Multi-modes switching problem, backward stochastic differential equations and partial differential equations

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Problèmes de Switching Optimal, Equation Différentielles Stochastiques Rétrogrades et Equations Différentielles Partielles

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Abstract

There are three main results in this thesis. The first is existence and uniqueness of the solution in viscosity sense for a system of nonlinear m variational integral-partial differential equations with inter-connected obstacles. From the probabilistic point of view, this system is related to optimal stochastic switching problem when the noise is driven by a Lévy process. As a by-product we obtain that the value function of the switching problem is continuous and unique solution of its associated Hamilton-Jacobi-Bellman system of equations. Next, we study a general class of min-max and max-min nonlinear second-order integral-partial variational inequalities with interconnected bilateral obstacles, related to a multiple modes zero-sum switching game with jumps. Using Perron's method and by the help of systems of penalized unilateral reflected backward SDEs with jumps, we construct a continuous with polynomial growth viscosity solution, and a comparison result yields the uniqueness of the solution. At last, we deal with the solutions of systems of PDEs with bilateral inter-connected obstacles of min-max and max-min types in the Brownian framework. These systems arise naturally in stochastic switching zero-sum game problems. We show that when the switching costs of one side are smooth, the solutions of the min-max and max-min systems coincide. Furthermore, this solution is identified as the value function of the zero-sum switching game.

Résumé

Cette thèse est composée de trois parties. Dans la première nous montrons l'existence et l'unicité de la solution continue et à croissance polynomiale, au sens viscosité, du système non linéaire de m équations variationnelles de type intégro-différentiel à obstacles unilatéraux interconnectés. Ce système est lié au problème du switching optimal stochastique lorsque le bruit est dirigé par un processus de Lévy. Un cas particulier du système correspond en effet à l'équation d'Hamilton-Jacobi-Bellman associé au problème du switching et la solution de ce système n'est rien d'autre que la fonction valeur du problème. Ensuite, nous étudions un système d'équations intégro-différentielles à obstacles bilatéraux interconnectés. Nous montrons l'existence et l'unicité des solutions continus à croissance polynomiale, au sens viscosité, des systèmes min-max et max-min. La démarche conjugue les systèmes d'EDSR réfléchies ainsi que la méthode de Perron. Dans la dernière partie nous montrons l'égalité des solutions des systèmes max-min et min-max d'EDP lorsque le bruit est uniquement de type diffusion. Nous montrons que si les coûts de switching sont assez réguliers alors ces solutions coincident. De plus elles sont caractérisées comme fonction valeur du jeu de switching de somme nulle.

Contents

Chapter 1

Introduction

1.1 General Results on Backward Stochastic Differential Equations

Let (Ω, F, P) be a probability space on which is defined a d-dimensional Brownian motion $B := (B_t)_{t\leq T}$. Let us denote by $(F_t^B)_{t\leq T}$ the natural filtration of B and $(F_t)_{t\leq T}$ its completion with the P-null sets of F. Define the following spaces:

- $\mathcal{P}_n$ the set of F_t-progressively measurable, R^n-valued processes on $\Omega \times [0,T]$;
- $L_n^2(F_t)$=$\{\eta : F_t - \text{measurable } R^n - \text{valued random variable s.t. } E[|\eta|^2] < \infty\}$;
- $S_n^2(0,T)$=$\{\varphi : \mathcal{P}_n - \text{measurable with continuous paths, s.t. } E[\sup_{s\leq T}|\varphi|^2] < \infty\}$;
- $H_n^2(0,T) = \{Z : \mathcal{P}_n - \text{measurable s.t. } E[\int_0^T |Z_s|^2 ds] < \infty\}$;
- $H_n^1(0,T) = \{Z : \mathcal{P}_n - \text{measurable s.t. } E[\int_0^T |Z_s|^2 ds]^{\frac{1}{2}} < \infty\}$;

Definition 1.1.1. *Let $\xi \in L_m^2(F_T)$ be an R^m-valued terminal condition and $g(t,\omega,y,z)$: $[0,T]\times\Omega\times R^m\times R^{m\times d} \to R^m$, $\mathcal{P}_m\otimes\mathcal{B}(R^m\times R^{m\times d})$-measurable. A solution for the m-dimensional BSDE associated with parameters (g,ξ) is a pair of progressively measurable processes $(Y,Z) := (Y_t, Z_t)_{t\leq T}$ with values in $R^m \times R^{m\times d}$ such that*

$$\begin{cases} Y \in \mathcal{S}_m^2, \quad Z \in H_{m\times d}^2; \\ Y_t = \xi + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s, \quad \forall 0 \leq t \leq T. \end{cases} \tag{1.1.1}$$

The differential from of this equation is

$$-dY_t = g(t,Y_t,Z_t)dt - Z_t dB_t, \quad Y_T = \xi. \tag{1.1.2}$$

Hereafter g is called the coefficient and ξ the terminal value of the BSDE. The BSDE (1.1.1) has a unique solution under the standard assumptions as follows:

(H1) $\begin{cases} (i) \quad (g(t,0,0))_{t\leq T} \in H_m^2 \\ (ii) \quad g \text{ is uniformly Lipschitz with respect to } (y,z), \textit{i.e.}, \text{there exists a constant } C \geq 0 \text{ such that} \\ \text{for any } (y,y',z,z') : \\ \qquad |g(\omega,t,y,z) - g(\omega,t,y',z')| \leq C(|y-y'| + |z-z'|), \quad dt\otimes dP - a.e. \end{cases}$

Theorem 1.1.1. *(Pardoux and Peng [49]) Under the assumption (**H1**), there exists a unique solution (Y,Z) of the BSDE with parameters (g,ξ).*

Using Itô's formula we obtain the following a priori estimate.

Proposition 1.1.1. *Let (Y,Z) be a solution of BSDE (1.1.1). Then there exists a constant $c > 0$ such that*

$$E[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_t|^2 dt] \leq cE[|\xi|^2 + \int_0^T |g(t,0,0)|^2 dt]. \tag{1.1.3}$$

When the coefficient is linear, we can get explicitly the component Y of the solution.

Proposition 1.1.2. *(El Karoui, Peng, and Quenez [22]) Let (β, μ) be a bounded (R, R^d)-valued progressively measurable process, ϕ be an element of $H_1^2(0,T)$ and $\xi \in L_1^2(F_T)$. Consider the following linear BSDE:*

$$dY_t = (\phi_t + Y_t\beta_t + Z_t\mu_t)dt - Z_t dB_t; \quad Y_T = \xi. \tag{1.1.4}$$

(i) Equation (1.1.4) has a unique solution $(Y,Z) \in S_1^2(0,T) \times H_d^2(0,T)$, and Y is given explicitly by

$$Y_t = E[\xi\Gamma_{t,T} + \int_t^T \Gamma_{t,s}\phi_s | F_t],$$

where $(\Gamma_{t,s})_{s\geq t}$ is the adjoint process defined by the forward linear SDE

$$\forall s \in [t,T], \qquad d\Gamma_{t,s} = \Gamma_{t,s}(\beta_s ds + \mu_s dB_s) \text{ and } \Gamma_{t,t} = 1.$$

(ii) If ξ and ϕ are both non-negative, then the process $(Y_t)_{t\leq T}$ is non-negative.

In one-dimensional case, i.e., when $m = 1$, we have a comparison result between the Y's as soon as we can compare the associated coefficient and terminal values. More precisely,

Theorem 1.1.2. *(El Karoui, Peng, and Quenez [22]) Let us consider the solutions (Y,Z) and (Y',Z') of two BSDEs associated with parameters (g,ξ) and (g',ξ'). We assume that g satisfies (**H1**), and $(g'(s,Y'_s,Z'_s))_{s\leq T}$ is element of H_1^2. If $\xi \leq \xi'$ $P-a.s.$ and $g(t,Y'_t,Z'_t) \leq g'(t,Y'_t,Z'_t)$, $dt\otimes dP - a.e.$, then,*

$$Y_t \leq Y'_t, \quad \forall t \in [0,T] \quad P-a.s..$$

When the coefficients of the BSDE are deterministic functions of a diffusion process, the solution (Y,Z) is also a deterministic function of the same process. If, in addition, a certain regularity on the coefficients is introduced, it is possible to relate these functions with the pair (solution, gradient) of some semi-linear PDE. The basic framework is the following: the randomness of the coefficient and the terminal value of a Markvian BSDE comes from a diffusion process $(X_s^{t,x})_{s\in[t,T]}$, which is the strong solution of a standard SDE:

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dB_s, & t \leq s \leq T \\ X_s^{t,x} = x, \qquad s \in [0,t]. \end{cases} \tag{1.1.5}$$

For any given $(t,x) \in [0,T]\times R^d$, we will denote by $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ the solution of the following BSDE:

$$\begin{cases} -dY_s = g(s,X_s^{t,x},Y_s,Z_s)ds - Z_s dB_s, \qquad s \leq T; \\ Y_T = \Psi(X_T^{t,x}). \end{cases} \tag{1.1.6}$$

In order to have good estimates of the solution, we assume that the following condition is satisfied.
(**H2**):
(i) b and σ are uniformly Lipschitz continuous with respect to x;
(ii) there exists a constant C s.t. for any (s,x),

$$|\sigma(s,x)| + |b(s,x)| \leq c(1+|x|);$$

(iii) The function $g : [0,T]\times R^d \times R^m \times R^{m\times d} \to R^m$ is uniformly Lipschitz in (y,z) with Lipschitz constant C, i.e.,

$$|g(s,x,y_1,z_1) - g(s,x,y_2,z_2)| \leq \quad C(|y_1-y_2| + |z_1-z_2|);$$

(iv) There exists two constants c and $p \geq 0$ such that,

$$|g(s,x,y,z)| + |\Psi(x)| \leq c(1+|x|^p);$$

(v) The mapping $x \to \quad (g(t,x,0,0),\Psi(x))$ is continuous.

Theorem 1.1.3. *(Dellacherie and Meyer [15]) Under (**H2**), there exist two measurable deterministic functions $u(t,x)$ and $d(t,x)$ such that the solution $(Y^{t,x}, Z^{t,x})$ of BSDE (1.1.6) is given by*

$$\forall t \leq s \leq T, Y_s^{t,x} = u(s,X_s^{t,x}) \text{ and } Z_s^{t,x} = d(s,X_s^{t,x})\sigma(s,X_s^{t,x}), \quad ds\otimes dP - a.e..$$

Let us now consider the following system of semilinear parabolic PDEs, where u is a R^m-valued function, defined on $[0,T]\times R^d$ satisfying

$$\begin{cases} \frac{\partial u}{\partial t} + Lu(t,x) + g(t,x,u(t,x),D_\sigma u(t,x)) = 0 \ \ \forall (t,x)\in[0,T]\times R^d, \\ u(T,x) = \Psi(x),\ \forall x \in R^d. \end{cases} \tag{1.1.7}$$

L is a second-order differential operator given by

$$L := \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{i,j}\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x^i}, \quad D_\sigma u := Du\sigma. \tag{1.1.8}$$

Under the assumptions (**H2**) on the coefficients, we can only consider the solution of PDE (1.1.7) in viscosity sense. Moreover, we need to make the following restriction: for $1 \le i \le m$, the i-th coordinate of g, denoted by g_i, depends only ont the i-th row of the matrix z. Therefore, the equation (1.1.7) can be written as

$$\begin{cases} \frac{\partial u_i}{\partial t} + Lu_i(t,x) + g_i(t,x,u(t,x),Du_i\sigma(t,x)) = 0, \quad i = 1,\cdots m, \\ u(T,x) = \Psi(x),\ \forall x \in R^d. \end{cases}$$

Now let us introduce the definition of a viscosity solution:

Definition 1.1.2. *Assume $u \in C[0,T]\times R^d; R^m)$ and $u(T,x) = \Psi(x)$, for all $x \in R^d$. u is called a viscosity subsolution (resp. supersolution) of PDE (1.1.7) if , for any $1 \le i \le m$, $\phi \in C^{1,2}([0,T]\times R^d)$ and $(t,x)\in[0,T]\times R^d$ such that $\phi(t,x) = u(t,x)$ and $u(t,x)$ is a local maximum (resp. minimum) of $u_i - \phi$,*

$$\frac{\partial \phi}{\partial t} + L\phi(t,x) + g_i(t,x,u(t,x),(D\phi\sigma)(t,x)) \le 0 \ (resp. \ge 0).$$

Moreover, $u \in C([0,T]\times R^d; R^m)$ is called a viscosity solution of PDE (1.1.7) if it is both a viscosity subsolution and a viscosity supersolution.

We now give the probabilistic interpretation of the viscosity solution of PDE (1.1.7) using $(Y_s^{t,x}, Z_s^{t,x})$ solution of the BSDE (1.1.5):

Theorem 1.1.4. *(Pardoux and Peng [50]) Under Assumptions (**H2**), $u := Y_t^{t,x}$ is a viscosity solution of PDE (1.1.7) and there exist two constants C and p, such that*

$$|u(t,x)| \le C(1+|x|^p), \qquad \forall (t,x)\in[0,T]\times R^d.$$

1.2 Systems of Integro-PDEs with Interconnected Obstacles and Multi-Modes Switching Problem Driven by Lévy Process

Let us introduce the following spaces:

$\mathcal{S}^2 := \{\{\varphi_t, 0\le t\le T\}$ is an $I\!R$-valued, $\mathcal{F}_t$-adapted RCLL (right continuous with left limits) process s.t. $\mathbb{E}(\sup_{0\le t\le T}|\varphi_t|^2) < \infty\}$; $\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of non-decreasing continuous processes null at $t=0$;
$H^2 := \{\{\varphi_t, 0\le t\le T\}$ is an $I\!R$-valued, $\mathcal{F}_t$-progressively measurable process s.t. $\mathbb{E}(\int_0^T|\varphi_t|^2) < \infty\}$;
$\ell^2 := \{x = (x_n)_{n\ge1}$ is an $I\!R$-valued sequence s.t. $\|x\|^2 := \sum_{i=1}^{\infty} x_i^2 < \infty\}$;
$\mathcal{H}^2(\ell^2) := \{\varphi = (\varphi_t)_{t\le T} = ((\varphi_t^n)_{n\ge1})_{t\le T}$ s.t. $\forall n \ge 1$, φ^n is $\mathcal{P}$-predictable process and

$$\mathbb{E}(\int_0^T \|\varphi_t\|^2 dt) = \sum_{i=1}^{\infty}\mathbb{E}(\int_0^T |\varphi_t^i|^2 dt) < \infty\};$$

$\mathcal{L}^2 := \{\varphi$ is an $I\!R$-valued, $\mathcal{F}_T$-random variable such that $\mathbb{E}|\varphi|^2 < \infty\}$. □

1.2.1 Recalling some results for RBSDE

We first recall the Lévy-Khintchine formula of a Lévy process $(L_t)_{t\leq T}$ whose characteristic exponent is Ψ, *i.e.*,

$$\forall\theta\in I\!R,\qquad E(e^{i\theta L_t})=e^{t\Psi(\theta)}$$

with

$$\begin{aligned}\Psi(\theta)&=ia\theta-\frac{1}{2}\varpi^2\theta^2+\int_{I\!R}(e^{i\theta x}-1-i\theta x\mathbb{1}_{(|x|<1)})\Pi(dx)\\&=ia\theta-\frac{1}{2}\varpi^2\theta^2+\int_{|x|>1}(e^{i\theta x}-1)\Pi(dx)+\int_{0<|x|<1}(e^{i\theta x}-1-i\theta x)\Pi(dx)\end{aligned}$$

where $a\in I\!R$, $\varpi\geq 0$ and Π is a measure concentrated on $I\!R$, setting $\Pi(0)=0$, so that the domain of integration is the whole space $I\!R$ and not only $E:=I\!R\backslash\{0\}$, called the Lévy measure of X, satisfying:

(i) $\int_{I\!R}(1\wedge x^2)\Pi(dx)<\infty$;
(ii) $\exists\epsilon>0,\lambda>0\ s.t.\int_{(-\epsilon,\epsilon)^c}e^{\lambda|x|}\Pi(dx)<+\infty$.

Those conditions (i)-(ii) imply that the Lévy process $(L_t)_{t\leq T}$ have moments of all orders. On the other hand we have,

$$\int_{-\infty}^{+\infty}|x|^i\Pi(dx)<\infty,\quad\forall i\geq 2. \tag{1.2.1}$$

Following Nualart-Schoutens (2000) we define, for every $i\geq 1$, the so-called power-jump processes $L^{(i)}$ and their compensated version $Y^{(i)}$, also called Teugels martingales, as follows:

$$\begin{aligned}&L_t^{(1)}=L_t\\&L_t^{(i)}=\textstyle\sum_{s\leq t}(\Delta L_s)^i, t\leq T\text{ and }i\geq 2\\&Y_t^{(i)}=L_t^{(i)}-t\mathbb{E}(L_1^{(i)}).\end{aligned}$$

Note that for any $t\leq T$, $\mathbb{E}(L_t^{(i)})=t\int_{I\!R}x^i\Pi(dx)$ is finite for any $i\geq 2$ ([46], pp.29).

An orthonormalization procedure can be applied to the martingales $Y^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $(H^{(i)})_{i=1}^{i=\infty}$ such that each $H^{(i)}$ is a linear combination of the $(Y^{(j)})_{j=1,i}$, i.e.,

$$H^{(i)}=c_{i,i}Y^{(i)}+...+c_{i,1}Y^{(1)}.$$

It has been shown in Nualart and Schoutens (2000) that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1,x,x^2,...$ with respect to the measure $\nu(dx)=x^2\Pi(dx)+\varpi^2\delta_0(dx)$, where δ_0 is the Diracmeasure in 0. Specifically the polynomials $(q_i)_{i\geq 0}$ defined by

$$q_{i-1}(x)=c_{i,i}x^{i-1}+c_{i,i-1}x^{i-2}+...+c_{i,1},i\geq 1$$

satisfy

$$\int_{I\!R}q_n(x)q_m(x)\nu(dx)=\delta_{nm},\ ,\forall n,m\geq 0.$$

Next let us set

$$\begin{aligned}p_i(x)&=xq_{i-1}(x)=c_{i,i}x^i+c_{i,i-1}x^{i-1}+...+c_{i,1}x\\\tilde{p}_i(x)&=x(q_{i-1}(x)-q_{i-1}(0))=c_{i,i}x^i+c_{i,i-1}x^{i-1}+...+c_{i,2}x^2.\end{aligned}$$

Then for any $i\geq 1$ and $t\leq T$ we have:

$$\begin{aligned}H_t^{(i)}&=\textstyle\sum_{0<s\leq t}\{c_{i,i}(\Delta L_s)^i+...+c_{i,2}(\Delta L_s)^2\}+c_{i,1}L_t-t\mathbb{E}[c_{i,i}(L_1)^{(i)}+\\&\quad ...+c_{i,2}(L_1)^{(2)}]-tc_{i,1}\mathbb{E}(L_1)\\&=q_{i-1}(0)L_t+\textstyle\sum_{0<s\leq t}\tilde{p}_i(\Delta L_s)-t\mathbb{E}[\sum_{0<s\leq 1}\tilde{p}_i(\Delta X_s)]-tq_{i-1}(0)\mathbb{E}(L_1).\end{aligned}$$

As a consequence, $\Delta H_t^{(i)}=p_i(\Delta L_t)$ for each $i\geq 1$. In the particular case of $i=1$, we obtain

$$H_t^{(1)}=c_{1,1}(L_t-t\mathbb{E}(L_1))$$

where

$$c_{1,1} = [\int_{I\!R} x^2 \Pi(dx) + \varpi^2]^{-\frac{1}{2}} \text{ and } \mathbb{E}[L_1] = a + \int_{|x|\geq 1} x\Pi(dx).$$

Finally note that for any $i, j \geq 1$ the predictable quadratic variation process is $< H^{(i)}, H^{(j)} >_t = \delta_{ij} t, \forall t \leq T$. □

The main result in [47] is the following representation property.

Theorem 1.2.1. *([47], Remark 2). Let ζ be a random variable of $\mathcal{L}^2$, then there exists a process $Z = (Z^i)_{i\geq 1}$ that belongs to $\mathcal{H}^2(\ell^2)$ such that:*

$$\zeta = \mathbb{E}(\zeta) + \sum_{i\geq 1} \int_0^T Z_s^i dH_s^{(i)}. \quad \square$$

As a consequence of Theorem 2.1.1, as in the framework of Brownian noise only, one can study standard BSDEs or reflected ones. The result below related to existence and uniqueness of a solution for a reflected BSDE driven by a Lévy process, is proved in [56]. Actually let us introduce a triplet (f, ξ, S) that satisfies:

Assumptions (A1)
(i) ξ a random variable of $\mathcal{L}^2$ which stands for the terminal value ;
(ii) f: $[0,T] \times \Omega \times I\!R \times \ell^2 \longrightarrow I\!R$ is a function such that the process $(f(t,0,0))_{t\leq T}$ belongs to H^2 and there exists a constant $\kappa > 0$ verifying

$$|f(t,y,z) - f(t,y',z')| \leq \kappa(|y-y'| + \|z-z'\|_{\ell^2}), \text{ for every } t, y, y', z \text{ and } z'.$$

(iii) $S := (S_t)_{0\leq t\leq T}$ is a process of $\mathcal{S}^2$ such that $S_T \leq \xi$, $P-a.s.$, and whose jumps are inaccessible stopping times. This in particular implies that for any $t \leq T$, $S_t^p = S_{t-}$, where S^p is the predictable projection of S (see e.g. [14], pp.58 for more details).

In [56], the authors have proved the following result related to existence and uniqueness of the solution of a reflected BSDE whose noise is driven by a Lévy process.

Theorem 1.2.2. *Assume that the triplet (f, ξ, S) satisfies Assumptions $(A1)$, then there exists a unique triplet of processes $(Y, U, K) := ((Y_t, U_t, K_t))_{t\leq T}$ with values in $I\!R \times \ell^2 \times I\!R^+$ such that:*

$$\begin{cases} (Y,U,K) \in \mathcal{S}^2 \times \mathcal{H}(\ell^2) \times \mathcal{A}^2; \\ Y_t = \xi + \int_t^T f(s,Y_s,U_s)ds + K_T - K_t - \sum_{i=1}^{\infty} \int_t^T U_s^i dH_s^{(i)}, \ \forall t \leq T; \\ Y_t \geq S_t, \quad \forall\ 0 \leq t \leq T, \quad \int_0^T (Y_t - S_t) dK_t = 0, \ P-a.s. \end{cases} \tag{1.2.2}$$

The triplet (Y, U, K) is called the solution of the reflected BSDE associated with (f, ξ, S). □

Let us now introduce the following assumption on the process V.

Assumptions (A2): The process $V = (V_t^i)_{i\geq 1}$ verifies:

$$\sum_{i=1}^{\infty} V_t^i p_i(\triangle L_t) > -1 \ \ dP \otimes dt - a.e$$

and there exists a constant C such that:

$$\sum_{i=1}^{\infty} |V_t^i|^2 \leq C, \ dP \otimes dt - a.e. \quad \square$$

We will give now a comparison theorem for RBSDE driven by a Lévy process.

Theorem 1.2.3. *For $i = 1, 2$, let (ξ_i, S^i, f_i) be a triple which satisfies the same Assumptions as in Theorem 1.1 and let $(Y_t^i, K_t^i, U_t^i)_{t\leq T}$ be the solution of the RBSDE associated with (ξ_i, S^i, f_i). Assume that:*

(i) $P-a.s.$, $\xi_1 \geq \xi_2$ and $\forall t \in [0,T]$, $\ f_1(t,y,u) \geq f_2(t,y,u)$, $S_t^1 \geq S_t^2$;

(ii) For any $U^1, U^2 \in \mathcal{H}^2(l^2)$, there exists $(V_j^{U^1,U^2})_{j\geq 1}$ which depends on U^1 and U^2, satisfies (A2) and such that f_1 verifies:

$$f_1(t, Y_t^2, U_t^1) - f_1(t, Y_t^2, U_t^2) \geq \langle V^{U^1,U^2}, (U^1 - U^2) \rangle_t^p, \ dP \otimes dt - a.e..$$

Then P-a.s. for any $t \leq T$, $Y_t^1 \geq Y_t^2$.

Next we are going to make a connection between reflected BSDEs and their associated PDEs with obstacle. Consider the following SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_{r-}^{t,x})dL_r, \qquad \forall t \leq s \leq T, \tag{1.2.3}$$

and $X_s^{t,x} = x$ if $s \leq t$, where we assume that the functions b and σ are jointly continuous, Lipschitz continuous *w.r.t.* x uniformly in t, i.e., there exists a constant $C \geq 0$ such that for any $t \in [0,T]$, x,$x' \in I\!R$, it holds,

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x - x'|. \tag{1.2.4}$$

σ is uniformly bounded, b is of linear growth, i.e., there exists a constant $C > 0$, such that for all $(t,x) \in [0,T] \times R$,

$$|b(t,x)| \leq C(1 + |x|), \quad |\sigma(t,x)| \leq C. \tag{1.2.5}$$

Under the above conditions, the process $X^{t,x}$ exists and is unique (see e.g. [42]), and satisfies:

$$\forall p \geq 1, \ E[\sup_{s \leq T} |X_s^{t,x}|^p] \leq C(1 + |x|^p). \tag{1.2.6}$$

Next let us consider the following functions:

$$h : x \in I\!R \mapsto h(x) \in I\!R;$$

$$f : (t,x,y,u) \in [0,T] \times I\!R \times I\!R \times l^2 \mapsto f(t,x,y,u) \in I\!R;$$

$$\Psi : (t,x) \in [0,T] \times I\!R \mapsto \Psi \in I\!R,$$

which satisfy the following assumptions:

Assumptions (A3):
(i) h, Ψ and $f(t,x,0,0)$ are jointly continuous and of polynomial growth, which we denote as h, Ψ and $f(t,x,0,0) \in \Pi_p$, i.e., there exist positive constants C and p such that: $\forall (t,x) \in [0,T] \times I\!R$,

$$|h(x)| + |\Psi(t,x)| + |f(t,x,0,0)| \leq C(1 + |x|^p).$$

(ii) the mapping $(y,z) \mapsto f(t,x,y,z)$ is Lipschitz continuous uniformly in (t,x) ;
(iii) For any $x \in I\!R$, $h(x) \geq \Psi(T,x)$.
(iv) The generator satisfies,

$$f(t,x,y,u) = h(t,x,y,\sum_{i \geq 1} \theta_t^i u_i), \ \forall (t,x,y,u) \in [0,T] \times I\!R \times I\!R \times \ell^2$$

where the mapping $\eta \in I\!R \longmapsto h(t,x,y,\eta)$ is non decreasing, and there exists a constant $C > 0$, such that $\forall t \in [0,T]$, $z, z' \in R$, $x, y \in R$,

$$|h(t,x,y,z) - h(t,x,y,z')| \leq C|z - z'|.$$

Further more $(\theta_t^i)_{i \geq 1}$ is uniformly bounded i.e.

$$\sum_{i \geq 1} \sup_{t \leq T} |\theta_t^i|^2 < \infty \ \ P - a.s.,$$

and moreover $\sum_{i \geq 1} \theta_t^i p_i(\Delta L_t) > 0, \ dt \otimes dP - a.e..$ □

Noting that the assumption (A3)(iv) satisfies the assumption (ii) in Theorem 1.7, which allows us to use comparison theorem in the proof of Theorem 1.8.

In the case of Markovian setting, i.e. when randomness stems from an exogenous process $(X_s^{t,x})_{s \leq T}$, Yong Ren and Mohamed El Otmani have shown in [56] the relationship between RBSDE and IPDE. Let $(t,x) \in [0,T] \times I\!R^k$ be fixed and let us consider the following reflected BSDE:

$$\begin{cases} (Y^{t,x}, U^{t,x}, K^{t,x}) \in \mathcal{S}^2 \times \mathcal{H}(\ell^2) \times \mathcal{A}^2; \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + K_T^{t,x} - K_s^{t,x} - \sum_{i=1}^{\infty} \int_s^T Z_r^{t,x,i} dH_r^{(i)}, \ s \leq T; \\ \forall s \leq T, \ \ Y_s^{t,x} \geq \Psi(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{t,x} - \Psi(s, X_s^{t,x}))dK_s^{t,x} = 0, \ P - a.s.. \end{cases} \tag{1.2.7}$$

There exists a continuous deterministic function $u(t,x)$ which satisfies

$$\forall (t,x) \in [0,T] \times I\!R^k, \forall s \in [t,T],\ Y_s^{t,x} = u(s, X_s^{t,x}). \tag{1.2.8}$$

Consider now the following IPDE:

$$\begin{cases} \min\Big\{u(t,x) - \Psi(t,x); -\partial_t u(t,x) - \mathcal{L}u(t,x) - f(t,x,u(t,x),\Phi(u)(t,x))\Big\} = 0 \\ u(T,x) = h(x) \end{cases} \tag{1.2.9}$$

where $\mathcal{L}$ is the generator which has the following expression:

$$\begin{aligned} \mathcal{L}u(t,x) = (E[L_1]\sigma(t,x) + b(t,x))\partial_x u(t,x) + \tfrac{1}{2}\sigma(t,x)^2\varpi^2\partial^2_{xx}u(t,x) \\ + \textstyle\int_{I\!R}[u(t,x+\sigma(t,x)y) - u(t,x) - \partial_x u(t,x)\sigma(t,x)y]\Pi(dy) \end{aligned} \tag{1.2.10}$$

and

$$\begin{aligned} \Phi(u)(t,x) \quad &= \Big(\tfrac{1}{c_{1,1}}\partial_x u(t,x)\sigma(t,x)\mathbb{1}_{k=1} \\ &+ \textstyle\int_{I\!R}(u(t,x+\sigma(t,x)y) - u(t,x) - \partial_x u(t,x)y)p_k(y)\Pi(dy)\Big)_{k\geq 1}. \end{aligned} \tag{1.2.11}$$

Theorem 1.2.4. *Under Assumption (A3), the function u defined in* (1.2.8) *is continuous and is a viscosity solution of* (1.2.9). □

1.2.2 Motivation

In this paper, we study the existence and uniqueness of a solution to the system of integro-partial differential equations (IPDEs in short) of the form: $\forall i = 1, \cdots, m$,

$$\begin{cases} min\{u_i(t,x) - \max\limits_{j\neq i}(u_i(t,x) - g_{ij}(t,x)); \\ \qquad -\frac{\partial u_i}{\partial t}(t,x) - \mathcal{L}u_i(t,x) - f_i(t,x,u_1,u_2,\cdots,u_m)\} = 0 \\ u_i(T,x) = h_i(x) \end{cases} \tag{1.2.12}$$

where $\mathcal{L}$ is a generator defined in (1.2.10) and associated with a stochastic differential equation whose noise is driven by a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\leq T}, P)$ and then $\mathcal{L}$ is a non local operator.

This system is related to a stochastic optimal switching problem since a particular case is actually its associated Hamilton-Jacobi-Bellman system.

The multi-modes switching problem of interest is related to investment of a capital in the most profitable economy in the globalization. More precisely, consider an agent that aims at investing a capital in one of several economies denoted by $\epsilon_1, \cdots, \epsilon_m$. His objective is to obtain the best return for the investment. The capital is invested in the economy ϵ_i up to the time when the agent makes the decision to switch it from ϵ_i to ϵ_j $(i \neq j)$ because there is no longer enough profitability in ϵ_i. Moving the capital from ϵ_i to ϵ_j incures expenditures which amount to g_{ij}. Therefore, the agent should deal with two main problem: what are the optimal successive times to move the capital, and when the decision to switch from current economy is made, in which new economy will the capital be invested. More precisely, let $(a_s)_{s\in[0,T]}$ be the following pure jump process:

$$a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1}\mathbb{1}_{]\theta_{j-1},\theta_j]}(s),\ \forall s \leq T,$$

where $\{\theta_j\}_{j\geq 0}$ is an increasing sequence of stopping times with values in $[0,T]$ and $(\alpha_j)_{j\geq 0}$ are random variable with values in $A := \{1,\ldots,m\}$ (the set of modes to which the controller can switch) such for any $j \geq 0$, α_j is $\mathcal{F}_{\theta_j}-$measurable. The pair $\Upsilon = ((\theta_j)_{j\geq 0}, (\alpha_j)_{j\geq 0})$ is called a strategy of switching and when it satisfies $P[\theta_n < T, \forall n \geq 0] = 0$ it is said admissible. Finally we denote by $\mathcal{A}_t^i$ the set of admissible strategies such that $\alpha_0 = i$ and $\theta_0 = t$.

Assume next that for any $i = 1,\ldots,m$, $f_i(t,x,(y_i)_{i=1,\ldots,m}) = f_i(t,x)$, i.e., f_i does not depend on $(y_i)_{i=1,m}$. Let Υ be an admissible strategy of $\mathcal{A}_t^i$ with which one associates a payoff given by:

$$J^a(t,x) = J(\Upsilon)(t,x) := \mathbb{E}[\int_t^T f_{a(s)}(s, X_s^{t,x})ds - \sum_{j\geq 1} g_{\alpha_{j-1},\alpha_j}(\theta_j, X_{\theta_j})\mathbb{1}_{\{\theta_j<T\}} + h_{a_T}(X_T^{t,x})]$$

where $f_{a(s)}(s, X_s^{t,x}) = \sum_{i\in A} f_i(s, X_s^{t,x}) 1_{[a(s)=i]}$, $s \in [t,T]$, (resp. $h_{a_T}(X_T^{t,x}) = \sum_{i\in A} h_i(X_T^{t,x}) 1_{[a_T=i]}$) is the instantaneous (resp. terminal) payoff when the strategy a (or Υ) is implemented while $g_{i\ell}$ is the switching cost function when moving from mode i to mode ℓ ($i, \ell \in A$, $i \neq \ell$). Next let us define the optimal payoff when starting from mode $i \in A$ at time t by

$$u^i(t,x) := \inf_{\Upsilon\in\mathcal{A}_t^i} J(\Upsilon)(t,x) \tag{1.2.13}$$

A similar problem has been already considered by Biswas et al. [8], however one should emphazise that in that work, the switching costs are constant and do not depend on (t,x). This latter feature makes the problem easier to handle since one can directly work with the functions u_i defined in (1.2.13).

Optimal switching problems are well documented in the literature (see e.g. [8, 13, 3, 11, 29, 32, 20, 34, 52, 60, 61, 19] etc. and the references therein), especially in connection with mathematical finance, energy market, etc.

1.2.3 Main results

The main objective and novelty of this paper is to study system (1.2.12) without the restrictions made by Biswas et al., [8] i.e., to allow the switching costs g_{ij} to depend on (t,x) and to show that (1.2.12) has a unique solution. First let us introduce the following functions f_i, h_i and g_{ij}, $i,j \in A$:

$$\begin{array}{rcll} f_i & : & [0,T]\times I\!R^k \times I\!R^m \times \ell^2 & \longrightarrow I\!R \\ & & (t,x,(y^i)_{i=1,m},u) & \longmapsto f_i(t,x,(y^i)_{i=1,m},u) \end{array}$$

$$\begin{array}{rcll} h_i \quad (\text{resp. } g_{ij}) & : & [0,T]\times I\!R^k & \longrightarrow I\!R \\ & & (t,x) & \longmapsto h_i(t,x) \text{ (resp. } g_{ij}(t,x)) \end{array}$$

which satisfy:

Assumption (A4)
(I) For any $i \in A$:

(i) the mapping $(t,x) \to f_i(t,x,\overrightarrow{y},u)$ is continuous uniformly with respect to $(\overrightarrow{y},u)$ where $\overrightarrow{y} = (y^i)_{i=1,m}$;

(ii) the mapping $(\overrightarrow{y},u) \mapsto f_i(t,x,\overrightarrow{y},u)$ is Lipschiz continuous uniformly *w.r.t.* (t,x) ;

(iii) $f_i(t,x,0,0)$ is of polynomial growth w.r.t. (t,x).

(iv) For any $U^1, U^2 \in \mathcal{H}^2(l^2)$, $X_t, Y_t \in \mathcal{S}^2$, there exists $(V_j^{U^1,U^2,i})_{j\geq 1}$ which depends on U^1 and U^2, satisfies (A2) such that :

$$f_i(t,X_t,Y_t,U_t^1) - f_i(t,X_t,Y_t,U_t^2) \geq \langle V^{U^1,U^2,i}, (U^1-U^2)\rangle_t^p,\ d\mathcal{P}\otimes dt - a.e.;$$

(v) For any $i \in A$, for any $k \neq i$, the mapping $y_k \to f_i(t,x,y_1,\cdots,y_{k-1},y_k,y_{k+1},\cdots,y_m,u)$ is non-decreasing whenever the other components $(t,x,y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m,u)$ are fixed.

(II) $\forall i,j \in A$, $g_{ii} \equiv 0$ and for $i \neq j$, $g_{jk}(t,x)$ is non-negative, continuous with polynomial growth and satisfy the following non-free loop property: $\forall (t,x) \in [0,T]\times R$ and for any sequence of indices $i_1,\cdots,i_k$ such that $i_1 = i_k$ and $card\{i_1,\cdots,i_k\} = k-1$ we have:

$$g_{i_1i_2}(t,x) + g_{i_2i_3}(t,x) + \cdots + g_{i_ki_1}(t,x) > 0,\ \forall (t,x) \in [0,T]\times I\!R^k.$$

(III) $\forall i \in A$, h_i is continuous with polynomial growth and satisfies the following coherance conditions:

$$h_i(x) \geq \max_{j\in A^{-i}} (h_j(x) - g_{ij}(T,x)), \forall x \in I\!R.$$

Our method is based on the link of (1.2.12) with systems of reflected BSDEs with inter-connected obstacles driven by a Lévy process, i.e., systems of the following form: $\forall j = 1, \dots, m$, $\forall s \leq T$,

$$\begin{cases} Y_s^{j,x,t} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in A}, (U_r^{j,x,t,i})_{i\geq 1})dr \\ - \sum_{i=1}^{\infty} \int_s^T U_r^{j,x,t,i} dH_r^{(i)} + K_T^{j,x,t} - K_s^{j,x,t},\ s \leq T \\ Y_s^{j,x,t} \geq \max_{k\neq j}\{Y_s^{k,x,t} - g_{jk}(s, X_s^{t,x})\},\ \forall s \leq T; \\ [Y_s^{j,x,t} - \max_{k\neq j}\{Y_s^{k,x,t} - g_{jk}(s, X_s^{t,x})\}]dK_s^{j,x,t} = 0. \end{cases} \tag{1.2.14}$$

Under assumption (A4) on the data $(f_i)_{i=1,\dots,m}$, $(h_i)_{i=1,\dots,m}$ and $(g_{ij})_{i,j=1,\dots,m}$ we show existence and uniqueness of $\mathcal{F}_t$-adapted processes $(Y_s^{j,x,t}, (U_s^{j,x,t,i})_{i\geq 1}, K_s^{j,x,t})_{s\leq T}$ which satisfy (1.2.14). The proof is given in two steps.

Step 1: Let us consider the following BSDEs :

$$\bar{Y}_s = \max_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \cdots, \bar{Y}_r, \bar{U}_r)dr - \sum_{i=1}^{\infty} \int_s^T \bar{U}_r^i dH_r^{(i)}$$

and

$$\underline{Y}_s = \min_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \min_{j=1,m} f_j(r, X_r^{t,x}, \underline{Y}_r, \cdots, \underline{Y}_r, \underline{U}_r)dr - \sum_{i=1}^{\infty} \int_s^T \underline{U}_r^i dH_r^{(i)}.$$

Foe $n \geq 0$ define $(Y^{j,n}, U^{j,n}, K^{j,n})$ by:

$$\begin{cases} Y^{j,n} \in \mathcal{S}^2,\ \ U^{j,n} \in \mathcal{H}^2(\ell^2),\ \ K^{j,n} \in \mathcal{A}^2 \\ Y^{j,0} = \underline{Y} \\ Y_s^{j,n} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^{1,n-1}, \cdots, Y_r^{j-1,n-1}, Y_r^{j,n}, Y_r^{j+1,n-1}, \\ \cdots, Y^{m,n-1}, U_r^{j,n})dr - \sum_{i=1}^{\infty} \int_s^T U_r^{i,j,n} dH_r^{(i)} + K_T^{j,n} - K_s^{j,n},\ s \leq T; \\ Y_s^{j,n} \geq \max_{k\in A_j}(Y_s^{k,n-1} - g_{jk}(s, X_s^{t,x})),\ \forall s \leq T; \\ \int_0^T [Y_r^{j,n} - \max_{k\in A_j}(Y_r^{k,n-1} - g_{jk}(r, X_r^{t,x}))]dK_r^j = 0 \end{cases} \tag{1.2.15}$$

For $i = 1, \cdots, m$, by induction we have: $\forall n, j, \forall s \leq T,\ \ Y_s^{j,n-1} \leq Y_s^{j,n} \leq \bar{Y}_s,\ \ P - a.s.$, and $E[\sup_{s\in[0,T]} |\bar{Y}_s|^2] < \infty$. The sequence $(Y^{j,n})_{n\geq 0}$ has a limit denoted by Y^j for $j = 1, \cdots, m$. By the monotonic limit theorem in [25], $Y^j \in \mathcal{S}^2$ and there exist $U^j \in \mathcal{H}^2(\ell^2), K^j \in \mathcal{A}^2$, such that

$$\begin{cases} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{Y}_r, U_r^j)dr - \sum_{i=1}^{\infty} \int_s^T U_r^{i,j} dH_r^{(i)} + K_T^j - K_s^j,\ s \leq T; \\ Y_s^j \geq \max_{k\in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})),\ s \leq T, \end{cases} \tag{1.2.16}$$

where for any $j \in A$, U^j is the weak limit of $(U^{j,n})_{n\geq 1}$ in $\mathcal{H}^2(\ell^2)$ and for any stopping time τ, K_τ^j is the weak limit of $K_\tau^{j,n}$ in $L^2(\Omega, \mathcal{F}_\tau, P)$. Finally note that K^j is predictable since the processes $K^{n,j}$ are so, $\forall n \geq 1$.

Let us now consider the following RBSDE:

$$\begin{cases} \hat{Y}^j \in \mathcal{S}^2,\ \ \hat{U}^j \in \mathcal{H}^2(\ell^2),\ \ \hat{K}^j \in \mathcal{A}^2; \\ \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^1, \cdots, \hat{Y}_r^{j-1}, \hat{Y}_r^j, \hat{Y}_r^{j+1}, \cdots, Y_r^m, \hat{U}_r^j)dr \\ - \sum_{i=1}^{\infty} \int_s^T \hat{U}_r^{i,j} dH_r^{(i)} + \hat{K}_T^j - \hat{K}_s^j,\ s \leq T; \\ \hat{Y}_s^j \geq \max_{k\in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})),\ \forall s \leq T;\ \ \int_0^T [\hat{Y}_{r-}^j - \max_{k\in A_j}(Y_{r-}^k - g_{jk}(r, X_{r-}^{t,x}))]d\hat{K}_r^j = 0. \end{cases} \tag{1.2.17}$$

Using Tanaka-Meyer's formula (see e.g.[55], pp.216) on $(\hat{Y}^j - Y^j)^+$ between s and T, we can show: P-a.s., $\hat{Y}^j \leq Y^j$ for any $j \in A$. On the other hand, since $\forall j \in A$, $Y^{j,n-1} \leq Y^j$, we have

$$\max_{k\in A_j}(Y_s^{k,n-1} - g_{jk}(s, X_s^{t,x})) \leq \max_{k\in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})), \forall s \leq T.$$

Then by Comparison Theorem, we obtain $Y^{j,n} \leq \hat{Y}^j$, thus by taking limit, $Y^j \leq \hat{Y}^j$ which implies $Y^j = \hat{Y}^j$, $\forall j \in A$.

Next using Itô's formula with $Y^j - \hat{Y}^j$ we obtain, for any $s \in [0,T]$,

$$\begin{array}{ll}(Y_s^j - \hat{Y}_s^j)^2 & = (Y_0^j - \hat{Y}_0^j)^2 + 2\int_0^s (Y_{r-}^j - \hat{Y}_{r-}^j) d(Y_r^j - \hat{Y}_r^j) \\ & + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \int_0^s (U_r^{i,j} - \hat{U}_r^{i,j})(U_r^{k,j} - \hat{U}_r^{k,j}) d[H^i, H^k]_r.\end{array}$$

As $Y^j = \hat{Y}^j$ and taking expectation in both-hand sides of the previous equality to obtain

$$\mathbb{E}[\int_0^T \sum_{i\geq 1} (U_r^{i,j} - \hat{U}_r^{i,j})^2 dr] = 0.$$

It implies that $U^j = \hat{U}^j$, $dt \otimes dP$ and finally $K^j = \hat{K}^j$ for any $j \in A$.

Next by the assumptions on g_{ij}, we can show that the predictable process K^j is continuous since it is predictable. As j is arbitrary in A, then the processes K^j is continuous and taking into account (1.2.17), we deduce that the triples (Y^j, U^j, K^j), $j \in A$, is a solution for system (1.2.14).

Step 2: Now we deal with the general case, and we introduce the operator $\Theta : [H^2]^m \to [H^2]^m$, $\Gamma \to Y$, such that:

$$\left\{\begin{array}{l} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \Gamma_r, U_r^j) dr - \sum\limits_{i=1}^{\infty} \int_s^T U_r^{i,j} dH_r^{(i)} + K_T^j - K_s^j, \ \forall s \leq T. \\ Y_s^j \geq \max\limits_{k \in A_j} \{Y_s^k - g_{jk}(s, Y_s^j)\}, \ \forall s \leq T; \quad \int_0^T [Y_s^j - \max\limits_{k \in A_j} \{Y_s^k - g_{jk}(s, Y_s^j)\}] dK_s^j = 0 \end{array}\right. \tag{1.2.18}$$

By Step 1, we have the existence of Y^j, $j \in A$. To get the uniqueness to the solution of (1.2.18), let $\Gamma := ((\Gamma_s^i)_{s\in[0,T]})_{i\in A}$ such that $\forall i \in A$, $\Gamma^i \in \mathcal{H}^2$. For $s \leq T$, let

$$V_s^{a(.)} = h_{a(T)}(X_T^{t,x}) + \int_s^T f^{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, N_r^a) dr - \sum_{i=1}^{\infty} \int_s^T N_r^{a,i} dH_r^{(i)} - A^a(T, X_T^{t,x}) + A^a(s, X_s^{t,x}).$$

We can prove that $Y_s^j = V_s^{a^*} = ess \sup\limits_{a \in \mathcal{A}_s^j} V_s^a$, $\forall (s,j) \in [0,T] \times \{1, \cdots, m\}$ and the uniqueness follows (see Appendix Theorem 5.1.1).

It follows that Θ is well defined. Next let us define the following norm:

$$\|Y\|_{2,\beta} := (E[\int_0^T e^{\beta s} |Y_s|^2 ds])^{\frac{1}{2}}.$$

Then we prove that ,

$$\|\Theta(\Gamma^1) - \Theta(\Gamma^2)\|_{2,\beta} \leq \sqrt{\frac{2LTm}{\beta}} \|\Gamma^1 - \Gamma^2\|_{2,\beta} \tag{1.2.19}$$

For β large enough, Θ is contraction on the Banach space $(([H^2])^m, \|.\|_{2,\beta})$, then the fixed point theorem ensures the existence of a unique Y such that $\Theta(Y) = Y$, which is the unique solution of system of RBSDE (1.2.14). On the other hand there exist deterministic functions $(u^j(t,x))_{j\in A}$ of polynomial growth such that:

$$\forall s \in [t,T], Y_s^{j,x,t} = u^j(s, X_s^{t,x}). \tag{1.2.20}$$

The next main result is the existence and uniqueness of a solution for the system of PDEs (1.2.12) with interconected obstacles. For this objective we use its link with the system of RBSDEs (1.2.14). However we are led to make, hereafter, the following additional assumption.

Assumption (A5): For any $i \in A$, f_i does not depend on the variable $u \in \ell^2$. □

In the Brownian framework of noise, the link between systems of PDEs with interconnected obstacles and systems of reflected BSDEs with oblique reflection has been already stated in several papers (see e.g. [22]). Therefore in this paper we extend this link to the setting where the noise is driven by a Lévy process. Recall the system of IPDEs: $\forall i \in A$,

$$\begin{cases} min\{u_i(t,x) - \max_{j\in A_i}(u_j(t,x) - g_{ij}(t,x)); -\frac{\partial u_i}{\partial t}(t,x) - \mathcal{L}u_i(t,x) - f_i(t,x,u_1,u_2,\cdots,u_m)\} = 0; \\ u_i(T,x) = h_i(x) \end{cases} \tag{1.2.21}$$

where

$$\begin{aligned}\mathcal{L}u(t,x) = (E[L_1]\sigma(t,x) + b(t,x))\partial_x u(t,x) + \tfrac{1}{2}Tr[(\sigma\sigma^T)\varpi^2 D^2_{xx}u(t,x)] + \\ \int_{I\!R}[u(t,x+\sigma(t,x)y) - u_i(t,x) - \tfrac{\partial u}{\partial x}(t,x)\sigma(t,x)y]\Pi(dy).\end{aligned}$$

We are going to give the definition of viscosity solution of (1.2.21). So let us define by $I^{1,\delta}$, $I^{2,\delta}$ the following non local terms:

$$I(t,x,\phi) := \int_{I\!R}[\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)\sigma(t,x)y]\Pi(dy);$$

$$I^1_\delta(t,x,\phi) = \int_{|y|\le\delta}[\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)\sigma(t,x)y]\Pi(dy);$$

$$I^2_\delta(t,x,q,\phi) = \int_{|y|\ge\delta}[\phi(t,x+\sigma(t,x)y) - \phi(t,x) - q\sigma(t,x)y]\Pi(dy);$$

$$\begin{aligned}\mathcal{L}_\phi u(t,x) = (E[L_1]\sigma(t,x) + b(t,x))\partial_x\phi(t,x) + \tfrac{1}{2}Tr[(\sigma\sigma^T)\varpi^2 D^2_{xx}\phi(t,x)] + \\ I^1_\delta(t,x,\phi) + I^2_\delta(t,x,\nabla\phi,u).\end{aligned}$$

By Lemma 5.1 in Appendix, $I^1_\delta(t,x,\phi)$ and $I^2_\delta(t,x,q,\phi)$ verify the Assumption (NLT) which is introduced by Barles et al.([6]).

Next, we give two definitions of the viscosity solution of (1.2.21), and according to [6] (pp.571), they are equivalent. For locally bounded function u: $(t,x) \in [0,T]\times R \to u(t,x) \in R$, we define its lower semi-continuous (lsc for short) envelope u_*, and upper semi-continuous (usc for short) envelope u^* as following:

$$u_*(t,x) = \underline{\lim}_{(t',x')\to(t,x),\ t'<T} u(t',x'), \quad u^*(t,x) = \overline{\lim}_{(t',x')\to(t,x),\ t'<T} u(t',x').$$

Definition 1.2.1. *A function $(u_1,\cdots,u_m) : [0,T]\times R \to R^m \in \Pi_g$ such that for any $i \in A$, u^i is lsc (resp. usc), is said to be a viscosity subsolution of (1.2.21) (resp. supsolution) if for any $i \in A$, $u^i(T,x) \le h_i(x)$ (resp. $u^i(T,x) \ge h_i(x)$); and for any test function $\varphi \in \Pi_g \bigcap C^{1,2}([0,T]\times R)$, if $(t_0,x_0) \in [0,T]\times R$ is global maximum (resp. minimum) point of $u^i - \varphi$,*

$$\begin{aligned} min\{u^i(t_0,x_0) - \max_{j\in A_i}(u^j(t_0,x_0) - g_{ij}(t_0,x_0)); \frac{\partial\varphi^i}{\partial t}(t_0,x_0) - \mathcal{L}\varphi^i(t_0,x_0) \\ - f_i((t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\} \le 0 \quad (resp. \ge 0),\end{aligned}$$

$(u^i)_{i=1}^m$ is called a viscosity solution of (1.2.21) if $(u^i_*)_{i=1}^m$ (resp. $(u^{i*})_{i=1}^m$) is a viscosity supersolution (resp. subsolution) of (1.2.21).

Definition 1.2.2. *A function $(u_1,\cdots,u_m) : [0,T]\times R \to R^m \in \Pi_g$ such that for any $i \in A$, u^i is lsc (resp. usc), is said to be a viscosity subsolution of (1.2.21) (resp. supsolution) if $u^i(T,x) \le h_i(x)$ (resp. $u^i(T,x) \ge h_i(x)$); and for any test function $\varphi \in C^{1,2}([0,T]\times R)$, if $(t_0,x_0) \in [0,T]\times R$ is a maximum (resp. minimum) point of $u^i - \varphi$ on $[0,T]\times B(x_0,C\delta)$, where C is the bound of σ, and $\delta > 0$,*

$$\begin{aligned} min\{u^i(t_0,x_0) - \max_{j\in A_i}(u^j(t_0,x_0) - g_{ij}(t_0,x_0)); \frac{\partial\varphi^i}{\partial t}(t_0,x_0) - \mathcal{L}_\varphi u^i(t_0,x_0) \\ - f_i((t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\} \le 0 \quad (resp. \ge 0).\end{aligned}$$

$(u^i)_{i=1}^m$ is called a viscosity solution of (1.2.21) if $(u^i_*)_{i=1}^m$ (resp. $(u^{i*})_{i=1}^m$) is a viscosity supersolution (resp. subsolution) of (1.2.21).

Using the first definition, we can prove the following lemma:

Lemma 1.2.1. *Let $(u^i)_{i=1}^m$ be a supersolution of (1.2.21) then $\forall \gamma \geq 0, \exists \lambda_0 > 0$ which does not depend on θ such that $\forall \lambda \geq \lambda_0$ and $\theta > 0$, $\overrightarrow{v} = (u_i(t,x) + \theta e^{-\lambda t}|x|^{2\gamma+2})_{i=1}^m$ is supersolution of (1.2.21).*

Remark 1.2.1. *If $(u^i)_{i=1}^m$ is a viscosity subsolution of (1.2.21) which belongs to Π_g, i.e. for some $\gamma > 0$ and $C > 0$,*

$$|u^i(t,x)| \leq C(1+|x|^\gamma),\ \forall (t,x) \in [0,T] \times I\!R^k \text{ and } i \in A.$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $\theta > 0$, $\overrightarrow{v}(t,x) = (u_i(t,x) - \theta e^{-\lambda t}(1+|x|^{2\gamma+2}))_{i=1}^m$ is subsolution of (1.2.21). □

The next theorem shows the relationship between (1.2.21) and (1.2.14), and so the existence of the viscosity solution for (1.2.21).

Theorem 1.2.5. *The function $(u_j(t,x))_{j\in A}$ defined in (1.2.20), is a viscosity solution of (1.2.21), with polynomial growth.*

For the sake of clarity, we divide the proof into two steps.
Step1. First we will show that $(u_j)_{j\in A}$ is a supersolution of (1.2.21). For all $j \in A$, as u_j is lsc, so $u_{j_*} = u_j$. Consider the sequence of function: $u_j^n(t,x) = Y_t^{j,n,t,x}$, where $Y_t^{j,n,t,x}$ is the unique solution of

$$\begin{cases} Y_s^{j,x,t,0} = \min_{j\in A} H^{(j)}(X_T^{t,x}) + \int_s^T \min_{j\in A} f_j(r, X_r^{t,x}, Y_r^{j,x,t,0}, \cdots, Y_r^{j,x,t,0})dr \\ -\sum_{i=1}^{\infty} \int_s^T U_r^{j,x,t,i,0} dH_r^{(i)} \\ Y_s^{j,x,t,n} = H^{(j)}(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^{1,x,t,n-1}, \cdots, Y_r^{i-1,x,t,n-1}, Y_r^{i,x,t,n}, \\ \cdots, Y_r^{m,x,t,n-1})dr - \sum_{i=1}^{\infty} \int_s^T U_r^{j,x,t,i,n} dH_r^{(i)} + K_T^{j,x,t,n} - K_s^{j,x,t,n} \quad n = 1,2,\cdots,m \\ Y_s^{j,x,t,n} \geq \max_{k\in A_j}\{Y_s^{k,x,t,n-1} - g_{jk}(s, X_s^{t,x})\},\ \forall s \leq T; \\ (Y_s^{j,x,t,n} - \max_{k\in A_j}\{Y_s^{k,x,t,n-1} - g_{jk}(s, X_s^{t,x})\})dK_s^{j,x,t,n} = 0. \end{cases} \tag{1.2.22}$$

By theorem 1.3 and induction, $u_j^n(t,x)$ is the unique viscosity solution of

$$\begin{cases} -\frac{\partial u^{j,0}}{\partial t}(t,x) - \mathcal{L}u^{j,0}(t,x) - \min_{j\in A} f_j(t,x,u^{j,0}) = 0; \\ u^{j,0}(T,x) = \min_{j\in A} h_j(x); \\ min\{u^{j,n}(t,x) - \max_{k\in A_j}(u^{j,n-1}(t,x) - g_{jk}(t,x)); \\ -\frac{\partial u^{j,n}}{\partial t}(t,x) - \mathcal{L}u^{j,n}(t,x) - f_j(t,x,u^{1,n-1},\cdots,u^{j-1,n-1},u^{j,n},\cdots,u^{m,n-1})\} = 0, \\ u^{j,n}(T,x) = h_j(x). \end{cases} \tag{1.2.23}$$

Also we know that, $\forall j \in A$, $u_j^n \nearrow u_j$, and for any $n = 1,2,\cdots$, u_j^n is continuous with polynomial growth. This together with the monotonic condition on f_j, i.e. for any $i \in A$, for any $k \neq i$, the mapping $y_k \to f_i(t,x,y_1,\cdots,y_{k-1},y_k,y_{k+1},\cdots,y_m,u)$ is nondecreasing whenever the other components $(t,x,y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m,u)$ are fixed, using the similar way with Theorem 1 in [6], we can show that

$$-\frac{\partial \phi}{\partial t}(t,x) - \mathcal{L}_\phi u^j(t,x) - f_j(t,x,u^1(t,x),\cdots,u^{j-1}(t,x),u^j(t,x),\cdots,u^m(t,x)) \geq 0.$$

We have know that $\forall j \in A$, $u^j \geq \max_{k\in A_j}(u^k(t,x) - g_{jk}(t,x))$ and $u^j(T,x) = h_j(x)$, so $(u_j)_{j=1}^m$ is a supersolution of (1.2.21).

Step2. Next we show that $(u_j^*)_{j\in A}$ is a subsolution of (1.2.21), using the same method as [30] we can prove that:

$$min\{u_j^*(T,x) - h_j(x);\ \ u_j^*(T,x) - \max_{k\in A_j}(u_k^*(T,x) - g_{jk}(T,x))\} = 0.$$

this together with the non-free loop assumption on the cost function g_{ij}, we can show that: $u_j^*(T,x) = h_j(x)$, $\forall j \in A$. Noting that since $u_j^n \nearrow u_j$ and u_j^n is continuous, we have

$$u_j^*(t,x) = \lim_{n\to\infty} sup^* u_j^n(t,x) = \overline{\lim_{n\to\infty, t'\to t, x'\to x}} u_j^n(t',x').$$

Besides $\forall j \in A$ and $n \geq 0$ we deduce from the construction of u_j^n that :

$$u_j^n(t,x) \geq \max_{l \in A_j}(u_l^n(t,x) - g_{jl}(t,x)),$$

take the limit to obtain: $\forall j \in A$, $\forall x \in R$,

$$u_j^*(t,x) \geq \max_{l \in A_j}(u_l^*(t,x) - g_{jl}(t,x)).$$

Next, fix $j \in A$, for $(t,x) \in [0,T[\times R$ such that

$$u_j^*(t,x) - \max_{l \in A_j}(u_l^*(t,x) - g_{jl}(t,x)) > 0. \tag{1.2.24}$$

By the same way as Step 1, we have:

$$-\frac{\partial \phi}{\partial t}(t,x) - \mathcal{L}_\phi u^{*,j}(t,x) - f_j(t,x,u^{*,1}(t,x),\cdots,u^{*,j-1}(t,x),u^{*,j}(t,x),\cdots,u^{*,m}(t,x)) \leq 0.$$

This together with (1.2.24) shows that $(u_j^*)_{j=1}^m$ is a subsolution of (1.2.21).

The second main result is a comparison theorem of subsolution and supersolution, from which we can get the continuity and uniqueness of the viscosity soluion of (1.2.21).

Theorem 1.2.6. *Let $(u_j)_{j\in A}$ be a subsolution of* (1.2.21)*, $(v_j)_{j\in A}$ be a supsolution of* (1.2.21) *such that $\forall j \in A$, $u_j, v_j \in \Pi_g$, then*

$$\forall (t,x) \in [0,T] \times R, \quad u_j(t,x) \leq v_j(t,x).$$

The proof is based on Jensen-Ishii's Lemma [6]. For $(\bar{t},\bar{x})$ which is the maximum point of $u_j(t,x) - w_j(t,x)$, for $\varepsilon > 0$ define test function as follows:

$$\Phi_\varepsilon^j(t,x,y) := u_j(t,x) - w_j(t,y) - \frac{|x-y|^2}{\varepsilon} - \psi(t,x),$$

where

$$\psi(t,x) := \rho|x-\bar{x}|^4 + |t-\bar{t}|^2.$$

Let $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ be such that

$$\Phi_\varepsilon^j(t_\varepsilon,x_\varepsilon,y_\varepsilon) = \max_{(t,x,y)\in[0,T]\times R^2} \Phi_\varepsilon^j(t,x,y).$$

Then we proved two facts:
(i)

$$\lim_\varepsilon (u_j(t_\varepsilon,x_\varepsilon), w_j(t_\varepsilon,y_\varepsilon)) = (u_j(\bar{t},\bar{x}), w_j(\bar{t},\bar{x})).$$

(ii)

$$\begin{aligned} l \quad &:= I^{1,\delta}(t_\varepsilon,x_\varepsilon,\phi_x) + I^{2,\delta}(t_\varepsilon,x_\varepsilon,q_u^\varepsilon,u_j) \\ &\leq I^{1,\delta}(t_\varepsilon,y_\varepsilon,-\phi_y) + I^{2,\delta}(t_\varepsilon,y_\varepsilon,q_w^\varepsilon,w_j) + O(\tfrac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon}) + o_\varepsilon(1) + \tfrac{1}{\varepsilon}o_\delta(1) + o_\rho(1). \end{aligned}$$

These with Jensen-Ishii's Lemma, by contradiction and doubling variable technique we can prove that

$$\forall (t,x) \in [0,T] \times R, \quad u_j(t,x) \leq v_j(t,x).$$

The last main result is the existence and uniqueness of systems of IPDE (1.2.21), when $(-f_j)_{j\in A}$ verify [**A4**] (1)-(iv), i.e. $\forall j \in A$, $k \in A_j$ f_j is non-increasing in y_k, which we rewrite as **Assumption (A4')**:
(1) For any $i \in A$:
(i) the mapping $(t,x) \to f_i(t,x,\overrightarrow{y})$ is continuous uniformly with respect to $\overrightarrow{y}$ where $\overrightarrow{y} = (y^i)_{i=1,m}$;

(ii)the mapping $\overrightarrow{y} \mapsto f_i(t,x,\overrightarrow{y})$ is Lipschiz continuous uniformly *w.r.t.* (t,x) ;

(iii) $f_i(t,x,0)$ is of polynomial growth w.r.t. (t,x).

(iv) For any $i \in A$, for any $k \neq i$, the mapping $y_k \to f_i(t,x,y_1,\cdots,y_{k-1},y_k,y_{k+1},\cdots,y_m)$ is **nonincreasing** whenever the other components $(t,x,y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m)$ are fixed.

(2) $\forall i,j \in A$, $g_{ii} \equiv 0$ and for $i \neq j$, $g_{jk}(t,x)$ is non-negative, continuous with polynomial growth and satisfy the following non-free loop property: $\forall (t,x) \in [0,T] \times R$ and for any sequence of indices $i_1,\cdots,i_k$ such that $i_1 = i_k$ and $card\{i_1,\cdots,i_k\} = k-1$ we have:

$$g_{i_1 i_2}(t,x) + g_{i_2 i_3}(t,x) + \cdots + g_{i_k i_1}(t,x) > 0,\ \forall (t,x) \in [0,T] \times I\!R^k.$$

(3) $\forall i \in A$, h_i is continuous with polynomial growth and satisfies the following coherance conditions:

$$h_i(x) \geq \max_{j \in A^{-i}} (h_j(x) - g_{ij}(T,x)), \forall x \in I\!R.$$

Theorem 1.2.7. *If $(f_j)_{j\in A}$ verify [**A4'**], then systems of IPDE (1.2.21) has a unique continuous viscosity solution $(u_j)_{j\in A}$ with polynomial growth.*

1.3 Viscosity solution of system of variational inequalities with interconnected bilateral obstacles and connections to multiple modes switching game of jump-diffusion processes

1.3.1 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null elements of $\mathcal{F}$, and $\mathcal{F}_{t^+} \triangleq \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, $t \geq 0$, and suppose that the filtration is generated by the following two mutually independent process:
- a d-dimensional standard Brownian motion $(W_t)_{t\geq 0}$
- a Poisson random measure N on $R_+ \times E$, where $E \triangleq R^l - \{0\}$ is equipped with its Borel field $\mathcal{B}_E$, with compensator $\nu(dtde) = dtn(de)$, such that $\{\hat{N}((0,t]\times A) = (N-\nu)((0,t]\times A)\}_{0\leq t\leq T}$ is and $\mathcal{F}_t$-martingale for all $A \in \mathcal{B}_E$ satisfying $n(A) < \infty$. n is assumed to be a σ-finite measure on $(E, \mathcal{B}_E)$ satisfying:

$$\int_E (1 \wedge x^2) n(dx) < \infty. \tag{1.3.1}$$

Let T be a fixed positive constant and A^1 (resp. A^2) denote the set of switching modes for player 1 (resp. player 2). Let m_1 (resp. m_2) be the cardinal of the set A^1 (resp. A^2) and for $(i,j) \in A^1 \times A^2$, $A^1_i := A^1 - \{i\}$ and $A^2_j := A^2 - \{j\}$. Next, for $\overrightarrow{y} = (y^{kl})_{(k,l)\in A^1\times A^2} \in R^{m_1\times m_2}$. For any $y_1 \in R$, denote by $[\overrightarrow{y}^{i,j}, y_1]$ the matrix which is obtained from $\overrightarrow{y}$ by replacing the element y^{ij} with y_1.
A function $\Phi : (t,x) \in [0,T] \times R \to \Phi(t,x) \in R$ is called of polynomial growth if there exist two non-negative real constant C and γ such that

$$|\Phi(t,x)| \leq C(1 + |x|^\gamma).$$

Hereafter, this class of functions is denoted by Π_g.

We define the following spaces of processes, let:

$\mathcal{P}$ be the σ-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0,T]$;
$\mathcal{L}^2 := \{\xi$ is an $I\!R$-valued, $\mathcal{F}_T$-random variable such that $||\xi||^2_{\mathcal{L}^2} := E|\xi|^2 < \infty\}$;
$\mathcal{H}^2 := \{\{\varphi_t, 0 \leq t \leq T\}$ is an $I\!R$-valued, $\mathcal{F}_t$-progressively measurable process s.t. $||\varphi||^2_{\mathcal{H}^2} := E(\int_0^T |\varphi_t|^2) < \infty\}$;
$\mathcal{S}^2 := \{\{\varphi_t, 0 \leq t \leq T\}$ is an $I\!R$-valued, $\mathcal{F}_t$-adapted RCLL process s.t. $||\varphi||^2_{\mathcal{S}^2} := E(\sup_{0\leq t\leq T} |\varphi_t|^2) < \infty\}$;
$\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of continuous non-decreasing processes null at $t = 0$;
$\mathcal{H}^2(\hat{N}) := \{U_t(e){:}\Omega\times[0,T]\times E \to R$ which are $\mathcal{P}\times\mathcal{B}_E$ measurable and s.t. $||U||^2_{\mathcal{H}^2(\hat{N})} := E(\int_0^T \int_E |U_t(e)|^2 n(de)dt) < \infty\}$.

In this paper, we investigate existence and uniqueness of viscosity solutions $\overrightarrow{v}(t,x) := (v^{ij}(t,x))_{(i,j)\in A^1\times A^2}$ of the following system of variational inequalities with upper and lower interconnected obstacles: $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} min\{(v^{ij} - L^{ij}[\overrightarrow{v}])(t,x), max\{(v^{ij} - U^{ij}[\overrightarrow{v}])(t,x), -\partial_t v^{ij}(t,x) - \mathcal{L}v^{ij}(t,x) \\ -g^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x v^{ij}(t,x), B^{ij}v^{ij}(t,x))\}\} = 0 \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{1.3.2}$$

where, for any $(t,x) \in [0,T] \times R$,

$$\begin{aligned} \mathcal{L}\phi(t,x) &:= b(t,x)D_x\phi(t,x) + \tfrac{1}{2}\sigma^2(t,x)D^2_{xx}\phi(t,x) \\ &\quad + \textstyle\int_E (\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de), \end{aligned}$$

$$\begin{aligned} B^{ij}\phi(t,x) \\ = \textstyle\int_E (\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de), \end{aligned}$$

and $\forall (i,j) \in A^1 \times A^2$,

$$L^{ij}[\overrightarrow{v}])(t,x) := \max_{k\in A^1_i}\{(v^{kj} - \underline{g}_{ik})(t,x)\} \text{ and } U^{ij}[\overrightarrow{v}])(t,x) := \min_{l\in A^2_j}\{(v^{il} - \overline{g}_{jl})(t,x)\}.$$

Denote by

$$I^1_\delta(t,x,\phi) = \int_{|e|\le\delta} (\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de);$$

$$I^2_\delta(t,x,q,\phi) = \int_{|e|\ge\delta} (\phi(t,x+\beta(x,e)) - \phi(t,x) - q\beta(x,e))n(de);$$

$$I^{1,B^{ij}}_\delta(t,x,\phi) = \int_{|e|\le\delta} (\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de);$$

$$I^{2,B^{ij}}_\delta(t,x,\phi) = \int_{|e|\ge\delta} (\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de);$$

$$I(t,x,\phi) = \int_E (\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de);$$

$$I^{B^{ij}}(t,x,\phi) = \int_E (\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de);$$

$$\mathcal{L}_\phi u(t,x) := b(t,x)D_x\phi(t,x) + \frac{1}{2}\sigma^2(t,x)D^2_{xx}\phi(t,x) + I^1(t,x,\phi) + I^2(t,x,D_x\phi,u),$$

The following assumptions will be in force throughout the rest of the paper.

(A0) The functions $b(t,x)$ and $\sigma(t,x)$: $[0,T]\times R \to R$ are jointly continuous in (t,x), of linear growth in (t,x) and Lipschitz continuous w.r.t. x, meaning that there exists a non-negative constant C such that for any $(t,x,x') \in [0,T]\times R$ we have:

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|), \quad |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \le C|x-x'|.$$

The function $\beta : R\times E \to R$ is measurable, continuous in x and such that for some real K and all $e \in E$, for any $x, x' \in R$,

$$|\beta(x,e)| \le K(1 \wedge |e|), \quad |\beta(x,e) - \beta(x',e)| \le K|x-x'|(1\wedge|e|).$$

(A1) For any $(i,j) \in A^1 \times A^2$, $g^{ij}(t,x,\overrightarrow{y},z,q) : R\times R\times R^{m_1\times m_2}\times R^d \times R \to R$,

(i) is continuous in (t,x) uniformly w.r.t. the other variables $(\overrightarrow{y},z,q)$ and for any (t,x) the mapping $(t,x) \to g^{i,j}(t,x,0,0,0)$ is of polynomial growth.

(ii) satisfies the standard hypothesis of Lipschitz continuity w.r.t. the variables $(\overrightarrow{y}, z, q)$, i.e. $\forall (t,x) \in [0,T] \times R, \forall (\overrightarrow{y}_1, \overrightarrow{y}_2) \in R^{m_1 \times m_2} \times R^{m_1 \times m_2}, (z_1, z_2) \in R^{d+d}, (q_1, q_2) \in R \times R$,

$$|g^{ij}(t,x,\overrightarrow{y}_1, z_1, q_1) - g^{ij}(t,x,\overrightarrow{y}_2, z_2, q_2)| \leq C(|\overrightarrow{y}_1 - \overrightarrow{y}_2| + |z_1 - z_2| + |q_1 - q_2|),$$

where, $|\overrightarrow{y}|$ stands for the standard Euclidean norm of $\overrightarrow{y}$ in $R^{m_1} \times R^{m_2}$.

(iii) $q \mapsto g^{ij}(t,x,y,z,q)$ is non-decreasing, for all $(t,x,y,z) \in [0,T] \times R \times R^{m_1 \times m_1} \times R$.

Futhermore, let $\gamma^{ij} : R \times \mathcal{B}_E \to R$ such that there exists $C > 0$,

$$0 \leq \gamma^{ij}(x,e) \leq C(1 \wedge |e|), \quad x \in R, e \in \mathcal{B}_E$$

$$|\gamma^{ij}(x,e) - \gamma^{ij}(x',e)| < C|x - x'|(1 \wedge |e|), \quad x, x' \in R, e \in E.$$

We set

$$f^{ij}(t,x,y,z,u) = g^{ij}(t,x,y,z, \int_E u(e)\gamma^{ij}(x,e)n(de)),$$

for $(t,x,y,z,u) \in [0,t] \times R \times R^{m_1 \times m_2} \times R \times \mathcal{L}^2(R, \mathcal{B}_E, n)$.
Noting that under Assumption (A0) and (A1), by ([5]), $I, I^{B^{ij}}, I^1_\delta, I^2_\delta, I^{1,B^{ij}}_\delta, I^{2,B^{ij}}_\delta$ satisfy the Assumption (NLT), which is given in appendix.

(A2) Monotonicity: For any $(i,j) \in A^1 \times A^2$ and any $(k,l) \neq (i,j)$ the mapping $y^{k,l} \to g^{i,j}(t,x,\overrightarrow{y},z,u)$ is non-decreasing.
(A3) The functions $h^{ij}(x) : R \to R$ are continuous w.r.t. x, belong to class Π_g and satisfy

$$\forall (i,j) \in A^1 \times A^2 and\ x \in R, \quad \max_{k \in A^1_i}(h^{kj}(x) - \underline{g}_{ik}(T,x)) \leq h^{ij}(x) \leq \min_{l \in A^2_j}(h^{il}(x) - \overline{g}_{jl}(T,x)),$$

where $\overline{g}_{ik}$ and $\underline{g}_{jl}$ are given in the next assumption.
(A4) The no free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t,x), belong to Π_g and satisfy the following condition:

For any loop in $A^1 \times A^2$, i.e., any sequence of pairs $(i_1,j_1), \ldots, (i_N,j_N)$ of $\Gamma^1 \times \Gamma^2$ such that $(i_N,j_N) = (i_1,j_1)$, $\text{card}\{(i_1,j_1), \ldots, (i_N,j_N)\} = N-1$ and $\forall\ q = 1, \ldots, N-1$, either $i_{q+1} = i_q$ or $j_{q+1} = j_q$, we have $\forall (t,x) \in [0,T] \times I\!R^k$,

$$\sum_{q=1,N-1} \varphi_{i_q i_{q+1}}(t,x) \neq 0, \tag{1.3.3}$$

where, $\forall\ q = 1, \ldots, N-1$, $\varphi_{i_q i_{q+1}}(t,x) = -\underline{g}_{i_q i_{q+1}}(t,x)\mathbb{1}_{i_q \neq i_{q+1}} + \bar{g}_{j_q i_{q+1}}(t,x)\mathbb{1}_{j_q \neq j_{q+1}}$.

Consider now the following SDE:

$$X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r)dW_r + \int_t^s \int_E \beta(X^{t,x}_{r-}, e)\hat{N}(drde), \quad s \in [t,T], x \in R.$$

The existence and uniqueness of the solution $X^{t,x}_s$ follows from [5].

Next, we give three definitions of the viscosity solution of (1.3.2), and according to [5] (pp.571), they are equivalent. For locally bounded function u: $(t,x) \in [0,T] \times R \to u(t,x) \in R$, we define its lower semi-continuous (lsc for short) envelope u_*, and upper semi-continuous(usc for short) envelope u^* as following:

$$u_*(t,x) = \underline{\lim}_{(t',x') \to (t,x),\ t' < T} u(t',x'), \quad u^*(t,x) = \overline{\lim}_{(t',x') \to (t,x),\ t' < T} u(t',x')$$

Definition 1.3.1. *A function $\overrightarrow{u} = (u^{ij}(t,x))_{(i,j) \in A^1 \times A^2} : [0,T] \times R \to R^{A^1 \times A^2}$ such that for any $(i,j) \in A^1 \times A^2$, $u^{ij} \in \Pi_g$ is lsc (resp. usc), is said to be a viscosity subsolution (resp. supsolution) of (1.3.2) if for any test function $\varphi \in C^{1,2}([0,T] \times R)$, if $(t_0,x_0) \in [0,T] \times R$ is a global maximum (resp.*

minimum) point of $u^{i,j}-\varphi$,

$$\begin{cases} min\{(u^{ij}-L^{ij}[\overrightarrow{u}])(t_0,x_0), max\{(u^{ij}-U^{ij}[\overrightarrow{u}])(t_0,x_0), -\partial_t\varphi(t_0,x_0)-b(t_0,x_0)\partial_x\varphi(t_0,x_0) \\ -\frac{1}{2}\sigma^2(t_0,x_0)\partial^2_{xx}\varphi(t_0,x_0)-I(t_0,x_0,\varphi) \\ -g^{ij}(t_0,x_0,(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2},\sigma(t_0,x_0))\partial_x\varphi(t_0,x_0), I^{B^{ij}}(t_0,x_0,\varphi)\}\}\le 0 \quad (resp.\ge 0); \\ v^{ij}(T,x)\le h^{ij}(x) \quad (resp.\ge). \end{cases}$$

Definition 1.3.2. *A function* $\overrightarrow{u}=(u^{ij}(t,x))_{(i,j)\in A^1\times A^2}:[0,T]\times R\to R^{A^1\times A^2}$ *such that for any* $(i,j)\in A^1\times A^2$, $u^{ij}\in\Pi_g$ *is lsc (resp. usc), is said to be a viscosity subsolution (resp. supsolution) of (1.3.2) if for any* $\delta>0$, $(t_0,x_0)\in(0,T)$ *and a function* $\varphi\in C^{1,2}([0,T]\times R)$, *such that* $(t_0,x_0)\in[0,T]\times R$ *is a maximum (resp. minimum) point of* $u^{i,j}-\varphi$ *on* $[0,T]\times B(x_0,K\delta)$, *where* K *is the bound of* β,

$$\begin{cases} min\{(u^{ij}-L^{ij}[\overrightarrow{u}])(t_0,x_0), max\{(u^{ij}-U^{ij}[\overrightarrow{u}])(t_0,x_0), -\partial_t\varphi(t_0,x_0)-b(t_0,x_0)\partial_x\varphi(t_0,x_0) \\ -\frac{1}{2}\sigma^2(t_0,x_0)\partial^2_{xx}\varphi(t_0,x_0)-I^1_\delta(t_0,x_0,\phi)-I^2_\delta(t_0,x_0,\partial_x\varphi,u^{ij}) \\ -g^{ij}(t_0,x_0,(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2},\sigma(t_0,x_0))\partial_x\varphi(t_0,x_0), I^{1,B^{ij}}_\delta(t_0,x_0,\varphi)+I^{2,B^{ij}}_\delta(t_0,x_0,u^{ij}))\}\}\le 0 \quad (resp.\ge 0); \\ v^{ij}(T,x)\le h^{ij}(x) \quad (resp.\ge). \end{cases}$$

Definition 1.3.3. *(i) For a function* $u:[0,T]\times R\to R$, *lsc (resp. usc), we denote* $J^-u(t,x)$ *the parabolic subjet (resp.* $J^+u(t,x)$ *the parabolic superjet) of* u *at (t,x)*$\in[0,T]\times R$, *as the set of triples (p,q,M)*$\in R\times R\times S^k$; *where* S^k *is the set of symmetric real matrices of dimension* k

$$u(t',x')\ge u(t,x)+p(t'-t)+\langle q,x'-x\rangle+\frac{1}{2}\langle x'-x,M(x'-x)\rangle+o(|t'-t|+|x'-x|)^2 \quad (resp.\le)$$

(ii) We denote $\bar{J}^-u(t,x)$ *(resp.* $\bar{J}^+u(t,x)$*) the parabolic limiting superjet (resp. superjet) of* u *at (t,x), as the set of triples (p,q,M)*$\in R\times R\times S^k$ *s.t.*

$$(p,q,M)=\lim_{n\to\infty}(p_n,q_n,M_n),\quad (t,x)=\lim_{n\to\infty}(t_n,x_n)$$

where $(p_n,q_n,M_n)\in J^-u(t_n,x_n)$ $(resp. J^+u(t_n,x_n))$ *and* $u(t,x)=\lim_{n\to\infty}u(t_n,x_n)$.

(iii) A function $\overrightarrow{u}=(u^{ij}(t,x))_{(i,j)\in A^1\times A^2}:[0,T]\times R\to R^{A^1\times A^2}$ *such that for any* $(i,j)\in A^1\times A^2$, $u^{ij}\in\Pi_g$ *is lsc (resp. usc), is said to be a viscosity subsolution (resp. supsolution) of (1.3.2) if for any* $\delta>0$, $(t_0,x_0)\in(0,T)\times R$ *and a function* $\phi\in C^{1,2}([0,T]\times R)$, *if* $(t_0,x_0)\in[0,T]\times R$ *is a maximum (resp. minimum) point of* $u^{i,j}-\phi$ *on* $(0,T)\times B(x_0,K\delta)$, *and if* $(p,q,M)\in\bar{J}^-u^{i,j}(t_0,x_0)(resp.\bar{J}^+u^{i,j}(t_0,x_0))$ *with* $q=D_t\phi(t_0,x_0)$, $p=D_x\phi(t_0,x_0)$, *and* $M\ge D^2_{xx}\phi(t_0,x_0)$ *(resp.* $M\le D^2_{xx}\phi(t_0,x_0)$*), then:*

$$\begin{cases} min\{(u^{ij}-L^{ij}[\overrightarrow{u}])(t_0,x_0), max\{(u^{ij}-U^{ij}[\overrightarrow{u}])(t_0,x_0), -p-b(t_0,x_0)q-\frac{1}{2}\sigma^2(t_0,x_0)M-I^1_\delta(t_0,x_0,\phi) \\ I^2_\delta(t_0,x_0,q,u^{ij})-g^{ij}(t_0,x_0,(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2},\sigma(t_0,x_0)q, I^{1,B^{ij}}_\delta(t_0,x_0,\phi)+I^{2,B^{ij}}_\delta(t_0,x_0,u^{ij}))\}\}\le 0 \\ (resp.\ge 0); \\ v^{ij}(T,x)\le h^{ij}(x) \quad (resp.\ge). \end{cases}$$

Definition 1.3.4. *A function* $\overrightarrow{u}=(u^{ij}(t,x))_{(i,j)\in A^1\times A^2}$ *such that for any* $(i,j)\in A^1\times A^2$, $u^{ij}\in\Pi_g$, *is called a viscosity solution of (1.3.2) if* $(u^{ij}_*(t,x))_{(i,j)\in A^1\times A^2}$ *(resp.* $(u^*_{ij}(t,x))_{(i,j)\in A^1\times A^2}$*) is a viscosity supersolution (resp. subsolution) of (1.3.2).*

1.3.2 Two approximating schemes

For $n,m\ge 0$, let $(Y^{i,j,n,m},Z^{i,j,n,m},U^{i,j,n,m})_{(i,j)\in A^1\times A^2}$ be the solution of the following system of BSDEs.

$$\begin{cases} (Y^{i,j,n,m},Z^{i,j,n,m},U^{i,j,n,m})\in\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2(\hat{N}); \\ dY^{i,j,n,m}_s=-f^{i,j,n,m}(s,X^{t,x}_s,(Y^{k,l,n,m}_s)_{(k,l)\in A^1\times A^2},Z^{i,j,n,m}_s,U^{i,j,n,m}_s)ds \\ \qquad\qquad +Z^{i,j,n,m}_s dB_s+\int_E U^{i,j,n,m}_s(e)\hat{N}(dsde),\ s\le T. \\ Y^{i,j,n,m}_T=h^{i,j}(X^{t,x}_T), \end{cases} \tag{1.3.4}$$

where,

$$
\begin{aligned}
&f^{i,j,n,m}(s, X_s^{t,x}, (y^{ij})_{(ij)\in A^1\times A^2}, z_s, u_s)\\
&:= g^{i,j,n,m}(s, X_s^{t,x}, (y^{kl})_{(kl)\in A^1\times A^2}, z_s, \int_E u_s(e)\gamma^{ij}(X_s^{t,x}, e)n(de))\\
&= g^{i,j}(s, X_s^{t,x}, (y^{kl})_{(kl)\in A^1\times A^2}, z_s, \int_E u_s(e)\gamma^{ij}(X_s^{t,x}, e)n(de))\\
&\quad + n(y^{ij} - \max_{k\in A_i^1}\{y^{kj} - \underline{g}_{ik}(s, X_s^{t,x})\})^- - m(y^{ij} - \min_{l\in A_j^2}\{y^{il} - \overline{g}_{jl}(s, X_s^{t,x})\})^+.
\end{aligned}
$$

Let us recall that under Assumption **(A1)**, the solution $(Y^{i,j,n,m}, Z^{i,j,n,m}, U^{i,j,n,m})_{(i,j)\in A^1\times A^2}$ of (1.3.4) exists and is unique (see [6]). By the assumption(**A1**)(iii), we have the comparison theorem for BSDE with jumps (see [58] Theorem 2.4). The we have:

Proposition 1.3.1. *For any $(i,j) \in A^1 \times A^2$ and $n, m \geq 0$ we have*

$$
P - a.s., \quad Y^{i,j,n,m} \leq Y^{i,j,n+1,m} \text{ and } \ Y^{i,j,n,m+1} \leq Y^{i,j,n,m}, \qquad (i,j) \in A^1 \times A^2. \tag{1.3.5}
$$

Moreover, for any $(i,j) \in A^1 \times A^2$ and $n, m \geq 0$, there exists a deterministic continuous function $v^{i,j,n,m} \in \Pi_g$ such that, for any $t \leq T$,

$$
Y_s^{i,j,n,m} = v^{i,j,n,m}(s, X_s^{t,x}), \quad s \in [t, T]. \tag{1.3.6}
$$

Finally, for any $(i,j) \in A^1 \times A^2$ and $n, m \geq 0$,

$$
v^{i,j,n,m}(t,x) \leq v^{i,j,n+1,m}(t,x) \text{ and } \ v^{i,j,n,m+1}(t,x) \leq v^{i,j,n,m}(t,x), \quad (t,x) \in [0,T] \times R \tag{1.3.7}
$$

The proof of first claim is based on the result by Xuehong Zhu (2010) ([62], Theorem 3.1) related to the comparison of solutions of multi-dimensional BSDEs. The second claim is just the representation of solutions of standard BSDEs with jumps by deterministic functions in the Markovian framework (see [6]). The inequalities of (1.3.7) are obtained by taking $s = t$ in (1.3.5) in view of the representation (1.3.6) of $Y^{i,j,n,m}$ by $v^{i,j,n,m}$ and $X^{t,x}$.

Now we will show two approximation schemes obtained from the sequence $Y^{i,j,m,n}, (i,j) \in A^1 \times A^2)_{n,m}$ of the solution of system (1.3.4). The first scheme is a sequence of decreasing reflected BSDEs with interconnected lower obstacles: $\forall (i,j) \in A^1 \times A^2$,

$$
\begin{cases}
(\bar{Y}^{i,j,m}, \bar{Z}^{i,j,m}, \bar{U}^{i,j,m}, \bar{K}^{i,j,m}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2;\\
\bar{Y}_s^{i,j,m} = h^{i,j}(X_T^{t,x}) + \int_s^T \bar{f}^{i,j,m}(r, X_r^{t,x}, (\bar{Y}_r^{k,l,m})_{(k,l)\in A^1\times A^2}, \bar{Z}_r^{i,j,m}, \bar{U}_r^{i,j,m})dr - \int_s^T \bar{Z}_r^{i,j,m} dB_r\\
- \int_s^T \int_E \bar{U}_r^{i,j,m}(e)\hat{N}(drde) + \bar{K}_T^{i,j,m} - \bar{K}_s^{i,j,m}, \ s \leq T;\\
\bar{Y}_s^{i,j,m} \geq \max_{k\in A_i^1}\{\bar{Y}_s^{k,j,m} - \underline{g}_{ik}(s, X_s^{t,x})\}, \quad s \leq T;\\
\int_0^T (\bar{Y}_s^{i,j,m} - \max_{k\in A_i^1}\{\bar{Y}_s^{k,j,m} - \underline{g}_{ik}(s, X_s^{t,x})\})d\bar{K}_s^{i,j,m} = 0,
\end{cases} \tag{1.3.8}
$$

where, $\forall (i,j) \in A^1 \times A^2$, $m \geq 0$ and $s \leq T$,

$$
\begin{aligned}
\bar{f}^{i,j,m}(s, X_s^{t,x}, \overrightarrow{y}, z, u) :=& g^{ij,+,m}(s, X_s^{t,x}, (y^{kl})_{(k,l)\in A^1\times A^2}, z, \int_E u(e)\gamma^{ij}(X_s^{t,x}, e)n(de))\\
=& g^{ij}(s, X_s^{t,x}, (y^{kl})_{(k,l)\in A^1\times A^2}, z, \int_E u^{ij}(e)\gamma^{ij}(X_s^{t,x}, e)n(de))\\
&- m(y^{ij} - \min_{l\in A_j^2}(y^{il} + \overline{g}_{jl}(s, X_s^{t,x})))^+.
\end{aligned}
$$

Thanks to the assumption (**A1**)-(**A3**) and non free loop assumption, by Theorem (5.4.1) in appendix, the solution of (1.3.8) exists and is unique. Moreover, we have the following properties.

Proposition 1.3.2. *For any $(i,j) \in A^1 \times A^2$ and $m \geq 0$, we have:*
(i)

$$\lim_{n\to\infty} E[\sup_{t\leq s\leq T} |Y_s^{i,j,n,m} - \bar{Y}_s^{i,j,m}|^2] \to 0 \tag{1.3.9}$$

(ii)

$$P-a.s., \quad \bar{Y}^{i,j,m} \geq \bar{Y}^{i,j,m+1}.$$

(iii)There exists a deterministic continuous functions $(\bar{u}^{k,l,m})_{(k,l)\in A^1\times A^2}$ in Π_g such that, for every $t \leq T$,

$$\bar{Y}_s^{i,j,m} = \bar{u}^{i,j,m}(s, X_s^{t,x}), \ s \in [t,T]. \tag{1.3.10}$$

Moreover, $\forall (i,j) \in A^1 \times A^2$ and $(t,x) \in [0,T] \times R^k$, $\bar{u}^{i,j,m}(t,x) \geq \bar{u}^{i,j,m+1}(t,x)$.
Finally, $(\bar{u}^{i,j,m})_{(i,j)\in A^1\times A^2}$ is the unique viscosity solution in the class Π_g of the following system of variational inequalities with inter-connected obstacles. $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} min\{\bar{u}^{i,j,m}(t,x) - \max\limits_{k\in A_i^1}(\bar{u}^{k,j,m}(t,x) - \underline{g}_{ik}(t,x)); -\partial_t \bar{u}^{i,j,m}(t,x) - \mathcal{L}\bar{u}^{i,j,m}(t,x) \\ \\ g^{ij,+,m}(t,x,(\bar{u}^{k,l,m}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x\bar{u}^{i,j,m}(t,x), B^{ij}\bar{u}^{i,j,m}(t,x))\} = 0; \\ \\ \bar{u}^{i,j,m}(T,x) = h^{i,j}(x). \end{cases} \tag{1.3.11}$$

The second scheme is the increasing approximating scheme: $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} (\underline{Y}^{i,j,n}, \underline{Z}^{i,j,n}, \underline{U}^{i,j,n}, \underline{K}^{i,j,n}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2; \\ \\ \underline{Y}_s^{i,j,n} = h^{i,j}(X_T^{t,x}) + \int_s^T \underline{f}^{i,j,n}(r, X_r^{t,x}, (\underline{Y}_r^{k,l,n})_{(k,l)\in A^1\times A^2}, \underline{Z}_r^{i,j,n}, \underline{U}_r^{i,j,n})dr - \int_s^T \underline{Z}_r^{i,j,n} dB_r \\ \\ - \int_s^T \int_E \underline{U}_r^{i,j,n}(e)\hat{N}(drde) + \underline{K}_T^{i,j,n} - \underline{K}_s^{i,j,n}, \ s \leq T; \\ \\ \underline{Y}_s^{i,j,n} \leq \min\limits_{l\in A_j^2}\{\underline{Y}_s^{i,l,n} + \overline{g}_{jl}(s, X_s^{t,x})\}, \quad s \leq T, \\ \\ \int_0^T (\underline{Y}_s^{i,j,n} - \min\limits_{l\in A_j^2}\{\underline{Y}_s^{k,j,n} + \overline{g}_{jl}(s, X_s^{t,x})\}) d\underline{K}_s^{i,j,n} = 0, \end{cases} \tag{1.3.12}$$

where, $\forall (i,j) \in A^1 \times A^2$, $n \geq 0$ and $s \leq T$,

$$\begin{aligned} \underline{f}^{i,j,n}(s, X_s^{t,x}, \overrightarrow{y}, z, u) :=& g^{ij,-,n}(s, X_s^{t,x}, (y^{kl})_{(k,l)\in A^1\times A^2}, z, \int_E u(e)\gamma^{ij}(X_s^{t,x}, e)n(de)) \\ =& g^{ij}(s, X_s^{t,x}, (y^{kl})_{(k,l)\in A^1\times A^2}, z, \int_E u(e)\gamma^{ij}(X_s^{t,x}, e)n(de)) \\ &+ n(y^{ij} - \max_{k\in A_i^1}(Y^{kj} - \underline{g}_{ik}(s, X_s^{t,x})))^-. \end{aligned}$$

Thanks to the assumption (**A1**)-(**A3**) and the non free loop assumption, by Theorem 5.4.1 in appendix, the solution of (1.3.12) exists and is unique.

Proposition 1.3.3. *For any $(i,j) \in A^1 \times A^2$ and $n \geq 0$, we have:*
(i)

$$\lim_{m\to\infty} E[\sup_{t\leq s\leq T} |Y_s^{i,j,n,m} - \underline{Y}_s^{i,j,n}|^2] \to 0 \tag{1.3.13}$$

(ii) For any $n \geq 0$,

$$P-a.s., \quad \underline{Y}^{i,j,n} \leq \underline{Y}^{i,j,n+1}.$$

(iii)There exits a unique $m_1 \times m_2$-uplet of deterministic continuous functions $(\underline{u}^{k,l,n})_{(k,l)\in A^1\times A^2}$ in Π_g such that, for every $t \leq T$,

$$\underline{Y}_s^{i,j,n} = \underline{u}^{i,j,n}(s, X_s^{t,x}), \ s \in [t,T]. \tag{1.3.14}$$

Moreover, $\forall (i,j) \in A^1 \times A^2$ and $(t,x) \in [0,T] \times R^k$, $\underline{u}^{i,j,n}(t,x) \leq \underline{u}^{i,j,n+1}(t,x)$.
Finally, $(\underline{u}^{i,j,n})_{(i,j)\in A^1\times A^2}$ is the unique viscosity solution in the class Π_g of the following system of

variational inequalities with inter-connected obstacles. $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} max\{\underline{u}^{i,j,n}(t,x) - \min\limits_{l \in A_j^2}(\underline{u}^{i,l,n}(t,x) + \overline{g}_{jl}(t,x)); -\partial_t \underline{u}^{i,j,n}(t,x) - \mathcal{L}\underline{u}^{i,j,n}(t,x) \\ \\ g^{ij,-,n}(t,x,(\underline{u}^{k,l,n}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x\underline{u}^{i,j,n}(t,x), B^{ij}\underline{u}^{i,j,n}(t,x))\} = 0; \\ \\ \underline{u}^{i,j,n}(T,x) = h^{i,j}(x). \end{cases} \tag{1.3.15}$$

We define

$$\bar{u}^{ij}(t,x) := \lim_{m\to\infty} \bar{u}^{i,j,m}(t,x), \quad \underline{u}^{ij}(t,x) := \lim_{n\to\infty} \underline{u}^{i,j,n}(t,x).$$

Then, as a by-product of Proposition 1.3.2 and 1.3.3, we have:

Corollary 1.3.1. *$\forall (i,j) \in A^1 \times A^2$, the function $\bar{u}^{ij}$ (resp. $\underline{u}^{ij}$) is usc (resp. lsc). Moreover, $\bar{u}^{ij}$ and $\underline{u}^{ij}$ belong to Π_g, for any $(t,x) \in [0,T] \times R$,*

$$\underline{u}^{ij}(t,x) \le \bar{u}^{ij}(t,x).$$

1.3.3 Main results

In this paper we will show the uniqueness and existence of solution for (1.3.2). To begin with, we need the following lemma for the proof of uniqueness.

Lemma 1.3.1. *Let $(\overrightarrow{u})_{(i,j)\in A^1\times A^2}$ (resp. $(\overrightarrow{w})_{(i,j)\in A^1\times A^2}$) be an usc subsolution (resp. lsc supersolution) of (1.3.2) which belongs to Π_g. For $(t,x) \in [0,T] \times R$ and let $\Gamma(t,x)$ be the following set:*

$$\Gamma(t,x) := \{(i,j) \in A^1 \times A^2, u^{ij}(t,x) - w^{ij}(t,x) = \max_{(k,l)\in A^1\times A^2}(u^{kl}(t,x) - w^{kl}(t,x))\}.$$

Then there exists $(i_0,j_0) \in \Gamma(t,x)$ such that

$$u^{i_0j_0}(t,x) > L^{i_0j_0}[\overrightarrow{u}](t,x), \quad w^{i_0j_0}(t,x) < U^{i_0j_0}[\overrightarrow{w}](t,x). \tag{1.3.16}$$

We then prove:

Theorem 1.3.1. *Let $(\overrightarrow{u})_{(i,j)\in A^1\times A^2}$ (resp. $(\overrightarrow{w})_{(i,j)\in A^1\times A^2}$) is an usc subsolution (resp. lsc supersolution) of (1.3.2) which belongs to Π_g. Then it holds that for any $(i,j) \in A^1 \times A^2$,*

$$u^{ij}(t,x) \le w^{ij}(t,x), \quad (t,x) \in [0,T] \times R.$$

Corollary 1.3.2. *System (1.3.2) has at most one viscosity solution belongs to Π_g, and it is necessary continuous.*

It will takes three steps to prove the existence.

Proposition 1.3.4. *The family $(\bar{u}^{ij})_{(i,j)\in A^1\times A^2}$ is a viscosity subsolution of the system (1.3.2).*

Proposition 1.3.5. *Let m_0 be fixed in N. Then the family $(\bar{u}^{ij,m_0})_{(i,j)\in A^1\times A^2}$ is a viscosity supersolution of the system (1.3.2).*

Consider now the set $\mathcal{U}_{m_0}$ defined as follows.

$$\mathcal{U}_{m_0} = \{\overrightarrow{u} := (u^{ij})_{(i,j)\in A^1\times A^2} s.t. \overrightarrow{u} \text{ is a subsolution of (1.3.2)}, \forall (i,j) \in A^1 \times A^2, \bar{u}^{i,j} \le u^{i,j} \le \bar{u}^{ij,m_0}\}.$$

$\mathcal{U}_{m_0}$ is not empty since it contains $(\bar{u}^{ij})_{(i,j)\in A^1\times A^2}$. Next for $(t,x) \in [0,T] \times R$ and $(i,j) \in A^1 \times A^2$, set:

$$^{m_0}u^{ij}(t,x) = sup\{u^{ij}(t,x),\ (u^{kl})_{(k,l)\in A^1\times A^2} \in \mathcal{U}_{m_0}\}.$$

Now we give the main result of this section, which mainly consists in adapting the Perron's method to construct a viscosity solution to (1.3.2).

Theorem 1.3.2. *The family $(^{m_0}u^{ij})_{(i,j)\in A^1\times A^2}$ does not depend on m_0 and is the unique continuous viscosity solution in the class Π_g of the system (3.1.2).* □

1.4 On the identity of min-max and max-min solutions of Systems of Variational Inequalities with Interconnected Bilateral Obstacles.

1.4.1 Assumptions and notations

Let T (resp. k, d) be a fixed positive constant (resp. two integers) and Γ^1 (resp. Γ^2) denote the set of switching modes for player 1 (resp. 2). For later use, we shall denote by Λ the cardinal of the product set $\Gamma^1\times\Gamma^2$ and for $(i,j)\in\Gamma^1\times\Gamma^2$, $(\Gamma^1)^{-i}:=\Gamma^1-\{i\}$ and $(\Gamma^2)^{-j}:=\Gamma^2-\{j\}$. For $\vec{y}=(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}\in I\!R^{\Lambda}$, $(i,j)\in\Gamma^1\times\Gamma^2$, and $\underline{y}\in I\!R$, we denote by $[(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2-\{i,j\}},\underline{y}]$ the matrix obtained from the matrix $\vec{y}=(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ by replacing the element y^{ij} with $\underline{y}$.

For any $(i,j)\in\Gamma^1\times\Gamma^2$, let

$$
\begin{array}{l}
b:(t,x)\in[0,T]\times I\!R^k\mapsto b(t,x)\in I\!R^k;\\
\sigma:(t,x)\in[0,T]\times I\!R^k\mapsto \sigma(t,x)\in I\!R^{k\times d};\\
f^{ij}:(t,x,\vec{y},z)\in[0,T]\times I\!R^{k+\Lambda+d}\mapsto f^{ij}(t,x,\vec{y},z)\ \in I\!R\,;\\
\underline{g}_{ik}:(t,x)\in[0,T]\times I\!R^k\mapsto \underline{g}_{ik}(t,x)\in I\!R\quad (k\in(\Gamma^1)^{-i});\\
\bar{g}_{jl}:(t,x)\in[0,T]\times I\!R^k\mapsto \bar{g}_{jl}(t,x)\in I\!R\quad (l\in(\Gamma^2)^{-j});\\
h^{ij}:x\in I\!R^k\mapsto h^{ij}(x)\in I\!R.
\end{array}
$$

A function $\Phi:(t,x)\in[0,T]\times I\!R^k\mapsto\Phi(t,x)\in I\!R$ is called of *polynomial growth* if there exist two non-negative real constants C and γ such that

$$|\Phi(t,x)|\le C(1+|x|^{\gamma}),\quad (t,x)\in[0,T]\times I\!R^k.$$

Hereafter, this class of functions is denoted by Π_g. Let $\mathcal{C}^{1,2}([0,T]\times I\!R^k)$ (or simply $\mathcal{C}^{1,2}$) denote the set of real-valued functions defined on $[0,T]\times I\!R^k$, which are once (resp. twice) differentiable w.r.t. t (resp. x) and with continuous derivatives.

The following assumptions on the data of the systems (1.4.7) and (1.4.8) are in force throughout the paper.

(**H0**) The functions b and σ are jointly continuous in (t,x) and Lipschitz continuous w.r.t. x uniformly in t, meaning that there exists a non-negative constant C such that for any $(t,x,x')\in[0,T]\times\mathbb{R}^{k+k}$ we have

$$|\sigma(t,x)-\sigma(t,x')|+|b(t,x)-b(t,x')|\le C|x-x'|.$$

Therefore, they are also of linear growth w.r.t. x, i.e., there exists a constant C such that for any $(t,x)\in[0,T]\times I\!R^k$,

$$|b(t,x)|+|\sigma(t,x)|\le C(1+|x|).$$

(**H1**) Each function f^{ij}

(i) is continuous in (t,x) uniformly w.r.t. the other variables $(\vec{y},z)$ and, for any (t,x), the mapping $(t,x)\to f^{ij}(t,x,0,0)$ is of polynomial growth.

(ii) is Lipschitz continuous with respect to the variables $(\vec{y}:=(y^{ij})_{(i,j)\in\Gamma_1\times\Gamma_2},z)$ uniformly in (t,x), i.e. $\forall\,(t,x)\in[0,T]\times\mathbb{R}^k$, $\forall\,(\vec{y}_1,\vec{y}_2)\in\mathbb{R}^{\Lambda}\times\mathbb{R}^{\Lambda}$, $(z^1,z^2)\in\mathbb{R}^d\times\mathbb{R}^d$,

$$|f^{ij}(t,x,\vec{y}_1,z_1)-f^{ij}(t,x,\vec{y}_2,z_2)|\le C\left(|\vec{y}_1-\vec{y}_2|+|z_1-z_2|\right),$$

where, $|\vec{y}|$ stands for the standard Euclidean norm of $\vec{y}$ in $I\!R^{\Lambda}$.

(**H2**) Monotonicity: Let $\vec{y}=(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$. For any $(i,j)\in\Gamma^1\times\Gamma^2$ and any $(k,l)\neq(i,j)$ the mapping $y^{kl}\to f^{ij}(s,\vec{y},z)$ is non-decreasing.

(**H3**) The functions h^{ij}, which are the terminal conditions in the systems (1.4.7) and (1.4.8), are continuous with respect to x, belong to class Π_g and satisfy $\forall(i,j)\in\Gamma^1\times\Gamma^2$ and $x\in I\!R^k$,

$$\max_{k\in(\Gamma^1)^{-i}}\left(h^{kj}(x)-\underline{g}_{ik}(T,x)\right)\le h^{ij}(x)\le\min_{l\in(\Gamma^2)^{-j}}\left(h^{il}(x)+\bar{g}_{jl}(T,x)\right).$$

(**H4**) The no free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t,x), belong to Π_g and satisfy the following condition:

For any loop in $\Gamma^1 \times \Gamma^2$, i.e., any sequence of pairs $(i_1,j_1),\dots,(i_N,j_N)$ of $\Gamma^1 \times \Gamma^2$ such that $(i_N,j_N)=(i_1,j_1)$, $\text{card}\{(i_1,j_1),\dots,(i_N,j_N)\} = N-1$ and any $q=1,\dots,N-1$, either $i_{q+1}=i_q$ or $j_{q+1}=j_q$, we have $\forall (t,x) \in [0,T]\times I\!R^k$,

$$\sum_{q=1,N-1} \varphi_{i_q i_{q+1}}(t,x) \neq 0, \tag{1.4.1}$$

where, $\forall\ q=1,\dots,N-1,\ \varphi_{i_q i_{q+1}}(t,x) = -\underline{g}_{i_q i_{q+1}}(t,x)\mathbb{1}_{i_q \neq i_{q+1}} + \bar{g}_{j_q i_{q+1}}(t,x)\mathbb{1}_{j_q \neq j_{q+1}}$.

This assumption implies in particular that

$$\forall\ (i_1,\dots,i_N) \in (\Gamma^1)^N \text{ such that } i_N = i_1 \text{ and } \text{card}\{i_1,\dots,i_N\} = N-1,\ \sum_{p=1}^{N-1} \underline{g}_{i_k,i_{k+1}} > 0 \tag{1.4.2}$$

and

$$\forall\ (j_1,\dots,j_N) \in (\Gamma^2)^N \text{ such that } j_N = j_1 \text{ and } \text{card}\{j_1,\dots,j_N\} = N-1,\ \sum_{p=1}^{N-1} \bar{g}_{j_k,j_{k+1}} > 0. \tag{1.4.3}$$

By convention we set $\bar{g}_{j,j} = \underline{g}_{i,i} = 0$.

Conditions (1.4.2) and (1.4.3) are classical in the literature of switching problems and usually referred to as the *no free loop property*. □

We now introduce the probabilistic tools we need later. Let $(\Omega,\mathcal{F},P)$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B=(B_t)_{0\le t\le T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s\le t\})_{0\le t\le T}$. Let $\mathbf{F}=(\mathcal{F}_t)_{0\le t\le T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\le t\le T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{0\le t\le T}$ satisfies the usual conditions, i.e., it is right continuous and complete. On the other hand let $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathbf{F}$-progressively measurable sets.

Next, let

(i) $\mathcal{H}^{2,\ell}$ $(\ell \ge 1)$ be the set of $\mathcal{P}$-measurable and $I\!R^\ell$-valued processes $w=(w_t)_{t\le T}$ such that $\mathbb{E}[\int_0^T |w_s|^2 ds] < \infty$;

(ii) $\mathcal{S}^2$ (resp. $\mathcal{S}_d^2$) be the set of $\mathcal{P}$-measurable continuous (resp. RCLL) processes such that $\mathbb{E}[\sup_{t\le T}|w_t|^2] < \infty$.

(iii) $\mathcal{A}_i^2$ be the subset of $\mathcal{S}^2$ of non-decreasing processes $K=(K_t)_{t\le T}$ such that $K_0=0$.

For $(t,x)\in[0,T]\times I\!R^k$, let $X^{t,x}$ be the diffusion process solution of the following standard SDE:

$$\forall s\in[t,T],\ \ X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dB_r;\ \ X_s^{t,x}=x,\ s\in[0,t]. \tag{1.4.4}$$

Under Assumption (H0) on b and σ, the process $X^{t,x}$ exists and is unique. Moreover, it satisfies the following estimates: For all $p\ge 1$,

$$\mathbb{E}[\sup_{s\le T}|X_s^{t,x}|^p] \le C(1+|x|^p). \tag{1.4.5}$$

Its infinitesimal generator $\mathcal{L}^X$ is given, for every $(t,x)\in[0,T]\times I\!R^k$ and $\phi\in\mathcal{C}^{1,2}$, by

$$\begin{array}{rcl} \mathcal{L}^X\phi(t,x) & := & \frac{1}{2}\sum\limits_{i,j=1}^{k} (\sigma\sigma^*(t,x))_{i,j}\partial^2_{x_ix_j}\phi(t,x) + \sum_{i=1,k} b_i(t,x)\partial_{x_i}\phi(t,x) \\ & = & \frac{1}{2}Tr[\sigma\sigma^\top(t,x)D^2_{xx}\phi(t,x)] + b(t,x)^\top D_x\phi(t,x). \quad \square \end{array} \tag{1.4.6}$$

1.4.2 Motivation

Let us consider the following two systems of partial differential equations (PDEs) with bilateral interconnected obstacles (i.e., the obstacles depend on the solution) of min-max and max-min types: for any $(i,j)\in\Gamma^1\times\Gamma^2$, $(t,x)\in[0,T]\times I\!R^k$,

$$\left\{\begin{array}{l} \min\Big\{v^{ij}(t,x) - L^{ij}(\vec{v})(t,x)\,;\, \max\Big\{v^{ij}(t,x) - U^{ij}(\vec{v})(t,x); \\ \qquad -\partial_t v^{ij} - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x v^{ij}(t,x))\Big\}\Big\} = 0\,; \\ v^{ij}(T,x) = h^{ij}(x) \end{array}\right. \tag{1.4.7}$$

and

$$\begin{cases} \max\Big\{\check{v}^{ij}(t,x) - U^{ij}(\vec{\check{v}})(t,x)\,; \min\Big\{\check{v}^{ij}(t,x) - L^{ij}(\vec{\check{v}})(t,x) \\ \qquad\qquad -\partial_t\check{v}^{ij} - \mathcal{L}^X(\check{v}^{ij})(t,x) - f^{ij}(t,x,(\check{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x\check{v}^{ij}(t,x))\Big\}\Big\} = 0\,; \\ \check{v}^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{1.4.8}$$

where
(i) Γ^1 and Γ^2 are finite sets (possibly different);
(ii) For any $(t,x)\in[0,T]\times I\!R^k$, $\vec{v}(t,x) = (v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}$ and for any $(i,j)\in\Gamma^1\times\Gamma^2$,

$$L^{ij}(\vec{v})(t,x) = \max_{k\in\Gamma^1, k\neq i}\{v^{kj}(t,x) - \underline{g}_{ik}(t,x)\},\ U^{ij}(\vec{v})(t,x) = \min_{p\in\Gamma^2, p\neq j}\{v^{ip}(t,x) + \bar{g}_{jp}(t,x)\}.$$

(iii) $\mathcal{L}^X$ is a second order generator associated with a diffusion process described below.

The systems (1.4.7) and (1.4.8) are of min-max and max-min types respectively. The barriers $L^{ij}(\vec{v}), U^{ij}(\vec{v})$ and $L^{ij}(\vec{\check{v}}), U^{ij}(\vec{\check{v}})$ depend on the solution $\vec{v}$ and $\vec{\check{v}}$ of (1.4.7) and (1.4.8) respectively. They are related to zero-sum switching game problems since actually, specific cases of these systems, stand for the Hamilton-Jacobi-Bellman-Isaacs equations associated with those games.

Switching problems have recently attracted a lot of research activities, especially in connection with mathematical finance, commodities, and in particular energy, markets, etc (see e.g. [13, 45, 10, 11, 23, 3, 18, 17, 20, 29, 30, 34, 36, 40, 43, 44, 54, 61, 19, 52, 60] and the references therein). Several points of view, mainly dealing with control problems have been considered (theoritical and applied [13, 45, 11, 18, 20, 29, 34, 54], numerics [10, 29], filtering and partial information [40]). However, except [34, 36], problems related to games did not attract that much interest in the literature.

In [17], by means of systems of reflected backward stochastic differential equations (BSDEs) with inter-connected obstacles in combination with Perron's method, Djehiche et al. ([17]) have shown that each of the systems (1.4.7) and (1.4.8) has a unique continuous solution with polynomial growth, under classical assumptions on the data f^{ij}, $\bar{g}_{ij}$, $\underline{g}_{ij}$, h^{ij}. The question of whether or not these solutions coincide was conjectured as an open problem, leaving a possible connection of the solution of system (1.4.7) and (1.4.8) with zero-sum switching games unanswered.

1.4.3 Main results

The main objective of this paper is two-fold: (i) to investigate under which additional assumptions on the data of these problems, the unique solutions of systems (1.4.7) and (1.4.8) coincide; (ii) to make a connection between this solution and the value function of the associated zero-sum switching game. Indeed, we show that if the switching costs of one side, i.e. either $(\bar{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ or $(\underline{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, are regular enough, then the solutions of the systems (1.4.7) and (1.4.8) coincide. Furthermore, we show that this solution has a representation as a value function of a zero-sum switching game. To the best of our knowledge these issues have not been addressed in the literature yet. The main strategy to obtain these results is to show that the barriers, which depend on the solution, are comparable and then to make use of Theorem 5.5.1 (whose proof in an appendix at the end of the paper) to conclude that the solutions of the min-max and max-min systems coincide. This comparison is obtained under a regularity assumption on $(\bar{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ or $(\underline{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$. Theorem 5.5.1 extends a result derived in [27] on min-max and max-min PDEs with fixed obstacles, where we relax the condition of strict separation between the obstacles. To get the result that (1.4.7) and (1.4.8) coincide, we should be able to compare the inter-connected obstacles of them, i.e.

$$\begin{array}{ll} \text{(i)} & \forall(i,j)\in\Gamma^1\times\Gamma^2,\quad L^{ij}(\vec{v}) \le U^{ij}(\vec{v}) \\ \text{or} & \\ \text{(ii)} & \forall(i,j)\in\Gamma^1\times\Gamma^2,\quad L^{ij}(\vec{\underline{v}}) \le U^{ij}(\vec{\underline{v}}). \end{array} \tag{1.4.9}$$

For that let us introduce the following assumption.

(H5):

(i) For any $(i,j)\in\Gamma^1\times\Gamma^2$, the functions $\bar{g}_{ij}$ are $\mathcal{C}^{1,2}$. Moreover, $D_x\bar{g}_{ij}$ and $D^2_{xx}\bar{g}_{ij}$ belong to Π_g. Furthermore, for any $j_1,j_2,j_3\in\Gamma_2$ such that $|\{j_1,j_2,j_3\}| = 3$,

$$\bar{g}_{j_1j_3}(t,x) < \bar{g}_{j_1j_2}(t,x) + \bar{g}_{j_2j_3}(t,x),\ \forall(t,x)\in[0,T]\times I\!R^k.$$

(ii) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} verifies the following estimate:

$$|f^{ij}(t,x,\vec{y},z^{ij})| \leq C(1+|x|^p)$$

for some real constants C and p.

Proposition 1.4.1. *Under Assumptions (H0)-(H5) we have, for every* $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$L^{ij}(\vec{v}) \leq v^{ij} \leq U^{ij}(\vec{v}).$$

The following estimate is the key to the proof of this proposition: For every $(i,j) \in \Gamma^1 \times \Gamma^2$ and $m \geq 0$,

$$\mathbb{E}\Big\{m \int_0^T \sum_{l \in (\Gamma^2)^{-j}} \{\check{Y}_s^{ij,m} - \check{Y}_s^{il,m} - \bar{g}_{jl}(s, X_s^{t,x})\}^+ ds\Big\} \leq C(1+|x|^p), \tag{1.4.10}$$

where, $\check{Y}$ is the unique solution of system of RBSDE (4.2.2) , and the constant C is independent of m and x. As a by product of Proposition 1.4.1 and Theorem 5.5.2 (displayed in the appendix), we have:

Theorem 1.4.1. *Under Assumptions (H0)-(H5), for any* $(i,j) \in \Gamma^1 \times \Gamma^2$, *it holds that*

$$v^{ij} = \check{v}^{ij}.$$

The next main result is the connection between this solution and the value function of the associated zero-sum switching game. Let us made the following assumption:

(H6):

(**i**) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} does not depend on z^{ij}.

(**ii**) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} does not depend on $(\vec{y}, z^{ij})$. □

We first describe briefly the zero-sum switching game. Assume we have two players π_1 and π_2 who intervene on a system with the help of switching strategies. An admissible switching strategy for π_1 (resp. π_2) is a sequence $\delta := (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu := (\tau_n, \zeta_n)_{n\geq 0}$) where for any $n \geq 0$,
(i) σ_n (resp. τ_n) is an $\mathbf{F}$-stopping times such that P-a.s., $\sigma_n \leq \sigma_{n+1} \leq T$ (resp. $\tau_n \leq \tau_{n+1} \leq T$) ;
(ii) ξ_n (resp. ζ_n) is a random variable with values in Γ^1 (resp. Γ^2) which is $\mathcal{F}_{\sigma_n}$ (resp. $\mathcal{F}_{\tau_n}$)-measurable ;
(iii) $P[\sigma_n < T, \forall n \geq 0] = P[\tau_n < T, \forall n \geq 0] = 0$;
(iv) If $(A_s^\delta)_{s\leq T}$ and $(B_s^\nu)_{s\leq T}$ are the $\mathbf{F}$-adapted RCLL processes defined by:

$$\forall\, s \in [t,T), \quad A_s^\delta = \sum_{n\geq 1} g_{\xi_{n-1}\xi_n}(\sigma_n, X_{\sigma_n}^{t,x}) 1_{[\sigma_n \leq s]} \quad \text{and} \quad A_T^\delta = \lim_{s\to T} A_s^\delta,$$

and

$$\forall\, s \in [t,T), \quad B_s^\nu = \sum_{n\geq 1} \bar{g}_{\zeta_{n-1}\zeta_n}(\tau_n, X_{\tau_n}^{t,x}) 1_{[\tau_n \leq s]} \quad \text{and} \quad B_T^\nu = \lim_{s\to T} B_s^\nu.$$

Then, $\mathbb{E}[(A_T^\delta)^2 + (B_T^\nu)^2] < \infty$. For any $s \leq T$, A_s^δ (resp. B_s^ν) is the cumulative switching cost at time s for π_1 (resp. π_2) when she implements the strategy δ (resp. ν).

Next, for $t \in I\!R$, $i \in \Gamma^1$ (resp. $j \in \Gamma^2$), we say that the admissible strategy $\delta := (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu := (\tau_n, \zeta_n)_{n\geq 0}$) belongs $\mathcal{A}_{\pi_1}^i(t)$ (resp. $\mathcal{A}_{\pi_2}^i(t)$) if

$$\sigma_0 = t,\ \xi_0 = i,\ \mathbb{E}[(A_T^\delta)^2] < \infty \quad (\text{resp. } \tau_0 = t,\ \zeta_0 = j,\ \mathbb{E}[(B_T^\nu)^2] < \infty).$$

Given an admissible strategy δ (resp. ν) of π_1 (resp. π_2) one associates a stochastic process $(u_s)_{s\leq T}$ (resp. $(v_s)_{s\leq T}$) which indicates along with time the current mode of π_1 (resp. π_2) and which is defined by:

$$\forall s \leq T,\ u_s = \xi_0 1_{\{\sigma_0\}}(s) + \sum_{n\geq 1} \xi_{n-1} 1_{]\sigma_{n-1},\sigma_n]}(s) \text{ (resp. } v_s = \zeta_0 1_{\{\tau_0\}}(s) + \sum_{n\geq 1} \zeta_{n-1} 1_{]\tau_{n-1},\tau_n]}(s)). \tag{1.4.11}$$

Let now $\delta = (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu = (\tau_n, \zeta_n)_{n\geq 0}$) be a strategy for π_1 (resp. π_2) which belongs to $\mathcal{A}^i_{\pi_1}(t)$ (resp. $\mathcal{A}^j_{\pi_2}(t)$). The interventions of the players are not free and generate a payoff which is a reward (resp. cost) for π_1 (resp. π_2) and whose expression is given by

$$J_t(\delta,\nu) := \mathbb{E}[h^{u_T v_T}(X_T) + \int_t^T f(r, X_r^{t,x}, u_r, v_r)dr - A_T^\delta + B_T^\nu|\mathcal{F}_t], \tag{1.4.12}$$

where, for any $(k,l) \in \Gamma^1 \times \Gamma^2$, we set $f(s,x,k,l) = f^{kl}(s,x)$, since f^{kl} is assumed to not depend on $(\vec{y}, z^{ij})$.

Theorem 1.4.2. *Suppose Assumptions (H0)-(H5) and (H6)-(ii) are satisfied. Then, for any* $(i_0, j_0) \in \Gamma^1 \times \Gamma^2$,

$$v^{i_0 j_0}(t,x) \quad = ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)} ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)} J_t(\delta,\nu) = ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)} ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)} J_t(\delta,\nu), \tag{1.4.13}$$

where $J_t(\delta,\nu)$ *is the payoff of the switching game defined in* (1.4.12).

As a by product of Theorem (1.4.1) and the uniqueness of the solution of system (1.4.7) we have the following result in the case when the functions f^{ij} depend also on $\vec{y}$.

Corollary 1.4.1. *Suppose Assumptions (H0)-(H5) and (H6)-(i) are satisfied and let* $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *be the unique solution of system (1.4.7) and (1.4.8). Then for any* $(t,x) \in [0,T]\times\mathbb{R}^k$ *and* $(i_0,j_0) \in \Gamma^1\times\Gamma^2$,

$$v^{i_0 j_0}(t,x) = ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}} ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}} \bar{J}_t(\delta,\nu) = ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}} ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}} \bar{J}_t(\delta,\nu). \tag{1.4.14}$$

where,

$$\bar{J}_t(\delta,\nu) := \mathbb{E}[h^{u_T v_T}(X_T) + \int_t^T f^{u_r v_r}(r, X_r^{t,x}, (v^{kl}(r, X_r^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2})dr - A_T^\delta + B_T^\nu|\mathcal{F}_t]. \tag{1.4.15}$$

Chapter 2

Systems of Integro-PDEs with Interconnected Obstacles and Multi-Modes Switching Problem Driven by Lévy Process.

This chapter is a joint work with Said Hamadène.

2.1 Preliminaries

A Lévy process is an $\mathbb{R}$-valued RCLL (for right continuous with left limits) stochastic process $L = \{L_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with stationary and independent increments ($L_0 = 0$) and stochastically continuous.

For $t \leq T$ let us set $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$ where $\mathcal{G}_t := \sigma\{L_s, 0 \leq s \leq t\}$ and $\mathcal{N}$ is the $\mathcal{P}$-null sets of $\mathcal{F}$, therefore $\{\mathcal{F}_t\}_{t\leq T}$ is complete and right continuous. Next by $\mathcal{P}$ we denote the σ-algebra of predictable processes on $[0,T] \times \Omega$ and finally for any RCLL process $(\Gamma_t)_{t\leq}$ we denote by $\Gamma_{t-} := \lim_{s\nearrow t} \Gamma_s$ and $\Delta\Gamma_t := \Gamma_t - \Gamma_{t-}$ its jump at t, $t \in (0,T]$.

We now introduce the following spaces:

(a) $\mathcal{S}^2 := \{\varphi := \{\varphi_t, 0 \leq t \leq T\}$ is an $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted RCLL process s.t. $\mathbb{E}(\sup_{0\leq t\leq T} |\varphi_t|^2) < \infty\}$; $\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of non-decreasing continuous processes null at $t = 0$;
(b) $H^2 := \{\varphi := \{\varphi_t, 0 \leq t \leq T\}$ is an $\mathbb{R}$-valued, $\mathcal{F}_t$-progressively measurable process s.t. $\mathbb{E}(\int_0^T |\varphi_t|^2 dt) < \infty\}$;
(c) $\ell^2 := \{x = (x_n)_{n\geq 1}$ is an $\mathbb{R}$-valued sequence s.t. $\|x\|^2 := \sum_{i=1}^{\infty} x_i^2 < \infty\}$;
(d) $\mathcal{H}^2(\ell^2) := \{\varphi = (\varphi_t)_{t\leq T} = ((\varphi_t^n)_{n\geq 1})_{t\leq T}$ such that $\forall n \geq 1$, φ^n is a $\mathcal{P}$-measurable process and

$$\mathbb{E}(\int_0^T \|\varphi_t\|^2 dt) = \sum_{i=1}^{\infty} \mathbb{E}(\int_0^T |\varphi_t^i|^2 dt) < \infty\};$$

$\mathcal{L}^2 := \{\xi$, an $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|\xi|^2] < \infty\}$;
(e) Π_g is the space of deterministic functions $u(t,x)$ from $[0,T] \times \mathbb{R}^k$ into $\mathbb{R}$ of polynomial growth, i.e., such that for some nonnegative constants p and C one has,

$$|u(t,x)| \leq C(1 + |x|^p),\ \forall (t,x) \in [0,T] \times \mathbb{R}^k. \quad \square$$

Let us now recall the Lévy-Khintchine formula of a Lévy process $(L_t)_{t\leq T}$ whose characteristic exponent is Ψ, *i.e.*,

$$\forall t \leq T \text{ and } \theta \in \mathbb{R},\ \mathbb{E}(e^{i\theta L_t}) = e^{t\Psi(\theta)}$$

with

$$\Psi(\theta) = ia\theta - \frac{1}{2}\varpi^2\theta^2 + \int_{I\!R}(e^{i\theta x} - 1 - i\theta x \mathbb{1}_{(|x|<1)})\Pi(dx)$$
$$= ia\theta - \frac{1}{2}\varpi^2\theta^2 + \int_{|x|\geq 1}(e^{i\theta x} - 1)\Pi(dx) + \int_{0<|x|<1}(e^{i\theta x} - 1 - i\theta x)\Pi(dx)$$

where $a \in I\!R$, $\varpi \geq 0$ and Π is a σ-finite measure on $I\!R^* := I\!R - \{0\}$ (we $\Pi(\{0\}) = 0$ and then the domain of integration is the whole space), called the Lévy measure of L, satisfying

$$\int_{I\!R}(1 \wedge x^2)\Pi(dx) < \infty \tag{2.1.1}$$

and

$$\exists \epsilon > 0, \lambda > 0 \ s.t. \int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}\Pi(dx) < +\infty. \tag{2.1.2}$$

Conditions (3.1.1)-(2.1.2) imply that for any $i \geq 2$,

$$\textstyle\int_{I\!R} |x|^i \Pi(dx) < \infty$$

and then the process $(L_t)_{t\leq T}$ have moments of any order.

Next following Nualart-Schoutens [48] we define, for every $i \geq 1$, the so-called power-jump processes $L^{(i)}$ and their compensated version $Y^{(i)}$, also called Teugels martingales, as follows: $\forall t \leq T$,

$$L_t^{(1)} = L_t \text{ and for } i \geq 2,\ L_t^{(i)} = \textstyle\sum_{s\leq t}(\Delta L_s)^i, Y_t^{(i)} = L_t^{(i)} - t\mathbb{E}(L_1^{(i)}).$$

Note that for any $t \leq T$, $\mathbb{E}(L_t^{(i)}) = t\int_{I\!R} x^i\Pi(dx) < \infty$ for any $i \geq 2$ ([46], pp.29).

An orthonormalization procedure can be applied to the martingales $Y^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $(H^{(i)})_{i\geq 1}$ such that each $H^{(i)}$ is a linear combination of $(Y^{(j)})_{j=1,i}$, i.e.,

$$H^{(i)} = c_{i,i}Y^{(i)} + ... + c_{i,1}Y^{(1)}.$$

It has been shown in [48] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, ...$ with respect to the measure $\nu(dx) = x^2\Pi(dx) + \varpi^2\delta_0(dx)$ (δ_0 is the Dirac measure at 0). Specifically the polynomials $(q_i)_{i\geq 0}$ defined by

$$q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + ... + c_{i,1}, i \geq 1$$

satisfy

$$\textstyle\int_{I\!R} q_n(x)q_m(x)\nu(dx) = \delta_{nm},\ \forall n, m \geq 0.$$

Next let us set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + ... + c_{i,1}x \text{ and}$$
$$\tilde{p}_i(x) = x(q_{i-1}(x) - q_{i-1}(0)) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + ... + c_{i,2}x^2.$$

Then for any $i \geq 1$ and $t \leq T$ we have:

$$H_t^{(i)} = \textstyle\sum_{0<s\leq t}\{c_{i,i}(\Delta L_s)^i + ... + c_{i,2}(\Delta L_s)^2\} + c_{i,1}L_t - t\mathbb{E}[c_{i,i}(L_1)^{(i)} + ... + c_{i,2}(L_1)^{(2)}] - tc_{i,1}\mathbb{E}(L_1)$$
$$= q_{i-1}(0)L_t + \textstyle\sum_{0<s\leq t}\tilde{p}_i(\Delta L_s) - t\mathbb{E}[\sum_{0<s\leq 1}\tilde{p}_i(\Delta L_s)] - tq_{i-1}(0)\mathbb{E}(L_1).$$

As a consequence, for any $t \leq T$ and $i \geq 1$, $\Delta H_t^{(i)} = p_i(\Delta L_t)$ for each $i \geq 1$. In the particular case of $i = 1$, we obtain

$$H_t^{(1)} = c_{1,1}(L_t - t\mathbb{E}(L_1))$$

where

$$c_{1,1} = [\textstyle\int_{I\!R} x^2\Pi(dx) + \varpi^2]^{-\frac{1}{2}} \text{ and } \mathbb{E}[L_1] = a + \int_{|x|\geq 1} x\Pi(dx). \tag{2.1.3}$$

Finally note that for any $i, j \geq 1$ the predictable quadratic variation process of $H^{(i)}$ and $H^{(j)}$ is $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t, \forall t \leq T$. □

Remark 2.1.1. *If $\Pi = 0$, we are in the classical Brownian case and all non-zero degree polynomials $q_i(x)$ will vanish, giving $H^{(i)} = 0$, $i \geq 2$. On the other hand, if Π only has mass at 1, we are in the Poisson case and once more $H^{(i)} = 0$, $i \geq 2$. Both cases are degenerate ones in this Lévy process framework.* □

The main result in the paper by Nualart-Schoutens [47] is the following representation property which allows for developping the BSDE theory in this Lévy framework.

Theorem 2.1.1. *([47], pp.118). Let ζ be a random variable of $\mathcal{L}^2$, then there exists a process $Z = (Z^i)_{i\geq 1}$ that belongs to $\mathcal{H}^2(\ell^2)$ such that:*

$$\zeta = \mathbb{E}(\zeta) + \sum_{i\geq 1} \int_0^T Z_s^i dH_s^{(i)}. \quad \square$$

2.2 Systems of Reflected BSDEs with Oblique Reflection driven by a Lévy process

2.2.1 Reflected BSDE driven by a Lévy process and their relationship with IPDEs

As a consequence of Theorem 2.1.1, and as in the framework of Brownian noise only, one can study standard BSDEs or reflected ones. The result below related to existence and uniqueness of a solution for a reflected BSDE driven by a Lévy process, is proved in [56]. Indeed let us introduce a triplet (f, ξ, S) that satisfies:

Assumptions (A1):
(i) ξ a random variable of $\mathcal{L}^2$ which stands for the terminal value ;
(ii) f: $[0,T] \times \Omega \times I\!R \times \ell^2 \longrightarrow I\!R$ is a function such that the process $(f(t,0,0))_{t\leq T}$ belongs to H^2 and there exists a constant $\kappa > 0$ verifying

$$|f(t,y,z) - f(t,y',z')| \leq \kappa(|y-y'| + \|z-z'\|_{\ell^2}), \text{ for every } t, y, y', z \text{ and } z'.$$

(iii) $S := (S_t)_{0\leq t\leq T}$ is a process of $\mathcal{S}^2$ such that $S_T \leq \xi$, $\mathcal{P} - a.s.$, and whose jump times are inaccessible stopping times. This in particular implies that for any $t \leq T$, $S_t^p = S_{t-}$, where S^p is the predictable projection of S (see e.g. [14], pp.58) for more details on those notions.

In [56], the authors have proved the following result related to existence and uniqueness of the solution of one barrier reflected BSDEs whose noise is driven by a Lévy process.

Theorem 2.2.1. *Assume that the triple (f, ξ, S) satisfies Assumptions $(A1)$. Then there exists a unique triplet of processes $(Y, U, K) := ((Y_t, U_t, K_t))_{t\leq T}$ with values in $I\!R \times \ell^2 \times I\!R^+$ such that:*

$$\begin{cases} (Y,U,K) \in \mathcal{S}^2 \times \mathcal{H}(\ell^2) \times \mathcal{A}^2; \\ Y_t = \xi + \int_t^T f(s,Y_s,U_s)ds + K_T - K_t - \sum_{i=1}^{\infty} \int_t^T U_s^i dH_s^{(i)}, \ \forall t \leq T; \\ Y_t \geq S_t, \ \ \forall\ 0 \leq t \leq T \text{ and } \int_0^T (Y_t - S_t)dK_t = 0, \ P - a.s. \end{cases} \tag{2.2.1}$$

The triple (Y, U, K) is called the solution of the reflected BSDE associated with (f, ξ, S). □

To proceed we need to compare solutions of reflected BSDEs of types (2.2.1). So let us consider a stochastic process $V = (V_t)_{t\leq T} = (V^i)_{i\geq 1} = ((V_t^i)_{t\leq T})_{i\geq 1}$ which belongs to $\mathcal{H}^2(\ell^2)$ and let $M := (M_t)_{t\leq T}$ be the stochastic integral defined by:

$$\forall t \leq T, \ M_t := \sum_{i=1}^{\infty} \int_0^t V_s^i dH_s^{(i)}.$$

We next denote by $\varepsilon(M) := (\varepsilon(M)_t)_{t\leq T}$ the process that satisfies: $\forall t \leq T$,

$$\varepsilon(M)_t = 1 + \int_0^t \varepsilon(M)_{s-} dM_s.$$

By Doléans-Dade's formula we have (see e.g. [55]):

$$\forall t \leq T,\ \varepsilon(M)_t = \exp\Big\{M_t - \frac{1}{2}[M,M]^c_t - \sum_{0\leq s\leq t} \triangle M_s\Big\} \prod_{0\leq s\leq t} \{1+\triangle M_s\}.$$

Let us now introduce the following assumption on the process V.

Assumptions (A2): The process $V = (V^i)_{i\geq 1} = ((V^i_t)_{t\leq T})_{i\geq 1}$ verifies:

$$\sum_{i=1}^{\infty} V^i_t p_i(\triangle L_t) > -1, \quad d\mathcal{P}\otimes dt - a.e \tag{2.2.2}$$

and there exists a constant C such that:

$$\sum_{i=1}^{\infty} |V^i_t|^2 \leq C,\ d\mathcal{P}\otimes dt - a.e. \tag{2.2.3}$$

We then have:

Proposition 2.2.1. *Assume that Assumption (A2) is fulfilled. Then, $\mathcal{P}$-a.s., for any $t\in[0,T]$, $\varepsilon(M)_t > 0$ and $\varepsilon(M)\in\mathcal{S}^2$.*

Proof. First note that for any $t\leq T$,

$$\triangle M_t = \sum_{i=1}^{\infty} V^i_t \triangle H^{(i)}_t = \sum_{i=1}^{\infty} V^i_t p_i(\triangle L_t) > -1,$$

therefore for any $t\leq T$, $\varepsilon(M_t) > 0$. Next by using Doléans-Dade's formula and since $d\langle H^{(i)}, H^{(j)}\rangle_s = \delta_{ij} ds$, we have: $\forall t\leq T$,

$$\begin{aligned}
\varepsilon(M)^2_t &= \varepsilon(2M + [M,M])_t \\
&= \varepsilon(2\sum_{i=1}^{\infty}\int_0^{\cdot} V^i_s dH^{(i)}_s + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\cdot} V^i_s V^j_s d[H^{(i)},H^{(j)}]_s)_t \\
&= \varepsilon(2\sum_{i=1}^{\infty}\int_0^{\cdot} V^i_s dH^{(i)}_s + \sum_{i=1}^{\infty}\int_0^{\cdot} |V^i_s|^2 ds + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\cdot} V^i_s V^j_s d([H^{(i)},H^{(j)}]_s - \langle H^{(i)},H^{(j)}\rangle_s))_t \\
&= \varepsilon(N)_t \exp\{\sum_{i=1}^{\infty}\int_0^t |V^i_s|^2 ds\}
\end{aligned}$$

where for $t\leq T$,

$$N_t = 2\sum_{i=1}^{\infty}\int_0^t V^i_s dH^{(i)}_s + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t V^i_s V^j_s d([H^{(i)},H^{(j)}]_s - \langle H^{(i)},H^{(j)}\rangle_s)$$

is a local martingale. On the other hand, the quantity $\sum_{i=1}^{\infty}\int_0^T |V^i_s|^2 ds$ is bounded and $\varepsilon(N)\geq 0$, then for any $t\leq T$,

$$\mathbb{E}[(\varepsilon(M)_t)^2] \leq C\mathbb{E}[\varepsilon(N)_0] \leq C$$

since $\varepsilon(N)$ is a supermartingale. It follows that $\varepsilon(M)$ is not only a local martingale but also a martingale and then by Doob's maximal inequality it belongs to $\mathcal{S}^2$. □

Remark 2.2.1. *The result of Proposition* 3.1 *still holds true if instead of (2.2.3) we only have*

$$\sum_{i=1}^{\infty}\int_0^T |V^i_s|^2 ds \leq C,\ \mathcal{P}-a.s. \quad \square \tag{2.2.4}$$

Next for two processes $U^i = (U^i_k)_{k\geq 1}$, $i=1,2$, of $\mathcal{H}^2(\ell^2)$ we define their scalar product in $\mathcal{H}^2(\ell^2)$ which we denote by $\langle U^1,U^2\rangle^p := (\langle U^1,U^2\rangle^p_t)_{t\leq T}$ as:

$$\forall t\leq T,\ \langle U^1,U^2\rangle^p_t = \sum_{k\geq 1} U^1_k(t)U^2_k(t).$$

Proposition 2.2.2. *: Let $\xi \in \mathcal{L}^2$, $\varphi := (\varphi_s)_{s\leq T} \in H^2$, $\delta := (\delta_s)_{s\leq T}$ a uniformly bounded process, and finally let $V = (V^i)_{i\geq 1} \in \mathcal{H}^2(\ell^2)$ satisfying (A2). Let $(Y, U) := (Y_t, U_t)_{t\leq T} \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$ be the solution of the following BSDE:*

$$\forall t \leq T,\ Y_t = \xi + \int_t^T (\varphi_s + \delta_s Y_s + \langle V, U\rangle_s^p)ds - \sum_{i=1}^{\infty} \int_t^T U_s^i dH_s^{(i)}. \tag{2.2.5}$$

For $t \leq T$, let $(X_s^t)_{s\in[t,T]}$ be the process defined as follows:

$$\forall s \in [t, T],\ X_s^t = e^{\int_t^s \delta_r dr} \frac{\varepsilon(M)_s}{\varepsilon(M)_t}. \tag{2.2.6}$$

Then for any $t \leq T$, Y_t satisfies:

$$Y_t = \mathbb{E}[X_T^t \xi + \int_t^T X_s^t \varphi_s ds|\mathcal{F}_t],\ \mathcal{P} - a.s..$$

On the other hand, if $(Y', U') \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$ is the solution of the BSDE:

$$Y_t' = \xi + \int_t^T f(s, Y_s', U_s')ds - \sum_{i=1}^{\infty} \int_t^T U_s'^i dH_s^{(i)}, \forall t \leq T \tag{2.2.7}$$

where

$$f(t, Y_t', U_t') \geq \varphi_t + \delta_t Y_t' + \langle V, U'\rangle_t^p,\ d\mathcal{P} \otimes dt - a.s.$$

then for any $t \leq T$,

$$Y_t' \geq \mathbb{E}[X_T^t \xi + \int_t^T X_s^t \varphi_s ds|\mathcal{F}_t],\ \mathcal{P} - a.s..$$

Proof. First note that the processes (Y, U) and (Y', U') exist thanks to Theorem 3.1. Let us now fix $t \in [0, T]$. Since V satisfies (A2) then $\varepsilon(M) > 0$ which implies that $(X_s^t)_{s\in[t,T]}$ is defined. On the other hand it satisfies

$$\forall s \in [t, T],\ dX_s^t = X_{s-}^t(\delta_s ds + dM_s)$$

and since δ is uniformly bounded then as in Proposition 3.1, one can show that $\mathbb{E}[\sup_{s\in[t,T]} |X_s^t|^2] < \infty$. Now by Itô's formula, for any $s \in [t, T]$, we have

$$\begin{aligned}
-d(Y_s X_s^t) =& - Y_{s-} dX_s^t - X_{s-}^t dY_s - d[Y, X^t]_s \\
=& - X_{s-}^t Y_{s-} \delta_s ds - Y_{s-} X_{s-}^t dM_s + X_{s-}^t \varphi_s ds + X_{s-}^t \delta_s Y_s ds \\
& - X_{s-}^t (\sum_{i\geq 1} U_s^i dH^{(i)}) - X_{s-}^t \{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} V_s^i U_s^j d([H^{(i)}, H^{(j)}]_s - \langle H^{(i)}, H^{(j)}\rangle_s)\} \\
=& X_s^t \varphi_s ds - dN_s
\end{aligned}$$

where for any $s \in [t, T]$

$$dN_s = Y_{s-} X_{s-}^t \sum_{i=1}^{\infty} V_s^i dH_s^{(i)} + X_{s-}^t \sum_{i\geq 1} U_s^i dH_s^{(i)} + X_{s-}^t \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} V_s^i U_s^j d([H^{(i)}, H^{(j)}]_s - \langle H^{(i)}, H^{(j)}\rangle_s).$$

Note that since X^t is uniformly square integrable, $Y \in \mathcal{S}^2$, $U \in \mathcal{H}^2(\ell^2)$ and finally taking into account Assumption (A2) on V, we get that N is a uniformly integrable martingale on $[t, T]$. Therefore taking conditional expectation to obtain:

$$Y_t = \mathbb{E}[X_T^t \xi + \int_t^T X_s^t \varphi_s ds|\mathcal{F}_t],\ \mathcal{P} - a.s.$$

which is the desired result.

We now focus on the second part of the claim. By Itô's formula we have: $\forall s \in [t,T]$,

$$\begin{aligned}-d(Y'_sX^t_s) =& -Y'_{s-}dX^t_s - X^t_{s-}dY'_s - d[Y',X^t]_s\\ =& -X^t_{s-}Y'_{s-}\delta_s ds - Y'_{s-}X^t_{s-}\sum_{i=1}^{\infty}V^i_s dH^{(i)}_s + X^t_{s-}f(s,Y'_s,U'_s)ds - X^t_{s-}\sum_{i=1}^{\infty}U'^i_s dH^{(i)}_s\\ &- X^t_{s-}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}V^i_sU'^j_s d[H^{(i)},H^{(j)}]_s.\end{aligned}$$

Next since $X^t \geq 0$ and taking into account the inequality satified by f to obtain:

$$-d(Y'_sX^t_s) \geq X^t_s\varphi_s ds - dN'_s \;\; \mathcal{P}-a.s.,$$

where for any $s \in [t,T]$,

$$dN'_s = Y'_{s-}X^t_{s-}\sum_{i=1}^{\infty}V^i_s dH^{(i)}_s - X^t_{s-}\sum U'^i_s dH^{(i)}_s - X^t_{s-}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}V^i_sU'^j_s d([H^{(i)},H^{(j)}]_s - \langle H^{(i)},H^{(j)}\rangle_s).$$

But once more N' is a uniformly integrable martingale then by taking the conditional expectation we obtain:

$$Y'_t \geq \mathbb{E}[X^t_T\xi + \int_t^T X^t_s\varphi_s ds|\mathcal{F}_t],\ \mathcal{P}-a.s.$$

which completes the proof. □

We are now ready to give a comparison result of solutions of two BSDEs of type (2.2.1).

Proposition 2.2.3. *For $i=1,2$, let (f_i,ξ_i) be a pair that satisfies Assumption (A1)-(i),(ii) and let $(Y^i,U^i) \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$ be the solution of the following BSDE: $\forall t \leq T$,*

$$Y^i_t = \xi_i + \int_t^T f_i(s,Y^i_s,U^i_s)ds - \sum_{j=1}^{\infty}\int_t^T U^{i,j}_s dH^{(j)}_s.$$

Assume that:

(i) For any $U^1,U^2 \in \mathcal{H}^2(l^2)$, there exists a process $V^{U^1,U^2} = (V_j^{U^1,U^2})_{j\geq 1}$ (which may depend on U^1 and U^2) satisfying (A2) such that f_1 verifies:

$$f_1(t,Y^2_t,U^1_t) - f_1(t,Y^2_t,U^2_t) \geq \langle V^{U^1,U^2},(U^1-U^2)\rangle^p_t,\ d\mathcal{P}\otimes dt - a.e.; \tag{2.2.8}$$

(ii) $\mathcal{P}-a.s.,\ \xi_1 \geq \xi_2$ and

$$f_1(t,Y^2_t,U^2_t) \geq f_2(t,Y^2_t,U^2_t), d\mathcal{P}\otimes dt - a.e.. \tag{2.2.9}$$

Then $\mathcal{P}$-a.s., $Y^1_t \geq Y^2_t,\ \forall t \in [0,T]$.

Proof. Let us set $\bar{Y} = Y^1 - Y^2$, $\bar{U} = U^1 - U^2$ and $\bar{\xi} = \xi^1 - \xi^2$, then $\forall t \in [0,T]$,

$$\bar{Y}_t = \bar{\xi} + \int_t^T\{f_1(s,Y^1_s,U^1_s) - f_2(s,Y^2_s,U^2_s)\}ds - \sum_{j=1}^{\infty}\int_t^T \bar{U}^j_s dH^{(j)}_s.$$

Next let us set:

$$\forall s \leq T,\ \delta_s = (f_1(s,Y^1_s,U^1_s) - f_1(s,Y^2_s,U^1_s)) \times (\bar{Y}_s)^{-1}\mathbb{1}_{\{\bar{Y}_s\neq 0\}} \text{ and } \varphi_s = f_1(s,Y^2_s,U^2_s) - f_2(s,Y^2_s,U^2_s).$$

Then by (2.2.9) we have, $\varphi_s \geq 0,\ d\mathcal{P}\otimes dt - a.e.$. On the other hand $(\delta_s)_{s\in[0,T]}$ is bounded since f_1 is uniformly Lipschitz. Finally we have

$$f_1(s,Y^1_s,U^1_s) - f_2(s,Y^2_s,U^2_s) \geq \varphi_s + \delta_s\bar{Y}_s + \langle V^{U^1,U^2},\bar{U}\rangle^p_s,\ d\mathcal{P}\otimes ds - a.e..$$

Therefore thanks to Proposition 3.2 we get,

$$\forall t \leq T,\ \bar{Y}_t \geq \mathbb{E}[X^t_T\bar{\xi} + \int_t^T X^t_s\varphi_s ds|F_s] \geq 0,\ \mathcal{P}-a.s.$$

where $(X^t_s)_{s\in[t,T]}$ is defined in the same way as in (2.2.6) with the new processes δ and φ. As X^t, $\bar{\xi}$ and φ are non-negative then for any $t \leq T$, $\bar{Y}_t \geq 0$ which implies that $\mathcal{P}-a.s., \forall t \leq T, Y^1_t \geq Y^2_t$ since Y^1 and Y^2 are RCLL. The proof of the claim is now complete. □

Remark 2.2.2. *Conditions (2.2.8) and (2.2.9) can be replaced respectively by*

$$f_2(t,Y_t^2,U_t^1)-f_2(t,Y_t^2,U_t^2)\geq \langle V^{U^1,U^2},(U^1-U^2)\rangle_t^p,\ d\mathcal{P}\otimes dt-a.e. \tag{2.2.10}$$

and

$$f_1(t,Y_t^1,U_t^1)\geq f_2(t,Y_t^1,U_t^1), d\mathcal{P}\otimes dt-a.e.. \tag{2.2.11}$$

In this case, with the other properties, one can show in the same way that we actually have $\mathcal{P}$-a.s., $Y^1\geq Y^2$. □

Remark 2.2.3. *Point (i) of Proposition 2.2.3 is satisfied in the following cases:*

(i) f does not depend on the component u ;
(ii) If L reduces to a Poisson process, we have $H^{(i)}\equiv 0$ for all $i\geq 2$, then Assumption (A2) reads: (a) $V=(V_t)_{t\in[0,T]}$ is bounded ; (b) for any stopping time τ, such that $\triangle L_\tau\neq 0$, $V_\tau>-1$, $\mathcal{P}-a.s.$.
(iii) The generator f satisfies

$$f(t,y,u)=h(t,y,\sum_{i\geq 1}\theta_t^i u^i),\ \forall (t,y,u)\in[0,T]\times I\!R^k\times\ell^2$$

where the mapping $\eta\in I\!R\mapsto h(t,y,\eta)$ is non decreasing and uniformly Lipschitz and, on the other hand, $((\theta_t^i)_{i\geq 1})_{t\leq T}$ satisfies

$$\sum_{i\geq 1}|\theta_t^i|^2\leq C\ \ and\ \sum_{i\geq 1}\theta_t^i p_i(\Delta L_t)\geq 0,\ dt\otimes d\mathcal{P}-a.e.\quad \square$$

We finally provide a comparison result of solutions of reflected BSDEs of type (2.2.1) which will be useful in the sequel.

Proposition 2.2.4. *For $i=1,2$, let (ξ_i,S^i,f_i) be a triple which satisfies Assumption (A1) and let $(Y_t^i,K_t^i,U_t^i)_{t\leq T}$ be the solution of the RBSDE associated with (ξ_i,S^i,f_i). Assume that:*
(i) $\mathcal{P}-a.s$, $\xi_1\geq\xi_2$ and $\forall t\in[0,T]$, $f_1(t,y,u)\geq f_2(t,y,u)$ and $S_t^1\geq S_t^2$;
(ii) f_1 verifies condition (2.2.8).
Then $\mathcal{P}$-a.s. for any $t\leq T$, $Y_t^1\geq Y_t^2$.

Proof. For $i=1,2$, let us consider the following sequence of processes $(Y^{i,n},U^{i,n})\in\mathcal{S}^2\times\mathcal{H}^2(\ell^2)$, $n\geq 0$, that satisfy:

$$Y_t^{i,n}=\xi_i+\int_t^T f_i(s,Y_s^{i,n},U_s^{i,n})ds+n\int_t^T(Y_s^{i,n}-S_s^i)^-ds-\sum_{j=1}^{\infty}\int_t^T U_s^{i,n,j}dH_s^{(j)},\ \forall t\leq T$$

and let us denote by

$$f_i^n(s,y,u):=f_i(s,y,u)+n(y-S_s^i)^-.$$

For any $n\geq 0$, f_1^n satisfies (2.2.8) and $f_1^n\geq f_2^n$. Therefore using the comparison result of Proposition 2.2.3, we deduce that: $\forall n\geq 0$,

$$\mathcal{P}-a.s.,\forall t\leq T, Y_t^{1,n}\geq Y_t^{2,n}. \tag{2.2.12}$$

But since f_1 verifies (2.2.8) then we can show, as in [31], Theorem 1.2.a, pp. 5, since the processes S^i do not have predictable jumps, that for $i=1,2$ $Y^{i,n}\nearrow Y^i$ in $\mathcal{S}^2$. Thus, inequality (2.2.12) implies that $\mathcal{P}$-a.s., $Y^1\geq Y^2$. □

We are now going to make a connection between reflected BSDEs and their associated IPDEs with obstacle. So let $(t,x)\in[0,T]\times I\!R^k$ and let $(X_s^{t,x})_{s\leq T}$ be the solution of the following standard SDE driven by the Lévy process L, i.e.,

$$X_s^{t,x}=x+\int_t^{t\vee s}b(r,X_r^{t,x})dr+\int_t^{t\vee s}\sigma(r,X_{r-}^{t,x})dL_r,\quad \forall s\leq T, \tag{2.2.13}$$

where we assume that the functions b and σ are jointly continuous, Lipschitz continuous $w.r.t.$ x uniformly in t, i.e., there exists a constant $C\geq 0$ such that for any $t\in[0,T]$, x,$x'\in I\!R$,

$$|\sigma(t,x)-\sigma(t,x')|+|b(t,x)-b(t,x')|\leq C|x-x'|. \tag{2.2.14}$$

As a consequence, the functions $b(t,x)$ and $\sigma(t,x)$ are of linear growth. We additionally assume that σ is bounded, i.e., there exists a constant C_σ such that

$$\forall (t,x) \in [0,T] \times I\!R, |\sigma(t,x)| \leq C_\sigma. \tag{2.2.15}$$

Under the above conditions on b and σ, the process $X^{t,x}$ exists and is unique (see e.g. [55], pp.249), and satisfies:

$$\forall p \geq 1,\ \mathbb{E}[\sup_{s \leq T} |X_s^{t,x}|^p] \leq C(1+|x|^p). \tag{2.2.16}$$

Next let us consider the following functions:

$$\begin{array}{l} h:\ x \in I\!R \mapsto h(x) \in I\!R; \\ f:\ (t,x,y,u) \in [0,T] \times I\!R^{1+1} \times l^2 \mapsto f(t,x,y,u) \in I\!R; \\ \Psi:\ (t,x) \in [0,T] \times I\!R \mapsto \Psi(t,x) \in I\!R, \end{array}$$

which we assume satisfying:

Assumptions (A3):
(i) h, Ψ and $f(t,x,0,0)$ are jointly continuous and belong to Π_g ;
(ii) the mapping $(y,z) \mapsto f(t,x,y,z)$ is Lipschitz continuous uniformly in (t,x) ;
(iii) For any $x \in I\!R$, $h(x) \geq \Psi(T,x)$.
(iv) The generator f has the following form,

$$f(t,x,y,u) = h(t,x,y,\sum_{i \geq 1} \theta_t^i u_i),\ \forall (t,x,y,u) \in [0,T] \times I\!R^{1+1} \times \ell^2$$

where the mapping $\eta \in I\!R \longmapsto h(t,x,y,\eta)$ is non decreasing, and there exists a constant $C > 0$, such that $\forall t \in [0,T]$, $x,y,z,z' \in I\!R$,

$$|h(t,x,y,z) - h(t,x,y,z') \leq C|z-z'|.$$

Moreover $(\theta_t^i)_{i \geq 1}$ satisfies

$$\sum_{i \geq 1} |\theta_t^i|^2 \leq C \ \text{ and } \sum_{i \geq 1} \theta_t^i p_i(\Delta L_t) > 0,\ dt \otimes d\mathcal{P} - a.e..$$

Next let $(t,x) \in [0,T] \times I\!R^k$ be fixed and let us consider the following reflected BSDE:

$$\left\{ \begin{array}{l} (Y^{t,x}, U^{t,x}, K^{t,x}) \in \mathcal{S}^2 \times \mathcal{H}(\ell^2) \times \mathcal{A}^2; \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, U_r^{t,x}) dr + K_T^{t,x} - K_s^{t,x} - \sum_{i=1}^{\infty} \int_s^T U_r^{t,x,i} dH_r^{(i)} \\ \forall s \leq T,\ Y_s^{t,x} \geq \Psi(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{t,x} - \Psi(s, X_s^{t,x})) dK_s^{t,x} = 0,\ \mathcal{P} - a.s. \end{array} \right. \tag{2.2.17}$$

Under assumptions (A3)-(i), (ii), (iii), the reflected BSDE (2.2.17) is well-posed and has a unique solution $(Y^{t,x}, U^{t,x}, K^{t,x})$, thanks to Theorem 3.1. Moreover the following estimate holds true:

$$\begin{array}{c} \mathbb{E}\Big[\sup_{0 \leq s \leq T} |Y_s^{t,x}|^2 + \int_0^T \{\sum_{i \geq 1} |U_s^{i,tx}|^2\} ds\Big] \\ \leq C\mathbb{E}\Big[|h(X_T^{t,x})|^2 + \int_0^T |f(s, X_s^{t,x}, 0, 0)|^2 ds + \sup_{0 \leq s \leq T} |\Psi(s, X_s^{t,x})|^2\Big]. \end{array} \tag{2.2.18}$$

On the other hand, the quantity

$$u(t,x) = Y_t^{t,x}, \tag{2.2.19}$$

is deterministic, continuous and satisfies

$$\forall (t,x) \in [0,T] \times I\!R^k, \forall s \in [t,T],\ Y_s^{t,x} = u(s, X_s^{t,x}).$$

Fore more details, one can see e.g. [57]. Finally note that under Assumptions (A3) and by (2.2.18) the function u belongs also to Π_g.

Next let us introduce the following IPDE with obstacle:

$$\begin{cases} \min\Big\{u(t,x)-\Psi(t,x); -\partial_t u(t,x)-\mathcal{L}u(t,x)-f(t,x,u(t,x),\Phi(u)(t,x))\Big\}=0,\ (t,x)\in[0,T)\times I\!R,\\ u(T,x)=h(x), \end{cases} \tag{2.2.20}$$

where $\mathcal{L}$ is the generator associated with the process $X^{t,x}$ of (2.2.13) and which has the following expression:

$$\begin{aligned}\mathcal{L}u(t,x) = (E[L_1]\sigma(t,x)+b(t,x))\partial_x u(t,x) + \tfrac{1}{2}\sigma(t,x)^2\varpi^2\partial^2_{xx}u(t,x)\\ + \textstyle\int_{I\!R}[u(t,x+\sigma(t,x)y)-u(t,x)-\partial_x u(t,x)\sigma(t,x)y]\Pi(dy)\end{aligned}$$

and

$$\Phi(u)(t,x)=\Big(\frac{1}{c_{1,1}}\partial_x u(t,x)\sigma(t,x)\mathbb{1}_{k=1}+\int_{I\!R}(u(t,x+\sigma(t,x)y)-u(t,x)-\partial_x u(t,x)y)p_k(y)\Pi(dy)\Big)_{k\geq 1}$$

where $c_{1,1}$ is defined in (2.1.3).

We are going to consider solutions of (2.2.20) in viscosity sense whose definition is as follows:

Definition 2.2.1. *A continuous function $u:[0,T]\times I\!R\to I\!R$ is said to be a viscosity subsolution (resp. supersolution) of* (2.2.20) *if for any $(t,x)\in[0,T)\times I\!R$ and for any $\varphi\in C^{1,2}([0,T]\times I\!R)\cap\Pi_g$ such that $\varphi(t,x)=u(t,x)$ and $\varphi-u$ attains its global minimum (resp. maximum) at (t,x),*

$$\begin{cases} min\{u(t,x)-\Psi(t,x); -\partial_t\varphi(t,x)-\mathcal{L}\varphi(t,x)-f(t,x,\varphi(t,x),\Phi(\varphi)(t,x))\}\leq 0,\\ u(T,x)\leq h(x); \end{cases}$$

(resp.

$$\begin{cases} min\{u(t,x)-\Psi(t,x); -\partial_t\varphi(t,x)-\mathcal{L}\varphi(t,x)-f(t,x,\varphi(t,x),\Phi(\varphi)(t,x))\}\geq 0,\\ u(T,x)\geq h(x);). \end{cases}$$

The function u is said to be a viscosity solution of (2.2.20) *if it is both its viscosity subsolution and supersolution.* □

In [56], Y.Ren and M.El Otmani have shown:

Theorem 2.2.2. *([56], Theorem 18) Assume that Assumption (A3) is fulfilled, then the function u defined in* (2.2.19) *is a viscosity solution of* (2.2.20). □

2.2.2 Systems of reflected BSDEs with inter-connected obstacles driven by a Lévy process and multi-modes switching problem.

We now introduce the following functions f_i, h_i and g_{ij}, $i,j\in A$:

$$\begin{aligned} f_i &: (t,x,(y^i)_{i=1,m},u)\in[0,T]\times I\!R^k\times I\!R^m\times\ell^2\longmapsto f_i(t,x,(y^i)_{i=1,m},u)\in I\!R\\ g_{ij} &: (t,x)\in[0,T]\times I\!R^k\longmapsto g_{ij}(t,x)\in I\!R\\ h_i &: x\in I\!R\longmapsto h_i(x)\in I\!R \end{aligned}$$

which we assume satisfying:

Assumption (A4)

(I) For any $i\in A$:

(i) The mapping $(t,x)\to f_i(t,x,\overrightarrow{y},u)$ is continuous uniformly with respect to $(\overrightarrow{y},u)$ where $\overrightarrow{y}=(y^i)_{i=1,m}$;

(ii) The mapping $(\overrightarrow{y},u)\mapsto f_i(t,x,\overrightarrow{y},u)$ is Lipschiz continuous uniformly *w.r.t.* (t,x) ;

(iii) $f_i(t,x,0,0)$ is measurable and of polynomial growth w.r.t. (t,x).

(iv) For any $U^1,U^2\in\mathcal{H}^2(l^2)$, $X,Y\in\mathcal{S}^2$, there exists $V^{U^1,U^2,i}=(V_j^{U^1,U^2,i})_{j\geq 1}$, $i=1,2$, which may depend on U^1 and U^2, that satisfies (A2) and such that :

$$f_i(t,X_t,Y_t,U_t^1)-f_i(t,X_t,Y_t,U_t^2)\geq\langle V^{U^1,U^2,i},(U^1-U^2)\rangle_t^p,\ d\mathcal{P}\otimes dt-a.e.,\ i=1,2; \tag{2.2.21}$$

(v) For any $i\in A$ and $k\in A_i:=A-\{i\}$, the mapping $y_k\to f_i(t,x,y_1,\cdots,y_{k-1},y_k,y_{k+1},\cdots,y_m,u)$ is nondecreasing whenever the other components $(t,x,y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m,u)$ are fixed.

(II) $\forall i,j \in A$, $g_{ii} \equiv 0$ and for $k \neq j$, $g_{jk}(t,x)$ is non-negative, continuous with polynomial growth and satisfy the following non-free loop property:

For any $(t,x) \in [0,T] \times I\!R$ and for any sequence of indices $i_1, \cdots, i_k$ such that $i_1 = i_k$ and $card\{i_1, \cdots, i_k\} = k-1$ we have

$$g_{i_1 i_2}(t,x) + g_{i_2 i_3}(t,x) + \cdots + g_{i_k i_1}(t,x) > 0.$$

(III) $\forall i \in A$, h_i is continuous with polynomial growth and satisfies the following coherence condition:

$$h_i(x) \geq \max_{j \in A_i}(h_j(x) - g_{ij}(T,x)), \forall x \in I\!R.$$

We now describe precisely the switching problem. Let $\Upsilon = ((\theta_j)_{j\geq 0}, (\alpha_j)_{j\geq 0})$ be an admissible strategy and let $a = (a_s)_{s\in[0,T]}$ be the process defined by

$$\forall s \leq T,\ a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s),$$

where $\{\theta_j\}_{j\geq 0}$ is an increasing sequence of $\mathcal{F}_t$-stopping times with values in [0,T] and for $j \geq 0$, α_j is a random variable F_{θ_j}-measurable with values in $A = \{1, ..., m\}$. If $\mathcal{P}[\lim_n \theta_n < T] = 0$, then the pair $\{\theta_j, \alpha_j\}_{j\geq 0}$ (or the process a) is called an admissible strategy of switching. Next we denote by $(A^a_s)_{s\leq T}$ the switching cost process associated with an admissible strategy a, which is defined as following:

$$\forall s < T,\ A^a_s = \sum_{j\geq 1} g_{\alpha_{j-1},\alpha_j}(\theta_j, X^{t,x}_{\theta_j}) \mathbb{1}_{[\theta_j \leq s]} \text{ and } A^a_T = \lim_{s\to T} A^a_s \tag{2.2.22}$$

where $X^{t,x}$ is the process given in (2.2.13). Next, for $\eta \leq T$ and $i \in A$, we denote by

$$\mathcal{A}^i_\eta := \{a \text{ admissible strategy such that } \alpha_0 = i,\ \theta_0 = \eta \text{ and } E[(A^a_T)^2] < \infty\}.$$

Assume momentarily that for $i \in A$, the function f_i introduced previously does not depend on $\overrightarrow{y}$ and u. For $t \leq T$ and a given admissible strategy $a \in \mathcal{A}^i_t$, we define the payoff $J^a_i(t,x)$ by:

$$J^a_i(t,x) := \mathbb{E}[\int_t^T f_{a(s)}(s, X^{t,x}_s)ds + h_{a(T)} - A^a_T]$$

where $f_{a(s)}(\dots) = f_k(\dots)$ (resp. $h_{a(T)}(.) = h_k(.)$) if at time s (resp. T) $a(s) = k$ (resp. $a(T) = k$) ($k \in A$). Finally let us define

$$J^i(t,x) := \sup_{a\in\mathcal{A}^i_t} J^a_i(t,x),\ i = 1, ..., m. \tag{2.2.23}$$

As a by-product of our main result which is given in Theorem 4.3 below, we get that the functions $(J^i(t,x))_{i=1,\dots,m}$ is the unique continuous viscosity solution of the Hamilton-Jacobi-Bellman system associated with this switching problem. □

Let $(t,x) \in [0,T] \times I\!R^k$ and let us consider the following system of reflected BSDEs with oblique reflection: $\forall j = 1, ..., m$

$$\begin{cases} Y^j \in \mathcal{S}^2,\ \ U^j \in \mathcal{H}^2(\ell^2),\ \ K^j \in \mathcal{A}^2 \\ Y^j_s = h_j(X^{t,x}_T) + \int_s^T f_j(r, X^{t,x}_r, Y^1_r, Y^2_r, \cdots, Y^m_r, U^j_r)dr - \sum_{i=1}^{\infty} \int_s^T U^{j,i}_r dH^{(i)}_r + K^j_T - K^j_s,\ \forall s \leq T; \\ \forall s \leq T,\ \ Y^j_s \geq \max_{k\in Aj}\{Y^k_s - g_{jk}(s, X^{t,x}_s)\} \text{ and } \int_0^T \{Y^j_s - \max_{k\in A_j}\{Y^k_s - g_{jk}(s, X^{t,x}_s)\}\}dK^j_s = 0. \end{cases} \tag{2.2.24}$$

Note that the solution of this BSDE depends actually on (t,x) which we will omit for sake of simplicity, as far as there is no confusion. We then have the following result related to existence and uniqueness of the solution of (2.2.24).

Theorem 2.2.3. *Assume that Assumption (A4)(I)(ii)-(iv), (A4)(II) and (A4)(III) are fulfilled. Then system of reflected BSDE with oblique reflection (2.2.24) has a unique solution.*

Proof. The proof follows the same lines as in [10] and [30]. It will be given in two steps.

Step 1: We will first assume that the functions f_i, $i \in A$, verify (A4)(I)(ii)-(v). The other assumptions remain fixed.

Let us introduce the following standard BSDEs : $\forall s \leq T$,

$$\begin{cases} \bar{Y} \in \mathcal{S}^2, \ \ \bar{U} \in \mathcal{H}^2(\ell^2); \\ \bar{Y}_s = \max\limits_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max\limits_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \cdots, \bar{Y}_r, \bar{U}_r)dr - \sum\limits_{i=1}^{\infty} \int_s^T \bar{U}_r^i dH_r^{(i)} \end{cases} \tag{2.2.25}$$

and

$$\begin{cases} \underline{Y} \in \mathcal{S}^2, \ \ \underline{U} \in \mathcal{H}^2(\ell^2); \\ \underline{Y}_s = \min\limits_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \min\limits_{j=1,m} f_j(r, X_r^{t,x}, \underline{Y}_r, \cdots, \underline{Y}_r, \underline{U}_r)dr - \sum\limits_{i=1}^{\infty} \int_s^T \underline{U}_r^i dH_r^{(i)}. \end{cases} \tag{2.2.26}$$

Note that thanks to Theorem 1 in [48], each one of the above BSDEs has a unique solutions. Next for $j = 1, \cdots, m$ and $n \geq 1$, let us define $(Y^{j,n}, U^{j,n}, K^{j,n})$ by:

$$\begin{cases} Y^{j,n} \in \mathcal{S}^2, \ \ U^{j,n} \in \mathcal{H}^2(\ell^2), \ \ K^{j,n} \in \mathcal{A}^2\,; \\ Y^{j,0} = \underline{Y} \\ Y_s^{j,n} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^{1,n-1}, \cdots, Y_r^{j-1,n-1}, Y_r^{j,n}, Y_r^{j+1,n-1}, \cdots, Y^{m,n-1}, U_r^{j,n})dr \\ \qquad\qquad - \sum\limits_{i=1}^{\infty} \int_s^T U_r^{i,j,n} dH_r^{(i)} + K_T^{j,n} - K_s^{j,n}, \quad \forall s \leq T; \\ Y_s^{j,n} \geq \max\limits_{k \in A_j}(Y_s^{k,n-1} - g_{jk}(s, X_s^{t,x})), \ \forall s \leq T; \ \ \int_0^T [Y_r^{j,n} - \max\limits_{k \in A_j}(Y_r^{k,n-1} - g_{jk}(r, X_r^{t,x}))]dK_r^j = 0. \end{cases} \tag{2.2.27}$$

By induction we can show that system (2.2.27) has a unique solution for any fixed $n \geq 1$ since when n is fixed, (2.2.27) reduces to m decoupled reflected BSDEs. On the other hand it is easily seen that $(\bar{Y}, \bar{U}, 0)$ is also a solution of :

$$\begin{cases} \bar{Y}_s = \max\limits_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max\limits_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \cdots, \bar{Y}_r, \bar{U}_r)dr - \sum\limits_{i=1}^{\infty} \int_s^T \bar{U}_r^i dH_r^{(i)} + \bar{K}_T - \bar{K}_s, \forall s \leq T; \\ \bar{Y}_s \geq \max\limits_{k \in A_j}(\bar{Y}_s - g_{jk}(s, X_s^{t,x})), \ \forall s \leq T; \ \ \int_0^T [\bar{Y}_r - \max\limits_{k \in A_j}(\bar{Y}_s - g_{jk}(s, X_s^{t,x}))]d\bar{K}_r = 0. \end{cases}$$

Next since for any $i \in A$, f_i verifies Assumption A4(I), by Proposition 2.2.4 and an induction argument, we get that $\mathcal{P}$-a.s. for any j, n and $s \leq T$, $Y_s^{j,n-1} \leq Y_s^{j,n} \leq \bar{Y}_s$. Then the sequence $(Y^{j,n})_{n \geq 0}$, has a limit which we denote by Y^j, for any $j \in A$. By the monotonic limit theorem in [25], $Y^j \in \mathcal{S}^2$ and there exist $U^j \in \mathcal{H}^2(\ell^2), K^j \in \mathcal{A}^2$, such that: $\forall s \leq T$,

$$\begin{cases} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{Y_r}, U_r^j)dr - \sum\limits_{i=1}^{\infty} \int_s^T U_r^{i,j} dH_r^{(i)} + K_T^j - K_s^j, \\ Y_s^j \geq \max\limits_{k \in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})), \end{cases} \tag{2.2.28}$$

where for any $j \in A$, U^j is the weak limit of $(U^{j,n})_{n \geq 1}$ in $\mathcal{H}^2(\ell^2)$ and for any stopping time τ, K_τ^j is the weak limit of $K_\tau^{j,n}$ in $L^2(\Omega, \mathcal{F}_\tau, \mathcal{P})$. Finally note that K^j is predictable since the processes $K^{n,j}$ are so, for any $n \geq 1$.

Let us now consider the following RBSE:

$$\begin{cases} \hat{Y}^j \in \mathcal{S}^2, \ \ \hat{U}^j \in \mathcal{H}^2(\ell^2), \ \ \hat{K}^j \in \mathcal{S}^2, \text{ non-decreasing and } \hat{K}_0^j = 0; \\ \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^1, \cdots, Y_r^{j-1}, \hat{Y}_r^j, Y_r^{j+1}, \cdots, Y_r^m, \hat{U}_r^j)dr \\ - \sum\limits_{i=1}^{\infty} \int_s^T \hat{U}_r^{i,j} dH_r^{(i)} + \hat{K}_T^j - \hat{K}_s^j, \forall s \leq T; \\ \hat{Y}_s^j \geq \max\limits_{k \in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})), \ \forall s \leq T; \ \ \int_0^T [\hat{Y}_{r-}^j - \max\limits_{k \in A_j}(Y_{r-}^k - g_{jk}(r, X_{r-}^{t,x}))]d\hat{K}_r^j = 0. \end{cases} \tag{2.2.29}$$

According to Theorem 3.3 in [1], this equation has a unique solution. By Tanaka-Meyer's formula (see e.g.[55], Theorem 68, pp. 216), for all $j \in A$:

$$\begin{aligned} (\hat{Y}_T^j - Y_T^j)^+ =& (\hat{Y}_s^j - Y_s^j)^+ + \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{Y}_r^j - Y_r^j) \\ &+ \sum_{s<r\leq T} [\mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{Y}_r^j - Y_r^j)^- + \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j \leq 0\}} (\hat{Y}_r^j - Y_r^j)^+] + \frac{1}{2} L_t^0 (\hat{Y}^j - Y^j) \end{aligned}$$

where the process $(L_t^0(\hat{Y}^j - Y^j))_{t\leq T}$ is the local time of the semi martingale $(\hat{Y}_s^j - Y_s^j)_{0\leq s\leq T}$ at 0 which is a nonnegative process. Then we have

$$\begin{aligned}(\hat{Y}_T^j - Y_T^j)^+ \geq&(\hat{Y}_s^j - Y_s^j)^+ + \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{Y}_r^j - Y_r^j)\\ =&(\hat{Y}_s^j - Y_s^j)^+ - \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} [f_j(r, X_r^{t,x}, Y_r^1, \cdots, \hat{Y}_r^j, \cdots, Y_r^m, \hat{U}_r^j)\\ &- f_j(r, X_r^{t,x}, Y_r^1, \cdots, Y_r^j, \cdots, Y_r^m, U_r^j)]dr - \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{K}_r^j - K_r^j)\\ &+ \sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{U}_r^{j,i} - U_r^{j,i}) dH_r^{(i)}.\end{aligned}$$

First note that by (2.2.29), $\int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{K}_r^j - K_r^j) \leq 0$. Now by Assumption (A4)(I)(iv), we obtain:

$$\begin{aligned}(\hat{Y}_s^j - Y_s^j)^+ \leq& \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} [f_j(r, X_r^{t,x}, Y_r^1, \cdots, \hat{Y}_r^j, \cdots, Y_r^m, \hat{U}_r^j) - f_j(r, X_r^{t,x}, Y_r^1, \cdots, Y_r^j, \cdots, Y_r^m, \hat{U}_r^j)\\ &+ f_j(r, X_r^{t,x}, Y_r^1, \cdots, Y_r^j, \cdots, Y_r^m, \hat{U}_r^j) - f_j(r, X_r^{t,x}, Y_r^1, \cdots, Y_r^j, \cdots, Y_r^m, U_r^j)]dr\\ &- \sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{U}_r^{j,i} - U_r^{j,i}) dH_r^{(i)}\\ \leq& \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} C(\hat{Y}_{r-}^j - Y_{r-}^j)^+ dr + \sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} V_r^{j,U,\hat{U},i} (\hat{U}_r^{j,i} - U_r^{j,i}) dr\\ &- \sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{U}_r^{j,i} - U_r^{j,i}) dH_r^{(i)}.\end{aligned}$$

Next for $t \leq T$, let us set $M_t = \sum_{i=1}^{\infty} \int_0^t V_r^{j,U^2,U^1,i} dH_r^{(i)}$ and $Z_t = \sum_{i=1}^{\infty} \int_0^t \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{U}_r^{j,i} - U_r^{j,i}) dH_r^{(i)}$. By Proposition 2.2.1, $\varepsilon(M) \in \mathcal{S}^2$, $\varepsilon(M) > 0$ and $\mathbb{E}[\varepsilon(M)_T] = 1$. Then using Girsanov's Theorem ([55], pp.136), under the probability measure $d\tilde{\mathcal{P}} := \varepsilon(M)_T d\mathcal{P}$, we obtain that the process

$$\tilde{Z}_t = Z_t - < M, Z >_t, \quad t \leq T,$$

is a martingale and then

$$\mathbb{E}_{\tilde{\mathcal{P}}}[\sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} V_r^{j,U^2,U^1,i} (\hat{U}_r^{j,i} - U_r^{j,i}) dr - \sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{U}_r^{j,i} - U_r^{j,i}) dH_r^{(i)}] = -\mathbb{E}_{\tilde{\mathcal{P}}}(\tilde{Z}_T - \tilde{Z}_s) = 0.$$

Thus for any $s \leq T$,

$$\mathbb{E}_{\tilde{\mathcal{P}}}(\hat{Y}_s^j - Y_s^j)^+ \leq \mathbb{E}_{\tilde{\mathcal{P}}}[\int_s^T C(\hat{Y}_r^j - Y_r^j)^+ dr]$$

and finally by Gronwall's Lemma, $\forall j \in A$, $\forall s \leq T$, $(\hat{Y}_s^j - Y_s^j)^+ = 0$ $\tilde{\mathcal{P}} - a.s.$ and then also $\mathcal{P} - a.s.$ since those probabilities are equivalent. It implies that $\mathcal{P}$-a.s., $\hat{Y}^j \leq Y^j$ for any $j \in A$. On the other hand, since $\forall n \geq 1$, $\forall j \in A$, $Y^{j,n-1} \leq Y^j$, then we have

$$\forall s \leq T, \ \max_{k \in A_j} (Y_s^{k,n-1} - g_{jk}(s, X_s^{t,x})) \leq \max_{k \in A_j} (Y_s^k - g_{jk}(s, X_s^{t,x})).$$

Therefore by comparison, we obtain $Y^{j,n} \leq \hat{Y}^j$, and then $Y^j \leq \hat{Y}^j$ which implies $Y^j = \hat{Y}^j$, $\forall j \in A$.

Next by Itô's formula applied to $(Y^j - \hat{Y}^j)^2$ we obtain: $\forall s \in [0, T]$,

$$(Y_s^j - \hat{Y}_s^j)^2 = (Y_0^j - \hat{Y}_0^j)^2 + 2\int_0^s (Y_{r-}^j - \hat{Y}_{r-}^j) d(Y_r^j - \hat{Y}_r^j) + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \int_0^s (U_r^{j,i} - \hat{U}_r^{j,i})(U_r^{j,k} - \hat{U}_r^{j,k}) d[H^i, H^k]_r.$$

As $Y^j = \hat{Y}^j$ and taking expectation in both-hand sides of the previous equality to obtain

$$\mathbb{E}[\int_0^T \sum_{i\geq 1}(U_r^{j,i} - \hat{U}_r^{j,i})^2 dr] = 0.$$

It implies that $U^j = \hat{U}^j$, $dt \otimes d\mathcal{P}$ and finally $K^j = \hat{K}^j$ for any $j \in A$, i.e. $(Y^j, U^j, K^j)_{j\in A}$ verify (2.2.29).

Next we will show that the predictable process K^j does not have jumps. So assume there exists $j_1 \in A$ and a predictable stopping time τ such that $\triangle Y_\tau^{j_1} = -\triangle K_\tau^{j_1} = -\triangle \hat{K}_\tau^{j_1} < 0$ (note that the process K^j is predictable). Then by the second equality in (2.2.29) we have

$$Y_{\tau-}^{j_1} = \max_{k\in A_{j_1}}(Y_{\tau-}^k - g_{j_1 k}(\tau, X_{\tau-}^{t,x})). \tag{2.2.30}$$

Now let $j_2 \in A_{j_1}$ be the optimal index in (2.2.30), i.e.,

$$Y_{\tau-}^{j_2} - g_{j_1,j_2}(\tau, X_\tau^{t,x}) = Y_{\tau-}^{j_1} > Y_\tau^{j_1} \geq Y_\tau^{j_2} - g_{j_1,j_2}(\tau, X_\tau^{t,x}).$$

Note that $g_{j_1,j_2}(\tau, X_{\tau-}^{t,x}) = g_{j_1,j_2}(\tau, X_\tau^{t,x})$ since the stopping time τ is predictable, and the process $(X_s^{t,x})_{t\leq s\leq T}$ does not have predictable jump times. Thus $\triangle Y_\tau^{j_2} < 0$ and once more we have,

$$Y_{\tau-}^{j_2} = \max_{k\in A_{j_2}}(Y_{\tau-}^k - g_{j_2 k}(\tau, X_{\tau-}^{t,x})). \tag{2.2.31}$$

We can now repeat the same argument as many times as necessary, to deduce the existence of a loop $\ell_1, ..., \ell_{p-1}, \ell_p = \ell_1$ ($p \geq 2$) and $l_2 \neq l_1$ such that

$$Y_{\tau-}^{\ell_1} = Y_{\tau-}^{\ell_2} - g_{\ell_1,\ell_2}(\tau, X_{\tau-}^{t,x}), \cdots, Y_{\tau-}^{\ell_{p-1}} = Y_{\tau-}^{\ell_p} - g_{\ell_{p-1},\ell_p}(\tau, X_{\tau-}^{t,x})$$

which implies that

$$g_{\ell_1,\ell_2}(\tau, X_{\tau-}^{t,x}) + \cdots + g_{\ell_{p-1},\ell_p}(\tau, X_{\tau-}^{t,x}) = 0$$

which is contradictory with Assumption (A4)(II). It implies that $\Delta K_\tau^{j_1} = 0$ and then K^{j_1} is continuous since it is predictable. As j is arbitrary in A, then the processes K^j are continuous and taking into account (2.2.29), we deduce that the triples $(Y^j, U^j, K^j)_{j\in A}$, is a solution for system (2.2.24). □

Step 2: We now deal with the general case i.e. we assume that f_i, $i \in A$, do no longer satisfy the monotonicity assumption (A4)(I)(v) but (A4)(I)(ii)-(iv) solely.

Let $i \in A$ and $t_0 \in [0, T]$ be fixed. For $a \in \mathcal{A}_{t_0}^i$ and $\Gamma := ((\Gamma_s^l)_{s\in[0,T]})_{l\in A} \in [H^2]^m := H^2 \times \cdots \times H^2$ (m times), we introduce the unique solution of the switched BSDE which is defined by: $\forall s \in [t_0, T]$,

$$V_s^a = h_{a(T)}(X_T^{t,x}) + \int_s^T f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma_r}, N_r^a)dr - \sum_{i=1}^\infty \int_s^T N_r^{a,i} dH_r^{(i)} - A_T^a + A_s^a \tag{2.2.32}$$

where $V^a \in \mathcal{S}^2$ and $N^a \in \mathcal{H}^2(\ell^2)$ ($\overrightarrow{\Gamma_r} = (\Gamma_r^i)_{i\in A}$). First note that the solution of this equation exists and is unique since in setting, for $s \in [t_0, T]$, $\tilde{V}_s^a = V_s^a - A_s^a$ and $\tilde{h}_T^a = h_{a(T)}(X_T^{t,x}) - A_T^a$ this equation becomes standard and has a unique solution by Nualart et al.'s result (see [48], Theorem 1, pp.765). Moreover as in [12] (see Appendix 5.1, Theorem 5.1.1) we have the following link between the BSDEs (2.2.24) and (2.2.32),

$$Y_{t_0}^i = \text{esssup}_{a\in\mathcal{A}_{t_0}^i}(V_{t_0}^a - A_{t_0}^a) = V_{t_0}^{a^*} - A_{t_0}^{a^*} \tag{2.2.33}$$

for some $a^* \in \mathcal{A}_{t_0}^i$.

Next let us introduce the following mapping Θ defined on $[H^2]^m$ by

$$\begin{array}{l} \Theta : [H^2]^m \to [H^2]^m \\ \Gamma = (\Gamma^j)_{j\in A} \mapsto (Y^j)_{j\in A} \end{array} \tag{2.2.34}$$

where $(Y^j, U^j, K^j)_{j\in A}$ is the unique solution of the following system of RBSDEs:

$$\left\{ \begin{array}{l} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{\Gamma_r}, U_r^j)dr - \sum_{i=1}^\infty \int_s^T U_r^{j,i} dH_r^{(i)} + K_T^j - K_s^j, \ \forall s \leq T. \\ Y_s^j \geq \max_{k\in A_j}\{Y_s^k - g_{jk}(s, Y_s^j)\}, \ \forall s \leq T; \quad \int_0^T [Y_s^j - \max_{k\in A_j}\{Y_s^k - g_{jk}(s, Y_s^j)\}]dK_s^j = 0 \end{array} \right. \tag{2.2.35}$$

By the result proved in Step 1, Θ is well defined. Next for $\eta \in H^2$ let us define $\|\cdot\|_{2,\beta}$ by

$$\|\eta\|_{2,\beta} := (\mathbb{E}[\int_0^T e^{\beta s}|\eta_s|^2 ds])^{\frac{1}{2}},$$

which is a norm of H^2, equivalent to $\|.\|$ and $(H^2, \|\cdot\|_{2,\beta})$ is a Banach space. Let now Γ^1 and Γ^2 be two processes of $[H^2]^m$ and for $k = 1, 2$, let $(Y^{k,j}, U^{k,j}, K^{k,j})_{j\in A} = \Theta(\Gamma^k)$, i.e., that satisfy: $\forall s \le T$,

$$\begin{cases} Y_s^{k,j} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{\Gamma_r^k}, U_r^{k,j})dr - \sum_{i=1}^{\infty} \int_s^T U_r^{k,j,i} dH_r^{(i)} + K_T^{k,j} - K_s^{k,j} \\ Y_s^{k,j} \ge \max_{q\in A_j}\{Y_s^{k,q} - g_{jq}(s, X_s^{t,x})\}; \quad \int_0^T [Y_s^{k,j} - \max_{q\in A_j}\{Y_s^{k,q} - g_{jq}(s, X_s^{t,x})\}]dK_s^{k,j} = 0. \end{cases}$$

Next let us define $(\hat{Y}^j)_{j\in A}$ through the following system of reflected BSDEs with oblique reflection: $\forall s \le T$,

$$\begin{cases} \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{\Gamma_r^1}, \hat{U}_r^j) \vee f_j(r, X_r^{t,x}, \overrightarrow{\Gamma_r^2}, \hat{U}_r^j)dr - \sum_{i=1}^{\infty} \int_s^T \hat{U}_r^{j,i} dH_r^{(i)} + \hat{K}_T^j - \hat{K}_s^j \\ \hat{Y}_s^j \ge \max_{q\in A_j}\{\hat{Y}_s^q - g_{jq}(s, \hat{Y}_s^j)\}; \quad \int_0^T [\hat{Y}_s^j - \max_{q\in A_j}\{\hat{Y}_s^q - g_{jq}(s, \hat{Y}_s^j)\}]d\hat{K}_s^j = 0, \end{cases}$$

Now let $t_0 \in [0, T]$ and a an admissible strategy such that $\theta_0 = t_0$ and $\mathbb{E}[(A_T^a)^2] < \infty$. Next let us define $V^{k,a}$, $k = 1, 2$, and $\hat{V}^a$, via BSDEs, by: $\forall s \in [t_0, T]$,

$$\begin{aligned} \hat{V}_s^a = & h_{a(T)}(X_T^{t,x}) + \int_s^T f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma_r^1}, \hat{N}_r^a) \vee f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma_r^2}, \hat{N}_r^a)dr \\ & - \sum_{i=1}^{\infty} \int_s^T \hat{N}_r^{a,i} dH_r^{(i)} - A_T^a + A_s^a \end{aligned}$$

and for $k = 1, 2$,

$$V_s^{k,a} = h_{a(T)}(X_T^{t,x}) + \int_s^T f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma_r^k}, N_r^{k,a})dr - A_T^a + A_s^a - \sum_{i=1}^{\infty} \int_s^T N_r^{k,a,i} dH_r^{(i)}.$$

By Proposition 5.1 in Appendix, we have:

$$Y_{t_0}^{k,j} = \operatorname{esssup}_{a\in\mathcal{A}_{t_0}^j}(V_{t_0}^{k,a} - A_{t_0}^a), \ k = 1, 2 \text{ and } \hat{Y}_{t_0}^j = \operatorname{esssup}_{a\in\mathcal{A}_{t_0}^j}(\hat{V}_{t_0}^a - A_{t_0}^a) := \hat{V}_{t_0}^{a^*} - A_{t_0}^{a^*}. \quad (2.2.36)$$

In addition for $s \in [t_0, T]$, $f_{a(s)}$ verifies the inequality (2.2.21) of Assumption (A4)(I)(iv). Actually let us set $a_s = \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s)$, $s \in [t_0, T]$, and let $U^1, U^2 \in \mathcal{H}^2(l^2)$, $X, Y \in \mathcal{S}^2$. For any $s \in [t_0, T]$ we have:

$$\begin{aligned} & f_{a(s)}(s, X_s, Y_s, U_s^1) - f_{a(s)}(s, X_s, Y_s, U_s^2) \\ & = [f_{\alpha_0}(s, X_s, Y_s, U_s^1) - f_{\alpha_0}(s, X_s, Y_s, U_s^2)]\mathbb{1}_{\{\theta_0 \le s \le \theta_1\}} \\ & + \sum_{j\ge 2}[f_{\alpha_{j-1}}(s, X_s, Y_s, U_s^1) - f_{\alpha_{j-1}}(s, X_s, Y_s, U_s^2)]\mathbb{1}_{]\theta_{j-1},\theta_j]}(s) \\ & \ge \langle V^{U^1,U^2,\alpha_0}, (U^1 - U^2)\rangle_s^p \mathbb{1}_{\{\theta_0 \le s \le \theta_1\}} + \sum_{j\ge 2} \langle V^{U^1,U^2,\alpha_{j-1}}, (U^1 - U^2)\rangle_s^p \mathbb{1}_{]\theta_{j-1},\theta_j]}(s) \\ & =: \langle V^{U^1,U^2,a}, (U^1 - U^2)\rangle_s^p. \end{aligned}$$

where for any $s \in [t_0, T]$,

$$V_s^{U^1,U^2,a} = V_s^{U^1,U^2,\alpha_0} \mathbb{1}_{\{\theta_0 \le s \le \theta_1\}} + \sum_{j\ge 2} V_s^{U^1,U^2,\alpha_{j-1}} \mathbb{1}_{]\theta_{j-1},\theta_j]}(s).$$

But on $[t_0, T] \times \Omega$,

$$ds \otimes d\mathcal{P}\{\sum_{i=1}^{\infty} V_s^{i,U^1,U^2,a}(\omega) p_i(\Delta L_s(\omega)) \le -1\} \le \sum_{j\in A} ds \otimes d\mathcal{P}\{\sum_{i=1}^{\infty} V_s^{i,U^1,U^2,j}(\omega) p_i(\Delta L_s(\omega)) \le -1\} = 0$$

which implies that on $[t_0,T]\times\Omega$ it holds

$$\sum_{i=1}^{\infty} V_s^{i,U^1,U^2,a(\omega)}(\omega)p_i(\Delta L_s(\omega)) > -1, ds\otimes d\mathcal{P}-a.e.$$

On the other hand, on $[t_0,T]\times\Omega$,

$$\sum_{i=1}^{\infty}|V_s^{i,U^1,U^2,a}|^2 \le \sum_{\ell\in A}\sum_{i=1}^{\infty}|V_s^{i,U^1,U^2,\ell}|^2 \le C,\ ds\otimes d\mathcal{P}-a.e.$$

Thus the process $V^{U^1,U^2,a}$ verifies Assumption (A2) and $f^{a(s)}$ satisfies Assumption (A4)(I)(iv) on $[t_0,T]$.

Consequently, by the comparison result (Proposition 2.2.3), for any strategy $a\in\mathcal{A}_{t_0}^j$, $\mathcal{P}$-a.s. for any $s\in[t_0,T]$, $\hat{V}_s^a \ge V_s^{1,a}\vee V_s^{2,a}$. This combined with (2.2.36) leads to $Y_{t_0}^{1,j}\vee Y_{t_0}^{2,j}\le \hat{Y}_{t_0}^j = \hat{V}_{t_0}^{a^*} - A_{t_0}^{a^*}$. We then deduce

$$V_{t_0}^{1,a^*} - A_{t_0}^{a^*} \le Y_{t_0}^{1,j} \le \hat{V}_{t_0}^{a^*} - A_{t_0}^{a^*} \text{ and } V_{t_0}^{2,a^*} - A_{t_0}^{a^*} \le Y_{t_0}^{2,j} \le \hat{V}_{t_0}^{a^*} - A_{t_0}^{a^*}$$

which implies

$$|Y_{t_0}^{1,j} - Y_{t_0}^{2,j}| \le |\hat{V}_{t_0}^{a^*} - V_{t_0}^{1,a^*}| + |\hat{V}_{t_0}^{a^*} - V_{t_0}^{2,a^*}|. \tag{2.2.37}$$

Next we first estimate the quantity $|\hat{V}_{t_0}^{a^*} - V_{t_0}^{1,a^*}|$. For $s\in[t_0,T]$ let us set $\triangle V_s^{a^*} := \hat{V}_s^{a^*} - V_s^{1,a^*}$ and $\triangle N_s^{a^*} := \hat{N}_s^{a^*} - N_s^{1,a^*}$. Applying Itô's Formula to the process $e^{\beta s}|\triangle V_s^{a^*}|^2$ we obtain: $\forall s\in[t_0,T]$,

$$\begin{aligned}
&e^{\beta s}|\triangle V_s^{a^*}|^2 + \int_s^T e^{\beta r}\|\triangle N_r^{a^*}\|^2 dr \\
&\quad = -\int_s^T \beta e^{\beta r}|\triangle V_{r-}^{a^*}|^2 dr - 2\sum_{i=1}^{\infty}\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}\triangle N_r^{i,a^*} dH_r^{(i)} \\
&\qquad +2\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}[f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{a^*})\vee f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^2,\hat{N}_r^{a^*}) - f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{1,a^*})]dr \\
&\qquad -\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\int_s^T e^{\beta r}\triangle N_r^{i,a^*}\triangle N_r^{l,a^*} d([H^{(i)},H^{(l)}]_r - \langle H^{(i)},H^{(l)}\rangle_r).
\end{aligned}$$

By the Lipschitz property of f_j, $j\in A$, and then of f_{a*} and the fact that for any $x,y\in\mathbb{R}$, $|x\vee y - y|\le |x-y|$ we have: $\forall s\in[t_0,T]$,

$$\begin{aligned}
&|f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{a^*})\vee f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^2,\hat{N}_r^{a^*}) - f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{1,a^*})| \\
&\qquad \le |f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{a^*})\vee f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^2,\hat{N}_r^{a^*}) - f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{a^*})| \\
&\qquad\qquad +|f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{a^*}) - f_{a^*(r)}(r,X_r^{t,x},\Gamma_r^1,\hat{N}_r^{1,a^*})| \\
&\qquad \le L(|\Gamma_r^1-\Gamma_r^2| + \|\hat{N}_r^{a^*} - N_r^{1,a^*}\|)
\end{aligned} \tag{2.2.38}$$

The inequality $2xy\le \frac{1}{\beta}x^2+\beta y^2$ ($\beta>0$ and $x,y\in\mathbb{R}$) and (2.2.38) yield: $\forall s\in[t_0,T]$,

$$\begin{aligned}
e^{\beta s}|\triangle \hat{V}_s^{a^*}|^2 \ &\le -\int_s^T e^{\beta r}\|\triangle N_r^{a^*}\|^2 dr - \int_s^T \beta e^{\beta r}|\triangle V_{r-}^{a^*}|^2 ds - 2\sum_{i=1}^{\infty}\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}\triangle N_r^{i,a^*} dH_r^{(i)} \\
&\quad +2L\int_s^T e^{\beta r}|\triangle V_{r-}^{a^*}|(|\Gamma_r^1-\Gamma_r^2| + |\hat{N}_r^{a^*} - N_r^{1,a^*}|)dr \\
&\quad -\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\int_s^T e^{\beta r}\triangle N_r^{i,a^*}\triangle N_r^{l,a^*} d([H^{(i)},H^{(l)}]_r - \langle H^{(i)},H^{(l)}\rangle_r) \\
&\le -\int_s^T e^{\beta r}\|\triangle N_r^{a^*}\|^2 dr - \int_s^T \beta e^{\beta r}|\triangle V_{r-}^{a^*}|^2 ds - 2\sum_{i=1}^{\infty}\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}\triangle N_r^{i,a^*} dH_r^{(i)} \\
&\quad +\int_s^T \beta e^{\beta r}|\triangle V_{r-}^{a^*}|^2 ds + \tfrac{L^2}{\beta}\int_s^T e^{\beta r}(|\Gamma_r^1-\Gamma_r^2| + |\hat{N}_r^{a^*} - N_r^{1,a^*}|)^2 dr \\
&\quad -\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\int_s^T e^{\beta r}\triangle N_r^{i,a^*}\triangle N_r^{l,a^*} d([H^{(i)},H^{(l)}]_r - \langle H^{(i)},H^{(l)}\rangle_r) \\
&\le \ \tfrac{2L^2}{\beta}\int_s^T e^{\beta r}|\Gamma_r^1-\Gamma_r^2|^2 dr - 2\textstyle\sum_{i=1}^{\infty}\int_s^T e^{\beta r}\triangle V_{r-}^{a^*} N_r^{i,a^*} dH_r^{(i)} \\
&\quad -\textstyle\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\int_s^T e^{\beta r}\triangle N_r^{i,a^*}\triangle N_r^{j,a^*} d([H^{(i)},H^{(l)}]_r - \langle H^{(i)},H^{(l)}\rangle_r),
\end{aligned}$$

for $\beta \geq 2L^2$. We deduce, in taking expectation,

$$\forall s \in [t_0, T],\ \mathbb{E}[e^{\beta s} |\triangle \hat{V}_s^{a^*}|^2] \leq \frac{2L^2}{\beta} E[\int_s^T e^{\beta r} |\Gamma_r^1 - \Gamma_r^2|^2 dr].$$

Similarly, we get also $\forall s \in [t_0, T]$,

$$\mathbb{E}[e^{\beta s} |\hat{V}_s^{a^*} - V_s^{2,a^*}|^2] \leq \frac{2L^2}{\beta} E[\int_s^T e^{\beta r} |\Gamma_r^1 - \Gamma_r^2|^2 dr].$$

Therefore by (2.2.37) we obtain:

$$\mathbb{E}[e^{\beta t_0} |Y_{t_0}^{1,j} - Y_{t_0}^{2,j}|^2] \leq \frac{8L^2}{\beta} \|\Gamma^1 - \Gamma^2\|_{2,\beta}^2. \tag{2.2.39}$$

As t_0 is arbitrary in $[0, T]$ then by integration *w.r.t.* t_0 we get

$$\|\Theta(\Gamma^1) - \Theta(\Gamma^2)\|_{2,\beta} \leq \sqrt{\frac{8L^2 Tm}{\beta}} \|\Gamma^1 - \Gamma^2\|_{2,\beta}. \tag{2.2.40}$$

It follows that for β large enough, Θ is contraction on the Banach space $([H^2]^m, \|.\|_{2,\beta})$, then it has a fixed point $(Y^j)_{j\in A}$ which has a version which is the unique solution of system of RBSDE (2.2.24). □

Remark 2.2.4. *As a consequence of (2.2.39), there exists a constant $C > 0$, such that $\forall j \in A$, $s \leq T$,*

$$\mathbb{E}[|Y_s^{1,j} - Y_s^{2,j}|^2] \leq C \|(Y^{1,j})_{j\in A} - (Y^{2,j})_{j\in A}\|_{2,\beta}^2. \tag{2.2.41}$$

This estimate will be useful later. □

Corollary 2.2.1. *Under Assumptions (A4)(I)(ii)-(iv), (A4)(II) and (A4)(III), there exist deterministic lower semi-continuous functions $(u^j(t,x))_{j\in A}$ of polynomial growth such that*

$$\forall (t,x) \in [0,T] \times I\!R^k,\ \forall s \in [t,T],\ Y_s^j = u^j(s, X_s^{t,x}),\ \forall j \in A.$$

Proof. This is a direct consequence of the construction by induction of the solution $(Y^j, U^j, K^j)_{j\in A}$ given in Step 1. Actually by Ren et al.'s result [56] there exist deterministic continuous functions of polynomial growth $\bar{u}(t,x)$, $\underline{u}(t,x)$ and $u^{j,n}(t,x)$, $n \geq 0$ and $j \in A$, such that $\forall (t,x) \in [0,T] \times I\!R^k,\ \forall s \in [t,T]$
(a)

$$\bar{Y}_s = \bar{u}(s, X_s^{t,x}) \text{ and } \underline{Y}_s = \underline{u}(s, X_s^{t,x}).$$

(b)

$$Y_s^{j,n} = u^{j,n}(s, X_s^{t,x}), \forall j \in A,$$

and

$$\underline{Y} \leq Y^{j,n} \leq Y^{j,n+1} \leq \bar{Y}.$$

This yields for any $n \geq 0$ and $(t,x) \in [0,T] \times I\!R^k$,

$$\underline{u}(t,x) \leq u^n(t,x) \leq u^{n+1}(t,x) \leq \bar{u}(t,x).$$

Thus $u^j(t,x) := \lim_{n\to\infty} u^{j,n}(t,x)$, $j \in A$, verify the required properties since $(Y^{j,n})_n$ converges to Y^j, $j \in A$, in $\mathcal{S}^2$. □

We now give a comparison result for solutions of systems (2.2.24). The induction argument allows to compare the solution of the approximating schemes, by Proposition 2.2.3, and then to deduce the same property for the limiting processes.

Remark 2.2.5. *Let $(\bar{Y}^j, \bar{U}^j, \bar{K}^j)_{j\in A}$ be a solution of the system of RBSDEs* (2.2.24) *associated with $((\bar{f}_j)_{j\in A}, (\bar{g}_{jk})_{j,k\in A}, (\bar{h}_j)_{j\in A})$ which satisfy* [**A4**]. *If for any $j, k \in A$,*

$$f_j \leq \bar{f}_j,\ h_j \leq \bar{h}_j,\ g_{jk} \geq \bar{g}_{jk}$$

then for any $j \in A$, $Y^j \leq \bar{Y}^j$. □

2.3 Existence and uniqueness of the solution for the system of IPDEs with inter-connected obstacles

This section focuses on the main result of this paper which is the proof of existence and uniqueness of the solution for a system of IPDEs. For this objective we use its link with the system of RBSDEs (2.2.24). However we are led to make, hereafter, the following additional assumption.

Assumption (A5): For any $i \in A$, f_i does not depend on the variable $u \in \ell^2$. □

So we are going to consider the following system of IPDEs: $\forall i \in A$,

$$\begin{cases} min\{u_i(t,x) - \max\limits_{j\in A_i}(u_j(t,x) - g_{ij}(t,x)); \\ \qquad -\partial_t u_i(t,x) - \mathcal{L}u_i(t,x) - f_i(t,x,u_1(t,x),\cdots,u_m(t,x))\} = 0,\ (t,x) \in [0,T]\times[0,T]\times I\!R^k; \\ u_i(T,x) = h_i(x) \end{cases} \tag{2.3.1}$$

where

$$\mathcal{L}u(t,x) = \mathcal{L}^1 u(t,x) + \mathcal{I}(t,x,u)$$

with

$$\mathcal{L}^1 u(t,x) := (E[L_1]\sigma(t,x) + b(t,x))\partial_x u(t,x) + \tfrac{1}{2}\sigma(t,x)^2\varpi^2 D^2_{xx}u(t,x) \tag{2.3.2}$$

$$\text{and } \mathcal{I}(t,x,u) := \textstyle\int_{I\!R}[u(t,x+\sigma(t,x)y) - u(t,x) - \partial_x u(t,x)\sigma(t,x)y]\Pi(dy).$$

Note that for any $\phi \in \mathcal{C}^{1,2}([0,T]\times I\!R^k) \cap \Pi_g$ and $(t,x) \in [0,T]\times I\!R^k$, the non-local term

$$\mathcal{I}(t,x,\phi) := \textstyle\int_{I\!R}[\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y]\Pi(dy) \tag{2.3.3}$$

is well-defined. Actually let $\delta > 0$ and let us define, for any $q \in I\!R$,

$$\mathcal{I}^{1,\delta}(t,x,\phi) := \textstyle\int_{|y|\le\delta}[\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y]\Pi(dy), \tag{2.3.4}$$

$$\mathcal{I}^{2,\delta}(t,x,q,u) := \textstyle\int_{|y|>\delta}[u(t,x+\sigma(t,x)y) - u(t,x) - q\sigma(t,x)y]\Pi(dy). \tag{2.3.5}$$

By application of Taylor's expansion we have

$$\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y = \textstyle\int_0^y \sigma(t,x)^2 D^2_{xx}\phi(t,x+\sigma(t,x)r)(y-r)dr.$$

But there exists a constant C_{tx} such that for any $|r| \le \delta$, $|D^2_{xx}\phi(t,x+\sigma(t,x)r)| \le C_{tx}$ since ϕ belongs to $\mathcal{C}^{1,2}$. Therefore

$$|\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y| \le C_{tx}|y|^2$$

which implies that $\mathcal{I}^{1,\delta}(t,x,\phi) \in I\!R$. Next for any (t,x), $\mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),\phi) \in I\!R$ since Π integrates any power function outside $[-\epsilon,\epsilon]$. Therefore $\mathcal{I}(t,x,\phi)$ is well defined. □

We are now going to give the definition of a viscosity solution of (2.3.1). First for a locally bounded function u: $(t,x) \in [0,T]\times I\!R \to u(t,x) \in I\!R$, we define its lower semi-continuous (lsc for short) envelope u_* and upper semi-continuous (usc for short) envelope u^* as following:

$$u_*(t,x) = \underset{(t',x')\to(t,x),\ t'<T}{\underline{\lim}} u(t',x'), \quad u^*(t,x) = \underset{(t',x')\to(t,x),\ t'<T}{\overline{\lim}} u(t',x')$$

Definition 2.3.1. *A function $(u_1,\cdots,u_m) : [0,T]\times I\!R \to I\!R^m$ which belongs to Π_g such that for any $i \in A$, u^i is lsc (resp. usc), is said to be a viscosity subsolution (resp. supersolution) of (2.3.1) if for any $i \in A$, $\varphi \in \Pi_g \bigcap C^{1,2}([0,T]\times I\!R)$, $u^i(T,x) \le h_i(x)$ (resp. $u^i(T,x) \ge h_i(x)$) and if $(t_0,x_0) \in (0,T)\times I\!R$ is a global maximum (resp. minimum) point of $u^i - \varphi$,*

$$\begin{aligned} min\Big\{ & u^i(t_0,x_0) - \max_{j\in A_i}\{u^j(t_0,x_0) - g_{ij}(t_0,x_0)\};\ -\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) \\ & - f_i(t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\Big\} \le 0 \quad (resp. \ge 0), \end{aligned}$$

The function $(u^i)_{i=1}^m$ is called a viscosity solution of (2.3.1) if $(u^i_*)_{i=1}^m$ and $(u^{i*})_{i=1}^m$) are respectively viscosity supersolution and subsolution of (2.3.1).

The following result is needed later.

Lemma 2.3.1. *Let $(u^i)_{i=1}^m$ be a supersolution of (2.3.1) which belongs to Π_g, i.e. for some $\gamma > 0$ and $C > 0$,*

$$|u^i(t,x)| \leq C(1+|x|^g amma),\ \forall (t,x) \in [0,T] \times I\!R^k \text{ and } i \in A.$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $\theta > 0$, $\overrightarrow{v}(t,x) = (u_i(t,x) + \theta e^{-\lambda t}(1+|x|^{2\gamma+2}))_{i=1}^m$ is supersolution of (4.1).

Proof. We use Definition 2.3.1. Let $i \in A$ be fixed and $\varphi^i \in C^{1,2} \cap \Pi_g$ such that $\varphi^i(s,y) - (u_i(s,y) + \theta e^{-\lambda s}(1+|y|^{2\gamma+2}))$ has a global maximum in $(t,x) \in (0,T) \times I\!R$ and $\varphi^i(t,x) = u_i(t,x) + \theta e^{-\lambda t}(1+|x|^{2\gamma+2})$. We then have:

$$\begin{aligned}
min\Big\{ & u_i(t,x) + \theta e^{-\lambda t}(1+|x|^{2\gamma+2}) - \max_{j \in A_i}(-g_{ij}(t,x) + (u_j(t,x) + \theta e^{-\lambda t}(1+|x|^{2\gamma+2}))); \\
& - \partial_t(\varphi^i(t,x) - \theta e^{-\lambda t}(1+|x|^{2\gamma+2})) - \frac{1}{2}\sigma(t,x)^2\varpi^2 D^2_{xx}(\varphi^i(t,x) - \theta e^{-\lambda t}(1+|x|^{2\gamma+2})) \\
& - (\sigma(t,x)E(L_1) + b(t,x))D_x(\varphi^i(t,x) - \theta e^{-\lambda t}(1+|x|^{2\gamma+2})) - \int_{I\!R}[\varphi^i(t,x+\sigma(t,x)y) \\
& - \theta e^{-\lambda t}|x+\sigma(t,x)y|^{2\gamma+2} - (\varphi^i(t,x) - \theta e^{-\lambda t}|x|^{2\gamma+2}) - D_x(\varphi^i(t,x) - \theta e^{-\lambda t}|x|^{2\gamma+2})\sigma(t,x)y]\Pi(dy) \\
& - f_i(t,x,\overrightarrow{u})\Big\} \geq 0.
\end{aligned}$$

Then

$$\begin{aligned}
& -\partial_t\varphi^i(t,x) - \mathcal{L}\varphi^i(t,x) - f_i(t,x,\overrightarrow{v}(t,x)) \\
& \geq \theta\lambda e^{-\lambda t}(1+|x|^{2\gamma+2}) - \tfrac{1}{2}\theta e^{-\lambda t}\sigma(t,x)^2\varpi^2 D^2_{xx}|x|^{2\gamma+2} - (\sigma(t,x)E(L_1) + b(t,x))D_x(\theta e^{-\lambda t}|x|^{2\gamma+2}) \\
& \quad - \textstyle\int_{I\!R}(\theta e^{-\lambda t}|x+\sigma(t,x)y|^{2\gamma+2} - \theta e^{-\lambda t}|x|^{2\gamma+2} - \theta e^{-\lambda t}D_x|x|^{2\gamma+2}\sigma(t,x)y)\Pi(dy) + f_i(t,x,\overrightarrow{u}(t,x)) \\
& \quad - f_i(t,x,\overrightarrow{v}(t,x)) \\
& \geq \theta e^{-\lambda t}\Big\{\lambda(1+|x|^{2\gamma+2}) - \tfrac{1}{2}\sigma(t,x)^2\varpi^2 D^2_{xx}|x|^{2\gamma+2} - (\sigma(t,x)E(L_1) + b(t,x))D_x|x|^{2\gamma+2} \\
& \quad - \textstyle\int_{I\!R}(|x+\sigma(t,x)y|^{2\gamma+2} - |x|^{2\gamma+2} - D_x|x|^{2\gamma+2}\sigma(t,x)y)\Pi(dy) + \sum_{k=1}^m C^{k,i}_{t,x,\theta,\lambda}|x|^{2\gamma+2}\Big\}
\end{aligned} \tag{2.3.6}$$

where $C^{k,i}_{t,x,\theta,\lambda}$ is bounded by the Lipschiz constant of f_i with respect to $(y^i)_{i=1,\cdots,m}$ which is independent of θ. But, since $\phi(y) = |y|^{2\gamma+2} \in C^{1,2} \cap \Pi_g$, then the non-local term is well defined. Now let us set $\psi(\rho) := \phi(x + \rho\sigma(t,x)y)$, for $\rho, x, y \in I\!R$. First note that for any t, x, y we have

$$\begin{aligned}
& |x+\sigma(t,x)y|^{2\gamma+2} - |x|^{2\gamma+2} - D_x|x|^{2\gamma+2}\sigma(t,x)y| \\
& \quad = |\psi(1) - \psi(0) - D_x\psi(0)| \\
& \quad = |\int_0^1 (1-\rho)\psi^{(2)}(\rho)d\rho| \\
& \quad \leq C|y|^2(|x|^{2\gamma} + |y|^{2\gamma}).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \textstyle\int_{I\!R}(|x+\sigma(t,x)y|^{2\gamma+2} - |x|^{2\gamma+2} - D_x|x|^{2\gamma+2}\sigma(t,x)y)\Pi(dy) \\
& = \textstyle\int_{|y|\leq\delta}|x+\sigma(t,x)y|^{2\gamma+2} - |x|^{2\gamma+2} - D_x|x|^{2\gamma+2}\sigma(t,x)y\Pi(dy) \\
& \quad + \textstyle\int_{|y|\geq\delta}|x+\sigma(t,x)y|^{2\gamma+2} - |x|^{2\gamma+2} - D_x|x|^{2\gamma+2}\sigma(t,x)y\Pi(dy)| \\
& \leq C(1+|x|^{2\gamma})
\end{aligned}$$

since the measure Π integrates any power function away from 0. Therefore there exists a constant $\lambda_0 \in I\!R^+$ which does not depend on θ such that if $\lambda \geq \lambda_0$ then the right-hand side of (2.3.6) is non-negative. Thus $\vec{v}$ is a viscosity supersolution of (2.3.1), which is the desired result. □

Remark 2.3.1. *In the same way one can show that if $(u^i)_{i=1}^m$ is a viscosity subsolution of (2.3.1) which belongs to Π_g, i.e. for some $\gamma > 0$ and $C > 0$,*

$$|u^i(t,x)| \leq C(1+|x|^\gamma),\ \forall (t,x) \in [0,T] \times I\!R^k \text{ and } i \in A.$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $\theta > 0$, $\overrightarrow{v}(t,x) = (u_i(t,x) - \theta e^{-\lambda t}(1+|x|^{2\gamma+2}))_{i=1}^m$ is subsolution of (2.3.1). □

2.3.1 Existence of the viscosity solution

In this section we deal with the issue of existence of the viscosity solution of (2.3.1). Recall that $(Y^j, U^j, K^j)_{j\in A}$ is the unique solution of (2.2.24) and let $(u_j(t,x))_{j\in A}$ be the functions defined in Corollary 2.2.1.

Theorem 2.3.1. *Assume Assumptions* [**A4**] *and* [**A5**] *and (2.2.14), (2.2.15) as well, then* $(u_j(t,x))_{j\in A}$ *is a viscosity solution of* (2.3.1).

Proof. The proof will be divided into two steps.

Step 1: We first show that $(u_j)_{j=1}^m$ is a supersolution of (2.3.1). Note that for all $j \in A$, as u_j is lsc, we then have $u_{j_*} = u_j$. Next let us set $u_j^n(t,x) = Y_t^{j,n,t,x}$, where $(Y^{j,n,t,x}; U^{j,n,t,x}, K^{j,n,t,x})_{j\in A}$ is the unique solution of (2.2.27). As pointed out in Corollary 2.2.1, for any $n \geq 0$, $(t,x) \in [0,T] \times I\!R^k$ and $s \in [t,T]$,

$$Y_s^{j,n,t,x} = u_j^n(s, X_s^{t,x}) \text{ and } u_j^n(t,x) \nearrow u_j(t,x).$$

Additionally by induction, $(u_j^n)_{j\in A}$, $n \geq 0$, are continuous, belong to Π_g and by Ren et al.'s result (Theorem 2.2.2) verify in viscosity sense the following system ($n \geq 1$): $\forall j \in A$,

$$\begin{cases} min\Big\{u^{j,n}(t,x) - \max\limits_{k\in A_j}(u^{j,n-1}(t,x) - g_{jk}(t,x)); \\ -\partial_t u^{j,n}(t,x) - \mathcal{L}u^{j,n}(t,x) - f_j(t,x,(u^{1,n-1},\cdots,u^{j-1,n-1},u^{j,n},u^{j+1,n-1},\cdots,u^{m,n-1})(t,x))\Big\} = 0; \\ u^{j,n}(T,x) = h_j(x). \end{cases} \tag{2.3.7}$$

First note that for any $j \in A$, u_j verifies

$$u_j(T,x) = h_j(x) \text{ and } u_j(t,x) \geq \max_{k\in A_j}\{u_k(t,x) - g_{jk}(t,x)\},\ \forall (t,x) \in [0,T] \times I\!R^k.$$

Next let $(t,x) \in (0,T)\times I\!R$ and let $j \in A$ be fixed. Let ϕ be a function which belongs to $\mathcal{C}^{1,2}([0,T]\times I\!R)\cap\Pi_g$ such that $u_j - \phi$ has a strict global minimum in (t,x) on $[0,T] \times I\!R^k$ and wlog we assume that $u_j(t,x) = \phi(t,x)$. Now let $\delta > 0$ be fixed. Then (t,x) is a global strict minimum of $u_j - \phi$ in $[0,T] \times B(x, C_\sigma\delta)$. Next let (t_n, x_n) be the global minimum of $u_j^n - \phi$ on $[0,T] \times \overline{B(x, C_\sigma\delta)}$. Therefore

$$\lim_n (t_n, x_n) = (t,x) \text{ and } u_j^n(t_n,x_n) \to u(t,x).$$

Actually let us consider a convergent subsequence of (t_n, x_n), which we still denote by (t_n, x_n), and let set (t^*, x^*) its limit. Then

$$u_j^n(t_n,x_n) - \phi(t_n,x_n) \leq u_j^n(t,x) - \phi(t,x). \tag{2.3.8}$$

Taking the limit wrt n and since $u_{j*} = u_j$ is lsc to obtain

$$u_j(t^*,x^*) - \phi(t^*,x^*) \leq u_j(t,x) - \phi(t,x).$$

As the minimum (t,x) of $u_j - \phi$ on $[0,T] \times I\!R^k$ is strict then $(t^*,x^*) = (t,x)$. It follows that the sequence $((t_n,x_n))_n$ converges to (t,x). Going back now to (2.3.8) and in sending n to infinite we obtain

$$u_{j*}(t,x) = u_j(t,x) \leq \liminf_n u_j^n(t_n,x_n) \leq \limsup_n u_j^n(t_n,x_n) \leq u_j(t,x)$$

which implies that $u_j^n(t_n,x_n) \to u_j(t,x)$ as $n \to \infty$.

Now for n large enough $(t_n,x_n) \in (0,T) \times B(x, C_\sigma\delta)$ and it is the global minimum of $u_j^n - \phi$ in $[0,T] \times B(x, C_\sigma\delta)$. As u_j^n is a supersolution of (2.3.7), then by Definition 5.2.1 (see Appendix 5.2) we have

$$\begin{array}{l} -\partial_t\phi(t_n,x_n) - \mathcal{L}^1\phi(t_n,x_n) - \mathcal{I}^{1,\delta}(t_n,x_n,\phi) - \mathcal{I}^{2,\delta}(t_n,x_n,D_x\phi(t_n,x_n),u^{j,n}) \geq \\ f_j(t_n,x_n,u^{1,n-1}(t_n,x_n),\cdots,u^{j-1,n-1}(t_n,x_n),u^{j,n}(t_n,x_n),u^{j+1,n-1}(t_n,x_n),\cdots,u^{m,n-1}(t_n,x_n)). \end{array} \tag{2.3.9}$$

But there exists a subsequence of $\{n\}$ such that:

(i) for any $k \in A_j$, $(u_k^{n-1}(t_n,x_n))_n$ is convergent and then $\lim_n u_k^{n-1}(t_n,x_n) \geq u_k(t,x) = u_{k*}(t,x)$;

(ii) $(\mathcal{I}^{1,\delta}(t_n,x_n,\phi))_n \to \mathcal{I}^{1,\delta}(t,x,\phi)$ as $n \to \infty$;

Sending now n to infinite (through the previous subsequence) in (2.3.9), using the fact that f_j is continuous and verifies (A4)(I)(v) and finally by Fatou's Lemma to obtain:

$$-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}^{1,\delta}(t,x,\phi) \geq \\ \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),u^j) + f_j(t,x,u_1(t,x),\cdots,u_{j-1}(t,x),u_j(t,x),u_{j+1}(t,x),\cdots,u_m(t,x)).$$

But $u_j(t,x) = \phi(t,x)$ and $u_j \geq \phi$, then $I^{2,\delta}(t,x,D_x\phi(t,x),u^j) \geq I^{2,\delta}(t,x,D_x\phi(t,x),\phi)$. Plugging now this inequality in the previous one to obtain

$$-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}(t,x,\phi) - f_j(t,x,u_1(t,x),\cdots,u_{j-1}(t,x),u_j(t,x),u_{j+1}(t,x),\cdots,u_m(t,x)) \geq 0.$$

Therefore u_j is a viscosity supersolution of

$$\begin{cases} min\{u_j(t,x) - \max\limits_{k\in A_j}(u_k(t,x) - g_{jk}(t,x)); \\ \qquad -\partial_t u_j(t,x) - \mathcal{L}u_j(t,x) - f_j(t,x,u_1(t,x),\cdots,u_m(t,x))\} = 0; \\ u_j(T,x) = h_j(x). \end{cases}$$

As j is arbitrary then $(u_j)_{j\in A}$ is a viscosity supersolution of (2.3.1). □

Step 2: We will now show that $(u_j^*)_{j\in A}$ is a subsolution of (2.3.1). As a first step we are going to show that

$$\forall j \in A,\ min\{u_j^*(T,x) - h_j(x);\ \ u_j^*(T,x) - \max_{k\in A_j}(u_k^*(T,x) - g_{jk}(T,x))\} = 0.$$

By definition of u_j^* and since $u_j^n \nearrow u_j$, we have

$$min\{u_j^*(T,x) - h_j(x);\ \ u_j^*(T,x) - \max_{k\in A_j}(u_k^*(T,x) - g_{jk}(T,x))\} \geq 0$$

Next suppose that for some $x_0 \in I\!R$, $\exists j > 0$, s.t.

$$min\{u_j^*(T,x_0) - h_j(x_0);\ \ u_j^*(T,x_0) - \max_{k\in A_j}(u_k^*(T,x_0) - g_{jk}(T,x_0))\} = 2\epsilon.$$

We will show that leads to a contradiction.
Let $(t_k,x_k)_{k\geq 1} \to (T,x_0)$ and $u_j(t_k,x_k) \to u_j^*(T,x_0)$. We can find a sequence of functions $(v^n)_{n\geq 0} \in C^{1,2}([0,T]\times I\!R)$ of compact support such that $v^n \to u_j^*$, since u_j^* is usc. On some neighborhood B_n of (T,x_0) we have,

$$\forall (t,x) \in B_n,\ min\{v^n(t,x) - h_j(x);\ \ v^n(t,x) - \max_{k\in A_j}(u_k^*(t,x) - g_{jk}(t,x))\} \geq \epsilon. \tag{2.3.10}$$

Let us denote by $B_k^n := [t_k,T]\times B(x_k,\delta_k^n)$, for some $\delta_n^k \in]0,1]$ small enough such that $B_k^n \subset B_n$. Since u_j^* is of polynomial growth, there exists $c > 0$, such that $|u_j^*| \leq c$ on B_n. We can then assume $v^n \geq -2c$ on B_n. Define

$$V_k^n(t,x) := v^n(t,x) + \frac{4c|x-x_k|^2}{{\delta_k^n}^2} + \sqrt{T-t}$$

Note that $V_k^n(t,x) \geq v^n(t,x)$ and

$$(u_j^* - V_k^n)(t,x) \leq -c\ \ \forall (t,x) \in [t_k,T]\times\partial B(x_k,\delta_k^n). \tag{2.3.11}$$

On the other hand, by Itô's formula we have

$$\begin{aligned} -\ & \{\partial_t V_k^n(t,x) + \mathcal{L}V_k^n(t,x)\} \\ & = -\Big\{\partial_t v^n(t,x) + \partial_t((T-t)^{\frac12}) + \{E(L_1)\sigma(t,x) + b(t,x)\}(\partial_x v^n(t,x) + \tfrac{8c(x-x_k)}{(\delta_k^n)^2}) \\ & \quad + \tfrac12\sigma(t,x)^2\varpi^2(D_{xx}^2 v^n(t,x) + \tfrac{8c}{(\delta_k^n)^2}) + \textstyle\int_{I\!R}[v^n(t,x+\sigma(t,x)y) + \frac{4c|x-x_k+\sigma(t,x)y|^2}{(\delta_k^n)^2} \\ & \quad + (T-t)^{\frac12} - v^n(t,x) - \frac{4c|x-x_k|^2}{(\delta_k^n)^2} - (T-t)^{\frac12} - \partial_x v^n(t,x)\sigma(t,x)y - \frac{8c(x-x_k)}{(\delta_k^n)^2}\sigma(t,x)y]\Pi(dy)\Big\} \\ & = -\Big\{\partial_t v^n(t,x) + \partial_t((T-t)^{\frac12}) + \{E(L_1)\sigma(t,x) + b(t,x)\}\{\partial_x v^n(t,x) + \tfrac{8c(x-x_k)}{(\delta_k^n)^2}\} \\ & \quad + \tfrac12\sigma(t,x)^2\varpi^2(D_{xx}^2 v^n(t,x) + \tfrac{8c}{(\delta_k^n)^2}) \\ & \quad + \textstyle\int_{I\!R}[v^n(t,x+\sigma(t,x)y) - v^n(t,x) - \partial_x v^n(t,x)\sigma(t,x)y]\Pi(dy) \\ & \quad + \textstyle\int_{I\!R}[\frac{4c|x-x_k+\sigma(t,x)y|^2}{(\delta_k^n)^2} - \frac{4c|x-x_k|^2}{(\delta_k^n)^2} - \frac{8c(x-x_k)}{(\delta_k^n)^2}\sigma(t,x)y]\Pi(dy)\Big\}. \end{aligned}$$

Note that $\Phi(x) := \frac{4c|x-x_k|^2}{(\delta_k^n)^2} \in C^2 \cap \Pi_g$ and $v^n \in C^{1,2}$ and of compact support, then the two non-local terms are bounded and $\partial_t v^n$, $\partial_x v^n$, $D^2_{xx} v^n$ are so. Since $\partial_t(\sqrt{T-t}) \to -\infty$, when $t \to T$, then we can choose t_k large enough in front of δ_k and the derivatives of v^n to ensure that

$$-(\partial_t V_k^n(t,x) + \mathcal{L}V_k^n)(t,x) \geq 0, \quad \forall (t,x) \in B_n^k. \tag{2.3.12}$$

Consider now the stopping time $\theta_n^k := inf\{s \geq t_k, (s, X_s^{t_k,x_k}) \in B_n^{k^c}\} \wedge T$, where $B_n^{k^c}$ is the complement of B_n^k and $\theta_k := inf\{s \geq t_k, u_j(s, X_s^{t_k,x_k}) = \max_{l \in A_j}(u_l(s, X_s^{t_k,x_k}) - g_{jl}(s, X_s^{t_k,x_k}))\} \wedge T$. Applying Itô's formula with $V_k^n(t,x)$ on $[t_k, \theta_n^k \wedge \theta_k]$ and taking into account (2.3.11), (2.3.10), (2.3.12) and the fact that $V_k^n \in C^{1,2}$, to obtain:

$$\begin{aligned} V_k^n(t_k,x_k) &= V_k^n(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k,x_k}) - \int_{t_k}^{\theta_n^k \wedge \theta_k} [b(r, X_r^{t_k,x_k}) \partial_x V_k^n(r, X_r^{t_k,x_k}) + \partial_t V_k^n(t,x)(r, X_r^{t_k,x_k})] dr \\ &- \int_{t_k}^{\theta_n^k \wedge \theta_k} \sigma(r, X_{r_-}^{t_k,x_k}) \partial_x V_k^n(r, X_{r_-}^{t_k,x_k}) dL_r - \tfrac{1}{2} \int_{t_k}^{\theta_n^k \wedge \theta_k} \sigma^2(r, X_r^{t_k,x_k}) \varpi^2 \partial^2_{xx} V_k^n(r, X_r^{t_k,x_k}) dr \\ &- \sum_{t_k < r \leq \theta_n^k \wedge \theta_k} \{V_k^n(r, X_r^{t_k,x_k}) - V_k^n(r, X_{r_-}^{t_k,x_k}) - \sigma(r, X_{r_-}^{t_k,x_k}) \partial_x V_k^n(r, X_{r_-}^{t_k,x_k}) \triangle L_r\} \end{aligned} \tag{2.3.13}$$

Next let us deal with the last term of the last equality and let us set

$$h(s,y) = V_k^n(s, X_{s-}^{t_k,x_k} + \sigma(s, X_{s-}^{t_k.x_k})y) - V_k^n(s, X_{s-}^{t_k,x_k}) - \partial_x V_k^n(s, X_{s-}^{t_k.x_k}) \sigma(s, X_{s-}^{t_k.x_k}) y.$$

By the mean value theorem we have

$$h(s,y) = \frac{1}{2} \partial^2_{xx} v^n(s, X_{s-}^{t_k,x_k} + \bar{X}\sigma(s, X_{s-}^{t_k.x_k})y)(\sigma(s, X_{s-}^{t_k.x_k})y)^2 + \frac{4c}{{\delta_k^n}^2}(\sigma(s, X_{s-}^{t_k.x_k})y)^2$$

where $\bar{X}$ is a stochastic processes which is valued in $(0,1)$. As v^n is of compact support and σ is bounded then

$$\mathbb{E}[\int_0^T \int_{I\!R} |h(s,y)| \Pi(dy) ds] < \infty.$$

It follows that

$$\begin{aligned} &\mathbb{E}[\sum_{t_k \leq r \leq \theta_n^k \wedge \theta_k} \{V_k^n(r, X_r^{t_k,x_k}) - V_k^n(r, X_{r_-}^{t_k,x_k}) - \sigma(r, X_{r_-}^{t_k,x_k}) \partial_x V_k^n(r, X_{r_-}^{t_k,x_k}) \triangle L_r\}] \\ =&\mathbb{E}[\int_{t_k}^{\theta_n^k \wedge \theta_k} \int_{I\!R} h(s,y) \Pi(dy) ds] < \infty. \end{aligned}$$

Next going back to (2.3.13) and taking expectation to obtain

$$\begin{aligned} V_k^n(t_k,x_k) &= \mathbb{E}[V_k^n(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k,x_k}) - \int_{t_k}^{\theta_n^k \wedge \theta_k} (\partial_t V_k^n(r, X_r^{t_k,x_k}) + \mathcal{L}V_k^n(r, X_r^{t_k,x_k})) dr] \\ &\geq \mathbb{E}[V_k^n(\theta_n^k, X_{\theta_n^k}^{t_k,x_k}) \mathbb{1}_{\{\theta_n^k \leq \theta_k\}} + V_k^n(\theta_k, X_{\theta_k}^{t_k,x_k}) \mathbb{1}_{\{\theta_n^k > \theta_k\}}] \\ &= \mathbb{E}[\{V_k^n(\theta_n^k, X_{\theta_n^k}^{t_k,x_k}) \mathbb{1}_{\{\theta_n^k < T\}} + V_k^n(T, X_T^{t_k,x_k}) \mathbb{1}_{\{\theta_n^k = T\}}\} \mathbb{1}_{\{\theta_n^k \leq \theta_k\}} + V_k^n(\theta_k, X_{\theta_k}^{t_k,x_k}) \mathbb{1}_{\{\theta_n^k > \theta_k\}}] \\ &\geq \mathbb{E}[\{(u_j^*(\theta_n^k, X_{\theta_n^k}^{t_k,x_k}) + c) \mathbb{1}_{\{\theta_n^k < T\}} + (\epsilon + h_j(X_T^{t_k,x_k})) \mathbb{1}_{\{\theta_n^k = T\}}\} \mathbb{1}_{\{\theta_n^k \leq \theta_k\}} \\ &\quad + \{\epsilon + \max_{k \in A_j}(u_k^*(\theta_k, X_{\theta_k}^{t_k,x_k}) - g_{jk}(\theta_k, X_{\theta_k}^{t_k,x_k}))\} \mathbb{1}_{\{\theta_n^k > \theta_k\}}] \\ &\geq \mathbb{E}[u_j(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k,x_k})] + c \wedge \epsilon \\ &= \mathbb{E}[u_j(t_k,x_k) - \int_{t_k}^{\theta_n^k \wedge \theta_k} f_j(s, X_s^{t_k,x_k}, (u_l(s, X_s^{t_k,x_k}))_{l=1,m} ds] + c \wedge \epsilon \end{aligned}$$

since the processes $(Y^j = u^j(., X.))_{j \in A}$ stopped at time $\theta_n^k \wedge \theta_k$ solves an explicit RBSDE system with triple of data given by $((f_j)_{j \in A}, (h_j)_{j \in A}, (g_{i,j})_{i,j \in A})$. In addition, $dK^{j,t,x} = 0$ on $[t_k, \theta_k]$. On the other hand, $(u^j)_{j \in A} \in \Pi_g$ and then taking into account (2.2.16) and Assumption (A4)(1)(iii), we deduce that

$$\lim_{k \to \infty} E[\int_{t_k}^{\theta_n^k \wedge \theta_k} f_j(s, X_s^{t_k,x_k}, (u_l(s, X_s^{t_k,x_k}))_{l=1,m}) ds] = 0.$$

Taking the limit in the previous inequalities yields:

$$\lim_{k\to\infty} V_k^n(t_k,x_k) = \lim_{k\to\infty} v^n(t_k,x_k) + \sqrt{T-t_k} = v^n(T,x_0)$$
$$\geq \lim_{k\to\infty} u_j(t_k,x_k) + c\wedge\epsilon = u_j^*(T,x_0) + c\wedge\epsilon.$$

As $v^n \to u_j^*$ pointwisely, then we get a contradiction, when taking the limit in the previous inequalities, and the result follows, i.e., $\forall x \in I\!R$, $\forall j \in A$,

$$min\{u_j^*(T,x) - h_j(x);\quad u_j^*(T,x) - \max_{l\in A_j}(u_l^*(T,x) - g_{jl}(T,x))\} = 0.$$

Finally the proof of

$$u_j^*(T,x) = h_j(x), \forall j \in A$$

is obtained in the same way as in ([30], pp.180) since the function $g_{ij}, i,j \in A$ verify the non-free loop property (A4)(II). □

Now let us show $(u_j^*)_{j\in A}$ is a subsolution of (2.3.1). First note that since $u_j^n \nearrow u_j$ and u_j^n is continuous, we have

$$u_j^*(t,x) = \limsup_{n\to\infty}{}^* u_j^n(t,x) = \overline{\lim_{n\to\infty, t'\to t, x'\to x}} u_j^n(t',x').$$

Besides $\forall j \in A$ and $n \geq 0$ we deduce from the construction of u_j^n that

$$u_j^n(t,x) \geq \max_{l\in A_j}(u_l^{n-1}(t,x) - g_{jl}(t,x))$$

and by taking the limit in n we obtain: $\forall j \in A$, $\forall x \in I\!R$,

$$u_j^*(t,x) \geq \max_{l\in A_j}(u_l^*(t,x) - g_{jl}(t,x)).$$

Next fix $j \in A$. Let $(t,x) \in (0,T)\times I\!R$ be such that

$$u_j^*(t,x) - \max_{l\in A_j}(u_l^*(t,x) - g_{jl}(t,x)) > 0. \tag{2.3.14}$$

Let ϕ be a $C^{1,2}([0,T]\times I\!R)\cap \Pi_g$ function such that $u_j^* - \phi$ has a global maximum at (t,x) in $[0,T]\times I\!R^k$ which wlog we suppose strict and $u_j(t,x) = \phi(t,x)$. Therefore (t,x) is a global strict maximum of $u_j - \phi$ in $[0,T]\times B(x, C_\sigma\delta)$. On the other hand there exist subsequences $\{n_k\}$ and $((t'_{n_k}, x'_{n_k}))_k$ such that

$$((t'_{n_k}, x'_{n_k}))_k \to_k (t,x) \text{ and } u_j^{n_k}(t'_{n_k}, x'_{n_k}) \to_k u_j^*(t,x).$$

Let now (t_{n_k}, x_{n_k}) be the global maximum of $u_j^{n_k} - \phi$ on $[0,T]\times\overline{B(x, C_\sigma\delta)}$. Therefore

$$(t_{n_k}, x_{n_k}) \to_k (t,x) \text{ and } u_j^{n_k}(t_{n_k}, x_{n_k}) \to_k u_j^*(t,x).$$

Actually let us consider a convergent subsequent of (t_{n_k}, x_{n_k}), which we still denote by (t_{n_k}, x_{n_k}), and let $(\bar t, \bar x)$ be its limit. Then for some k_0 and for $k \geq k_0$ we have

$$u_j^{n_k}(t_{n_k}, x_{n_k}) - \phi(t_{n_k}, x_{n_k}) \geq u_j^{n_k}(t'_{n_k}, x'_{n_k}) - \phi(t'_{n_k}, x'_{n_k}). \tag{2.3.15}$$

Taking the limit wrt k to obtain

$$u_j^*(\bar t, \bar x) - \phi(\bar t, \bar x) \geq u_j^*(t,x) - \phi(t,x).$$

As the maximum (t,x) of $u_j - \phi$ on $[0,T]\times I\!R^k$ is strict then $(\bar t, \bar x) = (t,x)$. It follows that the sequence $((t_{n_k}, x_{n_k}))_k$ converges to (t,x). Going back now to (3.3.24) and in sending k to infinite we obtain

$$u_j^*(t,x) \geq \limsup_k u_j^{n_k}(t_{n_k}, x_{n_k}) \geq \liminf_k u_j^{n_k}(t_{n_k}, x_{n_k}) \geq \liminf_k u_j^{n_k}(t'_{n_k}, x'_{n_k}) = u_j^*(t,x)$$

which implies that $u_j^{n_k}(t_{n_k}, x_{n_k}) \to u_j^*(t,x)$ as $k \to \infty$.

Now for k large enough,

(i) $(t_{n_k}, x_{n_k}) \in (0,T)\times B(x, C_\sigma\delta)$ and is the global maximum of $u_j^{n_k} - \phi$ in $(0,T)\times B(x, C_\sigma\delta)$;

(ii) $u_j^{n_k}(t_{n_k}, x_{n_k}) > \max_{l\in A_j}(u_l^{n_k-1}(t_{n_k}, x_{n_k}) - g_{jl}(t_{n_k}, x_{n_k}))$.

As $u_j^{n_k}$ is a subsolution of (2.3.7), then by Definition 5.2.1 (see Appendix 5.2) we have

$$-\partial_t\phi(t_{n_k},x_{n_k})-\mathcal{L}^1\phi(t_{n_k},x_{n_k})-\mathcal{I}^{1,\delta}(t_{n_k},x_{n_k},\phi)-\mathcal{I}^{2,\delta}(t_{n_k},x_{n_k},D_x\phi(t_{n_k},x_{n_k}),u^{j,n_k})\leq$$
$$f_j(t_{n_k},x_{n_k},u^{1,n_k-1}(t_{n_k},x_{n_k}),\cdots,u^{j-1,n_k-1}(t_{n_k},x_{n_k}),u^{j,n_k}(t_{n_k},x_{n_k}),u^{j+1,n_k-1}(t_{n_k},x_{n_k}),\cdots,u^{m,n_k-1}(t_{n_k},x_{n_k})). \tag{2.3.16}$$

But there exists a subsequence of $\{n_k\}$ (which we still denote by $\{n_k\}$) such that:
(i) for any $l\in A_j$, $(u_l^{n_k-1}(t_{n_k},x_{n_k}))_k$ is convergent and then $\lim_k u_l^{n_k-1}(t_{n_k},x_{n_k})\leq u_l^*(t,x)$;
(ii) $(\mathcal{I}^{1,\delta}(t_{n_k},x_{n_k},\phi))_{n_k}\to\mathcal{I}^{1,\delta}(t,x,\phi)$ as $k\to\infty$.

Sending now k to infinite (through the previous subsequence) in (3.3.25), using the fact that f_j is continuous and verifies (A4)(I)(v) and finally by Lebesgue's Theorem to obtain

$$-\partial_t\phi(t,x)-\mathcal{L}^1\phi(t,x)-\mathcal{I}^{1,\delta}(t,x,\phi)\leq$$
$$\mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),u_j)+f_j(t,x,u_1^*(t,x),\cdots,u_{j-1}^*(t,x),u_j^*(t,x),u_{j+1}^*(t,x),\cdots,u_m^*(t,x)).$$

But $u_j(t,x)=\phi(t,x)$ and $u_j\leq\phi$, then $I^{2,\delta}(t,x,D_x\phi(t,x),u_j)\leq I^{2,\delta}(t,x,D_x\phi(t,x),\phi)$. Plugging now this inequality in the previous one to obtain

$$-\partial_t\phi(t,x)-\mathcal{L}^1\phi(t,x)-\mathcal{I}(t,x,\phi)-f_j(t,x,u_1^*(t,x),\cdots,u_{j-1}^*(t,x),u_j^*(t,x),u_{j+1}^*(t,x),\cdots,u_m^*(t,x))\leq 0.$$

Therefore u_j is a viscosity subsolution of

$$\begin{cases} min\{u_j(t,x)-\max\limits_{k\in A_j}(u_k(t,x)-g_{jk}(t,x));\\ \qquad -\partial_t u_j(t,x)-\mathcal{L}u_j(t,x)-f_j(t,x,u_1(t,x),\cdots,u_m(t,x))\}=0;\\ u_j(T,x)=h_j(x).\end{cases}$$

As j is arbitrary then $(u_j)_{j\in A}$ is a viscosity subsolution of (2.3.1). □

2.3.2 Uniqueness of the viscosity solution

We now give a comparison result of subsolution and supersolution of system (2.3.1), from which we get the continuity and uniqueness of its solution.

Proposition 2.3.1. *Assume Assumptions* **(A4)** *fulfilled. Let $(u_j)_{j\in A}$ (resp. $(w_j)_{j\in A}$) be a subsolution (resp. supersolution) of* (2.3.1) *which belongs to Π_g. Then for any $j\in A$,*

$$\forall(t,x)\in[0,T]\times I\!R,\quad u_j(t,x)\leq w_j(t,x)$$

Proof. Let γ be a real constant such that for any $j\in A$ and $(t,x)\in[0,T]\times I\!R^k$,

$$|u_j(t,x)|+|w_j(t,x)|\leq C(1+|x|^\gamma).$$

To begin with we additionally assume the existence of a constant λ such that $\lambda<-m.\max\limits_{j\in A}\{C_j\}$ (C_j is the Lipschitz constant of f_j w.r.t $\overrightarrow{y}$) and $\forall j\in A$, $\forall t,x,y_1,\cdots,y_{j-1},y_{j+2},\cdots,y_m,y\geq y'$,

$$f_j(t,x,y_1,\cdots,y_{j-1},y,\cdots,y_m)-f_j(t,x,y_1,\cdots,y_{j-1},y',\cdots,y_m)\leq\lambda(y-y') \tag{2.3.17}$$

Thanks to Lemma 4.1 and Remark 2.3.1, we know there exists ν large enough such that for any $\theta>0$, $w_{j,\theta,\nu}(t,x)=w_j(t,x)+\theta e^{-\nu t}|x|^{2\gamma+2}$ (resp. $u_{j,\theta,\nu}(t,x)=u_j(t,x)-\theta e^{-\nu t}|x|^{2\gamma+2}$). So it is enough to show that

$$\forall j\in A,\ \forall(t,x)\in[0,T]\times I\!R,\quad u_{j,\theta,\nu}(t,x)\leq w_{j,\theta,\nu}(t,x),$$

then taking limits as $\theta\to 0$, the result follows. On the other hand by the growth condition there exists a constant $C>0$ such that

$$\forall j\in A,\forall(t,x)\in[0,T]\times I\!R,\quad s.t.\ |x|\geq C,\ u_{j,\theta,\nu}(t,x)<0<w_{j,\theta,\nu}(t,x). \tag{2.3.18}$$

Now for the sake of simplicity we merely denote $u_{j,\theta,\nu}$ (resp. $w_{j,\theta,\nu}$) by u_j (resp. w_j).

To get the comparison result, we proceed by contradiction assuming that

$$\exists(t_1,x_1)\in[0,T]\times I\!R,\ \text{such that}\ \max_{j\in A}(u_j(t_1,x_1)-w_j(t_1,x_1))>0.$$

Taking into account the values of the subsolution and the supersolution at T, there exist $(\bar{t},\bar{x}) \in [0,T[\times B(0,C)$ (wlog we assume that $\bar{t}>0$), such that :

$$\begin{aligned} 0 &< \max_{(t,x)\in[0,T]\times I\!R} \max_{j\in A}(u_j(t,x)-w_j(t,x)) \\ &= \max_{(t,x)\in[0,T[\times B(0,C)} \max_{j\in A}(u_j(t,x)-w_j(t,x)) \\ &= \max_{j\in A}(u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x})). \end{aligned}$$

We now define $\tilde{A}$ as follows:

$$\tilde{A} := \{j\in A, u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x}) = \max_{k\in A}(u_k(\bar{t},\bar{x})-w_k(\bar{t},\bar{x}))\}. \tag{2.3.19}$$

By the assumption **(A4)**(2), using the same argument as in ([30], pp. 171), we can prove that for some $j\in\tilde{A}$,

$$u_j(\bar{t},\bar{x}) > \max_{k\in A_j}(u_k(\bar{t},\bar{x})-g_{jk}(\bar{t},\bar{x})). \tag{2.3.20}$$

Let us now take such a $j\in\tilde{A}$. For $\varepsilon>0$ and $\rho>0$, let us define

$$\Phi^j_{\varepsilon,\rho}(t,x,y) := u_j(t,x)-w_j(t,y)-\frac{|x-y|^2}{\varepsilon}-|t-\bar{t}|^2-\rho|x-\bar{x}|^4.$$

For any $\varepsilon>0$ and $\rho>0$, let (t_0,x_0,y_0) be such that

$$\Phi^j_{\varepsilon,\rho}(t_0,x_0,y_0) = \max_{(t,x,y)\in[0,T]\times\bar{B}(0,C)^2}\Phi^j_{\varepsilon,\rho}(t,x,y) = \max_{(t,x,y)\in[0,T]\times I\!R^2}\Phi^j_{\varepsilon,\rho}(t,x,y).$$

Note that the maximum exists since $\Phi^j_{\varepsilon,\rho}$ is *usc* and $\bar{B}(0,C)^2$ is the closure of $B(0,C)^2$ and by (2.3.18) the maximum on $[0,T]\times I\!R^2$ can only be reached in $\bar{B}(0,C)^2$. Finally let us point out that (t_0,x_0,y_0) depends actually on ε and ρ which we omit for sake of simplicity. We then have,

$$\begin{aligned} \Phi^j_{\varepsilon,\rho}(\bar{t},\bar{x},\bar{x}) &= u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x}) \\ &\le u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x})+\frac{|x_0-y_0|^2}{\varepsilon}+|t_0-\bar{t}|^2+\rho|x_0-\bar{x}|^4 \\ &\le u_j(t_0,x_0)-w_j(t_0,y_0). \end{aligned} \tag{2.3.21}$$

The growth condition of u_j and w_j implies that $\frac{|x_0-y_0|^2}{\varepsilon}+|t_0-\bar{t}|^2+\rho|x_0-\bar{x}|^4$ is bounded and hence $\lim_{\varepsilon\to0}(x_0-y_0)=0$. Next by (3.3.6), for any subsequence $(t_{0_l},x_{0_l},y_{0_l})_l$ which converges to $(\tilde{t},\tilde{x},\tilde{x})$,

$$u_j(\bar{t},\bar{x})-w_j(\bar{t},\ \bar{x}) \le u_j(\tilde{t},\tilde{x})-w_j(\tilde{t},\tilde{x}),$$

since u_j is *usc* and w_j is *lsc*. By the definition of $(\bar{t},\bar{x})$ this last inequality is an equality. Using both the definiton of $\Phi^j_{\varepsilon,\rho}$ and (3.3.6), it implies that the sequence

$$\lim_{\varepsilon\to0}(t_0,x_0,y_0) = (\bar{t},\bar{x},\bar{x}) \tag{2.3.22}$$

and once more from (3.3.6) we deduce

$$\lim_{\varepsilon\to0}\frac{|x_0-y_0|^2}{\varepsilon}=0. \tag{2.3.23}$$

Finally classically (see e.g. [30], pp. 173) we have also

$$\lim_{\varepsilon\to0}(u_j(t_0,x_0),w_j(t_0,y_0)) = (u_j(\bar{t},\bar{x}),w_j(\bar{t},\bar{x})). \tag{2.3.24}$$

Next as the functions $(u_k)_{k\in A}$ are *usc* and $(g_{ij})_{i,j\in A}$ are continuous, and since the index j satisfies (2.3.19), there exists $r>0$ such that for $(t,x)\in B((\bar{t},\bar{x}),r)$ we have $u_j(t,x) > \max_{k\in A_j}(u_k(t,x)-g_{jk}(t,x))$. But by (3.3.9), (3.3.7) and once more since u_j is *usc* then there exists ε_0 such that for any $0<\varepsilon<\varepsilon_0$, we have:

$$u_j(t_0,x_0) > \max_{k\in A_j}(u_k(t_0,x_0)-g_{ij}(t_0,x_0)).$$

Now for ε small enough, we are able to apply Jensen-Ishii's Lemma for non local operators established by Barles and Imbert ([6], pp.583) (one can see also [8], Lemma 4.1, pp.64) with u_j, w_j and $\phi(t,x,y) := \frac{|x-y|^2}{\varepsilon} + |t-\bar{t}|^2 + \rho|x_0-\bar{x}|^4$ at the point (t_0,x_0,y_0). For any $\delta \in (0,1)$ there are p_u^0, q_u^0, p_w^0, q_w^0, M_u^0 and M_w^δ real constants such that:

(i)

$$p_u^0 - p_w^0 = \partial_t\phi(t_0,x_0,y_0), \quad q_u^0 = \partial_x\phi(t_0,x_0,y_0), \quad q_w^0 = -\partial_y\phi(t_0,x_0,y_0) \tag{2.3.25}$$

and

$$\begin{pmatrix} M_u^0 & 0 \\ 0 & -M_w^0 \end{pmatrix} \leq \frac{4}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 12\rho|x_0-\bar{x}|^2 & 0 \\ 0 & 0 \end{pmatrix}; \tag{2.3.26}$$

$$\begin{aligned} \text{(ii)} \quad & -p_u^0 - \{\sigma(t_0,x_0)\mathbb{E}(L_1) + b(t_0,x_0)\}q_u^0 - \frac{1}{2}\sigma(t_0,x_0)^2\varpi^2 M_u^0 - f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m) \\ & - I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0)) - I^{2,\delta}(t_0,x_0,q_u^0,u_j) \leq 0\,; \end{aligned} \tag{2.3.27}$$

$$\begin{aligned} \text{(iii)} \quad & -p_w^0 - \{\sigma(t_0,y_0)\mathbb{E}(L_1) + b(t_0,y_0)\}q_w^0 - \frac{1}{2}\sigma(t_0,y_0)^2\varpi^2 M_w^0 - f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m) \\ & - I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.)) - I^{2,\delta}(t_0,y_0,q_w^0,w_j) \geq 0. \end{aligned} \tag{2.3.28}$$

We are now going to provide estimates for the non-local terms. First let us define $\psi_\rho(t,x) := \rho|x-\bar{x}|^4 + |t-\bar{t}|^2$. By definition of (t_0,x_0,y_0), for any $d, d' \in \mathbb{R}$,

$$\begin{aligned} & u_j(t_0,x_0+d') - w_j(t_0,y_0+d) - \frac{|x_0+d'-y_0-d|^2}{\varepsilon} - \psi_\rho(t_0,x_0+d') \\ \leq & u_j(t_0,x_0) - w_j(t_0,y_0) - \frac{|x_0-y_0|^2}{\varepsilon} - \psi_\rho(t_0,x_0). \end{aligned}$$

Therefore for $z \in \mathbb{R}$, in taking $d' = \sigma(t_0,x_0)z$ and $d = \sigma(t_0,y_0)z$, we obtain

$$\begin{aligned} & u_j(t_0,x_0+\sigma(t_0,x_0)z) - u_j(t_0,x_0) - q_u^0\sigma(t_0,x_0)z \\ & \quad \leq w_j(t_0,y_0+\sigma(t_0,y_0)z) - w_j(t_0,y_0) - q_w^0\sigma(t_0,y_0)z + \tfrac{|\sigma(t_0,x_0)-\sigma(t_0,y_0)|^2z^2}{\varepsilon} \\ & \quad + \psi_\rho(t_0,x_0+\sigma(t_0,x_0)z) - \psi_\rho(t_0,x_0) - D_x\psi_\rho(t_0,x_0)\sigma(t_0,x_0)z. \end{aligned}$$

It implies that for any $\delta > 0$,

$$I^{2,\delta}(t_0,x_0,q_u^0,u_j) - I^{2,\delta}(t_0,y_0,q_w^0,w_j) \leq C\frac{|x_0-y_0|^2}{\varepsilon} + I^{2,\delta}(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho) \tag{2.3.29}$$

since $\sigma(t,x)$ is uniformly Lipschitz *w.r.t.* x. But it easy to check that

$$|I^{2,\delta}(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho)| \leq \rho\int_{|z|\geq\delta}\{|z|^2+|z|^4\}\Pi(dz).$$

On the other hand, since $\phi \in C^2$

$$\begin{aligned} & I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0)) \\ = \; & \textstyle\int_{|z|\leq\delta}\{\phi(t_0,x_0+\sigma(t_0,x_0)z,y_0) - \phi(t_0,x_0,y_0) - D_x\phi(t_0,x_0,y_0)\sigma(t_0,x_0)z\}\Pi(dz) \\ \leq \; & \sigma(t_0,x_0)^2\textstyle\int_{|z|\leq\delta}\{\varepsilon^{-1}+6\rho(1+|z|^2)\}|z|^2\Pi(dz), \end{aligned}$$

and

$$\begin{aligned} & I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.)) \\ = \; & \textstyle\int_{|z|\leq\delta}\{-\phi(t_0,x_0,y_0+\sigma(t_0,y_0)z) + \phi(t_0,x_0,y_0) + D_y\phi(t_0,x_0,y_0)\sigma(t_0,y_0)z\}\Pi(dz) \\ = \; & -\varepsilon^{-1}\sigma(t_0,y_0)^2\textstyle\int_{|z|\leq\delta}|z|^2 d\Pi(z). \end{aligned}$$

Therefore we have

$$\begin{aligned} & -I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0)) + I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.)) \\ & \quad \geq -\sigma(t_0,x_0)^2\textstyle\int_{|z|\leq\delta}\{\varepsilon^{-1}+6\rho(1+|z|^2)\}|z|^2\Pi(dz) - \varepsilon^{-1}\sigma(t_0,y_0)^2\int_{|z|\leq\delta}|z|^2 d\Pi(z). \end{aligned} \tag{2.3.30}$$

Making now the difference between (3.3.14) and (3.3.15) yields

$$
\begin{aligned}
&-(p_u^0-p_w^0)-[(\sigma(t_0,x_0)E(L_1)+b(t_0,x_0))q_u^0-(\sigma(t_0,y_0)E(L_1)+b(t_0,y_0))q_w^0]-\frac{1}{2}\varpi^2[\sigma(t_0,x_0)^2M_u^0\\
&\quad-\sigma(t_0,y_0)^2M_w^0]-[f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m)-f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m)]\\
&\quad-I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0))+I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.))-I^{2,\delta}(t_0,x_0,q_u^0,u_j)+I^{2,\delta}(t_0,y_0,q_w^0,w_j)\le 0.
\end{aligned}
$$

Taking now into account (3.3.16) and (3.3.17) we get

$$
\begin{aligned}
&-(p_u^0-p_w^0)-[(\sigma(t_0,x_0)E(L_1)+b(t_0,x_0))q_u^0-(\sigma(t_0,y_0)E(L_1)+b(t_0,y_0))q_w^0]-\frac{1}{2}\varpi^2[\sigma(t_0,x_0)^2M_u^0\\
&\quad-\sigma(t_0,y_0)^2M_w^0]-[f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m)-f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m)]\\
&\quad-\sigma(t_0,x_0)^2\int_{|z|\le\delta}\{\varepsilon^{-1}+6\rho(1+|z|^2)\}|z|^2\Pi(dz)-\varepsilon^{-1}\sigma(t_0,y_0)^2\int_{|z|\le\delta}|z|^2d\Pi(z)\\
&\quad-C\frac{|x_0-y_0|^2}{\varepsilon}-I^{2,\delta}(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho)\le 0.
\end{aligned}
$$

Next by using the properties satisfied by p_u^0, q_u^0, p_w^0, q_w^0, M_u^0 and M_w^0 and sending δ to 0 to obtain the existence of a constant $C_{\varepsilon,\rho}$ such that for any fixed ρ we have $\limsup_{\varepsilon\to 0} C_{\varepsilon,\rho}\le 0$ and

$$
-\{f_j(t_0,x_0,u_k(t_0,x_0))_{k=1}^m)-f_j(t_0,x_0,(w_k(t_0,y_0))_{k=1}^m,)\}\le C_{\varepsilon,\rho}+\rho\int_{I\!R}\{|z|^2+|z|^4\}\Pi(dz). \tag{2.3.31}
$$

Next since f_j is Lipschitz *w.r.t.* $(y_k)_{k=1}^m$ and by condition (2.3.17) we have

$$
-\lambda(u_j(t_0,x_0)-w_j(t_0,y_0))-\sum_{k\in A_j}\Upsilon_{\varepsilon,\rho}^{j,k}(u_k(t_0,x_0)-w_k(t_0,y_0))\le C_{\varepsilon,\rho}+\rho\int_{I\!R}\{|z|^2+|z|^4\}\Pi(dz),
$$

where $\Upsilon_{\varepsilon,\rho}^{j,k}$ stands for the increment rate of f_j with respect to y_k $(k\neq j)$, which, by monotonicity condition (A4)(1)(v) on f_j, is non-negative and bounded by C_j. Thus

$$
\begin{aligned}
-\lambda(u_j(t_0,x_0)-w_j(t_0,y_0))&\le\sum_{k\in A_j}\Upsilon_{\varepsilon,\rho}^{j,k}(u_k(t_0,x_0)-w_k(t_0,y_0))^+ +C_{\varepsilon,\rho}+\rho\int_{I\!R}\{|z|^2+|z|^4\}\Pi(dz)\\
&\le C_j\sum_{k\in A_j}(u_k(t_0,x_0)-w_k(t_0,y_0))^+ +C_{\varepsilon,\rho}+\rho\int_{I\!R}\{|z|^2+|z|^4\}\Pi(dz).
\end{aligned}
$$

Taking the limit superior in both hand-sides as $\varepsilon\to 0$, once again u_k (resp. w_k) is *usc* (resp. *lsc*) and $j\in\tilde{A}$, we get

$$
-\lambda(u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x}))\le C_j\sum_{k\in A_j}(u_k(\bar{t},\bar{x})-w_k(\bar{t},\bar{x}))^+ +\rho\int_{I\!R}\{|z|^2+|z|^4\}\Pi(dz),
$$

finally take $\rho\to 0$ to obtain,

$$
-\lambda(u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x}))\le C_j\sum_{k\in A_j}(u_k(\bar{t},\bar{x})-w_k(\bar{t},\bar{x}))^+\le(m-1)C_j(u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x})).
$$

But this is contradictory since $u_j(\bar{t},\bar{x})-w_j(\bar{t},\bar{x})>0$ and $-\lambda>(m-1)C_j$. Henceforth for any $j\in A$, $u_j\le w_j$.

We now consider the general case. Let $(u_j)_{j\in A}$ (resp. $(w_j)_{j\in A}$) be a subsolution (resp. supersolution) of (2.3.1). Denote $\tilde{u}_j(t,x)=e^{\lambda t}u_j(t,x)$ and $\tilde{w}_j(t,x)=e^{\lambda t}w_j(t,x)$. Then it is easy to show that $(\tilde{u}_j)_{j\in A}$ (resp. $(\tilde{w}_j)_{j\in A}$) is a subsolution (resp. supersolution) of the following system of variational inequalities which is similar to (2.2.20):

$$
\begin{cases}
min\{\tilde{u}_j(t,x)-\max\limits_{k\in A_j}(\tilde{u}_k(t,x)-e^{\lambda t}g_{jk}(t,x));\\[2ex]
-\partial_t\tilde{u}_j(t,x)-\mathcal{L}\tilde{u}_j(t,x)+\lambda\tilde{u}_j(t,x)-e^{\lambda t}f_j(t,x,(e^{-\lambda t}\tilde{u}_k)_{k=1}^m)\}=0\,;\\[2ex]
\tilde{u}_j(T,x)=e^{\lambda T}h_j(x).
\end{cases} \tag{2.3.32}
$$

Next let us set

$$F_j(t,x,\overrightarrow{y}) := -\lambda y + e^{\lambda t} f_j(t,x,(e^{-\lambda t}y)_{i=1}^m)$$

with λ is chosen such that $\lambda = m(1+\max_{k\in A} C_k)$ where C_k is the Lipschitz constant of f_k w.r.t. to $(y_l)_{l\in A}$. Then the functions F_k, $k\in A$, verify condition (2.3.17). It follows, from Step 1, that $\forall j\in A, \tilde{u}_j \le \tilde{w}_j$ and then $u_j \le w_j$. The proof is now complete. □

As a by-product we have:

Theorem 2.3.2. *Under Assumptions* [**A4**], [**A5**]*, and (2.2.14), (2.2.15) as well, the system of variational inequalities with inter-connected obstacles* (2.3.1) *has a unique continuous viscosity solution with polynomial growth.* □

In the case when f_j, $j\in A$, do not depend on $\vec{y}$, by the characterization (2.2.32)-(2.2.33) (see also Remark 5.1.1), we deduce that the functions $(u_j(t,x))_{j\in A}$ are nothing but $(J^j(t,x))_{j\in A}$. Thus, as a by product of Theorem 2.3.2, we have:

Corollary 2.3.1. *The value functions $(J^j(t,x))_{j\in A}$ defined in (2.2.23) are continuous, belong to Π_g and is the unique viscosity solution of the Hamilton-Jacobi-Bellman system associated with the stochastic optimal switching problem.* □

2.3.3 Second existence and uniqueness result

In this section we consider the issue of existence and uniqueness of a solution for the systems of IPDEs (2.3.1) when the functions $(-f_j)_{j\in A}$ verify [A4](I). This turns into assuming that $(f_j)_{j\in A}$ verify, instead of [A4](I)(v), the following:

[A4](†): For any $j\in A$, for any $k\neq j$, the mapping $y_k \to f_j(t,x,y_1,\cdots,y_{k-1},y_k,y_{k+1},\cdots,y_m)$ is nonincreasing whenever the other components $(t,x,y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m)$ are fixed.

The other assumptions on $(-f_j)_{j\in A}$ remain the same.

Theorem 2.3.3. *Assume that Assumptions [A1]-[A3], [A5] are fulfillef and $(-f_j)_{j\in A}$ verify [A4]. Then the system of IPDEs (2.2.20) has a unique continuous and of polynomial growth solution which is moreover unique.*

Proof. : We first focus on the issue of existence.

For any $j\in A$ and $\lambda\in R$ let us define F_j by:

$$F_j(t,x,y^1,\cdots,y^m) = e^{\lambda t} f_j(t,x,e^{-\lambda t}y^1,\cdots,e^{-\lambda t}y^m) - \lambda y^j.$$

Since f_j is uniformly Lipschitz w.r.t. $(y_k)_{k=1,m}$ then F_j is so and for λ large enough, F_j satisfies:

For any $k=1,m$, the mapping $y_k \to F_j(t,x,y_1,\cdots,y_{k-1},y_k,y_{k+1},\cdots,y_m)$ is **nonincreasing** whenever the other components $(t,x,y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m)$ are fixed.

Let us now consider the following iterative Picard sequence : $\forall j\in A$, $Y^{j,0}=0$ and for $n\ge 1$, define:

$$(Y^{1,n},\cdots,Y^{m,n}) = \Theta((Y^{1,n-1},\cdots,Y^{m,n-1}))$$

where Θ is the mapping defined in (2.2.34)-(2.2.35) where f_j is replaced with F_j. By (2.2.40), the sequence $(Y^{j,n})_{j\in A}$ converges in $([H^2]^m, \|.\|_{2,\beta})$ to the unique solution $(Y^j)_{j\in A}$ of the system of RBSDEs associated with

$$((F_j(s,X_s^{t,x},y^1,\cdots,y^m))_{j\in A}, (e^{\lambda T}h_j(X_T^{t,x}))_{j\in A}, (e^{\lambda t}g_{jk}(s,X_s^{t,x}))_{j,k\in A}).$$

So using an induction argument on n and Theorem 2.3.2, there exist deterministic continuous functions with polynomial growth $(u_j^n)_{j\in A}$ such that:

$$\forall\, n\ge 0,\ j\in A, \forall (t,x)\in[0,t]\times I\!R, \forall s\in[t,T], Y_s^{j,n} = u_j^n(s,X_s^{t,x}). \tag{2.3.33}$$

By (2.2.41), take $s=t$ we obtain

$$\forall j,n,q,t\le T, x\in I\!R,\ |u_j^n(t,x)-u_j^q(t,x)| = \mathbb{E}[|Y_t^{j,n}-Y_t^{j,q}|^2] \le C\|(Y^{j,n-1})_{j\in A}-(Y^{j,q-1})_{j\in A}\|_{2,\beta}^2.$$

Thus for any $j \in A$, $(u_j^n)_{n\geq 0}$ is of Cauchy type and converges pointwisely to a deterministic function u_j. But $(Y^j)_{j\in A} = \Theta((Y^j)_{j\in A})$, then once more by (2.2.41), we also have:

$$\forall s \in [0,T],\ \mathbb{E}[|Y_s^j - Y_s^{j,m}|^2] \leq C\|(Y^j)_{j\in A} - (Y^{j,m-1})_{j\in A}\|_{2,\beta}^2. \tag{2.3.34}$$

By (2.3.33) we then obtain

$$\forall j \in A, \forall s \in [t,T], \mathcal{P}-a.s., Y_s^j = u_j(s, X_s^{t,x}). \tag{2.3.35}$$

Next as Θ is a contraction then, by induction on n we have

$$\forall n, q \geq 0,\quad \|(Y^{j,n+q})_{j\in A} - Y^{j,n})_{j\in A}\|_{2,\beta} \leq \frac{C_\Theta^n}{1-C_\Theta}\|(Y^{j,1})_{j\in A}\|_{2,\beta}$$

where $C_\Theta \in]0,1[$ is the constant of contraction of Θ. Since the norms $\|.\|$ and $\|.\|_{2,\beta}$ are equivalent, then there exists a constan C_1 such that :

$$\forall n, q \geq 0,\quad \|(Y^{j,n+q})_{j\in A} - Y^{j,n})_{j\in A}\| \leq C_1 C_\Theta^n \|(Y^{j,1})_{j\in A}\|.$$

Take now the limit as q goes to $+\infty$ and in the view of (2.3.34) and (2.3.35), if we take $s = t$ we deduce that :

$$\forall (t,x) \in [0,T]\times I\!R,\quad |u_j(t,x) - u_j^n(t,x)| \leq C_2\|(Y^{j,1})_{j\in A}\|.$$

But it is easy to check that $\|(Y^{j,1})_{j\in A}\|_{2,\beta}(t,x)$ is of polynomial growth (by (2.2.18) and since $\mathbb{E}[\sup_{s\leq T}|X_s^{t,x}|^\gamma]$ is of polynomial growth for any $\gamma \geq 0$) and since for any fixed $n \geq 0$, u_j^n is so. Therefore for any $j \in A$, u_j is of polynomial growth, i.e., belongs to Π_g.

We will now show the continuity of u_j. For any $j \in A$, let us set

$$\bar{Y}_s^{j,0} = C(1 + |X_s^{t,x}|^p),\ s \leq T,$$

where C and p are related to polynomial growth of $(u_j)_{j\in A}$, i.e.,

$$\forall j \in A,\ |u_j(t,x)| \leq C(1+|x|^p),\quad \forall (t,x) \in [0,T]\times I\!R.$$

Next for any $n \geq 1$ and $j \in A$ let us set

$$(\bar{Y}^{1,n}, \cdots, \bar{Y}^{m,n}) = \Theta((\bar{Y}^{1,n-1}, \cdots, \bar{Y}^{m,n-1})).$$

As Θ is a contraction then once more the sequence $((\bar{Y}^{j,n})_{j\in A})_{n\geq 0}$ converges in $([H^2]^m, \|.\|_{2,\beta})$ to $(Y^{j,t,x})_{j\in A}$ the unique solution of the system of RBSDEs associated with

$$((F_j(s, X_s^{t,x}, y^1, \cdots, y^m))_{j\in A}, (e^{\lambda T}h_j(X_T^{t,x}))_{j\in A}, (e^{\lambda t}g_{jk}(s, X_s^{t,x}))_{j,k\in A}).$$

By the definition of $\bar{Y}^{j,0}$, we have

$$\mathcal{P}-a.s.,\ \forall j \in A, s \in [t,T],\quad Y_s^{j,t,x} \leq \bar{Y}_s^{j,0}$$

and taking into account of **[A4](†)** we obtain

$$\forall j \in A,\ \forall s \in [t,T],\ F_j(s, X_s^{t,x}, Y_s^{1,t,x}, \cdots, Y_s^{m,t,x}) \geq F_j(s, X_s^{t,x}, \bar{Y}_s^{1,0}, \cdots, \bar{Y}_s^{m,0}).$$

Next by the comparison result of Remark 2.2.5 and since $(\bar{Y}^{j,1})_{j\in A} = \Theta((\bar{Y}^{j,0})_{j\in A})$, $(Y^{j,t,x})_{j\in A} = \Theta((Y^{j,t,x})_{j\in A})$ we get

$$\forall j \in A,\ s \in [t,T],\ \bar{Y}_s^{j,1} \leq Y_s^{j,t,x}.$$

Now by an induction argument we obtain, for any $n \geq 0$ and $j \in A$,

$$\forall s \in [t,T],\quad \bar{Y}_s^{j,2n+1} \leq Y_s^{j,t,x} \leq \bar{Y}_s^{j,2n}. \tag{2.3.36}$$

In the same way as previously there exist deterministic continuous functions $\bar{u}_j^n$ with polynomial growth such that

$$\forall (t,x) \in [0,T]\times I\!R,\ s \in [t,T],\quad \bar{Y}_s^{j,n} = \bar{u}_j^n(s, X_s^{t,x}).$$

Moreover for any $j \in A$, the sequence $(\bar{u}_j^n)_n$ converges pointwisely to u and by (2.3.36) we have

$$\forall j \in A,\, \forall (t,x),\, u_j(t,x) = \lim_n \nearrow \bar{u}_j^{2n+1}(t,x) = \lim_n \searrow \bar{u}_j^{2n}(t,x).$$

Therefore, u_j, $j \in A$, is both *lsc* and *usc* and then continuous. Finally as $(Y^{j,t,x})_{j\in A} = \Theta((Y^{j,t,x})_{j\in A})$ and $\forall j \in A,\ Y_s^{j,t,x} = u_j(s, X_s^{t,x})$, $s \in [t,T]$, with u_j a deterministic continuous function with polynomial growth, then $(u_j)_{j\in A}$ is a viscosity solution for the corresponding system of IPDEs, thus $(e^{-\lambda t}u_j)_{j\in A}$ is a viscosity solution the system of IPDEs (4.1) with polynomial growth.

Let us now deal with the issue of uniqueness. Let $(\bar{u}_j)_{j\in A}$ be another solution of (2.3.1) which belongs to Π_g and $(\bar{Y}^j)_{j\in A} \in [H^2]^m$ such that for any $j \in A, s \in [t,T]$,

$$\bar{Y}_s^{j,t,x} = \bar{u}_j(s, X_s^{t,x}).$$

Define $(\tilde{Y}^{j,t,x})_{j\in A}$ as follow:

$$(\tilde{Y}^{j,t,x})_{j\in A} = \Theta((\bar{Y}^{j,t,x})_{j\in A})$$

Then there exist $(\tilde{u}_j)_{j\in A}$ deterministic continuous functions with polynomial growth $(\tilde{u}_j)_{j\in A}$ such that:

$$\forall j \in A, s \in [t,T],\ \tilde{Y}_s^{j,t,x} = \tilde{u}_j(s, X_s^{t,x}).$$

Moreover $(\tilde{u}_j)_{j\in A}$ is the unique viscosity solution of the following system of IPDEs : $\forall j \in A$

$$\begin{cases} min\{\tilde{u}_j(t,x) - \max\limits_{k\in A_j}(\tilde{u}_k(t,x) - g_{jk}(t,x)); \\ \qquad -\partial_t u_j(t,x) - \mathcal{L}\tilde{u}_j(t,x) - f_j(t,x,(\bar{u}_k)_{k\in A})\} = 0\,; \\ \tilde{u}_j(T,x) = h_j(x). \end{cases} \tag{2.3.37}$$

As $(\bar{u}_j)_{j\in A}$ is also a solution of (2.3.37), then by uniqueness of Theorem 2.3.3 we obtain $\tilde{u}_j = \bar{u}_j$, for any $j \in A$. Therefore

$$(\bar{Y}^{j,t,x})_{j\in A} = \Theta((\bar{Y}^{j,t,x})_{j\in A}).$$

As $(Y^j)_{j\in A}$ is the unique fixed point of Θ in $[H^2]^m$, we then have

$$\forall j \in A, s \in [t,T],\quad \bar{Y}_s^{j,t,x} = Y_s^j.$$

It follows that $\forall j \in A, \bar{u}_j = u_j$. Finally $(u_j(t,x))_{j\in A}$ is the unique continuous with polynomial growth functions viscosity solution of the system of IPDEs (4.1). □

Chapter 3

Viscosity solution of system of variational inequalities with interconnected bilateral obstacles and connections to multiple modes switching game of jump-diffusion processes

3.1 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis such that $\mathcal{F}_0$ contains all $\mathcal{P}$-null elements of $\mathcal{F}$, and $\mathcal{F}_{t^+} \triangleq \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, $t \geq 0$, and suppose that the filtration is generated by the following two mutually independent process:
- a d-dimensional standard Brownian motion $(W_t)_{t\geq 0}$
- a Poisson random measure N on $R_+ \times E$, where $E \triangleq R^l - \{0\}$ is equipped with its Borel field $\mathcal{B}_E$, with compensator $\nu(dtde) = dtn(de)$, such that $n(E) < \infty$, and $\{\hat{N}((0,t] \times A) = (N - \nu)((0,t] \times A)\}_{0\leq t\leq T}$ is and $\mathcal{F}_t$-martingale for all $A \in \mathcal{B}_E$ satisfying $n(A) < \infty$. n is assumed to be a σ-finite measure on $(E, \mathcal{B}_E)$ satisfying:

$$\int_E (1 \wedge x^2) n(dx) < \infty. \tag{3.1.1}$$

Let T be a fixed positive constant and A^1 (resp. A^2) denote the set of switching modes for player 1 (resp. player 2). Let m_1 (resp. m_2) be the cardinal of the set A^1 (resp. A^2) and for $(i,j) \in A^1 \times A^2$, $A^1_i := A^1 - \{i\}$ and $A^2_j := A^2 - \{j\}$. Next, for $\overrightarrow{y} = (y^{kl})_{(k,l)\in A^1\times A^2} \in R^{m_1\times m_2}$. For any $y_1 \in R$, denote by $[\overrightarrow{y}^{i,j}, y_1]$ the matrix which is obtained from $\overrightarrow{y}$ by replacing the element y^{ij} with y_1.
A function $\Phi : (t,x) \in [0,T] \times R \to \Phi(t,x) \in R$ is called of polynomial growth if there exist two non-negative real constant C and γ such that

$$|\Phi(t,x)| \leq C(1 + |x|^\gamma).$$

Hereafter, this class of functions is denoted by Π_g.

We now define the probabilistic tools and sets we need later. Let:

(i) $\mathcal{P}$ be the σ-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0,T]$;
(ii) $\mathcal{L}^2 := \{\varphi$ is an $I\!R$-valued, $\mathcal{F}_T$-random variable such that $\|\varphi\|^2_{\mathcal{L}^2} := \mathbb{E}[|\varphi|^2] < \infty\}$;
(iii) $\mathcal{H}^2 := \{\{\varphi_t, 0 \leq t \leq T\}$ is an $I\!R$-valued, $\mathcal{F}_t$-progressively measurable process s.t. $||\varphi||^2_{\mathcal{H}^2} := \mathbb{E}(\int_0^T |\varphi_t|^2 dt) < \infty\}$;
(iv) $\mathcal{S}^2 := \{\{\varphi_t, 0 \leq t \leq T\}$ is an $I\!R$-valued, $\mathcal{F}_t$-adapted RCLL process s.t. $||\varphi||^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < \infty\}$; $\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of continuous non-decreasing processes null at $t = 0$;

(v) $\mathcal{H}^2(\tilde{N}) := \{U : \Omega\times[0,T]\times E \to R, \mathcal{P}\otimes\mathcal{B}_E$-measurable and s.t. $\|U\|^2_{\mathcal{H}^2(\tilde{N})} := \mathbb{E}(\int_0^T \int_E |U_t(e)|^2 n(de)dt) < \infty\}$.

In this paper, we investigate existence and uniqueness of viscosity solutions $\vec{v}(t,x) := (v^{ij}(t,x))_{(i,j)\in A^1\times A^2}$ of the following system of variational inequalities with upper and lower interconnected obstacles: $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} \min\{(v^{ij} - L^{ij}[\vec{v}])(t,x); \max\{(v^{ij} - U^{ij}[\vec{v}])(t,x); \\ \\ -\partial_t v^{ij}(t,x) - \mathcal{L}v^{ij}(t,x) - g^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x v^{ij}(t,x), I^{B^{ij}}(t,x,v^{ij}))\}\} = 0\,; \\ \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{3.1.2}$$

where for any $(i,j) \in \Gamma^1 \times \Gamma^2$ and $(t,x) \in [0,T] \times R$ and $\phi \in \mathcal{C}^{1,2}$,

(a) $L^{ij}[\vec{v}](t,x) := \max_{k\in A^1_i}\{v^{kj}(t,x) - \underline{g}_{ik}(t,x)\}$ and $U^{ij}[\vec{v}](t,x) := \min_{l\in A^2_j}\{v^{il}(t,x) - \overline{g}_{jl}(t,x)\}$;

(b) $\mathcal{L}\phi(t,x) := b(t,x)D_x\phi(t,x) + \frac{1}{2}\sigma^2(t,x)D^2_{xx}\phi(t,x)$
$+ \int_E(\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de)$;

(c) $I^{B^{ij}}(t,x,\phi) = \int_E(\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de)$.

Next for $\delta > 0$, $(t,x) \in [0,T] \times I\!R^k$, $q \in R$, ϕ a $\mathcal{C}^{1,2}$ function and $(i,j) \in \Gamma^1 \times \Gamma^2$, let us set:

(d) $I^1_\delta(t,x,\phi) = \int_{|e|\le\delta}(\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de)$;

(e) $I^2_\delta(t,x,q,\phi) = \int_{|e|\ge\delta}(\phi(t,x+\beta(x,e)) - \phi(t,x) - q\beta(x,e))n(de)$;

(f) $I^{1,B^{ij}}_\delta(t,x,\phi) = \int_{|e|\le\delta}(\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de)$;

(g) $I^{2,B^{ij}}_\delta(t,x,\phi) = \int_{|e|\ge\delta}(\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de)$;

(h) $I(t,x,\phi) = \int_E(\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de) = I^1_\delta(t,x,\phi) + I^2_\delta(t,x,D_x\phi,\phi)$;

(i) $I^{B^{ij}}(t,x,\phi) = \int_E(\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma^{ij}(x,e)n(de) = I^{1,B^{ij}}_\delta(t,x,\phi) + I^{2,B^{ij}}_\delta(t,x,\phi)$;

(j) $\mathcal{L}_\phi u(t,x) := b(t,x)D_x\phi(t,x) + \frac{1}{2}\sigma^2(t,x)D^2_{xx}\phi(t,x) + I^1_\delta(t,x,\phi) + I^2_\delta(t,x,D_x\phi,u)$.

The following assumptions will be in force throughout the rest of the paper.

(A0) The functions $b(t,x)$ and $\sigma(t,x)$: $[0,T]\times R \to R$ are jointly continuous in (t,x), of linear growth in (t,x) and Lipschitz continuous w.r.t. x, meaning that there exists a non-negative constant C such that for any $(t,x,x') \in [0,T]\times R$ we have:

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|), \quad |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \le C|x-x'|.$$

The function $\beta : R\times E \to R$ is measurable, continuous in x and such that for some real K and all $e \in E$, for any $x, x' \in R$,

$$|\beta(x,e)| \le K(1\wedge|e|), \quad |\beta(x,e) - \beta(x',e)| \le K|x-x'|(1\wedge|e|).$$

(A1) For any $(i,j) \in A^1 \times A^2$, $g^{ij}(t,x,\overrightarrow{y},z,q) : R\times R\times R^{m_1\times m_2}\times R^d \times R \to R$,

(i) is continuous in (t,x) uniformly w.r.t. the other variables $(\overrightarrow{y},z,q)$ and for any (t,x) the mapping $(t,x) \to g^{i,j}(t,x,0,0,0)$ is of polynomial growth.

(ii) satisfies the standard hypothesis of Lipschitz continuity w.r.t. the variables $(\overrightarrow{y}, z, q)$, i.e. $\forall (t,x) \in [0,T]\times R, \forall (\overrightarrow{y}_1, \overrightarrow{y}_2) \in R^{m_1\times m_2}\times R^{m_1\times m_2}, (z_1,z_2)\in R\times R, (q_1,q_2)\in R\times R$,

$$|g^{ij}(t,x,\overrightarrow{y}_1,z_1,q_1) - g^{ij}(t,x,\overrightarrow{y}_2,z_2,q_2)| \le C(|\overrightarrow{y}_1-\overrightarrow{y}_2| + |z_1-z_2| + |q_1-q_2|),$$

where, $|\overrightarrow{y}|$ stands for the standard Euclidean norm of $\overrightarrow{y}$ in $R^{m_1}\times R^{m_2}$.

(iii) $q \mapsto g^{ij}(t,x,y,z,q)$ is non-decreasing, for all $(t,x,y,z)\in[0,T]\times R\times R^{m_1\times m_1}\times R$.

Futhermore, let $\gamma^{ij}: R\times \mathcal{B}_E \to R$ such that there exists $C>0$,

$$0 \le \gamma^{ij}(x,e) \le C(1\wedge|e|), \quad x\in R, e\in\mathcal{B}_E$$
$$|\gamma^{ij}(x,e)-\gamma^{ij}(x',e')| < C|x-x'|(1\wedge|e|), \quad x,x'\in R, e\in E.$$

We set

$$f^{ij}(t,x,y,z,u) = g^{ij}(t,x,y,z,\int_E u(e)\gamma^{ij}(x,e)n(de),$$

for $(t,x,y,z,u)\in[0,t]\times R\times R^{m_1\times m_2}\times R\times \mathcal{L}^2(R,\mathcal{B}_E,n)$.

(A2) Monotonicity: For any $(i,j)\in A^1\times A^2$ and any $(k,l)\neq(i,j)$ the mapping $y^{k,l}\to g^{i,j}(t,x,\overrightarrow{y},z,u)$ is non-decreasing.

(A3) The functions $h^{ij}(x): R\to R$ are continuous w.r.t. x, belong to class Π_g and satisfy

$$\forall (i,j)\in A^1\times A^2 and\ x\in R, \quad \max_{k\in A^1_i}(h^{kj}(x)-\underline{g}_{ik}(T,x)) \le h^{ij}(x) \le \min_{l\in A^2_j}(h^{il}(x)-\overline{g}_{jl}(T,x)),$$

where $\overline{g}_{ik}$ and $\underline{g}_{jl}$ are given in the next assumption.

(A4) The no free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t,x), belong to Π_g and satisfy the following condition:

For any loop in $A^1\times A^2$, i.e., any sequence of pairs $(i_1,j_1),\dots,(i_N,j_N)$ of $A^1\times A^2$ such that $(i_N,j_N)=(i_1,j_1)$, $\text{card}\{(i_1,j_1),\dots,(i_N,j_N)\}=N-1$ and $\forall\ q=1,\dots,N-1$, either $i_{q+1}=i_q$ or $j_{q+1}=j_q$, we have $\forall (t,x)\in[0,T]\times I\!R^k$,

$$\sum_{q=1,N-1}\varphi_{i_qi_{q+1}}(t,x)\neq 0, \tag{3.1.3}$$

where, $\forall\ q=1,\dots,N-1$, $\varphi_{i_qi_{q+1}}(t,x) = -\underline{g}_{i_qi_{q+1}}(t,x)1\!\!1_{i_q\neq i_{q+1}} + \bar{g}_{j_qi_{q+1}}(t,x)1\!\!1_{j_q\neq j_{q+1}}$.

To begin with let us point out that the non-local terms $\mathcal{I}(t,x,\phi)$ and $\mathcal{I}^{B^{ij}}(t,x,\phi)$ introduced previously are well defined under Assumptions (A0) and (A2) since for any function ϕ of class $\mathcal{C}^{1,2}$, by the mean value theorem, we have

$$|\phi(t,x+\beta(x,e))-\phi(t,x)-\partial_x\phi(t,x)\beta(e,x)| \le C^1_{t,x}|\beta(x,e)|^2 \le C^1_{t,x}(1\wedge|e|)^2,$$

and

$$|\phi(t,x+\beta(x,e))-\phi(t,x)| \le C^2_{t,x}|\beta(x,e)| \le C^2_{t,x}(1\wedge|e|)$$

where $C^1_{t,x}$ and $C^2_{t,x}$ are bounded constants. They are the bounds of the first and second derivatives of $y\to\phi(t,y)$ in $B(x,C_\beta)$ where C_β is a bound of the function β.

Now consider the following SDE:

$$X^{t,x}_s = x + \int_t^s b(r,X^{t,x}_r)dr + \int_t^s \sigma(r,X^{t,x}_r)dW_r + \int_t^s\int_E \beta(X^{t,x}_{r-},e)\hat{N}(drde), \quad s\in[t,T], x\in R. \tag{3.1.4}$$

The existence and uniqueness of the solution $X^{t,x}_s$ follows from [5].

Next, we give two definitions of the viscosity solution of (3.1.2), and according to [33](proposition 5.1 in appendix), they are equivalent. For locally bounded function u: $(t,x)\in[0,T]\times R\to u(t,x)\in R$, we define its lower semi-continuous (lsc for short) envelope u_*, and upper semi-continuous(usc for short) envelope u^* as following:

$$u_*(t,x) = \underline{\lim}_{(t',x')\to(t,x),\ t'<T} u(t',x'), \quad u^*(t,x) = \overline{\lim}_{(t',x')\to(t,x),\ t'<T} u(t',x')$$

Definition 3.1.1. *A function* $\overrightarrow{u} = (u^{ij}(t,x))_{(i,j)\in A^1\times A^2} : [0,T]\times R \to R^{A^1\times A^2}$ *such that for any* $(i,j)\in A^1\times A^2$, $u^{ij}\in \Pi_g$ *is lsc (resp. usc), is said to be a viscosity subsolution (resp. supsolution) of (3.1.2) if for any test function* $\varphi \in C^{1,2}([0,T]\times R)\bigcap \Pi_g$, *if* $(t_0,x_0)\in[0,T]\times R$ *is a global maximum (resp. minimum) point of* $u^{i,j}-\varphi$,

$$\begin{cases} min\{(u^{ij}-L^{ij}[\overrightarrow{u}])(t_0,x_0), max\{(u^{ij}-U^{ij}[\overrightarrow{u}])(t_0,x_0),\\ -\partial_t\phi(t_0,x_0)-b(t_0,x_0)\partial_x\phi(t_0,x_0)-\frac{1}{2}\sigma^2(t_0,x_0)\partial^2_{xx}\phi(t_0,x_0)-I^1_\delta(t_0,x_0,\phi)-I^2_\delta(t_0,x_0,\partial_x\phi,\phi)\\ -g^{ij}(t_0,x_0,(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2},\sigma(t_0,x_0))\partial_x\phi(t_0,x_0),I^{1,B^{ij}}_\delta(t_0,x_0,\phi)+I^{2,B^{ij}}_\delta(t_0,x_0,\phi))\}\}\le 0\ (resp.\ \ge 0)\\ v^{ij}(T,x)\le h^{ij}(x)\quad (resp.\ \ge). \end{cases}$$

Definition 3.1.2. *A function* $\overrightarrow{u} = (u^{ij}(t,x))_{(i,j)\in A^1\times A^2} : [0,T]\times R \to R^{A^1\times A^2}$ *such that for any* $(i,j)\in A^1\times A^2$, $u^{ij}\in \Pi_g$ *is lsc (resp. usc), is said to be a viscosity subsolution (resp. supsolution) of (3.1.2) if for any* $(t_0,x_0)\in(0,T)\times R$, $\delta>0$ *and a function* $\phi\in C^{1,2}([0,T]\times R)\bigcap\Pi_g$ *such that* $u^{ij}(t_0,x_0)=\phi(t_0,x_0)$ *and* $u^{ij}-\phi$ *has a global maximum (resp. minimum) at* (t_0,x_0) *on* $[0,T]\times B(x_0,K\delta)$ *where* K *is the constant such that for any* $x\in R$, $|\beta(x,e)|\le K(1\wedge|e|)$, *we have*

$$\begin{cases} min\{(u^{ij}-L^{ij}[\overrightarrow{u}])(t_0,x_0), max\{(u^{ij}-U^{ij}[\overrightarrow{u}])(t_0,x_0),\\ -\partial_t\phi(t_0,x_0)-b(t_0,x_0)\partial_x\phi(t_0,x_0)-\frac{1}{2}\sigma^2(t_0,x_0)\partial^2_{xx}\phi(t_0,x_0)-I^1_\delta(t_0,x_0,\phi)-I^2_\delta(t_0,x_0,\partial_x\phi,u^{ij})\\ -g^{ij}(t_0,x_0,(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2},\sigma(t_0,x_0))\partial_x\phi(t_0,x_0),I^{1,B^{ij}}_\delta(t_0,x_0,\phi)+I^{2,B^{ij}}_\delta(t_0,x_0,u^{ij}))\}\}\le 0\ (resp.\ \ge 0)\\ v^{ij}(T,x)\le h^{ij}(x)\quad (resp.\ \ge). \end{cases}$$

Definition 3.1.3. *A function* $\overrightarrow{u} = (u^{ij}(t,x))_{(i,j)\in A^1\times A^2}$ *such that for any* $(i,j)\in A^1\times A^2$, $u^{ij}\in\Pi_g$, *is called a viscosity solution of (3.1.2) if* $(u^{ij}_*(t,x))_{(i,j)\in A^1\times A^2}$ $(resp.(u^*_{ij}(t,x))_{(i,j)\in A^1\times A^2}$ *is a viscosity supersolution (resp. subsolution) of (3.1.2)*

3.2 Approximation schemes of the solution of systems of reflected BSDEs

For $n,m\ge 0$, let $(Y^{i,j,n,m},Z^{i,j,n,m},U^{i,j,n,m})_{(i,j)\in A^1\times A^2}$ be the solution of the following system of BSDEs.

$$\begin{cases} (Y^{i,j,n,m},Z^{i,j,n,m},U^{i,j,n,m})\in\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2(\hat{N});\\ dY^{i,j,n,m}_s=-f^{i,j,n,m}(s,X^{t,x}_s,(Y^{k,l,n,m}_s)_{(k,l)\in A^1\times A^2},Z^{i,j,n,m}_s,U^{i,j,n,m}_s)ds\\ +Z^{i,j,n,m}_s dB_s+\int_E U^{i,j,n,m}_s(e)\hat{N}(dsde)\\ Y^{i,j,n,m}_T=h^{i,j}(X^{t,x}_T), \end{cases} \tag{3.2.1}$$

where,

$$\begin{aligned} f^{i,j,n,m}(s,X^{t,x}_s,(y^{ij})_{(ij)\in A^1\times A^2},z_s,u_s) :=& g^{i,j,n,m}(s,X^{t,x}_s,(y^{kl})_{(kl)\in A^1\times A^2},z_s,\int_E u_s(e)\lambda^{ij}(X^{t,x}_s,e)n(de))\\ =& g^{i,j}(s,X^{t,x}_s,(y^{kl})_{(kl)\in A^1\times A^2},z_s,\int_E u_s(e)\lambda^{ij}(X^{t,x}_s,e)n(de))\\ &+n(y^{ij}-\max_{k\in A^1_i}\{y^{kj}-\underline{g}_{ik}(s,X^{t,x}_s)\})^-\\ &-m(y^{ij}-\min_{l\in A^2_j}\{y^{il}-\overline{g}_{jl}(s,X^{t,x}_s)\})^+. \end{aligned}$$

Let's recall that under the assumption **(A1)**, the solution $(Y^{i,j,n,m},Z^{i,j,n,m},U^{i,j,n,m})_{(i,j)\in A^1\times A^2}$ of (3.2.1) exists and is unique(see [5]). By the assumption(**A1**)(3), we have the comparison theorem for BSDE with jumps(see[58] Theorem 2.4). Next, let's show the monotonicity properties for the matrix $(Y^{i,j,n,m})_{n,m}$.

Proposition 3.2.1. *For any* $(i,j)\in A^1\times A^2$ *and* $n,m\ge 0$ *we have*

$$P-a.s.,\quad Y^{i,j,n,m}\le Y^{i,j,n+1,m}\ and\ \ Y^{i,j,n,m+1}\le Y^{i,j,n,m},\qquad (i,j)\in A^1\times A^2. \tag{3.2.2}$$

Moreover, for any $(i,j)\in A^1\times A^2$ *and* $n,m\ge 0$, *there exists a deterministic continuous function* $v^{i,j,n,m}\in\Pi_g$ *s.t., for any* $t\le T$,

$$Y^{i,j,n,m}_s=v^{i,j,n,m}(s,X^{t,x}_s),\quad s\in[t,T]. \tag{3.2.3}$$

Finally, for any $(i,j)\in A^1\times A^2$ *and* $n,m\ge 0$,

$$v^{i,j,n,m}(t,x)\le v^{i,j,n+1,m}(t,x)\ and\ \ v^{i,j,n,m+1}(t,x)\le v^{i,j,n,m}(t,x),\quad (t,x)\in[0,T]\times R \tag{3.2.4}$$

Proof. First, we recall the result by Xuehong Zhu (2010)([62] Theorem 3.1) related to the comparison of solutions of multi-dimensional BSDEs. Actually, it is enough to show the for any $t, (y^{ij})_{(i,j)\in A^1\times A^2}, (\overline{y}^{ij})_{(i,j)\in A^1\times A^2} \in R^{m_1\times m_2}, (z^{ij})_{(i,j)\in A^1\times A^2}$,
$(\overline{z}^{ij})_{(i,j)\in A^1\times A^2} \in R^{m_1\times m_2}$ and $(u^{ij})_{(i,j)\in A^1\times A^2}, (\overline{u}^{ij})_{(i,j)\in A^1\times A^2} \in \mathcal{L}^2(R,\mathcal{B}_E,n)^{m_1\times m_2}$, there exists a constant C,

$$
\begin{aligned}
&-4\sum_{(i,j)\in A^1\times A^2} y_{ij}^-(f^{i,j,n+1,m}(s,X_s^{t,x},(y_{kl}^+ + \overline{y}_{kl})_{(kl)\in A^1\times A^2},z_{ij},u_{ij})\\
&-(f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},\overline{u}_{ij}))\\
\le& 2\sum_{(i,j)\in A^1\times A^2}\mathbb{1}_{\{y^{ij}<0\}}|z_{ij}-\overline{z}_{ij}|^2 + C\sum_{(i,j)\in A^1\times A^2}(y_{ij}^-)^2\\
&+2\sum_{(i,j)\in A^1\times A^2}\int_E \mathbb{1}_{\{y^{ij}\ge 0\}}|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2 n(de)\\
&+2\sum_{(i,j)\in A^1\times A^2}\int_E \mathbb{1}_{\{y^{ij}<0\}}[|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2-|y_{ij}^-|^2-2y_{ij}(u_{ij}(e)-\overline{u}_{ij}(e))]n(de)
\end{aligned}
$$

This inequality follows from the fact that, for any $(i,j)\in A^1\times A^2$,

(i)$f^{i,j,n,m}(s,X_s^{t,x},(y_{kl})_{\in A^1\times A^2},z_{ij},u_{ij}) \le f^{i,j,n+1,m}(s,X_s^{t,x},(y_{kl})_{(kl)\in A^1\times A^2},z_{ij},u_{ij})$

(ii)For any $(u_{kl})_{(kl)\in A^1\times A^2}\ge 0$,

$$f^{i,j,n,m}(s,X_s^{t,x},(y_{kl}+u_{kl})_{\in A^1\times A^2},z_{ij},u_{ij}) \le f^{i,j,n,m}(s,X_s^{t,x},(y_{kl})_{(kl)\in A^1\times A^2},z_{ij},u_{ij}).$$

(iii)f^{ij} depends only on z_{ij},u_{ij} and not on the other components $z_{kl},u_{kl},(kl)\ne(ij)$.

(iv)$\forall(i,j)\in A^1\times A^2$,

$$
\begin{aligned}
&-4y_{ij}^-(f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},u_{ij})-f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},\overline{u}_{ij}))\\
\le& C(y_{ij}^-)^2+2\int_E \mathbb{1}_{\{y_{ij}\ge 0\}}|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2 n(de)\\
&+2\int_E \mathbb{1}_{\{y_{ij}<0\}}[|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2-|y_{ij}^-|^2-2y_{ij}(u_{ij}(e)-\overline{u}_{ij}(e))]n(de)
\end{aligned}
$$

(i),(ii) and (iii) are easy to check, now we proof (iv), in the case that $y_{ij}\ge 0$, (iv) is obvious, so we discuss only when $y_{ij}<0$. The right-hand side of the inequality is non-negative, in fact,

$$
\begin{aligned}
&\int_E[|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2-|y_{ij}^-|^2-2y_{ij}(u_{ij}(e)-\overline{u}_{ij}(e))]n(de)\\
=&\int_{u_{ij}(e)-\overline{u}_{ij}(e)<-y_{ij}}(u_{ij}(e)-\overline{u}_{ij}(e))^2 n(de)\\
&+\int_{u_{ij}(e)-\overline{u}_{ij}(e)\ge -y_{ij}}[-|y_{ij}^-|^2-2y_{ij}(u_{ij}(e)-\overline{u}_{ij}(e))]n(de)\\
\ge&0
\end{aligned}
$$

So we consider only the case that

$$
\begin{aligned}
&-4y_{ij}^-(f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},u_{ij})-f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},\overline{u}_{ij}))\\
=&-4y_{ij}((f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},\overline{u}_{ij})-f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},u_{ij}))\\
>&0
\end{aligned}
$$

By the assumption (**A1**)(iii), $\int_E \overline{u}_{ij}(e)\gamma^{ij}(X_s^{t,x},e)n(de) > \int_E u_{ij}(e)\gamma^{ij}(X_s^{t,x},e)n(de)$, otherwise the inequality above can not hold. Next, by the assumption (**A1**)(ii), there exists a constant $C>0$ such

that:

$$
\begin{aligned}
&-4y_{ij}((f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},\overline{u}_{ij})-f^{i,j,n,m}(s,X_s^{t,x},(\overline{y}_{kl})_{(kl)\in A^1\times A^2},\overline{z}_{ij},u_{ij}))\\
\leq&-4y_{ij}C\int_E\gamma^{ij}(X_s^{t,x},e)(\overline{u}_{ij}(e)-u_{ij}(e))n(de)\\
=&-4y_{ij}C\int_{u_{ij}(e)-\overline{u}_{ij}(e)<-y_{ij}}\gamma^{ij}(X_s^{t,x},e)(\overline{u}_{ij}(e)-u_{ij}(e))n(de)\\
&-4y_{ij}C\int_{u_{ij}(e)-\overline{u}_{ij}(e)\geq-y_{ij}}\gamma^{ij}(X_s^{t,x},e)(\overline{u}_{ij}(e)-u_{ij}(e))n(de)\\
\leq&4C^2y_{ij}^2\int_E\gamma_{ij}^2(X_s^{t,x},e)n(de)+\int_{u_{ij}(e)-\overline{u}_{ij}(e)<-y^{ij}}(\overline{u}_{ij}(e)-u_{ij}(e))^2n(de)\\
\leq&K(y_{ij}^-)^2+2\int_E\mathbb{1}_{\{y^{ij}\geq0\}}|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2n(de)\\
&+2\int_E\mathbb{1}_{\{y^{ij}<0\}}[|(y_{ij}+u_{ij}(e)-\overline{u}_{ij}(e))^-|^2-|y_{ij}^-|^2-2y_{ij}(u_{ij}(e)-\overline{u}_{ij}(e))]n(de),
\end{aligned}
$$

finally, we have (iv). Consequently, we have

$$P-a.s.,\quad Y^{i,j,n,m}\leq Y^{i,j,n+1,m}.$$

In the same way we can show that $P-a.s.,\quad Y^{i,j,n,m+1}\leq Y^{i,j,n,m}$. The second claim is just the representation of solutions of standard BSDEs with jumps by deterministic functions in the Markovian framework(see [5]). The inequalities of (3.2.4) are obtained by taking $s=t$ in (3.2.2) in view of the representation (3.2.3) of $Y^{i,j,n,m}$ by $v^{i,j,n,m}$ and $X^{t,x}$. □

We will show two approximation schemes obtained from the sequence $Y^{i,j,m,n},(i,j)\in A^1\times A^2)_{n,m}$ of the solution of system (3.2.1). The first scheme is a sequence of decreasing reflected BSDEs with interconnected lower obstacles:$\forall(i,j)\in A^1\times A^2$,

$$
\begin{cases}
(\bar{Y}^{i,j,m},\bar{Z}^{i,j,m},\bar{U}^{i,j,m},\bar{K}^{i,j,m})\in\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2(\hat{N})\times\mathcal{A}^2;\\
\bar{Y}_s^{i,j,m}=h^{i,j}(X_T^{t,x})+\int_s^T\bar{f}^{i,j,m}(r,X_r^{t,x},(\bar{Y}_r^{k,l,m})_{(k,l)\in A^1\times A^2},\bar{Z}_r^{i,j,m},\bar{U}_r^{i,j,m})dr-\int_s^T\bar{Z}_r^{i,j,m}dB_r\\
-\int_s^T\int_E\bar{U}_r^{i,j,m}(e)\hat{N}(drde)+\bar{K}_T^{i,j,m}-\bar{K}_s^{i,j,m}\\
\bar{Y}_s^{i,j,m}\geq\max\limits_{k\in A_i^1}\{\bar{Y}_s^{k,j,m}-\underline{g}_{ik}(s,X_s^{t,x})\},\quad s\leq T,\\
\int_0^T(\bar{Y}_s^{i,j,m}-\max\limits_{k\in A_i^1}\{\bar{Y}_s^{k,j,m}-\underline{g}_{ik}(s,X_s^{t,x})\})d\bar{K}_s^{i,j,m}=0,
\end{cases}
\tag{3.2.5}
$$

where, $\forall(i,j)\in A^1\times A^2$, $m\geq0$ and $(s,\overrightarrow{Y}^{ij},Z^{ij},U^{ij})$,

$$
\begin{aligned}
\bar{f}^{i,j,m}(s,X_s^{t,x},\overrightarrow{Y},Z^{ij},U^{ij}):=&g^{ij,+,m}(s,X_s^{t,x},(Y^{kl})_{(k,l)\in A^1\times A^2},Z^{ij},\int_E U^{ij}(e)\gamma^{ij}(X_s^{t,x},e)n(de))\\
=&g^{ij}(s,X_s^{t,x},(Y^{kl})_{(k,l)\in A^1\times A^2},Z^{ij},\int_E U^{ij}(e)\gamma^{ij}(X_s^{t,x},e)n(de))\\
&-m(Y^{ij}-\min_{l\in A_j^2}(Y^{il}+\overline{g}_{jl}(s,X_s^{t,x})))^+.
\end{aligned}
$$

Thanks to the assumption (**A1**)-(**A3**) and non free loop assumption, by Theorem (5.2) in appendix, the solution of (3.2.5) exists and is unique. Moreover, we have the following properties.

Proposition 3.2.2. *For any $(i,j)\in A^1\times A^2$ and $m\geq0$, we have:*
(i)

$$\lim_{n\to\infty}E[\sup_{t\leq s\leq T}|Y_s^{i,j,n,m}-\bar{Y}_s^{i,j,m}|^2]\to0 \tag{3.2.6}$$

(ii)

$$P-a.s.,\quad\bar{Y}^{i,j,m}\geq\bar{Y}^{i,j,m+1}.$$

(iii)There exsits a unique $A^1\times A^2$-uplet of deterministic continuous functions $(\bar{u}^{k,l,m})_{(k,l)\in A^1\times A^2}$ in Π_g such that, for every $t\leq T$,

$$\bar{Y}_s^{i,j,m}=\bar{u}^{i,j,m}(s,X_s^{t,x}),\ s\in[t,T]. \tag{3.2.7}$$

Moreover, $\forall (i,j) \in A^1 \times A^2$ *and* $(t,x) \in [0,T] \times R^k$, $\bar{u}^{i,j,m}(t,x) \geq \bar{u}^{i,j,m+1}(t,x)$.
Finally, $(\bar{u}^{i,j,m})_{(i,j)\in A^1\times A^2}$ *is a viscosity solution in the class* Π_g *of the following system of variational inequalities with inter-connected obstacles.* $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} min\{\bar{u}^{i,j,m}(t,x) - \max\limits_{k\in A_i^1}(\bar{u}^{k,j,m}(t,x) - \underline{g}_{ik}(t,x)); \\ -\partial_t \bar{u}^{i,j,m}(t,x) - \mathcal{L}\bar{u}^{i,j,m}(t,x) - g^{ij,+,m}(t,x,(\bar{u}^{k,l,m}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x\bar{u}^{i,j,m}(t,x), B^{ij}\bar{u}^{i,j,m}(t,x))\} = 0 \\ \bar{u}^{i,j,m}(T,x) = h^{i,j}(x). \end{cases} \tag{3.2.8}$$

Proof. (i)It is enough to consider the case $m = 0$, since for any $i,j \in A^1 \times A^2$, the function

$$(s,x,(y^{kl})_{(kl)\in A^1\times A^2}) \to -m(y^{ij} - \min_{l\in A_j^2}(y^{il} + \overline{g}_{jl}(s,x)))^+$$

has the same properties as f^{ij} displayed in (**A1**) and (**A2**). First, let us show that $\forall (i,j) \in A^1 \times A^2$ and $n \geq 0$,

$$P - a.s., \quad Y^{i,j,n,0} \leq \bar{Y}^{i,j,0}. \tag{3.2.9}$$

First and w.l.o.g we may assume that f^{ij} is non-decreasing w.r.t. $(y^{kl})_{(k,l)\in A_1\times A_2}$, since thanks to assumption (A2), it is enough to multiply the solution by $e^{\lambda t}$, where λ is appropriately chosen in order to fall in this latter case, since f^{ij} is Lipschitz in y^{ij}. Now, for fixed n, let us define recursively the sequence $(\tilde{Y}^{k,ij,n})_{k\geq 0}$ as follows: for $k = 0$ and any $(i,j) \in A_1 \times A_2$, we set $\tilde{Y}^{0,ij,n} := \bar{Y}^{ij,0}$ and, for any $k \geq 1$, let us define $(\tilde{Y}^{k,ij,n}, Z^{k,ij,n}, U^{k,ij,n}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N})$ as the solution of the following system of BSDEs: $\forall (i,j) \in A_1 \times A_2$,

$$\begin{cases} -d\tilde{Y}_s^{k,ij,n} = f^{ij}(s, X_s^{t,x}, (\tilde{Y}_s^{k-1,pq,n})_{(p,q)\in A^1\times A^2}, \tilde{Z}_s^{k,ij,n}, \tilde{U}_s^{k,ij,n})ds \\ +n(\tilde{Y}_s^{k,ij,n} - \max\limits_{l\in A_i^1}(\tilde{Y}_s^{k-1,lj,n} - \underline{g}_{il}(s, X_s^{t,x})))^- ds - \tilde{Z}_s^{k,ij,n}dB_s - \int_E \tilde{U}_s^{k,ij,n}(e)\hat{N}(dsde) \\ \tilde{Y}_T^{k,ij,n} = h^{i,j}(X_T^{t,x}), \end{cases} \tag{3.2.10}$$

The solution of (3.2.10) exists since it is a multi-dimensional standard BSDE with a Lipschiz coefficient, nothing that $(\tilde{Y}_s^{k-1,pq,n})_{(p,q)\in A^1\times A^2}$ is already given. Since, n is fixed and the coefficient

$$\phi^{ij,n}(s,\omega,(y^{pq})_{(p,q)\in A^1\times A^2}, z^{ij}, U^{ij}) := f^{ij}(s, X_s^{t,x}, (y^{pq})_{(p,q)\in A^1\times A^2}, z^{ij}, U^{ij}) + n(y^{ij} - \max_{l\in A_i^1}(y^{lj} - \underline{g}_{il}(s, X_s^{t,x})))^-$$

is Lipschitz w.r.t. $((y^{pq})_{(p,q)\in A^1\times A^2}, z^{ij}, U^{ij})$, the sequence $(\tilde{Y}^{k,ij,n})_{k\geq 0}$ converges in $\mathcal{S}^2$ to $Y^{ij,n,0}$ as $k \to \infty$, for any i,j and n.
Using an induction argument w.r.t.k, we prove that for any i,j and n,

$$P - a.s., \quad \tilde{Y}^{k,ij,n} \leq \bar{Y}^{ij,0}, \quad k \geq 0.$$

Indeed, for $k = 0$ the inequalities hold true through the definition of $\tilde{Y}^{0,ij,n}$. Assume not that these inequalities are valid for some $k-1$, i.e. for any i,j and n,

$$P - a.s., \quad \tilde{Y}^{k-1,ij,n} \leq \bar{Y}^{ij,0}. \tag{3.2.11}$$

Thus, taking into account that $\bar{Y}^{ij,0}$ satisfies (3.2.5) with $m = 0$ and the fact that f^{ij} is non-decreasing w.r.t. $(y^{pq})_{(p,q)\in A^1\times A^2}$ then for any i,j and n, it holds

$$\begin{aligned} &f^{ij}(s, X_s^{t,x}, (\tilde{Y}_s^{k-1,pq,n})_{(p,q)\in A^1\times A^2}, z^{ij}, U^{ij})ds + n(\bar{Y}_s^{ij,0} - \max_{l\in A_i^1}(\tilde{Y}_s^{k-1,lj,n} - \underline{g}_{il}(s, X_s^{t,x})))^- \\ &\leq f^{ij}(s, X_s^{t,x}, (\bar{Y}_s^{pq,0})_{(p,q)\in A^1\times A^2}, z^{ij}, U^{ij}). \end{aligned}$$

Using comparison result of solution of one dimensional BSDEs we obtain that for any i,j and n, $\tilde{Y}^{k,ij,n} \leq \bar{Y}^{ij,0}, a.s.$. Thus the property (3.2.11) is valid for any k. Taking the limit as k tends to ∞, we obtain (3.2.9).

(3.2.9) together with (3.2.2) imply that $\exists \hat{Y}^{ij} \in \mathcal{S}^2$ s.t. $Y^{i,j,n,0} \nearrow \hat{Y}^{ij}$. Now by Essaky's monotonic limit theorem in [24], there exist $(\hat{Z}^{i,j}, \hat{U}^{i,j}, \hat{K}^{i,j}) \in \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2$:

(a)$\hat{Y}^{ij}$ is RCLL and uniformly P-square integrable, for any stopping time τ, $\lim_{n\to\infty} \nearrow Y_\tau^{i,j,n,0} = \hat{Y}_\tau^{ij}$.

(b)$\hat{K}^{ij}$ is RCLL non-decreasing, $\hat{K}_0^{ij} = 0$ and for any stopping time τ, $\lim_{n\to\infty} K_\tau^{i,j,n,0} = \hat{K}_\tau^{ij}$, $P-a.s.$.

(c)$\hat{Z}^{ij} \in \mathcal{H}^2$, $\hat{U}^{ij} \in \mathcal{H}^2(\hat{N})$ and for any $p \in [1,2)$,

$$\lim_{n\to\infty} E[\int_0^T |Z_s^{i,j,n,0} - \hat{Z}^{ij}|^p ds] = 0, \quad \lim_{n\to\infty} E[\int_0^T \int_E |U_s^{i,j,n,0} - \hat{U}^{ij}|^{\frac{p}{2}} n(de)ds] = 0.$$

(d)For any $i,j \in A^1 \times A^2$, $s \le T$,

$$\begin{cases} \hat{Y}_s^{ij} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}(r, X_r^{t,x}, (\hat{Y}_r^{kl})_{(k,l)\in A^1\times A^2}, \hat{Z}_r^{ij}, \hat{U}_r^{ij}) + \hat{K}_T^{ij} - \hat{K}_s^{ij} \\ - \int_s^T \hat{Z}_r^{ij} dB_r - \int_s^T \int_E \hat{U}_r^{ij}(e) n(deds) \\ \hat{Y}_s^{ij} \ge \max_{k\in A_i^1}\{\hat{Y}_s^{kj} - \underline{g}_{ik}(s, X_s^{t,x})\}. \end{cases} \tag{3.2.12}$$

The remaining of the proof is in the same way with Theorem (5.2) in appendix, that is $\hat{Y}^{ij}$ is regular(see [33] page 14 for the definition of regular) and $\hat{K}^{ij}$ is continuous and the Skorohod condition is satisfied by using the no-free loop property (**A4**). So $(\hat{Y}^{ij}, \hat{Z}^{ij}, \hat{U}^{ij}, \hat{K}^{ij})$ satisfy (3.2.5) with $m = 0$, by the uniqueness of the solution of (3.2.5), $\hat{Y}^{ij} = \bar{Y}^{ij,0}$, $a.s.$, which completes the proof of (i).

(ii)is an immediate consequence of (i) and Proposition 3.1.

(iii)One can see Theorem 5.3 in Appendix for the proof of existence of $\bar{u}$ in (3.2.8). Also it is similar as theorem 4.1 in [33]. Actually, there is nonlocal-term $\mathcal{I}^{B^{ij}}$ in the generator g^{ij} which is different from the generator in [33], but since $\mathcal{I}^{B^{ij}}$ has the same proposition with the one in [33] and g^{ij} is non-decreasing and Lipschitz continuous in the nonlocal-term, uniformly with respect to all the other variables, we can use the same method to show the existence and uniqueness. Finally as $\forall s \in [t,T]$, $\bar{Y}_s^{i,j,m} \ge \bar{Y}_s^{i,j,m+1}$ a.s., let $s = t$ we get $\forall (t,x) \in [0,T]\times R$, $\bar{u}^{i,j,m}(t,x) \ge \bar{u}^{i,j,m+1}(t,x)$. □

We now consider the increasing approximating scheme: $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} (\underline{Y}^{i,j,n}, \underline{Z}^{i,j,n}, \underline{U}^{i,j,n}, \underline{K}^{i,j,n}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2; \\ \underline{Y}_s^{i,j,n} = h^{i,j}(X_T^{t,x}) + \int_s^T \underline{f}^{i,j,n}(r, X_r^{t,x}, (\underline{Y}_r^{k,l,n})_{(k,l)\in A^1\times A^2}, \underline{Z}_r^{i,j,n}, \underline{U}_r^{i,j,n}) dr - \int_s^T \underline{Z}_r^{i,j,n} dB_r \\ - \int_s^T \int_E \underline{U}_r^{i,j,n}(e) \hat{N}(drde) + \underline{K}_T^{i,j,n} - \underline{K}_s^{i,j,n} \\ \underline{Y}_s^{i,j,n} \le \min_{l\in A_j^2}\{\underline{Y}_s^{i,l,n} + \overline{g}_{jl}(s, X_s^{t,x})\}, \quad s \le T, \\ \int_0^T (\underline{Y}_s^{i,j,n} - \min_{l\in A_j^2}\{\underline{Y}_s^{k,j,n} + \overline{g}_{jl}(s, X_s^{t,x})\}) d\underline{K}_s^{i,j,n} = 0, \end{cases} \tag{3.2.13}$$

where, $\forall (i,j) \in A^1 \times A^2$, $n \ge 0$ and $(s, \overrightarrow{Y}^{ij}, Z^{ij}, U^{ij})$,

$$\begin{aligned} \underline{f}^{i,j,n}(s, X_s^{t,x}, \overrightarrow{Y}, Z^{ij}, U^{ij}) :=& g^{ij,-,n}(s, X_s^{t,x}, (Y^{kl})_{(k,l)\in A^1\times A^2}, Z^{ij}, \int_E U^{ij}(e)\gamma^{ij}(X_s^{t,x}, e) n(de)) \\ =& g^{ij}(s, X_s^{t,x}, (Y^{kl})_{(k,l)\in A^1\times A^2}, Z^{ij}, \int_E U^{ij}(e)\gamma^{ij}(X_s^{t,x}, e) n(de)) \\ & + n(Y^{ij} - \max_{k\in A_i^1}(Y^{kj} - \underline{g}_{ik}(s, X_s^{t,x})))^-. \end{aligned}$$

Thanks to the assumption (**A1**)-(**A3**), by Theorem 5.4.1 in appendix, the solution of (3.2.13) exists and is unique.

Next we give a analogous of Proposition 3.2, we do not give its proof since it is deduced from this latter proposition in considering the equation satisfied by $(-\underline{Y}^{i,j,n}, -\underline{Z}^{i,j,n}, -\underline{U}^{i,j,n}, -\underline{K}^{i,j,n})$

Proposition 3.2.3. *For any $(i,j) \in A^1 \times A^2$ and $n \ge 0$, we have:*
(i)

$$\lim_{m\to\infty} E[\sup_{t\le s\le T} |Y_s^{i,j,n,m} - \underline{Y}_s^{i,j,n}|^2] \to 0 \tag{3.2.14}$$

(ii) For any $n \ge 0$,

$$P-a.s., \quad \underline{Y}^{i,j,n} \le \underline{Y}^{i,j,n+1}.$$

(iii)There exsits a unique $A^1 \times A^2$-uplet of deterministic continuous functions $(\underline{u}^{k,l,n})_{(k,l)\in A^1\times A^2}$ in Π_g such that, for every $t \leq T$,

$$\underline{Y}_s^{i,j,n} = \underline{u}^{i,j,n}(s, X_s^{t,x}),\ s \in [t,T]. \tag{3.2.15}$$

Moreover, $\forall (i,j) \in A^1 \times A^2$ and $(t,x) \in [0,T] \times R^k$, $\underline{u}^{i,j,n}(t,x) \leq \underline{u}^{i,j,n+1}(t,x)$.
Finally, $(\underline{u}^{i,j,n})_{(i,j)\in A^1\times A^2}$ is a viscosity solution in the class Π_g of the following system of variational inequalities with inter-connected obstacles. $\forall (i,j) \in A^1 \times A^2$,

$$\begin{cases} max\{\underline{u}^{i,j,n}(t,x) - \min\limits_{l\in A_j^2}(\underline{u}^{i,l,n}(t,x) + \overline{g}_{jl}(t,x)); \\ -\partial_t \underline{u}^{i,j,n}(t,x) - \mathcal{L}\underline{u}^{i,j,n}(t,x) \\ -g^{ij,-,n}(t,x,(\underline{u}^{k,l,n}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x\underline{u}^{i,j,n}(t,x), B^{ij}\underline{u}^{i,j,n}(t,x))\} = 0 \\ \underline{u}^{i,j,n}(T,x) = h^{i,j}(x). \end{cases} \tag{3.2.16}$$

We define

$$\bar{u}^{ij}(t,x) := \lim_{m\to\infty} \bar{u}^{i,j,m}(t,x),\ \ \underline{u}^{ij}(t,x) := \lim_{n\to\infty} \underline{u}^{i,j,n}(t,x).$$

Then, we can prove a by-product of Proposition 3.2 and 3.3:

Corollary 3.2.1. *$\forall (i,j) \in A^1 \times A^2$, the function $\bar{u}^{ij}$ (resp. $\underline{u}^{ij}$ is usc (resp. lsc). Moreover, $\bar{u}^{ij}$ and $\underline{u}^{ij}$ belong to Π_g, for any $(t,x) \in [0,T] \times R$,*

$$\underline{u}^{ij}(t,x) \leq \bar{u}^{ij}(t,x).$$

Proof. For any $(i,j) \in A_1 \times A_2$, the function $\bar{u}^{ij}$ (resp. $\underline{u}^{ij}$) is obtained as a decreasing (resp. increasing) limit of continuous functions. Therefore, it is *usc* (resp. *lsc*). Next, for any (i,j) and n, m,

$$u^{ij,n,m}(t,x) \leq u^{ij,n,0}(t,x),\ \ (t,x) \in [0,T] \times R,$$

as the sequence $(u^{ij,n,m})_{m\geq 0}$ is decreasing. Thus, taking the limit as $m \to \infty$ we obtain,

$$\underline{u}^{ij,n} \leq u^{ij,n,0}.$$

Now using (3.2.3) and (3.2.6), it follows that, for any $t \leq T$ and $s \in [t,T]$, $Y_s^{ij,n,0} = u^{ij,n,0}(s, X_s^{t,x})$ and the processes $Y^{ij,n,0}$ converges in $\mathcal{S}^2$, as $n \to \infty$, to $\bar{Y}^{ij,0}$ which is solution of (3.2.5) with $m = 0$. Furthermore, by (3.2.15), there exists a deterministic continuous function $\bar{u}^{ij,0}$ with polynomial growth such that for any $t \leq T$ and $s \in [t,T]$, $Y_s^{ij,0} = \bar{u}^{ij,0}(s, X_s^{t,x})$. Then taking $s = t$ and the limit as $n \to \infty$ to obtain

$$\underline{u}^{ij}(t,x) := \lim_{n\to\infty} \underline{u}^{ij,n}(t,x) \leq \lim_{n\to\infty} u^{ij,n,0}(t,x) = \bar{u}^{ij,0}(t,x),\ \ \forall (t,x) \in [0,T] \times R.$$

But $\bar{u}^{ij,0}$ and $\underline{u}^{ij,n}$ belong to Π_g and $\underline{u}^{ij,n} \leq \underline{u}^{ij,n+1}$. Thus , $\underline{u}^{ij} \in \Pi_g$, for any $(i,j) \in A_1 \times A_2$. In the same way one can show that $\bar{u}^{ij} \in \Pi_g$, for any $(i,j) \in A_1 \times A_2$. The last inequality follows from (3.2.4) and the definitions of $\bar{u}^{ij}$ and $\underline{u}^{ij}$. □

3.3 Uniqueness and Existence of viscosity solution for system of IPDEs

In this section we will show the uniqueness of viscosity solution of (3.1.2) as the corollary of a comparison result. In the same way with [17], we can prove the following lemma.

Lemma 3.3.1. *Let $(\overrightarrow{u})_{(i,j} \in A^1 \times A^2$ (resp. $(\overrightarrow{w})_{(i,j} \in A^1 \times A^2$) is an usc subsolution (resp. lsc supersolution) of (3.1.2) which belongs to Π_g. For all $(t,x) \in [0,T] \times R$ and let $\Gamma(t,x)$ be the following set:*

$$\Gamma(t,x) := \{(i,j) \in A^1 \times A^2, u^{ij}(t,x) - w^{ij}(t,x) = \max_{(k,l)\in A^1\times A^2}(u^{kl}(t,x) - w^{kl}(t,x))\}.$$

Then there exists $(i_0,j_0) \in \Gamma(t,x)$ such that

$$u^{i_0j_0}(t,x) > L^{i_0j_0}[\overrightarrow{u}](t,x),\ \ w^{i_0j_0}(t,x) < U^{i_0j_0}[\overrightarrow{w}](t,x). \tag{3.3.1}$$

Now we give the comparison theorem.

Theorem 3.3.1. *Let $(\overrightarrow{u})_{(i,j)} \in A^1 \times A^2$ (resp. $(\overrightarrow{w})_{(i,j)} \in A^1 \times A^2$) is an usc subsolution (resp. lsc supersolution) of (3.1.2) which belongs to Π_g. Then it holds that for any $(i,j) \in A^1 \times A^2$,*

$$u^{ij}(t,x) \leq w^{ij}(t,x), \quad (t,x) \in [0,T] \times R.$$

Proof. We peoceed by contradiction, let $(t_0, x_0) \in [0,T] \times R$ such that there exists $\epsilon > 0$,

$$\max_{i,j}(u^{ij} - w^{ij})(t_0, x_0) \geq 0. \tag{3.3.2}$$

Next, w.l.o.g. assume that for any $(i,j) \in A^1 \times A^2$,

$$\lim_{x \to \infty}(u^{ij} - w^{ij})(t,x) = -\infty, \tag{3.3.3}$$

otherwise one may replace w^{ij} with $w^{i,j,\theta,\nu}$ defined by

$$w^{i,j,\theta,\nu}(t,x) = w^{ij}(t,x) + \theta e^{-\nu t}|x|^{2\gamma+2}(t,x), \ (t,x) \in [0,T] \times R,$$

which is still an *usc* supersolution of (3.1.2) for $\theta > 0$ and $\nu \geq \nu_0$ which satisdies (3.3.3). It suffices to show that $u^{ij}(t,x) \leq w^{i,j,\theta,\nu}(t,x), \ (t,x) \in [0,T] \times R$, since by taking the limit as $\theta \to 0$, we deduce that $u^{ij}(t,x) \leq w^{ij}(t,x), \ (t,x) \in [0,T] \times R$. Thus assume that (3.3.2) and (3.3.3) are satisfied. Then there exists $R > 0$ such that

$$\max_{(t,x) \in [0,T] \times R} \max_{ij}\{(u^{ij} - w^{ij})(t,x)\} = \max_{(t,x) \in [0,T] \times B(0,r)} \max_{ij}\{(u^{ij} - w^{ij})(t,x)\} = \max_{ij}\{(u^{ij} - w^{ij})(t^*, x^*)\}, \tag{3.3.4}$$

where, $(t^*, x^*) \in [0,T] \times B(0,r)$, where $B(0,r)$ denotes the ball in R with center the origin and radius r, since by definition $u^{ij}(T,x) \leq w^{ij}(T,x)$, for all $(i,j) \in A^1 \times A^2$. Next we will finish the proof in two steps: **Step 1.** First we make the following assumption on the functions $(g^{ij})_{(i,j) \in A^1 \times A^2}$. For all $\lambda > c_{ij}(m_1 \times m_2 - 1)$, $(i,j) \in A^1 \times A^2$, $(t,x,\overrightarrow{y},z,U) \in [0,T] \times R \times R^{m_1 \times m_2} \times R \times R$, and $(u,v) \in R^2 s.t. u \geq v$,

$$g^{ij}(t,x,[\overrightarrow{y}^{ij},u],z,U) - g^{ij}(t,x,[\overrightarrow{y}^{ij},v],z,U) \leq -\lambda(u-v), \tag{3.3.5}$$

where c_{ij} is the Lipschitz constant of g^{ij} w.r.t. $\overrightarrow{y}$. Next, let $(i_0, j_0) \in \Gamma(t^*, x^*)$ that satisfies (3.3.1). For $\epsilon > 0$, $\rho > 0$, let $\Phi^{i_0,j_0}_{\epsilon,\rho}$ be the function defined as follows:

$$\Phi^{i_0,j_0}_{\epsilon,\rho}(t,x,y) := (u^{i_0,j_0}(t,x) - w^{i_0,j_0}(t,y)) - \frac{|x-y|}{\epsilon} - |t-t^*|^2 - \rho|x-x^*|^4.$$

Let (t_0, x_0, y_0) be such that

$$\Phi^{i_0,j_0}_{\epsilon,\rho}(t_0,x_0,y_0) = \max_{(t,x,y) \in [0,T] \times \bar{B}(0,r)^2} \Phi^{i_0,j_0}_{\epsilon,\rho}(t,x,y) = \max_{(t,x,y) \in [0,T] \times I\!R^2} \Phi^{i_0,j_0}_{\epsilon,\rho}(t,x,y)$$

which exists since which exists since $\Phi^{i_0,j_0}_{\epsilon,\rho}$ is *usc* and $\bar{B}(0,r)^2$ is the closure of $B(0,r)^2$. Then we have

$$\begin{aligned}
\Phi^{i_0,j_0}_{\epsilon,\rho}(t^*,x^*,x^*) &= u^{i_0,j_0}(t^*,x^*) - w^{i_0,j_0}(t^*,x^*) \\
&\leq u^{i_0,j_0}(t^*,x^*) - w^{i_0,j_0}(t^*,x^*) + \frac{|x_0-y_0|^2}{\epsilon} + |t_0-t^*|^2 + \rho|x_0-x^*|^4 \\
&\leq u^{i_0,j_0}(t_0,x_0) - w^{i_0,j_0}(t_0,y_0)
\end{aligned} \tag{3.3.6}$$

The growth condition of u^{i_0,j_0} and w^{i_0,j_0} implies that $\frac{|x_0-y_0|^2}{\epsilon} + |t_0-t^*|^2 + \rho|x_0-x^*|^4$ is bounded and hence $\lim_{\epsilon \to 0}(x_0 - y_0) = 0$. Next by (3.3.6), for any subsequence $(t_{0_l}, x_{0_l}, y_{0_l})_l$ which converges to $(\tilde{t}, \tilde{x}, \tilde{x})$,

$$u^{i_0,j_0}(t^*,x^*) - w^{i_0,j_0}(t^*,x^*) \leq u^{i_0,j_0}(\tilde{t},\tilde{x}) - w^{i_0,j_0}(\tilde{t},\tilde{x}),$$

since u^{i_0,j_0} is *usc* and w^{i_0,j_0} is *lsc*. By the definition of (t^*,x^*) this last inequality is an equality. Using both the definiton of Φ^{i_0,j_0}_0 and (3.3.6), it implies that the sequence

$$\lim_{\epsilon \to 0}(t_0,x_0,y_0) = (t^*,x^*,x^*) \tag{3.3.7}$$

and once more from (3.3.6) we deduce

$$\lim_{\epsilon \to 0} \frac{|x_0-y_0|^2}{\epsilon} = 0. \tag{3.3.8}$$

Finally classically (see e.g. [17], pp. 14) we have also

$$\lim_{\epsilon\to 0}(u_j(t_0,x_0),w_j(t_0,y_0))=(u_j(t^*,x^*),w_j(t^*,x^*)). \tag{3.3.9}$$

Next recalling that u^{i_0,j_0} (resp. w^{i_0,j_0}) is *usc* (resp. *lsc*) and satisfies (3.3.1), then for ϵ small enough and there exists a subsequence which we still index by ϵ, we obtain

$$u^{i_0,j_0}(t_0,x_0)>\max_{k\in A^1_{i_0}}(u^{k,j_0}(t_0,x_0)-\underline{g}_{i_0k}(t_0,x_0)), \tag{3.3.10}$$

and

$$w^{i_0,j_0}(t_0,y_0)<\min_{l\in A^2_{j_0}}(w^{i_0,l}(t_0,y_0)-\overline{g}_{j_0l}(t_0,y_0)). \tag{3.3.11}$$

We are able to apply Jensen-Ishii's Lemma for non local operators established by Barles and Imbert ([5], pp.583) (one can see also [8], Lemma 4.1, pp.64) with u^{i_0,j_0}, w^{i_0,j_0} and $\phi(t,x,y):=\frac{|x-y|^2}{\epsilon}+|t-t^*|^2+\rho|x-x^*|^4$ at the point (t_0,x_0,y_0).

For any $\delta\in(0,1)$ there are $p^\epsilon_u,q^\epsilon_u,p^\epsilon_w$ and q^ϵ_w elements of $\mathbb{R}$ and M^ϵ_u, M^ϵ_w two non-negative constants such that:

(i)

$$p^\epsilon_u-p^\epsilon_w=\partial_t\phi(t_0,x_0,y_0),\quad q^\epsilon_u=\partial_x\phi(t_0,x_0,y_0),\quad q^\epsilon_w=-\partial_y\phi(t_0,x_0,y_0) \tag{3.3.12}$$

and

$$\begin{pmatrix}M^0_u & 0\\ 0 & -M^0_w\end{pmatrix}\le D^2_{xx}\phi(t_0,x_0,y_0)=\begin{pmatrix}12|x_0-x^*|^2 & 0\\ 0 & 0\end{pmatrix}+\frac{2}{\epsilon}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}; \tag{3.3.13}$$

$$\begin{aligned}\text{(ii)}\ &-p^\epsilon_u-b(t_0,x_0)q^\epsilon_u-\frac{1}{2}\sigma(t_0,x_0)^2M^\epsilon_u-g^{i_0,j_0}(t_0,x_0,(u^{ij}(t_0,x_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,x_0)q^\epsilon_u,\\ &I^{1,B^{i_0,j_0}}_\delta(t_0,x_0,\phi(t_0,.,y_0))+I^{2,B^{i_0,j_0}}_\delta(t_0,x_0,u^{i_0,j_0}))\\ &-I^1_\delta(t_0,x_0,\phi(t_0,.,y_0))-I^2_\delta(t_0,x_0,q^\epsilon_u,u^{i_0,j_0})\le 0;\end{aligned} \tag{3.3.14}$$

$$\begin{aligned}\text{(iii)}\ &-p^\epsilon_w-b(t_0,y_0)q^\epsilon_w-\frac{1}{2}\sigma(t_0,y_0)^2M^\epsilon_w-g^{i_0,j_0}(t_0,y_0,(w^{ij}(t_0,y_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,y_0)q^\epsilon_w,\\ &I^{1,B^{i_0,j_0}}_\delta(t_0,x_0,-\phi(t_0,x_0,.))+I^{2,B^{i_0,j_0}}_\delta(t_0,x_0,w^{i_0,j_0}))\\ &-I^1_\delta(t_0,y_0,-\phi(t_0,x_0,.))-I^2_\delta(t_0,y_0,q^\epsilon_w,w^{i_0,j_0})\ge 0.\end{aligned} \tag{3.3.15}$$

Next we are going to provide estimates for the non-local terms. Define $\psi_\rho(t,x):=\rho|x-x^*|^4+|t-t^*|^2$. By definition of (t_0,x_0,y_0), for any $d,d'\in\mathbb{R}$,

$$\begin{aligned}&u^{i_0,j_0}(t_0,x_0+d')-w^{i_0,j_0}(t_0,y_0+d)-\tfrac{|x_0+d'-y_0-d|^2}{\epsilon}-\psi_\rho(t_0,x_0+d')\\ &\le u^{i_0,j_0}(t_0,x_0)-w^{i_0,j_0}(t_0,y_0)-\tfrac{|x_0-y_0|^2}{\epsilon}-\psi_\rho(t_0,x_0).\end{aligned}$$

Therefore for $z\in E$, in taking $d=\beta(x_0,z)$ and $d'=\beta(y_0,z)$, we obtain

$$\begin{aligned}&u^{i_0,j_0}(t_0,x_0+\beta(x_0,z))-u^{i_0,j_0}(t_0,x_0)-q^\epsilon_u\beta(x_0,z)\\ &\quad\le w^{i_0,j_0}(t_0,y_0+\beta(y_0,z))-w^{i_0,j_0}(t_0,y_0)-q^\epsilon_w\beta(y_0,z)+\tfrac{|\beta(x_0,z)-\beta(y_0,z)|^2}{\epsilon}\\ &\quad+\psi_\rho(t_0,x_0+\beta(x_0,z))-\psi_\rho(t_0,x_0)-D_x\psi_\rho(t_0,x_0)\beta(x_0,z).\end{aligned}$$

By assumption (A0), for any $\delta>0$,

$$\begin{aligned}&I^2_\delta(t_0,x_0,q^\epsilon_u,u^{i_0,j_0})-I^2_\delta(t_0,y_0,q^\epsilon_w,w^{i_0,j_0})\le\tfrac{K|x_0-y_0|^2}{\epsilon}\textstyle\int_{|e|\ge\delta}(1\wedge|e|)^2n(de)+I^2_\delta(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho)\\ &\le\tilde{K}\tfrac{|x_0-y_0|^2}{\epsilon}+I^2_\delta(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho).\end{aligned} \tag{3.3.16}$$

It is easy to check that

$$|I^2_\delta(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho)|\le C\rho\int_{|z|\ge\delta}\{|z|^2+|z|^4\}n(dz).$$

On the other hand, since $\phi \in C^2$ and β is bounded,

$$\begin{aligned} I^1_\delta(t_0,x_0,\phi(t_0,.,y_0)) &= \int_{|e|\le\delta}\{\phi(t_0,x_0+\beta(x_0,e),y_0)-\phi(t_0,x_0,y_0)-D_x\phi(t_0,x_0,y_0)\beta(x_0,e)\}n(de) \\ &= C^1_{\epsilon,\rho}\int_{|e|\le\delta}\frac{\beta(x_0,e)^2}{\epsilon}dn(e) \le \frac{K^1_{\epsilon,\rho}}{\epsilon}\int_{|e|\le\delta}(1\wedge|e|)^2 n(de) \end{aligned}$$

and

$$\begin{aligned} I^1_\delta(t_0,y_0,-\phi(t_0,x_0,.)) &= \int_{|e|\le\delta}\{-\phi(t_0,x_0,y_0+\beta(y_0,e))+\phi(t_0,x_0,y_0)+D_y\phi(t_0,x_0,y_0)\beta(y_0,e)\}n(de) \\ &= -C^2_{\epsilon,\rho}\int_{|e|\le\delta}\frac{\beta(y_0,e)^2}{\epsilon}dn(e) \ge -\frac{K^2_{\epsilon,\rho}}{\epsilon}\int_{|e|\le\delta}(1\wedge|e|)^2 n(de). \end{aligned}$$

Therefore we have

$$-I^1_\delta(t_0,x_0,\phi(t_0,.,y_0))+I^1_\delta(t_0,y_0,-\phi(t_0,x_0,.)) \ge -\frac{K_{\epsilon,\rho}}{\epsilon}\int_{|e|\le\delta}(1\wedge|e|)^2 n(de). \tag{3.3.17}$$

Now by the definition of (t^*,x^*) and (3.3.9), for ϵ small enough, $d,d' \in I\!R$,

$$u^{i_0,j_0}(t_0,x_0+d')-w^{i_0,j_0}(t_0,y_0+d) \le u^{i_0,j_0}(t_0,x_0)-w^{i_0,j_0}(t_0,y_0).$$

Therefore for $z\in E$, in taking $d=\beta(x_0,z)$ and $d'=\beta(y_0,z)$, we obtain

$$\begin{aligned} &u^{i_0,j_0}(t_0,x_0+\beta(x_0,z))-u^{i_0,j_0}(t_0,x_0) \\ &\quad\le w^{i_0,j_0}(t_0,y_0+\beta(y_0,z))-w^{i_0,j_0}(t_0,y_0)+\tfrac{|\beta(x_0,z)-\beta(y_0,z)|^2-2(x_0-y_0)(\beta(x_0,z)-\beta(y_0,z))}{\epsilon} \\ &\quad+\psi_\rho(t_0,x_0+\beta(x_0,z))-\psi_\rho(t_0,x_0). \end{aligned}$$

Since γ^{i_0,j_0} is nonnegative, and by the assumption on β,for any $\delta>0$,

$$I^{2,B^{i_0,j_0}}_\delta(t_0,x_0,u^{i_0,j_0})-I^{2,B^{i_0,j_0}}_\delta(t_0,y_0,w^{i_0,j_0}) \le I^{2,B^{i_0,j_0}}_\delta(t_0,x_0,\phi_\rho)+O(\frac{|x_0-y_0|^2}{\epsilon}), \tag{3.3.18}$$

and it is easy to check that

$$|I^{2,B^{i_0,j_0}}_\delta(t_0,x_0,\psi_\rho)| \le C\rho\int_{|z|\ge\delta}\{|z|^2+|z|^4\}n(dz).$$

On the other hand, since $\phi\in C^2$,

$$\begin{aligned} I^{1,B^{i_0,j_0}}_\delta(t_0,x_0,\phi(t_0,.,y_0)) &= \int_{|e|\le\delta}\{\phi(t_0,x_0+\beta(x_0,e),y_0)-\phi(t_0,x_0,y_0)\}\gamma^{i_0,j_0}(e,x_0)n(de) \\ &\le C^{1,B}_{\epsilon,\rho}\int_{|e|\le\delta}|\beta(x_0,e)|\gamma^{i_0,j_0}(e,x_0)dn(e) \end{aligned}$$

and

$$\begin{aligned} I^{1,B^{i_0,j_0}}_\delta(t_0,y_0,-\phi(t_0,x_0,.)) &= \int_{|e|\le\delta}\{-\phi(t_0,x_0,y_0+\beta(y_0,e))+\phi(t_0,x_0,y_0)\}\gamma^{i_0,j_0}(e,y_0)n(de) \\ &\ge -C^{2,B}_{\epsilon,\rho}\int_{|e|\le\delta}|\beta(y_0,e)|\gamma^{i_0,j_0}(e,y_0)dn(e). \end{aligned}$$

By the assumption of β and for ϵ small enough,

$$I^{1,B^{i_0,j_0}}_\delta(t_0,x_0,\phi(t_0,.,y_0))-I^{1,B^{i_0,j_0}}_\delta(t_0,y_0,-\phi(t_0,x_0,.)) \le K^B_{\epsilon,\rho}\int_{|e|\le\delta}(1\wedge|e|)^2 n(de). \tag{3.3.19}$$

Making now the difference between (3.3.14) and (3.3.15) yields

$$\begin{aligned} &-(p^\epsilon_u-p^\epsilon_w)-[b(t_0,x_0)q^\epsilon_u-b(t_0,y_0)q^\epsilon_w]-\frac{1}{2}[\sigma(t_0,x_0)^2M^\epsilon_u-\sigma(t_0,y_0)^2M^\epsilon_w] \\ &-[g^{i_0,j_0}(t_0,x_0,(u^{ij}(t_0,x_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,x_0)q^\epsilon_u, \\ &I^{1,B^{i_0,j_0}}_\delta(t_0,x_0,\phi(t_0,.,y_0))+I^{2,B^{i_0,j_0}}_\delta(t_0,x_0,u^{i_0,j_0})) \\ &-g^{i_0,j_0}(t_0,y_0,(w^{ij}(t_0,y_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,y_0)q^\epsilon_w), \\ &I^{1,B^{i_0,j_0}}_\delta(t_0,y_0,-\phi(t_0,x_0,.))+I^{2,B^{i_0,j_0}}_\delta(t_0,y_0,w^{i_0,j_0}))] \\ &-I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0))+I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.))-I^{2,\delta}(t_0,x_0,q^\epsilon_u,u^{i_0,j_0})+I^{2,\delta}(t_0,y_0,q^\epsilon_w,w^{i_0,j_0}) \\ &\le 0. \end{aligned}$$

Taking now into account (3.3.16) and (3.3.17), let $\delta \to 0$, we get

$$
\begin{aligned}
&- (p_u^\epsilon - p_w^\epsilon) - [b(t_0,x_0)q_u^\epsilon - b(t_0,y_0)q_w^\epsilon] - \frac{1}{2}[\sigma(t_0,x_0)^2 M_u^\epsilon - \sigma(t_0,y_0)^2 M_w^\epsilon] \\
&- [g^{i_0,j_0}(t_0,x_0,(u^{ij}(t_0,x_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,x_0)q_u^\epsilon, I_\delta^{1,B^{i_0,j_0}}(t_0,x_0,\phi(t_0,.,y_0)) + I_\delta^{2,B^{i_0,j_0}}(t_0,x_0,u^{i_0,j_0})) \\
&- g^{i_0,j_0}(t_0,y_0,(w^{ij}(t_0,y_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,y_0)q_w^\epsilon, I_\delta^{1,B^{i_0,j_0}}(t_0,x_0,\phi(t_0,.,y_0)) + I_\delta^{2,B^{i_0,j_0}}(t_0,x_0,u^{i_0,j_0}))] \\
&\quad - \frac{K|x_0-y_0|^2}{\epsilon}\int_{|e|\geq\delta}(1\wedge|e|)^2 n(de) \leq C\rho\int_{I\!R}\{|z|^2+|z|^4\}n(dz).
\end{aligned}
$$

and finally as usual, by (3.3.18) and (3.3.19) and the assumption on g^{i_0,j_0}, the exists a constant $C_{\epsilon,\rho}$ such that for any fixed $\rho>0$, $\limsup_{\epsilon\to 0,\rho\to 0} C_{\epsilon,\rho} \leq 0$ and

$$
\begin{aligned}
&- \{g^{i_0,j_0}(t_0,x_0,(u^{ij}(t_0,x_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,x_0)q_u^\epsilon, I_\delta^{1,B^{i_0,j_0}}(t_0,x_0,\phi(t_0,.,y_0)) + I_\delta^{2,B^{i_0,j_0}}(t_0,x_0,u^{i_0,j_0})) \\
&- g^{i_0,j_0}(t_0,x_0,(w^{ij}(t_0,y_0))_{(i,j)\in A^1\times A^2},\sigma(t_0,y_0)q_u^\epsilon, I_\delta^{1,B^{i_0,j_0}}(t_0,x_0,\phi(t_0,.,y_0)) + I_\delta^{2,B^{i_0,j_0}}(t_0,x_0,u^{i_0,j_0}))\} \\
&\leq C_{\epsilon,\rho} + C\rho\int_{I\!R}\{|z|^2+|z|^4\}n(dz).
\end{aligned}
$$

By the convergence to zero of (x_0-y_0) and the Lipschitz condition of g^{ij} with respect to $(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2}$, and using condition (3.3.5) we obtain:

$$
\lambda(u^{i_0,j_0}(t_0,x_0)-w^{i_0,j_0}(t_0,y_0)) - \sum_{(i,j)\in A^1\times A^2\neq(i_0,j_0)} \beta_0^{ij}(u^{i,j}(t_0,x_0)-w^{i,j}(t_0,y_0)) \leq C_0 + C\rho\int_{I\!R}\{|z|^2+|z|^4\}n(dz),
$$

where $\beta_{\epsilon,\rho}^{ij}$ stands for the increment rate of f^{i_0,j_0} with respect to u^{ij} which is uniformly bounded with respect to ϵ, ρ, and non-negative, thanks to the monotonicity assumption (A2). Therefore,

$$
\begin{aligned}
&\lambda(u^{i_0,j_0}(t_0,x_0) - w^{i_0,j_0}(t_0,y_0)) \\
&\leq \sum_{(i,j)\in A^1\times A^2\neq(i_0,j_0)} \beta_{\epsilon,\rho}^{ij}(u^{i,j}(t_0,x_0) - w^{i,j}(t_0,y_0)) + C_{\epsilon,\rho} + C\rho\int_{I\!R}\{|z|^2+|z|^4\}n(dz) \\
&\leq K_{\epsilon,\rho}(f^{i_0,j_0}) \sum_{(i,j)\in A^1\times A^2\neq(i_0,j_0)} (u^{i,j}(t_0,x_0) - w^{i,j}(t_0,y_0))^+ + C_{\epsilon,\rho} + C\rho\int_{I\!R}\{|z|^2+|z|^4\}n(dz).
\end{aligned}
$$

Taking the limit as $\epsilon\to 0$, $\rho\to 0$,

$$
\begin{aligned}
&\lambda(u^{i_0,j_0}(t^*,x^*) - w^{i_0,j_0}(t^*,x^*)) \\
&\leq \overline{\lim_{\epsilon\to 0,\rho\to 0}} K_{\epsilon,\rho}(f^{i_0,j_0}) \sum_{(i,j)\in A^1\times A^2\neq(i_0,j_0)} (u^{i,j}(t_0,x_0) - w^{i,j}(t_0,y_0))^+ \\
&\leq K(f^{i_0,j_0}) \sum_{(i,j)\in A^1\times A^2\neq(i_0,j_0)} \overline{\lim_{n\to\infty}} (u^{i,j}(t_0,x_0) - w^{i,j}(t_0,y_0))^+ \\
&\leq K(f^{i_0,j_0}) \sum_{(i,j)\in A^1\times A^2\neq(i_0,j_0)} (u^{i,j}(t^*,x^*) - w^{i,j}(t^*,x^*))^+,
\end{aligned}
$$

since u^{ij} (resp. w^{ij}) is *usc* (resp. *lsc*). As $(i_0,j_0)\in\Gamma(t^*,x^*)$, we obtain

$$
\lambda(u^{i_0,j_0}(t^*,x^*) - w^{i_0,j_0}(t^*,x^*)) \leq C(f^{i_0,j_0})(m_1\times m_2 - 1)(u^{i_0,j_0}(t^*,x^*) - w^{i_0,j_0}(t^*,x^*)).
$$

But this is contradictory with (3.3.4) and (3.3.5), thus, for any $(i,j)\in A^1\times A^2$, $u^{ij}\leq w^{ij}$.

Step 2. Now we deal with the general case. For arbitarary $\lambda\in R$, if $(u^{ij})_{(i,j)\in A^1\times A^2}(resp.(w^{ij})_{(i,j)\in A^1\times A^2})$ be a subsolution (resp. supersolution) of (3.1.2). Denote $\tilde{u}^{ij} = e^{-\lambda t}u^{ij}(t,x)$ and $\tilde{w}^{ij} = e^{-\lambda t}w^{ij}(t,x)$. It is easy to show that $\tilde{u}^{ij}(t,x)(resp.\tilde{w}^{ij}(t,x))$ is a subsoltion (resp. supersolution) of the following system of variational inequalities with oblique reflection : $\forall(i,j)\in A^1\times A^2$,

$$
\left\{
\begin{array}{l}
min\{\tilde{v}^{ij}(t,x) - \max\limits_{l\in A_i^1}\{\tilde{v}^{lj}(t,x) - e^{-\lambda t}\underline{g}_{il}(t,x)\}, max\{\tilde{v}^{ij}(t,x) - \min\limits_{k\in A_j^2}\{\tilde{v}^{ik}(t,x) + e^{-\lambda t}\overline{g}_{jk}(t,x)\}, \\
-\partial_t\tilde{v}^{ij}(t,x) - \mathcal{L}\tilde{v}^{ij}(t,x) - \lambda\tilde{v}^{ij}(t,x) \\
-e^{-\lambda t}g^{ij}(t,x,(e^{\lambda t}\tilde{v}^{kl}(t,x))_{(k,l)\in A^1\times A^2},\sigma(t,x)e^{\lambda t}D_x\tilde{v}^{ij}(t,x),e^{\lambda t}B^{ij}\tilde{v}^{ij}(t,x))\}\} = 0 \\
\tilde{v}^{ij}(T,x) = e^{-T\lambda}h^{ij}(x)
\end{array}
\right. \tag{3.3.20}
$$

But, by choosing λ large enough the functions

$$\begin{aligned}&F^{ij}(t,x,u^{kl}(t,x))_{(k,l)\in A^1\times A^2},z,U)\\&:=\lambda u^{ij}+e^{-\lambda t}g^{ij}(t,x,(e^{\lambda t}u^{kl}(t,x))_{(k,l)\in A^1\times A^2},e^{\lambda t}z,e^{\lambda t}U),\quad (i,j)\in A^1\times A^2,\end{aligned}$$

satisfy condition (3.3.5). Hence thanks to the result stated in step 1, we have $\tilde{u}^{ij}\leq\tilde{w}^{ij}$, $(i,j)\in A^1\times A^2$. Thus, $u^{ij}\leq w^{ij}$, for any $(i,j)\in A^1\times A^2$, which is the desired result. □

Corollary 3.3.1. *System (3.1.2) has at most one viscosity solution belongs to Π_g, and it is necessary continuous.*

Now we prove the existence, this will be done in 3 steps.

Proposition 3.3.1. *The family $(\bar{u}^{ij})_{(i,j)\in A^1\times A^2}$ is a viscosity subsolution of the system (3.1.2).*

Proof. Recall that $\forall(i,j)\in A^1\times A^2$, $\bar{u}^{ij}=\lim_m\searrow u^{ij,m}$, so $\bar{u}^{ij}$ is *usc*, and

$$\bar{u}^{ij}(T,x)=h^{ij}(x),\quad x\in R.$$

For a fixed $(i,j)\in A^1\times A^2$, we suppose that there exists $\epsilon_0>0$ s.t.

$$\bar{u}^{ij}(t,x)\geq L^{ij}[\bar{u}](t,x)+\epsilon_0,$$

otherwise the subsolution property holds. Thanks to the decreasing convergence of $(\bar{u}^{ij,m})_{m\geq0}$ to $\bar{u}^{ij}$, there exists m_0 such that for any $m\geq m_0$, we have

$$\bar{u}^{ij,m}(t,x)\geq L^{ij}[(\bar{u}^{pq,m})_{(p,q)\in A^1\times A^2}](t,x)+\frac{\epsilon_0}{2}.\tag{3.3.21}$$

Next, since $\forall(i,j)\in A^1\times A^2,\ m\geq0$, $\bar{u}^{ij,m}$ and $L^{ij}[(\bar{u}^{pq,m})_{(p,q)\in A^1\times A^2}]$ are continuous, there exists a neighborhood Θ_m of (t,x) such that

$$\bar{u}^{ij,m}(t',x')\geq L^{ij}[(\bar{u}^{pq,m})_{(p,q)\in A^1\times A^2}](t',x')+\frac{\epsilon_0}{4},\quad (t',x')\in\Theta_m.\tag{3.3.22}$$

By the same argument of [17](page 17), we have

$$\bar{u}^{ij}\leq U^{ij}[\bar{u}].\tag{3.3.23}$$

Let ϕ be a $C^{1,2}([0,T]\times I\!R)\cap\Pi_g$ function such that $\bar{u}^{ij}-\phi$ has a global maximum at (t,x) in $[0,T]\times I\!R^k$ which wlog we suppose strict and $\bar{u}^{ij}(t,x)=\phi(t,x)$. Therefore (t,x) is a global strict maximum of $\bar{u}^{ij}-\phi$ in $[0,T]\times B(x,K\delta)$. On the other hand there exist subsequences $\{m_k\}$ and $((t'_{m_k},x'_{m_k}))_k$ such that

$$((t'_{m_k},x'_{m_k}))_k\to_k(t,x)\text{ and }\bar{u}^{ij,m_k}(t'_{m_k},x'_{m_k})\to_k\bar{u}^{ij}(t,x).$$

Let now (t_{m_k},x_{m_k}) be the global maximum of $\bar{u}^{ij,m_k}-\phi$ on $\overline{[0,T]\times B(x,K\delta)}$. Therefore

$$(t_{m_k},x_{m_k})\to_k(t,x)\text{ and }u_j^{m_k}(t_{m_k},x_{m_k})\to_k\bar{u}^{ij,m_k}(t,x).$$

Actually let us consider a convergent subsequent of (t_{m_k},x_{m_k}), which we still denote by (t_{m_k},x_{m_k}), and let $(\bar{t},\bar{x})$ be its limit. Then for some k_0 and for $k\geq k_0$ we have

$$\bar{u}^{ij,m_k}(t_{m_k},x_{m_k})-\phi(t_{m_k},x_{m_k})\geq\bar{u}^{ij,m_k}(t'_{m_k},x'_{m_k})-\phi(t'_{m_k},x'_{m_k}).\tag{3.3.24}$$

Taking the limit wrt k to obtain

$$\bar{u}^{ij}(\bar{t},\bar{x})-\phi(\bar{t},\bar{x})\geq\bar{u}^{ij}(t,x)-\phi(t,x).$$

As the maximum (t,x) of $\bar{u}^{ij}-\phi$ on $[0,T]\times I\!R^k$ is strict then $(\bar{t},\bar{x})=(t,x)$. It follows that the sequence $((t_{m_k},x_{m_k}))_k$ converges to (t,x). Going back now to (3.3.24) and in sending k to infinite we obtain

$$\bar{u}^{ij,m_k}(t,x)\geq\limsup_k\bar{u}^{ij,m_k}(t_{m_k},x_{m_k})\geq\liminf_k\bar{u}^{ij,m_k}(t_{m_k},x_{m_k})\geq\liminf_k\bar{u}^{ij,m_k}(t'_{m_k},x'_{m_k})=\bar{u}^{ij}(t,x)$$

which implies that $\bar{u}^{ij,m_k}(t_{m_k},x_{m_k})\to\bar{u}^{ij}(t,x)$ as $k\to\infty$.

Now for k large enough,

(i) $(t_{m_k}, x_{m_k}) \in (0,T) \times B(x, K\delta)$ and is the global maximum of $\bar{u}^{ij,m_k} - \phi$ in $(0,T) \times B(x, K\delta)$;
(ii) $\bar{u}^{ij,m_k}(t_{m_k}, x_{m_k}) > L^{ij}[(\bar{u}^{pq,m_k})_{(p,q)\in A^1\times A^2}](t_{m_k}, x_{m_k})$.

As $\bar{u}^{ij,m_k}$ is a subsolution of (3.2.8), then by Definition 2.2, we have

$$\begin{array}{l} -\partial_t\phi(t_{m_k}, x_{m_k}) - \mathcal{L}^1\phi(t_{m_k}, x_{m_k}) - \mathcal{I}^1_\delta(t_{m_k}, x_{m_k}, \phi) - \mathcal{I}^2_\delta(t_{m_k}, x_{m_k}, D_x\phi(t_{n_k}, x_{n_k}), \bar{u}^{ij,m_k}) \leq \\ g^{ij}(t_{m_k}, x_{m_k}, (\bar{u}^{k,l,m_k}(t_{m_k}, x_{m_k}))_{(k,l)\in A^1\times A^2}, \sigma(t_{m_k}, x_{m_k})D_x\bar{u}^{i,j,m_k}(t_{m_k}, x_{m_k}), \\ \mathcal{I}^{1,B^{ij}}_\delta(t_{m_k}, x_{m_k}, \phi) + \mathcal{I}^{2,B^{ij}}_\delta(t_{m_k}, x_{m_k}, \bar{u}^{i,j,m_k})). \end{array} \tag{3.3.25}$$

But there exists a subsequence of $\{m_k\}$ (which we still denote by $\{m_k\}$) such that:
(i) for any $(k,l) \in A^1_i \times A^2_j$, $(\bar{u}^{k,l,m_k-1}(t_{m_k}, x_{m_k}))_k$ is convergent and then $\lim_k \bar{u}^{k,l,m_k-1}(t_{m_k}, x_{m_k}) \leq \bar{u}^{kl}(t,x)$;
(ii) $(\mathcal{I}^1_\delta(t_{m_k}, x_{m_k}, \phi))_{m_k} \to \mathcal{I}^1_\delta(t,x,\phi)$ as $k \to \infty$;
(iii) $(\mathcal{I}^{1,B^{ij}}_\delta(t_{m_k}, x_{m_k}, \phi))_{m_k} \to \mathcal{I}^{1,B^{ij}}_\delta(t,x,\phi)$ as $k \to \infty$.

Sending now k to infinite, using the fact that g^{ij} is continuous and verifies (A2) and finally by Lebesgue's Theorem to obtain

$$\begin{array}{l} -\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}^1_\delta(t,x,\phi) \leq \\ \mathcal{I}^2_\delta(t,x,D_x\phi(t,x),\bar{u}^{ij}) + g^{ij}(t,x,(\bar{u}^{k,l}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x\bar{u}^{ij}(t,x), \\ \mathcal{I}^{1,B^{ij}}_\delta(t,x,\phi) + \mathcal{I}^{2,B^{ij}}_\delta(t,x,\bar{u}^{ij})). \end{array}$$

But $\bar{u}^{ij}(t,x) = \phi(t,x)$ and $\bar{u}^{ij} \leq \phi$, then $\mathcal{I}^2_\delta(t,x,D_x\phi(t,x),\bar{u}^{ij}) \leq \mathcal{I}^2_\delta(t,x,D_x\phi(t,x),\phi)$ and $I^{2,B^{ij}}_\delta(t,x,\bar{u}^{ij}) \leq \mathcal{I}^{2,B^{ij}}_\delta(t,x,\phi)$. Plugging now this inequality in the previous one to obtain

$$-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}(t,x,\phi) - g^{ij}(t,x,(\bar{u}^{k,l}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_x\bar{u}^{ij}(t,x), \mathcal{I}^{B^{ij}}(t,x,\phi)) \leq 0.$$

Therefore $\bar{u}^{ij}$ is a viscosity subsolution of

$$\left\{\begin{array}{l} min\{(v^{ij} - L^{ij}[\overrightarrow{v}])(t,x), max\{(v^{ij} - U^{ij}[\overrightarrow{v}])(t,x), \\ -\partial_t v^{ij}(t,x) - \mathcal{L}v^{ij}(t,x) - g^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in A^1\times A^2}, \sigma(t,x)D_xv^{ij}(t,x), \mathcal{I}^{B^{ij}}(t,x,v^{ij}))\}\} = 0 \\ v^{ij}(T,x) = h^{ij}(x) \end{array}\right.$$

As i,j is arbitrary then $(\bar{u}^{ij})_{(i,j)\in A^1\times A^2}$ is a viscosity subsolution of (3.1.2).

□

Proposition 3.3.2. *Let m_0 be fixed in N. Then the family $(\bar{u}^{ij,m_0})_{(i,j)\in A^1\times A^2}$ is a viscosity supersolution of the system (3.1.2).*

Proof. By the Proposition 3.2, the triples $(\bar{Y}^{ij,m_0}, \bar{Z}^{ij,m_0}, \bar{U}^{ij,m_0}, \bar{K}^{ij,m_0})_{(i,j)\in A^1\times A^2}$ introduced in (3.2.5), solve the following system of reflected BSDEss:$\forall (i,j) \in A^1 \times A^2$,

$$\left\{\begin{array}{l} (\bar{Y}^{i,j,m_0}, \bar{Z}^{i,j,m_0}, \bar{U}^{i,j,m_0}, \bar{K}^{i,j,m_0}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2; \\ \bar{Y}^{i,j,m_0}_s = h^{i,j}(X^{t,x}_T) - \int_s^T \bar{f}^{i,j,m_0}(r, X^{t,x}_r, (\bar{Y}^{k,l,m_0}_r)_{(k,l)\in A^1\times A^2}, \bar{Z}^{i,j,m_0}_r, \bar{U}^{i,j,m_0}_r)dr + \int_s^T \bar{Z}^{i,j,m_0}_r dB_r \\ + \int_s^T \int_E \bar{U}^{i,j,m_0}_r(e)\hat{N}(drde) + \bar{K}^{i,j,m_0}_T - \bar{K}^{i,j,m_0}_s \\ \bar{Y}^{i,j,m_0}_s \geq \max_{k\in A^1_i}\{\bar{Y}^{k,j,m_0}_s - \underline{g}_{ik}(s, X^{t,x}_s)\}, \quad s \leq T, \\ \int_0^T (\bar{Y}^{i,j,m_0}_s - \max_{k\in A^1_i}\{\bar{Y}^{k,j,m_0}_s - \underline{g}_{ik}(s, X^{t,x}_s)\})d\bar{K}^{i,j,m_0}_s = 0, \end{array}\right. \tag{3.3.26}$$

where, $\forall (i,j) \in A^1 \times A^2$, $m \geq 0$ and $(s, \overrightarrow{Y}^{ij}, Z^{ij}, U^{ij})$,

$$\begin{array}{rl} \bar{f}^{i,j,m_0}(s, X^{t,x}_s, \overrightarrow{Y}, Z^{ij}, U^{ij}) :=& f^{ij}(s, X^{t,x}_s, (Y^{kl})_{(k,l)\in A^1\times A^2}, Z^{ij}, U^{ij}) \\ & - m_0(Y^{ij} - \min_{l\in A^2_j}(Y^{il} + \overline{g}_{jl}(s, X^{t,x}_s)))^+ \\ =& g^{ij}(s, X^{t,x}_s, (Y^{kl})_{(k,l)\in A^1\times A^2}, Z^{ij}, \int_E U^{ij}(e)\gamma^{ij}(X^{t,x}_s, e)n(de)) \\ & - m_0(Y^{ij} - \min_{l\in A^2_j}(Y^{il} + \overline{g}_{jl}(s, X^{t,x}_s)))^+. \\ :=& g^{ij,+,m_0}(s, X^{t,x}_s, (Y^{kl})_{(k,l)\in A^1\times A^2}, Z^{ij}, \int_E U^{ij}(e)\gamma^{ij}(X^{t,x}_s, e)n(de)) \end{array}$$

Furthermore, there exsits a unique $A^1\times A^2$-uplet of deterministic continuous functions $(\bar{u}^{k,l,m_0})_{(k,l)\in A^1\times A^2}$ in Π_g such that, for every $t\leq T$,

$$\bar{Y}_s^{i,j,m_0}=\bar{u}^{i,j,m_0}(s,X_s^{t,x}),\ s\in[t,T].$$

By Theorem 5.1 in Appendix, it follows that $\bar{u}^{i,j,m_0}$ is a viscosity solution for the following PDE with two obstacles:

$$\begin{cases} min\{\theta(t,x)-\max_{k\in A_i^1}\{\bar{u}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x)\},\\ max\{\theta(t,x)-\bar{u}^{ij,m_0}(t,x)\vee min_{l\in A_j^2}(\bar{u}^{il,m_0}(t,x)-\bar{g}_{jl}(t,x)),\\ -\partial_t\theta(t,x)-\mathcal{L}\theta(t,x)-g^{ij}(t,x,(\bar{u}^{kl,m_0}(t,x))_{(k,l)\in A^1\times A^2},\sigma(t,x)D_x\theta(t,x),I^{B^{ij}}(t,x,\theta(t,x)))\}\}=0\\ \theta(T,x)=h^{ij}(x)\end{cases}$$

Next, let $(t,x)\in[t,T]\times R$ and $(p,q,M)\in\bar{J}^-\bar{u}^{ij,m_0}(t,x)$. As $\bar{u}^{ij,m_0}$ is a solution in a viscosity sense of the previous PDE with two obstacles then it holds that

$$\bar{u}^{ij,m_0}(t,x)\geq\max_{k\in A_i^1}\{\bar{u}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x)\}\tag{3.3.27}$$

and

$$\begin{aligned}&max\{\bar{u}^{ij,m_0}(t,x)-\bar{u}^{ij,m_0}(t,x)\vee min_{l\in A_j^2}(\bar{u}^{il,m_0}(t,x)-\bar{g}_{jl}(t,x));\\ &\quad -p-\mathcal{L}_{(q,M)}\bar{u}^{ij,m_0}(t,x)-g^{ij}(t,x,(\bar{u}^{kl,m_0}(t,x))_{(k,l)\in A^1\times A^2},\sigma(t,x)q,I^{B^{ij}}(t,x,\bar{u}^{ij,m_0}(t,x)))\}\geq 0.\end{aligned}$$

But, for any constants $a,b\in R$ we have $a-(a\vee b)\leq a-b$ and thus $a-(a\vee b)\geq 0\Rightarrow a-b\geq 0$. Therefore, we have

$$\begin{aligned}&max\{\bar{u}^{ij,m_0}(t,x)-min_{l\in A_j^2}(\bar{u}^{il,m_0}(t,x)-\bar{g}_{jl}(t,x));\\ &\quad -p-\mathcal{L}_{(q,M)}\bar{u}^{ij,m_0}(t,x)-g^{ij}(t,x,(\bar{u}^{kl,m_0}(t,x))_{(k,l)\in A^1\times A^2},\sigma(t,x)q,I^{B^{ij}}(t,x,\bar{u}^{ij,m_0}(t,x)))\}\geq 0.\end{aligned}$$

Combining this inequality with (3.3.27) and since $\bar{u}^{ij,m_0}(T,x)=h^{ij}(x)$ it follows that $\bar{u}^{ij,m_0}$ is a viscosity supersolution of the system

$$\begin{cases} min\{\theta(t,x)-\max_{k\in A_i^1}\{\bar{u}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x)\},max\{\theta(t,x)-min_{l\in A_j^2}(\bar{u}^{il,m_0}(t,x)-\bar{g}_{jl}(t,x)),\\ -\partial_t\theta(t,x)-\mathcal{L}\theta(t,x)-g^{ij}(t,x,(\bar{u}^{kl,m_0}(t,x))_{(k,l)\in A^1\times A^2},\sigma(t,x)D_x\theta(t,x)I^{,B^{ij}}(t,x,\theta(t,x)))\}\}=0\\ \theta(T,x)=h^{ij}(x)\end{cases}$$

Since (i,j) is arbitrary in $A^1\times A^2$, the system of continuous functions $(\bar{u}^{ij,m_0})_{(i,j)\in A^1\times A^2}$ is a supersolution of (3.1.2). □

Consider the set $\mathcal{U}_{m_0}$ defined as follows.

$$\mathcal{U}_{m_0}=\{\overrightarrow{u}:=(u^{ij})_{(i,j)\in A^1\times A^2}\,s.t.\,\overrightarrow{u}\ \textit{is subsolution of}\ (3.1.2)\ \textit{and}\ \forall(i,j)\in A^1\times A^2,\bar{u}^{i,j}\leq u^{i,j}\leq\bar{u}^{ij,m_0}\}.$$

$\mathcal{U}_{m_0}$ is not empty since it contains $(\overline{u}^{ij})_{(i,j)\in A^1\times A^2}$. Next for $(t,x)\in[0,T]\times R$ and $(i,j)\in A^1\times A^2$, set:

$$^{m_0}u^{ij}(t,x)=sup\{u^{ij}(t,x),\ (u^{kl})_{(k,l)\in A^1\times A^2}\in\mathcal{U}_{m_0}\}.$$

Now we give the main result of this section. To begin with we give a third definition of viscosity solution, in the same way with [5]Proposition 1), we can prove that they are equivalent.

Definition 3.3.1. *(i) For a function u: $[0,T]\times R\to R$, lsc (resp. usc), we denote $J^-u(t,x)$ the parabolic subjet (resp. $J^+u(t,x)$ the parabolic superjet) of u at (t,x)$\in[0,T]\times R$, as the set of triples (p,q,M)$\in R\times R\times S^k$; where S^k is the set of symmetric real matrices of dimension k*

$$u(t',x')\geq u(t,x)+p(t'-t)-\langle q,x'-x\rangle+\frac{1}{2}\langle x'-x,M(x'-x)\rangle+o(|t'-t|+|x'-x|)^2\quad(resp.\leq)$$

(ii) We denote $\bar{J}^-u(t,x)$ (resp. $\bar{J}^+u(t,x)$) the parabolic limiting superjet (resp. superjet) of u at (t,x), as the set of triples (p,q,M)$\in R\times R\times S^k$ s.t.

$$(p,q,M)=\lim_{n\to\infty}(p_n,q_n,M_n),\quad(t,x)=\lim_{n\to\infty}(t_n,x_n)$$

where $(p_n, q_n, M_n) \in J^- u(t_n, x_n)(resp. J^+ u(t_n, x_n))$ *and* $u(t,x) = \lim_{n\to\infty} u(t_n, x_n)$.
(iii) A function $\overrightarrow{u} = (u^{ij}(t,x))_{(i,j)\in A^1\times A^2} : [0,T]\times R \to R^{A^1\times A^2}$ *such that for any* $(i,j) \in A^1 \times A^2$, $u^{ij} \in \Pi_g$ *is lsc (resp. usc), is said to be a viscosity subsolution (resp. supsolution) of (3.1.2) if for any test function* $\phi \in C^{1,2}([0,T]\times R)$, $\delta > 0$, *if* $(t_0, x_0) \in [0,T]\times R$ *is a maximum (resp. minimum) point of* $u^{i,j} - \phi$ *on* $[0,T]\times B(x_0, K\delta)$ *where* K *is the same as in definition 2.2, and if* $(p,q,M) \in \bar{J}^- u^{i,j}(t_0,x_0)(resp. \bar{J}^+ u^{i,j}(t_0,x_0))$ *with* $q = D_t\phi(t_0,x_0)$, $p = D_x\phi(t_0,x_0)$, *and* $M \geq D^2_{xx}\phi(t_0,x_0)$ *(resp.* $M \leq D^2_{xx}\phi(t_0,x_0)$*), then:*

$$\begin{cases} min\{(u^{ij} - L^{ij}[\overrightarrow{u}])(t_0,x_0), max\{(u^{ij} - U^{ij}[\overrightarrow{u}])(t_0,x_0), \\ -p - b(t_0,x_0)q - \frac{1}{2}\sigma^2(t_0,x_0)M - I^1_\delta(t_0,x_0,\phi) - I^2_\delta(t_0,x_0,q,u^{ij}) \\ -g^{ij}(t_0,x_0,(u^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2}, \sigma(t_0,x_0)q, I^{1,B^{ij}}_\delta(t_0,x_0,\phi) + I^{2,B^{ij}}_\delta(t_0,x_0,u^{ij}))\}\} \leq 0 \quad (resp. \geq 0) \\ v^{ij}(T,x) \leq h^{ij}(x) \quad (resp. \geq) \end{cases}$$

Theorem 3.3.2. *The family* $({}^{m_0}u^{ij})_{(i,j)\in A^1\times A^2}$ *does not depend on* m_0 *and is the unique continuous viscosity solution in the class* Π_g *of the system (3.1.2).*

Proof. Note that for any $(i,j) \in A^1\times A^2$, $\bar{u}^{i,j} \leq^{m_0} u^{ij} \leq \bar{u}^{ij,m_0}$. Since $\bar{u}^{i,j}$ and $\bar{u}^{ij,m_0}$ are of polynomial growth, then $({}^{m_0}u^{ij})_{(i,j)\in A^1\times A^2}$ belongs to Π_g. The remaining of the proof is divided into two steps, to easy notation, we denote $({}^{m_0}u^{ij})_{(i,j)\in A^1\times A^2}$ by $(u^{ij})_{(i,j)\in A^1\times A^2}$ as no confusion is possible.

Step 1. First we show that $(u^{ij})_{(i,j)\in A^1\times A^2}$ is a subsolution of (3.1.2). It is clear that for any $(t,x) \in [0,T]\times R$, $\bar{u}^{i,j} \leq u^{ij} \leq \bar{u}^{ij,m_0}$. This implies that $\bar{u}^{i,j} \leq u^{ij,*} \leq \bar{u}^{ij,m_0}$ since $\bar{u}^{ij}$ is *usc* and $\bar{u}^{ij,m_0}$ is continuous. Therefore, for any $x \in R$, since $\bar{u}^{ij}(T,x) = \bar{u}^{ij,m_0} = h^{ij}(x)$, we have $u^{ij,*}(T,x) = h^{ij}(x)$. For the fixed $(i,j) \in A^1\times A^2$, let $(\tilde{u}^{ij})_{(i,j)\in A^1\times A^2}$ be an arbitrary element of $\mathcal{U}_{m_0}$. Then, for any $(t,x) \in [0,T]\times R$ and any $(p,q,M) \in \bar{J}^+\tilde{u}^{ij,*}(t,x)$ we have

$$\begin{cases} min\{(\tilde{u}^{ij,*} - L^{ij}[(\tilde{u}^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), max\{(\tilde{u}^{ij,*} - U^{ij}[(\tilde{u}^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), \\ -p - \mathcal{L}_{(q,M)}\tilde{u}^{ij,*}(t,x) - g^{ij}(t,x,(\tilde{u}^{kl,*})^{ij}_{(k,l)\in A^1\times A^2}, \sigma(t,x)q, I^{B^{ij}}(t,x,\tilde{u}^{ij,*}(t,x)))\}\} \geq 0 \end{cases}$$

By definition we have $\tilde{u}^{ij} \leq u^{ij}$ and then $\tilde{u}^{ij,*} \leq u^{ij,*}$. By the monotonicity property (**A2**), we have

$$\begin{cases} min\{(\tilde{u}^{ij,*} - L^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), max\{(\tilde{u}^{ij,*} - U^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), \\ -p - \mathcal{L}_{(q,M)}\tilde{u}^{ij,*}(t,x) - g^{ij}(t,x,[(u^{kl,*})^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, \tilde{u}^{ij}], \sigma(t,x)q, I^{B^{ij}}(t,x,\tilde{u}^{ij,*}(t,x)))\}\} \geq 0 \end{cases}$$

This means that $\tilde{u}^{ij}(t,x)$ is a subsotluion of the following equation.

$$\begin{cases} min\{(w - L^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), max\{(w - U^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), \\ -p - \mathcal{L}_{(q,M)}w - g^{ij}(t,x,[(u^{kl,*})^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, w], \sigma(t,x)q, I^{B^{ij}}(t,x,w))\}\} = 0 \end{cases}$$

Relying on the lower semi continuity of the function

$$\begin{cases} (t,x,w,p,q,M) \mapsto min\{(w - L^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), \\ max\{(w - U^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), \\ -p - \mathcal{L}_{(q,M)}w - g^{ij}(t,x,[(u^{kl,*})^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, w], \sigma(t,x)q, I^{B^{ij}}(t,x,w))\}\} \end{cases}$$

and using Lemma 4.2 in Crandall et al.(1992)([12],pp.23), related to suprema of subsolutions, combined with the above result, it holds that u^{ij} is a subsolution of the following equation

$$\begin{cases} min\{(w - L^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), max\{(w - U^{ij}[(u^{kl,*})_{(k,l)\in A^1\times A^2}])(t,x), \\ -p - \mathcal{L}_{(q,M)}w - g^{ij}(t,x,[(u^{kl,*})^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, w], \sigma(t,x)q, I^{B^{ij}}(t,x,w))\}\} = 0 \\ u^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{3.3.28}$$

Since (i,j) is arbitrary in $A^1\times A^2$, $(u^{ij})_{(i,j)\in A^1\times A^2}$ is a subsolution of (3.1.2).

Step 2. Now we will use Perron's method to show that $(u^{ij})_{(i,j)\in A^1\times A^2}$ is a viscosity supersolution of (3.1.2). Note that for any $(i,j) \in A^1\times A^2$,

$$\underline{u}^{ij} = \underline{u}^{ij}_* \leq \overline{u}^{ij}_* \leq u^{ij}_* \leq \overline{u}^{ij,m_0}_* = \overline{u}^{ij,m_0},$$

since $\overline{u}^{ij,m_0}$ is continuous and $\underline{u}^{ij}$ is lsc. Therefore, for any $x \in R$,

$$u_*^{ij}(T,x) = h^{ij}(x) \tag{3.3.29}$$

since $\underline{u}^{ij}(T,x) = h^{ij}(x) = \overline{u}^{ij,m_0}(T,x)$. Assume that $(u^{ij})_{(i,j)\in A^1\times A^2}$ is not a supersolution for (3.1.2), then taking into account of (3.3.29) and Remark 2.1, there exists at least on pair (i,j) such that u^{ij} does not satisfy the viscosity supersolution property: means that for some point $(t_0,x_0) \in [0,T]\times R$ there exists a triple (p,q,M) in $J^-(u_*^{ij})(t_0,x_0)$ such that

$$\begin{cases} min\{(u_*^{ij} - L^{ij}[\overrightarrow{u}_*])(t_0,x_0), max\{(u_*^{ij} - U^{ij}[\overrightarrow{u}_*])(t_0,x_0), \\ -p - \mathcal{L}_{(q,M)}u_*^{ij}(t_0,x_0) - g^{ij}(t_0,x_0,(u_*^{kl}(t_0,x_0))_{(k,l)\in A^1\times A^2}, \sigma(t_0,x_0)q, I^{B^{ij}}(t_0,x_0,u_*^{ij}(t_0,x_0)))\}\} < 0 \end{cases} \tag{3.3.30}$$

Now for any positive δ, γ and r, set $u_{\delta,\gamma}$ and B_r as follows:

$$u_{\delta,\gamma} = u_*^{ij}(t_0,x_0) + \delta + q(x-x_0) + p(t-t_0) + \frac{1}{2}(M-2\gamma)(x-x_0)^2,$$

and

$$B_r := \{(t,x) \in [0,T]\times R,\ s.t.\ |t-t_0| + |x-x_0| < r\}.$$

Using (3.3.30) and continuity of all the data, choosing δ, γ small enough we obtain

$$\begin{cases} min\{(u_*^{ij} + \delta - L^{ij}[\overrightarrow{u}_*])(t_0,x_0), max\{(u_*^{ij} + \delta - U^{ij}[\overrightarrow{u}_*])(t_0,x_0), \\ -p - \mathcal{L}_{(q,M-2\gamma)}(u_*^{ij}(t_0,x_0) + \delta) \\ -g^{ij}(t_0,x_0,[(u_*^{kl}(t_0,x_0))^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, u_*^{ij}(t_0,x_0)+\delta], \sigma(t_0,x_0)q, \\ I^{B^{ij}}(t_0,x_0,(u_*^{ij}(t_0,x_0)+\delta)))\}\} < 0, \end{cases} \tag{3.3.31}$$

noting that $\forall (t,x) \in [0,T]\times R$, $\delta \to I^{B^{ij}}(t,x,(u_*^{ij}(t,x)+\delta)$ is continuous. Next, let us define the function Υ as follows.

$$\Upsilon(t,x) = min\{u_{\delta,\gamma}(t,x) - \max_{k\neq i}(u_*^{kj} - \underline{g}_{ik})(t,x), max\{u_{\delta,\gamma}(t,x) - \min_{l\neq j}(u_*^{il} - \overline{g}_{jl})(t,x), \varpi(t,x)\}\},$$

where,

$$\varpi(t,x) = -p - \mathcal{L}_{(q,M-2\gamma)}u_{\delta,\gamma}(t,x) - g^{ij}(t,x,[(u_*^{kl}(t,x))^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, u_{\delta,\gamma}(t,x)], \sigma(t,x)q, I^{B^{ij}}(t,x,u_{\delta,\gamma}(t,x))).$$

First we note that from (3.3.31), $\Upsilon(t_0,x_0) < 0$, since $u_{\delta,\gamma}(t_0,x_0) = u_*^{ij}(t_0,x_0) + \delta$. On the other hand by the continuity of $u_{\delta,\gamma}$, Assumptions (A1), (A2) on f^{ij} and finally the lower semi-continuity of $(u_*^{kl})_{(k,l)\in A^1\times A^2}$, we can check that Υ is usc. Thus for any $\varepsilon > 0$, there exists $\eta > 0$ s.t. for any $(t,x) \in B_\eta$ we have $\Upsilon(t_0,x_0) \geq \Upsilon(t,x) - \varepsilon$. Since $\Upsilon(t_0,x_0) < 0$, choosing ε small enough we deduce that $\Upsilon(t,x) < 0$ for any $(t,x) \in B_\eta$ with an appropriate η. It follows that the function $u_{\delta,\gamma}$ is a viscosity subsolution on B_η of the following system.

$$\begin{cases} min\{(\varrho(t,x) - L^{ij}[\overrightarrow{u}_*])(t,x), max\{(\varrho(t,x) - U^{ij}[\overrightarrow{u}_*])(t,x), \\ -\partial_t\varrho(t,x) - \mathcal{L}\varrho(t,x) - g^{ij}(t,x,[(u_*^{kl}(t,x))^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, \varrho(t,x)], \\ \sigma(t,x)D_x\varrho(t,x), I^{B^{ij}}(t,x,\varrho(t,x)))\}\} = 0. \end{cases}$$

Since, for any $(k,l) \in A^1\times A^2$, $u_*^{kl} \leq u^{kl,m_0}$ and f^{ij} satisfies the monotonicity condition (A2), $u_{\delta,\gamma}$ is also a viscosity solution on B_η of the system

$$\begin{cases} min\{(\varrho(t,x) - L^{ij}[\overrightarrow{u}^*])(t,x), max\{(\varrho(t,x) - U^{ij}[\overrightarrow{u}^*])(t,x), \\ -\partial_t\varrho(t,x) - \mathcal{L}\varrho(t,x) - g^{ij}(t,x,[(u^{kl,*}(t,x))^{ij}_{(k,l)\neq(i,j)\in A^1\times A^2}, \varrho(t,x)], \\ \sigma(t,x)D_x\varrho(t,x), I^{B^{ij}}(t,x,\varrho(t,x)))\}\} = 0. \end{cases} \tag{3.3.32}$$

Next, as $(p,q,M) \in J^-(u_*^{ij}(t_0,x_0))$ then

$$u^{ij}(t,x) \geq u_*^{ij} \geq u_*^{ij}(t_0,x_0) + p(t-t_0) + q(x-x_0) + \frac{1}{2}M(x-x_0)^2 + o(|t-t_0|) + o(|x-x_0|^2).$$

Take $\delta = \frac{r^2}{8}\gamma$, it is easily seen that

$$u^{ij}(t,x) > u_{\delta,\gamma}(t,x),$$

as soon as $\frac{r}{2} < |x - x_0| \leq r$ and r small enough. Take $r \leq \eta$ and consider the function $\tilde{u}^{ij}$:

$$\tilde{u}^{ij}(t,x) = \begin{cases} max(u^{ij}(t,x), u_{\delta,\gamma}(t,x)), & if\ (t,x) \in B_r, \\ u^{ij}(t,x), & otherwise. \end{cases}$$

Then taking into account of (3.3.32), and using Lemma 4.2 in Crandall et al. (1992)([12]), it follows that $\tilde{u}^{ij}$ is a subsolution of (3.3.28). Next, as $\tilde{u}^{ij} \geq u^{ij}$ and using once more the monotonicity assumption (A2) on f^{kl}, we get that $[(u^{kl})_{(k,l)\neq(i,j)}, \tilde{u}^{ij}]$ a also a subsolution of (3.1.2) which belongs to Π_g. Thus, thanks to the comparison theorem 4.1, $[(u^{kl})_{(k,l)\neq(i,j)}, \tilde{u}^{ij}]$ belongs also to $\mathcal{U}_{m_0}$. Finally in view of the definition of u_*^{ij}, there exists a sequence $(t_n, x_n, u^{ij}(t_n, x_n))_{n\geq 1}$ that converges to $(t_0, x_0, u_*^{ij}(t_0, x_0))_{n\geq 1}$. This implies that

$$\lim_n (\tilde{u}^{ij} - u^{ij})(t_n, x_n) = (u_{\delta,\gamma} - u_*^{ij})(t_0, x_0) = u_*^{ij}(t_0, x_0) + \delta - u_*^{ij}(t_0, x_0) > 0.$$

It means that there are points (t_n, x_n) such that $\tilde{u}^{ij}(t_n, x_n) > u^{ij}(t_n, x_n)$. But this contradicts the definition of u^{ij}, since $[(u^{kl})_{(k,l)\neq(i,j)}, \tilde{u}^{ij}]$ belongs also to $\mathcal{U}_{m_0}$. Therefore, $(u^{ij})_{(i,j)\in A^1\times A^2}$ is also a supersolution for (3.1.2). Now, by the Corollary 4.1, $({}^{m_0}u^{ij})_{(i,j)\in A^1\times A^2}$ (i.e. $(u^{ij})_{(i,j)\in A^1\times A^2}$) dose not depend on m_0 is the unique continuous viscosity solution in the class Π_g of (3.1.2). □

Chapter 4

On the identity of min-max and max-min solutions of Systems of Variational Inequalities with Interconnected Bilateral Obstacles.

This chapter is a joint work with Boualem Djehiche, Said Hamadène and Marie-Amelie Morlais.

4.1 Notations and first results

Let us consider the following two systems of partial differential equations (PDEs) with bilateral interconnected obstacles (i.e., the obstacles depend on the solution) of min-max and max-min types: for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $(t,x) \in [0,T] \times I\!R^k$,

$$\begin{cases} \min\Big\{v^{ij}(t,x) - L^{ij}(\vec{v})(t,x)\,;\, \max\Big\{v^{ij}(t,x) - U^{ij}(\vec{v})(t,x); \\ \qquad -\partial_t v^{ij} - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x v^{ij}(t,x))\Big\}\Big\} = 0\,; \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{4.1.1}$$

and

$$\begin{cases} \max\Big\{\check{v}^{ij}(t,x) - U^{ij}(\vec{\check{v}})(t,x)\,;\min\Big\{\check{v}^{ij}(t,x) - L^{ij}(\vec{\check{v}})(t,x) \\ \qquad -\partial_t \check{v}^{ij} - \mathcal{L}^X(\check{v}^{ij})(t,x) - f^{ij}(t,x,(\check{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x \check{v}^{ij}(t,x))\Big\}\Big\} = 0\,; \\ \check{v}^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{4.1.2}$$

where
(i) Γ^1 and Γ^2 are finite sets (possibly different);
(ii) For any $(t,x) \in [0,T] \times I\!R^k$, $\vec{v}(t,x) = (v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}$ and for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$L^{ij}(\vec{v})(t,x) = \max_{k\in\Gamma^1, k\neq i}\{v^{kj}(t,x) - \underline{g}_{ik}(t,x)\},\ U^{ij}(\vec{v})(t,x) = \min_{p\in\Gamma^2, p\neq j}\{v^{ip}(t,x) + \bar{g}_{jp}(t,x)\}.$$

(iii) $\mathcal{L}^X$ is a second order generator associated with a diffusion process described below.

The systems (4.1.1) and (4.1.2) are of min-max and max-min types respectively. The barriers $L^{ij}(\vec{v}), U^{ij}(\vec{v})$ and $L^{ij}(\vec{\check{v}}), U^{ij}(\vec{\check{v}})$ depend on the solution $\vec{v}$ and $\vec{\check{v}}$ of (4.1.1) and (4.1.2) respectively. They are related to zero-sum switching game problems since actually, specific cases of these systems, stand for the Hamilton-Jacobi-Bellman-Isaacs equations associated with those games.

Let T (resp. k, d) be a fixed positive constant (resp. two integers) and Γ^1 (resp. Γ^2) denote the set of switching modes for player 1 (resp. 2). For later use, we shall denote by Λ the cardinal of the product set $\Gamma^1 \times \Gamma^2$ and for $(i,j) \in \Gamma^1 \times \Gamma^2$, $(\Gamma^1)^{-i} := \Gamma^1 - \{i\}$ and $(\Gamma^2)^{-j} := \Gamma^2 - \{j\}$. For $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} \in I\!R^\Lambda$,

$(i,j) \in \Gamma^1 \times \Gamma^2$, and $\underline{y} \in I\!R$, we denote by $[(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2-\{i,j\}}, \underline{y}]$ the matrix obtained from the matrix $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ by replacing the element y^{ij} with $\underline{y}$.

For any $(i,j) \in \Gamma^1 \times \Gamma^2$, let

$$
\begin{array}{l}
b : (t,x) \in [0,T] \times I\!R^k \mapsto b(t,x) \in I\!R^k; \\
\sigma : (t,x) \in [0,T] \times I\!R^k \mapsto \sigma(t,x) \in I\!R^{k\times d}; \\
f^{ij} : (t,x,\vec{y},z) \in [0,T] \times I\!R^{k+\Lambda+d} \mapsto f^{ij}(t,x,\vec{y},z) \ \in I\!R\,; \\
\underline{g}_{ik} : (t,x) \in [0,T] \times I\!R^k \mapsto \underline{g}_{ik}(t,x) \in I\!R \quad (k \in (\Gamma^1)^{-i}); \\
\bar{g}_{jl} : (t,x) \in [0,T] \times I\!R^k \mapsto \bar{g}_{jl}(t,x) \in I\!R \quad (l \in (\Gamma^2)^{-j}); \\
h^{ij} : x \in I\!R^k \mapsto h^{ij}(x) \in I\!R.
\end{array}
$$

A function $\Phi : (t,x) \in [0,T] \times I\!R^k \mapsto \Phi(t,x) \in I\!R$ is called of *polynomial growth* if there exist two non-negative real constants C and γ such that

$$|\Phi(t,x)| \le C(1+|x|^\gamma), \quad (t,x) \in [0,T] \times I\!R^k.$$

Hereafter, this class of functions is denoted by Π_g. Let $\mathcal{C}^{1,2}([0,T] \times I\!R^k)$ (or simply $\mathcal{C}^{1,2}$) denote the set of real-valued functions defined on $[0,T] \times I\!R^k$, which are once (resp. twice) differentiable w.r.t. t (resp. x) and with continuous derivatives.

The following assumptions on the data of the systems (4.1.1) and (4.1.2) are in force throughout the paper.

(**H0**) The functions b and σ are jointly continuous in (t,x) and Lipschitz continuous w.r.t. x uniformly in t, meaning that there exists a non-negative constant C such that for any $(t,x,x') \in [0,T] \times \mathbb{R}^{k+k}$ we have

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \le C|x - x'|.$$

Therefore, they are also of linear growth w.r.t. x, i.e., there exists a constant C such that for any $(t,x) \in [0,T] \times I\!R^k$,

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|).$$

(**H1**) Each function f^{ij}

(i) is continuous in (t,x) uniformly w.r.t. the other variables $(\vec{y},z)$ and, for any (t,x), the mapping $(t,x) \to f^{ij}(t,x,0,0)$ is of polynomial growth.

(ii) is Lipschitz continuous with respect to the variables $(\vec{y} := (y^{ij})_{(i,j)\in\Gamma_1\times\Gamma_2}, z)$ uniformly in (t,x), i.e. $\forall\ (t,x) \in [0,T] \times \mathbb{R}^k$, $\forall\ (\vec{y}_1,\vec{y}_2) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda, (z^1,z^2) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$|f^{ij}(t,x,\vec{y}_1,z_1) - f^{ij}(t,x,\vec{y}_2,z_2)| \le C\left(|\vec{y}_1 - \vec{y}_2| + |z_1 - z_2|\right),$$

where, $|\vec{y}|$ stands for the standard Euclidean norm of $\vec{y}$ in $I\!R^\Lambda$.

(**H2**) Monotonicity: Let $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$. For any $(i,j) \in \Gamma^1 \times \Gamma^2$ and any $(k,l) \ne (i,j)$ the mapping $y^{kl} \to f^{ij}(s,\vec{y},z)$ is non-decreasing.

(**H3**) The functions h^{ij}, which are the terminal conditions in the systems (4.1.1) and (4.1.2), are continuous with respect to x, belong to class Π_g and satisfy

$$\forall\ (i,j) \in \Gamma^1 \times \Gamma^2 \text{ and } x \in I\!R^k,\ \max_{k\in(\Gamma^1)^{-i}} \left(h^{kj}(x) - \underline{g}_{ik}(T,x)\right) \le h^{ij}(x) \le \min_{l\in(\Gamma^2)^{-j}} \left(h^{il}(x) + \bar{g}_{jl}(T,x)\right).$$

(**H4**) The no free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t,x), belong to Π_g and satisfy the following condition:

For any loop in $\Gamma^1 \times \Gamma^2$, i.e., any sequence of pairs $(i_1,j_1),\dots,(i_N,j_N)$ of $\Gamma^1 \times \Gamma^2$ such that $(i_N,j_N) = (i_1,j_1)$, $\text{card}\{(i_1,j_1),\dots,(i_N,j_N)\} = N-1$ and any $q = 1,\dots,N-1$, either $i_{q+1} = i_q$ or $j_{q+1} = j_q$, we have $\forall (t,x) \in [0,T] \times I\!R^k$,

$$\sum_{q=1,N-1} \varphi_{i_q i_{q+1}}(t,x) \ne 0, \tag{4.1.3}$$

where, $\forall\ q = 1,\dots,N-1,\ \varphi_{i_q i_{q+1}}(t,x) = -\underline{g}_{i_q i_{q+1}}(t,x)\mathbb{1}_{i_q \ne i_{q+1}} + \bar{g}_{j_q i_{q+1}}(t,x)\mathbb{1}_{j_q \ne j_{q+1}}$.

This assumption implies in particular that

$$\forall\,(i_1,\dots,i_N)\in(\Gamma^1)^N \text{ such that } i_N=i_1 \text{ and } \mathrm{card}\{i_1,\dots,i_N\}=N-1,\ \sum_{p=1}^{N-1}\underline{g}_{i_k,i_{k+1}}>0 \qquad (4.1.4)$$

and

$$\forall\,(j_1,\dots,j_N)\in(\Gamma^2)^N \text{ such that } j_N=j_1 \text{ and } \mathrm{card}\{j_1,\dots,j_N\}=N-1,\ \sum_{p=1}^{N-1}\bar{g}_{j_k,j_{k+1}}>0. \qquad (4.1.5)$$

By convention we set $\bar{g}_{j,j}=\underline{g}_{i,i}=0$.

Conditions (4.1.4) and (4.1.5) are classical in the literature of switching problems and usually referred to as the *no free loop property.* □

We now introduce the probabilistic tools we need later. Let $(\Omega,\mathcal{F},P)$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B=(B_t)_{0\le t\le T}$ whose natural filtration is $(\mathcal{F}_t^0:=\sigma\{B_s,s\le t\})_{0\le t\le T}$. Let $\mathbf{F}=(\mathcal{F}_t)_{0\le t\le T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\le t\le T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{0\le t\le T}$ satisfies the usual conditions, i.e., it is right continuous and complete. On the other hand let $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathbf{F}$-progressively measurable sets.

Next, let

(i) $\mathcal{H}^{2,\ell}$ $(\ell\ge 1)$ be the set of $\mathcal{P}$-measurable and $\mathbb{R}^\ell$-valued processes $w=(w_t)_{t\le T}$ such that $\mathbb{E}[\int_0^T|w_s|^2ds]<\infty$;

(ii) $\mathcal{S}^2$ (resp. $\mathcal{S}_d^2$) be the set of $\mathcal{P}$-measurable continuous (resp. RCLL) processes such that $\mathbb{E}[\sup_{t\le T}|w_t|^2]<\infty$.

(iii) $\mathcal{A}_i^2$ be the subset of $\mathcal{S}^2$ of non-decreasing processes $K=(K_t)_{t\le T}$ such that $K_0=0$.

For $(t,x)\in[0,T]\times\mathbb{R}^k$, let $X^{t,x}$ be the diffusion process solution of the following standard SDE:

$$\forall s\in[t,T],\ \ X_s^{t,x}=x+\textstyle\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dB_r;\ \ X_s^{t,x}=x,\ s\in[0,t]. \qquad (4.1.6)$$

Under Assumption (H0) on b and σ, the process $X^{t,x}$ exists and is unique. Moreover, it satisfies the following estimates: For all $p\ge 1$,

$$\mathbb{E}[\sup_{s\le T}|X_s^{t,x}|^p]\le C(1+|x|^p). \qquad (4.1.7)$$

Its infinitesimal generator $\mathcal{L}^X$ is given, for every $(t,x)\in[0,T]\times\mathbb{R}^k$ and $\phi\in\mathcal{C}^{1,2}$, by

$$\begin{array}{rcl}\mathcal{L}^X\phi(t,x) & := & \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^*(t,x))_{i,j}\partial^2_{x_ix_j}\phi(t,x)+\sum_{i=1,k}b_i(t,x)\partial_{x_i}\phi(t,x)\\ & = & \frac{1}{2}Tr[\sigma\sigma^\top(t,x)D^2_{xx}\phi(t,x)]+b(t,x)^\top D_x\phi(t,x).\quad\square\end{array} \qquad (4.1.8)$$

Under Assumptions (H0)-(H4), we have

Theorem 4.1.1. *([17], Theorems 5.4 and 5.5) There exists a unique continuous viscosity solution in the class* Π_g $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *(resp.* $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$*) of the following system:* $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\left\{\begin{array}{l}\min\Big\{(\bar{v}^{ij}-L^{ij}(\vec{v}))(t,x),\max\Big\{(\bar{v}^{ij}-U^{ij}(\vec{v}))(t,x),\\ -\partial_t\bar{v}^{ij}(t,x)-\mathcal{L}^X(\bar{v}^{ij})(t,x)-f^{ij}(t,x,(\bar{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\bar{v}^{ij}(t,x))\Big\}\Big\}=0,\\ \bar{v}^{ij}(T,x)=h^{ij}(x)\end{array}\right. \qquad (4.1.9)$$

(resp.

$$\left\{\begin{array}{l}\max\Big\{(\underline{v}^{ij}-U^{ij}(\vec{\underline{v}}))(t,x);\min\Big\{(\underline{v}^{ij}-L^{ij}(\vec{\underline{v}}))(t,x);\\ -\partial_t\underline{v}^{ij}(t,x)-\mathcal{L}^X(\underline{v}^{ij})(t,x)-f^{ij}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\underline{v}^{ij}(t,x))\Big\}\Big\}=0,\\ \underline{v}^{ij}(T,x)=h^{ij}(x).\end{array}\right. \qquad (4.1.10)$$

)

In order to obtain the solutions of the systems (4.1.9) and (4.1.10) Djehiche et al. ([17]) introduce the following sequences of backward reflected BSDEs with inter-connected obstacles: $\forall m,n \geq 0$, $\forall (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\left\{\begin{array}{l} \hat{Y}^{ij,m} \in \mathcal{S}^2,\ \hat{Z}^{ij,m} \in \mathcal{H}^{2,d} \text{ and } \hat{K}^{ij,m} \in \mathcal{A}_i^2\ ; \\ \hat{Y}_s^{ij,m} = h^{ij}(X_T^{t,x}) + \int_s^T \hat{f}^{ij,m}(r, X_r^{t,x}, (\hat{Y}_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2}, \hat{Z}_r^{ij,m})dr + \int_s^T d\hat{K}_s^{ij,m} - \int_s^T \hat{Z}_r^{ij,m} dB_r,\ s \leq T; \\ \hat{Y}_s^{ij,m} \geq \max\limits_{k\in(\Gamma^1)^{-i}} \{\hat{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\},\ s \leq T; \\ \int_0^T (\hat{Y}_s^{ij,m} - \max_{k\in(\Gamma^1)^{-i}}\{\hat{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\}) d\hat{K}_s^{ij,m} = 0 \end{array}\right. \tag{4.1.11}$$

and

$$\left\{\begin{array}{l} Y^{ij,n} \in \mathcal{S}^2,\ Z^{ij,n} \in \mathcal{H}^{2,d} \text{ and } K^{ij,n} \in \mathcal{A}_i^2\ ; \\ Y_s^{ij,n} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij,n}(r, X_r^{t,x}, (Y_r^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_r^{ij,n})dr - \int_s^T Z_r^{ij,n} dB_r - \int_s^T dK_r^{ij,n},\ s \leq T; \\ Y_s^{ij,n} \leq \min_{l\in(\Gamma^2)^{-j}} \big(Y_s^{il,n} + \bar{g}_{jl}(s, X_s^{t,x})\big),\ s \leq T; \\ \int_0^T (Y_s^{ij,n} - \min_{l\in(\Gamma^2)^{-j}} \big(Y_s^{il,n} + \bar{g}_{jl}(s, X_s^{t,x})\big)) dK_s^{ij,n} = 0 \end{array}\right. \tag{4.1.12}$$

where, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $n,m \geq 0$ and $(s,x,\vec{y},z^{ij})$,

$$\hat{f}^{ij,m}(s,x,\vec{y},z^{ij}) := f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z^{ij}) - m\big(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}(y^{il} + \bar{g}_{jl}(s,x))\big)^+ \tag{4.1.13}$$

and

$$f^{ij,n}(s,x,\vec{y},z^{ij}) := f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z^{ij}) + n\big(y^{ij} - \max_{k\in(\Gamma^1)^{-i}}(y^{kj} - \underline{g}_{ik}(s,x))\big)^-. \tag{4.1.14}$$

Under Assumptions (H0)-(H4) it is shown in [30] (see also [11] or [32]) that each one of the systems (4.1.11) and (4.1.12) has a unique solution $(\hat{Y}^{ij,m}, \hat{Z}^{ij,m}, \hat{K}^{ij,m})$ and $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})$ respectively. In addition, they enjoy the following properties:

(i) For any $m,n \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$

$$\hat{Y}^{ij,m} \geq \hat{Y}^{ij,m+1} \geq Y^{ij,n+1} \geq Y^{ij,n}. \tag{4.1.15}$$

(ii) For any $n,m \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$ there exist deterministic continuous functions $\hat{v}^{ij,m}$ and $v^{ij,n}$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and $s \in [t,T]$, we have

$$\hat{Y}_s^{ij,m} = \hat{v}^{ij,m}(s, X_s^{t,x}) \quad \text{and} \quad Y_s^{ij,n} = v^{ij,n}(s, X_s^{t,x}).$$

Moreover, from (4.1.15) we easily deduce that, for any $n,m \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\hat{v}^{ij,m} \geq \hat{v}^{ij,m+1} \geq v^{ij,n+1} \geq v^{ij,n}. \tag{4.1.16}$$

Finally, for any $m \geq 0$ (resp. $n \geq 0$), $\hat{v}_m := (\hat{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $v_n := (v^{ij,n})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is the unique continuous viscosity solution, in the class Π_g, of the following system of PDEs with inter-connected obstacles: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$, $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,

$$\left\{\begin{array}{l} \min\Big\{(\hat{v}^{ij,m} - L^{ij}(\vec{\hat{v}}_m)(t,x); -\partial_t \hat{v}^{ij,m} - \mathcal{L}^X(\hat{v}^{ij,m})(t,x) \\ -\hat{f}^{ij,m}(t,x,(\hat{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x \hat{v}^{ij,m}(t,x))\Big\} = 0, \\ \hat{v}^{ij,m}(T,x) = h^{ij}(x) \end{array}\right.$$

(resp.

$$\left\{\begin{array}{l} \max\Big\{(v^{ij,n} - U^{ij}(\vec{v}_n))(t,x); -\partial_t v^{ij,n}(t,x) - \mathcal{L}^X(v^{ij,n})(t,x) \\ -f^{ij,n}(t,x,(v^{kl,n}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x) D_x v^{ij,n}(t,x))\Big\} = 0, \\ v^{ij,n}(T,x) = h^{ij}(x)). \end{array}\right.$$

(iii) For $(i,j) \in \Gamma^1 \times \Gamma^2$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, let us set

$$\bar{v}^{ij}(t,x) := \lim_{m\to\infty} \searrow \hat{v}^{ij,m}(t,x) \quad \text{and} \quad \underline{v}^{ij}(t,x) := \lim_{n\to\infty} \nearrow v^{ij,n}(t,x).$$

Then, using Perron's method, it is shown that $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is continuous, belongs to Π_g and is the unique viscosity solution, in class Π_g, of system (4.1.9) (resp. (4.1.10)). Finally, by construction and in view of (4.1.16), it holds that, for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\underline{v}^{ij} \leq \bar{v}^{ij}. \tag{4.1.17}$$

4.2 Equality of min-max and max-min solutions

In [17], the question whether or not for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $\underline{v}^{ij} \equiv \bar{v}^{ij}$ was left open. This was mainly due to the fact we have not been able to compare the inter-connected obstacles neither in (4.1.9) nor in (4.1.10).

Actually, had we known that

$$\begin{array}{ll} \text{(i)} & \forall (i,j) \in \Gamma^1 \times \Gamma^2, \quad L^{ij}(\vec{v}) \leq U^{ij}(\vec{v}) \\ \text{or} & \\ \text{(ii)} & \forall (i,j) \in \Gamma^1 \times \Gamma^2, \quad L^{ij}(\underline{\vec{v}}) \leq U^{ij}(\underline{\vec{v}}) \end{array} \tag{4.2.1}$$

then we would have deduced from Theorem 5.5.1 in Appendix and the uniqueness of the solution of (4.1.9) or (4.1.10) that for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $\bar{v}^{ij} = \underline{v}^{ij}$. In this section we are going to investigate under which additional regularity assumptions on the data of the problem, one of the inequalities in (4.2.1) is satisfied to be able to conclude that $\bar{v}^{ij} = \underline{v}^{ij}$, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, i.e., the solutions of (4.1.9) and (4.1.10) are the same.

For that let us introduce the following assumption.
(H5):

(i) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the functions $\bar{g}_{ij}$ are $\mathcal{C}^{1,2}$. Moreover, $D_x \bar{g}_{ij}$ and $D^2_{xx} \bar{g}_{ij}$ belong to Π_g. Furthermore, for any $j_1, j_2, j_3 \in \Gamma_2$ such that $|\{j_1, j_2, j_3\}| = 3$,

$$\bar{g}_{j_1 j_3}(t,x) < \bar{g}_{j_1 j_2}(t,x) + \bar{g}_{j_2 j_3}(t,x), \ \forall (t,x) \in [0,T] \times I\!R^k.$$

(ii) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} verifies the following estimate:

$$|f^{ij}(t,x,\vec{y},z^{ij})| \leq C(1+|x|^p)$$

for some real constants C and p.

Remark 4.2.1. *By Itô's formula, for ant $(i,j) \in \Gamma^1 \times \Gamma^2$,*

$$\begin{array}{l} \bar{g}_{ij}(s, X_s^{t,x}) = \bar{g}_{ij}(t,x) + \int_t^S \mathcal{L}^X(\bar{g}_{ij})(s, X_s^{t,x}) ds + \int_t^s D_x \bar{g}_{ij}(s, X_s^{t,x}) \sigma(s, X_s^{t,x}) dB_s, \quad s \in [t,T] \\ \bar{g}_{ij}(s, X_s^{t,x}) = \bar{g}_{ij}(s,x), \quad s \leq t. \end{array}$$

Hereafter, we denote by

$$a^{ij}(s) := \mathcal{L}^X(\bar{g}_{ij})(s, X_s^{t,x}), \quad b^{ij}(s) := D_x \bar{g}_{ij}(s, X_s^{t,x}) \sigma(s, X_s^{t,x}), \quad s \leq T.$$

Proposition 4.2.1. *Under Assumptions (H0)-(H5) we have, for every $(i,j) \in \Gamma^1 \times \Gamma^2$,*

$$L^{ij}(\vec{v}) \leq \bar{v}^{ij} \leq U^{ij}(\vec{v}).$$

Proof. We derive this inequality through the following three steps.

Step 1: For any $m \geq 0$, $(i,j) \in \Gamma^1 \times \Gamma^2$ and $(t,x) \in [0,T] \times I\!R^k$, let us consider the system of reflected BSDEs with one inter-connected obstacles:

$$\left\{ \begin{array}{l} \check{Y}^{ij,m} \in \mathcal{S}^2, \ \check{Z}^{ij,m} \in \mathcal{H}^{2,d} \text{ and } \check{K}^{ij,m} \in \mathcal{A}_i^2 \ ; \\ \check{Y}_s^{ij,m} = h^{ij}(X_T^{t,x}) + \int_s^T \check{f}^{ij,m}(r, X_r^{t,x}, (\check{Y}_r^{kl,m})_{(k,l) \in \Gamma^1 \times \Gamma^2}, \check{Z}_r^{ij,m}) dr + \int_s^T d\check{K}_s^{ij,m} - \int_s^T \check{Z}_r^{ij,m} dB_r, \ s \leq T; \\ \check{Y}_s^{ij,m} \geq \max\limits_{k \in (\Gamma^1)^{-i}} \{\check{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\}, s \leq T; \\ \int_0^T (\check{Y}_s^{ij,m} - \max_{k \in (\Gamma^1)^{-i}} \{\hat{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\}) d\check{K}_s^{ij,m} = 0, \end{array} \right. \tag{4.2.2}$$

where,

$$\check{f}^{ij,m}(s,x,\vec{y},z^{ij}) := f^{ij}(s,x,\vec{y},z^{ij}) - m \sum_{l \in (\Gamma^2)^{-j}} (y^{ij} - y^{il} - \bar{g}_{jl}(s,x))^+. \tag{4.2.3}$$

By Corollary 2, in [30], the solution of this system exists and is unique and there exist deterministic continuous functions $(\check{v}^{ij,m})_{(i,j) \in \Gamma^1 \times \Gamma^2}$, which belong also to Π_g such that, for any i,j and $m \geq 0$, it holds that

$$\forall s \in [t,T], \quad \check{Y}_s^{ij,m} = \check{v}^{ij,m}(s, X_s^{t,x}).$$

Moreover, the family of functions $\check{v}_m := (\check{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique continuous solution in viscosity sense in Π_g of the following system of PDEs with obstacles:

$$\begin{cases} \min\Big\{(\check{v}^{i,j,m} - L^{i,j}(\vec{\check{v}}_m))(t,x); \\ \qquad -\partial_t\check{v}^{i,j,m}(t,x) - \mathcal{L}^X(\check{v}^{i,j,m})(t,x) - \check{f}^{ij,m}(t,x,(\check{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x\check{v}^{i,j,m}(t,x))\Big\} = 0, \\ \check{v}^{ij,m}(T,x) = h^{ij}(x). \end{cases}$$

Finally, by the Comparison Theorem (see [30], Remark 1), since $\check{f}^{ij,m+1} \le \check{f}^{ij,m}$ and $\hat{f}^{ij,|\Gamma_2|m} \le \check{f}^{ij,m} \le \hat{f}^{ij,m}$, we have, $\forall (i,j)\in\Gamma^1\times\Gamma^2$ and $m\ge 0$,

$$\check{Y}^{ij,m+1} \le \check{Y}^{ij,m} \quad \text{and} \quad \hat{Y}^{ij,|\Gamma_2|m} \le \check{Y}^{ij,m} \le \hat{Y}^{ij,m},$$

which implies that, for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $m\ge 0$,

$$\check{v}^{ij,m+1} \le \check{v}^{ij,m} \text{ and } \hat{v}^{ij,|\Gamma_2|m} \le \check{v}^{ij,m} \le \hat{v}^{ij,m}.$$

Then, for any $(i,j)\in\Gamma^1\times\Gamma^2$, the sequence $(\check{v}^{ij,m})_{m\ge0}$ is decreasing and converges, uniformly on compact subsets of $[0,T]\times I\!R^k$, to $\bar{v}^{ij}$ since $\lim_{m\to\infty}\hat{v}^{ij,m}(t,x) = \bar{v}^{ij}(t,x)$, for any $(t,x)\in[0,T]\times I\!R^k$.

Step 2: The following estimate holds: For every $(i,j)\in\Gamma^1\times\Gamma^2$ and $m\ge0$,

$$\mathbb{E}\Big\{m\int_0^T \sum_{l\in(\Gamma^2)^{-j}} \{\check{Y}_s^{ij,m} - \check{Y}_s^{il,m} - \bar{g}_{jl}(s,X_s^{t,x})\}^+ ds\Big\} \le C(1+|x|^p), \tag{4.2.4}$$

where, the constant C is independent of m and x.

We first give a representation of $\check{Y}^{ij,m}$ as the optimal payoff of a switching problem. Indeed, let $\delta := (\tau_n,\zeta_n)_{n\ge0}$ be an admissible strategy of switching, i.e.,

(a) $(\tau_n)_{n\ge0}$ is an increasing sequence of stopping times such that $P[\tau_n < T, \forall n\ge0] = 0$;
(b) $\forall n\ge0$, ζ_n is a random variable with values in Γ^1 and $\mathcal{F}_{\tau_n}$-measurable;
(c) Let $(A_s^\delta)_{s\le T}$ be the RCLL $\mathcal{F}_t$-adapted process defined by

$$\forall s\in[0,T), \quad A_s^\delta = \sum_{n\ge1} g_{\zeta_{n-1}\zeta_n}(\tau_n, X_{\tau_n}^{t,x})\mathbb{1}_{\{\tau_n\le s\}} \quad \text{and} \quad A_T^a = \lim_{s\to T} A_s^a.$$

Then, $\mathbb{E}[(A_T^a)^2] < \infty$. The quantity A_T^δ stands for the switching cost at terminal time T when the strategy δ is implemented.
Next, with an admissible strategy $\delta := (\tau_n,\zeta_n)_{n\ge0}$ we associate a piecewise constant process $a = (a_s)_{s\in[0,T]}$ defined by

$$a_s := \zeta_0\mathbb{1}_{\{\tau_0\}}(s) + \sum_{j=1}^\infty \zeta_{j-1}\mathbb{1}_{]\tau_{j-1},\tau_j]}(s), \quad s\le T. \tag{4.2.5}$$

For any $s\ge\tau_0$, a_s is the mode indicator at time s. Note that there is a bijection between the processes a and the admissible strategies δ, therefore hereafter A^a is nothing else but A^δ.
Finally, for any fixed $i\in\Gamma^1$ and a real constant $\theta\in[0,T]$, we denote by $\mathcal{A}_\theta^i$ the following set:

$$\mathcal{A}_\theta^i := \Big\{\delta = (\tau_n,\zeta_n)_{n\ge0} \text{ admissible strategy such that } \tau_0 = \theta \text{ and } \zeta_0 = i\Big\}.$$

Now, for an admissible strategy $\delta = (\tau_n,\alpha_n)_{n\ge0}$, or equivalently a, let us define the pair of processes $(\check{U}^{aj,m},\check{V}^{aj,m})$ which belongs to $\mathcal{S}_d^2\times\mathcal{H}^{2,d}$ solution of the following BSDE (which is of non standard form): For every $s\le T$,

$$\check{U}_s^{aj,m} = h^{a(T)j}(X_T) + \int_s^T 1_{\{r\ge\tau_0\}}\check{f}^{aj,m}(r,X_r^{t,x},\check{U}_r^{aj,m},\check{V}_r^{aj,m})dr - \int_s^T \check{V}_r^{aj,m}dB_r - \big(A_T^a - A_s^a\big), \tag{4.2.6}$$

where, for any $s\ge\tau_0$ and $(\bar{y},\bar{z})\in I\!R^{1+d}$, $\check{f}^{aj,m}(s,X_s^{t,x},\bar{y},\bar{z})$ (resp. $\check{f}^{aj}(s,X_s^{t,x},\bar{y},\bar{z})$) is equal to

$$\check{f}^{\ell j,m}(s,X_s^{t,x},[(\check{v}^{kl,m}(s,X_s^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2-\{(\ell,j)\}},\bar{y}],\bar{z})$$

(resp.
$$\check{f}^{\ell j}(s, X_s^{t,x}, [(\check{v}^{kl,m}(s, X_s^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2-\{(\ell,j)\}}, \bar{y}], \bar{z}))$$
if at time s, $a(s) = \ell$. Let us point out that since a is admissible, the solution of equation (4.2.6) exists and is unique. Furthermore, we have the following representation of $\check{Y}^{ij,m}$ (see e.g. [30, 35] for more details on this representation):

$$\check{Y}_\theta^{ij,m} = \text{ess sup}_{a\in\mathcal{A}_\theta^i}\{\check{U}_\theta^{a,j,m} - A_\theta^a\}, \quad \theta \le T. \tag{4.2.7}$$

Note that even though the function $\check{f}^{aj,m}$ depend on y^{kl}, $(k,l) \neq (i,j)$, the representation (4.2.7) still holds since the solution of system of reflected BSDEs (4.2.2) is unique. It follows that, for any $j, l \in \Gamma^2$ and $\theta \le T$,

$$(\check{Y}_\theta^{ij,m} - \check{Y}_\theta^{il,m} - \bar{g}_{jl}(\theta, X_\theta^{t,x}))^+ \le \text{ess sup}_{a\in\mathcal{A}_\theta^i}(\check{U}_\theta^{aj,m} - \check{U}_\theta^{al,m} - \bar{g}_{jl}(\theta, X_\theta^{t,x}))^+. \tag{4.2.8}$$

We now examine $(\check{U}_\theta^{aj,m} - \check{U}_\theta^{al,m} - \bar{g}_{jl}(\theta, X_\theta^{t,x}))^+$. Define the set $\mathcal{B}_{jl}$ as follows:

$$\mathcal{B}_{jl} = \{(s,\omega) \in [0,T]\times\Omega, \quad \text{such that } \check{U}_s^{aj,m} - \check{U}_s^{al,m} - \bar{g}_{jl}(s, X_s^{t,x}) > 0\}$$

and, for any $s \in [0,T]$,
$$W_s^{a,jl,m} := \check{U}_s^{aj,m} - \check{U}_s^{al,m} - \bar{g}_{jl}(s, X_s^{t,x}).$$

Then, by Itô-Tanaka's formula, we have, for every $s \in [\theta, T]$,

$$\begin{aligned}
&(W_s^{a,jl,m})^+ + \tfrac{1}{2}\int_s^T dL_r^{a,jl,m} + m\int_s^T dr\{\textstyle\sum_{j''\neq j}\mathbf{1}_{\mathcal{B}_{j,l}(r)}(W_r^{a,jj'',m})^+ - \sum_{j''\neq l}\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj'',m})^+\}\\
&\quad = \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(\check{f}^{aj}(r, X_r^{t,x}, \check{U}_r^{aj,m}) - \check{f}^{al}(r, X_r^{t,x}, U_r^{al,m}) - a_r^{jl})dr\\
&\qquad - \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)\big(V_r^{aj,m} - V_r^{al,m} - b_r^{jl}\big)dB_r
\end{aligned}$$

where, $L^{a,jl,m}$ is the local time at 0 of the semimartingale $W^{a,jl,m}$. Splitting the difference

$$\Delta_{a,j,l,m}(r) := m\sum_{j''\neq j}\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,jj'',m})^+ - m\sum_{j''\neq l}\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj'',m})^+$$

as

$$\Delta_{j,l,m}(r) = m\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,jl,m})^+ - \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj,m})^+ + m\sum_{j''\neq j, j''\neq l}\mathbf{1}_{\mathcal{B}_{j,l}}(r)\big((W_r^{a,jj'',m})^+ - (W_r^{a,lj'',m})^+\big),$$

the previous formula can be rewritten as follows: $\forall s \in [\theta, T]$,

$$\begin{aligned}
&(W_s^{a,jl,m})^+ + \tfrac{1}{2}\int_s^T dL_r^{a,jl,m} + m\int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,jl,m})^+dr\\
&\quad = \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(\check{f}^{aj}(r, X_r^{t,x}, \check{U}_r^{aj,m}, \check{V}_r^{aj,m}) - \check{f}^{al}(r, X_r^{t,x}, \check{U}_r^{al,m}, \check{V}_r^{al,m}) - a_r^{jl})dr\\
&\quad + m\int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj,m})^+dr - \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)\big(\check{V}_r^{aj,m} - \check{V}_r^{al,m} - b_r^{jl}\big)dB_r\\
&\quad - m\int_s^T dr\{\textstyle\sum_{j''\neq j,l}\mathbf{1}_{\mathcal{B}_{j,l}(r)}[(W_r^{a,j'',m})^+ - (W_r^{a,lj",m})^+]\}
\end{aligned} \tag{4.2.9}$$

But, $\bar{g}_{jl}(t,x) + \bar{g}_{lj}(t,x) > \bar{g}_{jj}(t,x) = 0$. Thus, we obtain that, for every $(t,x) \in [0,T]\times\mathbb{R}^k$,

$$\{y \in \mathbb{R}^m,\ y_j - y_l - \bar{g}_{j,l}(t,x) \ge 0\} \cap \{y \in \mathbb{R}^m,\ y_l - y_j - \bar{g}_{l,j}(t,x) \ge 0\} = \emptyset,$$

from which we deduce that
$$\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj,m})^+ = 0,\ \forall r \in [\theta, T]. \tag{4.2.10}$$

Relying next on the elementary inequality $a^+ - b^+ \le (a-b)^+$, it holds

$$\mathbf{1}_{\mathcal{B}_{j,l}(r)}[(W_r^{a,jj",m})^+ - (W_r^{a,lj",m})^+] \le \mathbf{1}_{\mathcal{B}_{j,l}}(r)\left(\check{U}_r^{al,m} - \check{U}_r^{aj,m} - \bar{g}_{lj''}(r, X_r^{t,x}) + \bar{g}_{jj''}(r, X_r^{t,x})\right)^+. \tag{4.2.11}$$

Using here that the family of penalty costs satisfies $\bar{g}_{j,j^{"}} < \bar{g}_{jl} + \bar{g}_{lj^{"}}$ we deduce that

$$\{y \in \mathbb{R}^m,\ y_j - y_l - \bar{g}_{jl}(t,x) \geq 0\} \cap \{y \in \mathbb{R}^m,\ y_l - y_j - \bar{g}_{lj''}(t,x) + \bar{g}_{jj''}(t,x) \geq 0\} = \emptyset$$

which therefore yields

$$\forall r \in [\theta, T],\ \mathbf{1}_{\mathcal{B}_{j,l}}(r)\left(\check{U}_r^{al,m} - \check{U}_r^{aj,m} - \bar{g}_{lj''}(r, X_r^{t,x}) + \bar{g}_{jj''}(r, X_r^{t,x})\right)^+ = 0. \tag{4.2.12}$$

Going back now to (4.2.9), applying Itô's formula to $e^{-ms}(W_s^{a,jl,m})^+$ and taking into account of (4.2.10), (4.2.11) and (4.2.12) to obtain: $\forall s \in [\theta, T]$,

$$(W_s^{a,jl,m})^+ \leq \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)e^{-m(r-s)}(\check{f}^{aj}(r, X_r^{t,x}, \check{U}_r^{aj,m}, \check{V}_r^{aj,m}) - \check{f}^{al}(r, X_r^{t,x}, \check{U}_r^{al,m}, \check{V}_r^{al,m}) - u_r^{jl})dr$$

$$- \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)e^{-m(r-s)}\left(\check{V}_r^{aj,m} - \check{V}_r^{al,m} - v_r^{jl}\right)dB_r.$$

Now in taking the conditional expectation and making use of estimates of Assumptions (H0)-(H5) (namely the polynomial growth of the functions) we obtain: $\forall s \in [\theta, T]$,

$$\begin{aligned}(W_s^{a,jl,m})^+ &\leq C\mathbb{E}[\int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)e^{-m(r-s)}(1 + |X_r^{t,x}|^p)dr|\mathcal{F}_s] \\ &\leq \tfrac{C}{m}\mathbb{E}[(1 + \sup_{r\leq T}|X_r^{t,x}|^p)|\mathcal{F}_s].\end{aligned}$$

Recall now (4.2.8) to obtain

$$m(\check{Y}_\theta^{ij,m} - \check{Y}_\theta^{il,m} - \bar{g}_{j,l}(\theta, X_\theta^{t,x}))^+ \leq C\mathbb{E}[(1 + \sup_{r\leq T}|X_r^{t,x}|^p)|\mathcal{F}_\theta] \tag{4.2.13}$$

and then in taking into account estimate (4.1.7) on $X^{t,x}$ we obtain

$$m\mathbb{E}\Big\{\sum_{l\neq j}(\check{Y}_\theta^{ij,m} - \check{Y}_\theta^{il,m} - \bar{g}_{j,l}(\theta, X_\theta^{t,x}))^+\Big\} \leq C(1 + |x|^p), \quad \forall \theta \leq T.$$

As θ is arbitrary in $[0,T]$ then by integration with respect to $d\theta$ in the previous inequality we obtain (4.2.4).

Step 3 : For any $(t_0, x_0) \in [0,T] \times I\!R^k$ and $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$L^{ij}(\vec{\bar{v}})(t_0,x_0) \leq \bar{v}^{ij}(t_0,x_0) \leq U^{ij}(\vec{\bar{v}})(t_0,x_0).$$

We first claim that $\bar{v}^{ij}(t_0,x_0) \geq L^{ij}(\vec{\bar{v}})(t_0,x_0)$ holds. Indeed, by construction of $\hat{v}^m := (\hat{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ one has $\hat{v}^{ij,m}(t_0,x_0) \geq L^{ij}(\vec{\hat{v}}_m)(t_0,x_0)$. Therefore, taking the limit w.r.t. m, we obtain $\bar{v}^{ij}(t_0,x_0) \geq L^{ij}(\vec{\bar{v}})(t_0,x_0)$.

We now show that $\bar{v}^{ij}(t_0,x_0) \leq U^{ij}(\vec{\bar{v}})(t_0,x_0)$. First, assume that $\bar{v}^{ij}(t_0,x_0) > L^{ij}(\vec{\bar{v}})(t_0,x_0)$. Then, relying on the viscosity subsolution property of $\bar{v}^{ij}$ yields

$$\begin{aligned}\min\Big\{(\bar{v}^{ij} - L^{ij}(\vec{\bar{v}}))(t_0,x_0); \max\Big\{&(\bar{v}^{ij} - U^{ij}(\vec{\bar{v}}))(t_0,x_0); \\ &-\partial_t\bar{v}^{ij}(t_0,x_0) - \mathcal{L}^X(\bar{v}^{ij})(t_0,x_0) - f^{ij}(t_0,x_0,(\bar{v}^{kl}(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t_0,x_0)^\top D_x\bar{v}^{ij}(t_0,x_0))\Big\}\Big\} \leq 0,\end{aligned}$$

which implies that

$$\max\Big\{(\bar{v}^{ij} - U^{ij})(t_0,x_0); -\partial_t\bar{v}^{ij}(t_0,x_0) - \mathcal{L}^X(\bar{v}^{ij}) - f^{ij}(t_0,x_0,(\bar{v}^{kl}(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2}))\Big\} \leq 0.$$

Hence, $(\bar{v}^{ij} - U^{ij}(\vec{\bar{v}}))(t_0,x_0) \leq 0$.

Suppose now that at (t_0,x_0) we have $\bar{v}^{ij}(t_0,x_0) = L^{ij}(\vec{\bar{v}})(t_0,x_0)$. Proceeding by contradiction we suppose in addition that

$$\exists\, \epsilon > 0, \quad (\bar{v}^{ij} - U^{ij}(\vec{\bar{v}}))(t_0,x_0) > \epsilon. \tag{4.2.14}$$

Using both the continuity of $(t,x) \mapsto \bar{v}^{i,j}(t,x)$ and $(t,x) \mapsto U^{ij}(\vec{\bar{v}})(t,x)$ and the uniform convergence on compact subsets of $(\check{v}^{ij,m})_m$ to $\bar{v}^{ij}$ we claim that for some strictly positive ρ and for m_0 large enough it holds that

$$\forall m \geq m_0,\ \forall (t,x) \in \mathcal{B}((t_0,x_0),\rho), \quad (\check{v}^{ij,m} - U^{ij}(\vec{\check{v}}_m))(t,x) \geq \frac{\epsilon}{2},$$

with $\mathcal{B}((t_0,x_0),\rho) = \{(t,x) \in [0,T] \times I\!R^k \text{ s.t. } |t-t_0| \leq \rho,\ |x-x_0| \leq \rho\}$.
Without loss of generality we can now assume $[t_0, t_0+\rho] \subset [t_0,T]$. By the definition of $U^{ij}(\vec{\check{v}}_m)$, there exists one index $l_0 \neq j$ such that the inequalities

$$\check{v}^{ij,m} - (\check{v}^{il_0,m} + \bar{g}_{jl_0}) \geq \frac{\epsilon}{2}$$

and

$$\sum_{l \in (\Gamma^2)^{-j}} \big(\check{v}^{ij,m} - \check{v}^{il,m} - \bar{g}_{jl}\big)^+ \geq \frac{\epsilon}{2}, \tag{4.2.15}$$

hold on the ball $\mathcal{B}((t_0,x_0),\rho)$.

Let us now introduce the following stopping time τ_X:

$$\tau_X = \inf\{s \geq t_0,\ X_s^{t_0,x_0} \notin \mathcal{B}((t_0,x_0),\rho)\} \wedge \big(t_0+\rho\big).$$

We then have, for all $m \geq m_0$,

$$m\mathbb{E}\left(\int_{t_0}^{\tau_X} \sum_{l \neq j} (\check{v}^{ij,m}(s, X_s^{t_0,x_0}) - \big(\check{v}^{il,m}(s, X_s^{t_0,x_0}) + \bar{g}_{jl}(s, X_s^{t_0,x_0})\big)^+ ds\right) \geq m\frac{\epsilon}{2}\mathbb{E}(\tau_X - t_0) \to \infty, \tag{4.2.16}$$

as $m \to \infty$. But, this is contradictory to (4.2.4). Then $\bar{v}^{ij}(t_0,x_0) \leq U^{ij}(\vec{\bar{v}}))(t_0,x_0)$ and the proof is complete. □

As a by product of Proposition 4.2.1 and Theorem 5.5.2 (displayed in the appendix), we have:

Theorem 4.2.1. *Under Assumptions (H0)-(H5), for any $(i,j) \in \Gamma^1 \times \Gamma^2$, it holds that*

$$\bar{v}^{ij} = \underline{v}^{ij}.$$

Remark 4.2.2. *(i) The result of Theorem 5.5.2 (see the appendix) is still valid if (H0)-(H4) are in force and the functions $(\underline{g}_{ij})_{(i,j) \in \Gamma^1 \times \Gamma^2}$ verify (H5).*
(ii) From (4.2.13) and Doob's inequality we have, for every $(t,x) \in [0,T] \times I\!R^k$,

$$m^2\mathbb{E}\Big\{\sum_{l \neq j} [(\check{Y}_s^{ij,m} - \check{Y}_s^{il,m} - \bar{g}_{jl}(s, X_s^{t,x}))^+]^2\Big\} \leq C(1+|x|^{2p}), \quad s \leq T, \tag{4.2.17}$$

where, C is a constant. □

4.3 The min-max solution as a the value of the zero-sum switching game

Let us consider now the following assumption which is used later:

(H6):

(**i**) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} does not depend on z^{ij}.

(**ii**) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} does not depend on $(\vec{y}, z^{ij})$. □

Once for all, in this section we suppose that Assumptions (H0)-(H5) hold.

Set

$$Y_s^{ij} = v^{ij}(s, X_s^{t,x}), \quad s \in [t,T] \text{ and } (i,j) \in \Gamma^1 \times \Gamma^2.$$

We then have the following representation of Y^{ij} as the value function of a Dynkin game. This is a by-product of Theorems 4.1.1, 5.5.2 and Propositions 4.2.1 and 5.5.10 (displayed in the appendix) since the barriers

$$L^{ij}(\vec{v}) = \max_{k\neq i}\left(v^{kj} - \underline{g}_{ik}\right) \text{ and } U^{ij}(\vec{v}) = \min_{l\neq j}\left(v^{il} + \bar{g}_{jl}\right)$$

are comparable, i.e., $L^{ij}(\vec{v}) \leq U^{ij}(\vec{v})$ for any i, j.

Proposition 4.3.1. *Assume that Assumptions (H0)-(H5) and (H6)-(i) are fulfilled. For any $(i,j) \in \Gamma^1 \times \Gamma^2$ and $s \in [t,T]$ we have,*

$$\begin{aligned} v^{ij}(s, X_s^{t,x}) &= Y_s^{ij} \\ &= ess\ sup_{\sigma\geq s} ess\ inf_{\tau\geq s} \mathbb{E}\Big\{ \int_s^{\sigma\wedge\tau} f^{ij}(r, X_r^{t,x}, (v^{kl}(r, X_r^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2})dr \\ &+\{\max_{k\in(\Gamma^1)^{-i}}\{v^{kj}(\sigma, X_\sigma^{t,x}) - \underline{g}_{ik}(\sigma, X_\sigma^{t,x})\}\}\mathbb{1}_{[\sigma<\tau]} \\ &\quad + \min_{l\in(\Gamma^2)^{-j}}\{v^{il}(\tau, X_\tau^{t,x}) + \bar{g}_{jl}(\tau, X_\tau^{t,x})\}\mathbb{1}_{[\tau\leq\sigma<T]} \\ &\quad +h_{ij}(X_T^{t,x})\mathbb{1}_{[\tau=\sigma=T]}|\mathcal{F}_s\Big\} \\ &= ess\ inf_{\tau\geq s} ess\ sup_{\sigma\geq s} \mathbb{E}\Big\{ \int_s^{\sigma\wedge\tau} f^{ij}(r, X_r^{t,x}, (v^{kl}(r, X_r^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2})dr \\ &+\{\max_{k\in(\Gamma^1)^{-i}}\{v^{kj}(\sigma, X_\sigma^{t,x}) - \underline{g}_{ik}(\sigma, X_\sigma^{t,x})\}\}\mathbb{1}_{[\sigma<\tau]} + \\ &\quad \min_{l\in(\Gamma^2)^{-j}}\{v^{il}(\tau, X_\tau^{t,x}) + \bar{g}_{jl}(\tau, X_\tau^{t,x})\}\mathbb{1}_{[\tau\leq\sigma<T]} \\ &\quad +h_{ij}(X_T^{t,x})\mathbb{1}_{[\tau=\sigma=T]}|\mathcal{F}_s\Big\}. \quad \square \end{aligned} \tag{4.3.1}$$

On the other hand, it is shown in ([28], Theorem 3.1) , that Y^{ij} is the unique local solution of the two barriers reflected BSDEs associated with $(f^{ij}(s, X_s^{t,x}, \vec{y})$, $h_{ij}(X_T^{t,x}), L^{ij}(\vec{v})(s, X_s^{t,x}), U^{ij}(\vec{v})(s, X_s^{t,x}))$. Precisely we have:

Proposition 4.3.2. *Let $(i,j) \in \Gamma^1 \times \Gamma^2$ be fixed. For any stopping time $\tau \geq t$, there exists another stopping time $\delta_\tau \geq \tau$, $P-a.s.$ (δ_τ depends also on i,j but we omit it as far as there is no confusion) and three processes $Z^{ij,\tau}$, $K^{ij,\pm,\tau}$ such that:*
(i) $Y_T^{ij} = h^{ij}(X_T^{t,x})$;
(ii)

$$\begin{cases} Z^{ij,\tau} \in \mathcal{H}^{2,d},\ K^{ij,\pm,\tau} \in \mathcal{A}_i^2 \text{ and non-decreasing} ; \\ \forall s \in [\tau, \delta_\tau], Y_s^{ij} = Y_{\delta_\tau}^{ij} + \int_s^{\delta_\tau} f^{ij}(r, X_r^{t,x}, (Y_r^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2})dr - \int_s^{\delta_\tau} Z_r^{ij}dB_r + \int_s^{\delta_\tau} dK_r^{ij,+,\tau} - \int_s^{\delta_\tau} dK_r^{ij,-,\tau} \\ L^{ij}(\vec{v})(s, X_s^{t,x}) \leq Y_s^{ij} \leq U^{ij}(\vec{v})(s, X_s^{t,x}),\ \forall s \in [t,T] ; \\ \int_\tau^{\delta_\tau}\left(Y_r^{ij} - L^{i,j}(\vec{v})(r, X_r^{t,x})\right)dK_r^{ij,+,\tau} = 0 \text{ and } \int_\tau^{\delta_\tau}\left(Y_r^{ij} - U^{ij}(\vec{v})(r, X_r^{t,x})\right)dK_r^{ij,-,\tau} = 0; \end{cases} \tag{4.3.2}$$

(iii) Let γ_τ and θ_τ be the following two stopping times:

$$\gamma_\tau := \inf\{s \geq \tau,\ Y_s^{ij} = L^{ij}(\vec{v})(s, X_s^{t,x})\} \wedge T \text{ and } \theta_\tau := \inf\{s \geq \tau,\ Y_s^{ij} = U^{ij}(\vec{v})(s, X_s^{t,x})\} \wedge T.$$

Then, $P-a.s.$, $\gamma_\tau \vee \theta_\tau \leq \delta_\tau$. $\square$

4.3.1 Description of the zero-sum switching game

We now address the issue of the relationship between the value function of a zero-sum switching game and the functions $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ solution of system (4.1.9). We first suppose that Assumption (H6)-(ii) is satisfied, i.e., f^{ij} does not depend on $(\vec{y}, z^{ij})$, for any $(i,j) \in \Gamma^1 \times \Gamma^2$.

To begin with let us describe briefly the zero-sum switching game. Assume we have two players π_1 and π_2 who intervene on a system with the help of switching strategies. An admissible switching strategy for π_1 (resp. π_2) is a sequence $\delta := (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu := (\tau_n, \zeta_n)_{n\geq 0}$) where for any $n \geq 0$,

(i) σ_n (resp. τ_n) is an $\mathbf{F}$-stopping times such that P-a.s., $\sigma_n \leq \sigma_{n+1} \leq T$ (resp. $\tau_n \leq \tau_{n+1} \leq T$) ;
(ii) ξ_n (resp. ζ_n) is a random variable with values in Γ^1 (resp. Γ^2) which is $\mathcal{F}_{\sigma_n}$ (resp. $\mathcal{F}_{\tau_n}$)-measurable ;
(iii) $P[\sigma_n < T, \forall n \geq 0] = P[\tau_n < T, \forall n \geq 0] = 0$;
(iv) If $(A^\delta_s)_{s\leq T}$ and $(B^\nu_s)_{s\leq T}$ are the $\mathbf{F}$-adapted RCLL processes defined by:

$$\forall\, s \in [t,T), \quad A^\delta_s = \sum_{n\geq 1} \underline{g}_{\xi_{n-1}\xi_n}(\sigma_n, X^{t,x}_{\sigma_n}) 1_{[\sigma_n \leq s]} \quad \text{and} \quad A^\delta_T = \lim_{s\to T} A^\delta_s,$$

and

$$\forall\, s \in [t,T), \quad B^\nu_s = \sum_{n\geq 1} \bar{g}_{\zeta_{n-1}\zeta_n}(\tau_n, X^{t,x}_{\tau_n}) 1_{[\tau_n \leq s]} \quad \text{and} \quad B^\nu_T = \lim_{s\to T} B^\nu_s.$$

Then, $\mathbb{E}[(A^\delta_T)^2 + (B^\nu_T)^2] < \infty$. For any $s \leq T$, A^δ_s (resp. B^ν_s) is the cumulative switching cost at time s for π_1 (resp. π_2) when she implements the strategy δ (resp. ν).

Next, for $t \in I\!R$, $i \in \Gamma^1$ (resp. $j \in \Gamma^2$), we say that the admissible strategy $\delta := (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu := (\tau_n, \zeta_n)_{n\geq 0}$) belongs $\mathcal{A}^i_{\pi_1}(t)$ (resp. $\mathcal{A}^i_{\pi_2}(t)$) if

$$\sigma_0 = t,\ \xi_0 = i,\ \mathbb{E}[(A^\delta_T)^2] < \infty \quad (\text{resp. } \tau_0 = t,\ \zeta_0 = j,\ \mathbb{E}[(B^\nu_T)^2] < \infty).$$

Given an admissible strategy δ (resp. ν) of π_1 (resp. π_2) one associates a stochastic process $(u_s)_{s\leq T}$ (resp. $(v_s)_{s\leq T}$) which indicates along with time the current mode of π_1 (resp. π_2) and which is defined by:

$$\forall s \leq T,\ u_s = \xi_0 1_{\{\sigma_0\}}(s) + \sum_{n\geq 1} \xi_{n-1} 1_{]\sigma_{n-1},\sigma_n]}(s) \text{ (resp. } v_s = \zeta_0 1_{\{\tau_0\}}(s) + \sum_{n\geq 1} \zeta_{n-1} 1_{]\tau_{n-1},\tau_n]}(s)). \quad (4.3.3)$$

Let now $\delta = (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu = (\tau_n, \zeta_n)_{n\geq 0}$) be a strategy for π_1 (resp. π_2) which belongs to $\mathcal{A}^i_{\pi_1}(t)$ (resp. $\mathcal{A}^j_{\pi_2}(t)$). The interventions of the players are not free and generate a payoff which is a reward (resp. cost) for π_1 (resp. π_2) and whose expression is given by

$$J_t(\delta,\nu) := \mathbb{E}[h^{u_T v_T}(X_T) + \int_t^T f(r, X^{t,x}_r, u_r, v_r)dr - A^\delta_T + B^\nu_T | \mathcal{F}_t], \quad (4.3.4)$$

where, for any $(k,l) \in \Gamma^1 \times \Gamma^2$, we set $f(s,x,k,l) = f^{kl}(s,x)$, since f^{kl} is assumed to not depend on $(\vec{y}, z^{ij})$.

As usual in the literature of zero-sum games, we are interested in the following issue:
Does this zero-sum switching game have a value, that is, does the following equality hold?

$$\text{ess inf}_{\nu\in\mathcal{A}^j_{\pi_2}(t)} \text{ess sup}_{\delta\in\mathcal{A}^i_{\pi_1}(t)} J_t(\delta,\nu) = \text{ess sup}_{\delta\in\mathcal{A}^i_{\pi_1}(t)} \text{ess inf}_{\nu\in\mathcal{A}^j_{\pi_2}(t)} J_t(\delta,\nu)$$

In the remaining part of this section, we focus on this issue.

For later use, let us introduce two new families of auxiliary processes $(\hat{U}^{\delta,j})_{j\in\Gamma^2}$ (resp. $(\hat{U}^{i,\nu})_{i\in\Gamma^1}$) associated with a given admissible strategy δ (resp. ν) of π_1 (resp π_2). They are defined by: $\forall j \in \Gamma^2$,

$$\begin{cases} \hat{U}^{\delta,j} \in \mathcal{S}^2_d,\ \hat{Z}^{\delta,j} \in \mathcal{H}^{2,d},\ K^{-,\delta,j} \in \mathcal{A}^2_i; \\ \hat{U}^{\delta,j}_s = \ h^{u(T)j}(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, u_r, j)dr - \int_s^T Z^{\delta,j}_r dB_r - (A^\delta_T - A^\delta_s) - (K^{-,\delta,j}_T - K^{-,\delta,j}_s),\ s \in [t,T]; \\ \forall s \in [t,T],\ \hat{U}^{\delta,j}_s \leq \min_{l\neq j}\left(\hat{U}^{\delta,l}_s + \bar{g}_{jl}(s, X^{t,x}_s)\right) \text{ and } \int_t^T \{\hat{U}^{\delta,j}_r - \min_{l\neq j}\{\hat{U}^{\delta,l}_r + \bar{g}_{jl}(r, X^{t,x}_r)\}\} dK^{-,\delta,j}_r = 0. \end{cases} \quad (4.3.5)$$

and for any $i \in \Gamma^1$

$$\begin{cases} U^{i,\nu} \in \mathcal{S}^2_d,\ Z^{i,\nu} \in \mathcal{H}^{2,d},\ K^{+,i,\nu} \in \mathcal{A}^2_i; \\ U^{i,\nu}_s = \ h^{iv(T)} + \int_s^T f(r, X^{t,x}_r, i, v_r)dr - \int_s^T Z^{i,\nu}_r dB_r + (B^\nu_T - B^\nu_s) + (K^{+,i,\nu}_T - K^{+,i,\nu}_s),\ s \in [t,T]\,; \\ \forall s \in [t,T],\ U^{i,\nu}_s \geq \max_{k\neq i}\{U^{k,\nu}_s - \underline{g}_{ik}(s, X^{t,x}_s)\} \text{ and } \int_s^T \left(\hat{U}^{\delta,j}_r - \max_{k\neq i}\{U^{k,\nu}_r - \underline{g}_{ik}(r, X^{t,x}_r)\}\right) dK^{+,i,\nu}_r = 0. \end{cases} \quad (4.3.6)$$

These equations are actually not of standard form, but by an obvious change of variables one can easily show that they have unique solutions. On the other hand, let us point out that thanks to the connection between the standard switching problem and multidimensional RBSDE with a lower interconnected obstacle (see e.g. [18] or [34]) the family $(\hat{U}^{\delta,j} - A^{\delta})_{j\in\Gamma^2}$ (resp. $(U^{i,\nu} + B^{\nu})_{i\in\Gamma^1}$) of processes verifies:

$$\hat{U}_t^{\delta,j} = \text{ess inf}_{\nu\in\mathcal{A}^j_{\pi_2}(t)}\{J_t(\delta,\nu) + A_t^{\delta}\} \text{ and } U_t^{i,\nu} = \text{ess sup}_{\delta\in\mathcal{A}^i_{\pi_1}(t)}\{J_t(\delta,\nu) - B_t^{\nu}\}. \tag{4.3.7}$$

4.3.2 the relationship between the zero-sum switching game and the min-max solution

We now give the main result of this section. It relates $(Y_s^{ij})_{s\leq T} = (v^{ij}(s, X_s^{t,x}))_{s\leq T}$, $(i,j) \in \Gamma^1 \times \Gamma^2$, with the value of the zero-sum switching game described above.

Theorem 4.3.1. *Suppose Assumptions (H0)-(H5) and (H6)-(ii) are satisfied. Then, for any $(i_0, j_0) \in \Gamma^1 \times \Gamma^2$,*

$$v^{i_0j_0}(t,x) = Y_t^{i_0j_0} \quad = ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)} ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)} J_t(\delta,\nu) = ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)} ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)} J_t(\delta,\nu). \tag{4.3.8}$$

Proof: Recall the definition of $(\check{Y}^{ij,m}, \check{Z}^{ij,m}, \check{K}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$, $m \geq 0$, given in (4.2.2). In order to alleviate notations, we denote it simply by $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$, $m \geq 0$: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} Y^{ij,m} \in \mathcal{S}^2,\ Z^{i,j,m} \in \mathcal{H}^{2,d} \text{ and } K^{ij,m} \in \mathcal{A}_i^2\ ; \\ Y_s^{ij,m} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij,m}(r, X_r^{t,x}, (Y_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2})dr + (K_T^{ij,m} - K_s^{ij,m}) - \int_s^T Z_r^{ij,m}dB_r,\ \forall s \in [t,T]; \\ Y_s^{ij,m} \geq \max\limits_{k\in(\Gamma^2)^{-i}}\{Y_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\},\ \forall s \in [t,T]; \\ \int_t^T (Y_s^{ij,m} - \max_{k\in(\Gamma^1)^{-i}}\{Y_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\})dK_s^{ij,m} = 0 \end{cases} \tag{4.3.9}$$

where, we recall that, for any $s \in [t,T]$, $m \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$f^{ij,m}(s, X_s^{t,x}, \vec{y}) = f^{ij}(s, X_s^{t,x}) - m \sum_{l\in(\Gamma^2)^{-j}} \big(y^{ij} - (y^{il} + \bar{g}_{jl}(s, X_s^{t,x}))\big)^+.$$

As already mentioned above, we know that, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $Y^{ij,m} \to_m Y^{ij}$ in $\mathcal{S}^2$. For sake of clarity, we divide the proof into two steps

Step 1: For any $(i_0, j_0) \in \Gamma^1 \times \Gamma^2$,

$$Y_t^{i_0j_0} = \text{ess sup}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)}\{\hat{U}_t^{\delta,j_0} - A_t^{\delta}\}. \tag{4.3.10}$$

Let $\delta = (\sigma_l, \xi_l)_{l\geq 0}$ be a strategy of $\mathcal{A}^{i_0}_{\pi_1}(t)$. We are going first to show that $Y_t^{i_0j_0} \geq \hat{U}_t^{\delta,j_0} - A_t^{\delta}$. So let us define the processes $(Y^{\delta j,m})_{j\in\Gamma^2}$ and $(\hat{U}^{\delta,j,m})_{j\in\Gamma^2}$ as follows:

(i) $\forall j \in \Gamma^2$,

$$\forall s \in [t,T), \quad Y_s^{\delta,j,m} = \sum_{l\geq 0} Y_s^{\xi_l j,m} \mathbb{1}_{[\sigma_l \leq s < \sigma_{l+1}]} \quad \text{and} \quad Y_T^{\delta,j,m} = h^{u(T)j}(X_T^{t,x}),$$

where,

$$\forall s \in [t,T], \quad Y_s^{\xi_l j,m} = \sum_{q\in\Gamma^1} Y_s^{qj,m} \mathbb{1}_{[\xi_l = q]}. \tag{4.3.11}$$

The process $Y^{\delta,j,m}$ is well defined since the sum contains only finite many terms since the strategy δ is admissible and then $P[\sigma_l < T, \forall l \geq 0] = 0$. On the other hand, at time $0 < \sigma_l < T$, $Y^{\delta,j,m}$ has a jump which is equal to $Y_{\sigma_l}^{\xi_l j,m} - Y_{\sigma_l}^{\xi_{l-1} j,m}$.

(ii) The processes $(\hat{U}^{\delta,j,m})_{j\in\Gamma^2}$ are defined as the solution of the following non standard multi-dimensional BSDE: $\forall j \in \Gamma^2$,

$$\begin{aligned} \hat{U}_s^{\delta,j,m} = \quad & h^{u(T)j}(X_T^{t,x}) + \int_s^T \Big\{ f(r, X_r^{t,x}, u_r, j) - m \textstyle\sum_{l\neq j} (\hat{U}_r^{\delta,j,m} - \hat{U}_r^{\delta,l,m} - \bar{g}_{jl})^+ \Big\} dr \\ & - (A_T^{\delta} - A_s^{\delta}) - \int_s^T \hat{V}_u^{\delta,j,m} dB_u,\ s \in [t,T]. \end{aligned} \tag{4.3.12}$$

Note that $(\hat{U}^{\delta,j,m}+A^{\delta})_{j\in\Gamma^2}$ is a solution of a standard multidimensional BSDE whose coefficient is Lipschitz. As those latter processes exist, then so are $(\hat{U}^{\delta,j,m})_{j\in\Gamma^2}$. On the other hand, as for the system given in (4.1.12), the sequence of processes $((\hat{U}^{\delta,j,m})_{j\in\Gamma^2})_{m\geq 0}$ converges in $\mathcal{S}_d^2$ toward $(\hat{U}^{\delta,j})_{j\in\Gamma^2}$. We now prove the following: for any $m\geq 0$, $j\in\Gamma^2$,

$$Y_0^{\delta,j,m}\geq \hat{U}_0^{\delta,j,m}. \tag{4.3.13}$$

For any $j\in\Gamma^2$ let us define $K^{\delta,j,m}$ and $Z^{\delta,j,m}$ as follows: $\forall s\in[t,T]$,

$$Z_s^{\delta,j,m}:=\sum_{l\geq 0}Z_s^{\xi_l j,m}\mathbb{1}_{[\sigma_l\leq s<\sigma_{l+1}[}\quad\text{and}\quad K_s^{\delta,j,m}=\sum_{l\geq 0}\int_{s\wedge\sigma_l}^{s\wedge\sigma_{l+1}}dK_s^{\xi_l j,m},$$

where, $Z_s^{\xi_l j,m}$ and $K_s^{\xi_l j,m}$ are defined in the same way as in (4.3.11). Once more there is no definition problem of those processes since δ is admissible. Therefore the triple of processes $(Y^{\delta,j,m},Z^{\delta,j,m},K^{\delta,j,m})_{j\in\Gamma^2}$ verifies: $\forall s\in[t,T)$,

$$\begin{aligned}Y_s^{\delta,j,m}&=Y_t^{\delta,j,m}-\int_t^s\Big\{f^{u_r j}(r,X_r^{t,x})dr+m\textstyle\sum_{l\neq j}\big(Y_r^{\delta,j,m}-Y_r^{\delta,l,m}-\bar{g}_{jl}(r,X_r^{t,x})\big)^+dr+Z_r^{\delta,j,m}dB_r-dK_r^{\delta,j,m}\Big\}\\&\quad+\textstyle\sum_{l\geq 1}(Y_{\sigma_l}^{\xi_l j,m}-Y_{\sigma_l}^{\xi_{l-1}j,m})\mathbb{1}_{[\sigma_l\leq s]}\\&=Y_t^{\delta,j,m}-\int_t^s\Big\{f^{u_r j}(r,X_r^{t,x})dr+m\textstyle\sum_{l\neq j}\big(Y_r^{\delta,j,m}-Y_r^{\delta,l,m}-\bar{g}_{jl}(r,X_r^{t,x})\big)^+dr+Z_r^{\delta,j,m}dB_r-dK_r^{\delta,j,m}\Big\}\\&\quad-\textstyle\sum_{l\geq 1}(Y_{\sigma_l}^{\xi_{l-1}j,m}-Y_{\sigma_l}^{\xi_l j,m}+\underline{g}_{\xi_{l-1}\xi_l}(\sigma_l,X_{\sigma_l}^{t,x}))\mathbb{1}_{[\sigma_l\leq s]}+A_s^{\delta}.\end{aligned}$$

Next, let us define $\tilde{A}^{\delta,j,m}$ by:

$$\tilde{A}_s^{\delta,j,m}:=\sum_{l\geq 1}(Y_{\sigma_l}^{\xi_{l-1}j,m}-Y_{\sigma_l}^{\xi_l j,m}+\underline{g}_{\xi_{l-1}\xi_l}(\sigma_l,X_{\sigma_l}^{t,x}))\mathbb{1}_{[\sigma_l\leq s]}\quad\text{for}\quad s\in[t,T)\quad\text{and}\quad\tilde{A}_T^{\delta,j}=\lim_{s\to T}\tilde{A}_s^{\delta,j}$$

which is an $\mathbf{F}$-adapted non-decreasing process. As the strategy δ is admissible, then writing backwardly between $s\wedge\sigma_k$ and $\sigma_k\vee s$ the equation for the process $Y^{\delta,j,m}$ and taking the limit $k\to\infty$, we obtain: $\forall j\in\Gamma^2$,

$$\begin{aligned}Y_s^{\delta,j,m}&=h^{u(T)j}(X_T^{t,x})+\int_s^T\Big\{f^{u_r j}(r,X_r^{t,x})dr-m\textstyle\sum_{l\neq j}\big(Y_r^{\delta,j,m}-Y_r^{\delta,l,m}-\bar{g}_{jl}(r,X_r^{t,x})\big)^+dr\\&\quad-Z_r^{\delta,j,m}dB_r+dK_r^{\delta,j,m}\Big\}-(A_T^{\delta}-A_s^{\delta})+(\tilde{A}_T^{\delta,j,m}-\tilde{A}_s^{\delta,j,m}),\ \forall s\in[t,T].\end{aligned}\tag{4.3.14}$$

This equation implies also that $\mathbb{E}[(\tilde{A}_T^{\delta,j,m})^2]<\infty$. Comparing now equation (4.3.14) for $(Y_s^{\delta,j,m},Z_s^{\delta,j,m})_{s\in[t,T]}$ and the one satisfied by $(\hat{U}_s^{\delta,j,m},\hat{V}_s^{\delta,j,m})_{s\in[t,T]}$ we have, by uniqueness of the solution of the multidimensional BSDE (4.3.12), that

$$Y_s^{\delta,j,m}-\mathbb{E}[(\tilde{A}_T^{\delta,j,m}-\tilde{A}_s^{\delta,j,m})+(K_T^{\delta,j,m}-K_s^{\delta,j,m})|\mathcal{F}_s]=\hat{U}_s^{\delta,j,m},\ \ \forall s\in[t,T]\quad\text{and}\quad j\in\Gamma^2.\tag{4.3.15}$$

As the processes $\tilde{A}^{\delta,j,m}$ and $K^{\delta,j,m}$ are non-decreasing then

$$Y_s^{\delta,j,m}\geq\hat{U}_s^{\delta,j,m},\quad\forall s\in[t,T].$$

Taking now the limit w.r.t. m, we obtain that

$$Y_t^{i_0 j}=\lim_{m\to\infty}Y_t^{i_0 j,m}\geq\lim_{m\to\infty}\{Y_t^{\delta,j,m}-A_t^{\delta}\}\geq\lim_{m\to\infty}\{\hat{U}_t^{\delta,j,m}-A_t^{\delta}\}=\hat{U}_t^{\delta,j}-A_t^{\delta},\quad\forall j\in\Gamma^2.$$

Step 2: In order to complete the proof of the claim we construct a strategy $\bar{\delta}$ of $\mathcal{A}_{\pi_1}^{i_0}(t)$ such that $Y_t^{i_0,j_0}=\hat{U}_t^{\bar{\delta},j_0}$.

Let us first define the strategy $\bar{\delta}=(\xi_l^*,\sigma_l^*)_{l\geq 0}$ as follows:
(i) $\xi_0^*=i$, $\sigma_0^*=t$.
(ii) Next, for any $l\geq 1$, we define σ_l^* and ξ_l^* by:

$$\begin{cases}\sigma_l^*=\inf\Big\{s\geq\sigma_{l-1}^*,\quad Y_s^{\xi_{l-1}^* j_0}=\max_{k\neq\xi_{l-1}^*}\Big(Y_s^{kj_0}-\underline{g}_{\xi_{l-1}^* k}(s,X_s^{t,x})\Big)\Big\}\wedge T,\\[2ex]\xi_l^*\in\ \operatorname{argmax}_{k,\,k\neq\xi_{l-1}^*}\Big\{Y_{\sigma_l^*}^{kj_0}-\underline{g}_{\xi_{l-1}^* k}(\sigma_l^*,X_{\sigma_l^*}^{t,x})\Big\}.\end{cases}\tag{4.3.16}$$

We first prove that $\bar{\delta}$ verifies

$$\mathbb{P}\left(\{\omega,\ \ \forall\, l \geq 0,\ \ \sigma_l^*(\omega) < T\}\right) = 0. \tag{4.3.17}$$

We proceed by contradiction. Assume that the last property does not hold. As the set Γ^1 is finite then one can find a loop $(i_1, i_2, \cdots, i_l = i_1)$ of exactly $l-1$ ($l \geq 2$) indices and a subsequence $(l_p)_{p\geq 0}$ (which may depend on ω) satisfying $l_{p+1} - l_p \geq l$ and such that:

$$\mathbb{P}\left(Y^{i_1 j_0}_{\sigma^*_{l_p}} = Y^{i_2 j_0}_{\sigma^*_{l_p}} - \underline{g}_{i_1 i_2}(\sigma^*_{l_p}, X^{t,x}_{\sigma^*_{l_p}}), \cdots, Y^{i_{l-1} j_0}_{\sigma^*_{l_p+l-1}} = Y^{i_l j_0}_{\sigma^*_{l_p+l-1}} - \underline{g}_{i_{l-1} i_l}(\sigma^*_{l_p+l-1}, X^{t,x}_{\sigma^*_{l_p+l-1}}), \quad \forall p \geq 0\right) > 0.$$

Next, let us set τ^* by $\tau^*(\omega) := \lim_p \sigma^*_{l_p}(\omega)$, then by taking the limit in the previous equalities we obtain

$$\mathbb{P}\left(Y^{i_1 j}_{\tau^*} = Y^{i_2 j_0}_{\tau^*} - \underline{g}_{i_1 i_2}(\tau^*, X^{t,x}_{\tau^*}), \cdots, Y^{i_{l-1} j_0}_{\tau^*} = Y^{i_l j_0}_{\tau^*} - \underline{g}_{i_{l-1} i_l}(\tau^*, X^{t,x}_{\tau^*})\right) > 0$$

Since $i_1 = i_l$, we obtain

$$\mathbb{P}\left(\sum_{k=1}^{l-1} \underline{g}_{i_k i_{k+1}}(\tau^*, X^{t,x}_{\tau^*}) = 0\right) > 0$$

which contradicts to the so called non free loop property and then $\bar{\delta}$ satisfies (4.3.17).

Let us show that $\mathbb{E}[(A^{\bar{\delta}}_T)^2] < \infty$. First note that due to the non-free loop property $\mathbb{E}[(A^{\bar{\delta}}_t)^2] < \infty$. Next let us introduce the process $Y^{\bar{\delta}, j_0}$ by setting

$$\forall\, s \in [t,T), \quad Y^{\bar{\delta}, j_0}_s = \sum_{l\geq 0} Y^{\xi^*_l j_0}_s \mathbf{1}_{\{\sigma^*_l \leq s < \sigma^*_{l+1}\}} \quad \text{and} \quad Y^{\bar{\delta}, j_0}_T = h^{u^{\bar{\delta}}(T) j_0}, \tag{4.3.18}$$

where, $(u^{\bar{\delta}}(s))_{s\in[t,T]}$, as in (4.3.12), is the RCLL process associated with $\bar{\delta}$ which indicates the mode of π_1 at time s when the strategy $\bar{\delta}$ is implemented. Next, by the local solution property of Proposition 4.3.2, for any $l \geq 0$, we have

$$\begin{cases} Y^{\xi^*_l j_0}_s = Y^{\xi^*_l j_0}_{\sigma^*_{l+1}} + \int_s^{\sigma^*_{l+1}} f^{\xi^*_l j_0}(r, X^{t,x}_r)dr - (K^{\xi^*_l j_0,-}_{\sigma^*_{l+1}} - K^{\xi^*_l j_0,-}_s) - \int_s^{\sigma^*_{l+1}} Z^{\xi^*_l j_0}_r dB_r, \ \ \forall s \in [\sigma^*_l, \sigma^*_{l+1}]; \\ Y^{\xi^*_l j_0}_s \leq \min_{p\in(\Gamma^2)^{-j_0}}\{Y^{\xi^*_l j_0}_s + \bar{g}_{j_0 p}(s, X^{t,x}_s)\}, \ \ \forall s \in [\sigma^*_l, \sigma^*_{l+1}]; \\ \int_{\sigma^*_l}^{\sigma^*_{l+1}} (Y^{\xi^*_l j_0}_s - \min_{p\in(\Gamma^2)^{-j_0}}\{Y^{\xi^*_l p}_s + \bar{g}_{j_0 p}(s, X^{t,x}_s)\})dK^{\xi^*_l j_0,-}_u = 0, \end{cases} \tag{4.3.19}$$

where, $Z^{\xi^*_l j_0}$ and $K^{\xi^*_l j_0,-}$ are fixed processes which depend actually on σ^*_l, for all $l \geq 0$. Let us now define $Z^{\bar{\delta}, j_0}$ and $K^{\bar{\delta}, j_0,-}$ by:

$$Z^{\bar{\delta}, j_0}_s := \sum_{l\geq 0} Z^{\xi^*_l j_0}_s \mathbb{1}_{[\sigma^*_l \leq s < \sigma^*_{l+1}[} \quad \text{and} \quad K^{\bar{\delta}, j_0,-}_s := \sum_{l\geq 0} \int_{s\wedge\sigma^*_l}^{s\wedge\sigma^*_{l+1}} dK^{\xi^*_l j_0,-}_s, \quad s \in [t,T].$$

We note that, by definition, we have, for any $l \geq 0$,

$$\{Y^{\xi^*_l j}_{\sigma^*_{l+1}} - Y^{\xi^*_{l+1} j}_{\sigma^*_{l+1}} + \underline{g}_{\xi^*_l \xi^*_{l+1}}(\sigma^*_{l+1}, X^{t,x}_{\sigma^*_{l+1}})\}\mathbf{1}_{\{\sigma^*_{l+1} < T\}} = 0.$$

Then, taking into account the jump of $Y^{\bar{\delta}, j_0}$ at σ^*_{l+1} (when smaller than T) which is equal to $Y^{\xi^*_{l+1} j_0}_{\sigma^*_{l+1}} - Y^{\xi^*_l j_0}_{\sigma^*_{l+1}}$ and by (4.3.17), we have, for every $s \in [t,T]$,

$$Y^{\bar{\delta}, j_0}_s = h^{u^{\delta}(T) j_0}(X^{t,x}_T) - (A^{\bar{\delta}}_T - A^{\bar{\delta}}_s) + \int_s^T f(r, X^{t,x}_r, u^{\bar{\delta}}_r, j_0)dr - (K^{\bar{\delta}, j_0,-}_T - K^{\bar{\delta}, j_0,-}_s) - \int_s^T Z^{\bar{\delta}, j_0}_r dB_r, \tag{4.3.20}$$

which implies that

$$Y^{\bar{\delta}, j_0}_s = Y^{\bar{\delta}, j_0}_t + (A^{\bar{\delta}}_s - A^{\bar{\delta}}_t) + K^{\bar{\delta}, j_0,-}_s - \int_t^s f(r, X^{t,x}_r, u^{\bar{\delta}}_r, j_0)dr + \int_t^s Z^{\bar{\delta}, j_0}_r dB_r, \quad \forall s \in [t,T].$$

As $Y^{\bar{\delta}, j_0}$ belongs to $\mathcal{S}^2_d$ then a localization procedure and Fatou's Lemma permit to deduce that

$$\mathbb{E}[A^{\bar{\delta}}_T + K^{\bar{\delta}, j_0,-}_T] < \infty.$$

Thus, for any $s \in [t,T]$,

$$\begin{aligned}\mathbb{E}[(A_T^{\bar\delta} - A_s^{\bar\delta}) \quad & +(K_T^{\bar\delta,j_0,-} - K_s^{\bar\delta,j_0,-})|\mathcal{F}_s] \\ & = \mathbb{E}[h^{u^{\bar\delta}(T),j_0}(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},u_r^{\bar\delta},j_0)dr|\mathcal{F}_s] - Y_s^{\bar\delta,j_0} \\ & = \mathbb{E}[h^{u^{\bar\delta}(T),j_0}(X_T^{t,x}) + \int_t^T f(r,X_r^{t,x},u_r^{\bar\delta},j_0)dr|\mathcal{F}_s] - \{Y_s^{\bar\delta,j_0} + \int_t^s f(r,X_r^{t,x},u_r^{\bar\delta},j_0)dr\}.\end{aligned} \tag{4.3.21}$$

Therefore, the right-hand side of the previous equality is a supermartingale which moreover, by Doob's inequality, belongs to $\mathcal{S}_d^2$. Hence, using a result by Dellacherie-Meyer ([15], pp. 220-221) we deduce that

$$\mathbb{E}[\{A_T^{\bar\delta} + K_T^{\bar\delta,j_0}\}^2] < \infty,$$

since the right-hand side of (4.3.21) belongs to $\mathcal{S}_d^2$. Thus, the strategy $\bar\delta$ is admissible.
It remains now to show that $Y_t^{i_0j_0} = \hat U_t^{\bar\delta,j_0} - A_t^{\bar\delta}$.

The equality (4.3.15) applied with $\bar\delta$ reads as:

$$Y_s^{\bar\delta,j_0,m} - \mathbb{E}[(\tilde A_T^{\bar\delta,j_0,m} - \tilde A_s^{\bar\delta,j_0,m}) + (K_T^{\bar\delta,j_0,m} - K_s^{\bar\delta,j_0,m})|\mathcal{F}_s] = \hat U_s^{\bar\delta,j_0,m}, \forall s \in [t,T]. \tag{4.3.22}$$

Therefore, the process $(Y_s^{\bar\delta,j_0,m} - \hat U_s^{\bar\delta,j_0,m})_{s\le T}$ is a supermartingale which satisfies

$$\sup_{m\ge 0}\mathbb{E}[\sup_{s\le T}|Y_s^{\bar\delta,j_0,m} - \hat U_s^{\bar\delta,j_0,m}|^2] < \infty,$$

since, for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\sup_{m\ge 0}\mathbb{E}[\sup_{s\le T}\{|Y_s^{ij,m}| + |\hat U_s^{ij,m}|\}^2] < \infty.$$

Once more, by a Dellacherie-Meyer's result ([15], pp. 220-221), we obtain

$$\sup_{m\ge 0}\mathbb{E}[\{\tilde A_T^{\bar\delta,j_0,m} + K_T^{\bar\delta,j_0,m}\}^2] < \infty. \tag{4.3.23}$$

But,

$$\tilde A_s^{\bar\delta,j_0,m} := \sum_{l\ge 1}(Y_{\sigma_l^*}^{\xi_{l-1}^* j_0,m} - Y_{\sigma_l^*}^{\xi_l^* j_0,m} + \underline{g}_{\xi_{l-1}^*\xi_l^*}(\sigma_l^*, X_{\sigma_l^*}^{t,x}))\mathbb{1}_{[\sigma_l^*\le s]} \quad \text{for} \quad s\in[t,T) \quad \text{and} \quad \tilde A_T^{\bar\delta,j_0,m} = \lim_{s\to T}\tilde A_s^{\bar\delta,j_0,m},$$

and, by definition of the strategy $\bar\delta$, for any $l \ge 0$, it holds

$$\{Y_{\sigma_{l+1}^*}^{\xi_l^* j_0} - Y_{\sigma_{l+1}^*}^{\xi_{l+1}^* j_0} + \underline{g}_{\xi_l^*\xi_{l+1}^*}(\sigma_{l+1}^*, X_{\sigma_{l+1}^*})\}\mathbf{1}_{\sigma_{l+1}^*<T} = 0.$$

As the strategy $\bar\delta$ is admissible (i.e. for ω fixed there is only a finite many σ_l^* such that $\sigma_l^* < T$) and $Y^{ij_0,m} \searrow Y^{ij_0}$ as $m\to\infty$ in $\mathcal{S}^2$ then $P-a.s.$, $\tilde A_s^{\bar\delta,j_0,m} \to 0$ as $m\to\infty$, for any $s\in[t,T]$. Therefore, with (4.3.23) we deduce that $\tilde A_T^{\bar\delta,j_0,m} \to 0$ in $L^1(dP)$.

Next, we shall show that there exists a subsequence of $\{m\}$ which we still denote by $\{m\}$ such that for any $l \ge 0$, the random variable

$$\sum_{p=0}^{p=l}\int_{\sigma_p^*}^{\sigma_{p+1}^*} dK_s^{\xi_p^* j_0,m} \to_m 0 \text{ weakly in } L^2(dP).$$

To begin with, by using (4.2.17), let $\{m\}$ be a subsequence such that for any $(i,j)\in\Gamma^1\times\Gamma^2$

$$(m\sum_{l\ne j}(Y_r^{ij,m} - Y_r^{il,m} - \bar g_{jl}(r,X_r^{t,x}))^+)_{r\in[t,T]}$$

converges weakly in $\mathcal{H}^{2,d}$ to a process $(\alpha_r^{ij})_{r\in[t,T]}$.

We only consider the sequence $(\int_{\sigma_0^*}^{\sigma_1^*} dK_s^{\xi_0^* j_0,m})_{m\ge 0} = (\int_t^{\sigma_1^*} dK_s^{ij_0,m})_{m\ge 0}$ since for the other cases a similar procedure applies (keep in mind that we should have $P[\sigma_1^* > \sigma_0^*] > 0$, otherwise this case is irrelevant and then one should begin with the next case, i.e, taking $p=1$). For $s\in[t,\sigma_1^*]$ and from (4.3.9) we have

$$\begin{aligned}Y_s^{ij_0,m} \quad & = Y_{\sigma_1^*}^{ij_0,m} + \int_s^{\sigma_1^*} f^{ij_0}(r,X_r^{t,x})dr - m\int_s^{\sigma_1^*}\sum_{l\ne j_0}(Y_r^{ij_0,m} - Y_r^{il,m} - \bar g_{j_0l}(r,X_r^{t,x}))^+dr \\ & - \int_s^{\sigma_1^*} Z_r^{ij_0,m}dB_r + K_{\sigma_1^*}^{ij_0,m} - K_s^{ij_0,m}.\end{aligned}$$

As $Y^{ij_0,m}$ converges to Y^{ij_0} in $\mathcal{S}^2$, by Itô's formula, we have:

(i) $\sup_{m\geq 0}\mathbb{E}[(K^{ij_0,m}_{\sigma_1^*})^2]<\infty$;

(ii) the sequence $(Z^{ij_0,m}_s\mathbb{1}_{[s\leq\sigma_1^*]})$ converges in $\mathcal{H}^{2,d}$ to some process $\bar{Z}^{ij_0}$.

Now, for $s\leq\sigma_1^*$, define

$$K^{ij_0}_s=Y^{ij_0}_t-Y^{ij_0}_s+\int_t^s\bar{Z}^{ij_0}_r dB_r+\int_t^s\alpha^{ij_0}_r dr-\int_t^s f^{ij_0}(r,X^{t,x}_r)dr.$$

Then, the process K^{ij_0} is continuous on $[t,\sigma_1^*]$. Moreover, using the weak convergence pointed out previously, for any any stopping time $\tau\in[t,\sigma_1^*]$, $K^{ij_0,m}_\tau\to_m K^{ij_0}_\tau$ weakly in $L^2(dP)$.

Next, let τ be a stopping time such that $t\leq\tau<\sigma_1^*$. The properties of $K^{ij_0,m}$ (especially the Skorokhod condition) combined with the uniform convergence of $(Y^{ij,m})_m$ to Y^{ij} and the definition of σ_1^*, i.e.,

$$\forall s<\sigma_1^*,\quad Y^{ij_0}_s>\max_{k\neq i}\{Y^{kj_0}_s-\underline{g}_{ik}(s,X^{t,x}_s)\}$$

imply the existence of some $m_0(\omega)$ such that if $m\geq m_0$ then $K^{ij_0,m}_\tau=0$. Therefore, the sequence $(K^{ij_0,m}_\tau)_m$ converges $P-a.s.$ to 0 and by (i) above it converges also in $L^{2-\epsilon}(dP)$ to 0 and then $K^{ij_0}_\tau=0$. Finally by continuity we have $K^{ij_0}_s=0$ for any $s\in[t,\sigma_1^*]$ and then the sequence $(\int_t^{\sigma_1^*}dK^{i,j_0,m}_s)_{m\geq 0}$ converges weakly in $L^2(dP)$ to 0. As we can do the same for the other sequences, the claim holds.

Let l be fixed. By using (4.3.22) between t and σ_l^* one obtains:

$$Y^{\bar{\delta},j_0,m}_t-\mathbb{E}[Y^{\bar{\delta},j_0,m}_{\sigma_l^*}+(\tilde{A}^{\bar{\delta},j_0,m}_{\sigma_l^*}-\tilde{A}^{\bar{\delta},j_0,m}_t)+K^{\bar{\delta},j_0,m}_{\sigma_l^*}|\mathcal{F}_t]=\hat{U}^{\bar{\delta},j_0,m}_t-\mathbb{E}[U^{\bar{\delta},j_0,m}_{\sigma_l^*}|\mathcal{F}_t].$$

Taking now the weak limit w.r.t. m (at least through the subsequence constructed above) we obtain that

$$Y^{\bar{\delta},j_0}_t-\mathbb{E}[Y^{\bar{\delta},j_0}_{\sigma_l^*}|\mathcal{F}_t]=\hat{U}^{\bar{\delta},j_0}_t-\mathbb{E}[U^{\bar{\delta},j_0}_{\sigma_l^*}|\mathcal{F}_t].$$

Finally, taking the limit as $l\to\infty$, noting that $Y^{\bar{\delta},j_0}_T=U^{\bar{\delta},j_0}_T=h^{u^{\bar{\delta}}(T)j_0}$, we obtain

$$Y^{\bar{\delta},j_0}_t=\hat{U}^{\bar{\delta},j_0}_t.$$

Thus, in view of the definition of $\bar{\delta}$, we have

$$Y^{i_0,j_0}_t=Y^{\bar{\delta},j_0}_t-A^{\bar{\delta}}_t=\hat{U}^{\bar{\delta},j_0}_t-A^{\bar{\delta}}_t.$$

Now, taking into account of (4.3.7), the first equality holds.

Finally, in order to obtain the second equality of (4.3.8), it is enough to consider the approximating increasing scheme (which is the opposite of (4.3.9)) and which can be transformed into a decreasing scheme by taking its opposite sign. Then, from the result of Step 1, we have

$$-v^{i_0j_0}(t,x)=-Y^{i_0j_0}_t=\text{ess sup}_{\nu\in\mathcal{A}^{j_0}_{\pi_2}}\text{ess inf}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}}-J_t(\delta,\nu)$$

and the proof is finished. □

As a by product of Theorem (4.3.1) and the uniqueness of the solution of system (4.1.9) we have the following result in the case when the functions f^{ij} depend also on $\vec{y}$.

Corollary 4.3.1. *Suppose Assumptions (H0)-(H5) and (H6)-(i) are satisfied and let $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the unique solution of system (4.1.9) and (4.1.10). Then for any $(t,x)\in[0,T]\times\mathbb{R}^k$ and $(i_0,j_0)\in\Gamma^1\times\Gamma^2$,*

$$v^{i_0j_0}(t,x)=ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}}ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}}\bar{J}_t(\delta,\nu)=ess\ inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}}ess\ sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}}\bar{J}_t(\delta,\nu).\tag{4.3.24}$$

where,

$$\bar{J}_t(\delta,\nu):=\mathbb{E}[h^{u_Tv_T}(X_T)+\int_t^T f^{u_rv_r}(r,X^{t,x}_r,(v^{kl}(r,X^{t,x}_r))_{(k,l)\in\Gamma^1\times\Gamma^2})dr-A^\delta_T+B^\nu_T|\mathcal{F}_t].\tag{4.3.25}$$

Proof: Let $(w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the unique solution in viscosity sense of the following system of PDEs with inter-connected obstacles: $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \min\Big\{(w^{ij}-L^{ij}(\vec{w}))(t,x); \max\Big\{(w^{ij}-U^{ij}(\vec{w}))(t,x); \\ \qquad\qquad -\partial_t w^{ij}(t,x)-\mathcal{L}^X(w^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2})\Big\}\Big\}=0; \\ w^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{4.3.26}$$

Then, by Theorem 4.3.1 we have, for any $(t,x)\in[0,T]\times I\!R^k$ and $(i_0,j_0)\in\Gamma^1\times\Gamma^2$,

$$w^{i_0j_0}(t,x)=\text{ess sup}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}}\text{ess inf}_{\nu\in\mathcal{A}^{j_0}_{\pi_2}}\bar{J}_t(\delta,\nu)=\text{ess inf}_{\nu\in\mathcal{A}^{j_0}_{\pi_2}}\text{ess sup}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}}\bar{J}_t(\delta,\nu). \tag{4.3.27}$$

But $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is also solution of the system (4.3.26), then by uniqueness for any $(i,j)\in\Gamma^1\times\Gamma^2$, $v^{ij}=w^{ij}$. Plug now this equality in (4.3.27) to obtain the desired result. □

Remark 4.3.1. *We have also the following relation:* $\forall(i_0,j_0)\in\Gamma^1\times\Gamma^2$,

$$v^{i_0j_0}(t,x)=\sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)}\inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)}\mathbb{E}[\bar{J}_t(\delta,\nu)]=\inf_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)}\sup_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)}\mathbb{E}[\bar{J}_t(\delta,\nu)]. \tag{4.3.28}$$

4.4 Conclusion

In this paper, we have given appropriate conditions on the data of both the min-max and max-min systems so that their respective unique viscosity solutions coincide. These unique continuous viscosity solution have been constructed by means of a penalization procedure in the recent paper [17]. The main difficulty faced in that paper is that the two obstacles are interconnected and therefore not comparable. For this reason and without the separation of the two barriers, we cannot apply the classical relationship between doubly reflected BSDEs, system of PDEs with lower and upper obstacles and the underlying game obtained e.g. in [27]). By providing appropriate regularity conditions so that comparison holds, we establish in the present paper that the solutions of the Min-Max and Max-Min systems coincide. Finally, under further conditions on the drivers, this solution can be interpreted as the value function of a switching game.

We note that to obtain the required condition of comparison, we rely on the regularity of penalty costs. We also need to get precise estimates of penalized terms which can be obtained by controlling the growth of the driver. Our analysis deeply relies on the Markovian setting, therefore it seems quite natural to ask whether one can study the switching game in the general non-Markovian case. We leave this question for future research. □

Chapter 5

APPENDIX

5.1 Representation of the value function of the stochastic optimal switching problem

Let $\Upsilon := (\theta_n, \alpha_n)_{n\geq 0}$ be an admissible strategy of switching and let $a = (a_s)_{s\in[0,T]}$ be the process defined by

$$\forall s \leq T,\ a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s). \tag{5.1.1}$$

Let $t_0 \in [0,T]$ and $\Gamma := ((\Gamma^j_s)_{s\in[0,T]})_{j\in A} \in [H^2]^m$. Let us define (when it exists) the pair of processes $(V^a, N^a) := (V^a_s, N^a_s)_{s\in[0,T]}$ as the solution of the following BSDE:

$$\begin{cases} V^a \in \mathcal{S}^2,\ N^a \in \mathcal{H}^2(l^2) \\ V^a_s = h_{a(T)}(X^{t,x}_T) + \int_s^T \mathbb{1}_{\{r\geq t_0\}} f_{a(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, N^a_r)dr - \sum_{i=1}^{\infty} \int_s^T N^{a,i}_r dH^{(i)}_r - A^a_T + A^a_s,\ s \in [0,T], \end{cases} \tag{5.1.2}$$

where $\overrightarrow{\Gamma}_r = (\Gamma^k_r)_{k\in A}$ and A^a is the cumulative switching cost associated with the strategy a or Υ (see (2.2.22) for its definition). This BSDE is not a standard one, but in assuming that $\mathbb{E}[(A^a_T)^2] < \infty$ and by setting up $\bar{V}^a = V^a - A^a$, it becomes a standard one and therefore we deduce the existence and uniqueness of the process (V^a, N^a) since the RCLL process A^a is adapted and square integrable. Obviously if for some $j \in A$, a belongs to $\mathcal{A}^j_{t_0}$ then (V^a, N^a) exists and is unique.

Theorem 5.1.1. *Under Assumption (**A4**), the solution of BSDE (5.1.2) satisfies: $\forall j \in A$,*

$$Y^{\Gamma,j}_{t_0} = esssup_{a\in\mathcal{A}^j_{t_0}} (V^a_{t_0} - A^a_{t_0}),\ \mathcal{P} - a.s. \tag{5.1.3}$$

where $(Y^{\Gamma,j})_{j\in A}$ is the first component of the solution of the BSDE (2.2.35). Thus the solution of (2.2.35) is unique. Moreover there exists $a^* \in \mathcal{A}^j_{t_0}$ such that $Y^{\Gamma,j}_{t_0} = V^{a^*}_{t_0} - A^{a^*}_{t_0}$.

Proof. By Theorem 3.4, let $(Y^{\Gamma,j}, U^{\Gamma,j}, K^{\Gamma,j})_{j\in A}$ be the solution of the system (2.2.35). Let $a \in \mathcal{A}^j_{t_0}$ and let us define

$$\tilde{K}^a_T = (K^{\Gamma,j}_{\theta_1} - K^{\Gamma,j}_{t_0}) + \sum_{n\geq 1} (K^{\Gamma,\alpha_n}_{\theta_{n+1}} - K^{\Gamma,\alpha_n}_{\theta_n}) \text{ and}$$

$$\forall i \geq 1 \text{ and } r \leq T, U^{a,i}_r = \sum_{n\geq 0} U^{\Gamma,\alpha_n,i}_r \mathbb{1}_{[\theta_n\leq r<\theta_{n+1}[} \text{ and } U^a := (U^{a,i})_{i\geq 1}.$$

Therefore

$$\begin{aligned}
Y_{t_0}^{\Gamma,j} &= Y_{\theta_1}^{\Gamma,j} + \int_{t_0}^{\theta_1} f_j(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^{\Gamma,j})dr - \sum_{i=1}^{\infty}\int_{t_0}^{\theta_1} U_r^{j,i}dH_r^{(i)} + (K_{\theta_1}^{\Gamma,j} - K_{t_0}^{\Gamma,j}) \\
&\geq (Y_{\theta_1}^{\Gamma,\alpha_1} - g_{i,\alpha_1}(\theta_1, X_{\theta_1}^{t,x}))1_{[\theta_1<T]} + 1_{[\theta_1=T]}h_{\alpha_0}(X_T^{t,x}) + \int_{t_0}^{\theta_1} f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^a)dr \\
&\quad - \sum_{i=1}^{\infty}\int_{t_0}^{\theta_1} U_r^{a,i}dH_r^{(i)} + (K_{\theta_1}^{\Gamma,j} - K_{t_0}^{\Gamma,j}) \\
&= Y_{\theta_2}^{\Gamma,\alpha_1}1_{[\theta_1<T]} + \int_{t_0}^{\theta_2} f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^a)dr - \sum_{i=1}^{\infty}\int_{t_0}^{\theta_2} U_r^{a,i}dH_r^{(i)} \\
&\qquad + (K_{\theta_1}^{\Gamma,i} - K_{t_0}^{\Gamma,i}) + (K_{\theta_2}^{\Gamma,\alpha_1} - K_{\theta_1}^{\Gamma,\alpha_1}) - g_{i,\alpha_1}(\theta_1, X_{\theta_1}^{t,x})1_{[\theta_1<T]}.
\end{aligned}$$

Repeat now this procedue as many times as necessary and since a is an admissible startegy (i.e. $\mathcal{P}[\theta_n < T, \forall n \geq 0] = 0$) we obtain:

$$Y_{t_0}^{\Gamma,j} \geq h_{a(T)}(X_T^{t,x}) + \int_{t_0}^{T} f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^a)dr - \sum_{i=1}^{\infty}\int_{t_0}^{T} U_r^{a,i}dH_r^{(i)} - A_T^a + \tilde{K}_T^a. \tag{5.1.4}$$

As $\tilde{K}_T^a \geq 0$ and by (5.1.2) we have

$$\begin{aligned}
Y_{t_0}^{\Gamma,j} - V_{t_0}^a + A_{t_0}^a &\geq \int_{t_0}^{T} (f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^a) - f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, N_r^a))dr - \sum_{i=1}^{\infty}\int_{t_0}^{T} (U_r^{a,i} - N_r^{a,i})dH_r^{(i)} \\
&\geq \int_{t_0}^{T} \langle V^{a,U^a,N^a}, U^a - N^a\rangle_s^p ds - \sum_{i=1}^{\infty}\int_{t_0}^{T} (U_r^{a,i} - N_r^{a,i})dH_r^{(i)}
\end{aligned}$$

Next by Girsanov's Theorem ([55], pp.136), under the probability measure $d\tilde{\mathcal{P}} := \varepsilon(\sum_{j=1}^{\infty}\int_{t_0}^{\cdot} V_r^{a,U^a,N^a,i}dH_r^{(i)})_T d\mathcal{P}$, $(M_t := \int_{t_0}^{t}\langle V^{a,U^a,N^a}, U^a - N^a\rangle_s^p ds - \sum_{i=1}^{\infty}\int_0^t (U_r^{a,i} - N_r^{a,i})dH_r^{(i)})_{t\in[t_0,T]}$ is a martingale, and by taking conditional expectation of $Y_{t_0}^{\Gamma,j} - V_{t_0}^a + A_{t_0}^a$, we obtain

$$\mathbb{E}_{\tilde{\mathcal{P}}}[Y_{t_0}^{\Gamma,j} - V_{t_0}^a + A_{t_0}^a|\mathcal{F}_s] \geq \mathbb{E}_{\tilde{\mathcal{P}}}[\int_{t_0}^{T} \langle V^{a,U^a,N^a}, U^a - N^a\rangle_s^p ds - \sum_{i=1}^{\infty}\int_{t_0}^{T} (U_r^{a,i} - N_r^{a,i})dH_r^{(i)}|\mathcal{F}_{t_0}] = 0.$$

Thus $Y_{t_0}^{\Gamma,j} \geq V_{t_0}^a$, $\tilde{\mathcal{P}} - a.s.$ and then, since $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are equivalent, for any $a \in \mathcal{A}_{t_0}^j$,

$$Y_{t_0}^{\Gamma,j} \geq V_{t_0}^a - A_{t_0}^a, \ \mathcal{P} - a.s.. \tag{5.1.5}$$

Next let us consider a^* the strategy defined by $a^*(r) = \alpha_0^* 1\!\!1_{\{t_0\}}(r) + \sum_{j=1}^{\infty}\alpha_{j-1}^* 1\!\!1_{]\theta_{j-1}^*\theta_j^*]}(r)$, $r \leq T$, where $\theta_0^* = t_0$, $\alpha_0^* = j$ and for $n \geq 0$,

$$\theta_{n+1}^* = \inf\{r \geq \theta_n^*, \ Y_r^{\Gamma,\alpha_n^*} = \max_{k\in A_{\alpha_n^*}}(Y_r^{\Gamma,k} - g_{\alpha_n^*,k}(r, X_r^{t,x}))\} \wedge T,$$

and

$$\alpha_{n+1}^* = arg\max_{k\in A_{\alpha_n^*}}\{Y_{\theta_{n+1}^*}^{\Gamma,k} - g_{\alpha_n^*,k}(\theta_{n+1}^*, X_{\theta_{n+1}^*}^{t,x})\}.$$

Let us show that $a^* \in \mathcal{A}_s^j$. We first prove that $\mathcal{P}[\theta_n^* < T, \ \forall n \geq 0] = 0$. We proceed by contradiction assuming that $\mathcal{P}[\theta_n^* < T, \ \forall n \geq 0] > 0$. By definition of θ_n^*, we then have

$$\mathcal{P}[Y_{\theta_{n+1}^*}^{\Gamma,\alpha_n^*} = Y_{\theta_{n+1}^*}^{\Gamma,\alpha_{n+1}^*} - g_{\alpha_n^*,\alpha_{n+1}^*}(\theta_{n+1}^*, X_{\theta_{n+1}^*}^{t,x}), \ \alpha_{n+1}^* \in A_{\alpha_n^*}, \ \forall n \geq 0] > 0.$$

But A is finite, then there is a loop $i_0, i_1, \cdots, i_k, i_0$ ($i_1 \neq i_0$) of elements of A and a subsequence $(n_q(\omega))_{q\geq 0}$ such that:

$$\mathcal{P}[Y_{\alpha_{n_{q+l}}^*}^{\Gamma,i_l} = Y_{\alpha_{n_{q+l}}^*}^{\Gamma,i_{l+1}} - g_{i_l,i_{l+1}}(\alpha_{n_{q+l}}^*, X_{\alpha_{n_{q+l}}^*}^{t,x}), \ l = 0, \cdots, k, \ (i_{k+1} = i_0), \ \forall q \geq 0] > 0. \tag{5.1.6}$$

Next let us consider $\theta^* = \lim_{n\to\infty}\theta_n^*$ and $\Theta = \{\theta_n^* < \theta^*, \forall n \geq 0\}$. Due to the non-free loop property $\mathcal{P}[(\theta^* < T) \cap \Theta^c] = 0$ and then θ^* is an accessible stopping time (see e.g. [16], pp.214 for more details). But for any $j \in A$, the process Y^j has only inaccessible jump times and θ^* is accessible. Therefore for any $j \in A$, $\Delta Y^j_{\theta^*} = 0, \mathcal{P} - a.s.$. Going back to (2.3.9) and take the limit w.r.t. q to obtain:

$$P[g_{i_0,i_1}(\theta^*, X^{t,x}_{\theta^*}) + \cdots + g_{i_k,i_0}(\theta^*, X^{t,x}_{\theta^*}) = 0] > 0,$$

which contradicts the no-loop property. We then have $\mathcal{P}[\theta_j^* < T,\ \forall j \geq 0] = 0$.

Now it remains to prove that $\mathbb{E}[(A^{a^*}_T)^2 < \infty]$ and a^* is optimal in $\mathcal{A}^j_s$ for the switching problem (5.1.3). Since $(Y^{\Gamma,j})_{j\in A}$ solves the RBSDE (2.2.35) and by the definition of a^*, it yields:

$$Y^{\Gamma,j}_{t_0} = Y^{\Gamma,j}_{\theta_1^*} + \int_{t_0}^{\theta_1^*} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta_1^*} U^{a^*,k}_r dH^{(k)}_r \tag{5.1.7}$$

since $K^{\Gamma,j}_r - K^{\Gamma,j}_{\theta_1^*} = 0$ holds for any $r \in [t_0, \theta_1^*]$. But

$$Y^{\Gamma,j}_{\theta_1^*} = (Y^{\Gamma,\alpha_1^*}_{\theta_1^*} - g_{j\alpha_1^*}(\theta_1^*, X^{t,x}_{\theta_1^*}))1_{[\theta_1^*<T]} + h_j(X^{t,x}_T)1_{[\theta_1^*=T]}$$

then

$$\begin{aligned} Y^{\Gamma,j}_{t_0} &= (Y^{\Gamma,\alpha_1^*}_{\theta_1^*} - g_{j\alpha_1^*}(\theta_1^*, X^{t,x}_{\theta_1^*}))1_{[\theta_1^*<T]} + h_j(X^{t,x}_T)1_{[\theta_1^*=T]} \\ &\quad + \textstyle\int_{t_0}^{\theta_1^*} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta_1^*} U^{a^*,k}_r dH^{(k)}_r \\ &= Y^{\Gamma,\alpha_1^*}_{\theta_1^*}1_{[\theta_1^*<T]} + h_j(X^{t,x}_T)1_{[\theta_1^*=T]} \\ &\quad + \textstyle\int_{t_0}^{\theta_1^*} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta_1^*} U^{a^*,k}_r dH^{(k)}_r - A^{a^*}_{\theta_1^*}. \end{aligned} \tag{5.1.8}$$

But we can do the same for the quantity $Y^{\Gamma,\alpha_1^*}_{\theta_1^*}1_{[\theta_1^*<T]}$ to obtain

$$Y^{\Gamma,\alpha_1^*}_{\theta_1^*}1_{[\theta_1^*<T]} = Y^{\Gamma,\alpha_1^*}_{\theta_2^*}1_{[\theta_2^*<T]} + h_{\alpha_1^*}(X^{t,x}_T)1_{[\theta_2^*=T]}1_{[\theta_1^*<T]} + \int_{\theta_1^*}^{\theta_2^*} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{\theta_1^*}^{\theta_2^*} U^{a^*,k}_r dH^{(k)}_r$$

Plug now this equality in the previous one and since α_2^* is the optimal index at θ_2^* to obtain:

$$\begin{aligned} Y^{\Gamma,j}_{t_0} &= (Y^{\Gamma,\alpha_2^*}_{\theta_2^*} - g_{\alpha_1^*\alpha_2^*}(\theta_2^*, X^{t,x}_{\theta_2^*}))1_{[\theta_2^*<T]} + h_{\alpha_1^*}(X^{t,x}_T)1_{[\theta_2^*=T]}1_{[\theta_1^*<T]} + h_j(X^{t,x}_T)1_{[\theta_1^*=T]} \\ &\quad + \textstyle\int_{t_0}^{\theta_2^*} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta_2^*} U^{a^*,k}_r dH^{(k)}_r - A^{a^*}_{\theta_1^*} \\ &= Y^{\Gamma,\alpha_2^*}_{\theta_2^*}1_{[\theta_2^*<T]} + h_{\alpha_1^*}(X^{t,x}_T)1_{[\theta_2^*=T]}1_{[\theta_1^*<T]} + h_j(X^{t,x}_T)1_{[\theta_1^*=T]} \\ &\quad + \textstyle\int_{t_0}^{\theta_2^*} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta_2^*} U^{a^*,k}_r dH^{(k)}_r - A^{a^*}_{\theta_2^*}. \end{aligned} \tag{5.1.9}$$

Repeating now this procedure as many times as necessary and since $\mathcal{P}[\theta_j^* < T,\ \forall j \geq 0] = 0$ to get

$$Y^{\Gamma,j}_s = h_{a^*(T)}(X^{t,x}_T) + \textstyle\int_{t_0}^{T} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{T} U^{a^*,k}_r dH^{(k)}_r - A^{a^*}_T. \tag{5.1.10}$$

Now since $\Gamma \in [H^2]^m$, $U^{a^*} \in \mathcal{H}^2(l^2)$ and $Y^{\Gamma,j} \in \mathcal{S}^2$, we deduce from (5.1.10) that $E[(A^{a^*}_T)^2] < \infty$. Next by (5.1.2),

$$\begin{aligned} V^{a^*}_{t_0} - A^{a^*}_{t_0} - Y^{\Gamma,j}_{t_0} &= \textstyle\int_{t_0}^{T} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, N^{a^*}_r)dr - \int_s^T f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr \\ &\quad - \sum_{k=1}^{\infty}\int_{t_0}^{T}(N^{a^*}_r - U^{a^*,k}_r)dH^{(k)}_r \\ &\geq \textstyle\int_{t_0}^{T}\langle V^{a^*,N^{a^*},U^{a^*}}, N^{a^*} - U^{a^*}\rangle^p_r dr - \sum_{k=1}^{\infty}\int_{t_0}^{T}(N^{a^*,k}_r - U^{a^*,k}_r)dH^{(k)}_r. \end{aligned}$$

Once more using Girsanov's theorem as previously, to obtain $\mathbb{E}_{\bar{\mathcal{P}}}[V^{a^*}_{t_0} - A^{a^*}_{t_0} - Y^{\Gamma,i}_{t_0}|F_{t_0}] \geq 0$ and then $V^{a^*}_{t_0} - A^{a^*}_{t_0} - Y^{\Gamma,i}_{t_0} \geq 0, \mathcal{P} - a.s.$ Taking now into account (5.1.5) leads to the desired result. □

Remark 5.1.1. *As a by product of (5.1.3) we have also:*

$$\forall j \in A,\ \mathbb{E}[Y^{\Gamma,j}_{t_0}] = \sup_{a\in\mathcal{A}^j_{t_0}} \mathbb{E}[V^a_{t_0} - A^a_{t_0}]. \quad \square$$

5.2 Second definition of a viscosity solution

Definition 5.2.1. *A function $(u_1, \cdots, u_m) : [0,T] \times I\!R \to I\!R^m \in \Pi_g$ such that for any $i \in A$, u^i is lsc (resp. usc), is said to be a viscosity subsolution of (4.1) (resp. supersolution) if: $\forall i \in A$,*

(i) $u^i(T,x) \leq h_i(x)$ (resp. $u^i(T,x) \geq h_i(x)$)

(ii) for any $\forall\,(t_0,x_0) \in (0,T) \times I\!R$, $\delta > 0$ and a function $\varphi \in C^{1,2}([0,T] \times I\!R) \cap \Pi_g$ such that $u^i(t_0,x_0) = \varphi(t_0,x_0)$ and $u_i - \varphi$ has a global maximum (resp. minimum) at (t_0,x_0) on $[0,T] \times B(x_0, C_\sigma \delta)$, we have

$$
\begin{aligned}
min\Big\{ & u^i(t_0,x_0) - \max_{j \in A_i}(u^j(t_0,x_0) - g_{ij}(t_0,x_0)); -\partial_t\varphi(t_0,x_0) - \mathcal{L}^1\varphi(t,x) - \mathcal{I}^{1,\delta}(t,x,\varphi) \\
& - I^{2,\delta}(t,x,D_x\varphi(t,x),u^i) \\
& - f_i(t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0)) \Big\} \leq 0 \quad (resp. \geq 0).
\end{aligned}
$$

The functions $(u^i)_{i=1}^m$ is called a viscosity solution of (2.3.1) *if $(u_*^i)_{i=1}^m$ and $u^{i*})_{i=1}^m$ are respectively viscosity supersolution and viscosity subsolution of* (2.3.1).

Proposition 5.2.1. *Definitions (5.2.1) and (2.3.1) are equivalent.*

Proof. First let us show that Def. 5.2.1 implies Def. 2.3.1. We prove it only for the subsolution case since the supersolution one is similar. Let $(u^i)_{i\in A}$ be a subsolution of system (2.2.20) according to Definition 5.2.1. Wlog assume that u^i, $i \in A$, is usc. Thus for any $i \in A$, $u^i(T,x) \leq h_i(x)$, for any $x \in I\!R$. Next let us fix $i \in A$, $(t_0,x_0) \in (0,T) \times I\!R$ and finally let us consider $\varphi \in \Pi_g \bigcap C^{1,2}([0,T] \times I\!R)$, such that $u^i - \varphi$ has a global maximum at (t_0,x_0) in $[0,T] \times I\!R^k$. Next let us set $\bar\varphi(t,x) := \varphi(t,x) + u_i(t_0,x_0) - \varphi(t_0,x_0)$. Then $\bar\varphi$ belongs to $\Pi_g \bigcap C^{1,2}([0,T] \times I\!R)$ and $u_i - \bar\varphi$ has a global maximum at (t_0,x_0) in $[0,T] \times I\!R^k$ and verifies $\bar\varphi(t_0,x_0) := u^i(t_0,x_0)$. Applying now Def. 5.2.1 with $\bar\varphi$ yields:

$$
\begin{aligned}
min\Big\{ & u^i(t_0,x_0) - \max_{j \in A_i}(u^j(t_0,x_0) - g_{ij}(t_0,x_0)); -\partial_t\varphi(t_0,x_0) - \mathcal{L}^1\varphi(t_0,x_0) \\
& - \mathcal{I}^{1,\delta}(t_0,x_0,\varphi) - \mathcal{I}^{2,\delta}(t_0,x_0,u_i,D_x\varphi(t_0,x_0)) \\
& - f_i((t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0)) \Big\} \leq 0
\end{aligned}
$$

for any $\delta > 0$. Next since $(t_0,x_0) \in (0,T) \times I\!R$ is global maximum point of $u^i - \varphi$, we then have

$$
u^i(t_0, x_0 + \sigma(t_0,x_0)y) - u^i(t_0,x_0) \leq \varphi(t_0, x_0 + \sigma(t_0,x_0)y) - \varphi(t_0,x_0)
$$

which implies that $I^{2,\delta}(t_0,x_0,D_x\varphi(t_0,x_0),u^i) \leq I^{2,\delta}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi)$ and then

$$
\begin{aligned}
min\{ & u^i(t_0,x_0) - \max_{j \in A_i}(u^j(t_0,x_0) - g_{ij}(t_0,x_0)); -\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) \\
& - f_i((t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\} \leq 0
\end{aligned}
$$

which means that $(u^i)_{i\in A}$ is a subsolution for (2.2.20) according to Def. 2.3.1.

We are going now to show that Def.2.3.1 implies Def. 5.2.1. Once more let us consider $(u^i)_{i\in A}$ a subsolution of system (2.2.20) according to Definition 2.3.1 which, wlog, we assume that they are *usc* functions. Then for any $i \in A$, $u^i(T,x) \leq h_i(x)$, for all $x \in I\!R$. Next let us fix $\delta > 0$, $i \in A$, $(t_0,x_0) \in (0,T) \times B(x_0, C_\sigma\delta)$ and finally let us consider $\varphi \in \Pi_g \bigcap C^{1,2}([0,T] \times I\!R)$, such that $u^i - \varphi$ has a global maximum at (t_0,x_0) on $[0,T] \times B(x_0, C_\sigma\delta)$ and $u^i(t_0,x_0) = \varphi(t_0,x_0)$. But there exists a function $\tilde\varphi$ which belongs to $\Pi_g \bigcap C^{1,2}([0,T] \times I\!R)$ such that $u - \tilde\varphi$ attains a global maximum in (t_0,x_0) on $[0,T] \times I\!R$ and satisfying $\tilde\varphi(s,y) = \varphi(s,y)$, for any (s,y) such that $|(s,y) - (t_0,x_0)| < \frac{C_\sigma\delta}{2}$. Consequently we have also

$$
\partial_t\tilde\varphi(t_0,x_0) = \partial_t\varphi(t_0,x_0),\ D_x\tilde\varphi(t_0,x_0) = D_x\varphi(t_0,x_0),\ D^2_{xx}\tilde\varphi(t_0,x_0) = D^2_{xx}\varphi(t_0,x_0),\ u(t_0,x_0) = \tilde\varphi(t_0,x_0). \tag{5.2.1}
$$

Next for any $\epsilon > 0$, there exists φ_ϵ element of $C^{1,2}([0,T] \times I\!R)$ such that $u_i \leq \varphi_\epsilon \leq \tilde\varphi$ and $\varphi_\epsilon \to u_i$ as $\epsilon \to 0$, a.e. (see e.g. Lemma 4.7 in [37] or [2]). It implies that $u_i - \varphi_\epsilon$ and $\varphi_\epsilon - \tilde\varphi$ have a global maximum at (t_0,x_0) on $[0,T] \times I\!R^k$. Therefore, on the one hand, we have

$$
\partial_t\varphi_\epsilon(t_0,x_0) = \partial_t\tilde\varphi(t_0,x_0),\ D_x\varphi_\epsilon(t_0,x_0) = D_x\tilde\varphi(t_0,x_0),\ D^2_{xx}\varphi_\epsilon(t_0,x_0) \leq D^2_{xx}\tilde\varphi(t_0,x_0) \tag{5.2.2}
$$

and, on the other hand, by Def. 2.3.1 it holds

$$\begin{aligned} min\Big\{ & u^i(t_0,x_0) - \max_{j\in A_i}\{u^j(t_0,x_0) - g_{ij}(t_0,x_0)\}\,; \\ & -\partial_t\varphi_\epsilon(t_0,x_0) - \mathcal{L}\varphi_\epsilon(t_0,x_0) - f_i(t_0,x_0,u^1(t_0,x_0),\cdots, \\ & u^{i-1}(t_0,x_0), u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\Big\} \le 0. \end{aligned} \tag{5.2.3}$$

Now (see (2.3.2) for the definition of $\mathcal{L}^1$)

$$\mathcal{L}\varphi_\epsilon(t_0,x_0) = \mathcal{L}^1\varphi_\epsilon(t_0,x_0) + \mathcal{I}(t_0,x_0,\varphi_\epsilon)$$

and by (5.2.1) and (5.2.2)

$$\mathcal{L}^1\varphi_\epsilon(t_0,x_0) \le \mathcal{L}^1\varphi(t_0,x_0). \tag{5.2.4}$$

On the other hand

$$\begin{aligned} \mathcal{I}(t_0,x_0,\varphi_\epsilon) &= \mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\varphi_\epsilon) + \mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi_\epsilon(t_0,x_0),\varphi_\epsilon) \\ &\le \mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\tilde{\varphi}) + \mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi_\epsilon) \\ &= \mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\varphi) + \mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi_\epsilon). \end{aligned} \tag{5.2.5}$$

Plug now (5.2.4) and (5.2.5) in (5.2.3) to get

$$\begin{aligned} & min\Big\{ u^i(t_0,x_0) - \max_{j\in A_i}\{u^j(t_0,x_0) - g_{ij}(t_0,x_0)\}\,; -\partial_t\varphi_\epsilon(t_0,x_0) - \mathcal{L}^1\varphi(t_0,x_0) - \mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\varphi) \\ & \quad -\mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi_\epsilon) - f_i(t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\Big\} \\ & \le 0. \end{aligned} \tag{5.2.6}$$

Take now the limit as $\epsilon \to 0$ in (5.2.1), using the Lebesgue dominated convergence theorem and by the following inequality (since $u_i \le \varphi$ in $[0,T]\times B(x_0, C_\sigma\delta)$)

$$\begin{aligned} & \textstyle\int_{\frac{\delta}{2}<|z|\le\delta}(\varphi(t_0,x_0+\sigma(t_0,x_0)z) - \varphi(t_0,x_0) - D_x\varphi(t_0,x_0)\sigma(t_0,x_0)z\}d\Pi(z) \ge \\ & \qquad \textstyle\int_{\frac{\delta}{2}<|z|\le\delta}(u_i(t_0,x_0+\sigma(t_0,x_0)z) - u_i(t_0,x_0) - D_x\varphi(t_0,x_0)\sigma(t_0,x_0)z\}d\Pi(z) \end{aligned}$$

we obtain

$$\begin{aligned} min\Big\{ & u^i(t_0,x_0) - \max_{j\in A_i}(u^j(t_0,x_0) - g_{ij}(t_0,x_0)); -\partial_t\varphi(t_0,x_0) - \mathcal{L}^1\varphi(t_0,x_0) \\ & - \mathcal{I}^{1,\delta}(t_0,x_0,\varphi) - \mathcal{I}^{2,\delta}(t_0,x_0,u_i,D_x\varphi(t_0,x_0)) \\ & - f_i((t_0,x_0,u^1(t_0,x_0),\cdots,u^{i-1}(t_0,x_0),u^i(t_0,x_0),\cdots,u^m(t_0,x_0))\Big\} \le 0 \end{aligned}$$

which is the desired result. □

5.3 BSDE with two reflected barriers

1 Representation of a penalization scheme of two barriers reflected BSDE

For $n \ge 0$ let (Y^n, Z^n, u^n, K^n) be the solution of the following one barrier reflected BSDE.

$$\begin{cases} (Y^n, Z^n, U^n, K^n) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2; \\ Y^n_s = \xi - \int_s^T g_r dr - n\int_s^T (Y^n_r - H_s)^+ ds + \int_s^T Z^n_r dB_r + \int_s^T \int_E U^n_r(e)\hat{N}(drde) + K^n_T - K^n_s \\ Y^n_s \ge L_s, \quad s \le T, \\ \int_0^T (Y^n_s - L_s)dK^n_s = 0, \end{cases} \tag{5.3.1}$$

where, the processes L and H belongs to $\mathcal{S}^2$, $(g_s)_{s\in[0,T]} \in \mathcal{H}^2$, ξ is square integrable and F_T-measurable. Moreover, we require that $L \le H$ and $L_T \le \xi$. The solution (Y^n, Z^n, U^n, K^n) exists and is unique (see e.g. [5]). Set $K^{n,-}_t = n\int_0^t (Y^n_s - H_s)^+ ds$, then $K^{n,-}_t \in \mathcal{A}^2$ and $\int_0^T (Y^n_s - Y^n_s \vee H_s)K^{n,-}_s = 0$. Therefore,

the equation (5.3.1) can be expressed as a BSDE with two reflecting barrier in the following manner. For all $t \leq T$,

$$\begin{cases} (Y^n, Z^n, U^n, K^n) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2(\hat{N}) \times \mathcal{A}^2; \\ Y_s^n = \xi - \int_s^T g_r dr + \int_s^T Z_r^n dB_r + \int_s^T \int_E U_r^n(e)\hat{N}(drde) + K_T^n - K_s^n - (K_T^{n,-} - K_s^{n,-}) \\ L_s \leq Y_s^n \leq Y_s^n \vee H_s, \quad s \leq T, \\ \int_0^T (Y_s^n - L_s) dK_s^n = \int_0^T (Y_s^n - Y_s^n \vee H_s) dK_s^{n,-} = 0, \end{cases} \tag{5.3.2}$$

Thus, a result by Hamadène and Hassani [27] allows to represent Y^n as a value function of a Dynkin game, i.e., it holds true that for any $t \leq T$,

$$\begin{aligned} Y_t^n &= ess\sup_{\sigma \geq t} ess\inf_{\tau \geq t} E[\int_t^{\sigma\wedge\tau} g_s ds + L_\sigma \mathbb{1}_{[\sigma<\tau]} + (Y_\tau^n \vee H_\tau)\mathbb{1}_{[\tau\leq\sigma<T]} + \xi \mathbb{1}_{[\tau=\sigma=T]}|F_t] \\ &= ess\inf_{\tau \geq t} ess\sup_{\sigma \geq t} E[\int_t^{\sigma\wedge\tau} g_s ds + L_\sigma \mathbb{1}_{[\sigma<\tau]} + (Y_\tau^n \vee H_\tau)\mathbb{1}_{[\tau\leq\sigma<T]} + \xi \mathbb{1}_{[\tau=\sigma=T]}|F_t], \end{aligned}$$

where τ and σ are F-stopping times.

2 Relation with double obstacle variational inequality

Let $g(t,x,y,z,u)$ satisfies the Assumption (A1), $L : (t,x) \mapsto L(t,x)$ and $H : (t,x) \mapsto H(t,x)$ are jointly continuous and of polynomial growth, i.e., there exist positve constant C and p such that: $\forall (t,x) \in [0,T] \times R$,

$$|L(t,x)| + H(t,x)| \leq C(1 + |x|^p).$$

The functions $h(x) : R \to R$ are continuous w.r.t. x, belong to class Π_g. Furthermore, for any $(s,x) \in [0,T] \times R$:

$$L(s,x) \leq H(s,x), and\ L(T,x) \leq h(x) \leq H(T,x).$$

Now let $(Y_s, Z_s, U_s, K_s^+, K_s^-)_{s\leq T}$ be the solution of the BSDE with two reflecting barriers associated with $(g(s, X_s^{t,x}, y, z, u), h(X_T^{t,x}), L(s, X_s^{t,x}), H(s, X_s^{t,x})$, which exists and is unique by Theorem 2.2 [27].(Their function g does not depend on u, the reason is to use comparison theorem, but in our case, if g satisfies the assumption (A1), we have also the comparison theorem by [5].) For $n \geq 0$, let $({}^nY_s)_{s\leq T}(resp.({}^n\bar{Y}_s)_{s\leq T})$ be the first component of the unique solution of the BSDE with on reflecting lower (resp. upper) barrier associated with $(g(s, X_s^{t,x}, y, z, u) - n(H(s, X_s^{t,x} - y)^-, h(X_T^{t,x}), L(s, X_s^{t,x}))$ (resp. $(g(s, X_s^{t,x}, y, z, u) - n(L(s, X_s^{t,x} - y)^-, h(X_T^{t,x}), H(s, X_s^{t,x}))))$. It has been shown in [5], that for any $n \geq 0$ there exist functions ${}^nu(t,x)$ and ${}^n\bar{u}$, $(t,x) \in [0,T] \times R$, such that

$$\forall s \in [t,T], \quad {}^nY_s = {}^n\, u(s, X_s^{t,x})\ and\ {}^n\bar{Y}_s = {}^n\, \bar{u}(s, X_s^{t,x}),$$

where ${}^nu$ (resp. ${}^n\bar{u}$) is continuous with polynomial growth and be a viscosity solution for the following obstacle prblem:

$$\begin{cases} min\{v(t,x) - L(t,x), -\partial_t v(t,x) - \mathcal{L}v(t,x) \\ -g(t,x,v(t,x), \sigma(t,x)D_x v(t,x), Bv(t,x)) + n(H(t,x) - v(t,x))^-\} = 0 \\ v(T,x) = h(x) \end{cases}$$

(resp.

$$\begin{cases} max\{v(t,x) - H(t,x), -\partial_t v(t,x) - \mathcal{L}v(t,x) \\ -g(t,x,v(t,x), \sigma(t,x)D_x v(t,x), Bv(t,x)) + n(L(t,x) - v(t,x))^-\} = 0 \\ v(T,x) = h(x), \end{cases}$$

where,

$$\mathcal{L}\phi(t,x) := b(t,x)D_x\phi(t,x) + \frac{1}{2}\sigma^2(t,x)D_{xx}^2\phi(t,x) + \int_E (\phi(t,x+\beta(x,e)) - \phi(t,x) - D_x\phi(t,x)\beta(x,e))n(de),$$

$$B\phi(t,x) = \int_E (\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma(x,e)n(de).$$

The comparison result allows us to infer that $({}^nY)_{n\geq 0}$ (resp. $({}^n\bar{Y})_{n\geq 0}$) is a decreasing (resp. increasing) sequence, moreover they converge in $\mathcal{S}^2$ to Y. Therefore for any $(t,x) \in [0,T] \times R$, the sequence $({}^nY(t,x))_{n\geq 0}$ (resp. $({}^n\bar{u}(t,x))_{n\geq 0}$) converges decreasingly (resp. increasingly) to the same limit $u(t,x) := Y_t$ which satisfies $Y_s = u(s, X_s^{t,x})$ for any $s \in [s,T]$. u is continuous with polynomial growth since ${}^nu$

and ${}^n\bar{u}$ are so. It implies that the convergence of ${}^nu$ and ${}^n\bar{u}$ to u are uniform on compact subsets of $[0,T]\times R$. Now consider the following obstacle problem:

$$\begin{cases} min\{v(t,x)-L(t,x), max[v(t,x)-H(t,x), \\ -\partial_t v(t,x)-\mathcal{L}v(t,x)-g(t,x,v(t,x),\sigma(t,x)D_xv(t,x),Bv(t,x))]\}=0 \\ v(T,x)=h(x) \end{cases} \tag{5.3.3}$$

Theorem 5.3.1. *The function u defined above is a viscosity solution of (5.5.4) and for any $(t,x)\in[0,T]\times R$ and any stopping times $\nu,\sigma\in \mathrm{T}_t$ where T_t is the set of F_t stopping time take values in $[t,T]$,*

$$u(t,x)=\inf_{\nu\in\mathrm{T}_t}\sup_{\sigma\in\mathrm{T}_t} J_{t,x}(\nu,\sigma)=\sup_{\sigma\in\mathrm{T}_t}\inf_{\nu\in\mathrm{T}_t} J_{t,x}(\nu,\sigma)=J_{t,x}(\delta_{t,x},\theta_{t,x}),$$

where

$$J_{t,x}(\nu,\sigma):=\begin{cases} E[\int_t^{\nu\wedge\sigma} g(s,X_s^{t,x},Y_s,Z_s,\int_E U(e)_s\lambda(X_s^{t,x},e)n(de))ds+L(\sigma,X_\sigma^{t,x})\mathbb{1}_{[\sigma\le\nu<T]} \\ \quad +H(\nu,X_\nu^{t,x})\mathbb{1}_{[\nu<\sigma]}+g(X_T^{t,x})\mathbb{1}_{[\nu=\sigma=T]}] \\ \quad if\ E[\int_t^{\nu\wedge\sigma}|g(s,X_s^{t,x},Y_s,Z_s,\int_E U(e)_s\lambda(X_s^{t,x},e)n(de))|ds]<\infty \\ +\infty\ \ else; \end{cases}$$

where, $\delta_{t,x}:=inf\{s\ge t, Y_s=H(s,X_s^{t,x})\}\wedge T$ and $\theta_{t,x}:=inf\{s\ge t, Y_s=L(s,X_s^{t,x})\}\wedge T$.

Proof. First we show that u is a viscosity subsolution of (5.5.4). Note that $u(T,x)=h(x)$ and $L(t,x)\le u(t,x)\le H(t,x)$, for any $\phi\in C^{1,2}([0,T]\times R)\bigcap\Pi_g$ and for any local maximum point $(t,x)\in[0,T]\times R$ of $u-\phi$ such that $u(t,x)>L(t,x)$, let (t_n,x_n) be a sequence of local maximum points of ${}^nu-\phi$ such that (t_n,x_n) converges to (t,x). For n large enough we have ${}^nu(t_n,x_n)>L(t_n,x_n)$, then we have

$$\begin{aligned} &-\partial_t\phi(t_n,x_n)-b(t_n,x_n)D_x\phi(t_n,x_n)+\frac{1}{2}\sigma^2(t_n,x_n)D_{xx}^2\phi(t_n,x_n) \\ &+\int_E(\phi(t_n,x_n+\beta(x_n,e))-\phi(t_n,x_n)-D_x\phi(t_n,x_n)\beta(x_n,e))n(de) \\ &-g(t_n,x_n,n^u(t_n,x_n),\sigma(t_n,x_n)D_x\phi(t_n,x_n),\int_E(\phi(t_n,x_n+\beta(x_n,e))-\phi(t_n,x_n))\gamma(x_n,e)n(de)) \\ &+n(H(t_n,x_n)-{}^n\,u(t_n,x_n))^-\le 0 \end{aligned}$$

Note that since $\lim_{n\to\infty}x_n=x$, and ϕ is a $C^{1,2}$ function

$$|\phi(t_n,x_n+\beta(x_n,e))-\phi(t_n,x_n)-D_x\phi(t_n,x_n)\beta(x_n,e)|\le C|\beta(x_n,e)|^2\le K^2(1+|e|)^2,$$

and

$$|(\phi(t_n,x_n+\beta(x_n,e))-\phi(t_n,x_n))\gamma(x_n,e)|\le C|\beta(x_n,e)|\gamma(x_n,e)\le K_1^2(1+|e|)^2+\frac{1}{4}K_2^2(1+|e|)^2.$$

These combine with Dominated convergence theorem,

$$\begin{aligned} &\lim_{n\to\infty}\int_E(\phi(t_n,x_n+\beta(x_n,e))-\phi(t_n,x_n)-D_x\phi(t_n,x_n)\beta(x_n,e))n(de) \\ =&\int_E(\phi(t,x+\beta(x,e))-\phi(t,x)-D_x\phi(t,x)\beta(x,e))n(de), \end{aligned}$$

and

$$\lim_{n\to\infty}\int_E(\phi(t_n,x_n+\beta(x_n,e))-\phi(t_n,x_n))\gamma(x_n,e)n(de)=\int_E(\phi(t,x+\beta(x,e))-\phi(t,x))\gamma(x,e)n(de).$$

Now from the continuity of the functions and the uniform convergence, we have

$$\begin{aligned} &-\partial_t\phi(t,x)-b(t,x)D_x\phi(t,x)+\frac{1}{2}\sigma^2(t,x)D_{xx}^2\phi(t,x) \\ &+\int_E(\phi(t,x+\beta(x,e))-\phi(t,x)-D_x\phi(t,x)\beta(x,e))n(de) \\ &-g(t,x,u(t,x),\sigma(t,x)D_x\phi(t,x),\int_E(\phi(t,x+\beta(x,e))-\phi(t,x))\gamma(x,e)n(de)) \\ &\le 0. \end{aligned}$$

In a similar way we can show that u is also a viscosity supersolution. □

5.4 Viscosity solution of system of variational inequalities with interconnected bilateral obstacles and connections to multiple modes switching game of jump-diffusion processes

First let $A := \{1, \cdots, m\}$, and let us introduce the following functions f_i, h_i and g_{ij}, $i, j \in A$:

$$\begin{array}{rccl} g^i & : & [0,T] \times I\!R^k \times I\!R^m \times I\!R & \longrightarrow I\!R \\ & & (t, x, (y^i)_{i=1,m}, u) & \longmapsto g^i(t, x, (y^i)_{i=1,m}, u) \end{array}$$

$$\begin{array}{rccl} h_i \quad (\text{resp. } g_{ij}) & : & [0,T] \times I\!R^k & \longrightarrow I\!R \\ & & (t, x) & \longmapsto h_i(t, x) \ (\text{resp. } g_{ij}(t, x)) \end{array}$$

which satisfy:

Assumption (H1)
(I) For any $i \in A$, $g^i(t, x, \overrightarrow{y}, z, q) : R \times R \times R^m \times R \times R \to R$,

(i) is continuous in (t, x) uniformly w.r.t. the other variables $(\overrightarrow{y}, z, q)$ and for any (t, x) the mapping $(t, x) \to g^i(t, x, 0, 0, 0)$ is of polynomial growth.

(ii) satisfies the standard hypothesis of Lipschitz continuity w.r.t. the variables $(\overrightarrow{y}, z, q)$, i.e. $\forall (t, x) \in [0, T] \times R, \forall (\overrightarrow{y}_1, \overrightarrow{y}_2) \in R^m \times R^m, (z_1, z_2) \in R \times R, (q_1, q_2) \in R \times R$,

$$|g^i(t, x, \overrightarrow{y}_1, z_1, q_1) - g^i(t, x, \overrightarrow{y}_2, z_2, q_2)| \leq C(|\overrightarrow{y}_1 - \overrightarrow{y}_2| + |z_1 - z_2| + |q_1 - q_2|),$$

where, $|\overrightarrow{y}|$ stands for the standard Euclidean norm of $\overrightarrow{y}$ in R^m.

(iii) $q \mapsto g^i(t, x, y, z, q)$ is non-decreasing, for all $(t, x, y, z) \in [0, T] \times R \times R^m \times R$.

Futhermore, let $\gamma^i : R \times \mathcal{B}_E \to R$ such that there exists $C > 0$,

$$0 \leq \gamma^i(x, e) \leq C(1 \wedge |e|), \quad x \in R, e \in \mathcal{B}_E$$
$$|\gamma^i(x, e) - \gamma^i(x', e')| < C|x - x'|(1 \wedge |e|), \quad x, x' \in R, e \in E.$$

We set

$$f^i(t, x, y, z, u) = g^i(t, x, y, z, \int_E u(e)\gamma^i(x, e)n(de),$$

for $(t, x, y, z, u) \in [0, t] \times R \times R^m \times R \times \mathcal{L}^2(R, \mathcal{B}_E, n)$.

(iv) Monotonicity: For any $i \in A$ and any $k \neq i$ the mapping $y^k \to g^i(t, x, y^1, \cdots, y^k, \cdots, y^m, z, u)$ is non-decreasing.

(II) $\forall i \in A$, $g_{ii} \equiv 0$ and for $i \neq j$, $g_{ij}(t, x)$ is non-negative, continuous with polynomial growth and satisfy the following non-free loop property: $\forall (t, x) \in [0, T] \times R$ and for any sequence of indices $i_1, \cdots, i_k$ such that $i_1 = i_k$ and $card\{i_1, \cdots, i_k\} = k - 1$ we have:

$$g_{i_1 i_2}(t, x) + g_{i_2 i_3}(t, x) + \cdots + g_{i_k i_1}(t, x) > 0, \ \forall (t, x) \in [0, T] \times I\!R^k.$$

(III) $\forall i \in A$, h_i is continuous with polynomial growth and satisfies the following coherance conditions:

$$h_i(x) \geq \max_{j \in A^{-i}} (h_j(x) - g_{ij}(T, x)), \forall x \in I\!R.$$

Now let $(t, x) \in [0, T] \times I\!R^k$ and let us consider the following system of reflected BSDEs with oblique reflection: $\forall j = 1, ..., m$

$$\left\{ \begin{array}{l} Y^j \in \mathcal{S}^2, Z^j \in \mathcal{H}^2, \quad U^j \in \mathcal{H}^2(\hat{N}), \quad K^j \in \mathcal{A}^2 \\ Y^j_s = h_j(X^{t,x}_T) + \int_s^T f_j(r, X^{t,x}_r, Y^1_r, Y^2_r, \cdots, Y^m_r, Z^j_r, U^j_r)dr - \int_s^T Z^j_r dB_r - \int_s^T \int_E U^j_r(e)\hat{N}(drde) \\ +K^j_T - K^j_s, \ \forall s \leq T; \\ \forall s \leq T, \quad Y^j_s \geq \max_{k \in Aj}\{Y^k_s - g_{jk}(s, X^{t,x}_s)\} \text{ and } \int_0^T \{Y^j_s - \max_{k \in A_j}\{Y^k_s - g_{jk}(s, X^{t,x}_s)\}dK^j_s = 0. \end{array} \right. \tag{5.4.1}$$

We then have the following result:

Theorem 5.4.1. *Assume that Assumption (H1) is fulfilled. Then system of reflected BSDE with oblique reflection (5.4.1) has a unique solution.*

Proof. It will be given in two steps.

Step 1: Let us consider the following BSDEs :

$$\bar{Y}_s = \max_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \cdots, \bar{Y}_r, \bar{Z}_r, \bar{U}_r)dr - \int_s^T \bar{Z}_r dB_r - \int_s^T \int_E \bar{U}_r(e)\hat{N}(drde)$$

and

$$\underline{Y}_s = \min_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \min_{j=1,m} f_j(r, X_r^{t,x}, \underline{Y}_r, \cdots, \underline{Y}_r, \underline{Z}_r, \underline{U}_r)dr - \int_s^T \underline{Z}_r dB_r - \int_s^T \int_E \underline{U}_r(e)\hat{N}(drde).$$

First note that the above BSDEs have unique solutions thanks to Theorem 2.1 in [5]. For $j = 1, \cdots, m$ and $n \geq 1$, let us define $(Y^{j,n}, U^{j,n}, K^{j,n})$ by:

$$\begin{cases} Y^{j,n} \in \mathcal{S}^2, Z^{j,n} \in \mathcal{H}^2, \quad U^{j,n} \in \mathcal{H}^2(\hat{N}), \quad K^{j,n} \in \mathcal{A}^2 \\ Y^{j,0} = \underline{Y}_s \\ Y_s^{j,n} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^{1,n-1}, \cdots, Y_r^{j-1,n-1}, Y_r^{j,n}, Y_r^{j+1,n-1}, \cdots, Y^{m,n-1}, Z_r^{j,n}, U_r^{j,n})dr \\ - \int_s^T Z_r^{j,n} dB_r - \int_s^T \int_E U_r^{j,n}(e)\hat{N}(drde) + K_T^{j,n} - K_s^{j,n}, \qquad \forall s \leq T; \\ Y_s^{j,n} \geq \max_{k \in A_j}(Y_s^{k,n-1} - g_{jk}(s, X_s^{t,x})), \ \forall s \leq T; \quad \int_0^T [Y_r^{j,n} - \max_{k \in A_j}(Y_r^{k,n-1} - g_{jk}(r, X_r^{t,x}))]dK_r^{j,n} = 0. \end{cases} \tag{5.4.2}$$

By induction on n we can show that system (5.4.2) has a unique soluton for any fixed $n \geq 1$. On the other hand it is easy to show that $(\bar{Y}, \bar{U}, 0)$ is also a solution of :

$$\begin{cases} \bar{Y}_s = \max_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \cdots, \bar{Y}_r, \bar{Z}_r, \bar{U}_r)dr - \int_s^T \bar{Z}_r dB_r - \int_s^T \int_E \bar{U}_r(e)\hat{N}(drde) + \bar{K}_T - \bar{K}_s \\ \bar{Y}_s \geq \max_{k \in A_j}(\bar{Y}_s - g_{jk}(s, X_s^{t,x})), \ \forall s \leq T; \quad \int_0^T [\bar{Y}_r - \max_{k \in A_j}(\bar{Y}_s - g_{jk}(s, X_s^{t,x}))]d\bar{K}_r = 0. \end{cases}$$

Next since $\forall i \in A$, f_i verifies the Assumption H1(I), by induction we have: $\forall n, j, \forall s \leq T, \quad Y_s^{j,n-1} \leq Y_s^{j,n} \leq \bar{Y}_s, \quad \mathcal{P} - a.s.$, and $E[\sup_{s \in [0,T]} |\bar{Y}_s|^2] < \infty$. Then sequence $(Y^{j,n})_{n \geq 0}$, has a limit which we denote by Y^j, for any $j \in A$. By the monotonic limit theorem in [24](Theorem 3.1, pp697), $Y^j \in \mathcal{S}^2$ and there exist $Z^j \in \mathcal{H}^2$, $U^j \in \mathcal{H}^2(\hat{N})$, $K^j \in \mathcal{A}^2$ such that

$$\begin{cases} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^1, Y_r^2, \cdots, Y_r^m, Z_r^j, U_r^j)dr - \int_s^T Z_r^j dB_r \\ - \int_s^T \int_E U_r^j(e)\hat{N}(drde) + K_T^j - K_s^j \\ Y_s^j \geq \max_{k \in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})); \end{cases} \tag{5.4.3}$$

where for any $j \in A$, U^j is the weak limit of $(U^{j,n})_{n \geq 1}$ in $\mathcal{H}^2(\hat{N})$, Z^j is the weak limit of $Z^{j,n}$ in $\mathcal{H}^2$ and for any stopping time τ, K_τ^j is the weak limit of $K_\tau^{j,n}$ in $L^2(\Omega, \mathcal{F}_\tau, \mathcal{P})$. Finally note that K^j is predictable since the processes $K^{n,j}$ are so, $\forall n \geq 1$.
Let us now consider the following RBSE:

$$\begin{cases} \hat{Y}^j \in \mathcal{S}^2, \hat{Z}j \in \mathcal{H}^2, \quad \hat{U}^j \in \mathcal{H}^2(\hat{N}), \quad \hat{K}^j \in \mathcal{A}^2 \\ \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^1, \cdots, Y_r^{j-1}, \hat{Y}^j, Y_r^{j+1}, \cdots, Y_r^m, \hat{Z}_r^j, \hat{U}_r^j)dr \\ - \int_s^T \hat{Z}_r^j dB_r - \int_s^T \int_E \hat{U}_r^j(e)\hat{N}(drde) \\ + \hat{K}_T^j - \hat{K}_s^j, \ \forall s \leq T; \\ \forall s \leq T, \quad \hat{Y}_s^j \geq \max_{k \in Aj}\{Y_s^k - g_{jk}(s, X_s^{t,x})\} \text{ and } \int_0^T \{\hat{Y}_s^j - \max_{k \in A_j}\{Y_s^k - g_{jk}(s, X_s^{t,x})\}dK_s^j = 0. \end{cases} \tag{5.4.4}$$

According to Theorem 1.2 in [31], this equation has a unique solution. Apply Tanaka-Meyer's formula (see e.g.[55] on $(\hat{Y}_t^j - Y_t^j)^+$ in $[s, T]$, we can prove that $\mathcal{P}$-a.s., $\hat{Y}^j \leq Y^j$ for any $j \in A$. On the other hand, since $\forall n \geq 1$, $\forall j \in A$, $Y^{j,n-1} \leq Y^j$, we have

$$\max_{k \in A_j}(Y_s^{k,n-1} - g_{jk}(s, X_s^{t,x})) \leq \max_{k \in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})), \forall s \leq T.$$

Then by Comparison Theorem, we obtain $Y^{j,n} \leq \hat{Y}^j$, thus $Y^j \leq \hat{Y}^j$ which implies $Y^j = \hat{Y}^j$, $\forall j \in A$.

Next using Itô's formula with $Y^j - \hat{Y}^j$ and taking expectation in both-hand sides, we obtain

$$\mathbb{E}[\int_0^T (Z_r^j - \hat{Z}_r^j)^2 dr + \int_0^T \int_E (U_r^j(e) - \hat{U}_r^j(e))^2 \hat{N}(dedr)] = 0.$$

It implies that $Z^j = \hat{Z}^j$, $U^j = \hat{U}^j$, $dt \otimes d\mathcal{P}$ and finally $K^j = \hat{K}^j$ for any $j \in A$, i.e. $(Y^j, Z^j, U^j, K^j)_{j\in A}$ verify (5.4.4).

Next we will show that the predictable process K^j does not have jumps. So assume there exists $j_1 \in A$ and a predictable stopping time τ such that $\triangle Y_\tau^{j_1} = -\triangle K_\tau^{j_1} = -\triangle \hat{K}_\tau^{j_1} < 0$. Then by the second equality in (5.4.4) we have

$$Y_{\tau-}^{j_1} = \max_{k\in A^{-j_1}} (Y_{\tau-}^k - g_{j_1 k}(\tau, X_{\tau-}^{t,x})). \tag{5.4.5}$$

Now let $j_2 \in A^{-j_1}$ be the optimal index in (5.4.5), i.e.,

$$Y_{\tau-}^{j_2} - g_{j_1,j_2}(\tau, X_\tau^{t,x}) = Y_{\tau-}^{j_1} > Y_\tau^{j_1} \geq Y_\tau^{j_2} - g_{j_1,j_2}(\tau, X_\tau^{t,x}).$$

Note that $g_{j_1,j_2}(\tau, X_{\tau-}^{t,x}) = g_{j_1,j_2}(\tau, X_\tau^{t,x})$ since the stopping time τ is predictable, and the process $(X_s^{t,x})_{t\leq s\leq T}$ does not have predictable jump. Thus $\triangle Y_\tau^{j_2} < 0$ and once more we have,

$$Y_{\tau-}^{j_2} = \max_{k\in A^{-j_2}} (Y_{\tau-}^k - g_{j_2 k}(\tau, X_{\tau-}^{t,x})). \tag{5.4.6}$$

We can now repeat the same argument as many times as necessary, and then we deduce the existence of a loop $\ell_1, ..., \ell_{p-1}, \ell_p = \ell_1$ and $l_2 \neq l_i$ such that

$$Y_{\tau-}^{\ell_1} = Y_{\tau-}^{\ell_2} - g_{\ell_1,\ell_2}(\tau, X_{\tau-}^{t,x}), \cdots, Y_{\tau-}^{\ell_{p-1}} = Y_{\tau-}^{\ell_p} - g_{\ell_{p-1},\ell_p}(\tau, X_{\tau-}^{t,x}).$$

Therefore

$$g_{\ell_1,\ell_2}(\tau, X_{\tau-}^{t,x}) + \cdots + g_{\ell_{p-1},\ell_p}(\tau, X_{\tau-}^{t,x}) = 0$$

which is contradictory with Assumption (A4)(II). It implies that $\Delta K_\tau^{j_1} = 0$ and then K^{j_1} is continuous since it is predictable. As j is arbitrary in A, then the processes K^j is continuous and taking into account (5.4.4), we deduce that the triples (Y^j, Z^j, U^j, K^j), $j \in A$, is a solution for system (5.4.1). □

Next for $s \leq T$ let us define the process $(a_s)_{s\leq T}$ for:

$$a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s), \tag{5.4.7}$$

where $\{\theta_j\}_{j\geq 0}$ is an increasing sequence of stopping times with values in [0,T] ($\theta_0 = 0$) and for $j \geq 0$, α_j is a random variable F_{θ_j}-measurable with values in $A = \{1, ..., m\}$. If $\mathcal{P}[\lim_n \theta_n < T] = 0$, then the pair $\{\theta_j, \alpha_j\}_{j\geq 0}$ is called an admissible strategy of switching . Next we denote by $(A_s^a)_{s\leq T}$ the switching cost process associated with the process a, which is defined as following:

$$\forall s < T,\ A_s^a = \sum_{j\geq 1} g_{\alpha_{j-1},\alpha_j}(\theta_j, X_{\theta_j}^{t,x}) \mathbb{1}_{[\theta_j<s]},\ A_T^a = \lim_{s\to T} A_s^a \text{ and } E[(A_T^a)^2] < \infty,$$

where $X^{t,x}$ is the process given in (3.1.4). Now, for $t \leq T$ we denote by $\mathcal{A}_t^i := \{a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s)$
is admissible strategy, $\alpha_0 = i, \theta_0 = 0, \theta_1 \geq t$ *and* $E[(A_T^a)^2] < \infty\}$. For $\Gamma := ((\Gamma_s^i)_{s\in[0,T]})_{i\in A}$, such that $\forall i \in A$, $\Gamma^i \in \mathcal{H}^2$, we introduce the unique solution of the switched BSDE:$\forall s \leq T$,

$$V_s^{a(.)} = h_{a(T)}(X_T^{t,x}) + \int_s^T f^{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, M_r^a, N_r^a) dr - \int_s^T M_r^a dB_r$$
$$- \int_s^T \int_E N_r^a(e) \hat{N}(dedr) - A^a(T, X_T^{t,x}) + A^a(s, X_s^{t,x}).$$

As the same way in [12], we can prove that $Y_s^j = V_s^{a^*} = ess\sup_{a\in\mathcal{A}_s^j} V_s^a$, $\forall (s,j)\in[0,T]\times\{1,\cdots,m\}$ and the uniqueness follows.

Step 2: Now we deal with the general case. Let $\Theta : [H^2]^m$ be $H^2\times\cdots\times H^2(m$ times) and we introduce the operator $\Theta : [H^2]^m \to [H^2]^m,\ \Gamma \to Y$:

$$\begin{cases} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \Gamma_r, Z_r^j, U_r^j)dr - \int_s^T Z_r^j dB_r - \int_s^T\int_E U_r^j(e)\hat{N}(drde) + K_T^j - K_s^j,\ \forall s\le T. \\ Y_s^j \ge \max_{k\in A_j}\{Y_s^k - g_{jk}(s, Y_s^j)\},\ \forall s\le T;\ \ \int_0^T[Y_s^j - \max_{k\in A_j}\{Y_s^k - g_{jk}(s,Y_s^j)\}]dK_s^j = 0 \end{cases} \tag{5.4.8}$$

By Step 1, Θ is well defined. Next for $Y\in H^2$ let us define $||\cdot||_{2,\beta}$ by:

$$\|Y\|_{2,\beta} := (E[\int_0^T e^{\beta s}|Y_s|^2 ds])^{\frac{1}{2}},$$

which is a norm of H^2. Let Γ^1 and Γ^2 be two processes in the Banach space $(([H^2])^m, \|.\|_{2,\beta})$, and for $k=1,2$, let $(Y^{k,j}, U^{k,j}, K^{k,j})_{j\in A}$ satisfy:

$$\begin{cases} Y_s^{k,j} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \Gamma_r^k, Z_r^{j,k}, U_r^{j,k})dr - \int_s^T Z_r^{j,k}dB_r - \int_s^T\int_E U_r^{j,k}(e)\hat{N}(drde) \\ +K_T^{k,j} - K_s^{k,j} \\ Y_s^{k,j} \ge \max_{q\in A_j}\{Y_s^{k,q} - g_{jq}(s, Y_s^{k,j})\};\ \ \int_0^T[Y_s^{k,j} - \max_{q\in A_j}\{Y_s^{k,q} - g_{jq}(s,Y_s^{k,j})\}]dK_s^{k,j} = 0. \end{cases}$$

Let $(a_s)_{t\le s\le T}$ be a admissible strategy defined in (5.4.7) and define $\hat{Y}_s^i, \hat{V}_s^a, V_s^{j,a}$ as follows:

$$\begin{cases} \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \Gamma_r^1, \hat{Z}_r^j, \hat{U}_r^j) \vee f_j(r, X_r^{t,x}, \Gamma_r^2, \hat{Z}_r^j, \hat{U}_r^j)dr \\ -\int_s^T \hat{Z}_r^j dB_r - \int_s^T\int_E \hat{U}_r^j(e)\hat{N}(drde) + \hat{K}_T^j - \hat{K}_s^j \\ \hat{Y}_s^j \ge \max_{q\in A_j}\{\hat{Y}_s^q - g_{jq}(s,\hat{Y}_s^j)\};\ \ \int_0^T[\hat{Y}_s^j - \max_{q\in A_j}\{\hat{Y}_s^q - g_{jq}(s,\hat{Y}_s^j)\}]d\hat{K}_s^j = 0, \end{cases}$$

$$\begin{aligned} \hat{V}_s^a = h_{a(T)}(X_T^{t,x}) + \int_s^T f^{a(r)}(r, X_r^{t,x}, \Gamma_s^1, \hat{M}_r^a, \hat{N}_r^a) \vee f^{a(r)}(r, X_r^{t,x}, \Gamma_s^2, \hat{M}_r^a, \hat{N}_r^a)dr \\ -\int_s^T \hat{M}_r^a dB_r - \int_s^T\int_E \hat{U}_r^a(e)\hat{N}(drde) - A^a(T, X_T^{t,x}) + A^a(s, X_s^{t,x}). \end{aligned}$$

For k=1,2:

$$\begin{aligned} V_s^{a,k} = h_{a(T)}(X_T^{t,x}) + \int_s^T f^{a(r)}(r, X_r^{t,x}, \Gamma_r^k, M_r^{k,a}, N_r^{k,a})dr - A^a(T, X_T^{t,x}) + A^a(s, X_s^{t,x}) \\ -\int_s^T M_r^{a,k}dB_r - \int_s^T\int_E U_r^{a,k}(e)\hat{N}(drde), \end{aligned}$$

by Theorem 5.1 in Appendix, we have

$$Y_s^{k,j} = ess\sup_{a\in\mathcal{A}_s^j} V_s^{k,a},\quad k=1,2,\quad \hat{Y}_s^j = ess\sup_{a\in\mathcal{A}_s^j}\hat{V}_s^a := \hat{V}_s^{a^*}. \tag{5.4.9}$$

In addition, for any admissible strategy $a_s = \alpha_0\mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty}\alpha_{j-1}\mathbb{1}_{]\theta_{j-1}\theta_j]}(s)$, it is easy to check that $f^{a(s)}$ satisfies the Assumption **Assumption (H1)**(1)(iii).

Now by Theorem 3.2, we have $\hat{V}^a \ge V^{1,a}\vee V^{2,a}$, for any strategy $a\in\mathcal{A}_t^j$, this combined with (2.2.36) leads to $Y_s^{1,j}\vee Y_s^{2,j} \le \hat{V}_s^{a^*} = \hat{Y}_s^j$, since a^* is an admissible strategy for the representation (2.2.36) of $Y^{1,j}$, $Y^{2,j}$ and the optimal strategy of $\hat{V}^a$, we deduce that $V_s^{1,a^*}\le Y_s^{1,j}\le\hat{V}_s^{a^*}$ and $V_s^{2,a^*}\le Y_s^{2,j}\le\hat{V}_s^{a^*}$, and this leads to

$$|Y_s^{1,i} - Y_s^{2,i}| \le |\hat{V}_s^{a^*} - V_s^{1,a^*}| + |\hat{V}_s^{a^*} - V_s^{2,a^*}|. \tag{5.4.10}$$

First we estimate $|\hat{V}^{a^*} - V^{1,a^*}|$. Denoting $\triangle V_s^{a^*} := \hat{V}_s^{a^*} - V_s^{1,a^*}$ and $\triangle N_s^{a^*} := \hat{N}_s^{a^*} - N_s^{1,a^*}$, we apply Itô's Formula to the process $e^{\beta s}|\triangle V_s^{a^*}|^2$ and compute

$$
\begin{aligned}
&e^{\beta s}|\triangle V_s^{a^*}|^2+\int_s^T\triangle|M_r^{a^*}|^2dr+\int_s^T\int_E e^{\beta r}|\triangle N_r^{a^*}(e)|^2n(de)dr\\
=&-\int_s^T\beta e^{\beta r}|\triangle V_{r-}^{a^*}|^2ds-2\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}M_r^{a^*}dB_r-2\int_s^T\int_E e^{\beta r}\triangle V_{r-}^{a^*}\triangle N_r^{a^*}(e)\hat{N}(drde)\\
&+2\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}[f^{a^*}(r,X_r^{t,x},\Gamma_r^1,\hat{Z}_r^{a^*},\hat{N}_r^{a^*})\vee f^{a^*}(r,X_r^{t,x},\Gamma_r^2,\hat{Z}_r^{a^*},\hat{N}_r^{a^*})\\
&-f^{a^*}(r,X_r^{t,x},\Gamma_r^1,\hat{Z}_r^{1,a^*},\hat{N}_r^{1,a^*})]dr
\end{aligned}
$$

By the Lipschitz property of f and the fact that $|x\vee y-y|\leq|x-y|$,

$$
\begin{aligned}
&|f^{a^*}(r,X_r^{t,x},\Gamma_r^1,\hat{Z}_r^{a^*},\hat{N}_r^{a^*})\vee f^{a^*}(r,X_r^{t,x},\Gamma_r^2,\hat{Z}_r^{a^*},\hat{N}_r^{a^*})-f^{a^*}(r,X_r^{t,x},\Gamma_r^1,\hat{Z}_r^{1,a^*},\hat{N}_r^{1,a^*})|\\
\leq&L(|\Gamma_r^1-\Gamma_r^2|+|\hat{Z}_r^{a^*}-\hat{Z}_r^{1,a^*}|+||\hat{N}_r^{a^*}-N_r^{1,a^*}||_{\mathcal{L}^2(\hat{N})})
\end{aligned}
$$

Combine these two estimates with the inequality $2xy\leq\frac{1}{\beta}x^2+\beta y^2(\beta>0)$

$$
e^{\beta s}|\triangle\hat{V}_s^{a^*}|^2\leq\frac{L}{\beta}\int_s^T e^{\beta r}|\Gamma_r^1-\Gamma_r^2|^2dr-2\int_s^T e^{\beta r}\triangle V_{r-}^{a^*}M_r^{a^*}dB_r-2\int_s^T\int_E e^{\beta r}\triangle V_{r-}^{a^*}\triangle N_r^{a^*}(e)\hat{N}(drde)
$$

for $\beta\geq L$.

We deduce $E[e^{\beta s}|\triangle\hat{V}_s^{a^*}|^2]\leq\frac{L}{\beta}E[\int_s^T e^{\beta r}|\Gamma_r^1-\Gamma_r^2|dr]$. Similarly, we get also $\forall s\leq T$, $E[e^{\beta s}|\hat{V}_s^{a^*}-V_s^{2,a^*}|^2]\leq\frac{L}{\beta}E[\int_s^T e^{\beta r}|\Gamma_r^1-\Gamma_r^2|dr]$. Since s is arbitary, we going back to (5.4.10), squaring and taking the expectation, we have

$$
E[\sup_{\eta\leq T}e^{\beta\eta}|Y_\eta^{1,i}-Y_\eta^{2,i}|^2]\leq\frac{2L}{\beta}\|\Gamma^1-\Gamma^2\|_{2,\beta} \tag{5.4.11}
$$

Therefore we have,

$$
\|\Theta(\Gamma^1)-\Theta(\Gamma^2)\|_{2,\beta}\leq\sqrt{\frac{2LTm}{\beta}}\|\Gamma^1-\Gamma^2\|_{2,\beta} \tag{5.4.12}
$$

For β large enough ,Θ is contraction on the Banach space $(([H^2])^m,\|.\|_{2,\beta})$, then the fixed point theorem ensures the existence of a unique Y such that $\Theta(Y)=Y$, which is the unique solution of system of RBSDE (5.4.1) □

Corollary 5.4.1. *Under Assumptions (H1), there exist deterministic lower semi-continuous functions $(u^j(t,x))_{j\in A}$ of polynomial growth such that*

$$
\forall(t,x)\in[0,T]\times I\!R^k,\ \forall s\in[t,T],\ Y_s^j=u^j(s,X_s^{t,x}),\ \forall j\in A.
$$

The proof is same as Corollary 3.1 in [33]. Now let us consider now the following system of IPDEs: $\forall i\in A$,

$$
\begin{cases}
min\{u_i(t,x)-\max\limits_{j\in A_i}(u_j(t,x)-g_{ij}(t,x));\\
-\partial_t u_i(t,x)-\mathcal{L}u_i(t,x)-g_i(t,x,(u_j)_{j=1}^m(t,x),\sigma(t,x)D_xu_i(t,x),I^{B^i}(t,x,u_i(t,x,)))\}=0,\\
(t,x)\in[0,T]\times[0,T]\times I\!R^k;\\
u_i(T,x)=h_i(x)
\end{cases} \tag{5.4.13}
$$

where

$$
\mathcal{L}u(t,x)=\mathcal{L}^1u(t,x)+\mathcal{I}(t,x,u)
$$

with

$$
\mathcal{L}^1u(t,x):=b(t,x)\partial_xu(t,x)+\tfrac{1}{2}\sigma(t,x)^2D_{xx}^2u(t,x)\text{ and}
$$

$$
I(t,x,\phi)=\int_E(\phi(t,x+\beta(x,e))-\phi(t,x)-D_x\phi(t,x)\beta(x,e))n(de);
$$

$$
I^{B^i}(t,x,\phi)=\int_E(\phi(t,x+\beta(x,e))-\phi(t,x))\gamma^i(x,e)n(de).
$$

And define:

$$I_\delta^1(t,x,\phi)=\int_{|e|\le\delta}(\phi(t,x+\beta(x,e))-\phi(t,x)-D_x\phi(t,x)\beta(x,e))n(de);$$

$$I_\delta^2(t,x,q,\phi)=\int_{|e|\ge\delta}(\phi(t,x+\beta(x,e))-\phi(t,x)-q\beta(x,e))n(de);$$

$$I_\delta^{1,B^i}(t,x,\phi)=\int_{|e|\le\delta}(\phi(t,x+\beta(x,e))-\phi(t,x))\gamma^i(x,e)n(de);$$

$$I_\delta^{2,B^i}(t,x,\phi)=\int_{|e|\ge\delta}(\phi(t,x+\beta(x,e))-\phi(t,x))\gamma^i(x,e)n(de),$$

and of cause, for any $\phi\in\Pi_g$, $(t,x,p)\in R^3$, $\mathcal{I}_\delta^2(t,x,p,\phi)$ and $\mathcal{I}_\delta^{2,B^{ij}}(t,x,\phi)$ are well defined.

Theorem 5.4.2. *Assume Assumptions (H1), then $(u_j(t,x))_{j\in A}$ is a viscosity solution of* (5.4.13).

Proof. We just show the main different steps with the proof of Theorem 4.1 in [33].

First we show that $(u_j)_{j=1}^m$ is a supersolution of (5.4.13). Noting that for all $j\in A$, as u_j is lsc, we have $u_{j_*}=u_j$. Next let us set $u_j^n(t,x)=Y_t^{j,n,t,x}$, where $(Y^{j,n,t,x};U^{j,n,t,x},K^{j,n,t,x})_{j\in A}$ is the unique solution of (5.4.2). As pointed out in Corollary 5.4.1, for any $n\ge0$, $(t,x)\in[0,T]\times I\!R^k$ and $s\in[t,T]$,

$$Y_s^{j,n,t,x}=u_j^n(s,X_s^{t,x}) \text{ and } u_j^n(t,x)\nearrow u_j(t,x).$$

Additionally by induction, $(u_j^n)_{j\in A}$, $n\ge0$, are continuous, belong to Π_g and by Barles et al.'s result ([5]) verify in viscosity sense the following system ($n\ge1$): $\forall j\in A$,

$$\begin{cases} min\Big\{u^{j,n}(t,x)-\max\limits_{k\in A_j}(u^{j,n-1}(t,x)-g_{jk}(t,x));\\ -\partial_t u^{j,n}(t,x)-\mathcal{L}u^{j,n}(t,x)-g_j(t,x,u^{1,n-1},\cdots,u^{j-1,n-1},u^{j,n},u^{j+1,n-1},\cdots,u^{m,n-1},\\ \sigma(t,x)D_xu^{j,n}(t,x),I^{B^i}(t,x,u^{j,n}(t,x,)))\Big\}=0;\\ u^{j,n}(T,x)=h_j(x). \end{cases} \tag{5.4.14}$$

First note that for any $j\in A$, u_j verifies

$$u_j(T,x)=h_j(x) \text{ and } u_j(t,x)\ge\max_{k\in A_j}\{u_k(t,x)-g_{jk}(t,x)\},\ \forall(t,x)\in[0,T]\times I\!R^k.$$

Next let $(t,x)\in(0,T)\times I\!R$ and let $j\in A$ be fixed. Let ϕ be a function which belongs to $\mathcal{C}^{1,2}([0,T]\times I\!R)\cap\Pi_g$ such that $u_j-\phi$ has a strict global minimum in (t,x) on $[0,T]\times I\!R^k$ and wlog we assume that $u_j(t,x)=\phi(t,x)$. Now let $\delta>0$ be fixed. Then (t,x) is a global strict minimum of $u_j-\phi$ in $[0,T]\times B(x,K\delta)$. Next let (t_n,x_n) be the global minimum of $u_j^n-\phi$ on $\overline{[0,T]\times B(x,K\delta)}$. Therefore

$$\lim_n(t_n,x_n)=(t,x) \text{ and } u_j^n(t_n,x_n)\to u(t,x).$$

Actually let us consider a convergent subsequence of (t_n,x_n), which we still denote by (t_n,x_n), and let set (t^*,x^*) its limit. Then

$$u_j^n(t_n,x_n)-\phi(t_n,x_n)\le u_j^n(t,x)-\phi(t,x). \tag{5.4.15}$$

Taking the limit wrt n and since $u_{j*}=u_j$ is lsc to obtain

$$u_j(t^*,x^*)-\phi(t^*,x^*)\le u_j(t,x)-\phi(t,x).$$

As the minimum (t,x) of $u_j-\phi$ on $[0,T]\times I\!R^k$ is strict then $(t^*,x^*)=(t,x)$. It follows that the sequence $((t_n,x_n))_n$ converges to (t,x). Going back now to (5.4.15) and in sending n to infintite we obtain

$$u_{j*}(t,x)=u_j(t,x)\le\liminf_n u_j^n(t_n,x_n)\le\limsup_n u_j^n(t_n,x_n)\le u_j(t,x)$$

which implies that $u_j^n(t_n,x_n)\to u_j(t,x)$ as $n\to\infty$.

Now for n large enough $(t_n, x_n) \in (0,T) \times B(x, K\delta)$ and it is the global minimum of $u_j^n - \phi$ in $[0,T] \times B(x, K\delta)$. As u_j^n is a supersolution of (5.4.14), then by Definition 2.2, we have

$$\begin{array}{l} -\partial_t\phi(t_n,x_n) - \mathcal{L}^1\phi(t_n,x_n) - \mathcal{I}^{1,\delta}(t_n,x_n,\phi) - \mathcal{I}^{2,\delta}(t_n,x_n,D_x\phi(t_n,x_n),u^{j,n}) \geq \\ f_j(t_n,x_n,u^{1,n-1}(t_n,x_n),\cdots,u^{j-1,n-1}(t_n,x_n),u^{j,n}(t_n,x_n),u^{j+1,n-1}(t_n,x_n),\cdots,u^{m,n-1}(t_n,x_n), \\ \sigma(t_n,x_n)D_xu^{j,n}(t_n,x_n), I_\delta^{1,B^j}(t_n,x_n,\phi) + I_\delta^{2,B^j}(t_n,x_n,u^{j,n})). \end{array} \tag{5.4.16}$$

But there exists a subsequence of $\{n\}$ such that:
(i) for any $k \in A_j$, $(u_k^{n-1}(t_n,x_n))_n$ is convergent and then $\lim_n u_k^{n-1}(t_n,x_n) \geq u_k(t,x)$;
(ii) $(\mathcal{I}_\delta^1(t_n,x_n,\phi))_n \to \mathcal{I}_\delta^1(t,x,\phi)$ as $n \to \infty$;
(iii) $(\mathcal{I}_\delta^{1,B^j}(t_n,x_n,\phi))_n \to \mathcal{I}_\delta^{1,B^j}(t,x,\phi)$ as $n \to \infty$.

Sending now n to infinite (through the previous subsequence) in (5.4.16), using the fact that g_j is continuous and verifies (A4)(I)(v) and finally by Fatou's Lemma to obtain:

$$\begin{array}{l} -\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}_\delta^1(t,x,\phi) \geq \\ \mathcal{I}_\delta^2(t,x,D_x\phi(t,x),u^j) + g_j(t,x,u_1(t,x),\cdots,u_{j-1}(t,x),u_j(t,x),u_{j+1}(t,x),\cdots,u_m(t,x), \\ \sigma(t,x)D_xu^j(t,x), I_\delta^{1,B^j}(t,x,\phi) + I_\delta^{2,B^j}(t,x,u^j)). \end{array}$$

But $u_j(t,x) = \phi(t,x)$ and $u_j \geq \phi$, then $I^{2,\delta}(t,x,D_x\phi(t,x),u_j) \geq I^{2,\delta}(t,x,D_x\phi(t,x),\phi)$ and $I^{2,B^j}(t,x,u_j) \geq I^{2,B^j}(t,x,\phi)$. Plugging now this inequality in the previous one to obtain

$$-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}(t,x,\phi) - g_j(t,x,(u_i)_{i\in A}(t,x),\sigma(t,x)D_xu_j, I^{B^j}(t,x,u_j)) \geq 0.$$

Therefore u_j is a viscosity supersolution of

$$\left\{ \begin{array}{l} min\{u_j(t,x) - \max\limits_{k\in A_j}(u_k(t,x) - g_{jk}(t,x)); \\ -\partial_t u_j(t,x) - \mathcal{L}u_j(t,x) - g_j(t,x,(u_i)_{i\in A}(t,x),\sigma(t,x)D_xu_j, I^{B^j}(t,x,u_j))\} = 0; \\ u_j(T,x) = h_j(x). \end{array} \right.$$

As j is arbitrary then $(u_j)_{j\in A}$ is a viscosity supersolution of (5.4.13). □

Now we will now show that $(u_j^*)_{j\in A}$ is a subsolution of (5.4.13). First we have:

$$\forall j \in A,\ min\{u_j^*(T,x) - h_j(x);\ \ u_j^*(T,x) - \max_{k\in A_j}(u_k^*(T,x) - g_{jk}(T,x))\} = 0,$$

and

$$u_j^*(T,x) = h_j(x), \forall j \in A.$$

The proof is same as Theorem 4.1 in [33]. □

Now let us show $(u_j^*)_{j\in A}$ is a subsolution of (5.4.13). First note that since $u_j^n \nearrow u_j$ and u_j^n is continuous, we have

$$u_j^*(t,x) = \limsup_{n\to\infty}{}^* u_j^n(t,x) = \overline{\lim_{n\to\infty, t'\to t, x'\to x}} u_j^n(t',x').$$

Besides $\forall j \in A$ and $n \geq 0$ we deduce from the construction of u_j^n that

$$u_j^n(t,x) \geq \max_{l\in A_j}(u_l^n(t,x) - g_{jl}(t,x))$$

and by taking the limit in n we obtain: $\forall j \in A$, $\forall x \in I\!R$,

$$u_j^*(t,x) \geq \max_{l\in A_j}(u_l^*(t,x) - g_{jl}(t,x)).$$

Next fix $j \in A$. Let $(t,x) \in (0,T) \times I\!R$ be such that

$$u_j^*(t,x) - \max_{l\in A_j}(u_l^*(t,x) - g_{jl}(t,x)) > 0. \tag{5.4.17}$$

Let ϕ be a $C^{1,2}([0,T] \times I\!R) \cap \Pi_g$ function such that $u_j^* - \phi$ has a global maximum at (t,x) in $[0,T] \times I\!R^k$ which wlog we suppose strict and $u_j(t,x) = \phi(t,x)$. Therefore (t,x) is a global strict maximum of $u_j - \phi$ in $[0,T] \times B(x, K\delta)$. On the other hand there exist subsequences $\{n_k\}$ and $((t'_{n_k}, x'_{n_k}))_k$ such that

$$((t'_{n_k}, x'_{n_k}))_k \to_k (t,x) \text{ and } u_j^{n_k}(t'_{n_k}, x'_{n_k}) \to_k u_j^*(t,x).$$

Let now (t_{n_k}, x_{n_k}) be the global maximum of $u_j^{n_k} - \phi$ on $\overline{[0,T] \times B(x, K\delta)}$. Therefore

$$(t_{n_k}, x_{n_k}) \to_k (t,x) \text{ and } u_j^{n_k}(t_{n_k}, x_{n_k}) \to_k u_j^*(t,x).$$

Actually let us consider a convergent subsequent of (t_{n_k}, x_{n_k}), which we still denote by (t_{n_k}, x_{n_k}), and let $(\bar{t}, \bar{x})$ be its limit. Then for some k_0 and for $k \geq k_0$ we have

$$u_j^{n_k}(t_{n_k}, x_{n_k}) - \phi(t_{n_k}, x_{n_k}) \geq u_j^{n_k}(t'_{n_k}, x'_{n_k}) - \phi(t'_{n_k}, x'_{n_k}). \tag{5.4.18}$$

Taking the limit wrt k to obtain

$$u_j^*(\bar{t}, \bar{x}) - \phi(\bar{t}, \bar{x}) \geq u_j^*(t,x) - \phi(t,x).$$

As the maximum (t,x) of $u_j - \phi$ on $[0,T] \times I\!R^k$ is strict then $(\bar{t}, \bar{x}) = (t,x)$. It follows that the sequence $((t_{n_k}, x_{n_k}))_k$ converges to (t,x). Going back now to (5.4.18) and in sending k to infinite we obtain

$$u_j^*(t,x) \geq \limsup_k u_j^{n_k}(t_{n_k}, x_{n_k}) \geq \liminf_k u_j^{n_k}(t_{n_k}, x_{n_k}) \geq \liminf_k u_j^{n_k}(t'_{n_k}, x'_{n_k}) = u_j^*(t,x)$$

which implies that $u_j^{n_k}(t_{n_k}, x_{n_k}) \to u_j^*(t,x)$ as $k \to \infty$.

Now for k large enough,

(i) $(t_{n_k}, x_{n_k}) \in (0,T) \times B(x, K\delta)$ and is the global maximum of $u_j^{n_k} - \phi$ in $(0,T) \times B(x, K\delta)$;

(ii) $u_j^{n_k}(t_{n_k}, x_{n_k}) > \max_{l \in A_j}(u_l^{n_k - 1}(t_{n_k}, x_{n_k}) - g_{jl}(t_{n_k}, x_{n_k}))$.

As $u_j^{n_k}$ is a subsolution of (5.4.14), then by Definition 2.2, we have

$$\begin{array}{l} -\partial_t \phi(t_{n_k}, x_{n_k}) - \mathcal{L}^1 \phi(t_{n_k}, x_{n_k}) - \mathcal{I}_\delta^1(t_{n_k}, x_{n_k}, \phi) - \mathcal{I}_\delta^2(t_{n_k}, x_{n_k}, D_x\phi(t_{n_k}, x_{n_k}), u^{j,n_k}) \leq \\ f_j(t_{n_k}, x_{n_k}, u^{1,n_k-1}(t_{n_k}, x_{n_k}), \cdots, u^{j-1,n_k-1}(t_{n_k}, x_{n_k}), u^{j,n_k}(t_{n_k}, x_{n_k}), u^{j+1,n_k-1}(t_{n_k}, x_{n_k}), \cdots, u^{m,n_k-1}(t_{n_k}, x_{n_k}), \\ \sigma(t_{n_k}, x_{n_k}) D_x u^{j,n_k}(t_{n_k}, x_{n_k}), \mathcal{I}_\delta^{1,B^j}(t_{n_k}, x_{n_k}, \phi) + \mathcal{I}_\delta^{1,B^j}(t_{n_k}, x_{n_k}, u^{j,n_k})). \end{array} \tag{5.4.19}$$

But there exists a subsequence of $\{n_k\}$ (which we still denote by $\{n_k\}$) such that:

(i) for any $l \in A_j$, $(u_l^{n_k-1}(t_{n_k}, x_{n_k}))_k$ is convergent and then $\lim_k u_l^{n_k-1}(t_{n_k}, x_{n_k}) \leq u_l^*(t,x)$;

(ii) $(\mathcal{I}_\delta^1(t_{n_k}, x_{n_k}, \phi))_{n_k} \to \mathcal{I}^{1,\delta}(t,x,\phi)$ as $k \to \infty$;

(ii) $(\mathcal{I}_\delta^{1,B^j}(t_{n_k}, x_{n_k}, \phi))_{n_k} \to \mathcal{I}^{1,B^j,\delta}(t,x,\phi)$ as $k \to \infty$

Sending now k to infinite (through the previous subsequence) in (5.4.19), using the fact that g_j is continuous and verifies (H1) and finally by Lebesgue's Theorem to obtain

$$\begin{array}{l} -\partial_t \phi(t,x) - \mathcal{L}^1 \phi(t,x) - \mathcal{I}^{1,\delta}(t,x,\phi) \leq \\ \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x), u_j^*) + g_j(t,x,(u_i^*)_{i\in A}(t,x), \sigma(t,x) D_x u_j^*(t,x), \mathcal{I}_\delta^{1,B^j}(t,x,\phi) + \mathcal{I}_\delta^{1,B^j}(t,x,u_j^*)). \end{array}$$

But $u_j(t,x) = \phi(t,x)$ and $u_j^* \leq \phi$, then $\mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),u_j^*) \leq \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),\phi)$ and $\mathcal{I}_\delta^{1,B^j}(t,x,u_j^*) \leq \mathcal{I}_\delta^{1,B^j}(t,x,\phi)$. Plugging now this inequality in the previous one to obtain

$$-\partial_t \phi(t,x) - \mathcal{L}^1 \phi(t,x) - \mathcal{I}(t,x,\phi) - g_j(t,x,(u_i^*)_{i\in A}(t,x), \sigma(t,x) D_x u_j^*(t,x), \mathcal{I}_\delta^{B^j}(t,x,\phi)) \leq 0.$$

Therefore u_j is a viscosity subsolution of

$$\left\{ \begin{array}{l} min\{u_j(t,x) - \max_{k\in A_j}(u_k(t,x) - g_{jk}(t,x)); \\ \qquad -\partial_t u_j(t,x) - \mathcal{L} u_j(t,x) - g_j(t,x,(u_i)_{i\in A}, \sigma(t,x,)D_x u_j(t,x), I^{B^j}(t,x,u_j))\} = 0; \\ u_j(T,x) = h_j(x). \end{array} \right.$$

As j is arbitrary then $(u_j)_{j\in A}$ is a viscosity subsolution of (5.4.13). □

5.5 Viscosity solution of PDE with two obstacle of min-max type

Let $(t,x) \in [0,T] \times I\!R^k$ and $(X_s^{t,x})_{s\leq T}$ be the solution of the standard SDE given in (4.1.6) where the functions b and σ satisfy Assumption (H0). Let us now consider the following functions:

$$\begin{array}{ll} g: & x \in I\!R^k \longmapsto g(x)\, I\!R \\ f: & (t,x,y,z) \in [0,T] \times I\!R^{k+1+d} \longmapsto f(t,x,y,z) \in I\!R \\ H: & (t,x) \in [0,T] \times I\!R^k \longmapsto H(t,x) \in I\!R \\ L: & (t,x) \in [0,T] \times I\!R^k \longmapsto L(t,x) \in I\!R \end{array}$$

We assume that all those functions are continuous and satisfy the following assumptions (A1)-(A2).

(A1): $\forall\ t \in [0,T]$, $x \in I\!R^k$, $y, y' \in I\!R$, $z, z' \in I\!R^d$,

$$\begin{cases} \text{(i) } |g(x)| + |f(t,x,0,0)| + |H(t,x)| + |L(t,x)| \le C(1+|x|^p), \\ \text{(ii) } |f(t,x,y,z) - f(t,x,y',z')| \le C(|y-y'| + |z-z'|), \\ \text{(iii) } L(t,x) \le H(t,x) \text{ and } L(T,x) \le g(x) \le H(T,x), \end{cases}$$

where C and p are some positive constants.

(A2): For each $R > 0$, there is a continuous function φ_R such that $\varphi_R(0) = 0$ and

$$|f(t,x,y,z) - f(t,x',y,z)| \le \varphi_R((1+|z|)|x-x'|)$$

for all $t \in (0,T)$, $|x|$, $|x'|$, $|y| \le R$ and $z \in I\!R^d$.

Next for $n \ge 0$, let $({}^nY_s^{t,x})_{s\le T}$ (resp. $({}^n\bar{Y}_s^{t,x})_{s\le T}$) be the first component of the unique solution of the BSDE with one reflecting lower (resp. upper) barrier associated with the triple
$(f(s,X_s^{t,x},y) - n(H(s,X_s^{t,x}) - y)^-, g(X_T^{t,x}), L(s,X_s^{t,x}))$
(resp. $(f(s,X_s^{t,x},y,z) + n(L(s,X_s^{t,x}) - y)^+, g(X_T^{t,x}), H(s,X_s^{t,x})))$, which exists and is unique (see e.g. [22]). It has been shown in [22] that, under Assumptions (H0) and (A1)-(A2), for any $n \ge 0$ there exist deterministic functions ${}^nu(t,x)$ and ${}^n\bar{u}$, $(t,x) \in [0,T] \times I\!R^k$, such that

$$\forall s \in [t,T],\quad {}^nY_s^{t,x} = {}^n\, u(s, X_s^{t,x}) \text{ and } {}^n\bar{Y}_s^{t,x} = {}^n\, \bar{u}(s, X_s^{t,x}),$$

where ${}^nu$ (resp. ${}^n\bar{u}$) is continuous with uniform polynomial growth i.e. there exist two non negative real constants C and p such that

$$|{}^nu(t,x)| \text{ (resp. } |{}^n\bar{u}(t,x)|) \le C(1+|x|^p),\ \forall (t,x) \in [0,T] \times I\!R^k.$$

Moreover it is a unique viscosity solution, in the classe Π_g, of the following PDE with obstacle:

$$\begin{cases} \min\Big\{v(t,x) - L(t,x)\ ; -\partial_t v(t,x) - \mathcal{L}^X v(t,x) \\ \qquad\qquad -f(t,x,v(t,x),\sigma(t,x)^\top D_x\phi(t,x)) + n(H(t,x) - v(t,x))^-\Big\} = 0, \\ v(T,x) = g(x), \end{cases} \tag{5.5.1}$$

(resp.

$$\begin{cases} \max\Big\{v(t,x) - H(t,x)\ ; -\partial_t v(t,x) - \mathcal{L}^X v(t,x) \\ \qquad\qquad -f(t,x,v(t,x),\sigma(t,x)^\top D_x v(t,x)) + n(L(t,x) - v(t,x))^-\Big\} = 0, \\ v(T,x) = g(x)). \end{cases} \tag{5.5.2}$$

By comparison (see e.g. [27]) we easily deduce that the sequence of processes $({}^nY^{t,x})_{n\ge0}$ (resp. $({}^n\bar{Y}^{t,x})_{n\ge0}$) is decreasing (resp. increasing), moreover they converge in $\mathcal{S}^2$ to a same processes $(\bar{Y}_s^{t,x})_{s\le T}$ which satisfies

$$L(s,X_s^{t,x}) \le Y_s^{t,x} \le H(s,X_s^{t,x}),\ \forall s \le T.$$

Therefore for any $(t,x) \in [0,T] \times I\!R^k$, the sequence $({}^nu(t,x))_{n\ge0}$ (resp. $({}^n\bar{u}(t,x))_{n\ge0}$) converges decreasingly (resp. increasingly) to the same limit

$$u(t,x) := Y_t^{t,x} \tag{5.5.3}$$

which verifies

$$u(T,x) = g(x) \text{ and } L(t,x) \le u(t,x) \le H(t,x), \forall (t,x) \in [0,T] \times I\!R^k.$$

Next as ${}^nu$ and ${}^n\bar{u}$ are continuous and belong to Π_g, then the function u belongs also to Π_g and is also continuous since it is both lsc and usc. By Dini's Theorem we deduce that the convergence of the sequence $({}^nu)_{n\ge0}$ (resp. $({}^n\bar{u})_{n\ge0}$) is uniform on compact subsets of $[0,T] \times I\!R^k$.

Next let us consider the following PDE with two obstacles of min-max type:

$$\begin{cases} \min\Big\{v(t,x) - L(t,x)\,;\, \max\Big[v(t,x) - H(t,x); \\ \qquad\qquad -\partial_t v(t,x) - \mathcal{L}^X v(t,x) - f(t,x,v(t,x),\sigma(t,x)^\top D_x v(t,x))\Big]\Big\} = 0\,; \\ v(T,x) = g(x). \end{cases} \tag{5.5.4}$$

To begin with, we are going to give the notion of viscosity solution of (5.5.4).

Definition 5.5.1. *Let v be a function which belongs to $\mathcal{C}([0,T]\times I\!R^k)$. It is called a viscosity:*
(i) subsolution of (5.5.4) if $v(T,x)\leq g(x)$ and for any $\phi\in\mathcal{C}^{1,2}([0,T]\times I\!R^k)$ and any local maximum point $(t,x)\in(0,T)\times I\!R^k$ of $v-\phi$, we have

$$\min\Big\{v(t,x)-L(t,x);max\Big[v(t,x)-H(t,x);-\partial_t\phi(t,x)-\mathcal{L}^X\phi(t,x)-f(t,x,v(t,x),\sigma(t,x)^\top D_x\phi(t,x))\Big]\Big\}\leq 0.$$

(ii) supersolution of (5.5.4) if $v(T,x)\geq g(x)$ and for any $\phi\in\mathcal{C}^{1,2}([0,T]\times I\!R^k)$ and any local minimum point $(t,x)\in(0,T)\times I\!R^k$ of $v-\phi$, we have

$$min\Big\{v(t,x)-L(t,x);max\Big[v(t,x)-H(t,x);-\partial_t\phi(t,x)-\mathcal{L}^X\phi(t,x)-f(t,x,v(t,x),\sigma(t,x)^\top D_x\phi(t,x))\Big]\Big\}\geq 0.$$

(iii) solution of (5.5.4) if it is both a viscosity subsolution and supersolution. □

Theorem 5.5.1. *The function u defined in (5.5.3) is a viscosity solution of (5.5.4).*

Proof. First we show that u is a viscosity subsolution of (5.5.4). Note that $u(T,x)=g(x)$ and $L(t,x)\leq u(t,x)\leq H(t,x)$. Let now $\phi\in\mathcal{C}^{1,2}$ and $(t,x)\in(0,T)\times I\!R^k$ be a local maximum of $u-\phi$ in $[0,T]\times I\!R^k$ such that $u(t,x)>L(t,x)$. Let (t_n,x_n) be a sequence of local maximum points of ${}^nu-\phi$ such that (t_n,x_n) converges to (t,x) (such a sequence exists because of the uniform convergence of ${}^nu$ to u on compact subsets (see e.g. [39], pp.117). For n large enough we have ${}^nu(t_n,x_n)>L(t_n,x_n)$ and since ${}^nu$ is a viscosity solution of (5.5.1) then

$$\begin{aligned}&-\partial_t\phi(t_n,x_n)-\mathcal{L}^X\phi(t_n,x_n)-f(t_n,x_n,{}^n u(t_n,x_n),\sigma(t_n,x_n)^\top D_x\phi(t_n,x_n))\\&\quad\leq -n(H(t_n,x_n)-{}^n u(t_n,x_n))^-\leq 0.\end{aligned}$$

Now by the continuity of the functions and the uniform convergence, we have

$$-\partial_t\phi(t,x)-\mathcal{L}^X\phi(t,x)-f(t,x,u(t,x),\sigma(t,x)^\top D_x\phi(t,x))\leq 0.$$

Thus u is a viscosity subsolution of (5.5.4). In a similar way we can show that u is also a viscosity supersolution. □

The following result is of comparison type between sub. and supersolutions of (5.5.4). Namely we have:

Proposition 5.5.1. *Assume that Assumptions (H0), (A1)-(A2) are in force. Then if v (resp. u) is a viscosity supersolution (resp. subsolution) of (5.5.4) with polynomial growth, then for all $(t,x)\in[0,T]\times I\!R^k$ we have $u(t,x)\leq v(t,x)$.*

Proof. Step (i): First we proof that $v\geq L$ and $u\leq H$.

By the definition of supersolution, it is clear that $v\geq L$. Let us now show that $u(t,x)\leq H(t,x)$. Suppose that for $t\leq T$, $u(t,x)>H(t,x)$. Therefore we have $t<T$ and $u(t,x)-L(t,x)>0$ since $H\geq L$. Now if ϕ is a test function for u at $(t,x)\in(0,T)\times I\!R^k$ then

$$\begin{aligned}&\min\Big\{u(t,x)-L(t,x),\max\Big[u(t,x)-H(t,x),-\partial_t\phi(t,x)\\&-\mathcal{L}\phi(t,x)-f(t,x,u(t,x),D_x\phi\sigma(t,x))\Big]\Big\}>0,\end{aligned}$$

which is contradictory. Thus $u\leq H$.

Step (ii): Let us define $v':=v\wedge H$ and $u'=u\vee L$, then v' is a viscosity supersolution and u' is a viscosity subsolution of (5.5.4).

In fact, since $H(T,x)\geq g(x)$, then $v'(T,x)=v(T,x)\wedge H(T,x)\geq g(x)$. Let now $(t,x)\in(0,T)\times I\!R^k$ and $\phi\in\mathcal{C}^{1,2}$ such that (t,x) is a local minimum point of $v'-\phi$ in $[0,T]\times I\!R^k$. If $v(t,x)<H(t,x)$, then $v'(t,x)=v(t,x)$ and by continuity, (t,x) is also a local minimum point of $v-\phi$. Since v is a supersolution of (5.5.4), then v' verifies

$$\begin{aligned}&\min\Big\{v'(t,x,)-L(t,x);\max\Big[v'(t,x)-H(t,x);-\partial_t\phi(t,x)\\&-\mathcal{L}\phi(t,x)-f(t,x,v'(t,x),\sigma(t,x)^\top D_x\phi(t,x))\Big]\Big\}\geq 0.\end{aligned}$$

Next if $v(t,x) \geq H(t,x)$, then $v'(t,x) = H(t,x)$. Since $H(t,x) - L(t,x) \geq 0$, then we have

$$\min\Big\{H(t,x) - L(t,x); \max\Big[H(t,x) - H(t,x); -\partial_t\phi(t,x) \\ -\mathcal{L}\phi(t,x) - f(t,x,H(t,x), D_x\phi\sigma(t,x))\Big]\Big\} \geq 0.$$

Thus v' is a viscosity supersolution of (5.5.4). In the same way we can prove u' is a subsolution of (5.5.4).

Step (iii): Modification of the problem.

Let $\lambda \in I\!R$ and ξ, η and κ be the functions defined on $I\!R^k$ as

$$\xi(x) := (1+|x|^2)^{\frac{p}{2}}, \\ \eta(x) := \xi(x)^{-1} D_x\xi(x) = px(1+|x|^2)^{-1} \\ \kappa(x) := \xi(x)^{-1} D^2_{xx}\xi(x) = p(1+|x|^2)^{-1} I_k - p(p-2)(1+|x|^2)^{-2} x \otimes x$$

where p is chosen in such a way that $\bar{u}$ and $\bar{v}$ below are bounded and converge uniformly to 0 as $\|x\| \to \infty$. It exists since u and v are both in Π_g. Next let us consider the followings

$$\bar{u}(t,x) := e^{\lambda t}\xi^{-1}(x)u'(t,x),\ \bar{v}(t,x) := e^{\lambda t}\xi^{-1}(x)v'(t,x), \\ \bar{L}(t,x) := e^{\lambda t}\xi^{-1}(x)L(t,x),\ \hat{H}(t,x) := e^{\lambda t}\xi^{-1}(x)H(t,x), \\ \bar{g}(x) := e^{\lambda T}\xi^{-1}(x)g(x), \\ \bar{\mathcal{L}}\varphi := \mathcal{L}^X\varphi + < \sigma\sigma^\top\eta, D_x\varphi > + \{\tfrac{1}{2}\mathrm{Tr}((\sigma\sigma^\top)\kappa) + < b, \eta > -\lambda\}\varphi \\ \bar{f}(t,x,y,z) := e^{\lambda t}\xi^{-1}(x) f(t, x, e^{-\lambda t}\xi(x)y, e^{-\lambda t}\xi(x)z + e^{-\lambda t}D_x\xi(x)\sigma(t,x)y).$$

Therefore one can easily check that $\bar{u}$ (resp. $\bar{v}$) is a viscosity subsolution (resp. supersolution) of

$$\begin{cases} \min\Big\{\tilde{u}(t,x) - \bar{L}(t,x); \max\Big[\tilde{u}(t,x) - \bar{H}(t,x); \\ \qquad -\partial_t\tilde{u}(t,x) - \bar{\mathcal{L}}\tilde{u}(t,x) - \bar{f}(t,x,\tilde{u}(t,x), \sigma(t,x)^\top D_x\tilde{u}(t,x))\Big]\Big\} = 0; \\ \tilde{u}(T,x) = \bar{g}(x). \end{cases} \tag{5.5.5}$$

Let now F be the function from $[0,T] \times I\!R^{k+1+d} \times \mathbb{S}_k$ ($\mathbb{S}_k$ is the space of symmetric real matrices of dimension k) which with (t,x,y,z,M) associates $F(t,x,y,z,M) \in I\!R$ and verifying

$$F(t,x,\tilde{u}(t,x), D_x\tilde{u}(t,x), D^2_{xx}\tilde{u}(t,x)) = \bar{\mathcal{L}}\tilde{u}(t,x) + \bar{f}(t,x,\tilde{u}(t,x), \sigma(t,x)^\top D_x\tilde{u}(t,x)).$$

We choose λ great enough in such a way that the mapping $y \in I\!R \longmapsto F(t,x,y,z,M) \in I\!R$ is strictly decreasing for all $(t,x,z,M) \in [0,T] \times I\!R^{k+1} \times \mathbb{S}_k$. Finally note that for all $\epsilon > 0$ the function $\bar{v} + \frac{\epsilon}{t}$ is also a supersolution solution of (5.5.5). Therefore in order to obtain the comparison result it is enough to show that $\bar{u} \leq \bar{v} + \frac{\epsilon}{t}$ and then to take the limit as $\epsilon \to 0$.

Step (iv): Last part of the proof.

We are going to show by contradiction that: $\forall R > 0$

$$\sup_{t\in[0,T], |x|\leq R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+ \leq \sup_{t\in[0,T], |x|=R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+ \tag{5.5.6}$$

where now L (resp. H) is $\bar{L}$ (resp. $\bar{H}$), $u' = \bar{u} \vee \bar{L}$, $v' = \bar{v} \wedge \bar{H}$, and finally f is $\bar{f}$ which is defined previously. Note that from Steps (i)-(ii), u' (resp. v') is a viscosity subsolution (resp. supersolution) of (5.5.5) and due to assumption (A1), $u' \leq H$ and $v' \geq L$.

So suppose that for some $R > 0$

$$\delta := \sup_{t\in[0,T], |x|\leq R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+ > \sup_{t\in[0,T], |x|=R} (u'(t,x) - v'(t,x) - \frac{\varepsilon}{t})^+ \geq 0.$$

For each $n > 0$, let (t_n, x_n, y_n) be a point in the compact set $[0,T] \times \bar{B}_R \times \bar{B}_R$ where $B_R \triangleq \{x \in I\!R^k; |x| < R\}$, and the continuous function

$$\Phi_n(t,x,y) = u'(t,x) - v'(t,y) - \frac{\varepsilon}{t} - n|x-y|^2$$

achieves its maximum. As u' and v' are bounded this maximum belongs to $(0,T)\times B_R\times B_R$. By Lemma 8.7 in [22], there exists $(p_n, X_n, Y_n)\in I\!R\times\mathbb{S}_k\times\mathbb{S}_k$ such that:

(i) $n|x_n-y_n|^2\to 0$ as $n\to\infty$;

(ii) $u'(t_n,x_n)\geq v'(t_n,y_n)+\frac{\varepsilon}{t_n}+\delta$;

(iii) $(p_n, n(x_n-y_n), X_n)\in \bar{J}^{2,+}(u'(t_n,x_n))$;

(iv) $(p_n, n(x_n-y_n), Y_n)\in \bar{J}^{2,-}(v'(t_n,y_n)+\frac{\varepsilon}{t_n})$;

(v) $\begin{pmatrix} X_n & 0\\ 0 & -Y_n\end{pmatrix}\leq 3n\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$

where $\bar{J}^{2,+}(u'(t,x))$ (resp. $\bar{J}^{2,-}(v'(t,x)+\frac{\epsilon}{t})$) is respectively the limiting superjet (resp. subjet) of u' (resp. $v'(t,x)+\frac{\epsilon}{t}$) (see e.g. [12], pp. 728 or [26], pp. 210).

Next as v' is continuous, combined with point (i), we deduce the existence of $N>0$ such that for any $n\geq N$, $|v'(t_n,x_n)-v'(t_n,y_n)|<\frac{\delta}{2}$. Therefore for $n\geq N_0$,

$$\begin{array}{l} u'(t_n,x_n)\geq v'(t_n,y_n)+\frac{\varepsilon}{t_n}+\delta\\ \geq v'(t_n,x_n)-|v'(t_n,x_n)-v'(t_n,y_n)|+\frac{\varepsilon}{t_n}+\delta\\ > v'(t_n,x_n)+\frac{\varepsilon}{t_n}+\frac{\delta}{2}\end{array}$$

which implies that for any $n\geq N_0$, $u'(t_n,x_n)>v'(t_n,x_n)$. On the other hand, by the results obtained in Steps (i)-(ii),

$$H(t_n,x_n)\geq u(t_n,x_n)\vee L(t_n,x_n)=u'(t_n,x_n)>v'(t_n,x_n)=v(t_n,x_n)\wedge H(t_n,x_n)\geq L(t_n,x_n)$$

and then $v'(t_n,x_n)<H(t_n,x_n)$ and $u'(t_n,x_n)>L(t_n,x_n)$. As u' (resp. $v'+\frac{\epsilon}{t}$) is a sub (resp. super) solution of ((5.5.4) modified), we then have

$$-p_n-\frac{1}{2}Tr(\sigma\sigma^\top(t_n,x_n)X_n)-<b,n(x_n-y_n)>-f(t_n,x_n,u'(t_n,x_n),n(x_n-y_n))\leq 0,$$

and

$$-p_n-\frac{1}{2}Tr(\sigma\sigma^\top(t_n,y_n)Y_n)-<b,n(x_n-y_n)>-f(t_n,y_n,v'(t_n,y_n)+\frac{\varepsilon}{t_n},n(x_n-y_n))\geq\frac{\varepsilon}{t_n^2}.$$

then

$$\begin{array}{rl}\frac{\varepsilon}{t_n^2}\leq\Lambda_n:=&\frac{1}{2}Tr(\sigma\sigma^\top(t_n,x_n)X_n-\sigma\sigma^\top(t_n,x_n)Y_n)\\ &+f(t_n,x_n,u'(t_n,x_n),n(x_n-y_n))-f(t_n,y_n,v'(t_n,y_n)++\frac{\varepsilon}{t_n},n(x_n-y_n)).\end{array}$$

With the same argument as in ([22], pp. 734), under (H0),(A1)-(A2), we obtain that $\liminf_{n\to\infty}\Lambda_n\leq 0$ and then $\varepsilon\leq 0$ which is contradictory. Finally taking the limits in (5.5.6), first when $R\to\infty$ then $\varepsilon\to\infty$, we obtain $u'\leq v'$ and then $u\leq v$. □

As a by-product we have:

Theorem 5.5.2. *Under (H0),(A1) and (A2) we have:*

(i) There is a unique continuous viscosity solution of (5.5.4) with polynomial growth ;

(ii) The function u is also a unique viscosity solution, in the class Π_g, for the following max-min problem:

$$\left\{\begin{array}{l}\max\Big\{v(t,x)-H(t,x)\ ;\ \min\Big[v(t,x)-L(t,x);\\ \qquad\qquad -\partial_t v(t,x)-\mathcal{L}^X v(t,x)-f(t,x,v(t,x),\sigma(t,x)^\top D_x v(t,x))\Big]\Big\}=0;\\ v(T,x)=g(x).\end{array}\right.\tag{5.5.7}$$

Proof. (i) The existence follows from Theorem 5.5.1 and uniqueness follows from Proposition 5.5.1.

(ii) The construction of the function u implies that $w = -u$ is the unique viscosity solution in the class Π_g of the following system:

$$\begin{cases} \min\Big\{w(t,x) + H(t,x), \min\Big[w(t,x) + L(t,x), \\ \qquad\qquad -\partial_t w(t,x) - \mathcal{L}w(t,x) + f(t,x,-w(t,x),-\sigma(t,x)^\top D_x w(t,x))\Big]\Big\} = 0; \\ w(T,x) = -g(x). \end{cases} \tag{5.5.8}$$

Thus $-w = u$ is the unique solution in the class Π_g of system (5.5.8) (see e.g. [6], pp.18). □

In terms of BSDEs the process $Y^{t,x}$ defined in (5.5.3) is a local solution for the two barriers reflected BSDE associated with $(f(s, X_s^{t,x}, y, z), g(X_T^{t,x}), L(s, X_s^{t,x}), H(s, X_s^{t,x}))$. Namely we have the following result:

Proposition 5.5.2. *([28], Theorem 3.1) For any stopping time τ, there exists another stopping time $\delta_\tau \geq \tau$, $P-a.s.$ and three processes Z^τ, $K^{\pm,\tau}$ such that:*
(i) $Y_T^{t,x} = g(X_T^{t,x})$;
(ii)

$$\begin{cases} Z^\tau \in \mathcal{H}^{2,d},\ K^{\pm,\tau} \in \mathcal{S}^2 \text{ and non-decreasing} ; \\ \forall s \in [\tau, \delta_\tau], Y_s^{t,x} = Y_{\delta_\tau}^{t,x} + \int_s^{\delta_\tau} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^\tau)dr - \int_s^{\delta_\tau} Z_r^\tau dB_r + \int_s^{\delta_\tau} dK_r^{+,\tau} - \int_s^{\delta_\tau} dK_r^{-,\tau} \\ L(s, X_s^{t,x}) \leq Y_s^{t,x} \leq H(s, X_s^{t,x}),\ \forall s \in [t,T] ; \\ \int_\tau^{\delta_\tau} \big(Y_r^\tau - L(r, X_r^{t,x})\big)dK_r^{+,\tau} = 0 \text{ and } \int_\tau^{\delta_\tau} \big(Y_r^\tau - U(r, X_r^{t,x})\big)dK_r^{-,\tau} = 0; \end{cases} \tag{5.5.9}$$

(iii) Let γ_τ and θ_τ be the following two stopping times:

$$\gamma_\tau := \inf\{s \geq \tau, Y_s^{t,x} = L^{i,j}(s, X_s^{t,x})\} \wedge T \text{ and } \theta_\tau := \inf\{s \geq \tau, Y_s^{t,x} = U(s, X_s^{t,x})\} \wedge T.$$

Then $P-a.s.$, $\gamma_\tau \vee \theta_\tau \leq \delta_\tau$.
The process $Y^{t,x}$ is unique to satisfy (i)-(iii). □

Finally in the case when f does not depend on z we have the following charaterization of $Y^{t,x}$ as the value function of a zero-sum Dynkin game.

Proposition 5.5.3. *([28], pp.894) The process $Y^{t,x}$ verifies: for any stopping time $\theta \geq t$,*

$$\begin{aligned} Y_\theta^{t,x} &= ess\ sup_{\sigma \geq \theta} ess\ inf_{\tau \geq \theta} \mathbb{E}\Big\{ \int_\theta^{\sigma\wedge\tau} f(r, X_r^{t,x}, Y_r^{t,x})dr \\ &\qquad + L(\sigma, X_\sigma^{t,x})\mathbb{1}_{[\sigma<\tau]} + H(\tau, X_\tau^{t,x})\mathbb{1}_{[\tau\leq\sigma<T]} + g(X_T^{t,x})\mathbb{1}_{[\tau=\sigma=T]} | \mathcal{F}_\theta \Big\} \\ &= ess\ inf_{\tau \geq \theta} ess\ sup_{\sigma \geq \theta} \mathbb{E}\Big\{ \int_\theta^{\sigma\wedge\tau} f(r, X_r^{t,x}, Y_r^{t,x})dr \\ &\qquad + L(\sigma, X_\sigma^{t,x})\mathbb{1}_{[\sigma<\tau]} + H(\tau, X_\tau^{t,x})\mathbb{1}_{[\tau\leq\sigma<T]} + g(X_T^{t,x})\mathbb{1}_{[\tau=\sigma=T]} | \mathcal{F}_\theta \Big\}. \end{aligned} \tag{5.5.10}$$

□

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PÔLE DE RECHERCHE ET D'ENSEIGNEMENT SUPÉRIEUR

Université du Maine
Le Mans - Laval

Thèse de Doctorat

Xuzhe ZHAO

Problèmes de Switching Optimal, Equation Différentielles Stochastiques Rétrogrades et Equations Différentielles Partielles

Résumé

Cette thèse est composée de trois parties. Dans la première nous montrons l'existence et l'unicité de la solution continue et à croissance polynomiale, au sens viscosité, du système non linéaire de *m* équations variationnelles de type intégro-différentiel à obstacles unilatéraux interconnectés. Ce système est lié au problème du switching optimal stochastique lorsque le bruit est dirigé par un processus de Lévy. Un cas particulier du système correspond en effet à l'équation d'Hamilton-Jacobi-Bellman associé au problème du switching et la solution de ce système n'est rien d'autre que la fonction valeur du problème. Ensuite, nous étudions un système d'équations intégro-différentielles à obstacles bilatéraux interconnectés. Nous montrons l'existence et l'unicité des solutions continus à croissance polynomiale, au sens viscosité, des systèmes min-max et max-min. La démarche conjugue les systèmes d'EDSR réfléchies ainsi que la méthode de Perron. Dans la dernière partie nous montrons l'égalité des solutions des systèmes max-min et min-max d'EDP lorsque le bruit est uniquement de type diffusion. Nous montrons que si les coûts de switching sont assez réguliers alors ces solutions coincident. De plus elles sont caractérisées comme fonction valeur du jeu de switching de somme nulle.

Mots clés : Equations intégro-différentielles à obstacles interconnectés; Processus de Lévy; Switching optimal ; Equations différentielles stochastiques rétrogrades; Jeu à somme nulle; Équation de Hamilton-Jacobi-Bellman-Isaacs; Méthode de Perron.

Abstract

There are three main results in this thesis. The first is existence and uniqueness of the solution in viscosity sense for a system of nonlinear m variational integral-partial differential equations with interconnected obstacles. From the probabilistic point of view, this system is related to optimal stochastic switching problem when the noise is driven by a Lévy process. As a by-product we obtain that the value function of the switching problem is continuous and unique solution of its associated Hamilton-Jacobi-Bellman system of equations. Next, we study a general class of min-max and max-min nonlinear second-order integral-partial variational inequalities with interconnected bilateral obstacles, related to a multiple modes zero-sum switching game with jumps. Using Perron's method and by the help of systems of penalized unilateral reflected backward SDEs with jumps, we construct a continuous with polynomial growth viscosity solution, and a comparison result yields the uniqueness of the solution. At last, we deal with the solutions of systems of PDEs with bilateral inter-connected obstacles of min-max and max-min types in the Brownian framework. These systems arise naturally in stochastic switching zero-sum game problems. We show that when the switching costs of one side are smooth, the solutions of the min-max and max-min systems coincide. Furthermore, this solution is identified as the value function of the zero-sum switching game.

Key Words : IPDE with interconnected obstacles ; Lévy process ; Multi-modes switching problem ; Backward stochastic differential equations; Switching zero-sum game; Hamilton-Jacobi-Bellman-Isaacs equation; Perron's method.