



**HAL**  
open science

# Some Contributions on Probabilistic Interpretation For Nonlinear Stochastic PDEs

Wissal Sabbagh

► **To cite this version:**

Wissal Sabbagh. Some Contributions on Probabilistic Interpretation For Nonlinear Stochastic PDEs. General Mathematics [math.GM]. Université du Maine, 2014. English. NNT : 2014LEMA1019 . tel-01223477

**HAL Id: tel-01223477**

**<https://theses.hal.science/tel-01223477v1>**

Submitted on 2 Nov 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Thèse de Doctorat

Wissal SABBAGH

Mémoire présenté en vue de l'obtention du  
**grade de Docteur de l'Université du Maine**  
sous le label de l'Université Nantes Angers Le Mans  
**et de l'Ecole Nationale d'Ingénieurs de Tunis**  
sous le label de l'Université de Tunis EL Manar

École doctorale : *STIM*

Discipline : 26  
Spécialité : *Mathématiques Appliquées*  
Unité de recherche : *LMM*  
Soutenue le : *08 Décembre 2014*

## Some Contributions on Probabilistic Interpretation For Nonlinear Stochastic PDEs

### JURY

Rapporteurs : **Antoine LEJAY**, Professeur, *INRIA, Université de Lorraine*  
**Jianfeng ZHANG**, Professeur, *Université South California*

Examineurs : **Amel BEN ABDA**, Professeur, *Ecole Nationale d'Ingénieurs de Tunis, Tunis*  
**Bruno BOUCHARD**, Professeur, *ENSAE et Université Paris Dauphine*  
**Anne DE BOUARD**, Professeur, *Ecole Polytechnique, Paris*  
**Laurent DENIS**, Professeur, *Université du Maine, Le Mans*  
**Emmanuel GOBET**, Professeur, *Ecole Polytechnique, Paris*  
**Said HAMADENE**, Professeur, *Université du Maine, Le Mans*  
**Marina KLEPSTYNA**, Professeur, *Université du Maine, Le Mans*  
**Nizar TOUZI**, Professeur, *Ecole Polytechnique, Paris*

Directeur de Thèse : **Anis MATOUSSI**, Professeur, *Université du Maine, Le Mans*

Co-directeur de Thèse : **Mohamed MNIF**, Professeur, *Ecole Nationale d'Ingénieurs de Tunis, Tunis*



## Remerciements

Mes premiers remerciements s'adressent à mon directeur de thèse, Anis Matoussi, pour la confiance qu'il m'a accordée en acceptant d'encadrer ce travail. Je lui suis également reconnaissante pour sa disponibilité, ses qualités pédagogiques et scientifiques. J'ai beaucoup appris à ses côtés et je lui adresse ma gratitude pour tout cela.

Je ne manquerais pas de remercier mon co-directeur de thèse, Mohamed Mnif, qui a été présent par ses conseils et ses remarques pour accompagner mes premiers pas en recherche depuis le master.

Je voudrais ensuite remercier Antoine Lejay et Jianfeng Zhang de m'avoir fait l'honneur d'être les rapporteurs de cette thèse. Je suis extrêmement reconnaissante du temps qu'ils m'ont consacré. Merci également à Antoine pour sa présence dans ce jury de soutenance. Je lui exprime ici ma plus haute considération.

Je tiens aussi à remercier Bruno Bouchard, Anne De Bouard, Emmanuel Gobet et Nizar Touzi d'avoir accepté de faire partie de mon jury malgré leurs emplois du temps très chargés. Je suis très reconnaissante à Laurent Denis et Saïd Hamadène pour leur présence dans mon jury et pour l'intérêt qu'ils portent à mon travail. Je remercie chaleureusement Amel Ben Abda pour sa gentillesse et ses conseils toujours avisés depuis le master.

Mes remerciements vont également à Marina Klepstyna pour avoir accepté de participer à ce jury de thèse et pour l'ambiance de travail très agréable qu'il a su créer au département de mathématiques grâce à sa très grande ouverture d'esprit en particulier avec les doctorants.

Je ne saurais oublier de remercier tous les membres du laboratoire LMM du Mans et LAMSIN du Tunis, qui ont beaucoup facilité ma vie de doctorante, avec une pensée particulière à Brigitte, Irène et Raoudha. Un merci particulier aux thésards du Laboratoire Manceau de Mathématiques avec qui j'ai passé des bons moments, en particulier Fanny, Maroua, Rui, Lin, Achref, Samvel, Lambert et Zhao.

Je remercie également l'IFT (Institut Français de Tunis) qui a financé une grande partie de cette thèse et mes déplacements entre la France et la Tunisie.

Enfin, mes remerciements ne sauraient être complets si je n'y incluais mes parents, mes frères et ma petite soeur, pour l'aide morale et la motivation qu'ils m'ont apportées pour achever ce travail et mes proches pour leur soutien, leur confiance et leurs encouragements tout au long de ces années de thèse. Un énorme MERCI à vous tous.

---

## Abstract

The objective of this thesis is to study the probabilistic representation (Feynman-Kac formula) of different classes of Stochastic Nonlinear PDEs (semilinear, fully nonlinear, reflected in a domain) by means of backward doubly stochastic differential equations (BDSDEs). This thesis contains four different parts. We deal in the first part with the second order BDSDEs (2BDSDEs). We show the existence and uniqueness of solutions of 2BDSDEs using quasi sure stochastic control technics. The main motivation of this study is the probabilistic representation for solution of Fully nonlinear SPDEs. First, under regularity assumptions on the coefficients, we give a Feynman-Kac formula for classical solution of fully nonlinear SPDEs and we generalize the work of Soner, Touzi and Zhang (2010-2012) for deterministic fully nonlinear PDE. Then, under weaker assumptions on the coefficients, we prove the probabilistic representation for stochastic viscosity solution of fully nonlinear SPDEs. In the second part, we study the Sobolev solution of obstacle problem for partial integro-differential equations (PIDEs). Specifically, we show the Feynman-Kac formula for PIDEs via reflected backward stochastic differential equations with jumps (BSDEs). Specifically, we establish the existence and uniqueness of the solution of the obstacle problem, which is regarded as a pair consisting of the solution and the measure of reflection. The approach is based on stochastic flow technics developed in Bally and Matoussi (2001) but the proofs are more technical. In the third part, we discuss the existence and uniqueness for RBDSDEs in a convex domain  $D$  without any regularity condition on the boundary. In addition, using the approach based on the technics of stochastic flow we provide the probabilistic interpretation of Sobolev solution of a class of reflected SPDEs in a convex domain via RBDSDEs. Finally, we are interested in the numerical solution of BDSDEs with random terminal time. The main motivation is to give a probabilistic representation of Sobolev solution of semilinear SPDEs with Dirichlet null condition. In this part, we study the strong approximation of this class of BDSDEs when the random terminal time is the first exit time of an SDE from a cylindrical domain. Thus, we give bounds for the discrete-time approximation error. We conclude this part with numerical tests showing that this approach is effective.

**Keywords:** Backward Doubly Stochastic Differential Equations, Second order Backward Doubly Stochastic Differential Equations, quasi-sure stochastic analysis, Semilinear Stochastic PDEs, Fully Nonlinear Stochastic PDEs, Reflected backward stochastic differential equation, partial parabolic integro-differential equations, jump diffusion process, obstacle problem, Stochastic flow, Skorohod problem, convex domains, regular measure, Forward-Backward System, Euler scheme, Monte Carlo method.

---

---

## Résumé

L'objectif de cette thèse est l'étude de la représentation probabiliste (formule de Feynman-Kac) des différentes classes d'EDP Stochastiques non-linéaires (semi-linéaires, complètement non-linéaires, réfléchies dans un domaine) en utilisant les équations différentielles doublement stochastiques rétrogrades (EDDSRs). Cette thèse contient quatre parties différentes. Nous traitons dans la première partie les EDDSRs du second ordre (2EDDSRs). Nous montrons l'existence et l'unicité des solutions des 2EDDSRs en utilisant des techniques de contrôle stochastique quasi-sure. La motivation principale de cette étude est la représentation probabiliste des EDP stochastiques complètement non-linéaires (ou Fully nonlinear). Tout d'abord, sous des hypothèses de régularité sur les coefficients, nous donnons une formule de Feynman-Kac pour les solutions classiques des EDP Stochastiques totalement non linéaires généralisant ainsi les travaux de Soner, Touzi et Zhang (2010-2012) pour les EDP totalement non-linéaire déterministes. Puis, sous des hypothèses plus faibles de type Lipschitz sur les coefficients, nous prouvons la représentation probabiliste des solutions de viscosité stochastique de EDPS totalement non linéaires. Dans la deuxième partie, nous étudions les solutions faibles de type Sobolev du problème d'obstacle pour les équations à dérivées partielles intégro-différentielles (EDPIDs). Plus précisément, nous montrons la formule de Feynman-Kac pour l'EDPIDs par l'intermédiaire des équations différentielles stochastiques rétrogrades réfléchies avec sauts (EDSRRs). Plus précisément, nous établissons l'existence et l'unicité de la solution du problème d'obstacle, qui est considérée comme un couple constitué de la solution et de la mesure de réflexion. L'approche utilisée est basée sur les techniques de flots stochastiques développées dans Bally et Matoussi (2001) mais les preuves sont beaucoup plus techniques. Dans la troisième partie, nous traitons l'existence et l'unicité pour les EDDSRs dans un domaine convexe  $D$  sans aucune condition de régularité sur la frontière. De plus, en utilisant l'approche basée sur les techniques du flot stochastiques nous démontrons l'interprétation probabiliste de la solution faible de type Sobolev d'une classe d'EDP Stochastiques réfléchies dans un domaine convexe via les EDDSRs. Enfin, nous nous intéressons à la résolution numérique des EDDSRs à temps terminal aléatoire. La motivation principale est de donner une représentation probabiliste des solutions de Sobolev d'EDP Stochastiques semi-linéaires avec condition de Dirichlet nul au bord. Dans cette partie, nous étudions l'approximation forte de cette classe d'EDDSRs quand le temps terminal aléatoire est le premier temps de sortie d'une EDS d'un domaine cylindrique. Ainsi, nous donnons les bornes pour l'erreur d'approximation en temps discret. Cette partie se conclut par des tests numériques qui démontrent que cette approche est effective.

**Mots-clés:** Équations Différentielles Doublement Stochastiques Rétrogrades, Équations Différentielles Doublement Stochastiques Rétrogrades du second ordre, Analyse stochastique quasi-sure, EDP Stochastiques semi-linéaires, EDP Stochastiques complètement non-linéaires, Flot stochastique, Problème de Skorohod, Schéma d'Euler, Méthode Monte Carlo.

---

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	2BDSDEs and Fully nonlinear SPDEs . . . . .	1
1.1.1	Backward Doubly SDEs and semilinear SPDEs . . . . .	1
1.1.2	Second Order BSDEs and Fully Nonlinear PDEs . . . . .	6
1.1.3	Second Order BDSDEs and Fully Nonlinear SPDEs . . . . .	7
1.1.3.1	Existence and uniqueness of 2BDSDEs . . . . .	7
1.1.3.2	Probabilistic interpretation of solution of SPDEs . . . . .	10
1.2	Reflection problem for PIDEs and SPDEs . . . . .	12
1.2.1	Obstacle problem for PIDEs . . . . .	12
1.2.1.1	Motivations and formulation . . . . .	13
1.2.1.2	New results . . . . .	14
1.2.2	BDSDEs and semilinear SPDEs in a convex domain . . . . .	18
1.2.2.1	Motivations and literature . . . . .	18
1.2.2.2	New results . . . . .	18
1.3	Numerical scheme for BDSDEs in domain . . . . .	22
1.3.1	Literature . . . . .	22
1.3.2	New results . . . . .	23
1.4	Work in progress and Perspectives . . . . .	28
<b>2</b>	<b>Probabilistic Interpretation for Fully Nonlinear SPDEs</b>	<b>29</b>
2.1	Introduction . . . . .	29
2.2	Preliminaries and Hypothesis . . . . .	30
2.2.1	The set of probability measures and the generator . . . . .	30
2.2.2	The Spaces and Norms . . . . .	32
2.2.3	Formulation . . . . .	32
2.2.4	Connection with standard BDSDEs . . . . .	33
2.2.5	Connection with G-expectation . . . . .	33
2.3	Uniqueness of the solution and other properties . . . . .	34
2.3.1	Representation and uniqueness of the solution . . . . .	34
2.3.2	A priori estimates . . . . .	37
2.4	A direct existence argument . . . . .	39
2.4.1	Notations . . . . .	39
2.4.2	Existence when $\xi$ is in $UC_b(\Omega)$ . . . . .	41
2.4.3	Existence result in the general case . . . . .	47
2.5	Probabilistic interpretation for Fully nonlinear SPDEs . . . . .	49
2.5.1	A nonlinear Feynman-Kac representation formula . . . . .	51
2.5.2	Stochastic viscosity solution for SPDE . . . . .	53
2.5.2.1	Definitions . . . . .	53
2.5.2.2	Doss transformation . . . . .	55
2.6	Appendix . . . . .	57
2.6.1	Doubly $f$ - supersolution and martingales . . . . .	57
2.6.2	Downcrossing inequality . . . . .	58
2.6.3	Reflected backward doubly stochastic differential equations . . . . .	59

<b>3</b>	<b>The obstacle problem for PIDEs</b>	<b>65</b>
3.1	Introduction . . . . .	65
3.2	Hypotheses and preliminaries . . . . .	67
3.2.1	Weak formulation for the partial differential-integral equations . . . . .	68
3.2.2	Stochastic flow of diffeomorphisms and random test functions . . . . .	69
3.3	Sobolev solutions for parabolic semilinear PIDEs . . . . .	72
3.4	Obstacle problem for PIDEs . . . . .	73
3.5	Appendix . . . . .	78
3.5.1	Proof of Proposition 3.2.3 . . . . .	78
3.5.2	Regularity of the solution of BSDE with jumps . . . . .	79
3.5.3	Proof of Theorem 3.3.1: existence and uniqueness of solution of PIDE (3.1.1) . . . . .	85
3.5.4	Proof of the tightness of the sequence $(\pi_n)_{n \in \mathbb{N}}$ . . . . .	88
<b>4</b>	<b>Semilinear Stochastic PDEs in a convex domain</b>	<b>91</b>
4.1	Introduction . . . . .	91
4.2	Backward Doubly Stochastic Differential Equations in a domain . . . . .	93
4.2.1	Hypotheses and preliminaries . . . . .	93
4.2.1.1	Convexity results . . . . .	93
4.2.1.2	Functional spaces and assumptions . . . . .	93
4.2.2	Existence and uniqueness of the solution . . . . .	95
4.3	Weak solution of semilinear SPDE in a convex domain . . . . .	102
4.3.1	Notations and Hypothesis . . . . .	102
4.3.2	Weak formulation for a solution of Stochastic PDEs . . . . .	103
4.3.3	Stochastic flow of diffeomorphisms and random test functions . . . . .	104
4.3.4	Existence and uniqueness of solutions for the reflected SPDE . . . . .	105
4.4	Appendix . . . . .	110
4.4.1	Some properties of convexity . . . . .	110
4.4.2	A priori estimates . . . . .	111
4.4.3	Proof of Lemma 4.2.4 . . . . .	112
4.4.4	Proof of the tightness of the sequence $(\pi_n)_{n \in \mathbb{N}}$ . . . . .	113
<b>5</b>	<b>Numerical Computation for BDSDEs in a domain</b>	<b>115</b>
5.1	Introduction . . . . .	115
5.2	Backward doubly stochastic differential equations with random terminal time . . . . .	116
5.3	Numerical scheme for Forward-Backward Doubly SDEs . . . . .	119
5.3.1	Formulation . . . . .	120
5.3.2	Euler scheme approximation of Forward-BDSDEs . . . . .	121
5.3.2.1	Forward Euler scheme . . . . .	121
5.3.2.2	Euler scheme for BDSDEs . . . . .	122
5.3.2.3	Upper bounds for the discrete-time approximation error . . . . .	123
5.4	Semilinear Stochastic PDEs with Dirichlet null condition . . . . .	128
5.4.1	Definitions and formulation . . . . .	128
5.4.2	Stochastic flow of diffeomorphism and random test functions . . . . .	130
5.4.3	Probabilistic representation of the solution of SPDE . . . . .	132
5.5	Implementation and numerical tests . . . . .	135
5.5.1	Notations and algorithm . . . . .	135



---

5.5.1.1	Forward Euler scheme . . . . .	135
5.5.1.2	Numerical scheme for BDSDEs . . . . .	136
5.5.1.3	Vector spaces of functions . . . . .	136
5.5.1.4	Description of the algorithm . . . . .	136
5.5.2	One-dimensional case (Case when $d = k = l = 1$ ) . . . . .	137
5.5.2.1	Function bases . . . . .	137
5.5.2.2	Comparison of numerical approximations of the solutions of the FBDSDE and the FBSDE: the general case . . . . .	138

**Bibliography**



# Introduction

---

The aim of this thesis is to study by a probabilistic approach different classes of nonlinear stochastic partial differential equations (SPDEs in short). More precisely, we provide the probabilistic representation of solution for SPDEs (classical, Sobolev and Viscosity solutions) by means of solution of some class of backward doubly stochastic differential equations (BDSDEs in short).

This thesis contains four chapters. In Chapter 2, we propose a wellposedness theory for a class of second order backward doubly stochastic differential equation (2BDSDE). We prove existence and uniqueness of the solution under a Lipschitz type assumption on the generator, and we investigate the links between our 2BDSDEs and a class of parabolic Fully non-linear Stochastic PDEs. Precisely, we show that the Markovian solution of 2BDSDEs provide a probabilistic interpretation of the classical and stochastic viscosity solution of Fully nonlinear SPDEs. Then, we present in Chapter 3 a probabilistic interpretation for the weak Sobolev solution of the obstacle problem for semilinear parabolic partial integro-differential equations (PIDEs). The results of Léandre [82] concerning the homeomorphic property for the solution of SDEs with jumps are used to construct random test functions for the variational equation for such PIDEs. These results are the natural connection with the associated Reflected Backward Stochastic Differential Equations with jumps (RBSDEs), namely Feynman Kac's formula for the solution of the PIDEs. In Chapter 4, we prove existence and uniqueness results for reflected backward doubly stochastic differential equations (in short RBDSDEs) in a convex domain  $D$  without any regularity conditions on the boundary. Moreover, using a stochastic flow approach a probabilistic interpretation for a class of reflected SPDEs in a domain is given via such RBDSDEs. The last Chapter (5) is about solving numerically backward doubly stochastic differential equations (BDSDEs) with random terminal time  $\tau$ . The main motivations are giving a probabilistic representation of the Sobolev's solution of Dirichlet problem for semi-linear SPDEs and providing the numerical scheme for such SPDEs. Thus, we study the strong approximation of this class of BDSDEs when  $\tau$  is the first exit time of a forward SDE from a cylindrical domain. We use the Euler scheme and we provide bounds for the discrete-time approximation error.

We begin by recalling the theory of BSDEs and BDSDEs.

## 1.1 2BDSDEs and Fully nonlinear SPDEs

### 1.1.1 Backward Doubly SDEs and semilinear SPDEs

Backward Stochastic Differential Equations (BSDEs) have been studied extensively in the last two decades as they naturally arise in the context of stochastic control problems [44, 62], and they provide a probabilistic representation for solutions to semilinear parabolic Partial Differential Equations (PDEs), via a nonlinear Feynman-Kac's formula. This class of equations were introduced by Bismut [16] as equation for the adjoint process in the stochastic version of Pontryagin maximum principle. Then Pardoux and Peng [110] generalized this notion in 1990

for a nonlinear general class of BSDEs. According to these authors, a solution to a BSDE consists of a pair of adapted processes  $(y, z)$  taking values in  $\mathbb{R}^n$  and  $\mathbb{R}^{d \times n}$ , respectively, such that

$$y_t = \xi + \int_t^T f(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T] \quad (1.1.1)$$

where  $T$  is a finite time horizon,  $(B_t)_{t \in [0, T]}$  a  $d$ -dimensional Brownian motion on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ ,  $\mathcal{F}$  is the  $\mathbb{P}$ -augmented natural filtration generated by  $B$ ,  $f$  a progressively measurable function from  $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{d \times n}$ , and  $\xi$  an  $\mathbb{R}^n$ -valued  $\mathcal{F}_T$ -measurable random variable. They showed the existence and uniqueness of the adapted solution under the condition that  $f$  is uniformly Lipschitz in  $y$  and  $z$  and that  $\xi$  and  $f(s, 0, 0)$  are square integrable. Since then, BSDEs have been studied with great interest. In particular, many efforts have been made to relax the assumptions on the generator. For example, Lepeltier and San Martin [85] have considered the case where the generator is continuous with linear growth. Then, Kobylanski [74] have showed the existence of a maximal solution in the case where the generator  $f$  has linear growth in  $y$  and quadratic growth in  $z$  with a bounded terminal condition. This latter BSDE is very useful in finance because it allows to solve problems in portfolio management constraints through a backward approach, where the utility function is exponential, (see e.g. El Karoui and Rouge [123], Hu, Imkeller and Muller [66]). El Karoui et al. [46] introduced the notion of one barrier reflected BSDE, which is actually a backward equation but the solution is forced to stay above a given barrier. This type of BSDEs is motivated by pricing American options and studying the mixed game problems (see e.g. Cvitanic and Karatzas [35], Hamadène, Lepeltier and Matoussi [63]).

The main motivation to introduce the nonlinear BSDEs was to give a probabilistic interpretation (Feynman-Kac's formula) for the solutions of semilinear parabolic PDEs. For smooth coefficients the PDE has a classical solution but if the coefficients are just Lipschitz continuous functions one has to consider weak solutions. This result was first obtained by Peng [115], see also Pardoux and Peng [108] by considering the viscosity and classical solutions of such PDEs. Later, Barles and Lesigne [13] proved that the same probabilistic interpretation holds for the variational formulation of the PDEs.

Since we deal with solutions of a class of Stochastic PDEs (SPDEs in short), we consider Backward Doubly Stochastic Differential Equations (BDSDEs, in short). Indeed, by introducing in standard BSDEs a second nonlinear term driven by an external noise, Pardoux and Peng introduced in [109] the so-called backward doubly stochastic differential equations (BDSDEs, in short),

$$y_t = \xi + \int_t^T f(s, y_s, z_s) ds + \int_t^T g(s, y_s, z_s) d\overleftarrow{W}_s - \int_t^T z_s dB_s, \quad t \in [0, T] \quad (1.1.2)$$

where  $(W_t)_{t \geq 0}$  and  $(B_t)_{t \geq 0}$  are two finite-dimensional independent Brownian motions. We note that the integral with respect to  $W$  is a "backward Itô integral". In order to establish the connection between BDSDEs and SPDEs (1.1.4), Pardoux and Peng [109] considered the following Markovian BDSDE, for all  $0 \leq t \leq s \leq T$

$$y_s^{t,x} = \Phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) dr + \int_s^T g(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) d\overleftarrow{W}_r - \int_s^T z_r^{t,x} dB_r. \quad (1.1.3)$$

These equations provide Feynman-Kac's representation for the solution of the following system of semilinear parabolic Stochastic PDEs

$$\begin{cases} du(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), (\nabla u \sigma)(t, x))dt \\ \quad + g(t, x, u(t, x), (\nabla u \sigma)(t, x))d\overleftarrow{W}_t = 0, \quad 0 \leq t < T \\ u(T, x) = \phi(x) \end{cases} \quad (1.1.4)$$

where  $a = (\sigma \sigma^*)$ ,  $\mathcal{L}u = (Lu^i)_{1 \leq i \leq n}$ ,

$$L = \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \quad (1.1.5)$$

$\mathcal{L}$  is the second order differential operator associated with a diffusion process  $\{X_s^{t,x}; t \leq s \leq T\}$ , which is the strong solution of a standard Itô (forward) stochastic differential equation (SDE):

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dB_r, \quad t \leq s \leq T.$$

These SPDEs (1.1.4) appear in various applications as, for instance, Zakai equations in filtering, pathwise stochastic control theory and stochastic control with partial observations. Several generalizations to investigate more general nonlinear SPDEs have been developed following different approaches of the notion of weak solutions, namely, Sobolev's solutions [43, 61, 75, 125, 137], and stochastic viscosity solutions [88, 89, 90, 24, 25].

Under smoothness assumptions on the coefficients, Pardoux and Peng [109] proved the existence and uniqueness of classical solution for SPDEs (1.1.4) and they related it to solutions of BDSDEs (1.1.3) given by

$$y_s^{t,x} = u(s, X_s^{t,x}) \quad \text{and} \quad z_s^{t,x} = (\nabla u \sigma)(s, X_s^{t,x}). \quad (1.1.6)$$

But, if the coefficients are just Lipschitz continuous one has to consider solutions in a weak sense. More precisely,  $u$  becomes a weak solution of equation (1.1.4) if the following relation holds almost surely, for each test function  $\varphi \in C^\infty([0, T]) \otimes C_c^\infty(\mathbb{R}^d)$ ,

$$\begin{aligned} & \int_t^T (u(s, \cdot), \partial_s \varphi(s, \cdot))ds + \int_t^T e(u(s, \cdot), \varphi(s, \cdot))ds + (u(t, \cdot), \varphi(t, \cdot)) - (\Phi(\cdot), \varphi(T, \cdot)) \\ & = \int_t^T (f(s, \cdot, u(s, \cdot), (\nabla u \sigma)(s, \cdot)), \varphi(s, \cdot))ds + \sum_{i=1}^n \int_t^T (g(s, \cdot, u(s, \cdot), (\nabla u \sigma)(s, \cdot)), \varphi(s, \cdot))d\overleftarrow{W}_s^i, \end{aligned} \quad (1.1.7)$$

where  $(\varphi, \phi) = \int_{\mathbb{R}^d} \varphi(x)\phi(x)dx$  denotes the scalar product in  $L^2(\mathbb{R}^d, dx)$  and

$$e(u, \varphi) = (Lu, \varphi) = \int_{\mathbb{R}^d} ((\nabla u \sigma)(\nabla \varphi \sigma) + \varphi \nabla \cdot (\frac{1}{2} \nabla \sigma \sigma + b)u)(x)dx$$

is the energy associated to the diffusion operator. This is a variational formulation for SPDE (1.1.4).

In [10], Bally and Matoussi proved that the probabilistic interpretation (1.1.6) holds for the variational formulation (1.1.7) and that the SPDE (1.1.4) has a unique solution  $u$  in a

Sobolev space. The key element in their approach is the theory of stochastic flows which has already been used by Kunita in [80] and [79] when the coefficients are linear and don't depend on the gradient of the solution. Next, Matoussi and Scheutzow [97] introduced another kind of external noise term given by the Itô-Kunita stochastic integral. This allows to give a probabilistic interpretation of classical and Sobolev's solutions of semilinear parabolic stochastic partial differential equations driven by a nonlinear coreled space-time noise.

An other approach of the notion of weak solution for PDEs and SPDEs is the viscosity solution. The notion of viscosity solution for PDEs was introduced by Crandall, Ishii and Lions [31] for certain first order Hamilton-Jacobi equations. Today this theory becomes an important tool in many applied fields, especially in optimal control theory and numerous subjects related to it, in particular mathematical finance. The stochastic viscosity solution for semilinear SPDEs was introduced firstly by Lions and Souganidis in some notes [88, 89, 90]. They used the so-called "stochastic characteristic" to remove the stochastic integrals from SPDEs and obtain PDEs with random coefficients. A few years later, Buckdahn and Ma [24, 25] have introduced the rigorous notion of stochastic viscosity solution for semilinear SPDEs and then they gave the probabilistic interpretation of such equation via BDSDEs where the intensity of the noise  $g$  in the SPDEs (1.1.4) does not depend on the gradient of the solution. They used the so-called Doss-Sussmann transformation and stochastic diffeomorphism flow technics to convert the semilinear SPDEs to PDEs with random coefficients. This transformation permits to remove the martingale term from the SPDEs.

The definition of the stochastic viscosity solution will depend on the following stochastic flow  $\eta \in C(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d \times \mathbb{R})$  defined as the unique solution of the stochastic differential equation (SDE):

$$\eta(t, x, y) = y + \int_t^T g(s, x, \eta(s, x, y)) \circ d\overleftarrow{W}_s, \quad 0 \leq t \leq T. \quad (1.1.8)$$

Under the smooth assumption on  $g$ , for fixed  $x$  the random field  $\eta(\cdot, x, \cdot)$  is continuously differentiable in the variable  $y$ ; and the mapping  $y \mapsto \eta(t, x, y, \omega)$  defines a diffeomorphism for all  $(t, x)$ ,  $\mathbb{P}$ -a.s. We denote by  $\mathcal{E}(t, x, y)$  the  $y$ -inverse of  $\eta(t, x, y)$ , so  $\mathcal{E}(t, x, y)$  is the solution of the following first-order SPDE:

$$\mathcal{E}(t, x, y) = y - \int_t^T D_y \mathcal{E}(s, x, y) g(s, x, y) \circ d\overleftarrow{W}_s, \quad \forall (t, x, y), \quad \mathbb{P} - a.s. \quad (1.1.9)$$

We note that  $\mathcal{E}(t, x, \eta(t, x, y)) = \mathcal{E}(T, x, \eta(T, x, y)) = y, \quad \forall (t, x, y)$ . Afterthat, we need to define the following spaces:

For  $p \geq 1$ ,  $L^p(\mathcal{F}_{\tau,T}^W; \mathbb{R}^d)$  denotes the space of all  $\mathbb{R}^d$ -valued  $\mathcal{F}_{\tau,T}^W$  measurable r.v.  $\xi$  such that  $\mathbb{E}[|\xi|^p] < +\infty$ .

$C^{l,k}([0, T] \times \mathbb{R}^d)$ , for  $k, l \geq 0$ , denotes the space of all  $\mathbb{R}$ -valued functions defined on  $[0, T] \times \mathbb{R}^d$  which are  $k$ -times continuously differentiable in  $t$  and  $l$ -times continuously differentiable in  $x$ .  $C^{l,k}(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$ , for  $k, l \geq 0$ , denotes the space of all  $C^{l,k}([0, T] \times \mathbb{R}^d)$ -valued random variables  $\varphi$  that are  $\mathcal{F}_{t,T}^W \otimes \mathcal{B}([0, T] \times \mathbb{R}^d)$  measurable.

$C^{l,k}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$ , for  $k, l \geq 0$ , denotes the space of random variables  $\varphi \in C^{l,k}(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$  such that for fixed  $x \in \mathbb{R}^d$  the mapping  $(t, \omega) \mapsto \varphi(t, x, \omega)$  is  $\mathbb{F}^W$ - progressively measurable.

We now define the notion of stochastic viscosity solution for SPDEs introduced first by Buckdahn and Ma [25, 24].

**Definition 1.1.1.** (i) A random field  $u \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  is called a stochastic viscosity subsolution (resp. supersolution) of SPDE  $(f, g)$ , if  $u(T, x) \leq$  (resp.  $\geq$ )  $\phi(x)$ ,  $\forall x \in \mathbb{R}^d$ ; and if for any  $\{\mathcal{F}_{t,T}^W\}_{t \geq 0}$ -stopping time  $\tau$  such that  $0 \leq \tau \leq t$ ,  $\zeta \in \mathbb{L}^0(\mathcal{F}_\tau^W; \mathbb{R}^d)$ , and any random field  $\varphi \in C^{1,2}(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$  satisfying

$$u(t, x) - \eta(t, x, \varphi(t, x)) \leq (\text{resp. } \geq) 0 = u(\tau, \zeta) - \eta(\tau, \zeta, \varphi(\tau, \zeta)),$$

for all  $(t, x)$  in a neighborhood of  $(\tau, \zeta)$ ,  $\mathbb{P}$ -a.e. on the set  $\{0 < \tau < T\}$ , it holds that

$$-\mathcal{L}\psi(\tau, \zeta) - f(\tau, \zeta, \psi(\tau, \zeta), (\nabla\psi\sigma)(\tau, \zeta)) \leq (\text{resp. } \geq) D_y\eta(\tau, \zeta, \varphi(\tau, \zeta))D_t\varphi(\tau, \zeta),$$

$\mathbb{P}$ -a.e. on  $\{0 < \tau < T\}$ , where  $\psi(t, x) \triangleq \eta(t, x, \varphi(t, x))$ .

(ii) A random field  $u \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  is called a stochastic viscosity solution of SPDE  $(f, g)$ , if it is both a stochastic viscosity subsolution and a supersolution.

**Definition 1.1.2.** A random field  $u \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  is called a  $\omega$ -wise viscosity (sub-, super-) solution if for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $u(\omega, \cdot, \cdot)$  is a (deterministic) viscosity (sub-, super-) solution of the SPDE  $(f, 0)$ .

**Remark 1.1.1.** If we assume that  $\varphi \in C^{1,2}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$ , and that  $g \in C^{0,0,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^l)$ , then a straightforward computation using the Itô-Ventzell formula shows that the random field  $\psi(t, x) = \eta(t, x, \varphi(t, x))$  satisfies

$$d\psi(t, x) = D_y\eta(t, x, \varphi(t, x))D_t\varphi(t, x)dt - \langle g(t, x, \psi(t, x)), \circ d\overleftarrow{W}_t \rangle, \quad t \in [0, T]. \quad (1.1.10)$$

Since  $g(\tau, \zeta, \psi(\tau, \zeta)) = g(\tau, \zeta, u(\tau, \zeta))$  by definition, it seems natural to compare  $-\mathcal{L}\psi(\tau, \zeta) - f(\tau, \zeta, \psi(\tau, \zeta), (\nabla\psi\sigma)(\tau, \zeta))$  with  $D_y\eta(\tau, \zeta, \varphi(\tau, \zeta))D_t\varphi(\tau, \zeta)$  to characterize a viscosity solution of SPDE  $(f, g)$ .

If the function  $g \equiv 0$  in SPDE  $(f, g)$ , the flow  $\eta$  becomes  $\eta(t, x, y) = y$ ,  $\forall (t, x, y)$  and  $\psi(t, x) = \varphi(t, x)$ . Thus the definition of a stochastic viscosity solution becomes the same as that of a deterministic viscosity solution (see, e.g. Crandall, Ishii and Lions [31]).

After extending the notion of viscosity solution to nonlinear SPDEs, Buckdahn and Ma [24] showed that a stochastic PDE can be converted to a PDE with random coefficients via a Doss-Sussmann transformation. Then, they proved the existence of the stochastic viscosity solution and obtained the nonlinear Feynmann-Kac formula. Here, we will recall briefly their main results in [24].

**Proposition 1.1.1.** Under Lipschitz continuous condition on  $f$  and smooth assumption on  $g$  and the coefficients  $\sigma$  and  $b$ , we have that a random field  $u$  is a stochastic viscosity sub- (resp. super-) solution to SPDE  $(f, g)$  (1.1.4) if and only if  $v(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot, u(\cdot, \cdot))$  is a stochastic viscosity sub- (resp. super-) solution to SPDE  $(\tilde{f}, 0)$ . Consequently,  $u$  is a stochastic viscosity solution to SPDE  $(f, g)$  (1.1.4) if and only if  $v(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot, u(\cdot, \cdot))$  is a stochastic viscosity solution to SPDE  $(\tilde{f}, 0)$ , with

$$\begin{aligned} \tilde{f}(t, x, y, z) &= \frac{1}{D_y\eta(t, x, y)} \{f(t, x, \eta(t, x, y), D_x\eta(t, x, y)\sigma(x) + D_y\eta(t, x, y)z) \\ &\quad + \mathcal{L}_x\eta(t, x, y) + D_{xy}\eta(t, x, y)\sigma(x)z + \frac{1}{2}D_{yy}\eta(t, x, y)|z|^2\}, \end{aligned} \quad (1.1.11)$$

for all  $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ ,  $\mathbb{P}$ -a.s. Here  $\mathcal{L}_x$  is the same as the operator in (1.1.5), with the emphasis that all the partial derivatives are with respect to  $x$ .

We will now apply Doss transformation to the BDSDE (1.1.3) in order that the backward integral vanishes. Thus, the BDSDE will become a BSDE with a new generator  $\tilde{f}$  which is of quadratic growth with respect to the gradient of the solution. Let us define the following two processes:

$$\begin{aligned} U_s^{t,x} &= \mathcal{E}(t, X_s^{t,x}, y_s^{t,x}) \\ V_s^{t,x} &= D_y \mathcal{E}(s, X_s^{t,x}, y_s^{t,x}) z_s^{t,x} + D_x \mathcal{E}(s, X_s^{t,x}, y_s^{t,x}) \sigma(x). \end{aligned} \quad (1.1.12)$$

It is easy to check the inverse transformation:

$$\begin{aligned} y_s^{t,x} &= \eta(s, X_s^{t,x}, U_s^{t,x}) \\ z_s^{t,x} &= D_y \eta(t, X_s^{t,x}, U_s^{t,x}) V_t + D_x \eta(t, X_s^{t,x}, U_s^{t,x}) \sigma(x). \end{aligned} \quad (1.1.13)$$

**Theorem 1.1.1.** *Under Lipschitz continuous condition on  $f$  and smooth assumption of  $g$  we have  $(U^{t,x}, V^{t,x})$  is the unique solution of the following BSDE,*

$$U_s^{t,x} = \phi(X_T^{t,x}) - \int_s^T \tilde{f}(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) dr - \int_s^T V_r^{t,x} dB_r \quad t \leq s \leq T, \quad (1.1.14)$$

where  $\tilde{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$  is defined by (1.1.11).

Next, Buckdahn and Ma proved the existence of the stochastic viscosity solution of the SPDE  $(f, g)$  (1.1.4). First, they introduced for each  $(t, x) \in [0, T] \times \mathbb{R}^d$  two random fields

$$u(t, x) = y_t^{t,x} \quad v(t, x) = U_t^{t,x},$$

where  $y$  and  $U$  are the solutions of the BDSDE (1.1.3) and BSDE (1.1.14). Then by (1.1.12) and (1.1.13) we know that, for  $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$

$$u(\omega, t, x) = \eta(\omega, t, x, v(\omega, t, x)); \quad v(\omega, t, x) = \mathcal{E}(\omega, t, x, u(\omega, t, x)). \quad (1.1.15)$$

Finally, thanks to Proposition 1.1.1, Buckdahn and Ma [24] need only to prove that the random field  $v$  defined in (1.1.15) is a stochastic viscosity solution to the SPDE  $(\tilde{f}, 0)$ .

**Theorem 1.1.2.** *Under the Assumptions of Proposition 1.1.1, the random field  $v$  is a stochastic viscosity solution of SPDE  $(\tilde{f}, 0)$ ; and  $u$  is a stochastic viscosity solution to SPDE  $(f, g)$ , respectively.*

## 1.1.2 Second Order BSDEs and Fully Nonlinear PDEs

Motivated by applications in financial mathematics and probabilistic numerical methods for PDEs, Cheridito, Soner, Touzi and Victoir [30] introduced the first formulation of second order BSDEs (in short 2BSDEs), which are connected to the larger class of fully nonlinear PDEs. Then, Soner, Touzi and Zhang [128] provided a different formulation of 2BSDEs based on the quasi sure analysis. Their key idea was to consider a family of BSDEs defined quasi-surely (q.s for short) under a non-dominated class of mutually singular probability measures, which means  $\mathbb{P}$ -a.s. for every probability measure  $\mathbb{P}$  in this class. This new point of view is inspired from the quasi-sure analysis of Denis and Martini [39] who established the connection between the so-called hedging problem in uncertain volatility models and the so-called Black-Scholes-Barremblatt PDE. The latter is fully nonlinear and has a simple piecewise linear dependance on the second order term.



These observations leads Soner, Touzi and Zhang [128] to consider the following formulation for solution to a 2BSDE, which consists of a triplet of processes  $(Y, Z, K)$  such that

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, 0 \leq t \leq T, \mathbb{P} - a.s. \text{ for all } \mathbb{P}. \quad (1.1.16)$$

The process  $K$  has a heuristic interpretation. Indeed, if there is a solution  $Y$  to the 2BSDE (1.1.16), it will be represented as a supremum of solutions  $y^\mathbb{P}$  to standard BSDEs. The role of the process  $K$  is then to push the process  $Y$  to ensure that this property holds for every probability measure  $\mathbb{P}$ .

Soner, Touzi and Zhang established a complete theory of existence and uniqueness for the solution to the equation (1.1.16), when the generator  $\hat{F}$  is uniformly Lipschitz in  $(Y, Z)$ , uniformly continuous with respect to the trajectory  $\omega$  and under conditions of integrability of  $\xi$  and  $\hat{F}$ . We note here that the uniform continuity condition is crucial in the proof of existence since the solution is constructed pathwise. Let us briefly recall the main ideas of their proof. First, the uniqueness is achieved through a representation of the solution  $Y$  as a supremum of solutions  $y^\mathbb{P}$  of classical BSDEs defined under different probability measures. This representation then provides a natural candidate for the solution. Nevertheless, since the probability measures are mutually singular, it is extremely difficult to tread over the problems related to negligible sets. Thus, Soner, Touzi and Zhang proposed a pathwise construction of the solution to a 2BSDE when the terminal condition is regular ( $\xi$  is bounded and uniformly continuous in  $\omega$ ), then they extended their results to the closure of this space.

The theory being very recent, the literature remains rather limited. However, we refer the interested reader to Possamai [118] and Possamai and Zhou [120] who respectively extended these wellposedness results to generators with linear and quadratic growth.

In the following, we will concentrate ourselves on this new formulation.

### 1.1.3 Second Order BDSDEs and Fully Nonlinear SPDEs

Our main motivation in this first part of the thesis is to provide a probabilistic interpretation for classical and stochastic viscosity solution of the following parabolic Fully nonlinear stochastic partial differential equation (in short FSPDE)

$$du_t(x) + \hat{h}(t, x, u_t(x), Du_t(x), D^2u_t(x)) dt + g(t, x, u_t(x), Du_t(x)) \circ \overleftarrow{dW}_t = 0 \quad (1.1.17)$$

over the time interval  $[0, T]$ , with a given final condition  $u_T = \Phi$  and  $g = (g_1, \dots, g_d)$  and  $h$  are nonlinear functions. While second order BSDEs are naturally connected to fully nonlinear PDEs, the solution of our SPDEs (1.2.15) will be represented via the solution of second order backward doubly SDEs (2BDSDEs in short). So, an extension of the theory of 2BSDEs, introduced by Soner, Touzi and Zhang [128], is needed in our context. We will briefly present our contributions in this framework (see e.g. [93]).

#### 1.1.3.1 Existence and uniqueness of 2BDSDEs

In our framework [93], we provide a complete theory of existence and uniqueness of Second order BDSDEs (2BDSDEs) under the Lipschitz-type hypotheses on the driver.

First of all, we shall work on the product space  $\Omega := \Omega^B \times \Omega^W$  where

- $\Omega^B := \{\omega \in C([0, T], \mathbb{R}^d), \omega_0 = 0\}$  equipped with the uniform norm  $\|\omega\|_\infty := \sup_{0 \leq t \leq T} |\omega_t|$ ,  $B$  the canonical process ( $B_t(\omega) = \omega_t$ ),  $\mathbb{P}_B^0$  the Wiener measure.

- $(W_t)_{0 \leq t \leq T}$  is a standard Brownian motion independent of  $B$  and defined on  $(\Omega^W, \mathcal{F}^W, \mathbb{P}_0^W)$ .

We then define  $\mathbb{F}^W = \{\mathcal{F}_{t,T}^W\}_{t \geq 0}$  a retrograde filtration generated by  $W$ , such that  $\mathcal{F}_{s,t}^W := \sigma\{W_r - W_s, s \leq r \leq t\}$  and let  $\mathcal{F}_T^W = \mathcal{F}_{0,T}^W$ . We also consider the following family of  $\sigma$ -fields  $\mathcal{F}_t^B := \sigma\{B_r, 0 \leq r \leq t\}$ . For each  $t \in [0, T]$ , we define

$$\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^W \quad \text{and} \quad \mathcal{G}_t := \mathcal{F}_t^B \vee \mathcal{F}_T^W.$$

Finally, we consider  $\mathcal{F} = \mathcal{F}^B \otimes \mathcal{F}^W$  let  $\mathcal{P}$  denotes the set of the non-dominated class of mutually singular probability measures  $\mathbb{P} := \mathbb{P}^\alpha \otimes \mathbb{P}_0^W$  such that

$$\mathbb{P}^\alpha := \mathbb{P}_B^0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0, T], \mathbb{P}_B^0 - a.s. \quad (1.1.18)$$

for some  $\mathbb{F}$ - progressively measurable process  $\alpha$  taking values in  $\mathbb{S}_d^{>0}$  with  $\int_0^T |\alpha_t| dt < \infty$   $\mathbb{P}_B^0 - a.s.$  Let  $H_t(w, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \rightarrow \mathbb{R}$  be  $\mathbb{F}$ - progressively measurable, where  $D_H \subset \mathbb{R}^{d \times d}$  is a given subset containing 0 and we define  $F$  as the corresponding conjugate of  $H$  w.r.t  $\gamma$  by

$$F_t(w, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} Tr(a\gamma) - H_t(w, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0},$$

where  $\mathbb{S}_d^{>0}$  denotes the space of all  $d \times d$  real valued positive definite matrices and

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t).$$

with  $\hat{a}_t := \overline{\lim}_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\langle B \rangle_t - \langle B \rangle_{t-\varepsilon})$ , where  $\langle B \rangle_t := B_t B_t^\top - 2 \int_0^t B_s dB_s^\top$  is defined pathwise and the  $\overline{\lim}$  is taken componentwise.

We consider also a function  $g_t(\omega, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^l$  which represents the intensity of the nonlinear external noise driven by  $(W_t)_{t \geq 0}$ .

We shall consider the following 2BDSDE .

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s) ds + \int_t^T g_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t, 0 \leq t \leq T, \mathcal{P} - q.s. \quad (1.1.19)$$

**Definition 1.1.3.** We say  $(Y, Z)$  is a solution to the 2BDSDE (1.1.19) if

- $Y_T = \xi$   $\mathcal{P} - q.s.$
- For each  $\mathbb{P} \in \mathcal{P}$ , the process  $K^\mathbb{P}$  defined below has nondecreasing paths,  $\mathbb{P} - a.s.$ :

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s) ds - \int_0^t g_s(Y_s, Z_s) d\overleftarrow{W}_s + \int_0^t Z_s dB_s, 0 \leq t \leq T, \mathbb{P} - a.s. \quad (1.1.20)$$

- The family  $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$  defined in (1.1.20) satisfies the following minimum condition:

$$K_t^\mathbb{P} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [K_T^{\mathbb{P}'}] \quad \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}, t \in [0, T] \quad (1.1.21)$$

where  $\mathcal{P}(t^+, \mathbb{P}) := \{\mathbb{P}' := \mathbb{P}'_B \otimes \mathbb{P}_0^W \in \mathcal{P}; \mathbb{P}'_B = \mathbb{P}_B \text{ on } \mathcal{F}_t^{B,+}\}$ .

Moreover, if the family  $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$  can be aggregated into a universal process  $K$ , we call  $(Y, Z, K)$  a solution of 2BDSDE (1.1.19)

The process  $K$  plays an important role, it forces  $Y$  to stay above every  $y^{\mathbb{P}}$ . To understand this formulation, we can consider the case where the set  $\mathcal{P}$  is reduced to a singleton  $\{\mathbb{P}\}$ . From the minimum condition (1.1.21), we know that  $K^{\mathbb{P}}$  is a non-decreasing martingale. Since  $\mathbb{P}$  satisfies the martingale representation property, this martingale is also continuous, and is therefore a constant. Thus, we have  $K^{\mathbb{P}} = 0$ ,  $\mathbb{P}$ -a.s. and the 2BDSDE (1.1.19) is equivalent to standard BDSDE (1.1.2). For 2BDSDEs, our main assumption on the functions  $F$  and  $g$  is as follows

- Assumption 1.1.1.** (i)  $\mathcal{P}$  is not empty, the domain  $D_{F_t(y,z)} = D_{F_t}$  is independent of  $(w, y, z)$ .  
(ii) For fixed  $(y, z, a)$ ,  $F$  is  $\mathcal{F}_t$  measurable in  $D_{F_t}$ , and  $g$  is  $\mathcal{F}_t$  measurable.  
(iii) We have the following uniform Lipschitz-type property in  $y$  and  $z$ : There exist constants  $C > 0$  and  $0 < \alpha < 1$  such that

$$\begin{aligned} \forall(y, y', z, z', t, w), \quad & |\hat{F}_t(w, y, z) - \hat{F}_t(w, y', z')| \leq C(|y - y'| + \|\hat{a}^{1/2}(z - z')\|) \\ \forall(y, y', z, z', t, w), \quad & \|g_t(w, y, z) - g_t(w, y', z')\|^2 \leq C|y - y'|^2 + \alpha\|(z - z')\|^2 \end{aligned}$$

- (iv) There exists a constant  $\lambda \in [0, 1[$  such that

$$(1 - \lambda)\hat{a} \geq \alpha I_d.$$

- (v)  $F$  is uniformly continuous in  $w$  for the  $\|\cdot\|_{\infty}$  norm.  
(vi)  $g$  is uniformly continuous in  $w$  for the  $\|\cdot\|_{\infty}$  norm.

We can now show as in [129] that the solution to 2BDSDE (1.1.19) can be represented as a supremum of solutions to BDSDE (1.1.2), which in turn implies the uniqueness of the solution.

**Theorem 1.1.3.** *Let assumption 1.1.1 holds. Assume  $\xi \in \mathbb{L}^2$  and that  $(Y, Z, K) \in \mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{I}^{\mathcal{P}}$  (see Chapter 2 for precise definition of the solution space) is a solution to 2BDSDE (1.1.19). Then, for any  $\mathbb{P} \in \mathcal{P}$  and  $0 \leq t_1 < t_2 \leq T$ ,*

$$Y_{t_1} = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})}^{\mathbb{P}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \quad \mathbb{P} - \text{a.s.} \quad (1.1.22)$$

Consequently, the 2BDSDE (1.1.2) has at most one solution in  $\mathbb{D}^2 \times \mathbb{H}^2$ .

As one can see from Theorem 1.1.3 above, if there is a solution to the 2BDSDE (1.1.19), it will be represented as a supremum of solutions to standard BDSDEs. However, since we are working under a family of non-dominated probability measures, we are not able to use the classical technics of BSDEs. So, we overcame this problem by constructing the solution pathwise using the so-called regular conditional probability distribution of Stroock and Varadhan [134]. This allows to construct a solution to the 2BDSDE when the terminal condition belongs to the space  $UC_b(\Omega)$ . Then, by passing to limit, we prove existence of solution when the terminal condition is in  $\mathcal{L}^2$ , the closure of  $UC_b(\Omega)$  under a certain norm defined in Chapter 2.

We are now in position to state the main result of this part.

**Theorem 1.1.4.** *Let  $\xi \in \mathcal{L}^2$  and assume that Assumption 1.1.1 holds. Then:*

- 1) *There exists a unique solution  $(Y, Z) \in \mathbb{D}^2 \times \mathbb{H}^2$  of the 2BDSDE (1.1.19).*
- 2) *Moreover, if in addition we choose to work under either of the following model of the theory*
  - (i) *Zermelo-Fraenkel set theory with axiom of choice (ZFC) plus the Continuum Hypothesis (CH).*
  - (ii) *ZFC plus the negation of CH plus Martin's axiom.*

Then there exists a unique solution  $(Y, Z, K) \in \mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$  of the 2BDSDE (1.1.19).

We point out that the 2BDSDE (1.1.19) is in fact a 2BSDE (1.1.16) perturbed by a nonlinear term driven by an extra Brownian motion. To prove the existence of the solution, we need to handle with the difficulties coming from this external noise. That's why we put some efforts to prove a control trajectorial for our candidate in order to show the dynamic programming principle. Let also mention that this proof requires us to extend the existing results on the theory of  $g$ -martingales of Peng [114] and the results of Karandikar [72] to our case (see Appendix of Chapter 2).

### 1.1.3.2 Probabilistic interpretation of solution of SPDEs

After establishing the result of uniqueness and existence of solution to 2BDSDEs, we are ready now to present the relation with a class of SPDEs. We consider the Markovian 2BDSDEs, whose solution can be represented by a deterministic function of  $t$  and  $B_t$ , and show the connection of these 2BDSDEs with fully nonlinear SPDEs.

Our main objective is to establish the connection  $Y_s^{t,x} = v(s, B_s^{t,x})$ ,  $s \in [t, T]$ ,  $\mathcal{P} - q.s.$ , where  $v$  solves, in some sense, the following fully nonlinear SPDE:

$$\begin{cases} du(t, x) + \hat{h}(t, x, u(t, x), Du(t, x), D^2u(t, x))dt + g(t, x, u(t, x), Du(t, x)) \circ d\overleftarrow{W}_t = 0, & 0 \leq t < T \\ u(T, x) = \phi(x). \end{cases} \quad (1.1.23)$$

We can write the SPDE (1.1.23) in the integral form

$$u(t, x) = \phi(x) + \int_t^T \hat{h}(t, x, u(t, x), Du(t, x), D^2u(t, x))dt + \int_t^T g(t, x, u(t, x), Du(t, x)) \circ d\overleftarrow{W}_t \quad (1.1.24)$$

where  $u$  is a mapping from  $[0, T] \times \Omega \times \mathbb{R}^d$  to  $\mathbb{R}$ .

We consider the case

$$H_t(\omega, y, z, \gamma) = h(t, B_t(\omega), y, z, \gamma),$$

where  $h : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times D_h \rightarrow \mathbb{R}$  is a deterministic map. Then the corresponding conjugate and bi-conjugate functions become

$$f(t, x, y, z, a) := \sup_{\gamma \in D_h} \left\{ \frac{1}{2} Tr(a\gamma) - h(t, x, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0}, \quad (1.1.25)$$

$$\hat{h}(t, x, y, z, \gamma) := \sup_{a \in \mathbb{S}_d^{>0}} \left\{ \frac{1}{2} Tr(a\gamma) - f(t, x, y, z, a) \right\} \text{ for } \gamma \in \mathbb{R}^{d \times d}. \quad (1.1.26)$$

Notice that  $-\infty < \hat{h} \leq h$  and  $\hat{h}$  is nondecreasing convex in  $\gamma$ . Also,  $\hat{h} = h$  if and only if  $h$  is convex and nondecreasing in  $\gamma$ .

The following is a slight strengthening of Assumption 1.1.1

#### Assumption 1.1.2.

- (i)  $\mathcal{P}$  is not empty, the domain  $D_{f_t(y,z)} = D_{f_t}$  is independent of  $(w, y, z)$ .
- (ii) There exist constants  $C > 0, 0 \leq \alpha < 1$  and modulus of continuity  $\rho$  with polynomial growth such that for all  $t \in [0, T], a \in D_{f_t}, x, x', z, z' \in \mathbb{R}^d, y, y' \in \mathbb{R}$ .

$$|f_t(x, y, z, a) - f_t(x, y', z', a)| \leq \rho(|x - x'|) + C(|y - y'| + \|a^{1/2}(z - z')\|)$$

$$\|g_t(x, y, z) - g_t(x, y', z')\|^2 \leq \rho(|x - x'|) + C|y - y'|^2 + \alpha\|(z - z')\|^2$$

- (iii) The function  $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^l)$

Next, let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a lebesgue measurable function. We denote, for any  $(t, x) \in [0, T] \times \mathbb{R}^d$

$$B_s^{t,x} := x + B_s^t \quad \text{for all } s \in [t, T].$$

and we consider the 2BDSDE (1.1.19) in this Markovian setting with terminal condition  $\xi = \phi(B_T^{t,x})$ :

$$\begin{aligned} Y_s^{t,x} &= \phi(B_T^{t,x}) - \int_s^T f(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \hat{a}_r) dr + \int_s^T g(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \circ d\overleftarrow{W}_r \\ &\quad - \int_s^T Z_r^{t,x} dB_r + K_T^{t,x} - K_s^{t,x}, \quad t \leq s \leq T, \quad \mathcal{P} - q.s \end{aligned} \quad (1.1.27)$$

We remark that the stochastic integral with respect to  $dW$  is the Stratonovich backward integral (see Kunita [78] page 194). Since  $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^l)$ , using the definition of the Stratonovich backward integral, we show easily that (1.1.27) is equivalent to the following 2BDSDE:

$$\begin{aligned} Y_s^{t,x} &= \phi(B_T^{t,x}) - \int_s^T \hat{f}(s, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \hat{a}_r) dr + \int_s^T g(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r \\ &\quad - \int_s^T Z_r^{t,x} dB_r + K_T^{t,x} - K_s^{t,x}, \quad t \leq s \leq T, \quad \mathcal{P} - q.s \end{aligned} \quad (1.1.28)$$

where  $\hat{f}(s, x, y, z, \hat{a}_s) = f(s, x, y, z, \hat{a}_s) + \frac{1}{2} \text{Tr}(g(s, x, y, z) D_y g(s, x, y, z))$ .

As we have explained above, our main motivation is to represent the classical and the stochastic viscosity solution for fully nonlinear SPDE (1.1.23) by means of solution of the Markovian 2BDSDE (1.1.27). We begin first by the classical solution for SPDEs.

### Classical solution for SPDEs

**Definition 1.1.4.** We define the classical solution of the SPDE (1.1.23) as a  $\mathbb{R}$ -valued random field  $\{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}^d\}$  such that  $u(t, x)$  is  $\mathcal{F}_{t,T}^W$ -measurable for each  $(t, x)$ , and whose trajectories belong to  $C^{0,2}([0, T] \times \mathbb{R}^d; \mathbb{R})$ .

**Theorem 1.1.5.** Let Assumption 1.1.2 hold true. Suppose further that  $\hat{h}$  is continuous in its domain,  $D_f$  is independent of  $t$  and is bounded both from above and away from 0. Let  $\{u(t, x); t \in [0, T] \times \mathbb{R}^d\}$  be a classical solution of (1.1.23) with  $\{(u, Du)(s, B_s^{t,x}), s \in [t, T]\} \in \mathbb{D}^2 \times \mathbb{H}^2$ . Then:

$$Y_s^{t,x} := u(s, B_s^{t,x}), \quad Z_s^{t,x} := Du(s, B_s^{t,x}), \quad K_s^{t,x} := \int_0^s k_r dr$$

$$\text{with } k_s := \hat{h}(s, B_s, Y_s, Z_s, \Gamma_s) - \frac{1}{2} \text{Tr}(\hat{a}_s \Gamma_s) + f(s, B_s, Y_s, Z_s, \hat{a}_s) \text{ and } \Gamma_s := D^2 u(s, B_s^{t,x})$$

is the unique solution of the 2BDSDE (1.1.27). Moreover,  $u(t, x) = Y_t^{t,x}$  for all  $t \in [0, T]$ .

### Stochastic viscosity solution for SPDE

Following the classical terminology in the BSDE literature, we say that the solution of the 2BDSDE is Markovian if it can be represented by means of a deterministic function of  $(t, B_t)$ . In this part, we construct a deterministic function  $u$ , by using a probabilistic representation in the spirit of (1.1.22), and show its connection with 2BDSDE (1.1.27). The connection between  $u$  and the SPDE (1.1.23) will be established in the next subsection.

We now introduce the random function  $u : [0, T] \times \Omega^W \times \mathbb{R}^d \rightarrow \mathbb{R}$  given by

$$u(t, x) := \sup_{\mathbb{P} \in \mathcal{P}^t} y_t^{\mathbb{P}, t, x}(T, \phi(B_T^{t,x})), \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^d. \quad (1.1.29)$$

By the Blumenthal zero-one law, it follows that  $u(t, x)$  is deterministic w.r.t  $B$  but still an  $\mathcal{F}_{t,T}^W$ -adapted process.

For the remainder of this part we will assume that the random field  $u$  is in  $C(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$ .

**Proposition 1.1.2.** *Let Assumption 1.1.2 holds true. A random field  $u$  is a stochastic viscosity sub- (resp. super-) solution to SPDE  $(\hat{h}, g)$  (1.1.23) if and only if  $v(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot, u(\cdot, \cdot))$  is a stochastic viscosity solution to SPDE  $(\tilde{h}, 0)$ , with*

$$\begin{aligned} \tilde{h}(t, x, y, z, \gamma) &= \frac{1}{D_y \eta(t, x, y)} \{ \hat{h}(t, x, \eta(t, x, y), D_x \eta(t, x, y) + D_y \eta(t, x, y)z, D_{xx} \eta(t, x, y) \\ &+ 2D_{xy} \eta(t, x, y)z + D_{yy} \eta(t, x, y)|z|^2 + D_y \eta(t, x, y)\gamma) \} \end{aligned} \quad (1.1.30)$$

We will now apply Doss transformation to the 2BDSDE (1.1.27) in order that the Stratonovich backward integral vanishes. Thus, the 2BDSDE will become a 2BSDE with a new generator  $\tilde{f}$ . Let us define the following three processes:

$$\begin{aligned} U_s^{t,x} &= \mathcal{E}(s, B_s^{t,x}, Y_s^{t,x}) \\ V_s^{t,x} &= D_y \mathcal{E}(s, B_s^{t,x}, Y_s^{t,x}) Z_s^{t,x} + D_x \mathcal{E}(s, B_s^{t,x}, Y_s^{t,x}) \\ \tilde{K}_s^{t,x} &= \int_0^s D_y \mathcal{E}(r, B_r^{t,x}, Y_r^{t,x}) d\tilde{K}_r^{t,x} \end{aligned} \quad (1.1.31)$$

It is easy to check the inverse transformation

$$\begin{aligned} Y_s^{t,x} &= \eta(s, B_s^{t,x}, U_s^{t,x}) \\ Z_s^{t,x} &= D_y \eta(s, B_s^{t,x}, U_s^{t,x}) V_t + D_x \eta(s, B_s^{t,x}, U_s^{t,x}) \\ K_s^{t,x} &= \int_0^t D_y \eta(s, B_s^{t,x}, U_s^{t,x}) d\tilde{K}_s^{t,x} \end{aligned} \quad (1.1.32)$$

**Theorem 1.1.6.** *Let Assumption 1.1.2 holds true. Then  $(U^{t,x}, V^{t,x}, \tilde{K}^{t,x})$  is the unique solution of the following 2BSDE,*

$$U_s^{t,x} = \phi(B_T^{t,x}) - \int_s^T \tilde{f}(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \hat{a}_r) dr - \int_s^T V_r^{t,x} dB_r + \tilde{K}_T^{t,x} - \tilde{K}_s^{t,x} \quad t \leq s \leq T \quad (1.1.33)$$

where  $\tilde{f}: [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times D_f \mapsto \mathbb{R}$  is defined by:

$$\begin{aligned} \tilde{f}(t, x, y, z, a) &\triangleq \frac{1}{D_y \eta(t, x, y)} \{ f(t, x, y, D_y \eta(t, x, y)z + D_x \eta(t, x, y)) - \frac{1}{2} \text{Tr}(a D_{xx} \eta(t, x, y)) \\ &- (a D_{xy} \eta(t, x, y)z) - \frac{1}{2} \text{Tr}(a D_{yy} \eta(t, x, y)|z|^2) \} \end{aligned} \quad (1.1.34)$$

We are now ready to prove that  $u$  defined by (1.1.29) is the stochastic viscosity solution of our SPDE  $(\hat{h}, g)$  (1.1.23). First, we introduce the random field  $v(t, x) = U_t^{t,x}$ , where  $U$  is the solution of 2BSDE (1.1.33). Then by (1.1.31) and (1.1.32) we know that, for  $(t, x) \in [0, T] \times \mathbb{R}^d$

$$u(t, x) = \eta(t, x, v(t, x)); \quad v(t, x) = \mathcal{E}(t, x, u(t, x)). \quad (1.1.35)$$

Finally, thanks to Proposition 1.1.2, we need only to prove that the random field  $v$  defined in (1.1.35) is a stochastic viscosity solution to the SPDE  $(\tilde{h}, 0)$ .

**Theorem 1.1.7.** *Under our Assumptions, the random field  $v$  is a stochastic viscosity solution of SPDE  $(\tilde{h}, 0)$ ; and  $u$  is a stochastic viscosity solution to SPDE  $(\hat{h}, g)$ , respectively.*

## 1.2 Reflection problem for PIDEs and SPDEs

### 1.2.1 Obstacle problem for PIDEs

We present in this part a probabilistic interpretation for the weak Sobolev solution of the obstacle problem for semilinear parabolic partial integro-differential equations (in short PIDEs), namely Feynman-Kac's formula for the solution of the PIDEs, via the associated Reflected Backward Stochastic Differential Equations with jumps (in short RBSDEs). This representation gives an application to the pricing and hedging of contingent claims with constraints in the wealth or portfolio processes in financial markets including jumps. We start first by giving the literature related to these equations.

### 1.2.1.1 Motivations and formulation

In the classical literature, the obstacle problem for semilinear PDEs and their connection with optimal stopping and control problems have been first studied by Mignot-Puel [103] (see also Bensoussan-Lions [15]), and then by Michel Pierre [116, 117]. They model such problems by means of variational inequalities.

Our main interest is to study the following partial integro-differential equations (in short PIDEs) of parabolic type:

$$(\partial_t + \mathcal{L})u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(x), u(t, x + \beta(x, \cdot)) - u(t, x)) = 0 \quad (1.2.1)$$

over the time interval  $[0, T]$ , with a given final condition  $u_T = g$ ,  $f$  is a nonlinear function and  $\mathcal{L} = \mathcal{K}_1 + \mathcal{K}_2$  is the second order integro-differential operator associated with a jump diffusion which is defined component by component with

$$\begin{aligned} \mathcal{K}_1\varphi(x) &= \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) \text{ and} \\ \mathcal{K}_2\varphi(x) &= \int_{\mathbb{E}} \left( \varphi(x + \beta(x, e)) - \varphi(x) - \sum_{i=1}^d \beta^i(x, e) \frac{\partial}{\partial x_i} \varphi(x) \right) \lambda(de), \quad \varphi \in C^2(\mathbb{R}^d). \end{aligned} \quad (1.2.2)$$

This class of PIDEs appears in the pricing and hedging contingent claims in financial markets including jumps (see [32] and [14]). Matache, von Petersdorff and Schwab [92] have studied a particular case where  $f$  is linear in  $(y, z)$  and not depends on  $v$  (the jump size variable). They have shown the existence and uniqueness of the Sobolev solution of the variational form of some types of PIDEs, stemming from pricing problems in Lévy markets. They used an analytic method in order to derive a numerical schema based on the wavelet Galerkin method. Our nonlinear PIDEs (1.2.1) include the case of pricing of contingent claims with constraints in the wealth or portfolio processes. As an example, hedging claims with higher interest rate for borrowing may be considered in a financial market with jumps. El Karoui, Peng and Quenez [49] have studied this example in a continuous financial market where the non linear source function  $f$  is given by  $f(t, x, y, \tilde{z}) = r_t y + \theta_t \sigma_t \tilde{z} - (R_t - r_t)(y - \sum_{i=1}^n \tilde{z}^i)$ . More recently, the pricing and hedging of American options in the Markovian case and the related obstacle problem for PDEs, was studied by El Karoui et al [46] and Bally et al [9].

If  $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a given function such that  $h(T, x) \leq g(x)$ , we may roughly say that the solution of the obstacle problem for (1.2.1) is a function  $u \in \mathbf{L}^2([0, T]; H^1(\mathbb{R}^d))$  such that the following conditions are satisfied in  $(0, T) \times \mathbb{R}^d$  :

$$\begin{aligned} (i) \quad & u \geq h, \quad dt \otimes dx - \text{a.e.}, \\ (ii) \quad & \partial_t u + \mathcal{L}u + f \leq 0 \\ (iii) \quad & (u - h)(\partial_t u + \mathcal{L}u + f) = 0. \\ (iv) \quad & u_T = g, \quad dx - \text{a.e.} \end{aligned} \quad (1.2.3)$$

The relation (ii) means that the distribution appearing in the LHS of the inequality is a non-positive measure. The relation (iii) is not rigorously stated. We may roughly say that one has  $\partial_t u + \mathcal{L}u + f = 0$  on the set  $\{u > h\}$ .

In the case of the obstacle problem for PDEs (when the non local term operator  $\mathcal{K}_2 = 0$ ), if one expresses the obstacle problem in terms of variational inequalities it should also be required that the solution has a minimality property (see Mignot-Puel [103] or Bensoussan-Lions [15] p.250). The work of El Karoui et al [46] treats the obstacle problem for (1.2.1) within the framework of backward stochastic differential equations (BSDEs in short). Namely the equation (1.2.1) is considered with

$f$  depending on  $u$  and  $\nabla u$ ,  $\lambda = 0$  and  $\beta = 0$  and the obstacle  $h$  is continuous. The solution is represented stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set  $\{u = h\}$ . This increasing process determines in fact the measure from the relation (ii). Bally et al [9] (see also Matoussi and Xu [99]) point out that the continuity of this process allows the classical notion of strong variational solution to be extended (see Theorem 2.2 of [15] p.238) and express the solution to the obstacle as a pair  $(u, \nu)$  where  $\nu$  equals the LHS of (ii) and is supported by the set  $\{u = h\}$ .

Barles, Buckdahn and Pardoux [12] (see also [136], [124] and [111]) have provided a probabilistic interpretation for the viscosity solution of (1.2.1) by using a forward BSDE with jumps. Situ [126] has studied the Sobolev solution of (1.2.1) via an appropriate BSDE with jumps, whose method is mainly based on Sobolev's embedding theorem. More recently, Matoussi and Stoica [98] studied the obstacle problem for parabolic quasilinear SPDEs and gave a probabilistic interpretation of the reflected measure  $\nu$  in terms of the associated increasing process which is a component of solution of reflected BSDEs. Such measures are called regular measures or Revuz measures. Their method is based on probabilistic quasi-sure analysis. On the other hand, Michel Pierre [116, 117] has studied parabolic PDEs with obstacles using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous with respect to so-called analytical capacity. Moreover he gave a representation of the reflected measure  $\nu$  in terms of the associated regular potential and the approach used is based on analytical quasi-sure analysis. Finally, Denis, Matoussi and Zhang [38] have extended the approach of Michel Pierre [116, 117] for the obstacle problem of quasilinear SPDEs.

### 1.2.1.2 New results

The main objective of this part is to prove existence and uniqueness results of the solution of the obstacle problem for the PIDEs and to give the associated probabilistic interpretation via the RB-SDEs with jumps. In the following, we present the main results obtained in the already published paper [96].

We consider the final condition to be a fixed function  $g \in \mathbf{L}_\rho^2(\mathbb{R}^d)$  (where  $\mathbf{L}_\rho^2(\mathbb{R}^d)$  is a weighted Hilbert space which is separable thanks to the assumptions verified by the weight function  $\rho$ ) and the obstacle  $h$  be a continuous function  $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ . Then the obstacle problem for the equation (1.2.1) is defined as a pair  $(u, \nu)$ , where  $\nu$  is a regular measure concentrated on  $\{u = h\}$  and  $u \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d; \mathbb{R})$  satisfies the following relations:

$$\begin{aligned}
 (i') \quad & u \geq h, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.}, \\
 (ii') \quad & \partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(x), u(t, x + \beta(x, \cdot)) - u(t, x)) = -\nu(dt, dx), \\
 (iii') \quad & \nu(u > h) = 0, \quad \text{a.s.}, \\
 (iv') \quad & u_T = g, \quad dx - \text{a.e.}
 \end{aligned}
 \tag{1.2.4}$$

$\nu$  represents the quantity which makes it possible to pass from inequality (ii) to equality (ii') and to get the uniqueness result for the obstacle problem. The rigorous sense of the relation (iii') which is based on the probabilistic representation of the measure  $\nu$  and plays the role of quasi-continuity of  $u$  in this context will be explained in Chapter 3.

The concern is to solve our problem (1.2.4) by developing in the jump setting a stochastic flow method which was first introduced by Bally and Matoussi [10] for parabolic SPDEs. The key element in [10] is to use the inversion of stochastic flow which transforms the variational formulation of the PDEs to the associated BSDEs. Thus it plays the same role as Itô's formula in the case of the classical solution of PDEs. More precisely, our approach will be based on the results of Léandre [82]



about the homeomorphic property for the solution of SDEs with jumps. We need to provide useful results on stochastic flow associated with the following forward SDEs with jumps

$$X_{t,s}(x) = x + \int_t^s b(X_{t,r}(x))dr + \int_t^s \sigma(X_{t,r}(x))dW_r + \int_t^s \int_E \beta(X_{t,r-}(x), e)\tilde{\mu}(dr, de). \quad (1.2.5)$$

where  $E = \mathbb{R}^l \setminus \{0\}$  and the coefficients of the diffusion satisfy the following assumption

**Assumption 1.2.1.**  $b \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^d)$ ,  $\sigma \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^{d \times d})$ ,  $\beta : \mathbb{R}^d \times E \rightarrow \mathbb{R}^d$  be measurable and for all  $e \in E$ ,  $\beta(\cdot, e) \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^d)$ , and for some  $K > 0$  and for all  $x \in \mathbb{R}^d$ ,  $e \in E$ ,

$$|\beta(x, e)| \leq K(1 \wedge |e|), \quad |D^\alpha \beta(x, e)| \leq K(1 \wedge |e|) \text{ for } 1 \leq |\alpha| \leq 3,$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$  is a multi-index and  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$ .  $D^\alpha$  is the differential operator  $D^\alpha = \frac{\partial^{|\alpha|}}{(\partial^{\alpha_1} x_1)(\partial^{\alpha_2} x_2) \dots (\partial^{\alpha_d} x_d)}$ .

The existence and uniqueness of this solution was proved in Fujiwara and Kunita [52]. It is well known that the stochastic flow associated to a continuous SDE with regular coefficients satisfies the homeomorphic property (see Bismut [16], Kunita [76, 80]). But this property fails for the solution of SDE with jumps in general. Léandre [82] gave a necessary and sufficient condition under which the homeomorphic property is preserved at the jump time, namely, for each  $e \in E$ , the maps  $H_e : x \mapsto x + \beta(x, e)$  should be one to one and onto. One can read also Fujiwara and Kunita [52] and Kunita [77] for more details on the subject. Therefore, we assume additionally that

**Assumption 1.2.2.** For each  $e \in E$ , the linkage operator:

$$H_e : x \mapsto x + \beta(x, e) \text{ is a } C^2\text{-diffeomorphism.}$$

We denote by  $H_e^{-1}$  the inverse map of  $H_e$ , and set  $h(x, e) := x - H_e^{-1}(x)$ . Under Assumptions 1.2.1 and 1.2.2, we have that  $\{X_{t,s}(x); x \in \mathbb{R}^d\}$  is a  $C^2$ -diffeomorphism a.s. stochastic flow. Moreover the inverse of the flow satisfies the following backward SDE (see Kunita [76])

$$\begin{aligned} X_{t,s}^{-1}(y) = & y - \int_t^s \widehat{b}(X_{r,s}^{-1}(y))dr - \int_t^s \sigma(X_{r,s}^{-1}(y))\overleftarrow{dW}_r - \int_t^s \int_E \beta(X_{r,s}^{-1}(y), e)\overleftarrow{\mu}(dr, de) \\ & + \int_t^s \int_E \widehat{\beta}(X_{r,s}^{-1}(y), e)\mu(\overleftarrow{dr}, de). \end{aligned} \quad (1.2.6)$$

for any  $t < s$ , where

$$\widehat{b}(x) = b(x) - \sum_{i,j} \frac{\partial \sigma^j(x)}{\partial x_i} \sigma^{ij}(x) \quad \text{and} \quad \widehat{\beta}(x, e) = \beta(x, e) - h(x, e). \quad (1.2.7)$$

**Remark 1.2.1.** Note that stochastic flow approach was used recently in other context. In [71] based on stochastic flow arguments, the author shows that the probabilistic equivalent formulation of Dupire's PDE is the Put-Call duality equality in local volatility models including exponential Lévy jumps. Also in [104], [48] and [47], the inversion of stochastic flow technics are used for building a family of forward utilities for a given optimal portfolio.

We also need equivalence of norms result which plays an important role in the proof of the existence of the solution for PIDE as a connection between the functional norms and "random process norms". For continuous SDEs, this result was first proved by Barles and Lesigne [13] by using an analytic method. Bally and Matoussi have proved in [10] the result with a probabilistic method. Note that Klimisiak [73] have extended this estimates for Markov process associated to a non-homogeneous divergence operator. The following result generalize Proposition 5.1 in [10] (see also [13]) in the case of a diffusion process with jumps, and the proof will be given in Appendix 3.5.1.

**Proposition 1.2.1.** *There exists two constants  $c > 0$  and  $C > 0$  such that for every  $t \leq s \leq T$  and  $\varphi \in L^1(\mathbb{R}^d, dx)$ ,*

$$c \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx \leq \int_{\mathbb{R}^d} \mathbb{E}(|\varphi(X_{t,s}(x))|) \rho(x) dx \leq C \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx. \quad (1.2.8)$$

Moreover, for every  $\Psi \in L^1([0, T] \times \mathbb{R}^d, dt \otimes dx)$ ,

$$c \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)| ds \rho(x) dx \leq \int_{\mathbb{R}^d} \int_t^T \mathbb{E}(|\Psi(s, X_{t,s}(x))|) ds \rho(x) dx \leq C \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)| ds \rho(x) dx. \quad (1.2.9)$$

As we have mentioned above our main interest in the work [96] is to establish the probabilistic representation of the Sobolev's solution of the obstacle problem for our PIDEs (1.2.4) via the following reflected BSDE with jumps (RBSDE with jumps for short) :

$$\begin{cases} Y_s^{t,x} = g(X_{t,T}(x)) + \int_s^T f(r, X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}) dr + K_T^{t,x} - K_s^{t,x} \\ \quad - \int_s^T Z_r^{t,x} dB_r - \int_s^T \int_E V_r^{t,x}(e) \tilde{\mu}(dr, de), \mathbb{P}\text{-a.s.}, \forall s \in [t, T] \\ Y_s^{t,x} \geq L_s^{t,x}, \quad \int_t^T (Y_s^{t,x} - L_s^{t,x}) dK_s^{t,x} = 0, \mathbb{P}\text{-a.s.} \end{cases} \quad (1.2.10)$$

The obstacle process  $L_s^{t,x} = h(s, X_{t,s}(x))$  is a càdlàg process which has only inaccessible jumps since  $h$  verifies the following hypothesis:

**Assumption 1.2.3.**  *$h \in C([0, T] \times \mathbb{R}^d; \mathbb{R})$  and there exist  $\iota, \kappa > 0$  such that  $|h(t, x)| \leq \iota(1 + |x|^\kappa)$ , for all  $x \in \mathbb{R}^d$ .*

According to Hamadène and Ouknine [65], there exists a unique quadruple  $(Y^{t,x}, Z^{t,x}, V^{t,x}, K^{t,x}) \in \mathcal{S}^2(t, T) \times \mathcal{H}_d^2(t, T) \times \mathcal{L}^2(t, T) \times \mathcal{A}^2(t, T)$  (see Chapter 3 for precise definition of the solution space) solution of the RBSDE with jumps (1.2.10).

We denote the space of solutions by

$$\mathcal{H}_T := \{u \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d) \mid \nabla u \sigma \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T} = \left( \int_{\mathbb{R}^d} \int_0^T [|u(s, x)|^2 + |\nabla u(s, x) \sigma(x)|^2] ds \rho(x) dx \right)^{1/2},$$

where we denote the gradient by  $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$ .

More precisely, we consider the following definition of weak solutions for the obstacle problem (1.2.4):

**Definition 1.2.1.** *We say that  $(u, \nu)$  is the weak solution of the PIDE with obstacle associated to  $(g, f, h)$ , if*

(i)  $\|u\|_{\mathcal{H}_T}^2 < \infty$ ,  $u \geq h$ , and  $u(T, x) = g(x)$ ,

(ii)  $\nu$  is a positive Radon regular measure in the following sense, i.e. for every measurable bounded and positive functions  $\phi$  and  $\psi$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) 1_{\{u=h\}}(s, x) \nu(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \phi(s, x) \psi(s, X_{t,s}(x)) dK_s^{t,x} dx, \text{ a.s.} \end{aligned} \quad (1.2.11)$$

where  $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$  is the solution of RBSDE with jumps (1.2.10) and such that

$$\int_0^T \int_{\mathbb{R}^d} \rho(x) \nu(dt, dx) < \infty,$$

(iii) for every test function  $\phi \in \mathcal{D}_T = \mathcal{C}^\infty([0, T]) \otimes \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u(s, x) \partial_s \phi(s, x) dx ds + \int_{\mathbb{R}^d} (u(t, x) \phi(t, x) - g(x) \phi(T, x)) dx + \int_t^T \int_{\mathbb{R}^d} u(s, x) \mathcal{L}^* \phi(s, x) dx ds \\ &= \int_t^T \int_{\mathbb{R}^d} f(s, x, u, \nabla u \sigma, u(s, x + \beta(x, \cdot)) - u(s, x)) \phi(s, x) dx ds + \int_t^T \int_{\mathbb{R}^d} \phi(s, x) 1_{\{u=h\}}(s, x) \nu(ds, dx), \end{aligned} \quad (1.2.12)$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ .

We give now the following result which allows us to link by a natural way the Sobolev solution to the obstacle problem with the associated reflected BSDE with jumps. Roughly speaking, if we choose in the variational formulation (1.2.12) the random functions  $\phi_t(\cdot, \cdot)$  defined by  $\phi_t(s, x) := \phi(X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x))$ , as a test functions, then we obtain the associated BSDE. In fact, this result plays the same role as Itô's formula used in [108] and [115] (see [108], Theorem 3.1, p. 20) to relate the solution of some semilinear PDEs with the associated BSDE.

**Proposition 1.2.2.** *Let  $\rho(x) = (1 + |x|)^{-p}$  with  $p \geq \gamma$  where  $\gamma = \kappa + d + 1$  and  $u \in \mathcal{H}_T$  be a weak solution of PIDE with obstacle associated to  $(g, f, h)$ . Then, under lipschitz condition on  $f$  and Assumptions 1.2.1, 1.2.2 and 1.2.3, we have for  $s \in [t, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r, x) d\phi_t(r, x) dx + (u(s, \cdot), \phi_t(s, \cdot)) - (g(\cdot), \phi_t(T, \cdot)) - \int_s^T (u(r, \cdot), \mathcal{L}^* \phi_t(r, \cdot)) dr \\ &= \int_{\mathbb{R}^d} \int_s^T f(r, x, u(r, x), \sigma^* \nabla u(r, x), u(r, x + \beta(x, \cdot)) - u(r, x)) \phi_t(r, x) dr dx \\ &+ \int_{\mathbb{R}^d} \int_s^T \phi_t(r, x) 1_{\{u=h\}}(r, x) \nu(dr, dx). \quad a.s. \end{aligned} \quad (1.2.13)$$

where  $\int_{\mathbb{R}^d} \int_s^T u(r, x) d\phi_t(r, x) dx$  is well defined thanks to the semimartingale decomposition result (Lemma 3.2.1).

The main result of this part is the following theorem, where we prove the existence and uniqueness results of the solution of our PIDEs (1.2.4) and we give the associated probabilistic interpretation via the reflected BSDEs with jumps.

**Theorem 1.2.1.** *Let  $\rho(x) = (1 + |x|)^{-p}$  with  $p \geq \gamma$  where  $\gamma = \kappa + d + 1$ . Then, under lipschitz condition on  $f$  and Assumptions 1.2.1, 1.2.2 and 1.2.3, there exists a weak solution  $(u, \nu)$  of the PIDE with obstacle (1.2.4) associated to  $(g, f, h)$  such that,  $ds \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.$ ,*

$$\begin{aligned} Y_s^{t,x} &= u(s, X_{t,s}(x)), Z_s^{t,x} = (\nabla u \sigma)(s, X_{t,s}(x)), \\ V_s^{t,x}(\cdot) &= u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x)) \end{aligned} \quad (1.2.14)$$

Moreover, the reflected measure  $\nu$  is a regular measure in the sense of the definition (ii) and satisfying the probabilistic interpretation (1.2.11).

If  $(\bar{u}, \bar{\nu})$  is another solution of the PIDE with obstacle (1.2.4) such that  $\bar{\nu}$  satisfies (1.2.11) with some  $\bar{K}$  instead of  $K$ , where  $\bar{K}$  is a continuous process in  $\mathcal{A}^2(t, T)$ , then  $\bar{u} = u$  and  $\bar{\nu} = \nu$ .

In other words, there is a unique Randon regular measure with support  $\{u = h\}$  which satisfies (1.2.11).

We show the existence and uniqueness of the solution  $(u, \nu)$  of the obstacle problem for (1.2.1) using a probabilistic method based on reflected BSDEs with jumps. The proof of this result differs from that of Bally et al [9] since we have to consider the stochastic flow associated with the forward jump diffusion process. In particular, the jump part appearing of the tightness result for the approximation measure has to be taken into account. The existence of the solution is established by an approximation penalization procedure, a priori estimates and the equivalence norm results. The uniqueness is a consequence of the variational formulation of the PIDEs written with random test functions and the uniqueness of the solution of the FBSDE.

## 1.2.2 BDSDEs and semilinear SPDEs in a convex domain

### 1.2.2.1 Motivations and literature

Our main interest is the following system of semilinear stochastic PDE with value in  $\mathbb{R}^k$ ,

$$du_t(x) + [\mathcal{L}u_t(x) + f_t(x, u_t(x), \nabla u_t \sigma(x))]dt + h_t(x, u_t(x), \nabla u_t \sigma(x)) \cdot d\overleftarrow{W}_t = 0, \quad (1.2.15)$$

over the time interval  $[0, T]$ , with a given final condition  $u_T = \Phi$ ,  $f, h$  are non-linear random functions and  $\mathcal{L}$  is the second order differential operator associated with a diffusion which is defined component by component with

$$\mathcal{L}\varphi(x) = \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x). \quad (1.2.16)$$

The differential term with  $d\overleftarrow{W}_t$  refers to the backward stochastic integral with respect to a  $d$ -dimensional Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P}, (W_t)_{t \geq 0})$ . We use the backward notation because in the proof we will employ the doubly stochastic framework introduced by Pardoux and Peng [109]. Such SPDEs appear in various applications like pathwise stochastic control problems, Zakai equations in filtering and stochastic control with partial observations. It is well known now that BSDEs provide a natural tools to give a probabilistic interpretation for the solution of a class of semi-linear PDEs. By introducing in standard BSDEs a second nonlinear term driven by an external noise, we obtain Backward Doubly SDEs (BDSDEs) [109] (see also [10], [97]), which can be seen as Feynman-Kac representation of SPDEs and form a powerful tool for probabilistic numerical schemes [6] for such SPDEs. Several generalizations to investigate more general nonlinear SPDEs have been developed following different approaches of the notion of weak solutions, namely, Sobolev's solutions [43, 61, 75, 125, 137], and stochastic viscosity solutions [88, 89, 90, 24, 25]. Given a convex domain  $D$  in  $\mathbb{R}^k$ , this part is concerned to study the reflection problem for SPDE (1.2.16) in such domain  $D$ . This problem is well known as a Skorohod problem for SPDEs.

In the case of diffusion processes, the reflection problem has been investigated in the literature by several authors. The case of a half-space domain was studied by Skorokod [127], McKean [100], Watanabe [139], El Karoui, Chaleyat-Maurel and Marchal [28]. In the case of a convex domain this reflection problem was treated by Tanaka [135] and Menaldi [101] by using the variational inequality and the convexity properties of the domain. Finally, Lions and Sznitman [91] have solved SDEs with reflecting boundary conditions in general domains by a direct pathwise approach based on the Skorohod problem.

### 1.2.2.2 New results

In this part of the thesis we study existence and uniqueness results for reflected backward doubly stochastic differential equations (in short RBDSDEs) in a convex domain  $D$  without any regularity conditions on the boundary. Moreover, using a stochastic flow approach a probabilistic interpretation

for the weak solution of a class of reflected SPDEs in a domain is given via such RBDSDEs. Inspired by the variational formulation of the obstacle problem for SPDEs and Menaldi's work [101] on reflected diffusion in a domain, we consider the solution of the reflection problem for the SPDEs (1.2.15) as a pair  $(u, \nu)$ , where  $\nu$  is a random regular measure and  $u \in \mathbf{L}^2(\Omega \times [0, T]; H_\rho^1(\mathbb{R}^d))$  (where  $H_\rho^1$  is the weighted Dirichlet space and  $\rho$  a weight function) satisfies the following relations:

$$\begin{aligned}
(i) \quad & u_t(x) \in \bar{D}, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.}, \\
(ii') \quad & du_t(x) + [\mathcal{L}u_t(x) + f_t(x, u_t(x), \nabla u_t \sigma(x))] dt + h_t(x, u_t(x), \nabla u_t \sigma(x)) \cdot d\overleftarrow{W}_t = -\nu(dt, dx), \quad \text{a.s.}, \\
(iii) \quad & \nu(u \notin \partial D) = 0, \text{ a.s.}, \\
(iv) \quad & u_T = \Phi, \quad dx - \text{a.e.}
\end{aligned} \tag{1.2.17}$$

$\nu$  is a random measure which acts only when the process  $u$  reaches the boundary of the domain  $D$ . The rigorous sense of the relation (iii) will be based on the probabilistic representation of the measure  $\nu$  in term of the bounded variation processes  $K$  component of the associated solution of the reflected BDSDE in the domain  $D$ .

Our first contribution in this work (see [94]) is to define the solution of Reflected BDSDE in a given convex domain  $D$  and to prove the existence and uniqueness of solution for such equations in order to relate it to the the solution of the reflection problem for our SPDEs (1.2.15).

Let us start with the following definition:

**Definition 1.2.2.** *The triplet of processes  $(Y_t, Z_t, K_t)_{\{0 \leq t \leq T\}}$  is solution of the backward doubly stochastic differential equation in a convex domain  $D$ , with terminal condition  $\xi$  and coefficients  $f$  and  $h$ , if the following hold:*

(i)  $Y \in \mathcal{S}_k^2([0, T])$ ,  $Z \in \mathcal{H}_{k \times d}^2([0, T])$  and  $K \in \mathcal{A}_k^2([0, T])$  (see Chapter 4 for precise definition of the solution space),

(ii)

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds + \int_t^T h_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \text{ a.s.} \tag{1.2.18}$$

(iii)  $Y_t \in \bar{D}$ ,  $0 \leq t \leq T$ , a.s.

(iv) for any continuous progressively measurable process  $(z_t)_{0 \leq t \leq T}$  valued in  $\bar{D}$ ,

$$\int_0^T (Y_t - z_t)^* dK_t \leq 0, \quad \text{a.s.} \tag{1.2.19}$$

The triplet  $(Y_t, Z_t, K_t)_{\{0 \leq t \leq T\}} \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T]) \times \mathcal{A}_k^2([0, T])$  is called a solution of RBDSDE with data  $(\xi, f, h)$ .

**Remark 1.2.2.** *From Lemma 2.1 in [53], the condition (1.2.19) implies that the bounded variation process  $K$  acts only when  $Y$  reaches the boundary of the convex domain  $D$  and the so-called Skorohod condition is satisfied:*

$$\int_0^T \mathbf{1}_{\{Y_t \in D\}} dK_t = 0. \tag{1.2.20}$$

Moreover there exists a  $\mathcal{F}_t$ -measurable process  $(\alpha_t)_{0 \leq t \leq T}$  valued in  $\mathbb{R}^k$  such that

$$K_t = \int_0^t \alpha_s d\|K_s\|_{VT} \quad \text{and} \quad -\alpha_s \in \nu(Y_s).$$

Next, results of existence and uniqueness of such RBDSDEs are established.

**Theorem 1.2.2.** *Let  $\xi \in \mathbf{L}_k^2(\mathcal{F}_T)$ ,  $\xi \in \bar{D}$  a.s.. Then, under Lipschitz continuous condition, the RBDSDE (1.2.18) has a unique solution  $(Y, Z, K) \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T]) \times \mathcal{A}_k^2([0, T])$ .*

We note that existence and uniqueness of the solution of RBDSDE is given under only convexity assumption for the domain without any regularity on the boundary. This result is proved by using penalization approximation. Thanks to the convexity properties we prove several technical lemmas, in particular the fundamental Lemma 4.2.2 of Chapter 4.

After establishing the existence and uniqueness results for the reflected BDSDE (1.2.18), we are able now to give a Feynman-Kac's formula for the weak solution of a semilinear reflected SPDEs (1.2.17) in a given convex domain  $D$  via Markovian class of RBDSDEs studied. As explained above, the solution of such SPDE is expressed as a pair  $(u, \nu)$  where  $u$  is a predictable continuous process which takes values in a Sobolev space and  $\nu$  is a random vector regular signed measure. The bounded variation processes  $K$  component of the solution of the reflected BDSDE controls the set when  $u$  reaches the boundary of  $D$ . In fact, this bounded variation process determines the measure  $\nu$  from a particular relation by using the inverse of the flow associated to the diffusion operator. In order to provide the probabilistic representation to the solution of the RSPDEs (1.2.17), we introduce the following Markovian RBDSDE:

$$\left\{ \begin{array}{l} (i) \ Y_s^{t,x} = \Phi(X_{t,T}(x)) + \int_s^T f_r(X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T h_r(X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r + K_T^{t,x} - K_s^{t,x} \\ \quad - \int_s^T Z_r^{t,x} dB_r, \ \mathbb{P}\text{-a.s.}, \ \forall s \in [t, T] \\ (ii) \ Y_s^{t,x} \in \bar{D} \quad \mathbb{P}\text{-a.s.} \\ (iii) \ \int_0^T (Y_s^{t,x} - v_s(X_{t,s}(x)))^* dK_s^{t,x} \leq 0, \ \mathbb{P}\text{-a.s.}, \\ \text{for any continuous } \mathcal{F}_t\text{-random function } v : [0, T] \times \Omega \times \mathbb{R}^d \longrightarrow \bar{D}. \end{array} \right. \quad (1.2.21)$$

where  $\{X_{t,s}(x), t \leq s \leq T\}$  is the diffusion process starting from  $x$  at time  $t$  and is the strong solution of the equation:

$$X_{t,s}(x) = x + \int_t^s b(X_{t,r}(x)) dr + \int_t^s \sigma(X_{t,r}(x)) dB_r. \quad (1.2.22)$$

Therefore under our assumptions and according to Theorem 1.2.2, there exists a unique triplet  $(Y^{t,x}, Z^{t,x}, K^{t,x})$  solution of the RBDSDE (1.2.21) associated to  $(\Phi, f, h)$ .

We denote by  $\mathcal{H}_T$  the space of  $\mathcal{F}_{t,T}^W$ -progressively measurable processes  $(u_t)$  with valued in the weighted Dirichlet space  $H_\rho^1(\mathbb{R}^d)$  where

$$H_\rho^1(\mathbb{R}^d) := \{v \in \mathbf{L}_\rho^2(\mathbb{R}^d) \mid \nabla v \sigma \in \mathbf{L}_\rho^2(\mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T}^2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |u_s(x)|^2 \rho(x) dx + \int_{\mathbb{R}^d} \int_0^T |\nabla u_s(x) \sigma(x)|^2 ds \rho(x) dx \right],$$

where we denote the gradient by  $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$ .

We now consider the following definition of weak solutions for the reflected SPDE (1.2.17):

**Definition 1.2.3.** *We say that  $(u, \nu) := (u^i, \nu^i)_{1 \leq i \leq k}$  is the weak solution of the reflected SPDE (1.2.17) associated to  $(\Phi, f, h)$ , if for each  $1 \leq i \leq k$*

(i)  $\|u\|_{\mathcal{H}_T} < \infty$ ,  $u_t(x) \in \bar{D}$ ,  $dx \otimes dt \otimes d\mathbb{P}$  a.e., and  $u(T, x) = \Phi(x)$ ,

(ii)  $\nu^i$  is a signed Radon regular measure in the following sense, i.e. for every measurable bounded and positive functions  $\varphi$  and  $\psi$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T \varphi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, X_{t,s}(x)) dK_s^{t,x,i} dx, \text{ a.s.} \end{aligned} \quad (1.2.23)$$

where  $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$  is the solution of RBDSDE (1.2.21) and such that  $\int_0^T \int_{\mathbb{R}^d} \rho(x) |\nu|(dt, dx) < \infty$ ,  
 (iii) for every test function  $\varphi \in \mathcal{D}_T = \mathcal{C}^\infty([0, T]) \otimes \mathcal{C}^\infty(\mathbb{R}^d)$

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u^i(s, x) \partial_s \varphi(s, x) dx ds + \int_{\mathbb{R}^d} (u^i(t, x) \varphi(t, x) - \Phi^i(x) \varphi(T, x)) dx - \int_t^T \int_{\mathbb{R}^d} u^i(s, x) \mathcal{L}^* \varphi(s, x) dx ds \\ &= \int_t^T \int_{\mathbb{R}^d} f_s(x, u(s, x), \nabla u(s, x) \sigma(x)) \varphi(s, x) dx ds + \int_t^T \int_{\mathbb{R}^d} h_s(x, u(s, x), \nabla u(s, x) \sigma(x)) \varphi(s, x) dx d\overleftarrow{W}_s \\ &+ \int_t^T \int_{\mathbb{R}^d} \varphi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx). \end{aligned} \quad (1.2.24)$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ . For the sake of simplicity we will omit in the sequel the subscript  $i$ .

We give now the following result which allows us to link by a natural way the solution of SPDE with the associated BDSDE. In fact, this result plays the same role as Itô's formula used in [109] to relate the solution of some semilinear SPDEs with the associated BDSDEs:

**Proposition 1.2.3.** *Let  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^k$  such that belongs to  $\mathbf{L}_\rho^2(\mathbb{R}^d)$ ,  $\Phi(x) \in \bar{D}$  a.e.  $\forall x \in \mathbb{R}^d$  and  $u \in \mathcal{H}_T$  be a weak solution of the reflected SPDE (1.2.17) associated to  $(\Phi, f, h)$ . Then, under lipschitz continuous condition on  $f$  and  $h$ , regularity condition on the coefficients of the diffusion, we have for  $s \in [t, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r, x) d\varphi_t(r, x) dx + (u(s, \cdot), \varphi_t(s, \cdot)) - (\Phi(\cdot), \varphi_t(T, \cdot)) - \int_{\mathbb{R}^d} \int_s^T u(r, x) \mathcal{L}^* \varphi_t(r, x) dr dx \\ &= \int_{\mathbb{R}^d} \int_s^T f_r(x, u(r, x), \nabla u(r, x) \sigma(x)) \varphi_t(r, x) dr dx + \int_{\mathbb{R}^d} \int_s^T h_r(x, u(r, x), \nabla u(r, x) \sigma(x)) \varphi_t(r, x) d\overleftarrow{W}_r dx \\ &+ \int_{\mathbb{R}^d} \int_s^T \varphi_t(r, x) 1_{\{u \in \partial D\}}(r, x) \nu(dr, dx). \text{ a.s.} \end{aligned} \quad (1.2.25)$$

where  $\int_{\mathbb{R}^d} \int_s^T u(r, x) d\varphi_t(r, x) dx$  is well defined thanks to the semimartingale decomposition result (Lemma 4.3.1).

Then, the main result of this part is the following theorem where we prove existence and uniqueness of the solution  $(u, \nu)$  of the reflection problem for (1.2.15) in a convex domain and we provide the probabilistic interpretation via the Reflected Forward-BDSDEs :

**Theorem 1.2.3.** *Let  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^k$  such that belongs to  $\mathbf{L}_\rho^2(\mathbb{R}^d)$ ,  $\Phi(x) \in \bar{D}$  a.e.  $\forall x \in \mathbb{R}^d$  and  $\rho(x) = (1 + |x|)^{-p}$  with  $p > d + 1$ . Then, under lipschitz continuous condition on  $f$  and  $h$ , and regularity condition on the coefficients of the diffusion there exists a weak solution  $(u, \nu)$  of the reflected SPDE (1.2.17) associated to  $(\Phi, f, h)$  such that,  $u(t, x) := Y_t^{t,x}$ ,  $dt \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.$ , and*

$$Y_s^{t,x} = u(s, X_{t,s}(x)), \quad Z_s^{t,x} = (\nabla u \sigma)(s, X_{t,s}(x)), \quad ds \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.. \quad (1.2.26)$$

Moreover, the reflected measure  $\nu$  is a signed regular measure in the sense of the Definition 1.2.3 (ii) and satisfying the probabilistic interpretation (1.2.23).

If  $(\bar{u}, \bar{\nu})$  is another solution of the reflected SPDE (1.2.17) such that  $\bar{\nu}$  satisfies (1.2.23) with some  $\bar{K}$  instead of  $K$ , where  $\bar{K}$  is a continuous process, then  $\bar{u} = u$  and  $\bar{\nu} = \nu$ .

In other words, there is a unique Randon regular measure with support  $\{u \in \partial D\}$  which satisfies (1.2.23).

The proof of this result is based on the probabilistic interpretation via the Reflected Forward-BDSDEs. The uniqueness is a consequence of the variational formulation of the SPDEs written with random test functions and the uniqueness of the solution of the Reflected FBSDE. The existence of the solution is established by an approximation penalization procedure, a priori estimates and the equivalence norm results. In the Appendix, technical lemmas for the existence of the solution of the Reflected BDSDEs and SPDEs in a convex domain are given.

## 1.3 Numerical scheme for BDSDEs in domain

### 1.3.1 Literature

As explained in Section 1.1.1 backward stochastic differential equations (BSDEs in short) are natural tools to give a probabilistic interpretation for the solution of a class of semi-linear PDEs (see [115], [108], [107], [36]). By introducing in standard BSDEs a second nonlinear term driven by an external noise, we obtain Backward Doubly SDEs (BDSDEs) [109] introduced in Section 1.1.1. In the Markovian setting, these equations can be seen as Feynman-Kac's representation of Stochastic PDEs and form a powerful tool for numerical schemes [6, 7]. Several generalizations to investigate more general nonlinear SPDEs have been developed following different approaches of the notion of weak solutions: the technique of stochastic flow (Bally and Matoussi [10], Matoussi et al. [97, 99], Kunita [80]); the approach based on Dirichlet forms and their associated Markov processes (Denis and Stoica [43], Bally, Pardoux and Stoica [11], Stoica [131], Denis, Matoussi and Stoica [40, 41, 42], Matoussi and Stoica [98]); stochastic viscosity solution for SPDEs (Buckdahn and Ma [24, 25], Lions and Souganidis [89, 90, 88]). Above approaches have allowed the study of numerical schemes for the Sobolev solution of semi-linear SPDEs via Monte-Carlo methods (time discretization and regression schemes [6, 5, 7]).

In the case when we consider the whole space  $\mathbb{R}^d$ , the numerical approximation of the BSDE has already been studied in the literature by Bally [8], Zhang [140], Bouchard and Touzi [22], Gobet, Lemor and Warin [57] and Bouchard and Elie [17]. Bouchard and Touzi [22] and Zhang [140] proposed a discrete-time numerical approximation, by step processes, for a class of decoupled FBSDEs with possible path-dependent terminal values. Zhang [140] proved a regularity result on  $Z$ , which allows the use of a regular deterministic time mesh. In Bouchard and Touzi [22], the conditional expectations involved in their discretization scheme were computed by using the kernel regression estimation. Therefore, they used the Malliavin approach and the Monte carlo method for its computation. Crisan, Manolarakis and Touzi [34] proposed an improvement on the Malliavin weights. Gobet, Lemor and Warin in [57] proposed an explicit numerical scheme base based on Monte Carlo regression on a finite basis of functions. Their approach is more efficient, because it requires only one set of paths to approximate all regression operators. These Monte Carlo type numerical schemes are investigated to solve numerically the solution of semilinear PDEs. These latter methods are tractable especially when the dimension of the state process is very large unlike the finite difference method.

For BDSDEs where the coefficient  $g$  does not depend in the control variable  $z$ , Aman [1] proposed a numerical scheme following the idea used by Bouchard and Touzi [22] and obtained a convergence of order  $h$  of the square of the  $L^2$ - error ( $h$  is the discretization step in time). Aboura [1] studied the same numerical scheme under the same kind of hypothesis, but following Gobet et al. [58]. He



obtained a convergence of order  $h$  in time and he attempt for a Monte Carlo solver. Bachouch et al [6] have studied the rate of convergence of the time discretization error for BDSDEs in the case when the coefficient  $g$  depending in  $(y, z)$ . They presented an implementation and numerical tests for such Euler scheme. Bachouch, Gobet and Matoussi [5] have recently analyzed the regression error arising from an algorithm approximating the solution of a discrete- time BDSDEs. They have studied the rate of converge of such error in the case when the coefficients of the BDSDEs are only depend in the variable  $y$ .

For BSDEs with finite random time horizon, namely, the first exit time of a forward SDEs from a domain  $\mathcal{O}$ , Bouchard and Menozzi [21] studied the Euler scheme of these equations and provided the upper bounds for the discrete time approximations error which is at most of order  $h^{1/2-\varepsilon}$  where  $\varepsilon$  is any positive parameter. This rate of convergence is due to the approximation error of the exit time. These results are obtained when the domain  $\mathcal{O}$  is piecewise smooth and under a non-characteristic boundary condition (without uniform ellipticity condition). Bouchard, Gobet and Geiss [20] have improved this error which is now at most of order  $h^{1/2}$  even if the time horizon is unbounded.

### 1.3.2 New results

We are interested in this part of the thesis in solving numerically backward doubly stochastic differential equations (BDSDEs) with random terminal time  $\tau$ . The main motivations are giving a probabilistic representation of the Sobolev's solution of Dirichlet problem for semi-linear SPDEs and providing the numerical scheme for such SPDEs. Thus, we study the strong approximation of this class of BDSDEs when  $\tau$  is the first exit time of a forward SDE from a cylindrical domain. We use the Euler scheme and we provide bounds for the discrete-time approximation error.

First, we define the solution of BDSDEs with random terminal time and associated to  $(\xi, f, g)$  as a couple  $\{(Y_s, Z_s); 0 \leq s \leq T\} \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  such that  $Y_t = \xi$  on the set  $\{t \geq \tau\}$ ,  $Z_t = 0$  on the set  $\{t > \tau\}$  and

$$Y_t = \xi + \int_t^{\tau \wedge T} f(s, Y_s, Z_s) ds + \int_t^{\tau \wedge T} g(s, Y_s, Z_s) d\overleftarrow{W}_s - \int_t^{\tau \wedge T} Z_s dB_s, \quad 0 \leq t \leq \tau. \quad (1.3.1)$$

where we need the following assumptions on the final condition and on the coefficients:

**Assumption (HT)** The final random time  $\tau$  is an  $\mathcal{F}_t^B$ -stopping time and the final condition  $\xi$  is an  $\mathcal{F}_\tau^B$ -measurable and  $k$ -dimensional random variable such that  $\mathbb{E}[e^{\lambda\tau}|\xi|^2] < \infty$ .

**Assumption (HL)** The two coefficients  $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$  and  $g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times l}$ , which for some real numbers  $\alpha, \mu, \lambda, K > 0, C > 0, \lambda > \frac{2K}{1-\alpha} - 2\mu + C$  and  $0 < \alpha < 1$  satisfy: for all  $t \in [0, T]$  and  $(y, z), (y', z') \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,

- (i)  $f(\cdot, y, z)$  and  $g(\cdot, y, z)$  are  $\mathcal{F}_t$  measurable,
- (ii)  $|f(t, y, z) - f(t, y', z')| \leq K(|y - y'| + \|z - z'\|)$ ,
- (iii)  $\langle y - y', f(t, y, z) - f(t, y', z) \rangle \leq -\mu|y - y'|^2$ ,
- (iv)  $\|g(t, y, z) - g(t, y', z')\|^2 \leq C|y - y'|^2 + \alpha\|z - z'\|^2$ ,
- (v)  $\mathbb{E} \int_0^\tau e^{\lambda s} |f(t, 0, 0)|^2 ds < \infty$  and  $\mathbb{E} \int_0^\tau e^{\lambda s} \|g(t, 0, 0)\|^2 ds < \infty$ .

This later equation (1.3.1) gives the probabilistic interpretation for the weak-Sobolev's solutions of a class of semilinear stochastic partial differential equations (SPDEs in short) with Dirichlet null condition on the boundary of some open smooth domain  $\mathcal{O} \subset \mathbb{R}^d$ . Let also mention that an alternative method to solve numerically nonlinear SPDEs is an analytic one, based on time- space

discretization of the SPDEs. The discretization in space can be achieved either by finite differences, or finite elements [138] and spectral Galerkin methods [70]. But most numerical works on SPDEs have concentrated on the Euler finite-difference scheme. Very interesting results have been obtained by Gyongy and Krylov [60]. The authors consider a symmetric finite difference scheme for a class of linear SPDE driven by an infinite dimensional Brownian motion.

To our best knowledge, there is no result concerning the existence and uniqueness of the solution for BDSDE (1.3.1) in the literature. So we need to prove existence and uniqueness of the solution for BDSDE (1.3.1) which is an extension of Peng's results [115] in the standard BSDE case:

**Theorem 1.3.1.** *Under the Assumptions (HT) and (HL), there exists a unique solution  $\{(Y_s, Z_s); 0 \leq s \leq T\} \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  of the BDSDE (1.3.1).*

The uniqueness of the solution is obtained directly by applying generalized Ito formula to the difference of two solutions and under the Lipschitz continuous assumption on the coefficients. For the existence, it is proved in two steps. The first one, we suppose that the intensity of the external noise  $g$  does not depend on the variables  $y$  and  $z$  in order to transform our BDSDE into a BSDE and we apply the result of Peng [115]. Then the nonlinear case is deduced by using the fixed point Banach theorem. Afterthat, we are interested in developing a discrete-time approximation of a Forward-Backward Doubly SDE with finite stopping time horizon, namely the first exit time of a forward SDE from a cylindrical domain  $D = [0, T] \times \mathcal{O}$ . For all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , let  $(X_s^{t,x})_{0 \leq s \leq t}$  be the unique strong solution of the following stochastic differential equation:

$$dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s, \quad s \in [t, T], \quad X_s^{t,x} = x, \quad 0 \leq s \leq t, \quad (1.3.2)$$

where  $b$  and  $\sigma$  are two functions on  $\mathbb{R}^d$  with values respectively in  $\mathbb{R}^d$  and  $\mathbb{R}^{d \times d}$ . We will omit the dependance of the forward process  $X$  in the initial condition if it starts at time  $t = 0$ . Let  $\tau^{t,x}$  is the first exit time of  $(s, X_s^{t,x})$  from a cylindrical domain  $D = [0, T] \times \mathcal{O}$  for some open bounded set  $\mathcal{O} \subset \mathbb{R}^d$ . We now consider the following Markovian BDSDE with terminal random time  $\tau$  associated to the data  $(\Phi, f, g)$ : For all  $t \leq s \leq T$ ,

$$\begin{cases} -dY_s^{t,x} &= \mathbf{1}_{\{s < \tau\}} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \mathbf{1}_{\{s < \tau\}} g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})d\overleftarrow{W}_s - Z_s^{t,x} dB_s, \\ Y_s^{t,x} &= \Phi(\tau, X_\tau^{t,x}), \quad \tau \leq s \leq T, \end{cases} \quad (1.3.3)$$

where  $f$  and  $\Phi$  are now two functions respectively on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$  and  $\mathbb{R}^d$  with values in  $\mathbb{R}^k$  and  $g$  is a function on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$  with values in  $\mathbb{R}^{k \times l}$ .

Now, we specify some conditions on the domain and the diffusion process:

**Assumption (D)**  $\mathcal{O}$  is an open bounded set of  $\mathbb{R}^d$  with a  $C^2$ -boundary.

**Assumption (MHD)**

- (i) The matrix  $a := \sigma\sigma^*$  is elliptic, i.e. there exist  $\Lambda > 0$  such that for all  $x, \zeta \in \bar{\mathcal{O}}$ ,

$$\Lambda \|\zeta\|^2 \leq \zeta a(x) \zeta^*. \quad (1.3.4)$$

- (ii) There exist a positive constant  $L$  such that

$$|b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \leq L|x - x'|, \quad \forall x, x' \in \mathbb{R}^d.$$

Besides, we assume that the terminal condition  $\Phi$  is sufficiently smooth:

**Assumption (MHT)**

$$\Phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \quad \text{and} \quad \|\partial_t \Phi\| + \|D\Phi\| + \|D^2\Phi\| \leq L \quad \text{on} \quad [0, T] \times \mathbb{R}^d.$$

We next state a strengthening of Assumption **(HL)** in the present Markov framework:

**Assumption (MHL)** There exist constants  $\alpha, \mu, \lambda, K > 0, C > 0, C' > 0, \lambda > \frac{2K}{1-\alpha} - 2\mu + C$  and  $0 < \alpha < 1$  such that for any  $(t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,

- (i)  $|f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2)| \leq K(\sqrt{|t_1 - t_2|} + |x_1 - x_2| + |y_1 - y_2| + \|z_1 - z_2\|),$
- (ii)  $\|g(t_1, x_1, y_1, z_1) - g(t_2, x_2, y_2, z_2)\|^2 \leq C(|t_1 - t_2| + |x_1 - x_2|^2 + |y_1 - y_2|^2) + \alpha\|z_1 - z_2\|^2,$
- (iii)  $\langle y_1 - y_2, f(t_1, x_1, y_1, z_1) - f(t_1, x_1, y_2, z_1) \rangle \leq -\mu|y_1 - y_2|^2,$
- (iv)  $\sup_{0 \leq t \leq T} (|f(t, 0, 0, 0)| + \|g(t, 0, 0, 0)\|) \leq C'.$

We note that the integrability condition given by Assumption **(HT)** is satisfied in this Markovian setting thanks to the smoothness of  $\Phi$  (Assumption **(MHT)**) and the fact that the exit time  $\tau$ , under the ellipticity condition (1.3.4) verified by the matrix  $a$  (see Stroock and Varadhan [133]), satisfy

$$\sup_{(t,x) \in [0,T] \times \mathcal{O}} \mathbb{E}[\exp(\lambda\tau^{t,x})] < \infty.$$

In order to approximate the forward diffusion process (1.3.2), we use a standard Euler scheme with time step  $h$ , associated to a grid

$$\pi := \{t_i = ih; i \leq N\}, \quad h := T/N, \quad N \in \mathbb{N},$$

This approximation is defined by

$$X_t^N = x + \int_0^t b(X_{\varphi(s)})ds + \int_0^t \sigma(X_{\varphi(s)})dB_s, \quad t \geq 0 \quad (1.3.5)$$

where  $\varphi(s) := \sup\{t \in \pi : t \leq s\}$ . Notice that  $\varphi(t) = t_i$ , for  $t \in [t_i, t_{i+1})$  and the continuous approximation (1.3.5) is equivalent to the following discrete approximation

$$\begin{cases} X_0^N = x, \\ X_{t_{i+1}}^N = X_{t_i}^N + b(X_{t_i}^N)(t_{i+1} - t_i) + \sigma(X_{t_i}^N)(B_{t_{i+1}} - B_{t_i}), \quad i \leq N. \end{cases} \quad (1.3.6)$$

Then, we approximate the exit time  $\tau$  by the first time of the Euler scheme  $(t, X_t^N)_{t \in \pi}$  from  $D$  on the grid  $\pi$ :

$$\bar{\tau} := \inf\{t \in \pi : X_t^N \notin \mathcal{O}\} \wedge T. \quad (1.3.7)$$

The upper bound estimates for the error due to the approximation of  $\tau$  by  $\bar{\tau}$  was proved by Bouchard and Menozzi [21] for the weak version of such estimate and Gobet [54, 55] for the strong one. Recently, Bouchard, Geiss and Gobet [20] have improved the following  $L^1$ -strong error:

**Theorem 1.3.2.** *Assume that **(MHD)** and **(D)** hold. Then, there exists  $C_L > 0$  such that*

$$\mathbb{E}[|\tau - \bar{\tau}|] \leq C_L h^{1/2}. \quad (1.3.8)$$

Let us mention that the upper bound estimates for the error due to the approximation of  $\tau$  by  $\bar{\tau}$  proved by Bouchard and Menozzi [21] for the weak version of such estimate is as following: for any  $\varepsilon \in (0, 1)$  and each positive random variable  $\xi$  satisfying  $\mathbb{E}[(\xi_L)^p] \leq C_L^p$  for all  $p \geq 1$ , there exists  $C_L^\varepsilon > 0$  such that

$$\mathbb{E}[\mathbb{E}[\xi_L | \tau - \bar{\tau} | \mathcal{F}_{\tau \wedge \bar{\tau}}^B]^2] \leq C_L^\varepsilon h^{1-\varepsilon}. \quad (1.3.9)$$

For the strong estimate error, Gobet [54, 55] have proved that, for each  $\varepsilon \in (0, 1/2)$ , there exists  $C_L^\varepsilon > 0$  such that

$$\mathbb{E}[|\tau - \bar{\tau}|] \leq C_L^\varepsilon h^{1/2-\varepsilon}. \quad (1.3.10)$$

Regarding the approximation of (1.3.3), we adapt the approach of [6]. We define recursively (in a backward manner) the discrete-time process  $(Y^N, Z^N)$  on the time grid  $\pi$  by

$$Y_T^N = \Phi(\bar{\tau}, X_{\bar{\tau}}^N), \quad (1.3.11)$$

and for  $i = N - 1, \dots, 0$ , we set

$$Z_{t_i}^N = h^{-1} \mathbb{E}_{t_i} \left[ (Y_{t_{i+1}}^N + g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i) \Delta B_i^\top \right], \quad (1.3.12)$$

$$Y_{t_i}^N = \mathbb{E}_{t_i} [Y_{t_{i+1}}^N] + \mathbf{1}_{\{t_i < \bar{\tau}\}} h \mathbb{E}_{t_i} [f(t_i, \Theta_i^N)] + \mathbf{1}_{\{t_i < \bar{\tau}\}} \mathbb{E}_{t_i} [g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i], \quad (1.3.13)$$

where

$$\Theta_i^N := (X_{t_i}^N, Y_{t_i}^N, Z_{t_i}^N), \quad \Delta W_i = W_{t_{i+1}} - W_{t_i}, \quad \Delta B_i = B_{t_{i+1}} - B_{t_i}.$$

$\top$  denotes the transposition operator and  $\mathbb{E}_{t_i}$  denotes the conditional expectation over the  $\sigma$ -algebra  $\mathcal{F}_{t_i}^0$ . The above conditional expectation are well defined at each step of the algorithm.

Then we consider a continuous-time extension of  $Y^N$  in  $\mathcal{S}^2$  defined on  $[0, T]$  by

$$Y_t^N = \Phi(\bar{\tau}, X_{\bar{\tau}}^N) + \int_t^T \mathbf{1}_{\{s < \bar{\tau}\}} f(\varphi(s), \Theta_{\varphi(s)}^N) ds + \int_t^T \mathbf{1}_{\{s < \bar{\tau}\}} g(\psi(s), \Theta_{\psi(s)}^N) d\overleftarrow{W}_s - \int_t^T Z_s^N dB_s, \quad (1.3.14)$$

where  $\psi(s) := \inf\{t \in \pi : t \geq s\}$ .

Afterthat, we provide bounds for the (square of the) discrete-time approximation error up to a stopping time  $\theta \leq T$   $\mathbb{P}$ -a.s. defined as

$$\text{Err}(h)_\theta^2 := \max_{i < N} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} \mathbf{1}_{\{t < \theta\}} |Y_t - Y_t^N|^2 \right] + \mathbb{E} \left[ \int_0^\theta \|Z_t - Z_{\varphi(t)}^N\|^2 dt \right], \quad (1.3.15)$$

where we recall  $\varphi(s) := \sup\{t \in \pi : t \leq s\}$ . We are interested in two important cases:  $\theta = T$  and  $\theta = \tau \wedge \bar{\tau}$ . This approximation error is naturally related to the error coming from the approximation of  $\tau$  by  $\bar{\tau}$  (1.3.8), the error due to the approximation of  $X$  by  $X_\varphi$  and the regularity of the solution  $(Y, Z)$  of (1.3.3) through the quantities

$$\mathcal{R}(Y)_{\mathcal{S}^2}^\pi := \max_{i < N} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} |Y_t - Y_{t_i}|^2 \right], \quad \mathcal{R}(Z)_{\mathcal{H}^2}^\pi := \mathbb{E} \left[ \int_0^T \|Z_t - \bar{Z}_{\varphi(t)}\|^2 dt \right]$$

where  $\bar{Z}_{t_i}$  be the best approximation of  $(Z_t)_{t_i \leq t \leq t_{i+1}}$  by  $\mathcal{F}_{t_i}$ -measurable random variable in the following sense

$$\bar{Z}_{t_i} := h^{-1} \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} Z_s ds \right], \quad i < N.$$

We finally obtain our main result, which provides an upper bound for the convergence rate of  $\text{Err}(h)_{\tau \wedge \bar{\tau}}^2$  (and thus for  $\text{Err}(h)_{\bar{\tau} \wedge \bar{\tau}}^2$  and  $\text{Err}(h)_T^2$ ) which is of order  $h^{1/2}$ .

**Theorem 1.3.3.** *Let the Assumptions (D), (MHT), (MHL) and (MHD) hold. Then, for each  $\varepsilon \in (0, 1/2)$ , there exists  $C_L^\varepsilon > 0$  such that*

$$\text{Err}(h)_{\tau \wedge \bar{\tau}}^2 \leq C_L h^{1/2} \quad \text{and} \quad \text{Err}(h)_T^2 \leq C_L h^{1/2} \quad (1.3.16)$$

Then, we end this part by giving a Feynman-Kac's formula for the weak solution of a class of semilinear SPDEs with Dirichlet null condition on the boundary via the associated Markovian class of BDSDEs with random terminal time studied above. This representation enables us to solve numerically by Monte Carlo method the solution of SPDEs in a domain. Indeed, for a given open bounded domain  $\mathcal{O}$  of  $\mathbb{R}^d$ , we are interested in the following semilinear SPDEs :

$$\begin{cases} du_t + \mathcal{L}u_t dt + f(t, x, u_t, D_\sigma u_t) dt + g(t, x, u_t, D_\sigma u_t) d\overleftarrow{W}_t = 0, \forall 0 \leq t \leq T, \forall x \in \mathcal{O}, \\ u(T, x) = \Phi(x), \quad \forall x \in \mathcal{O} \\ u(t, x) = 0, \quad \forall 0 \leq t \leq T, \forall x \in \partial\mathcal{O}. \end{cases} \quad (1.3.17)$$

where  $D_\sigma := \nabla u \sigma$  and  $\mathcal{L}$  is the second order differential operator associated with a diffusion which is defined component by component with

$$\mathcal{L}\varphi(x) = \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) \quad (1.3.18)$$

and  $a := \sigma \sigma^*$ .

We assume the following hypotheses on the coefficients of the diffusion and the final condition:

**Assumption (MHD')** The coefficients of the second order differential operator  $\mathcal{L}$  satisfy:

- $b$  is a bounded function and belongs to  $C_{l,b}^2(\mathbb{R}^d, \mathbb{R}^d)$ .
- $\sigma \in C_{l,b}^3(\mathbb{R}^d, \mathbb{R}^{k \times d})$  and satisfy the ellipticity condition (1.3.4).

**Assumption (MHT')**  $\Phi \in \mathbf{L}^2(\mathcal{O}; \mathbb{R}^k)$  with polynomial growth, namely there exists  $C > 0$  and  $p \in \mathbb{N}$  such that  $|\Phi(x)| \leq C(1 + |x|^p)$ .

We denote by  $\mathcal{H}$  the space of  $\mathcal{F}_{t,T}^W$ -progressively measurable processes  $(u_t)$  with valued in the Dirichlet space  $H_0^1(\mathcal{O})$  where

$$H_0^1(\mathcal{O}) := \{v \in \mathbf{L}^2(\mathcal{O}) \mid \nabla v \sigma \in \mathbf{L}^2(\mathcal{O})\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \mathbb{E} \left[ \sup_{0 \leq s \leq T} \|u_s\|_2^2 + \int_{\mathcal{O}} \int_0^T |\nabla u_s(x) \sigma(x)|^2 ds \rho(x) dx \right],$$

where we denote the gradient by  $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$ .

**Definition 1.3.1.** We say that  $u \in \mathcal{H}$  is a weak solution of the SPDE (1.3.17) if the following relation holds for each  $\Psi \in \mathcal{D}$ ,

$$\begin{aligned} & \int_t^T \int_{\mathcal{O}} u(s, x) \partial_s \Psi(s, x) dx ds - \int_{\mathcal{O}} \Phi(x) \Psi(T, x) dx + \int_{\mathcal{O}} u(t, x) \Psi(t, x) dx - \int_t^T \int_{\mathcal{O}} u(s, x) \mathcal{L}^* u(s, x) dx ds \\ &= \int_t^T \int_{\mathcal{O}} \Psi(s, x) f(s, x, u(s, x), D_\sigma u(s, x)) dx ds + \int_t^T \int_{\mathcal{O}} \Psi(s, x) g(s, x, u(s, x), D_\sigma u(s, x)) dx d\overleftarrow{W}_s. \end{aligned} \quad (1.3.19)$$

where  $\tilde{A}_i =: \frac{1}{2} \sum_{k=1}^d \frac{\partial a_{k,i}}{\partial x_k}$  and

$$(u(s, \cdot), \mathcal{L}^* \Psi(s, \cdot)) := \int_{\mathcal{O}} D_\sigma u(s, x) D_\sigma \Psi(s, x) dx + \int_{\mathcal{O}} u(s, x) \operatorname{div}((b - \tilde{A}) \Psi(s, x)) dx.$$

We consider now the Markovian BDSDE with random terminal time  $\tau^{t,x}$  which is the first exist time of the forward diffusion  $X^{t,x}$  from the domain  $\mathcal{O}$

$$\begin{aligned} Y_s^{t,x} &= \Phi(X_{T \wedge \tau^{t,x}}^{t,x}) + \int_s^T \mathbf{1}_{(\tau^{t,x} > r)} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dB_r \\ &+ \int_s^T \mathbf{1}_{(\tau^{t,x} > r)} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r. \end{aligned} \quad (1.3.20)$$

Then probabilistic representation for the weak solution of semilinear SPDEs with Dirichlet null condition on the boundary of the domain  $\mathcal{O}$  (1.3.17) is given by means of solution of BDSDEs with random terminal time (1.3.20). This is done by using localization procedure and stochastic flow technics (see e.g. [10], [97], [80, 79] for these flow technics).

**Theorem 1.3.4.** *Assume (MHT'), (D), (MHL) and (MHD') hold and let  $\{(Y_s^{t,x}, Z_s^{t,x}), t \leq s \leq T\}$  be the solution of BDSDE (1.3.20). Then,  $u(t, x) := Y_t^{t,x}$ ,  $dt \otimes dx$ , a.e. is the unique solution of the SPDE (1.3.19) and*

$$Y_s^{t,x} = u(s \wedge \tau^{t,x}, X_{s \wedge \tau^{t,x}}^{t,x}), \quad Z_s^{t,x} = D_\sigma u(s \wedge \tau^{t,x}, X_{s \wedge \tau^{t,x}}^{t,x}). \quad (1.3.21)$$

Thanks to representation (1.3.21), numerical resolution of BDSDEs can be applied to solve numerically the solution of semilinear SPDEs (1.3.17). Finally, we implement our numerical scheme (1.3.6)-(1.3.11)-(1.3.13)-(1.3.12) and we only test statically its convergence. Further analysis of the convergence of the used method and of the error bounds will be accomplished in a future work.

## 1.4 Work in progress and Perspectives

We end the introduction by presenting some work in progress and future perspectives.

1. In the first work, we are interested in the solution of the reflection problem for quasilinear Stochastic PDEs in a convex domain  $D$ . More precisely, we give probabilistic interpretation for these solutions via Reflected generalized BDSDEs in a convex domain. We prove in this work existence and uniqueness results of such RBDSDEs in a convex in order to relate it to the the solution of the refection problem for our quasilinear SPDEs.
2. In the second one, we prove existence result and representation of the robust problem associated to BSDE with weak terminal condition which was first introduced by Bouchard, Elie and Reiveillac [18]. This class of robust equations shall permit to get results on quantile hedging and hedging under loss constraints obtained by Folmer and Leukert [50] and in Bouchard, Elie and Touzi [19] under models uncertainty. We are able to get these robust results for a class of models in the dominated case, but the non dominated case (as the UVM = Uncertainty volatility models) seems to be a difficult question.
3. An other work in progress which is related to the obstacle problem for PIDEs (studied in Chapter 3) is to study the problem by using an analytical point of view (see Michel Pierre [116, 117], Denis, Matoussi and Zhang [38]). The question is how to define the analytical potential associated to the non local operator  $\mathcal{K}_2$  (1.2.16) in the jump case. Therefore, we are thinking of how to define the associated analytical capacity and extend the analytical quasi-sure approaches for the PIDEs and generally for the Stochastic PIDEs. Moreover, we also study this problem by probabilistic quasi-sure approach as in Matoussi and Stoica [98] for the obstacle problem of quasilinear parabolic stochastic PDEs.
4. For future research, an another topic is about quasilinear and fully nonlinear Stochastic PDEs driven by a general noise with less regularity condition that the Brownian motion. Thus we shall study Stochastic PDEs from a rough point view combining with BSDEs and 2BSDEs theory.

# Probabilistic Interpretation for Solutions of Fully Nonlinear Stochastic PDEs

---

## Contents

---

<b>2.1</b>	<b>Introduction</b>	<b>29</b>
<b>2.2</b>	<b>Preliminaries and Hypothesis</b>	<b>30</b>
2.2.1	The set of probability measures and the generator	30
2.2.2	The Spaces and Norms	32
2.2.3	Formulation	32
2.2.4	Connection with standard BDSDEs	33
2.2.5	Connection with G-expectation	33
<b>2.3</b>	<b>Uniqueness of the solution and other properties</b>	<b>34</b>
2.3.1	Representation and uniqueness of the solution	34
2.3.2	A priori estimates	37
<b>2.4</b>	<b>A direct existence argument</b>	<b>39</b>
2.4.1	Notations	39
2.4.2	Existence when $\xi$ is in $UC_b(\Omega)$	41
2.4.3	Existence result in the general case	47
<b>2.5</b>	<b>Probabilistic interpretation for Fully nonlinear SPDEs</b>	<b>49</b>
2.5.1	A nonlinear Feynman-Kac representation formula	51
2.5.2	Stochastic viscosity solution for SPDE	53
<b>2.6</b>	<b>Appendix</b>	<b>57</b>
2.6.1	Doubly $f$ - supersolution and martingales	57
2.6.2	Downcrossing inequality	58
2.6.3	Reflected backward doubly stochastic differential equations	59

---

## 2.1 Introduction

Our aim in this chapter is to provide a complete theory of existence and uniqueness of Second order BDSDEs (2BDSDEs) under the Lipschitz-type hypotheses on the driver. We will show that in this context, the definition of a 2BDSDE is very similar to that of a 2BSDE. The main motivation is to give the probabilistic interpretation of classical and stochastic viscosity of fully nonlinear Stochastic PDEs. Similarly to Buckdahn and Ma [24, 25], we use the Doss-Sussmann transformation to convert fully nonlinear SPDEs to fully nonlinear PDEs with random coefficients, then we use the solution of

2BDSDE to provide the Feynman-Kac's formula.

The outline is as follows. After recalling briefly some notations, we provide in Section 2.2 the precise definition of 2BDSDEs and show how they are connected to classical BDSDEs. Then, in Section 2.3, we show a representation formula for the solution of 2BDSDEs which in turn implies uniqueness and existence result is given in section 2.4 . In Section 2.5, we present first the link between the the markovian 2BDDSE with the associated fully nonlinear SPDEs. Then, we prove that stochastic viscosity solution of such SPDE's is given via the solution of 2BDSDEs. Finally, we give some technicals results needed for the existence of the solution of the 2BDDSEs . The starting point of this work is the following parabolic Fully nonlinear stochastic partial differential equation (in short FSPDE). This Chapter is based in [93].

## 2.2 Preliminaries and Hypothesis

The inner scalar product of the space  $\mathbb{R}^d (d \geq 2)$  will be denoted by  $\langle \cdot, \cdot \rangle$  and the associated Euclidean norm by  $\|\cdot\|$ .

In what follows let us fix a positive real number  $T > 0$ . First of all, we shall work on the product space  $\Omega := \Omega^B \times \Omega^W$  where

- $\Omega^B := \{\omega \in C([0, T], \mathbb{R}^d), \omega_0 = 0\}$  equipped with the uniform norm  $\|\omega\|_\infty := \sup_{0 \leq t \leq T} |\omega_t|$ ,  $B$  the canonical process  $(B_t(\omega) = \omega_t)$ ,  $\mathbb{P}_B^0$  the Wiener measure.
- $(W_t)_{0 \leq t \leq T}$  is a standard Brownian motion independent of  $B$  and defined on  $(\Omega^W, \mathcal{F}^W, \mathbb{P}_0^W)$ .

Let  $\mathbb{F}^W = \{\mathcal{F}_{t,T}^W\}_{t \geq 0}$  be a retrograde filtration generated by  $W$ , defined by  $\mathcal{F}_{s,t}^W := \sigma\{W_r - W_s, s \leq r \leq t\}$  and let  $\mathcal{F}_T^W = \mathcal{F}_{0,T}^W$ . We also consider  $\mathbb{F}^B = \{\mathcal{F}_t^B\}_{t \geq 0}$  a forward filtration generated by  $B$ , such tha  $\mathcal{F}_t^B := \sigma\{B_r, 0 \leq r \leq t\}$ . For each  $t \in [0, T]$ , we define

$$\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^W \quad \text{and} \quad \mathcal{G}_t := \mathcal{F}_t^B \vee \mathcal{F}_T^W.$$

Finally, we consider  $\mathcal{F} = \mathcal{F}^B \otimes \mathcal{F}^W$  and the probability measure  $\mathbb{P} := \mathbb{P}_B^0 \otimes \mathbb{P}_0^W$ .

The collection  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  is neither increasing nor decreasing and it does not constitute a filtration. However,  $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$  is a filtration.

### 2.2.1 The set of probability measures and the generator

We will say that a probability measure  $\mathbb{P}_B$  is a local martingale measure if the canonical process  $B$  is a local martingale under  $\mathbb{P}_B$ . By Karandikar [72], we know that we can give pathwise definition of the quadratic variation  $\langle B \rangle_t$  and its density  $\hat{a}_t$ .

$$\langle B \rangle_t := B_t B_t^\top - 2 \int_0^t B_s dB_s^\top \quad \text{and} \quad \hat{a}_t := \overline{\lim}_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}).$$

where  $\top$  denotes the transposition, and the  $\overline{\lim}$  is componentwise.

Let  $\overline{\mathcal{P}}$  denote the set of all measure  $\mathbb{P} := \mathbb{P}_B \otimes \mathbb{P}_0^W$  such that

$$\begin{aligned} \langle B \rangle_t & \text{ is absolutely continuous in } t \text{ and } \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}. \\ W & \text{ is a } \mathbb{P}_0^W \text{ - Brownian motion.} \end{aligned}$$

where  $\mathbb{S}_d^{>0}$  denotes the space of all  $d \times d$  real valued positive definite matrices.

This chapter concentrates on the subclass  $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}$  consisting of all probability measures  $\mathbb{P} := \mathbb{P}^\alpha \otimes \mathbb{P}_0^W$  such that

$$\mathbb{P}^\alpha := \mathbb{P}_B^0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0, T], \mathbb{P}_B^0 - a.s. \quad (2.2.1)$$



for some  $\mathbb{F}^B$ - progressively measurable process  $\alpha$  taking values in  $\mathbb{S}_d^{>0}$  with  $\int_0^T |\alpha_t| dt < \infty$ ,  $\mathbb{P}_B^0$ -a.s. For any  $\mathbb{P} \in \overline{\mathcal{P}}_S$ , it follows from the Lévy characterization that Itô's stochastic integral under  $\mathbb{P}$ ,

$$X_t^{\hat{a}} := \int_0^t \hat{a}_s^{1/2} dB_s, \quad t \in [0, T], \quad \mathbb{P}_B^0 - a.s. \quad (2.2.2)$$

defines a  $\mathbb{P}$ -Brownian motion.

**Remark 2.2.1.** We denote by  $\overline{\mathbb{F}}^{\mathbb{P}}$  (resp.  $\overline{\mathbb{F}}^{X^{\hat{a}}}$ ) the  $\mathbb{P}$ -augmentation of the right-limit filtration generated by  $B$  (resp. by  $X^{\hat{a}}$ ). We recall from [130] that

$$\overline{\mathcal{P}}_S = \left\{ \mathbb{P} \in \overline{\mathcal{P}} : \overline{\mathbb{F}}^{X^{\hat{a}}} = \overline{\mathbb{F}}^{\mathbb{P}} \right\}, \quad (2.2.3)$$

and every  $\mathbb{P} \in \overline{\mathcal{P}}_S$  satisfies the Blumenthal zero-one law and the martingale representation property.

**Remark 2.2.2.** We recall from [129] that for a fixed  $\mathbb{P} \in \overline{\mathcal{P}}_S$ , we have from the Blumenthal zero-one law that  $\mathbb{E}^{\mathbb{P}}[\xi | \mathcal{G}_t] = \mathbb{E}^{\mathbb{P}}[\xi | \mathcal{G}_t^+] - a.s.$  for any  $t \in [0, T]$  and  $\mathbb{P}$ -integrable  $\xi$ . In particular, this shows that any  $\mathcal{G}_t^+$ -measurable random variable has an  $\mathcal{G}_t$ -measurable  $\mathbb{P}$ -modification.

Let  $H_t(w, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \rightarrow \mathbb{R}$  be  $\mathbb{F}$ - progressively measurable, where  $D_H \subset \mathbb{R}^{d \times d}$  is a given subset containing 0.

Define the corresponding conjugate of  $H$  with respect to  $\gamma$  by

$$F_t(w, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} Tr(a\gamma) - H_t(w, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0},$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \text{ and } \hat{F}_t^0 := \hat{F}_t(0, 0).$$

We denote by  $D_{F_t(y, z)} := \{a, F_t(w, y, z, a) < +\infty\}$  the domain of  $F$  in  $a$  for a fixed  $(t, w, y, z)$ .

We consider also a function  $g_t(\omega, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^l$  which is  $\mathbb{F}$ - progressively measurable and denote  $g_t^0 = g_t(0, 0)$ .

**Definition 2.2.1.** We restrict the set of probability measures in  $\mathcal{P}$  which is the collection of all  $\mathbb{P} \in \overline{\mathcal{P}}_S$  such that

$$\begin{aligned} \underline{a}_{\mathbb{P}} \leq \hat{a} \leq \overline{a}_{\mathbb{P}}, \quad dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0}, \\ \mathbb{E}^{\mathbb{P}} \left[ \left( \int_0^T |\hat{F}_t^0|^2 dt \right) \right] < +\infty, \quad \mathbb{E}^{\mathbb{P}} \left[ \left( \int_0^T \|g_t^0\|^2 dt \right) \right] < +\infty. \end{aligned}$$

**Definition 2.2.2.** We say that a property holds  $\mathcal{P}$ -quasi-surely ( $\mathcal{P}$ -q.s. for short) if it holds  $\mathbb{P}$ -a.s. for all  $\mathbb{P} \in \mathcal{P}$ .

We now state our main assumptions on the functions  $F$  and  $g$

**Assumption 2.2.1.** (i)  $\mathcal{P}$  is not empty, the domain  $D_{F_t(y, z)} = D_{F_t}$  is independent of  $(\omega, y, z)$ .

(ii) For fixed  $(y, z, a)$ ,  $F$  is  $\mathcal{F}_t$  measurable in  $D_{F_t}$ , and  $g$  is  $\mathcal{F}_t$  measurable.

(iii) We have the following uniform Lipschitz-type property in  $y$  and  $z$ : There exist constants  $C > 0$  and  $0 \leq \alpha < 1$  such that  $\forall (y, y', z, z', t, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \Omega$ ,

$$\begin{aligned} |\hat{F}_t(\omega, y, z) - \hat{F}_t(\omega, y', z')| &\leq C(|y - y'| + \|\hat{a}^{1/2}(z - z')\|) \\ \|g_t(\omega, y, z) - g_t(\omega, y', z')\|^2 &\leq C|y - y'|^2 + \alpha\|(z - z')\|^2 \end{aligned}$$

(iv) There exists a constant  $\lambda \in [0, 1[$  such that

$$(1 - \lambda)\hat{a} \geq \alpha I_d.$$

(v)  $F$  is uniformly continuous in  $\omega$  for the  $\|\cdot\|_{\infty}$  norm.

(vi)  $H$  is uniformly continuous in  $\omega$  for the  $\|\cdot\|_{\infty}$  norm.

## 2.2.2 The Spaces and Norms

For the formulation of the second order BDSDEs, we will use the same spaces and norms introduced in the second order BSDEs [128].

For  $p \geq 1$ ,  $L^p$  denotes the space of all  $\mathcal{F}_T$ -measurable scalar r.v.  $\xi$  with

$$\|\xi\|_{L^p}^p := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty.$$

$\mathbb{H}^p$  denotes the space of all  $\mathcal{F}_t$ -measurable  $\mathbb{R}^d$ -valued processes  $Z$  with

$$\|Z\|_{\mathbb{H}^p}^p := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \left( \int_0^T |\hat{a}_t^{1/2} Z_t|^2 dt \right)^{\frac{p}{2}} \right] < +\infty.$$

$\mathbb{D}^p$  denotes the space of all  $\mathcal{F}_t$ -measurable  $\mathbb{R}$ -valued processes  $Y$  with

$$\mathcal{P}\text{-q.s. càdlàg paths, and } \|Y\|_{\mathbb{D}^p}^p := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] < +\infty.$$

$\mathbb{I}^p$  denotes the space of all  $\mathcal{F}_t$ -measurable  $\mathbb{R}$ -valued processes  $K$  null at 0 with

$$\mathcal{P}\text{-q.s. càdlàg paths, and nondecreasing paths, and } \|K\|_{\mathbb{I}^p}^p := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ (K_T)^p \right] < +\infty.$$

For each  $\xi \in L^1$ ,  $\mathbb{P} \in \mathcal{P}$  and  $t \in [0, T]$  denote  $\mathbb{E}_t^{H, \mathbb{P}}[\xi] := \text{ess sup}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi]$  where

$$\mathcal{P}(t+, \mathbb{P}) := \{\mathbb{P}' := \mathbb{P}'_B \otimes \mathbb{P}_0^W \in \mathcal{P}; \mathbb{P}'_B = \mathbb{P}_B \text{ on } \mathcal{F}_t^{B,+}\}.$$

Here  $\mathbb{E}_t^{\mathbb{P}}[\xi] := \mathbb{E}^{\mathbb{P}}[\xi | \mathcal{G}_t] = \mathbb{E}^{\mathbb{P}}[\xi | \mathcal{F}_t^B \vee \mathcal{F}_T^W]$ ,  $\mathbb{P}$ -a.s.

Then we define for each  $p \geq 2$ ,

$$\mathbb{L}^p := \{\xi \in L^p : \|\xi\|_{\mathbb{L}^p} < +\infty\} \text{ where } \|\xi\|_{\mathbb{L}^p}^p := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \text{ess sup}_{0 \leq t \leq T} (\mathbb{E}_t^{H, \mathbb{P}}[|\xi|^2])^{\frac{p}{2}} \right].$$

Finally, we denote by  $UC_b(\Omega)$  the collection of all bounded and uniformly continuous maps  $\xi : \Omega \rightarrow \mathbb{R}$  with respect to the  $\|\cdot\|_{\infty}$ -norm, and we let  $\mathcal{L}^p$  be the closure of  $UC_b(\Omega)$  under the norm  $\|\cdot\|_{\mathbb{L}^p}$ , for every  $2 \leq p$ .

## 2.2.3 Formulation

We shall consider the following second order backward doubly stochastic differential equation (2BDSDE for short)

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s) ds + \int_t^T g_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t, \quad (2.2.4)$$

$$0 \leq t \leq T, \mathcal{P}\text{-q.s.}$$

We note that the integral with respect to  $\{W_t\}$  is a "backward Itô integral" (see [78], Page 111-112) and the integral with respect to  $\{B_t\}$  is a standard forward Itô integral.

For any  $\mathbb{P} \in \mathcal{P}$ ,  $\mathbb{G}$ -stopping time  $\tau$ , and  $\mathcal{G}_\tau$ -measurable random variable  $\xi \in \mathbb{L}^2(\mathbb{P})$ , let  $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi))$  denote the unique solution to the following BDSDE (existence and uniqueness have been proved under our assumptions by Pardoux and Peng [109])

$$y_t^{\mathbb{P}} = \xi + \int_t^{\tau} \hat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}) ds + \int_t^{\tau} g_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}) d\overleftarrow{W}_s - \int_t^{\tau} z_s^{\mathbb{P}} dB_s \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \quad (2.2.5)$$

**Definition 2.2.3.** For  $\xi \in L^2$ , we say  $(Y, Z) \in \mathbb{D}^2 \times \mathbb{H}^2$  is a solution to the 2BDSDE (2.2.4) if

- $Y_T = \xi$   $\mathcal{P}$ -*q.s.*
- For each  $\mathbb{P} \in \mathcal{P}$ , the process  $K^\mathbb{P}$  defined below has nondecreasing paths,  $\mathbb{P}$ -*a.s.*:

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s) ds - \int_0^t g_s(Y_s, Z_s) d\overleftarrow{W}_s + \int_0^t Z_s dB_s, \quad (2.2.6)$$

$0 \leq t \leq T, \mathbb{P}$ -*a.s.*

- The family  $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$  defined in (2.2.6) satisfies the following minimum condition:

$$K_t^\mathbb{P} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [K_T^{\mathbb{P}'}] \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, t \in [0, T]. \quad (2.2.7)$$

Moreover, if the family  $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$  can be aggregated into a universal process  $K$ , we call  $(Y, Z, K)$  a solution of 2BDSDE (2.2.4).

## 2.2.4 Connection with standard BDSDEs

Let  $H$  be the following linear function of  $\gamma$ :

$$H_t(y, z, \gamma) = \frac{1}{2} I_d : \gamma - f_t(y, z),$$

where  $I_d$  is the identity matrix in  $\mathbb{R}^d$ .

Then,  $D_{F_t(w)} = \{I_d\}$ ,  $\hat{F}_t(y, z) = f_t(y, z)$  and  $\mathcal{P} = \{\mathbb{P}_0 := \mathbb{P}_B^0 \otimes \mathbb{P}_0^W\}$ . In this case, the minimum condition (2.2.7) implies

$$0 = K_0 = \mathbb{E}^{\mathbb{P}_0} [K_T] \quad \text{and thus} \quad K = 0, \quad \mathbb{P}_0\text{-a.s.}$$

since  $K_t$  is nondecreasing. Hence, the 2BDSDE (2.2.4) is equivalent to the following BDSDE:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds + \int_t^T g_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P}_0\text{-a.s.} \quad (2.2.8)$$

## 2.2.5 Connection with G-expectation

We consider  $H$  the following function:

$$H_t(y, z, \gamma) := G(\gamma) = \frac{1}{2} \sup_{\underline{a} \leq a \leq \bar{a}} (a : \gamma) \quad \underline{a}, \bar{a} \in \mathbb{S}_d^{>0},$$

and suppose that  $g$  does not depend on  $Y$  and  $Z$ , then

- $D_{F_t} = [\underline{a}, \bar{a}]$  and  $F_t(y, z, a) = 0$  for all  $a \in [\underline{a}, \bar{a}]$ ;
- $\mathcal{P} = \{\mathbb{P} \in \overline{\mathcal{P}}_S; \underline{a} \leq \hat{a} \leq \bar{a}, dt \times d\mathbb{P} - \text{a.s.} \quad \mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |g(t)|^2 dt \right) \right] < +\infty\}$ ,

and the 2BDSDE (2.2.4) becomes

$$Y_t = \xi + \int_t^T g(s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \quad \mathcal{P}\text{-q.s.}$$

Denoting

$$\bar{Y}_t = Y_t + \int_0^t g(s) d\overleftarrow{W}_s, \quad \bar{\xi} = \xi + \int_0^T g(s) d\overleftarrow{W}_s,$$

we have the following 2BSDE

$$\bar{Y}_t = \bar{\xi} - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \quad \mathcal{P} - q.s.$$

As in [128], we may decompose  $K$  into  $dK_t = k_t dt + dK_t^0$ , where  $k \geq 0$  and  $dK_t^0$  is a measure singular to the Lebesgue measure  $dt$ . Then, there exists a process  $\Gamma$  such that  $G(\Gamma_t) - \frac{1}{2}\hat{a}_t : \Gamma = k_t$  and we get

$$\bar{Y}_t = \bar{\xi} + \int_t^T \left( \frac{1}{2}\hat{a}_s : \Gamma - G(\Gamma_s) \right) ds - \int_t^T Z_s dB_s + K_T^0 - K_t^0, \quad 0 \leq t \leq T, \quad \mathcal{P} - q.s.$$

and it follows that

$$Y_t = \xi + \int_t^T \left( \frac{1}{2}\hat{a}_s : \Gamma - G(\Gamma_s) \right) ds + \int_t^T g(s) d\bar{W}_s - \int_t^T Z_s dB_s + K_T^0 - K_t^0, \quad 0 \leq t \leq T, \quad \mathcal{P} - q.s.$$

In addition to Assumption 2.2.1, we will assume

**Assumption 2.2.2.** *The processes  $\hat{F}^0$  and  $g^0$  satisfy the following integrability conditions*

$$\phi^2 := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \text{ess sup}_{0 \leq t \leq T}^{\mathbb{P}} \left( \mathbb{E}_t^{H, \mathbb{P}} \left[ \int_0^T |\hat{F}_s^0|^2 ds \right] \right) \right] < +\infty, \quad (2.2.9)$$

$$\psi^2 := \sup_{\mathbb{P} \in \mathcal{P}} \left( \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |g_s^0|^2 ds \right] \right) < +\infty. \quad (2.2.10)$$

## 2.3 Uniqueness of the solution and other properties

### 2.3.1 Representation and uniqueness of the solution

The aim of this section is to prove the uniqueness of solution for 2BDSDEs (2.2.4), which is a direct consequence from representation theorem. But, since in our 2BDSDEs (2.2.4) we have the extra backward integral we shall handle with the difficulties coming from this term. That's why, the following lemma is needed.

**Lemma 2.3.1.** *The minimum condition (2.2.7) implies that*

$$\inf_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} \left[ K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right] = 0.$$

**Proof.** Indeed, fix some  $\mathbb{P} \in \mathcal{P}$  and some  $\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})$  and observe first that (2.2.7) implies that  $K_t^{\mathbb{P}} = K_t^{\mathbb{P}'}$ ,  $\mathbb{P} - a.s.$  Hence, taking expectation under  $\mathbb{P}$  in (2.2.7), we obtain readily

$$\mathbb{E}^{\mathbb{P}} \left[ \text{ess inf}_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'} \left[ K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right] \right] = 0. \quad (2.3.1)$$

Then, we know (see [129]) that the family  $\mathcal{P}(t^+, \mathbb{P})$  is upward directed. Therefore, by classical results, there is a sequence  $(\mathbb{P}^n)_{n \geq 0} \subset \mathcal{P}(t^+, \mathbb{P})$  such that

$$\text{ess inf}_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'} \left[ K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right] = \lim_{n \rightarrow +\infty} \downarrow \mathbb{E}_t^{\mathbb{P}^n} \left[ K_T^{\mathbb{P}^n} - K_t^{\mathbb{P}^n} \right].$$

Using this in 2.3.1 and then the monotone convergence theorem under the fixed measure  $\mathbb{P}$ , we obtain

$$\begin{aligned}
0 &= \mathbb{E}^{\mathbb{P}} \left[ \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} [K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'}] \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[ \lim_{n \rightarrow +\infty} \downarrow \mathbb{E}_t^{\mathbb{P}^n} [K_T^{\mathbb{P}^n} - K_t^{\mathbb{P}^n}] \right] \\
&= \lim_{n \rightarrow +\infty} \downarrow \mathbb{E}^{\mathbb{P}} \left[ \mathbb{E}_t^{\mathbb{P}^n} [K_T^{\mathbb{P}^n} - K_t^{\mathbb{P}^n}] \right] \\
&= \lim_{n \rightarrow +\infty} \downarrow \mathbb{E}^{\mathbb{P}^n} \left[ \mathbb{E}_t^{\mathbb{P}^n} [K_T^{\mathbb{P}^n} - K_t^{\mathbb{P}^n}] \right] \\
&= \lim_{n \rightarrow +\infty} \downarrow \mathbb{E}^{\mathbb{P}^n} \left[ K_T^{\mathbb{P}^n} - K_t^{\mathbb{P}^n} \right] \\
&\geq \inf_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} \left[ K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \right].
\end{aligned}$$

Since  $K^{\mathbb{P}'}$  is a non-decreasing process, the result follows.  $\square$

We can now show as in Theorem 4.4 of [129] that the solution to 2BDSDE (2.2.4) can be represented as a supremum of solutions to BDSDE (5.3.2).

**Theorem 2.3.1.** *Let assumptions 2.2.1 and 2.2.2 hold. Assume  $\xi \in \mathbb{L}^2$  and that  $(Y, Z, K)$  is a solution to 2BDSDE (2.2.4). Then, for any  $\mathbb{P} \in \mathcal{P}$  and  $0 \leq t_1 < t_2 \leq T$ ,*

$$Y_{t_1} = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})}^{\mathbb{P}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \mathbb{P} - a.s. \quad (2.3.2)$$

Consequently, the 2BDSDE (2.2.4) has at most one solution in  $\mathbb{D}^2 \times \mathbb{H}^2$

**Proof.** Let first assume that  $(Y, Z, K)$  is a solution to 2BDSDE (2.2.4) and (2.3.2) is true then

$$Y_t = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t^+, \mathbb{P})}^{\mathbb{P}} y_t^{\mathbb{P}'}(T, \xi), \mathbb{P} - a.s., \text{ for all } \mathbb{P} \in \mathcal{P} \quad (2.3.3)$$

and thus  $Y$  is unique. Since we have that  $d\langle Y, B \rangle_t = Z_t d\langle B \rangle_t$ ,  $\mathcal{P}$ - $q.s.$ ,  $Z$  is unique.

Finally, the process  $K^{\mathbb{P}}$  is uniquely determined. We will now prove the representation (2.3.2).

(i) Fix  $0 \leq t_1 < t_2 \leq T$ , and  $\mathbb{P} \in \mathcal{P}$ . For any  $\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})$ , note that from (2.2.4)

$$Y_t = Y_{t_2} + \int_t^{t_2} \hat{F}_s(Y_s, Z_s) ds + \int_t^{t_2} g_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, t_1 \leq t \leq t_2, \mathbb{P}' - a.s.$$

and that  $K^{\mathbb{P}'}$  is nondecreasing,  $\mathbb{P}' - a.s.$ . Applying the comparison principle for BDSDE under  $\mathbb{P}$ , we have  $Y_{t_1} \geq y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \mathbb{P}' - a.s.$

Since  $\mathbb{P}' = \mathbb{P}$  on  $\mathcal{F}_{t_1}^+$ , we get  $Y_{t_1} \geq y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \mathbb{P} - a.s$  and thus

$$Y_{t_1} \geq \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})}^{\mathbb{P}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \mathbb{P} - a.s. \quad (2.3.4)$$

(ii) To prove the reverse inequality of representation (2.3.2), we use standard linearization techniques. Fix  $\mathbb{P} \in \mathcal{P}$ , for every  $\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})$ , denote:

$$\delta Y := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}) \text{ and } \delta Z := Z - z^{\mathbb{P}'}(t_2, Y_{t_2})$$

By the Lipschitz Assumption 2.2.1(iii), there exist  $(\mathcal{F}_t)$ -measurable bounded processes  $\lambda, \eta, \alpha, \beta$  such that

$$\begin{aligned}
\delta Y_t &= \int_t^{t_2} (\lambda_s \delta Y_s + \eta_s \hat{a}_s^{1/2} \delta Z_s) ds + \int_t^{t_2} (\alpha_s \delta Y_s + \beta_s \delta Z_s) d\overleftarrow{W}_s - \int_t^{t_2} \delta Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'} \\
& \quad t \leq t_2, \mathbb{P}' - a.s. \quad (2.3.5)
\end{aligned}$$

Define:

$$M_t := \exp\left(\int_0^t \eta_s \hat{\alpha}_s^{-1/2} dB_s + \int_0^t \lambda_s ds - \frac{1}{2} \int_0^t |\eta_s|^2 ds\right),$$

$$t_1 \leq t \leq t_2, \mathbb{P}' - a.s. \quad (2.3.6)$$

By Integration by parts we have

$$d(M_t \delta Y_t) = M_t (\delta Z_t + \delta Y_t \eta_t \hat{\alpha}_t^{-1/2}) dB_t - M_t \beta_t \delta Z_t d\overleftarrow{W}_t - M_t dK_t^{\mathbb{P}'} \quad (2.3.7)$$

so, we compute that:

$$\mathbb{E}^{\mathbb{P}'}[\delta Y_{t_1}] = \mathbb{E}^{\mathbb{P}'}\left[M_{t_1}^{-1} \int_{t_1}^{t_2} M_t dK_t^{\mathbb{P}'}\right] \leq \mathbb{E}^{\mathbb{P}'}\left[\sup_{t_1 \leq t \leq t_2} (M_{t_1}^{-1} M_t) (K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})\right]$$

by the nondecrease of  $K^{\mathbb{P}'}$  and where we used the fact that since  $\delta Y_{t_1}$  is  $\mathcal{F}_{t_1}^+$ -measurable, its expectation is the same under  $\mathbb{P}$  and  $\mathbb{P}'$ . By the boundedness of  $\lambda, \eta, \alpha, \beta$ , for every  $p \geq 1$  we have,

$$\mathbb{E}^{\mathbb{P}'}\left[\sup_{t_1 \leq t \leq t_2} (M_{t_1}^{-1} M_t)^p + \sup_{t_1 \leq t \leq t_2} (M_{t_1} M_t^{-1})^p\right] \leq C_p, t_1 \leq t \leq t_2, \mathbb{P}' - a.s. \quad (2.3.8)$$

Then it follows from the Hölder inequality that:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[Y_{t_1} - y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})] &\leq \left(\mathbb{E}^{\mathbb{P}'}\left[\sup_{t_1 \leq t \leq t_2} (M_{t_1}^{-1} M_t)^3\right]\right)^{1/3} \left(\mathbb{E}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^{3/2}]\right)^{2/3} \\ &\leq C \left(\mathbb{E}^{\mathbb{P}'}[K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}]\right)^{1/3} \mathbb{E}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2]^{1/3} \end{aligned}$$

From the definition of  $K^{\mathbb{P}'}$ , we have

$$\sup_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2] \leq C(\|Y\|_{\mathbb{D}^2}^p + \|Z\|_{\mathbb{H}^2}^p + \phi^2 + \psi^2) < \infty. \quad (2.3.9)$$

Then, by taking the infimum in  $\mathcal{P}(t_1^+, \mathbb{P})$  in the last inequality and using (2.3.9) and the result of Lemma 2.3.1, we obtain

$$\inf_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[Y_{t_1} - y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})\right] \leq 0.$$

But we clearly have

$$\inf_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[Y_{t_1} - y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})\right] \geq \mathbb{E}^{\mathbb{P}}\left[Y_{t_1} - \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})\right].$$

Hence

$$\mathbb{E}^{\mathbb{P}}\left[Y_{t_1} - \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})\right] \leq 0.$$

Since the quantity under the expectation is positive  $\mathbb{P}$ -a.s. by Step 1, we deduce that it is actually equal to 0,  $\mathbb{P} - a.s.$ , which is the desired result.  $\square$

As an immediate consequence of the representation formula (2.3.2) together with the comparison principle for BDSDEs, we have the following comparison principle for 2BDSDEs.

**Theorem 2.3.2.** *Let  $(Y, Z)$  and  $(Y', Z')$  be the solutions of 2BDSDEs with terminal conditions  $\xi$  and  $\xi'$  and generators  $\hat{F}$  and  $\hat{F}'$  respectively (with the corresponding functions  $H$  and  $H'$ ), and let  $(y^{\mathbb{P}}, z^{\mathbb{P}})$  and  $(y^{\mathbb{P}'}, z^{\mathbb{P}'})$  the solutions of the associated BDSDEs. Assume that they both verify Assumptions 2.2.1 and 2.2.2, and that we have  $\mathcal{P} - q.s.$*

$$\xi \leq \xi', \hat{F}(y_t^{\mathbb{P}}, z_t^{\mathbb{P}}) \leq \hat{F}'(y_t^{\mathbb{P}'}, z_t^{\mathbb{P}'})$$

Then  $Y \leq Y', \mathcal{P} - q.s.$

### 2.3.2 A priori estimates

In this section, we show some a priori estimates which will be useful in the sequel.

**Theorem 2.3.3.** *Let Assumptions 2.2.1 and 2.2.2 hold.*

- (i) *Assume  $\xi \in \mathbb{L}^2$  and that  $(Y, Z) \in \mathbb{D}^2 \times \mathbb{H}^2$  is a solution to the 2BDSDE (2.2.4). Then, there exist a constant  $C$  such that*

$$\|Y\|_{\mathbb{D}^2}^2 + \|Z\|_{\mathbb{H}^2}^2 + \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[|K_T^{\mathbb{P}}|^2] \leq C(\|\xi\|_{\mathbb{L}^2}^2 + \phi^2 + \psi^2). \quad (2.3.10)$$

- (ii) *Assume  $\xi^i \in \mathbb{L}^2$  and that  $(Y^i, Z^i) \in \mathbb{D}^2 \times \mathbb{H}^2$  is a corresponding solution to the 2BDSDE (2.2.4),  $i = 1, 2$ . Denote  $\delta\xi := \xi^1 - \xi^2$ ,  $\delta Y := Y^1 - Y^2$ ,  $\delta Z := Z^1 - Z^2$ , and  $\delta K^{\mathbb{P}} := K^{1, \mathbb{P}} - K^{2, \mathbb{P}}$ . Then, there exist a constant  $C$  such that*

$$\begin{aligned} \|\delta Y\|_{\mathbb{D}^2} &\leq C\|\delta\xi\|_{\mathbb{L}^2} \\ \|\delta Z\|_{\mathbb{H}^2}^2 + \sup_{\mathbb{P} \in \mathcal{P}} \left[ \sup_{0 \leq t \leq T} |\delta K_t^{\mathbb{P}}|^2 \right] &\leq C\|\delta\xi\|_{\mathbb{L}^2} (\|\xi^1\|_{\mathbb{L}^2} + \|\xi^2\|_{\mathbb{L}^2}) \\ &\quad + (\phi^2)^{1/2} + (\psi^2)^{1/2}. \end{aligned} \quad (2.3.11)$$

**Proof.** (i) Fix  $\mathbb{P} \in \mathcal{P}$ , for every  $\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})$ , we apply Itô's formula to  $|y_t^{\mathbb{P}'}|^2$  to obtain

$$\begin{aligned} |y_t^{\mathbb{P}'}|^2 &= |\xi|^2 + 2 \int_t^T \hat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}) y_s^{\mathbb{P}'} ds + 2 \int_t^T g_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}) y_s^{\mathbb{P}'} d\bar{W}_s \\ &\quad - 2 \int_t^T y_s^{\mathbb{P}'} z_s^{\mathbb{P}'} dB_s - \int_t^T |\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|^2 ds + \int_t^T |g_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})|^2 ds. \end{aligned}$$

Then, by taking expectation and from the Lipschitz Assumption 2.2.1(iii) we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}'}[|y_t^{\mathbb{P}'}|^2] &\leq C\mathbb{E}^{\mathbb{P}'}[|\xi|^2 + \int_t^T |y_s^{\mathbb{P}'}| (|\hat{F}_s^0| + |y_s^{\mathbb{P}'}| + |\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|) ds] - \mathbb{E}^{\mathbb{P}'}[|\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|^2 ds] \\ &\quad + C\mathbb{E}^{\mathbb{P}'}[\int_t^T (|g_s^0|^2 + |y_s^{\mathbb{P}'}|^2) ds] + \alpha\mathbb{E}^{\mathbb{P}'}[\int_t^T |z_s^{\mathbb{P}'}|^2 ds] \\ &\leq C\varepsilon^{-1}\mathbb{E}^{\mathbb{P}'}[|\xi|^2 + \int_t^T (|y_s^{\mathbb{P}'}|^2 + |\hat{F}_s^0|^2 + |g_s^0|^2) ds] \\ &\quad + (\varepsilon - 1)\mathbb{E}^{\mathbb{P}'}[\int_t^T |\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|^2 ds] + \alpha\mathbb{E}^{\mathbb{P}'}[\int_t^T |z_s^{\mathbb{P}'}|^2 ds], \end{aligned}$$

and, by setting  $\varepsilon = \frac{\lambda}{2}$  and the Assumption 2.2.1(iv), we get

$$\mathbb{E}^{\mathbb{P}'}[|y_t^{\mathbb{P}'}|^2] + \frac{\lambda}{2}\mathbb{E}^{\mathbb{P}'}[\int_t^T |\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|^2 ds] \leq C\mathbb{E}^{\mathbb{P}'}[|\xi|^2 + \int_t^T (|\hat{F}_s^0|^2 + |g_s^0|^2) ds] + \int_t^T |y_s^{\mathbb{P}'}|^2 ds.$$

It then follows, using Gronwall's lemma, that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}'}[|y_t^{\mathbb{P}'}|^2] &\leq C\left(\mathbb{E}^{\mathbb{P}'}[|\xi|^2] + \left(\mathbb{E}^{\mathbb{P}'}\left[\int_t^T (|\hat{F}_s^0|^2 ds)\right]\right) + \left(\mathbb{E}^{\mathbb{P}'}\left[\int_t^T (|g_s^0|^2 ds)\right]\right)\right) \\ &\quad \mathbb{P}' - a.s. \text{ for all } \mathbb{P}' \in \mathcal{P}(t+, \mathbb{P}), t \in [0, T]. \end{aligned}$$

Now, for every  $\mathbb{P} \in \mathcal{P}$ , we have

$$\begin{aligned}
 \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] &= \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} |y_t^{\mathbb{P}'}| \right)^2 \right] \\
 &\leq \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} [|\xi| + \int_0^T |\hat{F}_s^0| ds + \int_0^T (\lambda_s |y_s^{\mathbb{P}'}| + \mu_s |\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|) ds \right. \right. \\
 &\quad \left. \left. + \left| \int_0^T g_s^0 d\overleftarrow{W}_s \right| + \left| \int_0^T \alpha_s y_s^{\mathbb{P}'} d\overleftarrow{W}_s \right| + \left| \int_0^T \beta_s z_s^{\mathbb{P}'} d\overleftarrow{W}_s \right| + \left| \int_0^T z_s^{\mathbb{P}'} dB_s \right| \right)^2 \right] \\
 &\leq C \left( \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} [|\xi|^2] \right) \right] + \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} \left[ \int_0^T |\hat{F}_s^0|^2 ds \right] \right) \right] \right) \\
 &\quad + C \left( \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} \left[ \int_0^T |y_s^{\mathbb{P}'}|^2 ds \right] \right)^2 \right] + \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} z_s^{\mathbb{P}'}|^2 ds \right] \right)^2 \right] \right) \\
 &\quad + C \left( \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} \left[ \left| \int_0^T g_s^0 d\overleftarrow{W}_s \right|^2 \right] \right) \right] + \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \mathbb{E}_t^{H, \mathbb{P}} \left[ \left| \int_0^T z_s^{\mathbb{P}'} dB_s \right|^2 \right] \right) \right] \right),
 \end{aligned}$$

and, by the definition of the norms, we get

$$\|Y\|_{\mathbb{D}_H^2}^2 \leq C \left( \|\xi\|_{\mathbb{L}_H^2}^2 + \phi^2 + \psi^2 + \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |y_t^{\mathbb{P}}|^{2+\varepsilon} \right] + \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \left( \int_0^T |\hat{a}_s^{1/2} z_s^{\mathbb{P}}|^2 ds \right)^{\frac{2+\varepsilon}{2}} \right] \right),$$

where we used that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} \left( \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} |A| \right)^2 \right] \leq C \sup_{\mathbb{P} \in \mathcal{P}} \left( \mathbb{E}^{\mathbb{P}} [ |A|^{2+\varepsilon} ] \right)^{\frac{2}{2+\varepsilon}} \quad \forall \varepsilon > 0.$$

Finally, we obtain

$$\|Y\|_{\mathbb{D}_H^2}^2 \leq C (\|\xi\|_{\mathbb{L}^2}^2 + \phi^2 + \psi^2). \quad (2.3.12)$$

For the estimate for  $Z$ , we apply Itô's formula to  $|Y|^2$  under each  $\mathbb{P} \in \mathcal{P}$  and from the Lipschitz Assumption 2.2.1(iii) we have:

$$\begin{aligned}
 \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds \right] &\leq \mathbb{E}^{\mathbb{P}} \left[ |Y_0|^2 + \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds \right] \\
 &\leq C \mathbb{E}^{\mathbb{P}} \left[ |\xi|^2 + \int_0^T |Y_s| (|\hat{F}_s^0| + |Y_s| + |\hat{a}_s^{1/2} Z_s|) ds + \int_0^T |Y_s| dK_s^{\mathbb{P}} \right] \\
 &\quad + \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |g_s(Y_s, Z_s)|^2 ds \right] \\
 &\leq C \mathbb{E}^{\mathbb{P}} \left[ |\xi|^2 + \int_0^T |Y_s| (|\hat{F}_s^0| + |Y_s| + |\hat{a}_s^{1/2} Z_s|) ds + \int_0^T |Y_s| dK_s^{\mathbb{P}} \right] \\
 &\quad + C \mathbb{E}^{\mathbb{P}} \left[ \int_0^T (|g_s^0|^2 + |Y_s|^2) ds \right] + \alpha \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |Z_s|^2 ds \right] \\
 &\leq C \varepsilon^{-1} \mathbb{E}^{\mathbb{P}} \left[ |\xi|^2 + \sup_{0 \leq s \leq T} |Y_s|^2 + \int_0^T |\hat{F}_s^0|^2 ds + \int_0^T |g_s^0|^2 ds \right] \\
 &\quad + \varepsilon \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds + |K_T^{\mathbb{P}}|^2 \right] + \alpha \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |Z_s|^2 ds \right],
 \end{aligned}$$

for any  $\varepsilon \in (0, 1]$ . But by the definition of  $K_T^{\mathbb{P}}$ , it is clear that

$$\mathbb{E}^{\mathbb{P}} \left[ |K_T^{\mathbb{P}}|^2 \right] \leq C_0 \mathbb{E}^{\mathbb{P}} \left[ |\xi|^2 + \sup_{0 \leq s \leq T} |Y_s|^2 + \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds + \int_0^T |\hat{F}_s^0|^2 ds + \int_0^T |g_s^0|^2 ds \right], \quad (2.3.13)$$



for some constant  $C_0$  independent of  $\varepsilon$ . Then,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds \right] &\leq C\varepsilon^{-1} \mathbb{E}^{\mathbb{P}} \left[ |\xi|^2 + \sup_{0 \leq s \leq T} |Y_s|^2 + \left( \int_0^T |\hat{F}_s^0| ds \right)^2 + \int_0^T |g_s^0|^2 ds \right] \\ &\quad + (\varepsilon + C_0\varepsilon + 1 - \lambda) \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds \right]. \end{aligned}$$

Choosing  $\varepsilon := \frac{\lambda}{2(1+C_0)}$ , this provides

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} Z_s|^2 ds \right] \leq C \mathbb{E}^{\mathbb{P}} \left[ |\xi|^2 + \sup_{0 \leq s \leq T} |Y_s|^2 + \int_0^T |\hat{F}_s^0|^2 ds + \int_0^T |g_s^0|^2 ds \right].$$

By (2.3.12), we have

$$\|Z\|_{\mathbb{H}^2}^2 \leq C(\|\xi\|_{\mathbb{L}^2}^2 + \phi^2 + \psi^2). \quad (2.3.14)$$

(ii) First, we can follow the same proof of (i) to obtain that there exist a constant  $C$  depending only on  $T$  and the Lipschitz constant of  $\hat{F}$  and  $\hat{H}$  such that for all  $\mathbb{P}$

$$\mathbb{E}^{\mathbb{P}} [ |y_t^{\mathbb{P},1} - y_t^{\mathbb{P},2}|^2 ] \leq C \|\delta\xi\|_{\mathbb{L}^2}^2. \quad (2.3.15)$$

Then by definition of our norms, we get from (2.3.15) and (2.3.2) that

$$\|\delta Y\|_{\mathbb{D}^2} \leq C \|\delta\xi\|_{\mathbb{L}^2}. \quad (2.3.16)$$

Applying Itô formula to  $|\delta Y|^2$ , under each  $\mathbb{P} \in \mathcal{P}$ , leads to

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} \delta Z_s|^2 ds \right] &\leq C \mathbb{E}^{\mathbb{P}} \left[ |\delta\xi|^2 + \int_0^T |\delta Y_s| (|\delta Y_s| + |\hat{a}_s^{1/2} \delta Z_s|) ds + \int_0^T |\delta Y_s| d\delta K_s^{\mathbb{P}} \right] \\ &\quad + C \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\delta Y_s|^2 ds \right] \\ &\leq C \mathbb{E}^{\mathbb{P}} \left[ |\delta\xi|^2 + \sup_{0 \leq s \leq T} |\delta Y_s|^2 + \sup_{0 \leq s \leq T} |\delta Y_s|^2 [K_T^{1,\mathbb{P}} - K_T^{2,\mathbb{P}}] \right] \\ &\quad + \frac{1}{2} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |\hat{a}_s^{1/2} \delta Z_s|^2 ds \right]. \end{aligned}$$

The estimate for  $\delta Z$  is now obvious from the above inequality and the estimates of (i). Finally the estimate for the difference of the increasing processes is obvious by definition.  $\square$

## 2.4 A direct existence argument

As we have shown in Theorem 2.3.1, if there is a solution to the 2BDSDE (2.2.4), it will be represented as a supremum of solutions to standard BDSDEs. However, since we are working under a family of non-dominated probability measures, we are not able to use the classical technics of BSDEs. So, Soner, Touzi and Zhang [129] overcame this problem by constructing the solution pathwise using the so-called regular conditional probability distribution (in short r.c.p.d).

### 2.4.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [129].

- For  $0 \leq t \leq T$ , we denote  $\Omega_B^t := \{\omega \in C([t, T], \mathbb{R}^d); \omega(t) = 0\}$  the shifted canonical space;  $B^t$  the shifted canonical process;  $\mathbb{P}_0^{B,t}$  the shifted Wiener measure;  $\mathcal{F}_B^t$  the shifted filtration generated by  $B^t$ .

- For  $0 \leq t \leq T$ , we denote  $\Omega^t := \{\omega := (\omega^B, \omega^W) \text{ s.t. } \omega^B \in \Omega_B^t, \omega^W \in \Omega_W\}$ .

Exactly as in Section 2, we can define the set  $\bar{\mathcal{P}}_S^t$ , by restricting to the shifted space  $\Omega^t$

- For  $0 \leq s \leq t \leq T$  and  $\omega \in \Omega^s$ , define the shifted path  $\omega^t := (\omega^{B,t}, \omega^W) \in \Omega^t$  such that

$$\omega_r^{B,t} := \omega_r^B - \omega_t^B, \quad \forall r \in [t, T]$$

- For  $0 \leq s \leq t \leq T$  and  $\omega \in \Omega^s, \tilde{\omega} \in \Omega^t$  define the concatenation path  $\omega \otimes_t \tilde{\omega} := (\omega^B \otimes_t \tilde{\omega}^B, \omega^W \otimes_t \tilde{\omega}^W)$  by

$$\begin{aligned} (\omega^B \otimes_t \tilde{\omega}^B)(r) &:= \omega_r^B \mathbf{1}_{[s,t)}(r) + (\omega_t^B + \tilde{\omega}_r^B) \mathbf{1}_{[t,T)}(r), \quad \forall r \in [s, T] \\ (\omega^W \otimes_t \tilde{\omega}^W)(r) &:= \omega_r^W \mathbf{1}_{[s,t)}(r) + (\omega_t^W + \tilde{\omega}_r^W) \mathbf{1}_{[t,T)}(r), \quad \forall r \in [s, T]. \end{aligned}$$

- For  $0 \leq s \leq t \leq T$  and  $\mathcal{F}_T^s$ -measurable random variable  $\xi$  on  $\Omega^s$ , for each  $\omega \in \Omega^s$ , define the shifted  $\mathcal{F}_T^t$ -measurable random variable  $\xi^{t,\omega}$  on  $\Omega^t$  by

$$\xi^{t,\omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}), \quad \forall \tilde{\omega} \in \Omega^t.$$

Similarly, for an  $\mathbb{F}^s$ -progressively measurable process  $X$  on  $[s, T]$  and  $(t, \omega) \in [s, T] \times \Omega^s$ , the shifted process  $\{X_r^{t,\omega}, r \in [t, T]\}$  is  $\mathbb{F}^t$ - progressively measurable.

- For a  $\mathbb{G}$ -stopping time  $\tau$ , we use the following simplification

$$\omega \otimes_\tau \tilde{\omega} := \omega \otimes_{\tau(\omega)} \tilde{\omega}, \quad \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, \quad X^{\tau,\omega} := X^{\tau(\omega),\omega}.$$

- We define our "shifted " functions:

$$\begin{aligned} \hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) &:= F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^t(\tilde{\omega})) \quad \forall (s, \tilde{\omega}) \in [t, T] \times \Omega^t \\ g_s^{t,\omega}(\tilde{\omega}, y) &:= g_s(\omega \otimes_t \tilde{\omega}, y, z) \quad \forall (s, \tilde{\omega}) \in [t, T] \times \Omega^t. \end{aligned}$$

Then note that since  $F$  and  $g$  are assumed to be uniformly continuous in  $\omega$  under the  $\mathbb{L}^\infty$  norm, then so are  $\hat{F}_s^{t,\omega}$  and  $g_s^{t,\omega}$ . Notice that this implies that for any  $\mathbb{P} \in \bar{\mathcal{P}}_S^t$

$$\mathbb{E}^{\mathbb{P}} \left[ \left( \int_t^T |\hat{F}_s^{t,\omega}(0, 0)| 2ds \right) \right] < +\infty,$$

$$\mathbb{E}^{\mathbb{P}} \left[ \left( \int_t^T |g_s^{t,\omega}(0, 0)|^2 ds \right) \right] < +\infty$$

for some  $\omega$  if and only if it holds for all  $\omega \in \Omega$ .

- Finally, we extend Definition 2.2.1 in the shifted spaces

**Definition 2.4.1.**  $\mathcal{P}^t$  consists of all  $\mathbb{P} := \mathbb{P}^t \in \bar{\mathcal{P}}_S^t$  such that

$$\underline{a}_{\mathbb{P}} \leq \hat{a}^t \leq \bar{a}_{\mathbb{P}}, ds \times d\mathbb{P} - a.e \text{ on } [t, T] \times \Omega^t. \text{ for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0}$$

$$\mathbb{E}^{\mathbb{P}} \left[ \left( \int_t^T |\hat{F}_s^{t,\omega}(0, 0)|^2 ds \right) \right] < +\infty, \quad \mathbb{E}^{\mathbb{P}} \left[ \left( \int_t^T |g_s^{t,\omega}(0, 0)|^2 ds \right) \right] < +\infty \text{ for all } \omega \in \Omega$$

- By Stroock and Varadhan [134], there exist an r.c.p.d.  $\mathbb{P}_\tau^\omega$  on  $\mathcal{G}_T$  such that for a given  $\omega \in \Omega$ ,  $\mathbb{G}$ -stopping time  $\tau$ ,  $\mathbb{P} \in \mathcal{P}^t$  and every bounded  $\mathcal{G}_T$ -measurable random variable  $\xi$

$$\mathbb{E}_\tau^\mathbb{P}[\xi](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi], \text{ for } \mathbb{P} - a.e. \omega.$$

Furthermore,  $\mathbb{P}_\tau^\omega$  naturally induces a probability measure  $\mathbb{P}^{\tau,\omega}$  on  $\mathcal{G}_T^{\tau(\omega)}$ .

### 2.4.2 Existence when $\xi$ is in $\text{UC}_b(\Omega)$

When  $\xi$  is in  $\text{UC}_b(\Omega)$ , we know that there exists a modulus of continuity function  $\rho$  for  $\xi, F$  and  $g$  in  $\omega$ . Then, for any  $0 \leq t \leq s \leq T, (y, z) \in \mathbb{R} \times \mathbb{R}^d$  and  $\omega, \omega' \in \Omega, \tilde{\omega} \in \Omega^t$ ,

$$\begin{aligned} |\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})| &\leq \rho(\|\omega^B - \omega'^B\|_t), |\hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) - \hat{F}_s^{t,\omega'}(\tilde{\omega}, y, z)| \leq \rho(\|\omega^B - \omega'^B\|_t) \\ |g_s^{t,\omega}(\tilde{\omega}, y, z) - g_s^{t,\omega'}(\tilde{\omega}, y, z)| &\leq \rho(\|\omega^B - \omega'^B\|_t). \end{aligned}$$

We add the following assumption

**Assumption 2.4.1.**

$$\Lambda_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}^t} \left( \mathbb{E}^{\mathbb{P}} [|\xi^{t,\omega}|^2 + \int_t^T |\hat{F}_s^{t,\omega}(0,0)|^2 ds + \int_t^T |g_s^{t,\omega}(0,0)|^2 ds] \right)^{1/2} < \infty,$$

for all  $(t, \omega) \in [0, T] \times \Omega$ .

To prove existence, we define the following value process  $V_t$  pathwise

$$V_t(\omega^B, \cdot) := \text{ess sup}_{\mathbb{P} \in \mathcal{P}^t} \mathbb{P}_0^W \mathcal{Y}_t^{\mathbb{P}, t, \omega^B}(T, \xi)(\cdot), \mathbb{P}_0^W - a.s. \quad \text{with } \mathbb{P} := \mathbb{P}_B \otimes \mathbb{P}_0^W, \quad (2.4.1)$$

where, for any  $(t_1, \omega) \in [0, T] \times \Omega$ ,  $\mathbb{P} \in \mathcal{P}^{t_1}$ ,  $t_2 \in [t_1, T]$ , and any  $\mathcal{F}_{t_2}$ -measurable  $\eta \in L^2(\mathbb{P})$ , we denote  $\mathcal{Y}_{t_1}^{\mathbb{P}, t_1, \omega^B}(t_2, \eta) := y_{t_1}^{\mathbb{P}, t_1, \omega^B}$ , where  $(y^{\mathbb{P}, t_1, \omega^B}, z^{\mathbb{P}, t_1, \omega^B})$  is the solution of the following BDSDE on the shifted space  $\Omega^{t_1}$  under  $\mathbb{P}$ ,

$$\begin{aligned} y_s^{\mathbb{P}, t_1, \omega^B}(\omega^W) &= \eta^{t_1, \omega^B} + \int_s^{t_2} \hat{F}_r^{t_1, \omega^B}(y_r^{\mathbb{P}, t_1, \omega^B}, z_r^{\mathbb{P}, t_1, \omega^B})(\omega^W) dr - \int_s^{t_2} z_r^{\mathbb{P}, t_1, \omega^B}(\omega^W) dB_r^{t_1} \\ &+ \int_s^{t_2} g_r^{t_1, \omega^B}(y_r^{\mathbb{P}, t_1, \omega^B}, z_r^{\mathbb{P}, t_1, \omega^B})(\omega^W) d\overleftarrow{W}_r, \text{ for } \mathbb{P}_0^W - a.e. \omega^W \in \Omega_W. \end{aligned} \quad (2.4.2)$$

The following Lemma allows to give a link between BDSDEs on the shifted spaces.

**Lemma 2.4.1.** Fix some  $\mathbb{P}_B \in \mathcal{P}_S$ . For  $\mathbb{P}_B$ -a.e.  $\omega^B \in \Omega^B$ , the following inequality holds

$$y_t^{\mathbb{P}_B^t, \omega^B} \otimes \mathbb{P}_0^W(\omega^W) = y_t^{\mathbb{P}_B \otimes \mathbb{P}_0^W}(\omega^B, \omega^W), \quad t \in [0, T], \text{ for } \mathbb{P}_0^W - a.e. \omega^W \in \Omega_W,$$

**Proof.** We divide the proof in two steps.

*Step 1:* We start by showing the result in the case where  $F$  and  $g$  do not depend on  $(y, z)$ . In this case, we can solve directly the BDSDEs to find that for  $\mathbb{P}_B \otimes \mathbb{P}_0^W - a.e. (\omega^B, \omega^W) \in \Omega$

$$y_t^{\mathbb{P}_B \otimes \mathbb{P}_0^W}(\omega^B, \omega^W) = \mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} \left[ \xi + \int_t^T \hat{F}_s ds + \int_t^T g_s d\overleftarrow{W}_s \middle| \mathcal{G}_t \right] (\omega^B, \omega^W). \quad (2.4.3)$$

Then, since  $\xi$  is actually  $\mathcal{F}_T^B$ -measurable, we deduce immediately, using the definition of the r.c.p.d. that

$$\mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} [\xi | \mathcal{G}_t] (\omega^B, \omega^W) = \mathbb{E}^{\mathbb{P}_B} [\xi | \mathcal{F}_t^B] (\omega^B) = \mathbb{E}^{\mathbb{P}_B^t, \omega^B} [\xi^{t, \omega^B}].$$

Next, we know from the results of Stricker and Yor [132] that there we can define a measurable map from  $(\Omega^W \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]))$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  which coincides  $\mathbb{P}_0^W \otimes dt$ -a.e. with the conditional expectation of  $g_s$ , under  $\mathbb{P}^B$ , with respect to the  $\sigma$ -algebra  $\mathcal{F}_t^B$ . For notational simplicity, we still denote this map as

$$(\omega^W, s) \longmapsto \mathbb{E}^{\mathbb{P}_B} [g_s(\cdot, \omega^W) | \mathcal{F}_t^B].$$

In other words, the above map does indeed define a stochastic process. That being said, we claim that for  $\mathbb{P}_B$ -a.e.  $\omega^B \in \Omega^B$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} \left[ \int_t^T g_s d\overleftarrow{W}_s \middle| \mathcal{G}_t \right] (\omega^B, \omega^W) &= \left( \int_t^T \mathbb{E}^{\mathbb{P}_B} [g_s | \mathcal{F}_t^B] (\omega^B, \cdot) d\overleftarrow{W}_s \right) (\omega^W) \\ &= \left( \int_t^T \mathbb{E}^{\mathbb{P}_B^{t, \omega^B}} [g_s^{t, \omega^B}] (\cdot) d\overleftarrow{W}_s \right) (\omega^W), \text{ for } \mathbb{P}_0^W - a.e. \omega^W \in \Omega^W. \end{aligned} \quad (2.4.4)$$

To prove the claim, let us first show it in the case where  $g$  is a simple process on  $(\Omega^W, \mathcal{F}_T^W)$  with the following decomposition

$$g_t(\omega^B, \omega^W) = \sum_{i=0}^{n-1} g_{t_i} \mathbf{1}_{(t_i, t_{i+1}]}$$

Then, we have by definition of backward stochastic integrals, for  $\mathbb{P}_B \otimes \mathbb{P}_0^W$ -a.e.  $(\omega^B, \omega^W) \in \Omega$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} \left[ \int_t^T g_s d\overleftarrow{W}_s \middle| \mathcal{G}_t \right] (\omega^B, \omega^W) &= \sum_{i=0}^{n-1} \mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} [g_{t_{i+1}} (W_{t_{i+1} \wedge t} - W_{t_i \wedge t}) | \mathcal{G}_t] (\omega^B, \omega^W) \\ &= \sum_{i=0}^{n-1} \mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} [g_{t_{i+1}} | \mathcal{G}_t] (\omega^B, \omega^W) (W_{t_{i+1} \wedge t} - W_{t_i \wedge t}) (\omega^W). \end{aligned}$$

Notice next that for  $\mathbb{P}_B \otimes \mathbb{P}_0^W$ -a.e.  $(\omega^B, \omega^W) \in \Omega$

$$\mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} [g_{t_{i+1}} | \mathcal{G}_t] (\omega^B, \omega^W) = \mathbb{E}^{\mathbb{P}_B} [g_{t_{i+1}} (\cdot, \omega^W) | \mathcal{F}_t^B] (\omega^B).$$

Indeed, for any  $X$  which is  $\mathcal{G}_t$ -measurable, we have

$$\begin{aligned} &\int_{\Omega} \mathbb{E}^{\mathbb{P}_B} [g_{t_{i+1}} (\cdot, \omega^W) | \mathcal{F}_t^B] (\omega^B) X(\omega^B, \omega^W) d\mathbb{P}_B(\omega^B) d\mathbb{P}_0^W(\omega^W) \\ &= \int_{\Omega^W} \left( \int_{\Omega^B} \mathbb{E}^{\mathbb{P}_B} [g_{t_{i+1}} (\cdot, \omega^W) | \mathcal{F}_t^B] (\omega^B) X(\omega^B, \omega^W) d\mathbb{P}_B(\omega^B) \right) d\mathbb{P}_0^W(\omega^W) \\ &= \int_{\Omega^W} \left( \int_{\Omega^B} g_{t_{i+1}} (\omega^B, \omega^W) X(\omega^B, \omega^W) (\omega^B) d\mathbb{P}_B(\omega^B) \right) d\mathbb{P}_0^W(\omega^W) \\ &= \int_{\Omega} g_{t_{i+1}} (\omega^B, \omega^W) X(\omega^B, \omega^W) (\omega^B) d(\mathbb{P}_B \otimes \mathbb{P}_0^W) (\omega^B, \omega^W), \end{aligned}$$

where we have used the fact that since for every  $\omega^W \in \Omega^W$ ,  $\omega^B \mapsto X(\omega^B, \omega^W)$  is  $\mathcal{F}_t^B$ -measurable, we have by definition of the conditional expectation that

$$\int_{\Omega^B} \mathbb{E}^{\mathbb{P}_B} [g_{t_{i+1}} (\cdot, \omega^W) | \mathcal{F}_t^B] (\omega^B) X(\omega^B, \omega^W) d\mathbb{P}_B(\omega^B) = \int_{\Omega^B} g_{t_{i+1}} (\omega^B, \omega^W) X(\omega^B, \omega^W) d\mathbb{P}_B(\omega^B).$$

Hence, we deduce finally that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} \left[ \int_t^T g_s d\overleftarrow{W}_s \middle| \mathcal{G}_t \right] (\omega^B, \omega^W) &= \sum_{i=0}^{n-1} \mathbb{E}^{\mathbb{P}_B} [g_{t_{i+1}} (\cdot, \omega^W) | \mathcal{F}_t^B] (\omega^B) (W_{t_{i+1} \wedge t} - W_{t_i \wedge t}) (\omega^W) \\ &= \left( \int_t^T \mathbb{E}^{\mathbb{P}_B} [g_s | \mathcal{F}_t^B] (\omega^B, \cdot) d\overleftarrow{W}_s \right) (\omega^W). \end{aligned}$$

By a simple density argument, we deduce that the same holds for general processes  $g$ . Next, notice that by definition of r.p.c.d., we have for any  $\omega^W \in \Omega^W$

$$\mathbb{E}^{\mathbb{P}_B} [g_s | \mathcal{F}_t^B] (\omega^B, \omega^W) = \mathbb{E}^{\mathbb{P}_B^{t, \omega^B}} [g_s^{t, \omega^B}] (\omega^W), \text{ for } \mathbb{P}_B - a.e. \omega^B \in \Omega^B.$$

Therefore, the  $\mathbb{P}_B$ -negligible set outside of which the above does hold true depends, *a priori*, on  $\omega^W$ . However, since  $g$  is uniformly continuous in  $\omega$ , it is easily checked that the map  $\omega^W \mapsto \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [g_s^{t,\omega^B}] (\omega^W)$  is actually uniformly continuous. Hence, it is possible to choose the same  $\mathbb{P}_B$ -negligible set for all  $\omega^W \in \Omega^W$ . This finally proves (2.4.4).

Using similar argument, we show that we also have for  $\mathbb{P}_B$ -a.e.  $\omega^B \in \Omega^B$

$$\mathbb{E}^{\mathbb{P}_B \otimes \mathbb{P}_0^W} \left[ \int_t^T \widehat{F}_s ds \middle| \mathcal{G}_t \right] (\omega^B, \omega^W) = \int_t^T \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [\widehat{F}_s^{t,\omega^B}] (\omega^W) ds, \text{ for } \mathbb{P}_0^W - a.e. \omega^W \in \Omega^W.$$

To sum up, we have obtained that for  $\mathbb{P}_B$ -a.e.  $\omega^B \in \Omega^B$

$$\begin{aligned} y_t^{\mathbb{P}_B \otimes \mathbb{P}_0^W} (\omega^B, \omega^W) &= \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [\xi^{t,\omega^B}] + \left( \int_t^T \mathbb{E}^{\mathbb{P}_B} [g_s | \mathcal{F}_t^B] (\omega^B, \cdot) d\overleftarrow{W}_s \right) (\omega^W) \\ &\quad + \int_t^T \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [\widehat{F}_s^{t,\omega^B}] (\omega^W) ds, \text{ for } \mathbb{P}_0^W - a.e. \omega^W \in \Omega^W. \end{aligned}$$

But, we also have (remember that by the Blumenthal 0–1 law  $y_t^{\mathbb{P}_B^{t,\omega^B} \otimes \mathbb{P}_0^W, t, \omega^B}$  only depends on  $\omega^W$ ) for any  $\omega^B \in \Omega^B$

$$\begin{aligned} y_t^{\mathbb{P}_B^{t,\omega^B} \otimes \mathbb{P}_0^W, t, \omega^B} (\omega^W) &= \mathbb{E}^{\mathbb{P}_B^{t,\omega^B} \otimes \mathbb{P}_0^W} \left[ \xi^{t,\omega^B} + \int_t^T \widehat{F}_s^{t,\omega^B} ds + \int_t^T g_s^{t,\omega^B} d\overleftarrow{W}_s \middle| \mathcal{G}_t \right] (\omega^W) \\ &= \mathbb{E}^{\mathbb{P}_B^{t,\omega^B} \otimes \mathbb{P}_0^W} \left[ \xi^{t,\omega^B} + \int_t^T \widehat{F}_s^{t,\omega^B} ds + \int_t^T g_s^{t,\omega^B} d\overleftarrow{W}_s \middle| \mathcal{F}_{t,T}^W \right] (\omega^W). \end{aligned}$$

Using the same arguments as above, we obtain

$$\begin{aligned} y_t^{\mathbb{P}_B^{t,\omega^B} \otimes \mathbb{P}_0^W, t, \omega^B} (\omega^W) &= \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [\xi^{t,\omega^B}] + \int_t^T \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [\widehat{F}_s^{t,\omega^B}] ds \\ &\quad + \left( \int_t^T \mathbb{E}^{\mathbb{P}_B^{t,\omega^B}} [g_s^{t,\omega^B}] d\overleftarrow{W}_s \right) (\omega^W), \end{aligned}$$

which proves the desired result.

*Step 2:* Since we are in a Lipschitz setting, solutions to BDSDEs can be constructed via Picard iterations. Hence, using Step 1, the results holds at each step of the iteration and therefore also when passing to the limit.  $\square$

We point out that for classical 2BSDEs, Soner, Touzi and Zhang have proved in Lemma 4.6 [129] a regularity result for the value process, precisely the uniform continuity with respect to the trajectory  $\omega^B$  and this is crucial to prove their dynamic programming principle (Proposition 4.7 in [129]).

Since in our context, the value process  $V$  defined in (2.4.1) is a random field depending on two source of randomness, we prove the following regularity result which is weaker than Lemma 4.6 [129].

**Lemma 2.4.2.**

$$\mathbb{E}^{\mathbb{P}_0^W} \left[ (V_t(\omega^{B,1,\cdot}) - V_t(\omega^{B,2,\cdot}))^2 \right] \leq \rho^2 (\|\omega^{B,1} - \omega^{B,2}\|_t).$$

*In particular, this implies that the map  $\omega^B \mapsto V_t(\omega^B, \cdot)$  is uniformly continuous in probability (with respect to  $\mathbb{P}_0^W$ ), which implies that there is a  $\mathbb{P}_0^W$ -version, which we still denote  $V$  for simplicity such that  $V_t$  is  $\mathcal{F}_t$ -measurable.*

**Proof.** The estimate is an easy consequence of classical *a priori* estimates for BDSDEs, using in particular the uniform continuity in  $\omega$  of both  $F$  and  $g$ . As for the existence of measurable version, this is a classical result (see for instance Dellacherie and Meyer [37]).  $\square$

Now, we present the main result concerning the dynamic programming principle in our context. We follow the new approach (work in progress) of Possamai, Xiaolu and Zhou [119] where they proved existence result for 2BSDEs with only measurable parameters. Their proof is based on dynamic programming principle without regularity on the terminal condition and the generator.

We are now in position to show the following dynamic programming principle in our context as in [119].

**Theorem 2.4.1.** *Under the Assumptions 2.2.1, 2.2.2 and for  $\xi \in UC_b(\Omega)$ , we have for all  $0 \leq t \leq t_1 \leq t_2 \leq T$  and for all  $\omega = (\omega^B, \omega^W) \in \Omega$*

$$V_{t_1}(\omega^B, \cdot) = \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P}, t_1, \omega^B}(t_2, V_{t_2}^{t_1, \omega^B}(\cdot)), \quad \mathbb{P}_0^W - a.e. \omega^W \in \Omega^W \quad (2.4.5)$$

**Proof.** For simplicity we will omit the dependence of  $\mathcal{Y}$  with respect to  $(t, \omega^B)$  thanks to Lemma 2.4.1. Then, the dynamic programming principle (DPP in short) is a direct consequence of the comparison principle, once we have the measurability result given in Lemma 2.4.2. First, for every  $\mathbb{P} \in \mathcal{P}^t$ , we have

$$\mathcal{Y}_{t_1}^{\mathbb{P}}(T, \xi) = \mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, \mathcal{Y}_{t_2}^{\mathbb{P}}), \quad \mathbb{P}_0^W - a.s.$$

It follows by the comparison principle that

$$V_{t_1}(\omega) := \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P}}(T, \xi) = \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, \mathcal{Y}_{t_2}^{\mathbb{P}}) \leq \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2}^{\mathbb{P}}).$$

Next, for every  $\varepsilon > 0$ , by the measurable selection theorem together with concatenation technique, we can construct, for every  $\mathbb{P} \in \mathcal{P}^{t_1}$ , a new probability  $\mathbb{Q}_\varepsilon \in \mathcal{P}^{t_1}$ , such that

$$\mathcal{Y}_{t_1}^{\mathbb{Q}_\varepsilon}(t_2, \mathcal{Y}_{t_2}^{\mathbb{Q}_\varepsilon}) \geq \mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2}) - \varepsilon.$$

It follows that

$$V_{t_1}(\omega) = \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{Q}}(t_2, \mathcal{Y}_{t_2}^{\mathbb{Q}}) \geq \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2}) - \varepsilon.$$

And hence the other inequality of the DPP holds true by the arbitrariness of  $\varepsilon > 0$ .  $\square$

Next, we introduce the right limit of the  $V$  which is clearly  $\mathcal{F}_t$ -measurable:

$$V_t^+ := \overline{\lim}_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} V_r. \quad (2.4.6)$$

**Lemma 2.4.3.** *Under the Assumptions 2.2.1, 2.2.2, we have*

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} V_r, \quad \mathcal{P}^t - q.s.$$

and thus  $V^+$  is càdlàg  $\mathcal{P}^t - q.s.$

**Proof.** For each  $\mathbb{P} \in \mathcal{P}^t$ , let  $(\bar{\mathcal{Y}}^{\mathbb{P}}(T, \xi), \bar{\mathcal{Z}}^{\mathbb{P}}(T, \xi))$  be the solution of the BDSDE with generators  $\hat{F}$  and  $g$ , and terminal condition  $\xi$  at time  $T$ . We define  $\tilde{V}^{\mathbb{P}} := V - \bar{\mathcal{Y}}^{\mathbb{P}}(T, \xi)$ . Then,  $\tilde{V}^{\mathbb{P}} \geq 0, \mathbb{P} - a.s.$  For any  $0 \leq t_1 \leq t_2 \leq T$ , let  $(y^{\mathbb{P}, t_2}, z^{\mathbb{P}, t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2, V_{t_2}))$ .

Note that

$$\mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2})(\omega) = \mathcal{Y}_{t_1}^{\mathbb{P}, t_1, \omega}(t_2, V_{t_2}^{t_1, \omega}), \quad \mathbb{P} - a.s.$$

Then by the dynamic programming principle (Theorem 2.4.1) we get

$$V_{t_1} \geq y_{t_1}^{\mathbb{P}, t_2}, \mathbb{P} - a.s.$$

Denote  $\tilde{y}_t^{\mathbb{P},t_2} := y_t^{\mathbb{P},t_2} - \bar{Y}_t^{\mathbb{P}}$ ,  $\tilde{z}_t^{\mathbb{P},t_2} := \hat{a}_t^{-1/2}(z_t^{\mathbb{P},t_2} - \bar{Z}_t^{\mathbb{P}})$ . Then  $(\tilde{y}^{\mathbb{P},t_2}, \tilde{z}^{\mathbb{P},t_2})$  is solution of the following BDSDE on  $[0, t_2]$

$$\tilde{y}_t^{\mathbb{P},t_2} = \tilde{V}_t^{\mathbb{P}} + \int_t^{t_2} f_s^{\mathbb{P}}(\tilde{y}_s^{\mathbb{P},t_2}, \tilde{z}_s^{\mathbb{P},t_2}) ds + \int_t^{t_2} \hat{g}_s^{\mathbb{P}}(\tilde{y}_s^{\mathbb{P},t_2}, \tilde{z}_s^{\mathbb{P},t_2}) d\overleftarrow{W}_s - \int_t^{t_2} \tilde{z}_s^{\mathbb{P},t_2} dX_s^{\hat{a}},$$

where

$$\begin{aligned} f_t^{\mathbb{P}}(\omega, y, z) &:= \hat{F}_t(\omega, y + \bar{Y}_t^{\mathbb{P}}(\omega), \hat{a}^{1/2}(\omega)z + \bar{Z}_t^{\mathbb{P}}(\omega)) - \hat{F}_t(\omega, \bar{Y}_t^{\mathbb{P}}(\omega), \bar{Z}_t^{\mathbb{P}}(\omega)) \\ \hat{g}_t^{\mathbb{P}}(\omega, y, z) &:= g_t(\omega, y + \bar{Y}_t^{\mathbb{P}}(\omega), \hat{a}^{1/2}(\omega)z + \bar{Z}_t^{\mathbb{P}}(\omega)) - g_t(\omega, \bar{Y}_t^{\mathbb{P}}(\omega), \bar{Z}_t^{\mathbb{P}}(\omega)). \end{aligned}$$

Then  $\tilde{V}_{t_1}^{\mathbb{P}} \geq \tilde{y}_{t_1}^{\mathbb{P},t_2}$ . Therefore,  $\tilde{V}^{\mathbb{P}}$  is a positive weak doubly  $f^{\mathbb{P}}$ -supermartingale under  $\mathbb{P}$  by Definition 2.6.3 (given in the Appendix). Now applying the downcrossing inequality Theorem 2.6.1, one can easily see that  $\mathbb{P}$ -a.e.  $\omega$ , the limit  $\lim_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} \tilde{V}_r^{\mathbb{P}}$  exists for all  $t \in [0, T]$ . Note that  $y^{\mathbb{P}}$  is continuous,  $\mathbb{P}$ -a.s. We get that the  $\overline{\lim}$  in the definition of  $V^+$  is in fact the  $\lim$   $\mathbb{P}$ -a.s. Then,

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} V_r, \quad \mathcal{P}^t - q.s.$$

and therefore  $V^+$  is càdlàg  $\mathcal{P}^t - q.s.$   $\square$

Thanks to the dynamic programming principle for  $V$  and regularity results for the value process  $V^+$ , we have the following decomposition for  $V^+$ .

**Proposition 2.4.1.** *Under the Assumptions 2.2.1, 2.2.2, the process  $V^+$  defined by (2.4.6) verify for all  $0 \leq t \leq s \leq T$ , the following decomposition*

$$V_s^+ = \xi + \int_s^T \hat{F}_r(V_r^+, \tilde{Z}_r^{\mathbb{P}}) ds + \int_s^T g_r(V_r^+, \tilde{Z}_r^{\mathbb{P}}) d\overleftarrow{W}_r - \int_s^T \tilde{Z}_r^{\mathbb{P}} dB_r + \tilde{K}_T^{\mathbb{P}} - \tilde{K}_s^{\mathbb{P}}, \quad \mathbb{P} - a.s.$$

**Proof.** We introduce first the following RBDSDE with lower obstacle  $V^+$  under each  $\mathbb{P} \in \mathcal{P}^t$ ,

$$\begin{cases} \tilde{Y}_t^{\mathbb{P}} = \xi + \int_t^T \hat{F}_s(\tilde{Y}_s^{\mathbb{P}}, \tilde{Z}_s^{\mathbb{P}}) ds + \int_t^T g_s(\tilde{Y}_s^{\mathbb{P}}, \tilde{Z}_s^{\mathbb{P}}) d\overleftarrow{W}_s - \int_t^T \tilde{Z}_s^{\mathbb{P}} dB_s + \tilde{K}_T^{\mathbb{P}} - \tilde{K}_t^{\mathbb{P}} \\ \tilde{Y}_t^{\mathbb{P}} \geq V_t^+, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s. \\ \int_0^T (\tilde{Y}_{s^-}^{\mathbb{P}} - V_{s^-}^+) d\tilde{K}_{s^-}^{\mathbb{P}} = 0, \quad \mathbb{P} - a.s. \end{cases}$$

At the best of knowledge, there are no results in the literature for the existence and uniqueness of such RBDSDE with càdlàg obstacle. The proofs of these results are postponed to section 2.6.3 in the Appendix for completeness.

As mentioned in Remark 4.9 in [129], and for a fixed  $\mathbb{P} \in \mathcal{P}^t$ , we shall use the solution of the above RBDSDEs and the notion of  $\hat{F}$ -weak doubly supermartingale whis is introduced in the Appendix. This notion is a natural extension of nonlinear  $f$ -supermartingale introduced first by Peng [113] in the context of standard BSDEs.

For this end, we argue by contradiction and suppose that  $\tilde{Y}^{\mathbb{P}}$  is not equal  $\mathbb{P} - a.s.$  to  $V^+$ . Then we can assume without loss of generality that  $\tilde{Y}_0^{\mathbb{P}} > V_0^+$ ,  $\mathbb{P} - a.s.$  For each  $\varepsilon > 0$ , define the following  $\overline{\mathbb{G}}$ -stopping time :

$$\tau^\varepsilon := \inf\{t \geq 0, \tilde{Y}_t^{\mathbb{P}} \leq V_t^+ + \varepsilon\}.$$

Then  $\tilde{Y}^{\mathbb{P}}$  is strictly above the obstacle before  $\tau^\varepsilon$ , and therefore  $\tilde{K}^{\mathbb{P}}$  is identically equal to 0 in  $[0, \tau^\varepsilon]$ . Hence, we have for all  $0 \leq t \leq s \leq T$

$$\tilde{Y}_s^{\mathbb{P}} = \tilde{Y}_{\tau^\varepsilon}^{\mathbb{P}} + \int_s^{\tau^\varepsilon} \hat{F}_r(\tilde{Y}_r^{\mathbb{P}}, \tilde{Z}_r^{\mathbb{P}}) dr + \int_s^{\tau^\varepsilon} g_r(\tilde{Y}_r^{\mathbb{P}}, \tilde{Z}_r^{\mathbb{P}}) d\overleftarrow{W}_r - \int_s^{\tau^\varepsilon} \tilde{Z}_r^{\mathbb{P}} dB_r, \quad \mathbb{P} - a.s.$$

Let us now define the following BDSDE on  $[0, \tau^\varepsilon]$

$$y_s^{+, \mathbb{P}} = V_{\tau^\varepsilon}^+ + \int_s^{\tau^\varepsilon} \hat{F}_r(y_r^{+, \mathbb{P}}, z_r^{+, \mathbb{P}}) dr + \int_s^{\tau^\varepsilon} g_r(y_r^{+, \mathbb{P}}, z_r^{+, \mathbb{P}}) d\overleftarrow{W}_r - \int_s^{\tau^\varepsilon} z_r^{+, \mathbb{P}} dB_r, \quad \mathbb{P} - a.s.$$

By comparison theorem and the standard a priori estimates, we obtain that

$$\mathbb{E}[\tilde{Y}_0^{\mathbb{P}}] \leq \mathbb{E}[y_0^{+, \mathbb{P}}] + C\mathbb{E}[|V_{\tau^\varepsilon}^+ - \tilde{Y}_{\tau^\varepsilon}^{\mathbb{P}}|] \leq \mathbb{E}[y_0^{+, \mathbb{P}}] + C\varepsilon,$$

by definition of  $\tau^\varepsilon$ .

Moreover, we can show similarly to the proof of Lemma 2.4.3 (see also the arguments in Step 1 of the proof of Theorem 4.5 in [129] page 328-329) that  $V^+$  is a strong  $\hat{F}$  doubly supermartingale under each  $\mathbb{P} \in \mathcal{P}^t$ . Thus, we obtain particularly that  $y_0^{+, \mathbb{P}} \leq V_0^+$  which in turn implies

$$\mathbb{E}[\tilde{Y}_0^{\mathbb{P}}] \leq \mathbb{E}[V_0^+] + C\varepsilon,$$

hence a contradiction by arbitrariness of  $\varepsilon$ . Therefore, we have obtained the following decomposition

$$V_s^+ = \xi + \int_s^T \hat{F}_r(V_r^+, \tilde{Z}_r^{\mathbb{P}}) dr + \int_s^T g_r(V_r^+, \tilde{Z}_r^{\mathbb{P}}) d\overleftarrow{W}_r - \int_s^T \tilde{Z}_r^{\mathbb{P}} dB_r + \tilde{K}_T^{\mathbb{P}} - \tilde{K}_s^{\mathbb{P}}, \quad \mathbb{P} - a.s., \forall \mathbb{P} \in \mathcal{P}^t$$

□

We next prove a representation for  $V^+$  similar to (2.3.2).

**Proposition 2.4.2.** *Assume that Assumptions 2.2.1, 2.2.2 hold. Then we have*

$$V_t^+ = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_t^{\mathbb{P}'}(T, \xi), \quad \mathbb{P} - a.s., \forall \mathbb{P} \in \mathcal{P}^t \quad (2.4.7)$$

**Proof.** Fix  $\mathbb{P} \in \mathcal{P}^t$ . Denote

$$V_t^{\mathbb{P}, +} = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_t^{\mathbb{P}'}(T, \xi)$$

We want to prove the equality  $V_t^{\mathbb{P}, +} = V_t^+, \mathbb{P} - a.s.$ . First, for each  $\mathbb{P}' \in \mathcal{P}(t, \mathbb{P}) \subset \mathcal{P}^t$  and  $r \in \mathbb{Q} \cap (t, T]$ , we have  $y_r^{\mathbb{P}'} \leq V_r, \mathbb{P}' - a.s.$ . Sending  $r \downarrow t$ , we obtain  $y_t^{\mathbb{P}'} \leq V_t^+, \mathbb{P}' - a.s.$  Since both  $y_t^{\mathbb{P}'}$  and  $V_t^+$  are  $\mathcal{G}_t^+$ -measurable and  $\mathbb{P}' = \mathbb{P}$  on  $\mathcal{G}_t^+$ , then  $y_t^{\mathbb{P}'} \leq V_t^+, \mathbb{P} - a.s.$  and, thus,  $V_t^{\mathbb{P}, +} \leq V_t^+, \mathbb{P} - a.s.$

On the other hand, for each  $r \in \mathbb{Q} \cap (t, T]$ , since  $V_r = V_r^{\mathbb{P}}, \mathbb{P} - a.s.$  and following the same arguments in [128] Theorem 4.3, Step (iii), we have

$$\text{there exist } \mathbb{P}_n \in \mathcal{P}(r, \mathbb{P}) \text{ such that } \mathcal{Y}_r^{\mathbb{P}_n}(T, \xi) \uparrow V_r, \quad \mathbb{P} - a.s.$$

Then, it follows from the stability of BDSDE that

$$\mathcal{Y}_t^{\mathbb{P}}(r, V_r) = \mathcal{Y}_t^{\mathbb{P}}(r, \lim_{n \rightarrow \infty} \mathcal{Y}_r^{\mathbb{P}_n}(T, \xi)) = \lim_{n \rightarrow \infty} \mathcal{Y}_t^{\mathbb{P}}(r, \mathcal{Y}_r^{\mathbb{P}_n}(T, \xi)).$$

Since  $\mathbb{P}_n \in \mathcal{P}(r, \mathbb{P}) \subset \mathcal{P}(t+, \mathbb{P})$ , we have

$$\mathcal{Y}_t^{\mathbb{P}}(r, V_r) = \lim_{n \rightarrow \infty} \mathcal{Y}_t^{\mathbb{P}}(r, \mathcal{Y}_r^{\mathbb{P}_n}(T, \xi)) = \lim_{n \rightarrow \infty} \mathcal{Y}_t^{\mathbb{P}_n}(r, \mathcal{Y}_r^{\mathbb{P}_n}(T, \xi)) \leq V_t^{\mathbb{P}, +}, \quad \mathbb{P} - a.s.$$

Sending  $r \downarrow t$ , by the stability of BDSDEs again we obtain  $V_t^+ \leq V_t^{\mathbb{P}, +}, \mathbb{P} - a.s.$  □



### 2.4.3 Existence result in the general case

We are now in position to state the main result of this section.

**Theorem 2.4.2.** *Let  $\xi \in \mathcal{L}^2$  and assume that Assumptions 2.2.1, 2.2.2 hold. Then:*

1. *There exists a unique solution  $(Y, Z) \in \mathbb{D}^2 \times \mathbb{H}^2$  of the 2BDSDE (2.2.4).*
2. *Moreover, if in addition we choose to work under either of the following model of the theory*
  - (i) *Zermelo-Fraenkel set theory with axiom of choice (ZFC) plus the Continuum Hypothesis (CH).*
  - (ii) *ZFC plus the negation of CH plus Martin's axiom.*

*Then there exists a unique solution  $(Y, Z, K) \in \mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$  of the 2BDSDE (2.2.4).*

**Proof.** The proof is divided in three steps. In the first one we prove that the value process  $V^+$  defined by (2.4.6) is the solution of our 2BDSDE in the case when  $\xi$  belongs in  $UC_b(\Omega)$  and show the aggregation result for the solution. Then, in the second step we verify the minimality condition for the increasing process. Finally, we deal with the general case.

*Step 1: Existence and aggregation results for  $\xi$  belongs in  $UC_b(\Omega)$*

As we have mentioned above, the natural candidate for the  $Y$  solution for our 2BDSDE is given by  $Y_t = V_t^+ := \lim_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} V_r$ , where  $V$  is the value process defined by (2.4.1). First, we know that  $V^+$  is a càdlàg process defined pathwise and using the same notations in Proposition 2.4.1 our solution  $Y$  verifies

$$V_t^+ = V_0^+ - \int_0^t \widehat{F}_s(V_s^+, \widetilde{Z}_s^{\mathbb{P}}) ds - \int_0^t g_s(V_s^+, \widetilde{Z}_s^{\mathbb{P}}) d\overleftarrow{W}_s + \int_0^t \widetilde{Z}_s^{\mathbb{P}} dB_s - \widetilde{K}_t^{\mathbb{P}}, \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}^t.$$

We note that  $V^+$  is a càdlàg generalized semimartingale (studied by Pardoux and Protter in [112] and Pardoux and Peng [109]). Therefore, we can adapt Karandikar's results obtained for càdlàg semimartingale in our context to define a universal process  $\widetilde{Z}$  which aggregates the family  $\{\widetilde{Z}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}^t\}$ .

Concerning the fact that we can aggregate the family  $(\widetilde{K}^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}^t}$ , it can be deduced as follows. We have from (2.4.6) that  $V^+$  is defined pathwise, and so is the Lebesgue integral  $\int_0^t \widehat{F}_s(V_s^+, \widetilde{Z}_s) ds$ . In order to give a pathwise definition of the two stochastic integrals, we would like to use the recent results of Nutz [105]. However, the proof in this paper relies on the notion of medial limits, which may or may not exist depending on the model of set theory chosen. They exists in the model (i) above, which is the one considered by Nutz, but we know from [51] (see statement 22O(1) page 55) that they also do in the model (ii). Therefore, provided we work under either one of these models, the stochastic integrals  $\int_0^t \widetilde{Z}_s dB_s$  and  $\int_0^t g_s(V_s^+, \widetilde{Z}_s^{\mathbb{P}}) d\overleftarrow{W}_s$  can also be defined pathwise. We can therefore define pathwise

$$\widetilde{K}_t := V_0^+ - V_t^+ - \int_0^t \widehat{F}_s(V_s^+, \widetilde{Z}_s) ds - \int_0^t g_s(V_s^+, \widetilde{Z}_s^{\mathbb{P}}) d\overleftarrow{W}_s + \int_0^t \widetilde{Z}_s dB_s,$$

and  $\widetilde{K}$  is an aggregator for the family  $(\widetilde{K}^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}^t}$ , that is to say that it coincides  $\mathbb{P} - a.s.$  with  $\widetilde{K}^{\mathbb{P}}$ , for every  $\mathbb{P} \in \mathcal{P}^t$ . Thus, the triplet  $(Y, \widetilde{Z}, \widetilde{K})$  satisfy the equation (2.2.4) and from the a priori estimates in Theorem 2.3.1 we get that  $(Y, \widetilde{Z}, \widetilde{K})$  belongs to  $\mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$ . It remains to prove the minimality condition (2.5.7) for the increasing process  $\widetilde{K}$ .

*Step 2: The minimality condition of  $\tilde{K}^{\mathbb{P}}$*

Now, we have to check that the minimum condition (2.2.7) holds. We follow the arguments in the proof of Theorem 2.3.1. For  $t \in [0, T]$ ,  $\mathbb{P} \in \mathcal{P}^t$  and  $\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})$ , we denote  $\delta Y := V^+ - y^{\mathbb{P}'}(T, \xi)$  and  $\delta Z := \tilde{Z} - z^{\mathbb{P}'}(T, \xi)$  and we introduce the process  $M$  of (2.3.6). We first observe that the non decrease of  $\tilde{K}^{\mathbb{P}'}$  implies that

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [\tilde{K}_T^{\mathbb{P}'} - \tilde{K}_t^{\mathbb{P}'}] \geq 0.$$

Then, it suffices to prove that  $\mathbb{E}^{\mathbb{P}} \left[ \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [\tilde{K}_T^{\mathbb{P}'} - \tilde{K}_t^{\mathbb{P}'}] \right] \leq 0$ .

We know that the family  $\mathcal{P}(t+, \mathbb{P})$  is upward directed. Therefore, by classical results, there is a sequence  $(\mathbb{P}^n)_{n \geq 0} \subset \mathcal{P}(t+, \mathbb{P})$  such that

$$\mathbb{E}^{\mathbb{P}} \left[ \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [\tilde{K}_T^{\mathbb{P}'} - \tilde{K}_t^{\mathbb{P}'}] \right] = \lim_{n \rightarrow +\infty} \downarrow \mathbb{E}^{\mathbb{P}^n} [\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n}]. \quad (2.4.8)$$

On the other hand, by (2.3.8), we estimate by the Hölder inequality that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^n} [\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n}] \\ &= \mathbb{E}^{\mathbb{P}^n} \left[ \left( \inf_{t \leq s \leq T} (M_t^{-1} M_s) \right)^{1/3} (\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n})^{1/3} \left( \inf_{t \leq s \leq T} (M_t^{-1} M_s) \right)^{-1/3} (\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n})^{2/3} \right] \\ &\leq \left( \mathbb{E}^{\mathbb{P}^n} \left[ \left( \inf_{t \leq s \leq T} (M_t^{-1} M_s) \right) (\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n}) \right] \mathbb{E}^{\mathbb{P}^n} \left[ \sup_{t \leq s \leq T} (M_t^{-1} M_s) \right] \mathbb{E}^{\mathbb{P}^n} [(\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n})^2] \right)^{1/3} \\ &\leq C \left( \mathbb{E}^{\mathbb{P}^n} [(\tilde{K}_T^{\mathbb{P}^n})^2] \mathbb{E}^{\mathbb{P}^n} \left[ \left( \inf_{t \leq s \leq T} (M_t^{-1} M_s) \right) (\tilde{K}_T^{\mathbb{P}^n} - \tilde{K}_t^{\mathbb{P}^n}) \right] \right)^{1/3} \\ &\leq C \left( \mathbb{E}^{\mathbb{P}^n} [(\tilde{K}_T^{\mathbb{P}^n})^2] \mathbb{E}^{\mathbb{P}^n} \left[ M_t^{-1} \int_t^T M_s d\tilde{K}_s^{\mathbb{P}^n} \right] \right)^{1/3} \\ &\leq C \left( \mathbb{E}^{\mathbb{P}^n} [(\tilde{K}_T^{\mathbb{P}^n})^2] \right)^{1/3} (\mathbb{E}^{\mathbb{P}^n} [\delta Y_t])^{1/3}. \end{aligned} \quad (2.4.9)$$

where we have used in the last inequality the non decrease of  $\tilde{K}^{\mathbb{P}^n}$  and argument of the proof of Theorem 2.3.1 (ii). Plugging (2.4.9) in (2.4.8), we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[ \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [\tilde{K}_T^{\mathbb{P}'} - \tilde{K}_t^{\mathbb{P}'}] \right] &\leq C \lim_{n \rightarrow +\infty} \downarrow (\mathbb{E}^{\mathbb{P}^n} [\delta Y_t])^{1/3} \\ &\leq C \left( \operatorname{ess\,inf}_{\mathbb{P}^n \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}^{\mathbb{P}^n} [\delta Y_t] \right)^{1/3} = 0. \end{aligned}$$

which is the desired result.

*Step 3: Existence and aggregation results for  $\xi$  in  $\mathcal{L}^2$*

For  $\xi \in \mathcal{L}^2$ , there exists by definition a sequence  $(\xi_n)_{n \geq 0} \subset UC_b(\Omega)$  such that

$$\lim_{n \rightarrow +\infty} \|\xi_n - \xi_m\|_{L^2} = 0 \quad \text{and} \quad \sup_{n \geq 0} \|\xi_n\|_{L^2} < +\infty.$$

Let  $(Y^n, Z^n) \in \mathbb{D}^2 \times \mathbb{H}^2$  be the solution to 2BDSDE (2.2.4) with terminal condition  $\xi_n$ , and

$$K_t^{n, \mathbb{P}} := Y_0^n - Y_t^n - \int_0^t \hat{F}(Y_s^n, Z_s^n) ds - \int_0^t g(Y_s^n, Z_s^n) d\tilde{W}_s + \int_0^t Z_s^n dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

By the estimates of Theorem 2.3.3, we have

$$\begin{aligned} & \|Y^n - Y^m\|_{\mathbb{D}^2}^2 + \|Z^n - Z^m\|_{\mathbb{H}^2}^2 + \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |K_t^{n, \mathbb{P}} - K_t^{m, \mathbb{P}}|^2 \right] \\ &\leq C \|\xi_n - \xi_m\|_{\mathbb{L}^2}^2 + C \|\xi_n - \xi_m\|_{\mathbb{L}^2} (\|\xi_n\|_{\mathbb{L}^2} + \|\xi_m\|_{\mathbb{L}^2} + (\phi^2)^{1/2} + (\psi^2)^{1/2}) \xrightarrow{n, m \rightarrow +\infty} 0. \end{aligned}$$

Extracting a subsequence if necessary, we may assume that

$$\|Y^n - Y^m\|_{\mathbb{D}^2}^2 + \|Z^n - Z^m\|_{\mathbb{H}^2}^2 + \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |K_t^{n,\mathbb{P}} - K_t^{m,\mathbb{P}}|^2 \right] \leq 2^{-n} \quad (2.4.10)$$

for all  $m \geq n \geq 0$ . This implies by Markov inequality that for all  $\mathbb{P}$  and all  $m \geq n \geq 0$

$$\mathbb{P} \left[ \sup_{0 \leq t \leq T} [|Y_t^n - Y_t^m|^2 + |K_t^{n,\mathbb{P}} - K_t^{m,\mathbb{P}}|^2] + \int_0^T |Z_t^n - Z_t^m|^2 dt > \frac{1}{n} \right] \leq Cn2^{-n}. \quad (2.4.11)$$

Define

$$Y := \overline{\lim}_{n \rightarrow \infty} Y^n, \quad Z := \overline{\lim}_{n \rightarrow \infty} Z^n, \quad K^{\mathbb{P}} := \overline{\lim}_{n \rightarrow \infty} K^{n,\mathbb{P}}, \quad (2.4.12)$$

where the  $\overline{\lim}$  for  $Z$  is taken componentwise. It is clear that  $Y, Z, K^{\mathbb{P}}$  are all  $\mathbb{F}^+$ -progressively measurable. By (2.4.11), it follows from Borel Cantelli Lemma that for all  $\mathbb{P}$  we have  $\mathbb{P} - a.s.$

$$\lim_{n \rightarrow \infty} \left[ \sup_{0 \leq t \leq T} [|Y_t^n - Y_t^m|^2 + |K_t^{n,\mathbb{P}} - K_t^{m,\mathbb{P}}|^2] + \int_0^T |Z_t^n - Z_t^m|^2 dt \right] = 0.$$

It follows that  $Y$  is càdlàg,  $\mathcal{P}$ -q.s., and that  $K^{\mathbb{P}}$  is càdlàg nondecreasing process,  $\mathbb{P} - a.s.$  Furthermore, for all  $\mathbb{P}$ , sending  $m$  to infinity in (2.4.10) and applying Fatou's lemma under  $\mathbb{P}$  gives us that  $(Y, Z, K) \in \mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$ .  $\square$

## 2.5 Probabilistic interpretation for Fully nonlinear SPDEs

Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a lebesgue measurable function. In this section we denote, for any  $(t, x) \in [0, T] \times \mathbb{R}^d$

$$B_s^{t,x} := x + B_s^t \quad \text{for all } s \in [t, T],$$

where  $(B_s^t)_{s \in [t, T]}$  is the shifted canonical process on  $\Omega_B^t$ .

Let us first define the following functional spaces:

- $\mathcal{M}_{0,T}^W$  denotes all the  $\mathbb{F}^W$ -stopping times  $\tau$  such that  $0 \leq \tau \leq T$ .
- $L^p(\mathcal{F}_{\tau,T}^W; \mathbb{R}^d)$ , for  $p \geq 1$ , denotes the space of all  $\mathbb{R}^d$ -valued  $\mathcal{F}_{\tau,T}^W$  measurable r.v.  $\xi$  such that  $\mathbb{E}[|\xi|^p] < +\infty$ .
- $C^{l,k}([0, T] \times \mathbb{R}^d)$ , for  $k, l \geq 0$ , denotes the space of all  $\mathbb{R}$ -valued functions defined on  $[0, T] \times \mathbb{R}^d$  which are  $k$ -times continuously differentiable in  $t$  and  $l$ -times continuously differentiable in  $x$ .
- $C^{l,k}(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$ , for  $k, l \geq 0$ , denotes the space of all  $C^{l,k}([0, T] \times \mathbb{R}^d)$ -valued random variables  $\varphi$  that are  $\mathcal{F}_{t,T}^W \otimes \mathcal{B}([0, T] \times \mathbb{R}^d)$  measurable.
- $C^{l,k}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$ , for  $k, l \geq 0$ , denotes the space of random variables  $\varphi \in C^{l,k}(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$  such that for fixed  $x \in \mathbb{R}^d$  the mapping  $(t, \omega) \mapsto \varphi(t, x, \omega)$  is  $\mathbb{F}^W$ -progressively measurable.

Furthermore, for  $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ , we denote  $\partial/\partial y = D_y, \partial/\partial t = D_t, D = D_x = (\partial/\partial x_1, \dots, \partial/\partial x_d)$ , and  $D^2 = D_{xx} = (\partial_{x_i x_j}^2)_{i,j=1}^d$ . The meaning of  $D_{xy}, D_{yy}$ , etc should be clear. The following is a slight strengthening of Assumption 2.2.1

### Assumption 2.5.1.

- (i)  $\mathcal{P}^t$  is not empty, the domain  $D_{f_t(y,z)} = D_{f_t}$  is independent of  $(w, y, z)$ .

(ii) There exist constants  $C > 0$ ,  $0 \leq \alpha < 1$  and modulus of continuity  $\rho$  with polynomial growth such that for all  $t \in [0, T]$ ,  $a \in D_{f_t}$ ,  $x, x', z, z' \in \mathbb{R}^d$ ,  $y, y' \in \mathbb{R}$ .

$$|f_t(x, y, z, a) - f_t(x, y', z', a)| \leq \rho(|x - x'|) + C(|y - y'| + \|a^{1/2}(z - z')\|),$$

$$\|g_t(x, y, z) - g_t(x, y', z')\|^2 \leq \rho(|x - x'|) + C|y - y'|^2 + \alpha\|(z - z')\|^2.$$

(iii) The function  $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^l)$ .

We consider the 2BDSDE (2.2.4) in this Markovian setting with terminal condition  $\xi = \phi(B_T^{t,x})$ :

$$\begin{aligned} Y_s^{t,x} &= \phi(B_T^{t,x}) - \int_s^T f(s, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \hat{a}_r) dr + \int_t^T g(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \circ d\overleftarrow{W}_r \\ &\quad - \int_s^T Z_r^{t,x} dB_r + K_T^{t,x} - K_s^{t,x}, \quad t \leq s \leq T, \quad \mathcal{P}^t - q.s \end{aligned} \quad (2.5.1)$$

We remark that the stochastic integral with respect to  $dW$  is the Stratonovich backward integral (see Kunita [78] page 194). Since  $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^l)$ , using the definition of the Stratonovich backward integral, we show easily that (2.5.1) is equivalent to the following 2BDSDE:

$$\begin{aligned} Y_s^{t,x} &= \phi(B_T^{t,x}) - \int_s^T \hat{f}(s, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \hat{a}_r) dr + \int_t^T g(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r \\ &\quad - \int_s^T Z_r^{t,x} dB_r + K_T^{t,x} - K_s^{t,x}, \quad t \leq s \leq T, \quad \mathcal{P}^t - q.s \end{aligned} \quad (2.5.2)$$

where  $\hat{f}(s, x, y, z, \hat{a}_s) = f(s, x, y, z, \hat{a}_s) + \frac{1}{2} \text{Tr}(g(s, x, y, z) D_y g(s, x, y, z))$ .

Our main objective is to establish the connection  $Y_s^{t,x} = v(s, B_s^{t,x})$ ,  $s \in [t, T]$ ,  $\mathcal{P}^t - q.s$ , where  $v$  solves, in some sense (classical or viscosity solutions), the following fully nonlinear SPDE: for all  $0 \leq t < T$

$$\begin{cases} du(t, x) + \hat{h}(t, x, u(t, x), Du(t, x), D^2u(t, x)) dt + g(t, x, u(t, x), Du(t, x)) \circ d\overleftarrow{W}_t = 0, \\ u(T, x) = \phi(x). \end{cases} \quad (2.5.3)$$

We can write the SPDE (2.5.3) in the integral form in the case when  $\{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}^d\} \in C^{0,2}(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$  is a classical solution, namely

$$u(t, x) = \phi(x) + \int_t^T \hat{h}(t, x, u(t, x), Du(t, x), D^2u(t, x)) dt + \int_t^T g(t, x, u(t, x), Du(t, x)) \circ d\overleftarrow{W}_t. \quad (2.5.4)$$

We consider the case

$$H_t(\omega, y, z, \gamma) = h(t, B_t(\omega), y, z, \gamma),$$

where  $h : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times D_h \rightarrow \mathbb{R}$  is a deterministic map. Then the corresponding conjugate and bi-conjugate functions become

$$f(t, x, y, z, a) := \sup_{\gamma \in D_h} \left\{ \frac{1}{2} \text{Tr}(a\gamma) - h(t, x, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0}, \quad (2.5.5)$$

$$\hat{h}(t, x, y, z, \gamma) := \sup_{a \in \mathbb{S}_d^{>0}} \left\{ \frac{1}{2} \text{Tr}(a\gamma) - f(t, x, y, z, a) \right\} \text{ for } \gamma \in \mathbb{R}^{d \times d}. \quad (2.5.6)$$

Notice that  $-\infty < \hat{h} \leq h$  and  $\hat{h}$  is nondecreasing convex in  $\gamma$ . Also,  $\hat{h} = h$  if and only if  $h$  is convex and nondecreasing in  $\gamma$ .

### 2.5.1 A nonlinear Feynman-Kac representation formula

We define a similar notation to (5.3.2). Let  $\tau$  be an  $\mathbb{G}^t$ -stopping time,  $\mathbb{P} \in \mathcal{P}^t$ , and  $\eta$  a  $\mathbb{P}$ -square integrable  $\mathcal{G}_\tau^t$ -measurable random variable. We denote by  $(y^\mathbb{P}, z^\mathbb{P}) := (y^{\mathbb{P}, t, x}(\tau, \eta), z^{\mathbb{P}, t, x}(\tau, \eta))$  the solution of the following BDSDE:

$$y_s^\mathbb{P} = \eta - \int_s^\tau f(r, B_r^{t, x}, y_r^\mathbb{P}, z_r^\mathbb{P}, \hat{a}_r) dr - \int_s^\tau g(r, B_r^{t, x}, y_r^\mathbb{P}, z_r^\mathbb{P}) d\overleftarrow{W}_r - \int_s^\tau z_r^\mathbb{P} dB_r, \quad t \leq r \leq \tau, \quad \mathbb{P} - a.s.$$

Under our assumptions the above BDSDE has a unique solution.

**Definition 2.5.1.** We define the classical solution of the SPDE (2.5.3) as a  $\mathbb{R}$ -valued random field  $\{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}^d\}$  such that  $u(t, x)$  is  $\mathcal{F}_{t, T}^W$ -measurable for each  $(t, x)$ , and whose trajectories belong to  $C^{0,2}([0, T] \times \mathbb{R}^d; \mathbb{R})$ .

**Theorem 2.5.1.** Let Assumption 2.5.1 hold true. Suppose further that  $\hat{h}$  is continuous in its domain,  $D_f$  is independent of  $t$  and is bounded both from above and away from 0. Let  $\{u(t, x); t \in [0, T] \times \mathbb{R}^d\}$  be a classical solution of (2.5.3) with  $\{(u, Du)(s, B_s^{t, x}), s \in [t, T]\} \in \mathbb{D}^2 \times \mathbb{H}^2$ . Then:

$$Y_s^{t, x} := u(s, B_s^{t, x}), \quad Z_s^{t, x} := Du(s, B_s^{t, x}), \quad K_s^{t, x} := \int_0^s k_r dr$$

$$\text{with } k_s := \hat{h}(s, B_s, Y_s, Z_s, \Gamma_s) - \frac{1}{2} Tr(\hat{a}_s \Gamma_s) + f(s, B_s, Y_s, Z_s, \hat{a}_s) \text{ and } \Gamma_s := D^2 u(s, B_s^{t, x})$$

is the unique solution of the 2BDSDE (2.5.1). Moreover,  $u(t, x) = Y_t^{t, x}$  for all  $t \in [0, T]$ .

**Proof.** It suffices to show that  $(Y, Z, K)$  solves the 2BDSDE (2.5.1). For notational simplicity, we shall write  $B$  instead of  $B^{t, x}$  and let  $s = t_0 < t_1 < t_2 < \dots < t_n = T$ ,

$$\begin{aligned} \sum_{i=0}^{n-1} [u(t_i, B_{t_i}) - u(t_{i+1}, B_{t_{i+1}})] &= \sum_{i=0}^{n-1} [u(t_i, B_{t_i}) - u(t_i, B_{t_{i+1}})] + \sum_{i=0}^{n-1} [u(t_i, B_{t_{i+1}}) - u(t_{i+1}, B_{t_{i+1}})] \\ &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Du(t_i, B_r) dB_r - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2} Tr(\hat{a}_r D^2 u(t_i, B_r)) dr \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \hat{h}(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}}), D^2 u(r, B_{t_{i+1}})) dr \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} g(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}})) \circ d\overleftarrow{W}_s, \end{aligned}$$

where we have used the Itô formula and the equation satisfied by  $u$  (2.5.4). Now, the transformation from Stratonovich to Itô integral yields to

$$\begin{aligned} \sum_{i=0}^{n-1} [u(t_i, B_{t_i}) - u(t_{i+1}, B_{t_{i+1}})] &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Du(t_i, B_r) dB_r - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2} Tr(\hat{a}_r D^2 u(t_i, B_r)) dr \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \hat{h}(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}}), D^2 u(r, B_{t_{i+1}})) dr \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} g(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}})) d\overleftarrow{W}_r \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}}), \hat{a}_r) dr \\ &\quad - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}}), \hat{a}_r) dr \\ &\quad - \frac{1}{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Tr(g(r, B_{t_{i+1}}, u(r, B_{t_{i+1}}), Du(r, B_{t_{i+1}})) Dg(r, B_{t_{i+1}}, u(r, B_{t_{i+1}})), Du(r, B_{t_{i+1}})) dr. \end{aligned}$$

It suffices to let the mesh size go to zero. It remains to prove the minimum condition:

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T k_s ds \right] = 0 \text{ for all } t \in [0, T] \text{ } \mathbb{P} \in \mathcal{P}^t \quad (2.5.7)$$

by which we can conclude that  $(Y, Z, K)$  is a solution of the 2BDSDE (2.5.1). Since  $\phi(B_T) \in \mathbb{L}^2$ , the uniqueness follows from Theorems 3.1 and 3.3 (i).

To prove (2.5.7), we follow the same argument as in the proof of Lemma 3.1 in [49]. For every  $\varepsilon > 0$ , notice that the set

$$A^\varepsilon := \{a \in D_f : \hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \leq \frac{1}{2} \operatorname{Tr}(a_t \Gamma_t) - f(t, B_t, Y_t, Z_t, a_t) + \varepsilon\}$$

is not empty. Then it follows from a measurable selection argument that there exists a  $\mathcal{F}_t^B$ -predictable process  $a^\varepsilon$  taking values in  $D_f$  such that

$$\hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \leq \frac{1}{2} \operatorname{Tr}(a_t^\varepsilon \Gamma_t) - f(t, B_t, Y_t, Z_t, a_t^\varepsilon) + \varepsilon$$

We note that this in particular implies that  $\Gamma_t \in D_{\hat{h}}$ .

In the remainder of this proof, we show the existence of an  $\mathbb{F}$ -progressively measurable process  $\alpha^\varepsilon$  with values in  $\mathbb{S}_d^{>0}$ ,  $\int_0^T |\alpha_t^\varepsilon| dt < \infty$ , such that,  $\mathbb{P}^{\alpha^\varepsilon}$ -a.s.,  $\hat{a} \in A^\varepsilon$ .

Let  $\mathbb{P} = \mathbb{P}^\alpha := \mathbb{P}^\alpha \otimes \mathbb{P}_W \in \mathcal{P}^t$  and  $t_0 \in [0, T]$  be fixed. Let

$$\tau_0^\varepsilon := T \wedge \inf\{t \geq t_0 | K_t \geq K_{t_0} + \varepsilon\},$$

and define:

$$\tau_{n+1}^\varepsilon := T \wedge \inf\{t \geq \tau_n^\varepsilon \mid \hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \geq \frac{1}{2} \operatorname{Tr}(a_{\tau_n^\varepsilon}^\varepsilon \Gamma_t) - f(t, B_t, Y_t, Z_t, a_{\tau_n^\varepsilon}^\varepsilon) + 2\varepsilon\}$$

for  $n \geq 0$ . Since  $K$  is continuous, notice that  $\tau_0^\varepsilon > 0$ ,  $\mathcal{P}^t$ -q.s. Also, since  $B, Y, Z, \Gamma$  are all continuous in  $t$ ,  $\tau_n^\varepsilon$  are  $\mathbb{G}$ -stopping times and, for any fixed  $\omega$ , are uniformly continuous in  $t$ .

Next, for any fixed  $a \in D_f$ , the function  $f(\cdot, a)$  is continuous. Also  $\hat{h}$  is continuous. Then for  $\mathcal{P}^t$ -q.s.  $\omega \in \Omega$ ,

$$\hat{h}(t, B_t(\omega), Y_t(\omega), Z_t(\omega), \Gamma_t(\omega)) - \frac{1}{2} \operatorname{Tr}(a_{\tau_n^\varepsilon}^\varepsilon(\omega) \Gamma_t(\omega)) + f(t, B_t(\omega), Y_t(\omega), Z_t(\omega), a_{\tau_n^\varepsilon}^\varepsilon(\omega))$$

is uniformly continuous in  $t$  for  $t \in [\tau_n^\varepsilon, T]$ . Then  $\tau_{n+1}^\varepsilon - \tau_n^\varepsilon \geq \delta(\varepsilon, \omega) > 0$  whenever  $\tau_{n+1}^\varepsilon < 1$ , where the constant  $\delta(\varepsilon, \omega)$  does not depend on  $n$ . This implies that  $\tau_n^\varepsilon = 1$  for  $n$  large enough. Applying the arguments in Example 4.5 of [130] on  $[\tau_0^\varepsilon, T]$ , it is easy to see that there exists an  $(\mathcal{F}_t^B)_{0 \leq t \leq T}$ -progressively measurable process  $\alpha^\varepsilon$  taking values in  $D_f$  such that

$$\alpha_t^\varepsilon = \alpha_t \quad \text{for } t \in [0, \tau_0^\varepsilon] \quad \text{and} \quad \hat{a}_t = \sum_{n=0}^{\infty} a_{\tau_n^\varepsilon}^\varepsilon \mathbf{1}_{[\tau_n^\varepsilon, \tau_{n+1}^\varepsilon)}(t),$$

$$dt \times d\mathbb{P}_B^{\alpha^\varepsilon} - a.s. \quad \text{on} \quad [\tau_0^\varepsilon, T] \times \Omega.$$

This implies that

$$\hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \leq \frac{1}{2} \operatorname{Tr}(\hat{a}_t \Gamma_t) - f(t, B_t, Y_t, Z_t, \hat{a}_t) + 2\varepsilon \quad dt \times d\mathbb{P}^{\alpha^\varepsilon} - a.s. \quad \text{on} \quad [\tau_0^\varepsilon, T] \times \Omega.$$

It is obvious that  $\mathbb{P}^{\alpha^\varepsilon} \in \mathcal{P}^t$ , then  $\mathbb{P}^{\alpha^\varepsilon} \in \mathcal{P}(t_0+, \mathbb{P})$  since  $\tau_0^\varepsilon > t_0$ . Therefore,

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t_0^+, \mathbb{P})} \mathbb{E}_{t_0}^{\mathbb{P}'} \left[ \int_{t_0}^T k_t dt \right] \leq \varepsilon + \mathbb{E}_{t_0}^{\mathbb{P}^{\alpha^\varepsilon}} \left[ \int_{\tau_0^\varepsilon}^T k_t dt \right] \leq \varepsilon + 2\varepsilon(1 - t_0), \quad \mathbb{P} - a.s.$$

By the arbitrariness of  $\varepsilon > 0$ , and the nonnegativity of  $k$ , this provides (2.5.7).  $\square$

### 2.5.2 Stochastic viscosity solution for SPDE

Buckdahn and Ma [24, 25] have introduced the rigorous notion of stochastic viscosity solution for semilinear SPDEs and then they gave the probabilistic interpretation of such equation via BDSDEs where the intensity of the noise  $g$  in the SPDEs (2.5.3) does not depend on the gradient of the solution. Following the classical terminology in the BSDE literature, we say that the solution of the 2BDSDE is Markovian if it can be represented by means of a deterministic function of  $(t, B_t)$ . In this subsection, we construct a deterministic function  $u$ , by using a probabilistic representation in the spirit of (2.3.1), and show its connection with 2BDSDE (2.5.1). The connection between  $u$  and the SPDE (2.5.3) will be also established.

We will use the shifted probability spaces defined in Section 2.4. We now introduce the random function  $u : [0, T] \times \Omega^W \times \mathbb{R}^d \rightarrow \mathbb{R}$  given by

$$u(t, x) := \sup_{\mathbb{P} \in \mathcal{P}^t} y_t^{\mathbb{P}, t, x}(T, \phi(B_T^{t, x})), \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d. \quad (2.5.8)$$

By the Blumenthal zero-one law, it follows that  $u(t, x)$  is deterministic w.r.t  $B$  but still an  $\mathcal{F}_{t, T}^W$ -adapted process.

As we have explained in the existence of the solution for 2BDSDE in the abstract setting, we suppose that our candidate  $u$  defined by (2.5.8) belongs to  $C(\mathcal{F}_{t, T}^W, [0, T] \times \mathbb{R}^d)$ .

We next state a strengthening of Assumption 2.2.2 in the present Markov framework.

**Assumption 2.5.2.** (i) *The function  $\phi$  is uniformly continuous and bounded function on  $\mathbb{R}^d$ .*

(ii) *There is a continuous positive function  $\Lambda(t, x)$  such that, for any  $(t, x)$ :*

$$\begin{aligned} \sup_{\mathbb{P} \in \mathcal{P}^t} \mathbb{E}^{\mathbb{P}} \left[ |\phi(B_T^{t, x})|^2 + \int_t^T |f(s, B_s^{t, x}, 0, 0, \hat{a}_s^t)|^2 ds + \int_t^T |g(s, B_s^{t, x}, 0, 0)|^2 ds \right] &\leq \Lambda^2(t, x), \\ \sup_{\mathbb{P} \in \mathcal{P}^t} \mathbb{E}^{\mathbb{P}} \left[ \sup_{t \leq s \leq T} \Lambda^2(s, B_s^{t, x}) \right] &< \infty. \end{aligned} \quad (2.5.9)$$

#### 2.5.2.1 Definitions

We follow Buckdahn and Ma [24]. The definition of our stochastic viscosity solution will depend on the following stochastic flow  $\eta \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d \times \mathbb{R})$  defined as the unique solution of the stochastic differential equation (SDE):

$$\eta(t, x, y) = y + \int_t^T g(s, x, \eta(s, x, y)) \circ d\overleftarrow{W}_s, \quad 0 \leq t \leq T \quad (2.5.10)$$

Under the Assumption 2.5.1, for fixed  $x$  the random field  $\eta(\cdot, x, \cdot)$  is continuously differentiable in the variable  $y$ ; and the mapping  $y \mapsto \eta(t, x, y, \omega)$  defines a diffeomorphism for all  $(t, x)$ ,  $\mathbb{P}$ -a.s. We denote by  $\mathcal{E}(t, x, y)$  the  $y$ -inverse of  $\eta(t, x, y)$ , so  $\mathcal{E}(t, x, y)$  is the solution of the following first-order SPDE:

$$\mathcal{E}(t, x, y) = y - \int_t^T D_y \mathcal{E}(s, x, y) g(s, x, y) \circ d\overleftarrow{W}_s, \quad \forall (t, x, y), \mathbb{P} - a.s. \quad (2.5.11)$$

We note that  $\mathcal{E}(t, x, \eta(t, x, y)) = \mathcal{E}(T, x, \eta(T, x, y)) = y, \quad \forall (t, x, y)$

We now define the notion of stochastic viscosity solution for SPDE  $(\hat{h}, g)$  (2.5.3).

**Definition 2.5.2.** (i) *A random field  $u \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  is called a stochastic viscosity subsolution (resp. supersolution) of SPDE  $(\hat{h}, g)$ , if  $u(T, x) \leq$  (resp.  $\geq$ )  $\phi(x), \forall x \in \mathbb{R}^d$ ; and if for any  $\tau \in \mathcal{M}_{0, T}^W, \zeta \in \mathbb{L}^0(\mathcal{F}_\tau^W; \mathbb{R}^d)$ , and any random field  $\varphi \in C^{1,2}(\mathcal{F}_{t, T}^W, [0, T] \times \mathbb{R}^d)$  satisfying*

$$u(t, x) - \eta(t, x, \varphi(t, x)) \leq \text{(resp. } \geq \text{)} 0 = u(\tau, \zeta) - \eta(\tau, \zeta, \varphi(\tau, \zeta)),$$

for all  $(t, x)$  in a neighborhood of  $(\tau, \zeta)$ ,  $\mathbb{P}$  - a.e. on the set  $\{0 < \tau < T\}$ , it holds that

$$-\hat{h}(\tau, \zeta, \psi(\tau, \zeta), D\psi(\tau, \zeta), D^2\psi(\tau, \zeta)) \leq (\text{resp. } \geq) D_y \eta(\tau, \zeta, \varphi(\tau, \zeta)) D_t \varphi(\tau, \zeta),$$

$\mathbb{P}$  - a.e. on  $\{0 < \tau < T\}$ , where  $\psi(t, x) \triangleq \eta(t, x, \varphi(t, x))$ .

(ii) A random field  $u \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  is called a stochastic viscosity solution of SPDE  $(\hat{h}, g)$ , if it is both a stochastic viscosity subsolution and a supersolution.

**Definition 2.5.3.** A random field  $u \in C(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  is called a  $\omega$ -wise viscosity (sub-, super-) solution if for  $\mathbb{P}$  - a.e.  $\omega \in \Omega$ ,  $u(\omega, \cdot, \cdot)$  is a (deterministic) viscosity (sub-, super-) solution of the SPDE  $(\hat{h}, 0)$ .

**Remark 2.5.1.** If we assume that  $\varphi \in C^{1,2}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$ , and that  $g \in C^{0,0,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^l)$ , then a straightforward computation using the Itô-Ventzell formula shows that the random field  $\psi(t, x) = \eta(t, x, \varphi(t, x))$  satisfies

$$d\psi(t, x) = D_y \eta(t, x, \varphi(t, x)) D_t \varphi(t, x) dt + \langle g(t, x, \psi(t, x)), \circ d\overleftarrow{W}_t \rangle, \quad t \in [0, T]. \quad (2.5.12)$$

Since  $g(\tau, \zeta, \psi(\tau, \zeta)) = g(\tau, \zeta, u(\tau, \zeta))$  by definition, it seems natural to compare

$\hat{h}(\tau, \zeta, \psi(\tau, \zeta), D\psi(\tau, \zeta), D^2\psi(\tau, \zeta))$  with  $D_y \eta(\tau, \zeta, \varphi(\tau, \zeta)) D_t \varphi(\tau, \zeta)$  to characterize a viscosity solution of SPDE  $(\hat{h}, g)$ .

If the function  $g \equiv 0$  in SPDE  $(f, g)$ , the flow  $\eta$  becomes  $\eta(t, x, y) = y$ ,  $\forall (t, x, y)$  and  $\psi(t, x) = \varphi(t, x)$ . Thus the definition of a stochastic viscosity solution becomes the same as that of a deterministic viscosity solution (see, e.g. Crandall, Ishii and Lions [31]).

Next, we give a generalized version of Itô-Ventzell formula that combines the generalized Itô formula of Paroux and Peng (1994) and the Itô-Ventzell formula of Ocone and Pardoux (1989).

**Lemma 2.5.1.** (Generalized Itô-Ventzell formula)

Suppose that  $F \in C^{0,2}(\mathbb{F}, [0, T] \times \mathbb{R}^k)$  is a semimartingale with spatial parameter  $x \in \mathbb{R}^k$ :

$$F(t, x) = F(0, t) + \int_0^t G(s, x) ds + \int_0^t H(s, x) dB_s + \int_0^t K(s, x) d\overleftarrow{W}_s, \quad t \in [0, T],$$

where  $G \in C^{0,2}(\mathbb{F}^B, [0, T] \times \mathbb{R}^k)$ ,  $H \in C^{0,2}(\mathbb{F}^B, [0, T] \times \mathbb{R}^k; \mathbb{R}^d)$  and  $K \in C^{0,2}(\mathbb{F}^W, [0, T] \times \mathbb{R}^k; \mathbb{R}^l)$ . Let  $\phi \in C(\mathbb{F}, [0, T]; \mathbb{R}^k)$  be a process of the form

$$\phi_t = \phi_0 + A_t + \int_0^t \gamma_s dB_s + \int_0^t \delta_s d\overleftarrow{W}_s, \quad t \in [0, T],$$

where  $\gamma \in \mathbb{H}_{k \times d}^2$ ,  $\delta \in \mathbb{H}_{k \times l}^2$  and  $A$  is a continuous  $\mathbb{F}$ -adapted process with paths of locally bounded variation. Then,  $\mathbb{P}$ -almost surely, it holds for all  $0 \leq t \leq T$  that

$$\begin{aligned} F(t, \phi_t) = & F(0, x) + \int_0^t G(s, \phi_s) ds + \int_0^t H(s, \phi_s) dB_s + \int_0^t K(s, \phi_s) d\overleftarrow{W}_s \\ & + \int_0^t D_x F(s, \phi_s) dA_s + \int_0^t D_x F(s, \phi_s) \gamma_s dB_s + \int_0^t D_x F(s, \phi_s) \delta_s d\overleftarrow{W}_s \\ & + \frac{1}{2} \int_0^t \text{Tr}(D_{xx} F(s, \phi_s) \gamma_s \gamma_s^*) ds - \frac{1}{2} \int_0^t \text{Tr}(D_{xx} F(s, \phi_s) \delta_s \delta_s^*) ds \\ & + \int_0^t \text{Tr}(D_x H(s, \phi_s) \gamma_s^*) ds - \int_0^t \text{Tr}(D_x F(s, \phi_s) \delta_s^*) ds. \end{aligned} \quad (2.5.13)$$



### 2.5.2.2 Doss transformation

In this subsection we use the so-called Doss-Sussmann transformation to convert the fully nonlinear SPDEs (2.5.3) to PDEs with random coefficients. This transformation permits to remove the martingale term from the SPDEs.

To begin with, let us note that, under Assumption 2.5.1 (iii), the random field  $\eta \in C^{0,2,2}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d \times \mathbb{R})$ , thus so is  $\mathcal{E}$ . Now for any random field  $\psi : [0, T] \times \mathbb{R}^d \times \Omega \mapsto \mathbb{R}$ , consider the transformation introduced in Definition 2.5.2:

$$\varphi(t, x) = \mathcal{E}(t, x, \psi(t, x)), \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

or equivalently,  $\psi(t, x) = \eta(t, x, \varphi(t, x))$ ,  $(t, x), \mathbb{P} - a.s.$  One can easily check that  $\psi \in C^{0,p}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$  if and only if  $\varphi \in C^{0,p}(\mathbb{F}^W, [0, T] \times \mathbb{R}^d)$ , for  $p = 0, 1, 2$ . Moreover, if  $\varphi \in C^{0,2}(\mathcal{F}^W, [0, T] \times \mathbb{R}^d)$ , then  $D_x \psi = D_x \eta + D_y \eta D_x \varphi$ ; and

$$D_{xx} \psi = D_{xx} \eta + 2(D_{xy} \eta)(D_x \varphi)^* + (D_{yy} \eta)(D_x \varphi)(D_x \varphi)^* + (D_y \eta)(D_{xx} \varphi). \quad (2.5.14)$$

Furthermore, since  $\mathcal{E}(t, x, \eta(t, x, y)) \equiv y$ ,  $\forall (t, x, y)$ ,  $\mathbb{P} - a.s.$ , differentiating the equation up to the second order we have (suppressing variables), for all  $(t, x, y)$  and  $\mathbb{P}$ -almost surely,

$$\begin{aligned} D_x \mathcal{E} + D_y \mathcal{E} D_x \eta &= 0, & D_y \mathcal{E} D_y \eta &= 1, \\ D_{xx} \mathcal{E} + 2(D_{xy} \mathcal{E})(D_x \eta)^* + (D_{yy} \mathcal{E})(D_x \eta)(D_x \eta)^* + (D_y \mathcal{E})(D_{xx} \eta) &= 0, \\ (D_{xy} \mathcal{E})(D_y \eta) + (D_{yy} \mathcal{E})(D_x \eta)(D_y \eta) + (D_y \mathcal{E})(D_{xy} \eta) &= 0, \\ (D_{yy} \mathcal{E})(D_y \eta)^2 + (D_y \mathcal{E})(D_{yy} \eta) &= 0. \end{aligned} \quad (2.5.15)$$

Now, we will use the Doss transformation to obtain the following proposition where the proof is exaltly the same as Proposition 3.1. in [24].

**Proposition 2.5.1.** *Let Assumptions 2.5.1 and 2.5.2 hold true. A random field  $u$  is a stochastic viscosity sub- (resp. super-) solution to SPDE  $(\hat{h}, g)$  (2.5.3) if and only if  $v(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot, u(\cdot, \cdot))$  is a stochastic viscosity solution to SPDE  $(\tilde{h}, 0)$ , with*

$$\begin{aligned} \tilde{h}(t, x, y, z, \gamma) &= \frac{1}{D_y \eta(t, x, y)} \{ \hat{h}(t, x, \eta(t, x, y), D_x \eta(t, x, y) + D_y \eta(t, x, y)z, D_{xx} \eta(t, x, y) \\ &+ 2D_{xy} \eta(t, x, y)z + D_{yy} \eta(t, x, y)|z|^2 + D_y \eta(t, x, y)\gamma) \}. \end{aligned} \quad (2.5.16)$$

We will now apply Doss transformation to the 2BDSDE (2.5.1) in order that the Stratonovich backward integral vanishes. Thus, the 2BDSDE will become a 2BSDE with a new generator  $\tilde{f}$  which is quadratic in  $z$  (studied by Possamaï and Zhou [121] and Lin [87] in the case of a bounded final condition  $\phi(B_T^{t,x})$ ). Let us define the following three processes:

$$\begin{aligned} U_s^{t,x} &= \mathcal{E}(t, B_s^{t,x}, Y_s^{t,x}) \\ V_s^{t,x} &= D_y \mathcal{E}(s, B_s^{t,x}, Y_s^{t,x}) Z_s^{t,x} + D_x \mathcal{E}(s, B_s^{t,x}, Y_s^{t,x}) \\ \tilde{K}_s^{t,x} &= \int_0^s D_y \mathcal{E}(r, B_r^{t,x}, Y_r^{t,x}) dK_r^{t,x}. \end{aligned} \quad (2.5.17)$$

**Theorem 2.5.2.** *Let Assumptions 2.5.1 and 2.5.2 hold true. Then  $(U^{t,x}, V^{t,x}, \tilde{K}^{t,x})$  is the unique solution of the following 2BSDE, for all  $t \leq s \leq T$ ,*

$$U_s^{t,x} = \phi(B_T^{t,x}) - \int_s^T \tilde{f}(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \hat{a}_r) dr - \int_s^T V_r^{t,x} dB_r + \tilde{K}_T^{t,x} - \tilde{K}_s^{t,x}, \quad (2.5.18)$$

where  $\tilde{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times D_f \mapsto \mathbb{R}$  is defined by:

$$\begin{aligned} \tilde{f}(t, x, y, z, a) &\triangleq \frac{1}{D_y \eta(t, x, y)} \{ f(t, x, y, D_y \eta(t, x, y)z + D_x \eta(t, x, y)) - \frac{1}{2} \text{Tr}(a D_{xx} \eta(t, x, y)) \\ &- (a D_{xy} \eta(t, x, y)z) - \frac{1}{2} \text{Tr}(a D_{yy} \eta(t, x, y)|z|^2) \}. \end{aligned} \quad (2.5.19)$$

**Proof.** It is easily checked that the mapping  $(B, Y, Z, K) \mapsto (B, U, V, \tilde{K})$  is 1-1, with the inverse transformation:

$$\begin{aligned} Y_t &= \eta(t, B_t, U_t) \\ Z_t &= D_y \eta(t, B_t, U_t) V_t + D_x \eta(t, B_t, U_t) \\ K_t &= \int_0^t D_y \eta(s, B_s, U_s) d\tilde{K}_s. \end{aligned} \quad (2.5.20)$$

Consequently, the uniqueness of (2.5.18) follows from that of 2BDSDE (2.5.1), thanks to (2.5.17) and (2.5.20). Thus we need only show that  $(U, V, \tilde{K})$  is a solution of the 2BSDE (2.5.18).

Applying the generalized Itô-Ventzell formula (Lemma 2.5.1) to  $\mathcal{E}(t, B_t, Y_t)$ , one derives that for  $(t, x) \in [0, T] \times \mathbb{R}^d$

$$\begin{aligned} U_t &= \mathcal{E}(t, B_t, Y_t) = \phi(B_T) - \int_t^T D_x \mathcal{E}(s, B_s, Y_s) dB_s - \int_t^T D_y \mathcal{E}(s, B_s, Y_s) f(s, B_s, Y_s, Z_s, \hat{a}_s) ds \\ &\quad - \int_t^T D_y \mathcal{E}(s, B_s, Y_s) Z_s dB_s + \int_t^T D_y \mathcal{E}(s, B_s, Y_s) dK_s - \frac{1}{2} \int_t^T \text{Tr}(D_{xx} \mathcal{E}(s, B_s, Y_s) \hat{a}_s) ds \\ &\quad - \frac{1}{2} \int_t^T \text{Tr}(D_{yy} \mathcal{E}(s, B_s, Y_s) \hat{a}_s |Z_s|^2) ds - \int_t^T \text{Tr}(D_{xy} \mathcal{E}(s, B_s, Y_s) \hat{a}_s Z_s) ds \\ &= \phi(B_T) - \int_t^T \mathcal{H}(s, B_s, Y_s, Z_s, \hat{a}_s) ds - \int_t^T V_s dB_s + \tilde{K}_T - \tilde{K}_t, \end{aligned} \quad (2.5.21)$$

where

$$\begin{aligned} \mathcal{H}(s, x, y, z, \hat{a}_s) &\triangleq (D_y \mathcal{E}) f(s, x, y, z, \hat{a}_s) + \frac{1}{2} \text{Tr}((D_{xx} \mathcal{E}) \hat{a}_s) + \frac{1}{2} \text{Tr}((D_{yy} \mathcal{E}) \hat{a}_s |z|^2) \\ &\quad + \text{Tr}((D_{xy} \mathcal{E}) \hat{a}_s z). \end{aligned} \quad (2.5.22)$$

Next, we can show that

$$\mathcal{H}(s, B_s, Y_s, Z_s, \hat{a}_s) = \tilde{f}(s, B_s, U_s, V_s, \hat{a}_s) \quad \forall s \in [0, T], \mathbb{P} - a.s. \quad (2.5.23)$$

similarly as done in Buckdahn and Ma [24] (proof of Theorem 5.1. page 198-199).

The process  $\tilde{K}$  is an increasing process which satisfies the minimum condition (2.5.7) thanks to (2.5.17), the fact that  $y \mapsto \eta(t, x, y)$  is strictly increasing and that  $K$  satisfies the minimum condition (2.5.7).  $\square$

We are now ready to prove that  $u$  defined by (2.5.8) is the stochastic viscosity solution of our SPDE  $(\hat{h}, g)$  (2.5.3). First, we introduce the random field  $v(t, x) = U_t^{t,x}$ , where  $U$  is the solution of 2BSDE (2.5.18). Then by (2.5.17) and (2.5.20) we know that, for  $(t, x) \in [0, T] \times \mathbb{R}^d$

$$u(t, x) = \eta(t, x, v(t, x)); \quad v(t, x) = \mathcal{E}(t, x, u(t, x)). \quad (2.5.24)$$

Finally, thanks to Proposition 5.1, we need only prove that the random field  $v$  defined in (2.5.24) is a stochastic viscosity solution to the SPDE  $(\tilde{h}, 0)$ .

**Theorem 2.5.3.** *Under our Assumptions, the random field  $v$  is a stochastic viscosity solution of SPDE  $(\tilde{h}, 0)$ ; and  $u$  is a stochastic viscosity solution to SPDE  $(\hat{h}, g)$ , respectively.*

**Remark 2.5.2.** *As mentioned in the beginning of Section 2.5.2, we suppose that our candidate  $u$  defined by (2.5.8) belongs to  $C(\mathcal{F}_{t,T}^W, [0, T] \times \mathbb{R}^d)$ . Thanks to the relation (2.5.24), this condition is equivalent to  $v(\cdot, \cdot, \omega)$  belongs to  $C([0, T] \times \mathbb{R}^d)$ ,  $\mathbb{P}$ -a.s. for all  $\omega \in \Omega$ . This latter can be proved by using regularity results for deterministic fully nonlinear PDEs ([27],[86]), but it is still a work in progress to make it rigourously.*

## 2.6 Appendix

### 2.6.1 Doubly $f$ - supersolution and martingales

In this section, we extend some of the results of Peng [114] concerning  $f$ - supersolution of BSDEs to the case of BDSDEs. In the following, we fix a probability measure  $\mathbb{P}$ . Let us given the following objects:

- (i) a terminal condition  $\xi$  which is  $\mathcal{F}_T$ -measurable and in  $L^2(\mathbb{P})$
- (ii)  $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  ,  $g : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^l$  are two functions verifying:

- $\mathbb{E}[\int_0^T |f(t, 0, 0)|^2 dt] < +\infty$  ,  $\mathbb{E}[\int_0^T \|g(t, 0, 0)\|^2 dt] < +\infty$
- There exist constants  $c > 0$  and  $0 < \alpha < 1$  such that for any  $(\omega, t) \in \Omega \times [0, T]$  ;  $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 &\leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ \|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 &\leq c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2. \end{aligned}$$

- (iii) a real-valued r.c.l.l. progressively measurable process  $\{V_t, 0 \leq t \leq T\}$  with

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |V_t|^2\right) < +\infty.$$

We want to study the following problem: to find a pair of processes  $(y, z) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$  satisfying

$$y_t = \xi_T + \int_t^T f_s(y_s, z_s) ds + \int_t^T g_s(y_s, z_s) d\overleftarrow{W}_s + V_T - V_t - \int_t^T z_s dB_s, \quad \mathbb{P} - a.s. \quad (2.6.1)$$

We have the following existence and uniqueness theorem

**Proposition 2.6.1.** *Under the above hypothesis there exists a unique pair of processes  $(y, z) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$  of solution BDSDE (2.6.1).*

**Proof.** In the case where  $V \equiv 0$ , the proof can be found in [109]. Otherwise, we can make the change of variable  $\bar{y}_t := y_t + V_t$  and treat the equivalent BDSDE

$$\bar{y}_t = \xi_T + V_T + \int_t^T f_s(\bar{y}_s - V_s, z_s) ds + \int_t^T g_s(\bar{y}_s - V_s, z_s) d\overleftarrow{W}_s - \int_t^T z_s dB_s. \quad (2.6.2)$$

□

We also have a comparison theorem in this context

**Proposition 2.6.2.** *Let  $\xi_1$  and  $\xi_2 \in L^2(\mathbb{P}), V^i, i = 1, 2$  be two adapted càdlàg processes and  $f_s^i(y, z), g_s^i(y, z)$  four functions verifying the above assumption. Let  $(y^i, z^i) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}), i = 1, 2$  be the solution of the following BDSDEs:*

$$y_t^i = \xi_T^i + \int_t^T f_s^i(y_s^i, z_s^i) ds + \int_t^T g_s^i(y_s^i, z_s^i) d\overleftarrow{W}_s + V_T^i - V_t^i - \int_t^T z_s^i dB_s, \quad \mathbb{P} - a.s., \quad i = 1, 2$$

respectively. If we have  $\mathbb{P} - a.s.$  that  $\xi_1 \geq \xi_2, V^1 - V^2$  is non decreasing, and  $f_s^1(y_s^1, z_s^1) \geq f_s^2(y_s^1, z_s^1)$  then it holds that for all  $t \in [0, T]$

$$y_t^1 \geq y_t^2 \quad \mathbb{P} - a.s.$$

**Remark 2.6.1.** *If we replace the deterministic time  $T$  by a bounded stopping time  $\tau$ , then all the above is still valid.*

For a given stopping time, we now consider the following BDSDE

$$y_t = \xi_T + \int_{t \wedge \tau}^{\tau} f_s(y_s, z_s) ds + \int_{t \wedge \tau}^{\tau} g_s(y_s, z_s) d\overleftarrow{W}_s + V_\tau - V_{t \wedge \tau} - \int_{t \wedge \tau}^{\tau} z_s dB_s, \quad \mathbb{P} - a.s. \quad (2.6.3)$$

where  $\xi \in L^2(\mathbb{P})$  and  $V \in \mathbb{I}^2(\mathbb{P})$ .

**Definition 2.6.1.** *If  $y$  is a solution of BDSDE of form (2.6.3), then we call  $y$  a doubly  $f$ - supersolution on  $[0, \tau]$ . If  $V \equiv 0$  in  $[0, \tau]$ , then we call  $y$  a doubly  $f$ - solution.*

**Proposition 2.6.3.** *Given  $y$  a doubly  $f$ - supersolution on  $[0, \tau]$ , there is a unique  $z \in \mathbb{H}^2(\mathbb{P})$  and a unique  $V \in \mathbb{I}^2(\mathbb{P})$  such that  $(y, z)$  satisfies (2.6.3).*

**Proof.** If both  $(y, z, V)$  and  $(y, z^1, V^1)$  satisfy (2.6.3), then we apply Itô's formula to  $(y - y)^2$  on  $[0, \tau]$  and take expectation:

$$\mathbb{E}\left[\int_0^\tau |z_s - z_s^1|^2 ds\right] + \mathbb{E}\left[\sum_{t \in (0, \tau]} (\Delta(V_t - V_t^1))^2\right] = 0$$

Thus  $z \equiv z^1$ . From this it follows that  $V \equiv V^1$ . □

Thus we can define

**Definition 2.6.2.** *Let  $y$  be a supersolution on  $[0, \tau]$  and let  $(y, z, V)$  be the related unique triple in the sense of BDSDE (2.6.3). Then we call  $(z, V)$  the decomposition of  $y$ .*

We now introduce the notion of doubly  $f$ - (super) martingales.

**Definition 2.6.3.**

- (i) *A doubly  $f$ - martingale on  $[0, T]$  is a doubly  $f$ - solution on  $[0, T]$ .*
- (ii) *A process  $(Y_t)$  is a doubly  $f$ - supermartingale in the strong (resp. weak) sense if for all stopping time  $\tau \leq t$  (resp. all  $t \leq T$ ), we have  $\mathbb{E}^\mathbb{P}[|Y_\tau|^2] < +\infty$  (resp.  $\mathbb{E}^\mathbb{P}[|Y_t|^2] < +\infty$ ) and if the doubly  $f$ - solution  $(y_s)$  on  $[0, \tau]$  (resp.  $[0, t]$ ) with terminal condition  $Y_\tau$  (resp.  $Y_t$ ) verifies  $y_\sigma \leq Y_\sigma$  for every stopping time  $\sigma \leq \tau$  (resp.  $y_s \leq Y_s$  for every  $s \leq t$ ).*

Certainly, a doubly  $f$ - supermartingale in the strong sense is also a doubly  $f$ - supermartingale in the weak sense. Under mild conditions, a doubly  $f$ - supermartingale in the weak sense corresponds to a doubly  $f$ - supermartingale in the strong sense.

## 2.6.2 Downcrossing inequality

In this section, we prove a downcrossing inequality for doubly  $f$ -supermartingales.

**Theorem 2.6.1.** *Assume that  $f(0, 0) = 0$ . Let  $(Y_t)$  be a positive doubly  $f$ -supermartingale in the weak sense and let  $0 = t_0 < t_1 < \dots < t_i = T$  be a subdivision of  $[0, T]$ . Let  $0 < a < b$ , then there exists  $C > 0$  such that  $D_a^b[Y, n]$ , the number of downcrossings of  $[a, b]$  by  $\{Y_{t_i}\}$ , verifies*

$$\mathcal{E}^{-\mu}[D_a^b[Y, n]] \leq \frac{C}{b-a} \mathcal{E}^\mu[Y_0 \wedge b]$$

**Proof.** Consider

$$y_t^i = Y_{t_i} + \int_t^{t_i} (\mu|y_s^i| + \mu|z_s^i|) ds + \int_t^{t_i} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s - \int_t^{t_i} z_s^i dB_s, \quad 0 \leq t \leq t_i$$

Since  $Y_{t_i} \geq 0$ , using comparison theorem, we have  $y_t^i \geq 0$ ,  $t \leq t_i$

$$y_t^i = Y_{t_i} + \int_t^{t_i} (\mu y_s^i + a_s z_s^i) ds + \int_t^{t_i} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s - \int_t^{t_i} z_s^i dB_s, \quad 0 \leq t \leq t_i$$

where  $a_s^i := \mu \operatorname{sgn}(z_s^i) \mathbf{1}_{t_{i-1} < s \leq t_i}$  and  $a_s := \sum_{i=0}^n a_s^i$ .

Define  $\tilde{y}_s^i = e^{-\lambda(t_i-s)} y_s^i$ , we obtain for all  $0 \leq t \leq t_i$

$$\tilde{y}_t^i = + \int_t^{t_i} e^{-\lambda(t_i-s)} a_s z_s^i ds + \int_t^{t_i} e^{-\lambda(t_i-s)} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s - \int_t^{t_i} e^{-\lambda(t_i-s)} z_s^i dB_s$$

Let  $\mathbb{Q}$  be the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T e^{-\lambda(t_i-s)} a_s dB_s\right).$$

and define  $\hat{y}_s^i := \tilde{y}_s^i + \int_0^s e^{-\lambda(t_i-s)} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s$ , we then have easily that

$$\hat{y}_t^i = \mathbb{E}^{\mathbb{Q}}[\hat{y}_{t_i}^i | \mathcal{G}_t] \quad 0 \leq t \leq t_i$$

Since  $Y$  is a doubly  $f$ -supermartingale (and thus also a  $f^{-\mu}$ -supermartingale where  $f_s^{-\mu}(y, z) := -\mu(|y| + |z|)$ ), we therefore obtain

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}[Y_{t_i} + \int_0^{t_i} e^{-\lambda(t_i-s)} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s | \mathcal{G}_{t_{i-1}}] = \hat{y}_{t_{i-1}}^i \\ & = \tilde{y}_{t_{i-1}}^i + \int_0^{t_{i-1}} e^{-\lambda(t_i-s)} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s \\ & \leq e^{-\lambda(t_i-t_{i-1})} Y_{t_{i-1}} + \int_0^{t_{i-1}} e^{-\lambda(t_i-s)} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s \end{aligned}$$

It means that the process  $X := (X_{t_i})_{i=0}^n$  where  $X_{t_i} = e^{\lambda t_i} Y_{t_i} + \int_0^{t_i} e^{\lambda s} g_s(y_s^i, z_s^i) d\overleftarrow{W}_s$ , is a  $\mathbb{Q}$ -supermartingale with respect to  $(\mathcal{G}_{t_i})_{i=0}^n$ . Then we can finish the proof exactly as in [29].

### 2.6.3 Reflected backward doubly stochastic differential equations

In this section, we want to study the problem of a reflected backward doubly stochastic differential equation (RBDSDE in short) with one càdlàg barrier. This is an extension of the work of Hamadene and Ouknine [64] for the standard reflected BDSDEs to our case. So we need the following objects:

- (i) a terminal condition  $\xi$  which is  $\mathcal{F}_T$ -measurable and in  $L^2(\mathbb{P})$
- (ii)  $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $g : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^l$  are two functions verifying:

- $\mathbb{E}\left[\int_0^T |f(t, 0, 0)|^2 dt\right] < +\infty$ ,  $\mathbb{E}\left[\int_0^T \|g(t, 0, 0)\|^2 dt\right] < +\infty$
- There exist constants  $c > 0$  and  $0 < \alpha < 1$  such that for any  $(\omega, t) \in \Omega \times [0, T]$ ;  $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 & \leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ \|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 & \leq c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2. \end{aligned}$$

- (iii) The barrier  $\{S_t, 0 \leq t \leq T\}$  is a real-valued r.c.l.l. progressively measurable process satisfying  $S_T \leq \xi$  and

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} (S_t^+)^2\right) < +\infty.$$

Now we present the definition of the solution of RBDSDEs with one lower r.c.l.l. barrier.

**Definition 2.6.4.** We call  $(Y, Z, K)$  a solution of the backward doubly stochastic differential equation with one r.c.l.l. reflecting lower barrier  $S(\cdot)$ , terminal condition  $\xi$  and coefficients  $f$  and  $g$ , which satisfy (i)-(iii) if the following hold:

(iv)  $Y \in \mathbb{D}^2(\mathbb{P})$ ,  $Z \in \mathbb{H}^2(\mathbb{P})$ .

(v)  $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t$ ,  $0 \leq t \leq T$ .

(vi)  $Y_t \geq S_t$ ,  $0 \leq t \leq T$ , a.s.

(vii) If  $K^c$  (resp.  $K^d$ ) is the continuous (resp. purely discontinuous) part of  $K$ , then

$$\int_0^T (Y_s - S_s) dK_s^c = 0, \text{ a.s. and } \forall t \leq T, \Delta K_t^d = (S_{t-} - Y_t)^+ \mathbf{1}_{[Y_{t-} = S_{t-}]}$$

The state-process  $Y(\cdot)$  is forced to remain above the barrier  $S(\cdot)$ , thanks to the cumulation action of the reflection process  $K(\cdot)$ , which acts only when necessary to prevent  $Y(\cdot)$  from crossing the barrier, and in this sense, its action can be considered minimal.

**Remark 2.6.2.** The condition (vii) implies in particular that

$$\int_0^T (Y_{s-} - S_{s-}) dK_s = 0$$

Actually

$$\begin{aligned} \int_0^T (Y_{s-} - S_{s-}) dK_s &= \int_0^T (Y_{s-} - S_{s-}) dK_s^c + \int_0^T (Y_{s-} - S_{s-}) dK_s^d \\ &= \int_0^T (Y_{s-} - S_s) dK_s^c + \sum_{s \leq T} (Y_{s-} - S_{s-}) \Delta K_s^d \\ &= 0. \end{aligned}$$

The last term of the second equality is null since  $K^d$  jumps only when  $Y_{s-} = S_{s-}$ . □

The main objective of this section is to prove the:

**Theorem 2.6.2.** Under the above hypotheses, the RBDSDE (v) has a unique solution  $(Y, Z, K)$ .

Before we start proving this theorem, let us establish the same result in case  $f$  and  $g$  do not depend on  $Y$  and  $Z$ . More precisely, given  $f$  and  $g$  such that

$$\mathbb{E} \left( \int_0^T |f(s)|^2 ds \right) < +\infty,$$

$$\mathbb{E} \left( \int_0^T \|g(s)\|^2 ds \right) < +\infty$$

and  $\xi$  as above, consider the reflected BDSDE

$$Y_t = \xi + \int_t^T f(s) ds + \int_t^T g(s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t. \tag{2.6.4}$$

**Proposition 2.6.4.** There exists a unique triplet  $(Y, Z, K)$  verifies conditions of Definition 4.2.5 and satisfies (2.6.4).

**Proof.** **a) Existence:** The method combines penalization and the Snell envelope method.

For each  $n \in \mathbb{N}^*$ , we set

$$f_n(s, y) = f(s) + n(y_s - S_s)^-$$

and consider the BDSDE

$$Y_t^n = \xi^n + \int_t^T f_n(s, Y_s^n) ds + \int_t^T g(s) d\overleftarrow{W}_s - \int_t^T Z_s^n dB_s. \quad (2.6.5)$$

It is well known (see Pardoux and Peng [109]) that BDSDE (2.6.5) has a unique solution  $(Y^n, Z^n) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$  such that for each  $n \in \mathbb{N}$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds \right) < \infty.$$

From now on the proof will be divided into four steps.

*Step 1:* For all  $n \geq 0$  and  $(s, y) \in [0, T] \times \mathbb{R}$ ,

$$f_n(s, y, z) \leq f_{n+1}(s, y, z),$$

which provide by Comparison Theorem,  $Y_t^n \leq Y_t^{n+1}$ ,  $t \in [0, T]$  a.s..

*Step 2:* For each  $n \in \mathbb{N}$ , denoting

$$\bar{Y}_t^n := Y_t^n + \int_0^t g(s) d\overleftarrow{W}_s,$$

$$\bar{\xi} := \xi + \int_0^T g(s) d\overleftarrow{W}_s, \quad \bar{S}_t := S_t + \int_0^t g(s) d\overleftarrow{W}_s,$$

we have

$$\bar{Y}_t^n = \bar{\xi} + \int_t^T f(s) ds + n \int_t^T (\bar{Y}_s^n - \bar{S}_s)^- ds - \int_t^T Z_s^n dB_s. \quad (2.6.6)$$

The process  $\bar{Y}_t^n$  satisfies

$$\forall t \leq T, \quad \bar{Y}_t^n = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau f(s) ds + (\bar{Y}_\tau^n \wedge \bar{S}_\tau) \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} \middle| \mathcal{G}_t \right]. \quad (2.6.7)$$

In fact, for any  $n \in \mathbb{N}$  and  $t \leq T$  we have

$$\bar{Y}_t^n = \bar{\xi} + \int_t^T f(s) ds + n \int_t^T (\bar{Y}_s^n - \bar{S}_s)^- ds - \int_t^T \bar{Z}_s^n dB_s. \quad (2.6.8)$$

Therefore for any stopping time  $\tau \geq t$  we have

$$\begin{aligned} \bar{Y}_t^n &= \mathbb{E}[\bar{Y}_\tau^n + \int_t^\tau f(s) ds + n \int_t^\tau (\bar{Y}_s^n - \bar{S}_s)^- ds | \mathcal{G}_t] \\ &\geq \mathbb{E}[(\bar{S}_\tau \wedge \bar{Y}_\tau^n) \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} + \int_t^\tau f(s) ds | \mathcal{G}_t], \end{aligned} \quad (2.6.9)$$

since  $\bar{Y}_\tau^n \geq (\bar{S}_\tau \wedge \bar{Y}_\tau^n) \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}}$ . On the other hand, let  $\tau_t^*$  be the stopping time defined as follows:

$$\tau_t^* = \inf\{s \geq t, \bar{K}_s^n - \bar{K}_t^n > 0\} \wedge T,$$

where  $\bar{K}_t^n = n \int_0^t (\bar{Y}_s^n - \bar{S}_s)^- ds$ . Let us show that  $\mathbf{1}_{[\tau_t^* < T]} \bar{Y}_{\tau_t^*}^n = (\bar{S}_{\tau_t^*} \wedge \bar{Y}_{\tau_t^*}^n) \mathbf{1}_{[\tau_t^* < T]}$ .

Let  $\omega$  be fixed such that  $\tau_t^*(\omega) < T$ . Then there exists a sequence  $(t_k)_{k \geq 0}$  of real numbers which decreases to  $\tau_t^*(\omega)$  such that  $\bar{Y}_{t_k}^n(\omega) \leq \bar{S}_{t_k}(\omega)$ . As  $\bar{Y}^n$  and  $\bar{S}$  are RCLL processes then taking the

limit as  $k \rightarrow \infty$  we obtain  $\bar{Y}_{\tau_t^*}^n \leq \bar{S}_{\tau_t^*}$  which implies  $\mathbf{1}_{[\tau_t^* < T]} \bar{Y}_{\tau_t^*}^n = (\bar{S}_{\tau_t^*} \wedge \bar{Y}_{\tau_t^*}^n) \mathbf{1}_{[\tau_t^* < T]}$ . Now from (2.6.8), we deduce that:

$$\begin{aligned} \bar{Y}_t^n &= \bar{Y}_{\tau_t^*}^n + \int_t^{\tau_t^*} f(s) ds - \int_t^{\tau_t^*} \bar{Z}_s^n dB_s \\ &= (\bar{S}_{\tau_t^*} \wedge \bar{Y}_{\tau_t^*}^n) \mathbf{1}_{[\tau_t^* < T]} + \bar{\xi} \mathbf{1}_{\{\tau_t^* = T\}} + \int_t^{\tau_t^*} f(s) ds - \int_t^{\tau_t^*} \bar{Z}_s^n dB_s. \end{aligned}$$

Taking the conditional expectation and using inequality (2.6.9) we obtain:  $\forall n \geq 0$ , and  $t \leq T$

$$\bar{Y}_t^n = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^{\tau} f(s) ds + (\bar{Y}_{\tau}^n \wedge \bar{S}_{\tau}) \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} | \mathcal{G}_t \right]. \quad (2.6.10)$$

*Step 3:* There exists a RCLL  $(Y_t)_{t \leq T}$  of  $\mathbb{D}^2(\mathbb{P})$  such that  $\mathbb{P}$ -a.s.

(i)  $Y = \lim_{n \rightarrow \infty} Y^n$  in  $\mathbb{H}^2(\mathbb{P})$ ,  $S \leq Y$ .

(ii) for any  $t \leq T$ ,

$$Y_t = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^{\tau} f(s) ds + \bar{S}_{\tau} \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} | \mathcal{G}_t \right] - \int_0^t g(s) d\bar{W}_s. \quad (2.6.11)$$

Actually for  $t \leq T$  let us set  $\tilde{Y}_t = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^{\tau} f(s) ds + \bar{S}_{\tau} \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} | \mathcal{G}_t \right]$ .

The process  $\tilde{Y}$  belongs to  $\mathbb{D}^2(\mathbb{P})$  since  $\bar{S}$  is so,  $f \in \mathbb{H}^2(\mathbb{P})$  and  $\bar{\xi}$  is square integrable. On the other hand for any  $n \geq 0$  and  $t \leq T$  we have  $\bar{Y}_t^n \leq \tilde{Y}_t$ . Thus there exist a  $\mathcal{G}_t$ -measurable process  $\bar{Y}$  such that  $\mathbb{P}$ -a.s. for any  $t \leq T$ ,  $\bar{Y}_t^n \nearrow \bar{Y}_t \leq \tilde{Y}_t$  and we have  $Y_t^n \nearrow Y_t = \bar{Y}_t - \int_0^t g(s) d\bar{W}_s$ , then  $Y = \lim_{n \rightarrow \infty} Y^n$  in  $\mathbb{H}^2(\mathbb{P})$ .

Besides the process  $(\bar{Y}_t^n + \int_0^t f(s) ds)_{t \leq T}$  is a RCLL supermartingale as a snell envelope of

$(\int_0^t f(s) ds + (\bar{S}_t \wedge \bar{Y}_t^n) \mathbf{1}_{\{t < T\}} + \bar{\xi} \mathbf{1}_{\{t = T\}})_{t \leq T}$  and it converges increasingly to  $(\bar{Y}_t + \int_0^t f(s) ds)_{t \leq T}$ . It follows that this latter process is an RCLL supermartingale. Henceforth the process  $Y$  is also a  $(\mathcal{G}_t)$ -RCLL with  $\mathbb{E}[\sup_{t \leq T} |Y_t|^2] < +\infty$ .

The process  $(Y_t)_{t \geq 0}$  is  $(\mathcal{F}_t)$ -measurable as the limit of a  $(\mathcal{F}_t)$ -measurable process  $(Y_t^n)_{t \geq 0}$ .

Next let us prove that  $Y \geq S$ . We have

$$\mathbb{E}[Y_0^n] = \mathbb{E}[\xi + \int_0^T f(s) ds] + \mathbb{E}[\int_0^T n(Y_s^n - S_s)^- ds].$$

Dividing the two hand-sides by  $n$  and taking the limit as  $n \rightarrow \infty$  to obtain  $\mathbb{E}[\int_0^T (Y_s^n - S_s)^- ds] = 0$ . As the processes  $Y$  and  $S$  are RCLL then  $\mathbb{P}$ -a.s.  $Y_t \geq S_t$ , for  $t < T$ . But  $Y_T = \xi \geq S_T$ , therefore  $Y \geq S$ .

Finally let us show that  $Y$  satisfies (2.6.11). But this is a direct consequence of the continuity of the Snell envelope through sequences of increasing RCLL processes. In fact on the one hand, the sequence of increasing RCLL processes  $((\bar{S}_t \wedge \bar{Y}_t^n) \mathbf{1}_{\{t < T\}} + \bar{\xi} \mathbf{1}_{\{t = T\}})_{t \leq T}$  converges increasingly to the RCLL  $(\bar{S}_t \mathbf{1}_{\{t < T\}} + \bar{\xi} \mathbf{1}_{\{t = T\}})_{t \leq T}$  since  $\bar{Y}_t \geq \bar{S}_t$ .

Therefore, the sequence  $(\int_0^t f(s) ds + \bar{Y}_t^n)_{t \leq T}$  converges to  $\operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}[\int_0^{\tau} f(s) ds + \bar{S}_{\tau} \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} | \mathcal{G}_t]$  which then is equal to  $(\int_0^t f(s) ds + \bar{Y}_t)_{t \leq T}$  and which implies that

$$\begin{aligned} Y_t &= \bar{Y}_t - \int_0^t g(s) d\bar{W}_s \\ &= \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^{\tau} f(s) ds + \bar{S}_{\tau} \mathbf{1}_{\{\tau < T\}} + \bar{\xi} \mathbf{1}_{\{\tau = T\}} | \mathcal{G}_t \right] - \int_0^t g(s) d\bar{W}_s. \end{aligned}$$



*Step 4:* There exist two processes  $Z \in \mathbb{H}^2(\mathbb{P})$ ,  $K \in \mathbb{I}^2(\mathbb{P})$  such that  $(Y, Z, K)$  is the solution of the RBDSDE (2.6.4).

We know from (2.6.11) that the process  $(\int_0^t f(s)ds + \bar{Y}_t^n)_{t \leq T}$  is a Snell envelope. Then, there exist a process  $K \in \mathbb{I}^2(\mathbb{P})$  and a  $(\mathcal{G}_t)$ -martingale such that

$$\int_0^t f(s)ds + Y_t + \int_0^t g(s)d\overleftarrow{W}_s = M_t - K_t, \quad 0 \leq t \leq T.$$

Additionally  $K = K^c + K^d$  where  $K^c$  is continuous non-decreasing and  $K^d$  non-decreasing purely discontinuous predictable such that for any  $t \leq T$ ,  $\Delta_t K^d = (S_{t-} - Y_t)\mathbf{1}_{\{Y_{t-} = S_{t-}\}}$ .

Now the martingale  $M$  belongs to  $\mathbb{S}^2(\mathbb{P})$  then the Itô's martingale representation theorem implies the existence of  $(\mathcal{G}_t)$ -measurable process  $Z$  such that  $\mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty$  and

$$M_t = M_0 + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

Hence

$$Y_t = Y_0 - \int_0^t f(s)ds - \int_0^t g(s)d\overleftarrow{W}_s + \int_0^t Z_s dB_s - K_t, \quad 0 \leq t \leq T.$$

Let us now prove that  $\int_0^T (Y_s - S_s)dK_s^c = 0$ .

It remains to show that  $Z_t$  and  $K_t$  are in fact  $\mathcal{F}_t$ -adapted. For  $K_t$ , it is obvious since it is the limit of  $K_t^n = \int_0^t n(Y_s^n - S_s)^- ds$  which is  $\mathcal{F}_t$ -measurable for each  $t \leq T$ .

Now

$$\int_t^T Z_s dB_s = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{W}_s - Y_t + K_T - K_t,$$

and the right side is  $\mathcal{F}_T^B \vee \mathcal{F}_{t,T}^W$ -measurable. Hence from the Itô's martingale representation theorem  $Z_s$ ,  $t < s < T$  is  $\mathcal{F}_s^B \vee \mathcal{F}_{t,T}^W$  adapted. Consequently  $Z_s$  is  $\mathcal{F}_s^B \vee \mathcal{F}_{s,T}^W$ -measurable for any  $t < s$ , so it is  $\mathcal{F}_s^B \vee \mathcal{F}_{s,T}^W$ -measurable.

**b) Uniqueness:** under Lipschitz continuous condition, the proof of uniqueness is a standard in BSDE theory (see e.g. proof of Proposition 2.1. in [2]).  $\square$

The existence of solution of RBDSDE (v) in Theorem 2.6.2 is obtained via a standard fixed Banach point theorem for reflected BSDEs (see for instance El Karoui, Hamadene and Matoussi [45]).



# The obstacle problem for semilinear parabolic PIDEs

## Contents

<b>3.1</b>	<b>Introduction</b>	<b>65</b>
<b>3.2</b>	<b>Hypotheses and preliminaries</b>	<b>67</b>
3.2.1	Weak formulation for the partial differential-integral equations	68
3.2.2	Stochastic flow of diffeomorphisms and random test functions	69
<b>3.3</b>	<b>Sobolev solutions for parabolic semilinear PIDEs</b>	<b>72</b>
<b>3.4</b>	<b>Obstacle problem for PIDEs</b>	<b>73</b>
<b>3.5</b>	<b>Appendix</b>	<b>78</b>
3.5.1	Proof of Proposition 3.2.3	78
3.5.2	Regularity of the solution of BSDE with jumps	79
3.5.3	Proof of Theorem 3.3.1: existence and uniqueness of solution of PIDE (3.1.1)	85
3.5.4	Proof of the tightness of the sequence $(\pi_n)_{n \in \mathbb{N}}$	88

## 3.1 Introduction

Our main interest in this chapter is to study the following partial integro-differential equations (in short PIDEs) of parabolic type:

$$(\partial_t + \mathcal{L})u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(x), u(t, x + \beta(x, \cdot)) - u(t, x)) = 0 \quad (3.1.1)$$

over the time interval  $[0, T]$ , with a given final condition  $u_T = g$ ,  $f$  is a nonlinear function and  $\mathcal{L} = \mathcal{K}_1 + \mathcal{K}_2$  is the second order integro-differential operator associated with a jump diffusion which is defined component by component with

$$\begin{aligned} \mathcal{K}_1\varphi(x) &= \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) \text{ and} \\ \mathcal{K}_2\varphi(x) &= \int_{\mathbb{E}} \left( \varphi(x + \beta(x, e)) - \varphi(x) - \sum_{i=1}^d \beta^i(x, e) \frac{\partial}{\partial x_i} \varphi(x) \right) \lambda(de), \quad \varphi \in C^2(\mathbb{R}^d). \end{aligned} \quad (3.1.2)$$

More precisely, our main interest is to consider the final condition to be a fixed function  $g \in \mathbf{L}^2(\mathbb{R}^d)$  and the obstacle  $h$  be a continuous function  $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ . Then the obstacle problem for the

equation (3.1.1) is defined as a pair  $(u, \nu)$ , where  $\nu$  is a regular measure concentrated on  $\{u = h\}$  and  $u \in \mathbf{L}^2([0, T] \times R^d; R)$  satisfies the following relations:

$$\begin{aligned}
(i') \quad & u \geq h, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.}, \\
(ii') \quad & \partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(x), u(t, x + \beta(x, \cdot))) - u(t, x) = -\nu(dt, dx), \\
(iii') \quad & \nu(u > h) = 0, \quad \text{a.s.}, \\
(iv') \quad & u_T = g, \quad dx - \text{a.e.}
\end{aligned} \tag{3.1.3}$$

$\nu$  represents the quantity which forces  $u$  to stay above the obstacle  $h$  and we get the uniqueness result for the obstacle problem. In Section 3.4, the rigorous sense of the relation (iii') which is based on the probabilistic representation of the measure  $\nu$  and plays the role of quasi-continuity of  $u$  in this context will be explained. The main result of the chapter is Theorem 3.4.1 which ensures the existence and uniqueness of the solution  $(u, \nu)$  of the obstacle problem for (3.1.1) using a probabilistic method based on reflected BSDEs with jumps. The proof is based on the penalization procedure. It can be noted that the quasi-sure approaches for the PIDEs (probabilistic [98] or analytical one [38]) are unsuccessful. It remains unclear, until now, how to define the analytical potential associated to the operator  $\mathcal{L}$  specially for the non local operator  $\mathcal{K}_2$ . Thus, the stochastic flow method developed by Bally and Matoussi in [10] (see also [97]) for a class of parabolic semilinear SPDEs is used in this context.

As a preliminary work, first we present the existence and uniqueness of Sobolev's solution of PIDE (3.1.1) (without obstacle) and provide a probabilistic interpretation by using solution of BSDEs driven by a Brownian motion and an independent random measure. The concern is to solve our problem by developing a stochastic flow method based on the results of Léandre [82] about the homeomorphic property for the solution of SDEs with jumps. The key element in [10] is to use the inversion of stochastic flow which transforms the variational formulation of the PDEs to the associated BSDEs. Thus it plays the same role as Itô's formula in the case of the classical solution of PDEs.

This chapter is organized as following: in section 4.2.1, first the basic assumptions and the definitions of the solutions for PIDEs are presented. We provide useful results on stochastic flow associated with the forward SDEs with jumps, then in this setting a class of random test functions and their semimartingale decomposition are introduced. Finally, an equivalence norm result is given in the jump diffusion case. In section 3.3, we prove the existence and uniqueness results of the solution of our PIDEs and give the associated probabilistic interpretation via the FBSDEs with jumps. The uniqueness is a consequence of the variational formulation of the PIDEs written with random test functions and the uniqueness of the solution of the FBSDE. The existence of the solution is established by an approximation penalization procedure, a priori estimates and the equivalence norm results. In section 3.4, we prove existence and uniqueness of the solution of the obstacle problem for the PIDEs. The proof of this result differs from that of Bally et al [9] since we have to consider the stochastic flow associated with the forward jump diffusion process. In particular, the jump part appearing of the tightness result for the approximation measure has to be taken into account. In the Appendix, we first give the proof of the equivalence norm results, then prove a regularity result for the BSDEs solution with respect to the time-state variable  $(t, x)$ , in order to relate the solution of BSDEs to the classical solution of our PIDEs. Finally, we give a proof of a technical lemma which is crucial for the existence of the regular measure part of the solution of our obstacle problem for PIDEs. This chapter is based on [96].

## 3.2 Hypotheses and preliminaries

Let  $T > 0$  be a finite time horizon and  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space on which is defined two independent processes:

- a  $d$ -dimensional Brownian motion  $W_t = (W_t^1, \dots, W_t^d)$ ;
- a Poisson random measure  $\mu = \mu(dt, de)$  on  $([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{B}_E)$ , where  $E = \mathbb{R}^d \setminus \{0\}$  is equipped with its Borel field  $\mathcal{B}_E$ , with compensator  $\nu(dt, de) = \lambda(de)dt$ , such that  $\{\tilde{\mu}([0, T] \times A) = (\mu - \nu)([0, T] \times A)\}_{t \geq 0}$  is a martingale for all  $A \in \mathcal{B}_E$  satisfying  $\lambda(A) < \infty$ .  $\lambda$  is assumed to be a  $\sigma$ -finite measure on  $(E, \mathcal{B}_E)$  satisfying

$$\int_{\mathbb{E}} (1 \wedge |e|^2) \lambda(de) < +\infty$$

Denote  $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}_E$  where  $\mathcal{P}$  is the predictable  $\sigma$ -field on  $\Omega \times [0, T]$ .

Let  $(\mathcal{F}_t)_{t \geq 0}$  be the filtration generated by the above two processes and augmented by the  $P$ -null sets of  $\mathcal{F}$ . Besides let us define:

- $|X|$  the Euclidean norm of a vector  $X$ ;
- $\mathbf{L}^2(E, \mathcal{B}_E, \lambda; \mathbb{R}^n)$  (noted as  $\mathbf{L}_\lambda^2$  for convenience) the set of measurable functions from  $(E, \mathcal{B}_E, \lambda)$  to  $\mathbb{R}^n$  endowed with the topology of convergence in measure and for  $v \in \mathbf{L}^2(E, \mathcal{B}_E, \lambda; \mathbb{R}^n)$

$$\|v\|^2 = \int_E |v(e)|^2 \lambda(de) \in \mathbb{R}^+ \cup \{+\infty\};$$

- $\mathbf{L}_n^p(\mathcal{F}_T)$  the space of  $n$ -dimensional  $\mathcal{F}_T$ -measurable random variables  $\xi$  such that

$$\|\xi\|_{L^p}^p := \mathbb{E}(|\xi|^p) < +\infty;$$

- $\mathcal{H}_{n \times d}^p([0, T])$  the space of  $\mathbb{R}^{n \times d}$ -valued  $\mathcal{P}$ -measurable process  $Z = (Z_t)_{t \leq T}$  such that

$$\|Z\|_{\mathcal{H}^p}^p := \mathbb{E}[(\int_0^T |Z_t|^2 dt)^{p/2}] < +\infty;$$

- $\mathcal{S}_n^p([0, T])$  the space of  $n$ -dimensional  $\mathcal{F}_t$ -adapted càdlàg processes  $Y = (Y_t)_{t \leq T}$  such that

$$\|Y\|_{\mathcal{S}^p}^p := \mathbb{E}[\sup_{t \leq T} |Y_t|^p] < +\infty;$$

- $\mathcal{A}_n^p(t, T)$  the space of  $n$ -dimensional  $\mathcal{F}_t$ -adapted non-decreasing càdlàg processes  $K = (K_t)_{t \leq T}$  such that

$$\|K\|_{\mathcal{A}^p}^p := \mathbb{E}[|K_T|^p] < +\infty;$$

- $\mathcal{L}_n^p([0, T])$  the space of  $\mathbb{R}^n$ -valued  $\tilde{\mathcal{P}}$ -measurable mappings  $V(\omega, t, e)$  such that

$$\|V\|_{\mathcal{L}^p}^p := \mathbb{E}[(\int_0^T \|V_t\|^2 dt)^{p/2}] = \mathbb{E}[(\int_0^T \int_E |V_t(e)|^2 \lambda(de) dt)^{p/2}] < +\infty.$$

- $C_{l,b}^k(\mathbb{R}^p, \mathbb{R}^q)$  the set of  $C^k$ -functions which grow at most linearly at infinity and whose partial derivatives of order less than or equal to  $k$  are bounded.

- $\mathbf{L}_\rho^2(\mathbb{R}^d)$  will be the basic Hilbert space of our framework. We employ the usual notation for its scalar product and its norm,

$$(u, v)_\rho = \int_{\mathbb{R}^d} u(x) v(x) \rho(x) dx, \quad \|u\|_2 = \left( \int_{\mathbb{R}^d} u^2(x) \rho(x) dx \right)^{\frac{1}{2}}.$$

where  $\rho$  is a continuous positive and integrable weight function. We assume additionally that  $\frac{1}{\rho}$  is locally integrable

In general, we shall use the notation

$$(u, v) = \int_{\mathbb{R}^d} u(x) v(x) dx,$$

where  $u, v$  are measurable functions defined in  $\mathbb{R}^d$  and  $uv \in \mathbf{L}^1(\mathbb{R}^d)$ .

Our evolution problem will be considered over a fixed time interval  $[0, T]$  and the norm for an element of  $\mathbf{L}^2_\rho([0, T] \times \mathbb{R}^d)$  will be denoted by

$$\|u\|_{2,2} = \left( \int_0^T \int_{\mathbb{R}^d} |u(t, x)|^2 \rho(x) dx dt \right)^{\frac{1}{2}}.$$

We usually omit the subscript when  $n = 1$ . We assume the following hypotheses :

**(A1)**  $g$  belongs to  $\mathbf{L}^2_\rho(\mathbb{R}^d)$ ;

**(A2)**  $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbf{L}^2(E, \mathcal{B}_E, \lambda; \mathbb{R}^m) \rightarrow \mathbb{R}^m$  is measurable in  $(t, x, y, z, v)$  and satisfies  $f^0 \in \mathbf{L}^2_\rho([0, T] \times \mathbb{R}^d)$  where  $f^0 := f(\cdot, \cdot, 0, 0, 0)$ .

**(A3)**  $f$  satisfies Lipschitz condition in  $(y, z, v)$ , i.e., there exists a constant  $C$  such that for any  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $(y, z, v), (y', z', v') \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbf{L}^2(E, \mathcal{B}_E, \lambda; \mathbb{R}^m)$ :

$$|f(t, x, y, z, v) - f(t, x, y', z', v')| \leq C(|y - y'| + |z - z'| + \|v - v'\|);$$

**(A4)**  $b \in C^3_{l,b}(\mathbb{R}^d; \mathbb{R}^d)$ ,  $\sigma \in C^3_{l,b}(\mathbb{R}^d; \mathbb{R}^{d \times d})$ ,  $\beta : \mathbb{R}^d \times E \rightarrow \mathbb{R}^d$  be measurable and for all  $e \in \mathbb{E}$ ,  $\beta(\cdot, e) \in C^3_{l,b}(\mathbb{R}^d; \mathbb{R}^d)$ , and for some  $K > 0$  and for all  $x \in \mathbb{R}^d, e \in E$ ,

$$|\beta(x, e)| \leq K(1 \wedge |e|), \quad |D^\alpha \beta(x, e)| \leq K(1 \wedge |e|) \text{ for } 1 \leq |\alpha| \leq 3,$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$  is a multi-index and  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$ .  $D^\alpha$  is the differential operator  $D^\alpha = \frac{\partial^{|\alpha|}}{(\partial^{\alpha_1} x_1)(\partial^{\alpha_2} x_2) \dots (\partial^{\alpha_d} x_d)}$ .

### 3.2.1 Weak formulation for the partial differential-integral equations

The space of test functions which we employ in the definition of weak solutions of the evolution equations (3.1.1) is  $\mathcal{D}_T = \mathcal{C}^\infty([0, T]) \otimes \mathcal{C}^\infty_c(\mathbb{R}^d)$ , where  $\mathcal{C}^\infty([0, T])$  denotes the space of real functions which can be extended as infinite differentiable functions in the neighborhood of  $[0, T]$  and  $\mathcal{C}^\infty_c(\mathbb{R}^d)$  is the space of infinite differentiable functions with compact support in  $\mathbb{R}^d$ . We denote the space of solutions by

$$\mathcal{H}_T := \{u \in \mathbf{L}^2_\rho([0, T] \times \mathbb{R}^d) \mid \nabla u \sigma \in \mathbf{L}^2_\rho([0, T] \times \mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T} = \left( \int_{\mathbb{R}^d} \int_0^T [|u(s, x)|^2 + |\nabla u(s, x) \sigma(x)|^2] ds \rho(x) dx \right)^{1/2},$$

where we denote the gradient by  $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$ .

**Definition 3.2.1.** We say that  $u \in \mathcal{H}_T$  is a Sobolev solution of PIDE (3.1.1) if the following relation holds, for each  $\phi \in \mathcal{D}_T$ ,

$$\begin{aligned} & \int_t^T (u(s, x), \partial_s \phi(s, x)) ds + (u(t, x), \phi(t, x)) - (g(x), \phi(T, x)) - \int_t^T (u(s, x), \mathcal{L}^* \phi(s, x)) ds \\ & = \int_t^T (f(s, x, u(s, x), \nabla u(s, x) \sigma(x), u(s, x + \beta(x, \cdot)) - u(s, x)), \phi(s, x)) ds. \end{aligned} \quad (3.2.1)$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ . We denote by  $u := \mathcal{U}(g, f)$  such a solution.

### 3.2.2 Stochastic flow of diffeomorphisms and random test functions

In this section, we shall study the stochastic flow associated to the forward jump diffusion component. The main motivation is to generalize in the jump setting the flow technics which was first introduced in [10] for the study of semilinear PDE's. Let  $(X_{t,s}(x))_{t \leq s \leq T}$  be the strong solution of the equation:

$$X_{t,s}(x) = x + \int_t^s b(X_{t,r}(x))dr + \int_t^s \sigma(X_{t,r}(x))dW_r + \int_t^s \int_E \beta(X_{t,r-}(x), e)\tilde{\mu}(dr, de). \quad (3.2.2)$$

The existence and uniqueness of this solution was proved in Fujiwara and Kunita [52]. Moreover, we have the following properties (see Theorem 2.2 and Theorem 2.3 in [52]):

**Proposition 3.2.1.** *For each  $t > 0$ , there exists a version of  $\{X_{t,s}(x); x \in \mathbb{R}^d, s \geq t\}$  such that  $X_{t,s}(\cdot)$  is a  $C^2(\mathbb{R}^d)$ -valued càdlàg process. Moreover:*

- (i)  $X_{t,s}(\cdot)$  and  $X_{0,s-t}(\cdot)$  have the same distribution,  $0 \leq t \leq s$ ;
- (ii)  $X_{t_0,t_1}, X_{t_1,t_2}, \dots, X_{t_{n-1},t_n}$  are independent, for all  $n \in \mathbb{N}$ ,  $0 \leq t_0 < t_1 < \dots < t_n$ ;
- (iii)  $X_{t,r}(x) = X_{s,r} \circ X_{t,s}(x)$ ,  $0 \leq t < s < r$ .

Furthermore, for all  $p \geq 2$ , there exists  $M_p$  such that for all  $0 \leq t < s$ ,  $x, x' \in \mathbb{R}^d$ ,  $h, h' \in \mathbb{R} \setminus \{0\}$ , where  $\Delta_h^i g(x) = \frac{1}{h}(g(x + he_i) - g(x))$ , and  $(e_1, \dots, e_d)$  is an orthonormal basis of  $\mathbb{R}^d$ .

It is also known that the stochastic flow solution of a continuous SDE satisfies the homeomorphic property (see Bismut [16], Kunita [76], [80]). But this property fails for the solution of SDE with jumps in general. P.-A. Meyer in [102] (Remark p.111), gave a counterexample with the following exponential equation:

$$X_{0,t}(x) = x + \int_0^t X_{0,s-} dZ_s$$

where  $Z$  is semimartingale,  $Z_0 = 0$ , such that  $Z$  has a jump of size  $-1$  at some stopping time  $\tau$ ,  $\tau > 0$  a.s. Then all trajectories of  $X$ , starting at any initial value  $x$ , become zero at  $\tau$  and stay there after  $\tau$ . This may be seen trivially by the explicit form of the solution given by the Doléans-Dade exponential:

$$X_{0,t}(x) = x \exp\left(Z_t - \frac{1}{2}[Z, Z]_t^c\right) \prod_{0 < s \leq t} (1 + \Delta Z_s) e^{-\Delta Z_s}.$$

In the general setting of non-linear SDE, at the jump time  $\tau$ , the solution jumps from  $X_{0,\tau-}(x)$  to  $X_{0,\tau-}(x) + \beta(X_{0,\tau-}(x))$ . Léandre [82] gave a necessary and sufficient condition under which the homeomorphic property is preserved at the jump time, namely, for each  $e \in \mathbb{E}$ , the maps  $H_e : x \mapsto x + \beta(x, e)$  should be one to one and onto. One can read also Fujiwara and Kunita [52], Protter [122] and Kunita [77] for more details on the subject. Therefore, we assume additionally that, for each  $e \in \mathbb{E}$ , the linkage operator:

$$(A5) \quad H_e : x \mapsto x + \beta(x, e) \text{ is a } C^2\text{-diffeomorphism.}$$

We denote by  $H_e^{-1}$  the inverse map of  $H_e$ , and set  $h(x, e) := x - H_e^{-1}(x)$ . We have the following result where the proof can be found in [81] (Theorem 3.13, p.359):

**Proposition 3.2.2.** *Assume the assumptions (A4) and (A5) hold. Then  $\{X_{t,s}(x); x \in \mathbb{R}^d\}$  is a  $C^2$ -diffeomorphism a.s. stochastic flow. Moreover the inverse of the flow satisfies the following backward SDE*

$$\begin{aligned} X_{t,s}^{-1}(y) &= y - \int_t^s \widehat{b}(X_{r,s}^{-1}(y))dr - \int_t^s \sigma(X_{r,s}^{-1}(y))\overleftarrow{dW}_r - \int_t^s \int_E \beta(X_{r,s}^{-1}(y), e)\overleftarrow{\mu}(dr, de) \\ &\quad + \int_t^s \int_E \widehat{\beta}(X_{r,s}^{-1}(y), e)\overleftarrow{\mu}(dr, de), \end{aligned} \quad (3.2.3)$$

for any  $t < s$ , where

$$\widehat{b}(x) = b(x) - \sum_{i,j} \frac{\partial \sigma^j(x)}{\partial x_i} \sigma^{ij}(x) \quad \text{and} \quad \widehat{\beta}(x, e) = \beta(x, e) - h(x, e). \quad (3.2.4)$$

The explicit form (3.2.3) will be used in the proof of the equivalence of norms (Proposition 3.2.3).

**Remark 3.2.1.** In (3.2.3), the three terms  $\int_t^s \sigma(X_{r,s}^{-1}(y)) d\overleftarrow{W}_r$ ,  $\int_t^s \int_E \beta(X_{r,s}^{-1}(y), e) \tilde{\mu}(\overleftarrow{dr}, de)$  and  $\int_t^s \int_E \widehat{\beta}(X_{r,s}^{-1}(y), e) \mu(\overleftarrow{dr}, de)$  are backward Itô integrals. We refer the readers to literature [81] for the definition ([81] p. 358). For convenience, we give the definition of the backward Itô integral with respect to a Brownian motion. Let  $f(r)$  be a right continuous backward adapted process, then the backward Itô integral is defined by

$$\int_t^s f(r) d\overleftarrow{W}_r := \lim_{|\Pi| \rightarrow 0} \sum_k f(t_{k+1})(W_{t_{k+1}} - W_{t_k}),$$

where  $\Pi = \{t = t_0 < t_1 < \dots < t_n = s\}$  are partitions of the interval  $[t, s]$ . The other two terms can be defined similarly. Note that the inverse flow  $X_{r,s}^{-1}$  is backward adapted, so we may define the backward integrals such as  $\int_t^s \sigma(X_{r,s}^{-1}(y)) d\overleftarrow{W}_r$  etc..

**Remark 3.2.2.** In the paper of Ouknine and Turpin [106], the authors have weakened the regularity of the coefficients  $b$  and  $\sigma$  of the diffusion, but added additional boundedness on them. Since this improvement is not essential and the same discussion is also valid for our case, we omit it.

We denote by  $J(X_{t,s}^{-1}(x))$  the determinant of the Jacobian matrix of  $X_{t,s}^{-1}(x)$ , which is positive and  $J(X_{t,t}^{-1}(x)) = 1$ . For  $\phi \in C_c^\infty(\mathbb{R}^d)$ , we define a process  $\phi_t : \Omega \times [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$\phi_t(s, x) := \phi(X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)). \quad (3.2.5)$$

We know that for  $v \in \mathbf{L}^2(\mathbb{R}^d)$ , the composition of  $v$  with the stochastic flow is

$$(v \circ X_{t,s}(\cdot), \phi) := (v, \phi_t(s, \cdot)).$$

In fact, by a change of variable, we have

$$(v \circ X_{t,s}(\cdot), \phi) = \int_{\mathbb{R}^d} v(X_{t,s}(x)) \phi(x) dx = \int_{\mathbb{R}^d} v(y) \phi(X_{t,s}^{-1}(y)) J(X_{t,s}^{-1}(y)) dy = (v, \phi_t(s, \cdot)).$$

Since  $(\phi_t(s, x))_{t \leq s}$  is a process, we may not use it directly as a test function because  $\int_t^T (u(s, \cdot), \partial_s \phi_t(s, \cdot))$  has no sense. However  $\phi_t(s, x)$  is a semimartingale and we have the following decomposition of  $\phi_t(s, x)$ :

**Lemma 3.2.1.** For every function  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\begin{aligned} \phi_t(s, x) &= \phi(x) + \int_t^s \mathcal{L}^* \phi_t(r, x) dr - \sum_{j=1}^d \int_t^s \left( \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma^{ij}(x) \phi_t(r, x)) \right) dW_r^j \\ &\quad + \int_t^s \int_E \mathcal{A}_e^* \phi_t(r-, x) \tilde{\mu}(dr, de), \end{aligned} \quad (3.2.6)$$

where  $\mathcal{A}_e u(t, x) = u(t, H_e(x)) - u(t, x)$ ,  $\mathcal{A}_e^* u(t, x) = u(t, H_e^{-1}(x)) J(H_e^{-1}(x)) - u(t, x)$  and  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ .

**Proof.** Assume that  $v \in C_c^\infty(\mathbb{R}^d)$ . Applying the change of variable  $y = X_{t,s}^{-1}(x)$ , we can get

$$\begin{aligned} \int_{\mathbb{R}^d} v(x) (\phi_t(s, x) - \phi(x)) dx &= \int_{\mathbb{R}^d} v(x) (\phi(X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) - \phi(X_{t,t}^{-1}(x)) J(X_{t,t}^{-1}(x))) dx \\ &= \int_{\mathbb{R}^d} (v(X_{t,s}(y)) \phi(y) - v(y) \phi(y)) dy \\ &= \int_{\mathbb{R}^d} \phi(y) (v(X_{t,s}(y)) - v(y)) dy. \end{aligned}$$

As  $v$  is smooth enough, using Itô's formula for  $v(X_{t,s}(y))$ , we have



$$\begin{aligned} v(X_{t,s}(y)) - v(y) &= \int_t^s \mathcal{L}v(X_{t,r-}(y))dr + \int_t^s \sum_{i=1}^d \frac{\partial v}{\partial x_i}(X_{t,r}(y)) \sum_{j=1}^d \sigma^{ij}(X_{t,r}(y))dW_r^j \\ &\quad + \int_t^s \int_E \mathcal{A}_e v(X_{t,r-}(y))\tilde{\mu}(dr, de). \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^d} v(x)(\phi_t(s, x) - \phi(x))dx \\ &= \int_{\mathbb{R}^d} \phi(y) \left\{ \int_t^s \mathcal{L}v(X_{t,r}(y))dr + \int_t^s \sum_{i=1}^d \frac{\partial v}{\partial x_i}(X_{t,r}(y)) \sum_{j=1}^d \sigma^{ij}(X_{t,r}(y))dW_r^j \right\} dy \\ &\quad + \int_{\mathbb{R}^d} \phi(y) \int_t^s \int_E \mathcal{A}_e v(X_{t,r-}(y))\tilde{\mu}(dr, de)dy. \end{aligned}$$

Since the first term has been dealt in [10], and the adjoint operator of  $\mathcal{L}$  exists thanks to [92], we focus only on the second term. Using the stochastic Fubini theorem and the change of variable  $x = X_{t,r}(y)$  we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(y) \int_t^s \int_E \mathcal{A}_e v(X_{t,r-}(y))\tilde{\mu}(dr, de)dy &= \int_t^s \int_E \int_{\mathbb{R}^d} \phi(y)\mathcal{A}_e v(X_{t,r-}(y))dy\tilde{\mu}(dr, de) \\ &= \int_t^s \int_E \int_{\mathbb{R}^d} \phi_t(r-, x)\mathcal{A}_e v(x)dx\tilde{\mu}(dr, de). \end{aligned}$$

Finally, we use the change of variable  $y = H_e^{-1}(x)$  in the right hand side of the previous expression

$$\begin{aligned} &\int_t^s \int_E \int_{\mathbb{R}^d} \phi_t(r-, x)\mathcal{A}_e v(x)dx\tilde{\mu}(dr, de) \\ &= \int_t^s \int_E \int_{\mathbb{R}^d} \phi_t(r-, x)(v(H_e(x)) - v(x))dx\tilde{\mu}(dr, de) \\ &= \int_t^s \int_E \int_{\mathbb{R}^d} v(x)\mathcal{A}_e^* \phi_t(r-, x)dx\tilde{\mu}(dr, de). \end{aligned}$$

Since  $v$  is an arbitrary function, the lemma is proved.

We also need equivalence of norms result which plays an important role in the proof of the existence of the solution for PIDE as a connection between the functional norms and random norms. For continuous SDEs, this result was first proved by Barles and Lesigne [13] by using an analytic method. In [10], the authors have proved the result with a probabilistic method. Note that Klimisiak [73] have extended this estimates for Markov process associated to a non-homogeneous divergence operator. The following result generalize Proposition 5.1 in [10] (see also [13]) in the case of a diffusion process with jumps, and the proof will be given in Appendix 3.5.1.

**Proposition 3.2.3.** *There exists two constants  $c > 0$  and  $C > 0$  such that for every  $t \leq s \leq T$  and  $\varphi \in L^1(\mathbb{R}^d, dx)$ ,*

$$c \int_{\mathbb{R}^d} |\varphi(x)|\rho(x)dx \leq \int_{\mathbb{R}^d} \mathbb{E}(|\varphi(X_{t,s}(x))|)\rho(x)dx \leq C \int_{\mathbb{R}^d} |\varphi(x)|\rho(x)dx. \quad (3.2.7)$$

Moreover, for every  $\Psi \in L^1([0, T] \times \mathbb{R}^d, dt \otimes dx)$ ,

$$c \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)|ds\rho(x)dx \leq \int_{\mathbb{R}^d} \int_t^T \mathbb{E}(|\Psi(s, X_{t,s}(x))|)ds\rho(x)dx \leq C \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)|ds\rho(x)dx. \quad (3.2.8)$$

We give now the following result which allows us to link by a natural way the solution of PIDE with the associated BSDE. Roughly speaking, if we choose in the variational formulation (3.2.1) the random functions  $\phi_t(\cdot, \cdot)$  defined by (3.2.5), as a test functions, then we obtain the associated BSDE. In fact, this result plays the same role as Itô's formula used in [108] and [115] (see [108], Theorem 3.1, p. 20) to relate the solution of some semilinear PDE's with the associated BSDE:

**Proposition 3.2.4.** *Assume that all the previous assumptions hold. Let  $u \in \mathcal{H}_T$  be a weak solution of PIDE(3.1.1), then for  $s \in [t, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r, x) d\phi_t(r, x) dx + (u(s, x), \phi_t(s, x)) - (g(x), \phi_t(T, x)) - \int_s^T (u(r, \cdot), \mathcal{L}^* \phi_t(r, \cdot)) dr \\ &= \int_{\mathbb{R}^d} \int_s^T f(r, x, u(r, x), \nabla u(r, x) \sigma(x), u(r, x + \beta(x, \cdot)) - u(r, x)) \phi_t(r, x) dr dx. \quad a.s. \end{aligned} \quad (3.2.9)$$

where  $\int_{\mathbb{R}^d} \int_s^T u(r, x) d\phi_t(r, x) dx$  is well defined thanks to the semimartingale decomposition result (Lemma 3.2.1).

**Remark 3.2.3.** *Note that  $\phi_t(r, x)$  is  $\mathbb{R}$ -valued. We consider that in (3.2.9), the equality holds for each component of  $u$ .*

**Remark 3.2.4.** *Under Brownian framework, this proposition is first proved by Bally and Matoussi in [10] for linear case via the polygonal approximation for Brownian motion (see Appendix A in [10] for more details). In fact, thanks especially to the fact that  $\lambda$  is finite, we can make the similar approximation for Itô-Lévy processes only by approximating polygonally the Brownian motion, the proof of this proposition follows step by step the proof of Proposition 2.3 in [10] (pp. 156), so we omit it.*

### 3.3 Sobolev solutions for parabolic semilinear PIDEs

In this section, we consider the PIDE (3.1.1) under assumptions (A1)-(A5). Moreover, we consider the following decoupled forward backward stochastic differential equation (FBSDE in short) :

$$\begin{cases} X_{t,s}(x) &= x + \int_t^s b(X_{t,r}(x)) dr + \int_t^s \sigma(X_{t,r}(x)) dW_r + \int_t^s \int_E \beta(X_{t,r-}(x), e) \tilde{\mu}(dr, de); \\ Y_s^{t,x} &= g(X_{t,T}(x)) + \int_s^T f(r, X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}) dr \\ &\quad - \int_s^T Z_r^{t,x} dW_r - \int_s^T \int_E V_r^{t,x}(e) \tilde{\mu}(dr, de). \end{cases} \quad (3.3.1)$$

According to Proposition 5.4 in [33] which deals with of reflected BSDE, we know that (3.3.1) has a unique solution. Moreover, we have the following estimate of the solution.

**Proposition 3.3.1.** *There exists a constant  $c > 0$  such that, for any  $s \in [t, T]$ :*

$$\sup_{s \in [t, T]} \mathbb{E} [\|Y_s^{t,\cdot}\|_2^2] + \mathbb{E} \left[ \int_t^T \|Z_s^{t,\cdot}\|_2^2 ds + \int_t^T \int_E |V_s^{t,\cdot}(e)|_2^2 \lambda(de) ds \right] \leq c [\|g\|_2^2 + \int_t^T \|f_s^0\|_2^2 ds]. \quad (3.3.2)$$

Our main result in this section is the following where the proof will be given in Appendix 3.5.3:

**Theorem 3.3.1.** *Assume that (A1)-(A5) hold. There exists a unique solution  $u \in \mathcal{H}_T$  of the PIDE (3.1.1). Moreover, we have the probabilistic representation of the solution:  $u(t, x) = Y_t^{t,x}$ , where  $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x})$  is the solution of BSDE (3.3.1) and, we have,  $ds \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.$ ,*

$$\begin{aligned} Y_s^{t,x} &= u(s, X_{t,s}(x)), & Z_s^{t,x} &= (\sigma^* \nabla u)(s, X_{t,s}(x)), \\ V_s^{t,x}(\cdot) &= u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x)). \end{aligned} \quad (3.3.3)$$

**Remark 3.3.1.** *Since  $u \in \mathcal{H}_T$ ,  $u$  and  $v = (\nabla u \sigma)$  are elements in  $L_\rho^2([0, T] \times \mathbb{R}^d)$  and they are determined  $\rho(x) dx$  a.e., but because of the equivalence of norms, there is no ambiguity in the definition of  $u(s, X_{t,s}(x))$  and the others terms of (3.3.3).*

**Remark 3.3.2.** *This stochastic flow method can be generalized to the study of Sobolev solution of stochastic partial integro-differential equations (SPIDEs for short) without essential difficulties (see e.g. [10] for Brownian framewrok). More precisely, as the authors have done in [10, 109], by introducing an appropriate backward doubly stochastic differential equation (BDSDE for short) with jump, we can provide a probabilistic interpretation for Sobolev solution of an SPIDE by the solution of the BDSDE with jump.*

### 3.4 Obstacle problem for PIDEs

In this part, we will study the obstacle problem (3.1.3) with obstacle function  $h$ , where we restrict our study in the one dimensional case ( $n = 1$ ). We shall assume the following hypothesis on the obstacle:

**(A6)**  $h \in C([0, T] \times \mathbb{R}^d; \mathbb{R})$  and there exist  $\iota, \kappa > 0$  such that  $|h(t, x)| \leq \iota(1 + |x|^\kappa)$ , for all  $x \in \mathbb{R}^d$ .

We first introduce the reflected BSDE with jumps (RBSDE with jumps for short) associated with  $(g, f, h)$  which has been studied by Hamadène and Ouknine [65]:

$$\begin{cases} Y_s^{t,x} = g(X_{t,T}(x)) + \int_s^T f(r, X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x})dr + K_T^{t,x} - K_s^{t,x} \\ \quad - \int_s^T Z_r^{t,x}dW_r - \int_s^T \int_E V_r^{t,x}(e)\tilde{\mu}(dr, de), \mathbb{P}\text{-a.s.}, \forall s \in [t, T] \\ Y_s^{t,x} \geq L_s^{t,x}, \quad \int_t^T (Y_s^{t,x} - L_s^{t,x})dK_s^{t,x} = 0, \mathbb{P}\text{-a.s.} \end{cases} \quad (3.4.1)$$

The obstacle process  $L_s^{t,x} = h(s, X_{t,s}(x))$  is a càdlàg process which has only inaccessible jumps since  $h$  is continuous and  $(X_{t,s}(x))_{t \leq s \leq T}$  admits inaccessible jumps. Moreover, using assumption **(A1)** and **(A2)** and equivalence of norm results (3.2.7) and (3.2.8), we get

$$g(X_{t,T}(x)) \in \mathbf{L}^2(\mathcal{F}_T), \text{ and } f(s, X_{t,s}(x), 0, 0, 0) \in \mathcal{H}_d^2(t, T).$$

Therefore according to [65], there exists a unique quadruple  $(Y^{t,x}, Z^{t,x}, V^{t,x}, K^{t,x}) \in \mathcal{S}^2(t, T) \times \mathcal{H}_d^2(t, T) \times \mathcal{L}^2(t, T) \times \mathcal{A}^2(t, T)$  solution of the RBSDE with jumps (3.4.1).

More precisely, we consider the following definition of weak solutions for the obstacle problem (3.1.3):

**Definition 3.4.1.** *We say that  $(u, \nu)$  is the weak solution of the PIDE with obstacle associated to  $(g, f, h)$ , if*

- (i)  $\|u\|_{\mathcal{H}_T}^2 < \infty$ ,  $u \geq h$ , and  $u(T, x) = g(x)$ ,
- (ii)  $\nu$  is a positive Radon regular measure in the following sense, i.e. for every measurable bounded and positive functions  $\phi$  and  $\psi$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s, x)1_{\{u=h\}}(s, x)\nu(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \phi(s, x)\psi(s, X_{t,s}(x))dK_s^{t,x} dx, \text{ a.s.} \end{aligned} \quad (3.4.2)$$

where  $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$  is the solution of RBSDE with jumps (3.4.1) and such that

$$\int_0^T \int_{\mathbb{R}^d} \rho(x)\nu(dt, dx) < \infty,$$

(iii) for every  $\phi \in \mathcal{D}_T$

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u(s, x) \partial_s \phi(s, x) dx ds + \int_{\mathbb{R}^d} (u(t, x) \phi(t, x) - g(x) \phi(T, x)) dx + \int_t^T \int_{\mathbb{R}^d} u(s, x), \mathcal{L}^* \phi(s, x) dx ds \\ &= \int_t^T \int_{\mathbb{R}^d} f(s, x, u, \nabla u \sigma, u(s, x + \beta(x, \cdot)) - u(s, x)) \phi(s, x) dx ds + \int_t^T \int_{\mathbb{R}^d} \phi(s, x) 1_{\{u=h\}}(s, x) \nu(ds, dx). \end{aligned} \quad (3.4.3)$$

First, we give a weak Itô's formula similar to the one given in Proposition 3.2.4. This result is essential to show the link between a Sobolev solution to the obstacle problem and the associated reflected BSDE with jumps, which in turn insures the uniqueness of the solution. The proof of this proposition is the same as Proposition 3.2.4.

**Proposition 3.4.1.** *Assume that conditions (A1)-(A6) hold and  $\rho(x) = (1 + |x|)^{-p}$  with  $p \geq \gamma$  where  $\gamma = \kappa + d + 1$ . Let  $u \in \mathcal{H}_T$  be a weak solution of PIDE with obstacle associated to  $(g, f, h)$ , then for  $s \in [t, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r, x) d\phi_t(r, x) dx + (u(s, \cdot), \phi_t(s, \cdot)) - (g(\cdot), \phi_t(T, \cdot)) - \int_s^T (u(r, \cdot), \mathcal{L}^* \phi_t(r, \cdot)) dr \\ &= \int_{\mathbb{R}^d} \int_s^T f(r, x, u(r, x), \sigma^* \nabla u(r, x), u(r, x + \beta(x, \cdot)) - u(r, x)) \phi_t(r, x) dr dx \\ &+ \int_{\mathbb{R}^d} \int_s^T \phi_t(r, x) 1_{\{u=h\}}(r, x) \nu(dr, dx). \quad a.s. \end{aligned} \quad (3.4.4)$$

where  $\int_{\mathbb{R}^d} \int_s^T u(r, x) d\phi_t(r, x) dx$  is well defined thanks to the semimartingale decomposition result (Lemma 3.2.1).

The main result of this section is the following

**Theorem 3.4.1.** *Assume that conditions (A1)-(A6) hold and  $\rho(x) = (1 + |x|)^{-p}$  with  $p \geq \gamma$  where  $\gamma = \kappa + d + 1$ . There exists a weak solution  $(u, \nu)$  of the PIDE with obstacle (3.1.3) associated to  $(g, f, h)$  such that,  $ds \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.$ ,*

$$\begin{aligned} Y_s^{t,x} &= u(s, X_{t,s}(x)), Z_s^{t,x} = (\sigma^* \nabla u)(s, X_{t,s}(x)), \\ V_s^{t,x}(\cdot) &= u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x)), \quad a.s. \end{aligned} \quad (3.4.5)$$

Moreover, the reflected measure  $\nu$  is a regular measure in the sense of the definition (ii) and satisfying the probabilistic interpretation (3.4.2).

If  $(\bar{u}, \bar{\nu})$  is another solution of the PIDE with obstacle (3.1.3) such that  $\bar{\nu}$  satisfies (3.4.2) with some  $\bar{K}$  instead of  $K$ , where  $\bar{K}$  is a continuous process in  $\mathcal{A}^2(t, T)$ , then  $\bar{u} = u$  and  $\bar{\nu} = \nu$ .

In other words, there is a unique Randon regular measure with support  $\{u = h\}$  which satisfies (3.4.2).

**Remark 3.4.1.** *The expression (3.4.2) gives us the probabilistic interpretation (Feymann-Kac's formula) for the measure  $\nu$  via the nondecreasing process  $K^{t,x}$  of the RBSDE with jumps. This formula was first introduced in Bally et al. [9] (see also [99]). Here we generalize their results to the case of PIDE's.*

From Lemma 3.1 in [33], we know that if we have more information on the obstacle  $L$ , we can give a more explicit representation for the processes  $K$ . Then as a result of the above theorem, we have when  $h$  is smooth enough, the reflected measure  $\nu$  is Lebesgue absolute continuous, moreover there exist a unique  $\tilde{\nu}$  and a measurable function  $(\alpha_s)_{s \geq 0}$  such that  $\nu(ds, dx) = \alpha_s \tilde{\nu}_s(dx) ds$ .

**Proof.** **a) Existence:** The existence of a solution will be proved in two steps. For the first step, we suppose that  $f$  does not depend on  $y, z, w$ , then we are able to apply the usual penalization method. In the second step, we study the case when  $f$  depends on  $y, z, w$  with the result obtained in the first step.

*Step 1 :* We will use the penalization method. For  $n \in \mathbb{N}$ , we consider for all  $s \in [t, T]$ ,

$$\begin{aligned} Y_s^{n,t,x} &= g(X_{t,T}(x)) + \int_s^T f(r, X_{t,r}(x)) dr + n \int_s^T (Y_r^{n,t,x} - h(r, X_{t,r}(x)))^- dr \\ &\quad - \int_s^T Z_r^{n,t,x} dW_r - \int_s^T \int_E V_r^{n,t,x}(e) \tilde{\mu}(dr, de). \end{aligned}$$

From Theorem (3.3.1) in section 3, we know that  $u_n(t, x) := Y_t^{n,t,x}$ , is solution of the PIDE  $(g, f_n)$ , where  $f_n(t, x, y) = f(t, x) + n(y - h(t, x))^-$ , i.e. for every  $\phi \in \mathcal{D}_T$

$$\begin{aligned} \int_t^T (u^n(s, \cdot), \partial_s \phi(s, \cdot)) ds + (u^n(t, \cdot), \phi(t, \cdot)) - (g(\cdot), \phi(T, \cdot)) + \int_t^T (u^n(s, \cdot), \mathcal{L}^* \phi(s, \cdot)) ds \\ = \int_t^T (f(s, \cdot), \phi(s, \cdot)) ds + n \int_t^T ((u^n - h)^-(s, \cdot), \phi(s, \cdot)) ds. \end{aligned} \quad (3.4.6)$$

Moreover

$$\begin{aligned} Y_s^{n,t,x} &= u_n(s, X_{t,s}(x)), \quad Z_s^{n,t,x} = \nabla u_n(s, X_{t,s}(x)) \sigma(X_{t,s}(x)), \\ V_s^{n,t,x}(\cdot) &= u_n(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u_n(s, X_{t,s-}(x)) \end{aligned} \quad (3.4.7)$$

Set  $K_s^{n,t,x} = n \int_t^s (Y_r^{n,t,x} - h(r, X_{t,r}(x)))^- dr$ . Then by (3.4.7), we have that  $K_s^{n,t,x} = n \int_t^s (u_n - h)^-(r, X_{t,r}(x)) dr$ .

Following the estimates and convergence results for  $(Y^{n,t,x}, Z^{n,t,x}, V^{n,t,x}, K^{n,t,x})$  in the step 3 and step 5 of the proof of Theorem 1.2. in [65], we get as  $m, n$  tend to infinity :

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_s^{n,t,x} - Y_s^{m,t,x}|^2 \right] + \mathbb{E} \left[ \int_t^T |Z_s^{n,t,x} - Z_s^{m,t,x}|^2 ds \right] \\ + \mathbb{E} \left[ \int_t^T \int_E |V_s^{n,t,x}(e) - V_s^{m,t,x}(e)|^2 \lambda(de) ds \right] + \mathbb{E} \left[ \sup_{t \leq s \leq T} |K_s^{n,t,x} - K_s^{m,t,x}|^2 \right] \longrightarrow 0, \end{aligned}$$

and

$$\begin{aligned} \sup_n \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_s^{n,t,x}|^2 + \int_t^T (|Z_s^{n,t,x}|^2 ds) + \int_t^T \int_E |V_s^{n,t,x}(e)|^2 \lambda(de) ds + (K_T^{n,t,x})^2 \right] \\ \leq C (1 + |x|^{2\kappa}). \end{aligned} \quad (3.4.8)$$

By the equivalence of norms (3.2.8), we get

$$\begin{aligned} \int_{\mathbb{R}^d} \int_t^T \rho(x) (|u_n(s, x) - u_m(s, x)|^2 + |\sigma^* \nabla u_n(s, x) - \sigma^* \nabla u_m(s, x)|^2) ds dx \\ \leq \frac{1}{k_2} \int_{\mathbb{R}^d} \rho(x) \mathbb{E} \int_t^T (|Y_s^{n,t,x} - Y_s^{m,t,x}|^2 + |Z_s^{n,t,x} - Z_s^{m,t,x}|^2) ds dx \rightarrow 0. \end{aligned}$$

Thus  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{H}_T$ , and the limit  $u = \lim_{n \rightarrow \infty} u_n$  belongs to  $\mathcal{H}_T$ . Denote  $\nu_n(dt, dx) = n(u_n - h)^-(t, x) dt dx$  and  $\pi_n(dt, dx) = \rho(x) \nu_n(dt, dx)$ , then by (3.2.8)

$$\begin{aligned} \pi_n([0, T] \times \mathbb{R}^d) &= \int_{\mathbb{R}^d} \int_0^T \rho(x) \nu_n(dt, dx) = \int_{\mathbb{R}^d} \int_0^T \rho(x) n(u_n - h)^-(t, x) dt dx \\ &\leq \frac{1}{k_2} \int_{\mathbb{R}^d} \rho(x) \mathbb{E} |K_T^{n,0,x}| dx \leq C \int_{\mathbb{R}^d} \rho(x) (1 + |x|^\kappa) dx < \infty. \end{aligned}$$

It follows that

$$\sup_n \pi_n([0, T] \times \mathbb{R}^d) < \infty. \quad (3.4.9)$$

Moreover by Lemma 3.5.1 (see Appendix 3.5.4), the sequence of measures  $(\pi_n)_{n \in \mathbb{N}}$  is tight. Therefore, there exists a subsequence such that  $(\pi_n)_{n \in \mathbb{N}}$  converges weakly to a positive measure  $\pi$ . Define  $\nu = \rho^{-1}\pi$ ;  $\nu$  is a positive measure such that  $\int_0^T \int_{\mathbb{R}^d} \rho(x)\nu(dt, dx) < \infty$ , and so we have for  $\phi \in \mathcal{D}_T$  with compact support in  $x$ ,

$$\int_{\mathbb{R}^d} \int_t^T \phi d\nu_n = \int_{\mathbb{R}^d} \int_t^T \frac{\phi}{\rho} d\pi_n \rightarrow \int_{\mathbb{R}^d} \int_t^T \frac{\phi}{\rho} d\pi = \int_{\mathbb{R}^d} \int_t^T \phi d\nu.$$

Now passing to the limit in the PIDE  $(g, f_n)$  (3.4.6), we get that  $(u, \nu)$  satisfies the PIDE with obstacle associated to  $(g, f, h)$ , i.e. for every  $\phi \in \mathcal{D}_T$ , we have

$$\begin{aligned} & \int_t^T (u(s, \cdot), \partial_s \phi(s, \cdot)) ds + (u(t, \cdot), \phi(t, \cdot)) - (g(\cdot), \phi(T, \cdot)) + \int_t^T (u(s, \cdot), \mathcal{L}^* \phi(s, \cdot)) ds \\ &= \int_t^T (f(s, \cdot), \phi(s, \cdot)) ds + \int_t^T \int_{\mathbb{R}^d} \phi(s, x) \nu(ds, dx). \end{aligned} \quad (3.4.10)$$

The last point is to prove that  $\nu$  satisfies the probabilistic interpretation (3.4.2). Since  $K^{n,t,x}$  converges to  $K^{t,x}$  uniformly in  $t$ , the measure  $dK^{n,t,x} \rightarrow dK^{t,x}$  weakly in probability.

Fix two continuous functions  $\phi, \psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^+$  which have compact support in  $x$  and a continuous function with compact support  $\theta : \mathbb{R}^d \rightarrow \mathbb{R}^+$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) \theta(x) \nu(ds, dx) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) \theta(x) n(u_n - h)^-(s, x) ds dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_t^T \phi(s, x) \psi(s, X_{t,s}(x)) \theta(X_{t,s}(x)) n(u_n - h)^-(t, X_{t,s}(x)) dt dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_t^T \phi(s, x) \psi(s, X_{t,s}(x)) \theta(X_{t,s}(x)) dK_s^{n,t,x} dx \\ &= \int_{\mathbb{R}^d} \int_t^T \phi(s, x) \psi(s, X_{t,s}(x)) \theta(X_{t,s}(x)) dK_s^{t,x} dx. \end{aligned}$$

We take  $\theta = \theta_R$  to be the regularization of the indicator function of the ball of radius  $R$  and pass to the limit with  $R \rightarrow \infty$ , it follows that

$$\int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) \nu(ds, dx) = \int_{\mathbb{R}^d} \int_t^T \phi(s, x) \psi(s, X_{t,s}(x)) dK_s^{t,x} dx. \quad (3.4.11)$$

Since  $(Y_s^{n,t,x}, Z_s^{n,t,x}, V_s^{n,t,x}, K_s^{n,t,x})$  converges to  $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})$  as  $n \rightarrow \infty$  in  $\mathcal{S}^2(t, T) \times \mathcal{H}_d^2(t, T) \times \mathcal{L}^2(t, T) \times \mathcal{A}^2(t, T)$ , and  $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})$  is the solution of RBSDE with jumps  $(g(X_{t,T}(x)), f, h)$ , then we have

$$\int_t^T (Y_s^{t,x} - L_s^{t,x}) dK_s^{t,x} = \int_t^T (u - h)(s, X_{t,s}(x)) dK_s^{t,x} = 0, \text{ a.s.}$$

it follows that  $dK_s^{t,x} = 1_{\{u=h\}}(s, X_{t,s}(x)) dK_s^{t,x}$ . In (3.4.11), setting  $\psi = 1_{\{u=h\}}$  yields

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) 1_{\{u=h\}}(s, x) \nu(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \nu(ds, dx), \text{ a.s.} \end{aligned}$$

Note that the family of functions  $A(\omega) = \{(s, x) \rightarrow \phi(s, X_{t,s}^{-1}(x)) : \phi \in C_c^\infty\}$  is an algebra which separates the points (because  $x \rightarrow X_{t,s}^{-1}(x)$  is a bijection). Given a compact set  $G$ ,  $A(\omega)$  is dense in  $C([0, T] \times G)$ . It follows that  $J(X_{t,s}^{-1}(x))1_{\{u=h\}}(s, x)\nu(ds, dx) = J(X_{t,s}^{-1}(x))\nu(ds, dx)$  for almost every  $\omega$ . While  $J(X_{t,s}^{-1}(x)) > 0$  for almost every  $\omega$ , we get  $\nu(ds, dx) = 1_{\{u=h\}}(s, x)\nu(ds, dx)$ , and (3.4.2) follows.

Then we get easily that  $Y_s^{t,x} = u(s, X_{t,s}(x))$ ,  $Z_s^{t,x} = \nabla u(s, X_{t,s}(x))\sigma(X_{t,s}(x))$  and  $V_s^{t,x}(\cdot) = u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x))$ , in view of the convergence results for  $(Y_s^{n,t,x}, Z_s^{n,t,x}, V_s^{n,t,x})$  and the equivalence of norms. So  $u(s, X_{t,s}(x)) = Y_s^{t,x} \geq h(s, X_{t,s}(x))$ . Specially for  $s = t$ , we have  $u(t, x) \geq h(t, x)$ .

*Step 2 : The nonlinear case where  $f$  depends on  $y, z$  and  $w$ .*

Let define  $F(s, x) \triangleq f(s, x, Y_s^{s,x}, Z_s^{s,x}, V_s^{s,x})$ . By plugging into the facts that  $f^0 \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$  and  $f$  is Lipschitz with respect to  $(y, z, v)$ , then thanks to Proposition 3.3.1 we have  $F(s, x) \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$ . Since  $F$  is independent of  $y, z, w$ , by applying the result of Step 1 yields that there exists  $(u, \nu)$  satisfying the PIDE with obstacle  $(g, F, h)$ , i.e. for every  $\phi \in \mathcal{D}_T$ , we have

$$\begin{aligned} & \int_t^T (u(s, \cdot), \partial_s \phi(s, \cdot)) ds + (u(t, \cdot), \phi(t, \cdot)) - (g(\cdot), \phi(T, \cdot)) + \int_t^T (u(s, \cdot), \mathcal{L}^* \phi(s, \cdot)) ds \\ &= \int_t^T (F(s, \cdot), \phi(s, \cdot)) ds + \int_t^T \int_{\mathbb{R}^d} \phi(s, x) 1_{\{u=h\}}(s, x) \nu(ds, dx). \end{aligned} \quad (3.4.12)$$

Then by the uniqueness of the solution to the RBSDE with jumps  $(g(X_{t,T}(x)), f, h(X_{t,s}(x)))$ , we get easily that  $Y_s^{t,x} = u(s, X_{t,s}(x))$ ,  $Z_s^{t,x} = (\nabla u \sigma)(s, X_{t,s}(x))$ ,  $V_s^{t,x}(\cdot) = u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x))$ , and  $\nu$  satisfies the probabilistic interpretation (3.4.2). So  $u(s, X_{t,s}(x)) = Y_s^{t,x} \geq h(s, X_{t,s}(x))$ . Specially for  $s = t$ , we have  $u(t, x) \geq h(t, x)$ , which is the desired result.

**b) Uniqueness** : Set  $(\bar{u}, \bar{\nu})$  to be another weak solution of the PIDE with obstacle (3.4.3) associated to  $(g, f, h)$ ; with  $\bar{\nu}$  verifies (3.4.2) for a nondecreasing process  $\bar{K}$ . We fix  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ , a smooth function in  $C_c^2(\mathbb{R}^d)$  with compact support and denote  $\phi_t(s, x) = \phi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))$ . From Proposition 3.4.1, one may use  $\phi_t(s, x)$  as a test function in the PIDE  $(g, f, h)$  with  $\partial_s \phi(s, x) ds$  replaced by a stochastic integral with respect to the semimartingale  $\phi_t(s, x)$ . Then we get, for  $t \leq s \leq T$

$$\begin{aligned} & \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, x) d\phi_t(r, x) dr dx + (\bar{u}(s, \cdot), \phi_t(s, \cdot)) - (g(\cdot), \phi_t(T, \cdot)) - \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, x) \mathcal{L}^* \phi_t(r, x) dr dx \\ &= \int_s^T \int_{\mathbb{R}^d} f(r, x, \bar{u}(r, x), (\nabla \bar{u} \sigma)(r, x), \bar{u}(r, x + \beta(x, \cdot)) - \bar{u}(r, x)) \phi_t(r, x) dr dx \\ &+ \int_s^T \int_{\mathbb{R}^d} \phi_t(r, x) 1_{\{\bar{u}=h\}}(r, x) \bar{\nu}(dr, dx). \end{aligned} \quad (3.4.13)$$

By (3.2.6) in Lemma 3.2.1, we have

$$\begin{aligned} & \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, x) d\phi_t(r, x) dr dx = \int_s^T \left( \int_{\mathbb{R}^d} (\nabla \bar{u} \sigma)(r, x) \phi_t(r, x) dx \right) dW_r \\ &+ \int_s^T \int_E \int_{\mathbb{R}^d} \bar{u}(r, x) \mathcal{A}_e^* \phi_t(r-, x) dx \tilde{\mu}(dr, de) + \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, x) \mathcal{L}^* \phi_t(r, x) dr dx. \end{aligned}$$

Substitute this equality in (3.4.13), we get

$$\begin{aligned}
& \int_{\mathbb{R}^d} \bar{u}(s, x) \phi_t(s, x) dx = (g(\cdot), \phi_t(T, \cdot)) - \int_s^T \left( \int_{\mathbb{R}^d} (\nabla \bar{u} \sigma)(r, x) \phi_t(r, x) dx \right) dW_r \\
& + \int_s^T \int_{\mathbb{E}} \int_{\mathbb{R}^d} \mathcal{A}_e \bar{u}(r-, x) \phi_t(r, x) dx \tilde{\mu}(dr, de) \\
& + \int_s^T \int_{\mathbb{R}^d} f(r, x, \bar{u}(r, x), (\nabla \bar{u} \sigma)(r, x), \bar{u}(r, x + \beta(x, \cdot)) - \bar{u}(r, x)) \phi_t(r, x) dr dx \\
& + \int_s^T \int_{\mathbb{R}^d} \phi_t(r, x) 1_{\{\bar{u}=h\}}(r, x) \bar{\nu}(dr, dx).
\end{aligned}$$

Then by changing of variable  $y = X_{t,r}^{-1}(x)$  and applying (3.4.2) for  $\bar{\nu}$ , we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^d} \bar{u}(s, X_{t,s}(y)) \phi(y) dy = \int_{\mathbb{R}^d} g(X_{t,T}(y)) \phi(y) dy \\
& + \int_s^T \int_{\mathbb{R}^d} \phi(y) f(r, X_{t,r}(y), \bar{u}(r, X_{t,r}(y)), (\nabla \bar{u} \sigma)(r, X_{t,r}(y)), \mathcal{A}_e \bar{u}(r, X_{t,r-}(y))) dr dy \\
& + \int_s^T \int_{\mathbb{R}^d} \phi(y) 1_{\{\bar{u}=h\}}(r, X_{t,s}(y)) d\bar{K}_r^{t,y} dy - \int_s^T \left( \int_{\mathbb{R}^d} (\nabla \bar{u} \sigma)(r, X_{t,r}(y)) \phi(y) dy \right) dW_r \\
& + \int_s^T \int_E \left( \int_{\mathbb{R}^d} \mathcal{A}_e \bar{u}(r, X_{t,r-}(y)) \phi(y) dy \right) \tilde{\mu}(dr, de).
\end{aligned}$$

Since  $\phi$  is arbitrary, we can prove that for  $\rho(y) dy$  almost every  $y$ ,  $(\bar{u}(s, X_{t,s}(y)), (\nabla \bar{u} \sigma)(s, X_{t,s}(y)), \bar{u}(s, X_{t,s-}(y)) + \beta(X_{t,s-}(y), \cdot) - \bar{u}(s, X_{t,s-}(y)), \hat{K}_s^{t,y})$  solves the RBSDE with jumps  $(g(X_{t,T}(y)), f, h)$ . Here  $\hat{K}_s^{t,y} = \int_t^s 1_{\{\bar{u}=h\}}(r, X_{t,r}(y)) d\bar{K}_r^{t,y}$ . Then by the uniqueness of the solution of the RBSDE with jumps, we know  $\bar{u}(s, X_{t,s}(y)) = Y_s^{t,y} = u(s, X_{t,s}(y))$ ,  $(\sigma^* \nabla \bar{u})(s, X_{t,s}(y)) = Z_s^{t,y} = (\nabla u \sigma)(s, X_{t,s}(y))$ ,  $\bar{u}(s, X_{t,s-}(y)) + \beta(X_{t,s-}(y), \cdot) - \bar{u}(s, X_{t,s-}(y)) = V_s^{t,y}(\cdot) = u(s, X_{t,s-}(y)) + \beta(X_{t,s-}(y), \cdot) - u(s, X_{t,s-}(y))$  and  $\hat{K}_s^{t,y} = K_s^{t,y}$ . Taking  $s = t$  we deduce that  $\bar{u}(t, y) = u(t, y)$ ,  $\rho(y) dy$ -a.s. and by the probabilistic interpretation (3.4.2), we obtain

$$\int_s^T \int_{\mathbb{R}^d} \phi_t(r, x) 1_{\{\bar{u}=h\}}(r, x) \bar{\nu}(dr, dx) = \int_s^T \int_{\mathbb{R}^d} \phi_t(r, x) 1_{\{u=h\}}(r, x) \nu(dr, dx).$$

So  $1_{\{\bar{u}=h\}}(r, x) \bar{\nu}(dr, dx) = 1_{\{u=h\}}(r, x) \nu(dr, dx)$ .

## 3.5 Appendix

### 3.5.1 Proof of Proposition 3.2.3

In order to prove (5.4.10), it is sufficient to prove that

$$c \leq \mathbb{E} \left[ \frac{J(X_{t,s}^{-1}(x)) \rho(X_{t,s}^{-1}(x))}{\rho(x)} \right] \leq C.$$

In fact, making the change of variable  $y = X_{t,s}(x)$ , we can get the following relation:

$$\int_{\mathbb{R}^d} \mathbb{E}(|\varphi(X_{t,s}(x))|) \rho(x) dx = \int_{\mathbb{R}^d} |\varphi(y)| \mathbb{E} \left[ \frac{J(X_{t,s}^{-1}(y)) \rho(X_{t,s}^{-1}(y))}{\rho(y)} \right] \rho(y) dy.$$

We differentiate with respect to  $y$  in (5.4.6) in order to get:

$$\begin{aligned}
\nabla X_{t,s}^{-1}(y) &= I - \int_t^s \nabla \hat{b}(X_{r,s}^{-1}(y)) \nabla X_{t,r}^{-1}(y) dr - \int_t^s \nabla \sigma(X_{r,s}^{-1}(y)) \nabla X_{t,r}^{-1}(y) \overleftarrow{dW}_r \\
&\quad - \int_t^s \int_E \nabla \beta(X_{r,s}^{-1}(y), e) \nabla X_{t,r}^{-1}(y) \tilde{\mu}(\overleftarrow{dr}, de) \\
&\quad + \int_t^s \int_E \nabla \hat{\beta}(X_{r,s}^{-1}(y), e) \nabla X_{t,r}^{-1}(y) \mu(\overleftarrow{dr}, de) \\
&:= I + \Gamma_{t,s}(y)
\end{aligned} \tag{3.5.1}$$



where  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla \beta$  and  $\nabla \widehat{\beta}$  are the gradient of  $b$ ,  $\sigma, \beta$  and  $\widehat{\beta}$ , respectively and  $I$  is the identity matrix. Since  $J(X_{t,s}^{-1}(y)) := \det \nabla X_{t,s}^{-1}(y) = \inf_{\|\xi\|=1} \langle \nabla X_{t,s}^{-1}(y)\xi, \xi \rangle$ , we obtain

$$1 - \|\Gamma_{t,s}(y)\| \leq J(X_{t,s}^{-1}(y)) \leq 1 + \|\Gamma_{t,s}(y)\|.$$

Writing  $\Gamma_{t,s}(y) := C_{t,s}(y) + D_{t,s}(y)$ , where  $D_{t,s}(y)$  denotes the integration with respect to the random measure and  $C_{t,s}(y)$  denotes the others.

According to Bally and Matoussi [10], we know that for any  $y$ ,  $\mathbb{E}[|C_{t,s}(y)|^2] \leq K(s-t)$ . Now we are going to prove the similar relation for  $D_{t,s}(y)$ . We only deal with the first term of  $D_{t,s}(y)$  because another one can be treated similarly without any difficulty. In fact, by Burkholder-David-Gundy inequality (of course in the backward sense), we have

$$\begin{aligned} & \mathbb{E}\left[\left|\int_t^s \int_E \nabla \beta(X_{r,s}^{-1}(y), e) \nabla X_{t,r}^{-1}(y) \widetilde{\mu}(dr, de)\right|^2\right] \\ & \leq C \mathbb{E}\left[\int_t^s \int_E |\nabla \beta(X_{r,s}^{-1}(y), e) \nabla X_{r,s}^{-1}(y)|^2 \lambda(de) dr\right] \\ & \leq C_1 \mathbb{E}\left[\int_t^s \int_E (1 \wedge |e|)^2 |\nabla X_{r,s}^{-1}(y)|^2 \lambda(de) dr\right] \\ & = C_1 \mathbb{E}\left[\int_t^s |\nabla X_{r,s}^{-1}(y)|^2 \int_E (1 \wedge |e|^2) \lambda(de) dr\right] \\ & \leq K(s-t), \end{aligned}$$

since  $\mathbb{E}[|\nabla X_{r,s}^{-1}(y)|^2] < \infty$ . Therefore,  $\mathbb{E}[|\Gamma_s^{t,y}|^2] \leq 4K(s-t)$ . Hence,

$$1 - 2\sqrt{K(s-t)} \leq \mathbb{E}[J(X_{t,s}^{-1}(y))] \leq 1 + 2\sqrt{K(s-t)},$$

and the desired result follows.

### 3.5.2 Regularity of the solution of BSDE with jumps

In this section, we are going to prove regularity results for the solution of BSDE's with jumps with respect to the parameter  $(t, x)$  in order to relate the solution of BSDE to the classic solution of PIDE. We note that some part of the results given in this section were established in a preprint of Buckdahn and Pardoux (1994) [26]. However, for convenience of the reader and for completeness of the chapter, we give the whole proofs. We first start by giving the  $L^p$ -estimates for the solution of the following BSDE's with jumps:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E V_s(e) \widetilde{\mu}(ds, de). \quad (3.5.2)$$

**Theorem 3.5.1.** *Assume that  $f$  is uniformly Lipschitz with respect to  $(y, z, v)$  and additionally that for some  $p \geq 2$ ,  $\xi \in L_m^p(\mathcal{F}_T)$  and*

$$\mathbb{E} \int_0^T |f(t, 0, 0, 0)|^p dt < \infty. \quad (3.5.3)$$

Then

$$\mathbb{E} \left[ \sup_t |Y_t|^p + \left( \int_0^T |Z_t|^2 dt \right)^{p/2} + \left( \int_0^T \left( \int_E |V_t(e)|^2 \lambda(de) \right) dt \right)^{p/2} \right] < \infty.$$

**Proof.** We follow the idea of Buckdahn and Pardoux [26]. The proof is divided into 3 steps.

Step 1: From (3.5.2),

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s, V_s) ds + \int_0^t Z_s dW_s + \int_0^t \int_E V_s(e) \widetilde{\mu}(ds, de).$$

then by Itô's formula,

$$\begin{aligned} |Y_t|^2 &= |Y_0|^2 - 2 \int_0^t (f(s, Y_s, Z_s, V_s), Y_s) ds + \int_0^t (|Z_s|^2 + \|V_s\|^2) ds \\ &\quad + 2 \int_0^t (Y_s, Z_s dW_s) + \int_0^t \int_E (|Y_{s-} + V_s(e)|^2 - |Y_{s-}|^2) \tilde{\mu}(ds, de). \end{aligned}$$

Let  $\phi_{n,p}(x) = (x \wedge n)^p + pn^{p-1}(x-n)^+$  for all  $p \geq 1$ . Then  $\phi_{n,p} \in C^1(\mathbb{R}_+)$ ,  $\phi'_{n,p}(x) = p(x \wedge n)^{p-1}$  bounded and absolutely continuous with

$$\phi''_{n,p}(x) = p(p-1)(x \wedge n)^{p-2} \mathbf{1}_{[0,n]}(x).$$

Again by applying Itô's formula to  $\phi_{n,p}(|Y_t|^2)$ ,

$$\begin{aligned} \phi_{n,p}(|Y_T|^2) &= \phi_{n,p}(|Y_t|^2) - 2 \int_t^T \phi'_{n,p}(|Y_s|^2) (f(s, Y_s, Z_s, V_s), Y_s) ds \\ &\quad + 2 \int_t^T \phi'_{n,p}(|Y_s|^2) (Y_s, Z_s dW_s) + \int_t^T \phi''_{n,p}(|Y_s|^2) (Z_s Z_s^* Y_s, Y_s) ds \\ &\quad + \int_t^T \phi'_{n,p}(|Y_s|^2) |Z_s|^2 ds + \int_t^T \phi'_{n,p}(|Y_s|^2) \|V_s\|^2 ds \\ &\quad + \int_t^T \int_E [\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2)] \tilde{\mu}(ds, de) \\ &\quad + \int_t^T \int_E [\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2) - (|Y_{s-} + V_s(e)|^2 - |Y_{s-}|^2) \phi'_{n,p}(|Y_{s-}|^2)] \lambda(de) ds. \end{aligned}$$

Since  $(Y, Z, V) \in \mathcal{B}^2$ , it follows from Burkholder-Davis-Gundy's inequality,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \phi'_{n,p}(|Y_s|^2) (Y_s, Z_s dW_s) \right| \right] \leq Cpn^{p-1} \|Y\|_{\mathcal{S}^2} \|Z\|_{\mathcal{H}^2},$$

that the  $dW$  integral above is uniformly integrable, hence it is a martingale with zero expectation. From the boundedness of  $\phi'_{n,p}$ , the integrand in the  $\tilde{\mu}$ -integral can be written as follows

$$\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2) = \psi_s(e)(2(Y_{s-}, V_s(e)) + |V_s(e)|^2),$$

where  $\psi_s(e)$  is a bounded and predictable process. By BDG's inequality

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \int_E \psi_s(e)(Y_{s-}, V_s(e)) \tilde{\mu}(ds, de) \right| \right] &\leq c \mathbb{E} \left[ \left( \int_0^T \int_E |\psi_s(e)(Y_{s-}, V_s(e))|^2 \mu(ds, de) \right)^{1/2} \right] \\ &\leq cpn^{p-1} \|Y\|_{\mathcal{S}^2} \|V\|_{\mathcal{L}^2}, \end{aligned}$$

and by the decomposition  $\tilde{\mu}(ds, de) = \mu(ds, de) - ds\lambda(de)$ ,

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \int_E \psi_s(e) |V_s(e)|^2 \tilde{\mu}(ds, de) \right| \right] &\leq 2pn^{p-1} \mathbb{E} \left[ \int_0^T \int_E |V_s(e)|^2 \lambda(de) ds \right] \\ &= 2pn^{p-1} \|V\|_{\mathcal{L}^2}^2. \end{aligned}$$

We have also

$$\mathbb{E} \left[ \int_0^t \int_E (\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2)) \tilde{\mu}(ds, de) \right] = 0.$$

From Taylor's expansion of  $\phi_{n,p}$  and the positivity of  $\phi''_{n,p}$ , we conclude

$$\int_t^T \int_{\mathbb{E}} [\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2) - (|Y_{s-} + V_s(e)|^2 - |Y_{s-}|^2) \phi'_{n,p}(|Y_{s-}|^2)] \lambda(de) ds \geq 0.$$

Using again  $\phi''_{n,p} \geq 0$  we conclude that:

$$\begin{aligned} \mathbb{E}\phi_{n,p}(|Y_t|^2) &+ \mathbb{E} \int_t^T \phi'_{n,p}(|Y_s|^2)(|Z_s|^2 + \|V_s\|^2) ds \\ &\leq \mathbb{E}\phi_{n,p}(|\xi|^2) + 2\mathbb{E} \int_t^T \phi'_{n,p}(|Y_s|^2)(f(s, Y_s, Z_s, V_s), Y_s) ds \\ &\leq \mathbb{E}\phi_{n,p}(|\xi|^2) + C\mathbb{E} \int_t^T \phi'_{n,p}(|Y_s|^2)|Y_s|(|f(s, 0, 0, 0)| + |Y_s| + |Z_s| + \|V_s\|) ds. \end{aligned}$$

Hence we deduce that

$$\begin{aligned} \mathbb{E}\phi_{n,p}(|Y_t|^2) &+ \frac{1}{2}\mathbb{E} \int_t^T \phi'_{n,p}(|Y_s|^2)(|Z_s|^2 + \|V_s\|^2) ds \\ &\leq \mathbb{E}\phi_{n,p}(|\xi|^2) + C\mathbb{E} \int_t^T \phi'_{n,p}(|Y_s|^2)(|f(s, 0, 0, 0)|^2 + |Y_s|^2) ds \\ &\leq \mathbb{E}\phi_{n,p}(|\xi|^2) + C'\mathbb{E} \int_t^T (|f(s, 0, 0, 0)|^{2p} + \phi'_{n,p}(|Y_s|^2)^{\frac{p}{p-1}} + \phi'_{n,p}(|Y_s|^2)|Y_s|^2) ds \\ &\leq \mathbb{E}\phi_{n,p}(|\xi|^2) + \bar{C}\mathbb{E} \int_t^T (|f(s, 0, 0, 0)|^{2p} + \phi_{n,p}(|Y_s|^2)) ds. \end{aligned}$$

Then it follows from Gronwall's lemma that there exists a constant  $C(p, T)$  independent of  $n$  such that

$$\sup_{0 \leq t \leq T} \mathbb{E}\phi_{n,p}(|Y_t|^2) \leq C(p, T)\mathbb{E}[|\xi|^{2p} + \int_0^T |f(t, 0, 0, 0)|^{2p} dt],$$

hence from Fatou's lemma

$$\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t|^{2p}] < \infty, \quad (3.5.4)$$

and also

$$\mathbb{E} \int_0^T |Y_s|^{2(p-1)}(|Z_s|^2 + \|V_s\|^2) ds < \infty, \quad (3.5.5)$$

$$\mathbb{E} \int_0^T \int_E [ |Y_{s-} + V_s(e)|^{2p} - |Y_{s-}|^{2p} - p(|Y_{s-} + V_s(e)|^2 - |Y_{s-}|^2)|Y_{s-}|^{2(p-1)}] \lambda(de) ds < \infty, \quad (3.5.6)$$

and this holds for any  $p \geq 1$ .

Step 2: Now, again from Itô's formula,

$$\begin{aligned} |Y_T|^{2p} &\geq |Y_t|^{2p} - 2p \int_t^T |Y_s|^{2(p-1)}(f(s, Y_s, Z_s, V_s), Y_s) ds \\ &+ p \int_t^T |Y_s|^{2(p-1)}(|Z_s|^2 + \|V_s\|^2) ds + 2 \int_t^T |Y_s|^{2(p-1)}(Y_s, Z_s dW_s) \\ &+ \int_t^T \int_{\mathbb{E}} (|Y_{s-} + V_s(e)|^{2p} - |Y_{s-}|^{2p}) \tilde{\mu}(ds, de). \end{aligned}$$

It follows from (3.5.4) and (3.5.5) that the above  $dW$ -integral is a uniformly integral martingale, and from (3.5.5) and (3.5.6) that the  $\tilde{\mu}$ -integral is a uniformly integrable martingale. It is then easy to conclude that

$$\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^{2p}] < \infty, \quad p \geq 1.$$

Step 3: Finally,

$$\int_s^t Z_r dW_r + \int_s^t \int_E V_r(e) \tilde{\mu}(dr, de) = Y_t - Y_s + \int_s^t f(r, Y_r, Z_r, V_r) dr,$$

and from BDG inequality, (3.5.4) and (3.5.3), for any  $p \geq 2$ , there exists  $C_p$  such that for all  $0 \leq s \leq t \leq T$ ,  $n \geq 1$  if  $\tau_n = \inf\{u \geq s, \int_s^u (|Z_r|^2 + \|V_r\|^2) dr \geq n\} \wedge t$ ,

$$\begin{aligned} \mathbb{E} \left[ \left( \int_s^{\tau_n} (|Z_r|^2 + \|V_r\|^2) dr \right)^{p/2} \right] &\leq C_p \mathbb{E} \left[ 1 + \left( \int_s^{\tau_n} (|Z_r| + \|V_r\|) dr \right)^p \right] \\ &\leq C_p (1 + (t-s)^{p/2}) \mathbb{E} \left[ \left( \int_s^{\tau_n} (|Z_r|^2 + \|V_r\|^2) dr \right)^{p/2} \right]. \end{aligned}$$

Hence if  $C_p(t-s)^{p/2} < 1$ ,  $n \geq 1$ ,  $\mathbb{E} \left[ \left( \int_s^{\tau_n} (|Z_r|^2 + \|V_r\|^2) dr \right)^{p/2} \right] \leq \frac{C_p}{1 - C_p(t-s)^{p/2}}$ . It clearly follows that

$$E \left[ \left( \int_0^T |Z_t|^2 dt \right)^{p/2} + \left( \int_0^T \|V_t\|^2 dt \right)^{p/2} \right] < \infty, \quad p \geq 2.$$

□

From now on, we denote by  $\Sigma = (Y, Z, V)$  and  $\mathcal{B}^p$  the space of solutions, *i.e.*,

$$\|\Sigma\|_{\mathcal{B}^p}^p \triangleq \mathbb{E} \left[ \sup_t |Y_t|^p + \left( \int_0^T |Z_t|^2 dt \right)^{p/2} + \left( \int_0^T \|V_t\|^2 dt \right)^{p/2} \right].$$

In the sequel, we will consider a specific class of BSDE where

$$\xi = g(X_{t,T}(x)) \text{ and } f(s, y, z, v) = f(s, X_{t,s}(x), y, z, v)$$

and we assume

$$(\mathbf{H}) \begin{cases} g \in C_p^3(\mathbb{R}^d, \mathbb{R}^m), \\ \forall s \in [0, T], (x, y, z, v) \mapsto f(s, x, y, z, v) \in C^3 \\ \text{and all their derivatives are bounded.} \end{cases}$$

Note that  $f$  is differentiable w.r.t.  $v$  in the sens of Fréchet and its Fréchet differential is bounded with the norm in  $L^2(\mathbb{E}, \lambda; \mathbb{R}^k)$ .

Let  $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x})_{t \leq s \leq T}$  denote the unique solution of the following BSDE:

$$Y_s^{t,x} = g(X_{t,T}(x)) + \int_s^T f(r, X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r - \int_s^T \int_E V_r^{t,x}(e) \tilde{\mu}(dr, de). \quad (3.5.7)$$

It follows easily from the existence result in [12]:

**Corollary 3.5.1.** *For each  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ , the BSDE(3.5.7) has a unique solution*

$$\Sigma^{t,x} = (Y^{t,x}, Z^{t,x}, V^{t,x}) \in \mathcal{B}^2,$$

and  $Y_t^{t,x}$  defines a deterministic mapping from  $[0, T] \times \mathbb{R}^d$  into  $\mathbb{R}^m$ .

Now we are going to deal with the regularity of the solution with respect to the parameter  $x$ . Let us establish the following proposition:

**Proposition 3.5.1.** *Under the assumption in the previous theorem and assume moreover that  $(\mathbf{H})$  holds. Then, for any  $p \geq 2$ , there exists  $C_p, q$  such that for any  $0 \leq t \leq T$ ,  $x, x' \in \mathbb{R}^d$ ,  $h, h' \in \mathbb{R} \setminus \{0\}$ ,  $1 \leq i \leq d$ ,*

$$(i) \quad \|\Sigma^{t,x} - \Sigma^{t,x'}\|_{\mathcal{B}^p}^p \leq C_p (1 + |x| + |x'|)^q |x - x'|^p;$$

$$(ii) \|\Delta_h^i \Sigma^{t,x} - \Delta_{h'}^i \Sigma^{t,x'}\|_{\mathcal{B}^p}^p \leq C_p(1 + |x| + |x'| + |h| + |h'|)^q(|x - x'|^p + |h - h'|^p)$$

where  $\Delta_h^i \Sigma_s^{t,x} = \frac{1}{h}(Y_s^{t,x+he_i} - Y_s^{t,x}, Z_s^{t,x+he_i} - Z_s^{t,x}, V_s^{t,x+he_i} - V_s^{t,x})$ , and  $(e_1, \dots, e_d)$  is an orthonormal basis of  $\mathbb{R}^d$ .

**Proof.** Note that after applying  $L^p$ -estimation of the solution to the present situation, we can deduce that  $\forall p \geq 2$ , there exist  $C_p, q$  such that

$$\mathbb{E} \left[ \sup_s |Y_s^{t,x}|^p + \left( \int_t^T |Z_s^{t,x}|^2 ds \right)^{p/2} + \left( \int_t^T \|V_s^{t,x}\|^2 ds \right)^{p/2} \right] \leq C_p(1 + |x|^q).$$

For  $t \leq s \leq T$ ,

$$\begin{aligned} & Y_s^{t,x} - Y_s^{t,x'} \\ &= g(X_{t,T}(x)) - g(X_{t,T}(x')) + \int_s^T (f(r, X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}) - f(r, X_{t,r}(x'), Y_r^{t,x'}, Z_r^{t,x'}, V_r^{t,x'})) dr \\ &\quad - \int_s^T (Z_r^{t,x} - Z_r^{t,x'}) dW_r - \int_s^T \int_{\mathbb{E}} (V_r^{t,x}(e) - V_r^{t,x'}(e)) \tilde{\mu}(dr, de) \\ &= \int_0^1 g'(\theta X_{t,T}(x) + (1-\theta)X_{t,T}(x'))(X_{t,T}(x) - X_{t,T}(x')) d\theta + \int_s^T \left[ \varphi_r(t, x, x')(X_{t,r}(x) - X_{t,r}(x')) \right. \\ &\quad \left. + \psi_r(t, x, x')(Y_r^{t,x} - Y_r^{t,x'}) + \xi_r(t, x, x')(Z_r^{t,x} - Z_r^{t,x'}) + \langle \eta_r(t, x, x'), (V_r^{t,x} - V_r^{t,x'}) \rangle \right] dr \\ &\quad - \int_s^T (Z_r^{t,x} - Z_r^{t,x'}) dW_r - \int_s^T \int_{\mathbb{E}} (V_r^{t,x}(e) - V_r^{t,x'}(e)) \tilde{\mu}(dr, de), \end{aligned}$$

where

$$\begin{aligned} \varphi_r(t, x, x') &= \int_0^1 \frac{\partial f}{\partial x}(\Xi_{r,\theta}^{t,x,x'}) d\theta, & \psi_r(t, x, x') &= \int_0^1 \frac{\partial f}{\partial y}(\Xi_{r,\theta}^{t,x,x'}) d\theta \\ \xi_r(t, x, x') &= \int_0^1 \frac{\partial f}{\partial z}(\Xi_{r,\theta}^{t,x,x'}) d\theta, & \eta_r(t, x, x') &= \int_0^1 \frac{\partial f}{\partial v}(\Xi_{r,\theta}^{t,x,x'}) d\theta, \\ & & \Xi_{r,\theta}^{t,x,x'} &= \theta \Sigma_r^{t,x} + (1-\theta) \Sigma_r^{t,x'}, \end{aligned}$$

and  $\langle \cdot, \cdot \rangle$  stands for the scalar product in  $L^2(\mathbb{E}, \lambda; \mathbb{R}^m)$ .

Then given the boundedness of the derivatives, the previous proposition and

$$|\varphi_r(t, x, x')| \leq C(1 + |X_{t,r}(x)| + |X_{t,r}(x')|)^q,$$

we know that (i) holds true after applying the  $L^p$ -estimates of the solution.

Now we turn to (ii). In fact, we have

$$\begin{aligned} \Delta_h^i Y_s^{t,x} &= \frac{1}{h}(Y_s^{t,x+he_i} - Y_s^{t,x}) \\ &= \int_0^1 g'(\theta X_{t,T}(x + he_i) + (1-\theta)X_{t,T}(x)) \Delta_h^i X_{t,T}(x) d\theta \\ &\quad + \int_s^T \left[ \varphi_r(t, x + he_i, x) \Delta_h^i X_{t,r}(x) + \psi_r(t, x + he_i, x) \Delta_h^i Y_r^{t,x} \right. \\ &\quad \left. + \xi_r(t, x + he_i, x) \Delta_h^i Z_r^{t,x} + \langle \eta_r(t, x + he_i, x), \Delta_h^i V_r^{t,x} \rangle \right] dr \\ &\quad - \int_s^T \Delta_h^i Z_r^{t,x} dW_r - \int_s^T \int_{\mathbb{E}} \Delta_h^i V_r^{t,x}(e) \tilde{\mu}(dr, de). \end{aligned}$$

Note that for each  $p \geq 2$ , there exists  $C_p$  such that  $\mathbb{E}[\sup_s |\Delta_h^i X_{t,s}(x)|^p] \leq C_p$ . Then same calculation as in (i) implies that

$$\|\Delta_h^i \Sigma^{t,x}\|_{\mathcal{B}^p}^p \leq C_p(1 + |x|^q + |h|^q).$$

Finally, we consider

$$\begin{aligned}
\Delta_h^i Y_s^{t,x} &- \Delta_{h'}^i Y_s^{t,x'} = \Gamma^{i,t,s}(h, x; h', x') \\
&+ \int_s^T \left[ \varphi_r(t, x + he_i, x) (\Delta_h^i X_{t,r}(x) - \Delta_{h'}^i X_{t,r}(x')) + \psi_r(t, x + he_i, x) (\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_r^{t,x'}) \right. \\
&+ \xi_r(t, x + he_i, x) (\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t,x'}) + \langle \eta_r(t, x + he_i, x), \Delta_h^i V_r^{t,x} - \Delta_{h'}^i V_r^{t,x'} \rangle \Big] dr \\
&- \int_s^T (\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t,x'}) dW_r - \int_s^T \int_{\mathbb{E}} (\Delta_h^i V_r^{t,x}(e) - \Delta_{h'}^i V_r^{t,x'}(e)) \tilde{\mu}(dr, de),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^{i,t,s}(h, x; h', x') &= \int_0^1 (g'(\theta X_{t,T}(x + he_i) + (1-\theta)X_{t,T}(x)) \Delta_h^i X_{t,T}(x) \\
&- g'(\theta X_{t,T}(x' + h'e_i) + (1-\theta)X_{t,T}(x')) \Delta_{h'}^i X_{t,T}(x')) d\theta \\
&+ \int_s^T \left[ (\varphi_r(t, x + he_i, x) - \varphi_r(t, x' + h'e_i, x')) \Delta_{h'}^i X_{t,r}(x') \right. \\
&+ (\psi_r(t, x + he_i, x) - \psi_r(t, x' + h'e_i, x')) \Delta_{h'}^i Y_r^{t,x'} \\
&+ (\xi_r(t, x + he_i, x) - \xi_r(t, x' + h'e_i, x')) \Delta_{h'}^i Z_r^{t,x'} \\
&+ \langle \eta_r(t, x + he_i, x) - \eta_r(t, x' + h'e_i, x'), \Delta_{h'}^i V_r^{t,x'} \rangle \Big] dr.
\end{aligned}$$

Since  $f$  has bounded derivatives, we have

$$\begin{aligned}
&|\psi_r(t, x + he_i, x) - \psi_r(t, x' + h'e_i, x')| + |\xi_r(t, x + he_i, x) - \xi_r(t, x' + h'e_i, x')| \\
&+ |\eta_r(t, x + he_i, x) - \eta_r(t, x' + h'e_i, x')| \leq C(|\Sigma_r^{t,x+he_i} - \Sigma_r^{t,x'+h'e_i}| + |\Sigma_r^{t,x} - \Sigma_r^{t,x'}|)
\end{aligned}$$

and

$$\begin{aligned}
&|\varphi_r(t, x + he_i, x) - \varphi_r(t, x' + h'e_i, x')| \\
&\leq C \left( 1 + |\Sigma_r^{t,x+he_i}| + |\Sigma_r^{t,x'+h'e_i}| + |\Sigma_r^{t,x}| + |\Sigma_r^{t,x'}| \right)^q \left( |\Sigma_r^{t,x+he_i} - \Sigma_r^{t,x'+h'e_i}| + |\Sigma_r^{t,x} - \Sigma_r^{t,x'}| \right),
\end{aligned}$$

it follows from the previous proposition that there exist a constant  $C_p$  and some  $\alpha_p$  such that

$$\mathbb{E}[\sup_{t \leq s \leq T} |\Gamma^{i,t,s}(h, x; h', x')|^p] \leq C_p(1 + |x| + |x'| + |h| + |h'|)^{\alpha_p} (|x - x'|^p + |h - h'|^p).$$

Then, statement (ii) follows from the same estimate, which ends the proof.  $\square$

Therefore, using Kolmogorov's criterion, we can get that  $Y, Z, V$  are a.s. differentiable w.r.t.  $x$ . Iterating the same argument, we get in fact:

**Theorem 3.5.2.** *For all  $0 \leq t \leq T$ , there exists a version of  $\Sigma^{t,x} = (Y^{t,x}, Z^{t,x}, V^{t,x})$  such that  $x \mapsto \Sigma^{t,x}$  is a.e. of class  $C^2$  from  $\mathbb{R}^d$  into  $D([t, T]; \mathbb{R}^m) \times L^2([t, T]; \mathbb{R}^{m \times d}) \times L^2([t, T] \times \mathbb{E}, ds \lambda(de); \mathbb{R}^m)$ , where  $D([t, T]; \mathbb{R}^m)$  denotes the set of  $\mathbb{R}^m$ -valued càdlàg functions on  $[t, T]$ . Moreover, for any  $p \geq 2$ , there exist  $C_p$  and  $q$  such that*

$$\begin{aligned}
&\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \Sigma^{t,x} \right\|_{\mathcal{B}^p}^p \leq C_p(1 + |x|^q), \quad 0 \leq |\alpha| \leq 2, \\
&\left\| \frac{\partial^{|\alpha|}}{\partial x_i \partial x_j} \Sigma^{t,x} - \frac{\partial^{|\alpha|}}{\partial x_i \partial x_j} \Sigma^{t,x'} \right\|_{\mathcal{B}^p}^p \leq C_p(1 + |x| + |x'|)^q |x - x'|^p, \quad 0 \leq t \leq T.
\end{aligned}$$

Additionally, we have the following regularity in  $t$ :

**Proposition 3.5.2.** *For all  $p \geq 2$  and  $s \geq t \vee t'$ , there exist  $C_p$  and  $q$  such that*

$$\begin{aligned}
&\left\| \Sigma_s^{t,x} - \Sigma_s^{t',x'} \right\|_{\mathcal{B}^p}^p \leq C_p(1 + |x| + |x'|)^q (|t - t'|^{p/2} + |x - x'|^p), \\
&\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \Sigma_s^{t,x} - \frac{\partial^{|\alpha|}}{\partial x^\alpha} \Sigma_s^{t',x'} \right\|_{\mathcal{B}^p}^p \leq C_p(1 + |x| + |x'|)^q (|t - t'|^{p/2} + |x - x'|^p), \quad 0 \leq |\alpha| \leq 2.
\end{aligned}$$

Hence, we have

**Corollary 3.5.2.** *The function  $(t, x) \mapsto Y_t^{t,x} \in C_p^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^m)$ .*

### 3.5.3 Proof of Theorem 3.3.1: existence and uniqueness of solution of PIDE (3.1.1)

(i) *Uniqueness* : Let  $u^1, u^2 \in \mathcal{H}_T$  be two weak solutions of (3.1.1), then Prop. 3.2.4 implies that for  $i = 1, 2$

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u^i(r, x) d\phi_t(r, x) dx + (u^i(s, x), \phi_t(s, x)) - (g(x), \phi_t(T, x)) - \int_s^T (u^i(r, \cdot), \mathcal{L}^* \phi_t(r, \cdot)) dr \\ &= \int_{\mathbb{R}^d} \int_s^T f(r, x, u^i(r, x), \sigma^* \nabla u^i(r, x), u^i(r, x + \beta(x, \cdot)) - u^i(r, x)) \phi_t(r, x) dr dx. \end{aligned}$$

By the decomposition of the semimartingale  $\phi_t(s, x)$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u^i(r, x) d\phi_t(r, x) dx \\ &= \int_{\mathbb{R}^d} \int_s^T u^i(r, x) \mathcal{L}^* \phi_t(r, x) dr - \sum_{j=1}^d \int_{\mathbb{R}^d} \int_s^T \left( \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma^{ij}(x) u^i(r, x) \phi_t(r, x)) \right) dW_r^j \\ &+ \int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{E}} u^i(r, x) \mathcal{A}_e^* \phi_t(r-, x) \tilde{\mu}(dr, de). \end{aligned} \quad (3.5.8)$$

We substitute this in the above equation and get

$$\begin{aligned} \int_{\mathbb{R}^d} u^i(s, x) \phi_t(s, x) dx &= \int_{\mathbb{R}^d} g(x) \phi_t(T, x) dx - \int_s^T \int_{\mathbb{R}^d} (\nabla u^i \sigma)(r, x) \phi_t(r, x) dx dW_r \\ &- \int_s^T \int_E \int_{\mathbb{R}^d} u^i(r, x) \mathcal{A}_e^* \phi_t(r-, x) dx \tilde{\mu}(dr, de) \\ &+ \int_s^T \int_{\mathbb{R}^d} f(r, x, u^i(r, x), (\nabla u^i \sigma)(r, x), u^i(r, x + \beta(x, \cdot)) - u^i(r, x)) \phi_t(r, x) dr dx. \end{aligned}$$

Then by the change of variable, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} u^i(s, X_{t,s}(x)) \phi(x) dx \\ &= \int_{\mathbb{R}^d} g(X_{t,T}(x)) \phi(x) dx - \int_{\mathbb{R}^d} \int_s^T \phi(x) (\nabla u^i \sigma)(r, X_{t,r}(x)) dx dW_r \\ &- \int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{E}} \phi(x) (u^i(s, X_{t,r-}(x) + \beta(X_{t,r-}(x), e)) - u(s, X_{t,r-}(x))) \tilde{\mu}(dr, de) dx \\ &+ \int_{\mathbb{R}^d} \int_s^T \phi(x) f(r, X_{t,r}(x), u^i(r, X_{t,r}(x)), (\nabla u^i \sigma)(r, X_{t,r}(x)), \mathcal{A}_e u^i(r, X_{t,r-}(x))) dr dx. \end{aligned}$$

Since  $\phi$  is arbitrary, we deduce that  $(Y_s^{i,t,x}, Z_s^{i,t,x}, V_s^{i,t,x})$  solve the BSDE associated with  $(g(X_{t,T}(x)), f), \rho(x) dx$ -a.e., where

$$\begin{aligned} Y_s^{i,t,x} &= u^i(s, X_{t,s}(x)), Z_s^{i,t,x} = (\nabla u^i \sigma)(s, X_{t,s}(x)) \text{ and} \\ V_s^{i,t,x} &= u^i(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u^i(s, X_{t,s-}(x)). \end{aligned}$$

This means  $\rho(x) dx$ -a.e., we have

$$Y_s^{i,t,x} = g(X_{t,T}(x)) + \int_s^T f(r, X_{t,r}(x), Y_r^{i,t,x}, Z_r^{i,t,x}, V_r^{i,t,x}) dr - \int_s^T Z_r^{i,t,x} dW_r - \int_s^T \int_E V_r^{i,t,x}(e) \tilde{\mu}(dr, de).$$

Then the uniqueness follows from the uniqueness of the BSDE.

(ii) *Existence* : Let us set

$$F(s, x) \triangleq f(s, x, Y_s^{s,x}, Z_s^{s,x}, V_s^{s,x}).$$

Plugging into the facts that  $f^0 \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$  and  $f$  is Lipschitz with respect to  $(y, z, v)$ , then thanks to Prop. 3.3.1 we have  $F(s, x) \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$ . Since  $g \in \mathbf{L}_\rho^2(\mathbb{R}^d)$ , we can approximate them by a sequence of smooth functions with compact support  $(g_n, F_n)$  such that

$$\begin{cases} g_n \rightarrow g \text{ in } \mathbf{L}_\rho^2(\mathbb{R}^d), \\ F_n(s, x) \rightarrow F(s, x) \text{ in } \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d). \end{cases} \quad (3.5.9)$$

Denote  $(Y_s^{n,t,x}, Z_s^{n,t,x}, V_s^{n,t,x}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{L}^2$  the solution of the BSDE associated with  $(\xi_n, F_n)$ , where  $\xi_n = g_n(X_{t,T}(x))$ , i.e.

$$Y_s^{n,t,x} = g_n(X_{t,T}(x)) + \int_s^T F_n(r, X_{t,r}(x)) dr - \int_s^T Z_r^{n,t,x} dW_r - \int_s^T \int_E V_r^{n,t,x}(e) \tilde{\mu}(dr, de). \quad (3.5.10)$$

Since  $F_n$  and  $g_n$  are smooth enough, from the regularity result of the solution with respect to  $(t, x)$  (see the proof in Appendix), we know that  $u_n(t, x) := Y_t^{n,t,x} \in C^{1,2}([0, T] \times \mathbb{R}^d)$  is the classic solution for the following PIDE:

$$\begin{cases} (\partial_t + \mathcal{L})u(t, x) + F_n(t, x) = 0 \\ u(T, x) = g_n(x). \end{cases} \quad (3.5.11)$$

Moreover, we know that  $v_n(t, x) := Z_t^{n,t,x} = (\nabla u_n \sigma)(t, x)$ . Besides, from the flow property, we can deduce that  $Y_s^{n,t,x} = u_n(s, X_{t,s}(x))$  and  $Z_s^{n,t,x} = (\nabla u_n \sigma)(s, X_{t,s}(x))$ , as well as the representation of the jump part

$$V_s^{n,t,x} := w_n(s, X_{t,s}(x)) := u_n(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u_n(s, X_{t,s-}(x)), \text{ a.s.}$$

Applying the equivalence of norms result and the Proposition 3.3.1, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T (|u_n(s, x)|^2 + |\sigma^* \nabla u_n(s, x)|^2) ds \rho(x) dx \\ & \leq c \int_{\mathbb{R}^d} \int_t^T E(|u_n(s, X_{t,s}(x))|^2 + |\sigma^* \nabla u_n(s, X_{t,s}(x))|^2) ds \rho(x) dx \\ & = c \int_{\mathbb{R}^d} \int_t^T \mathbb{E}(|Y_s^{n,t,x}|^2 + |Z_s^{n,t,x}|^2) ds \rho(x) dx \\ & \leq c \left( \int_{\mathbb{R}^d} E|g_n(X_{t,T}(x))|^2 \rho(x) dx + \int_{\mathbb{R}^d} \int_t^T \mathbb{E}|F_n(s, X_{t,s}(x))|^2 ds \rho(x) dx \right) \\ & \leq c \left( \int_{\mathbb{R}^d} |g_n(x)|^2 \rho(x) dx + \int_{\mathbb{R}^d} \int_t^T |F_n(s, x)|^2 ds \rho(x) dx \right) < +\infty, \end{aligned}$$

where  $c$  is a constant which changes from line to line. Therefore, we have shown that  $\forall n, u_n \in \mathcal{H}_T$  solves the PIDE associated with  $(g_n, f_n)$ , and for every  $\phi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$ ,

$$\begin{aligned} & \int_t^T (u_n(s, \cdot), \partial_s \phi(s, \cdot)) ds + (u_n(t, \cdot), \phi(t, \cdot)) - (g_n(\cdot), \phi(T, \cdot)) - \int_t^T (\mathcal{L}u_n(s, \cdot), \phi(s, \cdot)) ds \\ & = \int_t^T (F_n(s, \cdot), \phi(s, \cdot)) ds. \end{aligned} \quad (3.5.12)$$

Now for  $m, n \in N$ , applying Itô's formula to  $|Y_s^{m,t,x} - Y_s^{n,t,x}|^2$ , we get for any  $s \in [t, T]$ ,

$$\begin{aligned} & \mathbb{E}|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 + \mathbb{E} \int_s^T |Z_r^{m,t,x} - Z_r^{n,t,x}|^2 dr + \mathbb{E} \int_s^T \int_E |V_r^{m,t,x}(e) - V_r^{n,t,x}(e)|^2 \lambda(de) dr \\ & = \mathbb{E}|g_m(X_{t,T}(x)) - g_n(X_{t,T}(x))|^2 \\ & \quad + 2\mathbb{E} \int_s^T (Y_r^{m,t,x} - Y_r^{n,t,x})(F_m(r, X_{t,r}(x)) - F_n(r, X_{t,r}(x))) dr. \end{aligned}$$



Combining the properties of  $f_m$  and  $f_n$  with the basic inequality  $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ , we have:

$$\begin{aligned} & \mathbb{E}|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 + \mathbb{E} \int_s^T |Z_r^{m,t,x} - Z_r^{n,t,x}|^2 dr \\ & \leq \mathbb{E}|g_m(X_{t,T}(x)) - g_n(X_{t,T}(x))|^2 + \varepsilon \mathbb{E} \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \\ & \quad + \frac{1}{\varepsilon} \mathbb{E} \int_s^T |F_m(r, X_{t,r}(x)) - F_n(r, X_{t,r}(x))|^2 dr. \end{aligned}$$

It follows again from the equivalence of norms that

$$\begin{aligned} & \int_{\mathbb{R}^d} \mathbb{E}|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 \rho(x) dx \\ & \leq \int_{\mathbb{R}^d} \mathbb{E}|g_m(X_{t,T}(x)) - g_n(X_{t,T}(x))|^2 \rho(x) dx + \varepsilon \int_{\mathbb{R}^d} \mathbb{E} \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \rho(x) dx \\ & \quad + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \mathbb{E} \int_s^T |F_m(r, X_{t,r}(x)) - F_n(r, X_{t,r}(x))|^2 dr \rho(x) dx \\ & \leq \varepsilon C \int_{\mathbb{R}^d} \mathbb{E} \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \rho(x) dx + C \int_{\mathbb{R}^d} |g_m(x) - g_n(x)|^2 \rho(x) dx \\ & \quad + \frac{C}{\varepsilon} \int_{\mathbb{R}^d} \int_s^T |F_m(r, x) - F_n(r, x)|^2 dr \rho(x) dx. \end{aligned}$$

Choosing  $\varepsilon$  appropriately, by Gronwall's inequality and the convergence of  $F_n$  and  $g_n$ , we get as  $m, n \rightarrow \infty$ ,

$$\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \mathbb{E}|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 \rho(x) dx \rightarrow 0,$$

which implies immediately as  $m, n \rightarrow \infty$ ,

$$\int_{\mathbb{R}^d} \mathbb{E} \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \rho(x) dx + \int_{\mathbb{R}^d} \mathbb{E} \int_s^T |Z_r^{m,t,x} - Z_r^{n,t,x}|^2 dr \rho(x) dx \rightarrow 0.$$

Thanks again to the equivalence of norms (5.4.11), we get as  $m, n \rightarrow \infty$ :

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} (|u_m(s, x) - u_n(s, x)|^2 + |\sigma^* \nabla u_m(s, x) - \sigma^* \nabla u_n(s, x)|^2) \rho(x) dx ds \\ & \leq \frac{1}{C} \int_t^T \int_{\mathbb{R}^d} \mathbb{E} (|(u_m - u_n)(s, X_{t,s}(x))|^2 + |(\sigma^* \nabla u_m - \sigma^* \nabla u_n)(s, X_{t,s}(x))|^2) \rho(x) dx ds \\ & = \frac{1}{C} \int_t^T \int_{\mathbb{R}^d} \mathbb{E} (|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 + |Z_s^{m,t,x} - Z_s^{n,t,x}|^2) \rho(x) dx ds \rightarrow 0, \end{aligned}$$

which means that  $u_n$  is Cauchy sequence in  $\mathcal{H}_T$ . Denote its limit as  $u$ , then  $u \in \mathcal{H}_T$ . Moreover, since  $X_{t,s}(x)$  has at most countable jumps, using again the equivalence of norms, we can deduce that

$$\int_t^T \int_{\mathbb{R}^d} |w_m(s, x) - w_n(s, x)|^2 \rho(x) dx ds \rightarrow 0,$$

henceforth there exists at least a subsequence  $\{u_{n_k}\}$ , such that  $dt \otimes dP \otimes \rho(x) dx$ -a.e.,

$$\begin{cases} Y_s^{n_k,t,x} &= u_{n_k}(s, X_{t,s}(x)) \rightarrow u(s, X_{t,s}(x)) =: Y_s^{t,x}, \\ Z_s^{n_k,t,x} &= \sigma^* \nabla u_{n_k}(s, X_{t,s}(x)) \rightarrow \sigma^* \nabla u(s, X_{t,s}(x)) =: Z_s^{t,x}, \\ V_s^{n_k,t,x} &= w_{n_k}(s, X_{t,s}(x)) \rightarrow u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x)) := V_s^{t,x}. \end{cases}$$

Then we get the desired probabilistic representations (5.4.14). Passing limit in (3.5.12), we have for every  $\phi$ ,

$$\begin{aligned} & \int_t^T (u(s, \cdot), \partial_s \phi(s, \cdot)) ds + (u(t, \cdot), \phi(t, \cdot)) - (g(\cdot), \phi(T, \cdot)) - \int_t^T (u(s, \cdot), \mathcal{L}^* \phi(s, \cdot)) ds \\ & = \int_t^T (f(s, \cdot, u(s, \cdot), \sigma^* \nabla u(s, \cdot), u(s, x + \beta(x, \cdot)) - u(s, x)), \phi(s, \cdot)) ds, \end{aligned} \tag{3.5.13}$$

which means that  $u \in \mathcal{H}_T$  is the weak solution of (3.1.1).  $\square$

### 3.5.4 Proof of the tightness of the sequence $(\pi_n)_{n \in \mathbb{N}}$

Recall first that  $\nu_n(dt, dx) = n(u_n - h)^-(t, x) dt dx$  and  $\pi_n(dt, dx) = \rho(x) \nu_n(dt, dx)$  where  $u_n$  is the solution of the PIDE's (3.4.6).

**Lemma 3.5.1.** *The sequence of measure  $(\pi_n)_{n \in \mathbb{N}}$  is tight.*

**Proof.** Since here we need to deal with the additional jump part, we adapt the proof of Theorem 4 in [9]. We shall prove that for every  $\varepsilon > 0$ , there exists some constant  $K$  such that

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \pi_n(ds, dx) \leq \varepsilon, \quad \forall n \in \mathbb{N}. \quad (3.5.14)$$

We first write

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \pi_n(ds, dx) \\ &= \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \left( \mathbf{1}_{\{|X_{0,s}^{-1}(x) - x| \leq K\}} + \mathbf{1}_{\{|X_{0,s}^{-1}(x) - x| \geq K\}} \right) \pi_n(ds, dx) \\ &:= I_K^n + L_K^n, \quad P - a.s. \end{aligned}$$

Taking expectation yields

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \pi_n(ds, dx) = \mathbb{E}I_K^n + \mathbb{E}L_K^n.$$

By (4.3.17) and for  $K \geq 2\|b\|_\infty T$ , we get thanks to large deviation results (see [67], [68] and [69])

$$\begin{aligned} \mathbb{E}L_K^n &\leq \int_0^T \int_{\mathbb{R}^d} \mathbb{P} \left( \sup_{0 \leq r \leq T} |X_{0,r}^{-1}(x) - x| \geq K \right) \pi_n(ds, dx) \\ &\leq (C_1 \exp(-C_2 K^2) + C_3 \exp(-C_4 K)) \pi_n([0, T] \times \mathbb{R}^d) \\ &\leq C'_1 \exp(-C_2 K^2) + C'_3 \exp(-C_4 K), \end{aligned}$$

so  $\mathbb{E}L_K^n \leq \varepsilon$  for  $K$  sufficiently large.

On the other hand, if  $|x| \geq 2K$  and  $|X_{0,s}^{-1}(x) - x| \leq K$  then  $|X_{0,s}^{-1}(x)| \geq K$ . Therefore

$$\begin{aligned} \mathbb{E}I_K^n &\leq \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|X_{0,s}^{-1}(x)| \geq K\}} \rho(x) \nu_n(ds, dx) \\ &= \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|X_{0,s}^{-1}(x)| \geq K\}} \rho(x) n(u_n - h)^-(s, x) ds dx \end{aligned}$$

which, by the change of variable  $y = X_{0,s}^{-1}(x)$ , becomes

$$\begin{aligned} & \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \geq K\}} \rho(X_{0,s}(y)) J(X_{0,s}(y)) n(u_n - h)^-(s, X_{0,s}(y)) ds dy \\ &\leq \mathbb{E} \int_{\mathbb{R}^d} \rho(x) \left( \rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right) K_T^{n,0,x} dx \\ &\leq \left( \mathbb{E} \int_{\mathbb{R}^d} (K_T^{n,0,x})^2 \rho(x) dx \right)^{1/2} \\ &\quad \left( \mathbb{E} \int_{\mathbb{R}^d} \left( \rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \rho(x) dx \right)^{1/2} \\ &\leq C \left( \mathbb{E} \int_{\mathbb{R}^d} \left( \rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \rho(x) dx \right)^{1/2} \end{aligned}$$

where the last inequality is a consequence of (3.4.8). It is now sufficient to prove that

$$\int_{\mathbb{R}^d} \rho(x)^{-1} \mathbb{E} \left[ \left( \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \right] dx < \infty. \quad (3.5.15)$$

Note that

$$\begin{aligned} & \mathbb{E} \left[ \left( \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \right] \\ & \leq \left[ \mathbb{E} \left( \sup_{0 \leq r \leq T} |\rho(X_{0,r}(x))| \right)^4 \right]^{1/2} \left[ \mathbb{E} \left( \sup_{0 \leq r \leq T} |J(X_{0,r}(x))| \right)^4 \right]^{1/2} \\ & \leq C \left[ \mathbb{E} \left( \sup_{0 \leq r \leq T} |\rho(X_{0,r}(x))| \right)^4 \right]^{1/2}. \end{aligned}$$

Therefore it is sufficient to prove that:

$$\int_{\mathbb{R}^d} \frac{1}{\rho(x)} \left( \mathbb{E} \left[ \sup_{t \leq r \leq T} |\rho(X_{t,r}(x))|^4 \right] \right)^{1/2} dx < \infty.$$

Since  $\rho(x) \leq 1$ , we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq r \leq T} |\rho(X_{t,r}(x))|^4 \right] & \leq \mathbb{E} \left[ \sup_{t \leq r \leq T} |\rho(X_{t,r}(x))|^4 \mathbf{1}_{\left\{ \sup_{t \leq r \leq T} |X_{t,r}(x) - x| \leq \frac{|x|}{2} \right\}} \right] \\ & \quad + \mathbb{P} \left( \sup_{t \leq r \leq T} |X_{t,r}(x) - x| \geq \frac{|x|}{2} \right) \\ & = : A(x) + B(x). \end{aligned}$$

If  $\sup_{t \leq r \leq T} |X_{t,r}(x) - x| \leq \frac{|x|}{2}$  then  $|X_{t,r}(x)| \geq \frac{|x|}{2}$  and so  $|\rho(X_{t,r}(x))| \leq \left(1 + \frac{|x|}{2}\right)^{-p}$ . Thus we have that  $A(x) \leq \left(1 + \frac{|x|}{2}\right)^{-4p}$  and so  $\int_{\mathbb{R}^d} (1 + |x|)^p A(x)^{1/2} dx < \infty$ . On the other hand, if  $|x| \geq 4\|b\|_\infty T$ , then (the same argument as in the existence proof step 2 of Theorem 4 in [9] for the Itô integral with respect to the Brownian motion; and see e.g. Theorem 5.2.9 in [3] for the integral with respect to the compensated Poisson random measure)

$$\begin{aligned} B(x) & \leq \mathbb{P} \left( \sup_{t \leq s \leq T} \left| \int_0^s \sigma(X_{0,r}(x)) dW_r \right| \geq \frac{|x|}{8} \right) \\ & \quad + \mathbb{P} \left( \sup_{t \leq s \leq T} \left| \int_0^s \int_{\mathbb{E}} \beta(X_{0,r-}(x), e) \tilde{\mu}(dr, de) \right| \geq \frac{|x|}{8} \right) \\ & \leq C_1 \exp(-C_2|x|^2) + C_3 \exp(-C_4|x|) \end{aligned}$$

and so  $\int_{\mathbb{R}^d} (1 + |x|)^p B(x)^{1/2} dx < \infty$ .



# Semilinear Stochastic PDEs in a convex domain

## Contents

<b>4.1 Introduction</b>	<b>91</b>
<b>4.2 Backward Doubly Stochastic Differential Equations in a domain</b>	<b>93</b>
4.2.1 Hypotheses and preliminaries	93
4.2.2 Existence and uniqueness of the solution	95
<b>4.3 Weak solution of semilinear SPDE in a convex domain</b>	<b>102</b>
4.3.1 Notations and Hypothesis	102
4.3.2 Weak formulation for a solution of Stochastic PDEs	103
4.3.3 Stochastic flow of diffeomorphisms and random test functions	104
4.3.4 Existence and uniqueness of solutions for the reflected SPDE	105
<b>4.4 Appendix</b>	<b>110</b>
4.4.1 Some properties of convexity	110
4.4.2 A priori estimates	111
4.4.3 Proof of Lemma 4.2.4	112
4.4.4 Proof of the tightness of the sequence $(\pi_n)_{n \in \mathbb{N}}$	113

## 4.1 Introduction

We are interested in this chapter in the following system of semilinear stochastic PDE with value in  $\mathbb{R}^k$ ,

$$du_t(x) + [\mathcal{L}u_t(x) + f_t(x, u_t(x), \nabla u_t \sigma(x))]dt + h_t(x, u_t(x), \nabla u_t \sigma(x)) \cdot d\overleftarrow{W}_t = 0, \quad (4.1.1)$$

over the time interval  $[0, T]$ , with a given final condition  $u_T = \Phi$ ,  $f, h$  are non-linear random functions and  $\mathcal{L}$  is the second order differential operator associated with a diffusion which is defined component by component with

$$\mathcal{L}\varphi(x) = \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x). \quad (4.1.2)$$

The differential term with  $d\overleftarrow{W}_t$  refers to the backward stochastic integral with respect to a  $d$ -dimensional Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P}, (W_t)_{t \geq 0})$ . We use the backward notation because in the proof we will employ the doubly stochastic framework introduced by Pardoux and Peng [109]. It is well known now that BSDEs provide a natural tools to give a probabilistic interpretation for

the solution of a class of semi-linear PDEs. By introducing in standard BSDEs a second nonlinear term driven by an external noise, we obtain Backward Doubly SDEs (BDSDEs) [109] (see also [10], [97]), which can be seen as Feynman-Kac representation of SPDEs and form a powerful tool for probabilistic numerical schemes [6] for such SPDEs. Given a convex domain  $D$  in  $\mathbb{R}^k$ , the concern is to study the reflection problem for SPDE (4.1.1) in such domain  $D$ . This problem is well known as a Skorohod problem for SPDEs.

In this chapter we shall study the weak solution of SPDE (4.1.1) in a given convex domain  $D$  by introducing the associated BDSDE. Inspired by the variational formulation of the obstacle problem for SPDEs and Menaldi's work [101] on reflected diffusion, we consider the solution of the reflection problem for the SPDEs (4.1.1) as a pair  $(u, \nu)$ , where  $\nu$  is a random regular measure and  $u \in \mathbf{L}^2(\Omega \times [0, T]; H^1(\mathbb{R}^d))$  satisfies the following relations:

$$\begin{aligned}
 (i) \quad & u_t(x) \in \bar{D}, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.}, \\
 (ii') \quad & du_t(x) + [\mathcal{L}u_t(x) + f_t(x, u_t(x), \nabla u_t \sigma(x))] dt + h_t(x, u_t(x), \nabla u_t \sigma(x)) \cdot d\overleftarrow{W}_t = -\nu(dt, dx), \quad \text{a.s.}, \\
 (iii) \quad & \nu(u \notin \partial D) = 0, \text{a.s.}, \\
 (iv) \quad & u_T = \Phi, \quad dx - \text{a.e.}
 \end{aligned}
 \tag{4.1.3}$$

$\nu$  is a random measure which acts only when the process  $u$  reaches the boundary of the domain  $D$ . The rigorous sense of the relation (iii) will be based on the probabilistic representation of the measure  $\nu$  in term of the bounded variation processes  $K$  component of the associated solution of the reflected BDSDE in the domain  $D$ .

Our contributions are as following: first of all, reflected BDSDEs in the convex domain  $D$  are introduced and results of existence and uniqueness of such RBDSDEs are established. Next, the existence and uniqueness results of the solution  $(u, \nu)$  of the reflection problem for (4.1.1) are given in Theorem 4.3.1. Indeed, a probabilistic method based on reflected BDSDEs and stochastic flow technics are investigated in our context (see e.g. [10], [97], [80, 79] for these flow technics). The key element in [10] is to use the inversion of stochastic flow which transforms the variational formulation of the SPDEs to the associated BDSDEs. Thus it plays the same role as Itô's formula in the case of the classical solution of SPDEs.

The outline is as following: in section 4.2.1, first the basic assumptions and the definitions of the solutions for Reflected BDSDE in a convex domain are presented. Then, existence and uniqueness of solution of RBDSDE (Theorem 4.2.1) is given under only convexity assumption for the domain without any regularity on the boundary. This result is proved by using penalization approximation. Thanks to the convexity properties we prove several technical lemmas, in particular the fundamental Lemma 4.2.2. In section 4.3, we study semilinear SPDEs in a convex domain. We first provide useful results on stochastic flow associated with the forward SDEs, then in this setting as in Bally and Matoussi [10], an equivalence norm result associated to the diffusion process is given. The main result of this section Theorem 4.3.1 is the existence and uniqueness results of the solution of reflected SPDEs in a convex domain. The proof of this result is based on the probabilistic interpretation via the Reflected Forward-BDSDEs. The uniqueness is a consequence of the variational formulation of the SPDEs written with random test functions and the uniqueness of the solution of the Reflected FBSDE. The existence of the solution is established by an approximation penalization procedure, a priori estimates and the equivalence norm results. In the Appendix, technical lemmas for the existence of the solution of the Reflected BDSDEs and SPDEs in a convex domain are given.

## 4.2 Backward Doubly Stochastic Differential Equations in a domain

### 4.2.1 Hypotheses and preliminaries

The euclidean norm of a vector  $x \in \mathbb{R}^k$  will be denoted by  $|x|$ , and for a  $k \times k$  matrix  $A$ , we define  $\|A\| = \sqrt{\text{Tr}AA^*}$ . In what follows let us fix a positive number  $T > 0$ .

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability product space, and let  $\{W_s, 0 \leq s \leq T\}$  and  $\{B_s, 0 \leq s \leq T\}$  be two mutually independent standard Brownian motion processes, with values respectively in  $\mathbb{R}^d$  and in  $\mathbb{R}^l$ . For each  $t \in [0, T]$ , we define

$$\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^W \vee \mathcal{N}$$

where  $\mathcal{F}_t^B = \sigma\{B_r, 0 \leq r \leq t\}$ ,  $\mathcal{F}_{t,T}^W = \sigma\{W_r - W_t, t \leq r \leq T\}$  and  $\mathcal{N}$  the class of  $\mathbb{P}$  null sets of  $\mathcal{F}$ . Note that the collection  $\{\mathcal{F}_t, t \in [0, T]\}$  is neither increasing nor decreasing, and it does not constitute a filtration.

#### 4.2.1.1 Convexity results

Besides, we need to recall properties related to the convexity of a nonempty domain  $D$  in  $\mathbb{R}^k$ . Let  $\partial D$  denotes the boundary of  $D$  and  $\pi(x)$  the projection of  $x \in \mathbb{R}^k$  on  $D$ . We have the following properties:

$$(x' - x)^*(x - \pi(x)) \leq 0, \quad \forall x \in \mathbb{R}^d, \forall x' \in \bar{D} \quad (4.2.1)$$

$$(x' - x)^*(x - \pi(x)) \leq (x' - \pi(x'))^*(x - \pi(x)), \quad \forall x, x' \in \mathbb{R}^k \quad (4.2.2)$$

$$\exists a \in D, \gamma > 0, \text{ such that } (x - a)^*((x - \pi(x)) \geq \gamma|x - \pi(x)|, \quad \forall x \in \mathbb{R}^k. \quad (4.2.3)$$

For  $x \in \partial D$ , we denote by  $\nu(x)$  the set of normal unit vectors at the point  $x$ . Since  $D$  is not regular, we define a sequence of regular convex which approximate uniformly  $D$ . Indeed, the function  $h(x) = d(x, D)$  is convex and uniformly continuous in  $\mathbb{R}^k$ . If we denote  $(g_\delta)_{0 \leq \delta \leq \delta_0}$  the approximation identity with compact support, then  $h_\delta = g_\delta * h$  is a sequence of regular convex functions which tends uniformly to  $h$  as  $\delta \rightarrow 0$ . For a fixed  $\eta > 0$ ,  $\{x, h_\delta(x) < \eta\}$  are regular convex domains that converge uniformly to  $\{x, d(x, D) < \eta\}$  when  $\delta$  tends to 0. Letting  $\eta \rightarrow 0$ , we conclude that for all  $\varepsilon > 0$  there exists a regular convex  $D_\varepsilon$  such that

$$\sup_{x \in D} d(x, D_\varepsilon) < \varepsilon \quad \text{and} \quad \sup_{x \in D_\varepsilon} d(x, D) < \varepsilon \quad (4.2.4)$$

One can find all these results in Menaldi [101].

#### 4.2.1.2 Functional spaces and assumptions

Hereafter, let us define the spaces and the norms which will be needed for the formulation of the BDSDE in a domain.

-  $\mathbf{L}_k^p(\mathcal{F}_T)$  the space of  $k$ -dimensional  $\mathcal{F}_T$ -measurable random variables  $\xi$  such that

$$\|\xi\|_{L^p}^p := \mathbb{E}(|\xi|^p) < +\infty;$$

-  $\mathcal{H}_{k \times d}^2([0, T])$  the space of  $\mathbb{R}^{k \times d}$ -valued  $\mathcal{F}_t$ -measurable process  $Z = (Z_t)_{t \leq T}$  such that

$$\|Z\|_{\mathcal{H}^2}^2 := \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < +\infty;$$

- $\mathcal{S}_k^2([0, T])$  the space of  $\mathbb{R}^k$ -valued  $\mathcal{F}_t$ -adapted processes  $Y = (Y_t)_{t \leq T}$ , with continuous paths such that

$$\|Y\|_{\mathcal{S}^2}^2 := \mathbb{E}[\sup_{t \leq T} |Y_t|^2] < +\infty;$$

- $\mathcal{A}_k^2([0, T])$  the space of  $\mathbb{R}^k$ -valued  $\mathcal{F}_t$ -adapted processes  $K = (K_t)_{t \leq T}$ , with continuous and bounded variation paths such that  $K_0 = 0$  and

$$\|K\|_{\mathcal{A}^2}^2 := \mathbb{E}[\sup_{t \leq T} |K_t|^2] < +\infty.$$

We next state our main assumptions on the terminal condition  $\xi$  and the functions  $f$  and  $h$ :

**Assumption 4.2.1.**  $\xi \in \mathbf{L}_k^2(\mathcal{F}_T)$  and  $\xi \in \bar{D}$  a.s..

**Assumption 4.2.2.**  $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$ ,  $h : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times l}$  are two random functions verifying:

- (i) For all  $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,  $f_t(\omega, y, z)$  and  $h_t(\omega, y, z)$  are  $\mathcal{F}_t$  - progressively measurable.
- (ii)  $\mathbb{E}[\int_0^T |f_t(0, 0)|^2 dt] < +\infty$ ,  $\mathbb{E}[\int_0^T \|h_t(0, 0)\|^2 dt] < +\infty$ .
- (iii) There exist constants  $c > 0$  and  $0 < \alpha < 1$  such that for any  $(\omega, t) \in \Omega \times [0, T]$ ;  $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |f_t(y_1, z_1) - f_t(y_2, z_2)|^2 &\leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ \|h_t(y_1, z_1) - h_t(y_2, z_2)\|^2 &\leq c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2. \end{aligned}$$

We denote by  $f_t^0 := f_t(\omega, 0, 0)$  and  $h_t^0 := h_t(\omega, 0, 0)$ .

Now we introduce the definition of the solution of BDSDEs in a domain.

**Definition 4.2.1.** The triplet of processes  $(Y_t, Z_t, K_t)_{\{0 \leq t \leq T\}}$  is solution of the backward doubly stochastic differential equation in a convex domain  $D$ , with terminal condition  $\xi$  and coefficients  $f$  and  $h$ , if the following hold:

- (i)  $Y \in \mathcal{S}_k^2([0, T])$ ,  $Z \in \mathcal{H}_{k \times d}^2([0, T])$  and  $K \in \mathcal{A}_k^2([0, T])$ ,
- (ii)

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds + \int_t^T h_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T \text{ a.s.} \quad (4.2.5)$$

- (iii)  $Y_t \in \bar{D}$ ,  $0 \leq t \leq T$ , a.s.

- (iv) for any continuous  $\mathcal{F}_t$ -measurable process  $(z_t)_{0 \leq t \leq T}$  valued in  $\bar{D}$ ,

$$\int_0^T (Y_t - z_t)^* dK_t \leq 0, \text{ a.s.} \quad (4.2.6)$$

The triplet  $(Y_t, Z_t, K_t)_{\{0 \leq t \leq T\}}$  is called a solution of RBDSDE with data  $(\xi, f, h)$ .

**Remark 4.2.1.** One may ask the question if there exists a solution  $Y \in \bar{D}$  to the following classical BDSDE introduced first in [109] :

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds + \int_t^T h_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s. \quad (4.2.7)$$



In the particular case when  $f \equiv 0$  and  $h$  does not depend on  $(y, z)$ , the solution  $Y$  of the BDDSE is given by

$$Y_t = \mathbb{E} \left[ \xi + \int_0^T h_s d\overleftarrow{W}_s \mid \mathcal{G}_t \right]. \quad (4.2.8)$$

By the generalized Ito-martingale representation theorem, there exists a unique  $\mathcal{G}_t$ -adapted process  $(Z_t)_{0 \leq t \leq T}$  such that  $(Y, Z)_{0 \leq t \leq T}$  is solution of the equation (4.2.7). Moreover, by the same arguments developed in [109] we can prove that  $(Y_t)_{0 \leq t \leq T}$  and  $(Z_t)_{0 \leq t \leq T}$  are precisely  $\mathcal{F}_t$ -adapted processes. But, since  $\xi \in \bar{D}$  and if we assume that  $\int_0^t h_s d\overleftarrow{W}_s \in \bar{D}$ ,  $\mathbb{P}$ -a.s., Jensen's inequality applied to the convex function  $\varphi(x) := d^2(x, D)$  gives that  $Y \in \bar{D}$ . When  $f$  is not identically equal to zero, the bounded variation term  $\int_t^T f_s(Y_s, Z_s) ds$  will push  $Y$  out of the domain  $D$ , therefore one need to reflect the process  $Y$  when it reaches the boundary of  $D$ , this yields to the Definition 4.2.5.

**Remark 4.2.2.** From Lemma 2.1 in [53], the condition (4.2.6) implies that the bounded variation process  $K$  acts only when  $Y$  reaches the boundary of the convex domain  $D$  and the so-called Skorohod condition is satisfied:

$$\int_0^T \mathbf{1}_{\{Y_t \in D\}} dK_t = 0. \quad (4.2.9)$$

Moreover there exists a  $\mathcal{F}_t$ -measurable process  $(\alpha_t)_{0 \leq t \leq T}$  valued in  $\mathbb{R}^k$  such that

$$K_t = \int_0^t \alpha_s d\|K_s\|_{VT} \quad \text{and} \quad -\alpha_s \in \nu(Y_s).$$

In the following,  $C$  will denote a positif constant which doesn't depend on  $\varepsilon$  nor on  $n$  and can vary from line to line.

### 4.2.2 Existence and uniqueness of the solution

In this section we establish existence and uniqueness results for RBDSDE (4.2.5).

**Theorem 4.2.1.** *Let Assumptions 4.2.1 and 4.2.2 hold. Then, the RBDSDE (4.2.5) has a unique solution  $(Y, Z, K) \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T]) \times \mathcal{A}_k^2([0, T])$ .*

**Proof.**

**a) Uniqueness:** Let  $(Y^1, Z^1, K^1)$  and  $(Y^2, Z^2, K^2)$  be two solutions of the RBDSDE (4.2.5). Applying Lemma 1.3 in [109] yields

$$\begin{aligned} |Y_t^1 - Y_t^2|^2 &+ \int_t^T \|Z_s^1 - Z_s^2\|^2 ds = 2 \int_t^T (Y_s^1 - Y_s^2)^* (f_s(Y_s^1, Z_s^1) - f_s(Y_s^2, Z_s^2)) ds \\ &+ 2 \int_t^T (Y_s^1 - Y_s^2)^* (h_s(Y_s^1, Z_s^1) - h_s(Y_s^2, Z_s^2)) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^1 - Y_s^2) (Z_s^1 - Z_s^2) dB_s \\ &+ 2 \int_t^T (Y_s^1 - Y_s^2)^* (dK_s^1 - dK_s^2) + \int_t^T \|h_s(Y_s^1, Z_s^1) - h_s(Y_s^2, Z_s^2)\|^2 ds. \end{aligned} \quad (4.2.10)$$

Therefore, since  $(Y_s^1 - Y_s^2)$  belongs to  $\bar{D}$ , under the minimality condition (iv) we have

$$\int_t^T (Y_s^1 - Y_s^2)^* (dK_s^1 - dK_s^2) \leq 0, \quad \text{for all } t \in [0, T]. \quad (4.2.11)$$

Then, plugging (4.2.11) in (4.2.10) and taking expectation we obtain

$$\begin{aligned} \mathbb{E}[|Y_t^1 - Y_t^2|^2] &+ \mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] \leq 2\mathbb{E}\left[\int_t^T (Y_s^1 - Y_s^2)^* (f_s(Y_s^1, Z_s^1) - f_s(Y_s^2, Z_s^2)) ds\right] \\ &+ \mathbb{E}\left[\int_t^T \|h_s(Y_s^1, Z_s^1) - h_s(Y_s^2, Z_s^2)\|^2 ds\right]. \end{aligned}$$

Hence from the Lipschitz property

$$\begin{aligned} \mathbb{E}[|Y_t^1 - Y_t^2|^2] + \mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] &\leq c(\alpha)\mathbb{E}\left[\int_t^T |Y_s^1 - Y_s^2|^2 ds\right] + \frac{1-\alpha}{2}\mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] \\ &+ \alpha\mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right], \end{aligned}$$

where  $0 < \alpha < 1$ . Consequently

$$\mathbb{E}[|Y_t^1 - Y_t^2|^2] + \left(\frac{1-\alpha}{2}\right)\mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] \leq c(\alpha)\mathbb{E}\left[\int_t^T |Y_s^1 - Y_s^2|^2 ds\right].$$

From Gronwall's lemma,  $\mathbb{E}[|Y_t^1 - Y_t^2|^2] = 0$ ,  $0 \leq t \leq T$ , and  $\mathbb{E}\left[\int_0^T \|Z_s^1 - Z_s^2\|^2 ds\right] = 0$ .

**b) Existence:** The existence of a solution will be proved by penalisation method. For  $n \in \mathbb{N}$ , we consider for all  $t \in [0, T]$ ,

$$Y_t^n = \xi + \int_t^T f_s(Y_s^n, Z_s^n) ds + \int_t^T h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s - n \int_t^T (Y_s^n - \pi(Y_s^n)) ds - \int_t^T Z_s^n dB_s. \quad (4.2.12)$$

Denote by  $K_t^n := -n \int_t^s (Y_s^n - \pi(Y_s^n)) ds$ . In order to prove the convergence of the sequence  $(Y^n, Z^n, K^n)$  to the solution of our RBDSDE (4.2.1), we need several lemmas. For this end, we add the following further assumption:

**Assumption 4.2.3.** (i)  $0 \in D$  and  $\xi \in \mathbf{L}_k^4(\mathcal{F}_T)$ .

(ii) There exist  $c > 0$  and  $0 \leq \beta < 1$  such that for all  $(t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$h_t h_t^*(y, z) \leq c(Id_{\mathbb{R}^k} + yy^*) + \beta zz^*.$$

(iii)  $f$  and  $h$  are uniformly bounded in  $(y, z)$ .

The Assumption 4.2.3 (i) and (ii) are needed to prove the uniform  $L^4$ -estimate for  $(Y^n, Z^n)$  solution of BDSDE (4.2.12) (see estimate (4.4.7) in the Appendix 4.4.3). This is crucial for our proof of the fundamental lemma 4.2.2. The Assumption 4.2.3 (iii) is only added for simplicity and it can be removed by standard technics of BSDEs. The natural condition instead of (iii) is  $f^0$  and  $h^0$  in  $\mathbf{L}^4(\Omega, \mathcal{F}, \mathbb{P})$ .

We assume also for the sake of simplicity that  $D$  is a convex set with class  $C^2$  boundary. If not we can approximate our convex domain  $D$  by regular convex domains as mentioned in section 4.2.1.1.

We start with the following lemma:

**Lemma 4.2.1.** *There exists a constant  $C > 0$  such that*

$$\forall n \in \mathbb{N} \quad n\mathbb{E}\left[\int_0^T d^2(Y_s^n, D) ds\right] \leq C. \quad (4.2.13)$$

**Proof.** We apply generalized Itô's formula [109] to  $\rho(Y_t^n) = d^2(Y_t^n, D)$  to obtain

$$\begin{aligned} \rho(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\rho(Y_s^n)] ds &= \rho(\xi) + \int_t^T (\nabla\rho(Y_s^n))^* f_s(Y_s^n, Z_s^n) ds \\ &- \int_t^T (\nabla\rho(Y_s^n))^* Z_s^n dB_s + \int_t^T (\nabla\rho(Y_s^n))^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &+ \frac{1}{2} \int_t^T \text{trace}[(h_s h_s^*)(Y_s^n, Z_s^n) \text{Hess}\rho(Y_s^n)] ds - 2n \int_t^T (Y_s^n - \pi(Y_s^n))^* (Y_s^n - \pi(Y_s^n)) ds. \end{aligned} \quad (4.2.14)$$

Since  $\xi \in \bar{D}$  a.s., it is obvious that  $\rho(\xi) = 0$ . We get from the boundedness of  $h$  and the Hessian of  $\rho$

$$\begin{aligned} \rho(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\rho(Y_s^n)] ds + 2n \int_t^T d^2(Y_s^n, D) ds \\ \leq 2 \int_t^T (\rho(Y_s^n))^{1/2} |f_s(Y_s^n, Z_s^n)| ds - 2 \int_t^T (Y_s^n - \pi(Y_s^n))^* Z_s^n dB_s \\ + 2 \int_t^T (Y_s^n - \pi(Y_s^n))^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s + C. \end{aligned} \quad (4.2.15)$$

Now the algebraic inequality  $2ab \leq a^2 + b^2$  with  $a = \sqrt{\frac{n}{2}}\rho(Y_s^n)$  yields

$$(\rho(Y_s^n))^{1/2} |f_s(Y_s^n, Z_s^n)| \leq \frac{n}{4} \rho(Y_s^n) + \frac{1}{n} |f_s(Y_s^n, Z_s^n)|^2.$$

Then it follows that,

$$\begin{aligned} \rho(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\rho(Y_s^n)] ds + \frac{3n}{2} \int_t^T d^2(Y_s^n, D) ds \\ \leq 2 \int_t^T \frac{1}{n} |f_s(Y_s^n, Z_s^n)|^2 ds - 2 \int_t^T (Y_s^n - \pi(Y_s^n))^* Z_s^n dB_s \\ + 2 \int_t^T (Y_s^n - \pi(Y_s^n))^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s + C. \end{aligned} \quad (4.2.16)$$

By taking expectation and using the boundedness of  $f$ , we have

$$\mathbb{E}[\rho(Y_t^n)] + \frac{1}{2} \mathbb{E} \left[ \int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\rho(Y_s^n)] ds \right] + \frac{n}{2} \mathbb{E} \left[ \int_t^T d^2(Y_s^n, D) ds \right] \leq C. \quad (4.2.17)$$

Hence, the required result is obtained.  $\square$

The next lemma plays a crucial role to prove the strong convergence of  $(Y^n, Z^n, K^n)$ .

**Lemma 4.2.2.**

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (d(Y_t^n, D))^4 \right] \xrightarrow{n \rightarrow +\infty} 0. \quad (4.2.18)$$

**Proof.** We denote by  $\rho(x) = d^2(x, D)$  and  $\varphi(x) = \rho^2(x)$ . By applying Itô's formula to  $\varphi(Y_t^n) = d^4(Y_t^n, D)$ , we obtain that

$$\begin{aligned} \varphi(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\varphi(Y_s^n)] ds = \varphi(\xi) + \int_t^T (\nabla\varphi(Y_s^n))^* f_s(Y_s^n, Z_s^n) ds \\ - \int_t^T (\nabla\varphi(Y_s^n))^* Z_s^n dB_s + \int_t^T (\nabla\varphi(Y_s^n))^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ + \frac{1}{2} \int_t^T \text{trace}[(h_s h_s^*)(Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)] ds - n \int_t^T \nabla\varphi(Y_s^n)^* (Y_s^n - \pi(Y_s^n)) ds. \end{aligned} \quad (4.2.19)$$

Since  $\xi \in \bar{D}$  a.s., we have that  $\varphi(\xi) = 0$  and it is easy to see that

$$\nabla\varphi(x) = 2\rho(x)\nabla\rho(x) = 4\rho(x)(x - \pi(x)) \quad (4.2.20)$$

$$\text{Hess}\varphi(x) = 2\nabla\rho(x) \otimes \nabla\rho(x) + 2\rho(x)\text{Hess}\rho(x). \quad (4.2.21)$$

Then it follows that

$$\begin{aligned} \varphi(Y_t^n) + \frac{1}{2} \int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\varphi(Y_s^n)] ds = 4 \int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* f_s(Y_s^n, Z_s^n)) ds \\ - 4 \int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* Z_s^n) dB_s + 4 \int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* h_s(Y_s^n, Z_s^n)) d\overleftarrow{W}_s \\ + \frac{1}{2} \int_t^T \text{tr}[(h_s h_s^*)(Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)] ds - 4n \int_t^T \rho^2(Y_s^n) ds. \end{aligned} \quad (4.2.22)$$

By taking expectation we have

$$\begin{aligned}
 \mathbb{E}[\varphi(Y_t^n)] &+ \frac{1}{2}\mathbb{E}\left[\int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\varphi(Y_s^n)]ds\right] + 4n\mathbb{E}\left[\int_t^T \varphi(Y_s^n)ds\right] \\
 &= 4\mathbb{E}\left[\int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* f_s(Y_s^n, Z_s^n))ds\right] \\
 &+ \frac{1}{2}\mathbb{E}\left[\int_t^T \text{trace}[(h_s h_s^*)(Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)]ds\right].
 \end{aligned} \tag{4.2.23}$$

For the last term, we get from the boundedness of  $h$  and  $\text{Hess}\rho$

$$\begin{aligned}
 \mathbb{E}\left[\int_t^T \text{trace}[(h_s h_s^*)(Y_s^n, Z_s^n) \text{Hess}\varphi(Y_s^n)]ds\right] &\leq 2\mathbb{E}\left[\int_t^T \langle h_s(Y_s^n, Z_s^n), \nabla\rho(Y_s^n) \rangle^2 ds\right] \\
 &+ \mathbb{E}\left[\int_t^T 2\rho(Y_s^n) \text{trace}[(h_s h_s^*)(Y_s^n, Z_s^n) \text{Hess}\rho(Y_s^n)]ds\right] \\
 &\leq C\mathbb{E}\left[\int_t^T |\nabla\rho(Y_s^n)|^2 ds\right] + C\mathbb{E}\left[\int_t^T 2\rho(Y_s^n)ds\right] \\
 &\leq C\mathbb{E}\left[\int_0^T (d(Y_s^n, D))^2 ds\right].
 \end{aligned} \tag{4.2.24}$$

Now the algebraic inequality  $2ab \leq a^2 + b^2$  with  $a = (d(Y_s^n, D))^2$  and the boundedness of  $f$  yield

$$\begin{aligned}
 4(d(Y_s^n, D))^3 |f_s(Y_s^n, Z_s^n)| &\leq 2(d(Y_s^n, D))^4 + 2(d(Y_s^n, D))^2 |f_s(Y_s^n, Z_s^n)| \\
 &\leq 2\varphi(Y_s^n) + 2C(d(Y_s^n, D))^2.
 \end{aligned} \tag{4.2.25}$$

By plugging the estimate (4.2.25) and (4.2.24) in (4.2.23), we obtain thanks to lemma 4.2.1

$$\begin{aligned}
 \mathbb{E}[\varphi(Y_t^n)] &+ \frac{1}{2}\mathbb{E}\left[\int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\varphi(Y_s^n)]ds\right] + (4n-2)\mathbb{E}\left[\int_t^T \varphi(Y_s^n)ds\right] \\
 &\leq C\mathbb{E}\left[\int_0^T (d(Y_s^n, D))^2\right] \leq \frac{C}{n}.
 \end{aligned} \tag{4.2.26}$$

Notice also that Hessienne of  $\varphi(Y_s^n)$  is a positive definite matrix since  $\varphi$  is a convex function, so we get that

$$\sup_{0 \leq t \leq T} \mathbb{E}[\varphi(Y_t^n)] \leq \frac{C}{n}. \tag{4.2.27}$$

Moreover, we can deduce from (4.2.26) that, for every  $t \in [0, T]$

$$\mathbb{E}\left[\int_t^T \text{trace}[Z_s^n Z_s^{n*} \text{Hess}\varphi(Y_s^n)]ds\right] \longrightarrow 0, \text{ as } n \rightarrow \infty. \tag{4.2.28}$$

In the other hand, we take the supremum over  $t$  in the equation (4.2.22) and by Burkholder-Davis-Gundy's inequality and the previous calculations it follows that

$$\begin{aligned}
 \mathbb{E}\left[\sup_{0 \leq t \leq T} \varphi(Y_t^n)\right] &\leq C\mathbb{E}\left[\int_0^T \varphi(Y_s^n)ds\right] + C\mathbb{E}\left[\int_0^T (d(Y_s^n, D))^2 ds\right] \\
 &+ C\mathbb{E}\left[\sup_{0 \leq t \leq T} \int_t^T (\rho(Y_s^n) \nabla\rho(Y_s^n))^* Z_s^n dB_s\right] \\
 &+ C\mathbb{E}\left[\sup_{0 \leq t \leq T} \int_t^T (\rho(Y_s^n) \nabla\rho(Y_s^n))^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s\right] \\
 &\leq C\mathbb{E}\left[\int_0^T \varphi(Y_s^n)ds\right] + C\mathbb{E}\left[\int_0^T (d(Y_s^n, D))^2 ds\right] \\
 &+ C\mathbb{E}\left[\left(\int_0^T (\rho(Y_s^n))^2 \langle \nabla\rho(Y_s^n), Z_s^n \rangle^2 ds\right)^{1/2}\right] \\
 &+ C\mathbb{E}\left[\left(\int_0^T (\rho(Y_s^n))^2 \langle \nabla\rho(Y_s^n), h_s(Y_s^n, Z_s^n) \rangle^2 ds\right)^{1/2}\right].
 \end{aligned} \tag{4.2.29}$$

From the boundedness of  $h$  and the fact that  $\nabla\rho^2(x) = 4\rho(x)$ , we have

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T (\rho(Y_s^n))^2 \langle \nabla\rho(Y_s^n), h_s(Y_s^n, Z_s^n) \rangle^2 ds\right)^{1/2}\right] &\leq C\mathbb{E}\left[\left(\int_0^T (\rho(Y_s^n))^2 \rho(Y_s^n) ds\right)^{1/2}\right] \\ &\leq C\mathbb{E}\left[\sup_{0 \leq s \leq T} (\varphi(Y_s^n))^{1/2} \left(\int_0^T \rho(Y_s^n) ds\right)^{1/2}\right] \\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{0 \leq s \leq T} \varphi(Y_s^n)\right] + C^2\mathbb{E}\left[\int_0^T (d(Y_s^n, D))^2 ds\right]. \end{aligned} \quad (4.2.30)$$

By the Holder's inequality, we obtain

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T (\rho(Y_s^n))^2 \langle \nabla\rho(Y_s^n), Z_s^n \rangle^2 ds\right)^{1/2}\right] &\leq C\mathbb{E}\left[\sup_{0 \leq s \leq T} (\varphi(Y_s^n))^{1/2} \left(\int_0^T \langle \nabla\rho(Y_s^n), Z_s^n \rangle^2 ds\right)^{1/2}\right] \\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{0 \leq s \leq T} \varphi(Y_s^n)\right] + C^2\mathbb{E}\left[\int_0^T \langle \nabla\rho(Y_s^n), Z_s^n \rangle^2 ds\right]. \end{aligned} \quad (4.2.31)$$

Reporting (4.2.30) and (4.2.31) in (4.2.29) leads to

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} \varphi(Y_t^n)\right] &\leq C\mathbb{E}\left[\int_0^T \varphi(Y_s^n) ds\right] + C\mathbb{E}\left[\int_0^T (d(Y_s^n, D))^2 ds\right] \\ &\quad + C^2\mathbb{E}\left[\int_0^T \langle \nabla\rho(Y_s^n), Z_s^n \rangle^2 ds\right]. \end{aligned} \quad (4.2.32)$$

Since each term of (4.2.21) is positive definite and from (4.2.28), we get

$$\mathbb{E}\left[\int_0^T \langle \nabla\rho(Y_s^n), Z_s^n \rangle^2 ds\right] \longrightarrow 0 \text{ as } n \rightarrow \infty.$$

Finally, by using (4.2.27) and Lemma 4.2.1, we get the desired result.  $\square$

Therefore, we deduce the strong convergence of  $(Y^n, Z^n)$ :

**Lemma 4.2.3.** *The sequence  $(Y^n, Z^n)$  is a Cauchy sequence in  $\mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$ , i.e.*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 + \int_0^T \|Z_t^n - Z_t^m\|^2 dt\right] \longrightarrow 0 \text{ as } n, m \rightarrow +\infty.$$

**Proof.** For all  $n, m \geq 0$ , we apply Itô formula to  $|Y_t^n - Y_t^m|^2$

$$\begin{aligned} |Y_t^n - Y_t^m|^2 + \int_t^T \|Z_s^n - Z_s^m\|^2 ds &= 2 \int_t^T (Y_s^n - Y_s^m)^* (f_s(Y_s^n, Z_s^n) - f_s(Y_s^m, Z_s^m)) ds \\ &\quad + 2 \int_t^T (Y_s^n - Y_s^m)^* (h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m)) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^n - Y_s^m) (Z_s^n - Z_s^m) dB_s \\ &\quad + \int_t^T \|h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m)\|^2 ds - 2n \int_t^T (Y_s^n - Y_s^m)^* (Y_s^n - \pi(Y_s^n)) ds \\ &\quad + 2m \int_t^T (Y_s^n - Y_s^m)^* (Y_s^m - \pi(Y_s^m)) ds. \end{aligned} \quad (4.2.33)$$

By the property (4.2.2), we have

$$\begin{aligned} -2n \int_t^T (Y_s^n - Y_s^m)^* (Y_s^n - \pi(Y_s^n)) ds + 2m \int_t^T (Y_s^n - Y_s^m)^* (Y_s^m - \pi(Y_s^m)) ds \\ \leq 2(n+m) \int_t^T (Y_s^n - \pi(Y_s^n))^* (Y_s^m - \pi(Y_s^m)) ds. \end{aligned} \quad (4.2.34)$$

Hence, from the Lipschitz continuous assumption on  $f$  and  $h$ , and taking expectation yields to

$$\begin{aligned} \mathbb{E}[|Y_t^n - Y_t^m|^2] + \mathbb{E}\left[\int_t^T \|Z_s^n - Z_s^m\|^2 ds\right] &\leq 2\mathbb{E}\left[\int_t^T C(|Y_s^n - Y_s^m|^2 + |Y_s^n - Y_s^m| \|Z_s^n - Z_s^m\|) ds\right] \\ &+ \mathbb{E}\left[\int_t^T C(|Y_s^n - Y_s^m|^2 + \alpha \|Z_s^n - Z_s^m\|) ds\right] \\ &+ 2(n+m)\mathbb{E}\left[\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m)) ds\right]. \end{aligned} \quad (4.2.35)$$

For the last term, we need to claim the following lemma whose proof is postponed in the Appendix.

**Lemma 4.2.4.** *There exists a constant  $C > 0$  such that, for each  $n \geq 0$ ,*

$$\mathbb{E}\left[\left(n \int_0^T d(Y_s^n, D) ds\right)^2\right] \leq C \quad (4.2.36)$$

Now we can deduce from the Holder inequality and Lemma 4.2.4 that

$$\begin{aligned} n\mathbb{E}\left[\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m)) ds\right] &\leq n\mathbb{E}\left[\int_t^T d(Y_s^n, D) d(Y_s^m, D) ds\right] \\ &\leq n\mathbb{E}\left[\sup_{0 \leq s \leq T} d(Y_s^m, D) \int_t^T d(Y_s^n, D) ds\right] \\ &\leq \left(\mathbb{E}\left[\left(n \int_0^T d(Y_s^n, D) ds\right)^2\right]\right)^{1/2} \left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2} \\ &\leq C \left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2}. \end{aligned} \quad (4.2.37)$$

Reporting (4.2.37) in the previous inequality, we have

$$\begin{aligned} \mathbb{E}[|Y_t^n - Y_t^m|^2] + (1 - \alpha - C\gamma)\mathbb{E}\left[\int_t^T \|Z_s^n - Z_s^m\|^2 ds\right] &\leq C\left(1 + \frac{1}{\gamma}\right)\mathbb{E}\left[\int_t^T |Y_s^n - Y_s^m|^2 ds\right] \\ &+ C\left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2}. \end{aligned}$$

Choosing  $1 - \alpha - C\gamma > 0$ , by Gronwall's lemma, we obtain

$$\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^n - Y_t^m|^2] \leq C\left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2}. \quad (4.2.38)$$

We deduce similarly

$$\mathbb{E}\left[\int_0^T \|Z_s^n - Z_s^m\|^2 ds\right] \leq C\left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2}. \quad (4.2.39)$$

Next, we apply again Itô's formula to  $|Y_t^n - Y_t^m|^2$ , and we have by the Burkholder-Davis-Gundy inequality and the previous calculations

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2\right] &\leq C\mathbb{E}\left[\int_0^T |Y_s^n - Y_s^m| |f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)| ds\right] \\ &+ C\mathbb{E}\left(\int_0^T |Y_s^n - Y_s^m|^2 \|h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m)\|^2 ds\right)^{1/2} + C\mathbb{E}\left(\int_0^T |Y_s^n - Y_s^m|^2 \|Z_s^n - Z_s^m\|^2 ds\right)^{1/2} \\ &+ \mathbb{E}\left[\int_t^T C(|Y_s^n - Y_s^m|^2 + \alpha \|Z_s^n - Z_s^m\|^2) ds\right] + 2(n+m)\mathbb{E}\left[\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m)) ds\right]. \end{aligned}$$

Then, it follows by the Lipschitz Assumption 4.2.2 on  $f$  and  $h$  and (4.2.37) that for any  $n, m \geq 0$

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2\right] &\leq C\left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2} + C\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 \int_0^T \|Z_s^n - Z_s^m\|^2 ds\right)^{1/2} \\ &\leq C\left(\mathbb{E}\left[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)\right]\right)^{1/2} + C\varepsilon\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2\right) + C\varepsilon^{-1}\mathbb{E}\left(\int_0^T \|Z_s^n - Z_s^m\|^2 ds\right). \end{aligned}$$

Choosing  $1 - C\varepsilon > 0$  and from the inequality (4.2.39) we conclude that

$$\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2] \leq C \left( \mathbb{E}[\sup_{0 \leq s \leq T} d^2(Y_s^m, D)] \right)^{1/2} \leq C \left( \mathbb{E}[\sup_{0 \leq s \leq T} d^4(Y_s^m, D)] \right)^{1/2} \rightarrow 0,$$

as  $n, m \rightarrow \infty$ , where Lemma 4.2.2 has been used.  $\square$

Finally, we conclude that  $(Y^n, Z^n)$  is a Cauchy sequence in  $\mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  and therefore there exists a unique pair  $(Y_t, Z_t)$  of  $\mathcal{F}_t$ -measurable processes which valued in  $\mathbb{R}^k \times \mathbb{R}^{k \times d}$ , satisfying

$$\mathbb{E}(\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.2.40)$$

Consequently, since for any  $n \geq 0$  and  $0 \leq t \leq T$ ,

$$\begin{aligned} K_t^n - K_t^m &= Y_0^n - Y_0^m - Y_t^n + Y_t^m - \int_0^t (f_s(Y_s^n, Z_s^n) - f_s(Y_s^m, Z_s^m)) ds \\ &\quad - \int_0^t (h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m)) d\overleftarrow{W}_s + \int_0^t (Z_s^n - Z_s^m) dB_s. \end{aligned} \quad (4.2.41)$$

we obtain from (4.2.40) and Burkholder-Davis-Gundy inequality,

$$\mathbb{E}(\sup_{0 \leq t \leq T} |K_t^n - K_t^m|^2) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \quad (4.2.42)$$

Hence, there exists a  $\mathcal{F}_t$ -adapted continuous process  $(K_t)_{0 \leq t \leq T}$  (with  $K_0 = 0$ ) such that

$$\mathbb{E}(\sup_{0 \leq t \leq T} |K_t - K_t^n|^2) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Passing to the limit in (4.2.12), the processes  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$  satisfy

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds + \int_t^T h_s(Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T.$$

Since we have from Lemma 4.2.2 that  $Y_t$  is in  $\bar{D}$ , it remains to check the minimality property for  $(K_t)$ , namely i.e., for any continuous progressively measurable process  $(z_t)$  valued in  $\bar{D}$ ,

$$\int_0^T (Y_t - z_t)^* dK_t \leq 0.$$

We note that (4.2.1) gives us

$$\int_0^T (Y_t^n - z_t)^* dK_t^n = -n \int_0^T (Y_t^n - z_t)^* (Y_t^n - \pi(Y_t^n)) dt \leq 0.$$

Therefore, we will show that we can extract a subsequence such that  $\int_0^T (Y_t^n - z_t)^* dK_t^n$  converge a.s. to  $\int_0^T (Y_t - z_t)^* dK_t$ . Following the proof of Lemma 4.4.3 in Appendix, we have

$$\begin{aligned} 2\gamma \|K^n\|_{VT} &\leq |\xi - a|^2 + 2 \int_0^T (Y_s^n - a) f_s(Y_s^n, Z_s^n) ds + 2 \int_0^T (Y_s^n - a) h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad + \int_0^T \|h_s(Y_s^n, Z_s^n)\|^2 ds - 2 \int_0^T (Y_s^n - a) Z_s^n dB_s. \end{aligned} \quad (4.2.43)$$

Notice that the right hand side tends in probability as  $n$  goes to infinity to

$$|\xi - a|^2 + 2 \int_0^T (Y_s - a) f_s(Y_s, Z_s) ds + 2 \int_0^T (Y_s - a) h_s(Y_s, Z_s) d\overleftarrow{W}_s + \int_0^T \|h_s(Y_s, Z_s)\|^2 ds - 2 \int_0^T (Y_s - a) Z_s dB_s.$$

Thus, there exists a subsequence  $(\phi(n))_{n \geq 0}$  such that the convergence is almost surely and  $\|K^{\phi(n)}\|_{VT}$  is bounded. Moreover, due to the convergence in  $\mathbb{L}^2$  of  $\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2$  to 0, we can extract a subsequence from  $(\phi(n))_{n \geq 0}$  such that  $Y^{\phi(\psi(n))}$  converges uniformly to  $Y$ . Hence, we apply Lemma 5.8 in [53] and we obtain

$$\int_0^T (Y_t^{\phi(\psi(n))} - z_t)^* dK_t^{\phi(\psi(n))} \longrightarrow \int_0^T (Y_t - z_t)^* dK_t \quad a.s. \text{ as } n \rightarrow \infty$$

which is the required result.  $\square$

### 4.3 Weak solution of semilinear SPDE in a convex domain

The aim of this section is to give a Feynman-Kac's formula for the weak solution of a semilinear reflected SPDEs (4.1.3) in a given convex domain  $D$  via Markovian class of RBDSDEs studied in the last section. As explained in the introduction, the solution of such SPDE is expressed as a pair  $(u, \nu)$  where  $u$  is a predictable continuous process which takes values in a Sobolev space and  $\nu$  is a random vector regular signed measure. The bounded variation processes  $K$  component of the solution of the reflected BDSDE controls the set when  $u$  reaches the boundary of  $D$ . In fact, this bounded variation process determines the measure  $\nu$  from a particular relation by using the inverse of the flow associated to the diffusion operator.

#### 4.3.1 Notations and Hypothesis

Let us first introduce some notations:

- $C_{l,b}^n(\mathbb{R}^p, \mathbb{R}^q)$  the set of  $C^n$ -functions which grow at most linearly at infinity and whose partial derivatives of order less than or equal to  $n$  are bounded.
- $\mathbf{L}_\rho^2(\mathbb{R}^d)$  will be a Hilbert weighted  $L^2$ -space of our framework. We employ the following notation for its scalar product and its norm,

$$(u, v)_\rho = \int_{\mathbb{R}^d} u(x)v(x)\rho(x)dx, \quad \|u\|_2 = \left( \int_{\mathbb{R}^d} u^2(x)\rho(x)dx \right)^{\frac{1}{2}}.$$

**Assumption 4.3.1.** *We assume that  $\rho$  is the weight function that satisfy the following conditions:*

- $\rho$  is a continuous positive function.
- $\rho$  is integrable and  $\frac{1}{\rho}$  is locally integrable.

In general, we shall use for the usual  $L^2$ -scalar product

$$(u, v) = \int_{\mathbb{R}^d} u(x)v(x) dx,$$

where  $u, v$  are measurable functions defined in  $\mathbb{R}^d$  and  $uv \in \mathbf{L}^1(\mathbb{R}^d)$ .

Our evolution problem will be considered over a fixed time interval  $[0, T]$  and the norm for an element of  $\mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$  will be denoted by

$$\|u\|_{2,2} = \left( \int_0^T \int_{\mathbb{R}^d} |u(t, x)|^2 \rho(x) dx dt \right)^{\frac{1}{2}}.$$

We assume the following hypotheses :

**Assumption 4.3.2.**  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^k$  is such that belongs to  $\mathbf{L}_\rho^2(\mathbb{R}^d)$  and  $\Phi(x) \in \bar{D}$  a.e.  $\forall x \in \mathbb{R}^d$ ;



**Assumption 4.3.3.** (i)  $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^n$  and  $h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times l}$  are measurable in  $(t, x, y, z)$  and satisfy  $f^0, h^0 \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$  where  $f_t^0(x) := f(t, x, 0, 0)$ ,  $h_t^0 := h(t, \cdot, x, 0, 0)$ .

(ii) There exist constants  $c > 0$  and  $0 < \alpha < 1$  such that for any  $(\omega, t) \in \Omega \times [0, T]$ ;  $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |f_t(x, y_1, z_1) - f_t(x, y_2, z_2)|^2 &\leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ \|h_t(x, y_1, z_1) - h_t(x, y_2, z_2)\|^2 &\leq c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2. \end{aligned}$$

**Assumption 4.3.4.** The coefficients  $b$  and  $\sigma$  of the second order differential operator  $\mathcal{L}$  (4.1.2) satisfy  $b \in C_{l,b}^2(\mathbb{R}^d; \mathbb{R}^d)$ ,  $\sigma \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^{d \times d})$ .

**Assumption 4.3.5.** (i)  $0 \in D$  and  $\Phi \in \mathbf{L}_\rho^4(\mathbb{R}^d)$ .

(ii) There exists  $0 \leq \beta < 1$  such that for all  $(t, x, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$h_t h_t^*(x, y, z) \leq c(\text{Id}_{\mathbb{R}^k} + yy^*) + \beta zz^*.$$

(iii)  $f$  and  $h$  are uniformly bounded in  $(x, y, z)$ .

### 4.3.2 Weak formulation for a solution of Stochastic PDEs

The space of test functions which we employ in the definition of weak solutions of the evolution equations (4.1.1) is  $\mathcal{D}_T = \mathcal{C}^\infty([0, T]) \otimes \mathcal{C}_c^\infty(\mathbb{R}^d)$ , where  $\mathcal{C}^\infty([0, T])$  denotes the space of real functions which can be extended as infinite differentiable functions in the neighborhood of  $[0, T]$  and  $\mathcal{C}_c^\infty(\mathbb{R}^d)$  is the space of infinite differentiable functions with compact support in  $\mathbb{R}^d$ . We denote by  $\mathcal{H}_T$  the space of  $\mathcal{F}_{t,T}^W$ -progressively measurable processes  $(u_t)$  with valued in the weighted Dirichlet space  $H_\rho^1(\mathbb{R}^d)$  where

$$H_\rho^1(\mathbb{R}^d) := \{v \in \mathbf{L}_\rho^2(\mathbb{R}^d) \mid \nabla v \sigma \in \mathbf{L}_\rho^2(\mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T}^2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|u_s\|_2^2 + \int_{\mathbb{R}^d} \int_0^T |\nabla u_s(x) \sigma(x)|^2 ds \rho(x) dx \right],$$

where we denote the gradient by  $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$ .

**Definition 4.3.1.** We say that  $u \in \mathcal{H}_T$  is a Sobolev solution of SPDE (4.1.1) if the following relation holds, for each  $\varphi \in \mathcal{D}_T$ ,

$$\begin{aligned} &\int_t^T (u(s, x), \partial_s \varphi(s, x)) ds + (u(t, x), \varphi(t, x)) - (\Phi(x), \varphi(T, x)) - \int_t^T (u(s, x), \mathcal{L}^* \varphi(s, x)) ds \\ &= \int_t^T (f_s(x, u(s, x), \nabla u(s, x) \sigma(x)), \varphi(s, x)) ds + \int_t^T (h_s(x, u(s, x), \nabla u(s, x) \sigma(x)), \varphi(s, x)) d\overleftarrow{W}_s, \end{aligned} \tag{4.3.1}$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ . We denote by  $u := \mathcal{U}(\Phi, f, h)$  solution of SPDEs with data  $(\Phi, f, h)$ .

The existence and uniqueness of weak solution for SPDEs (4.3.1) is ensured by Theorem 3.1 in Bally and Matoussi [10] or Denis and Stoica [43].

### 4.3.3 Stochastic flow of diffeomorphisms and random test functions

We are concerned in this part with solving our problem by developing a stochastic flow method which was first introduced in Kunita [76], [78] and Bally, Matoussi [10]. We recall that  $\{X_{t,s}(x), t \leq s \leq T\}$  is the diffusion process starting from  $x$  at time  $t$  and is the strong solution of the equation:

$$X_{t,s}(x) = x + \int_t^s b(X_{t,r}(x))dr + \int_t^s \sigma(X_{t,r}(x))dB_r. \quad (4.3.2)$$

The existence and uniqueness of this solution was proved in Kunita [76]. Moreover, we have the following properties:

**Proposition 4.3.1.** *For each  $t > 0$ , there exists a version of  $\{X_{t,s}(x); x \in \mathbb{R}^d, s \geq t\}$  such that  $X_{t,s}(\cdot)$  is a  $C^2(\mathbb{R}^d)$ -valued continuous process which satisfy the flot property:  $X_{t,r}(x) = X_{s,r} \circ X_{t,s}(x)$ ,  $0 \leq t < s < r$ . Furthermore, for all  $p \geq 2$ , there exists  $M_p$  such that for all  $0 \leq t < s$ ,  $x, x' \in \mathbb{R}^d$ ,  $h, h' \in \mathbb{R} \setminus \{0\}$ ,*

$$\begin{aligned} \mathbb{E} \left( \sup_{t \leq r \leq s} |X_{t,r}(x) - x|^p \right) &\leq M_p(s-t)(1 + |x|^p), \\ \mathbb{E} \left( \sup_{t \leq r \leq s} |X_{t,r}(x) - X_{t,r}(x') - (x - x')|^p \right) &\leq M_p(s-t)(|x - x'|^p), \\ \mathbb{E} \left( \sup_{t \leq r \leq s} |\Delta_h^i [X_{t,r}(x) - x]|^p \right) &\leq M_p(s-t), \\ \mathbb{E} \left( \sup_{t \leq r \leq s} |\Delta_h^i X_{t,r}(x) - \Delta_{h'}^i X_{t,r}(x')|^p \right) &\leq M_p(s-t)(|x - x'|^p + |h - h'|^p), \end{aligned}$$

where  $\Delta_h^i g(x) = \frac{1}{h}(g(x + he_i) - g(x))$ , and  $(e_1, \dots, e_d)$  is an orthonormal basis of  $\mathbb{R}^d$ .

Under regular conditions (Assumption 4.3.4) on the diffusion, it is known that the stochastic flow solution of a continuous SDE satisfies the homeomorphic property (see Bismut [16], Kunita [76], [78]). We have the following result where the proof can be found in [76].

**Proposition 4.3.2.** *Let Assumption 4.3.4 holds. Then  $\{X_{t,s}(x); x \in \mathbb{R}^d\}$  is a  $C^2$ -diffeomorphism a.s. stochastic flow. Moreover the inverse of the flow satisfies the following backward SDE*

$$X_{t,s}^{-1}(y) = y - \int_t^s \widehat{b}(X_{r,s}^{-1}(y))dr - \int_t^s \sigma(X_{r,s}^{-1}(y))d\overleftarrow{W}_r. \quad (4.3.3)$$

for any  $t < s$ , where

$$\widehat{b}(x) = b(x) - \sum_{i,j} \frac{\partial \sigma^j(x)}{\partial x_i} \sigma^{ij}(x). \quad (4.3.4)$$

We denote by  $J(X_{t,s}^{-1}(x))$  the determinant of the Jacobian matrix of  $X_{t,s}^{-1}(x)$ , which is positive and  $J(X_{t,t}^{-1}(x)) = 1$ . For  $\varphi \in C_c^\infty(\mathbb{R}^d)$ , we define a process  $\varphi_t : \Omega \times [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$  by

$$\varphi_t(s, x) := \varphi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x)). \quad (4.3.5)$$

We know that for  $v \in \mathbf{L}^2(\mathbb{R}^d)$ , the composition of  $v$  with the stochastic flow is

$$(v \circ X_{t,s}(\cdot), \varphi) := (v, \varphi_t(s, \cdot)).$$

In fact, by a change of variable, we have (see Kunita [80], [79], Bally and Matoussi [10])

$$(v \circ X_{t,s}(\cdot), \varphi) = \int_{\mathbb{R}^d} v(X_{t,s}(x))\varphi(x)dx = \int_{\mathbb{R}^d} v(y)\varphi(X_{t,s}^{-1}(y))J(X_{t,s}^{-1}(y))dy = (v, \varphi_t(s, \cdot)).$$

Since  $(\varphi_t(s, x))_{t \leq s}$  is a process, we may not use it directly as a test function because  $\int_t^T (u(s, \cdot), \partial_s \varphi_t(s, \cdot))$  has no sense. However  $\varphi_t(s, x)$  is a semimartingale and we have the following decomposition of  $\varphi_t(s, x)$

**Lemma 4.3.1.** For every function  $\varphi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\varphi_t(s, x) = \varphi(x) + \int_t^s \mathcal{L}^* \varphi_t(r, x) dr - \sum_{j=1}^d \int_t^s \left( \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma^{ij}(x) \varphi_t(r, x)) \right) dB_r^j, \quad (4.3.6)$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ .

We also need equivalence of norms result which plays an important role in the proof of the existence of the solution for SPDE as a connection between the functional norms and random norms. For continuous SDEs, this result was first proved by Barles and Lesigne [13] by using an analytic method and Bally, Matoussi [10] by a probabilistic method.

**Proposition 4.3.3.** There exists two constants  $c > 0$  and  $C > 0$  such that for every  $t \leq s \leq T$  and  $\varphi \in L^1(\mathbb{R}^d, dx)$ ,

$$c \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx \leq \int_{\mathbb{R}^d} \mathbb{E}(|\varphi(X_{t,s}(x))|) \rho(x) dx \leq C \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx. \quad (4.3.7)$$

Moreover, for every  $\Psi \in L^1([0, T] \times \mathbb{R}^d, dt \otimes dx)$ ,

$$c \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)| ds \rho(x) dx \leq \int_{\mathbb{R}^d} \int_t^T \mathbb{E}(|\Psi(s, X_{t,s}(x))|) ds \rho(x) dx \leq C \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)| ds \rho(x) dx. \quad (4.3.8)$$

#### 4.3.4 Existence and uniqueness of solutions for the reflected SPDE

In order to provide a probabilistic representation to the solution of the RSPDEs (4.1.3), we introduce the following Markovian RBDSDE:

$$\left\{ \begin{array}{l} (i) Y_s^{t,x} = \Phi(X_{t,T}(x)) + \int_s^T f_r(X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T h_r(X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r + K_T^{t,x} - K_s^{t,x} \\ \quad - \int_s^T Z_r^{t,x} dB_r, \quad \mathbb{P}\text{-a.s.}, \quad \forall s \in [t, T] \\ (ii) Y_s^{t,x} \in \bar{D} \quad \mathbb{P}\text{-a.s.} \\ (iii) \int_0^T (Y_s^{t,x} - v_s(X_{t,S}(x)))^* dK_s^{t,x} \leq 0., \quad \mathbb{P}\text{-a.s.}, \\ \text{for any continuous } \mathcal{F}_t\text{-random function } v : [0, T] \times \Omega \times \mathbb{R}^d \longrightarrow \bar{D}. \end{array} \right. \quad (4.3.9)$$

Moreover, using Assumptions 4.3.2 and 4.3.3 and the equivalence of norm results (4.3.7) and (4.3.8), we get

$$\begin{aligned} \Phi(X_{t,T}(x)) &\in \mathbf{L}^2(\mathcal{F}_T) \text{ and } \Phi(X_{t,T}(x)) \in \bar{D}, \\ f_s^0(X_{t,s}(x)) &\in \mathcal{H}_k^2(t, T) \text{ and } h_s^0(X_{t,s}(x)) \in \mathcal{H}_{k \times d}^2(t, T). \end{aligned}$$

Therefore under Assumption 4.3.2-4.3.5 and according to Theorem 4.2.1, there exists a unique triplet  $(Y^{t,x}, Z^{t,x}, K^{t,x})$  solution of the RBDSDE (4.3.9) associated to  $(\Phi, f, h)$ .

We now consider the following definition of weak solutions for the reflected SPDE (4.1.3):

**Definition 4.3.2.** We say that  $(u, \nu) := (u^i, \nu^i)_{1 \leq i \leq k}$  is the weak solution of the reflected SPDE (4.1.3) associated to  $(\Phi, f, h)$ , if for each  $1 \leq i \leq k$

(i)  $\|u\|_{\mathcal{H}_T} < \infty$ ,  $u_t(x) \in \bar{D}$ ,  $dx \otimes dt \otimes d\mathbb{P}$  a.e., and  $u(T, x) = \Phi(x)$ ,

(ii)  $\nu^i$  is a signed Radon regular measure in the following sense, i.e. for every measurable bounded and positive functions  $\varphi$  and  $\psi$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_t^T \varphi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, X_{t,s}(x)) dK_s^{t,x,i} dx, \quad a.s.. \end{aligned} \quad (4.3.10)$$

where  $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$  is the solution of RBDSDE (4.3.9) and such that  $\int_0^T \int_{\mathbb{R}^d} \rho(x) |\nu|(dt, dx) < \infty$ ,  
 (iii) for every  $\varphi \in \mathcal{D}_T$

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u^i(s, x) \partial_s \varphi(s, x) dx ds + \int_{\mathbb{R}^d} (u^i(t, x) \varphi(t, x) - \Phi^i(x) \varphi(T, x)) dx - \int_t^T \int_{\mathbb{R}^d} u^i(s, x) \mathcal{L}^* \varphi(s, x) dx ds \\ &= \int_t^T \int_{\mathbb{R}^d} f_s(x, u(s, x), \nabla u(s, x) \sigma(x)) \varphi(s, x) dx ds + \int_t^T \int_{\mathbb{R}^d} h_s(x, u(s, x), \nabla u(s, x) \sigma(x)) \varphi(s, x) dx d\overleftarrow{W}_s \\ &+ \int_t^T \int_{\mathbb{R}^d} \varphi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu^i(ds, dx). \end{aligned} \quad (4.3.11)$$

For the sake of simplicity we will omit in the sequel the subscript  $i$ .

We give now the following result which allows us to link by a natural way the solution of SPDE with the associated BDSDE. Roughly speaking, if we choose in the variational formulation (4.3.11) the random functions  $\varphi_t(\cdot, \cdot)$  defined by (4.3.5), as a test functions, then we obtain the associated BDSDE. In fact, this result plays the same role as Itô's formula used in [109] to relate the solution of some semilinear SPDEs with the associated BDSDEs:

**Proposition 4.3.4.** *Let Assumptions 4.3.2-4.3.5 hold and  $u \in \mathcal{H}_T$  be a weak solution of the reflected SPDE (4.1.3) associated to  $(\Phi, f, h)$ , then for  $s \in [t, T]$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r, x) d\varphi_t(r, x) dx + (u(s, \cdot), \varphi_t(s, \cdot)) - (\Phi(\cdot), \varphi_t(T, \cdot)) - \int_{\mathbb{R}^d} \int_s^T u(r, x) \mathcal{L}^* \varphi_t(r, x) dr dx \\ &= \int_{\mathbb{R}^d} \int_s^T f_r(x, u(r, x), \nabla u(r, x) \sigma(x)) \varphi_t(r, x) dr dx + \int_{\mathbb{R}^d} \int_s^T h_r(x, u(r, x), \nabla u(r, x) \sigma(x)) \varphi_t(r, x) d\overleftarrow{W}_r dx \\ &+ \int_{\mathbb{R}^d} \int_s^T \varphi_t(r, x) 1_{\{u \in \partial D\}}(r, x) \nu(dr, dx). \quad a.s. \end{aligned} \quad (4.3.12)$$

where  $\int_{\mathbb{R}^d} \int_s^T u(r, x) d\varphi_t(r, x) dx$  is well defined thanks to the semimartingale decomposition result (Lemma 4.3.1).

The main result of this section is the following:

**Theorem 4.3.1.** *Let Assumptions 4.3.2-4.3.5 hold and  $\rho(x) = (1 + |x|)^{-p}$  with  $p > d + 1$ . Then there exists a weak solution  $(u, \nu)$  of the reflected SPDE (4.1.3) associated to  $(\Phi, f, h)$  such that,  $u(t, x) := Y_t^{t,x}$ ,  $dt \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.$ , and*

$$Y_s^{t,x} = u(s, X_{t,s}(x)), \quad Z_s^{t,x} = (\nabla u \sigma)(s, X_{t,s}(x)), \quad ds \otimes d\mathbb{P} \otimes \rho(x) dx - a.e.. \quad (4.3.13)$$

Moreover, the reflected measure  $\nu$  is a signed regular measure in the sense of the definition (ii) and satisfying the probabilistic interpretation (4.3.10).

If  $(\bar{u}, \bar{\nu})$  is another solution of the reflected SPDE (4.1.3) such that  $\bar{\nu}$  satisfies (4.3.10) with some  $\bar{K}$  instead of  $K$ , where  $\bar{K}$  is a continuous process, then  $\bar{u} = u$  and  $\bar{\nu} = \nu$ .

In other words, there is a unique Radon regular measure with support  $\{u \in \partial D\}$  which satisfies (4.3.10).

**Remark 4.3.1.** The expression (4.3.10) gives us the probabilistic interpretation (Feymann-Kac's formula) for the measure  $\nu$  via the nondecreasing process  $K^{t,x}$  of the RBDSDE. This formula was first introduced in Bally et al. [9] (see also [99]) in the context of obstacle problem for PDEs. Here we adapt this notion to the case of SPDEs in a convex domain.

**Proof.** **a) Uniqueness** : Set  $(\bar{u}, \bar{\nu})$  to be another weak solution of the reflected SPDE (4.1.3) associated to  $(\Phi, f, h)$ ; with  $\bar{\nu}$  verifies (4.3.10) for a continuous process  $\bar{K}$ . We fix  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^k$ , a smooth function in  $C_c^2(\mathbb{R}^d)$  with compact support and denote  $\varphi_t(s, x) = \varphi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))$ . From Proposition 4.3.4, one may use  $\varphi_t(s, x)$  as a test function in the SPDE  $(\Phi, f, h)$  with  $\partial_s \varphi(s, x) ds$  replaced by a stochastic integral with respect to the semimartingale  $\varphi_t(s, x)$ . Then we get, for  $t \leq s \leq T$

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T \bar{u}(r, x) d\varphi_t(r, x) dx + \int_{\mathbb{R}^d} \bar{u}(s, x) \varphi_t(s, x) dx - \int_{\mathbb{R}^d} \Phi(x) \varphi_t(T, x) dx - \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, x) \mathcal{L}^* \varphi_t(r, x) dr dx \\ &= \int_s^T \int_{\mathbb{R}^d} f_r(x, \bar{u}(r, x), (\nabla \bar{u} \sigma)(r, x)) \varphi_t(r, x) dr dx + \int_s^T \int_{\mathbb{R}^d} h_r(x, \bar{u}(r, x), (\nabla \bar{u} \sigma)(r, x)) \varphi_t(r, x) dx d\overleftarrow{W}_r \\ &+ \int_s^T \int_{\mathbb{R}^d} \varphi_t(r, x) 1_{\{\bar{u} \in \partial D\}}(r, x) \bar{\nu}(dr, dx). \end{aligned} \quad (4.3.14)$$

By (4.3.6) in Lemma 4.3.1, we have

$$\int_{\mathbb{R}^d} \int_s^T \bar{u}(r, x) d\varphi_t(r, x) dx dr = \int_s^T \left( \int_{\mathbb{R}^d} (\nabla \bar{u} \sigma)(r, x) \varphi_t(r, x) dx \right) dB_r + \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, x) \mathcal{L}^* \varphi_t(r, x) dr dx.$$

Substitute this equality in (4.3.14), we get

$$\begin{aligned} & \int_{\mathbb{R}^d} \bar{u}(s, x) \varphi_t(s, x) dx = \int_{\mathbb{R}^d} \Phi(x) \varphi_t(T, x) dx - \int_s^T \left( \int_{\mathbb{R}^d} (\nabla \bar{u} \sigma)(r, x) \varphi_t(r, x) dx \right) dB_r \\ &+ \int_s^T \int_{\mathbb{R}^d} f_r(x, \bar{u}(r, x), (\nabla \bar{u} \sigma)(r, x)) \varphi_t(r, x) dr dx + \int_s^T \int_{\mathbb{R}^d} h_r(x, \bar{u}(r, x), (\nabla \bar{u} \sigma)(r, x)) \varphi_t(r, x) dx d\overleftarrow{W}_r \\ &+ \int_s^T \int_{\mathbb{R}^d} \varphi_t(r, x) 1_{\{\bar{u} \in \partial D\}}(r, x) \bar{\nu}(dr, dx). \end{aligned}$$

Then by changing of variable  $y = X_{t,r}^{-1}(x)$  and applying (4.3.10) for  $\bar{\nu}$ , we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} \bar{u}(s, X_{t,s}(y)) \varphi(y) dy = \int_{\mathbb{R}^d} \Phi(X_{t,T}(y)) \varphi(y) dy \\ &+ \int_{\mathbb{R}^d} \int_s^T \varphi(y) f_r(X_{t,r}(y), \bar{u}(r, X_{t,r}(y)), (\nabla \bar{u} \sigma)(r, X_{t,r}(y))) dr dy \\ &+ \int_{\mathbb{R}^d} \int_s^T \varphi(y) h_r(X_{t,r}(y), \bar{u}(r, X_{t,r}(y)), (\nabla \bar{u} \sigma)(r, X_{t,r}(y))) dy d\overleftarrow{W}_r \\ &+ \int_s^T \int_{\mathbb{R}^d} \varphi(y) 1_{\{\bar{u} \in \partial D\}}(r, X_{t,s}(y)) d\overleftarrow{K}_r^{t,y} dy - \int_s^T \int_{\mathbb{R}^d} \varphi(y) (\nabla \bar{u} \sigma)(r, X_{t,r}(y)) dy dB_r \end{aligned}$$

Since  $\varphi$  is arbitrary, we can prove that for  $\rho(y) dy$  almost every  $y$ ,  $(\bar{u}(s, X_{t,s}(y)), (\nabla \bar{u} \sigma)(s, X_{t,s}(y)), \widehat{K}_s^{t,y})$  solves the RBDSDE  $(\Phi(X_{t,T}(y)), f, h)$ . Here  $\widehat{K}_s^{t,y} = \int_t^s 1_{\{\bar{u} \in \partial D\}}(r, X_{t,r}(y)) d\overleftarrow{K}_r^{t,y}$ . Then by the uniqueness of the solution of the RBDSDE, we know  $\bar{u}(s, X_{t,s}(y)) = Y_s^{t,y} = u(s, X_{t,s}(y))$ ,  $(\nabla \bar{u} \sigma)(s, X_{t,s}(y)) = Z_s^{t,y} = (\nabla u \sigma)(s, X_{t,s}(y))$ , and  $\widehat{K}_s^{t,y} = K_s^{t,y}$ . Taking  $s = t$  we deduce that  $\bar{u}(t, y) = u(t, y)$ ,  $\rho(y) dy$ -a.s. and by the probabilistic interpretation (4.3.10), we obtain

$$\int_s^T \int_{\mathbb{R}^d} \varphi_t(r, x) 1_{\{\bar{u} \in \partial D\}}(r, x) \bar{\nu}(dr, dx) = \int_s^T \int_{\mathbb{R}^d} \varphi_t(r, x) 1_{\{u \in \partial D\}}(r, x) \nu(dr, dx).$$

So  $1_{\{\bar{u} \in \partial D\}}(r, x) \bar{\nu}(dr, dx) = 1_{\{u \in \partial D\}}(r, x) \nu(dr, dx)$ .

**b) Existence:** The existence of a solution will be proved in two steps. For the first step, we suppose that  $h$  doesn't depend on  $y, z$ , then we are able to apply the classical penalization method. In the second step, we study the case when  $h$  depends on  $y, z$  with the result obtained in the first step.

*Step 1 :* We will use the penalization method. For  $n \in \mathbb{N}$ , we consider for all  $s \in [t, T]$ ,

$$\begin{aligned} Y_s^{n,t,x} &= \Phi(X_{t,T}(x)) + \int_s^T f_r(X_{t,r}(x), Y_r^{n,t,x}, Z_r^{n,t,x}) dr + \int_s^T h_r(X_{t,r}(x)) d\overleftarrow{W}_r \\ &\quad - n \int_s^T (Y_r^{n,t,x} - \pi(Y_r^{n,t,x})) dr - \int_s^T Z_r^{n,t,x} dB_r. \end{aligned}$$

From Theorem 3.1 in Bally and Matoussi [10], we know that  $u_n(t, x) := Y_t^{n,t,x}$ , is solution of the SPDE  $(\Phi, f_n, h)$  (4.1.1), with  $f_n(t, x, y) = f(t, x, y, z) - n(y - \pi(y))$ , i.e. for every  $\varphi \in \mathcal{D}_T$

$$\begin{aligned} \int_t^T (u^n(s, \cdot), \partial_s \varphi(s, \cdot)) ds + (u^n(t, \cdot), \varphi(t, \cdot)) - (\Phi(\cdot), \varphi(T, \cdot)) - \int_t^T (u^n(s, \cdot), \mathcal{L}^* \varphi(s, \cdot)) ds \\ = \int_t^T (f_s(\cdot, u^n(s, \cdot), \sigma^* \nabla u_n(s, \cdot)), \varphi(s, \cdot)) ds + \int_t^T (h_s(\cdot), \varphi(s, \cdot)) d\overleftarrow{W}_s \\ - n \int_t^T ((u^n - \pi(u^n))(s, \cdot), \varphi(s, \cdot)) ds. \end{aligned} \quad (4.3.15)$$

Moreover from Theorem 3.1 in Bally and Matoussi [10], we also have

$$Y_s^{n,t,x} = u_n(s, X_{t,s}(x)), \quad Z_s^{n,t,x} = (\nabla u_n \sigma)(s, X_{t,s}(x)), \quad ds \otimes d\mathbb{P} \otimes \rho(x) dx - a.e. \quad (4.3.16)$$

Set  $K_s^{n,t,x} = -n \int_t^s (Y_r^{n,t,x} - \pi(Y_r^{n,t,x})) dr$ . Then by (4.3.16), we have that

$$K_s^{n,t,x} = n \int_t^s (u_n - \pi(u_n))(r, X_{t,r}(x)) dr.$$

Following the estimates and convergence results for  $(Y^{n,t,x}, Z^{n,t,x}, K^{n,t,x})$  in Section 2 and estimate (4.4.3), we get :

$$\sup_n \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_s^{n,t,x}|^2 + \int_t^T \|Z_s^{n,t,x}\|^2 ds + \|K^{n,t,x}\|_{VT} \right] \leq C(T, x),$$

where

$$C(T, x) := \mathbb{E} \left[ |\Phi(X_{t,T}(x))|^2 + \int_t^T (|f_s^0(X_{t,s}(x))|^2 + |h_s^0(X_{t,s}(x))|^2) ds \right],$$

and

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_s^{n,t,x} - Y_s^{m,t,x}|^2 \right] + \mathbb{E} \left[ \int_t^T \|Z_s^{n,t,x} - Z_s^{m,t,x}\|^2 ds \right] \\ + \mathbb{E} \left[ \sup_{t \leq s \leq T} |K_s^{n,t,x} - K_s^{m,t,x}|^2 \right] \longrightarrow 0, \quad \text{as } n, m \longrightarrow +\infty. \end{aligned}$$

Moreover, the equivalence of norms results (4.3.8) yield:

$$\begin{aligned} \int_{\mathbb{R}^d} \int_t^T \rho(x) (|u_n(s, x) - u_m(s, x)|^2 + |(\nabla u_n \sigma)(s, x) - (\nabla u_m \sigma)(s, x)|^2) ds dx \\ \leq \frac{1}{k_2} \int_{\mathbb{R}^d} \rho(x) \mathbb{E} \int_t^T (|Y_s^{n,t,x} - Y_s^{m,t,x}|^2 + \|Z_s^{n,t,x} - Z_s^{m,t,x}\|^2) ds dx \longrightarrow 0. \end{aligned}$$

Thus  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{H}_T$ , and the limit  $u = \lim_{n \rightarrow \infty} u_n$  belongs to  $\mathcal{H}_T$ . Denote  $\nu_n(dt, dx) = -n(u_n - \pi(u_n))(t, x)dt dx$  and  $\pi_n(dt, dx) = \rho(x)\nu_n(dt, dx)$ , then by (4.3.8)

$$\begin{aligned} |\pi_n|([0, T] \times \mathbb{R}^d) &= \int_{\mathbb{R}^d} \int_0^T \rho(x) |\nu_n|(dt, dx) = \int_{\mathbb{R}^d} \int_0^T \rho(x) n |u_n - \pi(u_n)|(t, x) dt dx \\ &\leq \frac{1}{k_2} \int_{\mathbb{R}^d} \rho(x) \mathbb{E} \|K^{n,0,x}\|_{VT} dx \leq C \int_{\mathbb{R}^d} \rho(x) dx < \infty. \end{aligned}$$

It follows that

$$\sup_n |\pi_n|([0, T] \times \mathbb{R}^d) < \infty. \quad (4.3.17)$$

Moreover by Lemma 4.4.4 (see Appendix 4.4.4), the sequence of measures  $(\pi_n)_{n \in \mathbb{N}}$  is tight. Therefore, there exists a subsequence such that  $(\pi_n)_{n \in \mathbb{N}}$  converges weakly to a measure  $\pi$ . Define  $\nu = \rho^{-1}\pi$ ;  $\nu$  is a measure such that  $\int_0^T \int_{\mathbb{R}^d} \rho(x) |\nu|(dt, dx) < \infty$ , and so we have for  $\varphi \in \mathcal{D}_T$  with compact support in  $x$ ,

$$\int_{\mathbb{R}^d} \int_t^T \varphi d\nu_n = \int_{\mathbb{R}^d} \int_t^T \frac{\phi}{\rho} d\pi_n \rightarrow \int_{\mathbb{R}^d} \int_t^T \frac{\phi}{\rho} d\pi = \int_{\mathbb{R}^d} \int_t^T \varphi d\nu.$$

Now passing to the limit in the SPDE  $(\Phi, f_n, h)$  (4.3.15), we get that that  $(u, \nu)$  satisfies the reflected SPDE associated to  $(\Phi, f, h)$ , i.e. for every  $\varphi \in \mathcal{D}_T$ , we have

$$\begin{aligned} &\int_t^T (u(s, \cdot), \partial_s \varphi(s, \cdot)) ds + (u(t, \cdot), \varphi(t, \cdot)) - (\Phi(\cdot), \varphi(T, \cdot)) - \int_t^T (u(s, \cdot), \mathcal{L}^* \varphi(s, \cdot)) ds \\ &= \int_t^T (f_s(\cdot, u(s, \cdot), \sigma^* \nabla u(s, \cdot)), \varphi(s, \cdot)) ds + \int_t^T (h_s(\cdot), \varphi(s, \cdot)) d\overleftarrow{W}_s + \int_t^T \int_{\mathbb{R}^d} \varphi(s, x) \nu(ds, dx). \end{aligned} \quad (4.3.18)$$

The last point is to prove that  $\nu$  satisfies the probabilistic interpretation (4.3.10). Since  $K^{n,t,x}$  converges to  $K^{t,x}$  uniformly in  $t$ , the measure  $dK^{n,t,x} \rightarrow dK^{t,x}$  weakly in probability. Fix two continuous functions  $\varphi, \psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^+$  which have compact support in  $x$  and a continuous function with compact support  $\theta : \mathbb{R}^d \rightarrow \mathbb{R}^+$ , from Bally et al [9] (The proof of Theorem 4), we have (see also Matoussi and Xu [99])

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_t^T \varphi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) \theta(x) \nu(ds, dx) \\ &= \lim_{n \rightarrow \infty} - \int_{\mathbb{R}^d} \int_t^T \phi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) \theta(x) n(u_n - \pi(u_n))(s, x) ds dx \\ &= \lim_{n \rightarrow \infty} - \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, X_{t,s}(x)) \theta(X_{t,s}(x)) n(u_n - \pi(u_n))(t, X_{t,s}(x)) dt dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, X_{t,s}(x)) \theta(X_{t,s}(x)) dK_s^{n,t,x} dx \\ &= \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, X_{t,s}(x)) \theta(X_{t,s}(x)) dK_s^{t,x} dx. \end{aligned}$$

We take  $\theta = \theta_R$  to be the regularization of the indicator function of the ball of radius  $R$  and pass to the limit with  $R \rightarrow \infty$ , it follows that

$$\int_{\mathbb{R}^d} \int_t^T \varphi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \psi(s, x) \nu(ds, dx) = \int_{\mathbb{R}^d} \int_t^T \varphi(s, x) \psi(s, X_{t,s}(x)) dK_s^{t,x} dx. \quad (4.3.19)$$

From Section 2, it follows that  $dK_s^{t,x} = 1_{\{u \in \partial D\}}(s, X_{t,s}(x)) dK_s^{t,x}$ . In (4.3.19), setting  $\psi = 1_{\{u \in \partial D\}}$  yields

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_t^T \varphi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) 1_{\{u \in \partial D\}}(s, x) \nu(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \varphi(s, X_{t,s}^{-1}(x)) J(X_{t,s}^{-1}(x)) \nu(ds, dx), \text{ a.s.} \end{aligned}$$

Note that the family of functions  $A(\omega) = \{(s, x) \rightarrow \phi(s, X_{t,s}^{-1}(x)) : \phi \in C_c^\infty\}$  is an algebra which separates the points (because  $x \rightarrow X_{t,s}^{-1}(x)$  is a bijection). Given a compact set  $G$ ,  $A(\omega)$  is dense in  $C([0, T] \times G)$ . It follows that  $J(X_{t,s}^{-1}(x))1_{\{u \in \partial D\}}(s, x)\nu(ds, dx) = J(X_{t,s}^{-1}(x))\nu(ds, dx)$  for almost every  $\omega$ . While  $J(X_{t,s}^{-1}(x)) > 0$  for almost every  $\omega$ , we get  $\nu(ds, dx) = 1_{\{u \in \partial D\}}(s, x)\nu(ds, dx)$ , and (4.3.10) follows. Then we get easily that  $Y_s^{t,x} = u(s, X_{t,s}(x))$  and  $Z_s^{t,x} = (\nabla u \sigma)(s, X_{t,s}(x))$ , in view of the convergence results for  $(Y_s^{n,t,x}, Z_s^{n,t,x})$  and the equivalence of norms. So  $u(s, X_{t,s}(x)) = Y_s^{t,x} \in \bar{D}$ . Specially for  $s = t$ , we have  $u(t, x) \in \bar{D}$ .

*Step 2 : The nonlinear case where  $h$  depends on  $y$  and  $z$ .*

Let define  $H(s, x) \triangleq h(x, Y_s^{s,x}, Z_s^{s,x})$ . By plugging into the facts that  $h^0 \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$  and  $h$  is Lipschitz with respect to  $(y, z)$ , then we have  $H_s(x) \in \mathbf{L}_\rho^2([0, T] \times \mathbb{R}^d)$ . Since  $H$  is independent of  $y, z$ , by applying the result of Step 1 yields that there exists  $(u, \nu)$  satisfying the PIDE with obstacle  $(\Phi, f, H)$ , i.e. for every  $\varphi \in \mathcal{D}_T$ , we have

$$\begin{aligned} & \int_t^T (u(s, \cdot), \partial_s \varphi(s, \cdot)) ds + (u(t, \cdot), \varphi(t, \cdot)) - (\Phi(\cdot), \varphi(T, \cdot)) - \int_t^T (u(s, \cdot), \mathcal{L}^* \varphi(s, \cdot)) ds \\ &= \int_t^T (f_s(\cdot, u(s, \cdot), \sigma^* \nabla u(s, \cdot)), \varphi(s, \cdot)) ds + \int_t^T (H_s(\cdot), \varphi(s, \cdot)) d\overleftarrow{W}_s \\ &+ \int_t^T \int_{\mathbb{R}^d} \phi(s, x) 1_{\{u \in \partial D\}}(s, x) \nu(ds, dx). \end{aligned} \quad (4.3.20)$$

Then by the uniqueness of the solution to the RBDSDE  $(\Phi(X_{t,T}(x)), f, h)$ , we get easily that  $Y_s^{t,x} = u(s, X_{t,s}(x))$ ,  $Z_s^{t,x} = (\nabla u \sigma)(s, X_{t,s}(x))$ , and  $\nu$  satisfies the probabilistic interpretation (4.3.10). So  $u(s, X_{t,s}(x)) = Y_s^{t,x} \in \bar{D}$ . Specially for  $s = t$ , we have  $u(t, x) \in \bar{D}$ , which is the desired result.  $\square$

## 4.4 Appendix

### 4.4.1 Some properties of convexity

Denote  $\varphi(x) \triangleq (d(x, D))^2 = |x - \pi(x)|^2$ , the square of the distance to the domain  $D$  and  $\pi$  the orthogonal projection on the closure  $\bar{D}$ . If  $D$  is a convex domain, then the function  $\varphi$  is convex. Moreover, if  $D$  is a regular domain,  $\varphi$  is two time differentiable on the complement of  $D$  and we obtain:

$$\forall x \notin D, \quad \nabla \varphi(x) = 2(x - \pi(x)).$$

From this expression of gradient  $\nabla \varphi$ , we remark that the hessienne matrix  $Hess\varphi(x)$  has the following form:

$$Hess\varphi(x) = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & (M) & & \\ 0 & & & \end{pmatrix}$$

where  $M$  is a positive semi definite matrix. We deduce also that:

$$\forall z \in \mathbb{R}^{k \times d} \quad \text{trace}[zz^* Hess\varphi(x)] \geq 0 \quad (4.4.1)$$

Since  $Hess$  is a positive matrix we have for every unit outward normal  $\nu(x)$

$$\begin{aligned} |z\nu^*(x)|^2 &\leq C \frac{1}{2} \text{trace} \left[ zz^* \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & (M) & & \\ 0 & & & \end{pmatrix} \right] \\ &\leq C \text{trace}[zz^* Hess\varphi(x)] \end{aligned} \quad (4.4.2)$$



**Lemma 4.4.1.** *If  $D_\varepsilon$  is a convex which satisfy (4.2.4), then  $\exists c > 0$  such that  $\forall \varepsilon < 1, \forall x \in \mathbb{R}^k$*

$$|\pi(x) - \pi_\varepsilon(x)| < c\sqrt{\varepsilon^2 + \varepsilon d(x, D_\varepsilon)} \quad \text{and} \quad |\pi(x) - \pi_\varepsilon(x)| < c\sqrt{\varepsilon^2 + \varepsilon d(x, D)}$$

where  $\pi_\varepsilon(x)$  is the orthogonal projection of  $x \in \mathbb{R}^k$  on the closure  $\bar{D}_\varepsilon$

**Lemma 4.4.2.**

$$\exists c > 0 \quad \text{such that} \quad \forall \varepsilon < 1, \forall x \in \mathbb{R}^k \quad |\pi(x) - \pi_\varepsilon(x)| < c\sqrt{\varepsilon}(1 + d(x, D_\varepsilon))$$

$$\exists c > 0 \quad \text{such that} \quad \forall \varepsilon < 1, \forall x \in \mathbb{R}^k \quad |\pi(x) - \pi_\varepsilon(x)| \mathbf{1}_{\{d(x, D_\varepsilon) > \varepsilon\}} < c\sqrt{\varepsilon} \sqrt{d(x, D_\varepsilon)} \mathbf{1}_{\{d(x, D_\varepsilon) > \varepsilon\}}$$

## 4.4.2 A priori estimates

In this section, we provide a priori estimates which are uniform in  $n$  on the solutions of (4.2.12).

**Lemma 4.4.3.** *There exists a constant  $C > 0$ , independent of  $n$ , such that for all  $n$  large enough*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds + \|K^n\|_{VT} \right] \leq C. \quad (4.4.3)$$

**Proof.** For a given point  $a \in D$ , that satisfy condition (4.2.3), we apply Itô's formula to get

$$\begin{aligned} |Y_t^n - a|^2 + \int_t^T \|Z_s^n\|^2 ds &= |\xi - a|^2 + 2 \int_t^T (Y_s^n - a)^* f_s(Y_s^n, Z_s^n) ds + 2 \int_t^T (Y_s^n - a)^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad - 2 \int_t^T (Y_s^n - a)^* Z_s^n dB_s + \int_t^T \|h_s(Y_s^n, Z_s^n)\|^2 ds - 2n \int_t^T (Y_s^n - a)^* (Y_s^n - \pi(Y_s^n)) ds. \end{aligned} \quad (4.4.4)$$

The stochastic integrals have both zero expectations since  $(Y^n, Z^n)$  belongs to  $\mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$ . We take expectation in (4.4.4) and we use the condition (4.2.3) and the Lipschitz Assumption 4.2.2 in order to obtain

$$\begin{aligned} \mathbb{E}[|Y_t^n - a|^2] + \mathbb{E} \left[ \int_t^T \|Z_s^n\|^2 ds \right] &\leq \mathbb{E}[|\xi - a|^2] + 2C \mathbb{E} \left[ \int_t^T (Y_s^n - a)^* (|f_s(a, 0)| + |Y_s^n - a| + |Z_s^n|) ds \right] \\ &\quad + \mathbb{E} \left[ \int_t^T |h_s(a, 0)|^2 ds \right] + C \mathbb{E} \left[ \int_t^T |Y_s^n - a|^2 ds \right] + \alpha \mathbb{E} \left[ \int_t^T |Z_s^n|^2 ds \right] \\ &\leq C(1 + \mathbb{E} \left[ \int_t^T (|f_s(a, 0)|^2 + |h_s(a, 0)|^2) ds \right]) + C(1 + \varepsilon^{-1}) \mathbb{E} \left[ \int_t^T |Y_s^n - a|^2 ds \right] \\ &\quad + (\alpha + \varepsilon) \mathbb{E} \left[ \int_t^T |Z_s^n|^2 ds \right]. \end{aligned} \quad (4.4.5)$$

Thus, if we choose  $\varepsilon = \frac{1 - \alpha}{2}$ , we have

$$\mathbb{E}[|Y_t^n - a|^2] + \left(\frac{1 - \alpha}{2}\right) \mathbb{E} \left[ \int_t^T \|Z_s^n\|^2 ds \right] \leq C(1 + \mathbb{E} \left[ \int_t^T |Y_s^n - a|^2 ds \right]).$$

Then, it follows from Gronwall's lemma that

$$\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^n - a|^2] \leq Ce^{CT}.$$

Therefore we can deduce

$$\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^n|^2] \leq C \quad \text{and} \quad \mathbb{E} \left[ \int_0^T \|Z_s^n\|^2 ds \right] \leq C \quad (4.4.6)$$

In the other hand, the uniform estimate on  $Y$  is obtained by taking the supremum over  $t$  in the equation (4.4.4), using the previous calculations and Burkholder-Davis-Gundy inequality. Thus, we get for all  $n \geq 0$

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^n - a|^2\right] \leq C \quad \text{and} \quad \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^n|^2\right] \leq C.$$

Finally, the total variation of the process  $K^n$  is defined by

$$\mathbb{E}[\|K^n\|_{VT}] = \mathbb{E}\left[n \int_0^T |Y_s^n - \pi(Y_s^n)| ds\right].$$

But from the property (4.2.3) and the equation (4.4.4) we have

$$\begin{aligned} 2n \int_t^T \gamma |Y_s^n - \pi(Y_s^n)| ds &\leq 2n \int_t^T |(Y_s^n - a)^*(Y_s^n - \pi(Y_s^n))| ds \\ &\leq |\xi - a|^2 + 2 \int_t^T (Y_s^n - a)^* f_s(Y_s^n, Z_s^n) ds + 2 \int_t^T (Y_s^n - a)^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad - 2 \int_t^T (Y_s^n - a)^* Z_s^n dB_s + \int_t^T \|h_s(Y_s^n, Z_s^n)\|^2 ds. \end{aligned}$$

Hence it follows from previous estimates that

$$\mathbb{E}[\|K^n\|_{VT}] \leq C,$$

and the proof of Lemma 4.4.3 is complete.  $\square$

### 4.4.3 Proof of Lemma 4.2.4

Let first recall that  $(Y^n, Z^n)$  is solution of the BDSDE (4.2.12) associated to  $(\xi, f^n, h)$  where  $f_t^n(y, z) = f_s(y, z) - n(y - \pi(y))$ , for each  $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ . Note that, since we have assumed that  $0 \in D$ ,  $f_t^n(0, 0) = f_s(0, 0) := f_s^0$ . Therefore, thanks to  $L^p$ -estimate for BDSDE (Theorem 4.1 in [109] applied for  $p = 4$ ), we have the following estimate

$$\sup_n \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^n|^4 + \left(\int_0^T \|Z_s^n\|^2 ds\right)^2\right] \leq C \mathbb{E}\left[|\xi|^4 + \int_0^T (|f_s^0|^4 + |h_s^0|^4) ds\right] < \infty. \quad (4.4.7)$$

Now, we apply generalized Itô's formula to get

$$\begin{aligned} |Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + 2n \int_t^T (Y_s^n)^*(Y_s^n - \pi(Y_s^n)) ds &= |\xi|^2 + 2 \int_t^T (Y_s^n)^* f_s(Y_s^n, Z_s^n) ds \\ &\quad + 2 \int_t^T (Y_s^n)^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s - 2 \int_t^T (Y_s^n)^* Z_s^n dB_s + \int_t^T \|h_s(Y_s^n, Z_s^n)\|^2 ds. \end{aligned} \quad (4.4.8)$$

From the property (4.2.3) and since  $0 \in D$  we have

$$\begin{aligned} 2n \int_0^T \gamma |Y_s^n - \pi(Y_s^n)| ds &\leq 2n \int_0^T |(Y_s^n)^*(Y_s^n - \pi(Y_s^n))| ds \\ &\leq |\xi|^2 + 2 \int_0^T (Y_s^n)^* f_s(Y_s^n, Z_s^n) ds + 2 \int_0^T (Y_s^n)^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s \\ &\quad - 2 \int_0^T (Y_s^n)^* Z_s^n dB_s + \int_0^T \|h_s(Y_s^n, Z_s^n)\|^2 ds. \end{aligned}$$

Then, taking the square and the expectation yields to

$$\begin{aligned} \mathbb{E}\left[\left(n \int_0^T |Y_s^n - \pi(Y_s^n)| ds\right)^2\right] &\leq C \mathbb{E}\left[|\xi|^4\right] + C \mathbb{E}\left[\left(\int_0^T (Y_s^n)^* f_s(Y_s^n, Z_s^n) ds\right)^2\right] \\ &\quad + C \mathbb{E}\left[\left(\int_0^T (Y_s^n)^* h_s(Y_s^n, Z_s^n) d\overleftarrow{W}_s\right)^2\right] + C \mathbb{E}\left[\left(\int_0^T (Y_s^n)^* Z_s^n dB_s\right)^2\right] \\ &\quad + C \mathbb{E}\left[\left(\int_0^T \|h_s(Y_s^n, Z_s^n)\|^2 ds\right)^2\right]. \end{aligned}$$

By using the isometry property and the boundedness of  $f$  and  $h$ , we obtain

$$\begin{aligned} \mathbb{E}\left[\left(n \int_0^T |Y_s^n - \pi(Y_s^n)| ds\right)^2\right] &\leq C\mathbb{E}\left[|\xi|^4\right] + C\mathbb{E}\left[\left(\int_0^T Y_s^n ds\right)^2\right] \\ &\quad + C\mathbb{E}\left[\int_0^T |Y_s^n h_s(Y_s^n, Z_s^n)|^2 ds\right] + C\mathbb{E}\left[\int_0^T |Y_s^n Z_s^n|^2 ds\right] + C. \end{aligned}$$

Finally, we deduce from Holder inequality and boundedness of  $h$  that

$$\mathbb{E}\left[\left(n \int_0^T |Y_s^n - \pi(Y_s^n)| ds\right)^2\right] \leq C\mathbb{E}\left[|\xi|^4 + \int_0^T |Y_s^n|^2 ds + \sup_{0 \leq t \leq T} |Y_t^n|^4 + \left(\int_0^T \|Z_s^n\|^2 ds\right)^2\right].$$

Thus, from the estimate (4.4.7) we get the desired result.  $\square$

#### 4.4.4 Proof of the tightness of the sequence $(\pi_n)_{n \in \mathbb{N}}$

Recall first that  $\nu_n(dt, dx) = -n(u_n - \pi(u_n))(t, x) dt dx$  and  $\pi_n(dt, dx) = \rho(x)\nu_n(dt, dx)$  where  $u_n$  is the solution of the SPDEs (4.3.15).

**Lemma 4.4.4.** *The sequence of measure  $(\pi_n)_{n \in \mathbb{N}}$  is tight.*

**Proof.** We shall prove that for every  $\varepsilon > 0$ , there exists some constant  $K$  such that

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} |\pi_n|(ds, dx) \leq \varepsilon, \quad \forall n \in \mathbb{N}. \quad (4.4.9)$$

We first write

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} |\pi_n|(ds, dx) \\ &= \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \left( \mathbf{1}_{\{|X_{0,s}^{-1}(x) - x| \leq K\}} + \mathbf{1}_{\{|X_{0,s}^{-1}(x) - x| \geq K\}} \right) |\pi_n|(ds, dx) \\ &:= I_K^n + L_K^n, \quad \mathbb{P} - a.s. \end{aligned}$$

Taking expectation yields

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geq 2K\}} \pi_n(ds, dx) = \mathbb{E}[I_K^n] + \mathbb{E}[L_K^n]. \quad (4.4.10)$$

By (4.3.17) and for  $K \geq 2\|b\|_\infty T$ , we get

$$\begin{aligned} \mathbb{E}[L_K^n] &\leq \int_0^T \int_{\mathbb{R}^d} \mathbb{P}\left(\sup_{0 \leq r \leq T} |X_{0,r}^{-1}(x) - x| \geq K\right) |\pi_n|(ds, dx) \\ &\leq (C_1 \exp(-C_2 K^2) + C_3 \exp(-C_4 K)) |\pi_n|([0, T] \times \mathbb{R}^d) \\ &\leq C'_1 \exp(-C_2 K^2) + C'_3 \exp(-C_4 K), \end{aligned}$$

so  $\mathbb{E}[L_K^n] \leq \varepsilon$  for  $K$  sufficiently large. On the other hand, if  $|x| \geq 2K$  and  $|X_{0,s}^{-1}(x) - x| \leq K$  then  $|X_{0,s}^{-1}(x)| \geq K$ . Therefore

$$\begin{aligned} \mathbb{E}[I_K^n] &\leq \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|X_{0,s}^{-1}(x)| \geq K\}} \rho(x) |\nu_n|(ds, dx)\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|X_{0,s}^{-1}(x)| \geq K\}} \rho(x) n |u_n - \pi(u_n)|(s, x) ds dx\right] \end{aligned}$$

which, by the change of variable  $y = X_{0,s}^{-1}(x)$ , becomes

$$\begin{aligned}
 & \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \geq K\}} \rho(X_{0,s}(y)) J(X_{0,s}(y)) n |u_n - \pi(u_n)|(s, X_{0,s}(y)) ds dy \right] \\
 & \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} \rho(x) \left( \rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right) \|K^{n,0,x}\|_{VT} dx \right] \\
 & \leq \left( \mathbb{E} \int_{\mathbb{R}^d} \|K^{n,0,x}\|_{VT}^2 \rho(x) dx \right)^{1/2} \\
 & \quad \left( \mathbb{E} \int_{\mathbb{R}^d} \left( \rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \rho(x) dx \right)^{1/2} \\
 & \leq C \left( \mathbb{E} \int_{\mathbb{R}^d} \left( \rho(x)^{-1} \mathbf{1}_{\{|x| \geq K\}} \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \rho(x) dx \right)^{1/2}.
 \end{aligned}$$

where the last inequality is a consequence of (4.4.3). It is now sufficient to prove that

$$\int_{\mathbb{R}^d} \rho(x)^{-1} \mathbb{E} \left[ \left( \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \right] dx < \infty. \quad (4.4.11)$$

Note that

$$\begin{aligned}
 & \mathbb{E} \left[ \left( \sup_{0 \leq r \leq T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \right] \\
 & \leq \left[ \mathbb{E} \left( \sup_{0 \leq r \leq T} |\rho(X_{0,r}(x))| \right)^4 \right]^{1/2} \left[ \mathbb{E} \left( \sup_{0 \leq r \leq T} |J(X_{0,r}(x))| \right)^4 \right]^{1/2} \\
 & \leq C \left[ \mathbb{E} \left( \sup_{0 \leq r \leq T} |\rho(X_{0,r}(x))| \right)^4 \right]^{1/2}.
 \end{aligned}$$

Therefore it is sufficient to prove that:

$$\int_{\mathbb{R}^d} \frac{1}{\rho(x)} \left( \mathbb{E} \left[ \sup_{t \leq r \leq T} |\rho(X_{t,r}(x))|^4 \right] \right)^{1/2} dx < \infty.$$

Since  $\rho(x) \leq 1$ , we have

$$\begin{aligned}
 \mathbb{E} \left[ \sup_{t \leq r \leq T} |\rho(X_{t,r}(x))|^4 \right] & \leq \mathbb{E} \left[ \sup_{t \leq r \leq T} |\rho(X_{t,r}(x))|^4 \mathbf{1}_{\left\{ \sup_{t \leq r \leq T} |X_{t,r}(x) - x| \leq \frac{|x|}{2} \right\}} \right] \\
 & \quad + \mathbb{P} \left( \sup_{t \leq r \leq T} |X_{t,r}(x) - x| \geq \frac{|x|}{2} \right) \\
 & = : A(x) + B(x).
 \end{aligned}$$

If  $\sup_{t \leq r \leq T} |X_{t,r}(x) - x| \leq \frac{|x|}{2}$  then  $|X_{t,r}(x)| \geq \frac{|x|}{2}$  and so  $|\rho(X_{t,r}(x))| \leq \left(1 + \frac{|x|}{2}\right)^{-p}$ . Thus we have that  $A(x) \leq \left(1 + \frac{|x|}{2}\right)^{-4p}$  and so  $\int_{\mathbb{R}^d} (1 + |x|)^p A(x)^{1/2} dx < \infty$ . On the other hand, if  $|x| \geq 4\|b\|_\infty T$ , then (the same argument as in the existence proof step 2 of Theorem 4 in [9] for the Itô integral with respect to the Brownian motion)

$$\begin{aligned}
 B(x) & \leq \mathbb{P} \left( \sup_{t \leq s \leq T} \left| \int_0^s \sigma(X_{0,r}(x)) dW_r \right| \geq \frac{|x|}{8} \right) \\
 & \leq C_1 \exp(-C_2|x|^2)
 \end{aligned}$$

and so  $\int_{\mathbb{R}^d} (1 + |x|)^p B(x)^{1/2} dx < \infty$ .

# Numerical Computation for Backward Doubly SDEs in a domain

## Contents

<b>5.1 Introduction</b>	<b>115</b>
<b>5.2 Backward doubly stochastic differential equations with random terminal time</b>	<b>116</b>
<b>5.3 Numerical scheme for Forward-Backward Doubly SDEs</b>	<b>119</b>
5.3.1 Formulation	120
5.3.2 Euler scheme approximation of Forward-BDSDEs	121
<b>5.4 Semilinear Stochastic PDEs with Dirichlet null condition</b>	<b>128</b>
5.4.1 Definitions and formulation	128
5.4.2 Stochastic flow of diffeomorphism and random test functions	130
5.4.3 Probabilistic representation of the solution of SPDE	132
<b>5.5 Implementation and numerical tests</b>	<b>135</b>
5.5.1 Notations and algorithm	135
5.5.2 One-dimensional case (Case when $d = k = l = 1$ )	137

## 5.1 Introduction

Backward stochastic differential equations (BSDEs in short) are natural tools to give a probabilistic interpretation for the solution of a class of semi-linear PDEs (see [115], [108], [107], [36]). By introducing in standard BSDEs a second nonlinear term driven by an external noise, we obtain Backward Doubly SDEs (BDSDEs) [109], namely,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\overleftarrow{W}_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad (5.1.1)$$

where  $(W_t)_{t \geq 0}$  and  $(B_t)_{t \geq 0}$  are two finite-dimensional independent Brownian motions. We note that the integral with respect to  $W$  is a "backward Itô integral". In the Markovian setting, these equations can be seen as Feynman-Kac's representation of Stochastic PDEs and form a powerful tool for numerical schemes [6, 7]. These SPDEs appear in various applications as, for instance, Zakai equations in filtering, pathwise stochastic control theory and stochastic control with partial observations.

Several generalizations to investigate more general nonlinear SPDEs have been developed following different approaches of the notion of weak solutions: the technique of stochastic flow (Bally and Matoussi [10], Matoussi et al. [97, 99], Kunita [80]); the approach based on Dirichlet forms and their associated Markov processes (Denis and Stoica [43], Bally, Pardoux and Stoica [11], Stoica [131],

Denis, Matoussi and Stoica [40, 41, 42], Matoussi and Stoica[98]); stochastic viscosity solution for SPDEs (Buckdahn and Ma [24, 25], Lions and Souganidis [89, 90, 88]). Above approaches have allowed the study of numerical schemes for the Sobolev solution of semi-linear SPDEs via Monte-Carlo methods (time discretization and regression schemes [6, 5, 7].

In this chapter, we are concerned with numerical scheme for backward doubly SDEs with random terminal time. These later equations give the probabilistic interpretation for the weak-Sobolev's solutions of a class of semilinear stochastic partial differential equations (SPDEs in short) with Dirichlet null condition on the boundary of some open smooth domain  $\mathcal{O} \subset \mathbb{R}^d$ . Let also mention that an alternative method to solve numerically nonlinear SPDEs is an analytic one, based on time- space discretization of the SPDEs. The discretization in space can be achieved either by finite differences, or finite elements [138] and spectral Galerkin methods [70]. But most numerical works on SPDEs have concentrated on the Euler finite-difference scheme. Very interesting results have been obtained by Gyongy and Krylov [60]. The authors consider a symmetric finite difference scheme for a class of linear SPDE driven by an infinite dimensional Brownian motion.

Our contributions are as following: first of all, BDSDEs with random terminal time are introduced and results of existence and uniqueness of such BDSDEs are established by means of some transformation to classical BSDEs studied by Peng [115], Darling and Pardoux [36], Pardoux [107] and Briand et al [23]. Next, Euler numerical scheme for a Forward-BDSDEs is developed where we provide upper bounds for the discrete time approximations error which is at most of order  $h^{1/2}$ . Then probabilistic representation for the weak solution of semilinear SPDEs with Dirichlet null condition on the boundary of the domain  $\mathcal{O}$  is given by means of solution of BDSDEs with random terminal time. This is done by using localization procedure and stochastic flow technics (see e.g. [10], [97], [80, 79] for these flow technics).

This chapter is organized as following: in section 5.2, first the basic assumptions and the definitions of the solutions for BDSDEs with random terminal time are presented. Then, existence and uniqueness results of such equations are given by using fixed point theorem. In section 5.3, we develop a discrete-time approximation of a Forward-Backward Doubly SDE with finite stopping time horizon, namely the first exit time of a forward SDE from a domain  $\mathcal{O}$ . The main result of this section is providing a rate of convergence of order  $h^{1/2}$  for the square of Euler time discretization error for Forward-Backward Doubly SDE scheme (5.3.7)-(5.3.13). Moreover, we relate the BDSDE in the Markovian setting to Sobolev semilinear SPDEs with Dirichlet null condition by proving Feynman-Kac's formula in Section 5.4. Finally, the last section is devoted to numerical implementations and tests. This chapter is based on [95].

## 5.2 Backward doubly stochastic differential equations with random terminal time

Any element  $x \in \mathbb{R}^d$ ,  $d \geq 1$ , will be identified with a line vector with  $i$ th component  $x^i$  and its Euclidean norm defined by  $|x| = (\sum_i |x_i|^2)^{1/2}$ . For each real matrix  $A$ , we denote by  $\|A\|$  its Frobenius norm defined by  $\|A\| = (\sum_{i,j} a_{i,j}^2)^{1/2}$ .

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $\{W_t, 0 \leq t \leq T\}$  and  $\{B_t, 0 \leq t \leq T\}$  be two mutually independent standard Brownian motions with values in  $\mathbb{R}^l$  and  $\mathbb{R}^d$ . For each  $0 \leq s \leq T$ , we define

$$\mathcal{F}_s := \mathcal{F}_s^B \vee \mathcal{F}_{s,T}^W,$$

with  $\mathcal{F}_s^B := \sigma(B_r; 0 \leq r \leq s)$  and  $\mathcal{F}_{s,t}^W := \sigma(W_r - W_s; s \leq r \leq t) \vee \mathcal{N}$  where  $\mathcal{N}$  is the class of  $\mathbb{P}$ -null sets of  $\mathcal{F}$ . Note that  $(\mathcal{F}_t)_{t \leq T}$  is not an increasing family of  $\sigma$ -fields, so it is not a filtration. Hereafter, let us define the spaces and the norms which will be needed for the formulation of the BDSDE with random terminal time.

-  $\mathbf{L}^p(\mathcal{F}_\tau^W)$  the space of  $\mathbb{R}^k$  valued  $\mathcal{F}_\tau^W$ -measurable random variables  $\xi$  such that

$$\|\xi\|_{L^p}^p := \mathbb{E}(e^{\lambda\tau}|\xi|^p) < +\infty;$$

-  $\mathcal{H}_{k \times d}^2([0, T])$  the space of  $\mathbb{R}^{k \times d}$ -valued  $\mathcal{F}_t$ -measurable process  $Z = (Z_t)_{t \leq T}$  such that

$$\|Z\|_{\mathcal{H}^2}^2 := \mathbb{E}\left[\int_0^\tau e^{\lambda t} |Z_t|^2 dt\right] < +\infty;$$

-  $\mathcal{S}_k^2([0, T])$  the space of  $\mathbb{R}^k$  valued  $\mathcal{F}_t$ -adapted processes  $Y = (Y_t)_{t \leq T}$ , with continuous paths such that

$$\|Y\|_{\mathcal{S}^2}^2 := \mathbb{E}\left[\sup_{t \leq \tau} e^{\lambda t} |Y_t|^2\right] < +\infty;$$

We need the following assumptions:

**Assumption (HT)** The final random time  $\tau$  is an  $\mathcal{F}_t^B$ -stopping time and the final condition  $\xi$  is an  $\mathcal{F}_\tau^B$ -measurable and  $k$ -dimensional random variable such that  $\mathbb{E}[e^{\lambda\tau}|\xi|^2] < \infty$ .

**Assumption (HL)** The two coefficients  $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$  and  $g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times l}$ , which for some real numbers  $\alpha, \mu, \lambda, K > 0, C > 0, \lambda > \frac{2K}{1-\alpha} - 2\mu + C$  and  $0 < \alpha < 1$  satisfy: for all  $t \in [0, T]$  and  $(y, z), (y', z') \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,

- (i)  $f(\cdot, y, z)$  and  $g(\cdot, y, z)$  are  $\mathcal{F}_t$  measurable,
- (ii)  $|f(t, y, z) - f(t, y', z')| \leq K(|y - y'| + \|z - z'\|)$ ,
- (iii)  $\langle y - y', f(t, y, z) - f(t, y', z) \rangle \leq -\mu |y - y'|^2$ ,
- (iv)  $\|g(t, y, z) - g(t, y', z')\|^2 \leq C|y - y'|^2 + \alpha \|z - z'\|^2$ ,
- (v)  $\mathbb{E} \int_0^\tau e^{\lambda s} |f(t, 0, 0)|^2 ds < \infty$  and  $\mathbb{E} \int_0^\tau e^{\lambda s} \|g(t, 0, 0)\|^2 ds < \infty$ .

Now we introduce the definition of BDSDEs with random terminal time  $\tau$  and associated to  $(\xi, f, g)$ .

**Definition 5.2.1.** A solution of BDSDE  $(\tau, \xi, f, g)$  is a couple  $\{(Y_s, Z_s); 0 \leq s \leq T\} \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  such that  $Y_t = \xi$  on the set  $\{t \geq \tau\}$ ,  $Z_t = 0$  on the set  $\{t > \tau\}$  and

$$Y_t = \xi + \int_t^{\tau \wedge T} f(s, Y_s, Z_s) ds + \int_t^{\tau \wedge T} g(s, Y_s, Z_s) d\overleftarrow{W}_s - \int_t^{\tau \wedge T} Z_s dB_s, \quad 0 \leq t \leq \tau. \quad (5.2.1)$$

We note that the integral with respect to  $W$  is a "backward Itô integral" (see Kunita [76] for the definition) and the integral with respect to  $B$  is a standard forward Itô integral. We establish existence and uniqueness of the solution for BDSDE (5.2.1) which is an extension of Peng's results [115] in the standard BSDE case:

**Theorem 5.2.1.** Under the Assumptions (HT) and (HL), there exists a unique solution  $\{(Y_s, Z_s); 0 \leq s \leq T\} \in \mathcal{S}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  of the BDSDE (5.2.1).

**Proof.**

**a) Uniqueness:** Let  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  be two solutions of the BDSDE (5.2.1) and denote by  $(\bar{Y}, \bar{Z}) := (Y^1 - Y^2, Z^1 - Z^2)$ . Applying generalized Ito formula (see Lemma 1.3 in [109]) to  $e^{\lambda s} |\bar{Y}_s|^2$  yields

$$\begin{aligned} e^{\lambda t} |\bar{Y}_t|^2 &+ \int_t^{\tau \wedge T} e^{\lambda s} (\lambda |\bar{Y}_s|^2 + \|\bar{Z}_s\|^2) ds = 2 \int_t^{\tau \wedge T} e^{\lambda s} \langle \bar{Y}_s, f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2) \rangle ds \\ &+ 2 \int_t^{\tau \wedge T} e^{\lambda s} \langle \bar{Y}_s, g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2) \rangle d\overleftarrow{W}_s - 2 \int_t^{\tau \wedge T} e^{\lambda s} \langle \bar{Y}_s, \bar{Z}_s \rangle dB_s \\ &+ \int_t^{\tau \wedge T} e^{\lambda s} \|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\|^2 ds. \end{aligned} \quad (5.2.2)$$

Then, taking expectation we obtain

$$\begin{aligned} \mathbb{E}[e^{\lambda t}|\bar{Y}_t|^2] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s}(\lambda|\bar{Y}_s|^2 + \|\bar{Z}_s\|^2)ds\right] &= 2\mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \langle \bar{Y}_s, f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2) \rangle ds\right] \\ &+ \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\|^2 ds\right]. \end{aligned}$$

From Assumption **(HL)** there exists  $0 < \varepsilon < 1$  such that

$$2 \langle \bar{Y}_s, f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2) \rangle \leq (-2\mu + \frac{K}{1-\varepsilon})|\bar{Y}_s|^2 + (1-\varepsilon)\|\bar{Z}_s\|^2,$$

which together with the Lipschitz continuous assumption on  $g$  provide

$$\begin{aligned} \mathbb{E}[e^{\lambda t}|\bar{Y}_t|^2] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s}(\lambda|\bar{Y}_s|^2 + \|\bar{Z}_s\|^2)ds\right] &\leq \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s}(-2\mu + \frac{K}{1-\varepsilon} + C)|\bar{Y}_s|^2 ds\right] \\ &+ \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s}(\alpha + 1 - \varepsilon)\|\bar{Z}_s\|^2 ds\right], \end{aligned}$$

where  $0 < \alpha < 1$ . Consequently

$$\mathbb{E}[e^{\lambda t}|\bar{Y}_t|^2] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s}((\lambda + 2\mu - \frac{K}{1-\varepsilon} - C)|\bar{Y}_s|^2 + (\varepsilon - \alpha)\|\bar{Z}_s\|^2)ds\right] \leq 0.$$

Next, choosing  $\varepsilon = \frac{1+\alpha}{2}$  such that  $\lambda + 2\mu - \frac{2K}{1-\alpha} - C > 0$ , we conclude that

$$Y^1 = Y^2 \quad \text{and} \quad Z^1 = Z^2, \quad \mathbb{P} - a.s., \quad \forall t \in [0, T].$$

**b) Existence:** The existence of a solution will be proved in two steps. In the first step, we suppose that  $g$  does not depend on  $y, z$ , then we are able to transform our BDSDE with data  $(\tau, \xi, f, g)$  into a BSDE  $(\tau, \bar{\xi}, \bar{f})$ , where  $\bar{\xi}$  and  $\bar{f}$  are explicited below. Thus, the existence is proved by appealing to the existence result for BSDEs with random terminal time established by Peng 1991. In the second step, we study the case when  $g$  depends on  $y, z$  using Picard iteration.

*Step 1 :* Suppose that  $g := g^0$  does not depend on  $y, z$ , and the BDSDE (5.2.1) becomes

$$Y_t = \xi + \int_t^{\tau \wedge T} f(s, Y_s, Z_s)ds + \int_t^{\tau \wedge T} g(s)d\overleftarrow{W}_s - \int_t^{\tau \wedge T} Z_s dB_s, \quad 0 \leq t \leq T. \quad (5.2.3)$$

Denoting

$$\bar{Y}_t := Y_t + \int_0^t g(s)d\overleftarrow{W}_s, \quad \bar{\xi} := \xi + \int_0^\tau g(s)d\overleftarrow{W}_s,$$

we have the following BSDE

$$\bar{Y}_t = \bar{\xi} + \int_t^{\tau \wedge T} \bar{f}(s, \bar{Y}_s, Z_s)ds - \int_t^{\tau \wedge T} Z_s dB_s, \quad 0 \leq t \leq T. \quad (5.2.4)$$

where  $\bar{f}(s, y, z) := f(s, y - \int_0^t g(s)d\overleftarrow{W}_s, z)$ . We can easily check that  $\bar{\xi}$  and  $\bar{f}$  satisfy the same assumptions that Peng [115] (Theorem 2.2) have proved for the existence and uniqueness of the solution for the standard BSDE (5.2.4). Thus, we get the existence of the solution for the BDSDEs (5.2.3).

*Step 2 :* The nonlinear case when  $g$  depends on  $y, z$ . The solution is obtained by using the fixed point Banach theorem. For any given  $(\bar{Y}, \bar{Z}) \in \mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$ , let consider the BDSDE with random terminal time:

$$Y_t = \xi + \int_t^{\tau \wedge T} f(s, Y_s, Z_s)ds + \int_t^{\tau \wedge T} g(s, \bar{Y}_s, \bar{Z}_s)d\overleftarrow{W}_s - \int_t^{\tau \wedge T} Z_s dB_s, \quad 0 \leq t \leq T. \quad (5.2.5)$$



It follows from Step 1 that the BDSDE (5.2.5) has a unique solution  $(Y, Z) \in \mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$ . Therefore, the mapping:

$$\begin{aligned} \Psi : \mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T]) &\longrightarrow \mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T]) \\ (\bar{Y}, \bar{Z}) &\longmapsto \Psi(\bar{Y}, \bar{Z}) = (Y, Z) \end{aligned}$$

is well defined.

Next, let  $(Y, Z), (Y', Z'), (\bar{Y}, \bar{Z})$  and  $(\bar{Y}', \bar{Z}') \in \mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  such that  $(Y, Z) = \Psi(\bar{Y}, \bar{Z})$  and  $(Y', Z') = \Psi(\bar{Y}', \bar{Z}')$  and set  $\Delta\eta = \eta - \eta'$  for  $\eta = Y, \bar{Y}, Z, \bar{Z}, K$ . Applying Ito formula and taking expectation yield to

$$\begin{aligned} \mathbb{E}[e^{\lambda t} |\Delta Y_t|^2] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} (\lambda |\delta Y_s|^2 + \|\delta Z_s\|^2) ds\right] &= 2\mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \rangle ds\right] \\ &\quad + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \|g(s, \bar{Y}_s, \bar{Z}_s) - g(s, \bar{Y}'_s, \bar{Z}'_s)\|^2 ds\right]. \end{aligned}$$

From Assumption **(HL)** there exists  $\alpha < \varepsilon < 1$  such that

$$\langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \rangle \leq \left(-2\mu + \frac{K}{1-\varepsilon}\right) |\Delta Y_s|^2 + (1-\varepsilon) \|\Delta Z_s\|^2,$$

which together with the Lipschitz continuous assumption on  $g$  provide

$$\begin{aligned} \mathbb{E}[e^{\lambda t} |\Delta Y_t|^2] + \left(\lambda + 2\mu - \frac{K}{1-\varepsilon}\right) \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} |\delta Y_s|^2 ds\right] + \varepsilon \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\mu s} \|\Delta Z_s\|^2 ds\right] \\ \leq C \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\mu s} |\Delta \bar{Y}_s|^2 ds\right] + \alpha \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\mu s} \|\Delta \bar{Z}_s\|^2 ds\right]. \end{aligned}$$

Next, choosing  $\varepsilon$  such that  $\lambda + 2\mu - \frac{K}{1-\varepsilon} = \frac{\varepsilon C}{\alpha}$ , we obtain

$$\begin{aligned} \varepsilon \left[\frac{C}{\alpha} \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} |\Delta Y_s|^2 ds\right] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \|\Delta Z_s\|^2 ds\right]\right] \\ \leq \alpha \left[\frac{C}{\alpha} \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} |\Delta \bar{Y}_s|^2 ds\right] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \|\Delta \bar{Z}_s\|^2 ds\right]\right]. \end{aligned}$$

Since  $\frac{\alpha}{\varepsilon} < 1$ , then  $\Psi$  is a strict contraction on  $\mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  equipped with the norm

$$\|(Y, Z)\|^2 = \frac{C}{\alpha} \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} |\Delta Y_s|^2 ds\right] + \mathbb{E}\left[\int_t^{\tau \wedge T} e^{\lambda s} \|\Delta Z_s\|^2 ds\right].$$

Thus from Banach fixed point theorem there exists a unique pair  $(Y, Z) \in \mathcal{H}_k^2([0, T]) \times \mathcal{H}_{k \times d}^2([0, T])$  solution of BDSDE associated to  $(\tau, \xi, f, g)$ . Moreover, thanks to Assumption **(HL)** and standard calculations and estimates we show that  $Y$  belongs to  $\mathcal{S}_k^2([0, T])$ .  $\square$

## 5.3 Numerical scheme for Forward-Backward Doubly SDEs

In this section, we are interested in developing a discrete-time approximation of a Forward-Backward Doubly SDE with finite stopping time horizon, namely the first exit time of a forward SDE from a cylindrical domain  $D = [0, T] \times \mathcal{O}$ . As usual, since we will state in the Markovian framework, we need a slight modification of the filtration. So, we fix  $t \in [0, T]$  and for each  $s \in [t, T]$ , we define

$$\mathcal{F}_s^t := \mathcal{F}_{t,s}^B \vee \mathcal{F}_{s,T}^W \vee \mathcal{N} \quad \text{and} \quad \mathcal{G}_s^t := \mathcal{F}_{t,s}^B \vee \mathcal{F}_{t,T}^W \vee \mathcal{N},$$

where  $\mathcal{F}_{t,s}^B = \sigma\{B_r - B_t, t \leq r \leq s\}$ ,  $\mathcal{F}_{s,T}^W = \sigma\{W_r - W_s, s \leq r \leq T\}$  and  $\mathcal{N}$  the class of  $\mathbb{P}$  null sets of  $\mathcal{F}$ . Note that the collection  $\{\mathcal{F}_s^t, s \in [t, T]\}$  is neither increasing nor decreasing and it does not constitute a filtration. However,  $\{\mathcal{G}_s^t, s \in [t, T]\}$  is a filtration. We will omit the dependance of the filtration with respect to the time  $t$  if  $t = 0$ .

### 5.3.1 Formulation

For all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , let  $(X_s^{t,x})_{0 \leq s \leq t}$  be the unique strong solution of the following stochastic differential equation:

$$dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s, \quad s \in [t, T], \quad X_s^{t,x} = x, \quad 0 \leq s \leq t, \quad (5.3.1)$$

where  $b$  and  $\sigma$  are two functions on  $\mathbb{R}^d$  with values respectively in  $\mathbb{R}^d$  and  $\mathbb{R}^{d \times d}$ . We will omit the dependance of the forward process  $X$  in the initial condition if it starts at time  $t = 0$ .

Let  $\tau^{t,x}$  is the first exit time of  $(s, X_s^{t,x})$  from a cylindrical domain  $D = [0, T] \times \mathcal{O}$  for some open bounded set  $\mathcal{O} \subset \mathbb{R}^d$ .

We now consider the following Markovian BDSDE with terminal random time  $\tau$  associated to the data  $(\Phi, f, g)$ : For all  $t \leq s \leq T$ ,

$$\begin{cases} -dY_s^{t,x} &= \mathbf{1}_{\{s < \tau\}} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \mathbf{1}_{\{s < \tau\}} g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})d\overleftarrow{W}_s - Z_s^{t,x} dB_s, \\ Y_s^{t,x} &= \Phi(\tau, X_\tau^{t,x}), \quad \tau \leq s \leq T, \end{cases} \quad (5.3.2)$$

where  $f$  and  $\Phi$  are now two functions respectively on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$  and  $\mathbb{R}^d$  with values in  $\mathbb{R}^k$  and  $g$  is a function on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$  with values in  $\mathbb{R}^{k \times l}$ .

Now, we specify some conditions on the domain and the diffusion process:

**Assumption (D)**  $\mathcal{O}$  is an open bounded set of  $\mathbb{R}^d$  with a  $C^2$ -boundary.

**Assumption (MHD)**

- (i) The matrix  $a := \sigma\sigma^*$  is elliptic, i.e. there exists  $\Lambda > 0$  such that for all  $x, \zeta \in \bar{\mathcal{O}}$ ,

$$\Lambda \|\zeta\|^2 \leq \zeta a(x) \zeta^*. \quad (5.3.3)$$

- (ii) There exists a positive constant  $L$  such that

$$|b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \leq L|x - x'|, \quad \forall x, x' \in \mathbb{R}^d.$$

**Remark 5.3.1.** We mention that this smoothness assumption (D) on the domain could be weakened by considering the domain  $\mathcal{O}$  as a finite intesection of smooth domains with compact boundaries and further conditions on the set of corners (see conditions (D1) and (D2) in [21]). Under this weakened hypotheses, one may just assume the the matrix  $a$  satisfies a non-characteristic boundary condition outside the set of corners  $\mathcal{C}$  and a uniform ellipticity condition on a neighborhood of  $\mathcal{C}$ .

Besides, we assume that the terminal condition  $\Phi$  is sufficiently smooth:

**Assumption (MHT)**

$$\Phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \quad \text{and} \quad \|\partial_t \Phi\| + \|D\Phi\| + \|D^2\Phi\| \leq L \quad \text{on} \quad [0, T] \times \mathbb{R}^d.$$

We next state a strengthening of Assumption (HL) in the present Markov framework:

**Assumption (MHL)** There exist constants  $\alpha, \mu, \lambda, K > 0, C > 0, C' > 0, \lambda > \frac{2K}{1-\alpha} - 2\mu + C$  and  $0 < \alpha < 1$  such that for any  $(t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,

- (i)  $|f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2)| \leq K(\sqrt{|t_1 - t_2|} + |x_1 - x_2| + |y_1 - y_2| + \|z_1 - z_2\|),$
- (ii)  $\|g(t_1, x_1, y_1, z_1) - g(t_2, x_2, y_2, z_2)\|^2 \leq C(|t_1 - t_2| + |x_1 - x_2|^2 + |y_1 - y_2|^2) + \alpha \|z_1 - z_2\|^2,$
- (iii)  $\langle y_1 - y_2, f(t_1, x_1, y_1, z_1) - f(t_1, x_1, y_2, z_1) \rangle \leq -\mu |y_1 - y_2|^2,$

$$(iv) \quad \sup_{0 \leq t \leq T} (|f(t, 0, 0, 0)| + \|g(t, 0, 0, 0)\|) \leq C'.$$

**Remark 5.3.2.** We note that the integrability condition given by Assumption **(HT)** in section 5.2 is satisfied in this Markovian setting thanks to the smoothness of  $\Phi$  (Assumption **(MHT)**) and the fact that the exit time  $\tau$ , under the ellipticity condition (5.3.3) verified by the matrix  $a$  (see Stroock and Varadhan [133]), satisfy

$$\sup_{(t,x) \in [0,T] \times \bar{O}} \mathbb{E}[\exp(\lambda \tau^{t,x})] < \infty.$$

From [109] and [76], the standard estimates for the solution of the Forward-Backward Doubly SDE (5.3.1)-(5.3.2) hold and we remind the following theorem:

**Theorem 5.3.1.** Under Assumptions **(MHT)** and **(MHL)**, there exist, for any  $p \geq 2$ , two positive constants  $C$  and  $C_p$  and an integer  $q$  such that :

$$\mathbb{E}\left[\sup_{t \leq s \leq \tau} |X_s^{t,x}|^2\right] \leq C(1 + |x|^2), \quad (5.3.4)$$

$$\mathbb{E}\left[\sup_{t \leq s \leq \tau} |Y_s^{t,x}|^p + \left(\int_t^\tau \|Z_s^{t,x}\|^2 ds\right)^{p/2}\right] \leq C_p(1 + |x|^q). \quad (5.3.5)$$

From now on,  $C_L^\eta$  denotes a generic constant whose value may change from line to line, but which depends only on  $X_0$ ,  $L$  and some extra parameter  $\eta$  (we simply write  $C_L$  if it depends only on  $X_0$  and  $L$ ). Similarly,  $\xi_L^\eta$  denotes a generic non-negative random variable such that  $\mathbb{E}[|\xi_L^\eta|^p] \leq C_L^{\eta,p}$  for all  $p \geq 1$  (we simply write  $\xi_L$  if it does not depend on the parameter  $\eta$ ).

## 5.3.2 Euler scheme approximation of Forward-BDSDEs

### 5.3.2.1 Forward Euler scheme

In order to approximate the forward diffusion process (5.3.1), we use a standard Euler scheme with time step  $h$ , associated to a grid

$$\pi := \{t_i = ih; i \leq N\}, \quad h := T/N, \quad N \in \mathbb{N},$$

This approximation is defined by

$$X_t^N = x + \int_0^t b(X_{\varphi(s)}) ds + \int_0^t \sigma(X_{\varphi(s)}) dB_s, \quad t \geq 0 \quad (5.3.6)$$

where  $\varphi(s) := \sup\{t \in \pi : t \leq s\}$ . Notice that  $\varphi(t) = t_i$ , for  $t \in [t_i, t_{i+1})$  and the continuous approximation (5.3.6) is equivalent to the following discrete approximation

$$\begin{cases} X_0^N = x, \\ X_{t_{i+1}}^N = X_{t_i}^N + b(X_{t_i}^N)(t_{i+1} - t_i) + \sigma(X_{t_i}^N)(B_{t_{i+1}} - B_{t_i}), \quad i \leq N. \end{cases} \quad (5.3.7)$$

Then, we approximate the exit time  $\tau$  by the first time of the Euler scheme  $(t, X_t^N)_{t \in \pi}$  from  $D$  on the grid  $\pi$ :

$$\bar{\tau} := \inf\{t \in \pi : X_t^N \notin \mathcal{O}\} \wedge T.$$

**Remark 5.3.3.** One may approximate the exit time  $\tau$  by its continuous version  $\tilde{\tau}$  which is defined as the first exit time of the Euler scheme  $(t, X_t^N)$ , namely

$$\tilde{\tau} := \inf\{t \in [0, T] : X_t^N \notin \mathcal{O}\} \wedge T.$$

However, this approximation requires more regularity on the boundary of  $\mathcal{O}$  (see e.g. [54, 55]).

The upper bound estimates for the error due to the approximation of  $\tau$  by  $\bar{\tau}$  was proved by Bouchard and Menozzi [21] for the weak version of such estimate and Gobet [54, 55] for the strong one. Recently, Bouchard, Geiss and Gobet [20] have improved the following  $L^1$ -strong error:

**Theorem 5.3.2.** *Assume that (MHD) and (D) hold. Then, there exists  $C_L > 0$  such that*

$$\mathbb{E}[|\tau - \bar{\tau}|] \leq C_L h^{1/2}. \quad (5.3.8)$$

**Remark 5.3.4.** *Let us mention that the upper bound estimates for the error due to the approximation of  $\tau$  by  $\bar{\tau}$  proved by Bouchard and Menozzi [21] for the weak version of such estimate is as following: for any  $\varepsilon \in (0, 1)$  and each positive random variable  $\xi$  satisfying  $\mathbb{E}[(\xi_L)^p] \leq C_L^p$  for all  $p \geq 1$ , there exists  $C_L^\varepsilon > 0$  such that*

$$\mathbb{E}[\mathbb{E}[\xi_L | \tau - \bar{\tau} | \mathcal{F}_{\bar{\tau} \wedge \tau}^B]^2] \leq C_L^\varepsilon h^{1-\varepsilon}. \quad (5.3.9)$$

For the strong estimate error, Gobet [54, 55] have proved that, for each  $\varepsilon \in (0, 1/2)$ , there exists  $C_L^\varepsilon > 0$  such that

$$\mathbb{E}[|\tau - \bar{\tau}|] \leq C_L^\varepsilon h^{1/2-\varepsilon}. \quad (5.3.10)$$

### 5.3.2.2 Euler scheme for BDSDEs

Regarding the approximation of (5.3.2), we adapt the approach of [6]. We define recursively (in a backward manner) the discrete-time process  $(Y^N, Z^N)$  on the time grid  $\pi$  by

$$Y_T^N = \Phi(\bar{\tau}, X_{\bar{\tau}}^N), \quad (5.3.11)$$

and for  $i = N - 1, \dots, 0$ , we set

$$Z_{t_i}^N = h^{-1} \mathbb{E}_{t_i} \left[ (Y_{t_{i+1}}^N + g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i) \Delta B_i^\top \right], \quad (5.3.12)$$

$$Y_{t_i}^N = \mathbb{E}_{t_i} [Y_{t_{i+1}}^N] + \mathbf{1}_{\{t_i < \bar{\tau}\}} h \mathbb{E}_{t_i} [f(t_i, \Theta_i^N)] + \mathbf{1}_{\{t_i < \bar{\tau}\}} \mathbb{E}_{t_i} [g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i], \quad (5.3.13)$$

where

$$\Theta_i^N := (X_{t_i}^N, Y_{t_i}^N, Z_{t_i}^N), \quad \Delta W_i = W_{t_{i+1}} - W_{t_i}, \quad \Delta B_i = B_{t_{i+1}} - B_{t_i}.$$

$\top$  denotes the transposition operator and  $\mathbb{E}_{t_i}$  denotes the conditional expectation over the  $\sigma$ -algebra  $\mathcal{F}_{t_i}^0$ . The above conditional expectation are well defined at each step of the algorithm.

Observe that  $Y_{t_i}^N \mathbf{1}_{\{t_i \geq \bar{\tau}\}} = \Phi(\bar{\tau}, X_{\bar{\tau}}^N) \mathbf{1}_{\{t_i \geq \bar{\tau}\}}$  and  $Z_{t_i}^N \mathbf{1}_{\{t_i \geq \bar{\tau}\}} = 0$ . One can easily check that

$$Y_{t_{i+1}}^N + g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i \in L^2(\mathcal{F}_{t_{i+1}})$$

for all  $0 \leq i < N$  under the Lipschitz continuous assumption. Then an obvious extension of Itô martingale representation theorem yields the existence of the  $\mathcal{G}_t$ -progressively measurable and square integrable process  $Z^N$  satisfying, for all  $i < N$

$$Y_{t_{i+1}}^N + g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i = \mathbb{E}_{t_i} [Y_{t_{i+1}}^N + g(t_{i+1}, \Theta_{i+1}^N) \Delta W_i] + \int_{t_i}^{t_{i+1}} Z_s^N dB_s.$$

Following the arguments of Pardoux and Peng [109] (see page 213), we can prove that in fact  $Z^N$  is  $\mathcal{F}_t$ -progressively measurable thanks to the independance of the increments of  $B$  and the two Brownian motions  $B$  and  $W$ .

This allows us to consider a continuous-time extension of  $Y^N$  in  $\mathcal{S}^2$  defined on  $[0, T]$  by

$$Y_t^N = \Phi(\bar{\tau}, X_{\bar{\tau}}^N) + \int_t^T \mathbf{1}_{\{s < \bar{\tau}\}} f(\varphi(s), \Theta_{\varphi(s)}^N) ds + \int_t^T \mathbf{1}_{\{s < \bar{\tau}\}} g(\psi(s), \Theta_{\psi(s)}^N) d\overleftarrow{W}_s - \int_t^T Z_s^N dB_s, \quad (5.3.14)$$

where  $\psi(s) := \inf\{t \in \pi : t \geq s\}$ .

**Remark 5.3.5.** Observe that  $Z_s = 0$  on  $] \tau, T]$  and  $Z_s^N = 0$  on  $] \bar{\tau}, T]$ . For later use, note also that

$$Z_{t_i}^N = h^{-1} \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} Z_s^N ds \right], \quad i < N. \quad (5.3.15)$$

In order to prove (5.3.25) of Proposition 5.3.2, we need the following lemma.

**Lemma 5.3.1.** Let Assumptions (MHL) and (MHT) hold. Then,

$$\max_{i < N} (|Y_{t_i}^N| + \sqrt{h} \|Z_{t_i}^N\|) \leq \xi_L \quad \text{and} \quad \|Y^N\|_{\mathcal{S}^2} + \|Z_\varphi^N\|_{\mathcal{H}^2} + \|Z_\psi^N\|_{\mathcal{H}^2} \leq C_L. \quad (5.3.16)$$

### 5.3.2.3 Upper bounds for the discrete-time approximation error

In this section, we provide bounds for the (square of the) discrete-time approximation error up to a stopping time  $\theta \leq T$   $\mathbb{P}$ -a.s. defined as

$$\text{Err}(h)_\theta^2 := \max_{i < N} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} \mathbf{1}_{\{t < \theta\}} |Y_t - Y_t^N|^2 \right] + \mathbb{E} \left[ \int_0^\theta \|Z_t - Z_{\varphi(t)}^N\|^2 dt \right], \quad (5.3.17)$$

where we recall  $\varphi(s) := \sup\{t \in \pi : t \leq s\}$ .

We first recall some standard controls on  $X$ ,  $(Y, Z)$  and  $X^N$ .

**Proposition 5.3.1.** Let Assumptions (MHL), (MHT) and (MHD) hold. Fix  $p \geq 2$ . Let  $\vartheta$  be a stopping time with values in  $[0, T]$ . Then,

$$\mathbb{E} \left[ \sup_{t \in [\vartheta, T]} |Y_t|^p + \left( \int_\vartheta^T \|Z_t\|^2 dt \right)^{p/2} \right] \leq C_L^p (1 + |X_\vartheta|^p),$$

and

$$\mathbb{E} \left[ \sup_{t \in [\vartheta, T]} (|X_t|^p + |X_t^N|^p) | \mathcal{F}_{0, \vartheta}^B \right] \leq \xi_L^p.$$

Moreover,

$$\max_{i < N} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} (|X_t - X_{t_i}|^p + |X_t^N - X_{t_i}^N|^p) \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} (|X_t - X_t^N|^p) \right] \leq C_L^p h^{p/2},$$

$$\mathbb{P} \left[ \sup_{t \in [0, T]} (|X_t^N - X_{\varphi(t)}^N| > r) \right] \leq C_L r^{-4} h, \quad r > 0,$$

and, if  $\theta$  is a stopping time with values in  $[0, T]$  such that  $\vartheta \leq \theta \leq \vartheta + h$   $\mathbb{P}$ -a.s., then

$$\mathbb{E} [ |X_\theta^N - X_\vartheta^N|^p + |X_\theta - X_\vartheta|^p | \mathcal{F}_{0, \vartheta}^B ] \leq \xi_L^p h^{p/2}.$$

**Remark 5.3.6.** Let  $\vartheta \leq \theta$   $\mathbb{P}$ -a.s. be two stopping times with values in  $\pi$  and  $\bar{Z}_{t_i}$  be the best approximation of  $(Z_t)_{t_i \leq t \leq t_{i+1}}$  by  $\mathcal{F}_{t_i}$ -measurable random variable in the following sense

$$\bar{Z}_{t_i} := h^{-1} \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} Z_s ds \right], \quad i < N. \quad (5.3.18)$$

Then, recalling that  $t_{i+1} - t_i = h$ , it follows from (5.3.18), (5.3.15) and Jensen's inequality that

$$\begin{aligned} \mathbb{E} \left[ \int_\vartheta^\theta \|\bar{Z}_{\varphi(s)} - Z_{\varphi(s)}^N\|^2 ds \right] &= \sum_{i < N} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{\vartheta \leq t_i \leq \theta\}} \left\| \mathbb{E}_{t_i} \left[ h^{-1} \int_{t_i}^{t_{i+1}} (Z_u - Z_u^N) du \right] \right\|^2 ds \right] \\ &\leq \sum_{i < N} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{\vartheta \leq t_i \leq \theta\}} h^{-1} \int_{t_i}^{t_{i+1}} \|Z_u - Z_u^N\|^2 du ds \right] \\ &\leq \mathbb{E} \left[ \int_\vartheta^\theta \|Z_s - Z_s^N\|^2 ds \right] \end{aligned} \quad (5.3.19)$$

Observe that the above inequality does not apply if  $\vartheta$  and  $\theta$  do not take values in  $\pi$ . This explains why it is easier to work with  $\tau_+$ , the next time after  $\tau$  in the grid  $\pi$  such that  $\tau_+ := \inf\{t \in \pi : \tau \leq t\}$ , instead of  $\tau$ , that is, work on  $\text{Err}(h)_{\tau_+ \wedge \bar{\tau}}^2$  instead of  $\text{Err}(h)_{\tau \wedge \bar{\tau}}^2$ .

Now we state an upper bound result for some stopping time  $\theta$  with values in  $\pi$ .

**Theorem 5.3.3.** *Assume that Assumptions (MHL), (MHD) and (MHT) hold, and define*

$$\mathcal{R}(Y)_{\mathcal{S}^2}^\pi := \max_{i < N} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} |Y_t - Y_{t_i}|^2 \right], \quad \mathcal{R}(Z)_{\mathcal{H}^2}^\pi := \mathbb{E} \left[ \int_0^T \|Z_t - \bar{Z}_{\varphi(t)}\|^2 dt \right]$$

Then for all stopping times  $\theta$  with values in  $\pi$ , we have

$$\begin{aligned} \text{Err}(h)_\theta^2 &\leq C_L \left( h + \mathbb{E}[|Y_\theta - Y_\theta^N|^2] + \mathcal{R}(Y)_{\mathcal{S}^2}^\pi + \mathcal{R}(Z)_{\mathcal{H}^2}^\pi \right. \\ &\quad \left. + \mathbb{E} \left[ \int_0^T \|Z_t - \bar{Z}_{\psi(t)}\|^2 dt \right] + \mathbb{E} \left[ \int_{\bar{\tau} \wedge \tau \wedge \theta}^{(\bar{\tau} \vee \tau) \wedge \theta} (\xi_L + \mathbf{1}_{\{\bar{\tau} < \tau\}} \|Z_s\|^2) ds \right] \right). \end{aligned} \quad (5.3.20)$$

**Proof.** The equations (5.3.2) and (5.3.14), the generalized Ito's lemma (see Lemma 1.3 in [109]) to  $(Y - Y^N)^2$  on  $[t \wedge \theta, t_{i+1} \wedge \theta]$  for  $t \in [t_i, t_{i+1}]$  and  $i < N$ , and taking expectation yield to

$$\begin{aligned} \Delta_{t, t_{i+1}}^\theta &:= \mathbb{E}[|Y_{t \wedge \theta} - Y_{t \wedge \theta}^N|^2] + \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \|Z_s - Z_s^N\|^2 ds \\ &= \mathbb{E}[|Y_{t_{i+1} \wedge \theta} - Y_{t_{i+1} \wedge \theta}^N|^2] + \mathbb{E} \left[ 2 \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} (Y_s - Y_s^N) (\mathbf{1}_{\{s < \tau\}} f(\Theta_s) - \mathbf{1}_{\{s < \bar{\tau}\}} f(\Theta_{\varphi(s)}^N)) ds \right] \\ &\quad + \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \|\mathbf{1}_{\{s < \tau\}} g(\Theta_s) - \mathbf{1}_{\{s < \bar{\tau}\}} g(\Theta_{\psi(s)}^N)\|^2 ds \right], \end{aligned}$$

where  $\Theta_s := (X_s, Y_s, Z_s)$ . Using the fact that  $\mathbf{1}_{\{s < \tau\}} \leq \mathbf{1}_{\{s < \bar{\tau}\}} + \mathbf{1}_{\{\tau \leq s < \bar{\tau}\}} + \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}}$  and the inequality  $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ , we then deduce that for  $\varepsilon > 0$  to be chosen later,

$$\begin{aligned} \Delta_{t, t_{i+1}}^\theta &\leq \mathbb{E}[|Y_{t_{i+1} \wedge \theta} - Y_{t_{i+1} \wedge \theta}^N|^2] + \varepsilon \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} |Y_s - Y_s^N|^2 ds \right] \\ &\quad + \varepsilon^{-1} \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{s < \bar{\tau}\}} (f(\Theta_s) - f(\Theta_{\varphi(s)}^N))^2 ds + \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} (f(\Theta_s))^2 ds \right] \\ &\quad + \varepsilon^{-1} \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\tau \leq s < \bar{\tau}\}} (f(\Theta_s))^2 ds \right] + \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{s < \bar{\tau}\}} \|g(\Theta_s) - g(\Theta_{\psi(s)}^N)\|^2 ds \right] \\ &\quad + \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|g(\Theta_s)\|^2 ds \right] + \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\tau \leq s < \bar{\tau}\}} \|g(\Theta_s)\|^2 ds \right]. \end{aligned}$$

Recall from Remark 5.3.5 that  $Z = 0$  on  $]\tau, T]$ . Since  $Y_t = \Phi(\tau, X_\tau)$  on  $\{t \geq \tau\}$ , we then deduce from the Lipschitz continuous assumption (MHL) that

$$\begin{aligned} \Delta_{t, t_{i+1}}^\theta &\leq \mathbb{E}[|Y_{t_{i+1} \wedge \theta} - Y_{t_{i+1} \wedge \theta}^N|^2] + \varepsilon \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} |Y_s - Y_s^N|^2 ds \right] \\ &\quad + C_L \varepsilon^{-1} \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{s < \bar{\tau}\}} (|X_s - X_{\varphi(s)}^N|^2 + |Y_s - Y_{\varphi(s)}^N|^2 + \|Z_s - Z_{\varphi(s)}^N\|^2) ds \right] \\ &\quad + C_L (\varepsilon^{-1} + 1) \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} (|X_s|^2 + |Y_s|^2) ds \right] \\ &\quad + C_L (\varepsilon^{-1} + 1) \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\tau \leq s < \bar{\tau}\}} (|X_\tau|^2 + |\Phi(\tau, X_\tau)|^2) ds \right] \\ &\quad + (C_L \varepsilon^{-1} + \alpha) \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|Z_s\|^2 ds \right] \\ &\quad + \mathbb{E} \left[ \int_{t \wedge \theta}^{t_{i+1} \wedge \theta} \mathbf{1}_{\{s < \bar{\tau}\}} (C_L |X_s - X_{\psi(s)}^N|^2 + C_L |Y_s - Y_{\psi(s)}^N|^2 + \alpha \|Z_s - Z_{\psi(s)}^N\|^2) ds \right]. \end{aligned}$$

Now, appealing to Proposition 5.3.1 yields to

$$\begin{aligned}
 \Delta_{t,t_{i+1}}^\theta &\leq \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - Y_{t_{i+1}\wedge\theta}^N|^2] + \varepsilon \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} |Y_s - Y_s^N|^2 ds\right] \\
 &+ C_L \varepsilon^{-1} \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (h + |Y_s - Y_{\varphi(s)}|^2 + |Y_{\varphi(s)} - Y_{\varphi(s)}^N|^2 + \|Z_s - \bar{Z}_{\varphi(s)}\|^2 + \|\bar{Z}_{\varphi(s)} - Z_{\varphi(s)}^N\|^2) ds\right] \\
 &+ C_L(\varepsilon^{-1} + 1) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\tau\wedge\tau \leq s < \tau\vee\bar{\tau}\}} \xi_L ds\right] + (C_L \varepsilon^{-1} + \alpha) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|Z_s\|^2 ds\right] \\
 &+ \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (C_L h + C_L |Y_s - Y_{\psi(s)}|^2 + C_L |Y_{\psi(s)} - Y_{\psi(s)}^N|^2 + \alpha \|Z_s - \bar{Z}_{\psi(s)}\|^2 + \alpha \|\bar{Z}_{\psi(s)} - Z_{\psi(s)}^N\|^2) ds\right].
 \end{aligned}$$

Next, we obtain from the definition of  $\varphi$

$$\begin{aligned}
 \Delta_{t,t_{i+1}}^\theta &\leq \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - Y_{t_{i+1}\wedge\theta}^N|^2] + \varepsilon \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} |Y_s - Y_s^N|^2 ds\right] \\
 &+ C_L(\varepsilon^{-1} + 1) \mathbb{E}[h|Y_{t_i\wedge\theta} - Y_{t_i\wedge\theta}^N|^2 + h|Y_{t_{i+1}\wedge\theta} - Y_{t_{i+1}\wedge\theta}^N|^2 + \int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (|Y_s - Y_{\varphi(s)}|^2 + |Y_s - Y_{\psi(s)}|^2) ds] \\
 &+ C_L(\varepsilon^{-1} + 1) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} h ds\right] + C_L \varepsilon^{-1} \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (\|Z_s - \bar{Z}_{\varphi(s)}\|^2 + \|\bar{Z}_{\varphi(s)} - Z_{\varphi(s)}^N\|^2) ds\right] \\
 &+ \alpha \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (\|Z_s - \bar{Z}_{\psi(s)}\|^2 + \|\bar{Z}_{\psi(s)} - Z_{\psi(s)}^N\|^2) ds\right] \\
 &+ C_L(\varepsilon^{-1} + 1) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\tau\wedge\tau \leq s < \tau\vee\bar{\tau}\}} \xi_L ds\right] + (C_L \varepsilon^{-1} + \alpha) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|Z_s\|^2 ds\right].
 \end{aligned} \tag{5.3.21}$$

It then follows from Gronwall's lemma that

$$\begin{aligned}
 \mathbb{E}[|Y_{t\wedge\theta} - Y_{t\wedge\theta}^N|^2] &\leq (1 + C_L(\varepsilon^{-1} + 1)h + C_L^\varepsilon h) \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - Y_{t_{i+1}\wedge\theta}^N|^2] \\
 &+ (C_L(\varepsilon^{-1} + 1) + C_L^\varepsilon h) \mathbb{E}[h|Y_{t_i\wedge\theta} - Y_{t_i\wedge\theta}^N|^2 + \int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (|Y_s - Y_{\varphi(s)}|^2 + |Y_s - Y_{\psi(s)}|^2) ds] \\
 &+ (C_L(\varepsilon^{-1} + 1) + C_L^\varepsilon h) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} h ds\right] \\
 &+ (C_L \varepsilon^{-1} + C_L^\varepsilon h) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (\|Z_s - \bar{Z}_{\varphi(s)}\|^2 + \|\bar{Z}_{\varphi(s)} - Z_{\varphi(s)}^N\|^2) ds\right] \\
 &+ (\alpha + C_L^\varepsilon h) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (\|Z_s - \bar{Z}_{\psi(s)}\|^2 + \|\bar{Z}_{\psi(s)} - Z_{\psi(s)}^N\|^2) ds\right] \\
 &+ (C_L(\varepsilon^{-1} + 1) + C_L^\varepsilon h) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\tau\wedge\tau \leq s < \tau\vee\bar{\tau}\}} \xi_L ds\right] \\
 &+ (C_L \varepsilon^{-1} + \alpha + C_L^\varepsilon h) \mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|Z_s\|^2 ds\right].
 \end{aligned} \tag{5.3.22}$$

Then, by taking  $t = t_i$  in (5.3.21), using (5.3.22) to estimate the second term in the right-hand side of (5.3.21) and recalling Remark 5.3.5 we have for  $\varepsilon > 0$  sufficiently large, depending on the constants  $C_L$ , and  $h$  small

$$\begin{aligned}
 \Delta_{t_i,t_{i+1}}^\theta &\leq (1 + C_L h) \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - Y_{t_{i+1}\wedge\theta}^N|^2] \\
 &+ C_L \mathbb{E}\left[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta} (h + |Y_s - Y_{\varphi(s)}|^2 + |Y_s - Y_{\psi(s)}|^2) ds\right] \\
 &+ C_L \mathbb{E}\left[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta} \|Z_s - \bar{Z}_{\varphi(s)}\|^2 ds\right] + C_L \mathbb{E}\left[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta} \|Z_s - \bar{Z}_{\psi(s)}\|^2 ds\right] \\
 &+ C_L \mathbb{E}\left[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\tau\wedge\tau \leq s < \tau\vee\bar{\tau}\}} \xi_L ds\right] + C_L \mathbb{E}\left[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|Z_s\|^2 ds\right].
 \end{aligned}$$

Thus, from the following estimate

$$\begin{aligned} \mathbb{E}[|Y_s - Y_{\psi(s)}|^2] &\leq \mathbb{E}\left[\sup_{t_i \leq s \leq t_{i+1}} |Y_s - Y_{\psi(s)}|^2\right] \\ &\leq C_L(1 + |x|)h. \end{aligned}$$

We conclude that

$$\begin{aligned} \Delta^\theta &:= \max_{i < N} \mathbb{E}[|Y_{t_i \wedge \theta} - Y_{t_i \wedge \theta}^N|^2 + \int_0^\theta \|Z_s - Z_s^N\|^2 ds] \\ &\leq C_L(\mathbb{E}[|Y_\theta - Y_\theta^N|^2] + h + \mathcal{R}(Y)_{\mathcal{S}^2}^\pi + \mathcal{R}(Z)_{\mathcal{H}^2}^\pi + E[\int_0^T \|Z_t - \bar{Z}_{\psi(t)}\|^2 dt]) \\ &\quad + C_L \mathbb{E}[\xi_L |\bar{\tau} \wedge \theta - \tau \wedge \theta| + \int_0^\theta \mathbf{1}_{\{\bar{\tau} \leq s < \tau\}} \|Z_s\|^2 ds]. \end{aligned}$$

We finish the proof by using again Remark 5.3.5 to obtain

$$\begin{aligned} &\mathbb{E}\left[\int_0^\theta \|Z_s - Z_{\varphi(s)}^N\|^2 ds\right] \\ &\leq C_L \left( \mathbb{E}\left[\int_0^\theta \|\bar{Z}_{\varphi(s)} - Z_{\varphi(s)}^N\|^2 ds\right] + \mathbb{E}\left[\int_0^T \|Z_s - \bar{Z}_{\varphi(s)}\|^2 ds\right] \right) \\ &\leq C_L \left( \mathbb{E}\left[\int_0^\theta \|Z_s - Z_s^N\|^2 ds\right] + \mathcal{R}(Z)_{\mathcal{H}^2}^\pi \right), \end{aligned} \tag{5.3.23}$$

which implies the required result, by the definition of  $\text{Err}(h)_\theta^2$  in (5.3.3).  $\square$

**Proposition 5.3.2.** *Let Assumptions (MHL), (MHD) and (MHT) hold. There then exist  $C_L > 0$  and a positive random variable  $\xi_L$  satisfying  $\mathbb{E}[(\xi_L)^p] \leq C_L^p$  for all  $p \geq 2$  such that*

$$\begin{aligned} \text{Err}(h)_T^2 &\leq C_L \left( h + \mathcal{R}(Y)_{\mathcal{S}^2}^\pi + \mathcal{R}(Z)_{\mathcal{H}^2}^\pi + \mathbb{E}[\xi_L |\tau - \bar{\tau}| + \mathbf{1}_{\{\bar{\tau} < \tau\}} \int_{\bar{\tau}}^\tau \|Z_s\|^2 ds] \right. \\ &\quad \left. + \mathbb{E}\left[\int_0^T \|Z_t - \bar{Z}_{\psi(t)}\|^2 dt\right] \right). \end{aligned} \tag{5.3.24}$$

and

$$\begin{aligned} \text{Err}(h)_{\tau \wedge \bar{\tau}}^2 &\leq \text{Err}(h)_{\tau_+ \wedge \bar{\tau}}^2 \leq C_L \left( h + \mathcal{R}(Y)_{\mathcal{S}^2}^\pi + \mathcal{R}(Z)_{\mathcal{H}^2}^\pi + \mathbb{E}[\xi_L |\tau - \bar{\tau}| + \mathbf{1}_{\{\bar{\tau} < \tau\}} \int_{\bar{\tau}}^\tau \|Z_s\|^2 ds] \right. \\ &\quad \left. + \mathbb{E}\left[\int_0^T \|Z_t - \bar{Z}_{\psi(t)}\|^2 dt\right] \right), \end{aligned} \tag{5.3.25}$$

where we recall  $\tau_+$  is the next time after  $\tau$  in the grid  $\pi$  such that  $\tau_+ := \inf\{t \in \pi : \tau \leq t\}$ .

**Remark 5.3.7.** *Note that we shall control  $\text{Err}(h)_{\tau \wedge \bar{\tau}}^2$  through the slightly stronger term  $\text{Err}(h)_{\tau_+ \wedge \bar{\tau}}^2$ . This will allow us to work with stopping times with values in the grid  $\pi$  in order to be able to apply (5.3.19), which will be technically easier.*

**Proof.**

(i) First to prove (5.3.24), it suffices to apply Theorem 5.3.3 for  $\theta = T$  and observe that the Lipschitz continuity of  $\Phi$  implies that

$$\begin{aligned} &\mathbb{E}[|\Phi(\tau, X_\tau) - \Phi(\bar{\tau}, X_{\bar{\tau}}^N)|^2] \\ &\leq C_L \mathbb{E}[|\tau - \bar{\tau}|^2 + |X_{\bar{\tau}} - X_{\bar{\tau}}^N|^2 + \left| \int_{\tau \wedge \bar{\tau}}^{\tau \vee \bar{\tau}} b(X_s) ds + \int_{\tau \wedge \bar{\tau}}^{\tau \vee \bar{\tau}} \sigma(X_s) dB_s \right|^2], \end{aligned}$$



where  $|\tau - \bar{\tau}|^2 \leq T|\tau - \bar{\tau}|$ ,  $\mathbb{E}[|X_{\bar{\tau}} - X_{\bar{\tau}}^N|^2] \leq C_L h$  by Proposition 5.3.1 and

$$\mathbb{E}\left[\left|\int_{\tau \wedge \bar{\tau}}^{\tau \vee \bar{\tau}} b(X_s) ds + \int_{\tau \wedge \bar{\tau}}^{\tau \vee \bar{\tau}} \sigma(X_s) dB_s\right|^2\right] \leq \mathbb{E}[\xi_L |\tau - \bar{\tau}|]$$

by Doob's inequality, **(MHD)** and Proposition 5.3.1 again.

(ii) We now prove the upper bound (5.3.25). We have by applying Theorem 5.3.3  $\theta = \tau_+ \wedge \bar{\tau}$

$$\text{Err}(h)_{\tau_+ \wedge \bar{\tau}}^2 \leq C_L (h + \mathbb{E}[|Y_{\tau_+ \wedge \bar{\tau}} - Y_{\tau_+ \wedge \bar{\tau}}^N|^2] + \mathcal{R}(Y)_{S^2}^\pi + \mathcal{R}(Z)_{\mathcal{H}^2}^\pi).$$

It remains to show that

$$\mathbb{E}[|Y_{\tau_+ \wedge \bar{\tau}} - Y_{\tau_+ \wedge \bar{\tau}}^N|^2] \leq C_L (h + \mathbb{E}[\xi_L |\tau - \bar{\tau}| + \mathbf{1}_{\{\bar{\tau} < \tau\}} \int_{\bar{\tau}}^{\tau} \|Z_s\|^2 ds]). \quad (5.3.26)$$

Observe that by (5.3.2) and (5.3.14)

$$\begin{aligned} Y_{\tau_+ \wedge \bar{\tau}} - Y_{\tau_+ \wedge \bar{\tau}}^N &= \Phi(\tau, X_\tau) - \Phi(\bar{\tau}, X_{\bar{\tau}}^N) \\ &+ \mathbf{1}_{\{\tau_+ < \bar{\tau}\}} \left( \int_{\tau_+}^{\bar{\tau}} f(X_{\varphi(s)}^N, Y_{\varphi(s)}^N, Z_{\varphi(s)}^N) ds + \int_{\tau_+}^{\bar{\tau}} g(X_{\psi(s)}^N, Y_{\psi(s)}^N, Z_{\psi(s)}^N) d\overleftarrow{W}_s - \int_{\tau_+}^{\bar{\tau}} Z_s^N dB_s \right) \\ &+ \mathbf{1}_{\{\bar{\tau} < \tau_+\}} \left( \int_{\bar{\tau}}^{\tau} f(X_s, Y_s, Z_s) ds + \int_{\bar{\tau}}^{\tau} g(X_s, Y_s, Z_s) d\overleftarrow{W}_s - \int_{\bar{\tau}}^{\tau} Z_s dB_s \right). \end{aligned} \quad (5.3.27)$$

Then

$$\begin{aligned} \mathbb{E}[|Y_{\tau_+ \wedge \bar{\tau}} - Y_{\tau_+ \wedge \bar{\tau}}^N|^2] &\leq \mathbb{E}[|\Phi(\tau, X_\tau) - \Phi(\bar{\tau}, X_{\bar{\tau}}^N)|^2] \\ &+ \mathbb{E}\left[\mathbf{1}_{\{\tau_+ < \bar{\tau}\}} \left| \int_{\tau_+}^{\bar{\tau}} f(X_{\varphi(s)}^N, Y_{\varphi(s)}^N, Z_{\varphi(s)}^N) ds \right|^2\right] + \mathbb{E}\left[\mathbf{1}_{\{\tau_+ < \bar{\tau}\}} \left| \int_{\tau_+}^{\bar{\tau}} g(X_{\psi(s)}^N, Y_{\psi(s)}^N, Z_{\psi(s)}^N) d\overleftarrow{W}_s \right|^2\right] \\ &+ \mathbb{E}\left[\mathbf{1}_{\{\bar{\tau} < \tau_+\}} \left| \int_{\bar{\tau}}^{\tau} f(X_s, Y_s, Z_s) ds \right|^2\right] + \mathbb{E}\left[\mathbf{1}_{\{\bar{\tau} < \tau_+\}} \left| \int_{\bar{\tau}}^{\tau} g(X_s, Y_s, Z_s) d\overleftarrow{W}_s \right|^2\right]. \end{aligned} \quad (5.3.28)$$

We start with the first term in the right hand side of (5.3.27). By using **(MHD)**, **(MHL)**, **(MHT)** and Proposition 5.3.1 and applying Itô's lemma to  $(\Phi(t, X_t))_{t \geq 0}$  between  $\bar{\tau}$  and  $\tau$ , we easily check that

$$\begin{aligned} \mathbb{E}[|\Phi(\tau, X_\tau) - \Phi(\bar{\tau}, X_{\bar{\tau}}^N)|^2] &\leq C_L (\mathbb{E}[|X_{\bar{\tau}} - X_{\bar{\tau}}^N|^2] + \mathbb{E}\left[\left| \int_{\bar{\tau}}^{\tau} \mathcal{L}\Phi(s, X_s) ds \right|^2\right]) \\ &\leq C_L (\mathbb{E}[|X_{\bar{\tau}} - X_{\bar{\tau}}^N|^2] + \mathbb{E}[\xi_L |\tau - \bar{\tau}|]). \end{aligned}$$

Then, by appealing to **(MHD)** and Proposition 5.3.1 we conclude that

$$\mathbb{E}[|\Phi(\tau, X_\tau) - \Phi(\bar{\tau}, X_{\bar{\tau}}^N)|^2] \leq C_L (h + \mathbb{E}[\xi_L |\tau - \bar{\tau}|]). \quad (5.3.29)$$

For the second term in (5.3.27), it follows from Jensen's inequality, the isometry property, the Lipschitz continuous assumption **(MHL)**, Lemma 5.3.1 and Proposition 5.3.1 that

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}_{\{\tau_+ < \bar{\tau}\}} \left| \int_{\tau_+}^{\bar{\tau}} f(X_{\varphi(s)}^N, Y_{\varphi(s)}^N, Z_{\varphi(s)}^N) ds \right|^2\right] + \mathbb{E}\left[\mathbf{1}_{\{\tau_+ < \bar{\tau}\}} \left| \int_{\tau_+}^{\bar{\tau}} g(X_{\psi(s)}^N, Y_{\psi(s)}^N, Z_{\psi(s)}^N) d\overleftarrow{W}_s \right|^2\right] \\ &\leq \mathbb{E}\left[|\bar{\tau} - \tau_+| \int_{\tau_+}^{\bar{\tau}} |f(X_{\varphi(s)}^N, Y_{\varphi(s)}^N, Z_{\varphi(s)}^N)|^2 ds\right] + \mathbb{E}\left[\int_{\tau_+}^{\bar{\tau}} |g(X_{\psi(s)}^N, Y_{\psi(s)}^N, Z_{\psi(s)}^N)|^2 ds\right] \\ &\leq C_L \mathbb{E}\left[\int_{\tau_+}^{\bar{\tau}} (|X_{\varphi(s)}^N|^2 + |Y_{\varphi(s)}^N|^2 + \|Z_{\varphi(s)}^N\|^2 + |X_{\psi(s)}^N|^2 + |Y_{\psi(s)}^N|^2 + \|Z_{\psi(s)}^N\|^2) ds\right] \\ &\leq C_L \mathbb{E}\left[\xi_L (|\bar{\tau} - \tau| + |\tau - \tau_+|)\right] \\ &\leq C_L \mathbb{E}[h + \xi_L |\bar{\tau} - \tau|]. \end{aligned} \quad (5.3.30)$$

The last term is easily controlled by using the same previous calculations.

$$\begin{aligned}
 & \mathbb{E} \left[ \mathbf{1}_{\{\bar{\tau} < \tau_+\}} \left| \int_{\bar{\tau}}^{\tau} f(X_s, Y_s, Z_s) ds \right|^2 \right] + \mathbb{E} \left[ \mathbf{1}_{\{\bar{\tau} < \tau_+\}} \left| \int_{\bar{\tau}}^{\tau} g(X_s, Y_s, Z_s) d\overleftarrow{W}_s \right|^2 \right] \\
 & \leq C_L \left( \mathbb{E} \left[ |\tau - \bar{\tau}| \int_{\bar{\tau}}^{\tau} |f(X_s, Y_s, Z_s)|^2 ds \right] + \mathbb{E} \left[ \mathbf{1}_{\{\bar{\tau} < \tau_+\}} \int_{\bar{\tau}}^{\tau} |g(X_s, Y_s, Z_s)|^2 ds \right] \right) \\
 & \leq C_L \left( \mathbb{E} \left[ |\tau - \bar{\tau}| \int_{\bar{\tau}}^{\tau} C_L (|X_s|^2 + |Y_s|^2 + \|Z_s\|^2) ds \right] + \mathbb{E} \left[ \mathbf{1}_{\{\bar{\tau} < \tau_+\}} \int_{\bar{\tau}}^{\tau} (|X_s|^2 + |Y_s|^2 + \|Z_s\|^2) ds \right] \right) \\
 & \leq C_L \mathbb{E} [|\tau - \bar{\tau}|^2 \xi_L] + \mathbb{E} [|\tau - \bar{\tau}| \int_{\bar{\tau}}^{\tau} C_L \|Z_s\|^2 ds] + C_L \mathbb{E} [\xi_L |\bar{\tau} - \tau| + \mathbf{1}_{\{\bar{\tau} < \tau\}} \int_{\bar{\tau}}^{\tau} \|Z_s\|^2 ds] \\
 & \leq C_L \mathbb{E} [\xi_L |\bar{\tau} - \tau| + \mathbf{1}_{\{\bar{\tau} < \tau\}} \int_{\bar{\tau}}^{\tau} \|Z_s\|^2 ds]. \tag{5.3.31}
 \end{aligned}$$

Finally, we finish the proof of (5.3.27) by combining the three estimates.  $\square$

Our next result concerns the regularity of  $(Y, Z)$  which was proved in [4]:

**Theorem 5.3.4.** *Let the Assumptions (D), (MHT), (MHL) and (MHD) hold. Then*

$$\mathcal{R}(Y)_{\mathcal{S}^2}^{\pi} + \mathcal{R}(Z)_{\mathcal{H}^2}^{\pi} \leq C_L h \quad \text{and} \quad \mathbb{E} \left[ \int_0^T \|Z_t - \bar{Z}_{\psi(t)}\|^2 dt \right] \leq C_L h. \tag{5.3.32}$$

Combining the estimates (5.3.8) and (5.3.32), we finally obtain our main result, which provides an upper bound for the convergence rate of  $\text{Err}(h)_{\tau_+ \wedge \bar{\tau}}^2$  (and thus for  $\text{Err}(h)_{\bar{\tau} \wedge \bar{\tau}}^2$  and  $\text{Err}(h)_T^2$ ).

**Theorem 5.3.5.** *Let the Assumptions (D), (MHT), (MHL) and (MHD) hold. Then, for each  $\varepsilon \in (0, 1/2)$ , there exists  $C_L^\varepsilon > 0$  such that*

$$\text{Err}(h)_{\tau_+ \wedge \bar{\tau}}^2 \leq C_L h^{1/2} \quad \text{and} \quad \text{Err}(h)_T^2 \leq C_L h^{1/2} \tag{5.3.33}$$

## 5.4 Semilinear Stochastic PDEs with Dirichlet null condition

The aim of this section is to give a Feynman-Kac's formula for the weak solution of a class of semilinear SPDEs with Dirichlet null condition on the boundary via the associated Markovian class of BDSDEs with random terminal time studied in the section 5.2. Indeed, for a given open connected domain  $\mathcal{O}$  of  $\mathbb{R}^d$ , we are interested in the following semilinear SPDEs :

$$\begin{cases} du_t + \mathcal{L}u_t dt + f(t, x, u_t, D_\sigma u_t) dt + g(t, x, u_t, D_\sigma u_t) d\overleftarrow{W}_t = 0, & \forall 0 \leq t \leq T, \forall x \in \mathcal{O}, \\ u(T, x) = \Phi(x), & \forall x \in \mathcal{O} \\ u(t, x) = 0, & \forall 0 \leq t \leq T, \forall x \in \partial\mathcal{O}, \end{cases} \tag{5.4.1}$$

where  $D_\sigma := \nabla u \sigma$  and  $\mathcal{L}$  is the second order differential operator which is defined component by component with

$$\mathcal{L}\varphi(x) = \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) \tag{5.4.2}$$

and  $a := \sigma \sigma^*$ .

### 5.4.1 Definitions and formulation

Let us first introduce some notations:

-  $C_{l,b}^n(\mathbb{R}^p, \mathbb{R}^q)$  the set of  $C^n$ -functions which grow at most linearly at infinity and whose partial

## 5.4. Semilinear Stochastic PDEs with Dirichlet null condition 129

derivatives of order less than or equal to  $n$  are bounded.

-  $\mathbf{L}^2(\mathcal{O})$  will be a Hilbert  $L^2$ -space of our framework. We employ the following notation for its scalar product and its norm,

$$(u, v) = \int_{\mathcal{O}} u(x) v(x) dx, \quad \|u\|_2 = \left( \int_{\mathcal{O}} u^2(x) dx \right)^{\frac{1}{2}}.$$

Our evolution problem will be considered over a fixed time interval  $[0, T]$  and the norm for an element of  $\mathbf{L}^2([0, T] \times \mathcal{O})$  will be denoted by

$$\|u\|_{2,2} = \left( \int_0^T \int_{\mathcal{O}} |u(t, x)|^2 dx dt \right)^{\frac{1}{2}}.$$

We assume the following hypotheses :

**Assumption (MHD')** The coefficients of the second order differential operator  $\mathcal{L}$  satisfy:

- $b$  is a bounded function and belongs to  $C_{l,b}^2(\mathbb{R}^d, \mathbb{R}^d)$ .
- $\sigma \in C_{l,b}^3(\mathbb{R}^d, \mathbb{R}^{k \times d})$  and satisfy the ellipticity condition (5.3.3).

**Assumption (MHT')**  $\Phi \in \mathbf{L}^2(\mathcal{O}; \mathbb{R}^k)$  with polynomial growth, namely there exists  $C > 0$  and  $p \in \mathbb{N}$  such that  $|\Phi(x)| \leq C(1 + |x|^p)$ .

The space of test functions which we employ in the definition of weak solutions of the evolution equations (5.4.1) is  $\mathcal{D} := \mathcal{C}^\infty([0, T]) \otimes \mathcal{C}_c^\infty(\mathcal{O})$ , where  $\mathcal{C}^\infty([0, T])$  denotes the space of real functions which can be extended as infinite differentiable functions in the neighborhood of  $[0, T]$  and  $\mathcal{C}_c^\infty(\mathcal{O})$  is the space of infinite differentiable functions with compact support in  $\mathcal{O}$ . We denote by  $\mathcal{H}$  the space of  $\mathcal{F}_{t,T}^W$ -progressively measurable processes  $(u_t)$  with valued in the Dirichlet space  $H_0^1(\mathcal{O})$  where

$$H_0^1(\mathcal{O}) := \{v \in \mathbf{L}^2(\mathcal{O}) \mid \nabla v \sigma \in \mathbf{L}^2(\mathcal{O})\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \mathbb{E} \left[ \sup_{0 \leq s \leq T} \|u_s\|_2^2 + \int_0^T \int_{\mathcal{O}} |\nabla u_s(x) \sigma(x)|^2 ds dx \right],$$

where we denote the gradient by  $\nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x))$ .

**Definition 5.4.1.** We say that  $u \in \mathcal{H}$  is a weak solution of the SPDE (5.4.1) if the following relation holds for each  $\Psi \in \mathcal{D}$ ,

$$\begin{aligned} & \int_t^T \int_{\mathcal{O}} u(s, x) \partial_s \Psi(s, x) dx ds - \int_{\mathcal{O}} \Phi(x) \Psi(T, x) dx + \int_{\mathcal{O}} u(t, x) \Psi(t, x) dx - \int_t^T \int_{\mathcal{O}} u(s, x) \mathcal{L}^* u(s, x) dx ds \\ &= \int_t^T \int_{\mathcal{O}} \Psi(s, x) f(s, x, u(s, x), D_\sigma u(s, x)) dx ds + \int_t^T \int_{\mathcal{O}} \Psi(s, x) g(s, x, u(s, x), D_\sigma u(s, x)) dx d\overleftarrow{W}_s. \end{aligned} \tag{5.4.3}$$

where

$$(u(s, \cdot), \mathcal{L}^* \Psi(s, \cdot)) := \int_{\mathcal{O}} D_\sigma u(s, x) D_\sigma \Psi(s, x) dx + \int_{\mathcal{O}} u(s, x) \operatorname{div}((b - \tilde{A}) \Psi(s, x)) dx,$$

$$\text{and } \tilde{A}_i =: \frac{1}{2} \sum_{k=1}^d \frac{\partial a_{k,i}}{\partial x_k}.$$

The existence and uniqueness of weak solution for such SPDEs with null Dirichlet condition is ensured by Denis and Stoica (Theorem 4 in [43]). Indeed, we can rewrite the second order differential operator  $\mathcal{L}$  as following:

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d \partial_i (a^{ij}(x) \partial_j) + \sum_{i=1}^d (b^i(x) - \frac{1}{2} \partial_i a^{ij}(x)) \partial_i. \quad (5.4.4)$$

Therefore, since  $b$  and  $\nabla a$  are bounded, the second term in the right hand side of (5.4.4) may be considered as an extra term in the nonlinear term coefficient  $f$  which still satisfy the uniform Lipschitz continuous condition in  $u$  and  $D_\sigma u$ .

Motivated by developing Euler numerical scheme for such solution, we are now interested in giving the probabilistic interpretation for the solution of SPDEs (5.4.1) within the framework of BDSDE with random terminal time. Thus, this connection between SPDEs and BDSDEs will be established by means of stochastic flow technics.

### 5.4.2 Stochastic flow of diffeomorphism and random test functions

We are concerned in this paper with solving SPDEs by developing a stochastic flow method which was first introduced in Kunita [76], and Bally, Matoussi [10]. We recall that  $\{X_s^{t,x}, t \leq s \leq T\}$  is the diffusion process starting from  $x$  at time  $t$  and is the strong solution of the equation:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dB_r. \quad (5.4.5)$$

The existence and uniqueness of this solution was proved in Kunita [76]. Moreover, we have the following properties:

**Proposition 5.4.1.** *For each  $t > 0$ , there exists a version of  $\{X_s^{t,x}; x \in \mathbb{R}^d, s \geq t\}$  such that  $X_s^{t,\cdot}$  is a  $C^2(\mathbb{R}^d)$ -valued continuous process which satisfy the flow property:  $X_r^{t,x} = X_r^{s,x} \circ X_s^{t,x}$ ,  $0 \leq t < s < r$ . Furthermore, for all  $p \geq 2$ , there exists  $M_p$  such that for all  $0 \leq t < s$ ,  $x, x' \in \mathbb{R}^d$ ,  $h, h' \in \mathbb{R} \setminus \{0\}$ ,*

$$\begin{aligned} \mathbb{E}(\sup_{t \leq r \leq s} |X_r^{t,x} - x|^p) &\leq M_p(s-t)(1 + |x|^p), \\ \mathbb{E}(\sup_{t \leq r \leq s} |X_r^{t,x} - X_r^{t,x'} - (x - x')|^p) &\leq M_p(s-t)(|x - x'|^p), \\ \mathbb{E}(\sup_{t \leq r \leq s} |\Delta_h^i [X_r^{t,x} - x]|^p) &\leq M_p(s-t), \\ \mathbb{E}(\sup_{t \leq r \leq s} |\Delta_h^i X_r^{t,x} - \Delta_{h'}^i X_r^{t,x'}|^p) &\leq M_p(s-t)(|x - x'|^p + |h - h'|^p), \end{aligned}$$

where  $\Delta_h^i g(x) = \frac{1}{h}(g(x + he_i) - g(x))$ , and  $(e_1, \dots, e_d)$  is an orthonormal basis of  $\mathbb{R}^d$ .

Under regular conditions Assumption (MHD') on the diffusion, it is known that the stochastic flow associated to a continuous SDE satisfies the homeomorphic property (see Kunita [76]). We have the following result where the proof can be found in [76].

**Proposition 5.4.2.** *Let Assumption (MHD') holds. Then  $\{X_s^{t,x}; x \in \mathbb{R}^d\}$  is a  $C^2$ -diffeomorphism a.s. stochastic flow. Moreover the inverse of the flow which denoted by  $\{X_{t,s}^{-1}(y); y \in \mathbb{R}^d\}$  satisfies the following backward SDE*

$$X_{t,s}^{-1}(y) = y - \int_t^s \widehat{b}(X_{r,s}^{-1}(y)) dr - \int_t^s \sigma(X_{r,s}^{-1}(y)) d\mathbb{B}_r \quad (5.4.6)$$

for any  $t < s$ , where

$$\widehat{b}(x) = b(x) - \sum_{i,j} \frac{\partial \sigma^j(x)}{\partial x_i} \sigma^{ij}(x). \quad (5.4.7)$$

## 5.4. Semilinear Stochastic PDEs with Dirichlet null condition 131

We denote by  $J(X_{t,s}^{-1}(x))$  the determinant of the Jacobian matrix of  $X_{t,s}^{-1}(x)$ , which is positive and  $J(X_{t,t}^{-1}(x)) = 1$ . For  $\varphi \in C_c^\infty(\mathbb{R}^d)$ , we define a process  $\varphi_t : \Omega \times [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$  by

$$\varphi_t(s, x) := \varphi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x)). \quad (5.4.8)$$

We know that for  $v \in \mathbf{L}^2(\mathbb{R}^d)$ , the composition of  $v$  with the stochastic flow is

$$(v \circ X_s^{t,\cdot}, \varphi) := (v, \varphi_t(s, \cdot)).$$

In fact, by a change of variable, we have (see Kunita [80], Bally and Matoussi [10])

$$(v \circ X_s^{t,\cdot}, \varphi) = \int_{\mathbb{R}^d} v(X_s^{t,x})\varphi(x)dx = \int_{\mathbb{R}^d} v(y)\varphi(X_{t,s}^{-1}(y))J(X_{t,s}^{-1}(y))dy = (v, \varphi_t(s, \cdot)).$$

Since  $(\varphi_t(s, x))_{t \leq s}$  is a process, we may not use it directly as a test function because

$\int_t^T (u(s, \cdot), \partial_s \varphi_t(s, \cdot))ds$  has no sense. However  $\varphi_t(s, x)$  is a semimartingale and we have the following decomposition of  $\varphi_t(s, x)$

**Lemma 5.4.1.** *For every function  $\varphi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\varphi_t(s, x) = \varphi(x) + \int_t^s \mathcal{L}^* \varphi_t(r, x)dr - \sum_{j=1}^d \int_t^s \left( \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma^{ij}(x)\varphi_t(r, x)) \right) dW_r^j, \quad (5.4.9)$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ .

We also need equivalence of norms result which plays an important role in the proof of the existence of the solution for SPDE as a connection between the functional norms and random norms. For continuous SDEs, this result was first proved by Barles and Lesigne [13] by using an analytic method and Bally and Matoussi [10] by a probabilistic method.

**Proposition 5.4.3.** *There exists two constants  $c > 0$  and  $C > 0$  such that for every  $t \leq s \leq T$  and  $\varphi \in L^1(\mathbb{R}^d)$ ,*

$$c \int_{\mathbb{R}^d} |\varphi(x)|dx \leq \int_{\mathbb{R}^d} \mathbb{E}(|\varphi(X_s^{t,x})|)dx \leq C \int_{\mathbb{R}^d} |\varphi(x)|dx. \quad (5.4.10)$$

Moreover, for every  $\Psi \in L^1([0, T] \times \mathbb{R}^d)$ ,

$$c \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)|dsdx \leq \int_{\mathbb{R}^d} \int_t^T \mathbb{E}(|\Psi(s, X_s^{t,x})|)dsdx \leq C \int_{\mathbb{R}^d} \int_t^T |\Psi(s, x)|dsdx. \quad (5.4.11)$$

We give now the following result which allows us to link by a natural way the solution of SPDE with the associated BDSDE. Roughly speaking, if we choose in the variational formulation (5.4.3) the random functions  $\varphi_t(\cdot, \cdot)$  defined by (5.4.8), as a test functions, then we obtain the associated BDSDE. In fact, this result plays the same role as Itô's formula used in [109] to relate the solution of some semilinear SPDEs with the associated BDSDEs:

**Proposition 5.4.4.** *Let Assumptions (MHT'), (MHL) and (MHD') hold and  $u \in \mathcal{H}$  be a weak solution of the SPDE (5.4.3) associated to  $(\Phi, f, g)$  on the hole domain  $\mathbb{R}^d$ , then for  $s \in [t, T]$  and  $\varphi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r, x)d\varphi_t(r, x)dx + (u(s, \cdot), \varphi_t(s, \cdot)) - (\Phi(\cdot), \varphi_t(T, \cdot)) - \int_{\mathbb{R}^d} \int_s^T u(r, x)\mathcal{L}^* \varphi_t(r, x)drdx \\ &= \int_{\mathbb{R}^d} \int_s^T f_r(x, u(r, x), D_\sigma u(r, x))\varphi_t(r, x)drdx + \int_{\mathbb{R}^d} \int_s^T g_r(x, u(r, x), D_\sigma u(r, x)\sigma(x))\varphi_t(r, x)d\overleftarrow{W}_r, dx, \end{aligned} \quad (5.4.12)$$

where  $\int_{\mathbb{R}^d} \int_s^T u(r, x)d\varphi_t(r, x)dx$  is well defined thanks to the semimartingale decomposition result (Lemma 5.4.1).

### 5.4.3 Probabilistic representation of the solution of SPDE

As introduced in the section 5.3, we consider now the Markovian BDSDE with random terminal time  $\tau^{t,x}$  which is the first exist time of the forward diffusion  $X^{t,x}$  from the domain  $\mathcal{O}$

$$\begin{aligned} Y_s^{t,x} &= \Phi(X_{T \wedge \tau^{t,x}}^{t,x}) + \int_s^T \mathbf{1}_{(\tau^{t,x} > r)} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dB_r \\ &+ \int_s^T \mathbf{1}_{(\tau^{t,x} > r)} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r. \end{aligned} \quad (5.4.13)$$

**Remark 5.4.1.** We have  $Y_s^{t,x} = Z_s^{t,x} = 0, \forall \tau^{t,x} \leq s \leq T$ . In fact, the process  $Z^{t,x}$  is the density which appears in the Ito's representation theorem of the random variable

$$\xi = \Phi(X_{T \wedge \tau^{t,x}}^{t,x}) + \int_s^T \mathbf{1}_{(\tau^{t,x} > r)} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr$$

But, the r.v  $\xi$  is  $\mathcal{F}_{\tau^{t,x}}^W$ -measurable, then  $Z_r^{t,x} = Z_r^{t,x} \mathbf{1}_{(\tau^{t,x} \geq r)}$ . Now, we look at (5.4.13) for  $T \geq s > \tau^{t,x}$ , all the terms in the right hand of (5.4.13) vanishes, then  $Y_s^{t,x}$  vanishes, for  $T \geq s > \tau^{t,x}$ .

The main result in this section is the following

**Theorem 5.4.1.** Assume (MHT'), (D), (MHL) and (MHD') hold and let  $\{(Y_s^{t,x}, Z_s^{t,x}), t \leq s \leq T\}$  be the solution of BDSDE (5.4.13). Then,  $u(t, x) := Y_t^{t,x}$ ,  $dt \otimes dx$ , a.e. is the unique solution of the SPDE (5.4.3) and

$$Y_s^{t,x} = u(s \wedge \tau^{t,x}, X_{s \wedge \tau^{t,x}}^{t,x}), \quad Z_s^{t,x} = D_\sigma u(s \wedge \tau^{t,x}, X_{s \wedge \tau^{t,x}}^{t,x}). \quad (5.4.14)$$

**Proof.** Step 1: local variational form of SPDE

Let  $u \in \mathcal{H}$  be weak solution of (5.4.1) and let  $\theta \in C_c^1(\mathcal{O})$ . Then, we apply the variational equation (5.4.3) for the test function  $\theta \Psi$ , with  $\Psi \in C^\infty([0, T]) \otimes C_c^\infty(\mathcal{O})$  to obtain

$$\begin{aligned} &\int_t^T \int_{\mathcal{O}} u(s, x) \theta(x) \partial_s \Psi(s, x) dx ds - \int_{\mathcal{O}} \Phi(x) \theta(x) \Psi(T, x) dx + \int_{\mathcal{O}} u(t, x) \theta(x) \Psi(t, x) dx \\ &- \int_t^T \int_{\mathcal{O}} D_\sigma u(s, x) \theta(x) D_\sigma \Psi(s, x) dx ds - \int_t^T \int_{\mathcal{O}} u(s, x) \operatorname{div}((b - \tilde{A}) \theta(x) \Psi(s, x)) dx ds \\ &= \int_t^T \int_{\mathcal{O}} \Psi(s, x) [\theta(x) f(s, x, u(s, x), D_\sigma u(s, x)) + D_\sigma u(s, x) D_\sigma \theta(x)] dx ds \\ &+ \int_t^T \int_{\mathcal{O}} \theta(x) \Psi(s, x) g(s, x, u(s, x), D_\sigma u(s, x)) dx d\overleftarrow{W}_s. \end{aligned} \quad (5.4.15)$$

Since  $\theta$  has a compact support on  $\mathcal{O}$ , we can rewrite the variational formulation (5.4.15) in the whole domain  $\mathbb{R}^d$

$$\begin{aligned} &\int_t^T \int_{\mathbb{R}^d} u(s, x) \theta(x) \partial_s \Psi(s, x) dx ds - \int_{\mathbb{R}^d} \Phi(x) \theta(x) \Psi(T, x) dx + \int_{\mathbb{R}^d} u(t, x) \theta(x) \Psi(t, x) dx \\ &- \int_t^T \int_{\mathbb{R}^d} D_\sigma u(s, x) \theta(x) D_\sigma \Psi(s, x) dx ds - \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div}((b - \tilde{A}) \theta(x) \Psi(s, x)) dx ds \\ &= \int_t^T \int_{\mathbb{R}^d} \Psi(s, x) [\theta(x) f(s, x, u(s, x), D_\sigma u(s, x)) + D_\sigma u(s, x) D_\sigma \theta(x)] dx ds \\ &+ \int_t^T \int_{\mathbb{R}^d} \theta(x) \Psi(s, x) g(s, x, u(s, x), D_\sigma u(s, x)) dx d\overleftarrow{W}_s. \end{aligned} \quad (5.4.16)$$

Then, from Proposition 5.4.4, which gives the weak variational formulation (5.4.16) applied to random test function  $\varphi_t(\cdot, \cdot)$  (5.4.8) yields to:

$$\begin{aligned}
 & \int_s^T \int_{\mathbb{R}^d} u(r, x) \theta(x) d_r \varphi_t(r, x) dx dr - \int_{\mathbb{R}^d} \Phi(x) \theta(x) \varphi_t(T, x) dx + \int_{\mathbb{R}^d} u(s, x) \theta(x) \varphi_t(s, x) dx \\
 & - \int_s^T \int_{\mathbb{R}^d} D_\sigma u(r, x) \theta(x) D_\sigma \varphi_t(r, x) dx dr - \int_s^T \int_{\mathbb{R}^d} u(r, x) \operatorname{div}((b - \tilde{A}) \theta(x) \varphi_t(r, x)) dx dr \\
 & = \int_s^T \int_{\mathbb{R}^d} \varphi_t(r, x) [\theta(x) f(r, x, u(r, x), D_\sigma u(r, x)) + D_\sigma u(r, x) D_\sigma \theta(x)] dx dr \\
 & + \int_s^T \int_{\mathbb{R}^d} \theta(x) \varphi_t(r, x) g(r, x, u(r, x), D_\sigma u(r, x)) dx d\overleftarrow{W}_r.
 \end{aligned} \tag{5.4.17}$$

Moreover, by Lemma 5.4.1, we have that

$$\begin{aligned}
 \int_s^T \int_{\mathbb{R}^d} u(r, x) \theta(x) d_r \varphi_t(r, x) dx dr & = \int_{\mathbb{R}^d} \int_s^T u(r, x) \theta(x) \mathcal{L}^* \varphi_t(r, x) dr dx \\
 & - \int_{\mathbb{R}^d} \int_s^T u(r, x) \theta(x) \nabla (\sigma^*(x) \varphi_t(r, x)) (x) dB_r dx.
 \end{aligned}$$

Using Integration by parts, we obtain

$$\begin{aligned}
 & \int_s^T \int_{\mathbb{R}^d} u(r, x) \theta(x) d_r \varphi_t(r, x) dx dr = \int_s^T \int_{\mathbb{R}^d} D_\sigma (u(r, x) \theta(x)) \varphi_t(r, x) dx dB_r \\
 & + \int_s^T \int_{\mathbb{R}^d} D_\sigma (u(r, x) \theta(x)) D_\sigma \varphi_t(r, x) dx dr + \int_s^T \int_{\mathbb{R}^d} u(r, x) \operatorname{div}((b - \tilde{A}) \theta(x) \varphi_t(r, x)) dx dr \\
 & = \int_s^T \int_{\mathbb{R}^d} \theta(x) (D_\sigma u(r, x)) \varphi_t(r, x) dx dB_r + \int_s^T \int_{\mathbb{R}^d} u(r, x) D_\sigma \theta(x) \varphi_t(r, x) dx dB_r \\
 & + \int_s^T \int_{\mathbb{R}^d} \theta(x) D_\sigma u(r, x) D_\sigma \varphi_t(r, x) dx dr + \int_s^T \int_{\mathbb{R}^d} u(r, x) D_\sigma \theta(x) D_\sigma \varphi_t(r, x) dx dr \\
 & + \int_s^T \int_{\mathbb{R}^d} u(r, x) \operatorname{div}((b - \tilde{A}) \theta(x) \varphi_t(r, x)) dx dr.
 \end{aligned}$$

Using again integration by parts for the fourth term in the right hand of the above equation, we get

$$\begin{aligned}
 & \int_s^T \int_{\mathbb{R}^d} u(r, x) \theta(x) d_r \varphi_t(r, x) dx dr = \int_s^T \int_{\mathbb{R}^d} \theta(x) (D_\sigma u(r, x)) \varphi_t(r, x) dx dB_r \\
 & + \int_s^T \int_{\mathbb{R}^d} u(r, x) D_\sigma \theta(x) \varphi_t(r, x) dx dB_r + \int_s^T \int_{\mathbb{R}^d} \theta(x) D_\sigma u(r, x) D_\sigma \varphi_t(r, x) dx dr \\
 & + \int_s^T \int_{\mathbb{R}^d} u(r, x) \operatorname{div}((b - \tilde{A}) \theta(x) \varphi_t(r, x)) dx dr \\
 & - \int_s^T \int_{\mathbb{R}^d} (D_\sigma u(r, x) D_\sigma \theta(x) + u(r, x) D_\sigma^2 \theta(x)) \varphi_t(r, x) dx dr.
 \end{aligned} \tag{5.4.18}$$

We substitute now the above equation in (5.4.17) to get

$$\begin{aligned}
 & \int_{\mathbb{R}^d} u(s, x) \theta(x) \varphi_t(s, x) dx - \int_{\mathbb{R}^d} \Phi(x) \theta(x) \varphi_t(T, x) dx \\
 & + \int_s^T \int_{\mathbb{R}^d} \theta(x) (D_\sigma u(r, x)) \varphi_t(r, x) dx dB_r + \int_s^T \int_{\mathbb{R}^d} u(r, x) D_\sigma \theta(x) \varphi_t(r, x) dx dB_r \\
 & = \int_s^T \int_{\mathbb{R}^d} \varphi_t(r, x) [\theta(x) f(r, x, u(r, x), D_\sigma u(r, x)) - u(r, x) D_\sigma^2 \theta(x)] dx dr \\
 & + \int_s^T \int_{\mathbb{R}^d} \theta(x) \varphi_t(r, x) g(r, x, u(r, x), D_\sigma u(r, x)) dx d\overleftarrow{W}_r.
 \end{aligned}$$

Now the change of variable  $y = X_{t,s}^{-1}(x)$  in the above equation gives

$$\begin{aligned}
 & \int_{\mathbb{R}^d} u(s, X_s^{t,x}) \theta(X_s^{t,x}) \varphi(x) dx - \int_{\mathbb{R}^d} \Phi(X_T^{t,x}) \theta(X_T^{t,x}) \varphi(x) dx \\
 & + \int_s^T \int_{\mathbb{R}^d} \theta(X_r^{t,x}) (D_\sigma u(r, X_r^{t,x})) \varphi(x) dx dB_r + \int_s^T \int_{\mathbb{R}^d} u(r, X_r^{t,x}) D_\sigma \theta(X_r^{t,x}) \varphi(x) dx dB_r \\
 & = \int_s^T \int_{\mathbb{R}^d} \varphi(x) [\theta(X_r^{t,x}) f(r, X_r^{t,x}, u(r, X_r^{t,x}), D_\sigma u(r, X_r^{t,x})) - u(r, X_r^{t,x}) D_\sigma^2 \theta(X_r^{t,x})] dx dr \\
 & + \int_s^T \int_{\mathbb{R}^d} \theta(X_r^{t,x}) \varphi(X_r^{t,x}) g(r, X_r^{t,x}, u(r, X_r^{t,x}), D_\sigma u(r, X_r^{t,x})) dx d\overleftarrow{W}_r.
 \end{aligned}$$

Define  $Y_s^{t,x} := u(s, X_s^{t,x})$ , a.e. and  $Z_s^{t,x} := D_\sigma u(s, X_s^{t,x})$  a.e.. In particular we have  $u(t, x) = Y_t^{t,x}$ , a.e. and  $D_\sigma u(t, x) = Z_t^{t,x}$ , a.e.. Thus, it follows from the last equation

$$\begin{aligned}
 & \int_{\mathbb{R}^d} [Y_s^{t,x} \theta(X_s^{t,x}) - Y_T^{t,x} \theta(X_T^{t,x})] \varphi(x) dx \\
 & = \int_{\mathbb{R}^d} \int_s^T [\theta(X_r^{t,x}) f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - u(r, X_r^{t,x}) D_\sigma^2 \theta(X_r^{t,x})] \varphi(x) dx dr \\
 & + \int_{\mathbb{R}^d} \int_s^T \theta(X_r^{t,x}) g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) d\overleftarrow{W}_r dx \\
 & - \int_{\mathbb{R}^d} \int_s^T [\theta(X_r^{t,x}) Z_r^{t,x} - Y_r^{t,x} D_\sigma \theta(X_r^{t,x})] dB_r \varphi(x) dx.
 \end{aligned}$$

Since  $\varphi \in C_c^\infty(\mathbb{R}^d)$  is arbitrary function, we get the following equation

$$\begin{aligned}
 Y_s^{t,x} \theta(X_s^{t,x}) & = Y_T^{t,x} \theta(X_T^{t,x}) + \int_s^T [\theta(X_r^{t,x}) f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - u(r, X_r^{t,x}) D_\sigma^2 \theta(X_r^{t,x})] dr \\
 & \int_s^T \theta(X_r^{t,x}) g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r - \int_s^T [\theta(X_r^{t,x}) Z_r^{t,x} - Y_r^{t,x} D_\sigma \theta(X_r^{t,x})] dB_r.
 \end{aligned} \tag{5.4.19}$$

*Step 2: Approximation of the random terminal time and BDSDE*

We denote by the set  $\mathcal{O}_\varepsilon := \{x \in \mathcal{O} : d(x, \mathcal{O}^c) > \varepsilon\}$  and the function

$$\theta_\varepsilon(x) := \begin{cases} 1, & x \in \mathcal{O}_\varepsilon, \\ 0, & x \in \mathcal{O}_\varepsilon^c. \end{cases}$$

So,  $0 \leq \theta_\varepsilon(x) \leq 1$  and  $\theta_\varepsilon \in C_c^\infty(\mathcal{O}_\varepsilon)$ . We define the exit stopping time from the set  $\mathcal{O}_\varepsilon$  by

$$\tau_\varepsilon^{t,x} := \inf\{t < s \leq T : X_s^{t,x} \notin \mathcal{O}_\varepsilon\} \wedge (T - \varepsilon(T - t)) \in [t, T].$$

Then, for  $t \leq s \leq \tau_\varepsilon^{t,x}$ , we have  $\theta_\varepsilon(X_s^{t,x}) = 1$  and  $D_\sigma \theta_\varepsilon(X_s^{t,x}) = D_\sigma^2 \theta_\varepsilon(X_s^{t,x}) = 0$ . Then, we use the localization function  $\theta_\varepsilon$  in the equation (5.4.19) to get

$$\begin{aligned}
 Y_{s \wedge \tau_\varepsilon^{t,x}}^{t,x} & = Y_{\tau_\varepsilon^{t,x}}^{t,x} + \int_{s \wedge \tau_\varepsilon^{t,x}}^{\tau_\varepsilon^{t,x}} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\
 & + \int_{s \wedge \tau_\varepsilon^{t,x}}^{\tau_\varepsilon^{t,x}} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r - \int_{s \wedge \tau_\varepsilon^{t,x}}^{\tau_\varepsilon^{t,x}} Z_r^{t,x} dB_r.
 \end{aligned} \tag{5.4.20}$$

Since the domain  $\mathcal{O}$  is smooth enough satisfying Assumption **D**, we have that the stopping time  $\tau_\varepsilon^{t,x}$  converge to the stopping time  $\tau^{t,x}$  a.s, where  $\tau^{t,x} := \inf\{t < s : X_s^{t,x} \notin \mathcal{O}\} \wedge T$  (see Chapter IV



page 119-120 in Gobet [56]).

So, passing to the limit in the BDSDE (5.4.20), we obtain

$$\begin{aligned} Y_{s \wedge \tau^{t,x}}^{t,x} &= Y_{\tau^{t,x}}^{t,x} + \int_{s \wedge \tau^{t,x}}^{\tau^{t,x}} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ &\quad + \int_{s \wedge \tau^{t,x}}^{\tau^{t,x}} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\overleftarrow{W}_r - \int_{s \wedge \tau^{t,x}}^{\tau^{t,x}} Z_r^{t,x} dB_r. \end{aligned} \quad (5.4.21)$$

In the other hand,  $Y_{T \wedge \tau^{t,x}}^{t,x} = \Phi(X_{T \wedge \tau^{t,x}}^{t,x})$ . Indeed, using the boundary condition of the solution  $u$  of the SPDE, we get  $Y_{T \wedge \tau^{t,x}}^{t,x} = u\left(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}\right) = 0$  which complete the proof of Theorem 5.4.1 and in particular the representation (5.4.14).  $\square$

**Remark 5.4.2.** *We may get the uniqueness of the solution for the SPDE (5.4.1) from the probabilistic representation. Indeed, let  $u$  and  $\bar{u}$  to be two solutions of The SPDE (5.4.1) and  $(Y, Z)$  and  $(\bar{Y}, \bar{Z})$  are the two associated solutions of the BDSDEs (5.4.21). We denote by  $\Delta u := u - \bar{u}$ ,  $\Delta Y := Y - \bar{Y}$  and  $\Delta Z := Z - \bar{Z}$ . By the usual calculus on the BSDEs, we obtain that  $\Delta u(t, x) = \Delta Y_{t \wedge \tau^{t,x}}^{t,x} = 0$ ,  $\forall x \in \mathcal{O}$ . So, the uniqueness of the solution of the SPDE is given by the uniqueness of the BDSDEs.*

## 5.5 Implementation and numerical tests

In this part, we are interested in implementing our numerical scheme. Our aim is only to test statically its convergence. Further analysis of the convergence of the used method and of the error bounds will be accomplished in a future work.

### 5.5.1 Notations and algorithm

Not forgetting that  $\pi := \{t_i = ih ; i \leq N\}$ ,  $h := T/N$ ,  $N \in \mathbb{N}$ , is the time grid of the interval  $[0, T]$ . We use a path-dependent algorithm, for every fixed path of the brownian motion  $W$ , we approximate by a regression method the solution of the associated PDE. Then, we replace the conditional expectations which appear in (5.5.3) and (5.5.4) by  $\mathbb{L}^2(\Omega, \mathcal{P})$  projections on the function basis approximating  $\mathbb{L}^2(\Omega, \mathcal{F}_{t_n})$ . We compute  $Z_{t_n}^N$  in an explicit manner and we use Picard iterations to compute  $Y_{t_n}^N$  in a implicit way. Actually, we proceed as in [57], except that in our case the solutions  $Y_{t_n}^N$  and  $Z_{t_n}^N$  are measurable functions of  $(X_{t_n}^N, (\Delta W_i)_{n \leq i \leq N-1})$ . So, each solution given by our algorithm depends on the fixed path of  $W$ .

#### 5.5.1.1 Forward Euler scheme

The discrete approximation of the forward diffusion process (5.3.1) is defined by

$$\begin{cases} X_0^N = x, \\ X_{t_{i+1}}^N = X_{t_i}^N + b(X_{t_i}^N)(t_{i+1} - t_i) + \sigma(X_{t_i}^N)(B_{t_{i+1}} - B_{t_i}), \quad i \leq N. \end{cases} \quad (5.5.1)$$

Then, we approximate the exit time  $\tau$  by the first time of the Euler scheme  $(t, X_t^N)_{t \in \pi}$  from  $D$  on the grid  $\pi$ :

$$\bar{\tau} := \inf\{t \in \pi : X_t^N \notin \mathcal{O}\} \wedge T.$$

The simulation of the diffusion stopped at the exit time is based on the approach of Gobet and Menozzi [59]. In this approach, we simulate the diffusion with an Euler scheme with step size  $h$  and stop it at discrete times  $(t_i)_{i \in \mathbb{N}^*}$  in a modified domain, whose boundary has been appropriately

shifted. The shift is locally in the direction of the inward normal  $n(t, x)$  at any point  $(t, x)$  on the parabolic boundary of  $\mathcal{O}$ , and its amplitude is equal to  $c_0|n^T \sigma|(t, x)\sqrt{h}$ , with

$$c_0 := \frac{\mathbb{E}[s_{\tau^+}^2]}{2\mathbb{E}[s_{\tau^+}]} = 0.5826\dots, \quad (5.5.2)$$

where  $s_0 = 0$ ,  $\forall n \geq 1$ ,  $s_n := \sum_{i=1}^n G^i$ , the  $G^i$  being i.i.d standard centered normal variables,  $\tau^+ := \inf\{n \geq 0 : s_n > 0\}$ .

### 5.5.1.2 Numerical scheme for BDSDEs

For each fixed path of  $W$ , the solution of (5.3.1)-(5.3.2) is approximated by  $(Y^N, Z^N)$  defined by the following algorithm, given in the multidimensional case.

For  $0 \leq n \leq N - 1$ :  $\forall j_1 \in \{1, \dots, k\}$ ,

$$Y_{t_n, j_1}^N = \mathbb{E}_{t_n} \left[ Y_{t_{n+1}, j_1}^N + hf_{j_1}(X_{t_n}^N, Y_{t_n}^N, Z_{t_n}^N) + \sum_{j=1}^l g_{j_1, j}(X_{t_{n+1}}^N, Y_{t_{n+1}}^N, Z_{t_{n+1}}^N) \Delta W_{n, j} \right], \quad (5.5.3)$$

$\forall j_1 \in \{1, \dots, k\}$  and  $\forall j_2 \in \{1, \dots, d\}$

$$hZ_{t_n, j_1, j_2}^N = \mathbb{E}_{t_n} \left[ Y_{t_{n+1}, j_1}^N \Delta B_{n, j_2} + \sum_{j=1}^l g_{j_1, j}(X_{t_{n+1}}^N, Y_{t_{n+1}}^N, Z_{t_{n+1}}^N) \Delta W_{n, j} \Delta B_{n, j_2} \right]. \quad (5.5.4)$$

We stress that at each discretization time, the solution of the algorithm depends on the fixed path of the brownian motion  $B$ .

### 5.5.1.3 Vector spaces of functions

At every  $t_n$ , we select  $k(d + 1)$  deterministic functions bases  $(p_{i, n}(\cdot))_{1 \leq i \leq k(d+1)}$  and we look for approximations of  $Y_{t_n}^N$  and  $Z_{t_n}^N$  which will be denoted respectively by  $y_n^N$  and  $z_n^N$ , in the vector space spanned by the basis  $(p_{j_1, n}(\cdot))_{1 \leq j_1 \leq k}$  (respectively  $(p_{j_1, j_2, n}(\cdot))_{1 \leq j_1 \leq k, 1 \leq j_2 \leq d}$ ). Each basis  $p_{i, n}(\cdot)$  is considered as a vector of functions of dimension  $L_{i, n}$ . In other words,  $P_{i, n}(\cdot) = \{\alpha \cdot p_{i, n}(\cdot), \alpha \in \mathbb{R}^{L_{i, n}}\}$  where  $\alpha$  is the coefficient of the projection on  $\mathbb{L}^2(\Omega, \mathcal{F}_{t_n})$ .

As an example, we cite the hypercube basis (**HC**) used in [57]. In this case,  $p_{i, n}(\cdot)$  does not depend nor on  $i$  neither on  $n$  and its dimension is simply denoted by  $L$ . A domain  $D \subset \mathbb{R}^d$  centered on  $X_0 = x$ , that is  $D = \prod_{i=1}^d (x_i - a, x_i + a]$ , can be partitionned on small hypercubes of edge  $\delta$ . Then,  $D = \bigcup_{i_1, \dots, i_d} D_{i_1, \dots, i_d}$  where  $D_{i_1, \dots, i_d} = (x_1 - a + i_1\delta, x_1 - a + i_1\delta] \times \dots \times (x_d - a + i_d\delta, x_d - a + i_d\delta]$ . Finally we define  $p_{i, n}(\cdot)$  as the indicator functions of this set of hypercubes.

### 5.5.1.4 Description of the algorithm

The main difference with the numerical scheme for FBDSDE in [6] is the simulation of the first exit time of the forward diffusion process from the domain  $\mathcal{O}$ . The computation of this exit time  $\bar{\tau}$  follows a simple and very efficient improved procedure given in [59]. The purpose is to stop the Euler scheme at its exit time of a smaller domain in order to compensate the underestimation of exits and to achieve an error of order  $o(\sqrt{h})$ . The smaller domain is defined by

$$\mathcal{O}^N := \{x \in \mathcal{O} : d(x, \partial\mathcal{O}) > c_0\sqrt{h}|n^T \sigma|(t, x)\},$$

where  $n(t, x)$  is the inward normal vector at the closest point of  $x$  on the boundary of  $\mathcal{O}$  and  $c_0$  is the constant given by (5.5.2). We shall interpret  $|n^T \sigma|(t, x)$  as the noise amplitude along the normal direction to the boundary. Thus the efficient exit time of the Euler scheme is given by

$$\hat{\tau}^N := \inf\{t_i > 0 : X_{t_i}^N \notin \mathcal{O}^N\} \leq \bar{\tau}.$$

For more details on this procedure see the book of Gobet [56] (page 142-144). We can also cite the work of Antoine Lejay [83] where he presents a C-library for simulating quantities and random variables related to where and when a Brownian motion hits the boundary of an interval, a square or a rectangle.

Now the projection coefficients  $\alpha$  are computed by using  $M$  independent Monte Carlo simulations of  $X_{t_n}^N$  and  $\Delta B_n$  which will be respectively denoted by  $X_{t_n}^{N,m}$  and  $\Delta B_n^m, m=1, \dots, M$ . The algorithm is explicite as follows:

→ Initialization: For  $n = N$ , take  $(y_N^{N,m,I}) = (\Phi(X_{\hat{r}^N}^{N,m}))$  and  $(z_N^{N,m}) = 0$ .

→ Iteration: For  $n = N - 1, \dots, 0$ :

- We approximate (5.5.4) by computing for all  $j_1 \in \{1, \dots, k\}$  and  $j_2 \in \{1, \dots, d\}$

$$\begin{aligned} \alpha_{j_1, j_2, n}^M &= \operatorname{arginf}_{\alpha} \frac{1}{M} \sum_{m=1}^M \left| y_{n+1, j_1}^{N, M, I}(X_{t_{n+1}}^{N, m}) \frac{\Delta B_{n, j_2}^m}{h} \right. \\ &\quad \left. + \sum_{j=1}^l g_{j_1, j} \left( X_{t_{n+1}}^{N, m}, y_{n+1}^{N, M, I}(X_{t_{n+1}}^{N, m}), z_{n+1}^{N, M}(X_{t_{n+1}}^{N, m}) \right) \frac{\Delta W_{n, j} \Delta B_{n, j_2}^m}{h} - \alpha \cdot p_{j_1, j_2, n}^m \right|^2. \end{aligned}$$

Then we set  $z_{n, j_1, j_2}^{N, M}(\cdot) = (\alpha_{j_1, j_2, n}^M \cdot p_{j_1, j_2, n}(\cdot))$ ,  $j_1 \in \{1, \dots, k\}$ ,  $j_2 \in \{1, \dots, d\}$ .

- We use  $I$  Picard iterations to obtain an approximation of  $Y_{t_n}$  in (5.5.3):

· For  $i = 0$ :  $\forall j_1 \in \{1, \dots, k\}$ ,  $\alpha_{j_1, n}^{M, 0} = 0$ .

· For  $i = 1, \dots, I$ : We approximate (5.5.3) by calculating  $\alpha_{j_1, n}^{M, i}$ ,  $\forall j_1 \in \{1, \dots, k\}$ , as the minimizer of:

$$\begin{aligned} \frac{1}{M} \sum_{m=1}^M \left| y_{n+1, j_1}^{N, M, I}(X_{t_{n+1}}^{N, m}) + h f_{j_1} \left( X_{t_n}^{N, m}, y_n^{N, M, i-1}(X_{t_n}^{N, m}), z_n^{N, M}(X_{t_n}^{N, m}) \right) \right. \\ \left. + \sum_{j=1}^l g_{j_1, j} \left( X_{t_{n+1}}^{N, m}, y_{n+1}^{N, M, I}(X_{t_{n+1}}^{N, m}), z_{n+1}^{N, M}(X_{t_{n+1}}^{N, m}) \right) \Delta W_{n, j} - \alpha_{j_1, n}^{M, i} \right|^2. \end{aligned}$$

Finally, we define  $y_n^{N, M, I}(\cdot)$  as:

$$y_{n, j_1}^{N, M, I}(\cdot) = (\alpha_{j_1, n}^{M, I} \cdot p_{j_1, n}(\cdot)), \forall j_1 \in \{1, \dots, k\}.$$

## 5.5.2 One-dimensional case (Case when $d = k = l = 1$ )

### 5.5.2.1 Function bases

We use the basis (HC) defined above. So we set:

$$d_1 = \min_{n, m} X_{t_n}^m, \quad d_2 = \max_{n, m} X_{t_n}^m \quad \text{and} \quad L = \frac{d_2 - d_1}{\delta}$$

where  $\delta$  is the edge of the hypercubes  $(D_j)_{1 \leq j \leq L}$  defined by  $D_j = [d + (j-1)\delta, d + j\delta)$ ,  $\forall j$ .

We take at each time  $t_n$

$$1_{D_j}(X_{t_n}^{N, m}) = 1_{[d+(j-1)\delta, d+j\delta)}(X_{t_n}^{N, m}), j = 1, \dots, L$$

and

$$(p_{i, n}^m(\cdot)) = \left\{ \sqrt{\frac{M}{\operatorname{card}(D_j)}} 1_{D_j}(X_{t_n}^{N, m}), 1 \leq j \leq L \right\}, i = 0, 1.$$

$\text{Card}(D_j)$  denotes the number of simulations of  $X_{t_n}^N$  which are in our cube  $D_j$ . This system is orthonormal with respect to the empirical scalar product defined by

$$\langle \psi_1, \psi_2 \rangle_{n,M} := \frac{1}{M} \sum_{m=1}^M \psi_1(X_{t_n}^{N,m}) \psi_2(X_{t_n}^{N,m}).$$

In this case, the solutions of our least squares problems are given by:

$$\begin{aligned} \alpha_{1,n}^M &= \frac{1}{M} \sum_{m=1}^M p_{1,n}(X_{t_n}^{N,m}) \left\{ y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}) \frac{\Delta B_n^m}{h} \right. \\ &\quad \left. + g\left(X_{t_{n+1}}^{N,m}, y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}), z_{n+1}^{N,M}(X_{t_{n+1}}^{N,m})\right) \frac{\Delta W_n^m \Delta B_n^m}{h} \right\}, \\ \alpha_{0,n}^{M,i} &= \frac{1}{M} \sum_{m=1}^M p_{0,n}(X_{t_n}^{N,m}) \left\{ y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}) + hf\left(X_{t_n}^{N,m}, y_n^{N,M,i-1}(X_{t_n}^{N,m}), z_n^{N,M}(X_{t_n}^{N,m})\right) \right. \\ &\quad \left. + g\left(X_{t_{n+1}}^{N,m}, y_{n+1}^{N,M,I}(X_{t_{n+1}}^{N,m}), z_{n+1}^{N,M}(X_{t_{n+1}}^{N,m})\right) \Delta W_n^m \right\}. \end{aligned}$$

**Remark 5.5.1.** We note that for each value of  $M$ ,  $N$  and  $\delta$ , we launch the algorithm 50 times and we denote by  $(Y_{0,m'}^{0,x,N,M,I})_{1 \leq m' \leq 50}$  the set of collected values. Then we calculate the empirical mean  $\bar{Y}_0^{0,x,N,M,I}$  and the empirical standard deviation  $\sigma^{N,M,I}$  defined by:

$$\bar{Y}_0^{0,x,N,M,I} = \frac{1}{50} \sum_{m'=1}^{50} Y_{0,m'}^{0,x,N,M,I} \quad \text{and} \quad \sigma^{N,M,I} = \sqrt{\frac{1}{49} \sum_{m'=1}^{50} |Y_{0,m'}^{0,x,N,M,I} - \bar{Y}_0^{0,x,N,M,I}|^2}. \quad (5.5.5)$$

We also note before starting the numerical examples that our algorithm converges after at most three Picard iterations. Finally, we stress that (5.5.5) gives us an approximation of  $u(0, x)$  the solution of the SPDE (5.4.1) at time  $t = 0$ .

### 5.5.2.2 Comparison of numerical approximations of the solutions of the FBDSDE and the FBSDE: the general case

Now we take

$$\begin{cases} \Phi(x) = -x + K, \\ f(t, x, y, z) = -\theta z - ry + (y - \frac{z}{\sigma})^-(R - r), \\ g_1(t, x, y, z) = 0.1z + 0.5y + \log(x) \end{cases}$$

and we set  $\theta = (\mu - r)/\sigma$ ,  $K = 115$ ,  $X_0 = 100$ ,  $\mu = 0.05$ ,  $\sigma = 0.2$ ,  $r = 0.01$ ,  $R = 0.06$ ,  $\delta = 1$ ,  $N = 20$ ,  $T = 0.25$  and we fix  $d_1 = 60$  and  $d_2 = 200$  as in [58]. We fix our domain  $\mathcal{O} = ]60, 200[$ , chosen large enough to compensate the rate of convergence of the exit time approximation which is slow (of order  $h^{1/2}$ ).

The functions  $g_1, g_2$  and  $g_3$  taken in the following are examples of the function  $g$ . They are sufficiently regular and Lipschitz on  $[60, 200] \times \mathbb{R} \times \mathbb{R}$  and could be extended to regular Lipschitz functions on  $\mathbb{R}^3$ . In this case, the continuous Lipschitz assumption is satisfied.

We compare the numerical solution of our BDSDE with terminal time  $\bar{\tau}$  (noted again  $\bar{Y}_t^{t,x,N,M,I}$ ), the BDSDE's one (noted here by  $\bar{Y}_{t,BDSDE}^{0,x,N,M}$ ) and the BSDE's one (noted here by  $\bar{Y}_{t,BSDE}^{0,x,N,M}$ ), without  $g$  and  $B$ .

When  $t$  is close to maturity  $t = t_{19}$

$M$	$\bar{Y}_{t_{19},BSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{19},BDSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{19}}^{0,x,N,M,I}(\sigma^{N,M,I})$
128	13.748(0.879)	15.453(0.948)	13.392(1.021)
512	13.827(0.384)	15.535(0.409)	12.210(0.3580)
2048	13.762(0.223)	15.465(0.240)	12.051(0.197)
8192	13.781(0.091)	15.485(0.097)	14.814 (0.107)
32768	13.796(0.054)	15.501(0.058)	14.729 (0.053)

when  $t = t_{15}$

$M$	$\bar{Y}_{t_{15},BSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{15},BDSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{15}}^{0,x,N,M,I}(\sigma^{N,M,I})$
128	14.168(0.905)	17.894(1.096)	13.049(1.116)
512	14.113(0.388)	17.774(0.429)	16.469(0.441)
2048	13.988(0.226)	17.607(0.270)	9.817(0.178)
8192	13.985(0.093)	17.623(0.104)	12.951(0.115)
32768	13.994(0.055)	17.627(0.064)	13.232(0.053)

when  $t = 0$

$M$	$\bar{Y}_{0,BSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{0,BDSDE}^{0,x,N,M,I}(\sigma^{N,M,I})$	$\bar{Y}_0^{0,x,N,M}(\sigma^{N,M})$
128	15.431(1.005)	13.571(1.146)	19.719(1.558)
512	15.029(0.428)	13.173(0.500)	24.371(0.659)
2048	14.763(0.243)	12.885(0.280)	13.433(0.233)
8192	14.718(0.098)	12.825(0.106)	12.543(0.122)
32768	14.715(0.060)	12.804(0.064)	13.458(0.057)

In the previous tables, we test our algorithm for different times (when they are close to the maturity and in initial time  $t = 0$ ) and we variate the number of Monte Carlo simulation  $M$  for fixed number of time discretization  $N$ . We note that the numerical value of the BDSDE with random terminal time  $\bar{\tau}$  converges to the value of classical BDSDE for  $M$  large and this can be explained by the fact that the approximated value of the exit time is close to the maturity  $T$ .

For  $g_2(y, z) = 0.1z + 0.5y$  when  $t = t_{19}$ .

$M$	$\bar{Y}_{t_{19},BSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{19}}^{0,x,N,M,I}(\sigma^{N,M,I})$
128	14.767(0.949)	13.545(1.020)
512	14.850(0.410)	12.862(0.358)
2048	14.781(0.240)	12.739(0.197)
8192	14.801(0.097)	14.401(0.107)
32768	14.818(0.058)	14.358(0.053)

when  $t = t_{15}$

$M$	$\bar{Y}_{t_{15},BSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{15}}^{0,x,N,M,I}(\sigma^{N,M,I})$
128	16.267(1.093)	13.607(1.111)
512	16.166(0.428)	15.191(0.443)
2048	16.007(0.270)	11.675(0.180)
8192	16.024(0.104)	13.551(0.114)
32768	16.029(0.064)	13.689(0.053)

when  $t = 0$

$M$	$\bar{Y}_{0,BDSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_0^{0,x,N,M,I}(\sigma^{N,M,I})$
128	13.821(0.063)	17.811(1.529)
512	14.555(1.132)	19.766(0.645)
2048	14.176(0.495)	13.976(0.241)
8192	13.899(0.277)	13.635(0.122)
32768	13.842(0.105)	14.139(0.058)

For  $g_3(x, y) = \log x + 0.5y$ : when  $t$  is close to maturity  $t = t_{19}$ .

$M$	$\bar{Y}_{t_{19},BDSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{19}}^{0,x,N,M,I}(\sigma^{N,M,I})$
128	15.452(0.948)	13.392(1.021)
512	15.534(0.409)	12.210(0.358)
2048	15.464(0.240)	12.051(0.197)
8192	15.484(0.097)	14.814(0.107)
32768	15.501(0.058)	14.729(0.053)

when  $t = t_{15}$

$M$	$\bar{Y}_{t_{15},BDSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_{t_{15}}^{0,x,N,M,I}(\sigma^{N,M,I})$
128	18.253(1.068)	12.782(1.003)
512	18.166(0.453)	17.383(0.454)
2048	18.010(0.266)	9.325(0.174)
8192	18.006(0.109)	12.490(0.097)
32768	18.017(0.065)	12.858(0.049)

when  $t = 0$

$M$	$\bar{Y}_{0,BDSDE}^{0,x,N,M}(\sigma^{N,M})$	$\bar{Y}_0^{0,x,N,M,I}(\sigma^{N,M,I})$
128	12.071(0.054)	20.496(1.421)
512	12.075(0.088)	27.093(0.654)
2048	12.122(0.218)	13.362(0.221)
8192	12.384(0.381)	11.878(0.101)
32768	12.791(0.903)	12.948(0.051)

In the previous tables, we test our algorithm for different examples of the function  $g$  ( $g_1$  and  $g_2$  are dependent in  $z$ ,  $g_3$  is independent of  $z$ ).

We see on Figure 5.1 and 5.2 the impact of the function  $g$  on the solution; we variate  $N$ ,  $M$  and  $\delta$  as in [84], by taking these quantities as follows: First we fix  $d_1 = 40$  and  $d_2 = 180$  (which means that  $x \in [d_1, d_2] = [40, 180]$  and in this case our continuous lipschitz assumptions are satisfied). Let  $j \in \mathbb{N}$ , we take  $\alpha_M = 3$ ,  $\beta = 1$ ,  $N = 2(\sqrt{2})^{(j-1)}$ ,  $M = 2(\sqrt{2})^{\alpha_M(j-1)}$  and  $\delta = 50/(\sqrt{2})^{(j-1)(\beta+1)/2}$ . Then, we draw the map of each solution at  $t = 0$  with respect to  $j$ . We remark from the figures that numerical values of the BDSDE wit random terminal time coincide with that of the classical BDSDE after just few variation of the parameters. This allow us to think about performing the rate of convergence of our algorithm by getting weaker estimates for the BDSDE (as Bouchard and Menozzi for the classical BSDEs [21]).

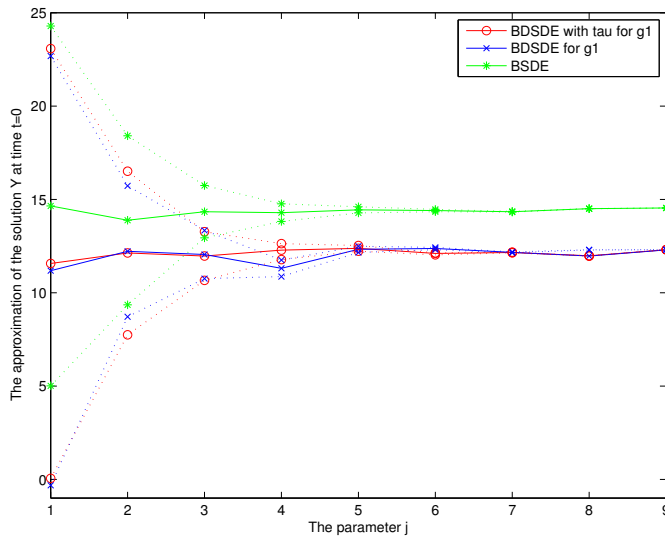


Figure 5.1: Comparison of the BSDE's solution, the BDSDE's one and the solution of BDSDE with random time for  $g_1(x, y, z) = \ln(x) + 0.5y + 0.1z$ . Confidence intervals are with dotted lines.

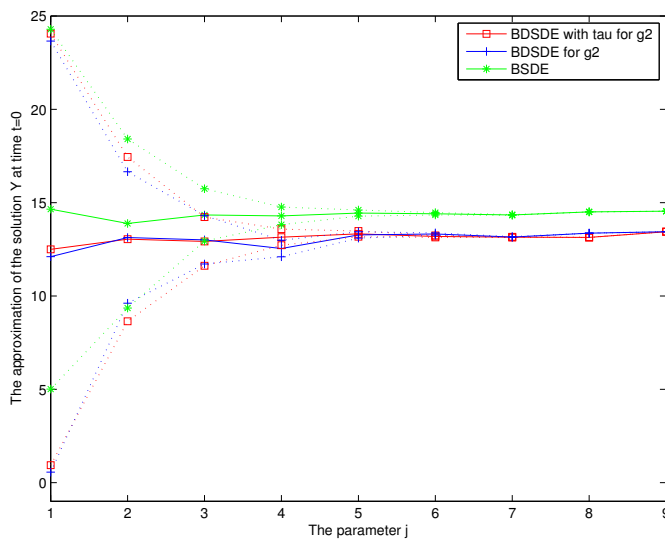


Figure 5.2: Comparison of the BSDE's solution, the BDSDE's one and the solution of BDSDE with random time for  $g_2(x, y, z) = 0.5y + 0.1z$ . Confidence intervals are with dotted lines.





# Bibliography

- [1] AMAN, A. A numerical scheme for backward doubly stochastic differential equations. *Bernoulli* 19, 1 (2013), 93–114. (Cited on page 22.)
- [2] AMAN, A., AND MRHARDY, N. Obstacle problem for SPDE with nonlinear neumann boundary condition via reflected generalized backward doubly spdes. (Cited on page 63.)
- [3] APPLEBAUM, D. Lévy processes and stochastic calculus (second edition). *Cambridge University Press* (2009). (Cited on page 89.)
- [4] BACHOUCH, A. Numerical computations for backward doubly stochastic differential equations and non-linear stochastic pdes. *Université du Maine, PhD Thesis*. (Cited on page 128.)
- [5] BACHOUCH, A., GOBET, E., AND MATOUSSI, A. Regression error for backward doubly SDEs and semilinear SPDEs. *Forthcoming paper* (2014). (Cited on pages 22, 23 and 116.)
- [6] BACHOUCH, A., LASMAR, A. B., MATOUSSI, A., AND MNIF, M. Numerical scheme for semilinear SPDEs via backward doubly SDEs. *arXiv :1302.0440*. (Cited on pages 18, 22, 23, 26, 92, 115, 116, 122 and 136.)
- [7] BACHOUCH, A., AND MATOUSSI, A. Numerical computation for quasilinear SPDEs via generalized backward doubly SDEs. *Forthcoming paper* (2014). (Cited on pages 22, 115 and 116.)
- [8] BALLY, V. Construction of asymptotically optimal controls for control and game problems. *Probab. Theory Related Fields* 111, 3 (1998), 453–467. (Cited on page 22.)
- [9] BALLY, V., CABALLERO, M., KAROUI, N. E., AND FERNANDEZ, B. Reflected bsde’s, pde’s and variational inequalities. *Rapport de Recherche INRIA*, 4455 (2002). (Cited on pages 13, 14, 18, 66, 74, 88, 89, 107, 109 and 114.)
- [10] BALLY, V., AND MATOUSSI, A. Weak solutions for SPDEs and backward doubly stochastic differential equations. *J. Theoret. Probab.* 14, 1 (2001), 125–164. (Cited on pages 3, 14, 15, 18, 22, 28, 66, 69, 71, 72, 73, 79, 92, 103, 104, 105, 108, 115, 116, 130 and 131.)
- [11] BALLY, V., PARDOUX, E., AND STOICA, L. Backward stochastic differential equations associated to a symmetric Markov process. *Potential Anal.* 22, 1 (2005), 17–60. (Cited on pages 22 and 115.)
- [12] BARLES, G., BUCKDAHN, R., AND PARDOUX, E. Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.* 60, 1-2 (1997), 57–83. (Cited on pages 14 and 82.)
- [13] BARLES, G., AND LESIGNE, E. SDE, BSDE and PDE. In *Backward stochastic differential equations (Paris, 1995–1996)*, vol. 364 of *Pitman Res. Notes Math. Ser.* Longman, Harlow, 1997, pp. 47–80. (Cited on pages 2, 15, 71, 105 and 131.)
- [14] BECHERER, D. Bounded solutions to backward SDE’s with jumps for utility optimization and indifference hedging. *Ann. Appl. Probab.* 16, 4 (2006), 2027–2054. (Cited on page 13.)
- [15] BENSOUSSAN, A., AND LIONS, J.-L. *Applications des Inéquations variationnelles en contrôle stochastique*. Dunod, Paris, 1978. (Cited on pages 13 and 14.)
- [16] BISMUT, J. Mécanique aléatoire. *Ecole d’été de Probabilité de Saint-Flour, Lect. Notes Math.*, 929 (1980), 5–100. (Cited on pages 1, 15, 69 and 104.)

- [17] BOUCHARD, B., AND ELIE, R. Discrete-time approximation of decoupled forward-backward SDE with jumps. *Stochastic Process. Appl.* 118, 1 (2008), 53–75. (Cited on page 22.)
- [18] BOUCHARD, B., ELIE, R., AND RÉVEILLAC, A. Bsdes with weak terminal condition. *arXiv:1210.5364*, to appear in *Annals of Probability*. (Cited on page 28.)
- [19] BOUCHARD, B., ELIE, R., AND TOUZI, N. Stochastic target problems with controlled loss. *SIAM J. Control Optim.* 48, 5 (2009/10), 3123–3150. (Cited on page 28.)
- [20] BOUCHARD, B., GEISS, S., AND GOBET, E. First time to exit of a continuous Itô process: general moment estimates and  $l_1$ -convergence rate for discrete time approximations. *arXiv:1307.4247*. (Cited on pages 23, 25 and 122.)
- [21] BOUCHARD, B., AND MENOZZI, S. Strong approximations of BSDEs in a domain. *Bernoulli* 15, 4 (2009), 1117–1147. (Cited on pages 23, 25, 120, 122 and 140.)
- [22] BOUCHARD, B., AND TOUZI, N. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* 111, 2 (2004), 175–206. (Cited on page 22.)
- [23] BRIAND, P., DELYON, B., HU, Y., PARDOUX, E., AND STOICA, L.  $L^p$  solutions of backward stochastic differential equations. *Stochastic Process. Appl.* 108, 1 (2003), 109–129. (Cited on page 116.)
- [24] BUCKDAHN, R., AND MA, J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I. *Stochastic Process. Appl.* 93, 2 (2001), 181–204. (Cited on pages 3, 4, 5, 6, 18, 22, 29, 53, 55, 56 and 116.)
- [25] BUCKDAHN, R., AND MA, J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. II. *Stochastic Process. Appl.* 93, 2 (2001), 205–228. (Cited on pages 3, 4, 18, 22, 29, 53 and 116.)
- [26] BUCKDAHN, R., AND PARDOUX, E. Bsdes with jumps and associated integro-partial differential equations. *Preprint* (1994). (Cited on page 79.)
- [27] CAFFARELLI, L. A., AND CABRÉ, X. *Fully nonlinear elliptic equations*, vol. 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995. (Cited on page 56.)
- [28] CHALEYAT-MAUREL, M., EL KAROUI, N., AND MARCHAL, B. Réflexion discontinue et systèmes stochastiques. *Ann. Probab.* 8, 6 (1980), 1049–1067. (Cited on page 18.)
- [29] CHEN, Z., AND PENG, S. A general downcrossing inequality for  $g$ -martingales. *Statist. Probab. Lett.* 46, 2 (2000), 169–175. (Cited on page 59.)
- [30] CHERIDITO, P., SONER, H. M., TOUZI, N., AND VICTOIR, N. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Comm. Pure Appl. Math.* 60, 7 (2007), 1081–1110. (Cited on page 6.)
- [31] CRANDALL, M. G., ISHII, H., AND LIONS, P.-L. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* 27, 1 (1992), 1–67. (Cited on pages 4, 5 and 54.)
- [32] CRÉPEY, S. About the pricing equation in finance. *Paris-Princeton Lectures in Mathematical Finance Lecture Notes in Mathematics*, Springer (2010). (Cited on page 13.)

- [33] CRÉPEY, S., AND MATOUSSI, A. Reflected and doubly reflected bsdes with jumps. *Annals of Applied Probability* 5, 18 (2008), 2041–2069. (Cited on pages 72 and 74.)
- [34] CRISAN, D., MANOLARAKIS, K., AND TOUZI, N. On the Monte Carlo simulation of BSDEs: an improvement on the Malliavin weights. *Stochastic Process. Appl.* 120, 7 (2010), 1133–1158. (Cited on page 22.)
- [35] CVITANIĆ, J., AND KARATZAS, I. Backward stochastic differential equations with reflection and Dynkin games. *Ann. Probab.* 24, 4 (1996), 2024–2056. (Cited on page 2.)
- [36] DARLING, R. W. R., AND PARDOUX, E. Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *Ann. Probab.* 25, 3 (1997), 1135–1159. (Cited on pages 22, 115 and 116.)
- [37] DELLACHERIE, C., AND MEYER, P. *Probabilités et Potentiel, Chapitres XII à XVI, Théorie du potentiel*. Hermann, Paris, 1980. (Cited on page 43.)
- [38] DENIS, L., A.MATOUSSI, AND ZHANG, J. The obstacle problem for quasilinear stochastic pdes: Analytical approach. *The Annals of Probability* (2012). (Cited on pages 14, 28 and 66.)
- [39] DENIS, L., AND MARTINI, C. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Ann. Appl. Probab.* 16, 2 (2006), 827–852. (Cited on page 6.)
- [40] DENIS, L., MATOUSSI, A., AND STOICA, L. Lp-estimates for the uniform norm of solutions quasilinear SPDE's. *Probability Theory and Related fields* 133 (2005), 437–463. (Cited on pages 22 and 116.)
- [41] DENIS, L., MATOUSSI, A., AND STOICA, L. Maximum principle and comparison theorems for solutions of quasilinear SPDE's. *Electronic Journal of Probability* 14 (2009), 500–530. (Cited on pages 22 and 116.)
- [42] DENIS, L., MATOUSSI, A., AND STOICA, L. Moser iteration applied to parabolic SPDE's: first approach. In *Stochastic partial differential equations and applications*, vol. 25 of *Quad. Mat.* Dept. Math., Seconda Univ. Napoli, Caserta, 2010, pp. 99–125. (Cited on pages 22 and 116.)
- [43] DENIS, L., AND STOICA, L. A general analytical result for non-linear SPDE's and applications. *Electron. J. Probab.* 9 (2004), no. 23, 674–709 (electronic). (Cited on pages 3, 18, 22, 103, 115 and 130.)
- [44] EL-KAROUI, N., AND HAMADÈNE, S. BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. *Stochastic Process. Appl.* 107, 1 (2003), 145–169. (Cited on page 1.)
- [45] EL KAROUI, N., HAMADÈNE, S., AND MATOUSSI, A. Backward stochastic differential equations and applications. *Chapter 8 in the book Indifference Pricing: Theory and Applications*, Springer-Verlag (2008), 267–320. (Cited on page 63.)
- [46] EL KAROUI, N., KAPOUDJIAN, C., PARDOUX, E., PENG, S., AND QUENEZ, M. Reflected solutions of backward sde and related obstacle problems for pde. *Annals of Probability*, 25 (1997), 702–737. (Cited on pages 2 and 13.)
- [47] EL KAROUI, N., AND MRAD, M. An exact connection between two solvable sdes and a nonlinear utility stochastic pde. *arXiv:1004.5191* (2012). (Cited on page 15.)
- [48] EL KAROUI, N., AND MRAD, M. Stochastic utilities with a given optimal portfolio : Approach by stochastic flows. *arXiv:1004.5192* (2013). (Cited on page 15.)

- [49] EL KAROUI, N., PENG, S., AND QUENEZ, M. Backward stochastic differential equations in finance. *Mathematical Finance* 7, 1 (1997), 1–71. (Cited on pages 13 and 52.)
- [50] FÖLLMER, H., AND LEUKERT, P. Quantile hedging. *Finance Stoch.* 3, 3 (1999), 251–273. (Cited on page 28.)
- [51] FREMLIN, D. H. *Consequences of Martin's axiom*, vol. 84 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1984. (Cited on page 47.)
- [52] FUJIWARA, T., AND KUNITA, H. Stochastic differential equations of jump type and lévy processes in diffeomorphisms group. *J. Math. Kyoto Univ.*, 25 (1985), 71–106. (Cited on pages 15 and 69.)
- [53] GEGOUT-PETIT, A., AND PARDOUX, E. Equations différentielles stochastiques rétrogrades réfléchies dans un convexe. *Stochastics and Stochastics Reports* 57 (1995), 11–128. (Cited on pages 19, 92, 95 and 102.)
- [54] GOBET, E. Schéma d'Euler continu pour des diffusions tuées et options barrière. *C. R. Acad. Sci. Paris Sér. I Math.* 326, 12 (1998), 1411–1414. (Cited on pages 25, 26, 121 and 122.)
- [55] GOBET, E. Weak approximation of killed diffusion using Euler schemes. *Stochastic Process. Appl.* 87, 2 (2000), 167–197. (Cited on pages 25, 26, 121 and 122.)
- [56] GOBET, E. *Méthodes de Monte-Carlo et processus stochastiques: du linéaire au non-linéaire*. École Polytechnique, 2013. (Cited on pages 135 and 137.)
- [57] GOBET, E., LEMOR, J.-P., AND WARIN, X. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Ann. Appl. Probab.* 15, 3 (2005), 2172–2202. (Cited on pages 22, 135 and 136.)
- [58] GOBET, E., LEMOR, J.-P., AND WARIN, X. Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli* 12, 5 (2006), 889–916. (Cited on pages 22 and 138.)
- [59] GOBET, E., AND MENOZZI, S. Stopped diffusion processes: boundary corrections and overshoot. *Stochastic Process. Appl.* 120, 2 (2010), 130–162. (Cited on pages 135 and 136.)
- [60] GYÖNGY, I., AND KRYLOV, N. Accelerated finite difference schemes for linear stochastic partial differential equations in the whole space. *SIAM J. Math. Anal.* 42, 5 (2010), 2275–2296. (Cited on pages 24 and 116.)
- [61] GYÖNGY, I., AND ROVIRA, C. On  $L^p$ -solutions of semilinear stochastic partial differential equations. *Stochastic Process. Appl.* 90, 1 (2000), 83–108. (Cited on pages 3 and 18.)
- [62] HAMADÈNE, S., AND LEPELTIER, J. P. Backward equations, stochastic control and zero-sum stochastic differential games. *Stochastics Stochastics Rep.* 54, 3-4 (1995), 221–231. (Cited on page 1.)
- [63] HAMADENE, S., LEPELTIER, J.-P., AND MATOUSSI, A. Double barrier backward SDEs with continuous coefficient. In *Backward stochastic differential equations (Paris, 1995–1996)*, vol. 364 of *Pitman Res. Notes Math. Ser.* Longman, Harlow, 1997, pp. 161–175. (Cited on page 2.)
- [64] HAMADENE, S., AND OUKNINE, Y. Reflected backward sdes with general jumps. *arXiv:0812.3965 to appear in Prob. theory and Applications*. (Cited on page 59.)

- [65] HAMADÈNE, S., AND OUKNINE, Y. Reflected backward stochastic differential equation with jumps and random obstacle. *Electronic Journal of Probability* 8, 2 (2003), 1–20. (Cited on pages 16, 73 and 75.)
- [66] HU, Y., IMKELLER, P., AND MÜLLER, M. Utility maximization in incomplete markets. *Ann. Appl. Probab.* 15, 3 (2005), 1691–1712. (Cited on page 2.)
- [67] IKEDA, N., AND WATANABE, S. Stochastic differential equations and diffusion processes. *North-Holland/Kodansha, second edition* (1989). (Cited on page 88.)
- [68] ISHIKAWA, Y., AND KUNITA, H. Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps. *Stochastic Process. Appl.* 116, 12 (2006), 1743–1769. (Cited on page 88.)
- [69] JACOD, J., AND SHIRYAEV, A. Limit theorems for stochastic processes. *Springer, Berlin Heidelberg New York* (2003). (Cited on page 88.)
- [70] JENTZEN, A., AND KLOEDEN, P. Taylor expansions of solutions of stochastic partial differential equations with additive noise. *Ann. Probab.* 38, 2 (2010), 532–569. (Cited on pages 24 and 116.)
- [71] JOURDAIN, B. Stochastic flows approach to Dupire’s formula. *Finance and Stochastics* 11, 4 (2007), 521–535. (Cited on page 15.)
- [72] KARANDIKAR, R. On pathwise stochastic integration. *Stochastic Processes and Their Applications*, 57:11-18 (1995). (Cited on pages 10 and 30.)
- [73] KLIMSIK, T. Reflected BSDEs and obstacle problem for semilinear PDEs in divergence form. *Stochastic Processes and their Applications* 1, 122 (2012). (Cited on pages 15 and 71.)
- [74] KOBYLANSKI, M. Résultats d’existence et d’unicité pour des équations différentielles stochastiques rétrogrades avec des générateurs à croissance quadratique. *C. R. Acad. Sci. Paris Sér. I Math.* 324, 1 (1997), 81–86. (Cited on page 2.)
- [75] KRYLOV, N. V. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, vol. 64 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 1999, pp. 185–242. (Cited on pages 3 and 18.)
- [76] KUNITA, H. Stochastic differential equations and stochastic flows of diffeomorphisms. In *École d’été de probabilités de Saint-Flour, XII—1982*, vol. 1097 of *Lecture Notes in Math.* Springer, Berlin, 1984, pp. 143–303. (Cited on pages 15, 69, 104, 117, 121 and 130.)
- [77] KUNITA, H. Convergence of stochastic flows with jumps and Lévy processes in diffeomorphisms group. *Annales de l’I.H.P.* 22, 3 (1986), 287–321. (Cited on pages 15 and 69.)
- [78] KUNITA, H. *Stochastic flows and stochastic differential equations*, vol. 24 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990. (Cited on pages 11, 32, 50 and 104.)
- [79] KUNITA, H. Generalized solutions of a stochastic partial differential equation. *J. Theoret. Probab.* 7, 2 (1994), 279–308. (Cited on pages 4, 28, 92, 104 and 116.)
- [80] KUNITA, H. Stochastic flow acting on Schwartz distributions. *Journal of Theoretical Probability* 7, 2 (1994), 279–308. (Cited on pages 4, 15, 22, 28, 69, 92, 104, 115, 116 and 131.)

- [81] KUNITA, H. Stochastic differential equations based on lévy processes and stochastic flows of diffeomorphisms. *Real and stochastic analysis Trends Math., Birkhäuser Boston, MA* (2004), 305–373. (Cited on pages 69 and 70.)
- [82] LÉANDRE, R. Flot d’une équation différentielle stochastique avec semimartingale directrice discontinue. *Séminaire de probabilités (Strasbourg) 19* (1985), 271–274. (Cited on pages 1, 14, 15, 66 and 69.)
- [83] LEJAY, A. exitbm: a library for simulating brownian motion’s exit times and positions from simple domains. *HAL : inria-00561409, version 2*. (Cited on page 137.)
- [84] LEMOR, J.-P., GOBET, E., AND WARIN, X. Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli* 12, 5 (2006), 889–916. (Cited on page 140.)
- [85] LEPELTIER, J. P., AND SAN MARTIN, J. Backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett.* 32, 4 (1997), 425–430. (Cited on page 2.)
- [86] LIEBERMAN, G. M. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. (Cited on page 56.)
- [87] LIN, Y. A new result for second order bsdes with quadratic growth and its applications. *arXiv:1301.0457*. (Cited on page 55.)
- [88] LIONS, P.-L., AND SOUGANIDIS, P. E. Fully nonlinear viscosity stochastic partial differential equations: non-smooth equations and applications. *C.R. Acad. Sci. Paris* 327, 1 (1998), 735–741. (Cited on pages 3, 4, 18, 22 and 116.)
- [89] LIONS, P.-L., AND SOUGANIDIS, P. E. Équations aux dérivées partielles stochastiques non-linéaires et solutions de viscosité. Exp. No. I, 15. (Cited on pages 3, 4, 18, 22 and 116.)
- [90] LIONS, P.-L., AND SOUGANIDIS, P. E. Viscosity solutions of fully nonlinear stochastic partial differential equations. *S = urikaisekikenky = usho K = oky = uroku*, 1287 (2002), 58–65. (Cited on pages 3, 4, 18, 22 and 116.)
- [91] LIONS, P.-L., AND SZNITMAN, A.-S. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* 37, 4 (1984), 511–537. (Cited on page 18.)
- [92] MATACHE, A. M., PETERSDORFF, T. V., AND SCHWAB, C. Fast deterministic pricing of options on lévy driven assets. *ESAIM: Mathematical Modelling and Numerical Analysis* 38, 1 (2004), 37–71. (Cited on pages 13 and 71.)
- [93] MATOUSSI, A., POSSAMAI, D., AND SABBAGH, W. Probabilistic interpretation for viscosity solutions of fully nonlinear stochastic pdes. *forthcoming paper*. (Cited on pages 7 and 30.)
- [94] MATOUSSI, A., AND SABBAGH, W. Backward doubly sdes and semilinear stochastic pdes in a convex domain. *arXiv:1405.5822*. (Cited on page 19.)
- [95] MATOUSSI, A., AND SABBAGH, W. Numerical computation for backward doubly sdes with random terminal time, date=2014, journal= arXiv:1409.2149,. (Cited on page 116.)
- [96] MATOUSSI, A., SABBAGH, W., AND ZHOU, C. The obstacle problem for semilinear parabolic partial integro-differential equations. *Stochastics and Dynamics* (to appear). (Cited on pages 14, 16 and 66.)

- [97] MATOUSSI, A., AND SCHEUTZOW, M. Semilinear stochastic pde's with nonlinear noise and backward doubly sde's. *Journal of Theoretical Probability* 15 (2002), 1–39. (Cited on pages 4, 18, 22, 28, 66, 92, 115 and 116.)
- [98] MATOUSSI, A., AND STOICA, L. The obstacle problem for quasilinear stochastic PDE's. *Ann. Probab.* 38, 3 (2010), 1143–1179. (Cited on pages 14, 22, 28, 66 and 116.)
- [99] MATOUSSI, A., AND XU, M. Sobolev solution for semilinear pde with obstacle under monotonicity condition. *Electronic Journal of Probability* 13, 35 (2008), 1035–1067. (Cited on pages 14, 22, 74, 107, 109 and 115.)
- [100] MCKEAN, JR., H. P. A. Skorohod's stochastic integral equation for a reflecting barrier diffusion. *J. Math. Kyoto Univ.* 3 (1963), 85–88. (Cited on page 18.)
- [101] MENALDI, J.-L. Stochastic variational inequality for reflected diffusion. *Indiana Univ. Math. J.* 32, 5 (1983), 733–744. (Cited on pages 18, 19, 92 and 93.)
- [102] MEYER, P.-A. Flot d'une équation différentielle stochastique (d'après Malliavin, Bismut, Kunita). In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, vol. 850 of *Lecture Notes in Math.* Springer, Berlin-New York, 1981, pp. 103–117. (Cited on page 69.)
- [103] MIGNOT, F., AND PUEL, J.-P. Solution maximum de certaines inéquations d'évolution paraboliques, et inéquations quasi variationnelles paraboliques. *C. R. Acad. Sci. Paris Sér. A-B* 280, 5 (1975), Ai, A259–A262. (Cited on page 13.)
- [104] MRAD, M. *Utilités progressives*. PhD thesis, École Polytechnique, 2009. (Cited on page 15.)
- [105] NUTZ, M. Pathwise construction of stochastic integrals. *Electron. Commun. Probab.* 17 (2012), no. 24, 7. (Cited on page 47.)
- [106] OUKNINE, Y., AND I.TURPIN. Weak solutions of semilinear pdes in sobolev spaces and their probabilistic interpretation via the fbsde. *Stochastic Analysis and Applications* 24 (2006), 871–888. (Cited on page 70.)
- [107] PARDOUX, É. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. In *Stochastic analysis and related topics, VI (Geilo, 1996)*, vol. 42 of *Progr. Probab.* Birkhäuser Boston, Boston, MA, 1998, pp. 79–127. (Cited on pages 22, 115 and 116.)
- [108] PARDOUX, É., AND PENG, S. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, vol. 176 of *Lecture Notes in Control and Inform. Sci.* Springer, Berlin, 1992, pp. 200–217. (Cited on pages 2, 17, 22, 71 and 115.)
- [109] PARDOUX, E., AND PENG, S. Backward doubly sde's and systems of quasilinear spdes. *Probab. Theory and Related Field* 98 (1994), 209–227. (Cited on pages 2, 3, 18, 21, 22, 32, 47, 57, 61, 73, 91, 92, 94, 95, 96, 106, 112, 115, 117, 121, 122, 124 and 131.)
- [110] PARDOUX, É., AND PENG, S. G. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* 14, 1 (1990), 55–61. (Cited on page 1.)
- [111] PARDOUX, E., PRADEILLES, F., AND RAO, Z. Probabilistic interpretation of systems of semilinear pdes. *Annales de l'Institut Henri Poincaré, série Probabilités Statistiques* 33 (1997), 467–490. (Cited on page 14.)

- [112] PARDOUX, É., AND PROTTER, P. A two-sided stochastic integral and its calculus. *Probab. Theory Related Fields* 76, 1 (1987), 15–49. (Cited on page 47.)
- [113] PENG, S. Backward SDE and related  $g$ -expectation. In *Backward stochastic differential equations (Paris, 1995–1996)*, vol. 364 of *Pitman Res. Notes Math. Ser.* Longman, Harlow, 1997, pp. 141–159. (Cited on page 45.)
- [114] PENG, S. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer’s type. *Probab. Theory Relat. Fields* 113, 4 (1999), 473–499. (Cited on pages 10 and 57.)
- [115] PENG, S. G. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.* 37, 1-2 (1991), 61–74. (Cited on pages 2, 17, 22, 24, 71, 115, 116, 117 and 118.)
- [116] PIERRE, M. Problèmes d’évolution avec contraintes unilatérales et potentiels parabolique. *Comm. in Partial Differential Equations* 4, 10 (1979), 1149–1197. (Cited on pages 13, 14 and 28.)
- [117] PIERRE, M. Représentant précis d’un potentiel parabolique. *Séminaire de Théorie du Potentiel Paris Lecture Notes in Math* 814, 5 (1980), 186–228. (Cited on pages 13, 14 and 28.)
- [118] POSSAMAÏ, D. Second order backward stochastic differential equations under a monotonicity condition. *Stochastic Process. Appl.* 123, 5 (2013), 1521–1545. (Cited on page 7.)
- [119] POSSAMAÏ, D., XIAOLU, T., AND ZHOU, C. Second order bsdes with measurable parameters. *forthcoming paper*. (Cited on page 44.)
- [120] POSSAMAÏ, D., AND ZHOU, C. Second order backward stochastic differential equations with quadratic growth. *Stochastic Process. Appl.* 123, 10 (2013), 3770–3799. (Cited on page 7.)
- [121] POSSAMAÏ, D., AND ZHOU, C. Second order backward stochastic differential equations with quadratic growth. *Stochastic Process. Appl.* 123, 10 (2013), 3770–3799. (Cited on page 55.)
- [122] PROTTER, P. *Stochastic integration and differential equations*. 2-nd Edition, Version 2.1, Springer B.H.N-Y, 2000. (Cited on page 69.)
- [123] ROUGE, R., AND EL KAROUI, N. Pricing via utility maximization and entropy. *Math. Finance* 10, 2 (2000), 259–276. INFORMS Applied Probability Conference (Ulm, 1999). (Cited on page 2.)
- [124] ROYER, M. BSDEs with jumps and related non linear expectations. *Stochastics Processes and their Applications* 116 (2006), 1358–1376. (Cited on page 14.)
- [125] SANZ-SOLÉ, M., AND VUILLERMOT, P.-A. Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Ann. Inst. H. Poincaré Probab. Statist.* 39, 4 (2003), 703–742. (Cited on pages 3 and 18.)
- [126] SITU, R. On solutions of backward stochastic differential equations with jumps and applications. *Stochastics Processes and their Applications* 66 (1997), 209–236. (Cited on page 14.)
- [127] SKOROHOD, A. V. Stochastic equations for diffusion processes with a boundary. *Teor. Veroyatnost. i Primenen.* 6 (1961), 287–298. (Cited on page 18.)
- [128] SONER, H., TOUZI, N., AND ZHANG, J. Wellposedness of second order backward sdes. *Prob. Theory and Relat. Fields* (2010). (Cited on pages 6, 7, 32, 34 and 46.)



- [129] SONER, H., TOUZI, N., AND ZHANG, J. Dual formulation of second order target problems. *Annals of Applied Probability* 23(1), 308–347 (2013). (Cited on pages 9, 31, 34, 35, 39, 43, 45 and 46.)
- [130] SONER, H. M., TOUZI, N., AND ZHANG, J. Quasi-sure stochastic analysis through aggregation. *Electron. J. Probab.* 16, 67 (2011), 1844–1879. (Cited on pages 31 and 52.)
- [131] STOICA, I. L. A probabilistic interpretation of the divergence and BSDE's. *Stochastic Process. Appl.* 103, 1 (2003), 31–55. (Cited on pages 22 and 115.)
- [132] STRICKER, C., AND YOR, M. Calcul stochastique dépendant d'un paramètre. *Z. Wahrsch. Verw. Gebiete* 45, 2 (1978), 109–133. (Cited on page 41.)
- [133] STROOCK, D., AND VARADHAN, S. R. S. On degenerate elliptic-parabolic operators of second order and their associated diffusions. *Comm. Pure Appl. Math.* 25 (1972), 651–713. (Cited on pages 25 and 121.)
- [134] STROOK, D.W., V. S. Multidimensional diffusion processes. *Spring-Verlag, Berlin, Heidelberg* (1979). (Cited on pages 9 and 40.)
- [135] TANAKA, H. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.* 9, 1 (1979), 163–177. (Cited on page 18.)
- [136] TANG, S., AND LI, X. Necessary condition for optimal control of stochastic systems with random jumps. *SIAM JCO* 332 (1994), 1447–1475. (Cited on page 14.)
- [137] WALSH, J. B. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, vol. 1180 of *Lecture Notes in Math.* Springer, Berlin, 1986, pp. 265–439. (Cited on pages 3 and 18.)
- [138] WALSH, J. B. Finite element methods for parabolic stochastic PDE's. *Potential Anal.* 23, 1 (2005), 1–43. (Cited on pages 24 and 116.)
- [139] WATANABE, S. On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. *J. Math. Kyoto Univ.* 11 (1971), 169–180. (Cited on page 18.)
- [140] ZHANG, J. A numerical scheme for BSDEs. *Ann. Appl. Probab.* 14, 1 (2004), 459–488. (Cited on page 22.)

# Thèse de Doctorat

Wissal SABBAGH

## Some Contributions on Probabilistic Interpretation For Nonlinear Stochastic PDEs

### Résumé

L'objectif de cette thèse est l'étude de la représentation probabiliste des différentes classes d'EDPSs non-linéaires (semi-linéaires, complètement non-linéaires, réfléchies dans un domaine) en utilisant les équations différentielles doublement stochastiques rétrogrades (EDDSRs). Cette thèse contient quatre parties différentes. Nous traitons dans la première partie les EDDSRs du second ordre (2EDDSRs). Nous montrons l'existence et l'unicité des solutions des 2EDDSRs en utilisant des techniques de contrôle stochastique quasi-sure. La motivation principale de cette étude est la représentation probabiliste des EDPSs complètement non-linéaires. Dans la deuxième partie, nous étudions les solutions faibles de type Sobolev du problème d'obstacle pour les équations à dérivées partielles intégro-différentielles (EDPIDs). Plus précisément, nous montrons la formule de Feynman-Kac pour l'EDPIDs par l'intermédiaire des équations différentielles stochastiques rétrogrades réfléchies avec sauts (EDSRRs). Plus précisément, nous établissons l'existence et l'unicité de la solution du problème d'obstacle, qui est considérée comme un couple constitué de la solution et de la mesure de réflexion. L'approche utilisée est basée sur les techniques de flots stochastiques développées dans Bally et Matoussi (2001) mais les preuves sont beaucoup plus techniques. Dans la troisième partie, nous traitons l'existence et l'unicité pour les EDSRRs dans un domaine convexe  $D$  sans aucune condition de régularité sur la frontière. De plus, en utilisant l'approche basée sur les techniques du flot stochastiques nous démontrons l'interprétation probabiliste de la solution faible de type Sobolev d'une classe d'EDPSs réfléchies dans un domaine convexe via les EDSRRs. Enfin, nous nous intéressons à la résolution numérique des EDDSRs à temps terminal aléatoire. La motivation principale est de donner une représentation probabiliste des solutions de Sobolev d'EDPSs semi-linéaires avec condition de Dirichlet nul au bord. Dans cette partie, nous étudions l'approximation forte de cette classe d'EDDSRs quand le temps terminal aléatoire est le premier temps de sortie d'une EDS d'un domaine cylindrique. Ainsi, nous donnons les bornes pour l'erreur d'approximation en temps discret. Cette partie se conclut par des tests numériques qui démontrent que cette approche est effective.

**Mots clés:** Equations différentielles doublement stochastiques rétrogrades, Equations aux dérivées partielles stochastiques non-linéaires, Problème d'obstacle, Domaine convexe, Simulations de Monte-Carlo.

### Abstract

The objective of this thesis is to study the probabilistic representation (Feynman-Kac formula) of different classes of Stochastic Nonlinear PDEs (semilinear, fully nonlinear, reflected in a domain) by means of backward doubly stochastic differential equations (BDSDEs). This thesis contains four different parts. We deal in the first part with the second order BDSDEs (2BDSDEs). We show the existence and uniqueness of solutions of 2BDSDEs using quasi sure stochastic control technics. The main motivation of this study is the probabilistic representation for solution of fully nonlinear SPDEs. First, under regularity assumptions on the coefficients, we give a Feynman-Kac formula for classical solution of fully nonlinear SPDEs and we generalize the work of Soner, Touzi and Zhang (2010-2012) for deterministic fully nonlinear PDE. Then, under weaker assumptions on the coefficients, we prove the probabilistic representation for stochastic viscosity solution of fully nonlinear SPDEs. In the second part, we study the Sobolev solution of obstacle problem for partial integro-differential equations (PIDEs). Specifically, we show the Feynman-Kac formula for PIDEs via reflected backward stochastic differential equations with jumps (BSDEs). Specifically, we establish the existence and uniqueness of the solution of the obstacle problem, which is regarded as a pair consisting of the solution and the measure of reflection. The approach is based on stochastic flow technics developed in Bally and Matoussi (2001) but the proofs are more technical. In the third part, we discuss the existence and uniqueness for RBDSDEs in a convex domain  $D$  without any regularity condition on the boundary. In addition, using the approach based on the technics of stochastic flow we provide the probabilistic interpretation of Sobolev solution of a class of reflected SPDEs in a convex domain via RBDSDEs. Finally, we are interested in the numerical solution of BDSDEs with random terminal time. The main motivation is to give a probabilistic representation of Sobolev solution of semilinear SPDEs with Dirichlet null condition. In this part, we study the strong approximation of this class of BDSDEs when the random terminal time is the first exit time of an SDE from a cylindrical domain. Thus, we give bounds for the discrete-time approximation error. We conclude this part with numerical tests showing that this approach is effective.

**Key Words:** Backward Doubly Stochastic Differential Equations, Second order Backward Doubly Stochastic Differential Equations, quasi-sure stochastic analysis, Nonlinear SPDEs, Obstacle problem, Stochastic flow, Skorohod problem, convex domains, Monte Carlo method.