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SORBONNE PARIS CITE**

**ÉCOLE DOCTORALE DE SCIENCES MATHÉMATIQUES  
DE PARIS CENTRE**

**THÈSE DE DOCTORAT**

Discipline : Mathématiques Appliquées

Présentée par

**Aser CORTINES**

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**Aspects dynamiques d'un modèle de propagation  
de front du type champ moyen**

**Dynamical aspects of a moving front model of  
mean-field type**

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Sous la direction de **Francis COMETS**

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*Para minha avó.*

*“A vida é um sopro.”*

Oscar Niemeyer



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## Résumé

On étudie un modèle stochastique en temps discret introduit par E. Brunet et B. Derrida en 2004 : un nombre fixe  $N$  de particules évolue sur la droite réelle selon un mécanisme de mutation, branchement et sélection. Les particules restent groupées et se déplacent comme un front soumis à un bruit aléatoire. Au-delà de leur intérêt mathématique, ces types de front décrivent, par exemple, l'évolution d'un système ayant deux différents types de particules  $X$  et  $Y$ , réagissant selon une règle d'auto catalyse irréversible  $X + Y \rightarrow 2X$ . Le modèle étudié est du type champ moyen et les particules peuvent être interprétées comme la percolation de dernier passage sur le graphe  $\{1, \dots, N\}$ . Il a été prouvé par F. Comets, J. Quastel et A. Ramirez en 2013 que le front évolue globalement à une vitesse déterminée et que les perturbations apparaissent dans une échelle de temps  $\sqrt{t}$ . Dans cette thèse, on calcule la vitesse asymptotique quand  $N \rightarrow \infty$  pour une large classe de bruits aléatoires. On prouve ainsi que le premier terme dans le développement limité de la vitesse satisfait des propriétés universelles ne dépendant que de la queue droite de la probabilité de  $\xi$ .

On peut également interpréter le modèle comme étant la dynamique d'une population de taille constante, les positions représentant alors les caractéristiques génétiques de chaque individu. Dans ce cas, on s'intéresse à la manière dont les individus sont reliés entre eux et combien de générations doit-on remonter dans le temps afin de leur trouver un ancêtre commun. Pour une loi particulière du bruit, on montre que l'échelle de temps de coalescence moyenne est donnée par  $\ln N$  et que l'arbre généalogique du modèle converge en loi vers le processus de coalescence de Bolthausen-Sznitman, ce qui confirme les prédictions physiques pour cette classe de modèles.

**Mots clés :** *Processus branchants; sélection; vitesse de propagation d'un front; coalescent; percolation de premier passage; théorie des valeurs extrêmes.*

## Abstract

We focus on the discrete-time stochastic model studied by E. Brunet and B. Derrida in 2004: a fixed number  $N$  of particles evolve on the real line according to a branching/selection mechanism. The particles remain grouped and move like a travelling-front driven by a random noise. Besides its the mathematical interest, moving fronts describe, for example, the evolution of systems having two different species  $X$  and  $Y$  of particles, reacting according to the irreversible auto-catalytic rule  $X + Y \rightarrow 2X$ . The model here is of mean-field type and the particles can also be interpreted as the last passage time in directed percolation on  $\{1, \dots, N\}$ . It has been proved by F. Comets, J. Quastel and A. Ramirez in 2013 that the front moves globally at a deterministic *speed* and that fluctuation occur in the diffusive scale  $\sqrt{t}$ . In this thesis, we compute the asymptotic speed as  $N \rightarrow \infty$  for a large class of random disorders. We prove that the finite-size correction to the *speed* satisfies *universal features* depending on the upper-tail probabilities. For a certain class of noise, the techniques we have developed also allow to compute the asymptotic diffusion constant.

From a different perspective, one can also interpret the model as the dynamics of

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a constant size population, the positions being the fitnesses of the individuals. In this case, we focus on how individuals are related and how many generation one has to go back in time in order to find a common ancestor. For a specific choice of disorder, we show that the average coalescence times scale like  $\ln N$  and that the limit genealogical trees are governed by the Bolthausen-Sznitman coalescent, which validates the physics predictions for this class of models.

**Keywords:** *Branching processes; selection; front propagation speed; coalescent; first-passage percolation; extreme value theory.*

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# Chapter 1

## Introduction

### 1.1 Stochastic models of front propagation

Front propagation phenomena can be observed in various non-equilibrium systems in physics, chemistry and biology. Roughly speaking, it describes the evolution of systems having two homogeneous steady states: a stable one and an unstable one. Initially, both states coexist, then as time passes the stable phase “invades” the unstable one and this dynamics is described by a moving boundary separating both phases. The boundary conditions at the interface are determined by the microscopic properties of the model and how the stable and unstable phases interact. It is impossible to provide an exhaustive list of references about the subject, but, for a general overview and different models, we recommend [bAH00, Pan04] and the references therein.

In this thesis, we focus on reaction-diffusion processes of the form



Here,  $X$  and  $Y$  represent the stable and unstable phases, respectively, that one may interpret as different molecules or reactants. In chemistry, the dynamics has the following description: in a certain medium, such as a sealed reaction tank, one places a certain amount of the two substances. Initially, the two substances are segregated and as time passes the molecules perform some motion determined by the internal properties of the medium. When a  $X$ -molecule encounters a  $Y$ -molecule a reaction takes place and we end up with two  $X$ -molecules. Since the tank is sealed, the total number of molecules remains constant and as time passes one observes the  $X$ -phase invading the  $Y$ -phase.

Various non-equilibrium systems can be described by (1.1), and the example of molecules reacting is far from being exhaustive. Reaction-diffusion processes of the form (1.1) can explain how an advantageous gene propagates in a population of individuals, as first demonstrated in [Fis37, KPP37]. It can also describe ordinary phenomena such as the propagation of an infection into a population or the combustion of a material, see Figure 1.2. In the latter example,  $X$  stands for the burning (or burned) surface and  $Y$  the non-burning part. If the material is not burning, it will stay in this state, the  $Y$ -phase, until an external agent starts a combustion. As soon as combustion starts, one observes the propagation of the flames towards

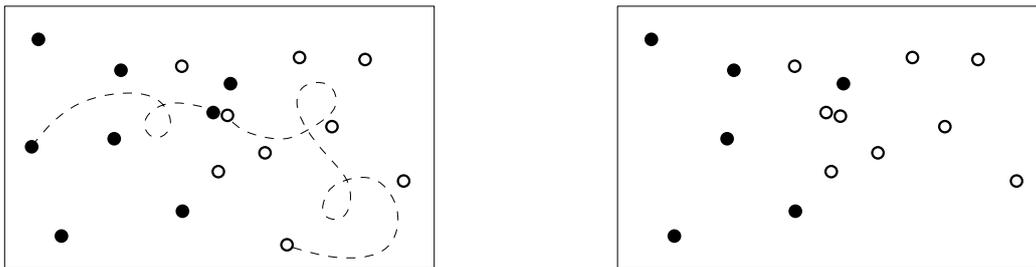


Figure 1.1: The molecules perform a random motion, determined by the medium (for example, independent Brownian motions). When a  $Y$ -molecule ( $\bullet$ ) hits a  $X$ -molecule ( $\circ$ ) the reaction occurs immediately.

the non-burning area. When the material is completely burned, the system reaches equilibrium once again (the  $X$ -phase) and it can not return to the previous state (irreversibility). Other motivations and models can be found in [bAH00, Pan04].

### 1.1.1 The F-KPP equation and continuous-time models

Although we will only focus on discrete-time models, the continuous-time analogues have been widely studied in the mathematical-physics [BCDM<sup>+</sup>86, BD97, BD99, BD01] and mathematical [Bra83, DR11, McK75, MMQ11, MS95] literatures. In the continuum, effective equations are available to describe front dynamics, and some of these results have been used as basis for predictions and conjectures concerning discrete-time models [BD97, BD99]. Here, we make a brief overview of the continuous theory, which will contribute to the general understanding of the theory.

We return to the preceding example of particles reacting in a sealed tank to provide a phenomenological picture, see Figure 1.1. Assume that the two species of particles ( $X$  and  $Y$ ) are within an inert, undisturbed fluid medium, and hence that their motion is Brownian. Let  $h_\varepsilon(t, x)$  be the local concentration in  $x$  of  $X$ -molecules at time  $t$ , so that the proportion of  $Y$ -molecules is given by  $1 - h_\varepsilon(t, x)$ . Here  $\varepsilon$  represents the ratio between the macroscopic and microscopic scales, hence, in this case,  $\varepsilon^{-1}$  can be seen as the average number of particles per unit of length. We will focus on the limit function

$$\lim_{\varepsilon \rightarrow 0} h_\varepsilon(t, x) = h(t, x),$$

which describes the macroscopic behaviour of the system. If the two species of particles do not react, then  $h$  will merely be solution to the heat-equation

$$\partial_t h = \partial_{xx} h, \tag{1.2}$$

satisfying the initial data  $h(0, x) = h_0(x)$  given by the initial concentration in  $x$  of  $X$ -molecules. Such solutions have the well-known probabilistic representation

$$h(t, x) = \mathbb{E}[h_0(x + B_t)],$$

where  $\{B_t, t \in \mathbb{R}\}$  is the standard Brownian motion. The heat equation (1.2) neglects the effects due to reaction, and one must add an extra term to it in order to capture this phenomenon.

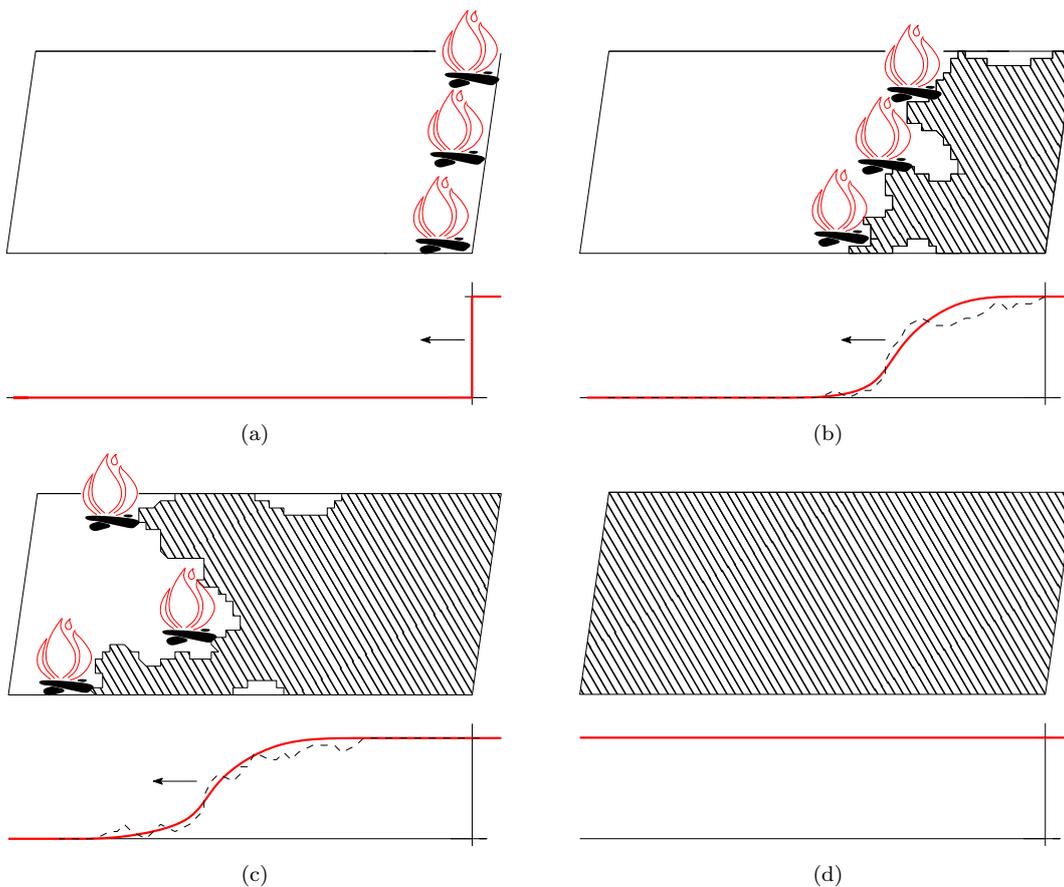


Figure 1.2: Flame propagation: the microscopic configuration of the medium can be represented by a lattice such as a rectangular box  $K \subset \varepsilon\mathbb{Z}^2$ , where  $\varepsilon$  represents the ratio between the macroscopic and microscopic scales. Each site may be burning or non-burning and the dynamics obeys a nearest neighbour constrain, that is, a non-burning site may change its state only if one of its neighbours is already burning. For a  $x \in \varepsilon\mathbb{Z}$ , let  $h_\varepsilon(t, x)$  be the proportion of burning sites in  $K$  along the vertical line passing through  $x$  (dashed line on the graphic). When the material is completely burned  $h_\varepsilon(t, x) \equiv 1$ , reversely  $h_\varepsilon(t, x) \equiv 0$  when it is not burning. Often as  $\varepsilon \rightarrow 0$ ,  $h_\varepsilon(t, x)$  can be approximated by a deterministic smooth front  $h(t, x)$  (in red).

The rate at which two different particles collide is proportional to

$$h(t, x)(1 - h(t, x)),$$

for simplicity we will assume that the proportionality constant is 1. By assumption, as soon as two different particles collide, the auto-catalytic reaction (1.1) takes place instantaneously, which yields a reaction term of the form  $h(1-h)$ . Adding it to (1.2), one gets the well-known Fisher Kolmogorov-Petrovskii-Piscounov (F-KPP) equation

$$\partial_t h = \partial_{xx} h + h - h^2. \quad (1.3)$$

This example indicates that in the continuum those reaction-diffusion processes are often described in the macroscopic level by partial-differential equations of the F-KPP type, see [GK04] for other examples and generalizations. For (1.3), the space homogeneous solutions  $h \equiv 1$  and  $h \equiv 0$  are respectively stable and unstable, and it

is known [Bra83, GK04] that for an initial data joining the two phases

$$\lim_{x \rightarrow -\infty} h(0, x) = 1, \quad \lim_{x \rightarrow +\infty} h(0, x) = 0, \quad \text{with } 0 \leq h(0, x) \leq 1 \quad \forall x \in \mathbb{R},$$

there exists a one-parameter family  $(F_v, v \geq 2)$  of travelling-wave solutions indexed by  $v \in [0, +\infty[$ , the *velocity* of the wave, such that

$$h(t, x) = F_v(x - tv), \quad t > 0.$$

Plugging  $F_v$  into (1.3), one sees that it solves the differential equation in  $z = x - tv$

$$F_v''(z) + vF_v'(z) + F_v(z)(1 - F_v(z)) = 0, \quad (1.4)$$

with boundary conditions  $F_v(+\infty) = 1$  and  $F_v(-\infty) = 0$ . The solution  $F_v$  of the above differential equation does not have an explicit analytic expression, but one can determine the possible velocities  $v$  for the system [BD97, BD99]. The idea is the following: since  $F_v(z) \rightarrow 0$  as  $z \rightarrow +\infty$ , for  $z$  sufficiently large, the differential equation (1.4) behaves like the linear differential equation

$$F'' + vF' + F = 0,$$

which solutions are spanned by  $Ae^{-\gamma z}$ . It is well-known that this linear equation admits positive solutions if and only if  $v \geq 2$ , in this case  $v = v(\gamma) = \gamma + 1/\gamma$ . The precise statement and a rigorous proof were given by Bramson [Bra83]. He also proves that if the initial data decays faster than  $e^{-x}$  at  $+\infty$  (for example, step functions), then the front moves asymptotically with the minimal speed  $v_{\min}$ , which is equal to 2 in this case.

The F-KPP equation has been introduced independently by Fisher [Fis37] and Kolmogorov *et al.* [KPP37] in the study of biological models and propagation of advantageous genes. Since then, the F-KPP equation has been extensively studied and it has connections with different themes in probabilities such as spin glasses theory [Bov06, Der85, DG86] and the branching Brownian motion [Bra78, Bra83, McK75, DS88].

Equations of the type (1.3) are often obtained either as the large-scale limit [BCDM<sup>+</sup>86, BD97, BD01, PL99] or the *mean-field* limit [DS88] of physical situations that are described in the microscopic level by random interactions between a finite number of particles. The deterministic equation (1.3) does not capture the fluctuation effects observed in the above examples. However, the microscopic picture is often represented as a noisy version of it:

$$\partial_t h_\varepsilon = \partial_{xx} h_\varepsilon + h_\varepsilon - h_\varepsilon^2 + \sqrt{\varepsilon(h_\varepsilon - h_\varepsilon^2)} \dot{W}(t, x), \quad (1.5)$$

where  $\dot{W}(t, x)$  is the space time white-noise and  $\varepsilon$  represents the ratio between the microscopic and macroscopic scales (the stochastic F-KPP).

For  $\varepsilon > 0$  sufficiently small, it has been proved [MS95] that for a continuous initial data  $h_0(x) := h_\varepsilon(0, x) \in [0, 1]$  such that  $h_0(1 - h_0)$  has compact support, there exists

a continuous (random) solutions  $h_\varepsilon(t, x, \omega) \in [0, 1]$  of (1.5), with a finite upper bound on the support

$$r_\varepsilon(t, \omega) := \sup\{x; h_\varepsilon(t, x, \omega) > 0\} < \infty; \quad \text{for almost every } \omega \in \Omega,$$

where  $(\Omega, \mathcal{H}, \mathbb{P})$  is the probability space in which  $h_\varepsilon$  is defined. Moreover, the *process seen from the leading edge*

$$h_\varepsilon^0(t, x, \omega) := h_\varepsilon(t, x - r_\varepsilon(t, \omega), \omega) \tag{1.6}$$

converges to a stationary limit as  $t \rightarrow \infty$ . In contrast to the stochastic equations of the F-KPP type, which admits a continuous family of solutions indexed by the *velocity*, in the stochastic models a single *velocity*  $v_\varepsilon$  is selected by the system, regardless of the initial conditions [Bra78, BD97, BD01]

$$\lim_{t \rightarrow \infty} \frac{r_\varepsilon(t)}{t} = v_\varepsilon \quad a.s.$$

It has been foreseen by Brunet and Derrida [BD97, BD01] and proved by Mueller *et al.* in the celebrated paper [MMQ11], that  $v_\varepsilon$  converges to the minimal *velocity*  $v_{\min}$ , which is 2 in (1.3),

$$v_\varepsilon = 2 - \frac{\pi^2}{|2 \ln \varepsilon|^2} + \mathcal{O}\left(\frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^3}\right) \quad \text{as } \varepsilon \rightarrow 0. \tag{1.7}$$

Notably, the corrections to the limit are extremely large in comparison to  $\varepsilon$ , which is a prominent feature of a whole class of models.

### 1.1.2 Brunet Derrida $N$ -particle system

In this thesis, we consider a discrete-time stochastic model in which a fixed number  $N$  of particles evolve on the real line. The state of the system at any time is described by its empirical distribution function, which is a non-increasing, random step function with jumps of size  $1/N$ . As the system evolves, we observe the stable phase (usually the “one” phase) invading the unstable one. We are interested in understanding what determines the motion and derive the universality properties of such models.

The process we focus on has been introduced by E. Brunet and B. Derrida [BD04]. Let  $X_1(0), \dots, X_N(0)$  be the initial positions of the particles. With

$$\{\xi_{ij}(s); 1 \leq i, j \leq N, s \geq 1\}$$

an i.i.d. family of real random variables (r.r.v.’s), the positions evolve as

$$X_i(t+1) := \max_{1 \leq j \leq N} \{X_j(t) + \xi_{ji}(t+1)\}. \tag{1.8}$$

A directed polymer formulation of (1.8) is given by Cook and Derrida in the earlier paper [CD90]. With some abuse of notation, we will often use  $\xi$  to denote the common law of the r.v.’s and also a generic r.v. having this distribution.

One can interpret the  $N$ -particle system (1.8) in many different ways. In Section 1.2, we will associate to it a first-passage percolation problem, which is simply the zero-temperature version of the polymer model studied in [CD90]. It can also be seen as a population dynamics under the *effects of selection*, see Section 1.3 below. In each context, the universal properties of the model have their respective meaning, and it may serve as a basis for predictions and conjectures to more general models.

**The velocity of the  $N$ -particle system.** As in the continuous case, the cloud of particles travels at a deterministic *velocity*. For  $\xi_{ij} \in L^1$ , it is proved in [CQR13] that the limits

$$\lim_{t \rightarrow \infty} \frac{1}{t} \max_{1 \leq i \leq N} X_i(t) = \lim_{t \rightarrow \infty} \frac{1}{t} \min_{1 \leq i \leq N} X_i(t) = v_N(\xi) \quad (1.9)$$

exist a.s. with  $v_N(\xi)$  a real constant depending on the law of  $\xi$ . The limit  $v_N(\xi)$  is called the *speed of the  $N$ -particle system*. We will summarize the general results in the field and give a brief proof of this result in Section 2.4.

An important part of this thesis is dedicated to the study of the *velocity* and its asymptotic behaviour for large  $N$ . We prove that for a large class of distributions the *finite-size corrections* to the speed satisfy *universal properties* depending only on the upper tail probabilities of  $\xi$ .

A related question concerns how the front deviates from this *velocity*. In this case, the authors in [CQR13] also prove that for  $\xi_{ij} \in L^2$ ,

$$\frac{\max \{X_i(t)\} - tv_N(\xi)}{\sqrt{t}}$$

converges in distribution to a Gaussian r.v. with variance  $D_N(\xi)$ . Unfortunately, the technique we have developed in order to compute the *finite-size corrections* to the speed does not apply in this case, and apart from a few examples the exact diffusion constant  $D_N(\xi)$  is known.

**Front shape and discrete noisy reaction-diffusion equations.** We associate to the positions of the particles the empirical distribution function

$$U_N(t, x) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\{X_i(t) \geq x\}}, \quad (1.10)$$

that is a front-like random step function satisfying

$$\lim_{x \rightarrow -\infty} U_N(t, x) = 1, \quad \lim_{x \rightarrow \infty} U_N(t, x) = 0 \quad a.s.$$

Here the ratio between the macroscopic and microscopic scales is given by  $1/N$ , the inverse of the number of particles. The front equation  $U_N(t, x)$  shares several common properties with the stochastic solution of the noisy F-KPP (1.5): independently from the initial data, the front moves globally at the deterministic *velocity*  $v_N(\xi)$ , and the whole motion is determined by what happens near the leading edge (*pulled front*). Thus, as time passes, one also observes the stable phase invading the unstable one, which occurs at the speed  $v_N(\xi)$ , and the front properties (such as its shape and the velocity) are essentially determined by the upper tails probabilities. Despite

the similarities, (1.10) has *mean-field* structure and it is intrinsically random and discrete. These peculiarities make the problem considered here rather different from the usual F-KPP problem, hence the techniques we have developed may be helpful to understand different front equations.

Only a few mathematical results concerning the front shape are available. In Theorem 1.2 of [CQR13], Comets *et al.* prove that, when  $\xi$  has exponential upper tail, the front seen from the leading edge converges to a stationary limit

$$\lim_{N \rightarrow \infty} U_N(t+1, \Phi(X(t)) + x) \stackrel{\text{law}}{=} 1 - \exp(-e^{-x}), \quad \forall t \in \mathbb{N} \quad (1.11)$$

where  $\Phi(X(t))$  is a random space-shift, which compensates the front position at time  $t$ , we refer to Section 2.5 for a detailed account. An analogy can be drawn between the  $N$ -particle system seen from the leading edge and (1.6), and the limit in (1.11) corresponds to the travelling wave  $F_v(x)$  from (1.4) with the minimum speed  $v = 2$ , thus (1.11) is a rigorous formulation for the particle system as a travelling front. However, this limit behaviour should not be expected for different  $\xi$ , that is, having an upper tail different from exponential. Consider, for example, negative  $\xi$  with polynomial density close to 0. In this case, a simple calculation shows that as  $N \rightarrow \infty$  the trivial convergence holds

$$U_N(t, x) \xrightarrow{\text{law}} \mathbf{1}_{\{x \leq 0\}}, \quad \forall t \in \mathbb{N}.$$

Instead, one should focus on the rescaled front equation  $U_N(t, x/a_N)$ , with  $a_N$  a real sequence converging to zero. The results obtained in Chapter 4 suggest that  $a_N$  is asymptotically equivalent to  $N^{-1/1+\alpha}$ , where  $\alpha$  is density exponent, but the asymptotic properties of the front for this kind of disorder remain unknown. It seems clear from simulations (see Section 2.6) that a limit shape exists, and that it is different from (1.11).

## 1.2 First passage percolation and the maxima of correlated random variables

From a different perspective, the model (1.8) can be interpreted as a first-passage percolation model on the oriented graph with vertex and edge sets given by

$$V := \mathbb{N} \times \{1, \dots, N\} \quad \text{and} \quad E := \{ \langle (i, t), (j, t+1) \rangle; 1 \leq i, j \leq N, t \geq 0 \}.$$

Here, the graph is oriented on the second coordinate ( $t$ -coordinate increases by one unit at each step of the path), though on the transverse direction jumps are allowed between all pairs of sites  $i, j$  ( $1 \leq i, j \leq N$ ), so the model is of *mean-field* type.

Assuming that  $\xi$  is negative,  $-\xi_{ij}(t+1)$  can be seen as the passage time on the oriented edge from  $(j, t)$  to  $(i, t+1)$ . By a simple induction argument, one obtains from (1.8) the formula

$$X_i(t) = \max \left\{ X_{j_0}(0) + \sum_{s=1}^t \xi_{j_{s-1}j_s}(s); 1 \leq j_s \leq N, \forall s = 0, \dots, t-1 \text{ and } j_t = i \right\}, \quad (1.12)$$

which yields a path representation of the interacting particle system (1.8). As (1.12) shows,  $-X_i(t)$  is the first-passage time from the line  $t = 0$  to the point  $(i, t)$ , in a model of first-passage percolation on the vertex set  $\{1, \dots, N\} \times \mathbb{N}$ . Similar random structures [FMS14, GNPT86, HM07] have been used in computer sciences and queueing theory to study task graphs for parallel processing. In this case,  $-\xi_{ij}(t+1)$  stands for the time required to complete a certain task. The direction constrain simply means that the task at step  $t-1 \rightarrow t$  must be completed before the task at step  $t \rightarrow t+1$  can start.

A natural question coming from this kind of problem is about the minimum time one should wait before all tasks are completed. In the case of (1.12), the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max_{1 \leq i \leq N} X_i(n) = v_N(\xi)$$

exists a.s. and  $-v_N(\xi)$  is the so-called time-constant of the model. Hence, the first-passage time grows, up to fluctuations, linearly with the number  $n$  of tasks, and it is important to precise the limit  $v_N(\xi)$ .

For general percolation models, the value of the time-constant is not available, but in the present case the *mean-field* feature allows us to compute the asymptotic in the limit of large graphs. The following theorem is a weaker version of Theorem 4.1 formulated in the framework of first passage percolation (with positive passage times).

**Theorem 1.1.** *Let  $\{\zeta_{ij}(t); 1 \leq i, j \leq N; t \in \mathbb{N}\}$  be a family of i.i.d. positive random variables with a polynomial density close to zero. Consider the first-passage percolation problem on the oriented graph  $\mathbb{N} \times \{1, \dots, N\}$*

$$m_n := \min \left\{ \sum_{t=1}^n \zeta_{j_{t-1}j_t}(t); 1 \leq j_t \leq N, \forall t = 0, \dots, n \right\}.$$

Let also  $\alpha > 0$  and  $C > 0$  such that

$$\mathbb{P}(\zeta_{ij} \leq x) \sim Cx^\alpha, \quad \text{for } x \searrow 0,$$

where “ $\sim$ ” means that the ratio of the sides approaches to 1 as  $x \searrow 0$ . Then, the limit

$$\lim_{n \rightarrow \infty} \frac{m_n}{n} = m_N$$

exists a.s. Moreover,

$$m_N = \frac{\alpha}{eN^{1/\alpha}} \left( \alpha \Gamma(\alpha) C \right)^{-\frac{1}{\alpha}} + o(N^{-1/\alpha}) \quad \text{as } N \rightarrow \infty,$$

where  $\Gamma(\cdot)$  is Euler’s gamma function and  $e = 2.718\dots$  is the Napier’s constant.

In the particular case of exponential passage times, Theorem 1.1 is in force and

$$m_N \sim \frac{1}{eN} \quad \text{as } N \rightarrow \infty.$$

We can relax the notion of “time” and allow it to be negative. Under this circumstances, the Gumbel distribution plays an important role. The Gumbel distribution is one of the three max-stable laws and it has remarkable properties, see Section 2.1. It was first noted by Brunet and Derrida [BD04] that for this choice of disorder the model is completely solvable and that the front position evolves like a random walk. For Gumbel passage times, the time-constant has the following asymptotic for large  $N$

$$v_N(\xi) = \ln N + \ln \ln N + \frac{\ln \ln N}{\ln N} + \frac{1 - \gamma_E}{\ln N} + o\left(\frac{1}{\ln N}\right), \quad (1.13)$$

the diffusion constant for the model is also explicit as we show in Theorem 2.13. It turns out that the technique used by Brunet and Derrida is also effective for other first-passage percolation models, as we show in Section 2.5.2.

### 1.2.1 A directed polymer model.

First-passage percolation models can be often seen as the zero-temperature version of a polymer problem. For (1.8), the lattice consists of planes in the transversal direction. In every plane there are  $N$  points that are connected to all points of the previous plane and the next one, see Figure 1.3. For each edge  $(i, j)$ , connecting the  $t$ th plane to the  $(t+1)$ th plane, a random energy  $\xi_{ij}(t+1)$  is sampled from a common probability distribution  $\xi$ . We associate to each directed path  $\omega = [\omega_0, \dots, \omega_L]$  of length  $L$  the energy

$$E_\omega := \sum_{s=1}^L \xi_{\omega_{s-1}\omega_s}(s).$$

We then define the probability measure  $\mu_L$  on the space of all directed paths of length  $L$  by

$$\mu_L(\omega) := Z_L(T)^{-1} \exp(-E_\omega/T),$$

where  $T$  is the temperature, which describes how strongly the polymer path  $\omega$  interacts with the medium, and  $Z_L(T)$  is the partition function that is, a normalizing constant making  $\mu_L$  a probability measure. In zero temperature, we are faced with an optimization problem: computing the ground state energy of the model that is, the lowest energy of all possible walks (first-passage percolation).

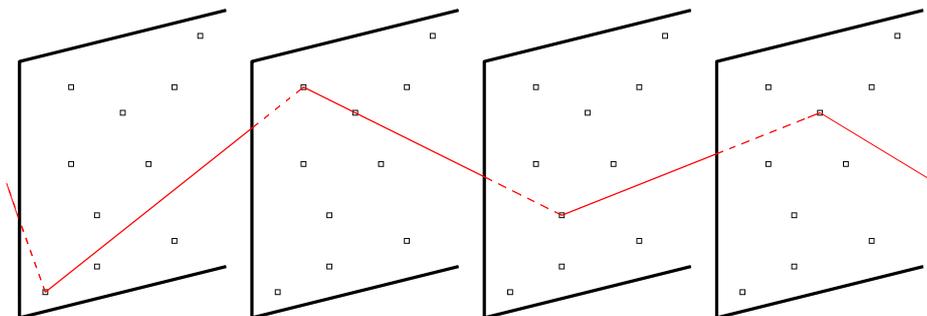


Figure 1.3: Mean-field polymer model

## 1.2. First-passage percolation

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In [CD90], Cook and Derrida considered, among other things, a particular case of disorder, that they named the “percolation distribution”. It consists in the inverse of a Bernoulli law with parameter  $\rho/N^{1+r}$ . In its zero temperature version, one can interpret that the edges for which  $\xi_{ij}(t) = -1$  are closed, whereas if  $\xi_{ij}(t) = 0$  the edge is open. For this kind of problem, one would like to know the average length of the largest path passing uniquely through open edges.

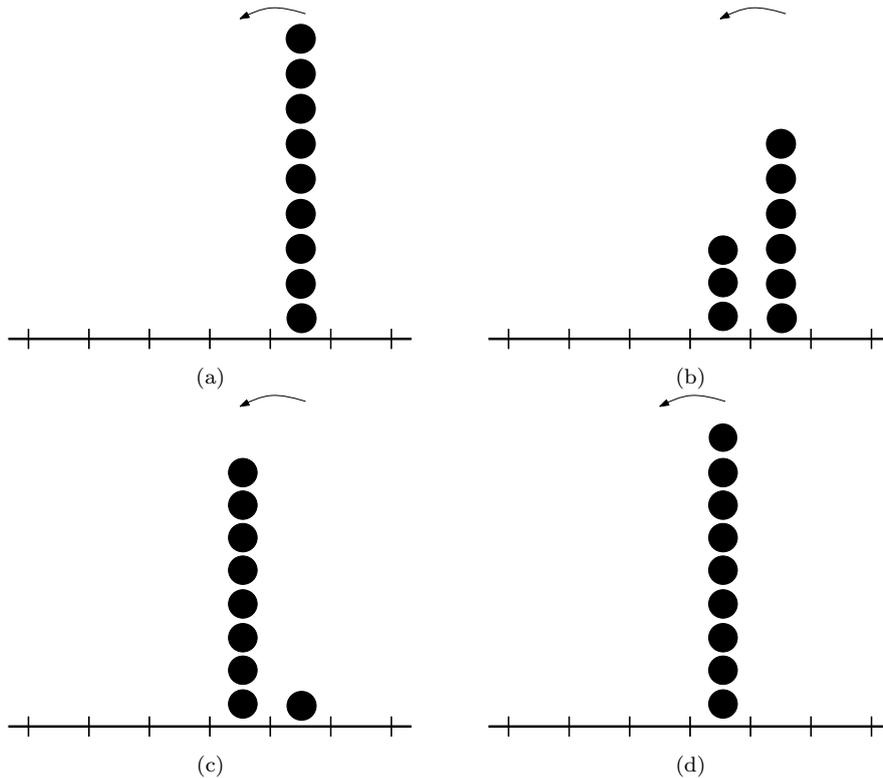


Figure 1.4: Renewal structure in the Cook and Derrida's dynamics.

They used heuristic arguments to study the asymptotic behaviour of  $v_N(\xi)$ . They first observe that each particle  $X_i(t)$  stays still or moves downward. Now, at each step  $t \rightarrow t+1$ , typically  $N^{-r}$  fraction of the leading particles remains to be the leader. Therefore, after  $[1/r] + 1$  steps, all  $N$  particles drop out and lie at a unit distance behind the leaders. Using a simple renewal theorem, one obtains the following formula

$$\lim_{N \rightarrow \infty} v_N(\xi) = \frac{1}{1 + [1/r]}.$$

However, if  $1/r \in \mathbb{N}$ , then the number of particles after  $1/r$  steps is of order of unity and the argument does not work. The above formula can not hold true in this case, and some rounding effect happens causing the limit speed to take values different from  $1/N^*$ . To obtain such result, a further analysis is necessary and one must know how many particles remain to be leaders at  $1/r \in \mathbb{N}$ .

In Chapter 3, we study among other things the front properties for this kind of disorder and prove that, as  $N \rightarrow \infty$ , the number of leaders at  $t = 1/r$  is distributed according to a Poisson r.v. with parameter one, obtaining a correction to the above

formula. The complete statement, the proof and a generalization for the next two theorems are in Chapter 3.

**Theorem 1.2.** *For the percolation distribution,*

$$\lim_{N \rightarrow \infty} v_N(\xi) = \begin{cases} (1 + [1/r])^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ (1 + [1/r] - e^{-\rho^{1/r}})^{-1}, & \text{if } 1/r \in \mathbb{N}. \end{cases}$$

The formula in Theorem 1.2 bridges the gap between the values of the speed in  $1/\mathbb{N}^*$  (case  $1/r \notin \mathbb{N}$  as described in [CD90]), as  $\rho$  ranges from 0 to  $+\infty$  and  $r \in 1/\mathbb{N}^*$ . The same techniques we have developed allow one to compute the asymptotic for the diffusion constant  $D_N(\xi)$ . For large  $N$  and  $1/r \notin \mathbb{N}$ , the system almost does not diffuse, hence the approximation by  $v_N(\xi)$  is sharp. On the other hand, when  $1/r \in \mathbb{N}$  the diffusion constant  $D_N(\xi)$  is positive, and the front position diffuses around  $tv_N(\xi)$  at the diffusive scale  $\sqrt{t}$ .

**Theorem 1.3.** *For the percolation distribution,*

$$\lim_{N \rightarrow \infty} D_N^2(\xi) = \begin{cases} 0 & \text{if } 1/r \notin \mathbb{N} \\ e^{-\rho^{1/r}} - e^{-2\rho^{1/r}}, & \text{if } 1/r \in \mathbb{N}. \end{cases}$$

## 1.2.2 The correlation structure and the overlap function

When studying the extremal properties of random structures of the form (1.12), one must take into account the correlation between variables. It is well-known that for a sequence of identically distributed r.v.'s  $Z_1, \dots, Z_n$ , the asymptotic behaviour of their extremum

$$M_n = \max_{1 \leq i \leq n} \{Z_i\}$$

depends strongly on the correlations between the variables. This dependence can be easily seen on the two opposite examples. If the  $Z_i$  are all equal (“maximal correlation”)

$$\mathbb{P}(M_n \leq x) = \mathbb{P}(Z_1 \leq x), \quad \forall n \in \mathbb{N},$$

and no scaling is necessary. On the other hand, when the variables are independent (no correlation)

$$\mathbb{P}(M_n \leq x) = \mathbb{P}(Z_1 \leq x)^n, \quad \forall n \in \mathbb{N},$$

and one must rescale in order to observe a non-trivial behaviour (see Section 2.1). Assuming that all particles start at zero, the path representation (1.12) yields to

$$\max\{X_i(n)\} = \max \left\{ \sum_{s=1}^t \xi_{j_{s-1}j_s}(s); 1 \leq j_s \leq N, \forall s = 0, \dots, n \right\}.$$

Let  $\omega := [j_0, j_1, \dots, j_n]$  and  $\omega' := [j'_0, j'_1, \dots, j'_n]$  be two directed paths of length  $n$ . A natural way to quantify the correlation between the r.v.'s

$$\sum_{s=1}^t \xi_{j_{s-1}j_s}(s) \quad \text{and} \quad \sum_{s=1}^t \xi_{j'_{s-1}j'_s}(s)$$

## 1.2. First-passage percolation

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is to count the *overlap* between the paths  $\omega$  and  $\omega'$

$$R(\omega, \omega') := \#\{s; (j_s, j_{s+1}) = (j'_s, j'_{s+1})\}. \quad (1.14)$$

If the *overlap* is zero, the r.v.'s are independent, while, if the *overlap* is  $n$  the r.v.'s are actually the same, see Figure 1.5.

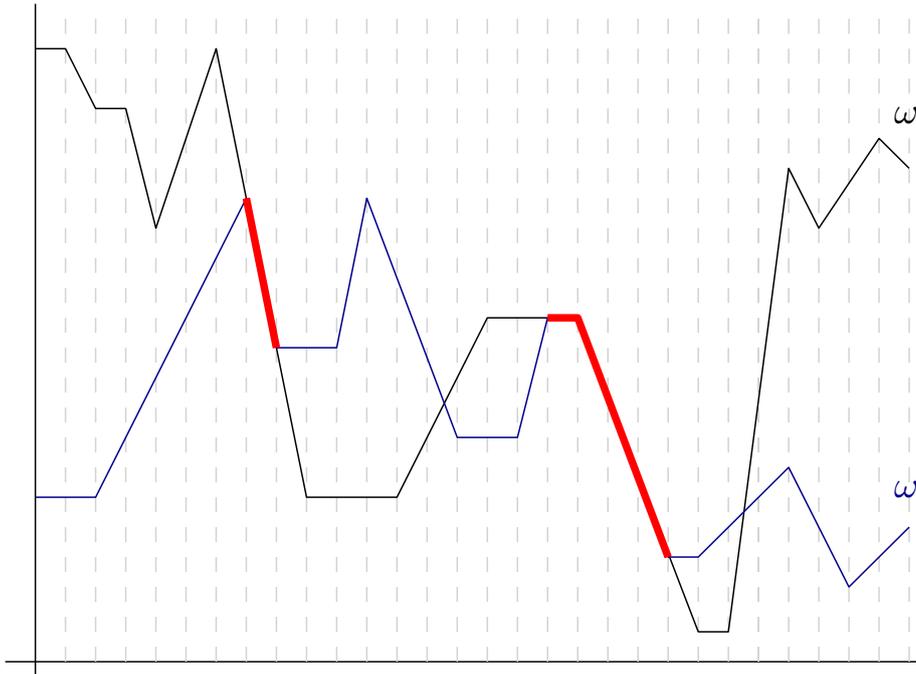


Figure 1.5: The vertical lines (in dashed grey) represent the vertical plans with  $N$  lattices. We represent two directed-paths  $\omega$  and  $\omega'$  in black and blue respectively. The *overlap* between the paths is represented in red. In contrast with classical branching models, the overlap segment is not connected.

Other classical models, such as *branching random walks*, have similar correlation structures. For branching random walks the detailed description of its extremal process is known [Aid13, Zei], we will recall the most relevant results and give a general overview of the theory in Section 2.2. The methods developed for studying branching random walks have become recently relevant in seemingly unrelated problems, such as the study of the maximum of certain Gaussian fields [BDZ13, BDZ14, BL13, BDZ11]. In the continuum, one can mention the recent studies of the point process generated by branching Brownian motion [ABK13, ABBS13, BH14b, BH14a]. In Gaussian models (*i.e.* when  $\xi$  is Gaussian distributed), the correlation structure determines completely the joint law (assuming that  $\xi$  has mean zero), so the *overlap* function characterizes completely the model.

As we will show in Section 2.4.2, the correlation structure in (1.8) is more involved when compared to the correlations in a branching random walk. Nevertheless, the techniques developed in the framework of branching random walk are also relevant to the study of (1.8), see Chapter 4.

## 1.3 Population Models and propagation of fronts

One can also see (1.8) in the context of population genetics, and interpret the positions of the particles as the individual fitnesses, which is transmitted to the offspring, up to variations due to mutations. An important question in the domain is to understand the *effect of selection* and mutation on the *genealogy*. For a given population, we would like to know how individuals are related and how many generations do we have to go back in time in order to find a common ancestor.

Kingman [Kin75, Kin82] was one of the first to give a mathematical formulation for this problem and study the ancestral history of a population. The genealogical relation between a group of  $n$  individuals can be described by a Markov process  $\Pi_t^n$  taking values on the set of partitions of  $\{1, \dots, n\}$ , the so-called *ancestral partition process*:

$$i \sim j \text{ on } \Pi_t^n \Leftrightarrow i \text{ and } j \text{ share the same ancestor } t \text{ generations ago,}$$

see Section 2.3 for the rigorous definition and a general overview of the theory. Kingman showed that in the absence of selection (neutral models) the genealogical trees converge to those of a Kingman's coalescent and that the average coalescence times scales like  $N$ , the size of the population. Two well-known examples lying in this *universality class* are the Wright-Fisher and Moran model [Mor58], see also [Möh99, Möh00, MS01] for other examples and generalizations.

It is well-known that some mathematical models do not lie in this universality class: Schweinsberg [Sch03] considers a Galton-Watson processes with selection of  $N$  offspring (they are chosen at random for survival). He proves that depending on the tail probabilities of the reproductive law, the limit may be Kingman's coalescent, a coalescent with multiple collisions ( $\Lambda$ -coalescent), or a coalescent with simultaneous multiple collisions ( $\Xi$ -coalescent). Discrete population models with unequal (skewed) fertilities, such as the skewed Wright-Fisher model and the Kimura model, are not necessarily in the domain of attraction of the Kingman's coalescent [HM11].

For several biological models selection plays an important role and there is an ongoing competition between the mutations, which makes individuals explore larger and larger regions of genome space, and selection, which tends to concentrate them at the optimal fitness genomes. Individuals with large fitness spawn a considerable fraction of the population, whereas the children of low fitness individuals tend to be eliminated. Therefore, these population models are sometimes referred to as "rapidly adapting". If we consider the evolution of the fitnesses along the real axis, it is simply a stochastic model of front propagation. The selection mechanism constrains the particles to stay together. Since individuals with large fitness quickly overrun the whole population, the front is essentially pulled by the leading edge. In the continuum these models are then related to noisy travelling wave equations of the F-KPP type [BD13, BDMM06, BDMM07], whereas for discrete-time models, equations of the form (1.8) are in force.

Recent results suggest that in rapidly adapting population models the genealogical correlations between individuals have universal features. It is conjectured [BD13, BDMM06, BDMM07] that the genealogical trees of these populations converge to the

Bolthausen-Sznitman coalescent and that the average coalescence times scales like the logarithmic of the populations size. The conjectures contrast with classical results in neutral population models, such as Wright-Fisher and Moran models. An example of rapidly adapting population model for which these conjectures have been proved is the “exponential model” [BD13, BDMM06, BDMM07]. It consists in a constant size population model, for which a complete mathematical treatment is possible. Each individual  $i$  in generation  $t$  carries a value  $x_i(t)$ , which represents its fitness. The offspring of the individuals are generated by independent Poisson point process with intensity measure

$$\chi(dy) = e^{-y+x_i(t)} dy.$$

One then selects the  $N$  rightmost individuals to form the next generation  $t + 1$ . The authors show that, after rescaling time by a factor  $\ln N$ , one obtains the convergence to the Bolthausen-Sznitman coalescent.

A dual problem is the *genealogy* of a branching population killed by a moving obstacle, *e.g.* a line. Berestycki *et al.* [BBS13] consider a system of particles, performing branching Brownian motion with negative drift and killed upon reaching zero. The authors choose the appropriate drift such that the model is in the near-critical regime and the initial population size  $N$  is roughly preserved. They show that the expected time to observe a merge is of order  $(\ln N)^3$  and that the *genealogy* of the particles is also governed by the Bolthausen-Sznitman coalescent.

In Section 2.3, we show that the  $N$ -particle system (1.8) leads naturally to a population dynamics, and in Chapter 5, we prove that when  $\xi_{ij}$  is Gumbel  $G(\rho, \beta)$ -distributed, the generations are independent and the model is mathematically solvable. In this case, the population dynamics has several similarities with the exponential model described above, notably, the only information one needs from the generation  $t$  to draw the next generation is the “front position”  $\Phi(X(t))$ , see Chapter 5 for a detailed account. Yet, the models differ in fundamental aspects, such as the selection and reproductive mechanisms. We obtain the following result concerning the limit in law of the *ancestral partition process*.

**Theorem 1.4.** *For  $\xi$  Gumbel distributed, the average coalescence times scale like  $\ln N$ . Moreover, re-scaling time by  $\ln N$ , the ancestral partition process converges in distribution to the Bolthausen-Sznitman coalescent.*

The above theorem validates the predictions by Brunet, Derrida, Mueller and Munier for this class of models. Although it has been predicted that the Bolthausen-Sznitman should appear as the limit genealogy of a whole class of system of particles which can be loosely described as branching random walks with selection, very few results in this direction have so far been proved.

**Polymer models and genealogies.** Polymers models in zero-temperature and population dynamics are close related. In its zero-temperature version, the polymer problem is reduced to find the path minimizing the energy in a random energy landscape. The optimal paths starting at the same point but arriving at different points give rise to a tree structure. Hence, an analogy can be drawn between the minimal energy of a directed polymer arriving on a site, and minus the fitness of an individual living on a site. Because of these similarities, it is expected that these models belong

to the same *universality class* [BDS08]. In strong disorder regime, *extreme* statistics are expected to play a crucial role [AL11, BK08]. The notion of ancestral lineages is replaced there by the optimal paths and *genealogy* by geodesics, but the correlation structures are more involved than in population dynamics.

## 1.4 Main contribution of the thesis

In this thesis, we have mainly studied the  $N$ -particles system (1.8) introduced by Brunet and Derrida [BD04] and we have obtained results concerning the *universal properties* of the model when the number of particles  $N$  diverges. These results can be equally found in the following articles [CC15, Cor14a, Cor14b]. In the next chapter (Chapter 2), we recall all background literature and necessary results used in the present thesis. Apart from Proposition 2.15 in Section 2.5.2, the results presented there are not original. In Section 2.5.2, we consider a specific last-passage percolation model studied by Foss *et al.* [FMS14]. We adapt the technique developed by Brunet and Derrida [BD04] and show that, when the passage times are Gumbel distributed, the explicit time-constant of the model can be computed.

A considerable part of this thesis focus on to the asymptotic *velocity*  $v_N(\xi)$  as  $N \rightarrow \infty$ . In Chapter 3, we study the case of percolation distribution, proving Theorems 1.2 and 1.3. We also consider a generalization of it obtaining results of the type of Theorem 1.2. In Chapter 4, we carry on with the study of the asymptotic velocity and consider the case of non-lattice and bounded from above distributions. We prove a generalized version of Theorem 1.1 showing that the finite-size correction to the speed have *universal features* depending only on the upper tails probabilities of  $\xi$ . The tools and techniques used in this chapter are rather different from the ones used in Chapter 3, a considerable part of it has been borrowed from the branching random walks literature, that we recall in Section 2.2.

The results of Chapters 3 and 4 are within the scope of the extreme value theory, since they concern the maximum of a family of r.v.'s. We give in Section 2.1 a general overview of the classical extreme value theory, in which one considers the maximum of i.i.d. r.v.'s. We also recall in this section the important theorems and the classical distributions that will often appear in the thesis.

Finally, in Chapter 5 we consider the genealogical aspects of the model, proving Theorem 1.4. All necessary results and a general overview of the related literature can be found in Section 2.3.



# Chapter 2

## Preliminary Results

In this chapter we give the background results entering this thesis and a general overview of the related topics. Several results we present are not original, so we will skip their proofs and indicate the references.

### 2.1 Extremes of independent random variables

In the probabilistic literature, the study of the maximum of a collection of random variables is often called *extreme value theory*. It is a remarkably rich and interesting subject with applications in several areas such as: statistical physics, applied statistics, earth sciences, mathematical finance and meteorology. Its roots can be traced back to the works of Tippett and Fisher [Tip25, FT28] and Fréchet [Fré27], where the idea of “max-stability” has been roughed out. In this introductory section, we give a general overview of the classical *extreme value theory*, a detailed account of the general theory and the proofs of the main results can be found in the books of Leadbetter, Lindgren and Rootzén [LLR83] and Resnick [Res87], from which we borrow the approach.

Let  $(Z_i)_{i \in \mathbb{N}}$  be a family of i.i.d. real random variables with common probability distribution function  $F(\cdot)$ . Given  $n \in \mathbb{N}$ , define the maximum up to time  $n$ ,

$$M_n = \max_{1 \leq i \leq n} \{Z_i\}.$$

We want to understand the asymptotic behaviour of  $M_n$  as  $n$  diverges. The independence between the r.v.'s yields

$$\mathbb{P}(M_n \leq x) = F^n(x),$$

hence if we want to find a non-trivial distribution function  $\Psi(x)$  for which

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq x) = \Psi(x),$$

we need to rescale the process. Otherwise, the following trivial limit always hold

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq x) = \begin{cases} 0 & \text{if } \mathbb{P}(Z_1 \leq x) < 1; \\ 1 & \text{if } \mathbb{P}(Z_1 \leq x) = 1. \end{cases}$$

## 2.1. Extremes of independent random variables

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Our goal is to find two sequences of real numbers  $(x_n)_{n \in \mathbb{N}}$ , with  $x_n > 0$  for each  $n \in \mathbb{N}$ , and  $(y_n)_{n \in \mathbb{N}}$ , for which the asymptotic behaviour of  $(M_n - y_n)x_n$  is non-trivial. Under mild assumptions, the limit of the rescaled maximum exists and it is stable under independent maxima. Precisely, the limit distribution  $\Psi(x)$  satisfies the so-called “max-stability” property

$$\Psi(\tilde{y}_n + x/\tilde{x}_n)^n = \Psi(x); \quad \text{for some sequences } \tilde{x}_n \text{ and } \tilde{y}_n.$$

**Theorem 2.1** (Fisher and Tippet[FT28], Gnedenko [Gne43]). *Suppose that there exist two sequence  $y_n \in \mathbb{R}$  and  $x_n > 0$  such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}((M_n - y_n)x_n \leq x) = \Psi_\alpha(x), \quad (2.1)$$

*a non-degenerated probability distribution. Then,  $\Psi_\alpha(\cdot)$  is of the one of the three following types:*

$$\begin{aligned} \text{Type I: } \Psi_\alpha(x) &= \exp(-e^{-x}). && \text{If } \alpha = 0 && \text{(Gumbel distribution)} \\ \text{Type II: } \Psi_\alpha(x) &= \begin{cases} 0, & \text{if } x \leq 0; \\ \exp(-x^\alpha), & \text{if } x > 0. \end{cases} && \text{For some } \alpha < 0 && \text{(Fréchet distribution)} \\ \text{Type III: } \Psi_\alpha(x) &= \begin{cases} \exp(-|x|^\alpha), & \text{if } x \leq 0; \\ 1, & \text{if } x > 0. \end{cases} && \text{For some } \alpha > 0 && \text{(Reverse-Weibull distribution)} \end{aligned}$$

We say that  $F(\cdot)$  belongs to the domain of attraction of  $\Psi_\alpha$  if the sequences  $x_n, y_n$ , for which (2.1) holds, exist. The existence of such sequences is directly related to the behaviour of  $F(\cdot)$  in a neighbourhood of its right-endpoint

$$x_F := \sup\{x; F(x) < 1\} \in \mathbb{R} \cup \{+\infty\}.$$

**Theorem 2.2** (Gnedenko [Gne43], de Haan [dH70]).  *$F(\cdot)$  belongs to the domain of attraction of  $\Psi_\alpha$  if and only if*

*Type I: There exists some strictly positive function  $g(t)$  such that  $\forall x \in \mathbb{R}$*

$$\lim_{t \rightarrow x_F^-} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x}.$$

*Type II:  $x_F = \infty$  and  $\forall x \in \mathbb{R}_+$*

$$\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^\alpha, \quad \text{where } \alpha < 0.$$

*Type III:  $x_F < \infty$  and  $\forall x \in \mathbb{R}_+$*

$$\lim_{t \rightarrow 0^+} \frac{1 - F(x_F - tx)}{1 - F(x_F - t)} = x^\alpha, \quad \text{where } \alpha > 0.$$

For these cases, let  $\varrho_n := \inf\{x; F(x) \geq 1 - 1/n\}$ , then the sequences  $x_n$  and  $y_n$  in Theorem 2.1 are of the form

Type I:  $x_n = g(\varrho_n)^{-1}$  and  $y_n = \varrho_n$ .

Type II:  $x_n = \varrho_n^{-1}$  and  $y_n = 0$ .

Type III:  $x_n = (x_F - \varrho_n)^{-1}$  and  $y_n = x_F$ .

To fix the ideas, it is useful to recall some particular cases.

*Example 2.3.* Assume that  $Z_1$  has a density  $p(x)$  such that:

Type I:  $x_F = \infty$  and

$$p(x) \sim e^{-x} \quad \text{as } x \rightarrow \infty,$$

then  $x_n \rightarrow 1$  and  $y_n \sim \ln n$  as  $n \rightarrow \infty$ .

Type II:  $x_F = \infty$  and there exists  $a < -1$  for which

$$p(x) \sim |a + 1|x^a \quad \text{as } x \rightarrow \infty,$$

then  $x_n \sim n^{1/(a+1)}$  as  $n \rightarrow \infty$  and  $y_n = 0$ .

Type III:  $x_F = 0$  and there exists  $a > -1$  for which

$$p(x) \sim (a + 1)x^a \quad \text{as } x \nearrow 0,$$

then  $x_n \sim n^{1/(a+1)}$  as  $n \rightarrow \infty$  and  $y_n = 0$ .

In the i.i.d. case the complete description of the order statistics is known. Denote by

$$Z^{(1;n)} \geq Z^{(2;n)} \geq \dots \geq Z^{(n;n)} \tag{2.2}$$

the order statistics of  $(Z_i; i = 1, \dots, n)$ . Then, the asymptotic relation between the  $(Z^{(i;n)}; i \leq n)$  is characterized by the point process

$$\sum_{i=1}^n \delta_{(Z^{(i;n)} - y_n)x_n}.$$

When  $F(\cdot)$  belongs to the domain of attraction of  $\Psi_\alpha$ , this point process converges in distribution to a Poisson point process with intensity measure  $\mu_\alpha$ :

Type I:  $\alpha = 0$  and  $\mu_0(dx) = e^{-x}dx$ .

Type II:  $\alpha < 0$  and  $\mu_\alpha(dx) = x^{\alpha-1}|\alpha|\mathbf{1}_{\{x \geq 0\}}dx$ , here one must only consider the points  $Z_i > 0$  in order to avoid a infinite mass at 0.

Type III:  $\alpha > 0$  and  $\mu_\alpha(dx) = |x|^{\alpha-1}\alpha\mathbf{1}_{\{x \leq 0\}}dx$ .

We bring to the reader's attention that the assumptions in Theorem 2.2 concern only the upper tails probabilities of  $F(\cdot)$ , hence one has no control on the behaviour of points that are "far away" from  $x_F$ .

### 2.1.1 Regularly varying functions and Karamata's representation

The characterization of the max-domain of attraction and the proof of Theorem 2.2 is based on the theory of regularly varying functions. In this subsection, we recall some basic notions, used in this thesis. Further information can be found in the following references: [BGT87, Fel71, dH70, Sen76].

A measurable function  $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be  $\alpha$ -regularly varying at infinity (*respectively* at zero) if

$$\lim_{t \rightarrow \infty} \frac{U(tx)}{U(t)} = x^\alpha, \quad \text{for every } x > 0;$$

*in the case of regular variation at zero we simply take the limit as  $t \rightarrow 0^+$ .* When  $\alpha = 0$  we say that the function is *slowly varying*. Both theories ( $t \rightarrow +\infty$  or  $t \rightarrow 0^+$ ) are equivalents and  $U(x)$  is  $\alpha$ -regularly varying at infinity if and only if  $U(x^{-1})$  is  $(-\alpha)$ -regularly varying at zero. Henceforth, we focus on the theory of regular variation at infinity.

From Theorem 2.2,  $F(\cdot)$  belongs to the max-domain of attraction of the Type II (*respectively* Type III) extreme value distributions if and only if

$$U(x) = 1 - F(x) \quad (\text{respectively } U(x) = 1 - F(x_F - x))$$

is  $\alpha$ -regularly varying at infinity (*respectively* zero), for some  $\alpha < 0$  (*respectively*  $\alpha > 0$ ). The Type I max-domain of attraction is characterized by the so-called de Haan's classes, see [dH74, dH76, Res87] for more details.

**Karamata's representation:** A function  $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is  $\alpha$ -regularly varying at infinity if and only if there exist two measurable functions  $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with

$$\lim_{x \rightarrow \infty} c(x) = c > 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \varepsilon(x) = 0,$$

such that

$$U(x) = x^\alpha c(x) \exp \left( \int_1^x \frac{\varepsilon(y)}{y} dy \right). \quad (2.3)$$

Although the representation (2.3) is not unique, we call it *Karamata's representation* of  $U$ .

### 2.1.2 Convergence of moments

It is well-known that weak convergence does not imply the convergence of moments. The latter convergence requires some control on the tail probabilities (the left tails also), which prevents improbable large values from disturbing the convergence of moments. Up to now, we have only imposed conditions on the upper tails of  $F(\cdot)$ , so one should not expect that the convergence of moments holds. Yet, in *extreme value theory*, one can obtain the convergence of moments under weak assumptions.

Denote by  $\mathcal{P}_k$  the  $k$ th largest point of a Poisson point process with intensity measure  $\mu_\alpha(dx)$ . We have seen that, if  $Z_i$  lies in the max-domain of attraction of  $\Psi_\alpha$ , then, the weak convergence

$$(Z^{(k;n)} - y_n)x_n \rightarrow \mathcal{P}_k$$

holds. For  $l > 0$ , a simple calculation shows that the  $l$ th moment of  $\mathcal{P}_k$  is finite if  $\alpha \geq 0$ , in the case  $\alpha < 0$  (Type II case) the  $l$ th moment exists for  $l < |\alpha|$ .

In *extreme value theory*, the only necessary condition to obtain the convergence of the  $l$ th moment is that  $Z_i$  itself has also finite  $l$ th moment. In the literature [LLR83, Res87], one can easily find proofs for the moment convergence of the maxima. Yet, we could not find any reference concerning the moment convergence of the  $k$ th order statistics. In fact, a rigorous proof can be obtained by a line-by-line adaptation of Proposition 2.1 in [Res87].

In Chapter 4, we use this convergence of moment in the case where  $Z_i$  lies in the max-domain of attraction of the Type III distribution. For the sake of completeness, we include here a demonstration of this result.

**Lemma 2.4.** *Assume that  $\{Z_i; i \in \mathbb{N}\}$  is a sequence of i.i.d. real r.v.'s, lying on the max domain of attraction of  $\Psi_\alpha$  for some  $\alpha > 0$  (Type III extreme value distribution). Assume further that  $Z_i$  has finite  $l$ th moment. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ |(Z^{(k;n)} - y_n)x_n|^l \right] = \mathbb{E}[\mathcal{P}_k^l].$$

*Proof.* Let  $L < 0$ , then

$$\mathbb{E} \left[ |(Z^{(k;n)} - y_n)x_n|^l \right] = \int_L^0 |x|^k \mathbf{Z}^{(k;n)}(dx) + \int_{-\infty}^L |x|^k \mathbf{Z}^{(k;n)}(dx), \quad (2.4)$$

where  $\mathbf{Z}^{(k;n)}(dx)$  is the distribution of  $(Z^{(k;n)} - y_n)x_n$  (note that in this case  $y_n \equiv x_F$ , which is finite). Since it converges in distribution to  $\mathcal{P}_k$ , the first integral in the right-hand side also converges as  $n \rightarrow \infty$ , moreover, by monotone convergence

$$\lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \int_L^0 |x|^k \mathbf{Z}^{(k;n)}(dx) = \int_{-\infty}^0 |x|^k \mathbf{P}_k(dx),$$

with  $\mathbf{P}_k(dx)$  the distribution of  $\mathcal{P}_k$ . Thus, it suffices to prove that if  $L$  is sufficiently large the second integral in the right-hand side of (2.4) is small.

$$\begin{aligned} \int_{-\infty}^L |x|^l \mathbf{Z}^{(k;n)}(dx) &= \int_{-\infty}^L \left( \int_x^0 l|y|^{l-1} dy \right) \mathbf{Z}^{(k;n)}(dx) \\ &= \int_{-\infty}^L \left( \int_L^0 l|y|^{l-1} dy \right) \mathbf{Z}^{(k;n)}(dx) + \int_{-\infty}^L \left( \int_x^L l|y|^{l-1} dy \right) \mathbf{Z}^{(k;n)}(dx) \\ &\leq |L|^l \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq L) + \int_{-\infty}^L \left( \int_x^L l|y|^{l-1} dy \right) \mathbf{Z}^{(k;n)}(dx). \end{aligned}$$

## 2.1. Extremes of independent random variables

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Again,  $(Z^{(k;n)} - y_n)x_n$  converges in distribution to  $\mathcal{P}_k$ , that has finite  $l$ th moment, then

$$\lim_{L \rightarrow -\infty} \limsup_{n \rightarrow \infty} |L|^l \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq L) = 0.$$

Finally, we focus on the integral

$$\int_{-\infty}^L \left( \int_x^L l|y|^{l-1} dy \right) \mathbf{Z}^{(k;n)}(dx) = \int_{-\infty}^L l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy.$$

For every  $y \in \mathbb{R}$ , a simple calculation yields to

$$\mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) = \sum_{i=0}^{k-1} \binom{n}{i} (1 - F(yx_n^{-1} + y_n))^i F(yx_n^{-1} + y_n)^{n-i}, \quad (2.5)$$

where  $F(\cdot)$  is the common probability distribution function of  $Z_i$ . Now fix  $\tilde{L} < x_F$  and assume that  $n$  is sufficiently large so that  $(\tilde{L} - y_n)x_n < L$  (note that in the Type III convergence the sequence  $x_n$  diverges and  $y_n = x_F$ ), then

$$\begin{aligned} & \int_{-\infty}^L l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy \\ &= \int_{-\infty}^{(\tilde{L} - y_n)x_n} l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy \\ &+ \int_{(\tilde{L} - y_n)x_n}^L l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy \\ &= \text{(I)} + \text{(II)}. \end{aligned}$$

We estimate each integral separately. For (I) we do a change of variables to obtain

$$\begin{aligned} \text{(I)} &= x_n^l \sum_{i=0}^{k-1} \binom{n}{i} \int_{-\infty}^{\tilde{L}} l|y|^{l-1} (1 - F(y))^{i-1} F(y)^{n-i} dy \\ &\leq x_n^l \sum_{i=0}^{k-1} \binom{n}{i} F(\tilde{L})^{n-i-1} \int_{-\infty}^{\tilde{L}} l|y|^{l-1} F(y) dy \\ &\leq \mathbb{E}[Z_i^l] F(\tilde{L})^{n-k} x_n^l \sum_{i=0}^{k-1} n^i. \end{aligned}$$

By Karamata's representation,  $x_n$  has a polynomial growth as  $n \rightarrow \infty$ . Moreover,  $\mathbb{E}[Z_i^l] < \infty$  and  $F(\tilde{L})^{n-k}$  has an exponential decay as  $n \rightarrow \infty$ , then

$$\limsup_{n \rightarrow \infty} \int_{-\infty}^{(\tilde{L} - y_n)x_n} l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy = 0.$$

Finally, we estimate (II). Let  $0 < \varepsilon < l/(k-1)$ , since  $\tilde{L}$  is fixed, one can find positive constants  $A < B$ , such that for every  $\tilde{L} \leq z \leq 0$

$$A|z|^{-\alpha} \leq 1 - F(x_F + z) \leq B|z|^{-\alpha - \varepsilon},$$

note that  $\alpha > 0$ . Then,

$$\begin{aligned}
 (\text{II}) &= \int_{(\tilde{L}-y_n)x_n}^L l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy \\
 &\leq \sum_{i=0}^{k-1} \binom{n}{i} \int_{(\tilde{L}-y_n)x_n}^L l|y|^{l-1} \frac{B^i |y|^{-(\alpha+\varepsilon)i}}{N^i} \left(1 - \frac{A|y|^{-\alpha}}{n}\right)^{n-i} dy \\
 &\leq \sum_{i=0}^{k-1} B^i \int_{(\tilde{L}-y_n)x_n}^L l|y|^{l-1-(\alpha+\varepsilon)i} \exp\left(-A|y|^{-\alpha} \frac{n-i}{n}\right) dy.
 \end{aligned}$$

Since  $l - i\varepsilon > 0$  for all  $i = 0, \dots, k-1$ , the integral

$$\int_{-\infty}^0 l|y|^{l-1-i(\alpha+\varepsilon)} e^{-A|y|^{-\alpha}} dy$$

converges. Hence,

$$\limsup_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{(\tilde{L}-y_n)x_n}^L l|y|^{l-1} \mathbb{P}((Z^{(k;n)} - y_n)x_n \leq y) dy = 0,$$

which proves the statement.  $\square$

## 2.2 Branching random walks and extreme of correlated random variables

When investigating the extremal process obtained from random structures of the form (1.12), one must take into account the correlation between variables. In Section 1.2.2, we have defined the overlap function  $R(\cdot, \cdot)$ , which is a natural way to measure the correlation in (1.12). A classical probabilistic model, having similar features is the branching random walks (BRW for short) [AN04]: at the beginning (generation 0), there is a single particle located at zero. This particle gives birth to new particles that are positioned according to a point process  $\mathcal{L}$ , and dies immediately afterwards. Its children, who form the first generation, give birth to new particles, that are positioned (with respect to their birth places) according to independent copies of  $\mathcal{L}$ , and die immediately afterwards; they form the second generation. The system goes on according to the same mechanism. Let  $\text{BRW}(\mathcal{L})$  denote any BRW defined by i.i.d. copies of  $\mathcal{L}$ .

Let  $\mathbb{T}$  be the Galton-Watson tree associated to the genealogical tree of the BRW defined as above. To each point (or individual) of the BRW one can associate a unique vertex  $w \in \mathbb{T}$ . Let  $e \in \mathbb{T}$  be the root of the Galton-Watson tree, then for a vertex  $w \in \mathbb{T}$ , let  $\llbracket e, w \rrbracket$  denote the shortest path connecting  $e$  with  $w$ , and  $|w|$  the length of this path. We will sometimes write its points  $\llbracket e, w \rrbracket = (e, w_1, \dots, w_k)$  with  $i = |w_i|$  and  $w_k = w$ . We denote by  $\eta(w)$ , the position of the individual  $w \in \mathbb{T}$ , which is obtained by summing the weights on the edges of  $\llbracket e, w \rrbracket$ , hence, the rightmost position is given by

$$M_n = \max_{|w|=n} \{\eta(w)\}.$$

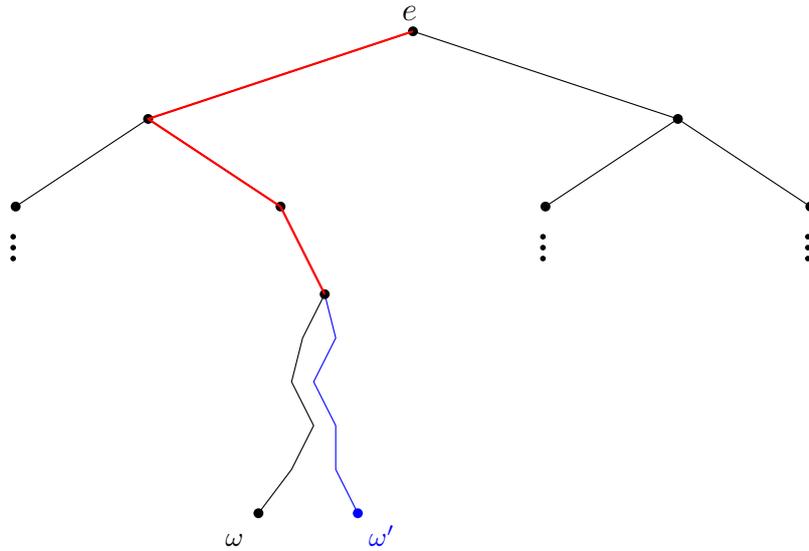


Figure 2.1: Binary tree and the overlap (in red) between two paths  $\omega$  and  $\omega'$ .

The random variables  $\eta(w)$  are not independent and the correlation between  $\eta(w)$  and  $\eta(w')$  can also be measured by the *overlap* between the paths  $[[e, w]]$  and  $[[e, w']]$ , see Figure 2.1. BRWs have the additional property that when the two paths  $[[e, w]]$  and  $[[e, w']]$  split, they can never merge again, which is clearly not the case in (1.8). Remarkably, this tree structure provides sufficient independence to the model and a complete description of its order statistics is known. The lecture notes [Zei] explain elegantly how one can use the independence inherited from this tree structure to study the extremal process of a BRW.

The first results concerning the extremal process of BRWs appeared in the 1970's in the works of Kingman [Kin75], Hammersley [Ham74] and Biggins [Big76, Big77], when the first-order correction to the maximum was computed. Assume that for some  $a > 0$ ,

$$\mathbb{E}[|\mathcal{L}|^{1+a}] < \infty, \tag{2.6}$$

where  $|\mathcal{L}|$  denotes the number of points of  $\mathcal{L}$  (*i.e.* the number of offspring). Condition (2.6) simply says that the reproductive law of the Galton-Watson tree  $\mathbb{T}$  associated to the BRW has finite moment of order  $1 + a$ .

Assume also that the logarithmic generating function for the branching random walk

$$\psi(u | \mathcal{L}) := \ln \mathbb{E} \left[ \int e^{uy} \mathcal{L}(dy) \right] \tag{2.7}$$

is finite in a neighbourhood of  $u = 0$  and that there exists a  $u^* = u(\mathcal{L}) > 0$  for which

$$\psi(u^* | \mathcal{L}) = u^* \psi'(u^* | \mathcal{L}). \tag{2.8}$$

If (2.6 – 2.8) hold, denote by  $\gamma(\mathcal{L}) = \psi'(u^* | \mathcal{L})$ , then, conditionally to the event that the BRW does not extinguish, the following limit exists

$$\lim_{n \rightarrow \infty} \frac{1}{n} M_n = \gamma(\mathcal{L}) \quad a.s. \tag{2.9}$$

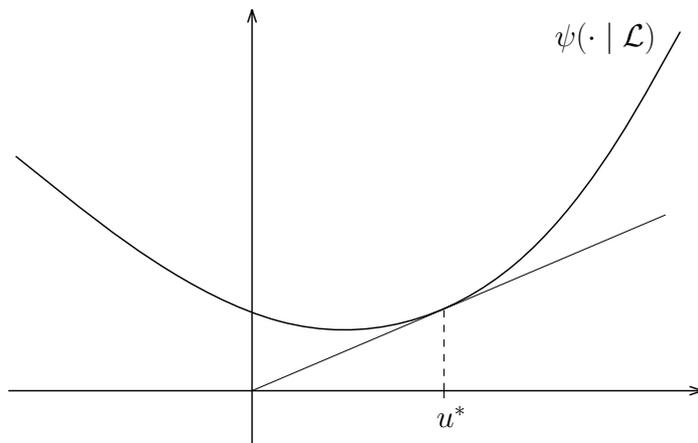


Figure 2.2: BRW's logarithmic generating function.

see [AN04] for a rigorous proof. Almost thirty years later, results concerning the second order expansion to the maximum have appeared [ABR09, Bac00]. The final picture and results about the convergence in law of the maximum have been obtained by Aïdékon [Aïd13]. With  $\gamma(\mathcal{L})$  as above and  $\gamma_1(\mathcal{L}) = 1/u^*$ , he proves that

$$M_n - \gamma(\mathcal{L})n - \frac{3}{2}\gamma_1(\mathcal{L}) \ln n$$

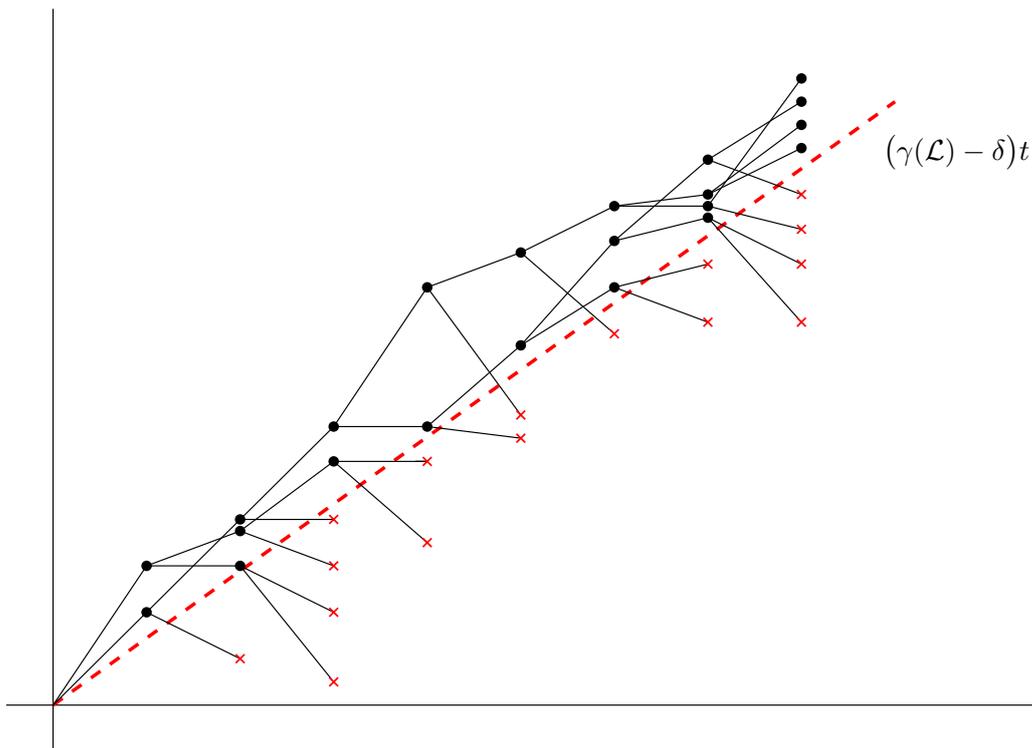
converges weakly to Gumbel distribution shifted by a random variable, see also [BDZ14] for the proof of the same result using basically the second moment method.

### 2.2.1 Surviving a ballistic absorbing barrier

In contrast with (1.8), in a BRW no constrain concerning the number of particles in the generations is imposed. Yet, one can easily come up with different ways to select individuals in a generation and control the population size of the BRW. Selection creates correlation between individuals of same generation and additional dependence in the whole process. In Section 2.2.2, we focus on the so-called  $M$ -branching random walk, where only the  $M$ -rightmost individuals are chosen to survive.

An other classical way to control the population size of a BRW is to add a ballistic absorbing barrier that is, we consider a line moving at a deterministic speed and when a particle lies bellow this moving line it is immediately absorbed (or killed) by it. This model is simply a discrete-time version of the “branching Brownian motion with a barrier” [BBS13, Mai13], for which sharp results have been obtained.

A natural question concerning this class of models is whether the system survives or not. Results in this direction were obtained by Gantert, Hu and Shi [GHS11]. They calculate the asymptotic decay for the probability that there exists an infinite ray in the BRW that always stays close to  $\gamma(\mathcal{L})$ . We recall that an infinite ray  $\llbracket e, w_\infty \rrbracket := \{e, w_1, w_2, \dots\} \subset \mathbb{T}$  is an infinite collection of vertices (or infinite path), such that  $w_i$  is the parent of  $w_{i+1}$ . It represents a family branch in the BRW that has not extinguished, and is parametrized by an element  $w_\infty \in \partial\mathbb{T}$  of the topological boundary  $\partial\mathbb{T}$  of the tree.


 Figure 2.3: BRW killed by the line of slope  $\gamma(\mathcal{L}) - \delta$ .

**Theorem 2.5** ([GHS11]). *Let  $\mathcal{L}$  be a point process satisfying (2.6 – 2.8) and*

$$(\eta(w); w \in \mathbb{T})$$

*be the particles positions in a BRW defined by  $\mathcal{L}$ . Given  $\delta > 0$ , denote by  $\rho(\infty, \delta)$  the probability that there exists an infinite ray in the branching random walk that always lies above the line of slope  $\gamma(\mathcal{L}) - \delta$ .*

$$\rho(\infty, \delta) := \mathbb{P}\left(\exists w_\infty \in \partial\mathbb{T} : \eta(w_t) \geq (\gamma(\mathcal{L}) - \delta)t, \forall w_t \in \llbracket e, w_\infty \rrbracket\right),$$

*where  $w_t \in \llbracket e, w_\infty \rrbracket$  is the vertex in generation  $t$ . Then, as  $\delta \searrow 0$*

$$\rho(\infty, \delta) \sim \exp\left(-\left[\frac{\chi(\mathcal{L}) + o(1)}{\delta}\right]^{1/2}\right),$$

*where  $\chi(\mathcal{L}) = \frac{\pi^2}{2} u^* \psi''(u^* | \mathcal{L})$  for  $u^*$  given by (2.8).*

In Chapter 4, we obtain the *finite-size corrections* to the speed  $v_N(\xi)$  of the  $N$ -particle system (1.8). Our result relies strongly on Theorem 2.5 above.

### 2.2.2 The $M$ -branching random walk

Recently, some attention has been paid to models of evolving particle systems under the effect of selection. In this subsection, we will focus on the branching random

walks with selection of the  $M$  rightmost individuals (the  $M$ -BRW) [BG10, BD97, BD99, Mal15]. A continuum version of it (the  $M$  branching Brownian motion) has been extensively studied by Maillard [Mai11].

The  $M$ -BRW is defined as follows: let  $\mathcal{L}$  be a point process that has more than one point a.s. (this condition is necessary, since it assures that the process will not extinguish). At generation 0, there are  $M$  particles located at zero. The generations are obtained by the following branching/selection step:

- *Branching*: Each particle gives birth to new particles, that are positioned according to independent copies of  $\mathcal{L}$ .
- *Selection*: One selects the  $M$ -rightmost individuals among all children to form the next generation (in average there are  $M|\mathcal{L}|$  individuals after the branching step).

Hence, the population size is kept constant and equal to  $M$ .

We denote by  $M$ -BRW( $\mathcal{L}$ ) any  $M$ -BRW defined as above. Bérard and Gouéré [BG10] focused on the binary  $M$ -BRW defined by the point process

$$\mathcal{L} = \delta_{p_1} + \delta_{p_2},$$

where  $p_i$  are i.i.d. and study its extremal process for large  $M$ . With  $y_1(t), \dots, y_M(t)$  the particles positions in generation  $t \in \mathbb{N}$ , it is not very difficult to prove that the cloud of particles travels up to fluctuations at a deterministic speed  $\gamma_M(\mathcal{L})$  and that its diameter

$$\max_{1 \leq i \leq M} y_i(t) - \min_{1 \leq i \leq M} y_i(t)$$

is of order  $\ln M$ . The existence of the asymptotic speed  $\gamma_M(\mathcal{L})$  is obtained by Kingman's sub-additive ergodic theorem, and by a coupling argument one can prove that  $\gamma_M(\mathcal{L})$  is increasing on  $M$ . The selection mechanism can only slow down the system, hence

$$\gamma_M(\mathcal{L}) \leq \gamma(\mathcal{L}),$$

which yields the convergence of  $\gamma_M(\mathcal{L})$  as  $M \rightarrow \infty$ , moreover, it is not difficult to see that the limit is precisely  $\gamma(\mathcal{L})$ . The striking result in [BG10] is that the authors prove the physics conjectures [BD97, BD99] for this model. Namely, the asymptotic velocity of the system  $\gamma_M(\mathcal{L})$  converges to  $\gamma(\mathcal{L})$  at the unexpectedly slow rate

$$\gamma_M(\mathcal{L}) = \gamma(\mathcal{L}) + \chi(\mathcal{L})(\ln M)^{-2} + o((\ln M)^{-2}) \quad \text{as } M \rightarrow \infty,$$

with  $\chi(\mathcal{L})$  from Theorem 2.5. This result is of the same nature as (1.7) and (1.13), indicating that all three models belong to the same *universality class*. Their proof makes use of ideas and results by Pemantle [Pem09], and by Gantert, Hu and Shi [GHS11], and relies on a comparison of the particle system with a family of  $M$  independent branching random walks killed below a linear space-time barrier.

Although their result is stated in the framework of binary branching with independent displacement, it can be generalized for more general reproductive point processes. We state here a weaker version of Theorem 1.1 in [Mal15], that holds for general point processes satisfying (2.6 – 2.8).

**Theorem 2.6.** *Let  $\mathcal{L}$  be a point process satisfying (2.6 – 2.8), then*

$$\gamma_M(\mathcal{L}) - \gamma(\mathcal{L}) \sim \frac{\chi(\mathcal{L})}{(\ln M)^2} \quad \text{as } M \rightarrow \infty.$$

where  $\chi(\mathcal{L}) = u^* \psi''(u^* | \mathcal{L})$  for  $u^*$  given by (2.8).

In Chapter 4, we show that the  $N$ -particle system (1.8) can be bounded from below by a family of  $M$ -BRW indexed by  $N$ . We will then adapt the arguments in [BG10] to obtain a uniform lower bound for the speeds of the BRWs.

## 2.3 Ancestral Partition Process and the Coalescent

As we have pointed out in Chapter 1, the  $N$ -particle system may be seen as a constant size population model evolving on discrete time. In this context, we are interested in knowing the genealogical relations between individuals. We analyse the genealogical tree of the population by observing the *ancestral partition process*, that is defined as follows: sample without replacement  $n \ll N$  individuals from a given generation  $T$ , say  $e_1, \dots, e_n$  and for  $0 \leq t \leq T$  we consider  $\Pi_t^{N,n}$  the random partition of  $[n] := \{1, \dots, n\}$  such that  $i$  and  $j$  belong to the same equivalent class if and only if  $e_i$  and  $e_j$  share the same ancestor at time  $T - t$ , see Figure 2.4. It is very important to realize that the direction of time for the ancestral process is the opposite of the direction of time for the “natural” evolution of the population.

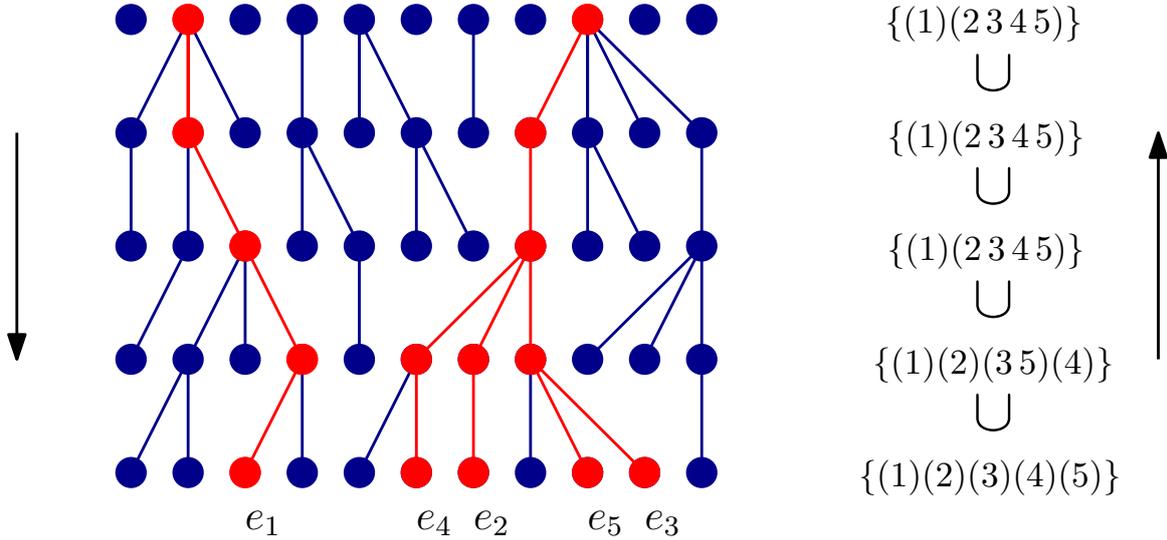


Figure 2.4: Ancestral partition process with  $N = 11$  and  $n = 5$  (its time parameter runs upwards).

Denote by  $P_n$  the finite set of all partitions of  $[n]$  and by  $P_\infty$  the set of partitions of  $\mathbb{N}^*$ . For  $\pi, \pi' \in P_n$  we say that  $\pi'$  is a refinement of  $\pi$  if every equivalent class of  $\pi$  is either a union of several equivalence classes of  $\pi'$  or coincides with an equivalence class of  $\pi'$ , we denote it by  $\pi' \subset \pi$ .

**Definition 2.7** (Coalescent). *We call a  $\mathbb{P}_n$ -valued Markov process  $(\Pi_t^n; t \geq 0)$  a  $n$ -coalescent if it has right-continuous step function paths and if  $\Pi_s^n$  is a refinement of  $\Pi_t^n$ , whenever  $s \leq t$ . We call a  $\mathbb{P}_\infty$ -valued process  $(\Pi_t; t \geq 0)$  a coalescent if it has càdlàg paths and if  $\Pi_s$  is a refinement of  $\Pi_t$  for all  $s < t$ .*

*The time variable  $t$  is not necessary in the continuum, and the coalescent can be equally defined for  $t \in \mathbb{N}$ .*

The *ancestral partition process* is a  $n$ -coalescent defined up to time  $T$ . We use the notation  $\Pi^{N,\cdot}$  to denote the ancestral partition of a constant size population with  $N$  individuals, while the notation  $\Pi^{\infty,\cdot}$ , or simply  $\Pi$ , stands for a coalescent process.

**$\Lambda$ -coalescent and  $\Xi$ -coalescent processes.** Pitman [Pit99] introduced the so-called  $\Lambda$ -coalescent: it consists in a continuous time  $\mathbb{P}_\infty$ -coalescent characterized by the rates  $\lambda_{b,k}$  at which  $k$  blocks merge into a single one when there are  $b$  blocks in total. The  $\Lambda$ -coalescent is obtained by Kolmogorov's extension theorem: for  $n \in \mathbb{N}$ , one can use the arrays  $(\lambda_{b,k})_{2 \leq k \leq b}$  to generate a  $n$ -coalescent (the projection of  $\Pi$  over  $\mathbb{P}_n$ ). Then, if  $\lambda_{b,k}$  satisfies the consistency condition

$$\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1},$$

Kolmogorov's extension theorem guaranties the existence of a probability space where the coalescent process  $\Pi$  is defined, see [Pit99]. Pitman also proves that the consistency condition holds if and only if there exists a non-negative and finite measure on the Borel subsets of  $[0, 1]$  such that

$$\lambda_{b,k} = \int_{[0,1]} u^{k-2}(1-u)^{b-k} \Lambda(du).$$

These coalescents are then characterized by and named after the measure  $\Lambda$ . When  $\Lambda$  is a unit mass at zero, we get the well-known Kingman's coalescent [Kin82], a coalescent process in which each pair of blocks merges at rate 1. Another notorious case is when  $\Lambda$  is the uniform distribution on  $[0, 1]$ , this process was first studied by Bolthausen and Sznitman [BS98] in the context of Ruelle's probability cascades, and it is named after the authors.

One can further generalize the  $\Lambda$ -coalescent and focus on coalescent processes that may undergo "simultaneous multiple collisions", the  $\Xi$ -coalescent, see Möhle and Sagitov [MS01] and Schweinsberg [Sch00]. Let  $b, b_1, \dots, b_a, s$  be non-negative integers such that  $b_1 \geq \dots \geq b_a \geq 2$  and  $b = s + \sum b_i$ . Then,  $\Xi$ -coalescent are  $\mathbb{P}_\infty$ -valued Markov processes characterized by the rates  $\lambda_{b;b_1, \dots, b_a; s}$  at which  $b$  blocks merge into  $a + s$  blocks, with  $s$  blocks that remain unchanged and  $a$  blocks that are obtained by the union of  $b_1, \dots, b_a$  blocks before the merging. In this case, the consistency condition that the rate  $\lambda_{b;b_1, \dots, b_a; s}$  have to satisfy is of the form

$$\lambda_{b;b_1, \dots, b_a; s+1} = \lambda_{b;b_1, \dots, b_a; s} - \sum_{j=1}^a \lambda_{b+1; b_1, \dots, b_{j+1}, \dots, b_a; s} - s \lambda_{b+1; b_1, \dots, b_a, 2; s-1}, \quad (2.10)$$

see, for example, Lemma 3.3 in [MS01] and [Sch00]. In particular, it implies that the distribution of the  $\Xi$ -coalescent is completely determined by the rates  $\lambda_{b;b_1, \dots, b_a}$ .

### 2.3.1 Weak convergence of ancestral processes

It is well-known that continuous-time coalescent processes may be obtained as the weak limit of *ancestral processes*, as the population size  $N \rightarrow \infty$ . Let  $\mathcal{D}([0, \infty); \mathbb{P}_n)$  be the space of càdlàg functions on  $[0, \infty)$  taking values in  $\mathbb{P}_n$ . Since  $\mathbb{P}_n$  endowed with the discrete metric is a separable complete metric space, the space  $\mathcal{D}([0, \infty); \mathbb{P}_n)$  is also separable and complete in the Skorokhod distance. We say that a process converges in the Skorokhod sense if the distribution of the process converges weakly in  $\mathcal{D}([0, \infty); \mathbb{P}_n)$  equipped with this metric.

Mölhe and Sagitov [Möh99, Möh00, MS01] obtain general conditions under which the *ancestral processes*  $\Pi_t^{N, \cdot}$  converge in the Skorokhod sense as  $N \rightarrow \infty$ . Consider the discrete-time population model defined by the family sizes

$$\nu(t) := (\nu_1(t), \dots, \nu_N(t)) \quad \text{with} \quad \nu_1(t) + \nu_2(t) + \dots + \nu_N(t) = N, \quad t \in \mathbb{Z},$$

where  $\nu_i(t)$  denotes the number of children of the  $i$ th individual in generation  $t$ . Assume that

- (i) The offspring vectors  $\nu(t)$ ,  $t \in \mathbb{Z}$  are i.i.d. copies of  $\nu$ .
- (ii) The offspring vector  $(\nu_1, \dots, \nu_N)$  is  $N$ -exchangeable.

Although (ii) may be weakened, the first assumption can not be avoided, since it ensures the Markov property of the *ancestral partition process*.

Under (i) and (ii), it is easy to compute the transition probability of  $\Pi^{N, n}$ . Let  $\pi' \subset \pi$  be two partitions of  $\mathbb{P}_n$  and denote by  $a$  and  $b$  the number of equivalent classes of  $\pi$  and  $\pi'$  respectively. Then  $b = b_1 + \dots + b_a$ , where  $b_i$ 's are positive integers denoting the number of equivalent classes of  $\pi'$  that one has to merge in order to obtain one equivalent class of  $\pi$ . By a combinatorial “putting balls into boxes” argument we obtain that the transition probability from  $\pi'$  to  $\pi$  is

$$\begin{aligned} p_N(\pi', \pi) &= \mathbb{P}(\Pi_{t+1}^{N, n} = \pi \mid \Pi_t^{N, n} = \pi') \\ &= \frac{1}{(N)_b} \sum_{\substack{i_1, \dots, i_a=1 \\ \text{all distinct}}}^N \mathbb{E}[(\nu_{i_1})_{b_1} \dots (\nu_{i_a})_{b_a}], \end{aligned} \quad (2.11)$$

where  $(N)_b := N(N-1)\dots(N-b+1)$ . Using (ii), one can further simplify (2.11) obtaining

$$p_N(\pi', \pi) = \frac{(N)_a}{(N)_b} \mathbb{E}[(\nu_1)_{b_1} \dots (\nu_a)_{b_a}].$$

We now state Mölhe and Sagitov result, we keep their notation and let  $c_N$  be the probability that two individuals, chosen randomly without replacement from some generation, have a common ancestor one generation backward in time

$$c_N := \frac{1}{N(N-1)} \sum_i^N \mathbb{E}[\nu_i(t)(\nu_i(t) - 1)] = \frac{1}{(N-1)} \mathbb{E}[\nu_1(t)(\nu_1(t) - 1)]. \quad (2.12)$$

**Theorem 2.8** ([MS01]). *Suppose that for all  $a \geq 1$  and  $b_1 \geq \dots \geq b_a \geq 2$ , the limit*

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_1)_{b_1} \cdots (\nu_a)_{b_a}]}{N^{b_1 + \dots + b_a - a} c_N} \quad (2.13)$$

*exists, and let  $b := b_1 + \dots + b_a$ . If*

$$\lim_{N \rightarrow \infty} c_N = 0,$$

*then the time-rescaled ancestral processes  $(\Pi_{[t/c_N]}^{N,n}, t \geq 0)$  converge weakly as  $N \rightarrow \infty$  to a process  $(\Pi_t^{\infty,n}, t \geq 0)$  that has the same law as the restriction to  $[n]$  of a  $\Xi$ -coalescent. Furthermore, the transition rates  $\lambda_{b;b_1, \dots, b_a}$ , that characterize the distribution of  $\Pi_t^{\infty,n}$ , are equal to the limits in (2.13). On the other hand, if*

$$\lim_{N \rightarrow \infty} c_N = c > 0,$$

*then the processes  $(\Pi_t^{N,n}, t \in \mathbb{N})$  converge weakly as  $N \rightarrow \infty$  to a process  $(\Pi_t^{\infty,n}, t \in \mathbb{N})$ , which has the same law as the restriction to  $[n]$  of a discrete-time  $\Xi$ -coalescent. The transition probabilities  $p_{b;b_1, \dots, b_a}$  satisfy*

$$p_{b;b_1, \dots, b_a} = \lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_1)_{b_1} \cdots (\nu_a)_{b_a}]}{N^{b_1 + \dots + b_a - a}}. \quad (2.14)$$

The existence of the limits in (2.13) implies that the finite-dimensional distributions of  $\Pi_{[t/c_N]}^{N,n}$  converge to those of the coalescent  $\Pi_t^n$ , as proved in [MS01]. The authors in [Möh99, MS01] prove that when  $c_N \rightarrow 0$  the sequence of processes  $\Pi_{[t/c_N]}^{N,n}$  is tight, which implies the weak convergence in the Skorokhod sense.

### 2.3.2 The genealogy of the $N$ -particle system

One can easily define a population model from (1.8) when the disorder  $\xi$  has no atoms, that is, for every  $x \in \mathbb{R}$  the probability

$$\mathbb{P}(\xi_{ij}(t+1) = x) = 0.$$

In this case, for all  $j$  the equation

$$X_j(t+1) = X_i(t) + \xi_{ij}(t+1). \quad (2.15)$$

has a.s. a unique solution  $i$ , and one can say that  $X_j(t+1)$  is an offspring or a descendant of  $X_i(t)$ . In the case where  $\xi$  has atoms, the equation (2.15) may have more than one solution  $i$ , and one must come up with a criteria to choose among the possible parents. We will only focus on the non-atomic case, though.

Denote by  $\nu_i(t)$  the number of descendants of  $X_i(t)$  in generation  $t+1$ . The fitnesses of the individuals are given by their positions  $X_1(t), \dots, X_N(t)$  and conditionally on

$$\mathcal{F}_t := \sigma\{\xi_{ij}(s) \text{ and } X_i(0); 0 \leq s \leq t, 1 \leq i, j \leq N\},$$

the probability that  $X_j(t+1)$  descends from  $X_i(t)$  is given by

$$\eta_i(t) := \mathbb{P}\left(\xi_{ij}(t+1) + X_i(t) \geq \xi_{kj}(t+1) + X_k(t); \text{ for every } 1 \leq k \leq N \mid \mathcal{F}_t\right). \quad (2.16)$$

Since  $\{\xi_{ij}(t+1); 1 \leq i, j \leq N\}$  are independent, it is easy to see that, for  $j_1, \dots, j_m$  distinct and  $i_1, \dots, i_m$  (not necessarily distinct),

$$\begin{aligned} & \mathbb{P}(X_{j_k}(t+1) \text{ descends from } X_{i_k}(t), \text{ for } 1 \leq k \leq m \mid \mathcal{F}_t) \\ &= \eta_{i_1}(t)\eta_{i_2}(t) \dots \eta_{i_m}(t). \end{aligned}$$

If  $i_k = i_l$ , the individuals  $j_k$  and  $j_l$  have a common ancestor in generation  $t$ . Hence, given  $\mathcal{F}_t$  the offspring vector  $\nu(t) := (\nu_1(t), \dots, \nu_N(t))$  is distributed according to a  $N$ -class multinomial with  $N$  trials and probabilities outcomes  $\eta(t) := (\eta_1(t), \dots, \eta_N(t))$ . In Chapter 5, we focus on this population model and show that under certain assumptions it converges to a continuous-time coalescent.

## 2.4 Properties of Brunet-Derrida $N$ -particle system

Finally, in this section, we state the basic properties of the  $N$ -particle system (1.8). Most of the results presented here have already been proved in [CQR13], but for the sake of completeness we propose a detailed overview of them.

It will be convenient to consider the process  $X^*(t)$  obtained by ordering the components of  $X(t)$  at each time  $t$ . Denote by

$$X^{(1)}(t) \geq X^{(2)}(t) \geq \dots \geq X^{(N)}(t)$$

the components of  $X^*(t)$ . Let, also  $\sigma = \sigma(t)$  be the random permutation of  $\{1, \dots, N\}$  such that

$$X^{(i)}(t) = X_{\sigma_i(t)}(t).$$

Such a ranking permutation is unique up to ties, which we break by choosing (without replacement) uniformly at random among the possibilities.

For  $t \geq 1$  and a given realization of the ordered process  $X^*(t)$ , the vector  $X(t)$  can be generated by a simple random permutation (chosen uniformly among the  $N!$  possibilities) of the vector  $X^*(t)$ .

The process  $X^*(t)$  is a discrete-time Markov chain on the state space

$$\Delta_N := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N; x_1 \geq x_2 \geq \dots \geq x_N\},$$

which we endow with the natural partial order  $x \leq y$  if and only if  $x_i \leq y_i$  for each  $1 \leq i \leq N$ . The process  $X^*$  preserves this partial order that is, for  $x \leq y$ , one can find a coupling  $(Y_1(t), Y_2(t))$  such that the marginal distribution has the law of the process  $X^*(t)$  starting from  $x$  and  $y$  respectively and

$$Y_1(t) \leq Y_2(t) \quad \forall t \in \mathbb{N}.$$

### 2.4.1 The process seen from the leading edge

It is trivial from its definition (1.8) that  $X(t)$  and  $X^*(t)$  are shift covariant, that is, for  $Y_1(t)$  and  $Y_2(t)$  as above starting respectively from  $0 \in \mathbb{R}^N$  and  $\mathbf{r} := (r, \dots, r) \in \mathbb{R}^N$

$$Y_1(t) + \mathbf{r} = Y_2(t) \quad \text{a.s.}$$

Hence, the location of the front at any time  $t$  can be described by a numerical function  $\Phi(X^*(t))$ , which commutes to space translations by constant vectors

$$\Phi(x + \mathbf{r}) = r + \Phi(x), \quad \forall x \in \Delta_N \text{ and } \mathbf{r} = (r, \dots, r), \quad (2.17)$$

and which is increasing for the partial order on  $\Delta_N$ . Among such, we mention the maximum or the minimum value and the arithmetic mean, but one can come up with different functions, which put forth the position of the bulk of the system rather than the leading edge. In Section 2.5, the function

$$\Phi(x) = \ln \left( \sum e^{x_i} \right)$$

will play an important role. For simplicity, in this section we will locate the front position by its rightmost particle and we focus on the *process seen from the leading edge*

$$X^0(t) := X(t) - X_{\sigma_1}(t), \quad (2.18)$$

and on its ordered  $X^{0*}(t)$  version, which takes values in  $\Delta_N^0 := \{x \in \Delta_N; x_1 = 0\}$ .

**Proposition 2.9.** *There exists a unique invariant measure  $\nu$  for the process seen from the leading edge  $X^{0*}$ . Furthermore, for any starting point  $X^{0*}(0) = x \in \Delta_N^0$ , let  $\nu_t$  be the law of  $X^{0*}(t)$ , then  $\nu_t$  converges in total variation distance to  $\nu$ ,*

$$\text{dist}_{T.V.}(\mathcal{L}(X^{0*}(t) \mid X(0) = x), \pi) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

*Proof.* We denote by  $\mathcal{F}_t$  the filtration

$$\mathcal{F}_t := \sigma\{X_i(0), \xi_{ij}(s); 1 \leq i, j \leq N \text{ and } s \leq t\}. \quad (2.19)$$

Then, one can easily check that the  $\mathcal{F}_t$ -stopping time

$$\tau := \inf \left\{ t \geq 1; \xi_{\sigma_1(t-1), i}(t) = \max_{1 \leq j \leq N} \{\xi_{ji}(t)\}; \forall i = 1 \dots N \right\}$$

is also a renewal times for the *process seen from the leading edge*, which implies that  $X^{0*}$  is an irreducible, aperiodic and Harris recurrent Markov chain. Thus, there exists a unique stationary measure  $\nu$  for it, and for every starting point  $X(0) = x$  the law of  $X(t)$  converges in total variation distance, see Proposition 2.1 in [CQR13] for more details.  $\square$

A consequence of the proof of Proposition 2.9 is that there exists a family of renewal times for  $X^{0*}$

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots$$

such that  $\tau_{i+1} - \tau_i$  is distributed according to  $\tau$ . By the law of large numbers and the renewal theorem, for any measurable function  $G : \Delta_N^0 \rightarrow \mathbb{R}$  the limit holds

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \sum G(X^{0*}(t)) &= \frac{1}{\mathbb{E}[\tau_1 - \tau_2]} \mathbb{E} \left[ \sum_{t=\tau_1}^{\tau_2-1} G(X^{0*}(t)) \right] \quad a.s. \\ &= \int G(y) \nu(dy). \end{aligned}$$

We use this simple observation to prove the existence of the *front speed*.

**Corollary 2.10** (Front Speed). *If  $\xi \in L^1$  the following limits*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \max_{1 \leq i \leq N} X_i(t) = \lim_{t \rightarrow \infty} \frac{1}{t} \min_{1 \leq i \leq N} X_i(t) = v_N(\xi)$$

*exist a.s. with  $v_N(\xi)$  the speed of the  $N$ -particle system given by*

$$v_N(\xi) = \int_{\Delta_N^0} \nu(dy) \mathbb{E} \left[ \max_{1 \leq i, j \leq N} \{y_i + \xi_{ij}(1)\} \right].$$

*Moreover, if  $\xi_{ij} \in L^2$ ,*

$$\frac{\max \{X_i(t)\} - tv_N(\xi)}{\sqrt{t}}$$

*converges in distribution to a Gaussian r.v. with variance  $D_N(\xi) \in (0, \infty)$ .*

*Proof.* The existence of the speed  $v_N(\xi)$  is a consequence of renewal theorem and the above remark. The equality of the two limits is due to the tightness and the fact that the front does not spread. Hence,

$$\frac{\max X_i(t) - \min X_i(t)}{t} \xrightarrow[t \rightarrow \infty]{} 0 \quad a.s.$$

Finally, the Gaussian limit is simply the central limit theorem for renewal processes. For more details see Corollary 2.1 in [CQR13].  $\square$

An immediate consequence of Corollary 2.10 is that for any  $\Phi$  satisfying (2.17)

$$\frac{1}{t} \Phi(X(t)) \xrightarrow[t \rightarrow \infty]{} v_N(\xi) \quad a.s.$$

Hence, the *front speed* does not depend on the way one measures the front position.

## 2.4.2 The correlation structure

The correlation structure between paths in (1.8) is quite involved and different from the BRW case. Paths that start together may merge and split several times, making the problem here rather difficult. To realize this complexity, one should distinguish a particular path (denoted by  $\mathbb{1}$ ) of length  $t$  and count the number  $r_k(t) = r_k$

$$r_k := \#\{\omega; |\omega| = t \text{ and } R(\omega, \mathbb{1}) = k\} \quad (2.20)$$

of paths  $\omega$  having *overlap*  $k$  with  $\mathbb{1}$ , for  $R(\cdot, \cdot)$  given by (1.14). Note that  $r_k$  does not depend on the path  $\mathbb{1}$  one chooses. In the BRW case, this calculation is straightforward, for example, for the binary branching  $r_k = 2^{t-k}$ .

Fixing  $\mathbb{1} = (1, \dots, 1)$  in the following two lemmas, we obtain a combinatorial expression for  $r_k$ , which highlights its complexity when compared to the BRW case. Such combinatorics are often useful when using moment methods.

**Lemma 2.11.** *Let  $r_0(t)$  be given by (2.20), then*

$$r_0(t) = \sum_{j=0}^{\lfloor (t+2)/2 \rfloor} (N-1)^{t+1-j} \binom{t+2-j}{j}.$$

*Proof.* We can write

$$r_0(t) = \sum_{j=0}^t r_0^{(j)}(t),$$

where  $r_0^{(j)}(t)$  is the number of paths  $\omega$  such that  $R(\omega, \mathbb{1}) = 0$  and there exist  $i_1 < i_2 < \dots < i_j$ , for which  $\omega_{i_s} = 1$  (and  $\omega_i \neq 1$  otherwise). Since  $R(\omega, \mathbb{1}) = 0$ , the integers  $i_s$  can not be consecutive.

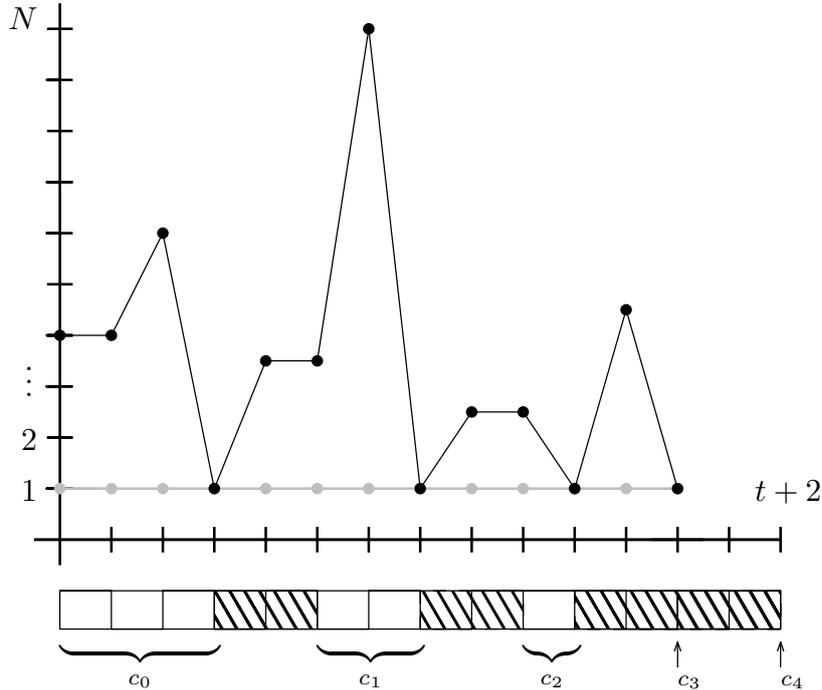


Figure 2.5:  $\omega$  such that  $R(\omega, \mathbb{1}) = 0$  and  $j = 4$

Given the integers  $i_1 < i_2 < \dots < i_j$  there are  $(N-1)^{t+1-j}$  distinct paths such that  $\omega_{i_s} = 1$ . Hence, it suffices to count the possible configurations of integers  $i_1 < i_2 < \dots < i_j$ . Let  $c_0$  be the distance from 0 to  $i_1$  and for  $s \geq 1$  define  $c_s$  to be the distance from  $i_s + 2$  to  $i_{s+1}$  (we set  $i_{j+1} = t+2$ ). Note that the distances  $c_s$  may be zero, see Figure 2.5.

## 2.4. Properties of Brunet-Derrida $N$ -particle system

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So the number of configurations  $i_1 < i_2 < \dots < i_j$  satisfying the desired constrain is equal to the number of integers  $c_0, \dots, c_j$  such that

$$\sum_{s=0}^j c_s = t + 2 - 2j,$$

which is equal to  $\binom{t+2-j}{j}$ , finishing the proof.  $\square$

We now obtain the exact expression for  $r_k(t)$  for  $1 \leq k \leq t$ .

**Lemma 2.12.** *Let  $r_k(t)$  be given by (2.20). Then,*

$$r_k(t) = \sum_{j=1}^{\lfloor (t-k+2)/2 \rfloor} (N-1)^{t-k+1-j} \binom{t-k+2-j}{j} \binom{k+j-1}{j-1}.$$

*Proof.* The strategy is to construct a bijection

$$\Pi \longleftrightarrow \{\omega, R(\omega, \mathbb{1}) = k\},$$

where  $\Pi$  is a set, whose cardinality is known. Define

$$\Pi^{(j)} := \{\tilde{\omega}; |\tilde{\omega}| = t-k; R(\tilde{\omega}, \mathbb{1}) = 0; \exists i_1 < \dots < i_j \text{ s.t. } \tilde{\omega}_{i_l} = 1 \text{ and } \tilde{\omega}_s \neq 1 \text{ otherwise}\}.$$

From the proof of Lemma 2.11

$$\#\Pi^{(j)} = (N-1)^{t-k+1-j} \binom{t-k+2-j}{j}.$$

Now, let

$$\Pi := \bigcup_{j=1}^{\lfloor (t-k+2)/2 \rfloor} \Pi^{(j)} \times \{(x_1, \dots, x_j) \in \mathbb{N}^j; x_1 + \dots + x_j = k\}$$

Note that for each  $\tilde{\omega} \in \Pi^{(j)}$  there exists

$$\binom{k+j-1}{j-1}$$

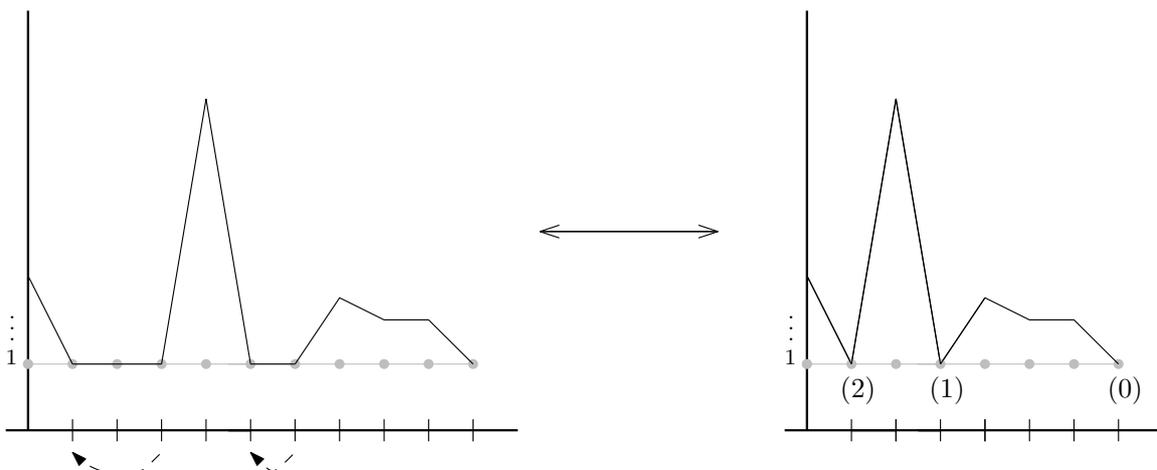
non negative integers  $x_1, \dots, x_j$  such that  $x_1 + \dots + x_j = k$ . Hence

$$\#\Pi = \sum_{j=1}^{\lfloor (t-k+2)/2 \rfloor} (N-1)^{t-k+1-j} \binom{t-k+2-j}{j} \binom{k+j-1}{j-1}.$$

We now prove that there exists a one-to-one correspondence

$$\Pi \longleftrightarrow \{\omega, R(\omega, \mathbb{1}) = k\},$$

which proves the statement.


 Figure 2.6:  $\omega$  such that  $R(\omega, \mathbb{1}) = 3$ .

Given a path  $\omega$  (such that  $|\omega| = t$  and  $R(\omega, \mathbb{1}) = k$ ), we can associate a path  $\tilde{\omega}$  of length  $t - k$  as follows. Let  $i_1, \dots, i_k$  be such that

$$\omega_{i_i} = \omega_{i_i+1} = 1.$$

Then we construct  $\tilde{\omega}$  as in Figure 2.6 by “removing” the components of  $\omega$  that have overlap with  $\mathbb{1}$ . In other words, one merges each connected components of

$$\{i; \omega_i = 1\}$$

into single vertices. By construction, the new path  $\tilde{\omega}$  has length  $t - k$ , it has no overlap with  $\mathbb{1}$  and  $\tilde{\omega}_s = 1$  for at least one  $s \leq t - k$ . Let  $j$  be the number of times  $\tilde{\omega}_s = 1$ , then  $\tilde{\omega} \in \Pi^{(j)}$ .

The correspondence  $\omega \rightarrow \tilde{\omega}$  is onto but not into. To make it into, one must assign labels  $0 \leq x_i \leq k$  to the vertices of  $\tilde{\omega}$ , for which  $\tilde{\omega}_{s_i} = 1$  ( $1 \leq i \leq j$ ). It suffices to take  $0 \leq x_i \leq k$  equal to the length of the connect component that has been merged in the construction of  $\tilde{\omega}$  (if there is no “merging” we set  $x_i = 0$ ). It is now simple to verify that the correspondence is one-to-one, which finishes the proof.  $\square$

## 2.5 The Gumbel distribution, a solvable case

Brunet and Derrida [BDMM06] discovered that the case of Gumbel disorder is remarkable. For this choice of disorder the model is completely solvable and precise calculations are possible. For simplicity, we will consider here  $G(0, 1)$ -Gumbel, but one can easily generalize for Gumbel laws with scaling parameter  $\lambda > 0$  and location parameter  $a \in \mathbb{R}$ , see for example Theorem 3.1 [CQR13].

As we have already mentioned, for Gumbel distribution the good way to describe the front position is

$$\Phi(x) = \ln \sum e^{x_i}. \quad (2.21)$$

In this case, the Markov process  $X(t) \in \mathbb{R}^N$  is lumpable with respect to the mapping  $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$ , and the new Markov process  $\Phi(X(t))$  is simply a random walk with

i.i.d. increments. One can write

$$\begin{aligned} X_i(t+1) &= -\ln \min_{1 \leq j \leq N} \left\{ \exp \left( -X_j(t) - \xi_{ji}(t+1) \right) \right\} \\ &= \Phi(X(t)) - \ln \min_{1 \leq j \leq N} \left\{ e^{-X_j(t) + \Phi(X(t))} e^{-\xi_{ji}(t+1)} \right\}. \end{aligned} \quad (2.22)$$

With  $\mathcal{F}_t$  from (2.19), one can easily show using the stability property of the exponential law under independent minimum (we point out that  $e^{-G(0,1)}$  is exponentially distributed with parameter one) that

$$\mathcal{E}_i := \min_{1 \leq j \leq N} \left\{ e^{-X_j(t) + \Phi(X(t))} e^{-\xi_{ji}(t+1)} \right\}$$

is independent from  $\mathcal{F}_t$  and that it has an exponential distribution with parameter one. It implies that  $\mathcal{E} := (\mathcal{E}_1, \dots, \mathcal{E}_N)$  is distributed according to a  $N$ -sample of the exponential law with parameter one. Denote by

$$\ln \mathcal{E}^{-1} := (\ln \mathcal{E}_1^{-1}, \dots, \ln \mathcal{E}_N^{-1}),$$

which is distributed as a  $N$ -sample of the  $G(0,1)$ -Gumbel law. The translation property (2.17) yields

$$\Phi(X(t+1)) = \Phi(X(t)) + \Phi(\ln \mathcal{E}^{-1}).$$

Hence, as we have claimed,  $\Phi(X(t+1))$  is a random walk with increments  $\Phi(\ln \mathcal{E}^{-1})$  and the following result holds.

**Theorem 2.13** ([BD04]). *Assume that  $\xi_{ij}$ 's in (1.8) are Gumbel  $G(0,1)$ -distributed. Then,  $\Phi(X(t))$  is a random walk, with increments  $\Phi(\ln \mathcal{E}^{-1})$ , with  $\ln \mathcal{E}^{-1}$  a  $N$ -sample of the Gumbel  $G(0,1)$ -distribution. Then,  $v_N$  and  $D_N$  are given by*

$$v_N(\xi) = \mathbb{E} [\Phi(\ln \mathcal{E}^{-1})] \quad \text{and} \quad D_N^2(\xi) = \text{Var} [\Phi(\ln \mathcal{E}^{-1})]$$

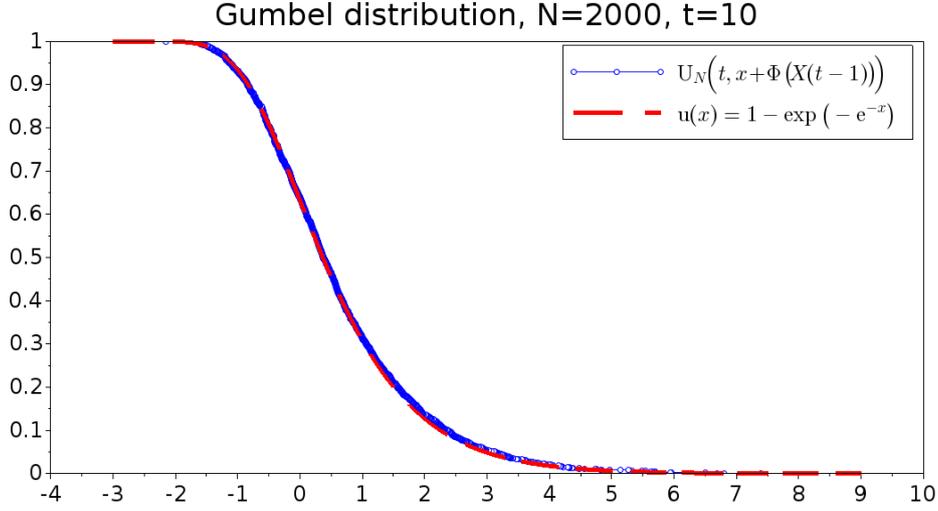
respectively,  $v_N(\xi)$  and  $D_N(\xi)$  have the following asymptotic for large  $N$

$$\begin{aligned} v_N &= \ln N + \ln \ln N + \frac{\ln \ln N}{\ln N} + \frac{1 - \gamma_E}{\ln N} + o\left(\frac{1}{\ln N}\right) \\ D_N &= \frac{\pi^2}{3 \ln N} + o\left(\frac{1}{\ln N}\right), \end{aligned}$$

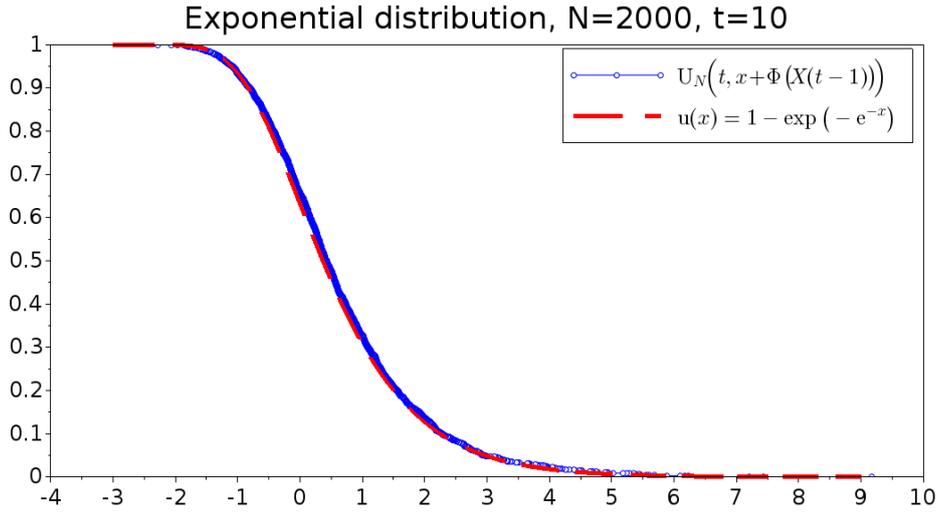
where  $\gamma_E$  is the Euler-gamma constant.

Moreover, the law of  $\nu$  the invariant measure from Proposition 2.9 is the law of the shift vector by  $\Phi$  of the ordered vector obtained from a  $N$ -sample from a Gumbel  $G(0,1)$ -distribution.

*Proof.* We have already proved that  $\Phi(X(t))$  is a random walk, which yields immediately the claims concerning the speed and diffusion constant. The asymptotic development can be found in [BD04]. Finally, the invariant measure  $\nu$  can be easily calculated from (2.22), see Theorem 3.1 in [CQR13] for more details.  $\square$



(a) Empirical distribution function for  $G(0,1)$ -Gumbel disorder with 2000 particles. Plot of the function at  $t = 10$ .



(b) Empirical distribution function for exponential disorder with 2000 particles. Plot of the function at  $t = 10$ .

Figure 2.7: In both cases the  $N$  particles start on zero. Despite the initial condition, when the disorder has exponential upper tail, the system mix in a few steps as  $N \rightarrow \infty$  as proved in [CQR13].

In the Gumbel case, regardless the positions of the particles at time  $t$ , the process  $X(t+1) - \Phi(X(t+1))$  mixes in one step. We can use this property to study the front equation (1.10). Shifting space by  $\Phi(X(t-1))$ , one gets from (2.22) that

$$U_N(t, x + \Phi(X(t-1))) \quad t \geq 1$$

is simply the empirical distribution function of a  $N$ -sample of the  $G(0,1)$ -Gumbel law. Hence, as  $N \rightarrow \infty$ , the law of large numbers yields

$$U_N(t, x + \Phi(X(t-1))) \rightarrow u(x) := 1 - \exp(-e^{-x}),$$

which proves (1.11) for the Gumbel disorder. Figure 2.7a shows a simulation of the empirical distribution function with 2000 particles, where one can already see this asymptotic front shape.

### 2.5.1 Exponential upper tail

Comets *et al.* [CQR13] prove that for perturbations of the Gumbel distribution the convergence

$$U_N\left(t, x + \Phi(X(t-1))\right) \rightarrow u(x) := 1 - \exp(-e^{-x}), \quad \text{as } N \rightarrow \infty,$$

still holds. It shows that  $u(x)$  is a universal scaling limit for the front shape and that all distribution with exponential upper tails lie in this universality class. As one can check from the simulation of the  $N$ -particle system with exponential disorder (Figure 2.7b), this universal limit shape already appears with 2000 particles.

**Theorem 2.14** (Theorem 1.2 in [CQR13]). *Assume that*

$$\mathbb{P}(\xi > x) \sim x^{-\alpha} \quad x \rightarrow \infty.$$

*Then, with  $\Phi$  from (2.21), for all initial configurations  $X(0) \in \mathbb{R}^N$ , all  $k \in \mathbb{N}$ ,  $K_N \subset \{1, \dots, N\}$  with cardinality  $k$  and all  $t \geq 2$ , we have*

$$(X_j(t) - \Phi(X(t-1)); j \in K_N) \xrightarrow{\text{law}} G(0, 1)^{\otimes k},$$

*where  $G(0, 1)^{\otimes k}$  is a  $k$  sample of Gumbel  $G(0, 1)$ -distribution. Moreover,*

$$U_N\left(t, x + \Phi(X(t-1))\right) \rightarrow u(x) = 1 - \exp(-e^{-x}), \quad \text{as } N \rightarrow \infty,$$

*uniformly in probability.*

### 2.5.2 Gumbel distribution and last-passage percolation

*In this chapter, we focus on a different model of passage percolation, with Gumbel passage times. Our goal is to show that the techniques developed in Section 2.5 are also effective in this case and that the exact time-constant can be computed.*

We focus here on the directed last-passage percolation (one can pass from last- to first-passage percolation by a simple change of signal) on  $\mathbb{Z}$  studied by Foss *et al.* [FMS14]. Consider the directed graph  $G = (\mathbb{Z}, E)$ , where every directed edge  $(i, j)$  from the vertex  $i$  to the vertex  $j > i$  is present. For each edge  $(i, j)$  the passage time  $v_{ij}$  is sampled from a given distribution  $v$ .

The authors in [FMS14] were interested in the last-passage percolation problem

$$w_{0,n} := \max_{\pi \in \Pi_{0,n}} \sum_{e \in \pi} v_e,$$

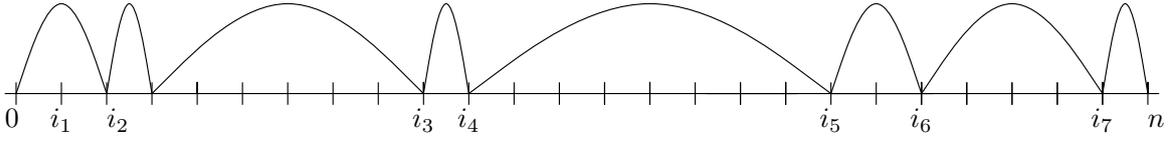


Figure 2.8: Last-passage percolation with arbitrary jumps.

where  $\Pi_{0,n}$  is the set of all increasing paths  $\pi = \{(i_0, i_1) \dots (i_{l-1}, i_l)\}$  from 0 to  $n$ . They show that there are two different regimes depending on whether the passage times have finite second moment or not. If the passage times have finite second moment, the process has a certain regenerative structure, and a strong law of large numbers holds

$$\lim_{n \rightarrow \infty} \frac{w_{0,n}}{n} = v,$$

but the exact expression for the time-constant is not available. In this section, we assume that the passages times  $v_{ij}$  are i.i.d. copies of a  $G(0, 1)$ -Gumbel distribution and obtain the exact expression of  $v$ . We decompose  $w_{0,n}$  as follows

$$w_{0,n} = \max_{1 \leq i \leq n-1} \{w_{0,i} + v_{i,n}\},$$

and we consider the filtration

$$\mathcal{G}_n := \sigma\{v_{i,j}; 0 \leq i < j \leq n\}.$$

Hence, for each  $i \leq n-1$  the last-passage time  $w_{0,i}$  is  $\mathcal{G}_{n-1}$ -measurable. By a simple computation, we rewrite  $w_{0,n}$  as

$$w_{0,n} = \ln \left( \sum_{i=1}^{n-1} e^{w_{0,i}} \right) - \ln \left( \min_{1 \leq i \leq n-1} \{e^{-w_{0,i}} e^{-v_{i,n}}\} \times \sum e^{w_{0,i}} \right). \quad (2.23)$$

The stability property of the exponential law under independent minima implies that conditionally on  $\mathcal{G}_{n-1}$

$$\min_{1 \leq i \leq n-1} \{e^{-w_{0,i}} e^{-v_{i,n}}\}$$

has an exponential distribution with parameter  $\sum e^{w_{0,i}}$ . Hence,

$$\mathcal{E}_n := \min\{e^{-w_{0,i}} e^{-v_{i,n}}\} \times \sum e^{w_{0,i}}$$

is exponentially distributed with parameter 1 and  $\mathcal{G}_{n-1}$ -independent. Now we focus on the sum on the right-hand side of (2.23). The same argument applied to  $w_{0,n-1}$  yields

$$\sum_{i=1}^{n-1} e^{w_{0,i}} = \sum_{i=1}^{n-2} e^{w_{0,i}} + \mathcal{E}_{n-1}^{-1} \sum_{i=1}^{n-2} e^{w_{0,i}},$$

with  $\mathcal{E}_{n-1}$  exponentially distributed with parameter 1 and  $\mathcal{G}_{n-2}$ -independent. By a conditioning argument, one can prove that  $\mathcal{E}_n$  and  $\mathcal{E}_{n-1}$  are also independent. Iterating the argument, we obtain the formula

$$w_{0,n} = \ln \mathcal{E}_n^{-1} + \sum_{i=2}^{n-1} \ln (1 + \mathcal{E}_i^{-1}) + w_{0,1}, \quad (2.24)$$

with  $\mathcal{E}_i$  an i.i.d. sequence. Summing up, for Gumbel passage times, one can write  $w_{0,n}$  as a sum of i.i.d. random variables, which yields the following result.

**Proposition 2.15.** *Assume that the passages times  $(v_e; e \in \Pi_{0,n})$  are Gumbel  $G(0, 1)$ -distributed. Then, by the law of large numbers*

$$\lim_{n \rightarrow \infty} \frac{w_{0,n}}{n} = v = \mathbb{E} \ln(1 + \mathcal{E}^{-1}) \quad a.s.$$

where  $\mathcal{E}$  is exponentially distributed with parameter 1. Moreover, by the central limit theorem

$$\frac{w_{0,n} - nv}{\sqrt{n}}$$

converges in distribution to a Gaussian random variable with variance

$$\sigma^2 = \text{Var} \ln(1 + \mathcal{E}^{-1}).$$

In contrast with (1.8), the model here is not of the *mean-field* type. Hence, the techniques and results obtained in [CQR13] for perturbations of the Gumbel law do not apply. Nevertheless, the method developed by Brunet and Derrida is general enough and may be effective for different passage percolation model for which the time-constant is unknown. Being able to solve the microscopic dynamics for a specific disorder allows one to have an idea of the expected behaviour of the system, which is in itself a good reason to present the technique in this generality.

## 2.6 Simulations

The definition of the  $N$ -particle system (1.8) already yields an algorithm to generate it. Nevertheless, the computational cost of such algorithm grows with the square of the number  $N$  of particles. Due to this constrain, we have only simulated systems with no more than  $2 \times 10^3$  particles.

**Front profile.** As stated in Theorem 2.14, when the disorder has an exponential upper tail the empirical distribution function converges uniformly in probability to a deterministic function  $u(x)$ . Such behaviour can already be observed in the Figures 2.7a and 2.7b.

No mathematical result for the front with other distributions is available. In Figure 2.9 we present simulations of the front profiles when the disorder is distributed respectively as the minus of the uniform law and exponential law. We rescale the distance between particles by  $N$  and in order to compare the curves with  $u(x)$  we centred them at the points where they are 1/2. The simulations indicate that  $u(x)$  is not the scaling limit for the front shape in those cases. Nevertheless, if we compare both curves it seems clear that a limit shape exists, see Figure 2.9c.

We have also simulated the  $N$ -particle system for other types of disorders, such as the Gaussian distribution. Nevertheless, due to the high computational cost, the simulations are not conclusive and hence we were not able to draw any interesting conclusion from it.

**Front velocity.** Theorem 1.1 states that when the disorder lies in the max domain of attraction of the Weibull law, then the finite size correction to the speed are of order  $N^{1/\alpha}$ . We also performed simulations in order to guess the second term in the expansion. We considered  $\xi_{ij}$  distributed according to the minus of an exponential law with parameter one and the minus of the uniform distribution. For both cases, the following limit holds

$$Nv_N(\xi) + 1/e \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

We simulate the  $N$ -particle system with  $N = 125, 250, 500, 10^3$  (Figure 2.10). For each  $N$  and  $200 \leq t \leq 250$ , we plot

$$\left( \frac{\max X_i(t)}{t} + \frac{1}{Ne} \right) N.$$

It seems clear from the simulation that the second order correction to the speed is much larger than  $N^{-2}$ .

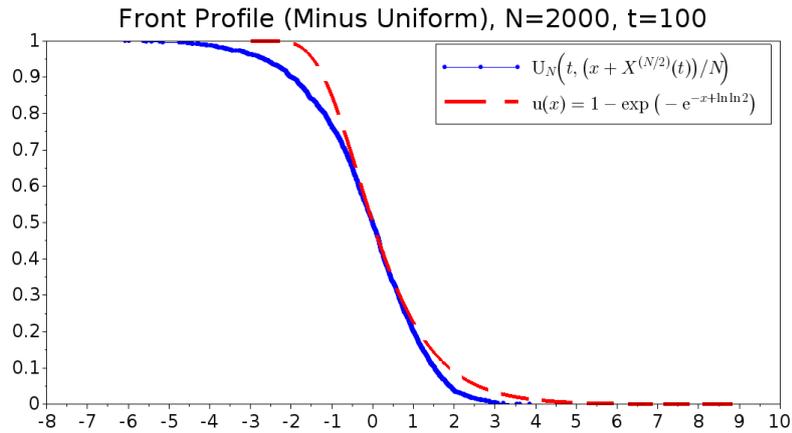
For  $N = 10^3$ , we see from both cases that

$$\left( \frac{\max X_i(t)}{t} + \frac{1}{10^3 e} \right) 10^3.$$

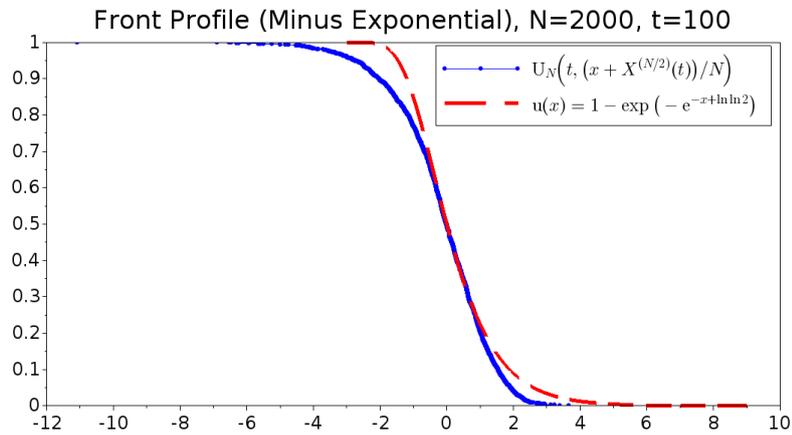
oscillates about  $-2 \times 10^{-2}$  which is approximatively  $-1/(\ln 10^3)^2$ . For  $N = 5 \times 10^2$ , the points oscillate between  $-3,5 \times 10^2$  and  $-2,5 \times 10^2$ . Once again, the points are not far from  $-1/(\ln 5 \times 10^2)^2 \cong -0,0258$ . Finally, for the smaller values of  $N$  the simulations are not conclusive.

Summarizing the discussion, it seems clear from Figure 2.10 that the second order expansion to the velocity does not have the order of  $N^{-2}$ , but it scales like  $N^{-1}$  times some logarithm correction.

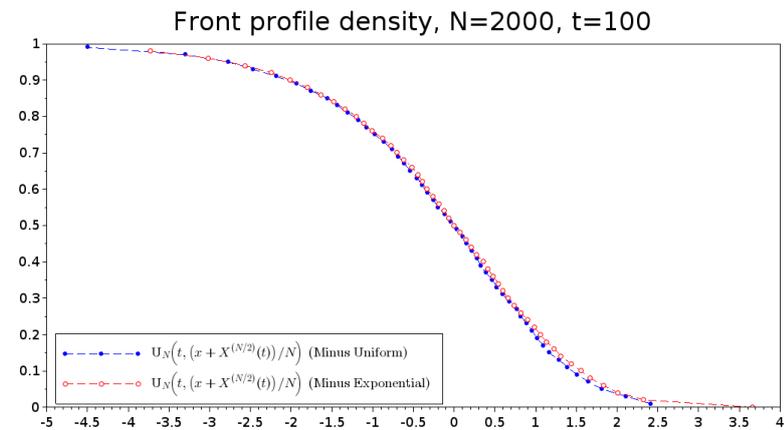
## 2.6. Simulations



(a) The  $N$  particles have started on zero. The disorder is distributed according to the minus of the uniform distribution. The curves have been centered at the values where they are  $1/2$



(b) The  $N$  particles have started on zero. The disorder is distributed according to the minus of the exponential distribution with parameter 1. The curves have been centered at the values where they are  $1/2$



(c) Comparison of  $-U$  and  $-Exp$ : the curves are so close to one another that we had to rarefy the plotted points, otherwise it would be impossible to distinguish between them.

Figure 2.9: Front profile, density case.

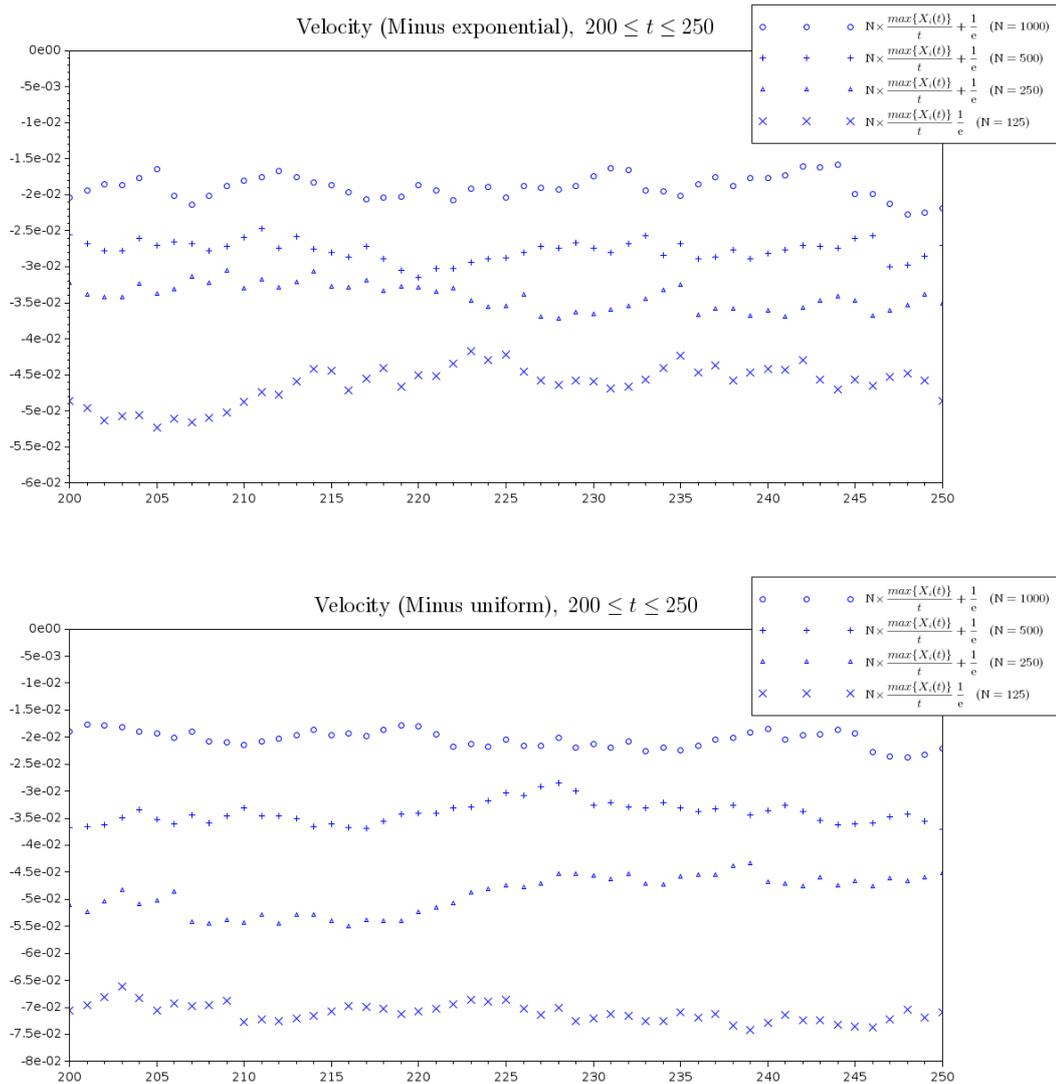


Figure 2.10: Since the particles start in zero, we consider only  $t \geq 200$ , so the system has the time to mix.



# Chapter 3

## Front speed on a discrete case, the percolation distribution

Apart from the introductory section that has been modified for a better readability and Section 3.5 that has been added later, this chapter is an article published in *Stochastic Processes and their Applications*, 2014, vol. 124, issue 11, pages 3698-3723.

### 3.1 Introduction

In this chapter, we focus on the case where the disorder  $\xi$  depends on  $N$  the number of particles. With  $r > 0$ ,  $\rho > 0$ , let

$$\begin{aligned}\mathbb{P}(\xi(N) = 0) &= p_0(N) \sim \rho/N^{1+r} \\ \mathbb{P}(\xi(N) = -1) &= 1 - \mathbb{P}(\xi(N) = 0),\end{aligned}\tag{3.1}$$

we will often omit  $N$  in the notation. As in [CD90], we will sometimes use the term “percolation distribution” to denote (3.1). Here, we will only consider the  $\xi_{ij}$  distributed according to the percolation distribution or variations of it, hence it will be convenient to denote respectively by  $v_N$  and  $D_N$  the speed and diffusion constant of the  $N$ -particle system (instead of using  $v_N(\xi)$  and  $D_N(\xi)$  as in Chapter 1).

For this choice of disorder, the front moves backwards, making the solution  $U_N \equiv 1$  unstable. If all particles start on zero, the lattice features of  $\xi$  constrains them to stay at a distance of at most one from the leaders, and when the front moves that is,

$$\max X(t) = \max X(t-1) - 1,$$

all particles jump to a same position. This particular behaviour hides a renewal structure that will be used when computing the *front speed*. Throughout this chapter, we denote by

$$\phi(X(t)) := \max_{1 \leq i \leq N} \{X_i(t)\}\tag{3.2}$$

the front’s position at time  $t$ . In [CD90], the authors study, among other things, the polymer model associated to (1.8) when the disorder  $\xi_{ij}$  is given by (3.1) with  $\rho = 1$ .

### 3.1. Introduction

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They focus on the so-called *ground state energy per unity of length*, which is simply the speed of the  $N$ -particle system and obtain the following asymptotic for large  $N$

$$v_N \rightarrow \left(1 + \lfloor 1/r \rfloor\right)^{-1}, \quad \text{as } N \rightarrow \infty. \quad (3.3)$$

Their statement is based on the observation that the typical number of sites on the  $t$ -th plane connected to the first plane by a path of zero energy is  $N^{1-tr}$ . Hence, if  $N$  is large enough and  $1 - tr$  positive there is a path of zero energy (which is necessarily a ground state) from 0 to  $t$ , whereas when  $1 - tr$  is negative there is no such path. Their argument, although informal, is correct, but the case where  $1/r$  is an integer (the critical case) requires a more careful analysis. In this chapter we formalize their argument and show that there is an additional term in (3.3) when  $1/r$  is an integer.

The case  $1/r \in \mathbb{N}$  is critical in the sense that the system displays a different behaviour when compared to  $1/r \notin \mathbb{N}$ . For  $1/r \in \mathbb{N}$  and  $N$  large enough, we show that at  $t = 1/r$  there is a Poissonian number of particles  $X_i$  that remain in position zero, while, when  $1/r \notin \mathbb{N}$  the typical number of such particles is  $N^{1-\lfloor 1/r \rfloor r}$ . The following theorem is the rigorous version of Theorem 1.2 stated in the Introduction, that we will prove in this chapter.

**Theorem 3.1.** *Let  $\xi$  be distributed according to (3.1). Then the front speed  $v_N$  satisfies*

$$\lim_{N \rightarrow \infty} v_N = \begin{cases} -(1 + \lfloor 1/r \rfloor)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ -(1 + \lfloor 1/r \rfloor - e^{-\rho^{1/r}})^{-1}, & \text{if } 1/r \in \mathbb{N}, \end{cases} \quad (3.4)$$

In the case where  $r = 0$

$$\lim_{N \rightarrow \infty} v_N = 0. \quad (3.5)$$

In Section 3.5 we focus on the diffusive properties of the model and obtain the following result concerning the diffusion constant  $D_N$ .

**Theorem 3.2.** *Let  $\xi$  be distributed according to (3.1), and  $v_N$  be the velocity of the front. Then, there exists a constant  $D_N^2$  such that*

$$\frac{\Phi(X(\lfloor ns \rfloor)) - ns v_N}{D_N v_N^{3/2} \sqrt{n}}$$

converges weakly to the standard Brownian motion  $(B_s; s \in \mathbb{R})$  as  $n \rightarrow \infty$ . Moreover,  $D_N$  satisfies the following limit

$$\lim_{N \rightarrow \infty} D_N^2 = \begin{cases} 0, & \text{if } 1/r \notin \mathbb{N} \\ \exp(-\rho^{1/r}) - \exp(-2\rho^{1/r}), & \text{if } 1/r \in \mathbb{N}. \end{cases}$$

In the case  $1/r \notin \mathbb{N}$ , let  $\lceil 1/r \rceil := \lfloor 1/r \rfloor + 1$ , then the following limit also holds

$$\lim_{N \rightarrow \infty} D_N^2 N^{r \lceil 1/r \rceil - 1} = \rho^{\lceil 1/r \rceil}.$$

We also consider a natural generalization of (3.1), and we calculate the speed of the  $N$ -particle system. First, we consider  $\xi$  taking values in the lattice  $\mathbb{Z}_0 = \{l \in \mathbb{Z}; l \leq 0\}$ . We set for  $i \in \mathbb{N}$

$$p_i(N) = \mathbb{P}(\xi(N) = -i), \quad (3.6)$$

and assume that  $p_0 \sim \rho/N^{1+r}$  where  $r$  and  $\rho$  are non-negative. Let

$$q_2(N) := \mathbb{P}(\xi(N) \leq -2) = 1 - p_0 - p_1. \quad (3.7)$$

We also assume that for  $i \geq 2$

$$\mathbb{P}(\xi = -i \mid \xi \leq -2) = \frac{p_i(N)}{q_2(N)} = \mathbb{P}(\vartheta = -i), \quad (3.8)$$

where  $\vartheta$  is an integrable distribution on the lattice  $\mathbb{Z}_{-2}$  that does not depend on  $N$ . We then compute the asymptotic of  $v_N$  as  $N \rightarrow \infty$ , that resembles to (3.4), but a different correction appears in the critical case, see Theorem 3.3 below.

The lattice constrain can be easily wakened (see Section 3.4), and one can further generalize the model considering  $\xi$  of the form

$$\xi = p_0(N)\delta_{\lambda_0} + p_1(N)\delta_{\lambda_1} + q_2(N)\vartheta(dx), \quad (3.9)$$

where  $\lambda_1 < \lambda_0$ ,  $\vartheta(dx)$  is an integrable probability distribution over  $(-\infty, \lambda_1 - (\lambda_0 - \lambda_1)]$  and  $\delta_{\lambda_i}$  is the mass distribution. Then, if we assume that  $p_0(N) \sim \rho/N^{1+r}$ , the velocity  $v_N$  obeys the following asymptotic.

**Theorem 3.3.** *Let  $\xi$  be distributed according to (3.9). Assume that*

$$p_0(N) \sim \frac{\rho}{N^{1+r}}, \quad \text{and} \quad \lim_{N \rightarrow \infty} q_2(N) = \theta,$$

where  $r > 0$  and  $0 < \theta < 1$ . Then the front speed  $v_N$  satisfies

$$\lim_{N \rightarrow \infty} v_N = \begin{cases} \lambda_0 - (\lambda_0 - \lambda_1)(1 + \lfloor 1/r \rfloor)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ \lambda_0 - (\lambda_0 - \lambda_1)(\lfloor 1/r \rfloor + 1 - 1/g(\theta))^{-1}, & \text{if } 1/r \in \mathbb{N}, \end{cases} \quad (3.10)$$

where  $g(\theta) \geq 1$  is a non-increasing function. The conclusion in the case  $1/r \notin \mathbb{N}$  still holds if  $\xi$  satisfies the weaker assumption  $q_2/(1 - p_0) \leq \theta'$  for some  $0 < \theta' < 1$ .

**Organization of the chapter:** in Subsection 3.2.1 we compute the typical number of leading particles, which corresponds to the number of paths of zero energy and in Subsection 3.2.2, we calculate the limit of  $v_N$  as  $N \rightarrow \infty$ , exhibiting in particular the additional term appearing in (3.3) in the critical case. In Subsection 3.3.1 we compute the typical number of leading particles, when  $\xi$  is distributed according to (3.9). Subsections 3.3.2 and 3.3.3 present some technical results and calculations. In Subsection 3.3.4 we compute the front velocity and prove the discrete version of Theorem 3.3. A sketch of the proof of Theorem 3.3 is given in Section 3.4. Finally, in the end of the chapter we include a section proving Theorem 3.3.

## 3.2 Front speed for the two-state percolation distribution

In this Section, we consider the case of  $\xi_{ij}$ 's distributed according to (3.1). For this choice of distribution, all  $N$  particles meet at a same location at a geometric time  $T$  regardless the initial configuration, see [CQR13] Section 5 for a rigorous proof. Due to the choice of  $\xi_{ij}$ 's and (1.8), at all later times  $t \geq T$  every  $X_i(t)$  is either at a leading position or it lies at a unit distance behind the leaders.

Since  $\phi(X(T))/t$  converges to 0 a.s. as  $t \rightarrow \infty$ , the front speed  $v_N$  is not affected by the particles' positions at time  $T$ . So henceforth, we will assume that at  $t = 0$

$$X_i(0) \in \{-1, 0\}, \quad \forall i \in \{1, \dots, N\}.$$

Under this hypothesis, we consider the following process.

**Definition 3.4.** Let  $\phi(X(t-1))$  be the front's position as in (3.2). Then, for  $t \in \mathbb{N}$  we define the stochastic process  $Z(t) := (Z_0(t), Z_1(t))$  as follows. For  $t = 0$  let  $Z_0(0)$  be the number of leading particles and  $Z_1(0)$  the number of particles that are at a unit distance behind the leaders. For  $t \geq 1$  define

$$\begin{aligned} Z_0(t) &:= \#\{1 \leq i \leq N; X_i(t) = \phi(X(t-1))\}; \\ Z_1(t) &:= \#\{1 \leq i \leq N; X_i(t) = \phi(X(t-1)) - 1\}, \end{aligned} \quad (3.11)$$

where  $\#$  denotes the number of elements in a set.

Note that for  $t \geq 1$   $Z_0(t)$  is equal to the number of leaders if the front has not moved backwards between times  $t-1$  and  $t$ , and to 0 if the front moved.  $Z$  is a homogeneous Markov chain on the set

$$\Omega(N) = \left\{ x \in \{0, 1, \dots, N\}^2; x_0 + x_1 = N \right\},$$

where  $x_0$  and  $x_1$  are the coordinates of  $x$ . The transition rates of the Markov chain  $Z_0(t)$  is given by the Binomial distributions

$$\begin{aligned} &\mathbb{P}(Z_0(t+1) = \cdot \mid Z(t) = x) \\ &= \mathbb{P}(Z_0(t+1) = \cdot \mid Z_0(t) = x_0) = \begin{cases} \mathcal{B}(N, 1 - (1 - p_0)^{x_0})(\cdot), & x_0 \geq 1 \\ \mathcal{B}(N, 1 - (1 - p_0)^N)(\cdot), & x_0 = 0. \end{cases} \end{aligned} \quad (3.12)$$

We will often consider Markov chains with different starting distributions. For this purpose we introduce the notation  $\mathbb{P}_\mu$  and  $\mathbb{E}_\mu$  for probabilities and expectations given that the Markov chain initial position has distribution given by  $\mu$ . Often, the initial distribution will be concentrated at a single state  $x$ . We will then simply write  $\mathbb{P}_x$  and  $\mathbb{E}_x$  for  $\mathbb{P}_{\delta_x}$  and  $\mathbb{E}_{\delta_x}$ .

In this Section,  $\oplus$  denotes the configuration  $(N, 0) \in \Omega(N)$ . Furthermore, we introduce the notation

$$1/r = m + \eta, \quad (3.13)$$

where  $m$  stands for the integer part of  $1/r$  and  $\eta$  its fractional part.

### 3.2.1 Number of Leading Particles

In this Subsection, we show that under a suitable normalization and initial conditions the process  $Z_0$  converges as  $N$  goes to infinity.

We consider the random variable

$$\tau := \inf \{t \geq 1; \phi(X(t)) < \phi(X(t-1))\}, \quad (3.14)$$

that is a stopping time for the filtration  $\mathcal{F}_t = \{\xi_{ij}(s); s \leq t \text{ and } 1 \leq i, j \leq N\}$ . It is not difficult to see that  $\tau$  is also the first time when  $Z_0$  visits zero

$$\tau = \inf \{t \geq 1; Z_0(t) = 0\},$$

as a consequence  $Z(\tau) = (0, N)$ . From (3.12), it is easy to conclude that the distribution of  $\{Z(\tau+t), t \geq 1\}$  is equal to the distribution of  $\{Z(t), t \geq 1\}$  under  $\mathbb{P}_\oplus$ . It yields the renewal structure that will be used when computing the front speed.

**Definition 3.5.** *Let  $Y(t)$  be the number of leading particles at time  $t$  if the front has not moved*

$$Y(t) := Z_0(t) \mathbf{1}_{\{t \leq \tau\}}. \quad (3.15)$$

Then,  $Y$  is a homogeneous Markov chain with absorption state at zero and transition rates given by the Binomial distributions

$$\mathbb{P}(Y(t+1) = \cdot \mid Y(t) = k) = \mathcal{B}(N, 1 - (1 - p_0)^k)(\cdot).$$

The advantage of working with  $Y$  rather than  $Z_0$  is that the above formula holds even if  $Y(t) = 0$ .

**Proposition 3.6.** *Let  $\xi$  be distributed according to (3.1). For  $k \in \{1, 2, \dots, N\}$  denote by  $G_k(s, t)$  the Laplace transform of  $Y(t)$  under  $\mathbb{P}_k$  at  $s \in \mathbb{R}$ . Then,*

$$G_k(s, t) := \mathbb{E}_k [e^{sY(t)}] = \exp \left\{ (e^s - 1)k(Np_0)^t (1 + o(1)) \right\} \quad (3.16)$$

as  $N \rightarrow \infty$ .

*Proof.* Conditioning on  $\mathcal{F}_{t-1} := \{\xi_{ij}(s); s \leq t-1\}$

$$\begin{aligned} \mathbb{E}_k [e^{sY(t)}] &= \mathbb{E}_k \left[ \mathbb{E} [e^{sY(t)} \mid Y(t-1)] \right] \\ &= \mathbb{E}_k \left[ \left( 1 + (e^s - 1)(1 - (1 - p_0)^{Y(t-1)}) \right)^N \right]. \end{aligned}$$

Since  $p_0 \sim \rho/N^{1+r}$  with  $r > 0$  and  $Y(t-1) \leq N$ , we obtain by first order expansion that

$$\left( 1 + (e^s - 1)(1 - (1 - p_0)^{Y(t-1)}) \right)^N = s_{(1)}(N)^{Y(t-1)},$$

where  $s_{(1)}(N) = \exp \left\{ (e^s - 1)(Np_0 + o(Np_0)) \right\}$  and  $o(Np_0)/Np_0 \rightarrow 0$  as  $N \rightarrow \infty$  independently from  $Y(t-1)$ . Repeating the argument

$$\mathbb{E}_k [e^{sY(t)}] = \mathbb{E}_k [s_{(1)}(N)^{Y(t-1)}] = \mathbb{E}_k [s_{(2)}(N)^{Y(t-2)}],$$

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with  $s_{(2)}(N) = \exp \left\{ (s_{(1)}(N) - 1)(Np_0 + o(Np_0)) \right\}$ . Expanding  $s_1(N) - 1$

$$\begin{aligned} s_{(1)}(N) - 1 &= \exp \left\{ (e^s - 1)(Np_0 + o(Np_0)) \right\} - 1 \\ &= (e^s - 1)(Np_0 + o(Np_0)). \end{aligned}$$

Hence,  $s_{(2)}(N) = \exp \left\{ (e^s - 1)(Np_0)^2 + o((Np_0)^2) \right\}$ . We proceed recursively and obtain the expression

$$\mathbb{E}_k [e^{sY(t)}] = \exp \left\{ k(e^s - 1)(Np_0)^t (1 + o(1)) \right\}, \quad \text{as } N \rightarrow \infty,$$

which proves the statement.  $\square$

We point out that the case  $k = N$  corresponds to  $Z(0) = \oplus$ . We now state two corollaries of Equation (3.16).

**Corollary 3.7.** *Let  $\xi$  be distributed according to (3.1) and  $k \in \{1, \dots, N\}$ . Then, for  $t \geq m + 1$*

$$\mathbb{P}_k(Y(t) = 0) \geq 1 - \rho^t N^{1-tr} + o(N^{1-tr}) \quad (3.17)$$

as  $N \rightarrow \infty$ .

*Proof.* Since  $\mathbb{P}_k(Y(t) = 0) = \lim_{s \rightarrow -\infty} \mathbb{E}_k [e^{sY(t)}]$ , Proposition 3.6 implies that

$$\mathbb{P}_k(Y(t) = 0) = \exp \left\{ -k(Np_0)^t (1 + o(1)) \right\} \geq \exp \left\{ -N(Np_0)^t (1 + o(1)) \right\}.$$

Then, we obtain (3.17) by first order expansion.  $\square$

**Corollary 3.8.** *Let  $\xi$  be distributed according to (3.1) with  $\eta = 0$  (i.e.  $r = 1/m$ ) and  $\kappa(N)$  be a sequence of random variables in  $\{1, \dots, N\}$  with some distribution  $\mu(N)$ . Suppose that  $\kappa(N)/N$  converges in distribution to  $U$  a positive random variable.*

*Then, under  $\mathbb{P}_{\mu(N)}$ ,  $Y(m)$  converges in distribution to  $Y_\infty$  a doubly stochastic Poisson random variable characterized by its Laplace transform*

$$\mathbb{E} [e^{sY_\infty}] = \mathbb{E} [\exp \{U(e^s - 1)\rho^m\}]. \quad (3.18)$$

*Proof.* We may assume that all  $\{\xi_{ij}(t); t, i, j \in \mathbb{N}\}$  and  $\{\kappa(N), N \in \mathbb{N}\}$  are constructed on the same probability space in such a way that  $\kappa(N)/N$  converges a.s. to  $U$ . Then we use (3.16) to get

$$\mathbb{E}_{\kappa(N)} [e^{sY(m)}] = \exp \left\{ (e^s - 1) \kappa(N) \rho^m N^{-1} (1 + o(1)) \right\}, \quad \text{as } N \rightarrow \infty.$$

The term  $o(1)$  converges to zero independently from  $\kappa(N)$  the initial position. Then, by dominated convergence, we obtain that

$$\lim_{N \rightarrow \infty} \mathbb{E} [e^{sY(m)}] = \mathbb{E} [\exp \{U(e^s - 1)\rho^m\}],$$

which concludes the proof.  $\square$

We now prove a large deviation principle for  $Y$ . As in [DZ10, DH00], we denote by

$$\Lambda_{k,t}(s) := \lim_{N \rightarrow \infty} \frac{1}{kN^{-rt}} \ln \mathbb{E}_k [e^{sY(t)}], \quad (3.19)$$

the cumulant generating function of  $Y$  under  $\mathbb{P}_k$ . From (3.16) we see that  $\Lambda_{k,t}(s) = (e^s - 1)\rho^t$ . Denoting by

$$\Lambda_{k,t}^*(x) := \sup_{s \in \mathbb{R}} \{xs - \Lambda_{k,t}(s)\}, \quad (3.20)$$

the Legendre transform of  $Y(t)$  under  $\mathbb{P}_k$ , we have that

$$\Lambda_{k,t}^*(x) = \begin{cases} x(\ln x - \ln \rho^t) + \rho^t - x, & \text{if } x \geq 0; \\ \infty, & \text{if } x < 0. \end{cases} \quad (3.21)$$

**Proposition 3.9.** (*Large Deviation Principle for  $Y$* ) *Let  $\xi$  be distributed according to (3.1). For  $t \leq m$ , let  $k(N) \leq N$  be a sequence of positive integers such that*

$$\lim_{N \rightarrow \infty} k(N) N^{-rt} = \infty.$$

*Then, under  $\mathbb{P}_{k(N)}$ ,  $Y(t)/(k(N)N^{-rt})$  satisfies a large deviation principle with rate function given by  $\Lambda_{k,t}^*$  as in (3.21) and speed  $k(N)N^{-rt}$ .*

*Proof.* In fact, it is a direct application of Gärtner-Ellis Theorem (see *e.g.* Theorem V.6 in [DH00]). Since  $\Lambda$  is smooth, it is a lower semi-continuous function, therefore the lower bound in the infimum can be taken over all points.  $\square$

The next Corollary formalizes the statement of Cook and Derrida in [CD90].

**Corollary 3.10.** *Let  $\xi$  be distributed according to (3.1) and  $\kappa(N)$  be a sequence of random variables in  $\{1, 2, \dots, N\}$ . Assume that*

$$\{\kappa(N); N \in \mathbb{N}\} \text{ and } \{\xi_{ij}(t); t, i, j \in \mathbb{N}\}$$

*are constructed in the same probability space in such a way that they are independent and denote by  $\mathbb{P}^{(1)}$  and  $\mathbb{P}^{(2)}$  their distributions.*

*Suppose also that in this probability space  $\kappa(N)/N$  converges a.s. to  $U$  a positive random variable. Then, for  $t < 1/r$*

$$\lim_{N \rightarrow \infty} \mathbb{P}^{(1)} \otimes \mathbb{P}_{\kappa(N)}^{(2)} \left( \left| \frac{Y(t)}{\rho^t U N^{1-tr}} - 1 \right| \geq \varepsilon \right) = 0, \quad (3.22)$$

*where  $\mathbb{P}^{(1)} \otimes \mathbb{P}_{\kappa(N)}^{(2)}$  is the distribution of  $\{Y(t); t \in \mathbb{N}\}$  started from  $\kappa(N)$ .*

*Proof.* We first consider the case where  $\kappa(N)$  is a deterministic sequence such that

$$\lim_{N \rightarrow \infty} \frac{\kappa(N)}{N} = u, \quad \text{for some } 0 < u \leq 1$$

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Then the conditions of Proposition 3.9 are satisfied and  $Y(t)/(\kappa(N) N^{-tr})$  satisfies a large deviation principle with rate function given by (3.21), which only zero is at  $\rho^t$ . This implies the desired convergence.

The random case is solved by conditioning on  $\kappa(N) = Y(0)$ .

$$\mathbb{P}^{(1)} \otimes \mathbb{P}_{\kappa(N)}^{(2)} \left( \left| \frac{Y(t)}{\rho^t U N^{1-tr}} - 1 \right| \geq \varepsilon \right) = \int \mathbb{P}_{\kappa(N)(\omega_1)}^{(2)} \left( \left| \frac{Y(t)}{\rho^t U(\omega_1) N^{1-tr}} - 1 \right| \geq \varepsilon \right) \mathbb{P}^{(1)}(d\omega_1).$$

Since  $\kappa(N)(\omega_1)/N$  converges to  $U(\omega_1)$   $\mathbb{P}^{(1)}$ -a.s.

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\kappa(N)(\omega_1)}^{(2)} \left( \left| \frac{Y(t)}{\rho^t U(\omega_1) N^{1-tr}} - 1 \right| \geq \varepsilon \right) = 0,$$

and we conclude by dominated convergence.  $\square$

Cook and Derrida [CD90] consider the particular case where  $\rho = 1$  in (3.1). From Corollary 3.10, we see that  $Y(t)/N^{1-rt}$  converges in probability to one. Since under  $\mathbb{P}_N$ ,  $Y(t)$  is equal to the number of paths with zero energy at time  $t$ , the typical number of such paths is  $N^{1-rt}$ .

#### 3.2.2 Front Speed

In this Subsection, we give the exact asymptotic for the front speed, proving Theorem 3.1. The front's position can be computed by counting the number of times  $Z$  visits  $(0, N)$ . Indeed, at a given time  $t$  either the front moves backwards and  $\phi(X(t)) = \phi(X(t-1)) - 1$  or it stays still and  $\phi(X(t)) = \phi(X(t-1))$ . Then,

$$\frac{\phi(X(t))}{t} = \frac{-N_t}{t},$$

where  $N_t$  is the stochastic process that counts the number of times that  $Z$  visited  $(0, N)$  until time  $t$ . A classic result in renewal theory (see *e.g.* [Dur10]) states that

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_{\oplus}[\tau]}. \quad (3.23)$$

Hence, to determine the front velocity, it suffices to determine  $\mathbb{E}_{\oplus}[\tau]$ .

$$\mathbb{E}_{\oplus}[\tau] = \sum_{t=0}^{\infty} \mathbb{P}_{\oplus}(\tau \geq t+1) = \sum_{t=0}^{\infty} \mathbb{P}_{\oplus}(Y(t) \geq 1). \quad (3.24)$$

A consequence of Corollaries 3.7, 3.8 and 3.10 is that if  $\xi$  is distributed according to (3.1) with  $\eta > 0$ , then

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\oplus}(Y(t) \geq 1) = \begin{cases} 1, & \text{if } t \leq m; \\ 0, & \text{if } t \geq m+1. \end{cases}$$

Whereas we have the following limits when  $\eta = 0$

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\oplus}(Y(t) \geq 1) = \begin{cases} 1, & \text{if } t \leq m - 1; \\ 1 - e^{-\rho^m}, & \text{if } t = m; \\ 0, & \text{if } t \geq m + 1. \end{cases}$$

Then, to finish the proof of Theorem 3.1, it suffices to show that

$$\lim_{N \rightarrow \infty} \sum_{t \geq m+1} \mathbb{P}_{\oplus}(Y(t) \geq 1) = 0. \quad (3.25)$$

Since  $Y$  is a homogeneous Markov chain we use the Markov property at time  $m + 1$  to obtain

$$\sum_{t \geq m+1} \mathbb{P}_{\oplus}(Y(t) \geq 1) = \sum_{t=0}^{\infty} \sum_{k=1}^N \mathbb{P}_k(Y(t) \geq 1) \mathbb{P}_{\oplus}(Y(m+1) = k).$$

It is not difficult to see that under  $\mathbb{P}_k$ ,  $Y$  is stochastically dominated by  $Y$  under  $\mathbb{P}_N$ , which implies that  $\mathbb{P}_k(Y(t) \geq 1) \leq \mathbb{P}_N(Y(t) \geq 1)$ . Then, applying this inequality in the above expression, we get

$$\sum_{t \geq m+1} \mathbb{P}_{\oplus}(Y(t) \geq 1) \leq \mathbb{P}_{\oplus}(Y(m+1) \geq 1) \mathbb{E}_{\oplus}[\tau]. \quad (3.26)$$

**Proposition 3.11.** *Let  $\xi$  be distributed according to (3.1). Then,  $\mathbb{E}_x[\tau]$  is bounded in  $N$*

$$\sup_{N \in \mathbb{N}} \sup_{x \in \Omega(N)} \{\mathbb{E}_x[\tau]\} < \infty. \quad (3.27)$$

*Proof.* By Corollary 3.10,  $\lim_{N \rightarrow \infty} \mathbb{P}_{\oplus}(\tau \geq m + 2) = 0$ . Therefore, there exists a constant  $c_1 < 1$  such that for  $N$  sufficiently large

$$\mathbb{P}_{\oplus}(\tau \geq m + 2) \leq c_1.$$

Coupling the chains started from  $\delta_x$  and  $\delta_{\oplus}$  we obtain that  $\mathbb{P}_x(\tau \geq m + 2) \leq \mathbb{P}_{\oplus}(\tau \geq m + 2)$  for every  $x \in \Omega(N)$  and therefore

$$\mathbb{P}_x(\tau \geq m + 2) \leq c_1. \quad (3.28)$$

Then, Proposition 3.11 follows as a consequence of the Markov property and (3.28). In Subsection 3.3.2 we present an equivalent argument in all details.  $\square$

Applying Proposition 3.11 and Corollary 3.7 in (3.26), we conclude that

$$\sum_{t \geq m+1} \mathbb{P}_{\oplus}(Y(t) \geq 1) = \mathcal{O}(N^{1-(m+1)r}).$$

Hence, from (3.24) we obtain the limits

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\oplus}[\tau] = \begin{cases} 1 + m, & \text{if } r \neq 1/m; \\ 1 + m - e^{-\rho^m}, & \text{if } r = 1/m, \end{cases} \quad (3.29)$$

proving Theorem 3.1 in the case  $r > 0$ .

To finish the proof of Theorem 3.1 it remains to study the case  $r = 0$ . For that we use a coupling argument. Up to the end of this Subsection we denote by  $\xi(r)$

$$\mathbb{P}(\xi(r) = 0) = 1 - \mathbb{P}(\xi(r) = -1) \sim \rho/N^{1+r}.$$

For  $r > 0$ , the random variables  $\xi(0)$  are stochastically larger than  $\xi(r)$  for  $N$  large enough. Denoting by  $X_i^r(t)$  the stochastic process defined by  $\xi(r)$  we construct the process in such a way that the following relation holds

$$0 \geq \frac{\phi(X^0(t))}{t} \geq \frac{\phi(X^r(t))}{t}.$$

From (3.29), if we choose  $r$  such that  $1/r$  is not an integer, we have the lower bound

$$0 \geq v_N(0) \geq v_N(r) \rightarrow -(1 + [1/r])^{-1},$$

whence taking  $r$  to 0, we obtain  $\lim_{N \rightarrow \infty} v_N(0) = 0$ , which concludes the proof of Theorem 3.1.

### 3.3 Front speed for the infinitely many states percolation distribution

In this Section, we prove a discrete version of Theorem 3.3. We consider the case of  $\xi_{ij}$  distributed according to (3.6).

**Assumption (A).** The random variable  $\xi$  distributed according to (3.6) satisfies Assumption (A) if there exists a constant  $0 < \theta < 1$  such that

$$\lim_{N \rightarrow \infty} q_2 = \theta,$$

and  $\vartheta$  defined in (3.8) is integrable.

In the non-critical case we do not need to assume the convergence of  $q_2$ . We prove Theorem 3.13 under the weaker condition.

**Assumption (A').** The random variable  $\xi$  distributed according (3.6) satisfies Assumption (A') if there exists a constant  $0 < \theta' < 1$  such that for  $N$  large enough

$$\frac{q_2}{(1 - p_0)} \leq \theta',$$

and  $\vartheta$  defined in (3.8) is integrable.

As in the “two-state percolation model”, the  $N$  particles meet at a same position at a geometric time despite the starting configuration. Since  $\xi_{ij}(s) \in \mathbb{Z}_0$  (for all  $s \in \mathbb{N}$ ), for all later times  $t$  every  $X_i(t)$  is at a distance  $d_i \in \{0, 1, 2, \dots\}$  behind the leaders. For this reason, we may assume that at  $t = 0$  the particles' relative positions to the leader already satisfy this property and we consider the process  $Z(t) := (Z_l(t); l \in \mathbb{N})$ , where

$$Z_l(t) := \#\{j; 1 \leq j \leq N, X_j(t) = \phi(X(t-1)) - l\}.$$

Then,  $Z$  is a homogeneous Markov chain on the set

$$\Omega(N) := \left\{ x \in \{0, 1, \dots, N\}^{\mathbb{N}}; \sum_{i=0}^{\infty} x_i = N \right\},$$

where  $x_i$  are the coordinates of  $x$ . If at time  $t$  we have that  $Z(t) = x \in \Omega(N)$ , for each  $k \in \mathbb{N}$  there are  $x_k$  particles in position  $-k$  with respect to the leader at time  $t-1$ . In this situation, suppose that  $x_0 \geq 1$ , then for every  $1 \leq i \leq N$  the probability that  $X_i(t+1)$  is in position  $-k$  with respect to the leader at time  $t$  is equal to

$$s_k(x) := \left( \sum_{i=1}^{\infty} p_i \right)^{x_{k-1}} \cdots \left( \sum_{i=k}^{\infty} p_i \right)^{x_0} - \left( \sum_{i=1}^{\infty} p_i \right)^{x_k} \cdots \left( \sum_{i=k+1}^{\infty} p_i \right)^{x_0}, \quad (3.30)$$

where we define  $x_{-1} = 0$ . So the probability that  $X_i(t+1) = \Phi(X(t))$  is given by

$$s_0(x) := 1 - (1 - p_0)^{x_0}.$$

If  $x_0 = 0$ , we shift  $(x_0, x_1, \dots)$  to get a nonzero first coordinate obtaining a vector  $\tilde{x} \in \Omega(N)$  such that  $\tilde{x}_0 \geq 1$ . Then, one can check that

$$s_k(x) = s_k(\tilde{x}).$$

The transition probability of the Markov chain  $Z$  is given by

$$\mathbb{P}(Z(t+1) = y \mid Z(t) = x) = \mathcal{M}(N; s(x))(y), \quad (3.31)$$

where  $s(x) = (s_0(x), s_1(x), \dots)$  and  $\mathcal{M}(N; s(x))$  denotes a Multinomial distribution with infinitely many classes, we refer to [CQR13] Section 6 for more details on the computations. It is clear that  $Z_0(t)$  has the same transition probability as the process studied in the two states model. In particular, the results proved in Subsection 3.2.1 hold with the obvious changes.

**Definition 3.12.** Let  $\mathcal{A} \subset \Omega(N)$  we denote by  $T_{\mathcal{A}}$  the first time that  $Z(t)$  visits  $\mathcal{A}$

$$T_{\mathcal{A}} := \inf\{t \geq 1; Z(t) \in \mathcal{A}\}, \quad (3.32)$$

that is a stopping time for the filtration  $\mathcal{F}_t = \sigma\{\xi_{ij}(s); s \leq t, 1 \leq i, j \leq N\}$ . Often  $\mathcal{A} = \{x\}$ , in this case we will simply write  $T_x$  for  $T_{\{x\}}$ .

For a stopping time  $T$ , we define recursively  $T^{(0)} = 0$  and for  $i \geq 1$

$$T^{(i)}(\omega) := \inf\{t > T^{(i-1)}(\omega); t = T \circ \Theta_{T^{(i-1)}(\omega)}(\omega)\}, \quad (3.33)$$

where  $\Theta_t$  is the time-shift operator. We adopt the convention that  $\inf\{\emptyset\} = \infty$ . Once more we denote by  $\tau$  the stopping time defined as

$$\tau := \inf\{t \geq 1; \phi(X(t)) < \phi(X(t-1))\}. \quad (3.34)$$

In contrast with the previous Section,  $\tau$  is not a renewal time for  $Z$ . We adapt the notation of Section 3.2 and define  $\oplus := (N, 0, \dots) \in \Omega(N)$  and  $\Delta := (0, N, 0, \dots) \in \Omega(N)$ . Finally, we keep notation (3.13) and let  $m$  be the integer part of  $1/r$  and  $\eta$  its fractional part. We now state the main result of the Section.

**Theorem 3.13.** *Assume that  $\xi$  satisfies Assumption (A). Then*

$$\lim_{N \rightarrow \infty} v_N = \begin{cases} -(1 + \lfloor 1/r \rfloor)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ -(\lfloor 1/r \rfloor + 1 - 1/g(\theta))^{-1}, & \text{if } 1/r = m \in \mathbb{N}, \end{cases} \quad (3.35)$$

where  $g(\theta) \geq 1$  is a non-increasing function. The conclusion in the case  $r \neq 1/m$  still holds if  $\xi$  satisfies the weaker Assumption (A').

### 3.3.1 The Distribution of $Z(\tau)$

In this Subsection we study the limit distribution of  $Z(\tau)$  as  $N \rightarrow \infty$ . When  $\eta > 0$  the limit is similar to the one obtained in the previous results.

**Proposition 3.14.** *Assume that  $\xi$  satisfies Assumption (A') and that  $\eta > 0$ . Then,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\oplus}(Z(\tau) = \Delta) = 1. \quad (3.36)$$

The case  $\eta = 0$  is critical. We show that  $Z_1(\tau)/N$  converges in distribution and that the limit distribution is a functional of a Poisson random variable.

**Proposition 3.15.** *Assume that  $\xi$  satisfies Assumption (A) with  $\eta = 0$ . Then under  $\mathbb{P}_{\oplus}$ ,  $Z_0(m)$  converges in distribution to  $\Pi(\rho^m)$  a Poisson random variable with parameter  $\rho^m$ .*

Moreover, there exists a function  $G : \mathbb{N} \rightarrow [0, 1]$  (see Definition 3.45) such that

$$\left( \frac{Z_1(\tau)}{N}, \sum_{i=2}^{\infty} \frac{Z_i(\tau)}{N} \right) \xrightarrow{d} \left( G(\Pi(\rho^m)), 1 - G(\Pi(\rho^m)) \right). \quad (3.37)$$

Before analyzing the cases  $\eta = 0$  and  $\eta > 0$  separately, we prove a technical Lemma that holds in both cases. It can be interpreted as follows: if at time  $t$  there are sufficiently many leading particles, then at time  $t + 1$ , with high probability, there is no particle at distance two or more to the leaders at time  $t$ .

**Lemma 3.16.** *Assume that  $\xi$  satisfies Assumption (A'). For  $x = x(N) \in \Omega(N)$  such that*

$$\ln N = o(x_0)$$

as  $N \rightarrow \infty$ , define  $s_i(x)$  as in (3.30) and let  $\mathcal{M}(N; s(x))$  be a Multinomial random variable with infinitely many classes as in (3.31). Then,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \mathcal{M}(N; s(x)) \in \left\{ y \in \Omega(N); \sum_{i=2}^{\infty} y_i = 0 \right\} \right) = 1. \quad (3.38)$$

*Proof.* We can write

$$\begin{aligned}
 & \mathbb{P}\left(\mathcal{M}(N; s(x)) \in \left\{y \in \Omega(N); \sum_{i=2}^{\infty} y_i = 0\right\}\right) \\
 &= \sum_{n=0}^N \mathbb{P}\left(\mathcal{M}(N; s(x)) \in \left\{y \in \Omega(N); y_0 = n, y_1 = N - n\right\}\right) \\
 &= \sum_{n=0}^N \frac{N!}{n!(N-n)!} s_0(x)^n s_1(x)^{N-n} \\
 &\geq (1 - \theta^{x_0})^N,
 \end{aligned}$$

where the last inequality holds for  $N$  large enough as a consequence of Assumption (A'). Since  $o(x_0) = \ln N$  we obtain that  $(1 - \theta^{x_0})^N \rightarrow 1$ , proving the result.  $\square$

**Case  $\eta > 0$**

*Proof of Proposition 3.14.* From Corollaries 3.7 and 3.10 we see that  $\mathbb{P}_{\oplus}(\tau \neq m+1) \rightarrow 0$ . Then, it suffices to prove that  $\mathbb{P}_{\oplus}(Z(m+1) = \Delta; \tau = m+1) \rightarrow 1$ .

$$\mathbb{P}_{\oplus}(Z(\tau) = \Delta; \tau = m+1) = \sum_{x \in \Omega(N)} \mathbb{P}_{\oplus}(Z(m+1) = \Delta; Z(m) = x; \tau = m+1).$$

Since  $\tau = m+1$  it suffices to consider  $x$  such that  $x_0 \geq 1$ . Fix  $0 < \varepsilon < \rho^m$  and take  $x \in \Omega(N)$  such that  $|x_0/N^{r\eta} - \rho^m| < \varepsilon$ . From (3.31),

$$\begin{aligned}
 & \mathbb{P}_{\oplus}(Z(m+1) = \Delta | Z(m) = x) \\
 &= \mathcal{M}(N; s(x))(\Delta) = s_1(x)^N \\
 &= \left( (1-p_0)^{x_0} - (1-p_0)^{x_1} (1-p_0-p_1)^{x_0} \right)^N \\
 &\geq (1-p_0)^{x_0 N} (1 - \theta^{x_0})^N,
 \end{aligned} \tag{3.39}$$

where the last inequality is a consequence of Assumption (A'). Due to the choice of  $x_0$ , (3.39) is bounded from below by

$$(1-p_0)^{(\rho^m + \varepsilon)N^{1+r\eta}} (1 - \theta^{(\rho^m - \varepsilon)N^{r\eta}})^N,$$

which converges to one as  $N \rightarrow \infty$ . Then, by Proposition 3.9 and Equation (3.39), we see that

$$\mathbb{P}_{\oplus}(Z(\tau) = \Delta) \geq \sum_{|x_0/N^{r\eta} - \rho^m| < \varepsilon} \mathbb{P}_{\oplus}(Z(\tau) = \Delta; Z(m) = x; \tau = m+1)$$

converges to one, proving the result.  $\square$

### 3.3. Infinitely many states distribution

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#### Case $\eta = 0$

In this paragraph, we prove Proposition 3.15 and also a generalization that allows us to compute the distribution of  $Z_1(\tau^{(i)})$ .

**Lemma 3.17.** *Assume that  $\xi$  satisfies Assumption (A') with  $\eta = 0$ . Fix  $0 < a < b$  and denote by  $\Omega_a^b(N)$  the subset of  $\Omega(N)$  defined as*

$$\Omega_a^b(N) := \{x \in \Omega(N); aN^{1/m} \leq x_0 \leq bN^{1/m}\}.$$

Then the following limit holds

$$\lim_{N \rightarrow \infty} \sup_{x \in \Omega_a^b(N)} \mathbb{P}_x(Z(1) \neq \Delta \mid Z_0(1) = 0) = 0. \quad (3.40)$$

*Proof.* It is not difficult to obtain the following inequality

$$\mathbb{P}_x(Z(1) \neq \Delta \mid Z_0(1) = 0) \leq \frac{\mathbb{P}_x\left(Z(1) \in \{y \in \Omega(N); \sum_{i=2}^{\infty} y_i \neq 0\}\right)}{\mathbb{P}_x(Z_0(1) = 0)}.$$

Under  $\mathbb{P}_x$ ,  $Z(1)$  is distributed according to  $\mathcal{M}(N, s(x))$ , then from the proof of Lemma 3.16

$$\lim_{N \rightarrow \infty} \sup_{x \in \Omega_a^b(N)} \mathbb{P}_x\left(Z(1) \in \left\{y \in \Omega(N); \sum_{i=2}^{\infty} y_i \neq 0\right\}\right) = 0.$$

To finish the proof it suffices to show that  $\mathbb{P}_x(Z_0(1) = 0)$  is bounded away from zero. Indeed,  $Z_0(1)$  is distributed according to a Binomial random variable of parameter  $N$  and  $s_0(x)$ . Using the hypotheses of the Lemma we obtain the lower bound

$$s_0(x) \geq 1 - (1 - p_0)^{bN^{1/m}}.$$

Coupling  $Z(1)$  with  $\mathcal{B}$  a Binomial of parameter  $N$  and  $1 - (1 - p_0)^{bN^{1/m}}$

$$\mathbb{P}_x(Z_0(1) = 0) \geq \mathcal{B}\left(N, 1 - (1 - p_0)^{bN^{1/m}}\right)(0) \rightarrow e^{-\rho b},$$

for every  $x \in \Omega_a^b(N)$ , which finishes the proof.  $\square$

From Corollary 3.10, we see that under  $\mathbb{P}_{\oplus}$ ,  $Z_0(m-1)/N^{1/m}$  converges in probability to  $\rho^{m-1}$ , as  $N \rightarrow \infty$ . Hence, from Lemma 3.17, we conclude that

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\oplus}(Z(\tau) = \Delta \mid Z_0(m) = 0) = 1. \quad (3.41)$$

This is the first step to prove Proposition 3.15. The second step is to study the conditional distribution of  $Z(\tau)$  under  $Z_0(m) = x_0$  for a positive integer  $x_0$ .

**Proposition 3.18.** *Assume that  $\xi$  satisfies Assumption (A) with  $\eta = 0$ . Let  $k$  be a nonzero integer and denote by  $\Omega_k(N)$  the subset of  $\Omega(N)$  defined as*

$$\Omega_k(N) := \{x \in \Omega(N); x_0 = k\}.$$

Then, for  $\varepsilon > 0$  the following limit holds

$$\lim_{N \rightarrow \infty} \sup_{x \in \Omega_k(N)} \mathbb{P}_x\left(\left|\left(\frac{Z_1(1)}{N}, \frac{\sum_{i \geq 2} Z_i(1)}{N}\right) - (1 - \theta^k, \theta^k)\right| > \varepsilon\right) = 0. \quad (3.42)$$

*Proof.* From (3.31), we see that under  $\mathbb{P}_x$ ,  $Z(1)$  is distributed according to an infinite class Multinomial of parameters  $N$  and  $s(x)$ . In particular, the triplet

$$(Z_0(1), Z_1(1), \sum_{i \geq 2} Z_i(1))$$

is distributed according to a three classes Multinomial of parameters

$$N \text{ and } \left( s_0(x), s_1(x), \sum s_i(x) \right).$$

If  $\xi$  satisfies Assumption (A) and  $x \in \Omega_k(N)$ , we have that

$$\lim_{N \rightarrow \infty} s_0(x) = 0; \quad \lim_{N \rightarrow \infty} s_1(x) = 1 - \theta^k; \quad \lim_{N \rightarrow \infty} \sum s_i(x) = \theta^k, \quad (3.43)$$

and the rate of convergence is uniform on  $x \in \Omega_k(N)$ . A three classes Multinomial random variable as above satisfies a large deviation principle (see *e.g.* [DZ10, DH00]) and the rate function is given by

$$\Lambda^*(y) = \begin{cases} y_1 \ln \left( \frac{(\theta^k) y_1}{(1-y_1)(1-\theta^k)} \right) - \ln \left( \frac{\theta^k}{1-y_1} \right), & \text{if } y_1 + y_2 = 1; \\ \infty, & \text{otherwise.} \end{cases} \quad (3.44)$$

The only zero of  $\Lambda^*$  is at  $y = (0, 1 - \theta^k, \theta^k)$ . Implying the convergence in probability

$$\frac{1}{N} \left( Z_0(1), Z_1(1), \sum_{i \geq 2} Z_i(1) \right) \rightarrow (0, 1 - \theta^k, \theta^k),$$

as  $N \rightarrow \infty$ , which proves the statement.  $\square$

We now give the definition of the function  $G(\cdot)$  appearing in Proposition 3.15.

**Definition 3.19.** Let  $G : \mathbb{N} \rightarrow [0, 1]$  be defined as

$$G(k) = \begin{cases} 1 - \theta^k, & \text{if } k \geq 1; \\ 1, & \text{if } k = 0, \end{cases} \quad (3.45)$$

where  $\theta$  is given by Assumption (A).

*Proof of Proposition 3.15.* From Corollary 3.8, we have that under  $\mathbb{P}_\oplus$ ,  $Z_0(m)$  converges in distribution to a Poisson random variable of parameter  $\rho^m$ . Hence, to prove Proposition 3.15 it suffices to show that

$$\begin{aligned} & \mathbb{P}_\oplus \left( \left| \left( \frac{Z_1(\tau)}{N}, \frac{\sum_{i \geq 2} Z_i(\tau)}{N} \right) - \left( G(Z_0(m)), 1 - G(Z_0(m)) \right) \right| > \varepsilon \right) \\ &= \sum_{k=0}^N \mathbb{P}_\oplus \left( \left| \left( \frac{Z_1(\tau)}{N}, \frac{\sum_{i \geq 2} Z_i(\tau)}{N} \right) - \left( G(k), 1 - G(k) \right) \right| > \varepsilon; Z_0(m) = k \right) \end{aligned}$$

converges to zero. From (3.41) and Proposition 3.18, we know that for each  $k \in \mathbb{N}$  the terms in the above sum converge to zero. Then, from the tightness of  $Z_0(m)$  we obtain that the sum itself converges to zero, proving the result.  $\square$

We finish the present Subsection by computing the limit distribution of  $Z(\tau^{(i)})$  for  $i \in \mathbb{N}$ . We also prove the convergence of some related processes that will appear when calculating the front velocity in Subsection 3.3.4.

**Proposition 3.20.** *Assume that  $\xi$  satisfies Assumption (A) and that  $\eta = 0$ . Let  $\kappa(N)$  be a sequence of random variables in  $\Omega(N)$  with some distribution  $\mu(N)$  and denote by  $\kappa_0(N)$  the first coordinate of  $\kappa(N)$ . Assume also that  $\kappa_0(N)/N$  converges in distribution to  $U$  a positive random variable. Then, under  $\mathbb{P}_{\mu(N)}$ , we have that*

1.  $Z_0(m)$  converges weakly to  $V$  a doubly stochastic Poisson random variable, which distribution is determined by the Laplace transform

$$\mathbb{E} [e^{sV}] = \mathbb{E} [\exp(e^s - 1) \rho^m U]. \quad (3.46)$$

2. Furthermore, the joint convergence also holds

$$\left( Z_0(m), \frac{Z_1(\tau)}{N}, \tau \right) \xrightarrow{d} (V, G(V), m + \mathbf{1}_{\{V \neq 0\}}). \quad (3.47)$$

*Proof.* We may assume that all  $\xi_{ij}(t)$ 's and  $\kappa(N)$ 's are constructed on the same probability space in such a way that  $\kappa_0(N)/N \rightarrow U$  a.s. The hypotheses of Corollary 3.8 are satisfied, implying the first statement of the Proposition. From Corollaries 3.7 and 3.10, we see that  $\mathbb{P}(m \leq \tau \leq m + 1)$  converges to one. Since  $\tau = m$  if and only if  $Z_0(m) = 0$  and  $\tau \notin \{1, \dots, m - 1\}$  we obtain that

$$\lim_{N \rightarrow \infty} \mathbb{P}(|\tau - m - \mathbf{1}_{\{Z_0(m) \neq 0\}}| > \varepsilon) = 0,$$

which implies the convergence in distribution  $\tau \xrightarrow{d} m + \mathbf{1}_{\{V \neq 0\}}$ . Finally, to prove that  $Z_1(\tau)/N$  converges to  $G(V)$ , we proceed as in Proposition 3.15 and show by dominated convergence that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \mathbb{P}_{\kappa(N)} \left( \left| \left( \frac{Z_1(\tau)}{N}, \frac{\sum_{i \geq 2} Z_i(\tau)}{N} \right) - (G(Z_0(m)), 1 - G(Z_0(m))) \right| > \varepsilon \right) \right] = 0,$$

we leave the details to the reader. □

As an application of Proposition 3.20 we can calculate the distribution of  $Z(\tau^{(2)})$ . Indeed we can consider the convergence in Proposition 3.15 as the stronger a.s. convergence. We do not lose any generality since we can construct a sequence of random variables (possibly in an enlarged probability space)  $\kappa(N) \stackrel{d}{=} Z(\tau)$  that converges a.s. Details about this construction can be found in [Bil99]. Passing to the appropriate product space we also consider that the  $\kappa$ 's and  $\xi_{ij}$ 's are independent, which implies that

$$\mathbb{P}_{\kappa(N)}(Z \circ \Theta_\tau(t) \in \cdot) \stackrel{d}{=} \mathbb{P}_{Z(\tau)}(Z \circ \Theta_\tau(t) \in \cdot), \quad (3.48)$$

for  $t \geq 0$ . We apply the strong Markov property and Proposition 3.20 to obtain the weak convergence

$$\left( Z_0(\tau + m), \frac{Z_1(\tau^{(2)})}{N}, \tau^{(2)} - \tau^{(1)} \right) \xrightarrow{d} (V^{(2)}, G(V^{(2)}), m + \mathbf{1}_{\{V^{(2)} \neq 0\}}),$$

where  $V^{(2)}$  is a doubly stochastic Poisson variable governed by  $V^{(1)}$  the limit distribution of  $Z_0(m)$ . This method can be iterated to obtain the following result.

**Lemma 3.21.** *Assume that  $\xi$  satisfies Assumption (A) with  $\eta = 0$ . Denote by  $\Delta\tau_N^{(i)} := \tau^{(i)} - \tau^{(i-1)}$  and let  $l \in \mathbb{N}$ . Then, under  $\mathbb{P}_\oplus$*

$$\left\{ \left( Z_0(\tau^{(i-1)} + m), Z_1(\tau^{(i)})/N, \Delta\tau_N^{(i)} \right); 1 \leq i \leq l \right\} \quad (3.49)$$

converges weakly to

$$\left\{ \left( V^{(i)}, G(V^{(i)}), m + \mathbf{1}_{\{V^{(i)} \neq 0\}} \right); 1 \leq i \leq l \right\}. \quad (3.50)$$

The distribution of  $V^{(i)}$  is determined by

$$\begin{aligned} \mathbb{P}(V^{(i+1)} = t_{i+1} \mid V^{(j)} = t_j, j \leq i) \\ = \mathbb{P}(V^{(i+1)} = t_{i+1} \mid V^{(i)} = t_i) = e^{-G(t_i)\rho^m} \frac{(G(t_i)\rho^m)^{t_{i+1}}}{t_{i+1}!}, \end{aligned} \quad (3.51)$$

where  $t_1, \dots, t_{i+1} \in \mathbb{N}$  and  $V^{(1)}$  is distributed according to a Poisson variable with parameter  $\rho^m$ .

*Proof.* It is a direct consequence of Proposition 3.20 and an induction argument.  $\square$

With a very small effort we can state Lemma 3.21 in a more general framework. We consider the space of real valued sequences  $\mathbb{R}^{\mathbb{N}}$  where we define the metric

$$d(a, b) = \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n}.$$

A complete description of this topological space can be found in [Bil99]. Since time is discrete, the following Proposition holds as a corollary of Lemma 3.21.

**Proposition 3.22.** *Assume that  $\xi$  satisfies Assumption (A) with  $\eta = 0$ . Then, under  $\mathbb{P}_\oplus$  the process*

$$\left\{ \left( Z_0(\tau^{(i-1)} + m), Z_1(\tau^{(i)})/N, \Delta\tau^{(i)} \right); i \in \mathbb{N} \right\} \quad (3.52)$$

converges weakly in  $((\mathbb{R}^{\mathbb{N}})^3, d)$ . The limit distribution  $\mathbb{W}_\theta$  is given by

$$\left\{ \left( V^{(i)}, G(V^{(i)}), m + \mathbf{1}_{\{V^{(i)} \neq 0\}} \right); i \in \mathbb{N} \right\}, \quad (3.53)$$

where  $V^{(i)}$  is a Markov chain with initial position at 0 and transition matrix given by

$$\mathbb{P}(V^{(i+1)} = l \mid V^{(i)} = k) = e^{-G(k)\rho^m} \frac{(G(k)\rho^m)^l}{l!}, \quad (3.54)$$

that is a Poisson distribution with parameter  $\rho^m G(k)$ .

### Process convergence in the case $\eta > 0$

For the sake of completeness, we state the result in the case  $\eta > 0$ . We omit the proof of the Proposition and leave the details to the reader .

**Proposition 3.23.** *Assume that  $\xi$  satisfies Assumption (A') and that  $\eta > 0$ . Then under  $\mathbb{P}_\oplus$  the process*

$$\{(Z_1(\tau^{(i)})/N, \Delta\tau^{(i)}); i \in \mathbb{N}\} \quad (3.55)$$

*converges weakly in  $((\mathbb{R}^{\mathbb{N}})^2, d)$ . The limit distribution is non-random, and concentrated on the sequence*

$$\{(a_i, b_i); \quad a_i = 1 \text{ and } b_i = m + 1 \ \forall i \in \mathbb{N}\}. \quad (3.56)$$

### 3.3.2 Uniform integrability and bounds for $T_\Delta$

In this Subsection, we show that if  $\xi$  satisfies Assumption (A'), then  $\mathbb{E}_x[T_\Delta]$  is bounded independently from the initial configuration  $x$

$$\sup_{N \in \mathbb{N}} \sup_{x \in \Omega(N)} \mathbb{E}_x[T_\Delta] < \infty. \quad (3.57)$$

We prove (3.57) through the following steps.

1. There exists a set  $\Xi \subset \Omega(N)$  such that for  $N$  large enough and every starting point  $x \in \Xi$  there is a positive probability to visit  $\Delta$  before  $m + 1$

$$\mathbb{P}_x(T_\Delta \leq m + 1) > c_2, \quad (3.58)$$

where  $c_2 > 0$  does not depend on  $x \in \Xi$ .

2. For  $N$  sufficiently large and every starting point  $x \in \Omega(N)$  there is a positive probability to visit  $\Xi$  before  $m + 1$

$$\mathbb{P}_x(T_\Xi \leq m + 1) > c_3, \quad (3.59)$$

where  $c_3$  does not depend on  $x \in \Omega(N)$ .

Before proving this two statements, we show that they indeed imply (3.57).

**Proposition 3.24.** *Assume that  $\xi$  satisfies Assumption (A'). Then, there exists  $K \in \mathbb{R}$  such that*

$$\sup_{N \in \mathbb{N}} \sup_{x \in \Omega(N)} \mathbb{E}_x[T_\Delta] < K. \quad (3.60)$$

*Proof.* If (3.58) and (3.59) hold, then for  $N$  large enough and any starting point  $x \in \Omega(N)$

$$\begin{aligned} & \mathbb{P}_x(T_\Delta \leq 2m + 2) \\ & \geq \mathbb{P}_x(T_\Delta \leq 2m + 2; T_\Xi \leq m + 1) \\ & \geq \mathbb{P}_x(T_\Delta - T_\Xi \leq m + 1; T_\Xi \leq m + 1) \\ & = \mathbb{E}_x[\mathbb{P}_{Z(T_\Xi)}[T_\Delta \leq m + 1] \mathbf{1}_{T_\Xi \leq m + 1}] \quad (\text{strong Markov property}) \\ & \geq c_2 c_3 > 0. \end{aligned}$$

Hence,  $\sup_{y \in \Omega(N)} \mathbb{P}_y(T_\Delta \geq 2m + 3) \leq 1 - c_2 c_3$ . For  $i \in \mathbb{N}$ , let  $j$  be such that  $(2m + 3)j \leq i < (2m + 3)(j + 1)$ . Using the Markov property  $j$  times we obtain the geometric upper bound for the tail probability

$$\mathbb{P}_x(T_\Delta \geq i) \leq \left( \sup_{y \in \Omega(N)} \{ \mathbb{P}_y(T_\Delta \geq (2m + 3)) \} \right)^j,$$

that is uniform in  $x \in \Omega(N)$ . As a consequence, (3.60) holds with  $K = (2m + 3)/(c_2 c_3)$ .  $\square$

We now present the formal definition of  $\Xi$ .

**Definition 3.25.** For  $x \in \Omega(N)$  define  $I(x) = \inf\{i \in \mathbb{N}; x_i \geq 1\}$ . Then,  $\Xi$  is the subset of  $\Omega(N)$  defined as follows

$$\Xi := \{x \in \Omega(N); x_{I(x)} \geq \alpha N\},$$

where  $0 < \alpha < 1 - \theta'$  and  $\theta'$  is given by Assumption (A'). Hence, if  $Z(t) \in \Xi$  there are at least  $\alpha N$  leaders at time  $t$ .

We prove (3.58) and (3.59) in the next two Lemmas.

**Lemma 3.26.** Assume that  $\xi$  satisfies Assumption (A'). Then, for  $\Xi$  given by Definition 3.25 there exists a positive constant  $c_2$  such that for  $N$  sufficiently large

$$\inf_{x \in \Xi} \mathbb{P}_x(T_\Delta \leq m + 1) > c_2.$$

*Proof.* Note that

$$\begin{aligned} \mathbb{P}_x(T_\Delta \leq m + 1) &\geq \mathbb{P}_x(Z(\tau) = \Delta; \tau \leq m + 1) \\ &= \mathbb{P}_x(Z(\tau) = \Delta) - \mathbb{P}_x(Z(\tau) = \Delta; \tau \geq m + 2), \end{aligned}$$

From Corollary 3.7, the second term in the lower bound converges to zero as  $N \rightarrow \infty$  and the rate of decay is uniform on  $x \in \Omega(N)$ . Hence it suffices to show that there exists a positive constant  $c_4$  such that uniformly on  $x \in \Xi$

$$\liminf_{N \rightarrow \infty} \mathbb{P}_x(Z(\tau) = \Delta) \geq c_4, \quad (3.61)$$

To prove (3.61) we distinguish between the cases  $\eta = 0$  and  $\eta > 0$ . We start with the latter case  $\eta > 0$ . Let  $Y(t) = Z_0(t) \mathbf{1}_{\{t \leq \tau\}}$  and denote by  $Y_k$  the process started from  $\delta_k$ . Then, for  $x \in \Xi$  we can couple the processes in such a way that

$$Y_{\lfloor \alpha N \rfloor}(t) \leq Y_{x_{I(x)}}(t) \leq Y_N(t), \quad (3.62)$$

where  $x_{I(x)}$  is the number of leaders when  $Z(0) = x$ . From the proof of Corollary 3.10 and (3.62) we obtain

$$\lim_{N \rightarrow \infty} \mathbb{P}_x((\rho^m - \varepsilon)\alpha N^m \leq Z_0(m) \leq (\rho^m + \varepsilon)N^m) = 1. \quad (3.63)$$

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Finally, applying the arguments of Lemma 3.16,

$$\lim_{N \rightarrow \infty} \mathbb{P}_x(Z(\tau) = \Delta) = 1.$$

In particular, any  $0 < c_4 < 1$  satisfies (3.61). The case where  $\eta = 0$  is similar but it requires an additional step. (3.62) still holds, hence by the same arguments we obtain

$$\lim_{N \rightarrow \infty} \mathbb{P}_x\left((\rho^{m-1} - \varepsilon)\alpha N^{1/m} \leq Z_0(m-1) \leq (\rho^{m-1} + \varepsilon)N^{1/m}\right) = 1.$$

From Lemma 3.17, we see that  $\mathbb{P}_x(Z(\tau) = \Delta \mid Z_0(m) = 0) \rightarrow 1$  and from the coupling argument (3.62) and Corollary 3.8 we obtain the following limit

$$\mathbb{P}_x(Y(m) = 0) \geq P_{\oplus}(Y(m) = 0) \rightarrow 1 - e^{-\rho^m}.$$

It implies that  $\liminf_{N \rightarrow \infty} \mathbb{P}_x(Z_0(m) = 0) \geq 1 - e^{-\rho^m}$ , hence every  $c_4 < 1 - e^{-\rho^m}$  satisfies (3.61), proving the statement.  $\square$

**Lemma 3.27.** *Assume that  $\xi$  satisfies Assumption (A'). Then, for  $\Xi$  given by Definition 3.25 there exists a positive constant  $c_3$  such that for  $N$  large enough*

$$\inf_{x \in \Omega(N)} \mathbb{P}_x(T_{\Xi} \leq m+1) > c_3.$$

*Proof.* Since  $\mathbb{P}_x(\tau \geq m+2)$  converges to zero uniformly on  $x \in \Omega(N)$ , it sufficient to show that for  $N$  sufficiently large

$$\mathbb{P}_x(Z(\tau) \in \Xi) \geq c_5,$$

and  $c_5 > 0$  does not depend on  $x \in \Omega(N)$ .

$$\begin{aligned} \mathbb{P}_x(Z(\tau) \in \Xi) &= \sum_{k=1}^{\infty} \mathbb{P}_x(Z(k) \in \Xi; \tau = k) \\ &= \sum_{k=1}^{\infty} \sum_{y \in \Omega(N)} \mathbb{E}_x \left[ \mathbb{P}_y(Z(1) \in \Xi; \tau = 1) \mathbf{1}_{\{Z(k-1)=y; \tau \geq k\}} \right] \quad (\text{Markov property}) \\ &\geq \inf_{y \in \Omega(N)} \left\{ \mathbb{P}_y(Z(1) \in \Xi \mid \tau = 1) \right\} \sum_{k=1}^{\infty} \sum_{y \in \Omega(N)} \mathbb{E}_x \left[ \mathbb{P}_y(\tau = 1) \mathbf{1}_{\{Z(k-1)=y; \tau \geq k\}} \right] \\ &= \inf_{y \in \Omega(N)} \left\{ \mathbb{P}_y(Z(1) \in \Xi \mid \tau = 1) \right\}. \end{aligned} \tag{3.64}$$

Then, it suffices to show that the infimum in (3.64) is larger than a  $c_5 > 0$ . We have that

$$\mathbb{P}_y(X_i(1) = \Phi(X(0)) - 1 \mid \tau = 1) = s_1(y)/(1 - s_0(y)),$$

hence  $\mathbb{P}_y(Z_1(1) = \cdot \mid \tau = 1)$  is binomially distributed with parameters  $N$  and  $s_1(y)/(1 - s_0(y))$ . Assuming that  $y_0 \geq 1$  (otherwise we must consider  $\tilde{y}$  the shifted vector)

$$\begin{aligned} \frac{s_1(y)}{1 - s_0(y)} &\geq \frac{(1 - p_0)^{y_0} - q_2^{y_0}}{(1 - p_0)^{y_0}} \\ &\geq 1 - (\theta')^{y_0} > \alpha, \end{aligned}$$

where the lower bound holds for  $N$  large enough as a consequence of Assumption (A') and the definition of  $\alpha$ . A large deviation argument allows us to conclude that

$$\mathbb{P}_y(Z(1) \in \Xi \mid Z_0(1) = 0) \geq \mathbb{P}_y(Z_1(1) \geq \alpha N \mid Z_0(1) = 0) \rightarrow 1.$$

Then, the infimum in (3.64) is larger than any  $c_5 < 1$  for  $N$  sufficiently large, finishing the proof.  $\square$

The next Corollary generalizes (3.57) to the later visiting times of  $\Delta$ .

**Corollary 3.28.** *Assume that  $\xi$  satisfies Assumption (A'). Then, for every  $i \in \mathbb{N}$ ,  $\sup_{x \in \Omega} \mathbb{E}_x [T_\Delta^{(i)}]$  and  $\sup_{x \in \Omega} \mathbb{E}_x [\tau^{(i)}]$  are bounded uniformly on  $N$ . In particular, under  $\mathbb{P}_x$  the families of random variables  $T_\Delta^{(i)}$  and  $\tau^{(i)}$  are uniformly integrable.*

*Proof.* Since  $\tau^{(i)} \leq T_\Delta^{(i)}$ , it suffices to prove the statements for  $T_\Delta^{(i)}$ . To prove that the expectation is bounded we proceed inductively and apply the strong Markov property at time  $T_\Delta^{(i-1)}$ . It is clear that

$$\sup_{x \in \Omega(N)} \mathbb{E}_x [T_\Delta^{(i)}] \leq K^i,$$

where  $K$  is given by (3.60). We now prove the uniform integrability. Applying the Markov property we obtain the upper bound

$$\mathbb{E}_x [T_\Delta^{(i)}; T_\Delta^{(i)} \geq l] \leq \left( \sup_{y \in \Omega(N)} \mathbb{E}_y [T_\Delta^{(i)}] + l \right) \mathbb{P}_x (T_\Delta^{(i)} \geq l). \quad (3.65)$$

It is not difficult to see that the right-hand side of (3.65) converges to zero, finishing the proof.  $\square$

### 3.3.3 Convergence of $\mathbb{E}_\oplus [T_\Delta]$ and $\mathbb{E}[\phi(X(T_\Delta))]$

To compute the front velocity in Subsection 3.3.4, we have to calculate two integrals

$$\mathbb{E}_\oplus [T_\Delta] \text{ and } \mathbb{E}[\phi(X(T_\Delta))].$$

As usual, we assume that all particles start from zero, then

$$\phi(X(T_\Delta)) = - \sum_{i=1}^{\infty} \min\{l \in \mathbb{N}; Z_l(\tau^{(i)}) \neq 0\} \mathbf{1}_{\{\tau^{(i)} \leq T_\Delta\}}.$$

In the next Lemma, we use for the first time the condition  $\mathbb{E}[|\vartheta|] < \infty$ , which appears in Assumption (A) and Assumption (A').

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**Lemma 3.29.** *Assume that  $\xi$  satisfies Assumption (A'). Then*

$$\sup_{x \in \Omega(N)} \mathbb{E}_x [\min\{l \in \mathbb{N}; Z_l(\tau) \neq 0\}] = 1 + o(1) \quad (3.66)$$

as  $N \rightarrow \infty$ .

*Proof.* By an argument similar to the one used in Lemma 3.27 we obtain that

$$\begin{aligned} & \mathbb{E}_x \left[ \min\{l \in \mathbb{N}; Z_l(\tau) \neq 0\} \right] \\ & \leq \sup_{y \in \Omega(N)} \mathbb{E}_y \left[ \min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \mid \tau = 1 \right] \\ & \leq 1 + \sup_{y \in \Omega(N)} \mathbb{E}_y \left[ \min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \mathbf{1}_{\{\min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \geq 2\}} \mid \tau = 1 \right]. \end{aligned} \quad (3.67)$$

Under the conditional probability  $Z$  is distributed according to a Multinomial with infinitely many classes and parameters  $s_i(y)/(1 - s_0(y))$ ,  $i \geq 1$ . Applying Assumption (A') we obtain that for  $N$  sufficiently large

$$\inf_{y \in \Omega(N)} \left\{ s_1(y)/(1 - s_0(y)) \right\} \geq 1 - \theta'.$$

Moreover, the minimum in (3.67) is bounded from above by some  $|\xi_{ij}|$ . Indeed, it suffices to choose  $i$  such that  $X_i(0)$  is a leader, then

$$- \min\{l \in \mathbb{N}; Z_l(1) \neq 0\} = \phi(X(1)) - X_i(0) \geq \xi_{ij}.$$

Hence, we can give an upper bound for the right-hand side in (3.67)

$$\begin{aligned} & \mathbb{E}_y \left[ \min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \mathbf{1}_{\{\min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \geq 2\}} \mid \tau = 1 \right] \\ & \leq \mathbb{E}_y \left[ |\xi_{ij}| \mathbf{1}_{\{\min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \geq 2\}} \mid \tau = 1 \right] \\ & \leq \mathbb{E} [|\vartheta_{ij}|] \mathbb{P}_y \left( \min\{l \in \mathbb{N}; Z_l(1) \neq 0\} \geq 2 \mid \tau = 1 \right) \\ & \leq \mathbb{E} [|\vartheta_{ij}|] (\theta')^N, \end{aligned}$$

which converges to zero independently from the initial position  $y \in \Omega(N)$ .  $\square$

With Lemma 3.29 at hand we prove the following result in the noncritical case.

**Proposition 3.30.** *Assume that  $\xi$  satisfies Assumption (A') with  $\eta > 0$ . Then*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\oplus} [T_{\Delta}] = (m + 1) \quad \text{and} \quad \lim_{N \rightarrow \infty} \mathbb{E} [\phi(X(T_{\Delta}))] = -1. \quad (3.68)$$

*Proof.* The first limit is a direct consequence of the uniform integrability of  $T_\Delta$  and  $\mathbb{P}_\oplus(T_\Delta = m + 1) \rightarrow 1$ , as  $N \rightarrow \infty$ . We now prove the second statement.

$$\begin{aligned}
 & \mathbb{E}[\phi(X(T_\Delta))] \\
 &= - \sum_{i=1}^{\infty} \mathbb{E}_\oplus \left[ \min\{l \in \mathbb{N}; Z_l(\tau^{(i)}) \neq 0\} \mathbf{1}_{\{T_\Delta \geq \tau^{(i)}\}} \right] \\
 &= - \sum_{i=1}^{\infty} \sum_{y \in \Omega(N)} \mathbb{E}_\oplus \left[ \mathbb{E}_y \left[ \min\{l \in \mathbb{N}; Z_l(\tau) \neq 0\} \right] \mathbf{1}_{\{Z(\tau^{(i-1)})=y; T_\Delta \geq \tau^{(i)}\}} \right] \\
 &= (-1 + o(1)) \sum_{i=1}^{\infty} \mathbb{P}_\oplus(T_\Delta \geq \tau^{(i)}), \quad \text{as } N \rightarrow \infty, \tag{3.69}
 \end{aligned}$$

the last equality in (3.69) is a consequence of Lemma 3.29. The sum in the right-hand side of (3.69) converges to one as  $N \rightarrow \infty$ . Indeed,  $\mathbb{P}_\oplus(T_\Delta \geq \tau) = 1$  and

$$\begin{aligned}
 \sum_{i \geq 2} \mathbb{P}_\oplus(T_\Delta \geq \tau^{(i)}) &= \sum_{i \geq 2} (i-1) \mathbb{P}_\oplus(T_\Delta = \tau^{(i)}) \\
 &\leq \mathbb{E}_\oplus [T_\Delta \mathbf{1}_{\{T_\Delta > \tau\}}]. \tag{3.70}
 \end{aligned}$$

Since  $T_\Delta$  is uniform integrable, it follows from Proposition 3.14 that the upper bound in (3.70) converges to zero, which proves the result.  $\square$

The critical case is more delicate and we prove the following result.

**Proposition 3.31.** *Assume that  $\xi$  satisfies Assumption (A) and that  $\eta = 0$ . Then*

$$\lim_{N \rightarrow \infty} \mathbb{E}_\oplus [T_\Delta] = (m+1) \mathbb{E}_0 [\mathcal{T}_0] - 1 \quad \text{and} \quad \lim_{N \rightarrow \infty} \mathbb{E} [\phi(X(T_\Delta))] = -\mathbb{E}_0 [\mathcal{T}_0], \tag{3.71}$$

where  $\mathcal{T}_0$  is the stopping time given by  $\mathcal{T}_0 := \min \{i \geq 1; V^{(i)} = 0\}$ , for  $V^{(i)}$  a Markov chain defined as in Proposition 3.22.

We split the proof of Proposition 3.31 in two parts. We first show that

$$\min \{i \geq 1; Z_1(\tau^{(i)})/N = 1\}$$

converges weakly and we characterize the limit distribution. Let

$$i^* = \min \{i \geq 1; Z_1(\tau^{(i)})/N = 1\},$$

then  $T_\Delta = \tau^{(i^*)}$ . So in the second part, we show that  $\mathbb{E}[\tau^{(i^*)}]$  and  $\mathbb{E}[\phi(X(\tau^{(i^*)}))]$  converge to the desired limits.

**Lemma 3.32.** *Assume that  $\xi$  satisfies Assumption (A) with  $\eta = 0$ . Then, under  $\mathbb{P}_\oplus$ , the following convergence in distribution*

$$\min \{i \geq 1; Z_1(\tau^{(i)})/N = 1\} \xrightarrow{d} \min \{i \geq 1; G(V^{(i)}) = 1\}$$

takes place. The process  $V^{(i)}$  is the Markov chain defined in Proposition 3.22 and

$$\mathcal{T}_0 := \min \{i \geq 1; V^{(i)} = 0\} = \min \{i \geq 1; G(V^{(i)}) = 1\}.$$

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*Proof.* Since  $x \rightarrow \min\{i \geq 1; x_i = 1\}$  is not continuous in  $\mathbb{R}^{\mathbb{N}}$  Lemma 3.32 is not a direct corollary of Proposition 3.22. On the other hand,  $\{Z_0(\tau^{(i-1)} + m); i \in \mathbb{N}\}$  converges in distribution to  $\{V^{(i)}; i \in \mathbb{N}\} \in \mathbb{N}^{\mathbb{N}}$  and the minimum becomes continuous when restricted to  $\mathbb{N}^{\mathbb{N}}$ , so

$$\min\{i \geq 1; Z_0(\tau^{(i-1)} + m) = 0\} \xrightarrow{d} \mathcal{T}_0 = \min\{i \geq 1; V^{(i)} = 0\},$$

we refer to [Bil99] for more details on convergence in distribution. We use Lemmas 3.17 and 3.21 to obtain that

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\oplus} \left( Z_0(\tau^{(j-1)} + m) = 0; Z(\tau^{(j)}) \neq \Delta \right) = 0,$$

for every  $j \geq 1$ . Then, we deduce from Lemma 3.21 that

$$\min\{i \geq 1; Z_0(\tau^{(i-1)} + m) = 0\} - \min\{i \geq 1; Z_1(\tau^{(i)})/N = 1\}$$

converges in probability to zero. It implies the convergence in distribution of  $\min\{i \geq 1; Z_1(\tau^{(i)})/N = 1\}$  to  $\mathcal{T}_0$ , finishing the proof of the Lemma.  $\square$

*Proof of Proposition 3.31.* We may write

$$\begin{aligned} \mathbb{E}_{\oplus}[T_{\Delta}] &= \sum_{k=1}^{\infty} \mathbb{E}_{\oplus} \left[ \tau^{(k)} \mathbf{1}_{\{T_{\Delta} = \tau^{(k)}\}} \right] \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{\oplus} \left[ \sum_{j=1}^k (\tau^{(j)} - \tau^{(j-1)}) \mathbf{1}_{\{\min_{i \geq 1} \{Z_1(\tau^{(i)})/N = 1\} = k\}} \right]. \end{aligned}$$

For a fixed  $k \in \mathbb{N}$  the random variable  $\sum_{j=1}^k (\tau^{(j)} - \tau^{(j-1)}) \mathbf{1}_{\{\min_{i \geq 1} \{Z_1(\tau^{(i)})/N = 1\} = k\}}$  converges in law to

$$\sum_{j=1}^k (m + \mathbf{1}_{\{V^{(j)} \neq 0\}}) \mathbf{1}_{\{\min\{i \in \mathbb{N}; G(V^{(i)}) = 1\} = k\}} = ((m + 1)k - 1) \mathbf{1}_{\{\mathcal{T}_0 = k\}}.$$

Since  $\tau^{(k)}$  is uniformly integrable the convergence also holds in  $L^1$ . From the uniform integrability of  $T_{\Delta}$  we obtain the convergence in  $L^1$  of the sum and the following limit holds.

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}_{\oplus}[T_{\Delta}] &= \sum_{k=1}^{\infty} ((m + 1)k - 1) \mathbb{P}_0(\mathcal{T}_0 = k) \\ &= (m + 1) \mathbb{E}_0[\mathcal{T}_0] - 1. \end{aligned}$$

This proves the first statement of Proposition 3.31. We now prove the second limit in (3.71). From the proof of Proposition 3.30 we obtain that

$$\mathbb{E}_{\oplus}[\phi(X(T_{\Delta}))] = -(1 + o(1)) \sum_{i=1}^{\infty} \mathbb{P}_{\oplus}(\tau^{(i)} \leq T_{\Delta}).$$

From the uniform integrability of  $T_\Delta$  we obtain that  $\sum_{i=1}^{\infty} \mathbb{P}_\oplus(\tau^{(i)} \leq T_\Delta) \rightarrow \mathbb{E}_0[\mathcal{T}_0]$ , which finishes the proof.  $\square$

The transition matrix of  $V^{(i)}$  depends on  $G$  and a fortiori on  $\theta$ . A coupling argument shows that  $\mathbb{E}_0[\mathcal{T}_0]$  is non-increasing in  $\theta$ . Nevertheless, we do not know how to calculate explicitly the integral. However the asymptotic behaviors as  $\theta \rightarrow 0$  and 1 are easy to compute.

**Proposition 3.33.** *Let  $V^{(i)}$  be the Markov chain whose transition matrix is given in Proposition 3.22, then*

$$\lim_{\theta \rightarrow 0} \mathbb{E}_0[\mathcal{T}_0] = \exp(\rho^m).$$

*Proof.* We write  $\mathbb{E}_0[\mathcal{T}_0] = \sum_{k=0}^{\infty} \mathbb{P}_0(\mathcal{T}_0 \geq k+1)$ . For  $l \geq 1$ , then  $1 \geq G(l) \geq G(1) = 1 - \theta$ , and

$$\begin{aligned} & \mathbb{P}_0(\mathcal{T}_0 \geq k+1) \\ &= \sum_{l_1=1}^{\infty} e^{-\rho^m} \frac{(\rho^m)^{l_1}}{l_1!} \dots \sum_{l_{k-1}=1}^{\infty} e^{-\rho^m G(l_{k-2})} \frac{(\rho^m G(l_{k-2}))^{l_{k-1}}}{l_{k-1}!} (1 - e^{-\rho^m G(l_{k-1})}). \end{aligned}$$

The last expression is bounded from above by  $(1 - e^{-\rho^m})^k$ . Since  $G(l) \rightarrow 1$  as  $\theta \rightarrow 0$  we can conclude by dominated convergence.  $\square$

We point out here that the case  $\theta \rightarrow 0$  corresponds to the “two-state percolation distribution” model studied in Section 3.2. Informally, when  $\theta$  is very small there is a high probability that  $Z(\tau)$  starts afresh from  $\Delta$ . A similar computation can be done in the case where  $\theta$  converges to one.

**Proposition 3.34.** *Let  $V^{(i)}$  be the Markov chain whose transition matrix is given in Proposition 3.22. Then*

$$\lim_{\theta \rightarrow 1} \mathbb{E}_0[\mathcal{T}_0] = 2 - \exp(-\rho^m).$$

*Proof.* The proof follows the same lines as that of Proposition 3.33 and we leave the details to the reader.  $\square$

### 3.3.4 Speed of the $N$ -particle system

As in Subsection 3.2.2, we explore the renewal structure of  $Z$  that starts afresh from  $\Delta$ . Let  $N(t) = \max\{i; T_\Delta^{(i)} \leq t\}$ . Then

$$\phi(X(t)) = - \sum_{i=1}^{N(t)} \left[ \phi\left(X\left(T_\Delta^{(i+1)}\right)\right) - \phi\left(X\left(T_\Delta^{(i)}\right)\right) \right] + o(t)$$

as  $t \rightarrow \infty$  almost surely. Taking the limit, as  $t \rightarrow \infty$ , we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\phi(X(t))}{t} &= \lim_{t \rightarrow \infty} -\frac{1}{t} \sum_{i=1}^{N(t)} \phi\left(X\left(T_\Delta^{(i+1)}\right)\right) - \phi\left(X\left(T_\Delta^{(i)}\right)\right) \\ &= \frac{\mathbb{E}[\phi(X(T_\Delta))]}{\mathbb{E}_\oplus[T_\Delta]} \quad a.s. \end{aligned} \tag{3.72}$$

The limit is a consequence of the ergodic Theorem and the renewal structure. In Subsection 3.3.3, we calculated the limits of the above expected values. We obtain that

$$\lim_{N \rightarrow \infty} v_N = \begin{cases} -(1 + \lfloor 1/r \rfloor)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ -(\lfloor 1/r \rfloor + 1 - 1/\mathbb{E}_0[\mathcal{T}_0])^{-1}, & \text{if } 1/r = m \in \mathbb{N}, \end{cases}$$

which proves Theorem 3.13 with  $g(\theta) = \mathbb{E}_0[\mathcal{T}_0]$ .

### 3.4 Conclusion and sketch of the proof of Theorem 3.3

Theorem 3.3 follows as a corollary of Theorem 3.13 proved in Section 3.3. We will not prove it in detail but we give a sketch of the proof. The constants  $\lambda_0$  and  $\lambda_1 - \lambda_0$  appearing in Theorem 3.3 are justified by an affine transformation. Then, it remains to explain how we pass from the distribution over the lattice to the more general one. In the proof of Theorem 3.13 we see that in the discrete case  $\vartheta$  contributes to the position of the leaders only in rare events. Indeed, if there are  $k$  leaders at time  $t$  the position of the front is determined by  $\vartheta$  at  $t+1$  only in the case where  $\xi_{ij}(t+1) \leq -2$  for at least  $N^k$  random variables. The probability of this event is of order  $\theta^{N^k}$ , as a consequence of Assumption (A). This behaviour still holds in the general case. For a complete proof we refer to [CQR13] Theorem 1.3, which applies also to our case with the obvious changes.

The position of the front depends basically on the tail distribution of  $\xi$ , that is determined by the point masses  $\lambda_0$  and  $\lambda_1$ . The only case where  $\vartheta$  could contribute to the position of the front in long time scales is in the non-integrable case. Then the mechanism responsible for propagation is of a very different nature and the front is no longer pulled by the leading edge. In the rare events, when the front moves backwards more than  $\lambda_0 - \lambda_1$  the contribution of  $\vartheta$  would be non-negligible depending on its tail and the global front profile. This problem is still open and much harder to solve.

### 3.5 Exact diffusion constant, percolation distribution

This section was written after discussion with Prof. Bernard Derrida. It does not appear in the paper “Front Velocity and Directed Polymers in Random Medium” published in Stochastic Processes and their Applications [Cor14a]. Here, we will only consider  $\xi$  distributed according to (3.1):

$$\begin{aligned} \mathbb{P}(\xi(N) = 0) &= p_0(N) \sim \rho/N^{1+r} \\ \mathbb{P}(\xi(N) = -1) &= 1 - \mathbb{P}(\xi(N) = 0), \end{aligned}$$

and we prove Theorem 3.2 for this kind of distribution. One can also generalize Theorem 3.2 for distributions of the form (3.9), however as in Theorem 3.3 the exact

limits are not explicit and the calculation are troublesome, for this reason, we decide not to develop it here.

Let  $0 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \dots$  be the sequence of  $\mathcal{F}_t$ -stopping times

$$\tau^{(i)} := \inf\{t \geq \tau^{(i-1)} + 1; Z(t) = 0\}. \quad (3.73)$$

As we have already seen,  $\tau^{(i)} - \tau^{(i-1)}$  are i.i.d. and we will denote by  $\tau := \tau^{(1)}$ , which is simply the first time that  $Z(t)$  is zero. We will assume without loss of generality that

$$X_i(0) = 0, \quad \forall i \in \{1, \dots, N\},$$

so that, at  $\tau^{(i)}$  every particle is located at  $-i$ . For  $t \in \mathbb{R}_+$  define

$$N(t) := \max \left\{ k; \sum_{i=0}^k \tau^{(i)} < t \right\},$$

then we have that  $\phi(X(\lfloor t \rfloor)) = -N(t)$ . We obtain Theorem 3.2 as an application of the following theorem, whose proof can be found in [Bil99] Theorem 17.3 p. 148.

**Theorem 3.35.** *Let  $\eta_1, \eta_2, \dots$  be a sequence of positive independent and identically distributed random variables with mean  $\mu$  and variance  $\sigma^2$  and*

$$\mathcal{N}(t) := \max \left\{ k; \sum_{i=1}^k \eta_i < t \right\}; \quad t > 0,$$

with  $\mathcal{N}(t) = 0$  if  $\eta_1 > t$ . Define  $\mathcal{Z}_n$  by

$$\mathcal{Z}_n(s, \omega) := \frac{\mathcal{N}(ns) - ns/\mu}{\sigma \mu^{-\frac{3}{2}} \sqrt{n}}.$$

Then  $\mathcal{Z}_n(s, \omega)$  converges in distribution to a standard Brownian motion as  $n \rightarrow \infty$ .

We will take

$$\eta_i = \tau^{(i)} - \tau^{(i-1)} \quad \text{for } i \geq 1,$$

that are positive i.i.d. random variables, whose mean is  $\mathbb{E}[\tau] = v_N^{-1}$ . Hence to prove Theorem 3.2 it suffices to show that  $\tau$  has finite second moment and compute its variance.

**Lemma 3.36.** *Let  $\xi$  be distributed according to (3.1) and assume that  $X_i(0) = 0$  for every  $1 \leq i \leq N$ . Then, as  $N \rightarrow \infty$*

$$\mathbb{P}(\tau \geq t + 1) = 1 - \exp \left\{ -N^{t+1} p_0^t + o(N^{t+1} p_0^t) \right\}; \quad t \in \mathbb{N}. \quad (3.74)$$

*Proof.* Let  $Y(t) = Z(t) \mathbf{1}_{\{t \leq \tau\}}$  be the number of particles that remain at zero if the front has not moved, then  $Y$  is a Markov chain in  $\{1, \dots, N\}$  with absorption state at zero and transition probabilities given by the Binomial distributions

$$\mathbb{P}(Y(s+1) = \cdot \mid Y(s) = k) = \mathcal{B}(N, 1 - (1 - p_0)^k)(\cdot), \quad 1 \leq k \leq N.$$

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It is trivial that  $\mathbb{P}(\tau \geq t + 1) = \mathbb{P}(Y(t) \geq 1)$  and that

$$\mathbb{P}(Y(t) = 0) = \lim_{s \rightarrow 0^+} \mathbb{E} [s^{Y(t)}].$$

Let  $s \in (0, 1)$ , then using the Markov property we obtain

$$\mathbb{E} [s^{Y(t)}] = \mathbb{E} \left[ \left( 1 + (s - 1) (1 - (1 - p_0)^{Y(t-1)}) \right)^N \right].$$

Since  $p_0 \sim \rho/N^{1+r}$  as  $N \rightarrow \infty$  and  $Y(t-1) \leq N$ , we obtain by Taylor expansion that

$$\left( 1 + (s - 1) (1 - (1 - p_0)^{Y(t-1)}) \right)^N = s_{(1)}(N)^{Y(t-1)},$$

where  $s_{(1)}(N) = \exp \{ (s - 1) N p_0 + \epsilon_{(1)}(s) \}$  and  $\epsilon_{(1)}(s)$  is the error term, which depends on  $s$ ,  $N$  and  $Y(t-1)$ . Moreover, there exists a constant  $c^{(1)}$  such that for every  $0 \leq Y(t-1) \leq N$

$$\sup_{s \in (0,1)} \frac{|\epsilon_{(1)}(s)|}{(N p_0)^2} \leq c^{(1)}, \quad (3.75)$$

that is negligible with respect to  $N p_0$  as  $N \rightarrow \infty$ . We use a similar argument to obtain

$$\mathbb{E} [s_{(1)}(N)^{Y(t-1)}] = \mathbb{E} [s_{(2)}(N)^{Y(t-2)}],$$

with  $s_{(2)}(N) = \exp \{ (s_{(1)}(N) - 1) (N p_0 + o(N p_0)) \}$  as  $N \rightarrow \infty$ . Expanding  $s_{(1)}(N) - 1$

$$s_{(1)}(N) - 1 = (s - 1) N p_0 + \epsilon_{(1)}(s) + \mathcal{O}((N p_0)^2) \quad \text{as } N \rightarrow \infty.$$

Hence,  $s_{(2)}(N) = \exp \{ (s - 1) (N p_0)^2 + \epsilon_{(2)}(s) \}$  and, similarly to (3.75) there exists a constant  $c^{(2)}$  such that

$$\sup_{s \in (0,1)} \frac{|\epsilon_{(2)}(s)|}{(N p_0)^3} \leq c^{(2)},$$

independently from  $0 \leq Y(t-2) \leq N$ . We proceed recursively and obtain the expression

$$\mathbb{E} [s^{Y(t)}] = \exp \left\{ (s - 1) N^{t+1} p_0^t + N \epsilon_{(t)}(s) \right\} \quad \text{as } N \rightarrow \infty,$$

then taking the limit as  $s \rightarrow 0$  we obtain

$$\mathbb{P}(Y(t) = 0) = \exp \left\{ - N^{t+1} p_0^t + N \epsilon_{(t)}(0) \right\},$$

that finishes the proof. □

A corollary of Lemma 3.36 is that  $\tau$  has finite  $m$ th moment for every  $m \in \mathbb{N}$ .

**Corollary 3.37.** *Let  $\xi$  be distributed according to (3.1) and assume that  $X_i(0) = 0$  for every  $1 \leq i \leq N$ . Then, there exists  $N_0 \in \mathbb{N}$  such that for  $N \geq N_0$*

$$\mathbb{E}[\tau^m] < \infty, \quad (3.76)$$

for every  $m \in \mathbb{N}$ .

*Proof.* We introduce the notation  $\mathbb{P}_k$  and  $\mathbb{E}_k$  for probabilities and expectations given that the Markov chain  $Z$  has initial distribution  $Z(0) = k$ . Then, using a simple coupling argument it is easy to obtain the inequality

$$\mathbb{P}_k(\tau \geq t) \leq \mathbb{P}_N(\tau \geq t), \quad \forall 0 \leq k \leq N. \quad (3.77)$$

Let  $0 < c < 1$ , then from (3.74) there exists  $N_0 \in \mathbb{N}$  such that for  $N \geq N_0$

$$\mathbb{P}_N(\tau \geq \lfloor 1/r \rfloor + 1) \leq 1 - c. \quad (3.78)$$

Let also  $s \in \mathbb{N}$ , then we have that

$$\begin{aligned} & \mathbb{P}_N(\tau \geq s\lfloor 1/r \rfloor + 1) \\ &= \mathbb{E}_N \left[ \sum_{n=1}^N \mathbb{P}_k(\tau \geq \lfloor 1/r \rfloor + 1) \mathbf{1}_{\{Z((s-1)\lfloor 1/r \rfloor) = n; \tau \geq (s-1)\lfloor 1/r \rfloor\}} \right] \quad (\text{Markov Property}) \\ &\leq \mathbb{P}_N(\tau \geq \lfloor 1/r \rfloor + 1) \mathbb{P}_N(\tau \geq (s-1)\lfloor 1/r \rfloor + 1), \end{aligned}$$

in the last inequality we used (3.77) with  $t = \lfloor 1/r \rfloor + 1$ . From a simple induction argument and (3.78), we obtain the geometric upper bound for the tail distribution of  $\tau$

$$\mathbb{P}_N(\tau \geq s\lfloor 1/r \rfloor + 1) \leq (1 - c)^s, \quad \text{for every } N \geq N_0,$$

which implies that  $\tau$  has finite  $m$ th moment if  $N \geq N_0$ .  $\square$

*Proof of Theorem 3.2.* Since  $\tau^{(i)}$  are i.i.d. copies of  $\tau$ , that has finite second moment (if  $N \geq N_0$ ) we may apply Theorem 3.35 to conclude that

$$\frac{\Phi(X(\lfloor ns \rfloor)) - ns/v_N}{D_N v_N^{3/2} \sqrt{n}}$$

converges in distribution to a Brownian motion as  $n \rightarrow \infty$ . It suffices now to compute  $D_N^2$  the variance of  $\tau$ . We distinguish between the two cases  $1/r \notin \mathbb{N}$  and  $1/r \in \mathbb{N}$ . Assume that  $1/r \notin \mathbb{N}$  then

$$\begin{aligned} \mathbb{E}[\tau^2] &= \sum_{i \geq 1} (2i - 1) \mathbb{P}(\tau \geq i) \\ &= \sum_{i=1}^{\lfloor 1/r \rfloor + 1} (2i - 1) \mathbb{P}(\tau \geq i) + \sum_{i=\lfloor 1/r \rfloor + 2}^{\infty} (2i - 1) \mathbb{P}(\tau \geq i) \end{aligned} \quad (3.79)$$

We use (3.74) to obtain

$$\sum_{i=1}^{\lfloor 1/r \rfloor + 1} (2i - 1) \mathbb{P}(\tau \geq i) = (\lfloor 1/r \rfloor + 1)^2 + \epsilon_N, \quad (3.80)$$

and the error  $\epsilon_N$  decays to zero like  $\exp(-\rho^{\lfloor 1/r \rfloor} N^{1-r\lfloor 1/r \rfloor})$ . Using a coupling argument and (3.74) we estimate the second sum in (3.79) as  $N \rightarrow \infty$

$$\sum_{i=\lfloor 1/r \rfloor + 2}^{\infty} (2i - 1) \mathbb{P}(\tau \geq i) = \frac{(2\lfloor 1/r \rfloor + 3)}{N^{r(\lfloor 1/r \rfloor + 1) - 1}} \rho^{\lfloor 1/r \rfloor + 1} + o(N^{1-r(\lfloor 1/r \rfloor + 1)}).$$

### 3.5. Exact diffusion constant, percolation distribution

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Note that  $\epsilon_N$  in (3.80) is  $o(N^{1-r(\lfloor 1/r \rfloor + 1)})$  as  $N \rightarrow \infty$ . To compute  $\mathbb{E}[\tau]^2$  we use the formula

$$\mathbb{E}[\tau] = \sum \mathbb{P}(\tau \geq i).$$

So, as  $N \rightarrow \infty$  we have that

$$\mathbb{E}[\tau] = (\lfloor 1/r \rfloor + 1) + \rho^{\lfloor 1/r \rfloor + 1} N^{1-r(\lfloor 1/r \rfloor + 1)} + o(N^{1-r(\lfloor 1/r \rfloor + 1)}).$$

Hence we obtain the exact asymptotic for  $D_N^2$

$$D_N^2 = \rho^{\lfloor 1/r \rfloor + 1} N^{1-r(\lfloor 1/r \rfloor + 1)} + o(N^{1-r(\lfloor 1/r \rfloor + 1)}) \quad \text{as } N \rightarrow \infty.$$

The case  $1/r \in \mathbb{N}$  is similar, but one must pay attention when computing  $\mathbb{P}(\tau \geq 1/r + 1)$ . In this case we have that

$$\begin{aligned} \mathbb{P}(\tau \geq 1/r + 1) &= 1 - \exp(-N^{1/r+1} p_0 + o(1)) \\ &= 1 - \exp(-\rho^{1/r}) + o(1), \quad \text{as } N \rightarrow \infty. \end{aligned}$$

From the proof of Lemma 3.36 we may only say that the error is  $\mathcal{O}(N^{-r})$  as  $N \rightarrow \infty$ , which converges to zero at the same rate of  $\mathbb{P}(\tau \geq 1/r + 2)$ . Using the similar arguments we obtain

$$\begin{aligned} \mathbb{E}[\tau^2] &= (1 + 1/r)^2 - (2/r + 1) e^{-\rho^{1/r}} + o(1) \quad \text{as } N \rightarrow \infty \\ \mathbb{E}[\tau] &= 1 + 1/r - e^{-\rho^{1/r}} + o(1) \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{N \rightarrow \infty} D_N^2 &= (1 - \exp(-\rho^{1/r})) - (1 - \exp(-\rho^{1/r}))^2 \\ &= \exp(-\rho^{1/r}) - \exp(-2\rho^{1/r}), \end{aligned}$$

which proves the result. □

It is easy to deduce from the proof of Theorem 3.2 the rate at which  $D_N^2$  decays to zero in the case  $1/r \notin \mathbb{N}$ .

**Corollary 3.38.** *Assume that the hypothesis of Theorem 3.2 hold and that  $1/r \notin \mathbb{N}$ . Then*

$$\lim_{N \rightarrow \infty} \frac{D_N^2}{N^{1-r(\lfloor 1/r \rfloor + 1)}} = \rho^{\lfloor 1/r \rfloor + 1}.$$

# Chapter 4

## Bounded jumps with polynomial density and more

This chapter is an article in collaboration with *Francis Comets* that we have submitted. The introduction has been modified for a better readability of the thesis.

### 4.1 Introduction

This chapter gives a partial answer to the problem raised by E. Brunet and B. Derrida [BD04], and we obtain the *finite-size corrections* to the speed for a large class of distributions that are bounded from above. To describe our framework, denote by  $\Lambda(u)$  the logarithmic generating function of  $\xi_{ij}$ ,

$$\Lambda(u) := \ln \mathbb{E}[\exp(u\xi_{ij})],$$

and let  $\mathcal{D}_\Lambda := \{u \in \mathbb{R}; \Lambda(u) < \infty\}$  be its domain. In this chapter, we will assume that the following hypothesis hold:

(H1)  $0 \in \mathcal{D}_\Lambda^0$  (the interior of  $\mathcal{D}_\Lambda$ ). In particular,  $\xi_{ij}$  has finite moments of all orders.

(H2) For every  $N \in \mathbb{N}$ , there exists  $u_N \in \mathcal{D}_\Lambda^0 \cap [0, \infty)$  such that

$$u_N \Lambda'(u_N) - \Lambda(u_N) = \ln N. \quad (4.1)$$

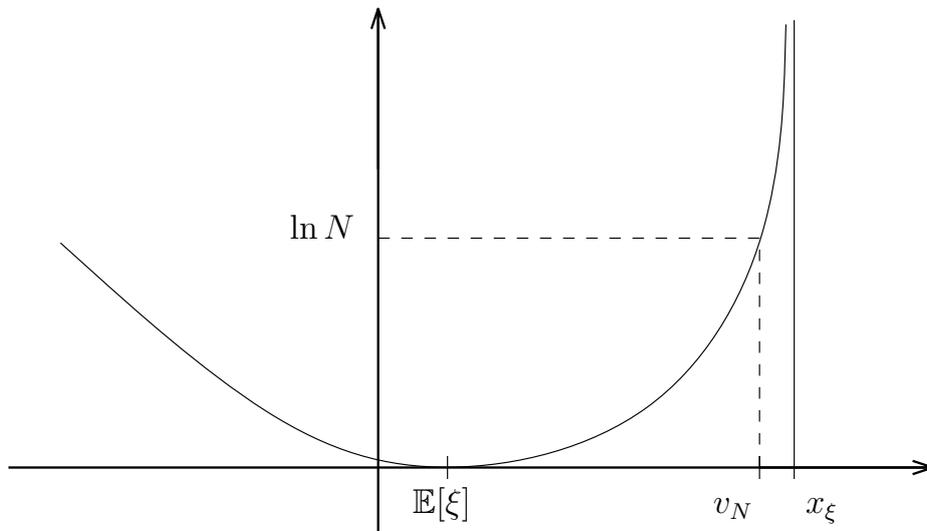
The function  $u\Lambda'(u) - \Lambda(u)$  is increasing on  $\mathcal{D}_\Lambda^0 \cap [0, \infty)$ , so that  $u_N$  is unique. Under these hypothesis the number

$$v_N := \Lambda'(u_N) \quad (4.2)$$

is well defined (*we bring to the reader's attention that the definition of  $v_N$  here differs from the  $v_N$  in Chapter 3*). If  $I_\xi(v)$  is the Cramer transform of  $\xi_{ij}$

$$I_\xi(v) := \sup_{x \in \mathbb{R}} \{vx - \Lambda(x)\},$$

then  $v_N$  is determined by  $I_\xi(v_N) = \ln N$ ,  $v_N > \mathbb{E}[\xi]$ , see Figure 4.1, and it holds  $I'_\xi(v_N) = u_N$ .


 Figure 4.1: Cramer transform  $I_\xi$  and  $v_N$ .

In Section 4.3, we show that  $v_N$  is an upper bound for the velocity  $v_N(\xi)$  of the  $N$ -particle system. To obtain a lower bound to  $v_N(\xi)$ , we assume that  $\xi$  lies in the domain of attraction of the Type III extreme value distribution see Section 2.1. Denoting by  $F(\cdot)$  the common probability distribution function

$$F(x) := \mathbb{P}(\xi_{ij} \leq x).$$

From Theorem 2.2,  $F(\cdot)$  belongs to the domain of attraction of  $\Psi_\alpha(\cdot)$ ;  $\alpha > 0$  if and only if  $F(\cdot)$  has a finite right-end

$$x_\xi := \sup\{x \in \mathbb{R}; F(x) < 1\} < \infty,$$

and for each  $x > 0$

$$\lim_{h \rightarrow 0^+} \frac{1 - F(x_\xi - hx)}{1 - F(x_\xi - h)} = x^\alpha,$$

In this case, let

$$a_N := x_\xi - \inf\{x; F(x) \geq 1 - 1/N\}, \quad (4.3)$$

then  $F^N(x_\xi + xa_N) \rightarrow \Psi_\alpha(x)$  as  $N \rightarrow \infty$  and

$$\lim_{N \rightarrow \infty} N(1 - F(x_\xi - a_N)) = 1.$$

The main result of this chapter is the following theorem concerning the speed of the  $N$ -particle system.

**Theorem 4.1.** *Assume that (H1), (H2) hold, and that  $\xi_{ij}$  belongs to the domain of attraction of the extreme value distribution  $\Psi_\alpha$ , for some  $\alpha > 0$ . Let*

$$c_\alpha := \frac{\alpha}{e} \left( \Gamma(\alpha) \alpha \right)^{-\frac{1}{\alpha}}, \quad (4.4)$$

where  $\Gamma(\cdot)$  is Euler's gamma function and  $e = 2.718\dots$  is the Napier's constant. Then, the speed  $v_N(\xi)$  of the  $N$ -particle system satisfies

$$v_N(\xi) = x_\xi - c_\alpha a_N + o(a_N) \quad \text{as } N \rightarrow \infty,$$

where  $a_N$  is given by (4.3).

**Warm-up calculations:** Let us explain how to determine the order of magnitude of the correction from elementary considerations. Assume in this paragraph that  $x_\xi = 0$ . On the one hand, we can bound from below our  $N$ -particle system with a single particle following the leader, i.e., the random walk with jumps law given by  $\max_{i \leq N} \xi_{1i}$ , resulting with a lower bound for  $v_N(\xi)$  of order  $a_N$ . On the other hand, a naive upper bound is given by the random walk with jumps  $\max_{i,j \leq N} \xi_{ji}$ , which leads to a different order  $O(a_{N^2})$  of the correction for the maximum is over  $N^2$  variables this time. One can improve the upper bound by using the first moment method of Section 4.3, leading to the same order  $O(a_N)$  as the lower bound. However, the multiplicative factors do not match, and some deeper understanding and improvement of the lower bound is needed. This is what we implement in Section 4.4, using a comparison with a branching random walk with selection.

**Organization of the chapter:** in Section 4.2, we present some point processes and branching random walks related to our model and we summarize the necessary results for our purpose. We prove the upper bound for the speed in Section 4.3 by a first moment estimate, and the lower bound in Section 4.4 by coupling, the two bounds resulting in Theorem 4.1.

## 4.2 Point processes and branching random walks

Sections 4.2.3 and 4.2.4 differ from the corresponding sections in the original paper and have been included here only to fix the notation used in this chapter. A more detailed version of the original section can be found in Chapter 2, Sections 2.2 and 2.4 respectively.

### 4.2.1 Point measures on $\mathbb{R}$

It is convenient to represent populations of particles by point measures on  $\mathbb{R}$ . Given a vector  $x \in \mathbb{R}^n$  with coordinates  $x_1, \dots, x_n$ , one can associate the point measure

$$\chi := \sum_{i=1}^n \delta_{\{x_i\}}.$$

We use the notation  $\mathcal{M}_b$  to represent the set of all simple point measures on  $\mathbb{R}$ , which are locally finite and have a maximum. Throughout this chapter, a point process is any random variable  $\mathcal{L}$  taking values on  $\mathcal{M}_b$ .

Conversely, an element  $\nu \in \mathcal{M}_b$  can be described as a sequence  $\nu = (\nu_i)_{i=1,2,\dots}$  (possibly finite) such that

$$\nu_1 \geq \nu_2 \geq \dots$$

We denote by  $\max(\nu) = \nu_1$  the maximum of the support of  $\nu$ , and by  $|\nu| = \nu(\mathbb{R}) \leq \infty$  the number of points in  $\nu$ . If two point measures  $\nu$  and  $\mu$  have the same number of points  $|\nu| = |\mu| = K$ , we can define the distance

$$\|\nu - \mu\| = \sup_{1 \leq i \leq K} \{|\nu_i - \mu_i|\}. \quad (4.5)$$

We use the notation “ $\prec$ ” to denote the usual stochastic ordering

$$\nu \prec \mu \quad \text{if and only if} \quad \nu[x, +\infty) \leq \mu[x, +\infty); \quad \forall x \in \mathbb{R},$$

and we will say that “ $\nu$  bounds  $\mu$  from below”, in this case  $|\nu| \leq |\mu|$ . If we represent  $\nu$  and  $\mu$  as an ordered sequence of points, then  $\nu \prec \mu$  implies that

$$\nu_i \leq \mu_i \quad \text{for every } i \leq |\nu|.$$

With a slight abuse of notation we will say that the vector  $x \in \mathbb{R}^n$  bounds  $y \in \mathbb{R}^m$  from below and denote “ $x \prec y$ ” if the point measures  $\chi, \gamma$  associated to  $x$  and  $y$  respectively satisfy  $\chi \prec \gamma$ .

### 4.2.2 Poisson point processes on $]-\infty, 0]$

In this section, we present some elementary facts concerning Poisson Point Process

$$\mathcal{P} = \{\mathcal{P}_1 > \mathcal{P}_2 > \dots\} \subset \mathbb{R}_-,$$

with intensity measure  $|z|^\beta C dz$  on  $\mathbb{R}_-$ ; we use the abbreviation PPP and assume that  $C > 0, \beta > -1$ . For  $K \geq 1$ , the point process

$$\mathcal{P}^{(K)} := (\mathcal{P}_i)_{i \leq K} \quad (4.6)$$

consisting in the  $K$  largest points of  $\mathcal{P}$  will play an important role in the next sections.

For  $\mathcal{L}$  a random point measure on  $\mathbb{R}_-$ , we denote by  $\psi(u \mid \mathcal{L})$  its logarithmic moments generating function

$$\psi(u \mid \mathcal{L}) := \ln \mathbb{E} \left[ \int e^{uy} \mathcal{L}(dy) \right],$$

We can easily compute the logarithmic generating function of the PPP and of its  $K$ -truncation.

**Lemma 4.2.** *For  $\beta > -1, C > 0$ , let  $\mathcal{P}$  be the Poisson point process on  $(-\infty, 0]$  with intensity measure  $\mu(dz) = |z|^\beta C dz$ , and  $\mathcal{P}^{(K)}$  its largest  $K$  points. For  $u > 0$  we have*

$$\mathbb{E} \left[ \int_{-\infty}^0 e^{uz} \mathcal{P}(dz) \right] = \frac{\Gamma(1 + \beta)C}{u^{1+\beta}}, \quad \mathbb{E} \left[ \int_{-\infty}^0 ze^{uz} \mathcal{P}(dz) \right] = \frac{-\Gamma(2 + \beta)C}{u^{2+\beta}},$$

and

$$\lim_{K \rightarrow \infty} \mathbb{E} \left[ \sum_{i=1}^K e^{u\mathcal{P}_i} \right] = \frac{\Gamma(1 + \beta)C}{u^{1+\beta}}, \quad \lim_{K \rightarrow \infty} \mathbb{E} \left[ \sum_{i=1}^K \mathcal{P}_i e^{u\mathcal{P}_i} \right] = \frac{-\Gamma(2 + \beta)C}{u^{2+\beta}}.$$

*Proof.* The first claim is obtained by the Campbell formula (see Chapter 9 in [DVJ03]) and the second claim is obtained by monotone convergence.  $\square$

**Corollary 4.3.** *Under the assumptions of Lemma 4.2, the sequence  $\psi(u \mid \mathcal{P}^{(K)})$  converges uniformly on the compacts of  $\mathbb{R}_+$  to  $\psi(u \mid \mathcal{P})$  as  $K \rightarrow \infty$ . In particular, if  $u_K > 0$  is such that*

$$\psi(u_K \mid \mathcal{P}^{(K)}) = \psi'(u_K \mid \mathcal{P}^{(K)})u_K$$

then

$$\lim_{K \rightarrow \infty} \psi'(u_K \mid \mathcal{P}^{(K)}) = -\frac{1+\beta}{e} \left( \frac{1}{C\Gamma(1+\beta)} \right)^{1/1+\beta}. \quad (4.7)$$

*Proof.* The compact convergence is a direct consequence of the pointwise convergence together with the monotonicity of  $\psi(u \mid \mathcal{P}^{(K)})$  and the continuity of  $\psi(u \mid \mathcal{P})$  in  $u$  (Dini's theorem). Let

$$u_\infty = e(\Gamma(1+\beta)C)^{1/1+\beta}$$

then from the first part of Lemma 4.2 we have that

$$\psi(u_\infty \mid \mathcal{P}) = \psi'(u_\infty \mid \mathcal{P})u_\infty \quad \text{and} \quad \psi'(u_\infty \mid \mathcal{P}) = -\frac{1+\beta}{e} (C\Gamma(1+\beta))^{-1/1+\beta}.$$

By point 2 of Lemma 4.2,  $u_K \rightarrow u_\infty$  and the second claim follows from the uniform convergence.  $\square$

### 4.2.3 Branching random walks

For a given point process  $\mathcal{L}$ , let  $\text{BRW}(\mathcal{L})$  be the branching random walk obtained by independent copies of  $\mathcal{L}$  as described in Section 2.2. Denote by  $\mathbb{T}$  the Galton-Watson tree obtained by the genealogical tree of the  $\text{BRW}(\mathcal{L})$ , thus, its offspring distribution is  $|\mathcal{L}|$ . To each point (or individual) of the  $\text{BRW}(\mathcal{L})$  one can associate a unique vertex  $w \in \mathbb{T}$ . Let  $e \in \mathbb{T}$  be the root of the Galton-Watson tree, then for a vertex  $w \in \mathbb{T}$ , let  $\llbracket e, w \rrbracket$  denote the shortest path connecting  $e$  with  $w$ , and  $|w|$  the length of this path. We will sometimes write its points  $\llbracket e, w \rrbracket = (e, w_1, \dots, w_k)$  with  $i = |w_i|$  and  $w_k = w$ . For two vertices  $w$  and  $w'$  in  $\mathbb{T}$  we denote by  $ww'$  the vertex in  $\mathbb{T}$  in generation  $|w|+|w'|$  obtained by concatenation.

We also denote by  $\eta(w)$  the positions of the individual  $w \in \mathbb{T}$ , and by  $y(t)$  the point measure associated to the  $\text{BRW}(\mathcal{L})$

$$y(t) := \sum_{w \in \mathbb{T}; |w|=t} \delta_{\{\eta(w)\}}.$$

Finally, an infinite ray  $\llbracket e, w_\infty \rrbracket := \{e, w_1, w_2, \dots\} \subset \mathbb{T}$  is an infinite collection of vertices (or infinite path), such that  $w_i$  is the parent of  $w_{i+1}$ . It represents a family branch in the BRW that has not extinguished, and is parametrized by an element  $w_\infty \in \partial\mathbb{T}$  of the topological boundary  $\partial\mathbb{T}$  of the tree.

We recall that, when  $\mathcal{L}$  satisfies (2.6–2.8), the limit

$$\lim_{t \rightarrow \infty} \frac{\max(y(t))}{t} = \gamma(\mathcal{L})$$

exists a.s. with  $\gamma(\mathcal{L})$  a non-random constant. Hypothesis (2.6) says that there exists  $a > 0$ , for which

$$\mathbb{E}[|\mathcal{L}|^{1+a}] < \infty, \quad (4.8)$$

a condition which can be weakened [Aid13], but in this chapter we will always have  $|\mathcal{L}| = K$  a constant, which trivially implies it. Hypothesis (2.7) concerns the logarithmic generating function for the branching random walk

$$\psi(u | \mathcal{L}) := \ln \mathbb{E} \left[ \int e^{uy} \mathcal{L}(dy) \right]. \quad (4.9)$$

It states that  $\psi(u | \mathcal{L})$  is finite in a neighbourhood of  $u = 0$ . Finally, (2.8) says that there exists a  $u^* = u^*(\mathcal{L}) > 0$  for which

$$\psi(u^* | \mathcal{L}) = u^* \psi'(u^* | \mathcal{L}). \quad (4.10)$$

We also recall that under (2.6 – 2.8), Theorem 2.5 in Section 2.2 holds and the precise decay for the probability

$$\rho(\infty, \delta) := \mathbb{P}(\exists w_\infty \in \partial\mathbb{T} : \eta(w_t) \geq (\gamma(\mathcal{L}) - \delta)t, \forall w_t \in \llbracket e, w_\infty \rrbracket) \quad \text{as } \delta \rightarrow 0$$

is known.

**Models with selection and the  $M$ -BRW:** In Section 4.4, we show that under the hypothesis of the Theorem 4.1, the  $N$ -particle system (1.8) can be bounded from below by a family of  $M$ -BRW indexed by  $N$ . We will then adapt the arguments in [BG10] to obtain a uniform lower bound for the speeds of the BRWs.

We will denote by  $y_M(t)$  the point process generated by a  $M$ -BRW. We recall Theorem 2.6 in Section 2.2.2 that states that under (2.6 – 2.8) there exists a constant  $\gamma_M(\mathcal{L})$  such that

$$\lim_{t \rightarrow \infty} t^{-1} \min(y_M(t)) = \lim_{t \rightarrow \infty} t^{-1} \max(y_M(t)) = \gamma_M(\mathcal{L}) \quad a.s.$$

#### 4.2.4 Elementary properties of Brunet-Derrida's $N$ -particle system

Finally we recall the notations and results of Section 2.4. Let  $X^*(t)$  be the process obtained by ordering the components of  $X(t)$  at each time  $t$ . Denote by

$$X^{(1)}(t) \geq X^{(2)}(t) \geq \dots \geq X^{(N)}(t)$$

the components of  $X^*(t)$ . Let, also  $\sigma = \sigma(t)$  be the random permutation of  $\{1, \dots, N\}$  such that

$$X^{(i)}(t) = X_{\sigma_i(t)}(t).$$

Such a ranking permutation is unique up to ties, which we break in the order of the original labels. We define the process seen from the leading edge

$$X_i^0(t) := X_i(t) - X_{\sigma_1(t)}(t).$$

It is Markov process on  $\mathbb{R}^N$ , which is irreducible, aperiodic and Harris recurrent (see Section 2.4).

### 4.3 Upper bound for the velocity

In this section, we show that if  $\xi_{ij}$  satisfies the hypothesis (H1), (H2), then

$$v_N(\xi) \leq v_N = \Lambda'(u_N),$$

where  $u_N > 0$  is the unique positive solution of  $u\Lambda'(u) - \Lambda(u) = \ln N$ . The idea is to use the so-called first moment method to bound the probability

$$\mathbb{P}\left(\max_{1 \leq i \leq N} \{X_i(t)\} > t\Lambda'(u_N)\right).$$

A first and simple observation is that the initial position of the particles does not change the speed of the  $N$ -particle system. Hence, we may assume without loss of generality that all  $N$ -particle start at zero. Using the representation (1.12) one gets

$$\max_{1 \leq i \leq N} \{X_i(t)\} = \max \left\{ \sum_{s=1}^t \xi_{j_{s-1}j_s}(s); 1 \leq j_s \leq N, \forall s = 0, \dots, t \right\}.$$

By the union bound and Chernoff bound we obtain, for  $v > v_N = \Lambda'(u_N)$  (which is larger than  $\mathbb{E}[\xi]$  for  $N$  sufficiently large),

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq i \leq N} \{X_i(t)\} \geq tv\right) &= \mathbb{P}\left(\exists j_0, j_1, \dots, j_t : \sum_{s=1}^t \xi_{j_{s-1}j_s}(s) \geq tv\right) \\ &\leq N^{t+1} \mathbb{P}\left(\sum_{s=1}^t \xi_{j_{s-1}j_s}(s) \geq tv\right) \\ &\leq N^{t+1} \exp(-tI_\xi(v)), \end{aligned} \quad (4.11)$$

for all  $N \in \mathbb{N}$ . Since (H1) and (H2) hold,  $I_\xi(v)$  exists and  $I_\xi(v) > \ln N$ . As a consequence, (4.11) has a geometrical decay as  $t \rightarrow \infty$ , which implies, by Borel-Cantelli lemma,

$$\mathbb{P}\left(\max_{1 \leq i \leq N} X_i(t) \geq tv \text{ for infinitely many } t \in \mathbb{N}\right) = 0,$$

hence,  $\limsup_{t \rightarrow \infty} t^{-1} \max\{X_i(t)\} \leq v$  a.s. for every  $v > v_N$ , finally yielding

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \max_{1 \leq i \leq N} X_i(t) \leq v_N \quad a.s. \quad (4.12)$$

We formalize this result in a proposition.

**Proposition 4.4.** *Assume that (H1), (H2) hold. Let  $u_N > 0$  such that  $u_N\Lambda'(u_N) - \Lambda(u_N) = \ln N$  and  $v_N = \Lambda'(u_N)$ . Then, for every  $N \in \mathbb{N}$ ,*

$$v_N(\xi) \leq v_N.$$

### 4.3. Upper bound for the velocity

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Hence the next step is to study the asymptotic of  $v_N$ , that we start with the case  $x_\xi = 0$ .

**Proposition 4.5.** *Assume that the hypothesis of Theorem 4.1 hold with  $x_\xi = 0$ . Let  $u_N > 0$  be the unique solution of  $u_N \Lambda'(u_N) - \Lambda(u_N) = \ln N$ ,  $c_\alpha$  be given by (4.4) and  $a_N$  by (4.3). Then, as  $N \rightarrow \infty$*

$$\Lambda'(u_N) = -c_\alpha a_N + o(a_N),$$

which implies that  $\limsup_{N \rightarrow \infty} a_N^{-1} v_N(\xi) \leq -c_\alpha$ .

*Proof.* By definition of  $\Lambda(\cdot)$ , we have that

$$\frac{\mathbb{E}[u_N \xi_{ij} e^{u_N \xi_{ij}}]}{\mathbb{E}[e^{u_N \xi_{ij}}]} - \ln(\mathbb{E}[e^{u_N \xi_{ij}}]) = \ln N.$$

Note that  $u_N \rightarrow \infty$  as  $N \rightarrow \infty$ , indeed it is a direct consequence of the monotonicity and continuity of  $u \Lambda'(u) - \Lambda(u)$ . Hence, the asymptotic behaviour of the Laplace transform of  $\xi_{ij}$  in  $u_N$  is determined by its behaviour in a neighbourhood of zero. Since  $\xi_{ij}$  is in the domain of attraction of  $\Psi_\alpha$ , the function  $1 - F(-x) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is  $\alpha$ -regularly varying at zero. By Karamata's representation (see Chapter 0 in [Res87])

$$1 - F(-x) = \mathbb{P}(\xi_{ij} > -x) = x^\alpha c(x^{-1}) \exp\left(\int_1^{x^{-1}} \frac{\epsilon(t)}{t} dt\right), \quad x > 0,$$

where  $c(\cdot)$  and  $\epsilon(\cdot)$  are positive functions such that  $c(t) \rightarrow c > 0$  and  $\epsilon(t) \rightarrow 0$  as  $t \rightarrow \infty$ . As a consequence, given  $\varepsilon > 0$ , one can find a  $u_\varepsilon > 0$  such that for  $0 < u \leq u_\varepsilon$

$$1 - F(-u) \geq (c - \varepsilon)u^\alpha.$$

Now, we compute the Laplace transform of  $\xi_{ij}$  in  $u_N$

$$\begin{aligned} \mathbb{E}[e^{u_N \xi_{ij}}] &= \mathbb{P}(\xi_{ij} u_N \geq -1) \int_0^\infty e^{-z} \frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} dz \\ &= (1 - F(-u_N^{-1})) \left( \int_0^{\sqrt{u_N}} \dots dz + \int_{\sqrt{u_N}}^\infty \dots dz \right). \end{aligned}$$

We analyse each integral separately. For  $N$  sufficiently large  $u_N^{-1} \leq u_\varepsilon$ , hence

$$\int_{\sqrt{u_N}}^\infty e^{-z} \frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} dz \leq \frac{u_N^\alpha}{(c - \varepsilon)} \int_{\sqrt{u_N}}^\infty e^{-z} dz,$$

which converges to zero as  $N \rightarrow \infty$ . Take  $L > 0$ , and assume that  $\sqrt{u_N} > L$ , then

$$\int_0^{\sqrt{u_N}} e^{-z} \frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} dz = \int_0^L \dots dz + \int_L^{\sqrt{u_N}} \dots dz.$$

Using dominated and monotone convergence we obtain that

$$\lim_{L \rightarrow \infty} \lim_{N \rightarrow \infty} \left( \int_0^L e^{-z} \frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} dz \right) = \Gamma(\alpha + 1).$$

Finally, we prove that the integral from  $L$  to  $\sqrt{u_N}$  vanishes as  $N \rightarrow \infty$  and  $L \rightarrow \infty$  (in this order). For  $L > 1$  and  $L \leq z \leq \sqrt{u_N}$ , Karamata's representation yields

$$\frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} = z^\alpha \frac{c(z^{-1} u_N)}{c(u_N)} \exp \left( \int_{u_N/z}^{u_N} \frac{\epsilon(t)}{t} dt \right).$$

Taking  $N$  sufficiently large so  $\epsilon(t) < \varepsilon$  and  $|c - c(t)| \leq \varepsilon$  for every  $t \geq \sqrt{u_N}$

$$\frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} \leq z^\alpha \frac{(c + \varepsilon)}{(c - \varepsilon)} z^\varepsilon,$$

which yields the upper bound

$$\int_L^{\sqrt{u_N}} e^{-z} \frac{\mathbb{P}(\xi_{ij} u_N \geq -z)}{\mathbb{P}(\xi_{ij} u_N \geq -1)} dz \leq \frac{(c + \varepsilon)}{(c - \varepsilon)} \int_L^\infty e^{-z} z^{\alpha + \varepsilon} dz.$$

The right-hand side of this inequality decays to zero as  $L \rightarrow \infty$ , and hence

$$\mathbb{E}[e^{u_N \xi_{ij}}] \sim (1 - F(-u_N^{-1})) \Gamma(1 + \alpha) \quad \text{as } N \rightarrow \infty.$$

By a similar argument one obtains that

$$\mathbb{E}[u_N \xi_{ij} e^{u_N \xi_{ij}}] \sim (1 - F(-u_N^{-1})) (\Gamma(1 + \alpha) - \Gamma(\alpha + 2)) \quad \text{as } N \rightarrow \infty.$$

The formula  $u_N \Lambda'(u_N) - \Lambda(u_N) = \ln N$  yields

$$(1 - F(-u_N^{-1})) N \sim \frac{1}{e^\alpha \alpha \Gamma(\alpha)} \quad \text{as } N \rightarrow \infty.$$

We now use (4.4) and Karamata's representation to conclude that

$$\lim_{N \rightarrow \infty} \frac{u_N^{-1}}{a_N} = \frac{1}{e} \left( \frac{1}{\alpha \Gamma(\alpha)} \right)^{1/\alpha},$$

and hence as  $N \rightarrow \infty$

$$\Lambda'(u_N) \sim -\frac{\alpha}{u_N} \sim -\frac{\alpha}{e} \left( \frac{1}{\alpha \Gamma(\alpha)} \right)^{1/\alpha} a_N,$$

which proves the statement. The second claim is a direct consequence of (4.12).  $\square$

If  $x_\xi \neq 0$ , we can simply translate the  $\xi_{ij}$  by  $x_\xi$ , so the hypothesis of Proposition 4.5 hold. In the next corollary, we prove the upper bound in Theorem 4.1.

**Corollary 4.6.** *Assume that the hypothesis of Theorem 4.1 hold. Then, as  $N \rightarrow \infty$*

$$\limsup_{N \rightarrow \infty} (v_N(\xi) - x_\xi) a_N^{-1} \leq -c_\alpha.$$

*Proof.* In the case  $x_\xi = 0$ , it is a straightforward consequence of Proposition 4.5 and (4.12). If  $x_\xi \neq 0$ , it suffices to translate the variables  $\xi_{ij}$  by  $x_\xi$ .  $\square$

## 4.4 Lower bound

In this Section, we show that for every  $\varepsilon > 0$  there exists a  $N_0$  such that  $\forall N \geq N_0$

$$\frac{(v_N(\xi) - x_\xi)}{a_N} \geq -c_\alpha - \varepsilon, \quad (4.13)$$

which proves the lower bound in Theorem 4.1. Throughout this section, we fix an arbitrary  $\varepsilon > 0$ , and we assume that  $x_\xi = 0$  without loss of generality. To prove (4.13), we construct a process  $x(t) \in \mathbb{R}^M$  that bounds  $X(t)$  from below, hence

$$\max(x(t)) \leq \max(X(t)).$$

Then, in Subsection 4.4.1, we check that the process  $x(t)$  is a M-BRW, and we show that for  $M$  large enough and the appropriate offspring distribution (see Subsection 4.4.2),

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \max(x(t)) \geq -(c_\alpha + \varepsilon)a_N \quad a.s.$$

which implies (4.13) for  $x_\xi = 0$ . The general case is obtained by a simple affine transformation.

### 4.4.1 Coupling with a branching random walk

We construct  $x(t)$  inductively as follows: let  $M, K \in \mathbb{N}$ , the appropriate values for  $K$  and  $M$  will be chosen later on, and assume that  $N \geq KM$  (in fact, we will take  $KM$  negligible compared to  $N$ ). For  $t = 0$ , we define

$$x_i(0) = X_{\sigma_i}(0),$$

with  $\sigma_i = \sigma_i(0)$ . Assuming that the process  $x(\cdot)$  has been constructed up to time  $t \in \mathbb{N}$ , the vector  $x(t+1) \in \mathbb{R}^M$  is obtained according to the inductive rule below.

1. *Branching step:* Every particle  $x_i(t)$  is replaced by  $K$  new particles (reproductive law), whose positions are defined by a point process  $\mathcal{L}^{(K)}(x_i(t))$  translated by  $x_i(t)$ .

The point processes  $(\mathcal{L}^{(K)}(x_i(t)); 1 \leq i \leq M)$  are also constructed according to an inductive rule, that we describe:

- For  $i = 1$ , let  $\mathcal{T}_1 := \{1, \dots, N - KM\}$  and denote by

$$\xi_{\sigma_1(t)}^{(1:\mathcal{T}_1)}(t+1) \geq \xi_{\sigma_1(t)}^{(2:\mathcal{T}_1)}(t+1) \geq \dots \geq \xi_{\sigma_1(t)}^{(K:\mathcal{T}_1)}(t+1),$$

the  $K$  largest values among  $\{\xi_{\sigma_1(t),j}(t+1); j \in \mathcal{T}_1\}$ . Let, also,

$$\mathcal{J}_1 = \mathcal{J}_1(t+1) := \{j_1^{(1)}, \dots, j_K^{(1)}\}$$

be the set of their indices, that is,

$$\xi_{\sigma_1(t)}^{(l:\mathcal{T}_1)}(t+1) = \xi_{\sigma_1(t),j_l^{(1)}}(t+1),$$

we will keep track of these labels. Note that the indices  $j_l^{(1)} = j_l^{(1)}(t+1)$ ;  $1 \leq l \leq K$  are random. Then,  $\mathcal{L}^{(K)}(x_1(t))$  is the point process

$$\mathcal{L}^{(K)}(x_1(t)) := \sum_{j \in \mathcal{J}_1(t+1)} \delta_{\{\xi_{\sigma_1 j}(t+1)\}},$$

obtained by the  $K$  largest values in  $\mathcal{T}_1$  and the descendants of  $x_1(t)$  are at the positions:

$$x_1(t) + \xi_{\sigma_1(t)}^{(l:\mathcal{T}_1)}(t+1) \quad \text{for } 1 \leq l \leq K.$$

- Assume that we have constructed  $(\mathcal{L}^{(K)}(x_j(t)); 1 \leq j \leq i-1)$  and that we have kept track of the sets  $\mathcal{J}_1, \dots, \mathcal{J}_{i-1}$ , appearing in the respective constructions. The sets  $\mathcal{J}_j = \mathcal{J}_j(t+1) \subset \{1, \dots, N\}$  are random and disjoint. Then, given  $\mathcal{J}_1 \cup \dots \cup \mathcal{J}_{i-1}$ , we choose

$$\mathcal{T}_i = \mathcal{T}_i(t+1) \subset \{1, \dots, N\} \setminus (\mathcal{J}_1 \cup \dots \cup \mathcal{J}_{i-1})$$

according to a deterministic rule. For example, one can choose the  $N - MK$  first elements (in the usual order of  $\mathbb{N}$ ) in  $\{1, \dots, N\} \setminus (\mathcal{J}_1 \cup \dots \cup \mathcal{J}_{i-1})$ . By construction,  $\mathcal{T}_i$  is a random set of  $\{1, \dots, N\}$  satisfying the property

$$\mathcal{T}_i \cap \mathcal{J}_1 = \emptyset = \mathcal{T}_i \cap \mathcal{J}_2 = \dots = \mathcal{T}_i \cap \mathcal{J}_{i-1}.$$

Let

$$\xi_{\sigma_i(t)}^{(1:\mathcal{T}_i)}(t+1) \geq \xi_{\sigma_i(t)}^{(2:\mathcal{T}_i)}(t+1) \geq \dots \geq \xi_{\sigma_i(t)}^{(K:\mathcal{T}_i)}(t+1)$$

be the  $K$  largest values among  $\{\xi_{\sigma_i(t),j}(t+1); j \in \mathcal{T}_i\}$ , and  $\mathcal{J}_i(t+1) = \{j_1^{(i)}, \dots, j_K^{(i)}\}$  be the set of their indices, that is,

$$\xi_{\sigma_i(t)}^{(l:\mathcal{T}_i)}(t+1) = \xi_{\sigma_i(t)j_l^{(i)}}(t+1).$$

Then,  $\mathcal{L}^{(K)}(x_i(t))$  is the point process formed by the these  $K$  points.

We end up the branching step with  $KM$  new particles.

2. *Selection:* We select the  $M$  rightmost particles among the  $KM$  obtained in the branching step.
3. *Ordering:* We reorder the  $M$  selected particles to obtain the vector  $x(t+1)$ .

In the next two lemmas, we show that  $x(t) \prec X(t)$  and that  $\mathcal{L}^{(K)}(\cdot)$  are i.i.d. which implies that the point process

$$\chi(t) := \sum_{i=1}^M \delta_{\{x_i(t)\}}$$

has the distribution of the point process obtained from a M-BRW  $(\mathcal{L}^{(K)})$ .

First, we prove that  $x(t)$  bounds  $X(t)$  from below. We bring to the reader's attention that the next lemma is a direct corollary of the construction of  $x(t)$  and it holds without any assumption on the family  $\{\xi_{ij}(s); 1 \leq i, j \leq N, s \geq 1\}$ .

**Lemma 4.7.** *For  $N \geq MK$ , let  $x(t)$  be the branching/selection process constructed as above. Then,  $x(t)$  bounds  $X(t)$  from below.*

*Proof.* It is immediate that  $x(0) \prec X(0)$ , hence assume that  $x(t) \prec X(t)$ . Before the selection step, there are  $MK$  points at the positions

$$x_i(t) + \xi_{\sigma_i(t), j_i^{(i)}}(t+1) \quad 1 \leq l \leq K \text{ and } 1 \leq i \leq M.$$

By the construction of  $x(\cdot)$  the  $j_i^{(i)}$  are all distinct. Since  $x_i(t) \leq X_{\sigma_i(t)}(t)$ , we have that

$$x_i(t) + \xi_{\sigma_i(t), j_i^{(i)}}(t+1) \leq X_{\sigma_i(t)}(t) + \xi_{\sigma_i(t), j_i^{(i)}}(t+1) \leq X_{j_i^{(i)}}(t+1).$$

Hence, the point process obtained from the branching step bounds  $X(t+1)$  from below, as a consequence, after the selection step  $x(t+1) \prec X(t+1)$ , proving the statement.  $\square$

Now, we prove that the point processes  $\mathcal{L}^{(K)}(\cdot)$  are i.i.d. Lemma 4.8 holds under the unique assumption that the family  $\{\xi_{ij}(s); 1 \leq i, j \leq N, s \geq 1\}$  is i.i.d.

**Lemma 4.8.** *Assume that  $N \geq KM$ , and let  $\{\mathcal{T}_i(t); t \in \mathbb{N}; i = 1, \dots, M\}$  be the set of indices obtained in the above construction. For  $t \geq 0$  denote by  $\Xi(x_i(t))$  the point process*

$$\Xi(x_i(t)) := \sum_{j \in \mathcal{T}_i(t+1)} \delta_{\{\xi_{\sigma_i(t), j}(t+1)\}},$$

then,  $\{\Xi(x_i(t)); 1 \leq i \leq M; t \in \mathbb{N}\}$  are i.i.d. In particular, the family of point processes

$$\{\mathcal{L}^{(K)}(x_i(t)); t \in \mathbb{N} \text{ and } i = 1, \dots, M\}$$

is also i.i.d.

*Proof.* Note that the families of random variables

$$\{\sigma(s); 0 \leq s \leq t\}, \quad \{\mathcal{T}_i(s); 0 \leq s \leq t; 1 \leq i \leq M\}$$

and  $\{\Xi(x_i(s)); 1 \leq s \leq t-1; 1 \leq i \leq M\}$  are  $\mathcal{F}_t$ -measurable with  $\mathcal{F}_t$  from (2.19). By assumption,  $\sigma\{\xi_{ij}(t+1); 1 \leq i, j \leq N\}$  is independent from  $\mathcal{F}_t$ , then, by successive conditioning, one easily checks that conditionally on  $\mathcal{F}_t$  the vector

$$(\xi_{\sigma_i(t), j}(t+1); i = 1, \dots, M \text{ and } j \in \mathcal{T}_i)$$

is distributed according to a  $M \times (N - KM)$  vector, whose entries are i.i.d. copies of  $\xi_{ij}$ , which implies the independence from  $\mathcal{F}_t$ . Moreover, the conditional independence of the  $\xi_{\sigma_i, j}(t+1)$  yields that  $(\Xi(x_i(t)); i = 1, \dots, M)$  are also independent, which proves the first claim.

The second claim is an immediate consequence of the first part of the lemma.  $\square$

Finally, we focus on the asymptotic distribution of  $\mathcal{L}^{(K)}(\cdot)$  after suitable rescaling,

$$\mathcal{P}^{(N,K)}(x_i(t)) := \sum_{z \in \mathcal{L}^{(K)}(x_i(t))} \delta_{\{za_N^{-1}\}}, \quad (4.14)$$

for  $a_N$  given by (4.3). With some abuse of notation, we denote by  $\mathcal{P}^{(N,K)}$  the common distribution of these point processes.

**Proposition 4.9.** *Assume that the hypothesis of Theorem 4.1 hold with  $x_\xi = 0$  and that  $M$  and  $K$  are fixed. Then, as  $N \rightarrow \infty$ ,*

$$\mathcal{P}^{(N,K)} \longrightarrow \mathcal{P}^{(K)} \quad \text{in law,}$$

with  $\mathcal{P}^{(K)}$  defined in Corollary (4.3) with  $\beta = \alpha - 1$  and  $C = \alpha$ . Moreover, for every  $\ell > 0$  the moment convergence

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \left| \min \mathcal{P}^{(N,K)} \right|^\ell \right] = \mathbb{E} \left[ \left| \min \mathcal{P}^{(K)} \right|^\ell \right] < \infty$$

also holds.

*Proof.* It suffices to prove the convergence for  $\mathcal{P}^{(N,K)}(x_1(t))$ . Since  $\xi_{ij}$  is in the domain of attraction of  $\Psi_\alpha$  and  $x_\xi = 0$ , for every  $z > 0$ ,

$$\mathbb{P}(\xi_{\sigma_1(t)j}(t) > -za_N) \sim z^\alpha \mathbb{P}(\xi_{\sigma_1(t)j}(t) > -a_N) \sim \frac{z^\alpha}{N} \quad \text{as } N \rightarrow \infty. \quad (4.15)$$

It is a classical result of extreme value theory [Res87] that, as  $N \rightarrow \infty$ , the point process

$$\mathcal{P}^{(N,K)} \stackrel{\text{law}}{=} \sum_{j=1}^{N-KM} \delta_{\{a_N^{-1}\xi_{1,j}(t)\}}.$$

converges in distribution to a PPP with intensity measure  $|z|^{\alpha-1} \alpha \mathbf{1}_{\{z < 0\}} dz$ , as we have claimed. A necessary and sufficient condition for the convergence of the  $\ell$ th moment is that the r.v.  $\xi_{ij}$  has itself finite  $\ell$ th moment, which is a consequence of (H1). Proposition 2.1 in [Res87] proves this statement for the maxima of i.i.d. random variables in the domain of attraction of  $\Psi_\alpha$ .

Now, a line-by-line adaptation of Proposition 2.1 in [Res87] yields the last claim.  $\square$

A straightforward consequence of Proposition 4.9 and the two previous lemmas is that

$$\chi^{(N)}(t) := \sum_{i=1}^M \delta_{\{a_N^{-1}x_i(t)\}}$$

converges in distribution to the point process obtained from a  $M$ -BRW( $\mathcal{P}^{(K)}$ ) at time  $t$ , moreover,

$$\frac{a_N}{t} \max(\chi^{(N)}(t)) \leq \frac{1}{t} \max_{1 \leq j \leq N} (X_j(t)).$$

We will prove in Subsection 4.4.2 that if one chooses  $K$  and  $M$  large enough (depending only on  $\varepsilon$  and the distribution  $\xi_{ij}$ ), then for  $N$  larger than some  $N_0 > 0$ ,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \max(\chi^{(N)}(t)) \geq -c_\alpha - \varepsilon \quad a.s. \quad (4.16)$$

which proves the lower bound (4.13).

### 4.4.2 Uniform lower bound for the velocities

In this subsection, we prove the lower bound (4.16), which concludes the proof of Theorem 4.1. The proof is divided in two main steps.

In the first one, we focus on the BRWs defined by  $\mathcal{P}^{(N,K)}$ . We prove that if  $N$  is sufficiently large, with positive probability there exists more than  $M$  vertices  $w$  in generation  $n$  (see Subsection 4.4.2 for its definition), such that

$$\text{position}(w_t) \geq -(c_\alpha + \varepsilon/2)t \quad \forall w_t \in \llbracket e, w \rrbracket.$$

In the second step, we use this result to obtain the uniform lower bound (4.16) for the  $M$ -BRWs.

Most of the arguments presented here have already been used by Bérard and Gouéré [BG10]. In our case, though, we deal with a family of BRWs indexed by  $N$ , whereas in [BG10] they compute the velocity for a unique  $M$ -BRW.

#### First step

Let  $\mathcal{P}^{(N,K)}$  be the distribution defined by (4.14) and  $\mathcal{P}^{(K)}$  denote the distribution of a point process obtained from the  $K$  largest points of a PPP with intensity measure  $|z|^{\alpha-1} \alpha \mathbf{1}_{\{z \leq 0\}} dz$ . Then,  $\text{BRW}(\mathcal{P}^{(N,K)})$  and  $\text{BRW}(\mathcal{P}^{(K)})$  generate the same Galton-Watson tree, in which every individual has a constant number  $K$  of offspring, denote by  $\mathbb{T}_K$  this tree. We will construct these BRWs on a same probability space.

Let  $\{\mathcal{P}^{(N,K)}(w); w \in \mathbb{T}_K\}$  be i.i.d. copies of  $\mathcal{P}^{(N,K)}$ ,  $\{\mathcal{P}^{(K)}(w); w \in \mathbb{T}_K\}$  be i.i.d. copies of  $\mathcal{P}^{(K)}$ , and  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space where those families of r.v. are defined. Since  $\mathcal{P}^{(N,K)}$  converges to  $\mathcal{P}^{(K)}$  in distribution (see Proposition 4.9), we can and we will assume that the stronger a.s. convergences

$$\lim_{N \rightarrow \infty} \mathcal{P}^{(N,K)}(w) = \mathcal{P}^{(K)}(w) \quad a.s. \quad (4.17)$$

hold for all  $w \in \mathbb{T}_K$ , which implies the point-to-point convergence

$$\lim_{N \rightarrow \infty} \|\mathcal{P}^{(N,K)}(w) - \mathcal{P}^{(K)}(w)\| = 0 \quad a.s.$$

where  $\|\cdot\|$  is the distance defined in (4.5). Note that we have not lost in generality, since we can always construct a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , for which the a.s. convergence holds.

Under these hypothesis, the construction goes as follows. Each individual  $w \in \mathbb{T}_K$  has  $K$  offspring, that we label according to some deterministic order. Let  $w^{(i)}$  be its

$i$ th children, then, its position  $\eta^{(N)}(ww^{(i)})$  and  $\eta^{(\infty)}(ww^{(i)})$  in the BRW( $\mathcal{P}^{(N,K)}$ ) and BRW( $\mathcal{P}^{(K)}$ ) are given by

$$\eta^{(N)}(ww^{(i)}) = \eta^{(N)}(w) + \mathcal{P}_i^{(N,K)}(w) \quad \text{and} \quad \eta^{(\infty)}(ww^{(i)}) = \eta^{(\infty)}(w) + \mathcal{P}_i^{(K)}(w),$$

where  $\mathcal{P}_i^{(N,K)}(w)$  and  $\mathcal{P}_i^{(K)}(w)$  denote respectively the  $i$ th largest point in the point processes  $\mathcal{P}^{(N,K)}(w)$  and  $\mathcal{P}^{(K)}(w)$ . This construction couples the BRWs and for  $w \in \mathbb{T}_K$  fixed

$$\lim_{N \rightarrow \infty} \eta^{(N)}(w) = \eta^{(\infty)}(w) \quad \text{a.s.}$$

A direct calculation shows that  $\mathcal{P}^{(K)}$  satisfies (2.6–2.8), which implies the existence of the asymptotic velocity  $\gamma(\mathcal{P}^{(K)})$ , with  $\gamma$  given by (2.9). Lemma 4.2 with  $C = \alpha$  and  $\beta = \alpha - 1 > -1$  yields

$$\lim_{K \rightarrow \infty} \gamma(\mathcal{P}^{(K)}) = -\frac{\alpha}{e} \left( \frac{1}{\Gamma(\alpha)\alpha} \right)^{\frac{1}{\alpha}} = -c_\alpha.$$

Let  $\delta = \varepsilon/12$ , then there exists  $K_0$  such that  $\forall K \geq K_0$

$$\gamma(\mathcal{P}^{(K)}) \geq -c_\alpha - \delta. \quad (4.18)$$

Fix  $K$  for which (4.18) holds; *we bring to the reader's attention that, as (H1), (H2) hold,  $\mathcal{P}^{(N,K)}$  also satisfies (2.6), (2.7). Moreover, a simple calculation shows that  $\gamma(\mathcal{P}^{(N,K)})$  converges to  $\gamma(\mathcal{P}^{(K)})$  as  $N \rightarrow \infty$ .*

We now prove that with positive probability there exists more than  $M$  individuals  $\tilde{w} \in \mathbb{T}$  in generation  $n$  such that

$$\eta^{(N)}(\tilde{w}_t) \geq -c_\alpha t - 6\delta, \quad \text{for every } \tilde{w}_t \in \llbracket e, \tilde{w} \rrbracket.$$

As it will become clearer in the sequel, we take  $n$  of the form  $n = s_M + m$ , with

$$s_M := \left\lceil \frac{\ln M}{\ln \varphi} \right\rceil + 1 \quad \text{and} \quad m = \left\lceil \frac{(|R| - c_\alpha - 6\delta)s_M}{3\delta} \right\rceil. \quad (4.19)$$

The constants  $\varphi > 1$  and  $R < -c_\alpha - 6\delta < 0$  are given by Lemma 4.10 and formula (4.20) below and depend only on the distribution  $\mathcal{P}^{(K)}$ . Although  $M$  may be very large, it will be kept constant throughout this section (while  $N \rightarrow \infty$ ), hence  $s_M$  and  $m$  are also constants.

First, we obtain a lower bound for the probability of the set

$$\left\{ \exists w \in \mathbb{T}_K \text{ in generation } m \text{ such that } \eta^{(N)}(w_t) \geq (-c_\alpha - 3\delta)t; \forall w_t \in \llbracket e, w \rrbracket \right\}.$$

Denote by  $A_{m,\delta}$  the set

$$A_{m,\delta} := \left\{ \left\| \mathcal{P}^{(N,K)}(w') - \mathcal{P}^{(K)}(w') \right\| \leq \delta; \forall w' \in \mathbb{T}_K \text{ such that } |w'| \leq m \right\}$$

#### 4.4. Lower bound

then, for  $m \in \mathbb{N}$  and  $\delta$  fixed, one obtains from (4.17) that  $\mathbb{P}(A_{m,\delta}) \rightarrow 1$  as  $N \rightarrow \infty$ . Since  $\gamma(\mathcal{P}^{(K)}) \geq c_\alpha - \delta$  we have the following set inclusions

$$\begin{aligned} & \{\exists w \in \mathbb{T}_K; |w| = m \text{ and } \eta^{(N)}(w_t) \geq (-c_\alpha - 3\delta)t; \forall w_t \in \llbracket e, w \rrbracket\} \\ & \supset \{\exists w \in \mathbb{T}_K; |w| = m \text{ and } \eta^{(N)}(w_t) \geq (-c_\alpha - 3\delta)t; \forall w_t \in \llbracket e, w \rrbracket\} \cap A_{m,\delta} \\ & \supset \{\exists w \in \mathbb{T}_K; |w| = m \text{ and } \eta^{(\infty)}(w_t) \geq (-c_\alpha - 2\delta)t; \forall w_t \in \llbracket e, w \rrbracket\} \cap A_{m,\delta} \\ & \supset \{\exists w \in \mathbb{T}_K; |w| = m \text{ and } \eta^{(\infty)}(w_t) \geq (-\gamma(\mathcal{P}^{(K)}) - \delta)t; \forall w_t \in \llbracket e, w \rrbracket\} \cap A_{m,\delta} \\ & \supset \{\exists w_\infty \in \partial\mathbb{T}_K; \eta^{(\infty)}(w_t) \geq (\gamma(\mathcal{L}) - \delta)t; \forall w_t \in \llbracket e, w_\infty \rrbracket\} \cap A_{m,\delta}, \end{aligned}$$

which yields the lower bound

$$\liminf_{N \rightarrow \infty} \mathbb{P}\left(\exists w \in \mathbb{T}_K; |w| = m \text{ and } \eta^{(N)}(w_t) \geq (-c_\alpha - 3\delta)t; \forall w_t \in \llbracket e, w \rrbracket\right) \geq \rho(\infty, \delta).$$

From Theorem 2.5,  $\rho(\infty, \delta) > 0$  is a constant depending only on  $\mathcal{P}^{(K)}$ . Then, there exists  $N_M \in \mathbb{N}$  depending only on  $m$  (and hence, on  $M$ ) such that  $\forall N \geq N_M$

$$\mathbb{P}\left(\exists w \in \mathbb{T}_K : |w| = m, \eta^{(N)}(w_t) \geq (-c_\alpha - 3\delta)t, \forall w_t \in \llbracket e, w \rrbracket\right) \geq \frac{\rho(\infty, \delta)}{2}.$$

Now, we choose  $R$  and  $\varphi$  in (4.19). Since  $\mathcal{P}^{(K)}(]-\infty, 0]) = K$ , one can take  $R < -c_\alpha - 6\delta < 0$  such that

$$\mathbb{P}(\mathcal{P}^{(K)}[R, 0] \geq 2) > \frac{2}{3}. \quad (4.20)$$

Using the convergence in distribution, there exists a  $N' > 0$  such that for  $N \geq N'$

$$\mathbb{P}(\mathcal{P}^{(N,K)}[R, 0] \geq 2) \geq \frac{2}{3}.$$

Without loss of generality, we can and we will assume that  $N_M \geq N'$ . The Galton-Watson tree whose offspring distribution is

$$p_i^{(N)} = \mathbb{P}(\mathcal{P}^{(N,K)}[R, 0] = i), \quad i = 0, 1, \dots, \quad (4.21)$$

has mean offspring larger than  $4/3$ . It is supercritical, and the following well-known result holds.

**Lemma 4.10** ([AN04] Theorem 2 Section 6 Chapter 1). *Let  $M_t$  denote the population size of a supercritical Galton-Watson process with square integrable offspring distribution (started with one individual). Then, there exists  $r > 0$  and  $\varphi > 1$  such that for all  $t \geq 0$*

$$\mathbb{P}(M_t \geq \varphi^t) \geq r.$$

Let  $M_t^{(N)}$  denote the population size of the Galton-Watson processes defined by  $(p_i^{(N)})_{i=0,1,\dots}$ . Using a simple coupling argument and Lemma 4.10, we can find a  $\varphi > 1$  and  $r > 0$  not depending on  $N \geq N_M$  such that for all  $t \geq 1$

$$\mathbb{P}(M_t^{(N)} \geq \varphi^t) > r.$$

With  $m$  and  $s_M$  from (4.19), we have that

$$M_{s_M}^{(N)} \geq M; \quad \text{with probability at least } r > 0,$$

and that

$$(-c_\alpha - 3\delta)m + R(t - m) \geq (-c_\alpha - 6\delta)t; \quad \text{for every } m \leq t \leq m + s_M.$$

Let  $ww' \in \mathbb{T}_K$  be a vertex in generation  $n$ , with the following properties:  $|w| = m$ ,

$$\eta^{(N)}(w_t) > (-c_\alpha - 3\delta)t, \quad \forall w_t \in \llbracket e, w \rrbracket,$$

$w'$  is in generation  $s_M$  in the  $\mathbb{T}_K$  sub-tree descending from  $w$  and

$$\eta^{(N)}(w'_{s+1}) - \eta^{(N)}(w'_s) \geq R, \quad \forall w'_s \in \llbracket w, w' \rrbracket.$$

Then, by a simple calculation one can conclude that the path  $\llbracket e, ww' \rrbracket \subset \mathbb{T}_K$  has always lain above the line of slope  $-c_\alpha - 6\delta$ . For  $N \geq N_M$ , a conditioning argument yields the lower bound for the probabilities

$$\begin{aligned} & \mathbb{P} \left( \# \left\{ \tilde{w} \in \mathbb{T}_K; |\tilde{w}| = n \text{ and } \eta^{(N)}(\tilde{w}_t) \geq - \left( c_\alpha + \frac{\varepsilon}{2} \right) t \forall \tilde{w}_t \in \llbracket e, \tilde{w} \rrbracket \right\} \geq M \right) \\ & \geq \mathbb{P} \left( \exists w \in \mathbb{T}_K \text{ such that } \eta^{(N)}(w_t) \geq (-c_\alpha - 3\delta)t; \forall w_t \in \llbracket e, w \rrbracket \text{ and } M_{s_M}^{(N)} \geq M \right) \\ & \geq r \frac{\rho(\infty, \delta)}{2}, \end{aligned}$$

in the second equation,  $w$  is a vertex in generation  $m$  and  $M_t^{(N)}$  is the population size of the Galton-Watson process generated by the descendants of  $w$  for which

$$\eta^{(N)}(w'_{s+1}) - \eta^{(N)}(w'_s) \geq R \quad \forall w'_s \in \llbracket w, w' \rrbracket.$$

In particular, we have just proved the following proposition.

**Proposition 4.11.** *Let  $(\eta^{(N)}(w); w \in \mathbb{T}_K)$  the BRW defined by the point processes  $\mathcal{P}^{(N,K)}$ . Given  $\varepsilon > 0$  let  $R$  be given by (4.20),  $r$  and  $\varphi$  as in Lemma 4.10. Then, take  $s_M$  and  $m$  as in (4.19). Then, with  $n = m + s_M$ , we can find some  $N_M$  (depending only on  $M$ ) such that if  $N \geq N_M$*

$$\begin{aligned} & \mathbb{P} \left( \# \left\{ w \in \mathbb{T}_K; |w| = n \text{ and } \eta^{(N)}(w_t) \geq - \left( c_\alpha + \frac{\varepsilon}{2} \right) t; \forall w_t \in \llbracket e, w \rrbracket \right\} \geq M \right) \\ & \geq \frac{\rho(\infty, \varepsilon/12)}{2} r \end{aligned}$$

### Second step: uniform lower bound for the speed

In this step, we obtain a uniform lower bound for the speed of  $a_N^{-1}x(t)$ , which is simply a M-BRW  $(\mathcal{P}^{(N,K)})$ . Let  $\chi^{(N)}(t)$  be the point process associated to  $a_N^{-1}x(t)$

$$\chi^{(N)}(t) := \sum_{i=1}^M \delta_{\{a_N^{-1}x_i(t)\}},$$

#### 4.4. Lower bound

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and  $\gamma_M(\mathcal{P}^{(N,K)})$  be the asymptotic velocity of the  $M$ -BRW( $\mathcal{P}^{(N,K)}$ ),

$$\lim_{t \rightarrow \infty} t^{-1} \min(\chi^{(N)}(t)) = \lim_{t \rightarrow \infty} t^{-1} \max(\chi^{(N)}(t)) = \gamma_M(\mathcal{P}^{(N,K)}) \quad a.s.$$

Then, we will prove that for  $\varepsilon > 0$  and  $K$  given by (4.18) the inequality

$$\liminf_{M \rightarrow \infty} \left( \liminf_{N \rightarrow \infty} \gamma_M(\mathcal{P}^{(N,K)}) \right) \geq -(c_\alpha + \varepsilon)$$

holds, which finishes the proof of Theorem 4.1. Following the strategy of [BG10], we construct a third point process  $W(t) = W^{(N)}(t)$  that bound  $\chi^{(N)}(t)$  from below. This new point process evolves like  $\chi^{(N)}(t)$  up to a certain random time  $\tau_i$ ,  $i \in \mathbb{N}$ , from which we shift the position of all particles to the minimal position, and start  $W(t)$  afresh.

Let  $n = m + s_M$ , where  $s_M$  and  $m$  are given by (4.19). We will construct the process  $W(t)$  and the stopping times  $0 = \tau_0 < \tau_1 < \dots$  together

$$\tau_1 := \inf \{ 1 \leq s \leq n; \min(\chi^{(N)}(s)) \geq (-c_\alpha - \varepsilon/2)s \},$$

where  $\inf\{\emptyset\} = n$ . Then,  $\tau_1 \leq n$  is a stopping time with respect to the filtration  $\mathcal{F}_t$ . For  $0 \leq t \leq \tau_1$  let

$$W(t) = \chi^{(N)}(t).$$

and  $m_1 := \min(W(\tau_1))$ , then at the time step  $\tau_1 \rightarrow \tau_1 + 1$  we shift all particles  $W_i$  to  $m_1$  and continue the construction up to  $\tau_2$  according to the induction step.

*Inductive step:* assume that  $\tau_1 < \dots < \tau_l$  and  $W(t)$  for  $t \leq \tau_l$  are defined. Then, for  $\tau_l + 1 \leq t \leq \tau_{l+1}$  (we will define  $\tau_{l+1}$  below),  $W(t)$  is the point process of a  $M$ -BRW( $\mathcal{P}^{(N,K)}$ ) starting from

$$m_l := \min(W(\tau_l)).$$

At each time step  $t \rightarrow t + 1$  the individuals  $(W_i(t))_{i=1, \dots, M}$  give birth to  $K$  new individuals, whose positions are determined by independent point process

$$(\mathcal{P}^{(N,K)}(x_i(t)); i = 1, \dots, M),$$

and die immediately afterwards. We assume that the point process defining  $\chi^{(N)}$  and  $W(\cdot)$  are the same. Moreover, we will also assume that the indices are organized in order to couple  $\chi^{(N)}$  by  $W(\cdot)$ . We then select the  $M$  rightmost particles to form the next generation.

The process evolves as above up to

$$\tau_{l+1} := \inf \{ \tau_l + 1 \leq s \leq \tau_l + n; \min(W(s)) - m_l \geq (-c_\alpha - \varepsilon/2)s \},$$

where we shift the positions of the  $M$  particles to  $m_{l+1}$ , the minimum of the positions. It is immediate from the construction of  $W(\cdot)$  that

$$W(t) \prec \chi^{(N)}(t) \quad \forall t \in \mathbb{N}.$$

For  $l \geq 1$ , the processes  $(W(t) - m_l; t \in [\tau_l + 1, \tau_{l+1}])$  and the random variables  $\tau_{l+1} - \tau_l$  are i.i.d. In the sequel, we use the notation  $\tau := \tau_1$ , then by the law of large numbers

$$\lim_{l \rightarrow \infty} \frac{1}{l} \min(\chi^{(N)}(\tau_l)) = \gamma_M(\mathcal{P}^{(N,K)}) \mathbb{E}[\tau] \quad a.s.$$

From the construction of  $W(\cdot)$  and the renewal theorem we also obtain that

$$\liminf_{l \rightarrow \infty} \frac{1}{l} \min(\chi^{(N)}(\tau_l)) \geq \liminf_{l \rightarrow \infty} \frac{1}{l} \min(W(\tau_l)) = \mathbb{E}[\min(W(\tau))] \quad a.s.$$

which implies that

$$\gamma_M(\mathcal{P}^{(N,K)}) \geq \frac{\mathbb{E}[\min(W(\tau))]}{\mathbb{E}[\tau]}. \quad (4.22)$$

With  $B = \{\min(W(\tau)) < (-c_\alpha - \varepsilon/2)\tau\}$ , we write

$$\begin{aligned} \min(W(\tau)) &\geq (-c_\alpha - \varepsilon/2)\tau \mathbf{1}_{B^c} + \min(W(n)) \mathbf{1}_B \\ &= (-c_\alpha - \varepsilon/2)\tau + (c_\alpha + \varepsilon/2)\tau \mathbf{1}_B + \min(W(n)) \mathbf{1}_B. \end{aligned}$$

Taking expected value we get

$$\mathbb{E}[\min(W(\tau))] \geq (-c_\alpha - \varepsilon/2)\mathbb{E}[\tau] + \mathbb{E}[\min(W(n)) \mathbf{1}_B]. \quad (4.23)$$

Let  $\min(\mathcal{P}^{(N,K)}(W_i(t)))$  be the smallest point of the point process generated by  $W_i(t)$  before the selection step, it has the law of the  $K$ th maxima of a  $N - KM$  sample of  $\xi_{ij}$ . Since  $\xi_{ij} \leq 0$ , one gets the lower bound

$$\min(W(n)) \geq \sum_{t=0}^n \sum_{i=1}^M \min(\mathcal{P}^{(N,K)}(W_i(t))),$$

which implies that

$$\mathbb{E}[\min(W(n)) \mathbf{1}_B] \geq -(n+1)M \mathbb{E}[|\min(\mathcal{P}^{(N,K)}) \mathbf{1}_B|].$$

The Cauchy-Schwarz inequality yields

$$\mathbb{E}[\min W(n) \mathbf{1}_B] \geq -(n+1)M \mathbb{E}\left[|\min(\mathcal{P}^{(N,K)})|^2\right]^{1/2} \mathbb{P}(B)^{1/2}.$$

By Proposition 4.9, the second moment of  $\min(\mathcal{P}^{(N,K)})$  converges as  $N \rightarrow \infty$  to a finite constant. Hence, there exists a constant  $\tilde{c}$ , depending only on  $\xi_{ij}$ , such that

$$\mathbb{E}[\min W(n) \mathbf{1}_B] \geq -\tilde{c}(n+1)M \mathbb{P}(B)^{1/2}.$$

Finally, the probability of  $B$  can be estimated using Proposition 4.11. The evolution of different individuals in the  $M$ -BRW is not independent. Yet, a  $M$ -BRW can be coupled with  $M$  independent BRWs, see Section 3.3 in [BG10], so that the event “the minimum of the  $M$ -BRW always lies below the line of slope  $-c_\alpha - \varepsilon/2$ ” implies that

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none of the  $M$  independent BRWs has more than  $M$  vertices in generation  $n$  that have always stayed above this line, hence

$$\mathbb{P}(B) \leq \left(1 - \frac{\rho(\infty, \varepsilon/12)}{2} r\right)^M.$$

From the definition of  $n$ , if  $M$  is large enough  $\tilde{c}(n+1)M < M^2$  and

$$\limsup_{M \rightarrow \infty} M^2 \left(1 - \frac{\rho(\infty, \varepsilon/12)}{2} r\right)^{M/2} = 0.$$

Then, choosing  $M$  properly (note that it depends only on  $\varepsilon$  and  $\xi_{ij}$  but not on  $N$ ), one gets

$$\mathbb{E}[\min W(n) \mathbf{1}_B] \geq -\frac{\varepsilon}{2}.$$

Combined with (4.22, 4.23), this ends the proof.

# Chapter 5

## The genealogical aspects of the model

This chapter is an article to appear in the Bernoulli journal. Only the introduction has been modified for a better readability.

### 5.1 Introduction

In this chapter we analyse the genealogical properties of (1.8) we focus on the case where  $\xi$  is Gumbel  $G(\rho, \beta)$ -distributed. Before stating the main result of the chapter, we recall the definitions and results of Section 2.3. We denote by  $\mathbb{P}_n$  and  $\mathbb{P}_\infty$  the set of partitions of  $[n]$  and  $\mathbb{N}^*$ , respectively.

A coalescent process is any right-continuous  $\mathbb{P}_n$ -valued (or  $\mathbb{P}_\infty$ -valued) Markov process  $(\Pi_t^n; t \geq 0)$  for which  $\Pi_s^n$  is a refinement of  $\Pi_t^n$ , whenever  $s \leq t$ . It is also important to recall the definition of the *ancestral partition process* of a population (see Section 2.3). We use the notation  $\Pi^{N,\cdot}$  to denote the ancestral partition of a constant size population with  $N$  individuals, while the notation  $\Pi^{\infty,\cdot}$ , or simply  $\Pi$ , stands for a coalescent process. We will focus on discrete-time population model defined by the family sizes

$$\nu(t) := (\nu_1(t), \dots, \nu_N(t)) \quad \text{with} \quad \nu_1(t) + \nu_2(t) + \dots + \nu_N(t) = N, \quad t \in \mathbb{Z},$$

where  $\nu_i(t)$  denotes the number of children of the  $i$ th individual in generation  $t$ . We will often assume that

- (i) The offspring vectors  $\nu(t)$ ,  $t \in \mathbb{Z}$  are i.i.d. copies of  $\nu$ .
- (ii) The offspring vector  $(\nu_1, \dots, \nu_N)$  is  $N$ -exchangeable.

We recall that if (i) and (ii) hold, one can easily compute the probability transitions  $p_N(\pi', \pi)$ , with  $\pi' \subset \pi$ , of the ancestral partition process  $\Pi^{N,n}$ , see equation (2.11). Moreover, the probability that two individuals, chosen randomly without replacement from some generation, have a common ancestor one generation backward in time is given by (equation (2.12) in Section 2.3)

$$c_N := \frac{1}{N(N-1)} \sum_i^N \mathbb{E}[\nu_i(t)(\nu_i(t) - 1)] = \frac{1}{(N-1)} \mathbb{E}[\nu_1(t)(\nu_1(t) - 1)]. \quad (5.1)$$

It is also important to keep in mind Theorem 2.8, that will be used to prove some of the results in this chapter.

Finally we recall from Section 2.3.2 that when

$$\mathbb{P}(\xi_{ij}(t+1) = x) = 0; \quad \forall x \in \mathbb{R},$$

the equation

$$X_j(t+1) = X_i(t) + \xi_{ij}(t+1). \quad (5.2)$$

has a.s. a unique solution  $i$ , and one can say that  $X_j(t+1)$  is an offspring or a descendant of  $X_i(t)$ . So that, we also denote by  $\nu_i(t)$  the number of descendants of  $X_i(t)$  in generation  $t+1$ . With

$$\mathcal{F}_t := \sigma\{\xi_{ij}(s) \text{ and } X_i(0); 0 \leq s \leq t, 1 \leq i, j \leq N\},$$

let

$$\eta_i(t) := \mathbb{P}\left(\xi_{ij}(t+1) + X_i(t) \geq \xi_{kj}(t+1) + X_k(t); \text{ for every } 1 \leq k \leq N \mid \mathcal{F}_t\right).$$

be the conditional probability that  $X_j(t+1)$  descends from  $X_i(t)$ . Since

$$\{\xi_{ij}(t+1); 1 \leq i, j \leq N\}$$

are independent, it is easy to see that, for  $j_1, \dots, j_m$  distinct and  $i_1, \dots, i_m$  (not necessarily distinct),

$$\begin{aligned} & \mathbb{P}(X_{j_k}(t+1) \text{ descends from } X_{i_k}(t), \text{ for } 1 \leq k \leq m \mid \mathcal{F}_t) \\ &= \eta_{i_1}(t) \eta_{i_2}(t) \dots \eta_{i_m}(t). \end{aligned}$$

Hence, given  $\mathcal{F}_t$  the offspring vector  $\nu(t) := (\nu_1(t), \dots, \nu_N(t))$  is distributed according to a  $N$ -class multinomial with  $N$  trials and probabilities outcomes

$$\eta(t) := (\eta_1(t), \dots, \eta_N(t)).$$

If we assume that the offspring vectors  $(\nu(t))_{t \in \mathbb{N}}$  are identically distributed and independent from generation to generation, then we obtain a “toy model”, in which generations are not correlated. In this chapter, we study the ancestral history of this population. We make two additional assumptions on the fitness  $\eta(t)$ . First, we assume that each  $\eta_i(t)$  is of the form

$$\eta_i(t) = Y_i(t) \Big/ \sum_{j=1}^N Y_j(t), \quad (5.3)$$

where  $Y_j(t)$  are i.i.d. positive random variables. Secondly, for some of our results, we assume that the tail distribution of  $Y_i(t)$  satisfies

$$\lim_{y \rightarrow \infty} \mathbb{P}(Y_i(t) \geq y) / y^{-\alpha} = C, \quad (5.4)$$

where  $\alpha$  and  $C$  are positive constants. To simplify the notation, the time parameter  $t$  is often omitted. Moreover,  $\eta_i(t)$  in (5.3) does not change if we replace  $Y_j(t)$  by  $Y_j(t)C^{-1/\alpha}$ , for this reason we may always assume that  $C = 1$ . Then, we show that the *ancestral partition processes* converge weakly and that the limit distribution depends on  $\alpha$ .

**Theorem 5.1.** *Consider the dynamics of a constant size  $N$  population with infinitely many generations backwards in time defined by the vectors*

$$\nu(t) = (\nu_1(t), \dots, \nu_N(t)), \quad t \in \mathbb{Z}$$

*of family sizes and denote by  $\Pi_t^{N,n}$  the ancestral partition process. Suppose that the family sizes  $\nu(t)$  are i.i.d. copies of  $\nu$  a doubly stochastic multinomial random variable with  $N$  trials and probability outcomes  $\eta = (\eta_1, \dots, \eta_N)$ :*

$$\mathbb{P}(\nu = (i_1, \dots, i_N) \mid \eta) = \frac{N!}{i_1! \dots i_N!} \eta_1^{i_1} \dots \eta_N^{i_N},$$

*where  $i_1, \dots, i_N \in \mathbb{N}$  and  $i_1 + \dots + i_N = N$ . Suppose also that  $\eta_i$  is of the form (5.3) with i.i.d.  $Y_i$ 's. Then, the following holds.*

- a. *If  $\mathbb{E}[Y_1^2] < \infty$  (in particular, if (5.4) holds and  $\alpha > 2$ ), then the processes  $(\Pi_{\lfloor t/c_N \rfloor}^{N,n}; t \geq 0)$  converge weakly as  $N \rightarrow \infty$  to the Kingman's  $n$ -coalescent. The scaling factor  $c_N$  is asymptotically equivalent to  $N$ , precisely*

$$\lim_{N \rightarrow \infty} N c_N = \frac{\mathbb{E}[Y_i^2]}{\mathbb{E}[Y_i]^2}.$$

- b. *If the  $Y_i$ 's satisfy (5.4) with  $\alpha = 2$ , then the processes  $(\Pi_{\lfloor t/c_N \rfloor}^{N,n}; t \geq 0)$  converge in the Skorokhod sense as  $N \rightarrow \infty$  to the Kingman's  $n$ -coalescent. The scaling factor  $c_N$  is asymptotically equivalent to  $N/\ln N$*

$$\lim_{N \rightarrow \infty} \frac{N c_N}{\ln N} = \frac{2}{\mathbb{E}[Y_i]^2}.$$

- c. *When (5.4) holds with  $1 \leq \alpha < 2$ , then the processes  $(\Pi_{\lfloor t/c_N \rfloor}^{N,n}; t \geq 0)$  converge in the Skorokhod sense as  $N \rightarrow \infty$  to a continuous-time process  $(\Pi_t^{\infty,n}; t \geq 0)$  that has the same law as the restriction to  $[n]$  of the  $\Lambda$ -coalescent, where  $\Lambda$  is the probability measure associated with the beta( $2 - \alpha; \alpha$ ) distribution. The transition rates are given by*

$$\lambda_{b,k} = \frac{B(k - \alpha; b - k + \alpha)}{B(2 - \alpha; \alpha)}, \quad (5.5)$$

*where  $B(c, d) = \Gamma(c)\Gamma(d)/\Gamma(c + d)$  is the beta function. The scaling factor  $c_N$  satisfies*

$$\lim_{N \rightarrow \infty} N^{\alpha-1} c_N = \frac{\alpha \Gamma(\alpha) \Gamma(2 - \alpha)}{\mathbb{E}[Y_i]^\alpha}, \quad \text{if } 1 < \alpha < 2,$$

$$\lim_{N \rightarrow \infty} c_N \ln N = 1, \quad \text{if } \alpha = 1.$$

d. When (5.4) holds with  $0 < \alpha < 1$ , then the processes  $(\Pi_t^{N,n}; t \in \mathbb{N})$  converge as  $N \rightarrow \infty$  to a discrete-time Markov chain  $(\Pi_t^{\infty,n}; t \in \mathbb{N})$  that has the same law as the restriction to  $[n]$  of a discrete-time  $\Xi_\alpha$ -coalescent. The transition probabilities are given by

$$p_{b;b_1;\dots;b_a;s} = \frac{\alpha^{a+s-1}(a+s-1)!}{(b-1)!} \cdot \prod_{i=1}^a \frac{\Gamma(b_i - \alpha)}{\Gamma(1 - \alpha)}. \quad (5.6)$$

If  $\xi_{ij}$  in (1.8) is Gumbel  $G(\rho, \beta)$ -distributed, the microscopic dynamics can be solved allowing precise calculations. In this case, see Proposition 5.3 in Section 5.2, the positions of the particles in generation  $t + 1$  can be obtained by a  $\mathcal{F}_t$ -measurable function  $\Phi(X(t))$  (that may be interpreted as the front position at time  $t$ ) and a  $\mathcal{F}_t$ -independent family of i.i.d random variables  $(\mathcal{E}_i(t+1))_{1 \leq i \leq N}$

$$X_i(t+1) = \rho + \Phi(X(t)) - \beta^{-1} \ln \mathcal{E}_i(t+1). \quad (5.7)$$

Hence, one only needs the information  $\Phi(X(t))$  from  $\mathcal{F}_t$  to generate the particle position  $X_i(t+1)$ . Then, the following weak limit for the ancestral partition process holds (Theorem 1.4 in the Introduction).

**Theorem 5.2.** *Assume that  $\xi_{ij}$  in (1.8) are Gumbel  $G(\rho, \beta)$ -distributed and that the initial position of particles  $(X_1(0), \dots, X_N(0))$  are distributed according to a probability distribution  $\mu$  on  $\mathbb{R}^N$ . Choose  $n$  particles  $e_1, \dots, e_n$  uniformly at random from the  $N$  particles in generation  $\lfloor T(\ln N) \rfloor$ . Let  $(\Pi_{\lfloor t(\ln N) \rfloor}^{N,n}; t \in [0, T])$  be the random partition of  $[n]$  such that  $i$  and  $j$  are in the same block if and only if  $e_i$  and  $e_j$  have the same ancestor in generation  $\lfloor (T-t)(\ln N) \rfloor$ .*

*Then, the processes  $(\Pi_{\lfloor t(\ln N) \rfloor}^{N,n}; t \in [0, T])$  converge weakly as  $N \rightarrow \infty$  to a continuous time process  $(\Pi_t^{\infty,n}; t \in [0, T])$  that has the same law as the restriction to  $[n]$  of the Bolthausen-Sznitman coalescent (up to time  $T^-$ ).*

**Organization of the chapter:** in Section 5.2 we study the case where the disorder  $\xi_{ij}$  is Gumbel distributed and we obtain Theorem 5.2 as an application of Theorem 5.1, that will be proved later in Section 5.3. In the end of the chapter we include two Appendix, in which we prove some technical results.

## 5.2 Relation with Brunet and Derrida's model.

In this section, we will assume that Theorem 5.1 holds and we show that when the  $\xi_{ij}$ 's are Gumbel distributed, then the family sizes  $\nu(t)$  of the model (1.8) are i.i.d. and the distribution satisfies the hypotheses of Theorem 5.1 with  $\alpha = 1$ , which implies Theorem 5.2. We bring to the reader's attention two important details.

The first one is that the time restriction in the statement of Theorem 5.2 is a necessary condition. One immediate reason is that the ancestral process is not even defined for  $t > T$ . A more subtle reason is that the partition  $\Pi_{\lfloor T(\ln N) \rfloor}^{N,n}$  depends on the

initial distribution  $X_1(0), \dots, X_N(0)$ . This dependence can be easily illustrated by the following example. One chooses an initial position of points:  $X_1(0), \dots, X_N(0)$ , for which  $X_1(0) \gg X_i(0)$ . Then, with an overwhelming probability, every individual in generation one descends from  $X_1(0)$  and

$$\Pi_{\lfloor T(\ln N) \rfloor}^{N,n} = \{(1, \dots, n)\},$$

in particular, as  $N \rightarrow \infty$  the partition  $\Pi_{\lfloor T(\ln N) \rfloor}^{N,n}$  does not converge in distribution to the  $n$ -Bolthausen-Sznitman coalescent at time  $T$ .

Secondly, we emphasize that, in the general case, the offspring vectors  $\nu(t)$  obtained from (1.8) may not be independent from generation to generation. We refer to [Cor14a] to provide a picture of a situation, in which the positions of the particles are highly related to the positions of their ancestors. It is considered the case, in which the distribution of  $\xi_{ij}$  depends on  $N$

$$\mathbb{P}(\xi_{ij} = 0) = 1 - \mathbb{P}(\xi_{ij} = -1) = 1/N^{1+r}.$$

In this model, the number of leaders  $\#\{i; X_i(t) = \max\{X_j(t)\}\}$  in generation  $t$  has a strong correlation with the number of leaders in generation  $t - 1$ . Therefore, the fitness vectors  $(\eta(t))_{t \in \mathbb{N}}$  between successive generations are correlated, and hence the offspring vectors  $\nu(t)$  are not independent (in particular (i) in page 107 does not hold).

Before proving Theorem 5.2, let us present some preliminary results and explain why the Gumbel case is particular. In [CQR13], it is shown that the particles remain grouped as  $t$  increases and that the position of the front at time  $t$  may be described by any numerical function  $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$  that is increasing for the partial order on  $\mathbb{R}^N$  and that commutes to space translations by constant vectors

$$\Phi(x + r\mathbf{1}) = r + \Phi(x), \tag{5.8}$$

where  $\mathbf{1}$  is the vector  $(1, 1, \dots, 1) \in \mathbb{R}^N$ . For a given function  $\Phi$ , we denote by  $x^0$  the vector  $x \in \mathbb{R}^N$  shifted by  $\Phi(x)$ .

$$x^0 = x - \Phi(x).$$

The authors also prove that there exists a non-random constant  $v_N$  (not depending on  $\Phi(\cdot)$ ) called speed of the front such that

$$\lim_{t \rightarrow \infty} \frac{\Phi(X(t))}{t} = v_N \quad a.s.$$

It is then clear that there is no invariant measure for  $X(t)$ . On the other hand, if we consider the shifted process  $X^0(t) := X(t) - \Phi(X(t))$ , then there exists a unique invariant measure (depending on  $\Phi(\cdot)$ ) for it. In the Gumbel case an appropriate measure of the front location is

$$\Phi(x) = \beta^{-1} \ln \sum_{i=1}^N \exp(\beta x_i). \tag{5.9}$$

In the proof of Proposition 5.3, we show that if the  $\xi$  are Gumbel  $G(\rho, \beta)$ -distributed, then  $\Phi(X(t))$  has all information needed to construct the next generation. The technique that we will present has been used in [BD04] to calculate the velocity and diffusion constant of the  $N$ -particle system. In [CQR13], the authors use the same argument to calculate explicitly the invariant measure for the process  $X^0(t)$ . It has the law of a shifted vector  $V^0 := V - \Phi(V)$  of a vector  $V$  obtained from a  $N$ -sample from a Gumbel  $G(0, \beta)$ . Summing up, when the disorder is Gumbel distributed the model is completely soluble, allowing exact computations.

**Proposition 5.3.** *Assume that  $\xi_{ij}$  in (1.8) are Gumbel  $G(\rho, \beta)$ -distributed and denote by  $\nu_i(t)$  the number of descendants of  $X_i(t)$  in generation  $t + 1$ .*

*Then, for every starting configuration  $\mu$  the family sizes*

$$\nu(t) = (\nu_1(t), \dots, \nu_N(t)), \quad t \geq 1$$

*are i.i.d. copies of  $\nu$  a doubly stochastic multinomial random variable with  $N$  trials and probability outcomes  $\eta_i$  given by*

$$\eta_i = \mathcal{E}_i^{-1} / \left( \sum_{k=1}^N \mathcal{E}_k^{-1} \right), \quad (5.10)$$

*where  $\{\mathcal{E}_i; 1 \leq i \leq N\}$  are independent and exponentially distributed with parameter 1. If  $\mu$  has the law of a shifted vector  $V^0 := V - \Phi(V)$  of a vector  $V$  obtained from a  $N$ -sample from a Gumbel  $G(0, \beta)$ , then we may take  $t \geq 0$ .*

*Proof.* Let  $\Phi(x)$  be given by (5.9), then  $\Phi(x)$  has all information one needs to construct the next generation and the process shifted by  $\Phi$ :  $X_j^0(t) = X_j(t) - \Phi(X(t))$ , are independent from generation to generation. Indeed, for  $t \geq 1$  we may write  $X_j(t)$  as follows, see [CQR13] (Theorem 3.1) and [BD04]

$$X_j(t) = \rho + \Phi(X(t-1)) - \beta^{-1} \ln \mathcal{E}_j(t), \quad (5.11)$$

where  $\mathcal{E}_j(t) := \min_{1 \leq i \leq N} \{ \exp(-\beta(\xi_{ij}(t) - \rho) - \beta X_i^0(t-1)) \}$ . Since  $\xi_{ij}(t)$  are Gumbel  $G(\rho, \beta)$ -distributed,  $\exp(-\beta(\xi_{ij}(t) - \rho))$  are exponentially distributed with parameter one. Hence, conditionally on  $\mathcal{F}_{t-1}$ ,

$$\exp(-\beta(\xi_{ij}(t) - \rho) - \beta X_i^0(t-1)); \quad 1 \leq i \leq N$$

are independent and  $\exp(-\beta(\xi_{ij}(t) - \rho) - \beta X_i^0(t-1))$  is distributed according to an exponential random variable with parameter  $\exp(\beta X_i^0(t-1))$ . Applying the stability property of the exponential law under independent minimum, we obtain that conditionally on  $\mathcal{F}_{t-1}$  each variable  $\mathcal{E}_i(t)$  is exponentially distributed with parameter one and, moreover, that the whole vector  $\mathcal{E}(t) := (\mathcal{E}_i(t), i \leq N)$  is conditionally independent. Therefore, the vector  $\mathcal{E}(t)$  is independent from  $\mathcal{F}_{t-1}$  and its coordinates  $\mathcal{E}_i(t)$ ,  $1 \leq i \leq N$  are i.i.d. having an exponential law with parameter one. Using once

again the stability property of the exponential law under independent minimum,

$$\begin{aligned}
 \eta_i(t) &:= \mathbb{P} \left( \xi_{ij}(t+1) + X_i(t) > \xi_{kj} + X_k(t), \text{ for every } k \neq i \mid \mathcal{F}_t \right) \\
 &= \mathbb{P} \left( e^{-\beta(\xi_{ij}(t+1)-\rho)} e^{-\beta X_i(t)} < \min_{k \neq i} e^{-\beta(\xi_{kj}(t+1)-\rho)} e^{-\beta X_k(t)} \mid \mathcal{F}_t \right) \\
 &= \exp(\beta X_i(t)) \Big/ \left( \sum_{k=1}^N \exp(\beta X_k(t)) \right). \tag{5.12}
 \end{aligned}$$

Then, from (5.11) we obtain that

$$\eta_i(t) = \mathcal{E}_i^{-1}(t) \Big/ \left( \sum_{k=1}^N \mathcal{E}_k^{-1}(t) \right), \tag{5.13}$$

which proves (5.10). In particular, the family sizes  $\nu(1), \nu(2), \dots$  have the same distribution. If at  $t = 0$  the particles are distributed according to the invariant measure the same argument holds and  $\nu(t), t \geq 0$  have the same distribution.

We now prove that the  $\nu(t)$ 's are independent. It suffices to show that

$$\mathbb{E} [f_1(\nu(1)) \dots f_{t+1}(\nu(t+1))] = \mathbb{E} [f_1(\nu(1)) \dots f_t(\nu(t))] \mathbb{E} [f_{t+1}(\nu(t+1))], \tag{5.14}$$

for all continuous bounded functions  $f_1(\cdot), \dots, f_t(\cdot), f_{t+1}(\cdot)$ . Let  $A_{i,j;t}$  be the event

$$A_{i,j;t} = \left\{ \xi_{ji}(t+1) + X_j(t) > \max_{k \neq i} \{ \xi_{ki}(t+1) + X_k(t) \} \right\}$$

that  $X_i(t+1)$  descends from  $X_j(t)$ . Denote by  $\mathcal{G}_t$  the  $\sigma$ -algebra generated by  $\mathcal{F}_t$  and  $A_{i,j;t}$  for every  $1 \leq i, j \leq N$ , then  $\nu(1), \dots, \nu(t)$  are  $\mathcal{G}_t$ -measurable. We claim that  $\nu(t+1)$  is independent from  $\mathcal{G}_t$ , which proves (5.14). Since  $\nu(t+1)$  is completely determined by  $\{ \mathcal{E}_k(t+1), 1 \leq k \leq N \}$  and  $\{ \xi_{kl}(t+2), 1 \leq k, l \leq N \}$ , it is immediate that it is independent from  $\mathcal{F}_t$ . Hence, we prove the claim once we show that  $\nu(t+1)$  and  $A_{i,j;t}$  are independent for every  $1 \leq i, j \leq N$ . Since

$$A_{i,j;t} \in \sigma \{ \mathcal{F}_t; \{ \xi_{ki}(t+1); 1 \leq k \leq N \} \} \subset \mathcal{F}_{t+1},$$

it suffices to show that  $A_{i,j;t}$  is independent from  $\sigma \{ \mathcal{E}_k(t+1), 1 \leq k \leq N \}$ . It is not hard to show that  $\mathcal{E}_k(t+1)$  and  $A_{i,j;t}$  are independent, whenever  $k \neq i$  and we leave the details to the reader. Let  $g(\cdot)$  be a bounded continuous function. Conditionally on  $\mathcal{F}_t$ ,  $\mathcal{E}_i(t+1)$  is the minimum of  $N$  independent random variables exponentially distributed with parameters  $\exp(\beta X_k^0(t-1))$  and the set  $A_{i,j;t}$  is the event that the minimum is attained by  $\exp(-\beta(\xi_{ji}(t) - \rho) - \beta X_j^0(t))$ . Then, using standard properties of exponential distributions, we obtain

$$\begin{aligned}
 \mathbb{E} [g(\mathcal{E}_i(t+1)) \mathbf{1}_{A_{i,j;t}} \mid \mathcal{F}_t] &= \mathbb{P}(A_{i,j;t} \mid \mathcal{F}_t) \int_{\mathbb{R}_+} g(y) \cdot \frac{\exp\left(-y \sum e^{\beta X_k^0(t-1)}\right)}{\sum e^{\beta X_k^0(t-1)}} \cdot dy \\
 &= \mathbb{P}(A_{i,j;t} \mid \mathcal{F}_t) \int_{\mathbb{R}_+} dy g(y) \exp(-y).
 \end{aligned}$$

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We used that  $X^0$  is the process shifted by  $\Phi$ , which satisfies  $\sum e^{\beta X_k^0(t-1)} = 1$ . Then,  $\mathcal{E}_i(t+1)$  and  $A_{i,j;t}$  are independent, which proves the claim and therefore the proposition.  $\square$

*Proof of Theorem 5.2.* By Proposition 5.3, the family sizes  $\nu(t)$  are independent and identically distributed for  $t \geq 1$  (and  $t \geq 0$  if the initial position of particles is distributed according to the invariant measure). Furthermore, it is easy to compute the tail distribution of  $\mathcal{E}_i^{-1}(t)$

$$\mathbb{P}(\mathcal{E}_i^{-1}(t) \geq x) = 1 - e^{-x^{-1}} \sim 1/x, \quad x \rightarrow \infty,$$

where “ $\sim$ ” means that the ratio of the sides approaches to one as  $x \rightarrow \infty$ , so (5.4) holds with  $\alpha = 1$ .

If  $T_0 < T$  and  $N$  is sufficient large such that  $(T - T_0)(\ln N) \geq 1$ , then the family sizes  $\nu(t)$ ,  $t \in \{[(T - T_0)(\ln N)], \dots, [T(\ln N)]\}$  are i.i.d. It is then possible to apply Theorem 5.1 with  $\alpha = 1$ , which concludes the proof.  $\square$

## 5.3 Proof of Theorem 5.1.

The proof of Theorem 5.1 will be divided in two main parts. In the first one, we focus on the case where  $Y_1$  has finite second moment, which generalize  $\alpha > 2$  in (5.4). The proof of the first part of Theorem 5.1 is an adaptation of the proof of part (a) of Theorem 4 in [Sch03]. In the second part, we prove Theorem 5.1 for  $\alpha \leq 2$ . We do so by studying the Laplace transform of  $Y_i$  and its derivatives.

Before proving Theorem 5.1, we prove a general statement about multinomial distributions. In the next lemma, we will denote by  $\nu$  a  $N$ -class multinomial random variable with  $N$  trials and by  $\eta_i$  the probability outcomes, that are not necessarily  $N$ -exchangeable.

**Lemma 5.4.** *Let  $\nu = (\nu_1, \dots, \nu_N)$  be a doubly stochastic multinomial random variable with probability outcomes  $\eta_1, \dots, \eta_N$ . Let also  $b_1 \geq \dots \geq b_a \geq 1$  and  $b = b_1 + \dots + b_a$  (we also assume that  $b \leq N$ ). Then,*

$$\mathbb{E}\left[(\nu_1)_{b_1} \dots (\nu_a)_{b_a}\right] = (N)_b \mathbb{E}\left[\eta_1^{b_1} \dots \eta_a^{b_a}\right]. \quad (5.15)$$

*Proof.* To simplify the notation, we assume that  $\eta_1, \dots, \eta_N$  are non-random. Then,  $\nu$  is distributed according to a standard multinomial distribution.

$$\begin{aligned} & \mathbb{E}\left[(\nu_1)_{b_1} \dots (\nu_a)_{b_a}\right] \\ &= \sum_{\substack{i_j \geq b_j \\ i_1 + \dots + i_a \leq N}} \frac{N! \eta_1^{i_1} \dots \eta_a^{i_a} (1 - \eta_{1, \dots, a})^{N - i_1 - \dots - i_a}}{i_1! \dots i_a! (N - i_1 - \dots - i_a)!} \cdot \frac{i_1!}{(i_1 - b_1)!} \dots \frac{i_a!}{(i_a - b_a)!}, \end{aligned} \quad (5.16)$$

where  $i_{1,\dots,a} := i_1 + \dots + i_a$  and  $\eta_{1,\dots,a} := \eta_1 + \dots + \eta_a$ . By a change of variables  $k_j = i_j - b_j$  we rewrite (5.16)

$$\begin{aligned} & \sum_{k_1+\dots+k_a \leq N-b} \frac{N!}{k_1! \dots k_a! (N-b-k_{1,\dots,a})!} \cdot \eta_1^{k_1+b_1} \dots \eta_a^{k_a+b_a} (1-\eta_{1,\dots,a})^{N-b-k_{1,\dots,a}} \\ &= (N)_b \eta_1^{b_1} \dots \eta_a^{b_a} \sum \frac{(N-b)!}{k_1! \dots k_a! (N-b-k_{1,\dots,a})!} \cdot \eta_1^{k_1} \dots \eta_a^{k_a} (1-\eta_{1,\dots,a})^{N-b-k_{1,\dots,a}} \\ &= (N)_b \eta_1^{b_1} \dots \eta_a^{b_a} (\eta_1 + \dots + \eta_a + (1-\eta_{1,\dots,a}))^{N-b}, \end{aligned}$$

proving the result in the non-random case. The random case is obtained by conditioning on  $\sigma\{\eta_1, \dots, \eta_N\}$ .  $\square$

### 5.3.1 Convergence to Kingman's coalescent $\mathbb{E}[Y_1^2] < \infty$ .

In [Möh00], Möhle shows that if the family sizes are not “too large” the processes  $\Pi_{\lfloor t/c_N \rfloor}^{N,n}$  converge to the Kingman's  $n$ -coalescent.

**Proposition 5.5** (Möhle [Möh00]). *Suppose that*

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_i)_3]}{N^2 c_N} = 0. \quad (5.17)$$

*Then, as  $N \rightarrow \infty$ , the processes  $\Pi_{\lfloor t/c_N \rfloor}^{N,n}$  converge to the Kingman's  $n$ -coalescent.*

We will use Proposition 5.5 to prove Theorem 5.1 in the case where the  $Y_i$ 's are square integrable. We first estimate  $c_N$ , the probability that two individuals have a common ancestor one generation backwards in time.

**Lemma 5.6.** *Assume that the hypotheses of Theorem 5.1 hold with  $\mathbb{E}[Y_1^2] < \infty$  and let  $c_N$  be as in (2.12). Then,*

$$\lim_{N \rightarrow \infty} N c_N = \frac{\mathbb{E}[Y_1^2]}{\mathbb{E}[Y_1]^2}. \quad (5.18)$$

*Proof.* From Lemma 5.15, we obtain that

$$N c_N = N^2 \mathbb{E}[\eta_1^2].$$

Let  $\delta_1 > 0$ , then by definition of  $\eta_1$

$$N^2 \mathbb{E}[\eta_1^2] = \mathbb{E} \left[ \frac{Y_1^2}{\left(N^{-1} \sum_{j=1}^N Y_j\right)^2} \right] \geq \mathbb{E} \left[ \frac{Y_1^2}{\delta_1 + \left(N^{-1} \sum_{j=1}^N Y_j\right)^2} \right]. \quad (5.19)$$

Since  $Y_1 > 0$ , we use dominated convergence in (5.19) to obtain that

$$\liminf_{N \rightarrow \infty} N c_N \geq \frac{\mathbb{E}[Y_1^2]}{\delta_1 + (\mathbb{E}[Y_1])^2}.$$

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The inequality holds for every  $\delta_1$  positive, which implies that the above  $\liminf$  is larger than  $\mathbb{E}[Y_1^2]/\mathbb{E}[Y_1]^2$ . We now obtain an upper bound for the  $\limsup$ . We use the Markov inequality to obtain that for all  $c > 0$

$$\lim_{x \rightarrow \infty} x^2 \mathbb{P}(Y_1 \geq cx) = 0. \quad (5.20)$$

Let  $S_{2,N} = \sum_{i=2}^N Y_i$  and take  $0 < \delta_2 < \mathbb{E}[Y_1]$  sufficiently small such that

$$\frac{\mathbb{E}[Y_1^2]}{(\mathbb{E}[Y_1] - \delta_2)^2} \leq \frac{\mathbb{E}[Y_1^2]}{\mathbb{E}[Y_1]^2} + \varepsilon/3, \quad (5.21)$$

for a fixed  $\varepsilon > 0$ . Then, we write

$$\begin{aligned} N^2 \mathbb{E} [\eta_1^2] &= \mathbb{E} \left[ \frac{Y_1^2}{(N^{-1}Y_1 + N^{-1}S_{2,N})^2}; S_{2,N} \geq N(\mathbb{E}[Y_1] - \delta_2) \right] \\ &\quad + \mathbb{E} \left[ \frac{Y_1^2}{(N^{-1}Y_1 + N^{-1}S_{2,N})^2}; S_{2,N} \leq N(\mathbb{E}[Y_1] - \delta_2) \right] \\ &= \text{(I)} + \text{(II)}. \end{aligned} \quad (5.22)$$

Since  $Y_i > 0$ , we may bound (II) in (5.22) as follows

$$\begin{aligned} \text{(II)} &\leq \mathbb{E} \left[ \frac{Y_1^2}{(N^{-1}Y_1)^2}; S_{2,N} \leq N(\mathbb{E}[Y_1] - \delta_2) \right] \\ &= N^2 \mathbb{P}(S_{2,N} \leq N(\mathbb{E}[Y_1] - \delta_2)). \end{aligned}$$

So we apply Chernoff inequality to conclude that if  $\delta_2$  is fixed and  $N$  sufficiently large, then (II) is smaller than  $\varepsilon/3$ .

$$\begin{aligned} \text{(I)} &\leq \mathbb{E} \left[ \frac{Y_1^2}{(\mathbb{E}[Y_1] - \delta_2)^2}; Y_1 \leq N(\mathbb{E}[Y_1] - \delta_2) \right] + N^2 \mathbb{P}(Y_1 \geq N(\mathbb{E}[Y_1] - \delta_2)) \\ &\leq \mathbb{E} \left[ \frac{Y_1^2}{(\mathbb{E}[Y_1] - \delta_2)^2} \right] + N^2 \mathbb{P}(Y_1 \geq N(\mathbb{E}[Y_1] - \delta_2)). \end{aligned}$$

From (5.20) with  $c = \mathbb{E}[Y_1] - \delta_2$ , the second term in the right-hand side converges to zero as  $N \rightarrow \infty$ , and we may choose  $N$  conveniently such that it is smaller than  $\varepsilon/3$ . It is implied that  $N$  is taken such that (II) is also smaller than  $\varepsilon/3$ . Then, applying the upper bounds in (5.22) we obtain

$$N^2 \mathbb{E} [\eta_1^2] \leq \frac{\mathbb{E}[Y_1^2]}{(\mathbb{E}[Y_1] - \delta_2)^2} + \frac{2}{3} \cdot \varepsilon < \frac{\mathbb{E}[Y_1^2]}{\mathbb{E}[Y_1]^2} + \varepsilon.$$

Since the inequality holds for every  $\varepsilon > 0$  and  $N$  large enough, we conclude that  $\limsup Nc_N \leq \mathbb{E}[Y_1^2]/\mathbb{E}[Y_1]^2$  proving the lemma.  $\square$

*Proof of Theorem 5.1 in the case  $\mathbb{E}[Y_1^2] < \infty$ .* In order to prove Theorem 5.1, it suffices to show that (5.17) holds and apply Proposition 5.5. From Lemma 5.6, there exists a constant  $c < 1$  such that for  $N$  sufficiently large  $Nc_N > c\mathbb{E}[Y_1^2]/\mathbb{E}[Y_1]^2$ , hence

$$0 \leq \frac{\mathbb{E}[(\nu_1)_3]}{N^2c_N} \leq \frac{\mathbb{E}[(\nu_1)_3]}{N} \cdot \frac{\mathbb{E}[Y_1]^2}{c\mathbb{E}[Y_1^2]}.$$

Then, to prove the convergence in (5.17), it suffices to show that  $N^{-1}\mathbb{E}[(\nu_1)_3] \rightarrow 0$ . From (5.15), it is equivalent to  $N^2\mathbb{E}[\eta_1^3] \rightarrow 0$  as  $N \rightarrow \infty$ . We proceed as in (5.22) and obtain

$$\begin{aligned} N^2\mathbb{E}[\eta_1^3] &= N^2\mathbb{E}\left[\frac{Y_1^3}{(Y_1 + S_{2,N})^3}; S_{2,N} \geq N(\mathbb{E}[Y_1] - \delta_2)\right] \\ &\quad + N^2\mathbb{E}\left[\frac{Y_1^3}{(Y_1 + S_{2,N})^3}; S_{2,N} \leq N(\mathbb{E}[Y_1] - \delta_2)\right] \\ &= \text{(I)} + \text{(II)}. \end{aligned} \tag{5.23}$$

Applying the same argument of Lemma 5.6, we conclude that (II) converges to zero as  $N$  diverges and we also obtain the following upper bound to (I)

$$\text{(I)} \leq N^2\mathbb{E}\left[\frac{Y_1^3}{(N(\mathbb{E}[Y_1] - \delta_2))^3}; Y_1 \leq N(\mathbb{E}[Y_1] - \delta_2)\right] + N^2\mathbb{P}(Y_1 \geq N(\mathbb{E}[Y_1] - \delta_2)). \tag{5.24}$$

We use the Markov inequality to show that the second term in the right-hand side of (5.24) converges to zero as  $N \rightarrow \infty$ . As a consequence, to finish the proof it suffices to show that the first term in the right-hand side of (5.24) converges to zero as  $N \rightarrow \infty$ . For  $\varepsilon > 0$  let  $L \in \mathbb{R}_+$  be such that

$$\mathbb{E}[Y_1^2; Y_1 \geq L]/(\mathbb{E}[Y_1] - \delta_2)^2 < \varepsilon/2.$$

Since  $L$ ,  $\delta_2$  and  $\varepsilon$  are fixed we may choose  $N$  sufficiently large such that

$$\frac{L\mathbb{E}[Y_1^2]}{N(\mathbb{E}[Y_1] - \delta_2)^3} < \varepsilon/2,$$

and we bound the first term in the right-hand side of (5.24)

$$\begin{aligned} &N^2\mathbb{E}\left[\frac{Y_1^3}{(N\mathbb{E}[Y_1] - \delta_2)^3}; Y_1 \leq N(\mathbb{E}[Y_1] - \delta_2)\right] \\ &\leq \frac{L}{N(\mathbb{E}[Y_1] - \delta_2)^3} \cdot \mathbb{E}[Y_1^2; Y_1 \leq L] + \frac{\mathbb{E}[Y_1^2; L \leq Y_1 \leq N(\mathbb{E}[Y_1] - \delta_2)]}{(\mathbb{E}[Y_1] - \delta_2)^2} \\ &\leq \frac{L}{N(\mathbb{E}[Y_1] - \delta_2)^3} \cdot \mathbb{E}[Y_1^2] + \frac{\mathbb{E}[Y_1^2 \mathbf{1}_{\{Y_1 \geq L\}}]}{(\mathbb{E}[Y_1] - \delta_2)^2} < \varepsilon, \end{aligned} \tag{5.25}$$

that finishes the proof.  $\square$

### 5.3.2 Proof of Theorem 5.1 when $\alpha \leq 2$ .

The strategy to prove Theorem 5.1 in the case  $\alpha \leq 2$  is to compute the limits (2.13) and apply Theorem 2.8. In the next proposition, we show how the moments of  $\eta_i$ 's are related to the Laplace transform of  $Y_i$ .

**Proposition 5.7.** *Let  $b_1 \geq b_2 \geq \dots \geq b_a \geq 2$  be positive integers,  $b = b_1 + \dots + b_a$  and for  $1 \leq i \leq N$*

$$\eta_i := \frac{Y_i}{\sum_{j=1}^N Y_j},$$

where  $Y_1, \dots, Y_N$  are i.i.d. random variables. Then,

$$\mathbb{E} [\eta_1^{b_1} \dots \eta_a^{b_a}] = \frac{1}{\Gamma(b)} \int_0^\infty u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du, \quad (5.26)$$

where  $\Gamma(\cdot)$  is the gamma function and

$$I_p(u) = \mathbb{E} [Y_1^p e^{-uY_1}], \quad p \in \mathbb{N}. \quad (5.27)$$

*Proof.* For every  $z \in \mathbb{R}_+^*$  we have the following integral representation

$$z^{-b} = \frac{1}{\Gamma(b)} \int_0^\infty u^{b-1} e^{-uz} du, \quad (5.28)$$

then applying (5.28) with  $z = \sum_{i=1}^N Y_i$  we obtain

$$\begin{aligned} \mathbb{E} [\eta_1^{b_1} \dots \eta_a^{b_a}] &= \mathbb{E} \left[ Y_1^{b_1} \dots Y_a^{b_a} \frac{1}{\Gamma(b)} \int_0^\infty u^{b-1} e^{-u \sum_{i=1}^N Y_i} du \right] \\ &= \int_0^\infty \frac{u^{b-1}}{\Gamma(b)} \mathbb{E} \left[ Y_1^{b_1} \dots Y_a^{b_a} e^{-u \sum_{i=1}^N Y_i} du \right] \quad (\text{Fubini}) \\ &= \int_0^\infty \frac{u^{b-1}}{\Gamma(b)} \mathbb{E} \left[ \exp(-uY_1) \right]^{N-a} \prod_{i=1}^a \mathbb{E} [Y_1^{b_i} \exp(-uY_1)] du. \quad (5.29) \end{aligned}$$

In the last equality, we used the fact that  $Y_i$  are i.i.d. Hence, from the definition of  $I_{b_i}$  we obtain that (5.29) and (5.26) are equal, proving the result.  $\square$

It is clear that the functions  $I_p(u)$  are decreasing and attain their maximum at zero. Moreover, the following relation can be easily deduced

$$\frac{d^p}{du^p} I_0(u) = (-1)^p I_p(u).$$

We now outline the strategy to prove Theorem 5.1.

- i We first obtain a precise asymptotic of  $I_p(u)$  in the neighbourhood of zero, where  $I_p(u)$  attains its maximum. As the reader will see, the behaviour of  $I_p(u)$  depends on  $\alpha$  and each case will be studied separately.

- ii We show that the integral in the right-hand side of (5.26) is essentially determined by the immediate neighbourhood of zero.
- iii We estimate  $\mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}]$ .
- iv We prove Theorem 5.1 using Lemma 5.4 that relates (2.13) with  $\mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}]$ .

**Lemma 5.8.** *Let  $I(u)$  be given by (5.27).*

a. *If  $Y_i$  satisfies (5.4) with  $\alpha = 2$  and  $C = 1$ . Then,*

$$\begin{aligned} I_0(u) &= 1 - u\mathbb{E}[Y_1] + o(u), & \text{when } u \rightarrow 0^+; \\ I_2(u) &= (-2 \ln u) + o(\ln(u^{-1})), & \text{when } u \rightarrow 0^+; \\ I_p(u) &= u^{2-p}(2\Gamma(p-2)) + o(u^{2-p}), & \text{when } p \geq 3 \text{ and } u \rightarrow 0^+. \end{aligned}$$

b. *When  $Y_i$  satisfies (5.4) with  $1 < \alpha < 2$  and  $C = 1$ . Then,*

$$\begin{aligned} I_0(u) &= 1 - u\mathbb{E}[Y_1] + o(u), & \text{when } u \rightarrow 0^+; \\ I_p(u) &= u^{\alpha-p}(\alpha \Gamma(p-\alpha)) + o(u^{\alpha-p}), & \text{when } p \geq 2 \text{ and } u \rightarrow 0^+. \end{aligned}$$

c. *If (5.4) holds with  $\alpha = 1$  and  $C = 1$ . Then,*

$$\begin{aligned} I_0(u) &= 1 + (u \ln u) + o(u \ln u), & \text{when } u \rightarrow 0^+; \\ I_p(u) &= u^{1-p}\Gamma(p-1) + o(u^{1-p}), & \text{when } p \geq 2 \text{ and } u \rightarrow 0^+. \end{aligned}$$

d. *Assume that  $Y_i$  satisfies (5.4) with  $0 < \alpha < 1$  and  $C = 1$ . Then,*

$$\begin{aligned} I_0(u) &= 1 - u^\alpha \Gamma(1-\alpha) + o(u^\alpha), & \text{when } u \rightarrow 0^+; \\ I_p(u) &= u^{\alpha-p}(\alpha \Gamma(p-\alpha)) + o(u^{\alpha-p}), & \text{when } p \geq 2 \text{ and } u \rightarrow 0^+. \end{aligned}$$

*Proof.* See Appendix A. □

In the next lemma, we show that only the immediate neighborhood of zero contributes to the integral in (5.26) of Proposition 5.7.

**Lemma 5.9.** *Let  $I(u)$  be given by (5.27) and  $\kappa_N := (\ln N)^2/N$ , assume also that  $Y_i$  satisfies (5.4) with  $\alpha \leq 2$  and  $C = 1$ . Then, for every  $K \in \mathbb{N}$*

$$\lim_{N \rightarrow \infty} N^K \int_{\kappa_N}^{\infty} u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du = 0, \quad (5.30)$$

where  $b_1 \geq \dots \geq b_a$  are fixed integers and  $b = b_1 + \dots + b_a$ . Hence, the integral in (5.30) decreases faster than any polynomial in  $N$ .

*Proof.* Since  $I_0$  is a decreasing function

$$\begin{aligned}
 & \int_{\kappa_N}^{\infty} u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du \\
 & \leq I_0(\kappa_N)^{N-a} \int_{\kappa_N}^{\infty} u^{b-1} I_{b_1}(u) \dots I_{b_a}(u) du \\
 & \leq I_0(\kappa_N)^{N-a} \int_0^{\infty} u^{b-1} \mathbb{E}[Y_1^{b_1} e^{-uY_1}] \dots \mathbb{E}[Y_a^{b_a} e^{-uY_a}] du \\
 & = I_0(\kappa_N)^{N-a} \Gamma(b) \mathbb{E} \left[ \frac{Y_1^{b_1} \dots Y_a^{b_a}}{(\sum_{i=1}^a Y_i)^b} \right]. \tag{5.31}
 \end{aligned}$$

In the last equality, we proceed as in Proposition 5.7 and use the integral representation (5.28) with  $z = \sum_{i=1}^a Y_i$ . The expected value in the right-hand side of (5.31) is bounded from above by one. Applying Lemma 5.8 with  $u = \kappa_N \rightarrow 0^+$  as  $N \rightarrow \infty$

$$\begin{aligned}
 I_0(\kappa_N)^{N-a} &= \exp \left\{ -\mathbb{E}[Y_i](\ln N)^2 + o(\ln^2 N) \right\}, & \text{if } 1 < \alpha \leq 2; \\
 I_0(\kappa_N)^{N-a} &= \exp \left\{ -(\ln N)^3 + (\ln N)^2(\ln 2 \ln N) + o(\ln^3 N) \right\}, & \text{if } \alpha = 1; \\
 I_0(\kappa_N)^{N-a} &= \exp \left\{ -\Gamma(1-\alpha)N^{1-\alpha}(\ln N)^{2\alpha} + o(N^{1-\alpha}(\ln N)^{2\alpha}) \right\}, & \text{if } 0 < \alpha < 1;
 \end{aligned}$$

that decreases faster than any polynomial in  $N$ .  $\square$

The  $\kappa_N$  in Lemma 5.9 is not optimal. The reason we have chosen such  $\kappa_N$  will be clear in the proof of Proposition 5.10 below, where we estimate  $\mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}]$ .

**Proposition 5.10.** *Let  $b_1 \geq b_2 \geq \dots \geq b_a \geq 2$  be positive integers,  $b = b_1 + \dots + b_a$ , and  $\eta_i$  be as in Proposition 5.7.*

a. *Suppose  $Y_i$  satisfies (5.4) with  $\alpha = 2$  and  $C = 1$ . Let  $g := \max\{i; b_i \geq 3\}$ , we adopt the convention that  $\max\{\emptyset\} = 0$ . Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}] \cdot \frac{N^{2a}}{(\ln N)^{a-g}} = \Gamma(2a) \cdot \frac{2^a \prod_{i=1}^g \Gamma(b_i - 2)}{\Gamma(b) \mathbb{E}[Y_1]^{2a}}. \tag{5.32}$$

b. *If (5.4) holds with  $1 < \alpha < 2$  and  $C = 1$ . Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}] N^{a\alpha} = \Gamma(a\alpha) \cdot \frac{\prod_{i=1}^a \alpha \Gamma(b_i - \alpha)}{\Gamma(b) \mathbb{E}[Y_1]^{a\alpha}}. \tag{5.33}$$

c. *If we assume that  $Y_i$  satisfies (5.4) with  $\alpha = 1$  and  $C = 1$ . Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}] (N \ln N)^a = \Gamma(a) \cdot \frac{\prod_{i=1}^a \Gamma(b_i - 1)}{\Gamma(b)}. \tag{5.34}$$

d. *If (5.4) holds with  $0 < \alpha < 1$  and  $C = 1$ . Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E}[\eta_1^{b_1} \dots \eta_a^{b_a}] N^a = \Gamma(a) \cdot \frac{\alpha^{a-1} \prod_{i=1}^a \Gamma(b_i - \alpha)}{\Gamma(1-\alpha)^a \Gamma(b)}. \tag{5.35}$$

*Proof.* See Appendix B. □

We now compute  $c_N$  the probability that two individuals randomly chosen have the same ancestor.

**Corollary 5.11.** *Assume that the hypotheses of Theorem 5.1 hold and let  $c_N$  be as in (2.12). Assume also that the  $Y_i$ 's satisfy (5.4) with  $\alpha \leq 2$  and  $C = 1$ . Then,*

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{N c_N}{\ln N} &= \frac{2}{\mathbb{E}[Y_1]^2}, & \text{if } \alpha = 2; \\ \lim_{N \rightarrow \infty} \frac{c_N}{N^{1-\alpha}} &= \frac{\alpha \Gamma(\alpha) \Gamma(2-\alpha)}{\mathbb{E}[Y_1]^\alpha}, & \text{if } 1 < \alpha < 2; \\ \lim_{N \rightarrow \infty} (\ln N) c_N &= 1, & \text{if } \alpha = 1. \end{aligned} \tag{5.36}$$

Finally, if  $Y_i$  satisfies (5.4) with  $0 < \alpha < 1$  and  $C = 1$ , then,

$$\lim_{N \rightarrow \infty} c_N = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}. \tag{5.37}$$

*Proof.* It is a direct application of Lemma 5.4 and Proposition 5.10. □

*Proof of Theorem 5.1 in the cases  $\alpha \leq 2$ .* We analyse each case separately and compute the limits

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_1)_{b_1} \dots (\nu_a)_{b_a}]}{N^{b-a} c_N}.$$

If  $\mathbb{P}(Y_i \geq x) \sim x^{-2}$  as  $x \rightarrow \infty$ , denote by  $g = \max\{i; b_i \geq 3\}$  (as in Proposition 5.10). Then, as  $N \rightarrow \infty$

$$\begin{aligned} & \frac{\mathbb{E}[(\nu_1)_{b_1} \dots (\nu_a)_{b_a}]}{N^{b-a} c_N} \\ &= \frac{(N)_b}{N^{b-a} c_N} \cdot \mathbb{E}[\eta^{b_1} \dots \eta^{b_a}] && \text{(Lemma 5.4)} \\ &\sim N^a \frac{N}{\ln N} \cdot \frac{\mathbb{E}[Y_1]^2}{2} \cdot \mathbb{E}[\eta^{b_1} \dots \eta^{b_a}] && \text{(Corollary 5.11)} \\ &\sim \frac{N^{a+1}}{\ln N} \cdot \frac{\mathbb{E}[Y_1]^2}{2} \cdot \frac{(\ln N)^{a-g}}{N^{2a}} \cdot \Gamma(2a) \cdot \frac{2^a \prod_{i=1}^g \Gamma(b_i - 2)}{\Gamma(b) \mathbb{E}[Y_1]^{2a}} && \text{(Proposition 5.10)} \\ &= \frac{(\ln N)^{a-g-1}}{N^{a-1}} \cdot \Gamma(2a) \cdot \frac{2^{a-1} \prod_{i=1}^g \Gamma(b_i - 2)}{\Gamma(b) \mathbb{E}[Y_1]^{2(a-1)}}, \end{aligned}$$

which converges to zero whenever  $a \geq 2$ . If  $a = 1 = g$ , which implies  $b_a = b \geq 3$ , then

$$\frac{\mathbb{E}[(\nu_1)_{b_1}]}{N^{b-1} c_N} \sim \frac{1}{\ln N} \cdot \frac{\Gamma(b-2)}{\Gamma(b) \mathbb{E}[Y_1]} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

On the other hand, if  $a = 1$  and  $g = 0$ , i.e.  $b = 2$ , then

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_1)_2]}{N^{2-1} c_N} = 1.$$

### 5.3. A toy model

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Hence, in the scaling limit we may only observe collisions of two distinct blocks that do not occur simultaneously, *i.e.* Kingman's coalescent.

*In the case  $1 < \alpha < 2$  we proceed as above obtaining*

$$\begin{aligned} & \frac{\mathbb{E}[(\nu_1)_{b_1} \cdots (\nu_a)_{b_a}]}{N^{b-a} c_N} \\ & \sim \frac{\Gamma(\alpha a)}{N^{(a-1)(\alpha-1)}} \cdot \frac{\mathbb{E}[Y_1]^\alpha}{\alpha \Gamma(\alpha) \Gamma(2-\alpha)} \cdot \frac{\prod_{i=1}^a \alpha \Gamma(b_i - \alpha)}{\Gamma(b) \mathbb{E}[Y_1]^{\alpha a}}, \quad \text{as } N \rightarrow \infty, \end{aligned}$$

that converges to zero whenever  $a \geq 2$ . If  $a = 1$  and *a fortiori*  $b_a = b$

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_1)_b]}{N^{b-1} c_N} &= \frac{\Gamma(b-\alpha)}{\Gamma(b) \Gamma(2-\alpha)} \\ &= \frac{(b-1-\alpha) \cdots (2-\alpha)}{(b-1)!} \\ &= \frac{B(b-\alpha, \alpha)}{B(2-\alpha, \alpha)} = \lambda_{b;b}, \end{aligned}$$

where  $B(c, d) = \Gamma(c)\Gamma(d)/\Gamma(c+d)$ , as defined in Theorem 5.1. Hence, using the recursive formula (2.10) for  $\lambda_{b;k}$

$$\begin{aligned} \lambda_{b;b-1;1} &= \lambda_{b-1,b-1} - \lambda_{b;b} \\ &= \frac{\Gamma(b-1-\alpha)}{\Gamma(b-1)\Gamma(2-\alpha)} - \frac{\Gamma(b-\alpha)}{\Gamma(b)\Gamma(2-\alpha)} \\ &= \frac{\alpha}{b-1} \cdot \frac{\Gamma(b-1-\alpha)}{\Gamma(b-1)\Gamma(2-\alpha)} \\ &= \frac{B(b-1-\alpha, 1+\alpha)}{B(2-\alpha, \alpha)} = \lambda_{b;b-1}. \end{aligned}$$

We may proceed by recurrence and conclude the convergence to the Beta-coalescent.

*In the case  $\alpha = 1$ , we have that*

$$\frac{\mathbb{E}[(\nu_1)_{b_1} \cdots (\nu_a)_{b_a}]}{N^{b-a} c_N} \sim \frac{\Gamma(a)}{(\ln N)^{a-1}} \cdot \frac{\prod_{i=1}^a \Gamma(b_i - 1)}{\Gamma(b)}, \quad \text{as } N \rightarrow \infty,$$

that converges to zero whenever  $a \geq 2$ , implying that we do not observe simultaneous collisions in the time scale. If  $a = 1$  and *a fortiori*  $b_a = b$

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\mathbb{E}[(\nu_1)_b]}{N^{b-1} c_N} &= \frac{\Gamma(b-1)}{\Gamma(b)} \\ &= \frac{1}{b-1} = \int_{[0,1]} x^{b-2} dx. \end{aligned}$$

Hence, using the recursive formula (2.10) for  $\lambda_{b;k}$ , we can conclude the convergence in distribution to the Bolthausen-Sznitman coalescent.

When  $\alpha < 1$ , by Corollary 5.11  $\lim c_N > 0$ . Then, as  $N \rightarrow \infty$

$$\begin{aligned}
\frac{\mathbb{E}[(\nu_1)_{b_1} \cdots (\nu_a)_{b_a}]}{N^{b-a}} &= \frac{(N)_b}{N^{b-a}} \cdot \mathbb{E}[\eta^{b_1} \cdots \eta^{b_a}] && \text{(Lemma 5.4)} \\
&\sim \Gamma(a) \cdot \frac{\alpha^{a-1} \prod_{i=1}^a \Gamma(b_i - \alpha)}{\Gamma(1 - \alpha)^a \Gamma(b)} && \text{(Proposition 5.10)} \\
&= \frac{\alpha^{a-1} (a-1)!}{(b-1)!} \cdot \prod \frac{\Gamma(b_i - \alpha)}{\Gamma(1 - \alpha)} \\
&= \frac{\alpha^{a-1} (a-1)!}{(b-1)!} \cdot \prod [1 - \alpha]_{b_i - 1; 1}, && (5.38)
\end{aligned}$$

where  $[x]_{m,y} := x(x+y) \cdots (x+(m-1)y)$ . We finish the proof by observing that the limit in (5.38) is exactly the same limit that Schweinsberg obtains when studying coalescent processes that govern the genealogical trees of supercritical Galton-Watson processes with selection, see Section 4 of [Sch03].  $\square$

## A Appendix: proof of Lemma 5.8.

In this appendix, we present the proof of Lemma 5.8. We first prove the expansion of  $I_0(u)$  and then of  $I_p(u)$  for  $p \geq 2$ . The idea of the proof is more or less the same for every  $0 < \alpha \leq 2$ , but some technical adaptations are required in specific cases.

The Laplace transform  $I_0$  of  $Y_i$  is differentiable, when  $1 < \alpha \leq 2$  and  $I'_0(0) = \mathbb{E}[Y_i]$ , then in this case, the expansion of  $I_0(u)$  is obtained by a simple Taylor development at zero. For  $\alpha \leq 1$ , the Laplace transform of  $Y_1$  is no longer differentiable at zero. On the other hand, we have that

$$\begin{aligned}
\mathbb{E}[e^{-uY_1}] &= \int_0^\infty e^{-x} \mathbb{P}(Y_1 \leq x/u) dx \\
&= 1 - \int_0^{c(u)} e^{-x} \mathbb{P}(Y_1 \geq x/u) dx - \int_{c(u)}^\infty e^{-x} \mathbb{P}(Y_1 \geq x/u) dx, && (A.39)
\end{aligned}$$

where  $c(u)$  is a function depending on  $u$  to be chosen. Let  $c(u) = u \ln \ln(u^{-1})$ , then

$$\frac{x}{u} \geq \ln \ln(u^{-1}), \quad \text{if } x \geq c(u);$$

that diverges if  $u \rightarrow 0^+$ . It is also trivial that  $c(u) = o(u^\alpha)$  (in the case  $\alpha < 1$ ) and  $c(u) = o(u \ln u)$  (in the case  $\alpha = 1$ ) as  $u \rightarrow 0^+$ . Hence, we can easily bound the first term in (A.39) by

$$\int_0^{c(u)} e^{-x} \mathbb{P}(Y_1 \geq x/u) dx \leq c(u),$$

that it is negligible as  $u \rightarrow 0^+$ . We study the second term in (A.39), since  $x/u$  diverges if  $x \geq c(u)$ , we can replace  $\mathbb{P}(Y_i \geq x/u)$  by its asymptotic equivalent  $u^\alpha/x^\alpha$

$$\int_{c(u)}^\infty e^{-x} \mathbb{P}(Y_1 \geq x/u) dx \sim u^\alpha \int_{c(u)}^\infty \frac{e^{-x}}{x^\alpha} dx \quad \text{as } u \rightarrow 0^+.$$

## A. Appendix A

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When  $\alpha < 1$ , we have that  $\int_{c(u)}^{\infty} \frac{e^{-x}}{x^\alpha} dx \rightarrow \Gamma(1 - \alpha) < \infty$ , that proves the statement in this case. For  $\alpha = 1$ , we use the following result, that may be found in [BO99] (Section 6.2 Example 4)

$$\int_z^{\infty} \frac{e^{-x}}{x} dx = -\gamma - \ln z - \sum_{m \geq 1} (-1)^m \frac{z^m}{m(m!)}, \quad z \rightarrow 0^+, \quad (\text{A.40})$$

where  $\gamma$  stands for the Euler-Mascheroni constant. Taking  $z = c(u)$ , we obtain that

$$\begin{aligned} \int_{c(u)}^{\infty} \frac{e^{-x}}{x} dx &= -\gamma - \ln(u \ln \ln(u^{-1})) - \sum_{m \geq 1} (-1)^m \frac{(u \ln \ln(u^{-1}))^m}{m(m!)} \\ &= -\ln u + o(\ln u), \quad \text{as } u \rightarrow 0^+, \end{aligned}$$

finishing the proof.

We now focus on the case  $p \geq 2$ . We start with the following relation

$$\begin{aligned} I_p(u) &= \int_0^{\infty} (px^{p-1}e^{-ux} - ux^pe^{-ux}) \mathbb{P}(Y_i \geq x) dx \\ &= \int_0^{c(u)} (pu^{-p}x^{p-1}e^{-x} - u^{-p}x^pe^{-x}) \mathbb{P}(Y_i \geq x/u) dx \end{aligned} \quad (\text{A.41})$$

$$+ \int_{c(u)}^{\infty} (pu^{-p}x^{p-1}e^{-x} - u^{-p}x^pe^{-x}) \mathbb{P}(Y_i \geq x/u) dx, \quad (\text{A.42})$$

where  $c(u)$  is a function depending on  $u$  to be chosen. As we did above, we will choose  $c(u)$  such that it is negligible in comparison to  $u^{\alpha-p}$ , but  $x/u$  diverges if  $x \geq c(u)$ .

Suppose that  $\alpha < 2$  or  $\alpha = 2$  and  $p \geq 3$ . Let  $\beta \in ]0, 1[$  such that  $\beta p > \alpha$  and choose  $c(u) = u^\beta$  (it is trivial that such  $\beta$  does not exist if  $p = \alpha = 2$ ). We bound (A.41) by

$$\begin{aligned} &\left| \int_0^{c(u)} (pu^{-p}x^{p-1}e^{-x} - u^{-p}x^pe^{-x}) \mathbb{P}(Y_i \geq x/u) dx \right| \\ &\leq u^p \int_0^{c(u)} pu^{-p}x^{p-1} + u^{-p}x^p dx \\ &= u^{(\beta+1)p} + \frac{u^{(\beta+1)p+1}}{p+1}, \end{aligned}$$

that is negligible in comparison to  $u^{\alpha-p}$  as  $u \rightarrow 0^+$ . We now turn our attention to (A.42), where  $x/u$  diverges as  $u \rightarrow 0^+$ . We may replace  $\mathbb{P}(Y_i \geq x/u)$  by its asymptotic equivalent  $u^\alpha/x^\alpha$ , then as  $u \rightarrow 0^+$

$$\begin{aligned} &\int_{c(u)}^{\infty} (pu^{-p}x^{p-1}e^{-x} - u^{-p}x^pe^{-x}) \mathbb{P}(Y_i \geq x/u) dx \\ &\sim u^{\alpha-p} \int_{c(u)}^{\infty} (px^{p-\alpha-1}e^{-x} - x^{p-\alpha}e^{-x}) dx \\ &= u^{\alpha-p} \alpha \Gamma(p - \alpha) - u^{\alpha-p} \int_0^{c(u)} (px^{p-\alpha-1}e^{-x} - x^{p-\alpha}e^{-x}) dx. \end{aligned} \quad (\text{A.43})$$

Finally, the second term in the right-hand side of (A.43) is  $o(u^{\alpha-p})$  as  $u \rightarrow 0^+$ , concluding the proof in the cases  $\alpha < 2$  and  $\alpha = 2$ , with  $p \geq 2$ .

The case  $p = 2$  and  $\alpha = 2$  is obtained as above, choosing  $c(u) = u \ln \ln(u^{-1})$  and using the asymptotic development (A.40). We leave the details to the reader.  $\square$

## B Appendix: proof of Proposition 5.10.

In this appendix, we prove Proposition 5.10. Once more, the main idea of the proof is roughly the same for every  $0 < \alpha \leq 2$ , but some technical adaptations are required in specific cases. For this reason we will present a detailed proof of the case  $\alpha = 2$  and only sketch the proofs of the other cases.

Let  $\kappa_N = (\ln N)^2/N$  be as in Lemma 5.9. By (5.26) and Lemma 5.9, we have that

$$\mathbb{E}\left[\eta_1^{b_1} \dots \eta_a^{b_a}\right] = \frac{1}{\Gamma(b)} \int_0^{\kappa_N} u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du + \epsilon_N,$$

where  $\epsilon_N$  decreases to zero faster than any polynomial in  $N$ . Hence, it suffices to show that

$$\lim_{N \rightarrow \infty} \frac{N^{2a}}{(\ln N)^{a-g}} \cdot \int_0^{\kappa_N} u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du = \frac{2^a \prod_{i=1}^g \Gamma(b_i - 2)}{\mathbb{E}[Y_1]^{2a}} \cdot \Gamma(2a). \quad (\text{B.44})$$

Let  $\varepsilon > 0$ , since  $\lim_{N \rightarrow \infty} \kappa_N = 0$  we apply Lemma 5.8 to conclude that there exists a  $N_0$  such that for  $N$  larger than  $N_0$  and  $u \leq \kappa_N$

$$(1 - \varepsilon)(2\Gamma(b_i - 2)) \leq I_{b_i}(u)/u^{2-b_i} \leq (1 + \varepsilon)(2\Gamma(b_i - 2)), \quad \text{if } b_i \geq 3;$$

$$2(1 - \varepsilon) \leq I_2(u)/\ln(u^{-1}) \leq 2(1 + \varepsilon), \quad \text{if } b_i = 2.$$

Since there are finitely many  $b_i$ 's, we may take  $N_0$  such that the inequalities hold for every  $i \in \{1, 2, \dots, a\}$ . As a consequence, for  $N > N_0$

$$\begin{aligned} & \int_0^{\kappa_N} u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du \\ & \geq (1 - \varepsilon)^a 2^a \prod_{i=1}^g \Gamma(b_i - 2) \int_0^{\kappa_N} u^{b-b_1-\dots-b_{g-1}+2g} (\ln(u^{-1}))^{a-g} I_0(u)^{N-a} du \\ & = (1 - \varepsilon)^a 2^a \prod_{i=1}^g \Gamma(b_i - 2) \int_0^{\kappa_N} u^{2a-1} (\ln(u^{-1}))^{a-g} I_0(u)^{N-a} du, \end{aligned} \quad (\text{B.45})$$

where we used  $b = b_1 + \dots + b_a = b_1 + \dots + b_g + 2(a - g)$  (a similar argument may be used to obtain a similar upper bound). Applying Lemma 5.8 for  $I_0$ , we get that

$$\lim_{u \rightarrow 0^+} \frac{I_0(u) - 1}{-u \mathbb{E}[Y_1]} = 1.$$

Hence, there exists a  $N_1$  such that for  $N \geq N_1$  and  $u \leq \kappa_N$  (we assume that  $N_1 \geq N_0$ )

$$(1 - u(1 + \varepsilon) \mathbb{E}[Y_1])^{N-a} \leq I_0(u)^{N-a} \leq (1 - u(1 - \varepsilon) \mathbb{E}[Y_1])^{N-a}.$$

## B. Appendix B

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Applying the above inequality in (B.45) to obtain a lower bound, and by the change of variables  $v = u(1 + \varepsilon)\mathbb{E}[Y_1]N$  we get

$$\begin{aligned} & (1-\varepsilon)^{a2^a} \prod_{i=1}^g \Gamma(b_i - 2) \int_0^{\kappa_N} u^{2a-1} (\ln(u^{-1}))^{a-g} I_0(u)^{N-a} du \\ & \geq \frac{(1-\varepsilon)^a}{(1+\varepsilon)^{2a}} \cdot \frac{1}{N^{2a}} \cdot \frac{2^a \cdot \prod_{i=1}^g \Gamma(b_i - 2)}{\mathbb{E}[Y_1]^{2a}} \\ & \quad \times \int_0^{\gamma_N} v^{2a-1} \left( -\ln \left( \frac{v}{N(1+\varepsilon)\mathbb{E}[Y_1]} \right) \right)^{a-g} \left( 1 - \frac{v}{N} \right)^{N-a} dv, \end{aligned}$$

where  $\gamma_N = N(1 + \varepsilon)\mathbb{E}[Y_1]\kappa_N$ , then,

$$-\ln \left( v / (N(1 + \varepsilon)\mathbb{E}[Y_1]) \right) = \ln N \left( 1 + \frac{\ln((1 + \varepsilon)\mathbb{E}[Y_1]) - \ln v}{\ln N} \right),$$

and for  $v \leq (1 + \varepsilon)\mathbb{E}[Y_1](\ln N)^2 = \gamma_N$

$$\frac{|\ln((1 + \varepsilon)\mathbb{E}[Y_1]) - \ln v|}{\ln N} \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (\text{B.46})$$

Moreover, (B.46) decays uniformly to zero for  $v \leq \gamma_N$ . We bring to the reader's attention the choice of  $\kappa_N$  in Lemma 5.9, because it was chosen such that (B.46) decays to zero uniformly. Then, there exists a  $N_2$  such that for  $N \geq N_2$  (we assume that  $N_2 \geq N_1$ )

$$(1 - \varepsilon) \ln N \leq -\ln \left( v / (N(1 + \varepsilon)\mathbb{E}[Y_1]) \right) \leq (1 + \varepsilon) \ln N, \quad \text{for every } v \leq \gamma_N.$$

Then, for  $N \geq N_2$  we may further bound (B.45) and obtain

$$\begin{aligned} & \int_0^{\kappa_N} u^{b-1} I_0(u)^{N-a} I_{b_1}(u) \dots I_{b_a}(u) du \\ & \geq \frac{(1-\varepsilon)^{2a-g}}{(1+\varepsilon)^{2a}} \cdot \frac{(\ln N)^{a-g}}{N^{2a}} \cdot \frac{2^a \prod_{i=1}^g \Gamma(b_i - 2)}{\mathbb{E}[Y_1]^{2a}} \cdot \int_0^{\gamma_N} v^{2a-1} \left( 1 - \frac{v}{N} \right)^{N-a} dv. \end{aligned} \quad (\text{B.47})$$

Since  $v \leq \gamma_N$ , both  $v/N$  and  $v^2/N$  decay to zero as  $N \rightarrow \infty$ . We also have that

$$\left( 1 - \frac{v}{N} \right)^{N-a} = \exp(-v + \mathcal{O}(v^2/N)), \quad \text{as } N \rightarrow \infty.$$

As a consequence, the following limit holds

$$\lim_{N \rightarrow \infty} \int_0^{\gamma_N} v^{2a-1} \left( 1 - \frac{v}{N} \right)^{N-a} dv = \Gamma(2a).$$

Since  $\varepsilon$  in (B.47) is arbitrary, we have that

$$\liminf_{N \rightarrow \infty} \mathbb{E} \left[ \eta_1^{b_1} \dots \eta_a^{b_a} \right] \cdot \frac{N^{2a}}{(\ln N)^{a-g}} \geq \frac{2^a \prod_{i=1}^g \Gamma(b_i - 2)}{\mathbb{E}[Y_1]^{2a}} \cdot \Gamma(2a).$$

We obtain an upper bound for the lim sup using a similar argument with the obvious changes, and we leave the details to the reader. Hence, the limit in (B.44) holds, which proves the statement.

We now sketch the proof of Proposition 5.10 in the remaining cases ( $\alpha < 2$ ), and we explain briefly how to overcome possible difficulties. *The case  $1 < \alpha < 2$  has no further difficulties and we leave the details of the proof to the reader. In the case  $\alpha = 1$  the relevant term to estimate is of the form:*

$$\Gamma(b_1 - 1) \dots \Gamma(b_a - 1) \cdot \int_0^{\kappa_N} u^{b-1} I_0(u)^{N-a} u^{1-b_i} \dots u^{1-b_a} du.$$

By Lemma 5.8,  $I_0(u)^{N-a} \cong (1 + u \ln u)^{N-a}$ . Then, by the change of variables  $v = uN \ln N$ , we obtain an expression of the form:

$$\frac{\prod \Gamma(b_i - 1)}{(N \ln N)^a} \cdot \int_0^{\kappa_N N \ln N} v^{a-1} \left( 1 + \frac{v}{N \ln N} \ln \frac{v}{N \ln N} \right)^{N-a} dv.$$

Since  $v \leq \kappa_N N \ln N = (\ln N)^3$ , the equation inside of the parentheses has the following asymptotic behaviour as  $N \rightarrow \infty$

$$\begin{aligned} 1 + \frac{v}{N \ln N} \ln \left( \frac{v}{N \ln N} \right) &= 1 - \frac{v}{N} \cdot \left( 1 + \frac{\ln \ln N - \ln v}{\ln N} \right) \\ &\cong 1 - \frac{v}{N}, \end{aligned}$$

then we may proceed as in the case  $\alpha = 2$  to prove the statement. *In the case  $\alpha < 1$ , we will arrive to an equation of the form*

$$\prod \alpha \Gamma(b_i - \alpha) \int_0^{\kappa_N} u^{a\alpha-1} I_0(u)^{N-a} du.$$

We then use the development of  $I_0(u)$  in a neighbourhood of zero and the change of variables  $v = u^\alpha \Gamma(1 - \alpha) N$  to obtain

$$\frac{\prod \alpha \Gamma(b_i - \alpha)}{\alpha \Gamma(1 - \alpha)^a N^a} \int_0^{\kappa_N^\alpha \Gamma(1 - \alpha) N} v^{a-1} \left( 1 - \frac{v}{N} \right)^{N-a} dv,$$

that finishes the proof. □



# Bibliography

- [ABBS13] E. Aïdékon, J. Berestycki, E Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields*, 157(1-2):405–451, 2013.
- [ABK13] L-P Arguin, A Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Related Fields*, 157(3-4):535–574, 2013.
- [ABR09] L. Addario-Berry and B. A. Reed. Minima in branching random walk. *Ann. Probab.*, 37(3):1044–1079, 2009.
- [Aïd13] E. Aïdékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, 41(3A):1362–1426, 2013.
- [AL11] A. Auffinger and O. Louidor. Directed polymers in random environment with heavy tails. *Comm. Pure Appl. Math.*, 64 (2):183–204, 2011.
- [AN04] K. B. Athreya and P. E. Ney. *Branching process*. Dover Publications, 2004.
- [Bac00] M. Bachman. Limit theorems for the minimal position in a branching random walk with independent logconcave displacements. *Adv. in Appl. Probab.*, 32(1):159–176, 2000.
- [bAH00] D. ben Avraham and S. Havlin. *Diffusion and reactions in fractals and disordered systems*. Cambridge University Press, 2000.
- [BBS13] J. Berestycki, N. Berestycki, and J. Schweinsberg. The genealogy of branching Brownian motion with absorption. *Ann. Probab.*, 41(2):527–618, 2013.
- [BCDM<sup>+</sup>86] M. Bramson, P. Calderoni, A. De Masi, P. Ferrari, J. Lebowitz, and R. H. Schonmann. Microscopic selection principle for a diffusion-reaction equation. *J. Stat. Phys.*, 45:905–920, 1986.
- [BD97] E. Brunet and B. Derrida. Shift in the velocity of a front due to a cut-off. *Phys. Rev. E*, 56:2597–2604, 1997.
- [BD99] E. Brunet and B. Derrida. Microscopic models of traveling wave equations. *Comput. Phys. Commun.*, (121-122):376–381, 1999.

- [BD01] E. Brunet and B. Derrida. Effect of microscopic noise on front propagation. *J. Stat. Phys.*, 103:269–282, 2001.
- [BD04] E. Brunet and B. Derrida. Exactly soluble noisy traveling-wave equation appearing in the problem of directed polymers in a random medium. *Phys. Rev. E*, 70(1):016106, 2004.
- [BD13] E. Brunet and B. Derrida. Genealogies in simple models of evolution. *J. Stat. Mech. Theory Exp.*, page P01006, 2013.
- [BDMM06] E. Brunet, B. Derrida, A.H. Mueller, and S. Munier. Noisy traveling waves: effect of selection on genealogies. *EPL*, 76(1):1–7, 2006.
- [BDMM07] E. Brunet, B. Derrida, A.H. Mueller, and S. Munier. Effect of selection on ancestry: an exactly soluble case and its phenomenological generalization. *Phys. Rev. E*, 76(4):041104, 2007.
- [BDS08] E. Brunet, B. Derrida, and D. Simon. Universal tree structures in directed polymers and models of evolving populations. *Phys. Rev. E*, 78:061102, 2008.
- [BDZ11] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni. Recursions and tightness for the maximum of the discrete two dimensional Gaussian free field. *Electron. Commun. Probab.*, 16(114-119), 2011.
- [BDZ13] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *arXiv:1301.6669*, 2013.
- [BDZ14] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of nonlattice branching random walk. *arXiv:1404.3423*, 2014.
- [BG10] J. Bérard and J.-B. Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.*, 298(2):323–342, 2010.
- [BGT87] N. Bingham, C. Goldie, and J. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 1987.
- [BH14a] A. Bovier and L Hartung. Extended convergence of the extremal process of branching Brownian motion. *arXiv:1412.5975*, 2014.
- [BH14b] A. Bovier and L Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19(18), 1-28 2014.
- [Big76] J. D. Biggins. The first- and last-birth problems for a multitype age-dependent branching process. *Adv. in Appl. Probab.*, 8:446–459, 1976.

- 
- [Big77] J. D. Biggins. Chernoff's theorem in the branching random walk. *J. Appl. Probability*, 14(3):630–636, 1977.
- [Bil99] P. Billingsley. *Convergence of probability measures*. Wiley-Interscience, 1999.
- [BK08] A. Bovier and A. Klimovsky. Fluctuations of the partition function in the GREM with external field. *J. Math. Phys.*, 49(12):125202, 2008.
- [BL13] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *arXiv:1306.2602*, 2013.
- [BO99] C. M. Bender and S. A. Orszag. *Advanced mathematical methods for scientists and engineers I*. Springer-Verlag, 1999.
- [Bov06] A. Bovier. *Statistical mechanics of disordered systems*. Cambridge University Press,, 2006.
- [Bra78] M. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.
- [Bra83] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.*, 44:iv+190, 1983.
- [BS98] E. Bolthausen and A.-S. Sznitman. On Ruelle's probability cascades and an abstract cavity method. *Comm. Math. Phys.*, 197(2):247–276, 1998.
- [CC15] F. Comets and A. Cortines. Finite-size corrections to the speed of a branching-selection process. *arXiv:1505.04971*, 2015.
- [CD90] J. Cook and B. Derrida. Directed polymers in a random medium:  $1/d$  expansion and the  $n$ -tree approximation. *J. Phys. A*, 23(9):1523–1554, 1990.
- [Cor14a] A. Cortines. Front velocity and directed polymers in random medium. *Stochastic Process. Appl.*, 124:3698–3723, 2014.
- [Cor14b] A. Cortines. The genealogy of a solvable population model under selection with dynamics related to directed polymers. *arXiv:1406.1432*, 2014.
- [CQR13] F. Comets, J. Quastel, and A.F. Ramírez. Last passage percolation and traveling fronts. *J. Stat. Phys.*, 152:419–451, 2013.
- [Der85] B. Derrida. A generalization of the random energy model which includes correlations between energies. *J. Physique Lett.*, 46(401-407), 1985.
- [DG86] B. Derrida and E. Gardner. Solution of the generalised random energy model. *J. Phys. C*, 19:2253–2274, 1986.

- [dH70] L. de Haan. On regular variation and its application to the weak convergence of sample extremes. *Amsterdam Math. Centre Tracts*, 32:1–124, 1970.
- [dH74] L. de Haan. Equivalence classes of regularly varying functions. *Stochastic Process. Appl.*, 2(3):243–259, 1974.
- [dH76] L. de Haan. Sample extremes: An elementary introduction. *Stat. Neerl.*, 30:161–172, 1976.
- [DH00] F. Den Hollander. *Large deviations*. American Mathematical Society, 2000.
- [DR11] R. Durrett and D. Remenik. Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations. *Ann. Probab.*, 39(6):2043–2078, 2011.
- [DS88] B. Derrida and H. Sphon. Polymers on disordered trees, spin glasses, and traveling waves. *J. Stat. Phys.*, 51(5-6):817–840, 1988.
- [Dur10] R. Durrett. *Probability: theory and examples*. Cambridge University Press, 2010.
- [DVJ03] D.J. Daley and D. Vere-Jones. *An introduction to the theory of point processes*. Springer-Verlag, 2003.
- [DZ10] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*. Springer-Verlag, 2010.
- [Fel71] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume II. Wiley, 1971.
- [Fis37] R. A. Fisher. The wave of advance of advantageous genes. *Ann. Hum. Genet.*, 7:355–369, 1937.
- [FMS14] S. Foss, J. B. Martin, and P. Schmidt. Long-range last-passage percolation on the line. *Ann. Appl. Probab.*, 24(1):198–234, 2014.
- [Fré27] M. Fréchet. Sur la loi de probabilité de l'écart maximum. *Ann. Soc. Pol. Math.*, 6:93–116, 1927.
- [FT28] R. A. Fisher and L. H. C. Tippett. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Math. Proc. Cambridge Philos. Soc.*, 24:180–190, 1928.
- [GHS11] N. Gantert, Y. Hu, and Z. Shi. Asymptotics for the survival probability in a killed branching random walk. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(1):111–129, 2011.

- 
- [GK04] B. H. Gilding and R Kersner. *Travelling waves in nonlinear diffusion-convection reaction*. Progress in Nonlinear Differential Equations and their Applications, 2004.
- [Gne43] B. V. Gnedenko. Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.*, 44:423–453, 1943.
- [GNPT86] E. Gelenbe, R. Nelson, T. Philips, and A. Tantawi. An approximation of the processing time for a random graph model of parallel computation. *ACM Fall Joint Computer Conference*, 1986. IEEE Computer Society Press, Los Alamitos, CA.
- [Ham74] J. M. Hammersley. Postulates for subadditive processes. *Ann. Probab.*, 2:652–680, 1974.
- [HM07] B. Hambly and J. B. Martin. Heavy tails in last-passage percolation. *Probab. Theory Related Fields*, 137(1-2):227–275, 2007.
- [HM11] T. Huillet and M. Möhle. Population genetics models with skewed fertilities: a forward and backward analysis. *Stoch. Models*, 27(3):521–554, 2011.
- [Kin75] J. Kingman. The first birth problem for an age-dependent branching process. *Ann. Probab.*, 3:790–801, 1975.
- [Kin82] J. Kingman. On the genealogy of large population. *J. Appl. Probab.*, 19:27–43, 1982.
- [KPP37] A. Kolmogorov, I. Petrovsky, and N. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Moscou*, A1(1-25), 1937.
- [LLR83] M. Leadbetter, G. Lindgren, and H. Rootzen. *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer-Verlag, 1983.
- [Mai11] P. Maillard. Branching Brownian motion with selection of the  $N$  right-most particles: An approximate model. *arXiv preprint arXiv:1112.0266*, 2011.
- [Mai13] P. Maillard. The number of absorbed individuals in branching brownian motion with a barrier. *Ann. Inst. H. Poincaré Probab. Statist.*, 49(2):428–455, 2013.
- [Mal15] B. Mallein.  $N$ -branching random walk with  $\alpha$ -stable spine. *arXiv:1503.03762*, 2015.
- [McK75] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975.

- [MMQ11] C. Mueller, L. Mytnik, and J. Quastel. Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Invent. Math.*, 184(2):405–453, 2011.
- [Möh99] M. Möhle. Weak convergence to the coalescent in neutral population models. *J. Appl. Probab.*, 36(2):446–460, 1999.
- [Möh00] M. Möhle. Total variation distances and rates of convergence for ancestral coalescent processes in exchangeable population models. *Adv. in Appl. Probab.*, 32(4):983–993, 2000.
- [Mor58] P. A. P. Moran. Random processes in genetics. *Proc. Cambridge Philos. Soc.*, 54:60–71, 1958.
- [MS95] C. Mueller and R. Sowers. Random traveling waves for the KPP equation with noise. *J. Funct. Anal.*, 128:439–498, 1995.
- [MS01] M. Möhle and S. Sagitov. A classification of coalescent processes for haploid exchangeable population models. *Ann. Probab.*, 29(4):1547–1562, 2001.
- [Pan04] D. Panja. Effects of fluctuations on propagating fronts. *Phys. Rep.*, 393:87–174, 2004.
- [Pem09] R. Pemantle. Search cost for a nearly optimal path in a binary tree. *Ann. Appl. Probab.*, 19(4):1273–1291, 2009.
- [Pit99] J. Pitman. Coalescents with multiple collisions. *Ann. Probab.*, 27(4):1870–1902, 1999.
- [PL99] L Pechenik and H Levine. Interfacial velocity correction due to multiplicative noise. *Phys. Rev. E*, 59:3893–3900, 1999.
- [Res87] S. I. Resnick. *Extreme values, regular variation, and point processes*. Springer-Verlag, 1987.
- [Sch00] J. Schweinsberg. Coalescents with simultaneous multiple collisions. *Electron. J. Probab.*, 5:1–50, 2000.
- [Sch03] J. Schweinsberg. Coalescent processes obtained from supercritical Galton-Watson processes. *Stochastic Process. Appl.*, 106(1):107–139, 2003.
- [Sen76] E. Seneta. *Regularly Varying Functions*, volume 508 of *Lecture Notes in Mathematics*. Springer, 1976.
- [Tip25] L. H. C. Tippett. On the extreme individuals and the range of samples taken from a normal population. *Biometrika*, 17(3/4):364–387, Dec 1925.
- [Zei] O. Zeitouni. Lecture notes: Branching random walks and Gaussian fields. Available at: [www.wisdom.weizmann.ac.il/~zeitouni](http://www.wisdom.weizmann.ac.il/~zeitouni).