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UNIVERSITÉ GRENOBLE ALPES

THÈSE

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Electronique, Electrotechnique, Automatique et Traitement du Signal

Stability analysis and Tikhonov approximation for linear singularly perturbed hyperbolic systems

Thèse soutenue publiquement le **18 Septembre 2015**,
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Résumé

Contexte et motivation

Les dynamiques des systèmes modélisés par des équations aux dérivées partielles (EDPs) en dimension infinie sont largement liées aux réseaux physiques. La synthèse de la commande et l'analyse de la stabilité de ces systèmes sont des questions importantes. De nombreux travaux de recherche ont été rapportés (par exemple [Li, 1994], [Coron et al., 2008], [Prieur and Mazenc, 2012] etc.). Les équations aux dérivées partielles singulièrement perturbées ont été prises en compte dans les travaux de recherche à partir de 1980. Ce type de systèmes est intéressant pour analyser de nombreux phénomènes avec des échelles de temps multiples dans les domaines divers de la physique et de l'ingénierie, tels que la dynamique des fluides, de la chimie, l'aérodynamique, etc. (voir [Kadalbajoo and Patidar, 2003]). Le travail présent concerne une classe des systèmes hyperboliques linéaires avec des échelles de temps multiples.

• Systèmes hyperboliques

Un système hyperbolique linéaire est décrit par l'équation aux dérivées partielles suivant

$$y_t(x,t) + \Lambda y_x(x,t) + My(x,t) = 0,$$

avec la condition aux bords

$$y(0,t) = Ky(1,t),$$

alors que la condition initiale est donnée par

$$y(x,0) = y_0(x),$$

où $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^n$, Λ est une matrice diagonale positive dans $\mathbb{R}^{n\times n}$, M est une matrice de dimensions appropriées, K est une matrice constante dans $\mathbb{R}^{n\times n}$ et $y_0 : [0,1] \to \mathbb{R}^n$.

Lois de conservation Lorsque la matrice M est nulle, il s'appelle système hyperbolique de lois de conservation. L'analyse de la stabilité de ces systèmes a été étudiée dans de nombreux travaux de recherche par des méthodes différentes. Un premier résultat important de la stabilité asymptotique pour un système de dimension 2 a été rapporté dans [Greenberg and Li, 1984]. Plus tard un résultat général pour systèmes de dimensions n a été donné par [Li, 1994]. Ces résultats sont basés sur l'évolution explicite de l'invariant Riemman sur des courbes caractéristiques. La plus faible condition suffisante

donnée par [Li, 1994, Théorème 1.3] a été formulée comme suit: $\rho(|K|) < 1$, où $\rho(K)$ est le rayon spectral de la matrice K dans $\mathbb{R}^{n\times n}$ et $|K|$ indique la matrice avec toutes les valeurs absolues des éléments de K.

Par suite de [Hale and Lunel, 1993], une condition de stabilité nécessaire et suffisante a été donnée dans le domaine fréquentiel. Plus précisément, le système hyperbolique linéaire a été considéré comme un système à retard. Il est exponentiellement stable si et seulement s'il existe $w > 0$ tel que

$$\left(\det(I_n - (\mathtt{diag}(e^{-\nu_1 s}, \cdots, e^{-\nu_n s})))K = 0, s \in \mathbb{C}\right) \Rightarrow (\Re(s) \leqslant -w),$$

où $\nu_j = \frac{1}{\Lambda_j}$, $j \in \{1, \cdots, n\}$ et I_n est la matrice identité dans $\mathbb{R}^n$, $\Re(s)$ est la partie réelle du nombre complexe s. De plus grâce à Silkowski, si $(\nu_1, \cdots, \nu_n)$ sont rationnellement indépendants, le système linéaire est exponentiellement stable si et seulement si $\rho_0(K) < 1$.
Avec

$$\rho_0(K) = \max\{\rho(\mathtt{diag}(e^{\iota\theta_1}, \cdots, e^{\iota\theta_n})K);\ (\theta_1, \cdots \theta_n)^\top \in \mathbb{R}^n\},$$

où $\iota = \sqrt{-1}$.

Une approche différente, l'approche de Lyapunov, est un outil puissant pour l'analyse de la stabilité du système hyperbolique. Dans [Coron et al., 1999] et [Coron et al., 2002], cette approche a été introduite pour contrôler l'équation de Saint-Venant modélisée par le système hyperbolique de lois de conservation. Dans [Coron et al., 2007] une fonction de Lyapunov stricte dans la norme de H^2 a été construite pour analyser la stabilité d'un système hyperbolique de lois de conservation de dimension 2 autour d'un point d'équilibre. Dans [Coron et al., 2008] une fonction de Lyapunov stricte dans la norme de H^2 a été utilisée pour prouver la stabilité exponentielle. La condition aux bords dissipative suffisante des $n \times n$ systèmes hyperboliques linéaires unidimensionnel a été présentée comme suit:

Théorème. *Si $\rho_1(K) < 1$, le système hyperbolique linéaire est exponentiellement stable à l'origine dans la norme de H^2.*

Dans le résultat précédent, les notations suivantes ont été utilisées

$$\rho_1(K) = \inf\{\|\Delta K \Delta^{-1}\|, \Delta \in D_{n,+}\},$$

$D_{n,+}$ désigne l'ensemble des matrices diagonales positives dans $\mathbb{R}^{n\times n}$.
Les comparaisons entre les deux conditions, $\rho_0(K) < 1$ et $\rho_1(K) < 1$, ont été faites dans le papier [Coron et al., 2008]. Pour chaque système de dimension $n \in \{1, 2, 3, 4, 5\}$, les deux conditions sont équivalentes. La proposition ci dessous est valable:

Proposition. *Notons* $\nu_i = \frac{1}{\Lambda_i}$ *avec* $i \in \{1, \cdots, n\}$, *si* $(\nu_1, \cdots, \nu_n)$ *sont rationnellement indépendants, le système linéaire de dimension* $n \in \{1, 2, 3, 4, 5\}$ *est exponentiellement stable si et seulement si* $\rho_1(K) < 1$.

Remarque. *Dans [Coron et al., 2008], la condition* $\rho_1(K) < 1$ *a été donnée pour la stabilité exponentielle des* $n \times n$ *systèmes hyperboliques quasilinéaires unidimensionnel. Elle peut donc être appliquée aux systèmes linéaires.*

Lois d'équilibre Lorsque la matrice M est non nulle, on parle de système hyperbolique de lois d'équilibre. [Xu and Sallet, 2002] a proposé une fonction de Lyapunov pour la stabilité exponentielle d'une classe générale des systèmes hyperboliques linéaires où la matrice M est symétrique. Une stabilité exponentielle dans la norme de L^2 pour le transport du gaz dans un pipeline géré par des équations d'Euler isothermes a été prouvée par une fonction de Lyapunov dans [Gugat and Herty, 2011]. [Gugat et al., 2012] a donné une fonction de Lyapunov dans la norme de H^2 pour les mêmes équations d'Euler. Dans [Diagne et al., 2012], les conditions aux bords dissipatives explicites ont été établies pour la stabilité exponentielle dans la norme de L^2 avec une matrice M qui est diagonale marginalement stable.

La synthèse de la commande pour les systèmes hyperboliques est généralement basée sur l'analyse de la stabilité. Par exemple, l'approche de backstepping est un outil utile pour la stabilisation de tels systèmes. Il a été développé pour des systèmes hyperboliques du deuxième ordre dans le temps et dans l'espace dans [Krstic et al., 2006b] et [Krstic et al., 2006a]. Dans [Krstic and Smyshlyaev, 2008], une stabilisation par retour d'état aux bords des systèmes hyperboliques du premier ordre en utilisant une approche de backstepping a été introduite. [Coron et al., 2013] a proposé une loi de retour d'état complet pour stabiliser un système 2×2 hyperbolique quasilinéaire du premier ordre dans la norme de H^2.

• Systèmes EDO-EDP couplés

L'approche de backstepping est largement utilisée pour étudier des systèmes modélisés par l'équation aux dérivées ordinaires couplée avec l'équation aux dérivées partielles (EDO-EDP). Par exemple, une EDP hyperbolique du premier ordre couplée avec une EDO du second ordre (dans l'espace) a été stabilisée par cette approche dans [Krstic and Smyshlyaev, 2008]. Dans [Krstic, 2009b], des lois de feedback prédictives et des observateurs ont été conçus pour une EDP de diffusion (équation de la chaleur) et une EDO linéaire invariante en temps en cascade. Une cascade d'une EDP du second ordre (dans le temps) et EDO a été étudiée par [Krstic, 2009a]. Dans [Susto and Krstic, 2010], les connexions internes de type Neumann ont été considérées pour une EDO-EDP en cascade. Une commande par

retour d'état aux bords a été proposée pour stabiliser une EDP couplée (équation de la chaleur) avec une EDO dans [Tang and Xie, 2011]. La stabilisation d'une EDO-Schrödinger en cascade a été présentée dans le papier [Ren et al., 2013].

La technique de Lyapunov est également utilisée pour analyser la stabilité de ces systèmes. Comme dans [Dos Santos et al., 2008], une fonction de Lyapunov stricte a été proposée pour montrer la stabilité d'un système hyperbolique avec les actions intégrales à la frontière, qui est une sorte de système EDO-EDP. Un système hyperbolique instable en boucle ouverte a été stabilisé par un contrôleur Proportionnelle Intégral (PI) aux bords, ce qui est prouvé dans le domaine fréquentiel par [Bastin et al., 2015].

• Théorie des systèmes singulièrement perturbés

Les modèles singulièrement perturbés, contenant des échelles de temps multiples sont utiles pour les systèmes physiques contenant de petits paramètres parasitaires, généralement de petites constantes de temps, les masses, les inductances, les moments d'inertie. Par exemple, les semi-conducteurs diodes [Smith, 1985], moteurs à courant continu et les régulateurs de tension [Kokotović et al., 1986], systèmes mécaniques flexibles [Da and Corless, 1993] et les réseaux de réactions chimiques [Breusegem and Bastin, 1991].

La théorie des perturbations singulières a été introduite pour le contrôle à la fin des années 1960, son assimilation dans la théorie du contrôle s'est rapidement développée et est devenue un outil pour l'analyse et la synthèse de la commande des systèmes. Les perturbations singulières sont une façon de négliger la transition rapide, en la considérant dans une échelle de temps rapide séparée. L'avantage important de cette technique est de réduire l'ordre du système. Par exemple, [Kokotović and Haddad, 1975] a calculé un contrôle optimal pour un système invariant en temps linéaire, constitué de la dynamique rapide et lente. Un contrôle optimal pour un système d'ordre plus élevé a été approché par un système d'ordre plus bas dans [Kokotović and Sannuti, 1968]. [Kokotović and Yackel, 1972] a établi des conditions suffisantes pour la solution d'un problème de régulateur d'ordre élevé vers la solution d'un système d'ordre bas. Il y a des littératures riches sur ces systèmes décrits par des équations aux dérivées ordinaires (EDOs) en dimension finie, comme indiqué dans les documents [Kokotović et al., 1976] et [Saksena et al., 1984] où des bibliographies complètes sont indiquées.

La décomposition d'un système singulièrement perturbé en sous-systèmes, à savoir le système réduit et la couche limite, est un outil puissant pour l'analyse de la stabilité (par exemple [Habets, 1974b], [Chow, 1978], [Grujic, 1981] et [Chow and Kokotović, 1981]). Plus précisément, rappelons un système

linéaire singulièrement perturbé présenté par des équations aux dérivées ordinaires en dimension finie:

$$\dot{x} = Ax + Bz,\ x \in \mathbb{R}^n,$$
$$\varepsilon\dot{z} = Cx + Dz,\ z \in \mathbb{R}^m,$$

avec $\varepsilon > 0$.
Le système réduit représentant la dynamique lente est décrit par

$$\dot{\bar{x}} = A_r\bar{x},$$

où $A_r = A - BD^{-1}C$. La barre désigne la variable qui appartient au système avec $\varepsilon = 0$.
La couche limite représentant la dynamique rapide est donnée par

$$\frac{dy}{d\tau} = Dy,$$

où $y = z + D^{-1}Cx$ et $\tau = \frac{t}{\varepsilon}$.
Le problème de l'analyse des propriétés de la stabilité de ces systèmes en dimension finie a reçu une attention considérable dans la littérature. Par exemple, dans [Kokotović et al., 1986, Chapter 2], la condition de la stabilité a été formulée comme suit:

Théorème. *Si A_r et D sont Hurwitz, il existe $\varepsilon^* > 0$ tel que pour $\varepsilon \in (0, \varepsilon^*]$, le système complet singulièrement perturbé est asymptotiquement stable.*

D'après le théorème ci-dessus, la stabilité des deux sous-systèmes assure la stabilité du système complet. Pour les systèmes non linéaires singulièrement perturbés, dans [Hoppensteadt, 1966] et [Habets, 1974a], les résultats se concentrent sur les résultats locaux ou des conditions de croissance imposées sur les non-linéarités du système. [Saberi and Khalil, 1984] a développé une fonction de Lyapunov quadratique pour énoncer des conditions d'interconnexion suffisantes afin de garantir la stabilité asymptotique. Dans [Marino et al., 1989], il a été donné des conditions suffisantes et un feedback de grand gain explicite pour les systèmes mono-entrée monosortie non linéaires en utilisant la méthode des perturbations singulières. [Da and Corless, 1993] a considéré la stabilité asymptotique pour une classe de systèmes non linéaires singulièrement perturbés dont la dynamique rapide est marginalement stable en utilisant la même approche. [Chen and Hsieh, 1994] a étudié une approche géométrique pour des systèmes singulièrement perturbés. Dans [Christofides and Teel, 1996], la stabilité entrée-état a été étudiée pour une classe de systèmes non linéaires singulièrement perturbés.

Le théorème de Tikhonov est un outil fondamental pour l'analyse des systèmes singulièrement perturbés. Il décrit le comportement limite de solutions du

système perturbé. Pour un système linéaire, l'approche de Tikhonov dans une intervalle de temps infinie a été donnée par [Kokotović et al., 1986, Chapitre 2] comme suit:

Théorème. *Si A_r et D sont Hurwitz, il existe $a > 0$ et $\varepsilon^* > 0$, tel que pour $\varepsilon \in (0, \varepsilon^*]$, les approximations suivantes sont valables pour tout $t \geqslant 0$,*

$$\begin{aligned} |x(t) - \bar{x}(t)| &\leqslant a\varepsilon, \\ |z(t) + D^{-1}C\bar{x}(t) - y(t/\varepsilon)| &\leqslant a\varepsilon. \end{aligned}$$

Pour le système non linéaire, l'approximation de Tikhonov dans un intervalle de temps infini est basée sur la stabilité exponentielle à la fois du système réduit et la couche limite. Les résultats ont été rapportés dans les travaux de recherche, par exemple [Khalil, 1996], [Verhulst, 2007].

Applications physiques et problèmes ouverts

Applications physiques Les systèmes hyperboliques de lois de conservation et de lois d'équilibre sont des modèles naturels pour représenter les réseaux physiques dans de nombreux domaines de l'ingénierie et des sciences (voir par exemple [Bastin and Coron, 2015, Chapitre 1]). Parmi les applications potentielles, les réseaux hydrauliques [Bastin et al., 2008], le transport du gaz dans des pipelines [Dick et al., 2010], les réseaux de transmission électrique [Gugat, 2014] ou les réseaux de trafic routier [Haut and Bastin, 2007] sont d'une importance significative.

Dans ce travail de thèse, nous sommes motivés par les deux applications suivantes.

• Canal représenté par la Figure 5.2.1

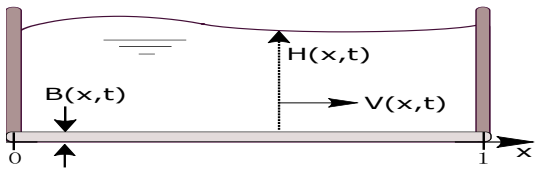

Figure 5.2.1 Dynamique de l'eau et du sédiment dans un bief

On considère un canal prismatique avec une section transversale rectangulaire et une largeur égale à l'unité. Il est présenté par les équations de Saint-Venant-Exner où l'effet du mouvement des sédiments sur l'écoulement

est pris en compte. Etant donné que la vitesse du sédiment est beaucoup plus lente que la vitesse de l'écoulement d'eau, ce système peut ainsi être présenté par un système hyperbolique singulièrement perturbé.
Le problème de régulation de canaux ouverts présente un intérêt économique et environnemental. Il a été considéré dans la littérature en utilisant une grande variété de techniques. Dans [Garcia et al., 1992] et [Malaterre, 1998], les approximations linéaires discrètes de l'équation Saint-Venant perturbée ont été utilisées. Un développement d'une synthèse de la commande H_∞ a été présenté dans [Litrico and Georges, 1999a], [Litrico and Georges, 1999b], et [Litrico and Georges, 2001]. [Prieur et al., 2008] a considéré des systèmes de lois de conservation pour la synthèse de la commande aux bords pour la stabilisation d'un canal, sous réserve de petites perturbations externes. Un contrôle aux bords de canaux ouverts avec des résultats expérimentaux a été proposé par [Dos Santos and Prieur, 2008].

• Système de transport du gaz représenté par la Figure 5.2.6

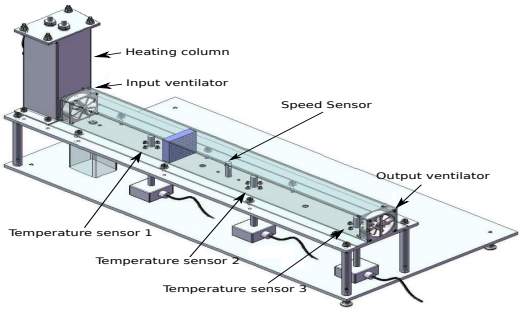

Figure 5.2.6 Dispositif expérimental pour le transport du gaz

Cet équipement se compose d'une colonne chauffante et un tube de section constante. Compte tenu du fait que la vitesse du gaz dans le tube est beaucoup plus lente que la vitesse du son, les dynamiques du gaz dans le tube sont d'abord modélisées par un système hyperbolique singulièrement perturbé. Ensuite, si l'on considère les dynamiques du gaz à la fois dans la colonne chauffante et le tube, un système EDO-EDP couplé peut être utilisé pour modéliser ce système.
Le transport du gaz dans des pipelines est d'un grand intérêt pour l'approvisionnement en énergie. La commande de ces systèmes a été rapportée dans de nombreux travaux de recherche (voir par exemple [Banda et al., 2006], [Colombo et al., 2009]).

Problèmes ouverts Les perturbations singulières pour les équations aux dérivées partielles ont été étudiées du point de vue mathématique (par exemple [Friedman, 1968], [Michelson, 1985], [Melenk and Schwab, 1999]). La

plupart de ces œuvres s'est concentrée sur l'existence de la solution, le comportement asymptotique ou un problème de valeur limite. À notre connaissance, il y a peu de travaux sur l'analyse de la stabilité et le contrôle de ce type de systèmes. Dans cette thèse, nous nous intéressons à des systèmes hyperboliques linéaires avec des échelles de temps multiples. Tout d'abord il a été étudié une classe de systèmes linéaires hyperboliques singulièrement perturbés. On a étudié la relation entre la stabilité du système complet et des sous-systèmes rapide et lent. Contrairement aux systèmes linéaires en dimension finie, les conditions de stabilité ne sont pas équivalentes. De plus, l'approximation de Tikhonov a été réalisée pour de tels systèmes en dimension infinie. Différents résultats sont obtenus par rapport aux systèmes singulièrement perturbés modélisés par des EDOs. Deuxièmement, nous avons considéré un système EDO-EDP couplé. L'analyse de la stabilité d'une EDO couplée avec une EDP rapide est similaire avec les systèmes linéaires en dimension finie. Cependant, il n'est pas valable pour une EDO rapide couplée avec une EDP.

Contributions principales

Les contributions principales sont présentées dans les trois sections suivantes.

Systèmes hyperboliques linéaires singulièrement perturbés

La première contribution principale est l'analyse de la stabilité et l'approximation de Tikhonov pour des systèmes hyperboliques linéaires de lois de conservation et de lois d'équilibre.

• *Chapitre 2:* Systèmes hyperboliques linéaires de lois de conservation singulièrement perturbés.
Le système complet est donné par:

$$\begin{cases} y_t(x,t) + \Lambda_1 y_x(x,t) = 0, \ x \in [0,1], \ t \in [0,+\infty), \\ \varepsilon z_t(x,t) + \Lambda_2 z_x(x,t) = 0, \ \varepsilon > 0, \\ \begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \ t \in [0,+\infty), \\ \begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix}, \ x \in [0,1], \end{cases}$$

où Λ_1 et Λ_2 sont des matrices diagonales positives, $K = \left(\begin{smallmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right)$ est une matrice constante de dimensions appropriées.
Le sous-système réduit représentant la dynamique lente est

$$\begin{cases} \bar{y}_t(x,t) + \Lambda_1 \bar{y}_x(x,t) = 0, \ x \in [0,1], \ t \in [0,+\infty), \\ \bar{y}(0,t) = K_r \bar{y}(1,t), \ t \in [0,+\infty), \\ \bar{y}(x,0) = \bar{y}_0(x) = y_0(x), \ x \in [0,1], \end{cases}$$

où $K_r = K_{11} + K_{12}(I_m - K_{22})^{-1}K_{21}$.
La couche limite représentant la dynamique rapide est

$$\begin{cases} \bar{z}_\tau(x,\tau) + \Lambda_2 \bar{z}_x(x,\tau) = 0, \ x \in [0,1], \ \tau \in [0,+\infty), \\ \bar{z}(0,\tau) = K_{22}\bar{z}(1,\tau), \ \tau \in [0,+\infty), \\ \bar{z}(x,0) = \bar{z}_0(x) = z_0(x) - (I_m - K_{22})^{-1}K_{21}y_0(1), \ x \in [0,1], \end{cases}$$

où $\tau = t/\varepsilon$.
La section 2.2 étudie la relation entre la stabilité du système complet et celle des deux sous-systèmes. La stabilité du système complet implique la stabilité des deux sous-systèmes comme indiquée dans la proposition 2.2.1. D'autre part, un contre-exemple est utilisé pour illustrer le fait que la stabilité des deux sous-systèmes ne garantit pas la stabilité du système complet. Cela montre une grande différence avec ce qui est bien connu pour des systèmes linéaires en dimension finie (par exemple le théorème 1.1.2).
Les sections 2.3 et 2.4 étudient l'approximation de Tikhonov pour de tels systèmes. Sous la condition de la stabilité $\rho_1(K) < 1$ (ce qui implique que le système complet est stable), la solution de la dynamique lente du système complet est approchée par la solution du sous-système réduit lorsque le paramètre de la perturbation est suffisamment petit. Ce résultat est d'abord prouvé par une fonction de Lyapunov dans la norme de L^2 dans la section 2.3. Les erreurs entre le système complet et les sous-systèmes sont estimées en fonction de l'ordre du paramètre de la perturbation. De plus, dans la section 2.4, nous proposons une fonction de Lyapunov dans la norme de H^2, où des estimations plus précises sont obtenues.

• *Chapitre 3:* Système hyperbolique linéaire de lois d'équilibre singulièrement perturbés
Par rapport au chapitre précédent, nous considérons le système avec des vitesses de transport non uniformes. Par ailleurs les termes source et la condition aux bords dépendent également de ε. La section 3.1 donne les descriptions du système complet et le sous-système réduit. Le système complet est

$$\begin{cases} y_t(x,t) + \Lambda_1(\varepsilon)y_x(x,t) = a(\varepsilon)y(x,t) + b(\varepsilon)z(x,t), \ x \in [0,1], \ t \in [0,+\infty) \\ \varepsilon z_t(x,t) + \Lambda_2(\varepsilon)z_x(x,t) = c(\varepsilon)y(x,t) + d(\varepsilon)z(x,t), \ \varepsilon > 0, \\ \begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K(\varepsilon) \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in [0,+\infty), \\ \begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix}, \quad x \in [0,1], \end{cases}$$

où $\Lambda_1(\varepsilon)$ et $\Lambda_2(\varepsilon)$ sont des matrices diagonales positives, $a(\varepsilon)$, $b(\varepsilon)$, $c(\varepsilon)$ et $d(\varepsilon)$ sont des matrices de dimensions appropriées qui s'annulent à $\varepsilon = 0$. $K(\varepsilon) = \begin{pmatrix} K_{11}(\varepsilon) & K_{12}(\varepsilon) \\ K_{21}(\varepsilon) & K_{22}(\varepsilon) \end{pmatrix}$ est une matrice constante.

Le système réduit est décrit par

$$\begin{cases} \bar{y}_t(x,t) + \Lambda_1(0)\bar{y}_x(x,t) = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \\ \bar{y}(0,t) = K_r\bar{y}(1,t), \quad t \in [0,+\infty), \\ \bar{y}(x,0) = \bar{y}_0(x) = y_0(x), \quad x \in [0,1], \end{cases}$$

où $\Lambda_1(0)$ est diagonale positive et $K_r = K_{11}(0)+K_{12}(0)(I_m-K_{22}(0))^{-1}K_{21}(0)$. La section 3.2 présente l'approximation de Tikhonov pour tel système. Sous la condition de la stabilité $\rho_1(K(0)) < 1$ et les hypothèses sur la continuité des vitesses de transport, des termes sources et de la condition aux bords, la solution du système complet est approchée par celle du sous-système réduit. L'estimation de l'erreur entre le système complet et le sous-système réduit est en fonction de l'ordre de ε.

Systèmes EDO-EDP couplés singulièrement perturbés

La deuxième contribution principale est l'analyse de la stabilité et l'approximation de Tikhonov pour des systèmes EDO-EDP couplés avec deux échelles de temps.

• *Chapitre 4:* Systèmes EDO-EDP couplés.
I) Une EDO couplée avec une EDP rapide
Section 4.1.1 considère le système couplé où le paramètre de la perturbation ε est introduit dans le système EDP:

$$\begin{cases} \dot{Z}(t) = AZ(t) + By(1,t), \ t \in [0,+\infty), \\ \varepsilon y_t(x,t) + \Lambda y_x(x,t) = 0, \ x \in [0,1], \ t \in [0,+\infty), \ \varepsilon > 0, \\ y(0,t) = G_1 y(1,t) + G_2 Z(t), \ t \in [0,+\infty), \\ Z(0) = Z_0, \\ y(x,0) = y_0(x), \ x \in [0,1], \end{cases}$$

où A et B sont des matrices de dimensions appropriées, Λ est une matrice diagonale positive. G_1 et G_2 sont des matrices constantes de dimensions appropriées.
Le système réduit est

$$\begin{cases} \dot{\bar{Z}}(t) = (A + BG_r)\bar{Z}(t), \ t \in [0,+\infty), \\ \bar{Z}_0 = Z_0, \end{cases}$$

où $G_r = (I_m - G_1)^{-1}G_2$.
La couche limite est

$$\begin{cases} \bar{y}_\tau(x,\tau) + \Lambda\bar{y}_x(x,\tau) = 0, \ x \in [0,1], \ \tau \in [0,+\infty), \\ \bar{y}(0,\tau) = G_1\bar{y}(1,\tau), \ \tau \in [0,+\infty), \\ \bar{y}_0(x) = y_0(x) - G_r Z_0, \ x \in [0,1], \end{cases}$$

où $\tau = t/\varepsilon$.
On prouve le même résultat de stabilité dans la section 4.1.2 que pour le système en dimension finie (par exemple le théorème 1.1.2). Précisément, la stabilité des deux sous-systèmes implique la stabilité du système complet. Ceci est prouvé par la méthode de Lyapunov. Le théorème de Tikhonov pour ce type de système est donné dans la section 4.1.3.

II) Une EDO rapide couplée avec Une EDP
Section 4.2.1 présente le système complet où ε est introduit dans la dynamique de l'EDO:

$$\begin{cases} \varepsilon\dot{Y}(t) = AY(t) + Bz(1),\ t \in [0, +\infty),\ \varepsilon > 0, \\ z_t(x,t) + \Lambda z_x(x,t) = 0,\ x \in [0,1],\ t \in [0, +\infty), \\ z(0,t) = K_1 z(1,t) + K_2 Y(t),\ t \in [0, +\infty), \\ Y(0) = Y_0, \\ z(x,0) = z_0(x),\ x \in [0,1], \end{cases}$$

où K_1 et K_2 sont des matrices constantes de dimensions appropriées.
Le système réduit est

$$\begin{cases} \bar{z}_t(x,t) + \Lambda \bar{z}_x(x,t) = 0,\ x \in [0,1],\ t \in [0, +\infty), \\ \bar{z}(0,t) = K_r \bar{z}(1,t),\ t \in [0, +\infty), \\ \bar{z}(x,0) = \bar{z}_0(x) = z_0(x),\ x \in [0,1], \end{cases}$$

où $K_r = (K_1 - K_2 A^{-1} B)$.
La couche limite est

$$\begin{cases} \frac{d\bar{Y}(\tau)}{d\tau} = A\bar{Y}(\tau),\ \tau \in [0, +\infty), \\ \bar{Y}(0) = \bar{Y}_0 = Y_0 + A^{-1} B z_0(1), \end{cases}$$

où $\tau = t/\varepsilon$.
A la différence du résultat de la stabilité pour le système où la dynamique lente est donnée par EDO, comme étudié dans la section 4.1, dans la section 4.2.2 un contre-exemple montre que le système complet n'est pas stable même si les deux sous-systèmes sont stables. De plus, une condition liée aux termes couplés est proposée pour assurer la stabilité de ce système. L'approximation de Tikhonov pour tel système EDO-EDP couplé est établie dans la section 4.2.3.

Synthèse du contrôle aux bords et applications

La troisième contribution principale est la synthèse du contrôle aux bords basée sur la méthode des perturbations singulières et la synthèse de la commande pour les deux applications précédents.

- *Chapitre 5:* Synthèse du contrôle aux bords pour le système hyperbolique

linéaire de lois d'équilibre et la synthèse de la commande aux bords pour deux applications.

I) Synthèse du contrôle aux bords

Prenons le système de lois d'équilibre singulièrement perturbé présentée dans le chapitre 3. Dans la section 5.1, le sous-système réduit est stabilisé en temps fini $T = \frac{1}{\underline{\lambda}(\Lambda_1(0))}$ en choisissant la condition aux bords $K_r = 0$. La condition aux bords du système complet $K(\varepsilon)$ est choisie avec K_r tel que $\rho_1(K(0)) < 1$. Ensuite, le système complet atteint un voisinage de taille proportionnelle au paramètre de la perturbation ε à l'origine en temps T.

II) Synthèse d'un contrôle aux bords pour un bief

Nous considérons un canal modélisé par des équations de Saint-Venant–Exner. Après linéarisation autour d'un point d'équilibre, le système est écrit en coordonnées de Riemann. Compte tenu du fait que le mouvement physique des sédiments est beaucoup plus lent que l'écoulement d'eau, ce système est finalement réécrit en système singulièrement perturbé. Les contrôles aux bords sont établis pour assurer la stabilité du système complet. De plus, la dynamique lente est stabilisée en temps fini. Les résultats principaux sont présentés dans la section 5.2.1.

III) Synthèse d'un contrôle aux bords du système de transport du gaz

Dynamiques du gaz dans un tube de section constante

Tout d'abord, considérons les dynamiques du gaz dans un tube de section constante qui sont régi par des équations d'Euler. Le paramètre de la perturbation de ce système est le ratio de la vitesse du gaz et de la vitesse du son. Les contrôles aux bords sont proposés pour stabiliser le système complet. Basée sur l'approche des perturbations singulières, la dynamique lente converge en temps fini. La section 5.2.2 présente les résultats principaux.

Dynamiques du gaz dans la colonne chauffante et le tube

Deuxièmement, considérons les dynamiques du gaz dans une colonne chauffante ainsi que dans un tube de section constante. Ce système est modélisé par un système EDO-EDP couplé. La dynamique du gaz dans la colonne chauffante est régie par une EDO et la dynamique du gaz dans le tube est présenté par une EDP. Les deux sous-systèmes sont connectés par la condition aux bords. Comme la dynamique du gaz dans le tube est beaucoup plus rapide que dans la colonne chauffante, le système complet est stabilisé par un contrôle aux bords basé sur la méthode des perturbations singulières. Les résultats principaux sont présentés dans la section 5.2.3.

Notation

To facilitate the reading, the following notations are used throughout this thesis. Given a matrix $A \in \mathbb{R}^{m\times m}$, A^{-1} and $A^\top$ represent the inverse and the transpose matrix of A respectively. The minimum eigenvalue of the matrix A is denoted by $\underline{\lambda}(A)$. For a partitioned symmetric matrix $B \in \mathbb{R}^{m\times m}$, the symbol $\star$ stands for the symmetric block. $B \geqslant 0$ stands for a matrix is positive semidefinite. Similarly $B \leqslant 0$ means negative semidefinite. For a positive integer n, I_n is the identity matrix in $\mathbb{R}^{n\times n}$. Given a function f, f_t (resp. f_x) stands for the partial derivative with respect to t (resp. x) (this is a shortcut for $\frac{\partial f}{\partial t}$, resp. $\frac{\partial f}{\partial x}$). When there is only one independent variable, $\dot{f}$ and f' stand respectively for the time and the space derivative. $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ is associated with the matrix norm. Consider a function $f : (0,1) \to \mathbb{R}^n$, $\|f\|_{L^2(0,1)}$ is defined by,

$$\|f\|_{L^2(0,1)} = \left(\int_0^1 |f(x)|^2 \, dx\right)^{1/2}. \tag{0.0.2}$$

For a function $f : (0,1) \to \mathbb{R}^n$, we define,

$$\|f\|_{H^2(0,1)} = \left(\int_0^1 (|f(x)|^2 + |f'(x)|^2 + |f''(x)|^2) \, dx\right)^{1/2}. \tag{0.0.3}$$

According to [Coron et al., 2008], for all matrices $G \in \mathbb{R}^{n\times n}$,

$$\rho_1(G) = \inf\{\|\Delta G \Delta^{-1}\|, \Delta \in D_{n,+}\},$$

where $D_{n,+}$ denotes the set of diagonal matrix in $\mathbb{R}^{n\times n}$ with strictly positive elements.

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Chapter 1

Introduction

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1.1 Background and motivations

Systems modeled by partial differential equations (PDEs) with infinite dimensional dynamics are relevant for a wide range of physical networks. The control and stability analysis of such systems become a challenge area. Many research works have been reported (e.g. [Li, 1994], [Coron et al., 2008], [Prieur and Mazenc, 2012] etc.). Singularly perturbed partial differential equations have been considered in research works from late 1980s. This kind of systems is interesting for analysis because of its relevance to many phenomena with multiple time scales in various fields in physics and engineering, such as fluid dynamics, chemical-reactor, aerodynamics etc. (see [Kadalbajoo and Patidar, 2003] as a survey). In the present work, our main concern is a class of linear hyperbolic partial differential equations with multiple time scales.

1.1.1 Hyperbolic systems

A linear hyperbolic system described by partial differential equation is given as follows

$$y_t(x,t) + \Lambda y_x(x,t) + My(x,t) = 0,$$

with the boundary condition

$$y(0,t) = Ky(1,t),$$

whereas the initial condition is given by

$$y(x,0) = y_0(x),$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^n$, Λ is a diagonal positive matrix in $\mathbb{R}^{n\times n}$, M is a matrix of appropriate dimensions, K is a constant matrix in $\mathbb{R}^{n\times n}$ and $y_0 : [0,1] \to \mathbb{R}^n$.

Conservation laws When the matrix M is null, this is the so-called hyperbolic system of conservation laws. The stability analysis for such hyperbolic partial differential equations has been referred in many research works by different methods. A first important result of asymptotic stability for a 2×2 system was reported in [Greenberg and Li, 1984]. A general result for $n \times n$ systems was given by [Li, 1994] later. These results were based on the explicit evolution of the Riemman invariant along the characteristic curves. The weakest sufficient condition given by [Li, 1994, Theorem 1.3] was formulated as: $\rho(|K|) < 1$, where $\rho(K)$ is the spectral radius of matrix $K \in \mathbb{R}^{n\times n}$ and $|K|$ denotes the matrix with all absolute values of the elements of K.

Due to [Hale and Lunel, 1993], a necessary and sufficient stability condition was given in frequency domain. More precisely, the linear hyperbolic system has been considered as a linear time-delay system. It is exponentially stable if and only if there exists $w > 0$ such that

$$\left(\det(I_n - (\mathtt{diag}(e^{-\nu_1 s}, \cdots, e^{-\nu_n s})))K = 0, s \in \mathbb{C}\right) \Rightarrow (\Re(s) \leqslant -w),$$

with $\nu_j = \frac{1}{\Lambda_j}$, $j \in \{1, \cdots, n\}$ and I_n is the identity matrix of $\mathbb{R}^n$, $\Re(s)$ is the real part of the complex number s. Moreover, due to Silkowski, if the $(\nu_1, \cdots, \nu_n)$ are rationally independent, the linear system is exponentially stable if and only if $\rho_0(K) < 1$.
Where

$$\rho_0(K) = \max\{\rho(\mathtt{diag}(e^{\iota\theta_1}, \cdots, e^{\iota\theta_n})K);\ (\theta_1, \cdots \theta_n)^\top \in \mathbb{R}^n\},$$

with $\iota = \sqrt{-1}$.

A different approach, namely Lyapunov approach, is a powerful tool for stability analysis for hyperbolic system. In [Coron et al., 1999] and [Coron et al., 2002], it was introduced for controlling Saint-Venant equation modeled by hyperbolic system of conservation laws where an entropy of the system was chosen as a Lyapunov function. In [Coron et al., 2007] a strict H^2-norm Lyapunov function was constructed to analyze the stability of a system of two hyperbolic conservation laws around equilibrium. In [Coron et al., 2008] a strict Lyapunov function in H^2-norm was employed to prove the exponential stability. The sufficient dissipative boundary condition for one-dimensional $n \times n$ linear hyperbolic systems was presented as follows:

Theorem 1.1.1. *If $\rho_1(K) < 1$, then the linear hyperbolic system is exponentially stable to the origin in H^2-norm.*

Where $\rho_1(K) = \inf\{\|\Delta K \Delta^{-1}\|, \Delta \in D_{n,+}\}$, with $D_{n,+}$ denotes the set of diagonal positive matrices in $\mathbb{R}^{n\times n}$.
The comparison of the two conditions, $\rho_0(K)$ and $\rho_1(K)$, was shown in [Coron et al., 2008]. For every dimension of the system $n \in \{1, 2, 3, 4, 5\}$, the two conditions are equivalent. Thus the following proposition holds:

Proposition 1.1.1. *Let $\nu_i = \frac{1}{\Lambda_i}$ with $i \in \{1, \cdots, n\}$, if $(\nu_1, \cdots, \nu_n)$ are rationally independent, the linear system for every $n \in \{1, 2, 3, 4, 5\}$ is exponentially stable if and only if $\rho_1(K) < 1$.*

Remark 1.1.1. In [Coron et al., 2008], the condition $\rho_1(K) < 1$ was given for exponential stability of one-dimensional $n \times n$ quasilinear hyperbolic systems. It can thus be applied to linear systems. ◦

Balance laws When the matrix M is non null, this is the so-called hyperbolic system of balance laws. [Xu and Sallet, 2002] proposed a Lyapunov function for exponential stability of a general class of linear hyperbolic systems where the matrix M is symmetric. An exponential stability in L^2-norm for the case of gas pipelines governed by isothermal Euler equations was proved by same kind of Lyapunov function in [Gugat and Herty, 2011]. [Gugat et al., 2012] gave a Lyapunov function in H^2-norm for the same Euler equations. In [Diagne et al., 2012], explicit boundary dissipative conditions were established for the exponential stability in L^2-norm with a marginally diagonally stable matrix M.

The control design for hyperbolic systems are usually based on the stability analysis. For instance, backstepping approach is a useful tool for stabilization of such system. It was developed for second-order in time and in space hyperbolic systems in [Krstic et al., 2006b] and [Krstic et al., 2006a]. In [Krstic and Smyshlyaev, 2008], a boundary feedback stabilization of first-order hyperbolic systems using a backstepping approach was introduced. In

[Coron et al., 2013], a full state feedback law was considered to stabilize a quasilinear 2×2 first-order hyperbolic system in H^2-norm.

1.1.2 Coupled ODE-PDE systems

Backstepping approach is widely used to study coupled ordinary differential equation-partial differential equation (ODE-PDE) systems. For example, a coupled first-order hyperbolic PDE and second-order (in space) ODE was stabilized by this approach in [Krstic and Smyshlyaev, 2008]. In [Krstic, 2009b], predictor-like feedback laws and observers were designed for a diffusion PDE (heat equation) and linear time invariant (LTI) ODE in cascade. A cascade of second order (in time) PDE and ODE was studied in [Krstic, 2009a]. In [Susto and Krstic, 2010], the Neumann type interconnections were considered for ODE-PDE cascades. A state feedback boundary controller was proposed to stabilize a coupled PDE (heat equation) and ODE in [Tang and Xie, 2011]. The stabilization of an ODE-Schrödinger cascade was given in [Ren et al., 2013].

Lyapunov technique is also used to analyze the stability for such systems. In [Dos Santos et al., 2008], a strict Lyapunov function was used to prove the stability of a hyperbolic system with the integral actions at the boundary, which is a kind of ODE-PDE system. An open-loop unstable hyperbolic system was stabilized by Proportional Integral (PI) boundary controller, which is proved in the frequency domain in [Bastin et al., 2015].

1.1.3 Singular perturbation theory

Singularly perturbed systems, containing multiple time scales, often occur naturally in physical systems due to the presence of small parasitic parameters, typically small time constants, masses, inductances, moments of inertia. For example semiconducting diodes [Smith, 1985], DC-motors and voltage regulators [Kokotović et al., 1986], flexible mechanical systems [Da and Corless, 1993] and chemical reaction networks [Breusegem and Bastin, 1991].

Singular perturbation was introduced in control engineering in late 1960s, its assimilation in control theory has rapidly developed and has become a tool for analysis and design of control systems. Singular perturbation is a way of neglecting the fast transition and considering them in a separate fast time scale. The significant advantage of this technique is to reduce the system order. For instance, [Kokotović and Haddad, 1975] was concerned with a time optimal control of a linear time invariant system consisting of fast and slow dynamics. An optimally sensitive control for higher order system was approximated by that for a lower order system in [Kokotović and Sannuti, 1968]. [Kokotović and Yackel, 1972] established

sufficient conditions for the solution of a higher order linear regulator problem to tend to the solution of a lower order problem. There are rich literature on such systems described by finite dimensional ordinary differential equations (ODEs), as reported in the survey paper [Kokotović et al., 1976] and [Saksena et al., 1984] where comprehensive bibliographies are included.
The decomposition of a singularly perturbed system into lower order subsystems, namely the reduced subsystem and the boundary-layer subsystem, provides a powerful tool for stability analysis (e.g. [Habets, 1974b], [Chow, 1978], [Grujic, 1981] and [Chow and Kokotović, 1981]). More precisely, let us recall a linear singularly perturbed system presented by finite dimensional ordinary differential equations:

$$\begin{aligned} \dot{x} &= Ax + Bz,\ x \in \mathbb{R}^n, \\ \varepsilon\dot{z} &= Cx + Dz,\ z \in \mathbb{R}^m,\ \varepsilon > 0. \end{aligned}$$

The reduced subsystem representing the slow dynamic is given by

$$\dot{\bar{x}} = A_r\bar{x},$$

where $A_r = A - BD^{-1}C$. The bar denotes the variable belongs to system with $\varepsilon = 0$.
The boundary-layer subsystem standing for the fast dynamics is given by

$$\frac{dy}{d\tau} = Dy,$$

where $y = z + D^{-1}Cx$ and $\tau = \frac{t}{\varepsilon}$.
The problem of analyzing the stability properties of such finite dimensional systems has received considerable attention in the literature. For example, in [Kokotović et al., 1986, Chapter 2], the stability condition was formulated as follows:

Theorem 1.1.2. *If A_r and D are Hurwitz matrices, then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, the full singularly perturbed system is asymptotically stable.*

From the above theorem, the stability of the two subsystems ensures the stability of the full system. For singularly perturbed nonlinear systems, in [Hoppensteadt, 1966] and [Habets, 1974a], the results focused on local results or imposed growth conditions on the nonlinearities of the system. [Saberi and Khalil, 1984] developed a quadratic Lyapunov function to state a set of sufficient interconnection conditions to guarantee asymptotic stability. In [Marino et al., 1989], it was given sufficient conditions and the explicit high gain feedback for nonlinear single-input single-output systems by using singular perturbation method. [Da and Corless, 1993] dealt with the asymptotic stability for a class of nonlinear singularly perturbed systems whose fast dynamics are marginally stable by using the same approach.

[Chen and Hsieh, 1994] was concerned with a geometric approach for singularly perturbed systems. In [Christofides and Teel, 1996], the input-to-state stability was studied for a class of singularly perturbation nonlinear systems.

Tikhonov theorem is a fundamental tool for analysis of singularly perturbed systems. It describes the limiting behaviour of solutions of the perturbed system. For linear system, the Tikhonov approach on infinite time interval was given by [Kokotović et al., 1986, Chapter 2] as follows:

Theorem 1.1.3. *If the matrices A_r and D are Hurwitz, there exist $a > 0$ and $\varepsilon^* > 0$, such that for all $\varepsilon \in (0, \varepsilon^*]$, the following approximations hold for all $t \geqslant 0$,*

$$\begin{aligned} |x(t) - \bar{x}(t)| &\leqslant a\varepsilon, \\ |z(t) + D^{-1}C\bar{x}(t) - y(t/\varepsilon)| &\leqslant a\varepsilon. \end{aligned}$$

For nonlinear system, the Tikhonov approximation on infinite time interval is based on the exponential stability of both reduced subsystem and boundary-layer subsystem. The results have been reported in research works, e.g. [Khalil, 1996], [Verhulst, 2007].

1.1.4 Physical applications and open problems

Physical applications Hyperbolic systems of conservation laws and balance laws are natural models for representing physical networks in many areas of engineering and sciences (see e.g.[Bastin and Coron, 2015, Chapter 1]). Among the potential applications, hydraulic networks [Bastin et al., 2008], gas flow in pipelines [Dick et al., 2010], electrical transmission network [Gugat, 2014] or road traffic networks [Haut and Bastin, 2007] are of significant importance.

In this thesis, we are motivated by the following two applications.
• Open channel problems presented in Figure 5.2.1

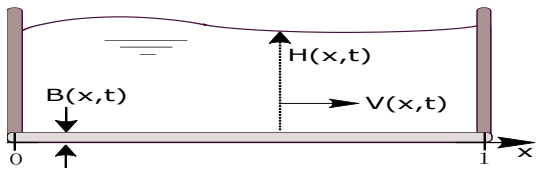

Figure 5.2.1 Open channel

We consider a prismatic channel with a rectangular cross section and a unit width. It is presented by Saint-Venant–Exner equation where the effect of the sediment movement to the flow is taken into account. Since the velocity of the sediment is much slower than the velocity of the water flow, this system can thus be presented by a singularly perturbed hyperbolic system. The regulation problem of open channels presents an economic and environmental interest. It has been considered in the literature using a wide variety of techniques. In [Garcia et al., 1992] and [Malaterre, 1998], discrete linear approximations of the perturbed Saint-Venant equation were used. A development of an H_∞ control design was presented in [Litrico and Georges, 1999a], [Litrico and Georges, 1999b], and [Litrico and Georges, 2001]. [Prieur et al., 2008] dealt with systems of conservation laws for the design of a stabilizing boundary control of a channel, subject to small external perturbations. A boundary control of open channels with experimental results was proposed by [Dos Santos and Prieur, 2008].

• Gas flow transport system presented in Figure 5.2.6

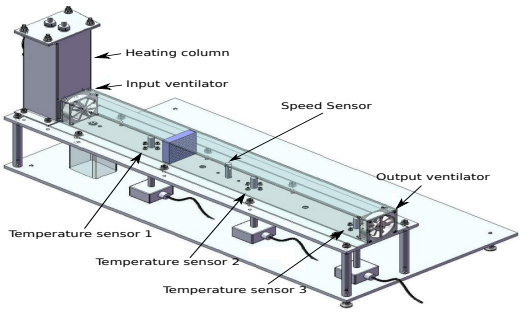

Figure 5.2.6 Gas transport setup

This setup consists of a heating column and a constant section tube. Based on the physical fact that the gas velocity in the tube is much slower than the sound speed, the gas dynamics in the tube is first modeled by a singularly perturbed hyperbolic system. Moreover, if it is considered the gas dynamics in both heating column and tube, then a coupled ODE-PDE system can be used to model this system.
The transportation of gas through pipelines is of great interest for energy supply. The control of such systems has been reported in many research works (see e.g. [Banda et al., 2006], [Colombo et al., 2009]).

Open problems The singular perturbations for partial differential equations have been studied from mathematical point of view in (e.g. [Friedman, 1968], [Michelson, 1985], [Melenk and Schwab, 1999]). Most of these works focused on the existence of the solution, asymptotic expansion or boundary value

problem. To our best knowledge, there are few works on stability analysis and control of this kind of systems. In this thesis, we are interested in linear hyperbolic systems with multiple time scales. Firstly, it has been concerned with a class of linear singularly perturbed hyperbolic systems. It has been studied the relation between the stability of the full system and the fast and slow subsystems. Contrary to finite dimensional linear systems (see Theorem 1.1.2), they are not equivalent. Moreover, the Tikhonov approximation has been achieved for such infinite dimensional systems. Different results are obtained compared to singularly perturbed systems modeled by ODEs (e.g. Theorem 1.1.3). Secondly it has been dealt with a coupled ODE-PDE system. The stability analysis of an ODE coupled with a singularly perturbed PDE is consistent with linear finite dimensional systems. However, it is not true for a singularly perturbed ODE coupled with a PDE system (see respectively Theorem 1.1.2 and Theorem 1.1.3).

1.2 Main contributions of the thesis

The main contributions are presented in the following three sections.

1.2.1 Singularly perturbed linear hyperbolic systems

The first main contribution is the stability analysis and Tikhonov approximation for linear hyperbolic systems of conservation laws and balance laws.

• *Chapter 2:* Singularly perturbed linear hyperbolic system of conservation laws.
The full system is given by:

$$\begin{cases} y_t(x,t) + \Lambda_1 y_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty), \\ \varepsilon z_t(x,t) + \Lambda_2 z_x(x,t) = 0,\ \varepsilon > 0, \\ \begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix},\ t \in [0,+\infty), \\ \begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix},\ x \in [0,1], \end{cases}$$

where Λ_1 and Λ_2 are diagonal positive matrices, $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is a constant matrix of appropriate dimensions.
The reduced subsystem representing the slow dynamics is

$$\begin{cases} \bar{y}_t(x,t) + \Lambda_1 \bar{y}_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty), \\ \bar{y}(0,t) = K_r \bar{y}(1,t),\ t \in [0,+\infty), \\ \bar{y}(x,0) = \bar{y}_0(x) = y_0(x),\ x \in [0,1], \end{cases}$$

where $K_r = K_{11} + K_{12}(I_m - K_{22})^{-1}K_{21}$.
The boundary-layer subsystem standing for the fast dynamics is

$$\begin{cases} \bar{z}_\tau(x,\tau) + \Lambda_2 \bar{z}_x(x,\tau) = 0, \ x \in [0,1], \ \tau \in [0,+\infty), \\ \bar{z}(0,\tau) = K_{22}\bar{z}(1,\tau), \ \tau \in [0,+\infty), \\ \bar{z}(x,0) = \bar{z}_0(x) = z_0(x) - (I_m - K_{22})^{-1}K_{21}y_0(1), \ x \in [0,1], \end{cases}$$

where $\tau = t/\varepsilon$.
Section 2.2 studies the relation between the stability of the full system and both subsystems. The stability of the full system implies the stability of both subsystems which is given in Proposition 2.2.1. On the other hand, a counterexample is used to illustrate that the stability of the two subsystems does not guarantee the stability of the full system. This shows a major difference with what is well known for linear finite dimensional systems (e.g. Theorem 1.1.2).
Sections 2.3 and 2.4 are concerned with Tikhonov approximation for such systems. Under the stability condition $\rho_1(K) < 1$ implying that the full system is stable, the solution of the slow dynamics of the full system is approximated by the solution of the reduced subsystem for sufficiently small perturbation parameter. This result is first proved by a Lyapunov function in L^2-norm in Section 2.3. The errors between the full system and the subsystems are estimated as the order of the perturbation parameter ε. Moreover, in Section 2.4, we propose a Lyapunov function in H^2-norm, where more precise estimates are obtained.

• *Chapter 3:* Singularly perturbed linear hyperbolic system of balance laws. Compared to the previous chapter, we consider the system with non uniform transport velocity and source terms, which are both dependent on ε as well as the boundary condition. Section 3.1 gives the descriptions of the full system and the reduced subsystem. The full system is

$$\begin{cases} y_t(x,t) + \Lambda_1(\varepsilon)y_x(x,t) = a(\varepsilon)y(x,t) + b(\varepsilon)z(x,t), \ x \in [0,1], \ t \in [0,+\infty) \\ \varepsilon z_t(x,t) + \Lambda_2(\varepsilon)z_x(x,t) = c(\varepsilon)y(x,t) + d(\varepsilon)z(x,t), \ \varepsilon > 0, \\ \begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K(\varepsilon) \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in [0,+\infty), \\ \begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix}, \quad x \in [0,1], \end{cases}$$

where $\Lambda_1(\varepsilon)$ and $\Lambda_2(\varepsilon)$ are positive diagonal matrices, $a(\varepsilon)$, $b(\varepsilon)$, $c(\varepsilon)$ and $d(\varepsilon)$ are matrices of appropriate dimensions and vanish at $\varepsilon = 0$. $K(\varepsilon) = \begin{pmatrix} K_{11}(\varepsilon) & K_{12}(\varepsilon) \\ K_{21}(\varepsilon) & K_{22}(\varepsilon) \end{pmatrix}$ is a matrix for each $\varepsilon > 0$.
The reduced subsystem is given by

$$\begin{cases} \bar{y}_t(x,t) + \Lambda_1(0)\bar{y}_x(x,t) = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \\ \bar{y}(0,t) = K_r\bar{y}(1,t), \quad t \in [0,+\infty), \\ \bar{y}(x,0) = \bar{y}_0(x) = y_0(x), \quad x \in [0,1], \end{cases}$$

where $\Lambda_1(0)$ is diagonal positive and $K_r = K_{11}(0)+K_{12}(0)(I_m-K_{22}(0))^{-1}K_{21}(0)$. Section 3.2 presents the Tikhonov approximation for such system. Under the stability condition $\rho_1(K(0)) < 1$ and the assumptions of the continuity of the transport velocity, the source terms and the boundary condition, the solution of the full system is approximated by that of the reduced subsystem. The estimate of the error between the full system and the reduced subsystem is the order of ε.

1.2.2 Singularly perturbed coupled ODE-PDE systems

The second main contribution is the stability analysis and the Tikhonov approximation for a coupled ODE-PDE system with two time scales.

• *Chapter 4:* Coupled ODE-PDE systems.
I) An ODE coupled with a fast PDE system
Section 4.1.1 considers the coupled system where the perturbation parameter ε is introduced in PDE system:

$$\begin{cases} \dot{Z}(t) = AZ(t) + By(1,t),\ t \in [0,+\infty), \\ \varepsilon y_t(x,t) + \Lambda y_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty),\ \varepsilon > 0, \\ y(0,t) = G_1 y(1,t) + G_2 Z(t),\ t \in [0,+\infty), \\ Z(0) = Z_0, \\ y(x,0) = y_0(x),\ x \in [0,1], \end{cases}$$

where A and B are matrices of appropriate dimensions, Λ is a diagonal positive matrix. G_1 and G_2 are constant matrices of appropriate dimensions. The reduced subsystem is

$$\begin{cases} \dot{\bar{Z}}(t) = (A + BG_r)\bar{Z}(t),\ t \in [0,+\infty), \\ \bar{Z}_0 = Z_0, \end{cases}$$

where $G_r = (I_m - G_1)^{-1}G_2$.
The boundary-layer subsystem is

$$\begin{cases} \bar{y}_\tau(x,\tau) + \Lambda \bar{y}_x(x,\tau) = 0,\ x \in [0,1],\ \tau \in [0,+\infty), \\ \bar{y}(0,\tau) = G_1 \bar{y}(1,\tau),\ \tau \in [0,+\infty), \\ \bar{y}_0(x) = y_0(x) - G_r Z_0,\ x \in [0,1], \end{cases}$$

where $\tau = t/\varepsilon$.
The same stability result as for finite dimensional systems is proved in Section 4.1.2 (compare Theorem 1.1.2 with Theorem 4.1.1). Precisely, the two subsystems' stability implies the stability of the full system. This is proved by Lyapunov method. The Tikhonov theorem for this kind of systems is given in Section 4.1.3.

II) A fast ODE coupled with a PDE system

Section 4.2.1 presents the full system where ε is in the dynamics of the ODE system:

$$\begin{cases} \varepsilon\dot{Y}(t) = AY(t) + Bz(1),\ t \in [0,+\infty),\ \varepsilon > 0, \\ z_t(x,t) + \Lambda z_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty), \\ z(0,t) = K_1 z(1,t) + K_2 Y(t),\ t \in [0,+\infty), \\ Y(0) = Y_0, \\ z(x,0) = z_0(x),\ x \in [0,1], \end{cases}$$

where K_1 and K_2 are real constant matrices of appropriate dimensions.
The reduced subsystem is

$$\begin{cases} \bar{z}_t(x,t) + \Lambda \bar{z}_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty), \\ \bar{z}(0,t) = K_r \bar{z}(1,t),\ t \in [0,+\infty), \\ \bar{z}(x,0) = \bar{z}_0(x) = z_0(x),\ x \in [0,1], \end{cases}$$

where $K_r = (K_1 - K_2 A^{-1} B)$.
The boundary-layer subsystem is

$$\begin{cases} \frac{d\bar{Y}(\tau)}{d\tau} = A\bar{Y}(\tau),\ \tau \in [0,+\infty), \\ \bar{Y}(0) = \bar{Y}_0 = Y_0 + A^{-1}Bz_0(1), \end{cases}$$

where $\tau = t/\varepsilon$.
Differently from the stability result for system where the slow dynamics is given by ODE, as considered in Section 4.1, in Section 4.2.2 a counterexample shows that the full system can be not stable even though both subsystems are stable. Moreover, a coupling condition is proposed to ensure the stability for such systems. The Tikhonov approximation for such coupled ODE-PDE systems is established in Section 4.2.3.

1.2.3 Boundary control synthesis and applications

The third main contribution is the boundary control synthesis based on singular perturbation method and the boundary control design to different applications.

• *Chapter 5:* Boundary control synthesis for linear hyperbolic system of balance laws and boundary control design for two applications.

I) Boundary control synthesis

Let us consider the singularly perturbed system of balance laws presented in Chapter 3. In Section 5.1, the reduced subsystem is stabilized in finite time $T = \frac{1}{\underline{\lambda}(\Lambda_1(0))}$ by choosing the boundary condition $K_r = 0$. The boundary condition for the full system $K(\varepsilon)$ is chosen based on K_r such that $\rho_1(K(0)) < 1$. Then the full system reaches a neighborhood of size proportional to the perturbation parameter ε to the origin at time T.

II) Boundary control design to an open channel
We consider an open channel modeled by Saint-Venant–Exner equation. After linerization around a steady state, the system is written in Riemann coordinates. Based on the physical fact that the movement of the sediment is much slower than the water flow, this system is finally written in singularly perturbed form. Boundary controls are established to ensure the stability of the full system. Moreover, the slow dynamics is stabilized in finite time. The main results are shown in Section 5.2.1.

III) Boundary control design to a gas flow transport system
Gas dynamics in a constant section tube
Firstly, let us consider the gas dynamics in a constant section tube which is governed by Euler equation. The perturbation parameter of this system is the ratio of the velocity of gas and of sound speed. The boundary controls are proposed to stabilize the full system. Based on the singular perturbation approach, the slow dynamics converges to the equilibrium in finite time. Section 5.2.2 shows the main results.

Gas dynamics in both heating column and tube
Secondly, let us consider the gas dynamics in a heating column as well as in a constant section tube. This system is modeled by a coupled ODE-PDE system. The gas dynamics in the heating column is governed by an ODE and the gas dynamics in the tube is presented by a PDE. The two subsystems are connected by boundary condition. Since the gas dynamics in the tube is much faster than that in the heating column, the full system is stabilized by a boundary control based on singular perturbation method. The main results are given in Section 5.2.3.

1.3 List of publications related to this work

Journal papers:

• Tang, Y., Prieur, C., and Girard, A. (2015). Tikhonov theorem for linear hyperbolic systems, Automatica, 57:1-10.

• Tang, Y., Prieur, C., and Girard, A. Singular perturbation approximation of linear hyperbolic systems of balance laws. Conditionally accepted for publication in IEEE Transactions on Automatic Control.

• Tang, Y., Prieur, C., and Girard, A. Singular perturbation approximation by means of a H2 Lyapunov function for linear hyperbolic systems. Submitted for publication in Systems and Control Letters.

International conference papers:

• Tang, Y., Prieur, C., and Girard, A. (2015a) Stability analysis of a singularly perturbed coupled ODE-PDE system. In Conference on Decision and Control, Osaka, Japan.

• Tang, Y., Prieur, C., and Girard, A. (2014b). Boundary control synthesis for hyperbolic systems: a singular perturbation approach, In Conference on Decision and Control, pages 2840-2845, Los Angeles, USA.

• Tang, Y., Prieur, C., and Girard, A. (2014a). Approximation of singularly perturbed linear hyperbolic systems. In 21th International Symposium on Mathematical Theory of Networks and Systems, pages 1620-1623, Groningen, The Netherlands.

• Tang, Y., Prieur, C., and Girard, A. (2013b). A new H^2-norm Lyapunov function for the stability of a singularly perturbed system of two conservation laws. In Conference on Decision and Control, pages 3026-3031, Florence, Italy.

• Tang, Y., Prieur, C., and Girard, A. (2013a). Lyapunov stability of a singularly perturbed system of two conservation laws. In 1st IFAC Workshop on Control of Systems Governed by Partial Differential Equations, Paris, France.

Chapter 2

Tikhonov theorem for linear hyperbolic systems of conservation laws

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This chapter is concerned with a class of linear hyperbolic systems of conservation laws with a perturbation parameter introduced in the dynamics. The two subsystems, representing the slow dynamics and the fast dynamics, are formally calculated. We first link the stability between the full system and both subsystems. The stability of the full system implies the stability of the two subsystems. However, the stability of the two subsystems is not enough to guarantee the full system's stability, as illustrated by a counterexample. The estimates of the errors between the full system and the subsystems are given in Tikhonov theorem. The estimates results are obtained by two kinds of Lyapunov functions. More precisely, we first use a L^2 Lyapunov function to prove the estimates. Then a H^2 Lyapunov function is considered to get more precise estimates.

This chapter is organized as follows. The full system under consideration is presented in Section 2.1 and the two subsystems are computed in the same section. The stability analysis for such system is given in the following Section 2.2. Sections 2.3 and 2.4 present the Tikhonov approximation by Lyapunov functions in L^2-norm and in H^2-norm respectively. The results based on the two methods are compared in Section 2.5. The numerical simulations of academic examples are shown in Section 2.6. All the proofs of the lemmas are given in Section 2.7.

The main results of Sections 2.2 and 2.3 are published in [Tang et al., 2015b]. The main result of Section 2.4 is submitted for publication in [Tang et al., a].

2.1 Introduction

2.1.1 Full system description

Let us consider the following linear singularly perturbed system of conservation laws

$$y_t(x,t) + \Lambda_1 y_x(x,t) = 0, \tag{2.1.1a}$$

$$\varepsilon z_t(x,t) + \Lambda_2 z_x(x,t) = 0, \tag{2.1.1b}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^n$, $z : [0,1] \times [0,+\infty) \to \mathbb{R}^m$, Λ_1 is a diagonal positive matrix in $\mathbb{R}^{n\times n}$, Λ_2 is a diagonal positive matrix in $\mathbb{R}^{m\times m}$, the perturbation parameter ε is a small positive value. Moreover, we consider the following boundary condition

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \ t \in [0,+\infty), \tag{2.1.2}$$

where $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is a constant matrix in $\mathbb{R}^{(n+m)\times(n+m)}$ with the matrices K_{11} in $\mathbb{R}^{n\times n}$, K_{12} in $\mathbb{R}^{n\times m}$, K_{21} in $\mathbb{R}^{m\times n}$ and K_{22} in $\mathbb{R}^{m\times m}$. Given two functions $y_0 : [0,1] \to \mathbb{R}^n$ and $z_0 : [0,1] \to \mathbb{R}^m$, the initial condition is

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix}, \ x \in [0,1]. \tag{2.1.3}$$

Remark 2.1.1. Let us recall the existence of the solutions to the Cauchy problem (2.1.1)-(2.1.3) in L^2-norm. According to Section 2.1 in [Coron, 2007], for all $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in L^2(0,1)$, the Cauchy problem (2.1.1)-(2.1.3) has a unique weak solution $\begin{pmatrix} y \\ z \end{pmatrix} \in C^0([0,+\infty), (L^2(0,1), \mathbb{R}^{n+m}))$. ∘

Remark 2.1.2. The existence of the solutions to the Cauchy problem (2.1.1)-(2.1.3) in H^2-norm is given by Proposition 2.1 in [Coron et al., 2008].

For every $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in H^2(0,1)$ satisfying the following compatibility conditions

$$\begin{pmatrix} y_0(0) \\ z_0(0) \end{pmatrix} = K \begin{pmatrix} y_0(1) \\ z_0(1) \end{pmatrix}, \tag{2.1.4a}$$

$$\begin{pmatrix} \Lambda_1 y_{0x}(0) \\ \varepsilon^{-1}\Lambda_2 z_{0x}(0) \end{pmatrix} = K \begin{pmatrix} \Lambda_1 y_{0x}(1) \\ \varepsilon^{-1}\Lambda_2 z_{0x}(1) \end{pmatrix}, \tag{2.1.4b}$$

the Cauchy problem (2.1.1)-(2.1.3) has a unique maximal classical solution $\begin{pmatrix} y \\ z \end{pmatrix} \in C^0([0,+\infty),(H^2(0,1),\mathbb{R}^{n+m}))$. ◦

2.1.2 Computation of two subsystems

Considering infinite dimensional systems described by partial differential equations (PDEs), let us compute the two subsystems for (2.1.1)-(2.1.2), namely the reduced and the boundary-layer subsystems. Inspired by the approach for finite dimensional systems described by ordinary differential equations (ODEs) in [Saberi and Khalil, 1984] and [Khalil, 1996, Chapter 9], the two subsystems are formally calculated as follows. Setting $\varepsilon = 0$ in (2.1.1b) yields

$$y_t(x,t) + \Lambda_1 y_x(x,t) = 0, \tag{2.1.5a}$$

$$z_x(x,t) = 0. \tag{2.1.5b}$$

Substituting (2.1.5b) into the second line of the boundary condition (2.1.2) and assuming $(I_m - K_{22})$ invertible yield

$$z(.,t) = (I_m - K_{22})^{-1} K_{21} y(1,t), \tag{2.1.6a}$$

$$y(0,t) = (K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}) y(1,t). \tag{2.1.6b}$$

The reduced subsystem is computed as

$$\bar{y}_t(x,t) + \Lambda_1 \bar{y}_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty), \tag{2.1.7}$$

with the boundary condition

$$\bar{y}(0,t) = K_r \bar{y}(1,t),\ t \in [0,+\infty), \tag{2.1.8}$$

where $K_r = K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}$, whereas the initial condition is given as the same as for the full system

$$\bar{y}(x,0) = \bar{y}_0(x) = y_0(x),\ x \in [0,1]. \tag{2.1.9}$$

Remark 2.1.3. The bar indicates the variable belongs to the system with $\varepsilon = 0$. ◦

To ensure the existence of H^2 solutions for the reduced subsystem, we consider the following compatibility conditions

$$\bar{y}_0(0) = K_r \bar{y}_0(1), \tag{2.1.10a}$$

$$\Lambda_1 \bar{y}_{0x}(0) = K_r \Lambda_1 \bar{y}_{0x}(1). \tag{2.1.10b}$$

Remark 2.1.4. In [Perrollaz and Rosier, 2014], a 2×2 quasilinear hyperbolic system is considered in $C^0([0,T] \times [0,1], \mathbb{R}^2)$ instead of $H^2(0,1)$, avoiding so strong compatibility conditions. However, in Section 2.3 of the present work, the H^2 convergence of a system is mandatory thus the compatibility conditions (2.1.10) should be considered as in Section 2.4 where the stability of the full system in H^2-norm is established. ◦

To compute the boundary-layer subsystem, let us first perform the following change of variable

$$\bar{z}(x,t) = z(x,t) - (I_m - K_{22})^{-1} K_{21} y(1,t). \tag{2.1.11}$$

This shifts the equilibrium of z to the origin. Let us use a new time variable $\tau = \frac{t}{\varepsilon}$. In the τ time scale, $y(1,t)$ in (2.1.11) is considered as a fixed parameter with respect to time. The boundary-layer subsystem is computed as

$$\bar{z}_\tau(x,\tau) + \Lambda_2 \bar{z}_x(x,\tau) = 0, \ x \in [0,1], \ \tau \in [0,+\infty), \tag{2.1.12}$$

with the boundary condition

$$\bar{z}(0,\tau) = K_{22} \bar{z}(1,\tau), \ \tau \in [0,+\infty), \tag{2.1.13}$$

whereas the initial condition is given as

$$\bar{z}(x,0) = \bar{z}_0(x) = z_0(x) - (I_m - K_{22})^{-1} K_{21} y_0(1), \ x \in [0,1]. \tag{2.1.14}$$

2.2 Stability analysis of two subsystems and counterexample

In this section, we first show how the stability of the full system (2.1.1)-(2.1.3) implies the stability of the two subsystems, the reduced subsystem (2.1.7)-(2.1.9) and the boundary-layer subsystem (2.1.12)-(2.1.14). We next give a counterexample to illustrate that the stability of both subsystems does not guarantee the stability of the full system. This shows a major difference with what is well known for linear finite dimensional systems (e.g. Theorem 1.1.2).

The following definition is adopted for the exponential stability of the full system (2.1.1)-(2.1.3) in L^2-norm.

Definition 2.2.1. *The linear system of conservation laws (2.1.1)-(2.1.3) is exponentially stable to the origin in L^2-norm if there exist $\gamma_1 > 0$ and $C_1 > 0$, such that for every $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in L^2(0,1)$, the solution to the system (2.1.1)-(2.1.3) satisfies*

$$\left\| \begin{pmatrix} y(.,t) \\ z(.,t) \end{pmatrix} \right\|_{L^2(0,1)} \leqslant C_1 e^{-\gamma_1 t} \left\| \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \right\|_{L^2(0,1)}, \ t \in [0, +\infty).$$

Similarly the exponential stability of the linear system of conservation laws (2.1.1)-(2.1.3) in H^2-norm is given by the following definition.

Definition 2.2.2. *The linear system of conservation laws (2.1.1)-(2.1.3) is exponentially stable to the origin in H^2-norm if there exist $\gamma_2 > 0$ and $C_2 > 0$, such that for every $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in H^2(0,1)$ satisfying the compatibility conditions (2.1.4), the solution to the system (2.1.1)-(2.1.3) satisfies*

$$\left\| \begin{pmatrix} y(.,t) \\ z(.,t) \end{pmatrix} \right\|_{H^2(0,1)} \leqslant C_2 e^{-\gamma_2 t} \left\| \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \right\|_{H^2(0,1)}, \ t \in [0, +\infty).$$

We can define the exponential stability in L^2-norm and H^2-norm of the reduced and boundary-layer subsystems in a similar way.
In the following, we are ready to state a proposition which is about the stability of both subsystems.

Proposition 2.2.1. *If $\rho_1(K) < 1$, then the reduced subsystem (2.1.7)-(2.1.9) and the boundary-layer subsystem (2.1.12)-(2.1.14) are exponentially stable to the origin in L^2-norm and in H^2-norm.*

Proof. Let us first prove the stability of the reduced subsystem (2.1.7)-(2.1.9). Let $\rho_1(K) \leqslant \alpha^* < 1$ and $\Delta \in D_{(n+m),+}$ such that $\|\Delta K \Delta^{-1}\| \leqslant \alpha^*$. Let $\Delta_1 \in D_{n,+}$ and $\Delta_2 \in D_{m,+}$, Δ be such that

$$\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}.$$

Consider $\tilde{Y} \in \mathbb{R}^{n+m}$ given by

$$\tilde{Y} = \begin{pmatrix} Y \\ \Delta_2 (I_m - K_{22})^{-1} K_{21} \Delta_1^{-1} Y \end{pmatrix},$$

where Y is an arbitrary vector of $\mathbb{R}^n$. It follows directly

$$\Delta K \Delta^{-1} \tilde{Y} = \begin{pmatrix} \Delta_1 (K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}) \Delta_1^{-1} Y \\ \Delta_2 (I_m - K_{22})^{-1} K_{21} \Delta_1^{-1} Y \end{pmatrix}. \tag{2.2.1}$$

Since $\|\Delta K \Delta^{-1}\| \leqslant \alpha^*$, it follows from (2.2.1)

$$|\Delta K \Delta^{-1} \tilde{Y}| \leqslant \alpha^* |\tilde{Y}|,$$

hence

$$|\Delta_1 (K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}) \Delta_1^{-1} Y| \leqslant \alpha^* |Y|.$$

The previous inequality holds, for all $Y \in \mathbb{R}^n$, therefore

$$\|\Delta_1 (K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}) \Delta_1^{-1}\| \leqslant \alpha^* < 1, \tag{2.2.2}$$

which implies $\rho_1(K_r) < 1$. Due to Theorem 1.1.1, the reduced subsystem (2.1.7)-(2.1.9) is exponentially stable in L^2-norm and in H^2-norm.
Next let prove the stability of the boundary-layer subsystem (2.1.12)-(2.1.14). Similarly, consider $\tilde{Z} \in \mathbb{R}^{n+m}$ given by

$$\tilde{Z} = \begin{pmatrix} 0 \\ Z \end{pmatrix},$$

where Z is an arbitrary vector of $\mathbb{R}^m$. It follows directly

$$\Delta K \Delta^{-1} \tilde{Z} = \begin{pmatrix} \Delta_1 K_{12} \Delta_2^{-1} Z \\ \Delta_2 K_{22} \Delta_2^{-1} Z \end{pmatrix}. \tag{2.2.3}$$

Since $\|\Delta K \Delta^{-1}\| \leqslant \alpha^*$, it follows from (2.2.3)

$$|\Delta K \Delta^{-1} \tilde{Z}| \leqslant \alpha^* |\tilde{Z}|,$$

hence

$$|\Delta_2 K_{22} \Delta_2^{-1} Z| \leqslant \alpha^* |Z|.$$

The previous inequality holds, for all $Z \in \mathbb{R}^m$, therefore

$$\|\Delta_2 K_{22} \Delta_2^{-1}\| \leqslant \alpha^* < 1, \tag{2.2.4}$$

which implies $\rho_1(K_{22}) < 1$. Due to Theorem 1.1.1, the boundary-layer subsystem (2.1.12)-(2.1.14) is exponentially stable in L^2-norm and in H^2-norm. This concludes the proof of Proposition 2.2.1. □

Counterexample: the stability criterion $\rho_1(K) < 1$ is a sufficient condition for stability of the reduced subsystem (2.1.7)-(2.1.9) and the boundary-layer subsystem (2.1.12)-(2.1.14). On the other hand, the stability of the two subsystems does not guarantee the stability of the full system (2.1.1)-(2.1.3). To see this, let us consider the following example. Let $\Lambda_1 = \Lambda_2 = 1$ in (2.1.1) with $n = m = 1$. The boundary condition of the full system (2.1.2) is chosen as $K_{11} = 2.5$, $K_{12} = -1$, $K_{21} = 1$, $K_{22} = 0.5$. The boundary condition of the reduced subsystem (2.1.8) is computed as $K_r = 0.5$. It holds $\rho_1(K_r) < 1$. By Theorem 1.1.1 the reduced subsystem (2.1.7)-(2.1.9) is exponentially stable in L^2-norm and in H^2-norm. The boundary condition of

the boundary-layer subsystem (2.1.13) is $K_{22} = 0.5$. It holds $\rho_1(K_{22}) < 1$. The boundary-layer subsystem (2.1.12)-(2.1.14) is exponentially stable in L^2-norm and in H^2-norm according to Theorem 1.1.1. Now let us check the stability condition $\rho_1(K) < 1$, which is equivalent to find a diagonal positive matrix Δ such that $\|\Delta K \Delta^{-1}\| < 1$ and it is in fact equivalent to $K^\top \Delta^2 K < \Delta^2$ (see [Coron et al., 2008] section 4). Let us consider a positive value β^* which is a little bit larger than 1. There is no loss of generality to look for $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ such that

$$K^\top \Delta^2 K < \beta^* \Delta^2. \tag{2.2.5}$$

Developing the above equation yields

$$\begin{pmatrix} 2.5^2 + b^2 & -2.5 + 0.5b^2 \\ -2.5 + 0.5b^2 & 1 + 0.5^2 b^2 \end{pmatrix} < \beta^* \begin{pmatrix} 1 & 0 \\ 0 & b^2 \end{pmatrix},$$

straightforward computations show that there is no $b > 0$ such that matrix $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ satisfies the condition (2.2.5), thus $\rho_1(K) > 1$. Due to Proposition 1.1.1, $\rho_1(K) < 1$ is a necessary and sufficient condition for stability of linear hyperbolic systems with dimension 1 to 5. Since this example is a linear system of two conservation laws, there exists $\varepsilon^* > 0$ such that for all irrational $\varepsilon \in (0, \varepsilon^*)$, the full system is not exponentially stable neither in L^2-norm nor in H^2-norm, although the reduced and boundary-layer subsystems are both exponentially stable.
Let us select the initial conditions: $y_0 = \bar{y}_0 = \cos(4\pi x) - 1$, $z_0 = \cos(2\pi x) - 1$, $\bar{z}_0 = z_0 - (1 - K_{22})^{-1} K_{21} y_0(1)$. The perturbation parameter $\varepsilon = \frac{1}{\sqrt{500}}$. The numerical simulation results are given in the following. Figures 2.2.1 and 2.2.2 are the time evolutions of the solutions of the reduced and boundary-layer subsystems respectively. The solutions of the two subsystems converge to the origin as time increases. On the other hand, the solutions of the full system, y and z, converge to infinity as time increases, which are shown in Figures 2.2.3 and 2.2.4. The numerical simulations are consistent with the above results.

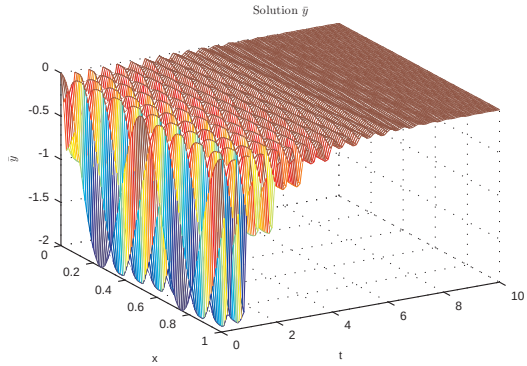

Figure 2.2.1: Time evolution of the solution of the reduced subsystem $\bar{y}$.

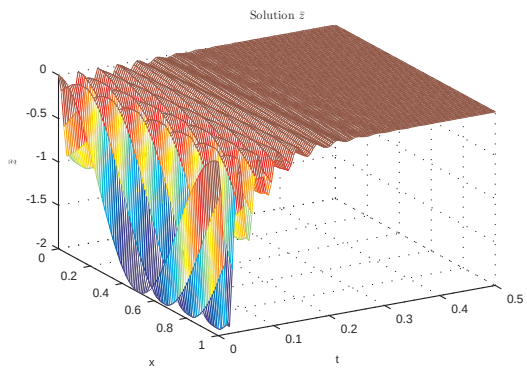

Figure 2.2.2: Time evolution of the solution of the boundary-layer subsystem $\bar{z}$.

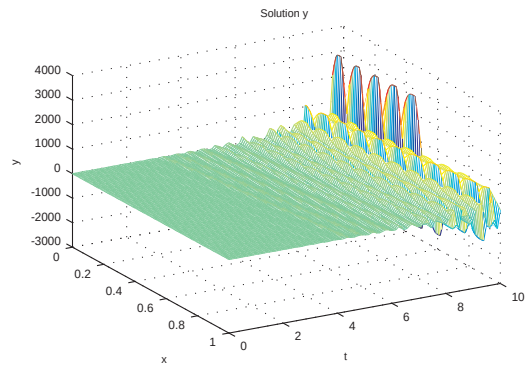

Figure 2.2.3: Time evolution of the solution of the slow dynamics y of the full system.

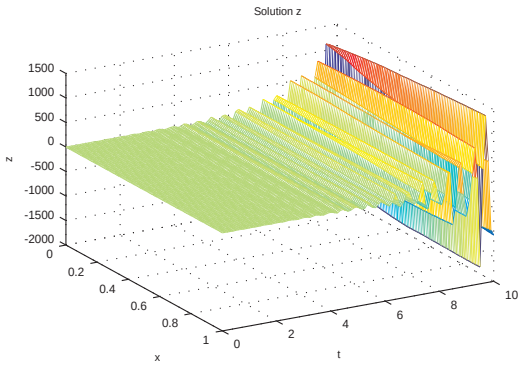

Figure 2.2.4: Time evolution of the solution of the fast dynamics z of the full system.

2.3 Tikhonov approximation by means of a L^2 Lyapunov function

In this section we present how solutions to the full system (2.1.1)-(2.1.3) can be approximated by solutions to the reduced subsystem (2.1.7)-(2.1.9). This approximation is proved based on a L^2 Lyapunov function.

2.3.1 Main result

Let us state Tikhonov approximation of the full system (2.1.1)-(2.1.3) in the following Theorem 2.3.1.

Theorem 2.3.1. *Consider the linear singularly perturbed system of conservation laws (2.1.1)-(2.1.3). Assuming that the condition $\rho_1(K) < 1$ is satisfied, there exist positive values ε^*, C, $\tilde{C}$ and ω such that for all $0 < \varepsilon < \varepsilon^*$, for any initial condition $y_0 \in H^2(0,1)$ satisfying the compatibility conditions (2.1.10) with $y_0 = \bar{y}_0$ and for any $z_0 \in L^2(0,1)$, the following holds for all $t \geqslant 0$,*

$$\begin{aligned} \|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \quad &\leqslant \\ \varepsilon C e^{-\omega t}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)\|^2_{L^2(0,1)} \Big), \end{aligned} \tag{2.3.1}$$

$$\begin{aligned} \int_0^{+\infty} \|z(.,t) - (I_m - K_{22})^{-1} K_{21} \bar{y}(1,t)\|^2_{L^2(0,1)} dt \quad &\leqslant \\ \varepsilon \tilde{C}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)\|^2_{L^2(0,1)} \Big). \end{aligned} \tag{2.3.2}$$

In the above theorem, the bound of ε^* could be explicitly calculated. However, the computations are complicated since it depends on many parameters which are specified in the proof of this theorem. The error between the slow dynamics y of the full system and the dynamics $\bar{y}$ of the reduced subsystem is estimated in L^2-norm. However, the estimate of the time integral of the L^2-norm of the error between the fast dynamics z of the full system and its equilibrium is actually in $L^2([0,+\infty)\times(0,1))$. To obtain the estimate of $\|z(.,t)-(I_m-K_{22})^{-1}K_{21}\bar{y}(1,t)\|_{L^2(0,1)}$, the particular initial condition z_0 has to be considered, which is given in the following corollary.

Corollary 2.3.1. *Assuming $\rho_1(K)<1$, there exist positive constants C^*, C_1^*, C_2^*, ε^* and ω, such that for all $0<\varepsilon<\varepsilon^*$, for any $\bar{y}_0\in H^2(0,1)$ satisfying the compatibility conditions (2.1.10) with $\bar{y}_0=y_0$ and if z_0 is the equilibrium point, that is $z_0=(I_m-K_{22})^{-1}K_{21}\bar{y}_0(1)$, the following estimates hold for all $t\geqslant 0$,*

$$\|y(.,t)-\bar{y}(.,t)\|^2_{L^2(0,1)}\leqslant \varepsilon^2 C^* e^{-\omega t}\|\bar{y}_0\|^2_{H^2(0,1)}, \tag{2.3.3}$$

$$\int_0^{+\infty}\|z(.,t)-(I_m-K_{22})^{-1}K_{21}\bar{y}(1,t)\|^2_{L^2(0,1)}dt\leqslant \varepsilon^2 C_1^*\|\bar{y}_0\|^2_{H^2(0,1)}, \tag{2.3.4}$$

$$\|z(.,t)-(I_m-K_{22})^{-1}K_{21}\bar{y}(1,t)\|^2_{L^2(0,1)}\leqslant \varepsilon C_2^* e^{-\omega t}\|\bar{y}_0\|^2_{H^2(0,1)}. \tag{2.3.5}$$

2.3.2 Proofs of Theorem 2.3.1 and Corollary 2.3.1

Theorem 2.3.1 and Corollary 2.3.1 are proved in this section.

Proof of Theorem 2.3.1 Let us use the following three steps to prove Theorem 2.3.1.
Step I) Let us perform the following change of variables

$$\eta = y-\bar{y}, \tag{2.3.6a}$$
$$\delta = z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1,\cdot), \tag{2.3.6b}$$

where η is the error between the slow dynamics y of the full system and $\bar{y}$ of the reduced subsystem, δ is the error between the fast dynamics z of the full system and its equilibrium point. Due to (2.3.6) and (2.1.7), system (2.1.1) is written in the new variables (η,δ) as follows

$$\eta_t+\Lambda_1\eta_x=0, \tag{2.3.7a}$$
$$\varepsilon\delta_t+\Lambda_2\delta_x=\varepsilon(I_m-K_{22})^{-1}K_{21}\Lambda_1\bar{y}_x(1,\cdot). \tag{2.3.7b}$$

The boundary conditions are calculated as follows

$$\begin{aligned}
\eta(0,t) &= y(0,t) - \bar{y}(0,t) \\
&= K_{11}y(1,t) + K_{12}z(1,t) - K_r\bar{y}(1,t) \\
&= K_{11}\Big(y(1,t) - \bar{y}(1,t)\Big) + K_{12}\Big(z(1,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t)\Big) \\
&= K_{11}\eta(1,t) + K_{12}\delta(1,t), \\
\delta(0,t) &= z(0,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t) \\
&= K_{21}y(1,t) + K_{22}z(1,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t) \\
&= K_{21}\Big(y(1,t) - \bar{y}(1,t)\Big) + K_{22}z(1,t) \\
&\quad + \Big(I_m - (I_m - K_{22})^{-1}\Big)K_{21}\bar{y}(1,t) \\
&= K_{21}\eta(1,t) + K_{22}\delta(1,t).
\end{aligned}$$

Thus the boundary condition for system (2.3.7) is written as

$$\begin{pmatrix} \eta(0,t) \\ \delta(0,t) \end{pmatrix} = K \begin{pmatrix} \eta(1,t) \\ \delta(1,t) \end{pmatrix}. \tag{2.3.8}$$

Let us consider the following candidate L^2 Lyapunov function for system (2.3.7)-(2.3.8)

$$V_\varepsilon(\eta,\delta) = \int_0^1 e^{-\mu x}\big(\eta^\top Q\eta + \varepsilon\delta^\top P\delta\big)dx, \tag{2.3.9}$$

with $\mu > 0$, Q a diagonal positive matrix in $\mathbb{R}^{n\times n}$ and P a diagonal positive matrix in $\mathbb{R}^{m\times m}$, which will be specified later.

Step II) Let us state the following two lemmas. The time derivative of $V_\varepsilon(\eta,\delta)$ is estimated in Lemma 2.3.1.

Lemma 2.3.1. *If the boundary condition satisfies $\rho_1(K) < 1$, then there exist a positive value μ and diagonal positive matrices P and Q such that for all $\varepsilon > 0$ and $\kappa > 0$, it holds*

$$\begin{aligned}
\dot{V}_\varepsilon(\eta,\delta) \leqslant\ & -\mu\int_0^1 e^{-\mu x}\eta^\top Q\Lambda_1\eta\, dx \\
& -\left(\mu - \frac{\varepsilon\kappa\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\|}{\underline{\lambda}(P\Lambda_2)}\right)\int_0^1 e^{-\mu x}\delta^\top P\Lambda_2\delta\, dx \\
& + \frac{\varepsilon\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\|}{\kappa}|\bar{y}_x(1,\cdot)|^2.
\end{aligned} \tag{2.3.10}$$

The term $|\bar{y}_x(1,\cdot)|$ is bounded as follows

$$\begin{aligned} |\bar{y}_x(1,\cdot)| &= \left|\int_0^1 \Big(x\bar{y}_{xx} + \bar{y}_x\Big)dx\right| \\ &\leqslant \int_0^1 \Big(|\bar{y}_{xx}| + |\bar{y}_x| + |\bar{y}|\Big)dx \leqslant \sqrt{3}\|\bar{y}(.,t)\|_{H^2(0,1)}. \end{aligned} \tag{2.3.11}$$

Thus, in order to bound the term $|\bar{y}_x(1,\cdot)|$, it is necessary to analyze the stability of the reduced subsystem in H^2-norm. Since we choose the initial condition of the reduced system $\bar{y}_0 = y_0$, thus $\bar{y}_0 \in H^2(0,1)$ and the compatibility conditions $\bar{y}_0(0) = K_r\bar{y}_0(1)$ and $\Lambda_1\bar{y}_{0x}(0) = K_r\Lambda_1\bar{y}_{0x}(1)$ are satisfied. The estimate of $\|\bar{y}(.,t)\|^2_{H^2(0,1)}$ is given in the following Lemma 2.3.2.

Lemma 2.3.2. *If $\rho_1(K) < 1$, let μ as in Lemma 2.3.1, there exists a positive value C_r such that for any initial condition $\bar{y}_0 \in H^2(0,1)$ satisfying the compatibility conditions $\bar{y}_0(0) = K_r\bar{y}_0(1)$ and $\Lambda_1\bar{y}_{0x}(0) = K_r\Lambda_1\bar{y}_{0x}(1)$, it holds*

$$\|\bar{y}(.,t)\|^2_{H^2(0,1)} \leqslant C_r e^{-\mu\underline{\lambda}(\Lambda_1)t}\|\bar{y}_0\|^2_{H^2(0,1)}, \quad t \geqslant 0. \tag{2.3.12}$$

All the proofs of lemmas are given in Section 2.7.

Step III) Combining (2.3.10), (2.3.11) and (2.3.12) yields

$$\begin{aligned} \dot{V}_\varepsilon(\eta,\delta) \leqslant & -\mu\int_0^1 e^{-\mu x}\eta^\top Q\Lambda_1\eta\, dx \\ & -\left(\mu - \frac{\varepsilon\kappa\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}{\underline{\lambda}(P\Lambda_2)}\right)\int_0^1 e^{-\mu x}\delta^\top P\Lambda_2\delta\, dx \\ & +\frac{3\varepsilon C_r\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}{\kappa}e^{-\mu\underline{\lambda}(\Lambda_1)t}\|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned} \tag{2.3.13}$$

By choosing $\kappa = \frac{\mu\underline{\lambda}(P\Lambda_2)}{2\varepsilon\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}$, we get

$$\begin{aligned} \dot{V}_\varepsilon(\eta,\delta) \leqslant & -\frac{\mu\beta}{2}V_\varepsilon(\eta,\delta) \\ & +\frac{6\varepsilon^2 C_r\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu\underline{\lambda}(P\Lambda_2)}e^{-\mu\underline{\lambda}(\Lambda_1)t}\|\bar{y}_0\|^2_{H_2(0,1)}, \end{aligned}$$

where $\beta = \min\Big(\underline{\lambda}(\Lambda_1), \underline{\lambda}(\Lambda_2)\Big)$. Due to $\underline{\lambda}(\Lambda_1) > \frac{\beta}{2}$, it follows

$$\begin{aligned} V_\varepsilon(\eta,\delta) \leqslant & \; e^{-\frac{\mu\beta}{2}t}V_\varepsilon(\eta_0,\delta_0) \\ & +\frac{6\varepsilon^2 C_r\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu^2\underline{\lambda}(P\Lambda_2)\left(\underline{\lambda}(\Lambda_1)-\frac{\beta}{2}\right)}e^{-\frac{\mu\beta}{2}t}\|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned} \tag{2.3.14}$$

Since $V_\varepsilon(\eta,\delta)$ is lower and upper estimated by

$$\begin{aligned} & e^{-\mu}\underline{\lambda}(Q)\|\eta\|^2_{L^2(0,1)} + \varepsilon e^{-\mu}\underline{\lambda}(P)\|\delta\|^2_{L^2(0,1)} \\ \leqslant\ & V_\varepsilon(\eta,\delta) \\ \leqslant\ & \|Q\|\ \|\eta\|^2_{L^2(0,1)} + \varepsilon\|P\|\ \|\delta\|^2_{L^2(0,1)}, \end{aligned} \tag{2.3.15}$$

it follows that

$$\begin{aligned} \|\eta(.,t)\|^2_{L^2(0,1)} \leqslant\ & \frac{e^{\mu}}{\underline{\lambda}(Q)}V_\varepsilon(\eta,\delta) \\ \leqslant\ & \frac{e^{\mu}}{\underline{\lambda}(Q)}\Bigg(e^{-\frac{\mu\beta}{2}t}V_\varepsilon(\eta_0,\delta_0) \\ & + \frac{6\varepsilon^2 C_r\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu^2\underline{\lambda}(P\Lambda_2)\left(\underline{\lambda}(\Lambda_1)-\frac{\beta}{2}\right)}e^{-\frac{\mu\beta}{2}t}\|\bar{y}_0\|^2_{H^2(0,1)}\Bigg) \\ \leqslant\ & \frac{e^{\mu}}{\underline{\lambda}(Q)}\Bigg(e^{-\frac{\mu\beta}{2}t}\Big(\|Q\|\ \|\eta_0\|^2_{L^2(0,1)} + \varepsilon\|P\|\ \|\delta_0\|^2_{L^2(0,1)}\Big) \\ & + \frac{6\varepsilon^2 C_r\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu^2\underline{\lambda}(P\Lambda_2)\left(\underline{\lambda}(\Lambda_1)-\frac{\beta}{2}\right)}e^{-\frac{\mu\beta}{2}t}\|\bar{y}_0\|^2_{H^2(0,1)}\Bigg), \end{aligned}$$

since $y_0=\bar{y}_0$, i.e. $\eta_0=0$, therefore

$$\begin{aligned} \|\eta(.,t)\|^2_{L^2(0,1)} \leqslant\ & \varepsilon C_a e^{-\frac{\mu\beta}{2}t}\Bigg(\|\delta_0\|^2_{L^2(0,1)} \\ & + \frac{6\varepsilon\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu^2\|P\|\underline{\lambda}(P\Lambda_2)\left(\underline{\lambda}(\Lambda_1)-\frac{\beta}{2}\right)}\|\bar{y}_0\|^2_{H^2(0,1)}\Bigg). \end{aligned} \tag{2.3.16}$$

where C_a is a given positive value. This proves (2.3.1).
Remarking the first term in the right hand side in (2.3.13) is negative and taking the same $\kappa=\frac{\mu\underline{\lambda}(P\Lambda_2)}{2\varepsilon\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}$ yield

$$\begin{aligned} \dot{V}_\varepsilon(\eta,\delta) \leqslant\ & -\frac{\mu}{2}e^{-\mu}\underline{\lambda}(P\Lambda_2)\|\delta(.,t)\|^2_{L^2(0,1)} \\ & + \frac{6\varepsilon^2 C_r\|P(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu\underline{\lambda}(P\Lambda_2)}e^{-\mu\underline{\lambda}(\Lambda_1)t}\|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned} \tag{2.3.17}$$

Performing the time integration of both sides of (2.3.17) and using $\lim_{t\to+\infty} V_\varepsilon(\eta,\delta)=$

0, we get

$$\begin{aligned} 0 - V_\varepsilon(\eta_0, \delta_0) &\leqslant -\frac{\mu}{2} e^{-\mu} \underline{\lambda}(P\Lambda_2) \int_0^{+\infty} \|\delta(.,t)\|^2_{L^2(0,1)} dt \\ &\quad + \frac{6\varepsilon^2 C_r \|P(I_m - K_{22})^{-1} K_{21}\Lambda_1\|^2}{\mu^2 \underline{\lambda}(P\Lambda_2)\underline{\lambda}(\Lambda_1)} \|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned} \tag{2.3.18}$$

Reorganizing (2.3.18), we have

$$\begin{aligned} \int_0^{+\infty} \|\delta(.,t)\|^2_{L^2(0,1)} dt &\leqslant \frac{2e^\mu}{\mu \underline{\lambda}(P\Lambda_2)} \Bigg(V_\varepsilon(\eta_0, \delta_0) \\ &\quad + \frac{6\varepsilon^2 C_r \|P(I_m - K_{22})^{-1} K_{21}\Lambda_1\|^2}{\mu^2 \underline{\lambda}(P\Lambda_2)\underline{\lambda}(\Lambda_1)} \|\bar{y}_0\|^2_{H^2(0,1)} \Bigg). \end{aligned}$$

Therefore with (2.3.15) and $\eta_0 = 0$, it follows

$$\int_0^{+\infty} \|\delta(.,t)\|^2_{L^2(0,1)} dt \leqslant \varepsilon C_{b1} \Big(\|\delta_0\|^2_{L^2(0,1)} + \varepsilon C_{b2} \|\bar{y}_0\|^2_{H^2(0,1)} \Big), \tag{2.3.19}$$

where C_{b1} and C_{b2} are positive constants. This proves (2.3.2). This concludes the proof of Theorem 2.3.1. □

Proof of Corollary 2.3.1 The proof of this corollary is similar to that of Theorem 2.3.1. Since $\delta_0 = 0$ in (2.3.16) and (2.3.19), the followings hold

$$\|\eta(.,t)\|^2_{L^2(0,1)} \leqslant \varepsilon^2 e^{-\frac{\mu\beta}{2}t} \left(\frac{6C_a \|P(I_m - K_{22})^{-1} K_{21}\Lambda_1\|^2}{\mu^2 \|P\| \underline{\lambda}(P\Lambda_2) \left(\underline{\lambda}(\Lambda_1) - \frac{\beta}{2} \right)} \|\bar{y}_0\|^2_{H^2(0,1)} \right),$$

$$\int_0^{+\infty} \|\delta(.,t)\|^2_{L^2(0,1)} dt \leqslant \varepsilon^2 C_{b2} \|\bar{y}_0\|^2_{H^2(0,1)}.$$

This proves (2.3.3) and (2.3.4).
Due to (2.3.14) and (2.3.15), with $\eta_0 = 0$ and $\delta_0 = 0$, therefore

$$\|\delta(.,t)\|^2_{L^2(0,1)} \leqslant \varepsilon e^{-\frac{\mu\beta}{2}t} \frac{6C_r C_b \|P(I - K_{22})^{-1} K_{21}\Lambda_1\|^2}{\mu^2 \underline{\lambda}(P\Lambda_2) \left(\underline{\lambda}(\Lambda_1) - \frac{\beta}{2} \right)} \|\bar{y}_0\|^2_{H^2(0,1)},$$

where $C_b > 0$. This proves (2.3.5). This concludes the proof of Corollary 2.3.1. □

2.4 Tikhonov approximation by means of a H^2 Lyapunov function

In this section, the estimates of system (2.1.1)-(2.1.3) are established based on a H^2 Lyapunov function.

2.4.1 Main result

The Tikhonov approximation for system (2.1.1)-(2.1.3) is stated as follows.

Theorem 2.4.1. *Consider the linear singularly perturbed hyperbolic system (2.1.1)-(2.1.3). If the boundary condition matrix K satisfies $\rho_1(K) < 1$, there exist positive values ε^*, l_1, l_2, θ such that for all $0 < \varepsilon < \varepsilon^*$, for all $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in H^2(0,1)$ satisfying the compatibility conditions (2.1.4) and (2.1.10) with $\bar{y}_0 = y_0$, the following holds for all $t \geqslant 0$,*

$$\begin{aligned} & \|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \\ \leqslant \; & \varepsilon l_1 e^{-\varepsilon\theta t}\Big(\|z_0 - (I_m - K_{22})^{-1}K_{21}\bar{y}_0(1)\|^2_{H^2(0,1)} + \|\bar{y}_0\|^2_{H^2(0,1)} \Big), \end{aligned} \tag{2.4.1}$$

$$\begin{aligned} & \int_0^{+\infty} \|z(.,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t)\|^2_{H^2(0,1)} dt \\ \leqslant \; & \varepsilon l_2 \Big(\|z_0 - (I_m - K_{22})^{-1}K_{21}\bar{y}_0(1)\|^2_{H^2(0,1)} + \|\bar{y}_0\|^2_{H^2(0,1)} \Big). \end{aligned} \tag{2.4.2}$$

By choosing the particular initial condition z_0, we establish more precise estimates in the following Corollary 2.4.1.

Corollary 2.4.1. *If $\rho_1(K) < 1$, there exist positive values l_1^*, l_2^*, l_3^*, ε^* and θ, such that for all $0 < \varepsilon < \varepsilon^*$ and for all $y_0 \in H^2(0,1)$ satisfying the compatibility conditions (2.1.4) and (2.1.10) with $\bar{y}_0 = y_0$, and let $z_0 = (I_m - K_{22})^{-1}K_{21}\bar{y}_0(1)$, the following holds for all $t \geqslant 0$,*

$$\|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \leqslant \varepsilon^2 l_1^* e^{-\varepsilon\theta t}\|\bar{y}_0\|^2_{H^2(0,1)}, \tag{2.4.3}$$

$$\int_0^{+\infty} \|z(.,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t)\|^2_{H^2(0,1)} dt \leqslant \varepsilon^2 l_2^* \|\bar{y}_0\|^2_{H^2(0,1)}, \tag{2.4.4}$$

$$\|z(.,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t)\|^2_{H^2(0,1)} \leqslant \varepsilon l_3^* e^{-\varepsilon\theta t}\|\bar{y}_0\|^2_{H^2(0,1)}. \tag{2.4.5}$$

2.4.2 Proofs of Theorem 2.4.1 and Corollary 2.4.1

We consider a more sophisticated system presented as follows

$$\begin{aligned} \eta_t + \Lambda_1 \eta_x &= 0, \\ \varepsilon z_t + \Lambda_2 z_x &= 0, \\ \bar{y}_t + \Lambda_1 \bar{y}_x &= 0, \end{aligned} \tag{2.4.6}$$

where $\eta = y - \bar{y}$ is chosen as the same as in Section 2.3, z is the fast dynamics in (2.1.1)-(2.1.2) and $\bar{y}$ is the dynamics in (2.1.7) and (2.1.8). According to (2.1.2) and (2.1.8), the boundary conditions for system (2.4.6) are computed for all $t \geqslant 0$,

$$\eta(0,t) = K_{11}\eta(1,t) + K_{12}z(1,t) - K_c\bar{y}(1,t), \tag{2.4.7a}$$
$$z(0,t) = K_{21}\eta(1,t) + K_{22}z(1,t) + K_{21}\bar{y}(1,t), \tag{2.4.7b}$$
$$\bar{y}(0,t) = K_r\bar{y}(1,t), \tag{2.4.7c}$$

where $K_c = K_{12}(I_m - K_{22})^{-1}K_{21}$.
The compatibility conditions (2.1.4) and (2.1.10) are written as follows

$$\eta_0(0) = K_{11}\eta_0(1) + K_{12}z_0(1) - K_c\bar{y}_0(1), \tag{2.4.8a}$$
$$z_0(0) = K_{21}\eta_0(1) + K_{22}z_0(1) + K_{21}\bar{y}_0(1), \tag{2.4.8b}$$
$$\bar{y}_0(0) = K_r\bar{y}_0(1), \tag{2.4.8c}$$
$$\eta_{0x}(0) = \Lambda_1^{-1}K_{11}\Lambda_1\eta_{0x}(1) + \frac{\Lambda_1^{-1}K_{12}\Lambda_2}{\varepsilon}z_{0x}(1) - \Lambda_1^{-1}K_c\Lambda_1\bar{y}_{0x}(1), \tag{2.4.8d}$$
$$z_{0x}(0) = \varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\eta_{0x}(1) + \Lambda_2^{-1}K_{22}\Lambda_2 z_{0x}(1) + \varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\bar{y}_{0x}(1), \tag{2.4.8e}$$
$$\bar{y}_{0x}(1) = \Lambda_1^{-1}K_r\Lambda_1\bar{y}_{0x}(1). \tag{2.4.8f}$$

In the following let us prove Theorem 2.4.1 by three steps.

Proof of Theorem 2.4.1

Step I) Let us introduce a candidate H^2 Lyapunov function for system (2.4.6)-(2.4.7)

$$\begin{aligned} V_\varepsilon(\eta, z, \bar{y}) &= a\int_0^1 e^{-\mu x}\left(\frac{1}{\varepsilon}\eta^\top\tilde{P}_1\eta + \varepsilon\eta_x^\top\tilde{P}_2\eta_x + \varepsilon^3\eta_{xx}^\top\tilde{P}_3\eta_{xx}\right)dx \\ &+ b\int_0^1 e^{-\mu x}\Bigg((z - (I_m - K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1(z - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)) \\ &+ z_x^\top\tilde{Q}_2 z_x + z_{xx}^\top\tilde{Q}_3 z_{xx}\Bigg)dx + \varepsilon c\int_0^1 e^{-\mu x}\left(\bar{y}^\top\tilde{P}_1\bar{y} + \bar{y}_x^\top\tilde{P}_2\bar{y}_x + \bar{y}_{xx}^\top\tilde{P}_3\bar{y}_{xx}\right)dx, \end{aligned}$$

where a, b, c and μ are positive values, $\tilde{P}_1$-$\tilde{P}_3$ and $\tilde{Q}_1$-$\tilde{Q}_3$ are diagonal positive, which will be specified later. If initial conditions satisfy the compatibility conditions (2.4.8), then $V_\varepsilon(\eta, z, \bar{y})$ is well defined along the solutions to (2.4.6) and (2.4.7). To simplify the analysis of time derivative of $V_\varepsilon(\eta, z, \bar{y})$, we rewrite it in the following way

$$V_\varepsilon(\eta, z, \bar{y}) = V_\varepsilon^0(\eta, z, \bar{y}) + V_\varepsilon^1(\eta, z, \bar{y}) + V_\varepsilon^2(\eta, z, \bar{y}), \tag{2.4.9}$$

with $V_\varepsilon^0(\eta, z, \bar{y})$, $V_\varepsilon^1(\eta, z, \bar{y})$ and $V_\varepsilon^2(\eta, z, \bar{y})$ selected respectively to the 0th, 1st and 2nd space derivative of the solutions, which are

$$\begin{aligned}
V_\varepsilon^0(\eta, z, \bar{y}) &= \frac{a}{\varepsilon}\int_0^1 e^{-\mu x}\eta^\top \tilde{P}_1\eta\, dx + \varepsilon c\int_0^1 e^{-\mu x}\bar{y}^\top \tilde{P}_1\bar{y}\, dx \\
&\quad + b\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top \tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\, dx, \\
V_\varepsilon^1(\eta, z, \bar{y}) &= \varepsilon a\int_0^1 e^{-\mu x}\eta_x^\top \tilde{P}_2\eta_x\, dx + b\int_0^1 e^{-\mu x}z_x^\top \tilde{Q}_2 z_x\, dx \\
&\quad + \varepsilon c\int_0^1 e^{-\mu x}\bar{y}_x^\top \tilde{P}_2\bar{y}_x\, dx, \\
V_\varepsilon^2(\eta, z, \bar{y}) &= \varepsilon^3 a\int_0^1 e^{-\mu x}\eta_{xx}^\top \tilde{P}_3\eta_{xx}\, dx + b\int_0^1 e^{-\mu x}z_{xx}^\top \tilde{Q}_3 z_{xx}\, dx \\
&\quad + \varepsilon c\int_0^1 e^{-\mu x}\bar{y}_{xx}^\top \tilde{P}_3\bar{y}_{xx}\, dx.
\end{aligned}$$

Step II) The estimates of $\dot{V}_\varepsilon^0(\eta, z, \bar{y})$, $\dot{V}_\varepsilon^1(\eta, z, \bar{y})$ and $\dot{V}_\varepsilon^2(\eta, z, \bar{y})$ are stated in the following lemmas. First, let us establish an estimate of $\dot{V}_\varepsilon^0(\eta, z, \bar{y})$.

Lemma 2.4.1. *If the boundary condition matrix K satisfies $\rho_1(K) < 1$, then there exist positive real values a, b and μ such that for all positive c and ε, along the solutions to (2.4.6)-(2.4.7), it holds*

$$\begin{aligned}
\dot{V}_\varepsilon^0(\eta, z, \bar{y}) &\leqslant -\frac{a\mu\underline{\lambda}(\Lambda_1)}{\varepsilon}\int_0^1 e^{-\mu x}\eta^\top \tilde{P}_1\eta\, dx - \varepsilon c\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\bar{y}^\top \tilde{P}_1\bar{y}\, dx \\
&\quad -\frac{b\mu\underline{\lambda}(\Lambda_2)}{\varepsilon}\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top \tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\, dx \\
&\quad +2b\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\, \tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\, \bar{y}_x(1)\, dx.
\end{aligned} \tag{2.4.10}$$

Next we establish an estimate of $\dot{V}_\varepsilon^1(\eta, z, \bar{y})$. Differentiating system (2.4.6) with respect to x yields

$$\begin{aligned}
\eta_{xt} + \Lambda_1\eta_{xx} &= 0, \\
\varepsilon z_{xt} + \Lambda_2 z_{xx} &= 0, \\
\bar{y}_{xt} + \Lambda_1\bar{y}_{xx} &= 0.
\end{aligned} \tag{2.4.11}$$

Differentiating (2.4.7) with respect to t and using (2.4.6), we obtain the following boundary conditions for all $t \geqslant 0$,

$$\begin{array}{rcl} \eta_x(0,t) & = & \Lambda_1^{-1}K_{11}\Lambda_1\eta_x(1,t) + \frac{\Lambda_1^{-1}K_{12}\Lambda_2}{\varepsilon}z_x(1,t) - \Lambda_1^{-1}K_c\Lambda_1\bar{y}_x(1,t), \\ z_x(0,t) & = & \varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\eta_x(1,t) + \Lambda_2^{-1}K_{22}\Lambda_2 z_x(1,t) + \varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\bar{y}_x(1,t), \\ \bar{y}_x(0,t) & = & \Lambda_1^{-1}K_r\Lambda_1\bar{y}_x(1,t). \end{array} \tag{2.4.12}$$

The estimate on $\dot{V}_\varepsilon^1(\eta, z, \bar{y})$ is given in the following lemma.

Lemma 2.4.2. *Assume $\rho_1(K) < 1$ and let a, b and μ as in Lemma 2.4.1. Then there exists a positive real value c' such that for all $c > c'$ and $\varepsilon > 0$, along the solutions to (2.4.11)-(2.4.12), it holds*

$$\begin{array}{rcl} \dot{V}_\varepsilon^1(\eta, z, \bar{y}) & \leqslant & -\varepsilon a\mu\underline{\lambda}(\Lambda_1)\displaystyle\int_0^1 e^{-\mu x}\eta_x^\top\tilde{P}_2\eta_x\,dx - \frac{b\mu\underline{\lambda}(\Lambda_2)}{\varepsilon}\int_0^1 e^{-\mu x}z_x^\top\tilde{Q}_2 z_x\,dx \\ & & -\varepsilon c\mu\underline{\lambda}(\Lambda_1)\displaystyle\int_0^1 e^{-\mu x}\bar{y}_x^\top\tilde{P}_2\bar{y}_x\,dx. \end{array} \tag{2.4.13}$$

Lastly, we establish an estimate of $\dot{V}_\varepsilon^2(\eta, z, \bar{y})$. Let us differentiate system (2.4.11) with respect to x. It follows

$$\begin{array}{rcl} \eta_{xxt} + \Lambda_1\eta_{xxx} & = & 0, \\ \varepsilon z_{xxt} + \Lambda_2 z_{xxx} & = & 0, \\ \bar{y}_{xxt} + \Lambda_1\bar{y}_{xxx} & = & 0. \end{array} \tag{2.4.14}$$

Differentiating (2.4.12) with respect to t and using (2.4.11), the boundary conditions are given as follows, for all $t \geqslant 0$,

$$\begin{array}{l} \eta_{xx}(0,t) = \Lambda_1^{-2}K_{11}\Lambda_1^2\eta_{xx}(1,t) + \frac{\Lambda_1^{-2}K_{12}\Lambda_2^2}{\varepsilon^2}z_{xx}(1,t) - \Lambda_1^{-2}K_c\Lambda_1^2\bar{y}_{xx}(1,t), \\ z_{xx}(0,t) = \varepsilon^2\Lambda_2^{-2}K_{21}\Lambda_1^2\eta_{xx}(1,t) + \Lambda_2^{-2}K_{22}\Lambda_2^2 z_{xx}(1,t) + \varepsilon^2\Lambda_2^{-2}K_{21}\Lambda_1^2\bar{y}_{xx}(1,t), \\ \bar{y}_{xx}(0,t) = \Lambda_1^{-2}K_r\Lambda_1^2\bar{y}_{xx}(1,t). \end{array} \tag{2.4.15}$$

The estimate on $\dot{V}_\varepsilon^2(\eta, z, \bar{y})$ is given in the following lemma.

Lemma 2.4.3. *Assume $\rho_1(K) < 1$ and let a, b, c' and μ as in Lemmas 2.4.1 and 2.4.2. Then, there exists a positive real value ε^*, such that for all $0 < \varepsilon < \varepsilon^*$ and $c > c'$, along the solutions to (2.4.14)-(2.4.15), it holds*

$$\begin{array}{rcl} \dot{V}_\varepsilon^2(\eta, z, \bar{y}) & \leqslant & -\varepsilon^3 a\mu\underline{\lambda}(\Lambda_1)\displaystyle\int_0^1 e^{-\mu x}\eta_{xx}^\top\tilde{P}_3\eta_{xx}\,dx \\ & & -\dfrac{b\mu\underline{\lambda}(\Lambda_2)}{\varepsilon}\displaystyle\int_0^1 e^{-\mu x}z_{xx}^\top\tilde{Q}_3 z_{xx}\,dx - \varepsilon c\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\bar{y}_{xx}^\top\tilde{P}_3\bar{y}_{xx}\,dx. \end{array} \tag{2.4.16}$$

The proofs of Lemmas 2.4.1, 2.4.2 and 2.4.3 are given in Section 2.7.

Step III) Let us compute the time derivative of $V_\varepsilon(\eta, z, \bar{y})$

$$\dot{V}_\varepsilon(\eta, z, \bar{y}) = \dot{V}_\varepsilon^0(\eta, z, \bar{y}) + \dot{V}_\varepsilon^1(\eta, z, \bar{y}) + \dot{V}_\varepsilon^2(\eta, z, \bar{y}).$$

Using Lemmas 2.4.1-2.4.3, it follows

$$\begin{aligned}
\dot{V}_\varepsilon(\eta, z, \bar{y}) \quad \leqslant \quad & -a\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\left(\frac{\eta^\top \tilde{P}_1\eta}{\varepsilon} + \varepsilon\eta_x^\top \tilde{P}_2\eta_x + \varepsilon^3\eta_{xx}^\top \tilde{P}_3\eta_{xx}\right)dx \\
& -\frac{b\mu\underline{\lambda}(\Lambda_2)}{\varepsilon}\int_0^1 e^{-\mu x}\Big((z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top \tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)) \\
& \qquad\qquad +z_x^\top \tilde{Q}_2 z_x + z_{xx}^\top \tilde{Q}_3 z_{xx}\Big)dx \\
& -\varepsilon c\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\Big(\bar{y}^\top \tilde{P}_1\bar{y} + \bar{y}_x^\top \tilde{P}_2\bar{y}_x + \bar{y}_{xx}^\top \tilde{P}_3\bar{y}_{xx}\Big)dx \\
& +2b\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\ \tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\ \bar{y}_x(1)\ dx.
\end{aligned}$$

Applying Young's inequality, for all positive values κ_1, it follows

$$\begin{aligned}
\dot{V}_\varepsilon(\eta, z, \bar{y}) \quad \leqslant \quad & -a\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\left(\frac{\eta^\top \tilde{P}_1\eta}{\varepsilon} + \varepsilon\eta_x^\top \tilde{P}_2\eta_x + \varepsilon^3\eta_{xx}^\top \tilde{P}_3\eta_{xx}\right)dx \\
& -\frac{b\mu\underline{\lambda}(\Lambda_2)}{\varepsilon}\int_0^1 e^{-\mu x}\Big((z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top \tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)) \\
& \qquad\qquad +z_x^\top \tilde{Q}_2 z_x + z_{xx}^\top \tilde{Q}_3 z_{xx}\Big)dx \\
& -\varepsilon c\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\Big(\bar{y}^\top \tilde{P}_1\bar{y} + \bar{y}_x^\top \tilde{P}_2\bar{y}_x + \bar{y}_{xx}^\top \tilde{P}_3\bar{y}_{xx}\Big)dx \\
& +\kappa_1 b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|\ |\bar{y}_x(1)|^2 + \frac{b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}{\kappa\underline{\lambda}(\tilde{Q}_1)} \\
& \times\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top \tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\ dx.
\end{aligned}$$

Using the estimate of $|\bar{y}_x(1)|$ in (2.3.11), it follows

$$\begin{aligned}
\dot{V}_\varepsilon(\eta,z,\bar{y}) \quad \leqslant \quad & -a\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\left(\frac{\eta^\top\tilde{P}_1\eta}{\varepsilon}+\varepsilon\eta_x^\top\tilde{P}_2\eta_x+\varepsilon^3\eta_{xx}^\top\tilde{P}_3\eta_{xx}\right)dx \\
& -\left(\frac{b\mu\underline{\lambda}(\Lambda_2)}{\varepsilon}-\frac{b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}{\kappa\underline{\lambda}(\tilde{Q}_1)}\right) \\
& \quad \times\int_0^1 e^{-\mu x}\Big((z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)) \\
& \qquad\qquad +z_x^\top\tilde{Q}_2z_x+z_{xx}^\top\tilde{Q}_3z_{xx}\Big)dx \\
& -\Big(\varepsilon c\mu\underline{\lambda}(\Lambda_1)-3e^\mu\kappa_1 b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|\Big) \\
& \quad \times\int_0^1 e^{-\mu x}\Big(\bar{y}^\top\tilde{P}_1\bar{y}+\bar{y}_x^\top\tilde{P}_2\bar{y}_x+\bar{y}_{xx}^\top\tilde{P}_3\bar{y}_{xx}\Big)dx.
\end{aligned}$$

By choosing $\kappa_1=\frac{2\varepsilon\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|}{\mu\underline{\lambda}(Q_1)\underline{\lambda}(\Lambda_2)}$, it follows

$$\begin{aligned}
\dot{V}_\varepsilon(\eta,z,\bar{y}) \quad \leqslant \quad & -a\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\left(\frac{\eta^\top\tilde{P}_1\eta}{\varepsilon}+\varepsilon\eta_x^\top\tilde{P}_2\eta_x+\varepsilon^3\eta_{xx}^\top\tilde{P}_3\eta_{xx}\right)dx \\
& -\frac{b\mu\underline{\lambda}(\Lambda_2)}{2\varepsilon}\int_0^1 e^{-\mu x}\Big((z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1 \\
& \qquad\qquad \times(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))+z_x^\top\tilde{Q}_2z_x+z_{xx}^\top\tilde{Q}_3z_{xx}\Big)dx \\
& -\left(\varepsilon c\mu\underline{\lambda}(\Lambda_1)-\frac{6\varepsilon e^\mu b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu\underline{\lambda}(\tilde{Q}_1)\underline{\lambda}(\Lambda_2)}\right) \\
& \quad \times\int_0^1 e^{-\mu x}\Big(\bar{y}^\top\tilde{P}_1\bar{y}+\bar{y}_x^\top\tilde{P}_2\bar{y}_x+\bar{y}_{xx}^\top\tilde{P}_3\bar{y}_{xx}\Big)dx. \qquad (2.4.17)
\end{aligned}$$

Let $c^*=\max\left(c',\frac{6\varepsilon e^\mu b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu^2\underline{\lambda}(\tilde{Q}_1)\underline{\lambda}(\Lambda_2)\underline{\lambda}(\Lambda_1)}\right)$, where c' is given in Lemma 2.4.2. There exists ε^* which is given in Lemma 2.4.3 such that for all $0<\varepsilon<\varepsilon^*$ and $c>c^*$, it follows $\dot{V}(\eta,z,\bar{y})\leqslant-\varepsilon\theta V(\eta,z,\bar{y})$, where $\theta=\left(c\mu\underline{\lambda}(\Lambda_1)-\frac{6e^\mu b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu\underline{\lambda}(\tilde{Q}_1)\underline{\lambda}(\Lambda_2)}\right)$. We get the following inequality for all $t\geqslant 0$.

$$V_\varepsilon(\eta,z,\bar{y})\leqslant e^{-\varepsilon\theta t}V_\varepsilon(\eta_0,z_0,\bar{y}_0). \qquad (2.4.18)$$

Using the fact that

$$\frac{ae^{-\mu}\underline{\lambda}(\tilde{P})}{\varepsilon}\|\eta\|^2_{L^2(0,1)} + be^{-\mu}\underline{\lambda}(\tilde{Q})\left\|z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)\right\|^2_{H^2(0,1)}$$
$$+\varepsilon ce^{-\mu}\underline{\lambda}(\tilde{P})\|\bar{y}\|^2_{H^2(0,1)} \leqslant V_\varepsilon(\eta,z,\bar{y}) \leqslant \frac{a\|\tilde{P}\|}{\varepsilon}\|\eta\|^2_{H^2(0,1)}$$
$$+b\|\tilde{Q}\|\left\|z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)\right\|^2_{H^2(0,1)} + \varepsilon c\|\tilde{P}\|\|\bar{y}\|^2_{H^2(0,1)}, \quad (2.4.19)$$

where

$$\underline{\lambda}(\tilde{Q}) = \min\left\{\underline{\lambda}(\tilde{Q}_1), \underline{\lambda}(\tilde{Q}_2), \underline{\lambda}(\tilde{Q}_3)\right\},$$
$$\underline{\lambda}(\tilde{P}) = \min\left\{\underline{\lambda}(\tilde{P}_1), \underline{\lambda}(\tilde{P}_2), \underline{\lambda}(\tilde{P}_3)\right\},$$
$$\|\tilde{P}\| = \max\left\{\|\tilde{P}_1\|, \|\tilde{P}_2\|, \|\tilde{P}_3\|\right\},$$
$$\|\tilde{Q}\| = \max\left\{\|\tilde{Q}_1\|, \|\tilde{Q}_2\|, \|\tilde{Q}_3\|\right\}.$$

It follows

$$\|\eta\|^2_{L^2(0,1)} \leqslant \frac{\varepsilon e^{\mu}}{a\underline{\lambda}(\tilde{P})}V_\varepsilon(\eta,z,\bar{y}) \leqslant \frac{\varepsilon e^{\mu}}{a\underline{\lambda}(\tilde{P})}e^{-\varepsilon\theta t}V_\varepsilon(\eta_0,z_0,\bar{y}_0)$$
$$\leqslant \frac{\varepsilon e^{\mu}}{a\underline{\lambda}(\tilde{P})}e^{-\varepsilon\theta t}\Bigg(\frac{a\|\tilde{P}\|}{\varepsilon}\|\eta_0\|^2_{H^2(0,1)} + b\|Q\|\left\|z_0-(I_m-K_{22})^{-1}K_{21}\bar{y}_0(1)\right\|^2_{H^2(0,1)}$$
$$+\varepsilon c\|\tilde{P}\|\|\bar{y}_0\|^2_{H^2(0,1)}\Bigg). \quad (2.4.20)$$

Due to $y_0 = \bar{y}_0$ i.e. $\eta_0 = 0$, therefore (2.4.1) holds.
Noting that the two following integrals in the right hand side of (2.4.17) are always negative

$$-a\mu\underline{\lambda}(\Lambda_1)\int_0^1 e^{-\mu x}\left(\frac{\eta^\top\tilde{P}_1\eta}{\varepsilon} + \varepsilon\eta_x^\top\tilde{P}_2\eta_x + \varepsilon^3\eta_{xx}^\top\tilde{P}_3\eta_{xx}\right)dx,$$

$$-\left(\varepsilon c\mu\underline{\lambda}(\Lambda_1) - \frac{6\varepsilon e^{\mu}b\|\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\|^2}{\mu\underline{\lambda}(Q)\underline{\lambda}(\Lambda_2)}\right)\int_0^1 e^{-\mu x}\left(\bar{y}^\top\tilde{P}_1\bar{y} + \bar{y}_x^\top\tilde{P}_2\bar{y}_x + \bar{y}_{xx}^\top\tilde{P}_3\bar{y}_{xx}\right)dx,$$

for all $c > c^*$, thus we can rewrite (2.4.17) as follows

$$\dot{V}_\varepsilon(\eta,z,\bar{y}) \leqslant -\frac{b\mu\underline{\lambda}(\Lambda_2)}{2\varepsilon}\int_0^1 e^{-\mu x}\Big((z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1$$
$$\times(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)) + z_x^\top\tilde{Q}_2z_x + z_{xx}^\top\tilde{Q}_3z_{xx}\Big)dx,$$

performing a time integration of both sides from 0 to $+\infty$ yields

$$\frac{b\mu e^{-\mu}\underline{\lambda}(\Lambda_2)}{2\varepsilon}\int_0^\infty\left\|z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)\right\|^2_{H^2(0,1)}dt \leqslant V_\varepsilon(\eta_0,z_0,\bar{y}_0) - \lim_{t\to+\infty}V_\varepsilon(\eta,z,\bar{y}),$$

since $\lim_{t\to+\infty} V_\varepsilon(\eta, z, \bar{y}) = 0$, it follows

$$\int_0^{+\infty} \left\| z - (I_m - K_{22})^{-1} K_{21}\bar{y}(1) \right\|^2_{H^2(0,1)} dt \leqslant \varepsilon \frac{2e^\mu}{b\mu\underline{\lambda}(\Lambda_2)} V_\varepsilon(\eta_0, z_0, \bar{y}_0)$$
$$\leqslant \varepsilon \frac{2e^\mu}{b\mu\underline{\lambda}(\Lambda_2)} \left(\frac{a\|\tilde{P}\|}{\varepsilon} \|\eta_0\|^2_{H^2(0,1)} + b\|Q\| \left\| z_0 - (I_m - K_{22})^{-1} K_{21}\bar{y}_0(1) \right\|^2_{H^2(0,1)} \right.$$
$$\left. + \varepsilon c \|\tilde{P}\| \|\bar{y}_0\|^2_{H^2(0,1)} \right), \tag{2.4.21}$$

since $\eta_0 = 0$, thus (2.4.2) holds. This concludes the proof of Theorem 2.4.1. □

Proof of Corollary 2.4.1 The proof of this corollary is based on Theorem 2.4.1. We get that (2.4.3) and (2.4.4) hold by choosing the initial condition $z_0 = (I_m - K_{22})^{-1} K_{21}\bar{y}_0(1)$ in (2.4.20) and (2.4.21). It is deduced from (2.4.18) and (2.4.19) that

$$\left\| z(.,t) - (I_m - K_{22})^{-1} K_{21}\bar{y}(1,t) \right\|^2_{H^2(0,1)} \leqslant \frac{e^\mu e^{-\varepsilon\theta t}}{b\underline{\lambda}(\tilde{Q})} \left(\frac{a\|\tilde{P}\|}{\varepsilon} \|\eta_0\|^2_{H^2(0,1)} \right.$$
$$\left. + b\|Q\| \left\| z_0 - (I_m - K_{22})^{-1} K_{21}\bar{y}_0(1) \right\|^2_{H^2(0,1)} + \varepsilon c \|\tilde{P}\| \|\bar{y}_0\|^2_{H^2(0,1)} \right),$$

due to the initial conditions $\eta_0 = \delta_0 = 0$, we get that (2.4.5) holds. This concludes the proof of Corollary 2.4.1. □

2.5 Comparison of the approximation results in Sections 2.3 and 2.4

In this section, η and δ denote

$$\eta = y(.,t) - \bar{y}(.,t),$$

$$\delta = z(.,t) - (I_m - K_{22})^{-1} K_{21} \bar{y}(1,t).$$

The comparison of the estimates in Theorem 2.3.1 and Theorem 2.4.1 are given in the following two tables.

	Estimate term	Initial condition	Estimate error	Lyapunov function type
Estimate (2.3.1)	$\|\eta(.,t)\|^2_{L^2}$	$y_0 = \bar{y}_0 \in H^2(0,1)$ $z_0 \in L^2(0,1)$	$\mathbf{O}(\varepsilon)$	L^2-norm
Estimate (2.4.1)				H^2-norm

Table 2.5.1: Comparison of the first estimates in Theorem 2.3.1 and Theorem 2.4.1

	Estimate term	Initial condition	Estimate error	Lyapunov function type
Estimate (2.3.2)	$\int_0^{+\infty} \|\delta(.,t)\|^2_{L^2} dt$	$y_0 = \bar{y}_0 \in H^2(0,1)$ $z_0 \in L^2(0,1)$	$\mathbf{O}(\varepsilon)$	L^2-norm
Estimate (2.4.2)	$\int_0^{+\infty} \|\delta(.,t)\|^2_{H^2} dt$			H^2-norm

Table 2.5.2: Comparison of the second estimates in Theorem 2.3.1 and Theorem 2.4.1

The following three tables show the differences of the estimates between Corollary 2.3.1 and Corollary 2.4.1.

	Estimate term	Initial condition	Estimate error	Lyapunov function type
Estimate (2.3.3)	$\|\eta(.,t)\|^2_{L^2}$	$y_0 = \bar{y}_0 \in H^2(0,1)$ $z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$	$\mathbf{O}(\varepsilon^2)$	L^2-norm
Estimate (2.4.3)				H^2-norm

Table 2.5.3: Comparison of the first estimates in Corollary 2.3.1 and Corollary 2.4.1

	Estimate term	Initial condition	Estimate error	Lyapunov function type
Estimate (2.3.4)	$\int_0^{+\infty} \|\delta(.,t)\|_{L^2}^2 dt$	$y_0 = \bar{y}_0 \in H^2(0,1)$	$\mathbf{O}(\varepsilon)$	L^2-norm
Estimate (2.4.4)	$\int_0^{+\infty} \|\delta(.,t)\|_{H^2}^2 dt$	$z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$	$\mathbf{O}(\varepsilon^2)$	H^2-norm

Table 2.5.4: Comparison of the second estimates in Corollary 2.3.1 and Corollary 2.4.1

	Estimate term	Initial condition	Estimate error	Lyapunov function type
Estimate (2.3.5)	$\|\delta(.,t)\|_{L^2}^2$	$y_0 = \bar{y}_0 \in H^2(0,1)$	$\mathbf{O}(\varepsilon)$	L^2-norm
Estimate (2.4.5)	$\|\delta(.,t)\|_{H^2}^2$	$z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$		H^2-norm

Table 2.5.5: Comparison of the third estimates in Corollary 2.3.1 and Corollary 2.4.1

2.6 Numerical simulations on academic examples

In this section, we give academic examples to illustrate the main results in Sections 2.3 and 2.4. Let us consider the linear hyperbolic system (2.1.1) with $n = m = 1$ and $\Lambda_1 = \Lambda_2 = 1$. The boundary condition matrix K is given by

$$K = \begin{pmatrix} 0.6 & 1.5 \\ 0.2 & -0.5 \end{pmatrix}. \tag{2.6.1}$$

Choosing $\Delta = \begin{pmatrix} \sqrt{0.1} & 0 \\ 0 & 1 \end{pmatrix}$, it holds $\|\Delta K \Delta^{-1}\| < 1$. Thus $\rho_1(K) < 1$ and Theorems 2.3.1 and 2.4.1 apply. To check the numerical solutions, let us use a two-step variant of the Lax-Wendroff method, which is presented in [Shampine, 2005a] and [Shampine, 2005b], to discretize the equation. Precisely, we divide the space domain $[0,1]$ into 100 intervals of identical length, and 50 as final time. We choose a time-step $dt = 0.9\varepsilon dx$ that satisfies the Courant–Friedrichs–Lewy (CFL) condition for the stability. First, let us select the following initial conditions

$$y(x,0) = \bar{y}(x,0) = 1 - \cos(4\pi x), \tag{2.6.2a}$$

$$z(x,0) = \sin(2\pi x), \tag{2.6.2b}$$

for all $x \in [0,1]$, such that the compatibility conditions (2.1.10) are satisfied. According to the computation of the subsystems in Section 2.1, the boundary condition for the reduced subsystem is computed as $K_r = 0.8$.

Remark 2.6.1. The singular perturbation approximation decreases the simulation cost. Precisely, instead of simulating the full system by using a small time-step which depends on ε and satisfies the CFL condition $dt < \varepsilon dx$,

we simulate the reduced subsystem where a longer time-step can be chosen satisfying the CFL condition $dt < dx$. ○

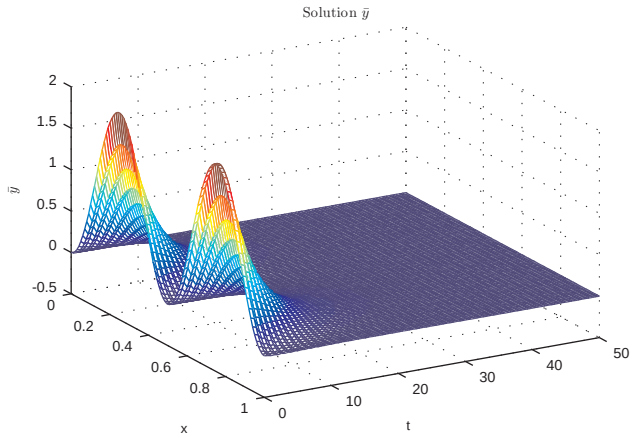

Figure 2.6.1: Time evolution of the solution $\bar{y}$ of the reduced system (2.1.7) with $K_r = 0.8$.

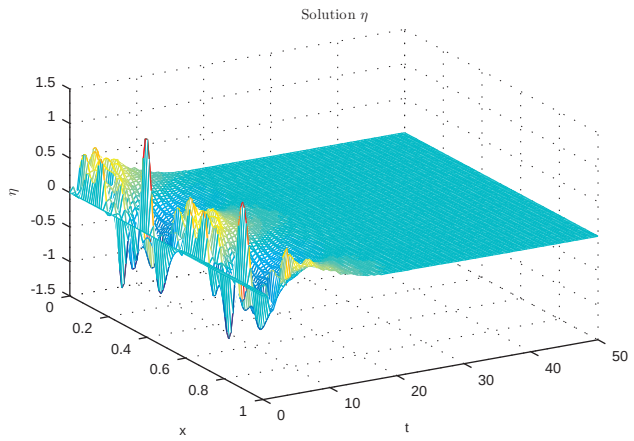

Figure 2.6.2: Time evolution of η which is the error between y of the full system and $\bar{y}$ of the reduced subsystem.

Figure 2.6.1 shows the time evolution of $\bar{y}$ of the reduced subsystem with boundary condition $K_r = 0.8$. It converges to the origin as time increases. Figure 2.6.2 is the time evolution of η, which is the error between y of the full system and $\bar{y}$ of the reduced subsystem. It decreases as time increases and is close to 0. In Table 2.6.1, it is shown that the errors between the full system and the subsystems, with $\|\eta(.,t=15)\|^2_{L^2(0,1)} = \|y(.,t) - \bar{y}(.,t=15)\|^2_{L^2(0,1)}$ and $\int_0^{50} \|\delta(.,t)\|^2_{L^2(0,1)} dt = \int_0^{50} \|z(.,t) - (I_m - K_{22})^{-1} K_{21} \bar{y}(1,t)\|^2_{L^2(0,1)} dt$,

ε	0.005	0.025	0.125
$\|\eta(.,t=15)\|^2_{L^2}$	2.1×10^{-5}	4.9×10^{-4}	1.7×10^{-3}
$\int_0^{50}\|\delta(.,t)\|^2_{L^2}dt$	6.6×10^{-6}	4.2×10^{-4}	4.1×10^{-2}

Table 2.6.1: Estimates of errors for different values of ε with initial conditions (2.6.2)

decrease as ε decreases and the values are near 0, as expected from Theorem 2.3.1. The estimates of $\|\delta(.,t=25)\|^2_{L^2(0,1)}$ with particular initial condition $z_0=(I_m-K_{22})^{-1}K_{21}\bar{y}_0(1)$ is given in Table 2.6.2. As ε decreases the value decreases which is consistent with Corollary 2.3.1. To check the numerical

ε	0.005	0.025	0.125
$\|\delta(.,t=25)\|^2_{L^2(0,1)}$	2.1×10^{-8}	7.1×10^{-7}	6.2×10^{-7}

Table 2.6.2: Estimates of errors with particular initial condition $z_0=(I_m-K_{22})^{-1}K_{21}\bar{y}_0(1)$

solutions for Section 2.4, let us consider the following initial conditions

$$y(x,0)=\bar{y}(x,0)=1-\cos(4\pi x), \tag{2.6.3a}$$
$$z(x,0)=\cos(6\pi x)-1, \tag{2.6.3b}$$

for all $x\in[0,1]$, such that the compatibility conditions (2.1.4) and (2.1.10) are satisfied. We perform these different simulations for $\varepsilon=10^{-3}$, $\varepsilon=10^{-2}$, and $\varepsilon=10^{-1}$. Table 2.6.3 shows the evolution of $\|\eta(.,t=15)\|^2_{L^2(0,1)}$ and

ε	10^{-3}	10^{-2}	10^{-1}
$\|\eta(.,t=15)\|^2_{L^2(0,1)}$	1.8×10^{-6}	9.9×10^{-5}	3.5×10^{-3}
$\int_0^{50}\|\delta(.,t)\|^2_{H^2(0,1)}dt$	1.7×10^{-6}	1.3×10^{-4}	2.3×10^{-2}

Table 2.6.3: Estimates of errors for different values of ε with initial conditions (2.6.3)

$\int_0^{50}\|\delta(.,t)\|^2_{H^2(0,1)}dt$ with different values of ε for $\begin{pmatrix}y_0\\z_0\end{pmatrix}\in H^2$ satisfying the compatibility conditions. It indicates that the two estimates are small and decrease as ε decreases, as expected from Theorem 2.4.1. However, in this simulation it is seen that the decay coefficient of the two estimates is roughly ε^2, it is different from the result in Theorem 2.4.1 which is ε. Figure 2.6.3 shows the time evolutions of $\log\|\eta(.,t)\|^2_{L^2(0,1)}$ for different ε.

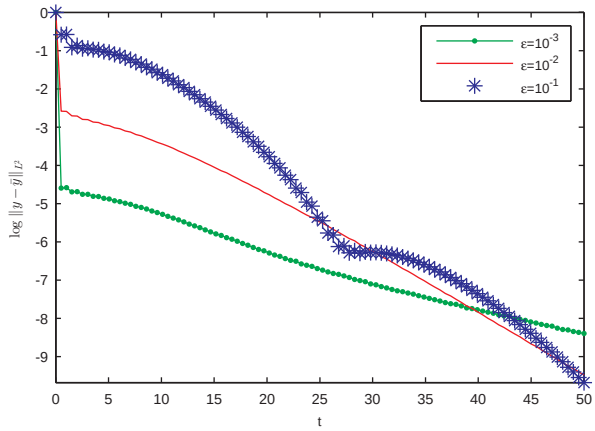

Figure 2.6.3: Time evolutions of $\log \|\eta(.,t)\|^2_{L^2(0,1)}$ for different values of ε with initial conditions (2.6.3).

It is observed that the error decreases as time increases. Let us examine Corollary 2.4.1 by choosing the equilibrium point as initial condition z_0, which is given by $z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$. Table 2.6.4 gives the estimates of $\|\eta(.,t=15)\|^2_{L^2(0,1)}$ and $\int_0^{50} \|\delta(.,t)\|^2_{H^2(0,1)} dt$ with different ε. When ε decreases, both estimates tend to zero. Moreover the decay coefficient is ε^2, as estimated in Corollary 2.4.1. The time evolutions of $\log \|\eta(.,t)\|^2_{L^2(0,1)}$ for different ε with particular $z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$ is given in Figure 2.6.4. After $t = 5$, the value decreases as time increases.

ε	10^{-3}	10^{-2}	10^{-1}
$\|\eta(.,t=15)\|^2_{L^2(0,1)}$	9.2×10^{-7}	8.4×10^{-5}	2.9×10^{-3}
$\int_0^{50} \|\delta(.,t)\|^2_{H^2(0,1)} dt$	2.0×10^{-7}	3.2×10^{-5}	1.6×10^{-2}

Table 2.6.4: Estimates of errors for different values of ε with particular initial condition $z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$

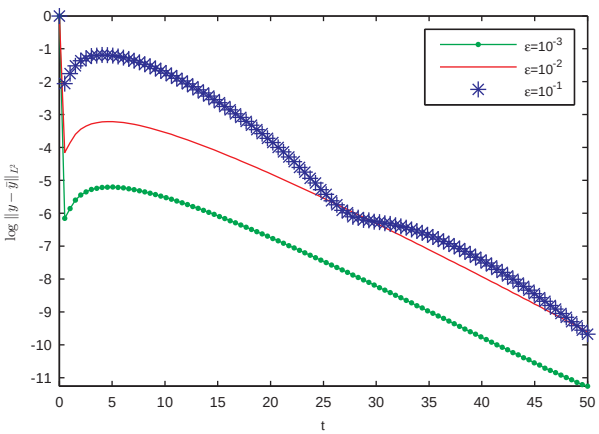

Figure 2.6.4: Time evolutions of $\log \|\eta(.,t)\|^2_{L^2}$ for different values of ε with particular initial condition $z_0 = (I_m - K_{22})^{-1} K_{21} \bar{y}_0(1)$

2.7 Technical proofs of Lemmas

Proof of Lemma 2.3.1 Computing the time derivative of $V_\varepsilon(\eta,\delta)$ along (2.3.7) and performing an integration by parts yield (see [Diagne et al., 2012])

$$\dot{V}_\varepsilon(\eta,\delta) = N_1 + N_2 + N_3$$

with:

$$\begin{aligned}
N_1 &= -\big(e^{-\mu x}\eta^\top Q\Lambda_1\eta\big)_{x=0}^{x=1} - \big(e^{-\mu x}\delta^\top P\Lambda_2\delta\big)_{x=0}^{x=1}, && (2.7.1)\\
N_2 &= -\mu\int_0^1 e^{-\mu x}\big(\eta^\top Q\Lambda_1\eta + \delta^\top P\Lambda_2\delta\big)dx, && (2.7.2)\\
N_3 &= 2\varepsilon\int_0^1 e^{-\mu x}\delta^\top P(I_m - K_{22})^{-1}K_{21}\Lambda_1\bar{y}_x(1,.)dx. && (2.7.3)
\end{aligned}$$

Using boundary condition (2.3.8), (2.7.1) follows

$$\begin{aligned}
N_1 = &-\Big(e^{-\mu}\eta^\top(1)Q\Lambda_1\eta(1) - (K_{11}\eta(1) + K_{12}\delta(1))^\top Q\Lambda_1\\
&\qquad (K_{11}\eta(1) + K_{12}\delta(1))\Big)\\
&-\Big(e^{-\mu}\delta^\top(1)P\Lambda_2\delta(1) - (K_{21}\eta(1) + K_{22}\delta(1))^\top P\Lambda_2\\
&\qquad (K_{21}\eta(1) + K_{22}\delta(1))\Big). \qquad (2.7.4)
\end{aligned}$$

Developing and reorganizing (2.7.4), we obtain

$$\begin{aligned}
N_1 = &-\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}^\top \begin{pmatrix} e^{-\mu}Q\Lambda_1 & 0\\ 0 & e^{-\mu}P\Lambda_2\end{pmatrix}\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}\\
&+\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}^\top \begin{pmatrix} K_{11}^\top Q\Lambda_1K_{11} + K_{21}^\top P\Lambda_2K_{21} & K_{11}^\top Q\Lambda_1K_{12} + K_{21}^\top P\Lambda_2K_{22}\\ \star & K_{22}^\top P\Lambda_2K_{22} + K_{12}^\top Q\Lambda_1K_{12}\end{pmatrix}\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}.
\end{aligned}$$

Since $\rho_1(K) < 1$, let $\Delta = \begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}$, such that $\|\Delta K\Delta^{-1}\| = \sigma^* < 1$, there exists $0 < \mu \leqslant -2\ln\sigma^*$. Choosing matrices Q and P such that $Q = \Delta_1^2\Lambda_1^{-1}$, $P = \Delta_2^2\Lambda_2^{-1}$, it follows

$$N_1 \leqslant -\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}^\top \big(e^{-\mu}\Delta^2 - \sigma^{*2}\Delta^2\big)\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix} \leqslant 0. \qquad (2.7.5)$$

The second term N_2 is always non positive.
According to Cauchy-Schwarz inequality, the third term N_3 is bounded as follows

$$N_3 \leqslant 2\varepsilon\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\| \, |\bar{y}_x(1,.)| \int_0^1 e^{-\mu x}|\delta| \, dx.$$

By Young's inequality, for all $\kappa > 0$, it holds

$$\begin{aligned} N_3 &\leqslant \varepsilon\kappa\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\| \int_0^1 e^{-\mu x}|\delta|^2 dx \\ &\quad + \frac{\varepsilon\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\|}{\kappa} \, |\bar{y}_x(1,.)|^2. \end{aligned} \tag{2.7.6}$$

Combining (2.7.2), (2.7.5) and (2.7.6), we obtain

$$\begin{aligned} \dot{V}_\varepsilon(\eta,\delta) &\leqslant -\mu \int_0^1 e^{-\mu x}\eta^\top Q\Lambda_1\eta \, dx \\ &\quad - \left(\mu - \frac{\varepsilon\kappa\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\|}{\underline{\lambda}(P\Lambda_2)}\right) \int_0^1 e^{-\mu x}\delta^\top P\Lambda_2\delta \, dx \\ &\quad + \frac{\varepsilon\|P(I_m - K_{22})^{-1}K_{21}\Lambda_1\|}{\kappa} \, |\bar{y}_x(1,.)|^2. \end{aligned}$$

This concludes the proof of Lemma 2.3.1. □

Proof of Lemma 2.3.2 The compatibility conditions (2.1.10) and $\bar{y}_0 \in H^2(0,1)$ imply that $\bar{y}(.,t) \in H^2(0,1)$ for all $t \geqslant 0$. Therefore we consider the following candidate Lyapunov function for reduced subsystem (2.1.7)-(2.1.9)

$$V_1(\bar{y}) = \int_0^1 e^{-\mu x}\big(\bar{y}^\top\bar{Q}_1\bar{y} + \bar{y}_x^\top\bar{Q}_2\bar{y}_x + \bar{y}_{xx}^\top\bar{Q}_3\bar{y}_{xx}\big)dx, \tag{2.7.7}$$

where $\bar{Q}_1$, $\bar{Q}_2$, $\bar{Q}_3$ are diagonal positive matrices in $\mathbb{R}^{n\times n}$. We rewrite $V_1(\bar{y})$ as

$$V_1(\bar{y}) = V_{10} + V_{11} + V_{12},$$

with

$$\begin{aligned} V_{10} &= \int_0^1 e^{-\mu x}\bar{y}^\top\bar{Q}_1\bar{y} \, dx, \\ V_{11} &= \int_0^1 e^{-\mu x}\bar{y}_x^\top\bar{Q}_2\bar{y}_x \, dx, \\ V_{12} &= \int_0^1 e^{-\mu x}\bar{y}_{xx}^\top\bar{Q}_3\bar{y}_{xx} \, dx. \end{aligned}$$

Computing the time derivative of V_{10} along (2.1.7) and performing an integration by parts we have

$$\dot{V}_{10} = -\big[e^{-\mu x}\bar{y}^\top\bar{Q}_1\Lambda_1\bar{y}\big]_{x=0}^{x=1} - \mu\int_0^1 e^{-\mu x}\bar{y}^\top\bar{Q}_1\Lambda_1\bar{y} \, dx. \tag{2.7.8}$$

Using the boundary condition (2.1.8), it follows from (2.7.8)

$$\dot{V}_{10} = -\Big(e^{-\mu}\bar{y}^\top(1)\bar{Q}_1\Lambda_1\bar{y}(1) - (K_r\bar{y}(1))^\top\bar{Q}_1\Lambda_1(K_r\bar{y}(1))\Big) - \mu\int_0^1 e^{-\mu x}\bar{y}^\top\bar{Q}_1\Lambda_1\bar{y}\,dx. \tag{2.7.9}$$

Developing and reorganizing (2.7.9) we get

$$\dot{V}_{10} = -\bar{y}^\top(1)\Big(e^{-\mu}\bar{Q}_1\Lambda_1 - K_r^\top\bar{Q}_1\Lambda_1K_r\Big)\bar{y}(1) - \mu\int_0^1 e^{-\mu x}\bar{y}^\top\bar{Q}_1\Lambda_1\bar{y}\,dx. \tag{2.7.10}$$

Similar to the proof of Proposition 2.2.1, we show that $\|\Delta K\Delta^{-1}\| < 1$ implies $\|\Delta_1K_r\Delta_1^{-1}\| < 1$ which is equivalent to $\Delta_1^2 - K_r^\top\Delta_1^2K_r > 0$. By selecting $\bar{Q}_1 = \Delta_1^2\Lambda_1^{-1}$, it is deduced from (2.7.10)

$$\dot{V}_{10} \leqslant -\mu\int_0^1 e^{-\mu x}\bar{y}^\top\bar{Q}_1\Lambda_1\bar{y}\,dx. \tag{2.7.11}$$

Differentiating system (2.1.7) with respect to x yields

$$\bar{y}_{xt} + \Lambda_1\bar{y}_{xx} = 0, \tag{2.7.12}$$

and differentiating (2.1.8) with respect to t and using (2.1.7), the boundary condition is calculated as

$$\bar{y}_x(0,t) = \Lambda_1^{-1}K_r\Lambda_1\bar{y}_x(1,t). \tag{2.7.13}$$

Computing the time derivative of V_{11} along (2.7.12) and performing an integration by parts, we have

$$\dot{V}_{11} = -\big[e^{-\mu x}\bar{y}_x^\top\bar{Q}_2\Lambda_1\bar{y}_x\big]_{x=0}^{x=1} - \mu\int_0^1 e^{-\mu x}\bar{y}_x^\top\bar{Q}_2\Lambda_1\bar{y}_x\,dx. \tag{2.7.14}$$

Using the boundary condition (2.7.13), it follows from (2.7.14)

$$\dot{V}_{11} = -\Big(e^{-\mu}\bar{y}_x^\top(1)\bar{Q}_2\Lambda_1\bar{y}_x(1) - (\Lambda_1^{-1}K_r\Lambda_1\bar{y}_x(1))^\top\bar{Q}_2\Lambda_1(\Lambda_1^{-1}K_r\Lambda_1\bar{y}_x(1))\Big) - \mu\int_0^1 e^{-\mu x}\bar{y}_x^\top\bar{Q}_2\Lambda_1\bar{y}_x\,dx. \tag{2.7.15}$$

Developing and reorganizing (2.7.15), we get

$$\dot{V}_{11} = -\bar{y}_x^\top(1)\Lambda_1\Big(e^{-\mu}\Lambda_1^{-1}\bar{Q}_2 - K_r^\top\Lambda_1^{-1}\bar{Q}_2K_r\Big)\Lambda_1\bar{y}_x(1) - \mu\int_0^1 e^{-\mu x}\bar{y}_x^\top\bar{Q}_2\Lambda_1\bar{y}_x\,dx. \tag{2.7.16}$$

Similarly, by selecting $\bar{Q}_2 = \Lambda_1 \Delta_1^2$, it is deduced from (2.7.16)

$$\dot{V}_{11} \leqslant -\mu \int_0^1 e^{-\mu x} \bar{y}_x^\top \bar{Q}_2 \Lambda_1 \bar{y}_x \, dx. \tag{2.7.17}$$

Differentiating system (2.7.12) with respect to x yields

$$\bar{y}_{xxt} + \Lambda_1 \bar{y}_{xxx} = 0, \tag{2.7.18}$$

and differentiating (2.7.13) with respect to t and using (2.7.12), the boundary condition is calculated as

$$\bar{y}_{xx}(0,t) = \Lambda_1^{-2} K_r \Lambda_1^2 \bar{y}_{xx}(1,t). \tag{2.7.19}$$

Computing the time derivative of V_{12} along (2.7.18) and performing an integration by parts, we have

$$\begin{aligned} \dot{V}_{12} = &-\left[e^{-\mu x} \bar{y}_{xx}^\top \bar{Q}_3 \Lambda_1 \bar{y}_{xx}\right]_{x=0}^{x=1} \\ &- \mu \int_0^1 e^{-\mu x} \bar{y}_{xx}^\top \bar{Q}_3 \Lambda_1 \bar{y}_{xx} \, dx. \end{aligned} \tag{2.7.20}$$

Using the boundary condition (2.7.19), it follows from (2.7.20)

$$\begin{aligned} \dot{V}_{12} = &-\Bigg(e^{-\mu} y_{xx}^\top(1) \bar{Q}_3 \Lambda_1 \bar{y}_{xx}(1) - (\Lambda_1^{-2} K_r \Lambda_1^2 \bar{y}_{xx}(1))^\top \bar{Q}_3 \Lambda_1 (\Lambda_1^{-2} K_r \Lambda_1^2 \bar{y}_{xx}(1))\Bigg) \\ &- \mu \int_0^1 e^{-\mu x} \bar{y}_{xx}^\top \bar{Q}_3 \Lambda_1 \bar{y}_{xx} \, dx. \end{aligned} \tag{2.7.21}$$

Developing and reorganizing (2.7.21), we get

$$\begin{aligned} \dot{V}_{12} = &-\bar{y}_{xx}^\top(1) \Lambda_1^2 \Big(e^{-\mu} \Lambda_1^{-2} \bar{Q}_3 \Lambda_1^{-1} - K_r^\top \Lambda_1^{-2} \bar{Q}_3 \Lambda_1^{-1} K_r\Big) \Lambda_1^2 \bar{y}_{xx}(1) \\ &- \mu \int_0^1 e^{-\mu x} \bar{y}_{xx}^\top \bar{Q}_3 \Lambda_1 \bar{y}_{xx} \, dx. \end{aligned} \tag{2.7.22}$$

By selecting $\bar{Q}_3 = \Lambda_1^2 \Delta_1^2 \Lambda_1$, it is deduced from (2.7.22)

$$\dot{V}_{12} \leqslant -\mu \int_0^1 e^{-\mu x} \bar{y}_{xx}^\top \bar{Q}_3 \Lambda_1 \bar{y}_{xx} \, dx. \tag{2.7.23}$$

Combining (2.7.11), (2.7.17) and (2.7.23), $\dot{V}_1(\bar{y})$ follows

$$\dot{V}_1(\bar{y}) \leqslant -\mu \underline{\lambda}(\Lambda_1) V_1(\bar{y}). \tag{2.7.24}$$

Therefore, there exists a positive value C_r such that

$$\|\bar{y}(.,t)\|_{H^2}^2 \leqslant C_r e^{-\mu \underline{\lambda}(\Lambda_1) t} \|\bar{y}_0\|_{H^2}^2.$$

This concludes the proof of Lemma 2.3.2. □

Proof of Lemma 2.4.1 Let us differentiate $V_\varepsilon^0(\eta, z, \bar{y})$ with respect to t along the solutions to (2.4.6), it follows

$$\begin{aligned}
\dot{V}_\varepsilon^0(\eta, z, \bar{y}) &= -\frac{a}{\varepsilon}\int_0^1 e^{-\mu x}(\eta^\top \tilde{P}_1\Lambda_1\eta)_x\, dx - \varepsilon c\int_0^1 e^{-\mu x}(\bar{y}^\top \tilde{P}_1\Lambda_1\bar{y})_x\, dx \\
&-\frac{b}{\varepsilon}\int_0^1 e^{-\mu x}\Big((z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\Big)_x\, dx \\
&+2b\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\bar{y}_x(1)\, dx.
\end{aligned}$$

Performing an integration by parts on the first three integrals yields

$$\dot{V}_\varepsilon^0(\eta, z, \bar{y}) = V_{01} + V_{02}$$

with

$$\begin{aligned}
V_{01} &= -\frac{a}{\varepsilon}[e^{-\mu x}\eta^\top\tilde{P}_1\Lambda_1\eta]_0^1 - \varepsilon c[e^{-\mu x}\bar{y}^\top\tilde{P}_1\Lambda_1\bar{y}]_0^1 \\
&\quad -\frac{b}{\varepsilon}\Big[e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1\Lambda_2(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\Big]_0^1, \\
V_{02} &= -\frac{a\mu}{\varepsilon}\int_0^1 e^{-\mu x}\eta^\top\tilde{P}_1\Lambda_1\eta\, dx - \varepsilon c\mu\int_0^1 e^{-\mu x}\bar{y}^\top\tilde{P}_1\Lambda_1\bar{y}\, dx \\
&-\frac{b\mu}{\varepsilon}\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1\Lambda_2(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))\, dx \\
&+2b\int_0^1 e^{-\mu x}(z-(I_m-K_{22})^{-1}K_{21}\bar{y}(1))^\top\tilde{Q}_1(I_m-K_{22})^{-1}K_{21}\Lambda_1\bar{y}_x(1)\, dx.
\end{aligned}$$

To simplify the computation, let us rewrite (2.4.7b) as follows

$$z(0)-(I_m-K_{22})^{-1}K_{21}\bar{y}(1) = K_{21}\eta(1)+K_{22}\Big(z(1)-(I_m-K_{22})^{-1}K_{21}\bar{y}(1)\Big). \tag{2.7.25}$$

Using (2.4.7a), (2.4.7c) and (2.7.25), V_{01} is rewritten as

$$\begin{aligned}
V_{01} \;=\; & -\frac{a}{\varepsilon}\Bigg[e^{-\mu}\eta^\top(1)\tilde{P}_1\Lambda_1\eta(1) - \Big(K_{11}\eta(1) + K_{12}\Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big)\Big)^\top \\
& \qquad \tilde{P}_1\Lambda_1\Big(K_{11}\eta(1) + K_{12}\Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big)\Big)\Bigg] \\
& -\frac{b}{\varepsilon}\Bigg[e^{-\mu}\Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big)^\top\tilde{Q}_1\Lambda_2\Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big) \\
& \qquad -\Big(K_{21}\eta(1) + K_{22}\Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big)\Big)^\top\tilde{Q}_1\Lambda_2 \\
& \qquad\qquad \Big(K_{21}\eta(1) + K_{22}\Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big)\Big)\Bigg] \\
& -\varepsilon c\Big[e^{-\mu}\bar{y}^\top(1)\tilde{P}_1\Lambda_1\bar{y}(1) - \bar{y}^\top(1)K_r^\top\tilde{P}_1\Lambda_1K_r\bar{y}(1)\Big].
\end{aligned}$$

Developing and reorganizing the above terms, we get

$$V_{01} \;=\; -\begin{pmatrix} \eta(1) \\ \Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big) \\ \bar{y}(1) \end{pmatrix}^\top M_0 \begin{pmatrix} \eta(1) \\ \Big(z(1) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1)\Big) \\ \bar{y}(1) \end{pmatrix},$$

with $M_0 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_{01}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & M_{02} \end{array}\right)$,

where $M_{01} = \begin{pmatrix} \frac{ae^{-\mu}\tilde{P}_1\Lambda_1 - (aK_{11}^\top\tilde{P}_1\Lambda_1K_{11} + bK_{21}^\top\tilde{Q}_1\Lambda_2K_{21})}{\varepsilon} & -\frac{aK_{11}^\top\tilde{P}_1\Lambda_1K_{12} + bK_{21}^\top\tilde{Q}_1\Lambda_2K_{22}}{\varepsilon} \\ * & \frac{be^{-\mu}\tilde{Q}_1\Lambda_2 - (bK_{22}^\top\tilde{Q}_1\Lambda_2K_{22} + aK_{12}^\top\tilde{P}_1\Lambda_1K_{12})}{\varepsilon} \end{pmatrix}$,

and $M_{02} = \varepsilon c(e^{-\mu}\tilde{P}_1\Lambda_1 - K_r^\top\tilde{P}_1\Lambda_1K_r)$.

To prove that V_{01} is negative definite let us check

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_{01}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & M_{02} \end{array}\right) > 0.$$

Let us consider the following matrix

$$\widehat{M_0} = e^{-\mu}\begin{pmatrix} a\tilde{P}_1\Lambda_1 & 0 \\ 0 & b\tilde{Q}_1\Lambda_2 \end{pmatrix} - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}^\top \begin{pmatrix} a\tilde{P}_1\Lambda_1 & 0 \\ 0 & b\tilde{Q}_1\Lambda_2 \end{pmatrix}\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

The straightforward calculations prove that (see e.g. [Coron et al., 2008, Section 4])

Claim 1. *If $\rho_1(K) < 1$, then for a suitable choice of diagonal positive matrices $\tilde{P}_1$, $\tilde{Q}_1$ and positive values a, b, and positive μ small enough, it holds $\widehat{M_0} > 0$.*

By using Claim 1, it holds $M_{01} = \varepsilon^{-1}\widehat{M_0} > 0$. Due to Proposition 2.2.1 in Section 2.2, $\rho_1(K_r) < 1$ implies $e^{-\mu}\tilde{P}_1\Lambda_1 - K_r^\top \tilde{P}_1\Lambda_1 K_r > 0$. Then, it follows $M_{02} > 0$, thus $V_{01} < 0$, the inequality (2.4.10) holds. This concludes the proof of Lemma 2.4.1. □

Proof of Lemma 2.4.2 Let us differentiate $V_\varepsilon^1(\eta, z, \bar{y})$ with respect to t along the solutions to (2.4.11), it follows

$$\begin{aligned}\dot{V}_\varepsilon^1(\eta, z, \bar{y}) &= -\varepsilon a \int_0^1 e^{-\mu x}(\eta_x^\top \tilde{P}_2\Lambda_1\eta_x)_x\, dx - \varepsilon c \int_0^1 e^{-\mu x}(\bar{y}_x\tilde{P}_2\Lambda_1\bar{y}_x)_x\, dx \\ &\quad -\frac{b}{\varepsilon}\int_0^1 e^{-\mu x}(z_x\tilde{Q}_2\Lambda_2 z_x)_x\, dx. \qquad (2.7.26)\end{aligned}$$

Performing an integration by parts, we obtain

$$\dot{V}_\varepsilon^1(\eta, z, \bar{y}) = V_{11} + V_{12},$$

with

$$\begin{aligned}V_{11} &= -\varepsilon a[e^{-\mu x}\eta_x^\top \tilde{P}_2\Lambda_1\eta_x]_0^1 - \frac{b}{\varepsilon}\left[e^{-\mu x}z_x^\top \tilde{Q}_2\Lambda_2 z_x\right]_0^1 - \varepsilon c[e^{-\mu x}\bar{y}_x^\top \tilde{P}_2\Lambda_1\bar{y}_x]_0^1, \\ V_{12} &= -\varepsilon a\mu\int_0^1 e^{-\mu x}\eta_x^\top \tilde{P}_2\Lambda_1\eta_x dx - \frac{b\mu}{\varepsilon}\int_0^1 e^{-\mu x}z_x^\top \tilde{Q}_2\Lambda_2 z_x dx \\ &\quad -\varepsilon c\mu\int_0^1 e^{-\mu x}\bar{y}_x^\top \tilde{P}_2\Lambda_1\bar{y}_x dx.\end{aligned}$$

Using the boundary conditions (2.4.12), V_{11} is rewritten as

$$\begin{aligned}
V_{11} &= -\varepsilon a\Bigg[e^{-\mu}\eta_x^\top(1)\tilde{P}_2\Lambda_1\eta_x(1) \\
&\qquad -\Big(\Lambda_1^{-1}K_{11}\Lambda_1\eta_x(1)+\frac{\Lambda_1^{-1}K_{12}\Lambda_2}{\varepsilon}z_x(1)-\Lambda_1^{-1}K_c\Lambda_1\bar{y}_x(1)\Big)^\top \\
&\qquad \tilde{P}_2\Lambda_1\Big(\Lambda_1^{-1}K_{11}\Lambda_1\eta_x(1)+\frac{\Lambda_1^{-1}K_{12}\Lambda_2}{\varepsilon}z_x(1)-\Lambda_1^{-1}K_c\Lambda_1\bar{y}_x(1)\Big)\Bigg] \\
&\quad -\frac{b}{\varepsilon}\Bigg[e^{-\mu}z_x^\top(1)\tilde{Q}_2\Lambda_2 z_x(1) \\
&\qquad -\Big(\varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\eta_x(1)+\Lambda_2^{-1}K_{22}\Lambda_2 z_x(1)+\varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\bar{y}_x(1)\Big)^\top \\
&\qquad \tilde{Q}_2\Lambda_2\Big(\varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\eta_x(1)+\Lambda_2^{-1}K_{22}\Lambda_2 z_x(1)+\varepsilon\Lambda_2^{-1}K_{21}\Lambda_1\bar{y}_x(1)\Big)\Bigg] \\
&\quad -\varepsilon c\Bigg[e^{-\mu}\bar{y}_x^\top(1)\tilde{P}_2\Lambda_1\bar{y}_x(1)-\Big(\Lambda_1^{-1}K_r\Lambda_1\bar{y}_x(1)\Big)^\top\tilde{P}_2\Lambda_1\Big(\Lambda_1^{-1}K_r\Lambda_1\bar{y}_x(1)\Big)\Bigg].
\end{aligned}$$

Developing and reorganizing the above terms, yield

$$V_{11} = -\begin{pmatrix}\Lambda_1\eta_x(1)\\ \Lambda_2 z_x(1)\\ \Lambda_1\bar{y}_x(1)\end{pmatrix}^\top \left(\begin{array}{c|c} M_{11} & M_{13}\\ \hline * & M_{12}\end{array}\right)\begin{pmatrix}\Lambda_1\eta_x(1)\\ \Lambda_2 z_x(1)\\ \Lambda_1\bar{y}_x(1)\end{pmatrix},$$

with

$$\begin{aligned}
M_{11} &= \begin{pmatrix}\sqrt{\varepsilon} & 0\\ 0 & \frac{1}{\sqrt{\varepsilon}}\end{pmatrix}^\top \widehat{M}_1\begin{pmatrix}\sqrt{\varepsilon} & 0\\ 0 & \frac{1}{\sqrt{\varepsilon}}\end{pmatrix},\\
M_{12} &= \varepsilon\Big(c(e^{-\mu}\Lambda_1^{-1}\tilde{P}_2-K_r^\top\Lambda_1^{-1}\tilde{P}_2K_r)-bK_{21}^\top\Lambda_2^{-1}\tilde{Q}_2K_{21}-aK_c^\top\Lambda_1^{-1}\tilde{P}_1K_c\Big),\\
M_{13} &= \begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\widehat{M}_{13},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{M}_1 &= e^{-\mu}\begin{pmatrix}a\Lambda_1^{-1}\tilde{P}_2 & 0\\ 0 & b\Lambda_2^{-1}\tilde{Q}_2\end{pmatrix}-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}^\top\begin{pmatrix}a\Lambda_1^{-1}\tilde{P}_2 & 0\\ 0 & b\Lambda_2^{-1}\tilde{Q}_2\end{pmatrix}\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix},\\
\widehat{M}_{13} &= \begin{pmatrix}\begin{pmatrix}aK_{11}^\top\Lambda_1^{-1}\tilde{P}_2K_c-bK_{21}^\top\Lambda_2^{-1}\tilde{Q}_2K_{21}\\ aK_{12}^\top\Lambda_1^{-1}\tilde{P}_2K_c-bK_{22}^\top\Lambda_2^{-1}\tilde{Q}_2K_{21}\end{pmatrix}\end{pmatrix}.
\end{aligned}$$

Similarly, we have the following claim

Claim 2. *If $\rho_1(K) < 1$, let a, b, μ as in Claim 1, then for a suitable choice of diagonal positive matrices $\tilde{P}_2$, $\tilde{Q}_2$, it holds $\widehat{M}_1 > 0$.*

To prove that V_{11} is negative definite, let us check $\left(\begin{array}{c|c} M_{11} & M_{13} \\ \hline * & M_{12} \end{array}\right) > 0$. By using Claim 2, it holds $M_{11} > 0$. Computing $M_{13}^T M_{11}^{-1} M_{13}$, we get

$$M_{13}^\top M_{11}^{-1} M_{13} \quad = \quad \varepsilon \widehat{M}_{13}^\top \widehat{M}_1^{-1} \widehat{M}_{13}.$$

Let us prove $M_{12} - M_{13}^\top M_{11}^{-1} M_{13} > 0$. Since $\rho_1(K_r) < 1$, let Δ_1 be diagonal positive matrix such that $\|\Delta_1 K_r \Delta_1^{-1}\| = l^* < 1$, it holds

$$e^{-\mu} \Lambda_1^{-1} \tilde{P}_2 - K_r^\top \Lambda_1^{-1} \tilde{P}_2 K_r > e^{-\mu} \Delta_1^2 - l^{*2} \Delta_1^2 > 0,$$

with $\tilde{P}_2 = \Lambda_1 \Delta_1^2$ which is consistent with the value chosen in Claim 2. There exists $c' > 0$ such that for all $c > c'$, it holds $M_{12} - M_{13}^\top M_{11}^{-1} M_{13} > 0$. Since $M_{11} > 0$ and $M_{12} - M_{13}^T M_{11}^{-1} M_{13} > 0$, using the Schur complement (see [Boyd et al., 1994]), it holds $\left(\begin{array}{c|c} M_{11} & M_{13} \\ \hline * & M_{12} \end{array}\right) > 0$. The inequality (2.4.13) holds. This concludes the proof of Lemma 2.4.2. □

Proof of Lemma 2.4.3 Let us differentiate $V_\varepsilon^2(\eta, z, \bar{y})$ with respect to t along the solutions to (2.4.14), it follows

$$\begin{aligned} \dot{V}_\varepsilon^2(\eta, z, \bar{y}) \quad = \quad & -\varepsilon^3 a \int_0^1 e^{-\mu x} (\eta_{xx}^\top \tilde{P}_3 \Lambda_1 \eta_{xx})_x \, dx - \varepsilon c \int_0^1 e^{-\mu x} (\bar{y}_{xx}^\top \tilde{P}_3 \Lambda_1 \bar{y}_{xx})_x \, dx \\ & + \frac{b}{\varepsilon} \int_0^1 e^{-\mu x} (z_{xx}^\top \tilde{Q}_3 \Lambda_2 z_{xx})_x \, dx. \end{aligned} \tag{2.7.27}$$

Performing an integration by parts, we obtain

$$\dot{V}_\varepsilon^2(\eta, z, \bar{y}) = V_{21} + V_{22},$$

with

$$\begin{aligned} V_{21} \quad = \quad & -\varepsilon^3 a [e^{-\mu x} \eta_{xx}^\top \tilde{P}_3 \Lambda_1 \eta_{xx}]_0^1 - \frac{b}{\varepsilon} \left[e^{-\mu x} z_{xx}^\top \tilde{Q}_3 \Lambda_2 z_{xx} \right]_0^1 - \varepsilon c [e^{-\mu x} \bar{y}_{xx}^\top \tilde{P}_3 \Lambda_1 \bar{y}_{xx}]_0^1, \\ V_{22} \quad = \quad & -\varepsilon^3 a \mu \int_0^1 e^{-\mu x} \eta_{xx}^\top \tilde{P}_3 \Lambda_1 \eta_{xx} \, dx - \frac{b\mu}{\varepsilon} \int_0^1 e^{-\mu x} z_{xx}^\top \tilde{Q}_3 \Lambda_2 z_{xx} \, dx \\ & -\varepsilon c \mu \int_0^1 e^{-\mu x} \bar{y}_{xx}^\top \tilde{P}_3 \Lambda_1 \bar{y}_{xx} \, dx. \end{aligned}$$

Using the boundary conditions (2.4.15), V_{21} follows

$$
\begin{aligned}
V_{21} \;=\; & -\varepsilon^3 a\Bigg[e^{-\mu}\eta_{xx}^\top(1)\tilde{P}_3\Lambda_1\eta_{xx}(1) - \Bigg(\Lambda_1^{-2}K_{11}\Lambda_1^2\eta_{xx}(1) + \frac{\Lambda_1^{-2}K_{12}\Lambda_2^2}{\varepsilon^2}z_{xx}(1) \\
& -\Lambda_1^{-2}K_c\Lambda_1^2\bar{y}_{xx}(1)\Bigg)^\top \tilde{P}_3\Lambda_1\Bigg(\Lambda_1^{-2}K_{11}\Lambda_1^2\eta_{xx}(1) + \frac{\Lambda_1^{-2}K_{12}\Lambda_2^2}{\varepsilon^2}z_{xx}(1) \\
& -\Lambda_1^{-2}K_c\Lambda_1^2\bar{y}_{xx}(1)\Bigg)\Bigg] \\
& -\frac{b}{\varepsilon}\Bigg[e^{-\mu}z_{xx}^\top(1)\tilde{Q}_3\Lambda_2 z_{xx}(1) - \Bigg(\varepsilon^2\Lambda_2^{-2}K_{21}\Lambda_1^2\eta_{xx}(1) + \Lambda_2^{-2}K_{22}\Lambda_2^2 z_{xx}(1) \\
& +\varepsilon^2\Lambda_2^{-2}K_{21}\Lambda_1^2\bar{y}_{xx}(1)\Bigg)^\top \tilde{Q}_3\Lambda_2\Bigg(\varepsilon^2\Lambda_2^{-2}K_{21}\Lambda_1^2\eta_{xx}(1) + \Lambda_2^{-2}K_{22}\Lambda_2^2 z_{xx}(1) \\
& +\varepsilon^2\Lambda_2^{-2}K_{21}\Lambda_1^2\bar{y}_{xx}(1)\Bigg)\Bigg] \\
& -\varepsilon c\Bigg[e^{-\mu}\bar{y}_{xx}^\top(1)\tilde{P}_3\Lambda_1\bar{y}_{xx}(1) - \Bigg(\Lambda_1^{-2}K_r^2\Lambda_1^2\bar{y}_{xx}(1)\Bigg)^\top \tilde{P}_3\Lambda_1\Bigg(\Lambda_1^{-2}K_r^2\Lambda_1^2\bar{y}_{xx}(1)\Bigg)\Bigg].
\end{aligned}
$$

Developing the above terms and reorganizing them, we obtain

$$
V_{21} \;=\; -\begin{pmatrix}\Lambda_1^2\eta_{xx}(1)\\ \Lambda_2^2 z_{xx}(1)\\ \Lambda_1^2\bar{y}_{xx}(1)\end{pmatrix}^\top \left(\begin{array}{c|c} M_{21} & M_{23} \\ \hline * & M_{22}\end{array}\right)\begin{pmatrix}\Lambda_1^2\eta_{xx}(1)\\ \Lambda_2^2 z_{xx}(1)\\ \Lambda_1^2\bar{y}_{xx}(1)\end{pmatrix},
$$

with

$$
\begin{aligned}
M_{21} &= \begin{pmatrix}\sqrt{\varepsilon^3} & 0\\ 0 & \frac{1}{\sqrt{\varepsilon}}\end{pmatrix}^\top \widehat{M}_2 \begin{pmatrix}\sqrt{\varepsilon^3} & 0\\ 0 & \frac{1}{\sqrt{\varepsilon}}\end{pmatrix},\\
M_{23} &= \begin{pmatrix}\varepsilon^3 & 0\\ 0 & \varepsilon\end{pmatrix}\widehat{M}_{23},\\
M_{22} &= \varepsilon\Bigg(c\Bigg(e^{-\mu}\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1} - K_r^\top\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1}K_r\Bigg) \\
&\qquad -\varepsilon^2 b K_{21}^\top\Lambda_2^{-2}\tilde{Q}_3\Lambda_2^{-1}K_{21} - \varepsilon^2 a K_c^\top\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1}K_c\Bigg),
\end{aligned}
$$

where

$$\begin{aligned}
\widehat{M_2} &= e^{-\mu}\begin{pmatrix} a\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1} & 0 \\ 0 & b\Lambda_2^{-2}\tilde{Q}_3\Lambda_2^{-1}\end{pmatrix} \\
&\quad - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{pmatrix}^\top \begin{pmatrix} a\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1} & 0 \\ 0 & b\Lambda_2^{-2}\tilde{Q}_3\Lambda_2^{-1}\end{pmatrix}\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{pmatrix}, \\
\widehat{M_{23}} &= \begin{pmatrix} \begin{pmatrix} aK_{11}^\top\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1}K_c - bK_{21}^\top\Lambda_2^{-2}\tilde{Q}_3\Lambda_2^{-1}K_{21}\end{pmatrix} \\ \begin{pmatrix} aK_{12}^\top\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1}K_c - bK_{22}^\top\Lambda_2^{-2}\tilde{Q}_3\Lambda_2^{-1}K_{21}\end{pmatrix}\end{pmatrix}.
\end{aligned}$$

Similarly, Let us state the following claim

Claim 3. *If $\rho_1(K) < 1$, let a, b, μ as in Claim 1, and c as in Claim 2, then for a suitable choice of diagonal positive matrices $\tilde{P}_3$, $\tilde{Q}_3$, it holds $\widehat{M_2} > 0$.*

To prove that V_{21} is negative definite, let us check $\left(\begin{array}{c|c} M_{21} & M_{23} \\ \hline * & M_{22}\end{array}\right) > 0$. Due to Claim 3, $M_{21} > 0$. Computing $M_{23}^\top M_{21}^{-1} M_{23}$, we obtain

$$M_{23}^\top M_{21}^{-1} M_{23} = \varepsilon^3 \widehat{M_{23}}^\top \widehat{M_2}^{-1}\widehat{M_{23}}$$

Let us prove $M_{22} - M_{23}^\top M_{21}^{-1} M_{23} > 0$. Since $\rho_1(K_r) < 1$, let Δ_1 be diagonal positive matrix such that $\|\Delta_1 K_r \Delta_1^{-1}\| = l^* < 1$, it holds

$$e^{-\mu}\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1} - K_r^\top\Lambda_1^{-2}\tilde{P}_3\Lambda_1^{-1}K_r > e^{-\mu}\Delta_1^2 - l^{*2}\Delta_1^2 > 0,$$

with $\tilde{P}_3 = \Lambda_1^2\Delta_1^2\Lambda_1$ which is consistent with the value chosen in Claim 3. There exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, it holds $M_{22} - M_{23}^\top M_{21}^{-1} M_{23} > 0$. Since $M_{21} > 0$ and $M_{22} - M_{23}^T M_{21}^{-1} M_{23} > 0$, using the Schur complement, it holds $\left(\begin{array}{c|c} M_{21} & M_{23} \\ \hline * & M_{22}\end{array}\right) > 0$. The inequality (2.4.16) holds. This concludes the proof of Lemma 2.4.3. $\square$

Chapter 3

Tikhonov theorem for linear hyperbolic systems of balance laws

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This chapter is concerned with linear hyperbolic systems of balance laws where the perturbation parameter ε is introduced in both dynamics and boundary conditions. Compared with Chapter 2, the transport velocity of the full system in this chapter depends on ε as well as the boundary conditions. The source terms of the full system depend also on ε. However, for the reduced subsystem, the transport velocity and the boundary condition do not depend on ε, and the source terms vanish at $\varepsilon = 0$. Under certain assumptions and conditions, the approximation between the full system and the reduced subsystem is achieved for ε sufficiently small.

This chapter is organized as follows. Section 3.1 presents the full system and the reduced subsystem. Tikhonov approximation for such system is given in Section 3.2. Precisely, the standing assumptions and the main result are stated in Section 3.2.1. The main result is proved by Lyapunov method in Section 3.2.2. In Section 3.3, an academic example is used to illustrate the main result. All the proofs of Lemmas are given in Section 3.4.

The main result of this chapter is conditionally accepted in IEEE Transactions on Automatic Control [Tang et al., b].

3.1 Full system and reduced subsystem descriptions

Let us consider the following linear hyperbolic system of balance laws

$$y_t(x,t) + \Lambda_1(\varepsilon)y_x(x,t) = a(\varepsilon)y(x,t) + b(\varepsilon)z(x,t), \tag{3.1.1a}$$
$$\varepsilon z_t(x,t) + \Lambda_2(\varepsilon)z_x(x,t) = c(\varepsilon)y(x,t) + d(\varepsilon)z(x,t), \tag{3.1.1b}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^n$, $z : [0,1] \times [0,+\infty) \to \mathbb{R}^m$, $0 < \varepsilon \ll 1$, $\Lambda_1(\varepsilon)$ and $\Lambda_2(\varepsilon)$ are positive diagonal matrices which are respectively, in $\mathbb{R}^{n\times n}$ and in $\mathbb{R}^{m\times m}$. Moreover, $\Lambda_1(0)$ and $\Lambda_2(0)$ are positive. The matrices $a(\varepsilon)$, $b(\varepsilon)$, $c(\varepsilon)$ and $d(\varepsilon)$ are of appropriate dimensions and vanish at $\varepsilon = 0$. The boundary condition under consideration is given by

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K(\varepsilon) \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in [0,+\infty), \tag{3.1.2}$$

where $K(\varepsilon) = \begin{pmatrix} K_{11}(\varepsilon) & K_{12}(\varepsilon) \\ K_{21}(\varepsilon) & K_{22}(\varepsilon) \end{pmatrix}$ is a matrix in $\mathbb{R}^{(n+m)\times(n+m)}$ with the matrices $K_{11}(\varepsilon)$ in $\mathbb{R}^{n\times n}$, $K_{12}(\varepsilon)$ in $\mathbb{R}^{n\times m}$, $K_{21}(\varepsilon)$ in $\mathbb{R}^{m\times n}$, $K_{22}(\varepsilon)$ in $\mathbb{R}^{m\times m}$. Given two functions $y_0 : [0,1] \to \mathbb{R}^n$ and $z_0 : [0,1] \to \mathbb{R}^m$, the initial condition is

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix}, \quad x \in [0,1]. \tag{3.1.3}$$

Remark 3.1.1. Motivated by the gas flow transport system which can be modeled by linear singularly perturbed hyperbolic systems with the transport velocities containing the perturbation parameter (as considering in Section 1.1, Page 7), in this chapter we consider that the transport velocities depend on ε. For the simplicity, we assume that all transport velocities are positive (i.e. $\Lambda_1(\varepsilon) > 0$ and $\Lambda_2(\varepsilon) > 0$), but we may consider a heterodirectional system such that $\Lambda_i(\varepsilon) = \begin{pmatrix} \Lambda_i^-(\varepsilon) & 0 \\ 0 & \Lambda_i^+(\varepsilon) \end{pmatrix}$ where $\Lambda_i^- < 0$ and $\Lambda_i^+ > 0$. The boundary condition (3.1.2) has to be adapted accordingly (see [Diagne et al., 2012]). The results remain valid but the proofs are more complex. ◦

Adopting the same computations as for the two subsystems in Section 2.1, the reduced subsystem for (3.1.1)-(3.1.3) is formally computed as follows.

$$\bar{y}_t(x,t) + \Lambda_1(0)\bar{y}_x(x,t) = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \tag{3.1.4}$$

with the boundary condition

$$\bar{y}(0,t) = K_r\bar{y}(1,t), \quad t \in [0,+\infty), \tag{3.1.5}$$

where $K_r = K_{11}(0) + K_{12}(0)(I_m - K_{22}(0))^{-1}K_{21}(0)$, whereas the initial condition is given as the same as for the full system,

$$\bar{y}(x,0) = \bar{y}_0(x) = y_0(x), \quad x \in [0,1]. \tag{3.1.6}$$

The compatibility conditions for existence of solutions of (3.1.4)-(3.1.6) in $H^2(0,1)$ are given by

$$\bar{y}_0(0) = K_r\bar{y}_0(1), \tag{3.1.7a}$$

$$\bar{y}_{0x}(0) = \Lambda_1^{-1}(0)K_r\Lambda_1(0)\bar{y}_{0x}(1). \tag{3.1.7b}$$

Remark 3.1.2. Compared with Chapter 2, the transport velocities of the full system in this chapter depend on ε as well as the boundary conditions. Moreover, we consider an additional source term which depend also on ε. Due to the presence of ε in both dynamics and boundary conditions, the full system becomes more complex. The assumptions on the continuity for such terms with respect to ε should be used to ensure that the Tikhonov approximation is valid for ε sufficiently small. The proof of the main result is then more sophisticated and is a non trivial extension. ○

3.2 Tikhonov approximation for systems of balance laws

In this section, we first state the standing assumptions. The approximation of the solutions to the full system by that to the reduced subsystem is given in Theorem 3.2.1. The main result is proved by a Lyapunov function.

3.2.1 Assumptions and main result

Let us consider the following assumptions:

Assumption 3.2.1. *The functions Λ_1 and Λ_2 are Lipschitz continuous at 0, that is there exist positive constants R_1 and $\bar{\varepsilon}$ such that for all $0 < \varepsilon < \bar{\varepsilon}$,*

$$\|\Lambda_1(\varepsilon) - \Lambda_1(0)\| \leqslant \varepsilon R_1,$$

$$\|\Lambda_2(\varepsilon) - \Lambda_2(0)\| \leqslant \varepsilon R_1.$$

Assumption 3.2.2. *Let $\bar{\varepsilon}$ as in Assumption 3.2.1, the functions a, b, c and d are Lipschitz continuous at 0, that is there exits a positive constant R_2, such that for all $0 < \varepsilon < \bar{\varepsilon}$,*

$$\|a(\varepsilon)\| \leqslant \varepsilon R_2,$$
$$\|b(\varepsilon)\| \leqslant \varepsilon R_2,$$
$$\|c(\varepsilon)\| \leqslant \varepsilon R_2,$$
$$\|d(\varepsilon)\| \leqslant \varepsilon R_2.$$

Assumption 3.2.3. *Let $\bar{\varepsilon}$ as in Assumption 3.2.1, the functions K_{11}, K_{12}, K_{21} and K_{22} are Lipschitz continuous at 0, that is there exists a positive value R_3, such that for all $0 < \varepsilon < \bar{\varepsilon}$,*

$$\begin{aligned}
\|K_{11}(\varepsilon) - K_{11}(0)\| &\leqslant \varepsilon R_3,\\
\|K_{12}(\varepsilon) - K_{12}(0)\| &\leqslant \varepsilon R_3,\\
\|K_{21}(\varepsilon) - K_{21}(0)\| &\leqslant \varepsilon R_3,\\
\|K_{22}(\varepsilon) - K_{22}(0)\| &\leqslant \varepsilon R_3.
\end{aligned}$$

With the above assumptions, we are ready to state the main result of this section in the following theorem.

Theorem 3.2.1. *Consider the linear hyperbolic system of balance laws (3.1.1)-(3.1.3), under Assumptions 3.2.1-3.2.3, if the condition $\rho_1(K(0)) < 1$ is satisfied, there exist positive values C_1, C_2, θ, ε^*, such that for any initial condition $y_0 \in H^2(0,1)$ satisfying compatibility conditions (3.1.7) with $\bar{y}_0 = y_0$, and $z_0 \in L^2(0,1)$ and for all $0 < \varepsilon < \varepsilon^*$, it holds for all $t \geqslant 0$*

$$\begin{aligned}
&\|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \leqslant \\
&\quad \varepsilon C_1 e^{-\theta t}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}_0(1)\|^2_{L^2(0,1)}\Big),
\end{aligned} \tag{3.2.1}$$

$$\begin{aligned}
&\int_0^{+\infty} \|z(.,t) - (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t)\|^2_{L^2(0,1)}dt \leqslant \\
&\quad \varepsilon C_2\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}_0(1)\|^2_{L^2(0,1)}\Big).
\end{aligned} \tag{3.2.2}$$

In the above theorem, the errors between the full system and the two subsystems are estimated in L^2-norm, for any initial condition $y_0 \in H^2(0,1)$ satisfying the compatibility conditions and for any $z_0 \in L^2(0,1)$. Compare with Corollary 2.3.1 in Chapter 2 where the term $\|z(.,t) - (I_m - K_{22})^{-1}K_{21}\bar{y}(1,t)\|^2_{L^2(0,1)}$ is estimated of the order of ε for particular initial condition $z_0 = (I_m - K_{22})^{-1}K_{21}\bar{y}_0(1)$, in this chapter we were not able to get this result even though z_0 is initially at the equilibrium point.

3.2.2 Proof of Theorem 3.2.1

In the following, Theorem 3.2.1 is proved by three steps.

Step I) Let us perform the following change of variables,

$$\eta = y - \bar{y}, \tag{3.2.3a}$$
$$\delta = z - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}(1,.), \tag{3.2.3b}$$

where η stands for the error between the slow dynamics y in (3.1.1) and the dynamics $\bar{y}$ in (3.1.4), and δ is the error between the fast dynamics z in (3.1.1) and its equilibrium point. In all the following, it is assumed $\varepsilon \in (0, \bar{\varepsilon})$. Due to (3.2.3) and (3.1.4), the system (3.1.1) is written in the new variables (η, δ) as follows

$$\begin{aligned} \eta_t + \Lambda_1(\varepsilon)\eta_x &= a(\varepsilon)\eta + b(\varepsilon)\delta - (\Lambda_1(\varepsilon) - \Lambda_1(0))\bar{y}_x \\ &\quad + a(\varepsilon)\bar{y} + b(\varepsilon)(I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,.), \end{aligned} \tag{3.2.4a}$$

$$\begin{aligned} \varepsilon\delta_t + \Lambda_2(\varepsilon)\delta_x &= c(\varepsilon)\eta + d(\varepsilon)\delta + c(\varepsilon)\bar{y} \\ &\quad + d(\varepsilon)(I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,.) \\ &\quad + \varepsilon(I_m - K_{22}(0))^{-1}K_{21}(0)\Lambda_1(0)\bar{y}_x(1,.). \end{aligned} \tag{3.2.4b}$$

Due to (3.1.2) and (3.1.5), the boundary condition for system (3.2.4) is computed as follows

$$\begin{aligned} \eta(0,t) &= y(0,t) - \bar{y}(0,t) \\ &= K_{11}(\varepsilon)y(1,t) + K_{12}(\varepsilon)z(1,t) \\ &\quad - \Big(K_{11}(0) + K_{12}(0)(I_m - K_{22}(0))^{-1}K_{21}(0)\Big)\bar{y}(1,t) \\ &= K_{11}(\varepsilon)\Big(\eta(1,t) + \bar{y}(1,t)\Big) \\ &\quad + K_{12}(\varepsilon)\Big(\delta(1,t) + (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t)\Big) \\ &\quad - \Big(K_{11}(0) + K_{12}(0)(I_m - K_{22}(0))^{-1}K_{21}(0)\Big)\bar{y}(1,t) \\ &= K_{11}(\varepsilon)\eta(1,t) + K_{12}(\varepsilon)\delta(1,t) + K_{d1}(\varepsilon)\bar{y}(1,t), \end{aligned}$$

$$
\begin{aligned}
\delta(0,t) &= z(0,t) - (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t) \\
&= K_{21}(\varepsilon)y(1,t) + K_{22}(\varepsilon)z(1,t) - (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t) \\
&= K_{21}(\varepsilon)\Big(\eta(1,t) + \bar{y}(1,t)\Big) \\
&\quad + K_{22}(\varepsilon)\Big(\delta(1,t) + (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t)\Big) \\
&\quad - (I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t) \\
&= K_{21}(\varepsilon)\eta(1,t) + K_{22}(\varepsilon)\delta(1,t) + \Big(K_{21}(\varepsilon) - K_{21}(0)\Big)\bar{y}(1,t) \\
&\quad + K_{22(\varepsilon)}(I_m - K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t) \\
&\quad - \Big((I_m - K_{22}(0))^{-1} - I_m\Big)K_{21}(0)\bar{y}(1,t) \\
&= K_{21}(\varepsilon)\eta(1,t) + K_{22}(\varepsilon)\delta(1,t) + K_{d2}(\varepsilon)\bar{y}(1,t).
\end{aligned}
$$

The boundary condition is written as

$$
\begin{pmatrix}\eta(0,t)\\ \delta(0,t)\end{pmatrix} = \begin{pmatrix}K_{11}(\varepsilon) & K_{12}(\varepsilon)\\ K_{21}(\varepsilon) & K_{22}(\varepsilon)\end{pmatrix}\begin{pmatrix}\eta(1,t)\\ \delta(1,t)\end{pmatrix} + \begin{pmatrix}K_{d1}(\varepsilon)\\ K_{d2}(\varepsilon)\end{pmatrix}\bar{y}(1,t), \tag{3.2.5}
$$

where $K_{d1}(\varepsilon) = (K_{11}(\varepsilon) - K_{11}(0)) + (K_{12}(\varepsilon) - K_{12}(0))(I_m - K_{22}(0))^{-1}K_{21}(0)$ and $K_{d2}(\varepsilon) = (K_{21}(\varepsilon) - K_{21}(0)) + (K_{22}(\varepsilon) - K_{22}(0))(I_m - K_{22}(0))^{-1}K_{21}(0)$.

Remark 3.2.1. Due to Assumption 3.2.3, there exists positive r_1, such that

$$
\begin{aligned}
\|K_{d1}(\varepsilon)\| &\leqslant \varepsilon r_1, \\
\|K_{d2}(\varepsilon)\| &\leqslant \varepsilon r_1.
\end{aligned}
$$

∘

The candidate Lyapunov function for system (3.2.4)-(3.2.5) is considered as follows,

$$
V(\eta,\delta,\varepsilon) = \int_0^1 e^{-\mu x}(\eta^\top Q\eta + \varepsilon\delta^\top P\delta)\,dx, \tag{3.2.6}
$$

with $\mu > 0$, Q a positive diagonal matrix in $\mathbb{R}^{n\times n}$ and P a positive diagonal matrix in $\mathbb{R}^{m\times m}$, which will be specified later. We rewrite $V(\eta,\delta,\varepsilon) = V_1 + V_2$ with

$$
V_1 = \int_0^1 e^{-\mu x}\eta^\top Q\eta\,dx, \quad V_2 = \varepsilon\int_0^1 e^{-\mu x}\delta^\top P\delta\,dx
$$

Let us compute the time derivative of V_1 along the solutions to (3.2.4a),

$$\begin{aligned}
\dot{V}_1 &= \int_0^1 e^{-\mu x}(2\eta^\top Q\eta_t)\,dx \\
&= -2\int_0^1 e^{-\mu x}\,\eta^\top Q\Lambda_1(\varepsilon)\eta_x\,dx + 2\int_0^1 e^{-\mu x}\,\eta^\top Qa(\varepsilon)\eta\,dx \\
&\quad +2\int_0^1 e^{-\mu x}\,\eta^\top Qb(\varepsilon)\delta\,dx + 2\int_0^1 e^{-\mu x}\,\eta^\top Qa(\varepsilon)\bar{y}\,dx \\
&\quad -2\int_0^1 e^{-\mu x}\,\eta^\top Q\,(\Lambda_1(\varepsilon)-\Lambda_1(0))\,\bar{y}_x\,dx \\
&\quad +2\int_0^1 e^{-\mu x}\,\eta^\top Qb(\varepsilon)\,(I_m-K_{22}(0))^{-1}\,K_{21}(0)\bar{y}(1,.)\,dx,
\end{aligned}$$

performing an integration by parts on the first integral yields

$$\begin{aligned}
\dot{V}_1 &= -[e^{-\mu x}\eta^\top Q\Lambda_1(\varepsilon)\eta]_{x=0}^{x=1} - \int_0^1 e^{-\mu x}\eta^\top\,(\mu Q\Lambda_1(\varepsilon)-2Qa(\varepsilon))\,\eta\,dx \\
&\quad +2\int_0^1 e^{-\mu x}\,\eta^\top Qb(\varepsilon)\delta\,dx + 2\int_0^1 e^{-\mu x}\,\eta^\top Qa(\varepsilon)\bar{y}\,dx \\
&\quad -2\int_0^1 e^{-\mu x}\,\eta^\top Q\,(\Lambda_1(\varepsilon)-\Lambda_1(0))\,\bar{y}_x\,dx \\
&\quad +2\int_0^1 e^{-\mu x}\,\eta^\top Qb(\varepsilon)\,(I_m-K_{22}(0))^{-1}\,K_{21}(0)\bar{y}(1,.)\,dx.
\end{aligned}$$

Similarly, we compute the time derivative of V_2 along the solutions to (3.2.4b),

$$\begin{aligned}
\dot{V}_2 &= \varepsilon\int_0^1 e^{-\mu x}(2\delta^\top P\delta_t)\,dx \\
&= -2\int_0^1 e^{-\mu x}\,\delta^\top P\Lambda_2(\varepsilon)\delta_x\;\,dx + 2\int_0^1 e^{-\mu x}\,\delta^\top Pc(\varepsilon)\eta\,dx \\
&\quad +2\int_0^1 e^{-\mu x}\,\delta^\top Pd(\varepsilon)\delta\,dx + 2\int_0^1 e^{-\mu x}\,\delta^\top Pc(\varepsilon)\bar{y}\,dx \\
&\quad +2\int_0^1 e^{-\mu x}\,\delta^\top Pd(\varepsilon)\,(I_m-K_{22}(0))^{-1}\,K_{21}(0)\bar{y}(1,.)\,dx \\
&\quad +2\varepsilon\int_0^1 e^{-\mu x}\,\delta^\top P\,(I_m-K_{22}(0))^{-1}\,K_{21}(0)\Lambda_1(0)\bar{y}_x(1,.)\,dx,
\end{aligned}$$

performing an integration by parts on the first integral yields

$$\begin{aligned}
\dot{V}_2 &= -[e^{-\mu x}\delta^\top P\Lambda_2(\varepsilon)\delta]_{x=0}^{x=1} - \int_0^1 e^{-\mu x}\delta^\top \left(\mu P\Lambda_2(\varepsilon) - 2Pd(\varepsilon)\right)\delta\, dx \\
&\quad +2\int_0^1 e^{-\mu x}\, \delta^\top Pc(\varepsilon)\eta\, dx + 2\int_0^1 e^{-\mu x}\, \delta^\top Pc(\varepsilon)\bar{y}\, dx \\
&\quad +2\int_0^1 e^{-\mu x}\, \delta^\top Pd(\varepsilon)\, (I_m - K_{22}(0))^{-1}\, K_{21}(0)\bar{y}(1,.)\, dx \\
&\quad +2\varepsilon\int_0^1 e^{-\mu x}\, \delta^\top P\, (I_m - K_{22}(0))^{-1}\, K_{21}(0)\Lambda_1(0)\bar{y}_x(1,.)\, dx.
\end{aligned}$$

Combining $\dot{V}_1$ and $\dot{V}_2$, we obtain

$$\dot{V}(\eta,\delta,\varepsilon) = \dot{V}_1 + \dot{V}_2 = T_1 + T_2 + T_3,$$

with:

$$\begin{aligned}
T_1 &= -[e^{-\mu x}\eta^\top Q\Lambda_1(\varepsilon)\eta]_{x=0}^{x=1} - [e^{-\mu x}\delta^\top P\Lambda_2(\varepsilon)\delta]_{x=0}^{x=1}, \\
T_2 &= -\int_0^1 e^{-\mu x}\eta^\top \left(\mu Q\Lambda_1(\varepsilon) - 2Qa(\varepsilon)\right)\eta\, dx \\
&\quad -\int_0^1 e^{-\mu x}\delta^\top \left(\mu P\Lambda_2(\varepsilon) - 2Pd(\varepsilon)\right)\delta\, dx \\
&\quad +2\int_0^1 e^{-\mu x}\eta^\top\Big(Qb(\varepsilon) + c^\top(\varepsilon)P\Big)\delta\, dx, \\
T_3 &= -2\int_0^1 e^{-\mu x}\, \eta^\top Q\, (\Lambda_1(\varepsilon) - \Lambda_1(0))\, \bar{y}_x\, dx \\
&\quad +2\int_0^1 e^{-\mu x}\, \eta^\top Qb(\varepsilon)\, (I_m - K_{22}(0))^{-1}\, K_{21}(0)\bar{y}(1,.)\, dx \\
&\quad +2\int_0^1 e^{-\mu x}\, \eta^\top Qa(\varepsilon)\bar{y}\, dx + 2\int_0^1 e^{-\mu x}\, \delta^\top Pc(\varepsilon)\bar{y}\, dx \\
&\quad +2\int_0^1 e^{-\mu x}\, \delta^\top Pd(\varepsilon)\, (I_m - K_{22}(0))^{-1}\, K_{21}(0)\bar{y}(1,.)\, dx \\
&\quad +2\varepsilon\int_0^1 e^{-\mu x}\, \delta^\top P\, (I_m - K_{22}(0))^{-1}\, K_{21}(0)\Lambda_1(0)\bar{y}_x(1,.)\, dx.
\end{aligned}$$

In order to deal with the terms $\bar{y}(1,.)$ and $\bar{y}_x(1,.)$ in T_3, let us consider the following estimates

$$\begin{aligned}
|\bar{y}(1,.)| &= \left|\int_0^1 \Big(\bar{y} + x\bar{y}_x\Big)\, dx\right| \\
&\leqslant \int_0^1 \Big(|\bar{y}| + |\bar{y}_x|\Big)\, dx \\
&\leqslant \sqrt{2}\|\bar{y}(.,t)\|_{H^2(0,1)},
\end{aligned} \tag{3.2.7}$$

$$\begin{aligned} |\bar{y}_x(1,.)| &= \left| \int_0^1 \left(\bar{y}_x + x\bar{y}_{xx} \right) dx \right| \\ &\leqslant \int_0^1 \left(|\bar{y}| + |\bar{y}_x| + |\bar{y}_{xx}| \right) dx \\ &\leqslant \sqrt{3} \|\bar{y}(.,t)\|_{H^2(0,1)}. \end{aligned} \tag{3.2.8}$$

Step II) To estimate the terms T_1-T_3, let us state the following lemmas. The stability of the reduced subsystem in H^2-norm is given in Lemma 3.2.1.

Lemma 3.2.1. *[Lemma 2.3.2 in Chapter 2] Consider the reduced subsystem (3.1.4)-(3.1.6), if $\rho_1(K(0)) < 1$, there exist $C_r > 0$, such that for any initial condition $\bar{y}_0 \in H^2(0,1)$ satisfying the compatibility conditions (3.1.7) and for all $t \geqslant 0$,*

$$\|\bar{y}(.,t)\|^2_{H^2(0,1)} \leqslant C_r e^{-\mu \underline{\lambda}(\Lambda_1(0))t} \|\bar{y}_0\|^2_{H^2(0,1)}. \tag{3.2.9}$$

Lemma 3.2.2. *If the boundary condition satisfies $\rho_1(K(0)) < 1$, under Assumptions 3.2.1 and 3.2.3, there exist positive values μ, C_{T_1} and ε_1^*, and diagonal positive matrices Q and P, such that for all $\varepsilon \in (0, \varepsilon_1^*)$ and $t \geqslant 0$,*

$$T_1 \leqslant \varepsilon C_{T_1} e^{-\mu \underline{\lambda}(\Lambda_1(0))t} \|\bar{y}_0\|^2_{H^2(0,1)}. \tag{3.2.10}$$

Lemma 3.2.3. *Under Assumptions 3.2.1 and 3.2.2, let μ, Q and P as in Lemma 3.2.2, there exist positive values C_{T_2} and ε_2^*, such that for all $\varepsilon \in (0, \varepsilon_2^*)$,*

$$T_2 \leqslant -C_{T_2} \int_0^1 e^{-\mu x} \left(\eta^\top Q \eta + \delta^\top P \delta \right) dx. \tag{3.2.11}$$

Lemma 3.2.4. *Under Assumptions 3.2.1 and 3.2.2, let μ as in Lemma 3.2.2, there exist positive constants $C_{T_{31}}$, $C_{T_{32}}$ and $C_{T_{33}}$, such that for all positive value ε and for all $t \geqslant 0$,*

$$\begin{aligned} T_3 \leqslant\ & \varepsilon C_{T_{31}} \int_0^1 e^{-\mu x} |\eta|^2 \, dx + \varepsilon C_{T_{32}} \int_0^1 e^{-\mu x} |\delta|^2 \, dx \\ & + \varepsilon C_{T_{33}} e^{-\mu \underline{\lambda}(\Lambda_1(0))t} \|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned} \tag{3.2.12}$$

The proofs of Lemmas 3.2.2-3.2.4 are given in Section 3.4.

Step III) Using Lemmas 3.2.2-3.2.4, we obtain

$$\begin{aligned} \dot{V}(\eta, \delta, \varepsilon) \leqslant\ & -(C_{T_2} - \varepsilon C_v) \int_0^1 e^{-\mu x} \left(\eta^\top Q \eta + \delta^\top P \delta \right) dx \\ & + \varepsilon (C_{T_1} + C_{T_{33}}) e^{-\mu \underline{\lambda}(\Lambda_1(0))t} \|\bar{y}_0\|^2_{H^2(0,1)}, \end{aligned} \tag{3.2.13}$$

where $C_v = \max\left(\frac{C_{T_{31}}}{\underline{\lambda}(Q)}, \frac{C_{T_{32}}}{\underline{\lambda}(P)}\right)$. Let $\varepsilon_3^* = \frac{C_{T_2}}{2C_v}$, ε_1^* in Lemma 3.2.2, ε_2^* in Lemma 3.2.3 and $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*)$, there exists $\varpi > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$,

$$\dot{V}(\eta, \delta, \varepsilon) \leqslant -\varpi V(\eta, \delta, \varepsilon) + \varepsilon(C_{T_1} + C_{T_{33}})e^{-\mu\underline{\lambda}(\Lambda_1(0))t}\|\bar{y}_0\|^2_{H^2(0,1)}.$$

In the above inequality, the term $\|\bar{y}_0\|^2_{H^2(0,1)}$ is seen as a disturbance and it follows

$$\begin{aligned} V(\eta, \delta, \varepsilon) &\leqslant e^{-\varpi t}V(\eta_0, \delta_0, \varepsilon) \\ &\quad +\varepsilon(C_{T_1} + C_{T_{33}})e^{-\varpi t}\frac{e^{(\varpi - \mu\underline{\lambda}(\Lambda_1(0)))t} - 1}{\varpi - \mu\underline{\lambda}(\Lambda_1(0))}\|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned}$$

Since $\varpi < C_{T_2}$, we may let $\varpi < \mu\underline{\lambda}(\Lambda_1(0))$, thus the above equation can be rewritten as follows

$$V(\eta, \delta, \varepsilon) \leqslant e^{-\varpi t}V(\eta_0, \delta_0, \varepsilon) + \varepsilon e^{-\varpi t}\frac{C_{T_1} + C_{T_{33}}}{\mu\underline{\lambda}(\Lambda_1(0)) - \varpi}\|\bar{y}_0\|^2_{H^2(0,1)}. \quad (3.2.14)$$

Since $V(\eta, \delta, \varepsilon)$ is lower and upper bounded by

$$\begin{aligned} & e^{-\mu}\underline{\lambda}(Q)\|\eta\|^2_{L^2(0,1)} + \varepsilon e^{-\mu}\underline{\lambda}(P)\|\delta\|^2_{L^2(0,1)} \\ \leqslant\ & V(\eta, \delta, \varepsilon) \\ \leqslant\ & \|Q\|\|\eta\|^2_{L^2(0,1)} + \varepsilon\|P\|\|\delta\|^2_{L^2(0,1)}, \end{aligned} \quad (3.2.15)$$

it follows

$$\|\eta(.,t)\|^2_{L^2(0,1)} \leqslant \frac{e^{\mu}e^{-\varpi t}}{\underline{\lambda}(Q)}V(\eta_0, \delta_0, \varepsilon) + \frac{\varepsilon e^{-\varpi t}e^{\mu}(C_{T_1} + C_{T_{33}})}{\underline{\lambda}(Q)\ (\mu\underline{\lambda}(\Lambda_1(0)) - \varpi)}\|\bar{y}_0\|^2_{H^2(0,1)}.$$

Due to the initial condition $y_0 = \bar{y}_0$ i.e. $\eta_0 = 0$, it is obtained

$$\begin{aligned} \|\eta(.,t)\|^2_{L^2(0,1)} &\leqslant \varepsilon e^{-\varpi t}\left(\frac{e^{\mu}\|P\|}{\underline{\lambda}(Q)}\|\delta_0\|_{L^2(0,1)}\right. \\ &\qquad \left. + \frac{e^{\mu}(C_{T_1} + C_{T_{33}})}{\underline{\lambda}(Q)\ (\mu\underline{\lambda}(\Lambda_1(0)) - \varpi)}\|\bar{y}_0\|^2_{H^2(0,1)}\right). \end{aligned}$$

This proves (3.2.1).

Noting that for $\varepsilon < \varepsilon^*$, the term $\left(-(C_{T_2} - \varepsilon C_v)\int_0^1 e^{-\mu x}\eta^\top Q\eta dx\right)$ in the right hand side of (3.2.13) is always negative, then $\dot{V}(\eta, \delta, \varepsilon)$ can be written as follows

$$\begin{aligned} \dot{V}(\eta, \delta, \varepsilon) &\leqslant -\varpi\int_0^1 e^{-\mu x}\delta^\top P\delta\ dx \\ &\quad +\varepsilon(C_{T_1} + C_{T_{33}})e^{-\mu\underline{\lambda}(\Lambda_1(0))t}\|\bar{y}_0\|^2_{H^2(0,1)}. \end{aligned}$$

Performing a time integration of both sides from 0 to $+\infty$, it follows

$$\begin{aligned}\int_0^{+\infty} \|\delta(.,t)\|^2_{L^2(0,1)}\, dt \;\leqslant\; & \frac{e^{\mu}}{\underline{\lambda}(P)\varpi}\Bigg(V(\eta_0,\delta_0,\varepsilon) - \lim_{t\to+\infty} V(\eta,\delta,\varepsilon) \\ & +\varepsilon\left(C_{T_1}+C_{T_{33}}\right)\|\bar{y}_0\|^2_{H^2(0,1)}\int_0^{+\infty} e^{-\mu\underline{\lambda}(\Lambda_1(0))t}dt\Bigg),\end{aligned}$$

since $\lim_{t\to+\infty} V(\eta,\delta,\varepsilon)=0$ and $\eta_0=0$, it follows

$$\int_0^{+\infty} \|\delta(.,t)\|^2_{L^2(0,1)}dt \;\leqslant\; \frac{\varepsilon e^{\mu}\|P\|}{\varpi\underline{\lambda}(P)}\|\delta_0\|_{L^2(0,1)} + \frac{\varepsilon e^{\mu}\left(C_{T_1}+C_{T_{33}}\right)}{\mu\varpi\underline{\lambda}(P)\underline{\lambda}(\Lambda_1(0))}\|\bar{y}_0\|^2_{H^2(0,1)}.$$

This proves (3.2.2) and concludes the proof of Theorem 3.2.1. □

3.3 Numerical simulations on academic example

In this section, we use an academic example to illustrate the main result. The full system is given by

$$y_t + (1+\varepsilon)y_x = 0.1\varepsilon\, y + 0.2\varepsilon\, z, \tag{3.3.1a}$$
$$\varepsilon z_t + (1-\varepsilon)z_x = 0.05\varepsilon\, y + 0.4\varepsilon\, z. \tag{3.3.1b}$$

The boundary condition for system (3.3.1) is considered as

$$K(\varepsilon) = \begin{pmatrix} 0.5+\varepsilon & 1+\varepsilon \\ 0.5+\varepsilon & -0.5+\varepsilon \end{pmatrix}. \tag{3.3.2}$$

Assumptions 3.2.1-3.2.3 hold for system (3.3.1) with boundary condition (3.3.2) for any $0<\bar{\varepsilon}<1$, $R_1=2$, $R_2=1$ and $R_3=2$. Let Δ be given as $\Delta = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.7 \end{pmatrix}$, we compute that $\|\Delta K(0)\Delta^{-1}\|<1$ holds, thus the condition $\rho_1(K(0))<1$ is satisfied. Theorem 3.2.1 applies. The reduced subsystem is computed as

$$\bar{y}_t + \bar{y}_x = 0, \tag{3.3.3}$$

with the boundary condition

$$K_r = 5/6. \tag{3.3.4}$$

To numerically compute the solutions of the full system (3.3.1) with (3.3.2) and of the reduced subsystem (3.3.3) with (3.3.4), we use the same method, which is presented in Section 2.6, to discretize the system. Precisely, the space domain [0,1] is divided into 100 intervals of identical length, the final time is chosen as 30. We take a time-step $0.9\varepsilon dt$ that satisfies the CFL

condition and select the initial conditions as follows, such that y_0 satisfies the compatibility condition (3.1.7), for all $x \in [0, 1]$,

$$y_0(x) = \bar{y}_0(x) = 1 - \cos(4\pi x),$$

$$z_0(x) = \sin(2\pi x).$$

Figure 3.3.1 is the time evolution of the solution $\bar{y}$ of the reduced subsystem. It converges to the origin when time increases. In Figure 3.3.2, it is shown that the error between the slow dynamics y of the full system and $\bar{y}$ of the reduced subsystem is small, and it tends to zero as time increases.

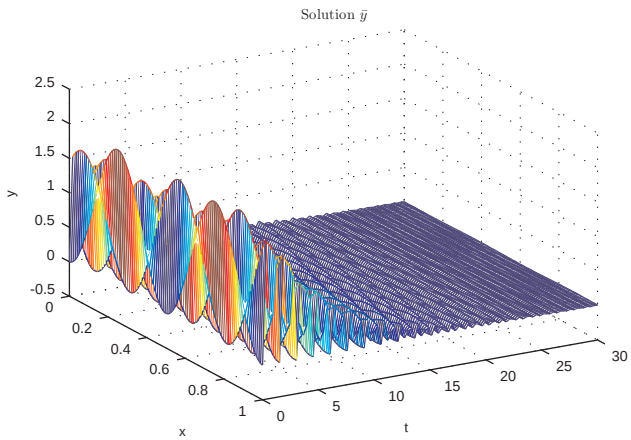

Figure 3.3.1: Time evolution of the solution $\bar{y}$ of the reduced subsystem

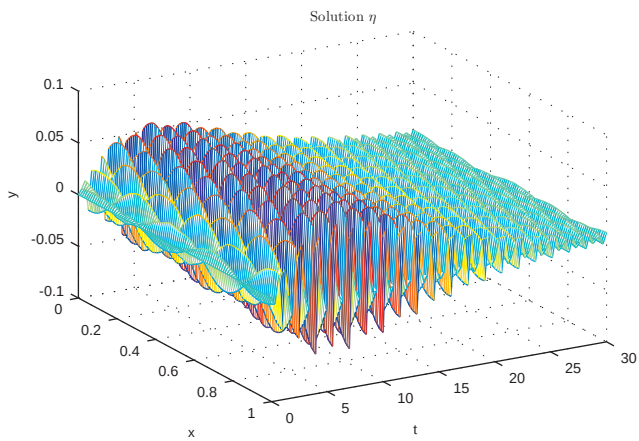

Figure 3.3.2: Time evolution of η which is the error between the slow dynamics y of the full system and $\bar{y}$ of the reduced subsystem

For the following estimates, the perturbation parameter are chosen as $\varepsilon =$

$(0.001, 0.002, 0.003)$ respectively. The estimates of the errors between the full system and the subsystems for different ε are given by Table 3.3.1, with $\|\eta(.,t=3)\|^2_{L^2(0,1)} = \|y-\bar{y}(.,t=3)\|^2_{L^2(0,1)}$ and $\int_0^{30}\|\delta(.,t)\|^2_{L^2(0,1)}dt = \int_0^{30}\|z(.,t)-(I_m-K_{22}(0))^{-1}K_{21}(0)\bar{y}(1,t)\|^2_{L^2(0,1)}dt$. The values are close to zero and decrease as ε decreases, as expected from Theorem 3.2.1.

ε	0.001	0.002	0.003
$\eta(.,t=3)\|^2_{L^2(0,1)}$	1.3×10^{-4}	5.2×10^{-4}	1.2×10^{-3}
$\int_0^{30}\|\delta(.,t)\|^2_{L^2(0,1)}dt$	2.8×10^{-4}	1.1×10^{-3}	2.5×10^{-3}

Table 3.3.1: Estimates of the errors for different values of ε

3.4 Technical proofs of Lemmas

Proof of Lemma 3.2.2 Using boundary condition (3.2.5) and after developing and reorganizing, T_1 is rewritten as

$$T_1 = T_{11} + T_{12},$$

with

$$\begin{aligned}
T_{11} &= -\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}^\top \begin{pmatrix} e^{-\mu}Q\Lambda_1(\varepsilon) & 0\\ 0 & e^{-\mu}P\Lambda_2(\varepsilon)\end{pmatrix}\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}\\
&\quad+\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}^\top \begin{pmatrix} K_{11}(\varepsilon) & K_{12}(\varepsilon)\\ K_{21}(\varepsilon) & K_{22}(\varepsilon)\end{pmatrix}^\top \begin{pmatrix} Q\Lambda_1(\varepsilon) & 0\\ 0 & P\Lambda_2(\varepsilon)\end{pmatrix}\\
&\quad\times\begin{pmatrix} K_{11}(\varepsilon) & K_{12}(\varepsilon)\\ K_{21}(\varepsilon) & K_{22}(\varepsilon)\end{pmatrix}\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix},\\
T_{12} &= \bar{y}^\top(1)\left(K_{d1}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}+K_{d2}^\top(\varepsilon)P\Lambda(\varepsilon)K_{d2}(\varepsilon)\right)\bar{y}(1)\\
&\quad+2\eta^\top(1)\left(K_{11}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon)+K_{21}^\top(\varepsilon)P\Lambda_2(\varepsilon)K_{d2}(\varepsilon)\right)\bar{y}(1)\\
&\quad+2\delta^\top(1)\left(K_{12}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon)+K_{22}^\top(\varepsilon)P\Lambda_2(\varepsilon)K_{d2}(\varepsilon)\right)\bar{y}(1).
\end{aligned}$$

Since $\rho_1(G(0))<1$, let $\Delta = \begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}$, such that $\|\Delta K(0)\Delta^{-1}\| = \sigma < 1$. Let

$$\Delta(\varepsilon) = \begin{pmatrix}\Delta_1\Lambda_1^{-\frac{1}{2}}(0)\Lambda_1^{\frac{1}{2}}(\varepsilon) & 0\\ 0 & \Delta_2\Lambda_2^{-\frac{1}{2}}(0)\Lambda_2^{\frac{1}{2}}(\varepsilon)\end{pmatrix},$$

under Assumption 3.2.1, due to the continuity of $\Lambda_1(\varepsilon)$ and $\Lambda_2(\varepsilon)$, there exists positive ε_{11}^* small enough such that for all $\varepsilon\in(0,\varepsilon_{11}^*)$, $\|\Delta(\varepsilon)K(\varepsilon)\Delta^{-1}(\varepsilon)\| =$

$\sigma^* < 1$. Let $Q = \Delta_1^2(\varepsilon)\Lambda_1^{-1}(\varepsilon)$, $P = \Delta_2^2(\varepsilon)\Lambda_2^{-1}(\varepsilon)$ and $0 < \mu < -2\ln\sigma^*$, there exists a positive value β such that it holds for all $\varepsilon \in (0, \varepsilon_{11}^*)$

$$\begin{aligned} T_{11} &= -\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix}^\top \left(e^{-\mu}\Delta^2(\varepsilon) - K^\top(\varepsilon)\Delta^2(\varepsilon)K(\varepsilon)\right)\begin{pmatrix}\eta(1)\\ \delta(1)\end{pmatrix} \\ &= -\beta(|\eta(1)|^2 + |\delta(1)|^2) < 0. \end{aligned} \tag{3.4.1}$$

Using Cauchy Schwarz inequality and Young's inequality, such that for all $k > 0$, T_{12} follows

$$\begin{aligned} T_{12} \leqslant\ & k\|K_{11}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon) + K_{21}^\top(\varepsilon)P\Lambda_2(\varepsilon)K_{d2}(\varepsilon)\|\ |\eta(1)|^2 \\ & + k\|K_{12}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon) + K_{22}^\top(\varepsilon)P\Lambda_2(\varepsilon)K_{d2}(\varepsilon)\|\ |\delta(1)|^2 \\ & + \frac{\|K_{11}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon) + K_{21}^\top(\varepsilon)P\Lambda_2(\varepsilon)K_{d2}(\varepsilon)\|}{k}\ |\bar{y}(1)|^2 \\ & + \frac{\|K_{12}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon) + K_{22}^\top(\varepsilon)P\Lambda_2(\varepsilon)K_{d2}(\varepsilon)\|}{k}\ |\bar{y}(1)|^2 \\ & + \|K_{d1}^\top(\varepsilon)Q\Lambda_1(\varepsilon)K_{d1}(\varepsilon) + K_{d2}^\top(\varepsilon)P\Lambda(\varepsilon)K_{d2}(\varepsilon)\|\ |\bar{y}(1)|^2. \end{aligned}$$

By choosing $k = 1$ and using Assumption 3.2.3 and Remark 3.2.1, it follows

$$T_{12} \leqslant \varepsilon R_1|\eta(1)|^2 + \varepsilon R_2|\delta(1)|^2 + \varepsilon R_3|\bar{y}(1)|^2, \tag{3.4.2}$$

where R_1-R_3 are positive. Combined with (3.4.1), it yields

$$T_1 \leqslant -(\beta - \varepsilon R_1)|\eta(1)|^2 - (\beta - \varepsilon R_2)|\delta(1)|^2 + \varepsilon R_3|\bar{y}(1)|^2.$$

Let $\varepsilon_{12}^* = \min\left(\frac{\beta}{R_1}, \frac{\beta}{R_2}\right)$, using the estimates in (3.2.7) and Lemma 3.2.1, then for all $0 < \varepsilon < \varepsilon_1^* = \min(\varepsilon_{11}^*, \varepsilon_{12}^*)$, T_1 follows

$$T_1 \leqslant \varepsilon C_{T_1} e^{-\mu\underline{\lambda}(\Lambda_1(0))t}\|\bar{y}_0\|_{H^2(0,1)}^2.$$

This concludes the proof of Lemma 3.2.2. □

Proof of Lemma 3.2.3 Due to the continuity of $\Lambda_1(\varepsilon)$ and $\Lambda_2(\varepsilon)$ in Assumption 3.2.1, we may assume, for all $\varepsilon \in (0, \bar{\varepsilon})$,

$$\|\Lambda_1(\varepsilon)\| \geqslant \frac{\underline{\lambda}(\Lambda_1(0))}{2},\ \|\Lambda_2(\varepsilon)\| \geqslant \frac{\underline{\lambda}(\Lambda_2(0))}{2}.$$

Under Assumption 3.2.2, it deduced from T_2

$$\begin{aligned} T_2 \leqslant\ & -\left(\frac{\mu\underline{\lambda}(Q)\underline{\lambda}(\Lambda_1(0))}{2} - 2\varepsilon R_2\|Q\|\right)\int_0^1 e^{-\mu x}|\eta|^2\,dx \\ & -\left(\frac{\mu\underline{\lambda}(P)\underline{\lambda}(\Lambda_2(0))}{2} - 2\varepsilon R_2\|P\|\right)\int_0^1 e^{-\mu x}|\delta|^2\,dx \\ & + 2\varepsilon R_2(\|Q\| + \|P\|)\int_0^1 e^{-\mu x}|\eta|\ |\delta|\,dx. \end{aligned}$$

Using Young's inequality to the third term, for all $k_2 > 0$, it holds

$$
\begin{aligned}
T_2 \leqslant & -\left(\frac{\mu\underline{\lambda}(Q)\underline{\lambda}(\Lambda_1(0))}{2} - 2\varepsilon R_2\|Q\| - \varepsilon k_2 R_2(\|Q\| + \|P\|)\right) \\
& \times \int_0^1 e^{-\mu x}|\eta|^2\, dx - \left(\frac{\mu\underline{\lambda}(P)\underline{\lambda}(\Lambda_2(0))}{2} - 2\varepsilon R_2\|P\| \right. \\
& \left. - \frac{\varepsilon R_2(\|Q\| + \|P\|)}{k_2}\right) \times \int_0^1 e^{-\mu x}|\delta|^2\, dx.
\end{aligned}
$$

By choosing $k_2 = 1$, let $\varepsilon_2^* = \min\left(\frac{\mu\underline{\lambda}(Q)\underline{\lambda}(\Lambda_1(0))}{2R_2(3\|Q\|+\|P\|)}, \frac{\mu\underline{\lambda}(P)\underline{\lambda}(\Lambda_2(0))}{2R_2(3\|P\|+\|Q\|)}\right)$, there exists a positive constant C_{T_2}, then for all $\varepsilon < \varepsilon_2^*$, it holds

$$
T_2 \leqslant -C_{T_2} \int_0^1 e^{-\mu x}(\eta^\top Q\eta + \delta^\top P\delta)\, dx.
$$

This concludes the proof of Lemma 3.2.3. □

Proof of Lemma 3.2.4 Under Assumptions 3.2.1-3.2.3, T_3 follows

$$
\begin{aligned}
T_3 \leqslant\ & 2\varepsilon e^{\mu} R_1\|Q\| \int_0^1 |\eta|\, |\bar{y}_x|\, dx + 2\varepsilon e^{\mu} R_2\|Q\| \int_0^1 |\eta|\, |\bar{y}|\, dx \\
& + 2\varepsilon e^{\mu} R_2\|Q\|\|(I_m - K_{22}(0))^{-1}K_{21}(0)\| \int_0^1 |\eta|\, |\bar{y}(1)|\, dx \\
& + 2\varepsilon e^{\mu} R_2\|P\|\|(I_m - K_{22}(0))^{-1}K_{21}(0)\| \int_0^1 |\delta|\, |\bar{y}(1)|\, dx \\
& + 2\varepsilon e^{\mu}\|P(I_m - K_{22}(0))^{-1}K_{21}(0)\Lambda_1(0)\| \int_0^1 |\delta|\, |\bar{y}_x(1)|\, dx \\
& + 2\varepsilon e^{\mu} R_2\|P\| \int_0^1 |\delta|\, |\bar{y}|\, dx.
\end{aligned}
$$

Using again Young's inequality, for all positive k_3, it holds

$$\begin{aligned}
T_3 \leqslant\ & \varepsilon k_3 e^{\mu}\|Q\|\Big(R_1+R_2+R_2\|(I_m-K_{22}(0))^{-1}K_{21}(0)\|\Big)\int_0^1 e^{-\mu x}|\eta|^2\,dx \\
& +\varepsilon k_3 e^{\mu}\|P\|\Big(R_2\|(I_m-K_{22}(0))^{-1}K_{21}(0)\| \\
& \qquad +R_2+\|(I_m-K_{22}(0))^{-1}K_{21}(0)\Lambda_1(0)\|\Big)\int_0^1 e^{-\mu x}|\delta|^2\,dx \\
& +\varepsilon e^{\mu}\|Q\|\left(\frac{R_1+R_2+R_2\|(I_m-K_{22}(0))^{-1}K_{21}(0)\|}{k_3}\right) \\
& \qquad \times\int_0^1 e^{-\mu x}(|\bar{y}_x|^2+|\bar{y}|^2+|\bar{y}(1)|^2)\,dx \\
& +\varepsilon e^{\mu}\|P\|\left(\frac{R_2\|(I_m-K_{22}(0))^{-1}K_{21}(0)\|+\|(I_m-K_{22}(0))^{-1}K_{21}(0)\Lambda_1(0)\|}{k_3}\right. \\
& \qquad \left.+\frac{R_2}{k_3}\right)\int_0^1 e^{-\mu x}(|\bar{y}_x(1)|^2+|\bar{y}|^2+|\bar{y}(1)|^2)\,dx.
\end{aligned}$$

Choosing $k_3=1$, using the estimates (3.2.7) and (3.2.8) and Lemma 3.2.1, we obtain

$$T_3 \leqslant \varepsilon C_{T_{31}}\int_0^1 e^{-\mu x}|\eta|^2\,dx+\varepsilon C_{T_{32}}\int_0^1 e^{-\mu x}|\delta|^2\,dx+\varepsilon C_{T_{33}}e^{-\mu\underline{\lambda}(\Lambda_1(0))t}\|\bar{y}_0\|^2_{H^2(0,1)},$$

where $C_{T_{31}}$, $C_{T_{32}}$ and $C_{T_{33}}$ are positive. This concludes the proof of Lemma 3.2.4. □

Chapter 4

Stability analysis of coupled singularly perturbed ODE-PDE systems

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In this chapter, we consider a coupled ordinary differential equation (ODE)-partial differential equation (PDE) system with two time scales. The scale of the time constant is modeled by a small perturbation parameter. In the first case, we study an ODE coupled with a fast linear hyperbolic system. The two subsystems, namely the reduced and the boundary-layer subsystems, are formally computed. Our first result is the stability analysis for such system. It is shown that the stability of the full system is guaranteed by that of both subsystems, which is consistent with the result for linear finite dimensional systems modeled by ODEs. Moreover, for particular initial condition $y_0 = G_r Z_0$, the estimate of the error between the slow dynamics of the full system and the reduced subsystem is the order of ε. In

the second case, we consider PDE coupled with fast ODE. Our second result shows that the full system can be unstable even though the two subsystems are stable which is different from the previous result (it is consistent with the result for linear singularly perturbed hyperbolic systems presented in Chapter 2). However, the stability of the full fast ODE-PDE system can be obtained via additional coupling condition. The Tikhonov approximation is established based on Lyapunov method.

This chapter is outlined as follows. In Section 4.1, it presents an ODE-fast PDE system. The reduced and the boundary-layer subsystems are computed in Section 4.1.1. Stability analysis between the full system and the two subsystems is given in Section 4.1.2. Section 4.1.3 shows the Tikhonov approximation for such system. Section 4.2 deals with a fast ODE coupled with a hyperbolic system. The descriptions of the full system and the subsystems are given in Section 4.2.1. In Section 4.2.2, a counterexample is used to indicate that the stability conditions given in Section 4.1.2 is not valid in the context of Section 4.2. The stability analysis between the full system and the subsystems is achieved by introducing an additional coupling condition. In Section 4.2.3, the error between the slow dynamics of the full system and the reduced subsystem is estimated as the order of ε. Numerical simulations given in Section 4.3 are used to illustrate the main results of Sections 4.1 and 4.2.

Some results of this chapter are accepted for publication at the CDC 2015.

4.1 An ODE coupled with a fast linear hyperbolic PDE system

In this section let us consider an ordinary differential equation coupled with a hyperbolic system described by a partial differential equation. A positive perturbation parameter ε is introduced into the dynamics of the PDE system.

4.1.1 Full system and subsystems descriptions

The full system under consideration in Section 4.1 is given by

$$\dot{Z}(t) = AZ(t) + By(1,t), \tag{4.1.1a}$$

$$\varepsilon y_t(x,t) + \Lambda y_x(x,t) = 0. \tag{4.1.1b}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $Z : [0,+\infty) \to \mathbb{R}^n$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}^m$, Λ is a diagonal positive matrix in $\mathbb{R}^{m\times m}$. In (4.1.1a) A and B are matrices of appropriate dimensions. The boundary condition for system (4.1.1) is given by

$$y(0,t) = G_1 y(1,t) + G_2 Z(t), \tag{4.1.2}$$

where G_1 and G_2 are constant matrices of appropriate dimensions. The initial conditions are considered as follows

$$Z(0) = Z_0, \tag{4.1.3a}$$
$$y(x,0) = y_0(x). \tag{4.1.3b}$$

Remark 4.1.1. Due to Theorem $A.6.$ in [Bastin and Coron, 2015], for every $Z_0 \in \mathbb{R}^n$, for every $y_0 \in L_2(0,1)$, the Cauchy problem (4.1.1)-(4.1.3) has one and only one solution $Z \in C^0([0,+\infty),\mathbb{R}^n)$, $y \in C^0([0,+\infty),(L^2(0,1),\mathbb{R}^m))$. ∘

Since system (4.1.1) contains two time scales, let us adopt the same calculations as for the two subsystems in Section 2.1, the two subsystems are formally computed as follows. By setting $\varepsilon = 0$ in system (4.1.1b), we obtain

$$y_x(x,t) = 0. \tag{4.1.4}$$

It implies $y(.,t) = y(1,t)$. Using this fact in the boundary condition (4.1.2) and assuming $(I_m - G_1)$ invertible yield

$$y(.,t) = G_r Z, \tag{4.1.5}$$

where $G_r = (I_m - G_1)^{-1}G_2$. Using the right hand side of (4.1.5) to replace $y(1,t)$ in (4.1.1a), the reduced subsystem is computed as follows,

$$\dot{\bar{Z}}(t) = (A + BG_r)\bar{Z}(t), \tag{4.1.6}$$

the initial condition is given by

$$\bar{Z}_0 = Z_0. \tag{4.1.7}$$

Performing the following change of variable

$$\bar{y} = y - G_r Z,$$

which shifts the equilibrium of y to the origin. The boundary-layer subsystem is computed in time scale $\tau = t/\varepsilon$

$$\bar{y}_\tau(x,\tau) + \Lambda \bar{y}_x(x,\tau) = 0, \tag{4.1.8}$$

with the boundary condition

$$\bar{y}(0,\tau) = G_1 \bar{y}(1,\tau), \tag{4.1.9}$$

whereas the initial condition is given by

$$\bar{y}_0 = y_0 - G_r Z_0. \tag{4.1.10}$$

4.1.2 Stability analysis of the full system and both subsystems

To state the stability of system (4.1.1)-(4.1.3), let us firstly present the following assumptions:

Assumption 4.1.1. *The matrix $A + BG_r$ is Hurwitz.*

Assumption 4.1.2. *The boundary condition G_1 satisfies $\rho_1(G_1) < 1$.*

Under Assumption 4.1.1 the reduced subsystem (4.1.6) is exponentially stable. Under Assumption 4.1.2, due to Theorem 1.1.1 in Chapter 1, the boundary-layer subsystem (4.1.8)-(4.1.9) is exponentially stable. The main stability result for system (4.1.1)-(4.1.3) is stated in the following theorem.

Theorem 4.1.1. *Under Assumptions 4.1.1-4.1.2, there exists $\varepsilon^* > 0$, such that for all $\varepsilon \in (0, \varepsilon^*)$, the full system (4.1.1)-(4.1.3) is exponentially stable in L^2-norm, that is for any $Z_0 \in \mathbb{R}^n$, $y_0 \in L^2(0,1)$, there exist positive values C and ν, it holds for all $t \geqslant 0$, $(|Z(t)| + \|y(.,t)\|_{L^2(0,1)}) \leqslant Ce^{-\nu t}(|Z_0| + \|y_0\|_{L^2(0,1)})$. Moreover, it has a strict Lyapunov function*

$$V(Z,y) \;=\; Z^\top PZ + \int_0^1 e^{-\mu x}(y - G_r Z)^\top Q(y - G_r Z)\, dx, \tag{4.1.11}$$

where $\mu > 0$, P is a symmetric positive matrix and Q is a diagonal positive matrix.

Proof. Let us consider the Lyapunov function V defined by (4.1.11) and write it as $V = V_1 + V_2$, with

$$V_1 = Z^\top PZ,$$

$$V_2 = \int_0^1 e^{-\mu x}(y - G_r Z)^\top Q(y - G_r Z)\, dx,$$

where P and Q will be specified later.
Computing the time derivative of V_1 along the solutions to system (4.1.1a)

yields

$$\begin{aligned}
\dot{V}_1 &= \dot{Z}^\top PZ + Z^\top P\dot{Z} \\
&= (AZ + By(1))^\top PZ + Z^\top P(AZ + By(1)) \\
&= \left(\left(A + BG_r\right)Z + B\left(y(1) - G_rZ\right)\right)^\top PZ \\
&\quad + Z^\top P\left(\left(A + BG_r\right)Z + B\left(y(1) - G_rZ\right)\right) \\
&= Z^\top\left((A + BG_r)^\top P + P(A + BG_r)\right)Z \\
&\quad + \left(y(1) - G_rZ\right)^\top B^\top PZ + Z^\top PB\left(y(1) - G_rZ\right).
\end{aligned}$$

Under Assumption 4.1.1, there exists symmetric positive matrix P such that

$$(A + BG_r)^\top P + P(A + BG_r) < -I_n. \tag{4.1.12}$$

Due to Cauchy Schwarz inequality, it follows

$$\dot{V}_1 \leqslant -|Z|^2 + 2\|B^\top P\|\,|y(1) - G_rZ|\,|Z|. \tag{4.1.13}$$

Similarly, computing the time derivative of V_2 along the solutions to system (4.1.1b) yields

$$\begin{aligned}
\dot{V}_2 &= \int_0^1 e^{-\mu x}\left((y_t - G_r\dot{Z})^\top Q(y - G_rZ) + (y - G_rZ)^\top Q(y_t - G_r\dot{Z})\right)dx \\
&= \int_0^1 e^{-\mu x}\left(\left(-\frac{\Lambda y_x}{\varepsilon} - G_r(AZ + By(1))\right)^\top Q(y - G_rZ)\right. \\
&\qquad \left. + (y - G_rZ)^\top Q\left(-\frac{\Lambda y_x}{\varepsilon} - G_r(AZ + By(1))\right)\right)dx \\
&= -\frac{1}{\varepsilon}\int_0^1 e^{-\mu x}\left(y_x^\top\Lambda^\top Q(y - G_rZ) + (y - G_rZ)^\top Q\Lambda y_x\right)dx \\
&\quad - \int_0^1 e^{-\mu x}(AZ + By(1))^\top G_r^\top Q(y - G_rZ)\,dx \\
&\quad - \int_0^1 e^{-\mu x}(y - G_rZ)^\top QG_r(AZ + By(1))\,dx.
\end{aligned}$$

Performing an integration by parts on the first integral yields

$$\begin{aligned}\dot{V}_2 &= -\frac{1}{\varepsilon}\Big[e^{-\mu x}(y-G_rZ)^\top Q\Lambda(y-G_rZ)\Big]_{x=0}^{x=1}\\ &\quad -\frac{\mu}{\varepsilon}\int_0^1 e^{-\mu x}(y-G_rZ)^\top Q\Lambda(y-G_rZ)\,dx\\ &\quad -\int_0^1 e^{-\mu x}(AZ+By(1))^\top G_r^\top Q(y-G_rZ)\,dx\\ &\quad -\int_0^1 e^{-\mu x}(y-G_rZ)^\top QG_r(AZ+By(1))\,dx. \qquad (4.1.14)\end{aligned}$$

The first term in (4.1.14) is noted by V_{21}. Using the boundary condition (4.1.2) and reorganizing, it follows

$$\begin{aligned}V_{21} &= -\frac{1}{\varepsilon}\Big[e^{-\mu}(y(1)-G_rZ)^\top Q\Lambda(y(1)-G_rZ)\\ &\quad -(y(0)-G_rZ)^\top Q\Lambda(y(0)-G_rZ)\Big]\\ &= -\frac{1}{\varepsilon}\Big[e^{-\mu}(y(1)-G_rZ)^\top Q\Lambda(y(1)-G_rZ)\\ &\quad -(G_1y(1)+G_2Z-G_rZ)^\top Q\Lambda(G_1y(1)+G_2Z-G_rZ)\Big].\end{aligned} \qquad (4.1.15)$$

Let us compute G_2-G_r

$$\begin{aligned}G_2-G_r &= G_2-(I_m-G_1)^{-1}G_2\\ &= \Big((I_m-G_1)(I_m-G_1)^{-1}-(I_m-G_1)^{-1}\Big)G_2\\ &= -G_1(I_m-G_1)^{-1}G_2=-G_1G_r. \qquad (4.1.16)\end{aligned}$$

Substituting the right hand side of the above equation into (4.1.15), we obtain

$$V_{21}=-\frac{1}{\varepsilon}\Big[(y(1)-G_rZ)^\top(e^{-\mu}Q\Lambda-G_1^\top Q\Lambda G_1)(y(1)-G_rZ)\Big].$$

Under Assumption 4.1.2, $\rho_1(G_1)<1$ implies there exists a diagonal positive matrix Δ such that $\|\Delta G_1\Delta^{-1}\|<1$. It is equivalent to $\Delta^2-G_1^\top\Delta^2G_1>0$. Let choose $Q=\Delta^2\Lambda^{-1}$, for $\mu>0$ small enough, it holds

$$e^{-\mu}Q\Lambda-G_1^\top Q\Lambda G_1>\underline{\lambda}(e^{-\mu}Q\Lambda-G_1^\top Q\Lambda G_1)>0. \qquad (4.1.17)$$

$$V_{21}\leqslant-\frac{\underline{\lambda}(e^{-\mu}Q\Lambda-G_1^\top Q\Lambda G_1)}{\varepsilon}|y(1)-G_rZ|^2. \qquad (4.1.18)$$

Let V_{22} denote the second term in (4.1.14), it follows

$$V_{22} \leqslant -\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|y-G_rZ\|^2_{L^2(0,1)}. \tag{4.1.19}$$

Let V_{23} denote the last two terms in (4.1.14), it follows

$$\begin{aligned} V_{23} &= -\int_0^1 e^{-\mu x}\Big((A+BG_r)Z+B(y(1)-G_rZ)\Big)^\top G_r^\top Q(y-G_rZ)\,dx \\ &\quad -\int_0^1 e^{-\mu x}(y-G_rZ)^\top QG_r\Big((A+BG_r)Z+B(y(1)-G_rZ)\Big)\,dx. \end{aligned}$$

Due to Cauchy Schwarz inequality, V_{23} follows

$$\begin{aligned} V_{23} &\leqslant 2\|(A+BG_r)^\top G_r^\top Q\|\,|Z|\int_0^1|y-G_rZ|\,dx \\ &\quad +2\|B^\top G_r^\top Q\|\,|y(1)-G_rZ|\int_0^1|y-G_rZ|\,dx \\ &\leqslant 2\sqrt{2}\|(A+BG_r)^\top G_r^\top Q\|\,|Z|\,\|y-G_rZ\|_{L^2(0,1)} \\ &\quad +2\sqrt{2}\|B^\top G_r^\top Q\|\,|y(1)-G_rZ|\,\|y-G_rZ\|_{L^2(0,1)}. \end{aligned} \tag{4.1.20}$$

Combining (4.1.18), (4.1.19) and (4.1.20) yields

$$\begin{aligned} \dot{V}_2 &\leqslant -\frac{\underline{\lambda}(e^{-\mu}Q\Lambda-G_1^\top Q\Lambda G_1)}{\varepsilon}|y(1)-G_rZ|^2 \\ &\quad -\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|y-G_rZ\|^2_{L^2(0,1)} \\ &\quad +2\sqrt{2}\|(A+BG_r)^\top G_r^\top Q\|\,|Z|\,\|y-G_rZ\|_{L^2(0,1)} \\ &\quad +2\sqrt{2}\|B^\top G_r^\top Q\|\,|y(1)-G_rZ|\,\|y-G_rZ\|_{L^2(0,1)}. \end{aligned} \tag{4.1.21}$$

Combine (4.1.13) with (4.1.21), we obtain

$$\dot{V} \leqslant -\begin{pmatrix} |y(1)-G_rZ| \\ |Z| \\ \|y-G_rZ\|_{L^2(0,1)} \end{pmatrix}^\top M \begin{pmatrix} |y(1)-G_rZ| \\ |Z| \\ \|y-G_rZ\|_{L^2(0,1)} \end{pmatrix},$$

where

$$M = \begin{pmatrix} M_1 & M_2 \\ \star & M_4 \end{pmatrix},$$

with

$$\begin{aligned} M_1 &= \begin{pmatrix} M_{11} & M_{12} \\ \star & M_{14} \end{pmatrix} = \begin{pmatrix} \frac{\underline{\lambda}(e^{-\mu}Q\Lambda-G_1^\top Q\Lambda G_1)}{\varepsilon} & -\|B^\top P\| \\ \star & 1 \end{pmatrix}, \\ M_2 &= \begin{pmatrix} -\sqrt{2}\|B^\top G_r^\top Q\| \\ -\sqrt{2}\|(A+BG_r)^\top G_r^\top Q\| \end{pmatrix}, \\ M_4 &= \left(\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\right). \end{aligned}$$

Let us first study matrix M_1. There exists ε_1^* such that for $\varepsilon \in (0, \varepsilon_1^*)$, $M_{11} - M_{12}M_{14}^{-1}M_{12}^\top > 0$. Due to Schur complement, we get $M_1 > 0$. The inverse of M_1 is computed as

$$M_1^{-1} = \frac{1}{\underline{\lambda}(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1) - \varepsilon\|B^\top P\|^2}\begin{pmatrix} \varepsilon & \varepsilon\|B^\top P\| \\ \star & \underline{\lambda}(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1) \end{pmatrix}.$$

There exists ε_2^*, such that for all $0 < \varepsilon < \min(\varepsilon_1^*, \varepsilon_2^*)$, $M_4 - M_2^\top M_1^{-1}M_2 > 0$. We get $M > 0$ according to Schur complement. Thus there exists $\alpha > 0$ such that

$$\dot{V} \leqslant -\alpha V.$$

This concludes the proof of Theorem 4.1.1. □

4.1.3 Tikhonov approximation for system (4.1.1)-(4.1.3)

To study the Tikhonov approximation for system (4.1.1)-(4.1.3), let us first perform the following change of variables

$$\eta = Z - \bar{Z},$$
$$\delta = y - G_r\bar{Z},$$

where η represents the error of the slow dynamics between the full system (4.1.1) and the reduced subsystem (4.1.6) while δ is the error of the fast dynamics of the full system and its equilibrium point. The error system is given as follows

$$\dot{\eta} = A\eta + B\delta(1), \tag{4.1.22a}$$
$$\varepsilon\delta_t + \Lambda\delta_x = -\varepsilon G_r(A + BG_r)\bar{Z}. \tag{4.1.22b}$$

Due to (4.1.2), the boundary condition is computed as follows

$$\begin{aligned} \delta(0,t) &= y(0,t) - G_r\bar{Z} = G_1y(1,t) + G_2Z - G_r\bar{Z} \\ &= G_1\Big(y(1,t) - G_r\bar{Z}\Big) + G_2Z - (I_m - G_1)G_r\bar{Z} \\ &= G_1\Big(y(1,t) - G_r\bar{Z}\Big) + G_2Z - (I_m - G_1)(I_m - G_1)^{-1}G_2\bar{Z}, \end{aligned}$$

the boundary condition is written as

$$\delta(0,t) = G_1\delta(1,t) + G_2\eta, \tag{4.1.23}$$

the initial condition is

$$\eta_0 = Z_0 - \bar{Z}_0 = 0, \tag{4.1.24a}$$
$$\delta_0 = y_0 - G_rZ_0. \tag{4.1.24b}$$

The Tikhonov approximation for system (4.1.1)-(4.1.3) is stated as follows

Theorem 4.1.2. *Consider the coupled ODE-PDE system (4.1.1)-(4.1.3), under Assumptions 4.1.1-4.1.2, there exist positive values C_1, θ, ε^* such that for all $0 < \varepsilon < \varepsilon^*$, for any initial conditions $Z_0 \in \mathbb{R}^n$ and $y_0 = G_r Z_0$, it holds for all $t \geqslant 0$*

$$|Z(t) - \bar{Z}(t)|^2 \quad \leqslant \quad \varepsilon C_1 e^{-\theta t} |\bar{Z}_0|^2. \tag{4.1.25}$$

Proof. Let us consider the following candidate Lyapunov function for system (4.1.22)-(4.1.24),

$$V(\eta, \delta) = \eta^\top P \eta + \int_0^1 e^{-\mu x} (\delta - G_r \eta)^\top Q (\delta - G_r \eta)\, dx, \tag{4.1.26}$$

with $\mu > 0$, matrices P and Q are selected in the proof of Theorem 4.1.1, such that (4.1.12) and (4.1.17) hold. The main proof of this theorem is similar to the proof of Theorem 4.1.1. We rewrite

$$V(\eta, \delta) = V_1 + V_2,$$

where $V_1 = \eta^\top P \eta$ and $V_2 = \int_0^1 e^{-\mu x} (\delta - G_r \eta)^\top Q (\delta - G_r \eta)\, dx$.
The time derivative of V_1 along the solutions to (4.1.22a) is computed as follows

$$\begin{aligned} \dot{V}_1 \quad &= \quad \eta^\top \Big[(A + BG_r)^\top P + P(A + BG_r) \Big] \eta \\ &\quad + (\delta(1) - G_r \eta)^\top B^\top P \eta + \eta^\top P B (\delta(1) - G_r \eta). \end{aligned}$$

Due to Cauchy Schwarz inequality and 4.1.12, it follows

$$\dot{V}_1 \quad \leqslant \quad -|\eta|^2 + 2\|B^\top P\|\, |\eta|\, |\delta(1) - G_r \eta|. \tag{4.1.27}$$

The time derivative of V_2 along the solutions to (4.1.22b) is computed as follows

$$\begin{aligned} \dot{V}_2 \quad &= \quad \int_0^1 e^{-\mu x} \Bigg\{ \Big[-\frac{1}{\varepsilon} \Lambda \delta_x - G_r (A + BG_r) \bar{Z} - G_r (A\eta + B\delta(1)) \Big]^\top Q (\delta - G_r \eta) \\ &\qquad + (\delta - G_r \eta)^\top Q \Big[-\frac{1}{\varepsilon} \Lambda \delta_x - G_r (A + BG_r) \bar{Z} - G_r (A\eta + B\delta(1)) \Big] \Bigg\}\, dx \\ &= \quad -\frac{1}{\varepsilon} \int_0^1 e^{-\mu x} \Big[(\delta - G_r \eta)_x^\top \Lambda^\top Q (\delta - G_r \eta) + (\delta - G_r \eta)^\top Q \Lambda (\delta - G_r \eta)_x \Big]\, dx \\ &\quad - \int_0^1 e^{-\mu x} \Big[\bar{Z}^\top (G_r (A + BG_r))^\top Q (\delta - G_r \eta) + (\delta - G_r \eta)^\top Q (G_r (A + BG_r)) \bar{Z} \Big]\, dx \\ &\quad - \int_0^1 e^{-\mu x} \Big[(A\eta + B\delta(1))^\top G_r^\top Q (\delta - G_r \eta) + (\delta - G_r \eta)^\top Q G_r (A\eta + B\delta(1)) \Big]\, dx. \end{aligned} \tag{4.1.28}$$

Performing an integration by parts on the first integral in (4.1.28), it is noted by V_{20},

$$\begin{aligned} V_{20} &= -\frac{1}{\varepsilon}\Big[e^{-\mu x}(\delta - G_r\eta)^\top Q\Lambda(\delta - G_r\eta)\Big]_{x=0}^{x=1} \\ &\quad -\frac{\mu}{\varepsilon}\int_0^1 e^{-\mu x}(\delta - G_r\eta)^\top Q\Lambda(\delta - G_r\eta)\, dx. \end{aligned}$$

Under the boundary condition (4.1.23), using (4.1.16) and (4.1.17), V_{20} follows

$$\begin{aligned} V_{20} &\leqslant -\frac{\underline{\lambda}(e^{-\mu}Q\Lambda - G_1^\top P\Lambda G_1)}{\varepsilon}|\delta(1) - G_r\eta|^2 \\ &\quad -\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon}\|\delta - G_r\eta\|^2_{L^2(0,1)}. \end{aligned} \tag{4.1.29}$$

Using Cauchy Schwarz inequality, the second integral in (4.1.28), noted by V_{21}, follows

$$V_{21} \leqslant 2\sqrt{2}\|(A + BG_r)^\top G_r^\top Q\|\ |\bar{Z}|\ \|\delta - G_r\eta\|_{L^2(0,1)}.$$

Due to Young's inequality, for any $\kappa > 0$, it holds

$$\begin{aligned} V_{21} &\leqslant \kappa\sqrt{2}\|(A + BG_r)^\top G_r^\top Q\|\ |\bar{Z}|^2 \\ &\quad +\frac{\sqrt{2}\|(A + BG_r)^\top G_r^\top Q\|}{\kappa}\|\delta - G_r\eta\|^2_{L^2(0,1)}. \end{aligned} \tag{4.1.30}$$

Similarly, the third integral in (4.1.28), denoted by V_{22}, follows

$$\begin{aligned} V_{22} &\leqslant 2\sqrt{2}\|(A + BG_r)^\top G_r^\top Q\|\ |\eta|\ \|\delta - G_r\eta\|_{L^2(0,1)} \\ &\quad +2\sqrt{2}\|B^\top G_r^\top Q\|\ |\delta(1) - G_r\eta|\ \|\delta - G_r\eta\|_{L^2(0,1)}. \end{aligned} \tag{4.1.31}$$

Combining (4.1.29)-(4.1.31) and (4.1.27), $\dot{V}$ follows

$$\begin{aligned} \dot{V} &\leqslant -\begin{pmatrix} |\delta(1) - G_r\eta| \\ |\eta| \\ \|\delta - G_r\eta\|_{L^2(0,1)} \end{pmatrix}^\top \tilde{M} \begin{pmatrix} |\delta(1) - G_r\eta| \\ |\eta| \\ \|\delta - G_r\eta\|_{L^2(0,1)} \end{pmatrix} \\ &\quad +\kappa\|(A + BG_r)^\top G_r^\top Q\|\ |\bar{Z}|^2, \end{aligned}$$

where

$$\tilde{M} = \begin{pmatrix} M_1 & M_2 \\ \star & \tilde{M}_4 \end{pmatrix},$$

with

$$\begin{aligned} M_1 &= \begin{pmatrix} M_{11} & M_{12} \\ \star & M_{14} \end{pmatrix} = \begin{pmatrix} \frac{\underline{\lambda}(e^{-\mu}Q\Lambda - G_1^\top Q\Lambda G_1)}{\varepsilon} & -\|B^\top P\| \\ \star & 1 \end{pmatrix}, \\ M_2 &= \begin{pmatrix} -\sqrt{2}\|B^\top G_r^\top Q\| \\ -\sqrt{2}\|(A + BG_r)^\top G_r^\top Q\| \end{pmatrix}, \\ \tilde{M}_4 &= \left(\frac{\mu e^{-\mu}\underline{\lambda}(Q\Lambda)}{\varepsilon} - \frac{\sqrt{2}\|(A+BG_r)^\top G_r^\top Q\|}{\kappa}\right). \end{aligned}$$

We have proved that $M_1 > 0$ for $\varepsilon \in (0, \varepsilon_1^*)$ in the proof of Theorem 4.1.1. By choosing $\kappa = n\varepsilon > 0$, there exist n sufficiently large and ε_2^*, such that for all $0 < \varepsilon < \min(\varepsilon_1^*, \varepsilon_2^*)$, we get $\tilde{M} > 0$. There exists positive value θ such that $\dot{V}$ follows

$$\dot{V} \leqslant -\theta V + n\varepsilon \|(A + BG_r)^\top G_r^\top Q\| \, |\bar{Z}|^2.$$

Under Assumption 4.1.1, the reduced subsystem (4.1.6) is exponentially stable, and there exists positive constant r, the following holds for all $t \geqslant 0$

$$|\bar{Z}(t)|^2 \leqslant e^{-rt} |\bar{Z}_0|^2.$$

Thus we get

$$\dot{V} \leqslant -\theta V + n\varepsilon e^{-rt} \|(A + BG_r)^\top G_r^\top Q\| \, |\bar{Z}_0|^2.$$

It holds

$$\begin{aligned} V &\leqslant e^{-\theta t} V(\eta_0, \delta_0) \\ &\quad + n\varepsilon \|(A + BG_r)^\top G_r^\top Q\| \, |\bar{Z}_0|^2 \int_0^t e^{-\theta(t-s)} e^{-rs} ds. \end{aligned}$$

Using the fact

$$\begin{aligned} & \underline{\lambda}(Q)|\eta|^2 + e^{-\mu} \underline{\lambda}(P) \|\delta - G_r \eta\|_{L^2(0,1)}^2 \\ \leqslant \; & V_\varepsilon(\eta, \delta) \\ \leqslant \; & \|Q\| |\eta|^2 + \|P\| \|\delta - G_r \eta\|_{L^2(0,1)}^2. \end{aligned}$$

Since the initial conditions are $\eta_0 = \delta_0 = 0$, we obtain

$$|\eta|^2 \leqslant \varepsilon C_1 e^{-\theta t} |\bar{Z}_0|^2.$$

for suitable positive value C_1. This concludes the proof of Theorem 4.1.2. $\square$

4.2 A fast ODE coupled with a linear hyperbolic PDE system

Compared with the previous Section 4.1, in this section we consider a coupled ODE-PDE system, where the perturbation parameter is introduced into the dynamics of the ODE.

4.2.1 Full system and subsystems descriptions

The full system under consideration in Section 4.2 is given by

$$\varepsilon \dot{Y}(t) = AY(t) + Bz(1), \tag{4.2.1a}$$

$$z_t(x,t) + \Lambda z_x(x,t) = 0, \tag{4.2.1b}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $Y : [0,+\infty) \to \mathbb{R}^n$, $z : [0,1] \times [0,+\infty) \to \mathbb{R}^m$, the matrices A, B and Λ are defined as the same as for Section 4.1. The boundary condition is given by

$$z(0,t) = K_1 z(1,t) + K_2 Y(t), \tag{4.2.2}$$

where K_1 and K_2 are constant matrices of appropriate dimension. The initial conditions are

$$Y(0) = Y_0, \tag{4.2.3a}$$

$$z(x,0) = z_0(x). \tag{4.2.3b}$$

By setting $\varepsilon = 0$ in (4.2.1a), we compute

$$Y = -A^{-1}Bz(1). \tag{4.2.4}$$

Substituting (4.2.4) into (4.2.2), the reduced subsystem is computed as

$$\bar{z}_t(x,t) + \Lambda \bar{z}_x(x,t) = 0, \tag{4.2.5}$$

with the boundary condition

$$\bar{z}(0,t) = K_r \bar{z}(1,t), \tag{4.2.6}$$

where $K_r = K_1 - K_2 A^{-1} B$, whereas the initial condition is

$$\bar{z}(x,0) = \bar{z}_0(x) = z_0(x). \tag{4.2.7}$$

To ensure the solution of the reduced subsystem in H^2-norm, the compatibility conditions are given by

$$\bar{z}_0(0) = K_r \bar{z}_0(1), \tag{4.2.8a}$$

$$\bar{z}_{0_x}(0) = \Lambda^{-1} K_r \Lambda \bar{z}_{0_x}(1). \tag{4.2.8b}$$

Performing a change of variable $\bar{Y} = Y + A^{-1}Bz(1)$, the boundary-layer subsystem is computed as

$$\frac{d\bar{Y}(\tau)}{d\tau} = A\bar{Y}(\tau), \tag{4.2.9}$$

with the time scale $\tau = t/\varepsilon$. The initial condition is given by

$$\bar{Y}(0) = \bar{Y}_0 = Y_0 + A^{-1}Bz_0(1). \tag{4.2.10}$$

To compute the error system in variable (η, δ), let us perform the following change of variables

$$\eta = z - \bar{z}, \tag{4.2.11a}$$

$$\delta = Y + A^{-1}B\bar{z}(1). \tag{4.2.11b}$$

Due to (4.2.1b) and (4.2.5), (4.2.1a) and (4.2.9), we obtain

$$\eta_t + \Lambda\eta_x = 0, \tag{4.2.12a}$$

$$\varepsilon\dot{\delta}(t) = A\delta + B\eta(1) - \varepsilon A^{-1}B\Lambda\bar{z}_x(1,t). \tag{4.2.12b}$$

Due to (4.2.2) and (4.2.6), we have

$$\eta(0,t) = K_1\eta(1,t) + K_2\delta(t). \tag{4.2.13}$$

4.2.2 Stability analysis of the full system and both subsystems

In this section, we first state the following two assumptions.

Assumption 4.2.1. *The boundary condition K_r satisfies $\rho_1(K_r) < 1$.*

Assumption 4.2.2. *The matrix A is Hurwitz.*

Under Assumption 4.2.1, due to Theorem 1.1.1 in Chapter 1, the reduced subsystem (4.2.5)-(4.2.6) is exponentially stable. Under Assumption 4.2.2 the boundary-layer subsystem (4.2.9) is exponentially stable. Surprisingly, Theorem 4.1.1 is not valid in the context of this section. To be more precise, it holds Proposition 4.2.1.

Proposition 4.2.1. *Assumptions 4.2.1 and 4.2.2 do not imply the exponential stability of system (4.2.1)-(4.2.3) for ε sufficiently small.*

To prove Proposition 4.2.1, we now provide an example to illustrate that the stability of the two subsystems does not guarantee the stability of the full system (4.2.1)-(4.2.2), which is different from the result in Section 4.1.2. We consider a 2×2 system (4.2.1) with $A = -0.1$, $B = -1$ and $\Lambda = 1$:

$$\varepsilon\dot{Y}(t) = -0.1Y(t) - z(1), \tag{4.2.14a}$$

$$z_t(x,t) + z_x(x,t) = 0. \tag{4.2.14b}$$

The boundary condition is given by $K_1 = 2$ and $K_2 = 0.2$,

$$z(0,t) = 2z(1,t) + 0.2Y(t). \tag{4.2.15}$$

The reduced subsystem is computed as

$$\bar{z}_t(x,t) + \bar{z}_x(x,t) = 0. \tag{4.2.16}$$

Let compute $K_r = 0$ for the $\bar{z}$ variable, and thus the boundary condition is given by

$$\bar{z}(0,t) = 0. \tag{4.2.17}$$

The explicit solution of (4.2.16)-(4.2.17) is calculated as

$$\bar{z}(x,t) = \begin{cases} \bar{z}_0(x-t), & t < x \\ 0, & t \geqslant x \end{cases}, \tag{4.2.18}$$

where $\bar{z}_0$ is the initial condition for $\bar{z}$ that will be specified later. The solution at $x = 1$ is

$$\bar{z}(1,t) = \begin{cases} \bar{z}_0(1-t), & t < 1 \\ 0, & t \geqslant 1 \end{cases}.$$

Differentiating $\bar{z}(1,t)$ with respect to time and using (4.2.16) yields

$$\bar{z}_x(1,t) = \begin{cases} \bar{z}_0'(1-t), & t < 1 \\ 0, & t \geqslant 1 \end{cases}, \tag{4.2.19}$$

we choose $\bar{z}_0 = 0$ which satisfies the compatibility conditions (4.2.8).
The boundary-layer subsystem is computed as

$$\frac{d\bar{Y}(\tau)}{d\tau} = -0.1\bar{Y}(\tau). \tag{4.2.20}$$

The solution to system (4.2.20) is

$$\bar{Y}(t/\varepsilon) = e^{-0.1t/\varepsilon}\bar{Y}_0. \tag{4.2.21}$$

Let us perform a change of variables as follows

$$\tilde{\eta} = z - \bar{z}, \tag{4.2.22a}$$

$$\tilde{\delta} = Y + A^{-1}B\bar{z}(1) - \bar{Y}(t/\varepsilon). \tag{4.2.22b}$$

The dynamics of $\tilde{\delta}$ is written as

$$\dot{\tilde{\delta}}(t) = -\frac{0.1}{\varepsilon}\tilde{\delta}(t) - 10\bar{z}_x(1,t). \tag{4.2.23}$$

Using (4.2.19), we compute $\tilde{\delta}(t)$ as follows, for all $t \in (0,1)$,

$$\tilde{\delta}(t) = e^{-0.1t/\varepsilon}\tilde{\delta}_0 - 10\int_0^t e^{-\frac{0.1(t-s)}{\varepsilon}}\bar{z}_0'(1-s)ds. \tag{4.2.24}$$

The dynamics of $\tilde{\eta}$ is written as

$$\tilde{\eta}_t + \tilde{\eta}_x = 0, \tag{4.2.25}$$

$$\tilde{\eta}(0,t) = 2\tilde{\eta}(1,t) + 0.2\tilde{\delta}(t) + 0.2\bar{Y}(t/\varepsilon). \tag{4.2.26}$$

Let denote $f(t) = 0.2\tilde{\delta}(t) + 0.2\bar{Y}(t/\varepsilon)$. Due to (4.2.25)-(4.2.26), we can compute, for all $t \geqslant 0$,

$$\tilde{\eta}(1,t) \quad = \quad 2\tilde{\eta}(1,t-1) + f(t-1).$$

Using a recursion, it can be proved that for all $t \in \mathbb{N}$,

$$\tilde{\eta}(1,t) \quad = \quad 2^{t-1}\sum_{s=0}^{t-1} f(s)2^{-s} + 2^t\tilde{\eta}(1,0).$$

Due to (4.2.7) and (4.2.22a), we get $\tilde{\eta}(1,0) = 0$, using (4.2.21) and (4.2.24) and since $\bar{z}_0 = 0$, it follows for all $t \in \mathbb{N}$,

$$\begin{aligned}\tilde{\eta}(1,t) &= 2^{t-1} \times 0.2\left(\sum_{s=0}^{t-1} 2^{-s}\tilde{\delta}(s) + \sum_{s=0}^{t-1} 2^{-s}\bar{Y}(s)\right)\\ &= 2^{t-1} \times 0.2(\tilde{\delta}_0 + \bar{Y}_0)\sum_{s=0}^{t-1} 2^{-s}e^{-0.1s/\varepsilon}.\end{aligned}$$

Due to (4.2.10) and (4.2.22b), if the initial conditions $\tilde{\delta}_0 + \bar{Y}_0 = \bar{Y}_0 \neq 0$, then $\tilde{\eta}(1,t)$ tends to infinity. Due to (4.2.18), we have $\bar{z}(x,t) = 0$. Then $z(1,t) = \tilde{\eta}(1,t)$ is divergent. It can be shown that the full system (4.2.14)-(4.2.15) is unstable in L^2-norm. This example proves that the stability of both subsystems does not ensure the stability of the full system as considered in Proposition 4.2.1.
To establish the stability of the full system (4.2.1)-(4.2.3), let us introduce the following assumption.

Assumption 4.2.3. *There exist a matrix P diagonal positive and a matrix Q symmetric positive such that the following holds*

$$P\Lambda - K_r^\top P\Lambda K_r > 0,$$

$$\begin{pmatrix} e^{-\mu}P\Lambda - K_1^\top P\Lambda K_1 & -(K_1^\top P\Lambda K_2 + B^\top Q) \\ \star & -(A^\top Q + QA) - K_2^\top P\Lambda K_2 \end{pmatrix} > 0.$$

Remark 4.2.1. The first inequality in the above assumption correspond to Assumption 4.2.1, which implies the stability of the reduced subsystem. The second inequality is concerned with the cross terms which is important for the stability of the full system. Note that it implies that Assumption 4.2.2 holds. ◦

Now we are ready to state the stability of system (4.2.1)-(4.2.3).

Theorem 4.2.1. *Under Assumption 4.2.3, the system (4.2.1)-(4.2.3) is exponentially stable for ε sufficiently small.*

In the following, we show the proof of Theorem 4.2.1.

Proof. Let us consider the following candidate Lyapunov function for the error system (4.2.12)-(4.2.13)

$$V_\varepsilon(\eta,\delta) = \int_0^1 e^{-\mu x}\eta^\top P\eta\, dx + \varepsilon\delta^\top Q\delta.$$

The time derivative of $V_\varepsilon(\eta,\delta)$ is computed along the solutions to (4.2.12)-(4.2.13)

$$\dot{V}_\varepsilon(\eta,\delta) \;=\; \int_0^1 e^{-\mu x}\Big(\eta_t^\top P\eta + \eta^\top P\eta_t\Big)\, dx + \varepsilon\Big(\dot{\delta}^\top Q\delta + \delta^\top Q\dot{\delta}\Big).$$

After performing an integration by parts and reorganizing, it follows

$$\begin{aligned}\dot{V}_\varepsilon(\eta,\delta) \;=\; & -\begin{pmatrix}\eta(1)\\ \delta\end{pmatrix}^\top L\begin{pmatrix}\eta(1)\\ \delta\end{pmatrix} - \mu\int_0^1 e^{-\mu x}\eta^\top P\Lambda\eta\, dx\\ & +\varepsilon\Big(\bar{z}_x^\top(1)(A^{-1}B\Lambda)^\top Q\delta + \delta^\top Q(A^{-1}B\Lambda)\bar{z}_x(1)\Big),\end{aligned}$$

where $L = \begin{pmatrix} e^{-\mu}P\Lambda - K_1^\top P\Lambda K_1 & -(K_1^\top P\Lambda K_2 + B^\top Q)\\ \star & -(A^\top Q + QA) - K_2^\top P\Lambda K_2\end{pmatrix}$. Under Assumptions 4.2.1-4.2.3, using Cauchy Schwarz inequality and Young's inequality, for any $\alpha > 0$ it follows

$$\begin{aligned}\dot{V}_\varepsilon(\eta,\delta) \;\leqslant\; & -\underline{\lambda}(L)\,|\delta|^2 - \mu e^{-\mu}\underline{\lambda}(P\Lambda)\,|\eta|^2 + \frac{\varepsilon}{\alpha}\|(A^{-1}B\Lambda)^\top Q\|\,|\delta|^2\\ & +\varepsilon\alpha\|(A^{-1}B\Lambda)^\top Q\|\,|\bar{z}_x(1)|^2.\end{aligned}$$

By choosing $\alpha = 1$, there exist $\varepsilon^* > 0$ and $\gamma > 0$, such that for all $\varepsilon \in (0,\varepsilon^*)$ it holds

$$\dot{V}_\varepsilon(\eta,\delta) \;\leqslant\; -\gamma V_\varepsilon(\eta,\delta) + \varepsilon\|(A^{-1}B\Lambda)^\top Q\|\,|\bar{z}_x(1)|^2.$$

Under Assumption 4.2.1, due to Theorem 1.1.1 in Chapter 1, the reduced subsystem (4.2.5)-(4.2.6) is exponentially stable in H^2-norm. Thus we get

$$\dot{V}_\varepsilon(\eta,\delta) \;\leqslant\; -\gamma V_\varepsilon(\eta,\delta) + C_r\varepsilon e^{-ct}\|(A^{-1}B\Lambda)^\top Q\|\,\|\bar{z}_0\|^2_{H^2(0,1)}, \tag{4.2.27}$$

where C_r and c are positive values. Since the error system (η,δ) is stable and the reduced subsystem $\bar{z}$ is stable in H^2-norm, due to (4.2.11), the full system (4.2.1)-(4.2.3) is exponentially stable. This concludes the proof of Theorem 4.2.1. □

4.2.3 Tikhonov approximation for system (4.2.1)-(4.2.3)

Let us state the Tikhonov theorem for system (4.2.1)-(4.2.3).

Theorem 4.2.2. *Consider the coupled ODE-PDE system (4.2.1)-(4.2.3), under Assumption 4.2.3, there exist positive values C_1, γ, ε^* such that for all $0 < \varepsilon < \varepsilon^*$, then for any initial condition $z_0 \in H^2(0,1)$ satisfying the compatibility conditions (4.2.8), with $\bar{z}_0 = z_0$, and for $Y_0 \in \mathbb{R}^n$, it holds for all $t \geqslant 0$*

$$\|z(.,t) - \bar{z}(.,t)\|^2_{L^2(0,1)} \leqslant \varepsilon C_1 e^{-\gamma t}\Big(\|\bar{z}_0\|^2_{H^2(0,1)} + |Y_0 + A^{-1}B\bar{z}_0(1)|^2\Big). \tag{4.2.28}$$

Proof. The proof of Theorem 4.2.2 is based on the proof of Theorem 4.2.1. Since the following is true

$$e^{-\mu}\underline{\lambda}(P)\|\eta\|^2_{L^2(0,1)} + \varepsilon\underline{\lambda}(Q)|\delta|^2 \leqslant V_\varepsilon(\eta,\delta) \leqslant \|P\|\|\eta\|^2_{L^2(0,1)} + \varepsilon\|Q\||\delta|^2.$$

By (4.2.27), we obtain for all $t \geqslant 0$,

$$\begin{aligned}\|\eta\|^2_{L^2(0,1)} &\leqslant \frac{e^{\mu}}{\underline{\lambda}(P)}V_\varepsilon(\eta,\delta)\\ &\leqslant C_1 e^{-\gamma t}\Big(\|\eta_0\|^2_{L^2(0,1)} + \varepsilon|\delta_0|^2 + \varepsilon\|\bar{z}_0\|^2_{H^2(0,1)}\Big).\end{aligned} \tag{4.2.29}$$

Since $\eta_0 = 0$, (4.2.28) holds. This concludes the proof of Theorem 4.2.2. □

4.3 Numerical simulations

In this section, we first show numerical simulations on academic examples, which are used to illustrate the stability results in Sections 4.1 and 4.2 respectively. Finally the simulations on the counterexample indicate that the full system in Section 4.2 is not stable even though the two subsystems are stable.

4.3.1 Numerical simulations illustrating the main result of Section 4.1

Let us consider a 2×2 full system (4.1.1) with $A = 1$, $B = -1$, $\Lambda = 1$. The boundary condition (4.1.2) is given by $G_1 = 0.5$, $G_2 = 1$. The initial conditions (4.1.3) are selected as $Z(0) = 1$ and $y(x,0) = \cos(4\pi x) - 1$. It is computed $A + BG_r = -1$ for the reduced subsystem (4.1.6). The initial condition (4.1.7) is chosen as the same as the full system $\bar{Z}(0) = Z(0) = 1$. The boundary condition for the boundary-layer subsystem is $G_1 = 0.5$. The initial condition is chosen as $\bar{y}_0 = \cos(4\pi x) - 3$. The perturbation parameter

ε is selected as $\varepsilon = 0.01$. Since $A + BG_r = -1$, thus Assumption 4.1.1 holds. By choosing $\Delta = 1$, it holds $\|\Delta G_1 \Delta^{-1}\| < 1$, which implies $\rho_1(G_1) < 1$, thus Assumption 4.1.2 holds. Therefore Theorem 4.1.1 applies. Figures 4.3.1 and 4.3.2 are the solutions of the boundary-layer and the reduced subsystems respectively. It is observed that the solutions of both subsystems converge to the origin as time increases. In Figures 4.3.3 and 4.3.4, the solutions of the slow and the fast dynamics of the full system tend to zero when time increases. It is observed that there is a difference at the beginning in Figures 4.3.2 and 4.3.3 which is due to the fast dynamics. After the fast dynamics stabilizing, the evolution of the solution of the reduced subsystem and the slow dynamics of the full system is almost the same.

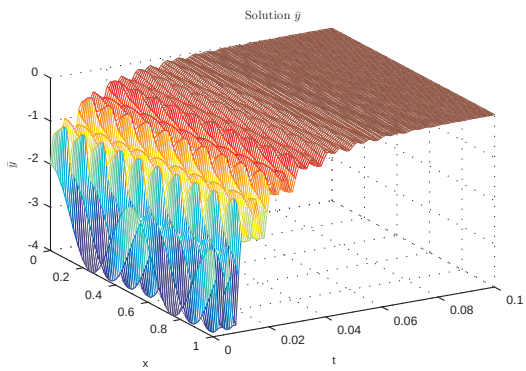

Figure 4.3.1: Solution of the boundary-layer subsystem

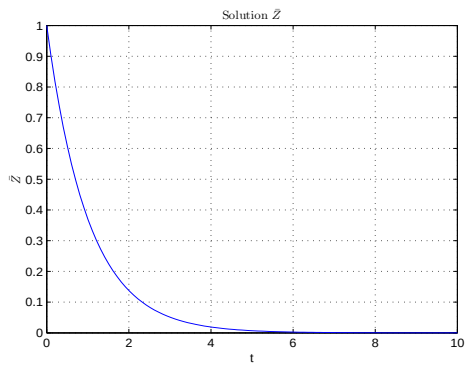

Figure 4.3.2: Solution of the reduced subsystem

Let us choose $\varepsilon = \{0.03, 0.02, 0.01\}$, the initial condition y_0 is selected as the equilibrium point $y_0 = G_r Z_0$. Table 4.3.1 shows that the errors between the full system and the two subsystems decrease as ε decreases.

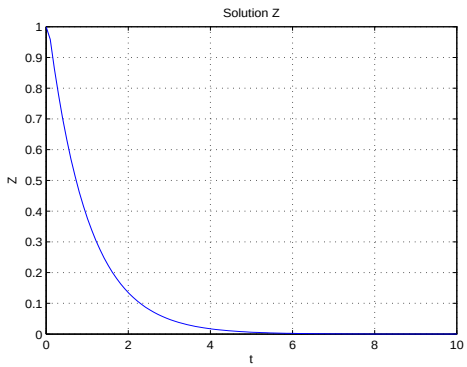

Figure 4.3.3: Solution of the slow dynamics of the full system

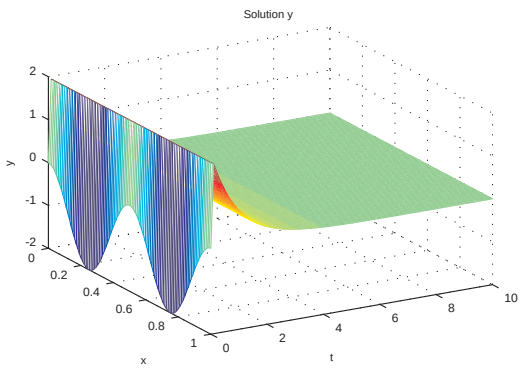

Figure 4.3.4: Solution of the fast dynamics of the full system

ε	0.03	0.02	0.01
$\lvert Z(t) - \bar{Z}(t)\rvert^2$	2×10^{-1}	9×10^{-2}	2×10^{-2}
$\lVert y(.,t) - G_r Z(t)\rVert^2_{L^2}$	5×10^{-3}	2×10^{-3}	4×10^{-4}

Table 4.3.1: Estimates of the errors between the full system and the subsystems for different values of ε with the initial condition $y_0 = G_r Z_0$

4.3.2 Numerical simulations illustrating the main result of Section 4.2

We consider a 2×2 full system (4.2.1) with $A = -1$, $B = 0.1$ and $\Lambda = 1$. The boundary condition (4.2.2) is given by $K_1 = K_2 = 0.5$ The initial conditions (4.2.3) are selected as $Y_0 = 2$ and $z_0(x) = \cos(4\pi x) - 1$ such that the compatibility conditions (4.2.8) are satisfied. The boundary condition matrix for the reduced subsystem is computed as $K_r = 0.5$. The initial

condition (4.2.7) is chosen as $\bar{z}_0 = z_0(x) = \cos(4\pi x) - 1$. The boundary-layer subsystem is $\frac{d\bar{Y}(\tau)}{d\tau} = -\bar{Y}(\tau)$. The perturbation parameter ε is selected as $\varepsilon = 0.01$. Since $K_r = 0.5$, by choosing $\Delta = 1$, it holds $\|\Delta K_r \Delta^{-1}\| < 1$, which implies $\rho_1(K_r) < 1$, thus Assumption 4.2.1 holds. Since $A = -1$, thus Assumption 4.2.2 holds. With $P = Q = 1$ Assumption 4.2.3 holds. Therefore Theorem 4.2.1 applies. Figures 4.3.5 and 4.3.6 are the solutions of the boundary-layer and the reduced subsystems respectively. It is observed that the solutions to both subsystems converge to the origin as time increases. In Figures 4.3.7 and 4.3.8, the solutions of the slow and the fast dynamics of the full system tend to zero when time increases.

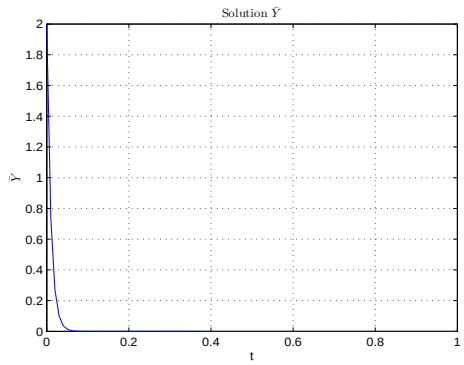

Figure 4.3.5: Solution of the boundary-layer subsystem

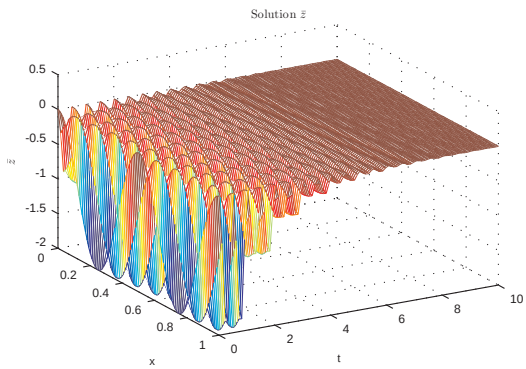

Figure 4.3.6: Solution of the reduced subsystem

Let us choose $\varepsilon = \{0.03, 0.02, 0.01\}$. Table 4.3.2 shows that the errors between the full system and the reduced subsystem decrease as ε decreases.

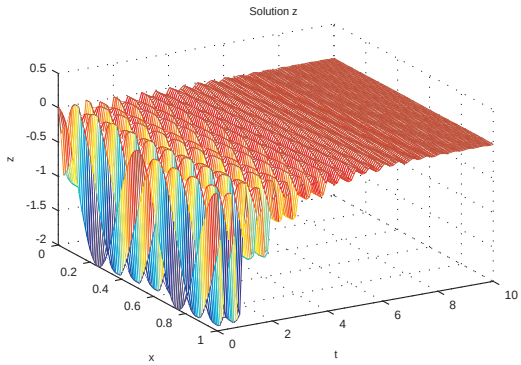

Figure 4.3.7: Solution of the slow dynamics of the full system

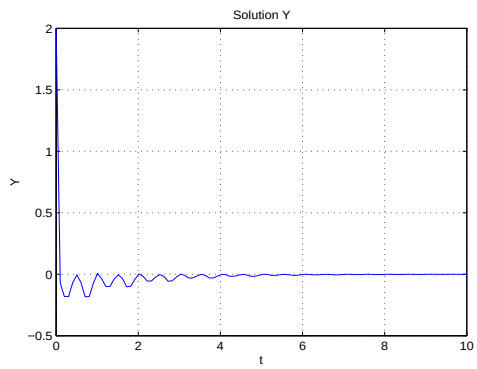

Figure 4.3.8: Solution of the fast dynamics of the full system

ε	0.03	0.02	0.01
$\|z(.,t) - \bar{z}(t)\|^2_{L^2}$	2×10^{-3}	1×10^{-3}	3×10^{-4}

Table 4.3.2: Estimates of the error between the full system and the reduced subsystem for different values of ε

4.3.3 Numerical simulations on counterexample

The initial conditions are selected as: $Y_0 = \bar{Y}_0 = 1$, $z_0 = \bar{z}_0 = 0$. The perturbation parameter ε is chosen as 0.01. Figures 4.3.9 and 4.3.10 show the solutions of the two subsystems. It is observed that the solutions of both subsystems converge to the origin when time increases. Figures 4.3.11 and 4.3.12 are the solutions of the full system. It is shown that the full system is divergent as time increases.

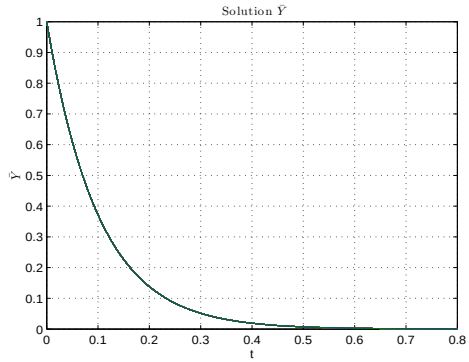

Figure 4.3.9: Solution of the boundary-layer subsystem (4.2.20)

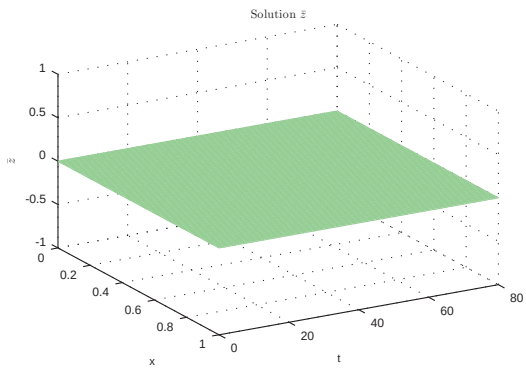

Figure 4.3.10: Solution of the reduced subsystem (4.2.16)-(4.2.17)

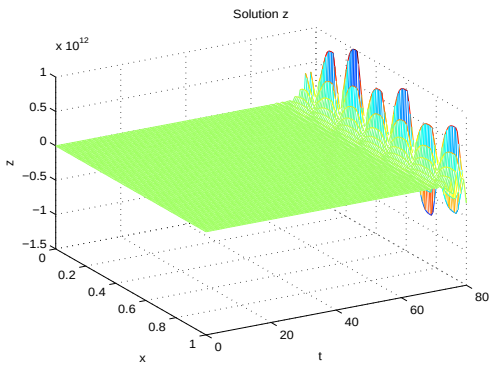

Figure 4.3.11: Solution of the slow dynamics of the full system (4.2.14)-(4.2.15)

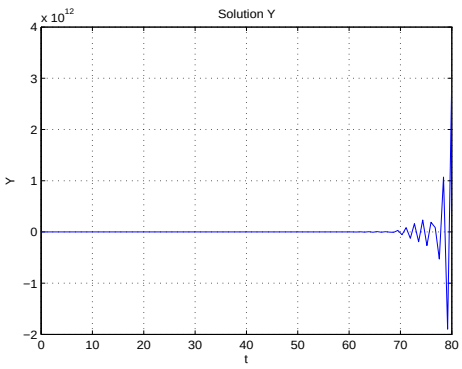

Figure 4.3.12: Solution of the fast dynamics of the full system (4.2.14)-(4.2.15)

Chapter 5

Boundary control synthesis based on singular perturbation method and applications

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In Section 5.1 we first show the boundary control synthesis of the general singularly perturbed hyperbolic systems presented in Chapter 3 based on singular perturbation approach. The reduced subsystem is stabilized in finite time. In Section 5.2 three applications are used to illustrate the main results in Chapters 2, 3 and 4 respectively. Moreover, the boundary control synthesis results given in Section 5.1 are used to design a new control strategy for the applications. The main results of this chapter are published in ([Tang et al., 2015b], [Tang et al., 2014b]).

5.1 Stabilization of the reduced subsystem in finite time

In this section, we consider a singular perturbation approach for the boundary condition synthesis. More precisely, we first choose the boundary conditions matrix K_r for the reduced subsystem (3.1.4)-(3.1.6) as 0, it makes

the slow dynamics converge to the equilibrium in finite time T, and the fast dynamics are not modified. Then the boundary conditions matrix $K(\varepsilon)$ for the full system (3.1.1)-(3.1.3) can be chosen based on the boundary conditions for the slow dynamics such that the stability condition $\rho_1(K(0)) < 1$ is satisfied. The full system reaches a neighborhood of the origin at time T.

5.1.1 Main results

Before introducing the main results in this section, let us first recall the full system and the reduced subsystem presented in Chapter 3. The full system is given by

$$y_t(x,t) + \Lambda_1(\varepsilon)y_x(x,t) = a(\varepsilon)y(x,t) + b(\varepsilon)z(x,t), \tag{3.1.1a}$$

$$\varepsilon z_t(x,t) + \Lambda_2(\varepsilon)z_x(x,t) = c(\varepsilon)y(x,t) + d(\varepsilon)z(x,t), \tag{3.1.1b}$$

with the boundary condition

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K(\varepsilon)\begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in [0,+\infty), \tag{3.1.2}$$

whereas the initial condition is

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y_0(x) \\ z_0(x) \end{pmatrix}, \quad x \in [0,1]. \tag{3.1.3}$$

The reduced subsystem computed from the full system is

$$\bar{y}_t(x,t) + \Lambda_1(0)\bar{y}_x(x,t) = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \tag{3.1.4}$$

with the boundary condition

$$\bar{y}(0,t) = K_r\bar{y}(1,t), \quad t \in [0,+\infty), \tag{3.1.5}$$

where $K_r = K_{11}(0) + K_{12}(0)(I_m - K_{22}(0))^{-1}K_{21}(0)$, whereas the initial condition is given as the same as for the full system

$$\bar{y}(x,0) = \bar{y}_0(x) = y_0(x), \quad x \in [0,1]. \tag{3.1.6}$$

In this section we adopt the following definition for the reduced subsystem convergent in finite time (see i.e. [Perrollaz and Rosier, 2014], [Bastin and Coron, 2015]),

Definition 5.1.1. *The reduced subsystem (3.1.4)-(3.1.6) is convergent in finite time if there exists positive value T such that for every initial condition $\bar{y}_0 \in H^2(0,1)$ satisfying the compatibility conditions $\bar{y}_0(0) = K_r\bar{y}_0(1)$, $\Lambda_1(0)\bar{y}_{0x}(0) = K_r\Lambda_1(0)\bar{y}_{0x}(1)$, the solution to the system (3.1.4)-(3.1.6) equals zero for all $t \geqslant T$:*

$$\bar{y}(.,t) = 0, \quad t \in [T,+\infty).$$

The stability result of the reduced subsystem in finite time is stated in Proposition 5.1.1.

Proposition 5.1.1. *If the boundary conditions matrix $K_r = 0$, then the reduced subsystem (3.1.4)-(3.1.6) is convergent in finite time T, where T is given by*

$$T = \frac{1}{\underline{\lambda}(\Lambda_1(0))}. \tag{5.1.2}$$

The estimates of the slow dynamics and the fast dynamics of the full system are given in the following corollary.

Corollary 5.1.1. *If the matrix $K_r = 0$ and if the condition $\rho_1(K(0)) < 1$ is satisfied, there exist positive values ε^*, C, $\tilde{C}$, ω and $T = \frac{1}{\underline{\lambda}(\Lambda_1(0))}$, such that for all $0 < \varepsilon < \varepsilon^*$, then for every initial condition $y_0 \in H^2(0,1)$ with $y_0 = \bar{y}_0$, satisfying the compatibility conditions $y_0(0) = 0$ and $y_{0x}(0) = 0$, and for all $z_0 \in L^2(0,1)$, it holds for all $t \geqslant T$,*

$$\begin{aligned} & \|y(.,t)\|^2_{L^2(0,1)} \\ \leqslant\ & \varepsilon C e^{-\omega t}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}_0(1)\|^2_{L^2(0,1)} \Big), \end{aligned} \tag{5.1.3}$$

$$\begin{aligned} & \int_T^\infty \|z(.,t)\|^2_{L^2(0,1)} dt \\ \leqslant\ & \varepsilon \tilde{C}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}_0(1)\|^2_{L^2(0,1)} \Big). \end{aligned} \tag{5.1.4}$$

5.1.2 Proof of Proposition 5.1.1 and Corollary 5.1.1

Proof of Proposition 5.1.1 First, let us prove that the reduced subsystem (3.1.4)-(3.1.6) is stable. The boundary conditions matrix is $K_r = 0$, which implies that the stability condition $\rho_1(K_r) < 1$ is satisfied. Due to Theorem 1.1.1 in Chapter 1, the reduced subsystem is stable.
Next, in order to show that the reduced subsystem converges to the origin in finite time, we can compute explicitly the solution of (3.1.4). By using the initial condition (3.1.6) and the boundary condition (3.1.5) with $K_r = 0$, the solution of the system is as follows:

$$\begin{cases} \bar{y}_i(x,t) &= y_{0i}(x - \lambda_i t), \quad 0 \leqslant t < \frac{x}{\lambda_i},\ i = 1, \cdots, n, \\ \bar{y}_i(x,t) &= 0, \quad t \geqslant \frac{x}{\lambda_i}. \end{cases}$$

Thus $\bar{y}(x,t) = 0$ for all $t \geqslant \frac{1}{\underline{\lambda}(\Lambda_1(0))}$. This concludes the proof of Proposition 5.1.1. $\square$

Proof of Corollary 5.1.1 The proof of this corollary is based on Theorem 3.2.1 and Proposition 5.1.1. Using Proposition 5.1.1 it follows

$$\bar{y}(x,t) = 0, \quad t \geqslant T.$$

Combined with (3.2.1) in Theorem 3.2.1, we get that (5.1.3) holds. Moreover it is deduced from (3.2.2)

$$\begin{aligned} & \int_0^T \|z(.,t) - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}(1,t)\|^2_{L^2(0,1)} dt \\ + & \int_T^\infty \|z(.,t) - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}(1,t)\|^2_{L^2(0,1)} dt \\ \leqslant & \ \varepsilon\tilde{C}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}_0(1)\|^2_{L^2(0,1)} \Big), \end{aligned}$$

and thus

$$\begin{aligned} & \int_T^\infty \|z(.,t) - (I - K_{22}(0))^{-1} K_{21}(0)\bar{y}(1,t)\|^2_{L^2(0,1)} dt \\ \leqslant & \ \varepsilon\tilde{C}\Big(\|\bar{y}_0\|^2_{H^2(0,1)} + \|z_0 - (I_m - K_{22}(0))^{-1} K_{21}(0)\bar{y}_0(1)\|^2_{L^2(0,1)} \Big). \end{aligned}$$

Similarly, using Proposition 5.1.1, we get that (5.1.4) holds. This concludes the proof of Corollary 5.1.1. □

5.2 Applications

In this section, we present three applications which illustrate the main results in Chapters 2, 3 and 4 respectively.

5.2.1 Open channel modeled by Saint-Venant–Exner equation

System description

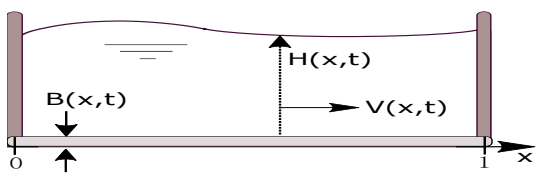

Figure 5.2.1: Open channel

In this section, we apply the main result in Chapter 2 to an open channel problem modeled by linearized Saint-Venant–Exner equation. Let us consider a prismatic open channel in Figure 5.2.1, with a rectangular cross-section and a unit width, where all the friction losses are neglected. The effect of the sediment on the flow is handled in this model. The dynamics of the system are described by the Saint-Venant equation in [Graf, 1998] and the Exner equation in [Graf, 1984], [Hudson and Sweby, 2003], [Diagne et al., 2012], for $x \in [0,1]$ and for $t \geqslant 0$:

$$H_t(x,t) + V(x,t)H_x(x,t) + H(x,t)V_x(x,t) = 0, \tag{5.2.1a}$$

$$V_t(x,t) + V(x,t)V_x(x,t) + gH_x(x,t) + gB_x(x,t) = 0, \tag{5.2.1b}$$

$$B_t(x,t) + \alpha V^2(x,t)V_x(x,t) = 0, \tag{5.2.1c}$$

where the state variables are the water level $H(x,t)$, the water velocity $V(x,t)$, and the bathymetry $B(x,t)$ which is the sediment layer above the channel bottom. The gravity constant is g and the constant parameter which represents the porosity and viscosity effects on the sediment dynamics is denoted by α. The space variable is $x \in [0,1]$ and the time variable is $t \geqslant 0$.

System linearization

Let us consider a steady-state (H^*, V^*, B^*). Since $B_t^* = 0$ and $V^* \neq 0$, (5.2.1c) gives $V_x^* = 0$. Similarly, since $H_t^* = V_x^* = 0$ and $V^* \neq 0$, we get $H_x^* = 0$ from (5.2.1a). In (5.2.1b), we obtain $B_x^* = 0$ due to $V_t^* = V_x^* = H_x^* = 0$. Thus the steady state is constant in space. Let us define the deviations of the state (H, V, B) with respect to the steady-state, for all $x \in [0,1]$ and $t \geqslant 0$,

$$\begin{aligned} h(x,t) &= H(x,t) - H^*, \\ v(x,t) &= V(x,t) - V^*, \\ b(x,t) &= B(x,t) - B^*. \end{aligned}$$

The linearized system is given by

$$\begin{aligned} h_t(x,t) + V^* h_x(x,t) + H^* v_x(x,t) &= 0, \\ v_t(x,t) + g h_x(x,t) + g b_x(x,t) + V^* v_x(x,t) &= 0, \\ b_t(x,t) + \alpha V^{*2} v_x(x,t) &= 0. \end{aligned} \tag{5.2.2}$$

Dynamics in Riemann coordinates

Let us perform a change of variable for the linearized system (5.2.2). Following [Hudson and Sweby, 2003], [Diagne et al., 2012], the characteristic coordinates are defined for each $k = 1,2,3$ by

$$W_k(x,t) = \frac{\Big((V^*-\lambda_i)(V^*-\lambda_j)+gH^*\Big)h(x,t)+H^*\lambda_k v(x,t)+gH^* b(x,t)}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)}, \quad k \neq i \neq j \in \{1,2,3\}. \tag{5.2.3}$$

The system (5.2.2) is written in the new variables $W_k(x,t)$ as follows

$$W_t(x,t) + \Lambda W_x(x,t) = 0, \tag{5.2.4}$$

where $W = (W_1 \quad W_2 \quad W_3)^\top$ and $\Lambda = \mathtt{diag}(\lambda_1 \quad \lambda_2 \quad \lambda_3)$, for all $x \in [0,1]$, $t \in [0,+\infty)$.
According to [Hudson and Sweby, 2003, Diagne et al., 2012], the three eigenvalues of Λ are such that

$$\lambda_1 < 0 < \lambda_2 < \lambda_3,$$

where λ_1 and λ_3 represent the velocities of the water flow and λ_2 represents the velocity of the sediment motion. The sediment motion is much slower than the water flow, then we get that $\lambda_2 << |\lambda_1|$ and $\lambda_2 << \lambda_3$. In this case by performing the change of spatial variable $W_1'(1-x,t) = W_1(x,t)$, we may assume without loss of generality that $\lambda_1 > 0$, thus Λ is diagonal positive. Let us define a small positive value $\varepsilon = \frac{\lambda_2}{\lambda_3}$, and a new time scale $\tilde{t} = \lambda_2 t$, the system (5.2.4) is rewritten as the following singularly perturbed system, for all $x \in [0,1]$ and for all $\tilde{t} \geqslant 0$,

$$\begin{array}{rcl} \varepsilon W'_{1\tilde{t}}(x,\tilde{t}) + \frac{\lambda_1}{\lambda_3} W'_{1x}(x,\tilde{t}) & = & 0, \\ W_{2\tilde{t}}(x,\tilde{t}) + W_{2x}(x,\tilde{t}) & = & 0, \\ \varepsilon W_{3\tilde{t}}(x,\tilde{t}) + W_{3x}(x,\tilde{t}) & = & 0. \end{array} \tag{5.2.5}$$

Boundary conditions

We assume that the channel is equipped with hydraulic control devices such as pumps, valves, spillways, gates, etc. The water levels at upstream and downstream of the channel are assumed to be measured. The control action is provided by the control devices. Let us introduce the following three boundary conditions (these are the same boundary conditions as in [Prieur and Mazenc, 2012]):
1) The first boundary condition describes the value of the channel inflow rate which is denoted by $c_0(\tilde{t})$. Here we consider $c_0(\tilde{t})$ as a control input (see [Bastin et al., 2009]):

$$H(0,\tilde{t})V(0,\tilde{t}) = c_0(\tilde{t}).$$

2) The second boundary condition is given by gate operation at outflow of the reach. A gate model can be expressed as follows (see [Bastin et al., 2009]):

$$H(1,\tilde{t})V(1,\tilde{t}) = \Gamma\sqrt{[H(1,\tilde{t}) - c_1(\tilde{t})]^3},$$

where Γ is a positive constant coefficient. The control input is denoted by $c_1(\tilde{t})$.

3) The third boundary condition is a physical constraint on the bathymetry (see [Diagne et al., 2012]):

$$B(0,\tilde{t}) = \mathcal{B},$$

where $\mathcal{B}$ is a constant value.
After the linearization of these boundary conditions, we derive the following boundary conditions for linearized system (5.2.2):

$$H^* v(0,\tilde{t}) + V^* h(0,\tilde{t}) = c_0(\tilde{t}) - c_0^*, \tag{5.2.6a}$$

$$H^* v(1,\tilde{t}) + V^* h(1,\tilde{t}) = \frac{3\Gamma(h(1,\tilde{t}) + c_1^* - c_1(\tilde{t}))\sqrt{H^* - c_1^*}}{2}, \tag{5.2.6b}$$

$$b(0,\tilde{t}) = 0, \tag{5.2.6c}$$

where c_0^* and c_1^* are constant control actions at the steady-state (H^*, V^*, B^*). The boundary conditions for the singularly perturbed system (5.2.5) are given as follows:

$$W_1'(0,\tilde{t}) = k_{12} W_2(1,\tilde{t}) + k_{13} W_3(1,\tilde{t}), \tag{5.2.7a}$$

$$W_2(0,\tilde{t}) = k_{21} W_1'(1,\tilde{t}), \tag{5.2.7b}$$

$$W_3(0,\tilde{t}) = \xi(k_{21}) W_1'(1,\tilde{t}), \tag{5.2.7c}$$

with

$$\xi(k_{21}) = -\frac{[(\lambda_1 - V^*)^2 - gH^*] + k_{21}[(\lambda_2 - V^*)^2 - gH^*]}{(\lambda_3 - V^*)^2 - gH^*}. \tag{5.2.8}$$

Proposition 5.2.1. *The boundary conditions (5.2.6) for the linearized system (5.2.2) are equivalent to the boundary conditions (5.2.7) for the singularly perturbed system (5.2.5) with the following boundary control inputs, for all $\tilde{t} \geqslant 0$,*

$$\begin{aligned} c_0(\tilde{t}) \;=\; & c_0^* + h(0,\tilde{t}) \Bigg[V^* + \frac{gH^*(k_{21}\phi_1 - \phi_2)}{\phi_2\lambda_2 - k_{21}\phi_1\lambda_1} \\ & + \frac{k_{21}\phi_1(V^* - \lambda_2)(V^* - \lambda_3) - \phi_2(V^* - \lambda_1)(V^* - \lambda_3)}{\phi_2\lambda_2 - k_{21}\phi_1\lambda_1} \Bigg], \end{aligned} \tag{5.2.9}$$

$$
\begin{aligned}
c_1(\tilde{t}) \;=\; & c_1^* + h(1,\tilde{t})\Bigg[1 - \frac{2V^*}{3\Gamma\sqrt{H^* - c_1^*}} \\
& - \frac{2H^*}{3\Gamma\sqrt{H^* - c_1^*}} \times \Bigg(\frac{gH^*(-\phi_1 + k_{12}\phi_2 + k_{13}\phi_3)}{H^*(\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3)} \\
& - \frac{\phi_1(V^* - \lambda_2)(V^* - \lambda_3) - k_{12}\phi_2(V^* - \lambda_1)(V^* - \lambda_3)}{H^*(\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3)} \\
& + \frac{k_{13}\phi_3(V^* - \lambda_1)(V^* - \lambda_2)}{H^*(\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3)}\Bigg)\Bigg] \\
& - b(1,\tilde{t})\left[\frac{2H^*}{3\Gamma\sqrt{H^* - c_1^*}}\,\frac{g(-\phi_1 + k_{12}\phi_2 + k_{13}\phi_3)}{\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3}\right],
\end{aligned}
\tag{5.2.10}
$$

where $H^* - c_1^* \neq 0$ *and* $\phi_k = \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}$ *for* (i,j,k) *in* $\{1,2,3\}^3$.

Proof. This proof is split in three parts where three equivalences are shown. Part 1: Equivalence of (5.2.6a) and (5.2.7b). Due to (5.2.6a), we can write the boundary control $c_0(\tilde{t})$ at $x = 0$ as follows

$$
c_0(\tilde{t}) = c_0^* + H^* v(0,\tilde{t}) + V^* h(0,\tilde{t}). \tag{5.2.11}
$$

Let us define $\phi_k = \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}$ for (i,j,k) in $\{1,2,3\}^3$. Then with (5.2.3) and (5.2.7b), we can compute $v(0,\tilde{t})$ as a function of $h(0,\tilde{t})$ and $b(0,\tilde{t})$,

$$
\begin{aligned}
v(0,\tilde{t}) \;=\; & \Bigg[\frac{-\phi_2\Big((V^* - \lambda_1)(V^* - \lambda_3) + gH^*\Big)}{H^*(\phi_2\lambda_2 - k_{21}\phi_1\lambda_1)} \\
& + \frac{k_{21}\phi_1\Big((V^* - \lambda_2)(V^* - \lambda_3) + gH^*\Big)}{H^*(\phi_2\lambda_2 - k_{21}\phi_1\lambda_1)}\Bigg] h(0,\tilde{t}) \\
& + \frac{g(k_{21}\phi_1 - \phi_2)}{\phi_2\lambda_2 - k_{21}\phi_1\lambda_1} b(0,\tilde{t}).
\end{aligned}
\tag{5.2.12}
$$

Combining (5.2.11) and (5.2.12), and using (5.2.6c), the control action $c_0(\tilde{t})$ is given by

$$
\begin{aligned}
c_0(\tilde{t}) \;=\; & c_0^* + h(0,\tilde{t})\Bigg[V^* + \frac{gH^*(k_{21}\phi_1 - \phi_2)}{\phi_2\lambda_2 - k_{21}\phi_1\lambda_1} \\
& + \frac{k_{21}\phi_1(V^* - \lambda_2)(V^* - \lambda_3) - \phi_2(V^* - \lambda_1)(V^* - \lambda_3)}{\phi_2\lambda_2 - k_{21}\phi_1\lambda_1}\Bigg],
\end{aligned}
$$

with a suitable choice of tuning parameter k_{21}, such that the condition $\phi_2\lambda_2 - k_{21}\phi_1\lambda_1 \neq 0$ holds. Thus (5.2.6a) and (5.2.7b) are equivalent.

Part 2: Equivalence of (5.2.6b) and (5.2.7a). Similarly, due to (5.2.6b), the boundary control $c_1(\tilde{t})$ at $x = 1$ is rewritten as

$$c_1(\tilde{t}) = c_1^* - \frac{2(H^* v(1,\tilde{t}) + V^* h(1,\tilde{t}))}{3\Gamma\sqrt{H^* - c_1^*}} + h(1,\tilde{t}). \tag{5.2.13}$$

With (5.2.3) and (5.2.7a), $v(1,\tilde{t})$ is computed as a function of $h(1,\tilde{t})$ and $b(1,\tilde{t})$

$$\begin{aligned} v(1,\tilde{t}) \;=\; & \Bigg[\frac{gH^*(k_{12}\phi_2 + k_{13}\phi_3 - \phi_1)}{H^*(\phi_1\lambda_1 - k_{12}\phi_2\lambda_2 - k_{13}\phi_3\lambda_3)} \\ & + \frac{k_{12}\phi_2(V^* - \lambda_1)(V^* - \lambda_3) + k_{13}\phi_3(V^* - \lambda_1)(V^* - \lambda_2)}{H^*(\phi_1\lambda_1 - k_{12}\phi_2\lambda_2 - k_{13}\phi_3\lambda_3)} \\ & - \frac{\phi_1(V^* - \lambda_2)(V^* - \lambda_3)}{H^*(\phi_1\lambda_1 - k_{12}\phi_2\lambda_2 - k_{13}\phi_3\lambda_3)}\Bigg] h(1,\tilde{t}) \\ & + \frac{g(k_{12}\phi_2 + k_{13}\phi_3)}{\phi_1\lambda_1 - k_{12}\phi_2\lambda_2 - k_{13}\phi_3\lambda_3} b(1,\tilde{t}). \end{aligned} \tag{5.2.14}$$

Combining (5.2.13) and (5.2.14), we obtain the control action at $x = 1$

$$\begin{aligned} c_1(\tilde{t}) \;=\; & c_1^* + h(1,\tilde{t})\Bigg[1 - \frac{2V^*}{3\Gamma\sqrt{H^* - c_1^*}} \\ & - \frac{2H^*}{3\Gamma\sqrt{H^* - c_1^*}} \times \Bigg(\frac{gH^*(-\phi_1 + k_{12}\phi_2 + k_{13}\phi_3)}{H^*(\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3)} \\ & - \frac{\phi_1(V^* - \lambda_2)(V^* - \lambda_3) - k_{12}\phi_2(V^* - \lambda_1)(V^* - \lambda_3)}{H^*(\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3)} \\ & + \frac{k_{13}\phi_3(V^* - \lambda_1)(V^* - \lambda_2)}{H^*(\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3)}\Bigg)\Bigg] \\ & - b(1,\tilde{t})\left[\frac{2H^*}{3\Gamma\sqrt{H^* - c_1^*}} \frac{g(-\phi_1 + k_{12}\phi_2 + k_{13}\phi_3)}{\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3}\right], \end{aligned}$$

where we assume that $H^* - c_1^* \neq 0$ and $\lambda_1\phi_1 - k_{12}\lambda_2\phi_2 - k_{13}\lambda_3\phi_3 \neq 0$ for a suitable choice of k_{12} and k_{13}. Thus (5.2.6b) and (5.2.7a) are equivalent.
Part 3: Equivalence of (5.2.6c) and (5.2.7c). In the Riemann coordinates, the variable b can be expressed by $\sum_i[(\lambda_i - V^*)^2 - gH^*]W_i$, for $i = 1, 2, 3$. Due to (5.2.6c), assuming $(\lambda_3 - V^*)^2 - gH^* \neq 0$ and using (5.2.7b), then we have

$$W_3(0,\tilde{t}) \;=\; -\frac{(\lambda_1 - V^*)^2 - gH^* + k_{21}[(\lambda_2 - V^*)^2 - gH^*]}{(\lambda_3 - V^*)^2 - gH^*} W_1(0,\tilde{t}). \tag{5.2.15}$$

Thus the boundary conditions (5.2.6c) and (5.2.7c) are equivalent. This concludes the proof of Proposition 5.2.1. □

Boundary control synthesis

Adopting the definitions of the reduced subsystem and the boundary-layer subsystem in Section 2.1, the two subsystems are computed as follows. The reduced subsystem is

$$\bar{W}_{2\tilde{t}}(x,\tilde{t}) + \bar{W}_{2x}(x,\tilde{t}) = 0, \tag{5.2.16}$$

with the boundary condition

$$\bar{W}_2(0,\tilde{t}) = K_r \bar{W}_2(1,\tilde{t}), \tag{5.2.17}$$

where $K_r = \frac{k_{12}k_{21}}{1-k_{13}\xi(k_{21})}$.
Let us perform the following change of variables:

$$\begin{aligned} \bar{W}_1'(x,\tilde{t}) &= W_1'(x,\tilde{t}) - \tfrac{k_{12}}{1-k_{13}\xi(k_{21})} W_2(1,\tilde{t}), \\ \bar{W}_3(x,\tilde{t}) &= W_3(x,\tilde{t}) - \tfrac{k_{12}\xi(k_{21})}{1-k_{13}\xi(k_{21})} W_2(1,\tilde{t}). \end{aligned} \tag{5.2.18}$$

The boundary-layer subsystem is

$$\begin{pmatrix} \bar{W}_1'(x,\tilde{\tau}) \\ \bar{W}_3(x,\tilde{\tau}) \end{pmatrix}_{\tilde{\tau}} + \begin{pmatrix} \frac{\lambda_1}{\lambda_3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{W}_1'(x,\tilde{\tau}) \\ \bar{W}_3(x,\tilde{\tau}) \end{pmatrix}_x = 0, \tag{5.2.19}$$

with the boundary condition

$$\begin{pmatrix} \bar{W}_1'(0,\tilde{\tau}) \\ \bar{W}_3(0,\tilde{\tau}) \end{pmatrix} = K_{22} \begin{pmatrix} \bar{W}_1'(1,\tilde{\tau}) \\ \bar{W}_3(1,\tilde{\tau}) \end{pmatrix}, \tag{5.2.20}$$

where $K_{22} = \begin{pmatrix} 0 & k_{13} \\ \xi(k_{21}) & 0 \end{pmatrix}$, $\tilde{\tau} = \frac{\tilde{t}}{\varepsilon}$.
The aim of the following work is to stabilize the reduced subsystem in finite time T and to stabilize the full system such that it reaches a neighborhood of the origin at time T. Due to Proposition 5.1.1, by choosing $K_r = 0$, the reduced subsystem (5.2.16)-(5.2.17) converges in finite time. According to Theorem 1.1.1, the boundary condition K is chosen such that $\rho_1(K) < 1$, then the full system (5.2.5) and (5.2.7) is stable. Assuming that $1 - k_{13}\xi(k_{21}) \neq 0$, $K_r = 0$ holds as soon as $k_{12} = 0$ or $k_{21} = 0$.

Proposition 5.2.2. *Consider the boundary conditions matrix*

$$K_1 = \begin{pmatrix} 0 & k_{12} & k_{13} \\ 0 & 0 & 0 \\ \xi(0) & 0 & 0 \end{pmatrix}, \tag{5.2.21}$$

with k_{12}, k_{13} in $\mathbb{R}$, assume that $k_{13}\xi(0) \neq 1$ and that there exist positive values d_2, d_3 such that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & d_3\xi(0) \\ 0 & d_2 & 0 & k_{12} & 0 & 0 \\ 0 & 0 & d_3 & k_{13} & 0 & 0 \\ 0 & k_{12} & k_{13} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2 & 0 \\ \xi(0) & 0 & 0 & 0 & 0 & d_3 \end{pmatrix} > 0 \tag{5.2.22}$$

is satisfied. Consider the boundary conditions matrix

$$K_2 = \begin{pmatrix} 0 & 0 & k_{13} \\ k_{21} & 0 & 0 \\ \xi(k_{21}) & 0 & 0 \end{pmatrix}, \tag{5.2.23}$$

with k_{21}, k_{13} in $\mathbb{R}$, assume that $k_{13}\xi(k_{21}) \neq 1$ and that there exist positive values d_2, d_3 such that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & k_{21}d_2 & \xi(k_{21})d_3 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & k_{13} & 0 & 0 \\ 0 & 0 & k_{13} & 1 & 0 & 0 \\ k_{21}d_2 & 0 & 0 & 0 & d_2 & 0 \\ \xi(k_{21})d_3 & 0 & 0 & 0 & 0 & d_3 \end{pmatrix} > 0 \tag{5.2.24}$$

is satisfied. Then Corollary 5.1.1 can be applied to system (5.2.5) with either boundary conditions matrix K_1 or K_2 defining the boundary conditions (5.2.7).

Proof. First let us prove that Corollary 5.1.1 applies to system (5.2.5) with the boundary conditions matrix K_1. The boundary conditions matrix K_r for the reduced subsystem (5.2.16)-(5.2.17) is 0 as soon as $k_{21} = 0$ in K_1. In order to check the boundary condition satisfies $\rho_1(K_1) < 1$, it is equivalent to find a diagonal positive matrix Δ such that the following condition is satisfied

$$\begin{pmatrix} \tilde{\Delta} & G^\top \\ G & \tilde{\Delta} \end{pmatrix} > 0, \tag{5.2.25}$$

with $G = \Delta^2 K_1$ and $\tilde{\Delta} = \Delta^2$. There is no loss of generality to look for

$$\Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{d_2} & 0 \\ 0 & 0 & \sqrt{d_3} \end{pmatrix},$$

where d_2 and d_3 are positive constants. Straightforward calculations show that (5.2.25) is satisfied as soon as (5.2.22) is satisfied. Thus Corollary 5.1.1 applies to (5.2.5) with K_1.
Next we prove that Corollary 5.1.1 applies to system (5.2.5) with K_2. Similarly, $K_r = 0$ due to $k_{12} = 0$ in K_2. Then we can show as a similar way that $\rho_1(K_2) < 1$ if (5.2.24) holds. This concludes the proof of Proposition 5.2.2. □

To solve (5.2.22), we can compute $\xi(0)$ from (5.2.8), then (5.2.22) is a linear matrix inequality (LMI) which can be solved. Similarly, by choosing $\xi(k_{21}) = 0$ in (5.2.24), k_{21} is computed from (5.2.8), then LMI (5.2.24) can be solved.

Numerical solutions

Using the numerical values in [Dos Santos and Prieur, 2008], the equilibrium is chosen as $H^* = 0.1365$, $V^* = 14.65$, $B^* = 0$. We take the gravity constant $g = 9.81$. The eigenvalues of matrix Λ are also given in [Dos Santos and Prieur, 2008] as $\lambda_1 = -10$, $\lambda_2 = 7.72 \times 10^{-4}$, $\lambda_3 = 13$. Using Yalmip toolbox [Löfberg, 2004] on Matlab to solve LMI (5.2.22) and (5.2.24). The obtained boundary conditions matrix K_1 is

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -14 & 0 & 0 \end{pmatrix},$$

and K_2 is

$$K_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0.095 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To numerically compute the solutions of system (5.2.5) with the boundary conditions matrix K_1 or K_2, the space domain [0,1] is divided into 100 intervals of identical length, the final time is chosen as 2000. The perturbation parameter is chosen as $\varepsilon = \frac{\lambda_2}{\lambda_3}$. We take $dt = 0.9\varepsilon dx$ that satisfies the CFL condition and select the initial conditions as follows such that the compatibility condition is satisfied, for all $x \in [0, 1]$,

$$\begin{aligned} W_1'^0(x) &= -1 + \cos(4\pi x), \\ W_2^0(x) &= -1 + \cos(2\pi x), \\ W_3^0(x) &= 1 - \cos(4\pi x). \end{aligned}$$

Figure 5.2.2 shows the solution of the reduced subsystem (5.2.16) with boundary condition matrix $K_r = 0$. It converges to the origin within time $T \simeq 1300s$. The finite time of convergence obtained in Proposition 5.2.2 is $T = \frac{1}{\lambda_2}$ which is close to the numerically computed finite time $T \simeq 1300s$. The slow dynamics of the full system (5.2.5) with the boundary conditions matrix K_1 in Figure 5.2.3 is roughly the same graph as Figure 5.2.2. Figures 5.2.4-5.2.5 show the time evolutions of the fast dynamics of the full system (5.2.5) with the boundary conditions matrix K_1. It is observed that the solutions converge to 0 as time increases. Similar results are obtained for system (5.2.5) with the boundary conditions matrix K_2 by numerical simulations.

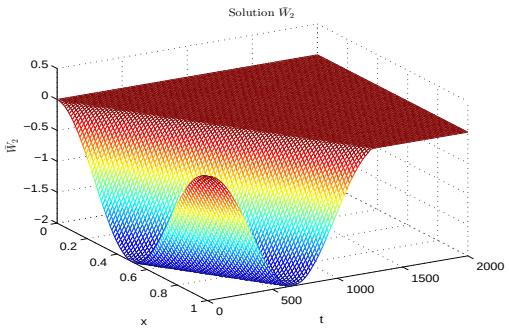

Figure 5.2.2: Time evolution of the reduced subsystem (5.2.16) with $K_r = 0$

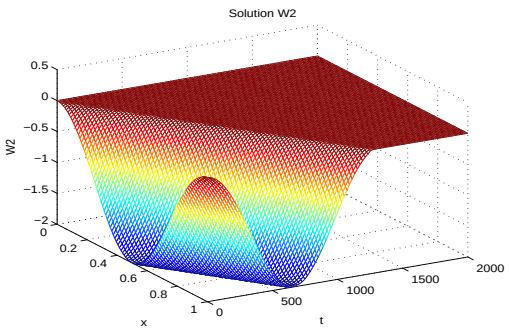

Figure 5.2.3: Time evolution of the slow dynamics of the full system (5.2.5) with K_1

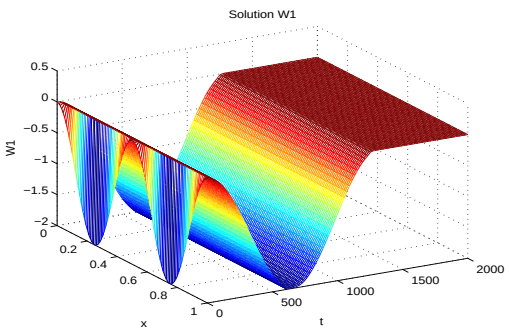

Figure 5.2.4: Time evolution of the fast dynamics of the full system (5.2.5) with K_1

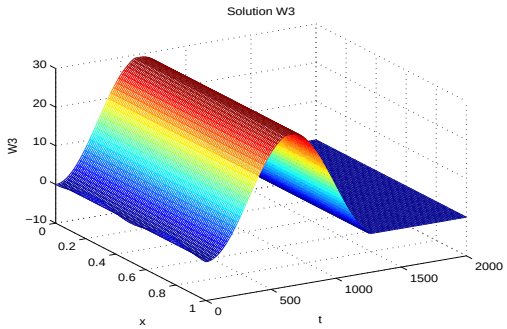

Figure 5.2.5: Time evolution of the fast dynamics of the full system (5.2.5) with K_1

5.2.2 Gas flow transport through a constant section tube governed by Euler equation

In this section we apply the main result in Chapter 3 to a boundary control design of a gas flow transport through a constant section tube.

System description

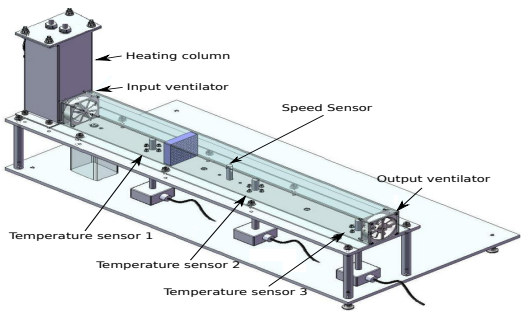

Figure 5.2.6: Gas transport setup

In Figure 5.2.6, we consider the gas dynamics through a constant cross section tube, where all the friction losses and heat transfers are neglected. This system can be modeled by the following Euler equation as considered in [Winterbone, 2000, Chapter 2], by considering a tube of length equals to 1.

$$u_t(x,t) + u(x,t)u_x(x,t) + \frac{1}{\rho(x,t)}p_x(x,t) = 0, \tag{5.2.26a}$$

$$\rho_t(x,t) + \rho(x,t)u_x(x,t) + u(x,t)\rho_x(x,t) = 0, \tag{5.2.26b}$$

$$p_t(x,t) + a^2(x,t)\rho(x,t)u_x(x,t) + u(x,t)p_x(x,t) = 0, \tag{5.2.26c}$$

where $u(x,t)$ stands for the gas velocity, the gas density is denoted by $\rho(x,t)$, the gas pressure is noted by $p(x,t)$, sound speed in ideal gas is presented by $a(x,t) = \sqrt{\frac{\gamma p(x,t)}{\rho(x,t)}}$, γ is the adiabatic index. The space variable is $x \in [0,1]$ and $t \geqslant 0$ denotes the time variable.

System linearization

The system (5.2.26) admits a steady-state (u^*, ρ^*, p^*). From (5.2.26a) and (5.2.26c), since $u_t^* = p_t^* = 0$, we can compute $(u^{*2} - a^{*2})u_x^* = 0$. It is obtained $u_x^* = 0$ since $u^* \neq a^*$. From (5.2.26b), since $u^*\rho^* = cst$, it is obtained $\rho_x^* = 0$. By (5.2.26a), we get $p_x^* = 0$ since $u_t^* = u_x^* = 0$. Thus the steady-state is space constant. The deviations of the state (u, ρ, p) around the steady-state are defined as

$$\begin{aligned}
\tilde{u}(x,t) &= u(x,t) - u^*, \\
\tilde{\rho}(x,t) &= \rho(x,t) - \rho^*, \\
\tilde{p}(x,t) &= p(x,t) - p^*.
\end{aligned}$$

Then the linearization of system (5.2.26) around the equilibrium is given by

$$\begin{aligned}
\tilde{u}_t(x,t) + u^*\tilde{u}_x(x,t) + \tfrac{1}{\rho^*}\tilde{p}_x(x,t) &= 0, \\
\tilde{\rho}_t(x,t) + \rho^*\tilde{u}_x(x,t) + u^*\tilde{\rho}_x(x,t) &= 0, \\
\tilde{p}_t(x,t) + a^{*2}\rho^*\tilde{u}_x(x,t) + u^*\tilde{p}_x(x,t) &= 0.
\end{aligned} \tag{5.2.27}$$

Dynamics in Riemann coordinates

We consider the following change of variables

$$\begin{pmatrix} S_1(x,t) \\ S_2(x,t) \\ S_3(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & -\frac{1}{a^{*2}} \\ \frac{1}{2} & 0 & -\frac{1}{2a^*\rho^*} \\ \frac{1}{2} & 0 & \frac{1}{2a^*\rho^*} \end{pmatrix} \begin{pmatrix} \tilde{u}(x,t) \\ \tilde{\rho}(x,t) \\ \tilde{p}(x,t) \end{pmatrix}. \tag{5.2.28}$$

The system (5.2.27) is written in Riemann coordinates

$$\begin{pmatrix} S_1(x,t) \\ S_2(x,t) \\ S_3(x,t) \end{pmatrix}_t + \begin{pmatrix} u^* & 0 & 0 \\ 0 & u^* - a^* & 0 \\ 0 & 0 & u^* + a^* \end{pmatrix} \begin{pmatrix} S_1(x,t) \\ S_2(x,t) \\ S_3(x,t) \end{pmatrix}_x = 0. \tag{5.2.29}$$

Assuming the propagation speed of gas is much slower than the sound speed, i.e. $u^* << a^*$, we define the perturbation parameter $\varepsilon = \frac{u^*}{a^*}$. In the following work, we use a new time scale $\tilde{t} = u^*t$ and perform a change of space variable $S_2'(x,t) = S_2(1-x,t)$, the system (5.2.29) is written in singularly perturbed form, for all $\tilde{t} \geqslant 0$

$$\begin{pmatrix} S_1(x,\tilde{t}) \\ \varepsilon S_2'(x,\tilde{t}) \\ \varepsilon S_3(x,\tilde{t}) \end{pmatrix}_{\tilde{t}} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\varepsilon & 0 \\ 0 & 0 & 1+\varepsilon \end{pmatrix} \begin{pmatrix} S_1(x,\tilde{t}) \\ S_2'(x,\tilde{t}) \\ S_3(x,\tilde{t}) \end{pmatrix}_x = 0. \tag{5.2.30}$$

Remark 5.2.1. Compared to the model in (5.2.30), the approximation $1 \pm \frac{1}{\varepsilon} \simeq \frac{1}{\varepsilon}$ is used to model the same system in [Tang et al., 2015b], for $\varepsilon > 0$ small enough. Such that the transport velocity Λ does not depend on ε. ○

Boundary conditions

The setup is provided with fans which are located at the two extremities of the tube. The rotation speed is considered as the control action. Let us consider the following three boundary conditions for system (5.2.26).
1. The first boundary condition describes the operation of the inflow fan (see the fan specification map in [Witrant et al., 2008]),

$$u(0,\tilde{t})S_b = \sigma c_0(\tilde{t})(p(0,\tilde{t}) - p_{in}), \tag{5.2.31}$$

where S_b stands for the tube's constant cross section, σ is a constant coefficient, the control input is denoted by $c_0(\tilde{t})$ and p_{in} is a constant pressure before the inflow fan.
2. Similarly, the second boundary condition is given by the outflow fan,

$$u(1,\tilde{t})S_b = \sigma c_1(\tilde{t})(p_{out} - p(1,\tilde{t})), \tag{5.2.32}$$

the control input is noted by $c_1(\tilde{t})$ and p_{out} is a constant pressure behind the outflow fan.
3. The third boundary condition is a physical constraint. Precisely, the gas pressure at the inflow fan is close to the atmospheric pressure (see [Castillo et al., 2012]),

$$\rho(0,\tilde{t}) = \underline{\rho} \tag{5.2.33}$$

where $\underline{\rho}$ is a constant value.
The boundary conditions for system (5.2.27) are obtained by linearizing the above three boundary conditions,

$$\tilde{u}(0,\tilde{t})S_b = \sigma[\tilde{c}_0(\tilde{t})(p^* - p_{in}) + c_0^*\tilde{p}(0,\tilde{t})], \tag{5.2.34a}$$
$$\tilde{u}(1,\tilde{t})S_b = \sigma[\tilde{c}_1(\tilde{t})(p_{out} - p^*) - c_1^*\tilde{p}(1,\tilde{t})], \tag{5.2.34b}$$
$$\tilde{\rho}(0,\tilde{t}) = 0, \tag{5.2.34c}$$

where c_0^*, c_1^* are the constant control actions at the steady-state (u^*, ρ^*, p^*).
The boundary conditions for the singularly perturbed system (5.2.30) are given by

$$S_1(0,\tilde{t}) = K_{12}S_2'(1,\tilde{t}), \tag{5.2.35a}$$
$$S_2'(0,\tilde{t}) = K_{21}S_1(1,\tilde{t}) + K_{23}S_3(1,\tilde{t}), \tag{5.2.35b}$$
$$S_3(0,\tilde{t}) = K_{32}S_2'(1,\tilde{t}), \tag{5.2.35c}$$

where $K_{12} = f(K_{32}) = \frac{\rho^*(1-K_{32})}{a^*}$.

Proposition 5.2.3. *For any values K_{23} and K_{32} in $\mathbb{R}$, such that $K_{23} \neq 1$ and $K_{32} \neq 1$, defining control actions by*

$$c_0(\tilde{t}) = c_0^* + \frac{\frac{S_b(1+K_{32})}{\sigma a^* \rho^*(K_{32}-1)} - c_0^*}{p^* - p_{in}} \tilde{p}(0,\tilde{t}), \tag{5.2.36}$$

$$c_1(\tilde{t}) = c_1^* + \frac{\frac{S_b(a^*(1+K_{23})-2\rho^* K_{21})}{\sigma a^{*2}\rho^*(1-K_{23})} + c_1^*}{p_{out} - p^*} \tilde{p}(1,\tilde{t}) + \frac{\frac{2S_b K_{21}}{\sigma(1-K_{23})}}{p_{out} - p^*} \tilde{\rho}(1,\tilde{t}), \tag{5.2.37}$$

the boundary conditions (5.2.35) are equivalent to (5.2.34).

Proof. Firstly let us prove the equivalence of (5.2.34a) and (5.2.35c). Due to (5.2.28) and (5.2.35c), under the condition $K_{32} - 1 \neq 0$, we can compute

$$\tilde{u}(0,\tilde{t}) = \frac{1 + K_{32}}{a^* \rho^*(K_{32} - 1)} \tilde{p}(0,\tilde{t}), \tag{5.2.38}$$

and substituting (5.2.38) into (5.2.34a), the control action at input of the tube is $c_0(\tilde{t})$ in (5.2.36). With a suitable choice of tuning parameter K_{32}, such that $K_{32}-1 \neq 0$ is satisfied. Thus (5.2.34a) and (5.2.35c) are equivalent. Secondly let us prove the equivalence of (5.2.34b) and (5.2.35b). Due to (5.2.28) and (5.2.35b), we compute

$$\tilde{u}(1,\tilde{t}) = \frac{2K_{21}}{1 - K_{23}} \tilde{\rho}(1,\tilde{t}) + \frac{a^*(1 + K_{23}) - 2\rho^* K_{21}}{a^{*2}\rho^*(1 - K_{23})} \tilde{p}(1,\tilde{t}), \tag{5.2.39}$$

and substituting (5.2.39) into (5.2.34b), the control action at input of the tube is $c_1(\tilde{t})$ in (5.2.37). With a suitable choice of tuning parameter K_{23}, such that $K_{23} - 1 \neq 0$ is satisfied. Thus (5.2.34b) and (5.2.35b) are equivalent.
Finally let us prove the equivalence of (5.2.34c) and (5.2.35a). Due to (5.2.28) and (5.2.34c), we compute

$$\tilde{u}(0,\tilde{t}) = \left(\frac{1}{a^* \rho^*} - \frac{2}{K_{12} a^{*2}} \right) \tilde{p}(0,\tilde{t}) + \frac{2}{K_{12}} \tilde{\rho}(0,\tilde{t}), \tag{5.2.40}$$

and substituting (5.2.34a) into (5.2.40), using (5.2.38), we get

$$K_{12} = \frac{\rho^*(1 - K_{32})}{a^*}.$$

This concludes the proof of Proposition 5.2.3 □

Boundary control synthesis

Adopting the definition of the reduced subsystem in Section 3.1, it is computed as follows,

$$S_{1\tilde{t}}(x,\tilde{t}) + S_{1x}(x,\tilde{t}) = 0, \tag{5.2.41}$$

with the boundary condition

$$S_1(0,\tilde{t}) = K_r S_1(1,\tilde{t}), \tag{5.2.42}$$

where $K_r = \frac{\rho^*(1-K_{32})K_{21}}{a^*(1-K_{23}K_{32})}$.
According to Proposition 5.1.1, the reduced subsystem (5.2.41) and (5.2.42) is convergent in finite time T by choosing $K_r = 0$. Assuming $1-K_{23}K_{32} \neq 0$, since $K_{32} \neq 1$ in Proposition 5.2.3, it holds $K_r = 0$ as soon as $K_{21} = 0$. The boundary condition for the full system (5.2.30), defining by (5.2.35), is given by

$$K = \begin{pmatrix} 0 & \frac{\rho^*(1-K_{32})}{a^*} & 0 \\ 0 & 0 & K_{23} \\ 0 & K_{32} & 0 \end{pmatrix}. \tag{5.2.43}$$

To ensure $\rho_1(K) < 1$, it is sufficient $\|K\| < 1$ which correspond with $\Delta = I_n$. In order to decrease the control cost, we can minimize $\|K\|$ which is equivalent to minimize $K_{32}^2 + \left(\frac{\rho^*(1-K_{32})}{a^*}\right)^2 + K_{23}^2$. K_{23} is chosen as zero. Let compute the derivative of $K_{32}^2 + \left(\frac{\rho^*(1-K_{32})}{a^*}\right)^2$ with respect to K_{32}, we obtain $K_{32} = \frac{\rho^{*2}}{\rho^{*2}+a^{*2}}$. Therefore the control actions become

$$c_0(\tilde{t}) = c_0^* - \frac{\frac{S_b(a^{*2}+2\rho^{*2})}{\sigma a^{*3}\rho^*} - c_0^*}{p^* - p_{in}}\tilde{p}(0,\tilde{t}),$$

$$c_1(\tilde{t}) = c_1^* + \frac{\frac{S_b}{\sigma a \rho^*} + c_1^*}{p_{out} - p^*}\tilde{p}(1,\tilde{t}).$$

Numerical results

Let us consider the following values for numerical simulation: $a^* = 300$, $u^* = 2$, $\rho^* = 2$, $K = 10^{-5}\begin{pmatrix} 0 & 600 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$. The perturbation parameter $\varepsilon = \frac{u^*}{a^*}$. The time evolution of the solution of the reduced subsystem (5.2.41) with $K_r = 0$ is shown in Figure 5.2.7. It is observed that the solution converges to the origin in finite time. Time evolution of η in Figure 5.2.8 shows that the difference between the full system (5.2.29) with K and the reduced subsystem (5.2.41) with $K_r = 0$ is close to 0 as time increases.

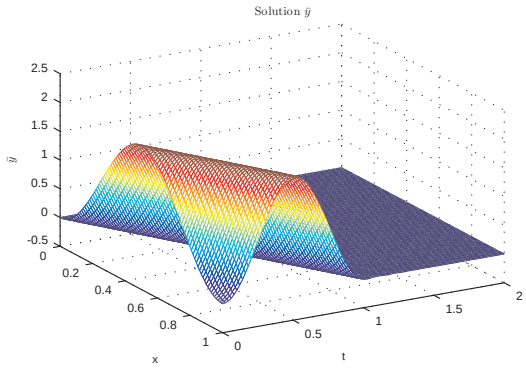

Figure 5.2.7: Time evolution of the reduced subsystem (5.2.41) with $K_r = 0$

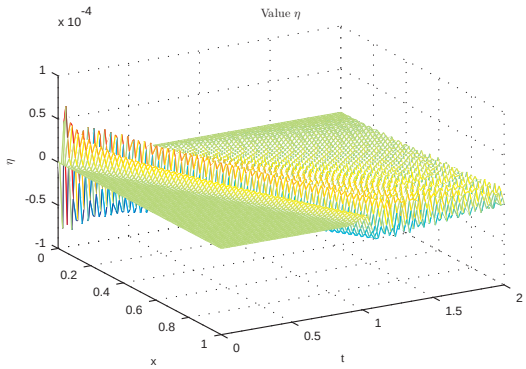

Figure 5.2.8: Time evolution of the difference between the full system (5.2.29) with K and the reduced subsystem (5.2.41) with $K_r = 0$

5.2.3 Gas flow transport governed by coupled ODE-PDE system

In this section let us consider again the setup in Figure 5.2.6. In contrast to Section 5.2.2, the gas flow transport in the heating column is taken into account as well as those in the tube. The application in this section is used to illustrate the main result in Chapter 4. The two subsystems, the heating column and the tube, are modeled as follows.

Model of the heating column

To model the gas dynamics in the heating column, we first consider the following assumptions.

Assumption 5.2.1. *The dynamics of the pressure in the gas control volume is much faster than that of the temperature, the pressure and the mass can be considered as quasi static.*

Assumption 5.2.2. *The pressure losses are neglected because of the low mass flow and of the sufficiently large input/output section of gas. This implies $p_0 \approx p_{in}$, where p_{in} is input pressure.*

Assumption 5.2.3. *There is no work done by gas.*

Under the above three assumptions and due to the first law of thermodynamics and ideal gas law, the gas dynamics in the heating column is modeled by (see [Castillo et al., 2012])

$$\dot{\rho}_0 = -\frac{R\gamma T_{in}\dot{m}_{in}}{p_{in}V_0}\rho_0 - \frac{R}{p_{in}V_0C_v}\rho_0 dQ + \frac{\gamma\dot{m}_{in}}{V_0}, \tag{5.2.44}$$

where ρ_0 is the gas density in the heating column, R is the specific gas constant, T_{in} denotes the gas temperature at input, the input mass flow is given by $\dot{m}_{in}$, V_0 is the volume of the heating column, C_v and C_p are the special heat of volume constant gas and of pressure constant gas respectively, dQ is the heating exchange that can be controlled and $\gamma = \frac{C_p}{C_v}$.

Model of the tube

To model the gas dynamics in the tube, let us state the following assumptions.

Assumption 5.2.4. *All the heat transfers and friction losses are negligible.*

Assumption 5.2.5. *The gas pressure in the tube is assumed to be constant, which is close to the atmosphere pressure.*

Under Assumptions 5.2.4-5.2.5, due to (5.2.26b), the gas dynamics in the tube is written as follows, for $x \in [0, 1]$ and for $t \geqslant 0$,

$$\rho_t(x,t) + u_b\rho_x(x,t) = 0, \tag{5.2.45}$$

where ρ represents the gas density in the tube. The propagation speed in the tube is denoted by u_b. With a scaling of the space domain, it may be assumed that the tube's length equals 1. Due to ideal gas law, $u_b = \frac{\dot{m}_{in}}{\rho_0 S_b}$, where S_b is the cross section of the tube. The boundary condition is given by

$$\rho(0,t) = \rho_0. \tag{5.2.46}$$

Control problem statement

In the following we state our control problem. Let us rewrite (5.2.44) and (5.2.45) as follows

$$\dot{\rho}_0 = -\frac{1}{\kappa}\rho_0 + U(t), \tag{5.2.47a}$$

$$\rho_t + u_b\rho_x = 0. \tag{5.2.47b}$$

where κ is the transport time constant in the heating column and $U(t)$ is the control (it could be defined from dQ). The boundary condition is the same as (5.2.46).
The control problem is formulated as: for any desired mass density in the tube $\rho^* \geqslant 0$, let the controller be

$$U(t) = c_1\rho(1,t) + c_2\rho^*, \tag{5.2.48}$$

such that the system is exponentially stable at the equilibrium point $\rho = \rho^*$ with an appropriate choice of real values c_1 and c_2. Replacing $U(t)$ in (5.2.47a) by the right hand side in (5.2.48), the closed-loop system is written as

$$\dot{\rho}_0 = -\frac{1}{\kappa}\rho_0 + c_1\rho(1,t) + c_2\rho^*, \tag{5.2.49a}$$

$$\rho_t + u_b\rho_x = 0, \tag{5.2.49b}$$

with the same boundary condition (5.2.46).
At the equilibrium point of the gas density inside of the tube ρ_0^*, from (5.2.49a) we get

$$\rho^* = \frac{\frac{1}{\kappa} - c_1}{c_2}\rho_0^*, \tag{5.2.50}$$

where c_2 has to be selected such that $c_2 \neq 0$. Due to (5.2.46), it holds at the equilibrium $\rho^* = \rho_0^*$. From (5.2.50), the values of c_1 and c_2 should satisfy

$$c_1 + c_2 = \frac{1}{\kappa}.$$

Let us define the state deviations with respect to the equilibrium point

$$\tilde{\rho}_0 = \rho_0 - \rho^*,$$
$$\tilde{\rho} = \rho - \rho^*.$$

The linearized system is

$$\dot{\tilde{\rho}}_0 = -\frac{1}{\kappa}\tilde{\rho}_0 + c_1\tilde{\rho}(1,t), \tag{5.2.51a}$$

$$\varepsilon\tilde{\rho}_t + \frac{1}{\kappa}\tilde{\rho}_x = 0, \tag{5.2.51b}$$

where the perturbation parameter is given by $\varepsilon = \frac{1/\kappa}{u_b^*}$, due to the transport velocity of gas in the heating column is much smaller than that in the tube. The boundary condition is

$$\tilde{\rho}(0,t) = \tilde{\rho}_0. \tag{5.2.52}$$

From (5.2.52), recalling the condition in (4.1.2) in Chapter 4, we compute $G_1 = 0$, $G_2 = 1$ and $G_r = 1$. Moreover, by (4.1.1a) and (4.1.6), Assumption 4.1.1 holds as soon as $c_1 < \frac{1}{\kappa}$. Assumption 4.1.2 holds since $\rho_1(G_1) = 0$. The reduced subsystem is

$$\dot{\bar{\rho}}_0 = \left(-\frac{1}{\kappa} + c_1\right)\bar{\rho}_0, \tag{5.2.53}$$

whereas the boundary-layer subsystem is

$$\bar{\rho}_\tau + \frac{1}{\kappa}\bar{\rho}_x = 0, \tag{5.2.54}$$

with the boundary condition

$$\bar{\rho}(0,\tau) = 0, \tag{5.2.55}$$

where $\tau = \frac{t}{\varepsilon}$.

Numerical results

Let us take the experimental data from [Castillo et al., 2012]: $\gamma = 1.4$, $R = 8.3J/(mol * K)$, $p_{in} = 1 \times 10^5 Pa$, $T_{in} = 300K$, $V_0 = 4 \times 10^{-3} m^3$, $\dot{m}_{in} = 0.01kg/s$, $S_b = 6.4 \times 10^{-3} m^2$. We compute $\kappa = 10$, $\varepsilon = 0.1$. We choose $c_1 = 0.01$. The initial conditions are given by: $\rho_0(0) = \bar{\rho}_0(0) = 2$, $\rho(0) = \cos(4\pi x) - 1$, $\bar{\rho}(0) = \rho(0) - \rho_0(0) = \cos(4\pi x) - 3$. Figures 5.2.9 and 5.2.10 are the solutions of the boundary-layer and reduced subsystems. It is observed that the solutions of the two subsystems converge to the origin. In Figures 5.2.11 and 5.2.12, it is shown that the solutions of the full system tend to zero as time increases.

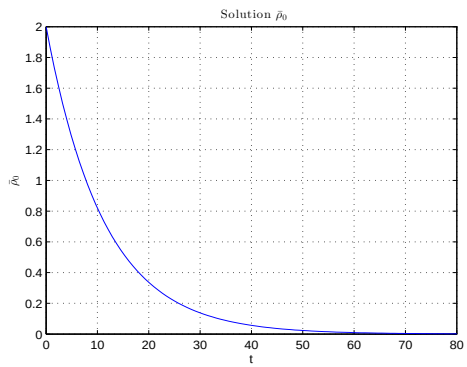

Figure 5.2.9: Solution of the reduced subsystem (5.2.53)

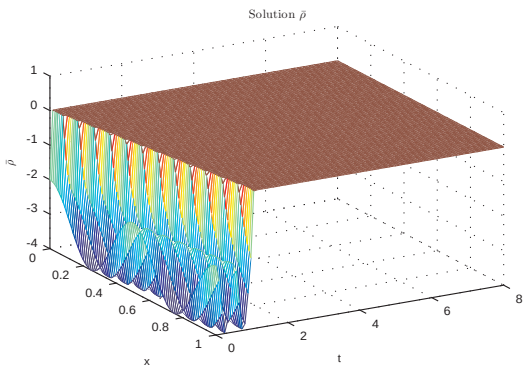

Figure 5.2.10: Solution of the boundary-layer subsystem (5.2.54)-(5.2.55)

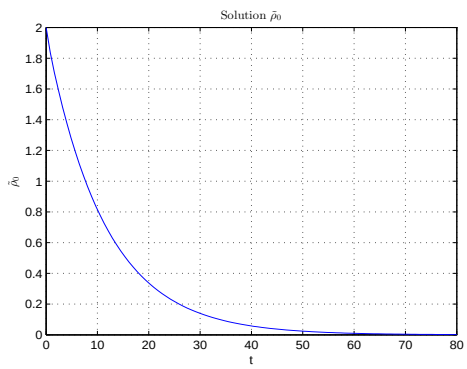

Figure 5.2.11: Solution of the slow dynamics of the full system (5.2.51)-(5.2.52)

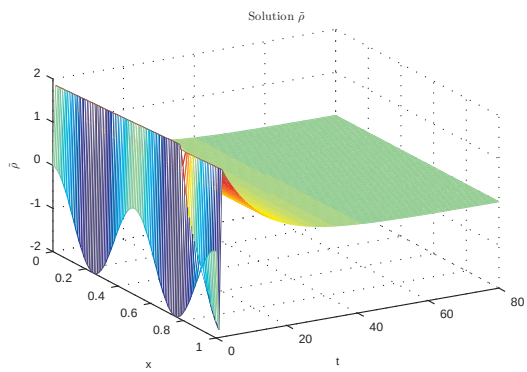

Figure 5.2.12: Solution of the fast dynamics of the full system (5.2.51)-(5.2.52)

Chapter 6

Conclusions and perspectives

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6.1 Conclusions

In this thesis, it has been studied a class of singularly perturbed linear hyperbolic systems of conservation laws and balance laws. By setting the perturbation parameter to zero, two subsystems are formally computed. The slow dynamics are represented by reduced subsystem while the fast dynamics are given by boundary-layer subsystem. A coupled ODE-PDE system with multiple time scales has also been considered in this work. The contributions of this work are concluded as follows:

• *Chapter 2* has been concerned with a class of linear hyperbolic systems of conservation laws. It has been established the link in terms of the stability between the full system and the two subsystems for such infinite dimensional systems. The stability of the full system has indicated the stability of the two subsystems. However, in contrast with the well known results for linear finite dimensional systems, the stability of both subsystems was not enough to guarantee the stability of the full system. The Tikhonov approximation in this chapter is achieved based on a Lyapunov method. First, the estimates of the error between the full system and the subsystems are the order of ε. A Lyapunov function in L^2-norm has been used to prove the above results. Moreover, by means of a Lyapunov function in H^2-norm, more precise estimates have been obtained.

• *Chapter 3* has dealt with a class of linear hyperbolic systems of balance laws. The source terms, the transport velocity and the boundary conditions depend all on the perturbation parameter ε. Under the assumptions of the continuity of the transport velocity, the source terms and the boundary conditions and if the condition $\rho_1(K(0)) < 1$ is satisfied, then the errors between the full system and the reduced subsystem have been estimated as the order of ε.

• *Chapter 4* has focused on a class of coupled ODE-PDE systems with multiple time scales. In the first case, the perturbation parameter has been introduced to the dynamics of the PDE system. The stability of the two subsystems has implied the full system's stability, which has been consistent with the results for linear finite dimensional systems. On the other hand, it was not valid for the case where the perturbation parameter has been introduced to the dynamics of the ODE system. A counterexample has been used to show that the full system diverges even though both subsystems converge. An additional condition has been given to ensure the full system's stability. The Tikhonov approximation has been established for these two cases.

• *Chapter 5* has presented the boundary control design for different applications. A boundary control synthesis has been studied based on the singular perturbation method. More precisely, the boundary condition for the reduced subsystem is firstly chosen as zero, which makes the slow dynamics converge to the equilibrium in finite time T. The boundary condition for the full system is chosen so as to guarantee stability. Then the full system reaches a neighbourhood of the origin of size proportional to the perturbation parameter at time T. The main result in Chapter 2 has been applied to a linearized Saint-Venant–Exner equation. The applications to gas transport systems modeled by linerized Euler equation and by coupled ODE-PDE system have illustrated the main results in Chapters 3 and 4 respectively.

6.2 Perspectives

• *Robust control of hyperbolic systems with multiple time scales.*
The presence of many uncertainties (disturbances, noises etc.) is often found in realistic studies of physical and engineering problems. For practical applications, the control systems should be robust when uncertain behaviours occur. The objective of this kind of control is to compensate the undesirable perturbations. It allows us to provide a threshold where the system becomes unstable. Robust stabilization for finite dimensional systems has been studied in many research works, see e.g. [Pan and Basar, 1993], [Belmiloudi, 2008]. In [Christofides and Daoutidis, 1997], the authors proposed a robust output tracking control for a two time scales nonlinear system. In [Prieur et al., 2008], a sufficient condition was stated for the stability of hyperbolic system of conservation laws perturbed by small nonhomogeneous terms. It will be interesting to consider the following singu-

larly perturbed hyperbolic system with disturbance:

$$\begin{aligned} y_t(x,t) + \Lambda_1(\varepsilon)y_x(x,t) &= a(\varepsilon)y(x,t) + b(\varepsilon)z(x,t) + d_1(t,\varepsilon), \\ \varepsilon z_t(x,t) + \Lambda_2(\varepsilon)z_x(x,t) &= c(\varepsilon)y(x,t) + d(\varepsilon)z(x,t) + d_2(t,\varepsilon), \end{aligned}$$

with the boundary condition

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = K(\varepsilon) \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix} + d_3(t,\varepsilon), \quad t \in [0,+\infty),$$

where $d_1(t,\varepsilon)$, $d_2(t,\varepsilon)$, $d_3(t,\varepsilon)$ are disturbances.

• *Boundary observers for linear hyperbolic systems with two time scales.*
Boundary observers are useful for distributed parameter systems since the measurements for such systems are not always available. Design of boundary observer for hyperbolic systems has been reported in some research papers. For example, the boundary stabilization and state estimation for a 2×2 linear hyperbolic system were studied by backstepping approach in [Vazquez et al., 2011]. In [Castillo et al., 2013], sufficient conditions for exponential convergence of the boundary observer for first order linear hyperbolic systems were established by Lyapunov method. It will be a challenge to study the convergence of the boundary observer for hyperbolic system based on singular perturbation approach.

• *Application to physical networks.*
Singular perturbed systems are used to model physical systems since the presence of small perturbation parameter. In this thesis, it has been considered two applications. The water shallow problems and gas flow transport systems. Another potential application is the telegrapher equations governed by two linear hyperbolic system of balance laws [Heaviside, 1892], [Bastin and Coron, 2015]:

$$\begin{aligned} (L_e I)_t + V_x + ReI &= 0, \\ (C_e V)_t + I_x + GeV &= 0, \end{aligned}$$

where $I(x,t)$ is the current intensity, $V(x,t)$ is the voltage, L_e, C_e, R_e are the self-inductance per unit length, capacitance per unit length, resistance of the two conductors per unit length respectively. G_e is the admittance per unit length of the dielectric material separating the conductors. The inductance L_e makes the electrons have inertia. Large inductance makes the wave move slowly. It is difficult to increase or decrease the current flow at any given point. So in practically, the value of the inductance L_e is small and can play the role of a perturbation parameter.

• *Stability analysis for a coupled singular perturbed ODE-PDE system in*

frequency domain.
In Section 4.2, it has been considered PDE coupled with fast ODE. The classical stability condition for finite dimensional systems does not apply to such systems. It means that the stability of both subsystems is not enough to guarantee the stability of the full system. An additional condition of the stability for such coupled ODE-PDE systems was obtained based on a Lyapunov function for the error system (error between the full system and the reduced subsystem). It would be nice to work on the general stability condition of such systems. For example considering the transfer function for the full system (see. e.g. [Bastin et al., 2015]).

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