mathrm{ram}(p)$ be the invariants introduced in Definition 7.4.6, 7.4.11 and 7.4.16. We have that*

$$2g^{\mathrm{an}}(C') - 2 = \deg(\phi)\Sigma_{p \in \Upsilon_{C^{\mathrm{an}}}}[\widetilde{\mathrm{ram}}(p) - 2/(c_1(p)\mathrm{ram}(p))].$$

Chapter 8

Finite morphisms and skeleta

8.1 The class of open sets $\mathcal{O}_L$

Let L/k be a complete non-Archimedean real valued algebraically closed field extension. Let $x \in \mathbb{P}_k^{n,\mathrm{an}}(L)$ be an L-point of the analytification of projective n-space. As outlined in Remark 1.1, the pair $x : \mathrm{Spec}(L) \to \mathbb{P}_k^n$, $\mathrm{id}_L : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ defines a closed point of the variety $V_L = V \times_k \mathrm{Spec}(L)$ which we denote x_L. We proceed to define the family $\mathcal{O}_{x_L,\mathbb{P}_L^{n,\mathrm{an}}}$ of open neighbourhoods of x_L

In Proposition 5.7.4 we showed that having chosen an affine chart of $\mathbb{P}_k^n$, the space $\mathbb{P}_k^{n,\mathrm{an}}$ can be seen to be the union of $n+1$ n-dimensional Berkovich closed disks defined over k. We denoted this collection $\{A_i\}_i$. Likewise $\mathbb{P}_L^{n,\mathrm{an}} = \bigcup_i A_{i,L}$ where $\{A_{i,L} := A_i \times_k L\}_i$ forms a collection of $n+1$ n-dimensional Berkovich closed disks defined over L. For some j, we must have that $x_L \in A_{j,L}$. Let $\mathcal{O}_{x_L,\mathbb{P}_L^{n,\mathrm{an}}}$ be the family of Berkovich open balls containing x_L and contained in $A_{j,L}$. Each such Berkovich open ball centered at x_L corresponds uniquely to an n-tuple of positive real numbers less than or equal to 1. We proceed below in greater detail.

In the proof of Proposition 5.7.4, we introduced the following notation concerning the $A_{j,L}$.

$$A_{j,L} = \mathcal{M}(\mathcal{B}_{j,L})$$

where

$$\mathcal{B}_j := k\{T_1/T_j, ..T_i/T_j, ..T_{n+1}/T_j\}$$

and $\mathcal{B}_{j,L} := \mathcal{B}_j \hat{\otimes}_k L$. The affinoid space $\mathcal{M}(\mathcal{B}_{j,L})$ is an n-dimensional Berkovich closed disk over L. The point $x_L \in \mathbb{P}_L^n$ is a closed point defined over L. Let it have coordinates $[x_{1,L} : \ldots : x_{n+1,L}]$ and i be any index such that $|x_{j,L}| \leq |x_{i,L}|$ for every $j \in \{1, .., n+1\}$. By definition of the space $A_{i,L}$, x_L belongs to it. Using the fact that $A_{i,L} = \mathcal{M}(\mathcal{B}_{i,L})$ we define a family of open neighbourhoods of x_L, namely the collection of Berkovich open balls defined by the equations $|(T_j/T_i - x_{j,L}/x_{i,L})(p)| < r_j$, where $j \neq i$ and $r_j \leq 1$. If $x_L \in A_{t,L}$ then $|x_{i,L}| = |x_{t,L}|$ and any such Berkovich open sub ball of $A_{i,L}$ will also be contained in

$A_{t,L}$. By this we mean that there exists $\mathbf{r}_t := (r'_1, \ldots, r'_{n+1}) \in (0,1]^{n+1}$ such that $r'_t = 1$ and $B = \{p \in A_{t,L} | \wedge_{j \neq t} [|(T_j/T_t - x_{j,L}/x_{t,L})(p)| < r'_j]\}$. It can be shown that as B varies through all the Berkovich open sub balls of $A_{i,L}$ which contain x_L, it also varies through all Berkovich open sub balls of $A_{s,L}$ which contain x_L i.e. for any s such that $|x_{s,L}| = |x_{i,L}|$. Hence, we may define the family $\mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}$ to be the collection $\{B(x_L, \mathbf{r}_i)\} \subset A_{i,L}$ for any i such that $|x_{j,L}| \leq |x_{i,L}|$ for every $j \in \{1, .., n+1\}$ and where $\mathbf{r}_i := (r_1, \ldots, r_i, \ldots, r_{n+1})$ is any $n+1$-tuple for which $r_i = 1$ and $r_h \leq 1$ for any other h. Let $\mathcal{O}_{L, \mathbb{P}^{n,\mathrm{an}}_L} := \bigcup_{x \in \mathbb{P}^{n,\mathrm{an}}_k(L)} \mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}$.

To an element W of this family we associate an $(n+1)^2$-tuple. That is, we define a function $h_{L, \mathbb{P}^{n,\mathrm{an}}_L} : \mathcal{O}_{L, \mathbb{P}^{n,\mathrm{an}}_L} \to \mathbb{R}^{(n+1)^2}$. If for an index t, $|x_{t,L}| = |x_{i,L}|$ then let $\mathbf{r}_t = (r_1, \ldots, r_{n+1})$ be such that $r_t = 1$ and the Berkovich open ball W is defined by the equations $|(T_j/T_t - x_{j,L}/x_{t,L})(p)| < r_j$ for $j \neq t$. If on the other hand $|x_{t,L}| < |x_{i,L}|$ for some $i \in \{1, \ldots, n+1\}$ then let $\mathbf{r}_t = (1, \ldots, 1)$. Let $h_{L, \mathbb{P}^{n,\mathrm{an}}_L}(W) := (\mathbf{r}_i)_i$.

If V is an arbitrary projective k-variety then by definition it admits an embedding $V \hookrightarrow \mathbb{P}^n_k$. Let $x \in V^{\mathrm{an}}(L) \subset \mathbb{P}^{n,\mathrm{an}}_k(L)$. We defined a family of open neighbourhoods of x_L in $\mathbb{P}^{n,\mathrm{an}}_L$ which we called $\mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}$. Along with this family of open neighbourhoods, we also defined a function $h_{L, \mathbb{P}^{n,\mathrm{an}}_L}$ which defines the polyradius of every element in $\mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}$. Restricting every element of the family $\mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}$ to V^{an}_L will define a family $\mathcal{O}_{x_L, V^{\mathrm{an}}_L}$ of open neighbourhoods of x_L in V^{an}_L. Hence $\mathcal{O}_{x_L, V^{\mathrm{an}}_L} = \{W \cap V^{\mathrm{an}}_L | W \in \mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}\}$. Let $W \in \mathcal{O}_{x_L, V^{\mathrm{an}}_L}$. We define Q_W to be the collection of $W' \in \mathcal{O}_{x_L, \mathbb{P}^{n,\mathrm{an}}_L}$ such that $W' \cap V^{\mathrm{an}}_L = W$. We set $h_{L, V^{\mathrm{an}}_L}(W \cap V^{\mathrm{an}}_L) := \inf_{W' \in Q_W} \{h_{L, \mathbb{P}^{n,\mathrm{an}}_L}(W')\}$. The infimum here is taken with respect to the coordinate wise partial ordering defined on $\mathbb{R}^{(n+1)^2}$ i.e. $(x_n)_n \leq (y_n)_n$ if and only if $x_i \leq y_i$ for every $1 \leq i \leq (n+1)^2$. By Lemma 8.1.1, the function h_{L, V^{an}_L} is well defined.

Lemma 8.1.1. *Let $x_L \in V^{\mathrm{an}}_L(L)$ and $W \in \mathcal{O}_{x_L, V^{\mathrm{an}}_L}$. The set of $(n+1)^2$-tuples $\{h_{L, \mathbb{P}^{n,\mathrm{an}}_L}(W') | W' \in Q_W\} \subset \mathbb{R}^{(n+1)^2}$ has a well defined infimum with respect to the coordinate wise partial ordering defined on $\mathbb{R}^{(n+1)^2}$.*

(Note : This infimum need not belong to the set $\{h_{L, \mathbb{P}^{n,\mathrm{an}}_L}(W') | W' \in Q_W\}$).

Proof. Let $P := \{h_{L, \mathbb{P}^{n,\mathrm{an}}_L}(W') | W' \in Q_W\}$. Let $t \in \{1, \ldots, n+1\}$ be such that $x_L \in A_{t,L}$. Over the course of this proof, we employ the following convention. An $(n+1)^2$-tuple $(x_1, \ldots, x_{(n+1)^2})$ belonging to $\mathbb{R}^{(n+1)^2}$ is denoted by $\mathbf{x}$.

We are required to show that there exists $\mathbf{s} \in \mathbb{R}^{(n+1)^2}$ such that

1. If $\mathbf{y} \in P$ then $s_j \leq y_j$ for every j.
2. If $\mathbf{z} \in \mathbb{R}^{(n+1)^2}$ is such that for any $\mathbf{y} \in P$ the inequality $z_j \leq y_j$ holds for every j, then $z_j \leq s_j$ for every j.

Let p_i denote the i-th projection morphism $\mathbb{R}^{(n+1)^2} \to \mathbb{R}$. As $P \subset \mathbb{R}^{(n+1)^2}_{\geq 0}$, there exists a unique $s_i \in \mathbb{R}_{\geq 0}$ which is the infimum of the set $\{p_i(\mathbf{y}) | \mathbf{y} \in P\}$. The $(n+1)^2$-tuple $\mathbf{s}$ can be checked to satisfy property (1).

To show that $\mathbf{s}$ satisfies property (2) as well, we exhibit a sequence of elements in P which converges to $\mathbf{s}$. By definition, for every $j \in \{1, \ldots, n+1\}$ there exists a sequence of elements $(W_{j,m})_m$ of Q_W such that $(p_j \circ h_{L, \mathbb{P}^{n,\mathrm{an}}_L}(W_{j,m}))_m$ converges to s_j. Let $W_m := \bigcap_j W_{j,m}$. It can be verified that the intersection

of a finite number of open polydisks contained in $A_{t,L}$ and centered at the same point is again an open polydisk in $A_{t,L}$ centered at that same point. It follows from this fact that W_m belongs to $\mathcal{O}_{x_L,\mathbb{P}^{n,\mathrm{an}}_L}$. Furthermore, for every j, $W = W_{j,m} \cap V^{\mathrm{an}}_L$ and hence $W = \cap_{1\leq j\leq n+1} W_{j,m} \cap V^{\mathrm{an}}_L$. As a result, W_m is an element of Q_W. Also, for every j, we have the following inequality.

$$s_j \leq (p_j \circ h_{L,\mathbb{P}^{n,\mathrm{an}}_L})(W_m) \leq (p_j \circ h_{L,\mathbb{P}^{n,\mathrm{an}}_L})(W_{j,m}).$$

By our choice of the sequence $(W_{j,m})_m$, we have that for every j, $\lim_{m\to\infty}(p_j \circ h_{L,\mathbb{P}^{n,\mathrm{an}}_L})(W_{j,m}) = s_j$ and hence the sequence $((p_j \circ h_{L,\mathbb{P}^{n,\mathrm{an}}_L})(W_m))_m$ converges to s_j. This is equivalent to saying that $(h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(W_m))_m$ converges to $\mathbf{s}$. □

In Remark 1.5, we introduced a family of functions $\mathbb{R}^{(n+1)^2}_{\geq 0} \to \mathbb{R}$ which we denoted by S.

Lemma 8.1.2. *Let L/k be an algebraically closed complete non-Archimedean real valued field extension and $g \in S$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to $\mathcal{O}_{x_L,V^{\mathrm{an}}_L}$ such that $\mathcal{O}_1 \subset \mathcal{O}_2$. The following inequality holds true.*

$$(g \circ h_{L,V^{\mathrm{an}}_L})(\mathcal{O}_1) \leq (g \circ h_{L,V^{\mathrm{an}}_L})(\mathcal{O}_2).$$

Proof. We first show that we may reduce to the case when $V = \mathbb{P}^n_k$ and $\mathcal{O}_i \in \mathcal{O}_{x_L,\mathbb{P}^{n,\mathrm{an}}_L}$. Suppose that the lemma is true for $\mathbb{P}^n_k$. Given $\mathcal{O}_1$ and $\mathcal{O}_2$ belonging to $\mathcal{O}_{x_L,V^{\mathrm{an}}_L}$ such that $\mathcal{O}_1 \subset \mathcal{O}_2$, we show that the following inequality holds true.

$$(g \circ h_{L,V^{\mathrm{an}}_L})(\mathcal{O}_1) \leq (g \circ h_{L,V^{\mathrm{an}}_L})(\mathcal{O}_2).$$

We make use of the notation introduced before Lemma 8.1.1. Let $\mathcal{O}'_2 \in Q_{\mathcal{O}_2}$ and $\mathcal{O}'_1 \in Q_{\mathcal{O}_1}$. It can be checked that $\mathcal{O}'_2 \cap \mathcal{O}'_1 \in Q_{\mathcal{O}_1}$. The inequalities

$$h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(\mathcal{O}'_2 \cap \mathcal{O}'_1) \leq h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(\mathcal{O}'_2),$$
$$h_{L,V^{\mathrm{an}}_L}(O_1) \leq h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(\mathcal{O}'_2 \cap \mathcal{O}'_1)$$

imply that $h_{L,V^{\mathrm{an}}_L}(O_1) \leq h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(\mathcal{O}'_2)$. Observe that our choice of $\mathcal{O}'_2$ in $Q_{\mathcal{O}_2}$ was arbitrary and hence $h_{L,V^{\mathrm{an}}_L}(O_1) \leq h_{L,V^{\mathrm{an}}_L}(O_2)$.

Let us now suppose that $V = \mathbb{P}^n_k$. In the course of the proof we will make use of this fact: Since the field L is algebraically closed and endowed with a non-trivial valuation, its value group is dense in $\mathbb{R}_{\geq 0}$. Let $h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(\mathcal{O}_1) := (\mathbf{r}_1, \ldots, \mathbf{r}_{n+1})$ and $h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(\mathcal{O}_2) := (\mathbf{r}'_1, \ldots, \mathbf{r}'_{n+1})$. If $\mathbf{r}_t = (r_{1,t}, \ldots, r_{n+1,t})$ and $\mathbf{r}'_t = (r'_{1,t}, \ldots, r'_{n+1,t})$ then we claim that for every i, t, $r_{i,t} \leq r'_{i,t}$. We proceed by assuming the contrary. If for some t there exists an i such that $r_{i,t} > r'_{i,t}$ then we can find an element $y_i \in L$ such that $r'_{i,t} < |(y_i/x_t - x_i/x_t)| < r_{i,t}$. The element $[x_1 : .. : y_i : .. : x_{n+1}]$ will belong to $\mathcal{O}_1$ but not $\mathcal{O}_2$. This is not possible and we must hence have $r_i \leq r'_i$. Our choice of the function g implies that the inequality $(g \circ h_{L,\mathbb{P}^{n,\mathrm{an}}_L})(\mathcal{O}_1) \leq (g \circ h_{L,\mathbb{P}^{n,\mathrm{an}}_L})(\mathcal{O}_2)$ holds. □

The following lemma implies that the family of open neighbourhoods $\mathcal{O}_{x_L,V^{\mathrm{an}}_L}$ for $x \in V^{\mathrm{an}}(L)$ does not depend on the extension L/k.

Lemma 8.1.3. *Let L'/k and L/k be complete non-Archimedean real valued algebraically closed field extensions such that L embeds into L'. Let $x \in V^{\mathrm{an}}(L)$. We have the following equality of sets.*

$$\mathcal{O}_{x_{L'},V^{\mathrm{an}}_{L'}} = \{O \times_L L' | O \in \mathcal{O}_{x_L,V^{\mathrm{an}}_L}\}.$$

Furthermore, if $O \in \mathcal{O}_{x_L,V^{\mathrm{an}}_L}$ then $h_{L',V^{\mathrm{an}}_{L'}}(O \times_L L') = h_{L,V^{\mathrm{an}}_L}(O)$.

Proof. It can be inferred from the discussion above concerning the family $\mathcal{O}_{x_L,V_L^{\mathrm{an}}}$ that it suffices to prove the lemma assuming $V = \mathbb{P}^n_k$ for some $n \in \mathbb{N}$. If $x \in \mathbb{P}^{n,\mathrm{an}}_k(L)$ then there exists exactly one point on the fibre over x_L for the projection morphism $\mathbb{P}^n_L \times_L L' \to \mathbb{P}^n_L$ and $(x_L)_{L'} = x_{L'}$. We assume without loss of generality that $x_L \in A_{i,L}$ with homogenous coordinates $[x_{1,L} : \ldots : x_{n+1,L}]$. If $x_{L'}$ has homogenous coordinates $[x_{1,L'} : \ldots : x_{n+1,L'}]$ then $x_{j,L}/x_{i,L} = x_{j,L'}/x_{i,L'}$ for all j.

Let $O \in \mathcal{O}_{x_L,\mathbb{P}^{n,\mathrm{an}}_L}$. By definition, O must be of the form $\{p \in A_{i,L} | \wedge_j |(T_j/T_i - x_{j,L}/x_{i,L})(p)| < r_j\}$. It follows that $O \times_L L' = \{p \in A_{i,L'} | \wedge_j |(T_j/T_i - x_{j,L'}/x_{i,L'})(p)| < r_j\}$ which is an element of $\mathcal{O}_{x_{L'},\mathbb{P}^{n,\mathrm{an}}_{L'}}$. Hence $\{O \times_L L' | O \in \mathcal{O}_{x_L,\mathbb{P}^{n,\mathrm{an}}}\} \subset \mathcal{O}_{x_{L'},\mathbb{P}^{n,\mathrm{an}}_{L'}}$ and $h_{L',\mathbb{P}^{n,\mathrm{an}}_{L'}}(O \times_L L') = h_{L,\mathbb{P}^{n,\mathrm{an}}_L}(O)$.

Let $O \in \mathcal{O}_{x_{L'},\mathbb{P}^{n,\mathrm{an}}_{L'}}$. By definition, O must be of the form $\{p \in A_{i,L'} | \wedge_j |(T_j/T_i - x_{j,L'}/x_{i,L'})(p)| < r_j\}$. Using the equality $x_{j,L}/x_{i,L} = x_{j,L'}/x_{i,L'}$, the image of this open set under the projection morphism $A_{i,L} \times_L L' \to A_{i,L}$ is of the form $O_0 := \{p \in A_{i,L} | \wedge_j |(T_j/T_i - x_{j,L}/x_{i,L})(p)| < r_j\}$. It follows that $O_0 \times_L L' = O$. Hence $\mathcal{O}_{x_{L'},\mathbb{P}^{n,\mathrm{an}}_{L'}} = \{O \times_L L' | O \in \mathcal{O}_{x_L,\mathbb{P}^{n,\mathrm{an}}_L}\}$. □

Remark 8.1.4. *We henceforth make no reference to $h_{L,V_L^{\mathrm{an}}}$ and $\mathcal{O}_{x_L,V^{\mathrm{an}}}$. We simplify notation and use $\mathcal{O}_{x_L}$ to denote $\mathcal{O}_{x_L,\mathbb{P}^{n,\mathrm{an}}_L}$, h_L in place of $h_{L,\mathbb{P}^{n,\mathrm{an}}_L}$ and $\mathcal{O}_L$ for $\mathcal{O}_{L,\mathbb{P}^{n,\mathrm{an}}_L} = \bigcup_{x_L \in \mathbb{P}^{n,\mathrm{an}}_L(L)} \mathcal{O}_{x_L}$.*

8.1.1 The family $\mathcal{O}_{x_L}$ when $x_L \in \mathbb{P}^{1,\mathrm{an}}_L$.

The space $\mathbb{P}^{1,\mathrm{an}}_L$ admits two descriptions, one of which was outlined in the proof of Proposition 5.7.4. We briefly describe these constructions.

Let S denote the set of all multiplicative seminorms on the polynomial algebra $L[T_1, T_2]$ which restrict to the valuation on the field L such that if $x \in S$ then it cannot be that $|T_1(x)| = |T_2(x)| = 0$. We define an equivalence relation $\sim$ on S as follows.

$$x \sim y \iff \text{There exists } c \in \mathbb{R}_{>0} \text{ such that for any homogenous } f \in k[T_1, T_2], |f(x)| = c^{deg(f)}|f(y)|.$$

Let

$$A_2 := \{x \in S | |T_1(x)| \leq |T_2(x)|\}$$

and

$$A_1 := \{x \in S | |T_2(x)| \leq |T_1(x)|\}.$$

The subspaces A_1 and A_2 are stable under the equivalence relation. We will abuse notation and refer to their images in $S/\sim$ as A_1 and A_2 as well.

One can also describe the space $\mathbb{P}^{1,\mathrm{an}}_L$ in the following manner. Following Example 5.4.13, the space $\mathbb{A}^{1,\mathrm{an}}_L$ can be realised as the set of multiplicative seminorms on the algebra $L[T]$ which restrict to the norm on L. We endow this set with the weakest topology such that if $f \in L[T]$ then the function from $\mathbb{A}^{1,\mathrm{an}}_L$ to $\mathbb{R}$ defined by $x \mapsto |f(x)|$ is continuous. Let $y \in \mathbb{A}^{1,\mathrm{an}}_L(L)$. This means that y corresponds to a morphism $L[T] \to L$. Such a morphism defines a seminorm on $L[T]$ and hence corresponds to a point on $\mathbb{A}^{1,\mathrm{an}}_L$. With this

topology the sets $B(y,r) := \{p \in \mathbb{A}_L^{1,\mathrm{an}} || (T-y)(p)| < r\}$ form a family of open neighbourhoods around the point y. These open sets are precisely the Berkovich open balls around y of radius r. As a set $\mathbb{P}_L^{1,\mathrm{an}} = \mathbb{A}_L^{1,\mathrm{an}} \cup \infty$. A basis of open neighbourhoods around the point ∞ is given by sets of the form $\{p \in \mathbb{P}_L^{1,\mathrm{an}} || T(p)| > r\}$.

We now identify these two descriptions of $\mathbb{P}_L^{1,\mathrm{an}}$ in order to relate the family $\mathcal{O}_{x_L}$ as x_L varies along the L-points of $\mathbb{A}_L^{1,\mathrm{an}}$. We go back to the first description of $\mathbb{P}_L^{1,\mathrm{an}}$. Let $S' \subset S$ denote the sub collection of seminorms such that if $p \in S'$ then $|T_2(p)| \neq 0$. The set S' is stable for the equivalence relation and set $A' := S'/\sim$. By definition, the elements of A' define multiplicative seminorms on the algebra $L[T_1/T_2]$. Hence we have a function from $A' \to \mathbb{A}_L^{1,\mathrm{an}}$. With the induced topology on A' it can be shown that we have a homeomorphism $H : A' \to \mathbb{A}_L^{1,\mathrm{an}}$.

An L-point of $S'/\sim$ can be uniquely described by means of homogenous coordinates as in the Proj construction. Let $x_L \in S'/\sim$ then x_L can be represented by homogenous coordinates $[a : b]$ where $a, b \in L$ and $b \neq 0$. By definition, $H([a : b]) = a/b$, where by a/b we mean the L-point on $\mathbb{A}_L^{1,\mathrm{an}}$ defined by the equation $T_1/T_2 = a/b$. For the calculations that follow, we will assume without loss of generality that $x_L = [a : 1]$.

Let $B(H(x_L), r)$ be a Berkovich open ball around the point $H(x_L)$ of radius $r \leq 1$. By definition $B(H(x_L), r) \in \mathcal{O}_{x_L}$. We will now write down its associated 4-tuple $h_L(B(H(x_L), r))$. We divide the problem into three cases.

(1). If $|a| < 1$. The point x_L does not belong to the closed subspace A_1. By definition of the function h_L we have that $h_L(B(H(x_L), r)) = ((1,1),(r,1))$.

(2). If $|a| = 1$. The point x_L belongs to both A_1 and A_2. We then have that $h_L(B(H(x_L), r)) = ((1,r),(r,1))$.

(3). If $|a| > 1$. The point x_L does not belong to the closed space A_2. It can be shown that $h_L(B(H(x_L), r)) = ((1, r/|a|^2),(1,1))$.

Similarly, every element of the family $\mathcal{O}_{x_L}$ corresponds to a Berkovich open ball around $H(x_L)$. Let $O \in \mathcal{O}_{x_L}$ and let $B(H(x_L), s)$ be the corresponding Berkovich open ball around x_L of radius s. The radius of this ball can be expressed in terms of the 4-tuple, $h_L(O)$ as follows.

(1). If $|a| < 1$. If $h_L(O) = ((1,1),(r,1))$ then the corresponding Berkovich open ball around x_L has radius $s = r$.

(2). If $|a| = 1$. If $h_L(O) = ((1,r),(r,1))$ then $s = r$.

(3). If $|a| > 1$ and $h_L(O) = ((1,r),(1,1))$. If $r \leq 1/|a|$ then $s = r|a|^2$. If $r > 1/|a|$ then O is an open neighbourhood of the point ∞. It is a Berkovich open ball around ∞ but is not a Berkovich open ball contained in $\mathbb{A}_L^{1,\mathrm{an}}$.

The calculations above will be made use of in Section 8.5. In future, we will not refer to the homeomorphism H and use x_L itself to denote $H(x_L)$.

8.2 An application of the reduction morphism

Our goal in this section is to prove Proposition 8.2.3 which we will make use of in the proof of Theorem 8.4.3. We do so using the reduction map described in Section 5.7. Let L be an algebraically closed, complete, non-Archimedean real valued field which is non-trivially valued. We will write the value group multiplicatively in this section. Let $A := L[T_1, \ldots, T_n]$ and I be an ideal in this polynomial algebra such that $B := L[T_1, \ldots, T_n]/I$ is an integral domain. Let $U := \mathrm{Spec}(B)$. Let $\mathcal{A} := L\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ and $\mathcal{B} := L\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}/I$ be strict L-affinoid algebras with $r_i \in |L^*|$. We use $\mathbf{r}$ to denote the n-tuple $(r_1, \ldots, r_n)$. Let $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ denote the reduction morphisms $\mathcal{M}(\mathcal{A}) \to \widetilde{\mathcal{M}(\mathcal{A})}$ and $\mathcal{M}(\mathcal{B}) \to \widetilde{\mathcal{M}(\mathcal{B})}$ respectively. We simplify notation and use X to denote the affinoid space $\mathcal{M}(\mathcal{B})$. We assume without loss of generality that the space X contains the point at the origin of $\mathrm{Spec}(A)^{\mathrm{an}}$. We have a closed immersion $i : \mathcal{M}(\mathcal{B}) \hookrightarrow \mathcal{M}(\mathcal{A})$. By Section 5.7, the reduction morphism is a functorial construction and we have an associated morphism between $\tilde{L}$-schemes $\tilde{i} : \widetilde{\mathcal{M}(\mathcal{B})} \to \widetilde{\mathcal{M}(\mathcal{A})}$. Note that this morphism is not necessarily a closed immersion, it is however finite [[BGR], Theorem 6.3.4/2].

Lemma 8.2.1. *Let $x := (a_1, \ldots, a_n) \in L^n$ be an L-point of the affinoid space X. Let $\mathbf{B}((a_1, \ldots, a_n), \mathbf{r}) \subset \mathcal{M}(\mathcal{A})$ denote the Berkovich open ball around $(a_1, \ldots, a_n)$ of polyradius $\mathbf{r}$. Let $\tilde{x} := \pi_{\mathcal{A}} \circ i(x)$. Let $\tilde{i}^{-1}(\tilde{x}) = \{\tilde{y}_1, \ldots, \tilde{y}_t\}$. We have that*

1. $$\mathbf{B}((a_1, \ldots, a_n), \mathbf{r}) \cap X = \bigcup_i \pi_{\mathcal{B}}^{-1}(\tilde{y}_i).$$

   The connected components of the open set $\mathbf{B}((a_1, \ldots, a_n), \mathbf{r}) \cap X$ are the $\pi_{\mathcal{B}}^{-1}(\tilde{y}_i)$ for all i.

2. *There exists a finite set of polynomials $\mathbf{F} := \{F_1, \ldots, F_t\} \subset L[T_1, \ldots, T_n]$ such that for any $y \in X(L)$, the open set $\pi_{\mathcal{B}}^{-1}(\pi_{\mathcal{B}}(y)) = \bigcap_{F \in \mathbf{F}} D_X(F, y)$ where $D_X(F, y)$ is the open set $\{p \in X || (F - F(y))(p)| < 1\}$.*

Proof. 1. The reduction map $X \mapsto \tilde{X}$ is functorial on the category of L-affinoid spaces. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{M}(\mathcal{B}) \arrow[r, "i"'] \arrow[d, "\pi_B"] & \mathcal{M}(\mathcal{A}) \arrow[d, "\pi_A"] \\
\widetilde{\mathcal{M}(\mathcal{B})} \arrow[r, "\tilde{i}"'] & \widetilde{\mathcal{M}(\mathcal{A})}
\end{tikzcd}$$

.

In the diagram, we used i to denote the closed immersion $\mathcal{M}(\mathcal{B}) \hookrightarrow \mathcal{M}(\mathcal{A})$ and $\tilde{i}$ to denote the morphism $\widetilde{\mathcal{M}(\mathcal{B})} \to \widetilde{\mathcal{M}(\mathcal{A})}$. By the commutativity of the diagram we need only show that $\pi_{\mathcal{A}}^{-1}(\pi_{\mathcal{A}}(x)) = \mathbf{B}((a_1, \ldots, a_n), \mathbf{r})$.

Let $p \in \mathcal{M}(\mathcal{A})$. The point p defines a morphism $\mathcal{A} \to \mathcal{H}(p)$ which induces a morphism $\tilde{\mathcal{A}} \to \widetilde{\mathcal{H}(p)}$. Let $h_i \in L$ be such that $|h_i| = r_i^{-1}$. It can be verified directly from the definition of $\tilde{\mathcal{A}}$ that $\tilde{\mathcal{A}} = \tilde{L}\left[\tilde{T}_1, .., \tilde{T}_n\right]$ where $\tilde{T}_i$ is the image of $h_i.T_i$ for the reduction map $\mathcal{A}^\circ \to \tilde{\mathcal{A}}$. The $\tilde{L}$-point $\pi_{\mathcal{A}}(x)$ is defined by the $\tilde{L}$-morphism $\tilde{\mathcal{A}} \to \tilde{L}$ which maps $\tilde{T}_i \mapsto \widetilde{h_i.a_i}$. It follows that $\pi_{\mathcal{A}}(p) = \pi_{\mathcal{A}}(x)$ if and only if $|(h_i.T_i - h_i.a_i)(p)| < 1$. Hence we have that $\pi_{\mathcal{A}}^{-1}(\pi_{\mathcal{A}}(x)) = \mathbf{B}((a_1, \ldots, a_n), \mathbf{r})$. By [[Bos], Kor 6.2] and Proposition 5.7.1, the sets $\pi_{\mathcal{B}}^{-1}(\tilde{y}_i)$ are connected and open.

2. By definition, $\mathcal{B}^\circ := \{x \in \mathcal{B} | \rho_{\mathcal{B}}(x) \leq 1\}$ where $\rho_{\mathcal{B}}$ denotes the spectral norm associated to the affinoid algebra $\mathcal{B}$. The ring $\mathcal{B}^\circ$ contains the ideal $\mathcal{B}^{\circ\circ} := \{x \in \mathcal{B} | \rho_{\mathcal{B}}(x) < 1\}$ and we denote the quotient $\tilde{\mathcal{B}}$. By [[BGR], 6.3.4/3], $\tilde{\mathcal{B}}$ is a finite type $\tilde{L}$-algebra. Let $\{\tilde{F}_1, \ldots, \tilde{F}_t\}$ be a set of generators of $\tilde{\mathcal{B}}$ and let $\mathbf{F} := \{F_1, \ldots, F_t\}$ be a set of elements in $\mathcal{B}^\circ$ such that the image of F_j in $\tilde{\mathcal{B}}$ is $\tilde{F}_j$. We can choose the F_j so that they are polynomials in $L[T_1, \ldots, T_n]$. Let $y \in X \cap \mathbf{B}((a_1, \ldots, a_n), \mathbf{r})$ be an L-point. Then $\tilde{y} := \pi_{\mathcal{B}}(y) \in \tilde{X}$ is an $\tilde{L}$-point. The point $\tilde{y}$ is uniquely defined by a morphism $\tilde{y} : \tilde{\mathcal{B}} \to \tilde{L}$. This morphism is in turn determined by its values at the generators $\tilde{F}_i$. The image of $\tilde{F}_j$ for the morphism $\tilde{y}$ is $\widetilde{F_j(y)} \in \tilde{L}$. Let $p \in X$. The point $\pi_{\mathcal{B}}(p) \in \tilde{X}$ defines a morphism $\tilde{\mathcal{B}} \to \widetilde{\mathcal{H}(p)}$. It follows that $\pi_{\mathcal{B}}(p) = \tilde{y}$ if and only if the images of the $\tilde{F}_j$ for the morphism defined by $\pi_{\mathcal{B}}(p)$ are equal to $\widetilde{F_j(y)}$ i.e. if and only if for every j, $|(F_j - F_j(y))(p)| < 1$. This proves (2).

□

Given a finite set of polynomials $\mathbf{F} \subset L[T_1, \ldots, T_n]$ and a point $x \in X(L)$, we define $\mathfrak{V}_X(\mathbf{F}, x) := \cap_{F \in \mathbf{F}} D_X(F, x)$ where $D_X(F, x) = \{p \in X || (F - F(x))(p)| < 1\}$. We now prove the result which we will use in Section 8.4. We make use of the notation introduced above. Let D be a finite B-algebra which contains B and is also an integral domain. Hence $D = A[S_1, \ldots, S_m]/\langle I, J\rangle$ where J is an ideal in the polynomial algebra $C := L[T_1, \ldots, T_n, S_1, \ldots, S_m]$. Let $U' := \mathrm{Spec}(D)$. Let ϕ denote the finite surjective morphism $U' \to U$. It induces a finite surjective morphism $\phi^{\mathrm{an}} : U'^{\mathrm{an}} \to U^{\mathrm{an}}$. The strict affinoid space X is an affinoid domain in U^{an}. By [[B3], 2.1.8, 2.1.9], the finiteness of the morphism ϕ^{an} implies that $\mathcal{B}[S_1, ...S_m]/J$ is a strict affinoid algebra and

$$(\phi^{\mathrm{an}})^{-1}(X) = \mathcal{M}(\mathcal{B}[S_1, ...S_m]/J).$$

Let $\mathcal{D}$ denote the affinoid algebra $\mathcal{B}[S_1, ...S_m]/J$. After a suitable change of coordinates we can assume that $\mathcal{M}(\mathcal{D})$ contains the point at the origin of $(\mathrm{Spec}(C))^{\mathrm{an}}$ and in addition that ϕ maps the point at the origin of $\mathcal{M}(\mathcal{D})$ to the origin in $X = \mathcal{M}(\mathcal{B})$. We hence assume that $\mathcal{D}$ is of the form $L\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, s_1^{-1}S_1, \ldots, s_m^{-1}S_m\}/\langle I, J\rangle$ where the s_i are non negative real numbers. We define $Y := \mathcal{M}(\mathcal{D})$ and $\mathcal{C} = L\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, s_1^{-1}S_1, \ldots, s_m^{-1}S_m\}$.

Let x be an L-point of X. Let W denote the Berkovich open ball around x of polyradius $\mathbf{l} := (l_1, \ldots, l_n)$ contained in $\mathcal{M}(\mathcal{A})$ where the l_i are positive real numbers less than or equal to r_i which belong to the value group $|L^*|$.

Remark 8.2.2. *Lemma 8.2.1 implies that there exists a finite set of L-points $P \subset X$ and polynomials $\{F_1, \ldots, F_t\} \subset L[T_1, \ldots, T_n]$ such that*

$$W \cap X = \bigcup_{x \in P} \mathfrak{V}_X(\mathbf{F}, x). \tag{8.1}$$

Although this is not the exact version of Lemma 8.2.1, we can derive this formulation as follows. Let $\mathcal{B}' := L\{l_1^{-1}T_1, \ldots, l_n^{-1}T_n\}/I$. By Lemma 8.2.1, we have that

$$W \cap X = \bigcup_{x \in P} \mathfrak{V}_{\mathcal{M}(\mathcal{B}')}(\mathbf{F}', x).$$

for some finite set of polynomials $\mathbf{F}'$ and a finite set of L-points P. For every l_i let $e_i \in L$ be such that $|e_i| = l_i^{-1}$. Extending the set $\mathbf{F}'$ by adding the polynomials e_iT_i will yield equation (2). The set P is chosen so that the right hand side of (2) is the disjoint union of open sets.

Proposition 8.2.3 below concerns itself with the nature of the preimage $(\phi^{\mathrm{an}})^{-1}(W \cap X)$.

Proposition 8.2.3. *There exists a finite set of polynomials $\mathbf{G} := \{G_1, \ldots, G_{t'}\} \subset L[T_1, \ldots, T_n, S_1, \ldots, S_m]$ and a finite set of points $Q \subset Y(L)$ such that*

1. *We have the following equality of sets*

$$(\phi^{\mathrm{an}})^{-1}(W \cap X) = \bigcup_{y \in Q} \mathfrak{V}_Y(\mathbf{G}, y).$$

2. *The $\{\mathfrak{V}_Y(\mathbf{G}, y)\}_{y \in Q}$ are the connected components of the space $(\phi^{\mathrm{an}})^{-1}(W \cap X)$.*

3. *When the restriction of the morphism ϕ^{an} to the open set $(\phi^{\mathrm{an}})^{-1}(W \cap X)$ is an open morphism, we can choose Q to be the set $\phi^{-1}(P)$ and for any $y \in \phi^{-1}(P)$, we have that $\phi^{\mathrm{an}}(\mathfrak{V}_Y(\mathbf{G}, y)) = \mathfrak{V}_X(\mathbf{F}, \phi(y))$.*

Proof. The Berkovich open ball $W \subset \mathcal{M}(\mathcal{A})$ has polyradius $(l_1, \ldots, l_n)$ with $l_i \in |L^*|$ for every i. We will assume without loss of generality that the point x has coordinates $(0, \ldots, 0)$. Let $\mathcal{B}' := L\{l_1^{-1}T_1, \ldots, l_n^{-1}T_n\}/I$. Observe that the affinoid space $\mathcal{M}(\mathcal{B}')$ is the intersection of the Berkovich closed disk centred at x in $\mathcal{M}(\mathcal{A})$ of polyradius $(l_1, .., l_n)$ and $(\mathrm{Spec}(B))^{\mathrm{an}}$. Let $\mathcal{D}' := \mathcal{B}'[S_1, .., S_m]/J$. By [[B3], 2.1.8, 2.1.9], $\mathcal{D}'$ is a strict L-affinoid algebra since it is a finite $\mathcal{B}'$-algebra. By definition $\mathcal{D}'$ contains $\mathcal{B}'$. Furthermore,

$$(\phi^{\mathrm{an}})^{-1}(\mathcal{M}(\mathcal{B}')) = \mathcal{M}(\mathcal{D}').$$

As with the affinoid algebra $\mathcal{D}$, we can write

$$\mathcal{D}' = L\{l_1^{-1}T_1, \ldots, l_n^{-1}T_n, {l'}_1^{-1}S_1, \ldots, {l'}_m^{-1}S_m\}/\langle I, J\rangle$$

where the l'_i are non negative real numbers belonging to $|L^*|$.

Consider the following commutative diagram.

$$\begin{tikzcd}
\mathcal{M}(\mathcal{D}') \arrow[r, "\phi^{\mathrm{an}}"'] \arrow[d, "\pi_{\mathcal{D}'}"] & \mathcal{M}(\mathcal{B}') \arrow[d, "\pi_{\mathcal{B}'}"] \\
\widetilde{\mathcal{M}(\mathcal{D}')} \arrow[r, "\widetilde{\phi^{\mathrm{an}}}"'] & \widetilde{\mathcal{M}(\mathcal{B}')}
\end{tikzcd}$$

The morphism $\mathcal{M}(\mathcal{D}') \to \mathcal{M}(\mathcal{B}')$ is finite. By [[BGR], Theorem 6.3.4/2], the induced morphism between the associated reductions $\widetilde{\mathcal{M}(\mathcal{D}')} \to \widetilde{\mathcal{M}(\mathcal{B}')}$ is finite as well. For every $x_i \in P$, let $\tilde{x}_i$ be the image of x_i for the reduction morphism $\pi_{\mathcal{B}'}$. Let $\tilde{Q} := \{\tilde{z}_1, \ldots, \tilde{z}_v\} \subset \widetilde{\mathcal{M}(\mathcal{D}')}$ be the set of preimages of the set $\{\tilde{x}_i | x_i \in P\}$ for the morphism $\widetilde{\phi^{\mathrm{an}}}$. From the commutative diagram above, we have the following equality

$$\bigcup_{\tilde{z}_i \in \tilde{Q}} \pi_{\mathcal{D}'}^{-1}(\tilde{z}_i) = \bigcup_{x_j \in P} (\phi^{\mathrm{an}})^{-1}(\pi_{\mathcal{B}'}^{-1}(\pi_{\mathcal{B}'}(x_j))). \tag{8.2}$$

By Proposition 5.7.1 and [[Bos], Kor 6.2], the sets $\pi_{\mathcal{D}'}^{-1}(\tilde{z}_i) \subset \mathcal{M}(\mathcal{D}')$ are connected and open. From the commutative diagram we can also infer the following inequality

$$\bigcup_{y_i \in \phi^{-1}(P)} \pi_{\mathcal{D}'}^{-1}(\pi_{\mathcal{D}'}(y_i)) \subseteq \bigcup_{\tilde{z}_i \in H} \pi_{\mathcal{D}'}^{-1}(\tilde{z}_i). \tag{8.3}$$

Let Q be a set of L-points of $\mathcal{M}(\mathcal{D}')$ which are in bijection with the set $\tilde{Q}$ via the reduction morphism $\pi_{\mathcal{D}'}$. By Remark 8.2.2, Lemma 8.2.1 and equation (3) there exists a finite set of polynomials $\mathbf{F} \subset L[T_1, \ldots, T_n]$ and $\mathbf{G}_0 \subset L[T_1, \ldots, T_n, S_1, \ldots, S_m]$ such that

$$W \cap X = \bigcup_{x \in P} \mathfrak{V}_{\mathcal{M}(\mathcal{B}')}(\mathbf{F}, x) = \bigcup_{x \in P} \mathfrak{V}_X(\mathbf{F}, x)$$

and

$$(\phi^{\mathrm{an}})^{-1}(W \cap X) = \bigcup_{z \in Q} \mathfrak{V}_{\mathcal{M}(\mathcal{D}')}(\mathbf{G}_0, z).$$

To complete the proof of the proposition, we enlarge the set $\mathbf{G}_0$ as follows. Recall that in Remark 8.2.2, we chose $e_i \in L$ such that $|e_i| = l_i^{-1}$. Likewise let $|e'_i| = {l'}_i^{-1}$. Such elements exist as we had assumed that l_i and l'_i belong to $|L^*|$. Let $T'_i := e_i T_i$ and $S'_i := e'_i S_i$. Let $\mathbf{G} := \mathbf{G}_0 \cup \{T'_i, \ldots, T'_n, S'_1, \ldots, S'_m\}$. Observe that since $\mathfrak{V}_{\mathcal{M}(\mathcal{D}')}(\mathbf{G}_0, y) = \pi_{\mathcal{D}'}^{-1}(\pi_{\mathcal{D}'}(y))$, by Lemma 8.2.1(1) we must have that $\mathfrak{V}_{\mathcal{M}(\mathcal{D}')}(\mathbf{G}_0, y) \subseteq B(y, (l_1, \ldots, l_n, l'_1, \ldots, l'_m))$ where $B(y, (l_1, \ldots, l_n, l'_1, \ldots, l'_m))$

is the Berkovich open ball in $\mathcal{M}(\mathcal{C})$ around y of polyradius $(l_1, \ldots, l_n, l'_1, \ldots, l'_m)$. It follows that

$$\mathfrak{V}_{\mathcal{M}(\mathcal{D}')}(\mathbf{G}_0, y) = \mathfrak{V}_Y(\mathbf{G}, y).$$

This concludes parts (1) and (2) of the proposition.

We now show that the inequality (4) above which is

$$\bigcup_{y_i \in \phi^{-1}(P)} \pi_{\mathcal{D}'}^{-1}(\pi_{\mathcal{D}'}(y_i)) \subseteq \bigcup_{\tilde{z}_i \in H} \pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$$

is in fact an equality when the restriction ϕ^{an} to $(\phi^{\mathrm{an}})^{-1}(W \cap X)$ is an open morphism. In this case, we claim that if $\tilde{z}_i \in \tilde{Q}$ then there exists $x_j \in P$ such that ϕ^{an} restricts to a surjection from $\pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$ onto $\pi_{\mathcal{B}'}^{-1}(\pi_{\mathcal{B}'}(x_j))$. The morphism ϕ^{an} restricts to a morphism between $\pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$ and the disjoint union of open sets - $\bigcup_{x \in P} \pi_{\mathcal{B}'}^{-1}(\pi_{\mathcal{B}'}(x))$. The connectedness of $\pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$ implies that there exists $x_j \in P$ such that the image $\phi^{\mathrm{an}}((\pi_{\mathcal{D}'})^{-1}(\tilde{z}_i))$ is contained in $\pi_{\mathcal{B}'}^{-1}(\pi_{\mathcal{B}'}(x_j))$. Also, the restriction $\phi^{\mathrm{an}} : \mathcal{M}(\mathcal{D}') \to \mathcal{M}(\mathcal{B}')$ is closed as it is a finite morphism. The set $\pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$ is both open and closed in $\bigcup_{\tilde{z}_i \in \tilde{Q}} \pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$. As $\pi_{\mathcal{B}'}^{-1}(\pi_{\mathcal{B}'}(x_j))$ is connected, we must have that ϕ^{an} restricts to a surjection from $\pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$ onto $\pi_{\mathcal{B}'}^{-1}(\pi_{\mathcal{B}'}(x_j))$. It follows that there exists $y \in \phi^{-1}(P)$ such that $y \in \pi_{\mathcal{D}'}^{-1}(\tilde{z}_i)$ from which we get the equality

$$\bigcup_{y_i \in \phi^{-1}(P)} \pi_{\mathcal{D}'}^{-1}(\pi_{\mathcal{D}'}(y_i)) = \bigcup_{\tilde{z}_i \in \tilde{Q}} \pi_{\mathcal{D}'}^{-1}(\tilde{z}_i).$$

This proves part (3) of the proposition. □

8.3 The theorem for $\widehat{V}$

We now reinterpret Theorem 8.4.3 for the spaces $\widehat{V}$ discussed in Section 4. Let V' and V be integral projective k-varieties such that V is normal and let $\phi : V' \to V$ be a finite surjective morphism. The morphism ϕ induces a pro-definable map $\widehat{\phi} : \widehat{V}' \to \widehat{V}$. We write the structure of the value group additively in this section.

As before, we fix an embedding $V \hookrightarrow \mathbb{P}^n_k$ and an affine chart of $\mathbb{P}^n_k$. We will regard the spaces V', V and $\mathbb{P}^n$ as k-definable sets in ACVF. As in Section 3, we fix $\mathbb{U}$ - a very large saturated model of ACVF and assume that every model of interest to us is a small sub structure of $\mathbb{U}$.

One of the advantages of working in the model theoretic setting is that we need no longer concern ourself with the process of extending scalars. In Section 2.1 of [HL] a brief discussion concerning definable sets is given. We reproduce a part of that discussion here. Let σ be a formula in ACVF with parameters contained in a structure C. The formula σ defines a functor from the category of models which contain C and elementary embeddings of ACVF to the category of sets i.e. given a model M of ACVF which contains C, the functor Z_σ associates M to the set $Z_\sigma(M) := \{a \in M | M \models \sigma(a)\}$. The functor Z_σ is completely determined by the large set $Z_\sigma(\mathbb{U})$. The set of L-points of $V \times_k L$ in the algebraic sense is the set $V(L)$ (in the model theoretic sense) where the latter is not to be confused with the scheme theoretic notion $\mathrm{Hom}_k(\mathrm{Spec}(L), V)$. The points

which are not closed in the variety V correspond to k-types which concentrate on V.

As in Proposition 5.7.4, the definable set $\mathbb{P}^n$ can be realised as the union of $n+1$ closed disks A_i^0 which are glued together definably. Each A_i^0 is a 0-definable sub set of $\mathbb{A}^n$ and comes equipped with definable functions $T_j/T_i : A_i^0 \to VF$ for $j \in \{1, \ldots, n+1\}$ and where VF denotes the value field sort. These functions define the coordinates of the points of A_i^0. In Section 8.1, we used these functions to define Berkovich open balls around points. We repeat that procedure to define for every $x \in \mathbb{P}^n$, a family of definable sets $\mathcal{O}_x^0$ which are $v+g$ open neighbourhoods of x. The notion of a $v+g$ topology was introduced in [[HL], Section 3.7].

Definition 8.3.1. *(The family $\mathcal{O}_x^0$)* Let $x \in \mathbb{P}^n$. Let $i \in \{1, \ldots, n+1\}$ be such that $x \in A_i^0$. The family $\mathcal{O}_x^0$ is defined to be the collection of definable sets $O \subset \mathbb{P}^n$ which are of the form $\{x \in \mathbb{P}^n | val(T_j/T_i - T_j/T_i(x)) > r_j\}$ where $r_j \in \Gamma$, $r_j \geq 0$ and val denotes the valuation $VF \to \Gamma_\infty$. It can be checked that the family $\mathcal{O}_x^0$ defined in this way is independent of the A_i^0 chosen i.e. if $x \in A_i^0 \cap A_j^0$ and $O = \{x \in \mathbb{P}^n | val(T_t/T_i - T_t/T_i(x)) > r_t\}$ then there exists $r_t' \in \Gamma$, $r_t' \geq 0$ such that $O = \{x \in \mathbb{P}^n | val(T_t/T_j - T_t/T_j(x)) > r_t'\}$.

As in Section 8.1, we have a function $h : \mathcal{O}_x^0 \to [0, \infty]^{(n+1)^2}$ which defines the poly radii of elements of $\mathcal{O}_x^0$. Precisely, if $x \in A_i^0$ and $O \in \mathcal{O}_x^0$ then O is uniquely defined by its polyradius $\mathbf{r}_i := (r_1, \ldots, r_{n+1})$ where we set $r_i = 0$. If $x \notin A_i^0$ then we set $\mathbf{r}_i := (0, \ldots, 0)$. We define $h(O) := (\mathbf{r}_i)_i$. Observe that if x and $h(O)$ are defined over a model M of ACVF then $\widehat{O}$ is an open pro-M definable subspace of $\widehat{\mathbb{P}}^n$. By definition the function h extends to a map $h : \mathcal{O}^0 := \bigcup_{x \in \mathbb{P}^n} \mathcal{O}_x^0 \to [0, \infty]^{(n+1)^2}$. Observe that for $x \in \mathbb{P}^n(L)$ where L is a non-Archimedean real valued extension of k and $O \in \mathcal{O}_x^0(\mathbb{R}_\infty)$, we have that $h(O) = \log(h_L(O^{\mathrm{an}}))$ (Remark 1.3, Section 8.1).

Definition 8.3.2. *(The set $R_{X,e} \subset \Gamma_\infty^{(n+1)^2} \times V$)* Let X be a projective k-variety and $e : X \hookrightarrow \mathbb{P}^n$ be an embedding. We define $R_{X,e} \subset \Gamma_\infty^{(n+1)^2} \times X$ to be the set of pairs $(\mathbf{r}, x)$ where $x \in X$ and $\mathbf{r} \in \Gamma_\infty^{(n+1)^2}$ is such that there exists an element $O \in \mathcal{O}_x^0$ for which $h(O) = \mathbf{r}$.

Lemma 8.3.3. *The set $R_{X,e}$ is k-definable.*

Proof. We write $\Gamma_\infty^{(n+1)^2} = P_1 \times \ldots \times P_{n+1}$ where $P_i = \Gamma_\infty^{(n+1)}$. By definition, we have that $R_{X,e} \subset (\prod_i P_i) \times X$. Let $R_{X,e,i} \subset R_{X,e}$ be the set of pairs $((\mathbf{r}_j)_j, x) \in R_{X,e}$ with $\mathbf{r}_j \in P_j$ and $x \in X \cap A_i^0$. Let $p_i : (\prod_j P_j) \times X \to P_i \times X$ denote the projection map. The map p_i is definable. Let $((\mathbf{r}_j)_j, x) \in R_{X,e,i}$. By assumption, there exists $O \in \mathcal{O}_x^0$ such that $h(O) = (\mathbf{r}_j)_j$. As $x \in A_i^0$, we must have that $h(O)$ is uniquely determined by the polyradius $\mathbf{r}_i$. We deduce from this that the map p_i restricts to a bijection from $R_{X,e,i}$ onto $P_i \times X$. The set $P_i \times X$ is k-definable and as p_i is definable, we must have that $R_{X,e,i}$ is k-definable. As $R_{X,e} = \bigcup_{1 \leq i \leq n+1} R_{X,e,i}$, we conclude that $R_{X,e}$ is k-definable. $\square$

As stated above, the goal of this section is to prove a version of Theorem 8.4.3 in the model theoretic setting. Recall that we are given a finite morphism $\phi : V' \to V$ between integral projective k-varieties with V normal. We fixed an

embedding $e : V \hookrightarrow \mathbb{P}^n$ for some $n \in \mathbb{N}$. For the remainder of this section, we use R in place $R_{V,e}$.

Remark 8.3.4. *In Remark 1.5, we introduced a collection S of functions from $\mathbb{R}_{>0}^{(n+1)^2}$ to $\mathbb{R}_{>0}$ an element of which extends to a function $\mathbb{R}_{\geq 0}^{(n+1)^2} \to \mathbb{R}_{\geq 0}$ naturally. When writing the value group additively, we adapt the family S as follows. Firstly,* $\log : (\mathbb{R}_{>0}, \times) \to (\mathbb{R}, +)$ *(Remark 1.3) is an isomorphism of abelian groups which reverses the ordering and whose inverse is the function* $\exp : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$ *which maps $x \mapsto c^x$. If $g \in S$, we define $g' : \mathbb{R}^{(n+1)^2} \to \mathbb{R}$. Let $\mathbf{r} = (r_{i,j})_{i,j} \in \mathbb{R}^{(n+1)^2}$. It follows that $c^{\mathbf{r}} \in \mathbb{R}_{>0}^{(n+1)^2}$ where $c^{\mathbf{r}} := (c^{r_{i,j}})_{i,j}$. Let $g'(\mathbf{r}) := \log(g(c^{\mathbf{r}}))$. The properties of the function $g \in S$ imply the following.*

1. *The function g' is continuous with respect to the topology induced by the ordering.*
2. *If $(r_{i,j})_{i,j}$ and $(s_{i,j})_{i,j}$ are $(n+1)^2$-tuples in $\mathbb{R}^{(n+1)^2}$ such that $r_{i,j} \leq s_{i,j}$ then $g'((r_{i,j})_{i,j}) \leq g'((s_{i,j})_{i,j})$.*
3. *g is a definable function in the language of Ordered Abelian groups.*

As in Remark 1.5, we extend the function g' so that it defines a function $\Gamma_\infty^{(n+1)^2} \to \Gamma_\infty$.

Let $g \in S$. As in Section 6, given a real valued model F of ACVF, let $\mathbf{F}$ denote the structure defined by the pair $(F, \mathbb{R}_\infty)$. As in Section 8.1, the function g' induces an ordering on the set $\mathcal{O}^0(\mathbf{F}) := \bigcup_{x \in \mathbb{P}^n(F)} \mathcal{O}_x^0(\mathbf{F})$ where $\mathcal{O}_x^0(\mathbf{F})$ are those elements of $\mathcal{O}_x^0$ which are defined over $\mathbf{F}$. More precisely, as in Lemma 8.1.2, the function $g' \circ h : \mathcal{O}^0(\mathbf{F}) \to \mathbb{R}_\infty$ has the following property. If $O_1, O_2 \in \mathcal{O}^0(\mathbf{F})$ such that $O_1 \subseteq O_2$ then $(g' \circ h)(O_1) \geq (g' \circ h)(O_2)$. The inequality above has been reversed owing to the fact that $h(O_i) = \log(h_F(O_i^{\mathrm{an}}))$. The functions $g \in S$ hence allow us to quantify the size of elements belonging to $\mathcal{O}^0$. The following elementary lemma is made use of at several instances later.

Lemma 8.3.5. *Let U and U' be integral k-varieties and $\phi : U' \to U$ be a finite surjective morphism between them. There exists integral affine k-varieties $W \subset U$, $W' \subset U'$ and W'' along with morphisms $\phi_1 : W' \to W''$ and $\phi_2 : W'' \to W$ such that*

1. *W and W' are Zariski open subsets of U and U' respectively.*
2. *$\phi = \phi_2 \circ \phi_1$.*
3. *The extension of function fields $k(W'') \hookrightarrow k(W')$ induced by ϕ_1 is purely inseparable.*
4. *The extension of function fields $k(W) \hookrightarrow k(W'')$ induced by ϕ_2 is separable.*

Proof. To begin, observe that the morphism ϕ is flat over a Zariski open subset of U and that U' is birational to its normalization. A flat morphism which is of finite type is open. It follows that there exists a Zariski open affine subset $W \subset U$ and a Zariski open set $W' \subset U'$ which is normal and in addition the restriction $\phi : W' \to W$ is flat and surjective.

The extension of function fields $k(W) \hookrightarrow k(W')$ can be realized as the composition of a purely inseparable extension and a separable extension. To be precise, there exists a field $k(W'')$ such that $k(W) \hookrightarrow k(W')$ factorizes into $k(W) \hookrightarrow k(W'')$ which is separable and $k(W'') \hookrightarrow k(W')$ which is purely inseparable.

Let A and A' be k-algebras of finite type such that $W = \mathrm{Spec}(A)$ and $W' = \mathrm{Spec}(A')$. Let A'' denote the normalisation of A in $k(W'')$ and set $W'' := \mathrm{Spec}(A'')$. We hence have a separable morphism $\phi_2 : W'' \to W$. Since A' was constructed to be integrally closed in $k(W')$ and to contain A, we have that the integral closure of A in $k(W')$ must be contained in A'. This implies A' contains A'' and hence we have a purely inseparable morphism $\phi_1 : W' \to W''$. This proves the lemma. □

Lemma 8.3.6. *Let d denote the separable degree of the finite morphism $\phi : V' \to V$. For $p \in \widehat{V}$, the cardinality of the set of preimages $\widehat{\phi}^{-1}(p)$ is bounded above by d and the set of simple points (cf. 4.2) x in V for which* $\mathrm{card}(\phi^{-1}(x)) = d$ *is dense in $\widehat{V}$.*

Proof. Let M be a model of ACVF which contains k and $x \in V(M)$. The point x defines a closed point of the variety $V \times_k M$. From algebraic geometry, the cardinality of fibre $\phi_M^{-1}(x)$ is bounded above by d where $\phi_M : V' \times_k M \to V \times_k M$. From our discussion above on definable sets, we conclude that the cardinality of $\phi^{-1}(x)$ is also bounded above by d when ϕ is viewed as a definable map between the definable sets V' and V.

Let $p \in \widehat{V}$. By definition, p is a stably dominated type which concentrates on V. Let us assume that it is defined over a model M of ACVF which contains k. Let a be a realization of the M - type $p_{|M}$. Our discussion above implies that there exists $\{a'_1, \ldots, a'_t\} \subset V'$ such that $\phi^{-1}(a) := \{a'_1, \ldots, a'_t\}$ and $t \leq d$. As the function ϕ is definable over k, we must have that the preimage of the M - type $p_{|M}$ extends to at most t M - types which concentrate on V' and hence at most t stably dominated types over M.

We now verify the remainder of the lemma. By Lemma 8.3.5, there exists affine open sets $U \subset V$, $U' \subset V'$ and an affine scheme U'' along with morphisms $\phi_1 : U' \to U''$ purely inseparable and $\phi_2 : U'' \to U$ separable of degree d such that the restriction of the morphism ϕ to U' factors as $\phi_2 \circ \phi_1$. As k is algebraically closed, there exists an open sub scheme $U_0 \subset U$ over which the morphism ϕ_2 is etale and the cardinality of the set $\phi_2^{-1}(y)$ for any $y \in U_0$ is equal to d. By Lemma 6.1.5, U_0 is dense in $\widehat{V}$. This completes the proof. □

For $O \in \mathcal{O}_x^0$ with $h(O) = \mathbf{r}$, let $N_V(\mathbf{r}, x)$ denote the number of connected components of the space $\widehat{O} \cap \widehat{V}$ and $N_{V'}(\mathbf{r}, x)$ denote the number of connected components of the space $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$.

Lemma 8.3.7. *Let d denote the separable degree of the morphism $\phi : V' \to V$. Let $x \in V$ and $O \in \mathcal{O}_x^0$. Let $\mathbf{r} := h(O)$. The preimage $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$ is the disjoint union of open sets of $\widehat{V}'$ each of which is homeomorphic to $\widehat{O} \cap \widehat{V}$ via $\widehat{\phi}$ if and only if $N_{V'}(\mathbf{r}, x) = d.N_V(\mathbf{r}, x)$. Furthermore, if $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$ is the disjoint union of open subsets of $\widehat{V}'$ each of which is homeomorphic to $\widehat{O} \cap \widehat{V}$ via $\widehat{\phi}$ then the cardinality of the number of preimages of a point in $\widehat{O} \cap \widehat{V}$ is d.*

Proof. Let $(\mathbf{r}, x) \in R$ (cf. Definition 8.3.2) and $O \in \mathcal{O}_x^0$ such that $h(O) = \mathbf{r}$. By 11.1.1 in [HL], there exists a continuous deformation retraction $H : I \times \widehat{O} \cap \widehat{V} \to \widehat{O} \cap \widehat{V}$ such that the image $H(e, \widehat{O} \cap \widehat{V})$ is a Γ-internal subset of $\widehat{O} \cap \widehat{V}$ which is definably homeomorphic to a definable subset Λ in Γ_∞^m for some $m \in \mathbb{N}$. The connected components of Λ are open and there are only finitely many of these. It follows that $\widehat{O} \cap \widehat{V}$ is the finite disjoint union of path connected open sets. Let $\{C_1, \ldots, C_t\}$ be the connected components of $\widehat{O} \cap \widehat{V}$. By a similar argument, $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$ is the disjoint union of a finite number of open sets each of which is path connected. Let $\{C'_1, \ldots, C'_{t'}\}$ denote the connected components of $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$. As the variety V is normal, by Corollary 9.7.2 in [HL], the morphism $\widehat{\phi} : \widehat{V}' \to \widehat{V}$ is open. By Lemma 4.2.25 in loc.cit, the map $\widehat{\phi}$ is closed as well. Hence the morphism $\widehat{\phi}$ is clopen when restricted $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$ and we see that for every j there exists a unique i such that $\widehat{\phi}$ maps C'_j surjectively onto C_i. A map that is clopen is a homeomorphism if it is in addition a bijection. Hence the preimage $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$ is the disjoint union of open sets of $\widehat{V}'$ each of which is homeomorphic to $\widehat{O} \cap \widehat{V}$ via $\widehat{\phi}$ if and only if for every j there exists a unique i such that $\widehat{\phi}$ restricts to a bijection from C'_j onto C_i and the number of preimages of an element $p \in \widehat{O} \cap \widehat{V}$ is constant. By Lemma 8.13, this constant must be d as the set of elements $p \in \widehat{V}$ for which $\mathrm{card}(\widehat{\phi}^{-1}(p)) = d$ is dense in $\widehat{V}$. Since by Lemma 8.13, for $p \in \widehat{V}$ the set $\widehat{\phi}^{-1}(p)$ has cardinality bounded by d it follows that the preimage $\widehat{\phi}^{-1}(\widehat{O} \cap \widehat{V})$ is the disjoint union of $\widehat{V}'$ open sets each of which is homeomorphic to $\widehat{O} \cap \widehat{V}$ via $\widehat{\phi}$ if and only if $t' = dt$ i.e $N_{V'}(\mathbf{r}, x) = dN_V(\mathbf{r}, x)$. □

The proof above can be adapted to show the following result.

Lemma 8.3.8. *Let M be a model of* ACVF*. Let $(\mathbf{r}, x) \in R(M)$ (cf. Definition 8.3.2) and $O \in \mathcal{O}_x^0$ such that $h(O) = \mathbf{r}$. Let $N_{V'}^M(\mathbf{r}, x)$ denote the number of connected components of the space $(\widehat{\phi}^{-1})(\widehat{O}(M) \cap \widehat{V}(M))$ and $N_V^M(\mathbf{r}, x)$ be the number of connected components of the space $\widehat{O}(M) \cap \widehat{V}(M)$. The preimage $\widehat{\phi}^{-1}(\widehat{O}(M) \cap \widehat{V}(M))$ is the disjoint union of open subsets of $\widehat{V}'(M)$ each of which is homeomorphic to $\widehat{O}(M) \cap \widehat{V}(M)$ via $\widehat{\phi}(M)$ if and only if $N_{V'}^M(\mathbf{r}, x) = d.N_V^M(\mathbf{r}, x)$.*

We now prove a lemma which is central to the proof of Theorem 8.3.11. We preserve the notation $N_{V'}^M(\mathbf{r}, x)$ and $N_V^M(\mathbf{r}, x)$ introduced in the preceding lemma.

Lemma 8.3.9. *There exists a k-definable subset D of R (cf. Definition 8.3.2) such that for M a model of* ACVF *with value group $\mathbb{R}_\infty$ we have that $(\mathbf{r}, x) \in D(M)$ if and only if $N_{V'}^M(\mathbf{r}, x) = dN_V^M(\mathbf{r}, x)$ where d denotes the separable degree of the morphism ϕ.*

Proof. Consider the set $X \subset V' \times R$ consisting of tuples $(z, \mathbf{r}, x)$ such that $(\mathbf{r}, x) \in R$, $z \in V'$ and $\phi(z) \in O$ where $O \in \mathcal{O}_x^0$ and $h(O) = \mathbf{r}$. The set X is definable (cf. Lemma 8.3.3) and if pr denotes the projection $X \to R$ then for any $(\mathbf{r}, x) \in R$ the fibre $pr^{-1}(\mathbf{r}, x)$ will be the definable set $\phi^{-1}(O \cap V)$ where $O \in \mathcal{O}_x^0$ such that $h(O) = \mathbf{r}$. For $\tau = (\mathbf{r}, x) \in R$, we will write O_τ for that element $O \in \mathcal{O}_x^0$ such that $h(O) = \mathbf{r}$. Given $\tau \in R$, we write $k(\tau)$ for the definable closure

of $k \cup \{\tau\}$. By Theorem 11.7.1 in [HL] there exists, uniformly in $\tau \in R$, a pro-definable family $H_\tau : I \times \widehat{\phi}^{-1}(\widehat{O}_\tau \cap \widehat{V}) \to \widehat{\phi}^{-1}(\widehat{O}_\tau \cap \widehat{V})$, a finite $k(\tau)$ -definable set $w(\tau)$, a $k(\tau)$- definable set $W_\tau \subset \Gamma_\infty^{w(\tau)}$ and $j_\tau : W_\tau \to H_\tau(e, \widehat{\phi}^{-1}(\widehat{O}_\tau \cap \widehat{V}))$, pro-definable uniformly in τ such that for each $\tau \in R$, H_τ is a deformation retraction and $j_\tau : W_\tau \to H_\tau(e, \widehat{\phi}^{-1}(\widehat{O}_\tau \cap \widehat{V}))$ is a definable homeomorphism and e denotes the end point of the interval I. Let $Z_\tau := H_\tau(e, \widehat{\phi}^{-1}(\widehat{O}_\tau \cap \widehat{V})$. By the claim in the proof of Theorem 14.3.1 [HL], there exists uniformly in τ a $k(\tau)$ -definable set $T_\tau \subset \Gamma_\infty^r$, a $k(\tau)$ - definable set $W(\tau)$ and for $w \in W(\tau)$, a definable homeomorphism $\psi_w : Z_\tau \to T_\tau$. We have in this manner obtained a family of definable subsets of Γ_∞^r parametrized by R. Observe that if M is a model of ACVF with value group $\mathbb{R}_\infty$ and $\tau \in R(M)$ then the image of the deformation retraction $H_\tau(M) : I(\mathbb{R}_\infty) \times \widehat{\phi}^{-1}(\widehat{O}_\tau(M) \cap \widehat{V}(M)) \to \widehat{\phi}^{-1}(\widehat{O}_\tau(M) \cap \widehat{V}(M))$ is definably homeomorphic to $T_\tau(\mathbb{R}_\infty)$.

Let Γ^* be an expansion of Γ to RCF and ACVF$'$ denote the extension of ACVF with the sort Γ^* in place of Γ.

Lemma 8.3.10. *There exists $N \in \mathbb{N}$ such that the set R (cf. Definition 8.3.2) can be partitioned into* ACVF$'$ *definable sets $\{E'_1, \ldots, E'_N\}$ such that if M is a model of* ACVF *whose value group is $\mathbb{R}_\infty$ and $\tau := (\mathbf{r}, x) \in E'_j(M)$ then $\widehat{\phi}^{-1}(\widehat{O}_\tau(M) \cap \widehat{V}(M))$ must have j connected components. In addition, the definable sets E'_j are* ACVF$'$ *- definable with parameters in k.*

Proof. Our discussion above implies the existence of a definable subset $Y \subset R \times \Gamma_\infty^r$ such that for $\tau \in R$, the fibre $Y_\tau := pr^{-1}(\tau)$ is the set T_τ where $pr : R \times \Gamma_\infty^r \to R$ is the projection map onto the first coordinate. In addition, there exists a definable homeomorphism $\psi_w : H_\tau(e, \widehat{\phi}^{-1}(\widehat{O}_\tau \cap \widehat{V})) \to T_\tau$. It suffices hence to show that R can be partitioned into N ACVF$'$ definable sets $\{E'_1, \ldots, E'_N\}$ such that if M is a model of ACVF whose value group is $\mathbb{R}_\infty$ and $\tau := (\mathbf{r}, x) \in E'_j(M)$ then T_τ must have j connected components.

We first show that there exists a uniform triangulation of the definable family $\{Y_\tau | \tau \in R\}$ by means of which we may reduce to proving the result when Y_τ is a complex in $(\Gamma_\infty^*)^r$. More precisely, we show that there exists a definable map $\lambda : Y \to (\Gamma_\infty^*)^r \times R$ such that for every $\tau \in R$, the induced map $\lambda_\tau := \lambda_{|Y_\tau}$ is a definable homeomorphism onto a complex contained in $(\Gamma_\infty^*)^r$. To prove this we make use of a compactness argument identical to that employed in 11.7.1 of [HL]. By loc.cit. it suffices to verify that for $a \in R$, the family of definable maps $\{\lambda_a : Y_a \to (\Gamma_\infty^*)^r\}$ for which λ_a is a definable homeomorphism onto a complex contained in $(\Gamma_\infty^*)^r$ is an ind-definable family. A complex K in $(\Gamma_\infty^*)^r$ is determined completely by its set of vertices and the subsets of this set which span simplexes of K (cf. [[LVD], 8.1]). It follows that the family of complexes contained in $(\Gamma_\infty^*)^r$ is an ind-definable family. Hence, the property that the image of λ_a is a complex is an ind-definable condition. Let $\lambda_a : Y_a \to (\Gamma_\infty^*)^r$ be a definable map whose image C_a is a complex. The property of λ_a being a bijection is definable. The continuity of λ_a can be expressed by asking that for every $x \in Y_a$, $\epsilon > 0$ there exists $\delta > 0$ such that $\{z \in Y_a | d(z, x) < \delta\} \subset \lambda_a^{-1}(\{y \in C_a | d(y, \lambda_a(x)) < \epsilon\})$ where d is a definable metric on $(\Gamma_\infty^*)^r$ which generates the linear topology. Likewise, the continuity of λ_a^{-1} is also a definable condition. Thus, there exists a definable map $\lambda : Y \to (\Gamma_\infty^*)^r \times R$ such that for every $\tau \in R$, the induced map $\lambda_\tau = \lambda_{|Y_\tau}$ is a definable homeomorphism onto

a complex C_τ contained in $(\Gamma^*_\infty)^r$. Let C denote the definable subset of $(\Gamma^*_\infty)^r$ which is the image of the map λ.

As said before, a complex K in $(\Gamma^*_\infty)^r$ is determined by its set of vertices and the subsets of this set that span simplexes contained in K. Let $Vert(n) \subset R$ denote the set of $\tau \in R$ such that the complex C_τ has n vertices. As the family $\{C_\tau | \tau \in R\}$ is a uniformly definable family of complexes. The set $Vert(n)$ is a definable subset of R. Also, $R = \bigcup_{n\in\mathbb{N}} Vert(n)$. By compactness, there exists $m \in \mathbb{N}$ such that for any $\tau \in R$, the complex C_τ has at most m vertices. Let $a \in R$, by 8.2.12 in [LVD] there exists a definable homeomorphism $\gamma_a : C_a \to S_a$ where S_a is the union of faces of a simplex $(e_1, \ldots, e_m)$ and $\{e_1, \ldots, e_m\}$ forms a 0-definable basis of $(\Gamma^*)^m$. Let $\mathfrak{S}_m$ denote the set of objects each of which is the union of faces of the simplex $(e_1, \ldots, e_m)$. The arguments above can be repeated to show that there exists a definable map $\gamma : C \to (\Gamma^*_\infty)^m \times R$ such that for $\tau \in R$ the induced map γ_τ is a definable homeomorphism onto an element $S_\tau \subset (\Gamma^*_\infty)^m$ where $S_\tau \in \mathfrak{S}_m$. Let S denote the image of the map γ. Let $J \in \mathfrak{S}_m$. We define $DF_J \subset R$ to be the set of $\tau \in R$ such that $S_\tau = J$. As the family $\{S_\tau | \tau \in R\}$ is uniformly definable for $\tau \in R$, the set DF_J is definable. Furthermore, by construction $R = \bigcup_{J\in\mathfrak{S}_m} DF_J$. By construction, as τ varies in DF_J, the homeomorphism type of Y_τ is constant. The first part of the lemma can be deduced from this and the fact that the set $\mathfrak{S}_m$ is finite.

We now show that the E'_j are definable with parameters in k. Indeed, the field k^{max} which is a maximally complete field extension of k with value group $\mathbb{R}_\infty$ and residue field $\tilde{k}$ (Section 5.7) can be extended to a model of ACVF$'$. Let $g \in \mathrm{Aut}(k^{max}/k)$. Let $\tau \in R(k^{max})$. As R is definable over k and the T_τ are definable uniformly over $k(\tau)$, it follows that $T_\tau(\mathbb{R}_\infty)$ and $T_{g(\tau)}(\mathbb{R}_\infty)$ have the same number of connected components which implies $g(\tau) \in E'_j(k^{max})$. As $E'_j(k^{max})$ is preserved by the action of $\mathrm{Aut}(k^{max}/k)$, we conclude that it is defined with parameters from k. $\square$

We have thus proved Lemma 8.3.10 and now use it to conclude a proof of 8.3.9. In Lemma 8.3.10, if we were to substitute the set $X \subset V' \times R$ with the k-definable set $X' \subset V \times R$ defined by tuples $(z, \mathbf{r}, x)$ such that $(\mathbf{r}, x) \in R$, $z \in V \cap O$ where $O \in \mathcal{O}^0_x$ and $h(O) = \mathbf{r}$, we would obtain a similar partition of R. That is, there exists N' and a collection of ACVF$'$ k-definable subsets $\{F'_1, \ldots, F'_{N'}\}$ which partition R such that if M is a model of ACVF whose value group is $\mathbb{R}_\infty$ and $\tau := (\mathbf{r}, x) \in F'_j(M)$ then $\widehat{O}(M) \cap \widehat{V}(M)$ must have j connected components.

The two ACVF$'$ partitions of R can be used to give an ACVF$'$ definable set $D' \subset R$ which is defined with parameters from k such that if M is a model of ACVF whose value group is $\mathbb{R}_\infty$ and $\tau := (\mathbf{r}, x) \in D'(M)$ then $N^M_{V'}(\mathbf{r}, x) = dN^M_V(\mathbf{r}, x)$. By Beth's theorem, there exists an ACVF definable set $D \subset R$ defined with parameters in k such that if M is a model of ACVF whose value group is $\mathbb{R}_\infty$ then $D(M) = D'(M)$ i.e. if $\tau \in D(M)$ then $N^M_{V'}(\mathbf{r}, x) = dN^M_V(\mathbf{r}, x)$. $\square$

The following is a version of Theorem 8.4.3 for the spaces $\widehat{V}$.

Theorem 8.3.11. *Let $\phi : V' \to V$ be a finite surjective morphism between irreducible, projective varieties with V normal. Let $g \in S$. There exists a pro-*

definable deformation retraction

$$\psi : I \times \widehat{V} \to \widehat{V}$$

which satisfies the following properties.

1. *Let I be a generalised interval of the form $[i, e]$. The image $T_g := \psi(e, \widehat{V})$ of the deformation retraction ψ is a Γ-internal subset of $\widehat{V}$* [[HL],Chapter 6] *and there exists a definable homeomorphism $j_g : T_g \to \Upsilon_g$ where $\Upsilon_g \subset \Gamma_\infty^n$ is a k-definable set.*

2. *There exists a well defined piecewise linear function $M_g : T_g \to \Gamma_\infty$ which satisfies the following conditions. The function M_g takes values other than ∞. In fact there exists $x \in T_g(k)$ such that $M_g(x) \neq \infty$. Let $\gamma \in T_g$ be a point for which $M_g(\gamma) \neq 0$ and $x \in \psi(e, _)^{-1}(\gamma)$ such that there exists L/k a complete non-Archimedean real valued algebraically closed field extension for which $\Gamma_\infty(L) = \mathbb{R}_\infty$ and $x \in V(L)$. There exists $W \in (g' \circ h)^{-1}(M_g(\gamma)) \cap \mathcal{O}_x^0$ such that the open set $(\widehat{\phi})^{-1}(\widehat{W}(L) \cap \widehat{V}(L)) \subset \widehat{V}'(L)$ decomposes into the disjoint union of open subsets of $\widehat{V}'(L)$, each of which is homeomorphic to $\widehat{W}(L) \cap \widehat{V}(L)$ via $\widehat{\phi}$. Furthermore, let $O \in \mathcal{O}_x^0$ be such that $h(O) \in \mathbb{R}_\infty^{(n+1)^2}$ and the preimage of $\widehat{O}(L) \cap \widehat{V}(L)$ under $\widehat{\phi}$ decomposes into the disjoint union of open sets in $\widehat{V}'(L)$, each homeomorphic to $\widehat{O}(L)$ via the morphism $\widehat{\phi}$. Then $(g' \circ h)(O) \geq M_g(\gamma)$.*

Proof. By Lemmas 8.3.8 and 8.3.9, there exists a k-definable subset D of R such that if M is a model of ACVF with value group $\mathbb{R}_\infty$ then $D(M)$ is the set of tuples $(\mathbf{r}, x)$ defined over M such that if $W \in \mathcal{O}_x^0(M)$ and $h(W) = \mathbf{r}$ then $\widehat{\phi}^{-1}(\widehat{W}(M) \cap \widehat{V}(M))$ is the disjoint union of open subsets of $\widehat{V}(M)$ each of which is homeomorphic to $\widehat{W}(M) \cap \widehat{V}(M)$ via the morphism $\widehat{\phi}$. The k-definable set $D \subset [0, \infty]^{(n+1)^2} \times V$ comes equipped with a projection map $pr : D \to V$. For $x \in V$, let $D_x := pr^{-1}(x)$. By definition, D_x is uniformly definable in x with parameters in $k(x)$. Hence $g' \circ h(D_x) \subset \Gamma_\infty$ is a $k(x)$ - definable set. For $x \in V$, let $g'_{\inf}(x)$ be the infimum of the definable set $g' \circ h(D_x)(\mathbb{U}) \subset \Gamma_\infty(\mathbb{U})$. We set $g'_{\inf}(x) = \infty$ when the set D_x is empty. As D_x is uniformly definable in x with parameters in $k(x)$, $g'_{\inf}$ extends to a k-definable function from $V \to \Gamma_\infty$ which can be extended to a pro-definable function $\widehat{V}$ to Γ_∞.

By Theorem 4.8.1, there exists a k-definable Γ-internal subset T_g of $\widehat{V}$, a pro-definable deformation retraction $H : I \times \widehat{V} \to \widehat{V}$ such that $H(e, \widehat{V}) = T_g$ where e denotes the end point of the interval I and the function $g'_{\inf}$ is constant along the fibres of the deformation retraction. Let M_g denote the restriction of $g'_{\inf}$ to T_g. It remains to verify part (2) of the statement of the theorem.

Let $\gamma \in T_g$ and $x \in V$ such that $H(e, x) = \gamma$. Furthermore, we suppose that L is a model of ACVF over which x is defined and $\Gamma_\infty(L) = \mathbb{R}_\infty$. Let $O \in \mathcal{O}_x^0(L)$ be such that the preimage of $\widehat{O}(L) \cap \widehat{V}(L)$ under $\widehat{\phi}$ decomposes into the disjoint union of open sets in $\widehat{V}'(L)$, each homeomorphic to $\widehat{O}(L)$ via the morphism $\widehat{\phi}$. It follows from Lemma 8.3.8 that $h(O) \in D_x(L)$. Hence from the definition of $g'_{\inf}(x)$ we get that $(g' \circ h)(O) \geq g'_{\inf}(x) = M_g(\gamma)$. Furthermore, by Lemma 8.19 there exists $W \in \mathcal{O}_x^0(L)$ such that $g' \circ h(W) = g'_{\inf}(x)$. This proves part (2) of the theorem.

We now show that the function M_g takes values other than ∞. By Lemma 8.3.5, there exists affine open sets $U \subset V$, $U' \subset V'$ and an affine scheme U'' along with morphisms $\phi_1 : U' \to U''$ purely inseparable and $\phi_2 : U'' \to U$ separable of degree d such that the restriction of the morphism ϕ to U' factors as $\phi_2 \circ \phi_1$. As ϕ_1 is purely inseparable, the induced map $\widehat{\phi_1}$ is a homeomorphism. There exists a smooth open sub scheme $U_0 \subset U$ over which the morphism ϕ_2 is etale. Let x be a k-point in U_0. By Lemma 7.4.1 in [HL], there exists $O \in \mathcal{O}_x^0(\mathbb{R}_\infty)$ with $h(O) \in D_x$ with finite polyradius which implies that $g'_{\inf}(x) < \infty$. $\square$

Let $x \in V$ and M be a model of ACVF which contains k such that $x \in V(M)$ and $\Gamma(M) \subseteq \mathbb{R}$. In the following lemma, we refer to the definable sets D_x which were introduced during the course of Theorem 8.3.11.

Lemma 8.3.12. *Let $x \in V$ and M be a model of* ACVF *which contains k such that $x \in V(M)$ and $\Gamma(M) \subseteq \mathbb{R}$. Let $\mathbf{M}$ denote the structure defined by $(M, \mathbb{R}_\infty)$. There exists $O \in \mathcal{O}_x^0(\mathbf{M})$ such that $h(O) \in D_x(\mathbf{M})$ and $g' \circ h(O) = \inf\{g' \circ h(O') | O' \in \mathcal{O}_x^0(\mathbf{M}) \wedge h(O') \in D_x(\mathbf{M})\}$ i.e. the set $\{g' \circ h(O') | O' \in \mathcal{O}_x^0(\mathbf{M}) \wedge h(O') \in D_x(\mathbf{M})\}$ contains its infimum.*

Proof. Firstly observe that $D_x(\mathbf{M}) = D_x(\mathbb{R}_\infty)$. By definition, the function $g' : \mathbb{R}_\infty^{(n+1)^2} \to \mathbb{R}_\infty$ is continuous with respect to the topology on $\mathbb{R}_\infty$ induced by its ordering. Hence to prove the lemma it suffices to show that $\{h(O') | O' \in \mathcal{O}_x^0(\mathbf{M}) \wedge h(O') \in D_x(\mathbf{M})\} \cup (\infty, \ldots, \infty)$ is compact. As $\{h(O') | O' \in \mathcal{O}_x^0(\mathbf{M}) \wedge h(O') \in D_x(\mathbf{M})\} \cup (\infty, \ldots, \infty) \subseteq [0, \infty]^{(n+1)^2}$, we need only show that it is closed. Let $(O_n)_n$ be a sequence of elements in $\{O' \in \mathcal{O}_x^0(\mathbf{M}) | h(O') \in D_x(\mathbf{M})\}$ such that $(h(O_n))_n$ converges to $\mathbf{r} \in \mathbb{R}_\infty^{(n+1)^2}$. By definition of the family $\mathcal{O}_x^0(\mathbf{M})$, it can be verified that there exists an element $W \in \mathcal{O}_x^0(\mathbf{M})$ such that $h(W) = \mathbf{r}$ and W is uniquely determined by $\mathbf{r}$ and x. We will show that $h(W) \in D_x(\mathbf{M})$. There exists a subsequence $(O_{m_n})_n$ of $(O_n)_n$ such that the sequence of $\mathbb{R}_\infty$ - tuples $((r_{i,m_n})_i)_n := h(O_{m_n})_n$ is either increasing or decreasing at each component i.e. for every i the sequence $(r_{i,m_n})_n$ is either increasing or decreasing. Observe that if $O \in \{O' \in \mathcal{O}_x^0(\mathbf{M}) | h(O') \in D_x(\mathbf{M})\}$ and $O'' \in \mathcal{O}_x^0(\mathbf{M})$ such that $h(O'') \geq h(O)$ then $O'' \in \{O' \in \mathcal{O}_x^0(\mathbf{M}) | h(O') \in D_x(\mathbf{M})\}$. It follows that we can assume the sequence $(O_n)_n$ is increasing i.e. that the sequence $(h(O)_n)_n$ is decreasing with respect to the point wise ordering. This implies that $W = \bigcup_n O_n$ and it can be verified that the ball W must also belong to $\{O' \in \mathcal{O}_x^0(\mathbf{M}) | h(W) \in D_x(\mathbf{M})\}$. $\square$

8.4 Proof of the main theorem

In this section we prove Theorem 8.4.3. Let V and V' be irreducible, projective k-varieties with V normal and $\phi : V' \to V$ be a finite surjective morphism. The morphism ϕ induces a morphism between the respective analytifications. Hence we have

$$\phi^{\mathrm{an}} : V'^{\mathrm{an}} \to V^{\mathrm{an}}.$$

Remark 8.4.1. *We introduced the collection of functions S (Remark 1.5) for the following reason. Let $x \in V^{\mathrm{an}}(L)$ (Remark 1.1). Associated to x is an*

L-point of V_L^{an} which we denoted x_L (Section 1.2). In Section 8.1, we defined a collection of open neighbourhoods $\mathcal{O}_{x_L}$ of x_L along with a function $h_L : \bigcup_{x \in V^{\mathrm{an}}(L)} \mathcal{O}_{x_L} \to \mathbb{R}_{\geq 0}^{(n+1)^2}$. An element $g \in S$ allows us to compare elements of the family $\mathcal{O}_{x_L}$. To be precise, the family $\bigcup_{x \in V^{\mathrm{an}}(L)} \mathcal{O}_{x_L}$ is partially ordered by the partial ordering defined by set theoretic inclusion. By Lemma 8.1.2, if $O_1, O_2 \in \bigcup_{x \in V^{\mathrm{an}}(L)} \mathcal{O}_{x_L}$ with $O_1 \subseteq O_2$ then $g \circ h_L(O_1) \leq g \circ h_L(O_2)$.

We could also avoid using the function g and instead do the following. Let S' denote the collection of 0-definable total orderings of the set $(\mathbb{R}_\infty)^{(n+1)^2}$ which satisfy the following property. If $\leq_p \in S'$ then given a pair of $(n+1)^2$-tuples $(x_i)_i$, $(y_j)_j$ such that $x_i \leq y_i$ for all i, we must have that $(x_i)_i \leq_p (y_i)_i$. We ask in addition that if C is a non empty, compact subspace of $(\mathbb{R}_{\geq 0})^{(n+1)^2}$ then it contains a supremum with respect to the ordering $\leq_p$.

One can prove the following version of the main result:

Theorem 8.4.2. *Let $\phi : V' \to V$ be a finite surjective morphism between irreducible, projective varieties with V normal. Let $\leq_g \in S'$. There exists a generalised real interval $I := [i, e]$ and a deformation retraction*

$$\psi : I \times V^{\mathrm{an}} \to V^{\mathrm{an}}$$

which satisfies the following properties.

1. *The image $\psi(e, V^{\mathrm{an}})$ of the deformation retraction ψ is a finite simplicial complex. Let Υ_g denote this finite simplicial complex.*

2. *There exists a well defined function $M_g : \Upsilon_g \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions. The function M_g is not identically zero and $\log(M_g)$ is piecewise linear. Let $\gamma \in \Upsilon_g$ be a point on the finite simplicial complex for which $M_g(\gamma) \neq 0$ and $x \in \psi(e, _)^{-1}(\gamma)$. Let L/k be any complete non-Archimedean real valued algebraically closed field extension such that $x \in V^{\mathrm{an}}(L)$. There exists $W_{x_L} \in (h_L)^{-1}(M_g(\gamma)) \cap \mathcal{O}_{x_L}$ such that the open set $(\phi_L^{\mathrm{an}})^{-1}(W_{x_L} \cap V_L^{\mathrm{an}}) \subset V_L'^{\mathrm{an}}$ decomposes into the disjoint union of open sets, each homeomorphic to $W_{x_L} \cap V_L^{\mathrm{an}}$ via ϕ_L^{an}. Furthermore, let $O \in \mathcal{O}_{x_L}$ be such that the preimage of $O \cap V_L^{\mathrm{an}}$ under ϕ_L^{an} decomposes into the disjoint union of open sets in $V_L'^{\mathrm{an}}$, each homeomorphic to $O \cap V_L^{\mathrm{an}}$ via the morphism ϕ_L^{an}. Then $(g \circ h_L)(O) \leq_g M_g(\gamma)$.*

The proof of the above result is similar to the proof of Theorem 8.4.3. We now state and prove Theorem 8.4.3. We make use of Theorem 8.3.11 wherein we viewed the value group additively and used the functions h in place of h_L. By definition, if $x \in V_L(L)$ where L is a real valued model of ACVF and $O \in \mathcal{O}_x^0(L)$ then $B_{\mathbf{L}}(O) \in \mathcal{O}_x$ where $B_{\mathbf{L}}(O)$ is the Berkovich analytification of O [Section 6]. We have that $h(O) = \log(h_L(B_{\mathbf{L}}(O)))$ where the function log (Remark 1.3) is applied component wise.

Theorem 8.4.3. *Let $\phi : V' \to V$ be a finite surjective morphism between irreducible, projective varieties with V normal. Let $g \in S$. There exists a generalised real interval $I := [i, e]$ and a deformation retraction*

$$\psi : I \times V^{\mathrm{an}} \to V^{\mathrm{an}}$$

which satisfies the following properties.

1. *The image $\psi(e, V^{\mathrm{an}}) \subset V^{\mathrm{an}}$ of the deformation retraction ψ is homeomorphic to a finite simplicial complex. Let Υ_g denote this finite simplicial complex.*

2. *There exists a well defined function $M_g : \Upsilon_g \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions. The function M_g takes values other 0 and $\log \circ M_g$ is piecewise linear (Remark 1.3). Let $\gamma \in \Upsilon_g$ be a point on the finite simplicial complex for which $M_g(\gamma) \neq 0$ and $x \in \psi(e, _)^{-1}(\gamma)$. Let L/k be any complete non-Archimedean real valued algebraically closed field extension such that $x \in V^{\mathrm{an}}(L)$. There exists $W_{x_L} \in (g \circ h_L)^{-1}(M_g(\gamma)) \cap \mathcal{O}_{x_L}$ such that $(\phi_L^{\mathrm{an}})^{-1}(W_{x_L} \cap V_L^{\mathrm{an}}) \subset V_L'^{\mathrm{an}}$ decomposes into the disjoint union of open sets, each homeomorphic to $W_{x_L} \cap V_L^{\mathrm{an}}$ via ϕ_L^{an}. Furthermore, let $O \in \mathcal{O}_{x_L}$ be such that the preimage of $O \cap V_L^{\mathrm{an}}$ under ϕ_L^{an} decomposes into the disjoint union of open sets in $V_L'^{\mathrm{an}}$, each homeomorphic to $O \cap V_L^{\mathrm{an}}$ via the morphism ϕ_L^{an}. Then $(g \circ h_L)(O) \leq M_g(\gamma)$.*

Proof. We apply Theorem 8.3.11 to the given data. Hence there exists a pro k-definable deformation retraction $H : I \times \widehat{V} \to \widehat{V}$ where I is a generalised interval defined over k, a k-definable Γ-internal set $Z \subset \widehat{V}$ which is k - definably homeomorphic to a finite simplicial complex Υ_g' and is the image of the deformation retraction H i.e. $H(e, \widehat{V}) = Z$. Furthermore, there exists a k-definable function M_g' on Z and hence a piecewise linear function on Υ_g' which satisfies properties (1) and (2) stated in Theorem 8.3.11.

Let $\mathbf{k}$ denote the substructure of ACVF defined by the pair $(k, \mathbb{R}_\infty)$. By Section 6, the space $B_{\mathbf{k}}(V)$ of weakly orthogonal $\mathbf{k}$ - types is canonically homeomorphic to the Berkovich space V^{an}. We will for the remainder of this proof use the notation $B_{\mathbf{k}}(V)$ for the Berkovich space V^{an}.

Given a valued field M whose value group is contained in $\mathbb{R}_\infty$, there exists a maximally complete valued field K which contains M and whose residue field is equal to the algebraic closure of the residue field of M. By Kaplansky's theorem this field is unique up to isomorphism over $\mathbf{M} = (M, \mathbb{R}_\infty)$ and we denote it M^{max}. By Lemma 14.1.1 and Corollary 14.1.6 in [HL], there exists a canonical continuous closed surjection $\widehat{V}(k^{max}) \to B_{\mathbf{k}}(V)$ which induces a deformation retraction $H : I(\mathbf{k}) \times B_{\mathbf{k}}(V) \to B_{\mathbf{k}}(V)$ with image $Z(\mathbf{k})$. As $Z(\mathbf{k})$ is homeomorphic to $\Upsilon_g(\mathbf{k})$ we identify it via this homeomorphism and set $\Upsilon_g := Z(\mathbf{k})$. Observe that $I(\mathbf{k})$ is a generalised real interval. By Theorem 8.3.11, the function M_g' restricted to Υ_g takes values different from ∞. It follows that $M_g := \exp \circ M_g'$ (Remark 8.3.4) takes values different from 0.

We verify part (2) of the theorem. Let $p \in B_{\mathbf{k}}(V)(L)$ where L is a real valued complete model of ACVF and $\mathcal{H}(p)$ embeds into L. This is equivalent to saying that $\mathcal{H}(p) \subseteq L$. This implies that p when viewed as a weakly orthogonal $\mathbf{k}$ - type on V admits a realisation defined over L. The point $x_L \in V(L)$ is such a realisation. Let $\gamma = H(e, x_L)$. By 8.3.11, there exists $W \in \mathcal{O}_{x_L}^0(L^{max})$ such that $g' \circ h(W) = M_g'(\gamma)$ and $h(O) \in D_x$ where D_x is as in the proof of Theorem 8.3.11. Since $\Gamma(L^{max}) = \Gamma(\mathbf{L})$, W is in fact $\mathbf{L}$ - definable. Hence we must have that $B_{\mathbf{L}}(W) \in \mathcal{O}_{x_L}$ and by Remark 8.3.4, $g \circ h_L(B_{\mathbf{L}}(W)) = M_g(\gamma)$. We have the following commutative diagram.

$$\begin{tikzcd}
\widehat{V'}(L^{max}) \arrow[r, "\widehat{\phi}"'] \arrow[d, "\pi_{V'_L}"] & \widehat{V}(L^{max}) \arrow[d, "\pi_{V_L}"'] \\
B_{\mathbf{L}}(V') \arrow[r, "\phi_L^{\mathrm{an}}"'] & B_{\mathbf{L}}(V)
\end{tikzcd}$$

As $h(W) \in D_x$, $\widehat{\phi}^{-1}(\widehat{W}(L^{max}) \cap \widehat{V}(L^{max}))$ is the disjoint union of open sets each of which is homeomorphic to $\widehat{W}(L^{max}) \cap \widehat{V}(L^{max})$ via the morphism $\widehat{\phi}$. Let d denote the separable degree of the morphism ϕ. By Lemma 8.3.7, there exists $W'_1, \ldots, W'_d \subset \widehat{V'}$ such that the $W'_i(L^{max})$ are open in $\widehat{V'}(L^{max})$, $\widehat{\phi}^{-1}(\widehat{W}(L^{max}) \cap \widehat{V}(L^{max})) = \bigcup_i W'_i(L^{max})$ and the morphism $\widehat{\phi}$ restricts to a homeomorphism from each of the $W'_i(L^{max})$ onto $\widehat{W}(L^{max}) \cap \widehat{V}(L^{max})$. Let $W_i'^0$ denote the subset of simple points (cf. 4.2) of W'_i. The set $W_i'^0$ is an ind-definable set [[HL],Section 2.2] and as $W_i'^0(L^{max})$ is in bijection with the definable set $W(L^{max}) \cap V(L^{max})$ via the definable map ϕ, we deduce by compactness that $W_i'^0(L^{max})$ and hence $W_i'^0$ must be definable as well. Since the morphism ϕ is defined over k and W is defined over $\mathbf{L} = (L, \mathbf{R}_\infty)$, it follows that $W_i'^0$ is defined over $\mathbf{L}$. As all models of ACVF which contain L^{max} are equivalent, we deduce that the $W_i'^0$ are disjoint and ϕ restricts to a bijection from $W_i'^0$ onto $W \cap V$. Also $\widehat{W_i'^0} = W'_i$. It can be deduced from the definition of the morphism $\pi_{V'_L}$ that it restricts to a morphism from $W'_i(L^{max})$ onto $B_{\mathbf{L}}(W_i'^0)$ i.e. $\pi_{V'_L}(W'_i(L^{max})) = B_{\mathbf{L}}(W_i'^0)$. We claim that the $B_{\mathbf{L}}(W_i'^0)$ are disjoint open subspaces of $B_{\mathbf{L}}(V')$. That they are disjoint follows from the fact that the $W'_i(L^{max})$ are disjoint. Indeed, let p be an $\mathbf{L}$ - type lying in the intersection of $B_{\mathbf{L}}(W_1'^0)$ and $B_{\mathbf{L}}(W_2'^0)$ which are distinct. Let c be a realisation of p. The type $\mathrm{tp}(c|L^{max})$ is an L^{max} - stably dominated type that belongs to both W'_1 and W'_2 which is not possible. We now show that for every i, $B_{\mathbf{L}}(W_i'^0)$ is an open subspace of $B_{\mathbf{L}}(V')$. The morphism $\pi_{V'_L}$ is closed and hence it restricts to a closed surjection from $\widehat{\phi}^{-1}(\widehat{W}(L^{max}) \cap \widehat{V}(L^{max}))$ onto $\bigcup_i B_{\mathbf{L}}(W_i'^0)$. For a fixed j, the set $\bigcup_{i \neq j} W'_i(L^{max})$ is a closed subspace of $\widehat{\phi}^{-1}(\widehat{W}(L^{max}) \cap \widehat{V}(L^{max}))$ whose image via the morphism $\pi_{V'_L}$ is the set $\bigcup_{i \neq j} B_{\mathbf{L}}(W_i'^0)$. Since the $B_{\mathbf{L}}(W_i'^0)$ are disjoint, the set $B_{\mathbf{L}}(W_j'^0)$ is open in $\bigcup_i B_{\mathbf{L}}(W_i'^0)$. The commutative diagram implies that

$$(\phi_L^{\mathrm{an}})^{-1}(B_{\mathbf{L}}(W) \cap B_{\mathbf{L}}(V)) = \bigcup_i B_{\mathbf{L}}(W_i'^0).$$

Hence the $B_{\mathbf{L}}(W_i'^0)$ are open in $B_{\mathbf{L}}(V')$.

We claim that ϕ^{an} restricts to a homeomorphism from each of the $B_{\mathbf{L}}(W_i'^0)$ onto $B_{\mathbf{L}}(W) \cap B_{\mathbf{L}}(V)$. We fix an index j. Since the vertical arrows of the commutative diagram above are closed and the restriction of $\widehat{\phi}$ to $W'_i(L^{max})$ is a homeomorphism onto $\widehat{W}(L^{max}) \cap \widehat{V}(L^{max})$ the morphism ϕ_L^{an} is a closed

surjection from $B_{\mathbf{L}}(W_j'^0)$ onto $B_{\mathbf{L}}(W) \cap B_{\mathbf{L}}(V)$. We now show that it is also a bijection. Let $p \in B_{\mathbf{L}}(W) \cap B_{\mathbf{L}}(V)$ and c be a realisation of p. The point c is simple in $\widehat{W} \cap \widehat{V}$. By Lemma 8.3.7, there exists exactly d preimages of c in V' each contained in exactly one $W_i'^0$. Let $\{c_1', \ldots, c_d'\}$ denote this set of preimages where $c_i' \in W_i'^0$. The type $\mathrm{tp}(c_i'|L^{max})$ is an L^{max} - stably dominated type contained in W_i' and its image in $\widehat{W} \cap \widehat{V}$ for the morphism $\widehat{\phi}$ is the stably dominated type $\mathrm{tp}(c|L^{max})$. As the $B_{\mathbf{L}}(W_i'^0)$ are mutually disjoint and $\pi_{V_L'}(W_i'(L^{max})) = B_{\mathbf{L}}(W_i'^0)$ it follows that there must be at least d weakly orthogonal types in $B_{\mathbf{L}}(V')$ which map to p. However, the cardinality of the fibre over p for the morphism ϕ_L^{an} is bounded above by d. It follows that there exists one unique element in $B_{\mathbf{L}}(W_i')$ which maps to p via ϕ^{an}. This implies that the morphism ϕ_L^{an} restricts to a closed bijection from $B_{\mathbf{L}}(W_i')$ onto $B_{\mathbf{L}}(W) \cap B_{\mathbf{L}}(V)$. It is hence a homeomorphism.

We now verify the remainder of the theorem. Let $O \in \mathcal{O}^0_{x_L}$ be such that $(\phi_L^{\mathrm{an}})^{-1}(B_{\mathbf{L}}(O) \cap B_{\mathbf{L}}(V))$ is the disjoint union of open sets in $B_{\mathbf{L}}(V')$ each of which is homeomorphic to $B_{\mathbf{L}}(O) \cap B_{\mathbf{L}}(V)$ via the morphism ϕ_L^{an}. From the definition of the functions h and h_L and Remark 8.3.4, $g' \circ h(O) = \log((g \circ h_L)(O))$. It can be deduced from the definition of the functions g, g' and M_g that to complete the proof we must show that $g' \circ h(O) \geq M_g'(\gamma)$. The field L is algebraically closed and non - trivially valued. Hence its value group $\Gamma(L)$ is dense in $\mathbb{R}_\infty$. This implies that $\{h(O)|O \in \mathcal{O}^0_{x_L}(L)\}$ which is the set of elements definable over L is dense in $\{h(O)|O \in \mathcal{O}^0_{x_L}(\mathbf{L})\}$. As $g'(\mathbb{R}_\infty)$ is a continuous function, we can assume that $O \in \mathcal{O}^0_{x_L}(L)$. To show $g' \circ h(O) \geq M_g'(\gamma)$ it suffices to prove that $h(O) \in D_{x_L}$. By assumption, $(\phi_L^{\mathrm{an}})^{-1}(B_{\mathbf{L}}(O) \cap B_{\mathbf{L}}(V))$ is the disjoint union of open sets in $B_{\mathbf{L}}(V')$. By Proposition 8.2.3, there exists d, L - definable semi-algebraic sets $O_i' \subset V'$ such that $(\phi_L^{\mathrm{an}})^{-1}(B_{\mathbf{L}}(O) \cap B_{\mathbf{L}}(V)) = \bigcup_i B_{\mathbf{L}}(O_i')$ and the morphism ϕ_L^{an} restricts to a homeomorphism from $B_{\mathbf{L}}(O_i')$ onto $B_{\mathbf{L}}(O) \cap B_{\mathbf{L}}(V)$ for every i. This implies in particular that ϕ restricts to a bijection from $O_i'(L)$ onto $O(L)$ for every i. As all models of ACVF which contain L are equivalent to L, we must have that the morphism ϕ restricts to a bijection between $O_i'(\mathbb{U})$ and $O(\mathbb{U}) \cap V(\mathbb{U})$ for every i and in addition $O_i'(\mathbb{U}) \cap O_j'(\mathbb{U})$ is empty. Furthermore, for any $z \in O \cap V$, there exists exactly d preimages of z in V', exactly one in each of the O_i'. We now show that the morphism $\widehat{\phi}$ induces a homeomorphism between $\widehat{O}_i$ and $\widehat{O} \cap \widehat{V}$. Firstly, our description of the sets O_i' from Proposition 8.2.3 was explicit, and it follows from this description that $\widehat{O_i'}$ is an open subset of $\widehat{V'}$. Since $\widehat{V}$ is normal, the morphism $\widehat{\phi}$ is open and the restriction of $\widehat{\phi}$ to the open set $\widehat{O}_i'$ is also an open map. This restriction is in fact bijective. Indeed, let $p \in (\widehat{O} \cap \widehat{V})(L^{max})$. By definition p is a stably dominated type. Let a be a realisation of the type $p_{|L}$. The arguments above imply that there exists exactly d preimages of a, one in each of the O_i'. But as O_i' is defined over L there exists at least d preimages of p. However the cardinality of the set $\widehat{\phi}^{-1}(p)$ is bounded above by d. Hence there exists exactly one preimage of p in each of the $\widehat{O_i'}$. It follows that the restriction of $\widehat{\phi}$ to each of the $\widehat{O_i'}$ is a bijective open morphism which in turn implies that the restriction of $\widehat{\phi}$ to $\widehat{O_i'}(L^{max})$ is a bijective open morphism onto $\widehat{O}(L^{max}) \cap \widehat{V}(L^{max})$. Hence $h(O) \in D_{x_L}$. $\square$

8.5 The tying up of loose ends

In the introduction we announced that the goal of this article was to prove a generalization of Theorem 1.2.2. In the previous section we proved Theorem 8.4.3. We now show that the main theorem implies Theorem 1.2.2. We begin by showing that Theorem 1.2.2 is equivalent to Theorem 1.2.6. We write the value group multiplicatively in this section.

Proposition 8.5.1. *Let $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a finite morphism. Given such a morphism, theorems* 1.2.2 *and* 1.2.6 *are equivalent.*

Proof. Let us assume that Theorem 1.2.2 is true. Let $x \in \mathbb{P}^{1,\mathrm{an}}_k$ and L/k be a complete, algebraically closed, non-Archimedean real valued field extension of k such that $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$. By definition, $f(x)$ is the minimum of the radius of the largest Berkovich open ball around x_L whose preimage is the disjoint union of homeomorphic copies of the ball via the morphism ϕ^{an}_L and 1. The assumption that Theorem 1.2.2 is true implies that there exists a finite simplicial complex $\Upsilon \subset \mathbb{P}^{1,\mathrm{an}}_k$ and a deformation retraction

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

with image Υ such that the function f is constant on the fibres of this retraction. We define $M : \Upsilon \to [0,1]$ as follows. Let $\gamma \in \Upsilon$. Pick any $x \in \mathbb{P}^{1,\mathrm{an}}_k$ which retracts to γ and set $M(\gamma) := f(x)$. Since the function f is constant along the fibres of the retraction, M is well defined. It is also not identically zero and $\log(M)$ is piecewise linear. It can be checked that the existence of the simplicial complex Υ, the deformation retraction ψ and the function $M : \Upsilon \to [0,1]$ imply that Theorem 1.2.6 is true.

We now assume Theorem 1.2.6 and show 1.2.2 is true. By assumption, there exists a finite, simplicial complex $\Upsilon \subset \mathbb{P}^{1,\mathrm{an}}_k$, a retraction

$$\psi : I \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$$

with image Υ and a function $M : \Upsilon \to [0,1]$ such that if $x \in \psi(e,_)^{-1}(\gamma)$ where $M(\gamma) > 0$ and L/k is a complete, algebraically closed, non-Archimedean real valued field extension of k such that $x \in \mathbb{P}^{1,\mathrm{an}}_k(L)$ then the Berkovich open ball around x_L of radius $M(\gamma)$ decomposes into the disjoint union of Berkovich open balls each homeomorphic to it. Furthermore, if O is any other Berkovich open ball around x_L whose radius is less than or equal to 1 such that its preimage for the morphism ϕ^{an}_L decomposes into the disjoint union of homeomorphic copies of the ball then its radius is less than or equal to $M(\gamma)$. If $\psi(e,x) = \gamma$ then it is clear that $f(x) = M(\gamma)$. Hence the function f is constant along the fibres of the retraction morphism $\psi(e,_)$. Furthermore, f is not identically zero and $\log(f)$ is piecewise linear on Υ. This proves Theorem 1.2.2. □

Proposition 8.5.2. *Theorem 8.4.3 implies Theorem 1.2.6.*

Proof. To apply Theorem 8.4.3, we need to choose a suitable definable function $g : \mathbb{R}^4_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let $(r_1,..,r_4) \in \mathbb{R}^4_{\geq 0}$. We set $g(r_1,..,r_4) := \Pi_i r_i$. By Theorem 8.4.3, there exists a finite simplicial complex Υ', a deformation retraction $\psi' : [i,e] \times \mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$ with image Υ' and a function $M' : \Upsilon' \to \mathbb{R}_{\geq 0}$ which satisfies the following property. Let L/k be a non-Archimedean real valued field

extension of k and $x \in \mathbb{P}_k^{1,\mathrm{an}}(L)$. Let $\gamma = \psi'(e, x_L)$. There exists $O \in \mathcal{O}_{x_L}$ such that $(g \circ h_L)(O) = M'(\gamma)$ and the open set $(\phi_L^{\mathrm{an}})^{-1}(O)$ decomposes into the disjoint union of homeomorphic copies of O via the morphism ϕ_L^{an}.

Let x_L have homogenous coordinates $[a : 1]$. By 8.1.1, if $|a| \leq 1$ then the family $\mathcal{O}_{x_L}$ is the set of Berkovich open balls around x_L whose radius is bounded by 1. If $|a| > 1$ then $\mathcal{O}_{x_L}$ contains the set of Berkovich open balls around x_L. As sketched in 8.1.1, the radius of these Berkovich open balls can be expressed in terms of the 4-tuple $h_L(O)$.

Using 8.1.1 we see that if $O \in \mathcal{O}_{x_L}$ is a Berkovich open ball $B(x_L, r)$ then the formula which relates the radius r to the tuple $h_L(O)$ varies according to the value $|a|$. For this reason we modify the simplicial complex suitably. Let Z_0 be the smallest path-connected closed subspace that contains the set $\{0, \infty\}$ and Υ be a finite simplicial complex that contains $\Upsilon' \cup Z_0$. The space $\mathbb{P}_k^{1,\mathrm{an}}$ admits a deformation retraction onto any non-empty finite sub graph. In particular there exists a deformation retraction $\psi : [i, e] \times \mathbb{P}_k^{1,\mathrm{an}} \to \mathbb{P}_k^{1,\mathrm{an}}$ with image Υ. The function M' extends to a function on Υ as follows. Let $p \in \Upsilon$. We set $M''(p) := M'(\psi'(e, p))$. The function $M'' : \Upsilon \to \mathbb{R}_{\geq 0}$ is well defined.

We now define $M : \Upsilon \to \mathbb{R}_{\geq 0}$ which will imply Theorem 1.2.6.

$$M(p) := \begin{cases} M''(p) & |T_1(p)| < |T_2(p)| \\ M''(p)^{1/2} & |T_1(p)| = |T_2(p)| \\ \mathrm{Min}\{1, M''(p)(|T_1(p)|/|T_2(p)|)^2\} & |T_1(p)| > |T_2(p)| \end{cases}$$

Using 8.1.1 it can be verified that the function M is bounded above by 1. It remains to check that the function M defined above satisfies the properties required by Theorem 1.2.6. Let $x_L \in \mathbb{P}_L^{1,\mathrm{an}}(L)$ have homogenous coordinates $[a : 1]$ and let x_L retract to the point $p \in \Upsilon$ via the retraction ψ.

Let $|a| > 1$. By Theorem 8.4.3, there exists $O \in \mathcal{O}_{x_L}$ such that the preimage of O is the disjoint union of copies of O for the morphism ϕ_L^{an} and also that $g \circ h_L$ achieves its maximal value at O amongst all elements of $\mathcal{O}_{x_L}$ which satisfy this property. Let $h_L(O) = ((1, r), (1, 1))$. It follows that that $M''(p) = r$. Observe that since x_L retracts to the point p via the retraction ψ, $|(T_1/T_2)(p)| = |a|$.

If $M(p) = 1$ we must show that the preimage of the Berkovich open ball $B(x_L, 1)$ decomposes into the disjoint union of copies of itself. It follows from the definition of the function $M(p)$ that $r \geq 1/|a|^2$. Any $O' \in \mathcal{O}_{x_L}$ such that $h_L(O') = ((1, s), (1, 1))$ with $s \leq r$ must be such that its preimage for the morphism ϕ_L^{an} is the disjoint union of homeomorphic copies of itself. In particular we may choose O' for which $h_L(O') = ((1, 1/|a|^2), (1, 1))$. By 8.1.1 the open neighbourhood O' is a Berkovich open ball around x_L of radius 1.

Let $M(p) < 1$. By definition of the function M we have that $r < 1/|a|^2$. Using 8.1.1, we see that the open set O corresponds to the Berkovich open ball around x_L of radius $r|a|^2$. Let $B(x_L, s)$ be a Berkovich open ball around x_L such that its preimage decomposes into the disjoint union of homeomorphic copies of itself via the morphism ϕ_L^{an}. By 8.1.1, we have that $h_L(B(x_L, s)) = ((1, s/|a|^2), (1, 1))$. Theorem 8.4.3 then implies that $s \leq r|a|^2$.

If $|a| \leq 1$ then from our construction of Υ the point x_L must retract to $p \in \Upsilon$ such that $|T_1(p)| \leq |T_2(p)|$. As done above, by our choice of g, the calculations in Section 8.1.1 and Theorem 8.4.3, it can be shown that the preimage of the

Berkovich open ball $B(x_L, M(p))$ for the morphism ϕ_L^{an} decomposes into the disjoint union of homeomorphic copies of $B(x_L, M(p))$ via the morphism ϕ_L^{an}. Furthermore, if $B(x_L, s)$ is a Berkovich open ball such that its preimage splits into the disjoint union of homeomorphic copies of $B(x_L, s)$ via ϕ_L^{an} then by 8.1.1, $s \leq M(p)$.

That the function $\log(M)$ is piecewise linear on Υ follows from the fact that the function $\log(M')$ is piecewise linear on the finite graph Υ'. $\square$

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