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THÈSE DE DOCTORAT

Présentée le 04/12/2014

par

Ali EL ATI

**Synchronization analysis of complex networks of
nonlinear oscillators**

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Résumé

Cette thèse porte sur l'analyse de la synchronisation des grands réseaux d'oscillateurs non linéaires et hétérogènes à l'aide d'outils et de méthodes issues de la théorie du contrôle. Nous considérons deux modèles de réseaux ; à savoir, le modèle de Kuramoto qui considère seulement les coordonnées de phase des oscillateurs et des réseaux composés d'oscillateurs non linéaires de Stuart-Landau connectés par un couplage linéaire. Pour le modèle de Kuramoto nous construisons un système linéaire qui conserve les informations sur les fréquences naturelles et sur les gains d'interconnexion du modèle original de Kuramoto. Nous montrons en suite que l'existence de solutions à verrouillage de phase du modèle de Kuramoto est équivalente à l'existence d'un tel système linéaire avec certaines propriétés. Ce système est utilisé pour formuler les conditions d'existence de solutions à verrouillage de phase et de leur stabilité pour des structures particulières de l'interconnexion. Ensuite, cette analyse s'est étendue au cas où des interactions attractives et répulsives sont présentes dans le réseau. Nous considérons cette situation lorsque les gains d'interconnexion peuvent être à la fois positif et négatif.

Dans le cadre de réseaux d'oscillateurs de Stuart-Landau, nous présentons une nouvelle transformation de coordonnées du réseau qui permet de réécrire le modèle du réseau en deux parties : une décrivant le comportement de l'oscillateur « moyenne » du réseau et la seconde partie présentant les dynamiques des erreurs de synchronisation par rapport à cet oscillateur « moyenne ». Cette transformation nous permet de caractériser les propriétés du réseau en termes de la stabilité des erreurs de synchronisation et du cycle limite de l'oscillateur « moyenne ». Pour ce faire, nous reformulons ce problème en un problème de stabilité de deux ensembles compacts et nous utilisons des outils issus de la stabilité de Lyapunov pour montrer la stabilité pratique de ces derniers pour des valeurs suffisamment grandes du gain d'interconnexion.

Mots clés : synchronisation, réseau d'oscillateurs non-linéaires, oscillateur à cycle limite, graphe orienté, graphe signé, modèle de Kuramoto, équation de Stuart-Landau, interactions attractives et répulsives, stabilité asymptotique, stabilité pratique.

Abstract

This thesis is devoted to the analysis of synchronization in large networks of heterogeneous nonlinear oscillators using tools and methods issued from control theory. We consider two models of networks; namely, the Kuramoto model which takes into account only phase coordinates of the oscillators and networks composed of nonlinear Stuart-Landau oscillators interconnected by linear coupling. For the Kuramoto model we construct an auxiliary linear system that preserves information on the natural frequencies and interconnection gains of the original Kuramoto model. We show next that existence of phase locked solutions of the Kuramoto model is equivalent to the existence of such a linear system with certain properties. This system is used to formulate conditions that ensure existence of phase-locked solutions and their stability for particular structures of network interconnections. Next, this analysis is extended to the case where both attractive and repulsive interactions are present in the network that is we consider the situation where some of the interconnection gains are allowed to be negative.

In the context of networks of Stuart-Landau oscillators, we present a new coordinate transformation of the network which allows to split the network model into two parts, one describing behaviour of an "averaged" network oscillator and the second one, describing dynamics of the synchronization errors relative to this "averaged" oscillator. This transformation allows us to characterize properties of the network in terms of stability of synchronization errors and limit cycle of the "averaged" oscillator. To do so, we recast this problem as a problem of stability of compact sets and use Lyapunov stability tools to ensure practical stability of both sets for sufficiently large values of the coupling strength.

Keywords : synchronization, network of nonlinear systems, limit-cycle oscillator, digraph, signed graphs, Kuramoto model, phase-locked solution, Stuart-Landau equation, attractive and repulsive interactions, asymptotic stability, practical stability.

*À la mémoire d'un homme courageux et généreux, à la mémoire de mon père,
Mohamed El-Ati paix à son âme.*

*"Pendant le vivant de votre père,
observez avec soin sa volonté;
Après sa mort,
ayez toujours les yeux fixés sur ses actions."*

Citation de Confucius; Les entretiens - VIe s. av. J.-C.

À la personne la plus chère à mon coeur, à une mère d'exception sans qui je n'aurai sans doute jamais tant accompli, à ma très chère maman.

"Le paradis est sous les pieds des mères."
Prophète Mohamed.

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Résumé français

1. Introduction

Les travaux de cette thèse sont consacrés à l'analyse de la synchronisation des réseaux à grand nombre d'oscillateurs non linéaires et hétérogènes, en utilisant des outils et des méthodes issues de la théorie du contrôle. Deux modèles sont utilisés pour décrire les réseaux d'oscillateurs : le modèle de Kuramoto et le modèle de Stuart-Landau. Notre analyse se concentre sur l'effet des interactions entre les éléments du réseau sur la synchronisation du système. La thèse contient des résultats d'analyse et leurs confirmations à travers des exemples de simulation démonstratifs.

La première partie du travail porte sur l'étude du modèle de Kuramoto. Tout d'abord, nous considérons un réseau complexe arbitraire des oscillateurs de Kuramoto. Pour ce dernier, nous donnons les conditions d'existence de solutions à verrouillage de phase ainsi que l'expression analytique de la fréquence de synchronisation. Ensuite, nous proposons une nouvelle généralisation du modèle de Kuramoto avec des interactions plus hétérogènes : nous supposons que chaque oscillateur est caractérisé par ses poids d'entrée et de sortie outre sa fréquence naturelle. Pour ce modèle, nous présentons les solutions à verrouillage de phase ainsi que leur analyse de stabilité. Notre étude s'est étendue au cas où le réseau contient des pondérations négatives. Ceci permet de modéliser à la fois les interactions attractives et répulsives entre les oscillateurs. Dans ce cas, les critères d'existence de solutions à verrouillage de phase sont formulés et leur stabilité locale est analysée.

Dans la seconde partie, nous étudions le problème de synchronisation d'un réseau hétérogène d'oscillateurs de Stuart-Landau. Ainsi, nous affirmons l'existence d'un comportement de synchronisation dans le cas d'une force du couplage suffisamment grande. Ensuite, en se basant sur la notion de stabilité pratique, une analyse de la stabilité du système a été fournie et une approximation de la variété de synchronisation a été formulée.

2. Modèle de Kuramoto

2.1. Généralités

Le modèle de Kuramoto est un modèle standard pour l'étude des phénomènes de synchronisation collective dans un grand réseau d'oscillateurs à cycle limite ([62], [109]). Ce modèle et ses généralisations englobent une très large classe de phénomènes de synchronisation allant de la physique [49] à la biologie [18], [63]. Le modèle de Kuramoto décrit chaque oscillateur par une équation différentielle de phase où le couplage entre les oscillateurs est modélisé par le sinus de différence de phases. Le modèle de Kuramoto dans sa forme la plus générale est décrit comme suit :

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^N k_{ij} \sin(\theta_j - \theta_i), \quad i \in 1 \dots N, \quad (0.1)$$

où les θ_i , $\dot{\theta}_i$ et $\omega_i \in \mathbb{R}$ représentent respectivement la phase instantanée, la fréquence instantanée et la fréquence naturelle de chaque oscillateur i . N est le nombre d'oscillateurs et la matrice $\mathcal{K} = [k_{ij}]$ est la matrice d'interconnexion des oscillateurs, dont les éléments $k_{ij} \in \mathbb{R}$ représentent la force du couplage entre les oscillateurs i et j . Dans sa présentation la plus simple (Modèle de Kuramoto classique ou "all-to-all"), le modèle a la forme suivante :

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad i \in 1 \dots N.$$

Ainsi, le graphe d'interconnexion est considéré complet et la force de couplage K est identique entre tous les oscillateurs.

Synchronisation : définitions et notions

Le modèle de Kuramoto (0.1) décrit chaque oscillateur en fonction de la phase instantanée et sa dérivée instantanée (fréquence instantanée). En conséquence, il y a deux concepts de synchronisation : la synchronisation en fréquence et la synchronisation en phase.

Définition 1 (Fréquence d'oscillation verrouillée) – *Les oscillateurs définis par (0.1) ont une solution à verrouillage de fréquence s'il existe une constante $\omega_s \in \mathbb{R}$ et si les conditions initiales $\theta_o \in \mathbb{R}^N$ sont telles que pour tout $i \in \{1, \dots, N\}$*

$$\dot{\theta}_i(t, \theta_o) = \omega_s, \quad \forall t \geq 0. \quad (0.2)$$

Définition 2 (Synchronisation en fréquence) – *Les oscillateurs définis par (0.1) sont synchronisés en fréquence s’il existe une solution à verrouillage de fréquence du système et si toutes les fréquences $\dot{\theta}_i(t)$ convergent asymptotiquement vers cette solution, c’est-à-dire :*

$$\lim_{t \rightarrow \infty} \dot{\theta}_i = \omega_s, \quad \forall i \in \{1, \dots, N\}. \quad (0.3)$$

Définition 3 (Verrouillage de phase) *Soit $\theta_o \in \mathbb{R}^N$ et $\theta(t, \theta_o)$, $t \geq 0$ la solution du modèle de Kuramoto pour les conditions initiales θ_o . La solution $\theta(t, \theta_o)$ est appelée solution à verrouillage de phase si*

$$\theta_i(t, \theta_o) - \theta_j(t, \theta_o) = \theta_{i_o} - \theta_{j_o},$$

pour tout $t \geq 0$ et tout $i, j = 1, \dots, N$.

Définition 4 (Synchronisation en phase) – *Les oscillateurs définis par (0.1) sont synchronisés en phase s’il existe une solution à verrouillage de phase du système $\theta^* \in \mathbb{R}^N$ et un ensemble $\Theta \subset \mathbb{R}^N$ sachant qu’asymptotiquement toutes les différences de phase $\theta_i(t, \theta_o) - \theta_j(t, \theta_o)$ convergent vers cette solution pour toutes les conditions initiales $\theta_o \in \Theta$, c’est-à-dire*

$$\lim_{t \rightarrow \infty} \theta_i(t, \theta_o) - \theta_j(t, \theta_o) = \theta_i^* - \theta_j^*, \quad \forall i, j \in \{1, \dots, N\}. \quad (0.4)$$

2.2. Résultats

2.2.1. Synchronisation dans un réseau complexe des oscillateurs de Kuramoto

Cette section résume les résultats principaux du Chapitre 3. Dans ce dernier, nous considérons le système (0.1) des oscillateurs de Kuramoto dans un réseau complexe. Ces oscillateurs sont couplés avec des gains hétérogènes. Nous caractérisons les conditions d’existence de solutions à verrouillage de fréquence et de phase.

Le théorème suivant présente le problème d’existence de solutions à verrouillage de fréquence sous la forme d’un problème d’existence de solutions pour un système non linéaire des équations algébriques. Ces conditions sont formulées en termes des fréquences naturelles d’oscillateurs et des éléments de la matrice d’adjacence $\mathcal{K} = [k_{ij}]$.

Théorème 1 *Considérons les oscillateurs de Kuramoto (0.1) couplés via un graphe orienté G et une force de couplage $\gamma > 0$:*

$$\dot{\theta}_i = \omega_i + \gamma \sum_{j=1}^N k_{ij} \sin(\theta_j - \theta_i), \quad i \in 1 \dots N. \quad (0.5)$$

Soit $\tilde{\omega}_i = \omega_s - \omega_i$ et $\omega_s \in \mathbb{R}$ une constante non définie. Le système a une solution à verrouillage de fréquence si et seulement si les conditions suivantes sont satisfaites :

1. *Le système d'équations algébriques suivant*

$$\begin{aligned} \sum_{j,j \neq i}^{N-1} \pm k_{i,j} \left(\xi_i \sqrt{1 - \xi_j^2} - \sqrt{1 - \xi_i^2} \xi_j \right) &= \frac{\tilde{\omega}_i}{\gamma}, \quad i = 1 \dots N-1 \\ \sum_{j=1}^{N-1} k_{N,j} \xi_j &= \frac{\tilde{\omega}_N}{\gamma}, \end{aligned} \quad (0.6)$$

a au moins une solution $[\xi_1^, \dots, \xi_{N-1}^*, \omega_s^*]$.*

2. *Pour tous $i = 1 \dots N-1$, les solutions ξ_i^* satisfont $|\xi_i^*| \leq 1$.*

Dans le cas des graphes d'interconnexion symétriques, il est connu que la fréquence de synchronisation est égale à la moyenne des fréquences naturelles. Toutefois, dans le cas de réseaux complexes, la fréquence de synchronisation dépend de la topologie du graphe d'interconnexion et de la force de couplage. Dans le théorème suivant, nous donnons une expression analytique de la fréquence de synchronisation limite dans le cas où la force de couplage est suffisamment grande.

Théorème 2 *Considérons le système (0.5) avec un graphe d'interconnexion fortement couplé et un gain de couplage γ . Si le système est synchronisé en fréquence, alors la fréquence de synchronisation ω_s converge vers $\frac{w_l^\top \Omega}{w_l^\top \mathbf{1}}$ si le gain de couplage γ est suffisamment grand, c'est-à-dire*

$$\lim_{\gamma \rightarrow \infty} \omega_s(\gamma) = \frac{w_l^\top \Omega}{w_l^\top \mathbf{1}}. \quad (0.7)$$

Notons que w_l est le vecteur propre gauche de la matrice Laplacienne du réseau L associé à la valeur propre $\lambda_1(L) = 0$ et Ω est le vecteur des fréquences naturelles.

Dans [106] et [25], les auteurs introduisent un modèle linéaire à variables complexes et ils prouvent que ce modèle linéaire possède les mêmes propriétés asymptotiques du modèle classique de Kuramoto. Nous commençons notre analyse par la généralisation de ces résultats pour le modèle de Kuramoto avec un réseau d'interconnexion complexe, et nous montrons que le problème d'existence des solutions à verrouillage de phase pour le modèle Kuramoto (0.1) avec une matrice d'interconnexion arbitraire $\mathcal{K} = [k_{ij}]$, peut être ramené à un problème d'existence d'un système linéaire à variables complexes satisfaisant un ensemble de propriétés.

L'idée sous-jacente de l'approche peut être résumée comme suit. Considérons un système linéaire paramétré (en termes de μ) dans cette forme :

$$\dot{x}_i = (i\omega_i - \mu_i)x_i + \sum_{j=1}^N k_{ij} x_j, \quad i = \{1 \dots N\}, \quad (0.8)$$

où $x_i \in \mathbb{C}$, $\mu_i \in \mathbb{R}$ et k_{ij} sont les éléments de la matrice d'interconnexion \mathcal{K} . On peut également mettre l'équation (0.8) sous la forme matricielle suivante :

$$\dot{\mathbf{x}} = (\mathcal{K} + i\Omega + \mathcal{M})\mathbf{x}, \quad (0.9)$$

où $\mathcal{K} = [K_{ij}]$ est la matrice des interconnexions du modèle de Kuramoto (0.1), $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ et $\mathcal{M} = \text{diag}(\mu_1, \dots, \mu_N) \in \mathbb{R}^{N \times N}$.

En utilisant les coordonnées polaires $x_j = R_j e^{i\theta_j}$, nous pouvons mettre le système (0.8) sous la forme suivante :

$$\begin{aligned} \frac{\dot{R}_i}{R_i} &= -\mu_i + \sum_{j=1}^N k_{ij} \frac{R_j}{R_i} \cos(\theta_j - \theta_i), \\ \dot{\theta}_i &= \omega_i + \sum_{j=1}^N k_{ij} \frac{R_j}{R_i} \sin(\theta_j - \theta_i). \end{aligned}$$

Dans la dernière équation (parties imaginaires), il est possible de prouver l'existence d'une constante $R > 0$ tel que $\lim_{t \rightarrow \infty} R_i = R$ pour tout $i \in \{1, \dots, N\}$. Ceci impliquerait que asymptotiquement, la dynamique de θ peut être décrite par le modèle de Kuramoto (0.1).

En utilisant le modèle linéaire à variable complexe (0.8), nous proposons un cadre alternatif cohérent à travers lequel nous pouvons résoudre analytiquement des problèmes de synchronisation pour des nouveaux types d'interactions, à savoir des réseaux asymétriques, pondérés ou encore signés. Cette démarche nous permet d'étendre l'analyse du modèle de Kuramoto classique proposé dans [4].

Dans le théorème ci-dessous, nous présentons les conditions sur la matrice A du système linéaire (0.9) garantissant l'existence de solutions à verrouillage de phase pour le modèle Kuramoto (0.1).

Théorème 3 *Considérons le modèle de Kuramoto (0.1), soit la matrice d'interconnexion $\mathcal{K} = [k_{ij}]$ et soit $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ la matrice des fréquences naturelles. Pour ce système, les deux instructions suivantes sont équivalentes :*

1) *Il existe une constante $\omega_s \in \mathbb{R}$ et une matrice $\mathcal{M} = \text{diag}(\mu_1, \dots, \mu_N) \in \mathbb{R}^{N \times N}$ tel que la matrice $A = \mathcal{K} + i\Omega + \mathcal{M}$ a une valeur propre $\lambda_o(A) = i\omega_s$ d'ordre de multiplicité égal à*

un et son vecteur propre droit associé a la forme suivante :

$$v_o(A) = \text{col}(e^{\xi_1}, \dots, e^{\xi_N}). \quad (0.10)$$

2) Le modèle Kuramoto (0.1) a une solution à verrouillage de phase.

2.2.2. Analyse du modèle de Kuramoto avec un graphe pondéré asymétrique

Cette section résume les résultats principaux du Chapitre 4. En premier lieu, nous proposons une généralisation du modèle de Kuramoto avec plus d'hétérogénéité au niveau des oscillateurs. Autrement dit, nous supposons que chaque élément est caractérisé à la fois par sa fréquence naturelle et par ses poids d'entrée et de sortie. Contrairement à la fréquence naturelle qui détermine la dynamique individuelle en l'absence de couplage, les deux nouvelles pondérations affectent la façon dont chaque oscillateur interagit avec l'ensemble. Ainsi, elles définissent l'hétérogénéité des interactions dans le réseau.

Avec ce choix particulier de couplage, le modèle Kuramoto (0.1) peut être écrit sous la forme suivante :

$$\dot{\theta}_i = \omega_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \{1, \dots, N\}, \quad (0.11)$$

où la force de couplage entre les paires d'oscillateurs connectés est définie par le gain de couplage $K > 0$, les vecteurs de pondération d'entrée $W_{in}^\top = [w_{in}^1, \dots, w_{in}^N] \in \mathbb{R}_+^N$, et de pondération de sortie $W_{out}^\top = [w_{out}^1, \dots, w_{out}^N] \in \mathbb{R}_+^N$. En utilisant ces notations, la matrice d'interconnexion \mathcal{K} peut être représentée comme suit : $\mathcal{K} = W_{out} W_{in}^\top$.

L'analyse de la synchronisation du modèle de Kuramoto (0.11) détaillée dans le Chapitre 4 concerne les points suivants :

- Définir les conditions nécessaires et suffisantes d'existence de solutions à verrouillage de phase.
- Donner la forme explicite des solutions à verrouillage de phase ainsi que l'expression de la fréquence de synchronisation.
- Analyser les propriétés de stabilité locale des solutions à verrouillage de phase.

Solutions à verrouillage de phase

Dans un premier temps, nous introduisons une hypothèse reliant les fréquences naturelles ω_j , le gain de couplage K et la matrice d'interconnexion $\mathcal{K} = W_{out} W_{in}^\top$. Cette hypothèse est déduite à partir des résultats du Théorème 3.

Hypothèse 1 *Il existe une valeur $r_\infty \in (0;1]$ telle que*

$$r_\infty = \frac{1}{N} \sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{K_j r_\infty}\right)^2}, \quad (0.12)$$

avec $-1 \leq \frac{\tilde{\omega}_j}{K_j r_\infty} \leq 1$ pour tous $j = \{1, \dots, N\}$, où $K_j = K w_{out}^j$, $\tilde{\omega}_j = \omega_j - \omega_s$ et $\omega_s = \frac{\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \omega_j}{\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j}}$.

Dans la littérature sur les systèmes de Kuramoto, l'expression (0.12) avec $W_{in} = W_{out} = \mathbb{1}_N$ est connue par la condition de consistance sur r_∞ des solutions à verrouillage de phase. Elle est introduite dans [4] et dans [84], où ils ont montré que la condition de consistance est une condition nécessaire et suffisante d'existence de solutions à verrouillage de phase. Ainsi, l'Hypothèse 1 peut être considéré comme la condition pondérée analogue de la condition de consistance utilisée pour le modèle de Kuramoto classique.

Dans le théorème suivant, nous montrons que l'Hypothèse 1 est une condition nécessaire et suffisante d'existence de solutions à verrouillage de phase pour le modèle de Kuramoto (0.11) et nous donnons la forme explicite des solutions à verrouillage de phase ainsi que l'expression de la fréquence de synchronisation.

Théorème 4 *Le modèle de Kuramoto (0.11) a une solution à verrouillage de phase si et seulement si l'Hypothèse 1 est satisfaite.*

En outre, la fréquence de synchronisation est définie par l'expression suivante

$$\omega_s = \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}} \quad (0.13)$$

et les solutions à verrouillage de phase ont la forme $\theta(t, \theta^) = \theta^* + \omega_s t \mathbb{1}_n + c \mathbb{1}_n$, où $c \in \mathbb{R}$ est une constante arbitraire tandis que le vecteur θ^* est défini par les équations suivantes :*

$$\cos(\theta_j^*) = \pm \sqrt{1 - \left(\frac{\tilde{\omega}_j}{K_j r_\infty}\right)^2} \quad ; \quad \sin(\theta_j^*) = \frac{\tilde{\omega}_j}{K_j r_\infty}. \quad (0.14)$$

Le choix des signes dans les expressions des cos correspond au choix des signes dans l'équation (0.12).

Analyse de stabilité

Dans cette section, nous analysons les propriétés de stabilité locale des solutions à verrouillage de phase définis dans la section précédente. En particulier, nous montrons

qu'une seule solution à verrouillage de phase est stable. Cette solution correspond à la solution r_∞ de l'équation (0.12) avec un choix des signes positifs. Nous résumons les résultats de notre analyse ci-dessous.

Dans un premier temps, nous considérons le modèle de Kuramoto (0.11) avec un changement de coordonnées $\theta_i(t) \rightarrow \theta_i(t) + \omega_s t$, le système (0.11) peut être réécrit sous la forme :

$$\dot{\theta}_i = f_i(\theta) = \tilde{\omega}_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \{1 \dots N\}, \quad (0.15)$$

avec $\tilde{\omega}_i = \omega_i - \omega_s$ et ω_s est la fréquence de synchronisation.

Etant donné que dans ce nouveau cadre de coordonnées la fréquence de synchronisation est égale à zéro, nous pouvons déduire donc que les solutions à verrouillage de phase du modèle de Kuramoto (0.11) correspondent aux points d'équilibre du système (0.15).

Supposons que l'ensemble d'équation (0.12) a m solutions $r_{\infty k}$, $k = \{1, \dots, m\}$, satisfassent l'Hypothèse 1. Notons $\theta_k^* \in \mathbb{R}^N$, $k = \{1, \dots, m\}$ l'ensemble de points d'équilibre qui définissent des solutions à verrouillage de phase du système (0.15).

Puisque le système (0.15) est invariant sous les transformations de phase $\theta \rightarrow \theta + c\mathbb{1}$, chaque point d'équilibre $\theta_k^* \in \mathbb{R}^N$ du système (0.15) appartient alors à une courbe \mathcal{D}_k^* de points d'équilibre définie par $\mathcal{D}_k^* = \{\theta \in \mathbb{R}^N : \theta = \theta_k^* + c\mathbb{1}, c \in \mathbb{R}\}$. Nous rappelons ici que chacune de ces courbes correspond à une solution à verrouillage de phase du système (0.11).

Comme il a été mentionné dans [4], la synchronisation asymptotique de phase du système (0.11) est vérifiée si les deux propriétés suivantes des solutions du système (0.15) sont satisfaites :

- la stabilité locale de chaque point d'équilibre sur la courbe \mathcal{D}_k^* .
- la stabilité locale de l'ensemble \mathcal{D}_k^* .

Les points d'équilibre du système (0.15) sont définis par l'équation (0.14). En linéarisant le système (0.15) autour de ces points d'équilibre, nous obtenons que les éléments hors-diagonale de linéarisation sont définis comme suit :

$$\begin{aligned} \frac{df_i}{d\theta_l} \Big|_{\theta=\theta^*} &= \frac{K}{N} w_{out}^i w_{in}^l \cos(\theta_l^* - \theta_i^*) \\ &= \frac{K}{N} w_{out}^i w_{in}^l [\cos(\theta_l^*) \cos(\theta_i^*) + \sin(\theta_l^*) \sin(\theta_i^*)] \\ &= \frac{K w_{out}^i}{N} \left[w_{in}^l \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_l}{w_{out}^l K r_\infty} \right)^2} \right) \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) + \frac{w_{in}^l \tilde{\omega}_l \tilde{\omega}_i}{w_{out}^i w_{out}^l (K r_\infty)^2} \right], \end{aligned} \quad (0.16)$$

tandis que les éléments diagonaux sont calculés comme suit :

$$\begin{aligned} \frac{df_i}{d\theta_i} \Big|_{\theta=\theta^*} &= -\frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \cos(\theta_j^* - \theta_i^*) \\ &= \frac{w_{out}^i K}{N} \left[-Nr_\infty \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i Kr_\infty} \right)^2} \right) + w_{in}^i \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i Kr_\infty} \right)^2} \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i Kr_\infty} \right)^2} + \frac{w_{in}^i \tilde{\omega}_i^2}{(w_{out}^i Kr_\infty)^2} \right]. \end{aligned} \quad (0.17)$$

Ainsi, la matrice jacobienne peut être écrite sous la forme suivante :

$$J = \frac{K}{N} \Psi \Upsilon^{-1} (\Upsilon A + bb^\top + cc^\top) = \frac{K}{N} \mathcal{S} J_1 \quad (0.18)$$

avec

$$J_1 = \Upsilon A + bb^\top + cc^\top ; \mathcal{S} = \Psi \Upsilon^{-1} \quad (0.19)$$

$$\Psi = \text{diag}(w_{out}^1, \dots, w_{out}^N) ; \Upsilon = \text{diag}(w_{in}^1, \dots, w_{in}^N)$$

$$A = -\text{diag}(a_1, \dots, a_N) , a_i = \pm Nr_\infty \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i Kr_\infty} \right)^2} \quad (0.20)$$

$$b = \begin{pmatrix} \pm w_{in}^1 \sqrt{1 - \left(\frac{\tilde{\omega}_1}{w_{out}^1 Kr_\infty} \right)^2} \\ \vdots \\ \pm w_{in}^N \sqrt{1 - \left(\frac{\tilde{\omega}_N}{w_{out}^N Kr_\infty} \right)^2} \end{pmatrix} ; c = \begin{pmatrix} w_{in}^1 \frac{\tilde{\omega}_1}{w_{out}^1 Kr_\infty} \\ \vdots \\ w_{in}^N \frac{\tilde{\omega}_N}{w_{out}^N Kr_\infty} \end{pmatrix}$$

En utilisant ces notations, le système linéarisé peut être écrit sous la forme :

$$\dot{\theta} = J \theta = \frac{K}{N} \mathcal{S} J_1 \theta, \quad (0.21)$$

où la matrice constante $J \in \mathbb{R}^{N \times N}$ est définie par les expressions (0.18 - 0.20) et la matrice J_1 est symétrique.

Pour le système (0.21), nous considérons le changement de coordonnées $y = \mathcal{S}^{-1/2} \theta$. Ainsi, nous obtenons

$$\dot{y} = \mathcal{S}^{-1/2} \dot{\theta} = \frac{K}{N} \mathcal{S}^{1/2} J_1 \theta = \frac{K}{N} \mathcal{S}^{1/2} J_1 \mathcal{S}^{1/2} y = \frac{K}{N} \tilde{J} y, \quad (0.22)$$

où $\tilde{J} = \mathcal{S}^{1/2} J_1 \mathcal{S}^{1/2}$.

En utilisant le fait que la matrice \mathcal{S} est diagonale et non singulière (avec des éléments positifs) et la symétrie de la matrice J_1 , nous déduisons que la matrice \tilde{J} est symétrique. En utilisant la loi de Sylvester d'inertie (voir l'annexe B), nous remarquons que les matrices \tilde{J} et J_1 sont deux matrices congruentes. Donc, elles ont la même inertie. En

conséquence, il suffit d'analyser les propriétés spectrales de la matrice J_1 pour étudier les propriétés de stabilité du système linéaire (0.22) ou encore le système linéarisé (0.21). Ainsi, le reste de notre démarche est consacré à l'analyse des propriétés spectrales de la matrice symétrique J_1 d'une dimension arbitraire N .

Dans la suite, nous montrons qu'il n'y a qu'une seule solution qui est localement stable parmi toutes les solutions à verrouillage de phase du modèle de Kuramoto (0.11).

Théorème 5 *Soit l'Hypothèse 1 satisfaite et soit $\mathbb{A} = \{\theta_k^* \in \mathbb{R}^N, k = \{1, \dots, M\}\}$ l'ensemble de toutes les solutions à verrouillage de phase définies par les équations (0.12)-(0.14). Alors, les assertions suivantes sont validées :*

1) *Au plus, il existe une seule solution à verrouillage de phase $\theta^* \in \mathbb{A}$ qui est localement stable.*

2) *Soit θ^+ est la solution à verrouillage de phase définie par l'équation (0.14) et l'équation (0.12) prise avec tous les signes positifs, à savoir*

$$r_\infty = \frac{1}{N} \sum_{j=1}^N \alpha_j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{\sigma_j K r_\infty}\right)^2},$$

alors l'ensemble $\mathcal{D}^+ = \{\theta \in \mathbb{R}^N : \theta = \theta^+ + c\mathbf{1}, c \in \mathbb{R}\}$ correspondant est localement asymptotiquement stable si et seulement si l'inégalité suivante est vérifiée :

$$\sum_{j=1}^N \frac{\alpha_j (1 - 2\left(\frac{\tilde{\omega}_j}{\sigma_j K r_\infty}\right)^2)}{\sqrt{1 - \left(\frac{\tilde{\omega}_j}{\sigma_j K r_\infty}\right)^2}} > 0. \quad (0.23)$$

2.2.3. Analyse de la synchronisation du modèle de Kuramoto avec des interactions attractives et répulsives

Dans le chapitre 4, nous considérons le problème de synchronisation de phase pour le modèle de Kuramoto avec des gains de couplage positifs. Ainsi, seulement les interactions attractives entre les oscillateurs ont été prises en compte. Néanmoins, l'existence des interactions attractives et répulsives dans un réseau peut jouer un rôle important sur la dynamique du système, comme c'est le cas par exemple, dans les circuits de synthèse génétiques [41] et dans les réseaux neuronaux [45]. Récemment, l'utilisation des couplages attractives et répulsives dans le modèle de Kuramoto a été étudiée, le plus souvent numériquement, dans [48], [70], [71].

Dans le chapitre 5, nous analysons la synchronisation de phase et de fréquence du modèle de Kuramoto avec un graphe d'interconnexion pondéré et signé. La motivation de ce travail est d'étudier l'effet des gains de couplage signés sur la dynamique du système et sur la fréquence de synchronisation. Comme dans la section précédente, nous considérons un scénario où les coefficients de couplage entre chaque paire d'oscillateurs peuvent être séparés en deux facteurs : les poids d'entrée et de sortie. En plus, nous supposons que ces pondérations peuvent avoir des valeurs négatives.

En particulier, nous montrons que dans le cas d'un graphe d'interconnexion signé symétriquement, l'existence de solutions à verrouillage de phase pour le modèle de Kuramoto avec des interactions positives garantit l'existence de solutions à verrouillage de phase pour une famille du modèle de Kuramoto avec des interactions signées. Leurs matrices d'interconnexion sont générées à l'aide de transformations de jauge. Dans le cas des graphes d'interconnexion signés d'une façon arbitraire, l'analyse du modèle de Kuramoto n'est pas une tâche triviale ; c'est pourquoi nous nous concentrons sur le cas du réseau structurellement équilibré (complet et signé symétriquement).

Dans cette section, nous commençons par généraliser les conditions d'existence de solutions à verrouillage de phase pour le modèle de Kuramoto formulé dans la section précédente avec des poids d'entrées-sorties mixtes (positifs et négatifs). Ainsi, nous considérons le système suivant :

$$\dot{\theta}_i = \omega_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \{1, \dots, N\}, \quad (0.24)$$

La matrice d'interconnexion $\mathcal{K} = W_{out} W_{in}^T$ est signée symétriquement, c'est-à-dire $k_{ij}k_{ji} > 0$. La dernière inégalité peut être réécrite sous la forme $k_{ij}k_{ji} = w_{in}^i w_{out}^i w_{in}^j w_{out}^j > 0$ pour tous $i, j \in \{1, \dots, N\}$. Ceci peut être satisfait seulement si w_{in}^i et w_{out}^i ont le même signe pour tout $i \in \{1, \dots, N\}$.

Dans la suite de notre analyse, nous utilisons les propriétés du graphe signé structurellement équilibré et l'invariance de jauge. L'ensemble des transformations de jauge dans \mathbb{R}^N est notée par :

$$\mathcal{G} = \{G \in \mathbb{R}^{N \times N}, G = \text{diag}(\delta), \delta = [\delta_1 \dots \delta_N], \delta_i \in \{\pm 1\}\}. \quad (0.25)$$

Soit $\overline{\mathcal{K}} = \overline{W}_{in} \overline{W}_{out}^T$ la matrice d'adjacence avec des éléments positifs. On note aussi que pour chaque graphe orienté avec des pondérations positives, il existe une famille de graphes signés et structurellement équilibrés qui possèdent les mêmes pondérations mais avec des signes différents. Toutes ces réalisations de graphes signés sont liées par les transformations de jauge. Ainsi, nous définissons l'ensemble des matrices d'adjacence

\mathcal{K}_G généré par les transformations de jauge comme suit :

$$\mathcal{K}_G = \{\mathcal{K}_g = G\overline{\mathcal{K}}G, G \in \mathcal{G}\}, \quad (0.26)$$

où \mathcal{G} est l'ensemble de toutes les transformations de jauge définies dans (0.25) et l'ensemble \mathcal{K}_G contient 2^{N-1} matrices distinctes .

Le théorème suivant donne la forme de solutions à verrouillage de phase pour une famille du modèle de Kuramoto avec des graphes d'interconnexion signés générés à l'aide des transformations de jauge.

Théorème 6 *Soit $\mathcal{G} = \{G = \text{diag}(\delta_1, \dots, \delta_N), \delta_i \in \{\pm 1\}\}$ l'ensemble de toutes les transformations de jauge dans \mathbb{R}^N . Alors, chaque modèle de Kuramoto (0.24) avec un graphe d'interconnexion signé symétriquement défini par la matrice d'adjacence $\mathcal{K}_g = G\mathcal{K}G$, a une solution à verrouillage de phase si et seulement si l'Hypothèse 1 est satisfaite. En plus, la fréquence de synchronisation est définie par (0.13) et les solutions à verrouillage de phase ont la forme suivante :*

$$\theta_g(t, \theta_g^*) = \theta_g^* + \omega_s t \mathbf{1}_n + c \mathbf{1}_n, \quad (0.27)$$

où $c \in \mathbb{R}$ est une constante arbitraire tandis que le vecteur θ_g^* est défini par les relations suivantes :

$$\cos(\theta_g^*) = G \cos(\theta^*) \quad ; \quad \sin(\theta_g^*) = G \sin(\theta^*), \quad (0.28)$$

et le vecteur θ^* est la solution à verrouillage de phase du modèle de Kuramoto (0.11), définie dans (0.14).

Remarque 1 *Le théorème ci-dessus stipule que tout modèle de Kuramoto défini par la matrice d'adjacence $\mathcal{K}_g = G\mathcal{K}G$ avec $G \in \mathcal{G}$, a une solution à verrouillage de phase si et seulement si le modèle de Kuramoto (0.11) avec seulement des pondérations positives a une solution à verrouillage de phase. De plus, les angles θ_g^* qui correspondent aux entrées négatives de la matrice de jauge G , ont un décalage de π par rapport aux angles du modèle (0.11), c'est à dire $\theta_{g_j}^* = \theta_j^* + \pi$.*

La deuxième partie de chapitre 5 est consacrée à l'étude de stabilité des solutions à verrouillage de phase pour le modèle de Kuramoto avec des interactions attractives et répulsives. Dans le théorème suivant, nous démontrons que parmi toutes les solutions à verrouillage de phase pour le système (0.24), il n'y a qu'une solution localement asymptotiquement stable.

Théorème 7 *Considérons les modèles de Kuramoto (0.11), (0.24). Soit $\mathcal{K} = \overline{W}_{out} \overline{W}_{out}^\top$ et l'Hypothèse 1 est satisfaite. Soit $\mathcal{K}_G = \{\mathcal{K}_g = G\mathcal{K}G, G \in \mathcal{G}\}$ la famille des matrices*

d'adjacence générées par la transformation de jauge. Pour une matrice de jauge donnée G , soit $\mathbb{A}_g = \{\theta_{gk}^* \in \mathbb{R}^N, k = \{1, \dots, M\}\}$ l'ensemble de toutes les solutions à verrouillage de phase possibles pour le système (0.24) définies dans le Théorème 6. Alors les assertions suivantes sont validées :

1) Au plus, il existe une seule solution à verrouillage de phase $\theta_g^* \in \mathbb{A}_g$ qui est localement stable.

2) Soit θ^+ la solution à verrouillage de phase définie par l'équation (0.14) et l'équation (0.12) prise avec tous les signes positifs. Soit θ_g^+ la solution à verrouillage de phase correspondante au système (0.24) définie par (0.28). Alors, l'ensemble $\mathcal{D}_g^+ = \{\theta \in \mathbb{R}^N : \theta = \theta_g^+ + c\mathbf{1}, c \in \mathbb{R}\}$ est localement asymptotiquement stable si et seulement si l'inégalité suivante est vérifiée :

$$\sum_{j=1}^N \frac{|w_{in}^j| (1 - 2(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty})^2)}{\sqrt{1 - (\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty})^2}} > 0. \quad (0.29)$$

3. Modèle de Stuart-Landau

Dans les chapitres précédents, nous avons utilisé un modèle de phase pour étudier le phénomène de synchronisation engendré par les interactions entre des oscillateurs périodiques dans un réseau. En fait, les modèles de phase représentent une bonne approximation de dynamique des systèmes avec des orbites à cycle limite lorsque le couplage dans le réseau est faible. Ceci est le cas du modèle de Kuramoto dans sa forme classique. Néanmoins, lorsque le couplage est assez fort, cette approximation perd de sa consistance. Il devient donc nécessaire de prendre en compte la dynamique entière de chaque élément oscillant, y compris sa phase et son amplitude. L'interaction de ces deux variables dans un réseau des oscillateurs à cycle limite permet d'analyser des nouveaux phénomènes dans l'évolution collective. Par conséquent, nous proposons dans le chapitre 6 d'étudier la synchronisation dans un réseau des oscillateurs de Stuart-Landau. Ce dernier permet de prendre en compte la dynamique des amplitudes ainsi que celle des phases.

3.1. Généralités

Nous supposons que la dynamique de chaque oscillateur est décrite par l'équation Stuart-Landau

$$\dot{z}_i = -|z_i|^2 z_i + \mu_i z_i + u_i = f(z_i, \mu_i) + u_i, \quad i \in I \quad (0.30)$$

où $z_i, u_i \in \mathbb{C}$ sont respectivement l'état et l'entrée d'oscillateur i , $\mu_i = \beta_i + i\omega_i \in \mathbb{C}$ est un paramètre complexe qui définit le comportement asymptotique de l'oscillateur i .

Nous supposons que les oscillateurs sont connectés via un couplage diffusif qui représente une relation statique entre les entrées et les états des oscillateurs, à savoir pour l'oscillateur i , le couplage est donné par la relation suivante :

$$u_i = -\gamma \left(d_{i1}(z_i - z_1) + d_{i2}(z_i - z_2) \dots + d_{iN}(z_i - z_N) \right), \quad (0.31)$$

où $\gamma > 0$ est un paramètre scalaire qui correspond à la force de couplage entre les oscillateurs.

Nous supposons que le graphe d'interconnexion est non orienté, connexe et pondéré par des poids positifs. Ainsi, la matrice Laplacienne L de ce graphe a une seule valeur propre égale à zéro et les autres sont strictement positives.

La dynamique d'ensemble du réseau peut être décrite par l'équation différentielle suivante :

$$\dot{\mathbf{z}} = F(\mathbf{z}) - \gamma L\mathbf{z}, \quad (0.32)$$

où $\gamma \in \mathbb{R}_+$ est le gain de couplage, la fonction $F : \mathbb{C}^N \rightarrow \mathbb{C}^N$ est donnée par

$$F(\mathbf{z}) = [f(z_i, \mu_i)]_{i \in I}. \quad (0.33)$$

et la matrice Laplacienne est définie comme suit :

$$L = \begin{bmatrix} \sum_{i=2}^N d_{1i} & -d_{12} & \dots & -d_{1N} \\ -d_{21} & \sum_{i=1, i \neq 2}^N d_{2i} & \dots & -d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{N1} & -d_{N2} & \dots & \sum_{i=1}^{N-1} d_{Ni} \end{bmatrix} \quad (0.34)$$

Nous montrons d'abord que les trajectoires du réseau d'oscillateurs de Stuart-Landau (0.32) sont ultimement bornées (ultimately bounded). Ceci signifie que toutes les solutions vont finir dans un domaine borné. Précisément, nous prouvons la proposition suivante :

Proposition 1 *Considérons le système (0.32) et soit le graphe d'interconnexion connexe et non orienté. Alors, les solutions du système (0.32) sont ultimement bornées et satisfont la borne suivante :*

$$|\mathbf{z}(t, \mathbf{z}_0)| \leq \sqrt{2\mu N}. \quad (0.35)$$

Dans le cas d'un réseau des oscillateurs identiques, la synchronisation est souvent décrite en terme d'une évolution identique (asymptotique) aux éléments du réseau et par la suite formulée comme la stabilité (asymptotique) d'une variété de synchronisation

$$\mathcal{S} = \{\mathbf{z} \in \mathbb{C}^N : \mathbf{z}_1 = \mathbf{z}_2 = \dots = \mathbf{z}_N\}. \quad (0.36)$$

Le comportement des réseaux avec des unités non identiques est plus complexe. En effet, la variété de synchronisation \mathcal{S} n'existe pas toujours à cause des différences de la dynamique des oscillateurs. Néanmoins, il est bien connu de la littérature que ces réseaux hétérogènes peuvent présenter un certain type de synchronisation. Une des approches possibles est de considérer la synchronisation «pratique» du réseau ; lorsque les différences entre l'évolution de la dynamique des différentes unités sont bornées et décroissent infiniment en augmentant la valeur du gain de couplage γ . Cette approche est adoptée dans notre travail. En particulier, nous montrons pour des grandes valeurs du gain de couplage γ que le comportement du réseau peut être décomposé en deux parties : une correspond à la dynamique d'un certain oscillateur "moyennisé" ("mean-field"), tandis que l'autre décrit la dynamique des unités du réseau par rapport à la dynamique "moyennisée".

3.2. Reformulation de la dynamique du réseau

Pour une meilleure présentation de notre approche, nous avons ensuite réécrit ce système dans un nouveau cadre de coordonnées. Ceci est plus pratique pour présenter notre analyse. Tous d'abord, nous séparons la dynamique du réseau en deux parties, une partie linéaire et l'autre non linéaire. Ainsi, le système est réécrit sous cette forme :

$$\dot{\mathbf{z}} = A_\gamma \mathbf{z} - C(\mathbf{z}) \mathbf{z}, \quad (0.37)$$

où

$$A_\gamma = \mathcal{M} - \gamma L \quad (0.38)$$

et les deux matrices diagonales $C(\mathbf{z})$ et \mathcal{M} sont définies comme suit :

$$C(\mathbf{z}) = \begin{bmatrix} |\mathbf{z}_1|^2 & 0 & \dots & 0 \\ 0 & |\mathbf{z}_2|^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & |\mathbf{z}_N|^2 \end{bmatrix} \quad \text{et} \quad \mathcal{M} = \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mu_N \end{bmatrix}$$

Ensuite, nous remarquons que la matrice A_γ peut être écrite sous la forme $A_\gamma = \gamma(-L + \varepsilon \mathcal{M})$ où $\varepsilon = 1/\gamma$. Alors, pour des valeurs suffisamment petites de ε (ou encore des valeurs suffisamment grandes du gain de couplage γ), la matrice A_γ peut être vue comme une matrice Laplacienne avec des perturbations. En utilisant des résultats de la théorie de la perturbation des matrices, nous pouvons montrer que la matrice A_γ a une valeur propre positive et les autres $N - 1$ valeurs propres négatives. Précisément,

$$\lambda_1(A_\gamma) = \gamma(-\lambda_1(L) + c_1\varepsilon + o(\varepsilon)) = \frac{1}{N} \sum_{i=1}^N \mu_i + O(\varepsilon),$$

et $\lambda_j(A)$ ($j = 2, \dots, N$) sont proportionnelles au γ : $Real(\lambda_j(A)) \rightarrow -\infty$ quand $\gamma \rightarrow \infty$.

Ainsi, nous assumons que la force de couplage γ est suffisamment grande. Donc, la matrice A_γ est diagonalisable. Précisément, nous proposons l'hypothèse suivante :

Hypothèse 2 *Il existe une $\gamma^* > 0$ sachant que pour tout $\gamma \geq \gamma^*$, la matrice A_γ définie dans (0.38) peut être factorisée comme le suivant :*

$$A_\gamma = V_\gamma \Lambda_\gamma V_\gamma^{-1}, \quad (0.39)$$

où $\Lambda_\gamma \in \mathbb{C}^{N \times N}$ est une matrice diagonale et ses éléments diagonaux sont les valeurs propres de A_γ . La matrice $V_\gamma \in \mathbb{C}^{N \times N}$ est une matrice orthogonale ($V_\gamma^{-1} = V_\gamma^\top$). En plus, il existe $k \in \{1, \dots, N\}$ sachant que $Re(\lambda_k) > \max_{j \in \{1, \dots, N\}, j \neq k} Re(\lambda_j)$.

Nous assumons que les valeurs propres λ_j de A_γ sont ordonnées comme suit $Re(\lambda_1) > Re(\lambda_2) \geq \dots \geq Re(\lambda_N)$. Ensuite, nous décomposons la matrice A_γ de cette façon. Nous écrivons d'abord la matrice Λ_γ sous cette forme :

$$\Lambda = \lambda_1(A_\gamma)\mathbb{I} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2(A_\gamma) - \lambda_1(A_\gamma) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_N(A_\gamma) - \lambda_1(A_\gamma) \end{pmatrix} = \Lambda_1 + \Lambda_2. \quad (0.40)$$

En utilisant cette notation, nous obtenons

$$A = V\Lambda_1V^\top + V\Lambda_2V^\top = \lambda_1\mathbb{I} + D, \quad (0.41)$$

où $D = V\Lambda_2V^\top$. Notons que si l'Hypothèse 2 est satisfaite, les $(N - 1)$ valeurs propres non nulles de la matrice Λ_2 ont une partie réelle négative : pour $i = 2, \dots, N$ nous avons $\lambda_i(\Lambda_2) \rightarrow -\infty$ quand $\gamma \rightarrow +\infty$. Donc, nous pouvons conclure que $D \leq 0$.

En utilisant ces notations, nous pouvons écrire le système (0.37) sous cette forme :

$$\dot{\mathbf{z}} = (\lambda_1 \mathbb{I} - C(\mathbf{z})) \mathbf{z} + D \mathbf{z}, \quad (0.42)$$

rappelons que l'équation (0.42) n'est qu'une autre forme d'écrire les équations du réseau des oscillateurs (0.32). Ainsi, les transformations que nous avons présentées nous permettent d'interpréter le réseau d'oscillateurs hétérogènes comme un réseau où tous les éléments ont des dynamiques identiques. Aussi, ces éléments sont couplés via une matrice d'interconnexion D possédant des propriétés similaires à la matrice Laplacienne L . En utilisant le modèle (0.42) et les propriétés spectrales des matrices A_γ et D , nous sommes capable de décomposer explicitement la dynamique du réseau en deux parties avec des rôles très différents : la première partie correspond à l'oscillateur "moyennisé" noté par \mathbf{z}_m , tandis que la deuxième partie reflète l'erreur de synchronisation entre les oscillateurs dans le réseau notée \mathbf{e} . Ces deux parties sont définies comme suit :

$$\mathbf{z}_m = \vartheta_{\mathbf{1}}^\top \mathbf{z} \quad (0.43)$$

et

$$\mathbf{e} = \mathbf{z} - \vartheta_{\mathbf{r}_1} \mathbf{z}_m = \mathbf{z} - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{1}}^\top \mathbf{z} = (\mathbb{I} - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{1}}^\top) \mathbf{z} = P \mathbf{z}, \quad (0.44)$$

où les vecteurs $\vartheta_{\mathbf{r}_1}$ et $\vartheta_{\mathbf{1}}$ sont les vecteurs propres à gauche et à droite de la matrice A_γ associés à la valeur propre $\lambda_1(A_\gamma)$. La matrice $P = \mathbb{I} - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{1}}^\top$ est une matrice de projection.

3.3. Analyse de stabilité

Dans la section précédente, nous avons vu que dans le cas d'un réseau hétérogène avec un nombre arbitraire N d'oscillateurs et pour une valeur grande du gain de couplage γ , le comportement du réseau peut être décomposé en deux parties :

- l'une d'elles décrit la dynamique de l'oscillateur "moyennisé" \mathbf{z}_m .
- l'autre décrit la dynamique des oscillateurs du réseau par rapport à la dynamique moyennisée.

C'est pourquoi nous caractérisons les propriétés de synchronisation du réseau en termes de stabilité des deux ensembles compacts :

- la variété de la synchronisation $\mathcal{S}(\gamma)$ que nous définissons sous la forme

$$\mathcal{S}(\gamma) = \{\mathbf{e} \in \mathbb{C}^N : \mathbf{e}_1 = \mathbf{e}_2 = \dots = \mathbf{e}_N = 0\}. \quad (0.45)$$

- l'ensemble invariant \mathcal{W} qui correspond à l'oscillateur "moyennisé". Il contient l'origine et le cycle limite de \mathbf{z}_m .

La dynamique de l'oscillateur moyennisé

A partir de la définition d'oscillateur moyennisé \mathbf{z}_m et après un simple calcul mais assez long, la dynamique $\dot{\mathbf{z}}_m$ peut être écrite en fonction de \mathbf{z}_m et \mathbf{e} comme suit :

$$\dot{\mathbf{z}}_m = (\lambda_1 - c|\mathbf{z}_m|^2) \mathbf{z}_m + \mathbf{f}_2(\mathbf{z}_m, \mathbf{e}), \quad (0.46)$$

où $c \in \mathbb{C}$ est une constante sous la forme suivante :

$$c = \sum_{i=1}^N \vartheta_{li}^* \vartheta_{ri}^* \vartheta_{ri}^2 \quad (0.47)$$

et la fonction $\mathbf{f}_2(\mathbf{z}_m, \mathbf{e})$ est donnée par l'expression suivante

$$\begin{aligned} \mathbf{f}_2(\mathbf{z}_m, \mathbf{e}) = & \\ & - \vartheta_{\mathbf{1}\mathbf{1}}^* \begin{pmatrix} |z_1|^2 + z_1^* v_{r1} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |z_N|^2 + z_N^* v_{rN} z_m \end{pmatrix} \mathbf{e} \\ & - \vartheta_{l_1}^* \begin{pmatrix} (v_{r1} z_m)^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{rN} z_m)^2 \end{pmatrix} \bar{\mathbf{e}}, \end{aligned} \quad (0.48)$$

où $v_{r_{ij}}$ correspondent aux éléments ij -th de la matrice V_γ .

La dynamique des erreurs de synchronisation

A partir de la définition du vecteur des erreurs de synchronisation \mathbf{e} et les propriétés des matrices P et D , la dérivée $\dot{\mathbf{e}}$ peut être écrite sous cette forme :

$$\dot{\mathbf{e}} = D \mathbf{e} + (\lambda_1 \mathbf{e} - P C(\mathbf{e}, z_m)(\mathbf{e} + z_m \mathbf{1})). \quad (0.49)$$

Ainsi, les équations (0.46) et (0.49) représentent les dynamiques du réseau dans les nouvelles coordonnées, c'est-à-dire en termes de \mathbf{z}_m et \mathbf{e} . Le système (0.32) peut être vu sous la forme des deux sous-systèmes suivants :

$$\dot{\mathbf{z}}_m = (\lambda_1 - c|\mathbf{z}_m|^2) \mathbf{z}_m + \mathbf{f}_2(\mathbf{z}_m, \mathbf{e}), \quad (0.50)$$

$$\dot{\mathbf{e}} = D \mathbf{e} + (\lambda_1 \mathbf{e} - P C(\mathbf{e}, z_m)(\mathbf{e} + z_m \mathbf{1})). \quad (0.51)$$

Dans la section suivante, nous présentons les résultats de notre analyse de stabilité du réseau. En premier lieu, nous donnons les propriétés de stabilité du sous-système (0.51). Puis, nous analysons la stabilité du sous-système (0.50).

3.3.1. Synchronisation pratique d'un réseau diffusif d'oscillateurs Stuart-Landau

Dans cette section, nous analysons le système (0.51) pour des grandes valeurs du gain de couplage γ , c'est-à-dire, nous supposons que l'Hypothèse 2 est satisfaite. Nous présentons une analyse de la stabilité de l'ensemble $\mathcal{S}(\gamma)$ et nous formulons les conditions qui garantissent la stabilité asymptotique globale pratique de cet ensemble. Ceci implique la synchronisation pratique du réseau. Précisément, nous montrons le théorème suivant :

Théorème 8 *Considérons le système (0.32) et soit l'Hypothèse 2 satisfaite. Alors, l'ensemble $\mathcal{S}(\gamma)$ est globalement pratiquement stable pour tout $\gamma \geq \gamma^*$. En plus, il existe $T^* > 0$ et une constante $c > 0$ indépendantes de γ sachant que l'erreur de synchronisation $\mathbf{e}(\mathbf{t})$ satisfait la borne suivante pour tout $t \geq T^*$*

$$|\mathbf{e}(\mathbf{t})|^2 \leq \frac{2c}{|\text{Real}(\lambda_2 - \lambda_1)|}. \quad (0.52)$$

Stabilité pratique de l'ensemble invariant d'oscillateur moyennisé

Dans cette section, nous présentons l'étude du comportement de z_m . Notons que la dynamique de z_m peut être vue comme un oscillateur de Stuart-Landau perturbé par une entrée qui dépend linéairement de $|\mathbf{e}|$, à savoir

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + u, \quad (0.53)$$

où $u = f_2(z_m, \mathbf{e})$.

C'est pourquoi, nous allons d'abord analyser les propriétés de stabilité et de robustesse d'un oscillateur de Stuart-Landau généralisé. Ensuite, nous allons appliquer ces résultats à l'analyse du système (0.53).

L'oscillateur de Stuart-Landau généralisé a la forme suivante :

$$\dot{z} = -\nu|z|^2 z + \mu z + u \quad (0.54)$$

où $z, u \in \mathbb{C}$ sont respectivement l'état et l'entrée de l'oscillateur. $\nu = \nu_1 + i\nu_2 \in \mathbb{C}$ et $\mu = \mu_1 + i\mu_2 \in \mathbb{C}$ sont des paramètres complexes et nous assumons que $\nu_1 > 0$.

L'ensemble invariant du système (0.54)

Nous considérons d'abord le système sans perturbations, c'est-à-dire $u \equiv 0$. L'ensemble invariant est composé de deux éléments : l'origine et le cycle limite d'amplitude $\sqrt{\mu_1/\nu_1}$, à savoir

$$\mathcal{W} = \{z = 0\} \cup \left\{ z \in \mathbb{C} : |z| = \sqrt{\frac{\mu_1}{\nu_1}} \right\}. \quad (0.55)$$

Analyse de stabilité de l'ensemble invariant \mathcal{W}

Nous prouvons le théorème suivant pour l'oscillateur de Stuart-Landau généralisé.

Théorème 9 *Considérons l'oscillateur de Stuart-Landau (0.54) avec une entrée $u \equiv 0$. Les assertions suivantes sont validées :*

(1) *Si $\operatorname{Re}(\nu) \leq 0$ alors l'origine $z \equiv 0$ est globalement asymptotiquement stable pour le système (0.54).*

(2) *Si $\operatorname{Re}(\nu) > 0$ alors le cycle limite $\mathcal{W}_1 = \{z : |z| = \sqrt{\nu_1/\mu_1}\}$ est presque globalement asymptotiquement stable et l'origine est anti-stable pour le système (0.54).*

En plus, la fréquence d'oscillation du cycle limite est définie comme suit :

$$\omega = \mu_2 - \frac{\nu_2}{\nu_1} \mu_1$$

Etude de robustesse d'oscillateur de Stuart-Landau : analyse de stabilité de l'ensemble invariant \mathcal{W} en présence de perturbations

En se basant sur les résultats de l'article [8], nous analysons les propriétés de stabilité et de robustesse du système (0.54). Tout d'abord, nous définissons la norme $|\cdot|_{\mathcal{W}}$ comme le suivant :

$$|z|_{\mathcal{W}} = \begin{cases} |z| & \text{if } |z| \leq \frac{1}{\sqrt{2}} \sqrt{\alpha} \\ \sqrt{|z|^2 - \frac{\mu_1}{\nu_1}} & \text{if } |z| \geq \frac{1}{\sqrt{2}} \sqrt{\alpha} \end{cases} \quad (0.56)$$

où $\frac{\mu_1}{\nu_1} = \alpha$.

Le théorème suivant assure que la propriété du gain asymptotique est vérifiée pour le système (0.54). C'est-à-dire, asymptotiquement, la distance entre la trajectoire de l'oscillateur et l'ensemble \mathcal{W} est proportionnelle à la taille des perturbations $\|d\|_{\infty}$.

Théorème 10 *Considérons le système (0.54) avec des conditions initiales z_0 et soit l'ensemble \mathcal{W} définie par (0.55). Alors, le système (0.54) a la propriété du gain asymptotique ; à savoir la borne supérieure suivante est satisfaite*

$$\limsup_{t \rightarrow +\infty} |z(t, z_0; u)|_{\mathcal{W}} \leq \eta(\|u\|_{\infty}). \quad (0.57)$$

Synchronisation pratique du réseau d'oscillateurs de Stuart-Landau

Considérant l'équation (0.53) qui décrit la dynamique de l'oscillateur "moyennisé" avec $u = f_2(z_m, \mathbf{e})$:

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + u,$$

En utilisant les résultats précédents sur la robustesse de l'ensemble invariant \mathcal{W} (Théorème 10) et sur la stabilité pratique des erreurs de synchronisation (Théorème 8), nous obtenons le résultat suivant :

Théorème 11 *Considérons le réseau des oscillateurs de Stuart-Landau décrit par les équations (0.32), (0.33). Considérons aussi l'oscillateur "moyennisé" du réseau défini par (0.43) et sa dynamique donnée par l'équation (0.53). Soit l'Hypothèse 2 satisfaite. Alors, le système (0.53) a la propriété du gain asymptotique. En plus, pour tout $\varepsilon > 0$, il existe un gain $\gamma \geq \gamma^*$ sachant que la borne suivante est satisfaite*

$$\limsup_{t \rightarrow +\infty} |z_m(t, z_0)|_{\mathcal{W}} \leq \varepsilon.$$

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Notation

Classes and sets

- \mathbb{R} represents the set of real numbers.
- \mathbb{R}_+ represents the set of real positive or null numbers.
- $\mathbb{R}_{>0}$ represents the set of real positive numbers.
- $\mathbb{R}^{m \times n}$ denotes the space of $m \times n$ real matrices.
- \mathbb{C} represents the set of complex numbers.
- $\mathbb{C}^{m \times n}$ denotes the space of $m \times n$ complex matrices.

Vectors and matrices

- Throughout the thesis, capital letters indicate a matrix.
- a_{ij} is the element of the i –*th* row and of the j –*th* column of the matrix A .
- A^\top , the symbol \top indicates the transpose operator.
- \bar{A} , the symbol $\bar{\cdot}$ indicates the conjugate operator.
- A^* , the symbol $*$ indicates the conjugate transpose operator.
- \mathbb{I}_N represents the matrix identity of size N .

- $\mathbf{1}$ denotes a column vector with appropriate size of 1's i.e. $\mathbf{1}^\top = [1 \dots 1]$.
- Given a vector $a^\top = [a_1 \dots a_n]$, we use the notation $A = \text{diag}(a)$ for the square diagonal matrix A with elements a_i on the diagonal i.e.

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_N \end{pmatrix}$$

- Denote the i -th eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$ by $\lambda_i(A)$.
- $A \vartheta_{r_i} = \lambda_i \vartheta_{r_i}$, ϑ_{r_i} is the i -th right eigenvector of matrix A corresponding to eigenvalue λ_i .
- $\vartheta_{l_i}^\top A = \lambda_i \vartheta_{l_i}^\top$, ϑ_{l_i} is the i -th left eigenvector of matrix A corresponding to eigenvalue λ_i .
- A matrix A , of real or complex elements, is orthogonal if $A^\top = A^{-1}$.

Norms

- $|\cdot|$ denotes the absolute value of a scalar.
- $\| \cdot \|$ represents the Euclidian norm of a vector a i.e.

$$\| a \|^2 = a^\top a \quad \text{if } a \in \mathbb{R}^N.$$

$$\| a \|^2 = a^* a \quad \text{if } a \in \mathbb{C}^N.$$

- The (Euclidian) quasi-norm of a vector $a \in \mathbb{C}^N$ is defined as $[a]^2 = a^\top a$.
- $\text{Re}(a)$ and $\text{Im}(a)$ represent the real and imaginary parts of $a \in \mathbb{C}$.
- $\|(\cdot)\|_\infty$ denotes the (essential) supremum of a function of time, i.e.

$$\|f\|_\infty = (\text{ess})\text{sup}\{|f(t)|, t \geq 0\}.$$

- For a compact set \mathcal{A} , $\|(\cdot)\|_{\mathcal{A}}$ denotes the standard point-to-set distance, i.e.

$$\|x\|_{\mathcal{A}} = \inf_{s \in \mathcal{A}} \|x - s\|.$$

Chapter 1

Introduction

Self-organization is a remarkable characteristic of nature, and it is also a base concept in science. One can easily identify the ordered patterns in biological, physical, and social systems. It is studied in the different branches of science and goes by various names : (self)-synchronization, orderliness, synergy, and emergent behaviour to name a few. Usually, it is viewed as a static aspect of structural organization ; nevertheless, it can also be viewed as an attitude of collective dynamics. In a dynamically ordered state, the individual components of a system are well coordinated ; therefore, the system is qualified to display a coherent performance. A classic example of a dynamical order are the systems of periodic nature. The repeated occurrence of events at regular intervals can be found at all scales. The functioning of all living organisms seems to be dependent on specified cycles. Periodic oscillations are also one of the most ubiquitous types of dynamical behavior in the real world. The presence of persisting oscillations can be understood in two different ways. One prevalent class of oscillations characterize the Hamiltonian systems : in the conservative systems, oscillations are maintained due to the conservation of energy (for example, the frictionless pendulum). The second interesting type of periodic oscillations is self-sustained oscillations. In this case, oscillations can be viewed as a result of instability of the static state, but also as the consequence of the existence of a stable periodic attractor.

Moreover in various physical systems coordinated action of individual components can be freely developed. In fact, emergence of collective organization is a fundamental problem in the theory of complex systems, and collective dynamic organization is also closely related to the network synchronization phenomena. Actually, the systems synchronize when some components of such systems are connected and strong interactions among their individual dynamics emerge. Synchronization can be generated by an external force producing the entrainment of the system. Furthermore, synchronization can also

be induced as a consequence of interaction among system elements only. This form of self-organized synchronization plays a fundamental role in various complex systems of several natures, i.e. especially in the field of life sciences [140] and it has been studied via a variety of mathematical models [63], [77].

When analyzing collective dynamics of biological or physical systems, one of the first questions to address is how to model dynamics of individual components in the population. Even when such components represent just a single neuron, or a biological cell, it is difficult to incorporate all their internal dynamics into a model; since analysis of the overall population model becomes intractable. Fortunately, investigations have displayed that essentially the same models are repeated in different phenomena in biological, neural, or social self-synchronization [82]. The prevalent use of canonical models of oscillatory periodic dynamics in diverse research areas is an example of the methodology that is dominant in the science of complex systems. The underlying perception is that the validity of results found in a certain canonical model may extend to a specific real system under some consideration [77].

The importance of these models lies in the fact that, while being simple, they capture fundamental characteristics of many different systems with oscillatory periodic behavior. For example, progressive wave activity can be observed inside simple biological cells and meet themselves in the cell populations. They are of an essential importance for the functioning of the heart and are also observed in the brain. Similar structures of waves exist in ecological populations, such as the movement of schools of fish. Each of these systems may be hard to describe mathematically, and this difficulty arises for different reasons. In the case of neuronal activity, the generation of action potentials in a neuron is the result of the collective action of millions of ion channels and transmissions along the synaptic membranes. The numerical simulation of a small neural network activity may require important computational power. The interactions between all the molecules in the cell cannot be known, and experiments on a simple molecule to study their behaviour with enough precision cannot be possible. Nevertheless, these kinds of systems have clear characteristics in common, namely that all of them act by repeating a functional cycle. Even though deviations in this cycle may exist from system to system, what is important for the analysis is that their operation requires the periodic and repeated execution of an ordered functional cycle. That is why in mathematical study of these kinds of synchronization phenomena, attention is mostly focused on the analysis of generic models describing self-synchronization processes.

Analysis of synchronization as collective behavior represents a large part of the study of complex systems. In the last decades, numerous research groups have actively been

working on these problems. There are many books devoted to the analysis of synchronization phenomena. The first mathematical formulation of the synchronization problem for nonlinear oscillators belongs to A. A. Andronov [6]. The textbook by N. Wiener [137] and A. T. Winfree [140] recognized its ubiquity in the natural world, and give a relevant introduction into the collective behavior of biological oscillators and the generation of characteristic rhythms in the brain. Based on the fact that a network of Andronov-Hopf oscillators synchronizes for the large values of interconnection gains; in the 1980s Y. Kuramoto proposed a reduced order model which characterize limit case behaviour of Andronov-Hopf oscillators [63]. More recently, an extensive monograph on collective dynamics, including self-synchronization, has been published, see [77], [121], [124].

As previously mentioned, collective synchronization plays an important role in a wide range of disciplines, such as physics [49], ecology [120], biology [131], chemical [136], social [2], and technological applications [142]. Within this rich modelling of synchronization among interconnected oscillators, one can remark that various systems essentially illustrate networks of coupled periodic elements. For this reason, our analysis is concentrated on the interactions among elements, more than the elements themselves, and this thesis focuses on the canonical model of a continuous limit-cycle oscillator network with directed, weighted, and signed interconnection graphs. Thus, we consider the functional cycle of these systems as a circle and the dynamic of such oscillators can be described by the radius and the velocity of rotation along this circle. The progress along this cycle can also be illustrated by an angle variable that increases at a constant rate. In this view, the oscillatory network model is often simplified by modelling each individual dynamic by a simple phase oscillator such as the case of the Kuramoto model [63], [64]. This model is often used to study the emergent collective behavior because it is conceptually simple and easy to implement numerically.

In the scientific literature on coupled phase oscillators, many studies can be found which focus on the existence conditions and the stability of the synchronization dynamic [3]. It is widely used in numerous applications in several domains such as biology [18], [61], [114], [115], chemistry [63], [136], and physics [38], [72], [138]. More recently, it has also attracted attention from the automatic control community, see e.g. [4], [24], [31], [86].

Y. Kuramoto in [63], was the first to derive phase equations for autonomous oscillators with averaging techniques and perturbation approach. This phase equation is called the Kuramoto oscillator and is coming by a reduction of limit cycle oscillators to the dynamics of a single parameter, the phase of the system (see Chapter 2). Many extensions of the Kuramoto model have been investigated, and its mathematical implications have been explored. An excellent review about phase transitions, the Kuramoto model and

its various generalizations has been published in [77], see also [11], [84], [133] for detailed overview.

In this thesis, we start by investigating several kinds of synchronization behaviors (frequency synchronization and phase synchronization) observed in systems of coupled phase oscillators with various interaction configurations among oscillators. Our intention is to establish the necessary and sufficient conditions for the existence of synchronized state and to study its stability properties. In particular, the emphasis will be placed on the effect of network structure (bidirectional, weighted, signed, and non-symmetric interconnection graphs) on ordered states.

Organization and contributions

The contents of this thesis are organized into five main chapters, followed by a conclusion. In the following paragraphs, we outline the contents of each chapter.

Chapter 2 serves as a reference where we give a generalized theoretical background and concepts that will be used in this thesis. Theoretical definitions are introduced and a mathematical framework is presented within which the work is developed. Moreover, a basic review of some research results in this area is provided and previous pertinent works are reported.

In Chapter 3, we study heterogeneous Kuramoto oscillator populations with distinct natural frequencies and a nontrivial coupling topology. Therefore, we consider a system of oscillators which interact through a network where each element is connected to a subset of the elements in the entire system and with heterogeneous gains. Our emphasis will be placed on the existence conditions of frequency synchronization in complex oscillator networks. Additionally, we give an analytical expression for the limit of synchronization frequency in the case when coupling strength is sufficiently large. In the case of symmetric graphs, it is always true that synchronization frequency is equal to the average of natural frequencies. In Section 3 we show that is not true in the case of non-symmetric interconnection graphs and for fixed natural frequencies and the given coupling structure, frequency of synchronization can also depend on the coupling strength. Finally, we introduce a novel linear model that possesses properties similar to those of the Kuramoto model and asymptotically it can be reduced to the Kuramoto model. Next we present new necessary and sufficient conditions for the existence of phase locked solutions for the Kuramoto model with a complex interconnection digraph.

In Chapter 4, we propose an original generalization of the Kuramoto phase equation by incorporating an additional source of heterogeneity to the Kuramoto model. Thus, we

assume that the individual contribution of each oscillator to the mean field is weighted by two factors. These factors present the input and output weights of the oscillator. Consequently, they define a heterogeneous interaction in the system. Therefore, we present the necessary and sufficient conditions for the existence of phase locked solutions of the Kuramoto model with a complete weighted asymmetric interconnection digraph. We also give an exact expression of synchronization frequency as a function of the weights of the interconnection graph and the natural frequencies. Finally, we present a local stability analysis of the phase locked solutions and similarly to the classical Kuramoto model we show that all phase locked solutions are unstable except for one.

In Chapter 5, we analyze phase and frequency synchronization of the Kuramoto model with a directed signed weighted graph of interconnection. Motivation for this work is to examine the effect of the signed coupling gains on the system's dynamics and on synchronization frequency. It turns out that the ideas behind stability analysis of phase-locking for the input-output weighted Kuramoto model introduced in Chapter 4 carry over to the case of weighted signed digraphs, although some results are different. We show that in the case of sign-symmetric graphs of interconnection, existence and stability of phase locked solutions for the Kuramoto model with positive weights guarantee the existence of the whole family of Kuramoto models defined by the initial model and by the family of gauge transformations. To the best of our knowledge, there are no similar results for the Kuramoto model with with these types of interconnections. We also give an exact expression for synchronization frequency and we present local stability analysis.

Dynamical systems with limit cycle orbits present a suitable model for the autonomous elementary oscillators that give rise to averaged oscillatory rhythms through synchronization. In the preceding chapters, we used the Kuramoto model as a model to study the synchronization of interacting periodic oscillators. In fact, the Kuramoto model represents an acceptable approximation to systems with limit-cycle orbits under certain conditions, i.e. the coupling strength is weak [63]. However, in the case of stronger coupling, it is required to take into account the full dynamics of each oscillator, including its phase and its amplitude. Following the results of Andronov on limit-cycle oscillators, more generalized versions of the coupled oscillators model, including both phase and amplitude variations, have been published. Among these, the complex Stuart-Landau equation displays the amplitude equation derived from a general ordinary differential equation near an Andronov-Hopf bifurcation point –[29]. The Stuart-Landau oscillator is used in a wide range of applications ; for instance, to describe chemical reaction diffusion systems [52], semiconductor lasers [22] as well as in neurophysiology [10].

In Chapter 6, we analyze stability of a network of diffusive coupled Stuart-Landau oscillators, based on the notion of practical stability which allows the study of synchronization

properties in cases when only approximations of the limit cycle and synchronization frequency are available.

Summary of publications

This thesis is mostly based on published conference papers.

This section summarizes the relationship between the papers and the chapters in this thesis. Notice that some papers are used in more than one chapter.

Chapter 3 contains results presented in :

- [34] : A. EL-ATI and E. PANTELEY, On frequency synchronization of Kuramoto model with non-symmetric interconnection structure, *IEEE Conference on Communications, Computing and Control Applications*, Marseilles, France 2012.
- [35] : A. EL-ATI and E. PANTELEY, Asymptotic phase synchronization of Kuramoto model with weighted non-symmetric interconnections : a case study, *IEEE Conference on Decision and Control*, Florence, Italy 2013.

Chapter 4 contains results presented in :

- [35] : A. EL-ATI and E. PANTELEY, Asymptotic phase synchronization of Kuramoto model with weighted non-symmetric interconnections : a case study, *IEEE Conference on Decision and Control*, Florence, Italy 2013.

Chapter 5 contains results presented in :

- [37] : A. EL-ATI and E. PANTELEY, Synchronization of phase oscillators with attractive and repulsive interconnections, *IEEE Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland 2013 .
- [36] : A. EL-ATI and E. PANTELEY, Phase locked synchronization for Kuramoto model with attractive and repulsive interconnections, *IEEE Conference on Systems, Man, and Cybernetics*, Manchester, UK 2013.

Chapter 6 contains results presented in :

- [98] : E. PANTELEY and A. EL-ATI, On practical stability of a network of coupled non-linear limit cycle oscillators, *IEEE Conference on Systems, Man, and Cybernetics*, Manchester, UK 2013.

Chapter 2

Synchronization of oscillators network : Frameworks, tools and definitions

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2.1 Limit cycle oscillator models

We start by introducing the Stuart-Landau equation,

$$\frac{dz}{dt} = (\alpha + i\omega - c|z|^2)z, \quad (2.1)$$

where $z = x + iy \in \mathbb{C}$, $c = c_1 + ic_2 \in \mathbb{C}$; $c_1 \in \mathbb{R}_{>0}$, $c_2 \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$ is a parameter specifying the distance from a the bifurcation.

Using the following change of coordinates $z = re^i$ and writing the system (2.1) in polar coordinates, the equations for the radial amplitude r and the angular variable θ can be decoupled :

$$\dot{r} = -r - c_1 r^3; \quad (2.2)$$

$$\dot{\theta} = c_2 r^2 \quad (2.3)$$

When $\alpha < 0$, equation (2.2) has only one stable fixed point at $r = 0$. However, if $\alpha > 0$, this equation has a stable fixed point $r = \frac{1}{c_1}$, while $r = 0$ becomes unstable. This implies, in this case, that the trajectories of the system converge to a circle of radius r , starting from initial conditions both inside and outside the circle. Thus, this circle represents an attractor and the system (2.1) exhibits periodic oscillations. In this case, z represents the position of the oscillator in the complex plane and $z(t)$ has a stable limit cycle at the amplitude $z = \frac{1}{c_1}$ on which it moves at its natural frequency. The bifurcation of the limit cycle from the origin that appears at the value $\alpha = 0$ is known in the literature as the Andronov-Hopf bifurcation. The curves $z(t)$, represent this limit cycle of the system as follows :

$$z(t) = \frac{1}{c_1} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \quad (2.4)$$

An example of different trajectories for both $\alpha < 0$ and $\alpha > 0$ can be seen in the following figure 2.1.

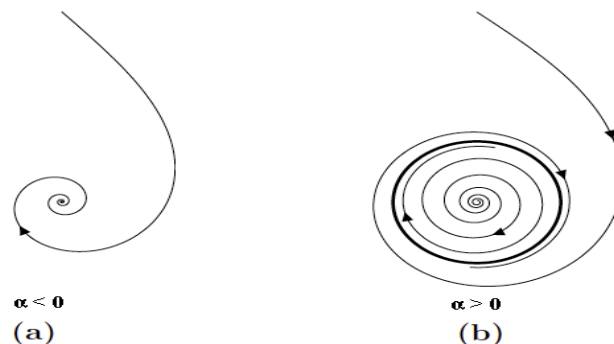


FIGURE 2.1 (a) The fixed point is stable for negative values of α and all orbits spiral towards it. (b) If $\alpha > 0$ the origin is unstable, but all trajectories go towards a stable limit cycle with radius $r = \frac{1}{c_1}$

The stability analysis of this limit cycle is usually done by using the Lyapunov exponent tools (see [65], [99] for more details overview), or using the second Lyapunov method (see e.g. [81], [101]).

In the literature, the system (2.20) with $\alpha > 0$, is known as the Stuart-Landau oscillator [10], [58], [80]. It is also known as the Andronov-Hopf oscillator [99]. The Stuart-Landau equation is a normal form, which means that the limit cycle dynamics of many other oscillators can be transformed onto or can be approximated by the dynamics given by equation (2.1), [51]. We cite, for example, the papers [44], [127] where the Van-der-Pol oscillator and the Haken-Kelso-Bunz (HKB) model in the neurophysiological applications are approximated by the equations (2.2) and (2.3).

Analysis of the synchronization behavior of coupled Stuart-Landau oscillators is more involved than that of a single oscillator. Stability analysis for a small network of two or three oscillators with different frequencies was done in e.g. [13], [16]. Numerical analysis of a general network of such heterogeneous oscillators was considered in [81]; however, we were unable to find in the literature theoretical results on stability analysis of such a network.

In Chapter 6, we analyze the synchronization behavior of networks consisting of many limit cycle oscillators described with the following model :

$$\frac{dz_j}{dt} = (\alpha - |z_j|^2 + i \omega_j) z_j + \gamma \sum_{i=1}^N a_{ji} (z_i - z_j) \quad j = 1 \dots N, \quad (2.5)$$

where j -th oscillator is described by the complex variable z_j and the oscillators are coupled in a network specified by the adjacency matrix $\mathcal{A} = [a_{ij}]$ and $\gamma > 0$ is the coupling strength between pairs of connected oscillators.

2.1.1 Phase reduction

In his book "Chemical Oscillations, Waves, and Turbulence", Y. Kuramoto studied the derivation of phase equations for systems, like coupled limit cycle oscillators or oscillatory reaction diffusion systems, by means of a perturbation analysis. Since the analysis of oscillatory systems is the main subject of this thesis, we roughly review, in this section, the phase reduction of coupled limit cycle oscillators proposed in [63].

Consider N mutually coupled oscillators described by the following equation :

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}_i(\mathbf{x}_i) - \sum_{j=1}^N \mathbf{V}_{i,j}(\mathbf{x}_i, \mathbf{x}_j), \quad i = 1 \dots N, \quad (2.6)$$

where \mathbf{x}_i is the state vector of the i -th oscillator, $\mathbf{F}_i(\mathbf{x}_i)$ is the autonomous dynamics corresponding to i -th oscillator and $\mathbf{V}_{i,j}(\mathbf{x}_i, \mathbf{x}_j)$ is a function describing the coupling between the i -th and j -th oscillators.

Additionally, assume that all oscillators have the similar dynamics that is $\mathbf{F}_i(\mathbf{x}_i) = \mathbf{F}(\mathbf{x}_i) +$

$\Delta\mathbf{F}_i(\mathbf{x}_i)$, with $\Delta\mathbf{F}_i(\mathbf{x}_i)$ small, and the coupling between the oscillators is weak, meaning $\mathbf{V}_{i,j}$ is small.

Moreover, assume that an uncoupled oscillator with nominal dynamics $\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$ has a stable limit cycle \mathcal{C} with an associated period T . Then, every state $\mathbf{x}_o \in \mathcal{C}$ on the orbit \mathcal{C} can be associated with a distinct scalar variable, namely its phase θ .

The mapping of the phase $\theta \in [0, T)$ associates to state $\mathbf{x} \in \mathcal{C}$, $\mathbf{x}_o : [0, T) \rightarrow \mathcal{C}$ is defined in such a way that the phase θ_o of the nominal oscillator linearly increases with time

$$\frac{d\theta_o}{dt} = \omega.$$

Next, in order to associate a phase with every state of the non-nominal oscillators, Kuramoto proposed the following reduction method based on perturbation theory [77] and considering that $\mathbf{V}_{i,j}$ and $\Delta\mathbf{F}_i$ are small perturbations to the nominal dynamics \mathbf{F} , for more details see [63], [68], [109]. In this case, the phase θ_i of the oscillator (2.6) can be approximated by :

$$\frac{d\theta_i}{dt} = \omega_i + \nabla(\theta_i)(\Delta\mathbf{F}_i(\mathbf{x}_o(\theta_i)) - \sum_{j=1}^N \mathbf{V}_{i,j}(\mathbf{x}_o(\theta_i), \mathbf{x}_o(\theta_j))), \quad i = 1 \dots N, \quad (2.7)$$

with $\nabla(\theta_i) = \text{grad}_{\mathbf{x}} \theta_i|_{\mathbf{x}=\mathbf{x}_o(\theta_i)}$.

Since the coupling $\mathbf{V}_{i,j}$ are weak, it was proved in [63] that the effect of the small perturbations on the oscillator phase θ_i can be averaged over one period T . Then, the system (2.7) can be presented in the following form :

$$\frac{d\theta_i}{dt} = \omega_i + \Delta\omega_i - \sum_{j=1}^N \Gamma_{i,j}(\theta_i - \theta_j), \quad i = 1 \dots N, \quad (2.8)$$

where $\Gamma_{i,j}(\cdot)$ are the coupling functions and $\omega_i = \omega + \Delta\omega_i$ is the natural frequency of the i -th oscillator.

This description of the coupled limit cycle oscillators is still too complex to be analyzed analytically ; therefore, Y. Kuramoto analysed a special case of (2.8), where the coupling function has a particular form :

$$\Gamma_{i,j}(\theta_i - \theta_j) = \frac{K}{N} \sin\left(\frac{\theta_j - \theta_i}{T}\right). \quad (2.9)$$

This model is well known in the literature as the Kuramoto model.

2.1.2 Kuramoto model

Like Winfree (1967) before him, Y. Kuramoto worked out a model to study the phenomenon of spontaneous collective synchronization that appears when a large number of oscillators are coupled ([62], [109]). In Kuramoto's original formulation, the only dynamical variable is the phase of the oscillators. He started by studying of the most simple case, a mean-field dynamic system resulting from a model of limit cycle coupled oscillators. The governing equations for the dynamics are given by the phase equations of the following general form :

$$\dot{\theta}_i = \omega_i + \frac{1}{N} \sum_{j=1}^N \Gamma_{i,j}(\theta_j - \theta_i) , \quad i \in 1 \dots N, \quad (2.10)$$

where θ_i is the phase of the i -th oscillator, ω_i its natural frequency and N is the number of oscillators. The phase interaction functions Γ_{ij} can be computed as integrals introducing certain terms from the original limit-cycle model (see [62], [122] for the detailed overview). This function can also be thought of as the phase response of oscillator i to input from oscillator j . However, these equations are too difficult to analyze since the form of the phase response curve is not specified and the interaction functions could have arbitrarily Fourier harmonics [63]. Moreover, the oscillators network could have various connection topologies.

In order to simplify the analysis, Kuramoto identified that purely sinusoidal coupling function (as a first-order approximation to the more general form (2.10)) should be the most tractable. The *classical* Kuramoto model corresponds to the simplest possible case when the network is supposed to be fully connected (all-to-all) with constant uniform coupling weight :

$$\dot{\theta}_i = \omega_i + \frac{1}{N} \sum_{j=1}^N k_{ij} \sin(\theta_j - \theta_i) , \quad i \in 1 \dots N, \quad (2.11)$$

The matrix $\mathcal{K} = [k_{ij}]$ represents the coupling matrix between the different oscillators, whose elements are k_{ij} . In the case of complex topologies of oscillators network, the coupling matrix $\mathcal{K} = [k_{ij}]$ can be considered as the adjacency matrix of the weighted directed interconnection's graph $G = (V, E, W)$ of the oscillators. For the *classical* Kuramoto model, $k_{ij} = K$ for all $i \neq j$ and $k_{ii} = 0$ for all $i \in \mathbb{I}_N$.

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) , \quad i \in 1 \dots N, \quad (2.12)$$

Roughly speaking, each oscillator attempts to oscillate independently with its natural frequency ω_i , while the phase coupling tends to synchronize it with all the others. When the

coupling is weak, the oscillators run independently, whereas beyond a certain threshold of coupling strength, collective synchronization emerges spontaneously. In his analysis, Kuramoto used the complex order parameter as a measure of phase synchronization in the network, defined as :

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}, \quad (2.13)$$

where r is the magnitude of the order parameter and ψ is the average phase.

The idea is to think of the phase θ_i of oscillator i as a vector $e^{i\theta_i} \in \mathbb{C}$. The order parameter then corresponds to the geometric centroid of the set of all vectors. The magnitude of the order parameter serves as a measure of the coherence in the system, in the sense that when more vectors are being aligned, r is closer to its maximal value 1, while vectors that are far from alignment will give rise to values of r significantly smaller than 1.

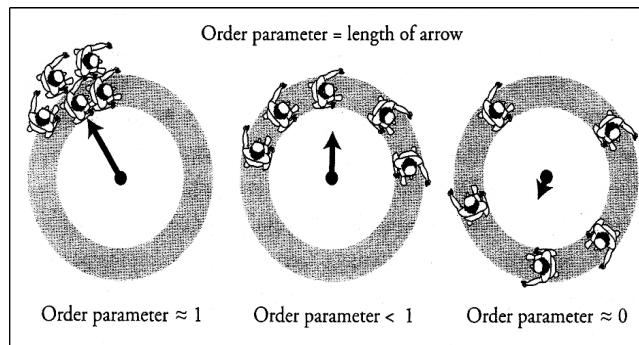


FIGURE 2.2 – Coupling and synchronisation of oscillators [125].

The classical Kuramoto model can be rewritten neatly in terms of the order parameter as follows :

$$\dot{\theta}_i = \omega_i + K r \sin(\psi - \theta_i), \quad i \in 1 \dots N, \quad (2.14)$$

In this form, the mean-field character of the model becomes obvious. Each oscillator appears to be uncoupled from all the others : of course they are interacting, but only through the mean-field quantities r and ψ . The complex order parameter is used to find the phase lock solutions of the classical Kuramoto model in [4].

Many extensions of this model have been investigated, and its mathematical implications have been explored extensively. Among others, issues related to dependency of the synchronization on the strength of the coupling between the oscillators attracted particular attention during the last few decades, see [122] and [31] for a detailed overview. In [4], the authors studied the local stability of phase locking behavior of uniform "all-to-all" interconnected Kuramoto oscillators, and they proved that there exists one unique asymptotically stable phase locked solution. In [54], the authors use the symmetry of the

graph to formulate the model in terms of incidence matrix and to study its stability properties. The case where oscillators are interconnected by undirected but not necessarily a complete graph was considered, e.g. in [54],[87]. Among others, we also cite articles [31], [54], [133] where an extension of the Kuramoto model with symmetric interconnection graph was considered. In all these papers it was supposed that the interconnections are symmetric and bidirectional, i.e the graph which defines the network interconnections is undirected. To the best of our knowledge, only a few theoretical results are available in the literature for the case of non-symmetric interconnection graphs. Mostly the case of identical natural frequencies was considered in this framework, we cite here [12], [21], [87].

In the case of an undirected network, the interconnection matrix $\mathcal{K} = [k_{ij}]$ is symmetric, i.e. $k_{ij} = k_{ji}$ for all $i, j = 1, \dots, N$. Consequently, the synchronization frequency of the oscillators is always constant and equal to the average of the natural frequencies. In this case, asymptotically the frequencies of all oscillators converge to the average frequency. Indeed, let us define the average phase $\theta_m = \frac{1}{N} \sum_{j=1}^N \theta_j(t)$. Then,

$$\begin{aligned} \dot{\theta}_m &= \frac{1}{N} \sum_{j=1}^N \dot{\theta}_j(t) \\ &= \frac{1}{N} \sum_{j=1}^N \omega_j + \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \sin(\theta_j - \theta_i) \\ &= \frac{1}{N} \sum_{j=1}^N \omega_j = \omega_m. \end{aligned}$$

In the case of a non-symmetric adjacency matrix, frequency of synchronization does not necessarily coincide with the average frequency, it was shown in [34] that for a non-symmetric interconnection graph synchronization frequency ω_s can depend on the gain of the coupling strength ; see also [31] for frequency synchronization in a particular case of weighted interconnection matrix.

2.2 Graph theory preliminaries

The network analysis plays an important role in a wide variety of disciplines, ranging from communications and power systems engineering to molecular and population biology [79]. The algebraic graph theory is a link between matrix theory and network theory and it is a powerful tool for analysis of large-scale interconnected systems.

In this section, we introduce some notation and preliminary results from the algebraic graph theory. We will use the standard terminology of graph theory and introduce the

essential concepts necessary to develop the results of this thesis. Therefore, we are interested mostly in directed graphs (digraphs); however, many basic properties proved for undirected graphs cannot be useful. We introduce some basic definitions and facts for directed graphs. A comprehensive survey on properties of matrix presentation of undirected graphs can be found in [43]. More information is available in [85], [135].

2.2.1 Digraphs, associated matrices and properties

A digraph $G = (V, E)$ consists of a non-empty set of vertices or nodes $V = \{v_1, \dots, v_n\}$ and set of edges $E(G) \subseteq V \times V$. For notational convenience, we will often write $e_{ij} \in E(G)$ for the edge connects the node v_i to the node v_j . An edge, e_{ij} in a directed graph G is said to be an edge starting at the vertices v_i and the vertices v_j is said to be adjacent to v_i . For the graph G we define adjacency matrix, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$, of G is defined in the following way :

$$a_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in E(G) \\ 0 & \text{if } e_{ij} \notin E(G). \end{cases} \quad (2.15)$$

For example, for the graph depicted on Figure 3.1, the adjacency matrix A has the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is easy to see that the adjacency matrix of an undirected graph is symmetric while this not the case for a general directed graph.

Given a node v , we define its in-degree $deg_{in}(v)$ and out-degree $deg_{out}(v)$ as the number of edges that start and terminate at this node, that is

$$deg_{in}(v_i) = \sum_{j=1}^n a_{ji}, \quad deg_{out}(v_i) = \sum_{j=1}^n a_{ij}. \quad (2.16)$$

The degree matrix of graph G is a diagonal matrix $D = diag(d_i) \in \mathbb{R}^{n \times n}$ where $d_i = deg_{out}(v_i)$.

Definition 1. A digraph $G = (V, E)$ is called complete if the degree of every vertex is equal to $n - 1$. This type of graph is known also as an all-to-all connected graph.

Definition 2. A directed graph is called strongly connected if any two distinct nodes of the graph can be connected via a path that follows the direction of the edges of the graph.

Definition 3. The node v_i of a graph $G = (V, E)$ is balanced if its in-degree deg_{in} and out-degree deg_{out} are equal, i.e. $deg_{in}(v_i) = deg_{out}(v_i)$. A graph $G = (V, E)$ is called balanced

if all of its nodes are balanced that is

$$\sum_j a_{ij} = \sum_j a_{ji}, \quad \text{for all } i. \quad (2.17)$$

Any undirected graph is balanced. Furthermore, any directed balanced graph is strongly connected.

Laplacian matrix

The Laplacian matrix of a graph and its spectral properties (vectors and eigenvalues) can be used in several areas of mathematical research and have a physical interpretation in various physical theories. The Laplacian matrix associated with the digraph G is defined as the difference between the degree matrix and the adjacency matrix

$$L = D - A. \quad (2.18)$$

From the definition of the Laplacian matrix it follows that every row sum is equal to zero, and it always has an eigenvalue $\lambda(L) = 0$ and the corresponding right eigenvector $w_r = \mathbf{1}$. Notice that the multiplicity of zero as an eigenvalue of $L(G)$ is equal to the number of connected components of G , so for strongly connected graphs there is always only one zero eigenvalue. Furthermore, the rank of the Laplacian matrix associates to a strongly connected digraph G of order n satisfies : $\text{rank}(L) = n - 1$. The second-smallest eigenvalue is known as the algebraic connectivity of the graph [135]. The eigenvector corresponding to that second-smallest eigenvalue is called the Fiedler vector, and has been used successfully for analysis of consensus convergence speed problem [95].

Moreover, for a strongly connected digraph G with a Laplacian matrix L , the following relations are valid for these vectors $L w_r = 0$, $w_l^\top L = 0$, $w_l^\top w_r = 1$ (see [97]).

2.2.2 Weighted and signed digraphs properties

If all the interactions in the network are alike, or in other words, when a link only establishes the presence of an interaction between two nodes, then the network is unweighted. Otherwise, if there are different types of interactions, for example some more important, then the links are weighted and so is the graph. Notice that the weighted graphs appear as a model for numerous problems where the agents are linked with links of different weights (distance, resistance, ...). In this case, in addition to giving the set of nodes and links of the network, we also need to define the weight of every link in order to properly define the graph. So now we have a pair (G, W) where $G = (V, E)$ is the directed graph

and $WE \rightarrow \mathbb{R}$ is a weight function, that are real numbers attached to the corresponding links. Often, they will be positive numbers, so stronger links between the pair of nodes, but negative links have also been used, describing the stronger of repulsive interaction [132]. A weighted graph can be represented as a triplet $G = (V, E, W)$. Note that the graph $G = (V, E)$ can be viewed as a weighted graph where all edges have weight 1.

Weighted digraph

Let $G = (V, E, W)$ be an weighted directed graph. Where every edge $e_{ij} \in E(G)$, ($i \neq j$) (we assume that the graph does not have loops) has a weight $W_{ij} \in \mathbb{R}$ and satisfies $W_{ij} \neq 0$ if and only if $e_{ij} \in E(G)$. The graph G is undirected if $W_{ij} = W_{ji}$ for all $i, j = 1 \dots N$. In this case, the adjacency matrix will be symmetric, and this is not the case for the directed graph in general.

The graph is balanced if $\sum_{j=1}^N W_{ij} = \sum_{j=1}^N W_{ji}$ for all $i = 1 \dots N$. The adjacency matrix $A = [a_{ij}]$ of the edge-weighted digraph is defined as :

$$a_{ij} = \begin{cases} W_{ij} & \text{if } e_{ij} \in E(G) \\ 0 & \text{if } e_{ij} \notin E(G), \end{cases} \quad (2.19)$$

The degree matrix of the edge-weighted digraph is a diagonal matrix $D = \text{diag}(A \mathbf{1}) = \text{diag}(\sum_{j=1}^N a_{1j}, \dots, \sum_{j=1}^N a_{Nj}) \in \mathbb{R}^{n \times n}$. The Laplacian matrix associated with the edge-weighted digraph $G = (V, E, W)$ is defined as the difference between the degree matrix and the adjacency matrix $L = D - A$. The Laplacian matrix is a zero row-sum matrix. Properties of weighted digraphs may be found in the comprehensive paper [135] see also [100] and [15] for a detailed overview.

Signed digraph

A signed digraph is a graph in which each arc has a sign associated with it. A mathematical model of a system can be represented by a signed directed graph in which the nodes represent the system model and directed arcs show how systems affect each other. The sign of a directed arc defines the effect of the interaction. Notice that interplay of attractive and repulsive interactions can play an important role in the context of dynamical networks, as it is the case in synthetic genetic circuits [41] and neuronal networks [45].

A signed weighted digraph $G = (V, E, A)$ where $A \in \mathbb{R}^{n \times n}$ is the matrix of the signed weights of G . This adjacency matrix alone completely specifies a signed digraph. In a digraph, a pair of edges sharing the same nodes $e_{ij}, e_{ji} \in E$ is called a digon.

A directed path \mathcal{P} of G is a concatenation of edges of E with respect to the direction and in which all nodes are distinct. A directed cycle \mathcal{C} of G is a directed path beginning and ending with the same node. A directed cycle is positive if it contains an even number of negative edge weights.

Definition 4. [144] *A signed digraph G associated to the adjacency matrix $A[a_{ij}]$, is called structurally balanced if it admits a bipartition of the nodes V_1, V_2 satisfying $V_1 \cup V_2 = V; V_1 \cap V_2 = \emptyset$ such that $a_{ij} \geq 0 \forall v_i, v_j \in V_q (q \in \{1, 2\})$ and $a_{ij} \leq 0 \forall v_i \in V_q, v_j \in V_r, q \neq r (r, q \in \{1, 2\})$. More clearly, it is proved that a signed digraph G is structurally balanced if all cycles of $G(A)$ are positive. It is called structurally unbalanced otherwise.*

The degree matrix \bar{D} (the row connectivity matrix of A) is the diagonal matrix having diagonal elements $d_{ii} = \sum_j |a_{ij}|$, where v_j are the nodes adjacent to v_i in E . The signed Laplacian matrix of a digraph G with adjacency matrix A is given by $L = \bar{D} - A$.

The analogy can be made rigorous by observing that all structurally balanced networks are equivalent, under a suitable change of orthant order, to nonnegative networks. By adopting the terminology of [5], we call the change of orthant order gauge transformation.

Definition 5. [5] – *A partial orthant order in \mathbb{R}^N is a vector $\delta = [\delta_1 \dots \delta_N]$, $\delta_i = \pm 1$. A gauge transformation is a change of orthant order in \mathbb{R}^N introduced by a matrix $G = \text{diag}(\delta)$. The set of all gauge transformations in \mathbb{R}^N is denoted as $\mathcal{G} = \{G \in \mathbb{R}^{N \times N}, G = \text{diag}(\delta), \delta = [\delta_1 \dots \delta_N], \delta_i \in \{\pm 1\}\}$.*

Given an adjacency matrix $A \in \mathbb{R}^{N \times N}$ and a gauge transformation G , an equivalent adjacency matrix A_g is defined as $A_g = G A G$. A detailed analysis of gauge transformations and signed directed graphs can be found in [5]. In the following propositions, we summarize several results from [5] and [144], that will be used later in the proof.

Proposition 1. – *Let A be an adjacency matrix for a directed graph. The family of all gauge equivalent adjacency matrices $A_g = \{A_g = G A G, G \in \mathcal{G}\}$ contains at most 2^{N-1} distinct matrices.*

– *Given a structurally balanced directed graph with its set of edge weights, there exists a family of structurally balanced signed directed graphs characterized by the same weights but with different signs. All these realizations of the signed graphs are related by gauge transformations and all are isospectral.*

– *Any sign-symmetric complete directed graph is structurally balanced.*

2.3 Synchronization of dynamical systems : concept and notions

2.3.1 Consensus protocol

In this section, we present the basic concepts of consensus problems and a few important results. In particular, we present some results and analysis methods for the consensus problem for networks of integrators, and we present how these methods can be extended to networks of identical oscillators. We refer to the textbook [19] for detailed treatments.

Similarly to the self-organizing systems, the consensus problem is a problem involving multiple agents whom reached an agreement about a static state value over a network. Agreement problems have a long history in the field of computer science, particularly in distributed computation and automata theory [97]. Many applications involving multi-agent systems need to agree upon certain quantities of interest. Areas of application of consensus problems include formation control of mobile robots [145], sensory networks, and flocking [96]. This problem has been studied by numerous methods and it can be solved quite adequately using algebraic graph theory.

Among the numerous directions of research on consensus and averaging, we would like to mention the following famous papers : consensus in small-world networks [93], [126], continuous-time consensus algorithms [42], [95] consensus problems with switching topology [97], consensus on manifolds [110], [113], and bipartite consensus [5]. Some results and methods presented in these papers will be used in our analysis of the synchronization of nonlinear oscillator networks.

2.3.1.1 Static networks of linear agents

Consider a system network of N agents. In the simplest setting, the state of each agent is characterized by a state variable $x_i \in \mathcal{R}$ and the objective is to achieve a consensus on a common state value. The interactions between the agents can be represented by a weighted digraph $G = (V, E, A)$, where V is the set of agents, E is the set of interconnections between them and $A \in \mathcal{R}^{n \times n}$ is the adjacency matrix describing the interaction between agents.

The equations governing the network represented by the graph G are :

$$\dot{x}_i = \sum_{j=1}^N a_{ij} (x_j - x_i) , \quad i \in \{1, \dots, N\}, \quad (2.20)$$

Since $x_i \in \mathbb{R}$, that is each agent evolves in a one-dimensional space, then in vector notation the consensus protocol (2.20) can be rewritten as

$$\dot{x} = -L x, \quad (2.21)$$

where $x = (x_1, \dots, x_N)^\top$ and L is the Laplacian matrix of G . The system 2.21 is linear and time-invariant ; therefore, its stability properties are entirely defined by the eigenvalues of the matrix $-L$. Let us assume that the digraph is strongly connected, then we have that $\text{Ker}(L) = \mathbf{1}_N$ and by the Gersgorin disk theorem (see Appendix B) we have that L is positive semi-definite ; therefore, the matrix $-L$ is negative semi-definite. Additionally, L has a simple zero eigenvalue associated to the eigenvector $\mathbf{1}_N$. Thus, we have that the system 2.21 is stable and that x will tend to the equilibrium-space of L asymptotically. Notice that the consensus dynamics (2.21) are invariant by the translation $x \longrightarrow x + c \mathbf{1}_N$ where $c \in \mathbb{R}$, that is, $\mathbf{1}_N$ is an equilibrium subspace i.e. *agreement subspace*. In other words, $x_i \longrightarrow \alpha$ as $t \longrightarrow \infty$ with $\alpha \in \mathbb{R}$. The interpretation here is that all x_i (i.e. the scalar positions of all the agents) will tend to the same value α . Hence the consensus problem is solved. In fact, it is easy to establish that if the digraph is symmetric, then α is equal to the static averaged $x_{avg} = \frac{1}{N} \sum_{i=1}^N x_i$.

The essential results regarding convergence and decision value of consensus protocol for directed networks with a fixed topology are summarized in the following theorem.

Theorem 1. (Consensus with linear protocol [95])

Consider a network of N agents with topology G applying the following consensus algorithm :

$$\dot{x}_i = \sum_{j=1}^N a_{ij} (x_j - x_i) , \quad i \in \{1, \dots, N\}, x(0) = z.$$

Suppose G is a strongly connected digraph. Let L be the Laplacian of G with a left eigenvector $w_l = [w_{l_1}, \dots, w_{l_N}]^\top$ satisfying $w_l^\top L = 0$. Then

- *a consensus is asymptotically reached for all initial states,*
- *the algorithm solves the consensus problem with the linear function $f(z) = \frac{w_l^\top z}{w_l^\top \mathbf{1}}$, i.e. the group decision is $\alpha = \sum_i w_{l_i} z_i$ with $\sum_i w_{l_i} = 1$,*
- *if the digraph is balanced, an average consensus is asymptotically reached and $\alpha = \sum_i \frac{z_i}{N}$.*

Moreover, it was shown in [95], [97] that the equilibrium subspace $\mathbf{1}_N$ is exponentially stable and that the rate of convergence is given by

$$\|x(t) - x_{avg} \mathbf{1}_N\| \leq \|x(0) - x_{avg} \mathbf{1}_N\| e^{-\lambda_2(L)t},$$

where $\lambda_2(L)$ is the second smallest eigenvalue of the Laplacian matrix.

The consensus protocol (2.20) is well-studied in the control literature [19], [73], [96], [104], various extensions of this protocol are considered, in particular, consensus for digraphs with time varying weighted $a_{ij}(t)$ was analysed in [42] while consensus for digraphs with signed weights (positive and negative weighted) $a_{ij}(t) \in \mathbb{R}$ was considered in [5].

2.3.1.2 Identical oscillators network as nonlinear consensus problem

Related to consensus protocols are synchronization phenomena arising in systems of coupled nonlinear oscillator. It was shown in [88] that the analysis of identical oscillators networks can be viewed as nonlinear extensions of the consensus system (2.20). Indeed, consider a network of identical Kuramoto oscillators for communication digraph $G = (V, E, A)$ and equal natural frequencies. Actually without loss of generality, we can assume that all oscillators have zero natural frequencies $\omega = 0$. Then, the model of coupled oscillator takes the form

$$\dot{\theta}_i = \sum_{j=1}^N a_{ij} \sin(\theta_j - \theta_i), \quad i \in \{1, \dots, N\}, \quad (2.22)$$

where coefficients a_{ij} are strictly positive.

If we assume that all angles lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then as it was proposed in [88], the dynamics (2.22) can be projected onto the real line via the local coordinates $(-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R} : \theta_i \mapsto x_i = \tan(\theta_i)$.

In this case, the Kuramoto model (2.22) can be rewritten as a consensus model

$$\dot{x}_i = \sum_{j=1}^N \phi_{ij}(x) (x_j - x_i), \quad i \in \{1, \dots, N\}, \quad (2.23)$$

where $\phi_{ij}(x) = a_{ij} \sqrt{\frac{1+x_i^2}{1+x_j^2}}$ are positive scalar functions. In this formulation, the system (2.23) can be viewed as a consensus algorithm (2.20) with strictly positive weights whose values are state-dependent. Notice that a similar viewpoint and other types of projections can be found in [54], [74], [116], [118].

The following theorem summarizes some results which follow from the analysis of nonlinear consensus protocols, (see [54], [74] and [88]).

Theorem 2. (Consensus with nonlinear protocol)

Consider the coupled oscillator model (2.22) with a weighted digraph $G = (V; E; A)$, and assume that G is strongly connected. Then, for any given initial condition $\theta_\circ \in \mathbb{R}$, the

vector $(\theta_{avg} \mathbf{1}_N)$ is an asymptotically stable equilibrium solution and the synchronized state is globally asymptotically stable over any compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^N$.

Moreover, if the digraph G is symmetric, then the convergence is exponential and the rate of approach to equilibrium is no worse than

$$\|\theta(t) - \theta_{avg} \mathbf{1}_N\| \leq \|\theta(0) - \theta_{avg} \mathbf{1}_N\| e^{-(2\lambda_2(L)/\pi)t},$$

where $\lambda_2(L)$ is the second smallest eigenvalue of the Laplacian matrix and $\theta_{avg} = \frac{1}{N} \sum_{i=1}^N \theta_i(0)$ is the average initial phases.

2.3.2 Synchronization for Kuramoto model

Within the framework of phase-reduction of limit cycle oscillators, the presentation of the interactions between the oscillatory systems is remarkably simplified. In this way, the Kuramoto model describes each oscillatory by an equation involving the derivative of the instantaneous phases (instantaneous frequency) and the instantaneous phases.

For the oscillatory systems there are two concepts of synchronization which are widely used in the literature : the *phase* synchronization and the *frequency* synchronization.

The main objects under study in most applications and theoretic analyses are phase-locked and frequency-synchronized oscillation, where asymptotically all oscillators rotate with the same frequency and all the pairwise phase differences are bounded and constant. In what follows, we restrict our attention to this case and we call a solution synchronized if it is frequency synchronized and phase locked. In what follows, we define these notions in detail.

Frequency synchronization

First of all, we recall that all known notions of synchronized solutions have a common property that the oscillator frequencies converge to a constant synchronization frequency. Thus, we define *frequency synchronization* in two steps. First, we define frequency locked solutions, i.e. the solutions have the common frequency of oscillation for **all** $t \geq 0$. Roughly speaking, the frequency locked solution can be seen as an analogue of equilibrium point.

Definition 6. (Frequency locked oscillation) – *The oscillators defined by (2.12) are frequency locked if there exists a constant $\omega_s \in \mathbb{R}$ and initial conditions $\theta_o \in \mathbb{R}^N$ such*

that for all $i \in \{1, \dots, N\}$

$$\dot{\theta}_i(t, \theta_o) = \omega_s, \quad \forall t \geq 0. \quad (2.24)$$

Next, frequency synchronization is usually defined as existence and asymptotic stability of the frequency locked solutions. Since the Kuramoto model is studied in many different contexts and application domains, sometimes, terms "frequency locking", "frequency synchronization", or also "frequency entrainment" are used.

Definition 7. (Frequency synchronization) – *The oscillators defined by (2.12) are frequency synchronized if there exists a frequency locked solution of the system and asymptotically all the frequencies $\dot{\theta}_i(t)$ converge to this solution, i.e. :*

$$\lim_{t \rightarrow \infty} \dot{\theta}_i = \omega_s, \quad \forall i \in \{1, \dots, N\}. \quad (2.25)$$

If the matrix $\mathcal{K} = [k_{ij}]$ of the interconnection gains is symmetric, i.e. $\mathcal{K} = \mathcal{K}^\top$, then the average frequency of the oscillators is constant and equals the average of the natural frequencies. In the case where matrix \mathcal{K} is non-symmetric, frequency of synchronization does not necessarily coincide with the average frequency. We analyse this case in Chapter 3 and show that for a non-symmetric interconnection graph, synchronization frequency can depend on the coupling strength ; see also [31], [35] for frequency synchronization in a particular case of weighted interconnection matrix. Numerically this issue was studied in [60], [75], [105].

The stability of the frequency synchronized solutions was analyzed, e.g. [24], [54] ; see also [31] and [122] for a detailed overview. Nonetheless, the structure of the synchronized solution was never addressed in these papers.

Phase synchronization

The definition of a frequency locking solution presented above allows for multiple possible phase configurations. That is why several terminologies are used in the literature to describe phase synchronization, for example : full or perfect synchronization, phase cohesiveness, and partial synchronization.

In what follows, we define *phase synchronization* in two steps. Similar to the frequency synchronization, in the first step we define phase locked solutions and later define phase synchronization as asymptotic stability of the phase locked solution.

Definition 8. (Phase locking) *Let $\theta_o \in \mathbb{R}^N$ and $\theta(t, \theta_o)$, $t \geq 0$ be a solution of the Kuramoto model with initial conditions θ_o . Solution $\theta(t, \theta_o)$ is called phase locked if*

$$\theta_i(t, \theta_o) - \theta_j(t, \theta_o) = \theta_{i_o} - \theta_{j_o},$$

for all $t \geq 0$ and all $i, j = 1, \dots, N$.

We define phase synchronization as the asymptotic stability of the phase locked solution.

Definition 9. (Phase synchronization) – *The oscillators defined by (2.12) are phase synchronized if there exists a phase locked solution $\theta^* \in \mathbb{R}^N$ of the system and a set $\Theta \subset \mathbb{R}^N$ such that asymptotically all phase differences $\theta_i(t, \theta_o) - \theta_j(t, \theta_o)$ converge to this solution for all initial conditions $\theta_o \in \Theta$, i.e.,*

$$\lim_{t \rightarrow \infty} \theta_i(t, \theta_o) - \theta_j(t, \theta_o) = \theta_i^* - \theta_j^*, \quad \forall i, j \in \mathbb{I}_N. \quad (2.26)$$

Next we present some results existing in the literature on the existence of phase locked solutions and synchronization for the Kuramoto model 2.12, with particular attention paid to the role that coupling gain K plays in the existence of synchronized solutions in the case of a complete (all-to-all) interconnection.

Synchronization results for the Kuramoto model with all-to-all uniform coupling

Qualitative analysis of the Kuramoto model (2.12) has been an active field of research during several decades and, in particular, analysis of the phase synchronization. Due to the complexity of the problem in the case of general coupling matrix, the majority of the theoretical results are concerned with a more simple particular case of all-to-all uniform coupling, i.e. model (2.12). As we mentioned before, the problem of phase synchronization (see Definition ??) can be viewed as a problem of existence and (asymptotic) stability of the phase locked solutions. Below we give a brief overview of the results in this area which served as a basis for our research on the Kuramoto model.

Existence of solutions

Explicit expressions for the phased locked solutions of the Kuramoto model (2.12) were given in [4] (see also [84] and [133]). It was shown that the following assumption on the natural frequencies and the interconnection gain is both necessary and sufficient for the existence of the phase locked solutions.

Assumption 1. (Consistency condition, [4])

There exists a $r_\infty \in (0; 1]$ such that

$$r_\infty = \frac{1}{N} \sum_{j=1}^N \pm \sqrt{1 - \left(\frac{\tilde{\omega}_j}{Kr_\infty}\right)^2}, \quad (2.27)$$

and the following bounds are satisfied $-1 \leq \frac{\tilde{\omega}_j}{Kr_\infty} \leq 1$, where $\tilde{\omega}_j = \omega_j - \frac{1}{N} \sum_{k=1}^N \omega_k$ for all $i = \{1, \dots, N\}$.

In particular, the authors of [4] showed that if r_∞ is a solution of equation (2.27) then the corresponding phase locked solution is given by $\theta_s(t) = \theta^* + \omega_m t$ where ω_m is a mean of natural frequencies $\omega_m = \frac{1}{N} \sum_{i=1}^N \omega_i$ and the phases θ^* are defined as

$$\sin(\theta_i^*) = \frac{\tilde{\omega}_i}{Kr_\infty} \quad ; \quad \cos(\theta_i^*) = \pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{Kr_\infty}\right)^2} \forall i \in \{1, \dots, N\}. \quad (2.28)$$

We remark that the system (2.12) is invariant under the angular translation, that is the oscillator network dynamics (2.12) remain invariant under a rotation of all oscillators by the same constant angle. Therefore, any solution of the system (2.12) with the initial conditions $\theta_\circ = \theta_\circ^* + \alpha \mathbf{1}$ will also be a phase locked solution. Actually, due to the invariance of the system under translation, it is more correct to speak about stability of a set of phase locked solutions.

Moreover, expression (2.27) represents a set of 2^N equations : each term in the summation can have a plus as well as a minus sign. Notice that given a choice of \pm signs in (2.27), every solution $r_\infty \in (0; 1]$ corresponds to a different phase locked solution of (2.27). As it was remarked in [4] and [84], not all 2^N equations, represented by (2.27), would have a solution for each fixed coupling gain K . On the other hand, some of the equations in (2.27) can have multiple solutions.

Stability results

Analysis of synchronized state for the classical Kuramoto model (2.12) has been studied using various control theoretic methods [4], [24], [30], [53], [54], [117], [134], to cite a few.

In [4], the authors proved that phase locked solutions can be asymptotically stable only if the phases differences belong to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and formulated conditions that guarantee local asymptotic stability of phase locked solutions in this case. The following theorem in [4] summarizes these results.

Theorem 3. *Consider the system of oscillators described by (2.12). If the amplitude r_∞ satisfies (2.27) containing minus signs, the corresponding phase locking solution of (2.12) is locally unstable in the variables θ_i , $i \in \{1, \dots, N\}$. Furthermore, the phase locking solution of (2.12) with the amplitude r_∞ satisfying (2.27) containing only plus signs is*

locally asymptotically stable if and only if the extra condition

$$\sum_{j=1}^N \frac{1 - 2\left(\frac{\tilde{\omega}_j}{Kr_\infty}\right)^2}{\sqrt{1 - \left(\frac{\tilde{\omega}_j}{Kr_\infty}\right)^2}} > 0. \quad (2.29)$$

is satisfied.

In [54], control and algebraic graph theory methods were used to analyze Kuramoto oscillators for a connected symmetric graph topology. The authors derived a threshold gain K_c necessary for the existence of the phase-locked state in the all-to-all Kuramoto model. The authors also show that there exists a large enough coupling gain K when the vector of phase differences locally $(-\frac{\pi}{2}, \frac{\pi}{2})$ converges to a unique constant. The following theorem presents these results

Theorem 4. [54]

Consider the Kuramoto model for non-identical coupled oscillators and an interconnection graph $G = (V; E)$ corresponding to the Laplacian matrix L . For $K \geq K_c = 2 \frac{\sqrt{N} \|\omega\|}{\lambda_2(L)}$, there exist at least one fixed point for $|\theta_i| < \frac{\pi}{4}$. Moreover, for $K \geq \frac{\pi^2 N \lambda_{\max}(L) \|\omega\|}{4 \lambda_2^2(L)}$ there is only one stable fixed-point and the order parameter is strictly increasing.

More recently, almost global exponential frequency synchronization for the traditional Kuramoto model has been studied in [24]. This result is presented in the following theorem

Theorem 5. [24]

Consider the system of oscillators described by (2.12). If there exists $T \geq 0$ such that $\forall t \geq T, |\theta_i - \theta_j| < \frac{\pi}{2} - \varepsilon \forall i, j$, where $0 < \varepsilon < \frac{\pi}{2}$, then the oscillator frequencies $\dot{\theta}_i$ synchronize exponentially to the mean frequency $\omega_{\text{avg}} = \frac{1}{N} \sum_{i=1}^N \omega_i$ and satisfy $|\dot{\theta}_i - \omega_{\text{avg}}| \leq \sigma_T e^{-K \sin(\varepsilon)(t-T)}$, $\sigma_T > 0 \forall i = 1, \dots, N$.

2.3.3 Set stability notions

Similar to local asymptotic stability of equilibrium point, weaker stability notions were introduced in the literature, in particular, stability notions for sets [128]. In this section we establish definitions, notation, and the key notions of sets stability used later in the following chapters.

A continuous function $\alpha: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K} ($\alpha \in \mathcal{K}$), if it is strictly increasing and

$\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if, in addition, $\alpha(s) \rightarrow \infty$ as $s \rightarrow \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{L} ($\sigma \in \mathcal{L}$) if it is decreasing and tends to zero as its argument tends to infinity.

In the following section, we introduce stability definitions of closed, not necessarily compact, set $\mathcal{A} \subseteq \mathbb{R}^n$ for time-invariant differential equations

$$\dot{x} = f(x), \quad (2.30)$$

where $x \in \mathbb{R}^n$ and $f(\cdot)$ is locally Lipschitz on x , that is, for each compact subset $\mathcal{B} \subset \mathbb{R}^n$ there is some constant c so that $|f(x) - f(z)| \leq c |x - z|$ for all $x, z \in \mathcal{B}$, where $|\cdot|$ denotes the usual Euclidian norm.

The system (2.30) is said to be forward complete if for each $x \in \mathbb{R}^n$ all elements of \dot{x} are defined on $[0, \infty)$.

We define uniform forms of stability, where uniformity is with respect to initial conditions that are at an arbitrary given distance from the set \mathcal{A} . We define the distance of x from \mathcal{A} as $|x|_{\mathcal{A}} = \inf_{z \in \mathcal{A}} |x - z|$. For the set \mathcal{A} we define the following stability notions

Definition 10. For the system (2.30), the closed set $\mathcal{A} \subset \mathbb{R}^n$ is uniformly stable if for each $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that, for all $x_0 \in \mathbb{R}^n$ satisfying $|x_0|_{\mathcal{A}} \leq \delta$, we have :

$$|x(t, x_0)|_{\mathcal{A}} \leq \varepsilon, \quad \forall t \geq 0.$$

Definition 11. For the system (2.30), the closed set \mathcal{A} is uniformly globally stable (UGS) if the system (2.30) is forward complete and there exists $\rho \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}^n$

$$|x(t, x_0)|_{\mathcal{A}} \leq \rho(|x_0|_{\mathcal{A}}), \quad \forall t \geq 0.$$

Definition 12. For the system (2.30), the closed set \mathcal{A} is uniformly globally asymptotically stable (UGAS) if it is UGS and for each $r > 0$ and $\varepsilon > 0$ there exists $T > 0$ such that, for all $x_0 \in \mathbb{R}^n$

$$|x_0|_{\mathcal{A}} \leq r, \quad t \geq T \quad \Rightarrow \quad |x(t, x_0)|_{\mathcal{A}} \leq \varepsilon.$$

In the case of a network where all nodes are identical (that is to say where $f_i(x) = f_j(x)$), synchronization (asymptotic synchronization) is often described in terms of evolution of the same elements. Therefore, it is formulated as stability (asymptotic stability) of the variety synchronization

$$S = \{x \in \mathbb{R}^N : x_1 = x_2 = \dots = x_N\}.$$

The stability of this variety can be derived using the results and methodologies developed for incremental input-output stable systems and also by the sets stability tools. The general theory of stability contains many other definitions and notions of stable movement. In particular, we cite the concepts of orbital and structural stability. Roughly speaking, orbital stability describes the behaviour of a closed trajectory under the action of small external perturbations. Let the system (2.30) and the function $\varphi(t)$ be a periodic solution (the form of closed trajectory) of the given autonomous system. If for any $\varepsilon > 0$ there is a constant $\delta = \delta(\varepsilon) > 0$ such that the trajectory of any solution $x(t)$ starting at the δ -neighbourhood of the trajectory $\varphi(t)$ remains in the δ -neighbourhood of the trajectory $\varphi(t)$ for all $t \geq 0$, then the trajectory $\varphi(t)$ is called orbitally stable.

However, the behaviour of the networks of non-identical elements is more complex, that is the variety synchronization \mathcal{S} does not exist considering the difference between the dynamic elements. However, a heterogeneous network can show a certain type of collective behaviour. In this case, we can talk about practical synchronization for which the differences between the dynamics of the different elements are bounded and become small when the values of interconnection gain is large enough.

The network behaviour can be decomposed into two parts. One of these two parts corresponds to the *averaged* dynamic and the second part describes the dynamics of the network elements with respect to the averaged dynamic element. For this purpose, we introduce the following definition of practical stability of a set, which is similar to that of practical stability of an equilibrium point –see [23, 129].

Consider a parametrized system of differential equations

$$\dot{x} = f(x, \varepsilon), \quad (2.31)$$

where $x \in \mathbb{R}^n$ and the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz and $\varepsilon > 0$ is a scalar parameter. We assume that $\varepsilon \in (0, \varepsilon_0]$. For such a family, we introduce the notion of almost global practical uniform asymptotic stability with respect to closed, not necessarily compact, sets $\mathcal{A} \in \mathbb{R}^n$.

Definition 13. *For the system (2.31), the closed set \mathcal{A} is practically almost globally uniformly asymptotically stable (practical almost UGAS) if the system (2.31) is forward complete and*

(1) *for any given $r > 0$ there exists $R > 0$ and $\varepsilon^* \in (0, \varepsilon_0]$ such that for all $\varepsilon \in (0, \varepsilon^*]$*

$$|x_0| \leq r \quad \implies \quad |x(t, x_0, \varepsilon)| \leq R, \quad \forall t \geq 0$$

(2) for any given (r, δ) with $0, \delta < r$, there exists $\epsilon^* \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_*]$

$$|x_0|_{\mathcal{A}} \leq \delta \quad \implies \quad |x(t, x_0, \epsilon)|_{\mathcal{A}} \leq r, \quad \forall t \geq 0$$

(3) for any given (r, δ) with $0, \delta < r$, there exists $T = T(r, \delta)$ and $\epsilon^* \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_*]$ and almost all $x_0 \in \mathbb{R}^n$ such that $|x_0|_{\mathcal{A}} \leq r$, we have

$$|x(t, x_0, \epsilon)|_{\mathcal{A}} \leq \delta \quad \forall t \geq T.$$

If the last inequality holds for all $x_0 \in \mathbb{R}^n$ then the set \mathcal{A} is practically globally uniformly asymptotically stable.

Remark 1. Similarly to the definition of UGAS of a set, the definition of practical almost UGAS includes three properties : uniform boundedness of the solutions with respect to the set (part 1), uniform stability of the set (part 2), and uniform practical convergence to the set for almost all initial conditions $x_0 \in \mathbb{R}^n$ (part 3).

Chapter 3

Synchronization in the complex Kuramoto oscillators network

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3.1 Introduction

In the previous chapter, we presented a system of Kuramoto oscillators where each element interacts with all other elements in a reciprocal way and with a homogeneous gain. This type of interaction is a useful approximation for many physical and engineering systems (see [14], [63]). However, in many situations this type of "all-to-all" interconnection is not realistic, rather each element interacts with a given subset of elements. Furthermore, this interaction is not necessarily reciprocal. It is the case, i.e. for systems of such

diverse nature as the synaptic neuronal networks [79], power networks [31], chemical reactions [68], and many others.

This chapter considers the Kuramoto coupled oscillator model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^N k_{ij} \sin(\theta_i - \theta_j) \quad i = 1 \dots N, \quad (3.1)$$

featuring dissimilar natural frequencies $\boldsymbol{\Omega}^\top = [\omega_1 \dots \omega_N]$ and the coupling gains $k_{ij} = \gamma a_{ij}$ through a graph $G(V; E; A)$ with a non-trivial topology defined by the adjacency matrix $A = [a_{ij}]$. The network science and physics communities coined the term *complex* for such non-trivial topologies to distinguish them from particular topologies of graphs which are usually symmetric. The interest of the control community in such complex oscillator networks has been sparked by the study of consensus and its applications [73], [83], [94], [104].

3.1.1 Motivation

Different aspects of the "all-to-all" coupled Kuramoto model - including bounds on the interconnection terms, explicit expressions for the asymptotic phase offset [4] and stability issues - were addressed during the last 30 years (see [31] and [122] for a detailed overview). Much fewer theoretical results are available for a more general setting where network topology differs from the "classical" all-to-all interconnection and many fundamental questions still remain unanswered. We cite here the articles [31], [54], and [133], where an extension of the Kuramoto model with a symmetric interconnection graph was considered and asymptotic synchronization to the averaged frequency was proved.

However, the networks that appear in different applications are usually directed and asymmetric. For example in biology, transcriptional regulatory networks and metabolic network are usually modelled as weighted digraphs [79]. In the same way in neuroscience, neuronal networks are typically modelled as directed graphs, where nodes represent individual neurons and the edges represent synaptic connections between neurons [3], [18], [28], [79].

With the growing interest in complex networks, several authors have become interested in investigating the synchronization phenomena of the Kuramoto model in complex wirings [122], [3]. In general, the results are known only for specific topologies such as the complete graphs discussed in the previous chapter, ring graphs [107], tree graphs [30], and complete bipartite graphs with uniform weights [134]. To the best of our knowledge, for arbitrary network topologies and weights only few theoretical results are available in the literature for the case of non-symmetric interconnection graphs. Mostly the case of

identical natural frequencies was considered in this framework, we cite here [12], [21], [87]. The reference [89] has studied numerically the conditions for the onset of synchronization in random scale-free networks. Soon afterwards, another author [69] used the mean-field approach to investigate the same problem from a theoretical perspective, as well as with numerical simulations. The results do not fully clarify whether or not the existence of phase locked solutions. The main difficulty comes from the fact that there is no unique consensus about what the order parameter should be.

In [105], the authors analyze the emergence of collective synchronization in directed networks of Kuramoto oscillators by generalizing the mean-field approach and demonstrating dependence of the critical coupling strength on the largest eigenvalue of the adjacency matrix describing the network connectivity. Their results are supported by numerical simulations; however, their approach is based on the assumption that both the size of the network and the number of connections into each node is large. This assumption is crucial for approximation of the order parameter, and unfortunately, this approach is not applicable for weakly connected networks, such as for example lattice network or also small-world network.

For arbitrary network topologies, the literature contains only sufficient conditions for frequency locking as well as statistical and numerical investigations for large random networks. In the recent reference [32], the authors give the existence conditions of frequency synchronization solutions. However, a complete analytic treatment is missing at this time, and many fundamental questions still remain unanswered. In particular, the analytical expression of the synchronization frequency and the conditions ensure the existence of a stable phase locked solution.

3.1.2 Organization and contributions

The remainder of this chapter is organized as follows. In Section 2, we consider a system of oscillators which interact through a network, where each element is connected to a subset of the elements in the entire system and with heterogeneous gains. Our emphasis will be placed on the existence conditions of frequency synchronization. We show that the problem of frequency synchronization is equivalent to the problem of existence of solutions for a non-linear system of algebraic equations.

In Section 3, we address a question which is rather common in the study of synchronization dynamic, namely what is the expression of the synchronization frequency? In the case of symmetric graphs, it is always true that synchronization frequency is equal to the average of natural frequencies. However, in the case of complex networks, frequency of synchronization can depend on the graph topology and the value of the coupling

strength. In this section, we give an analytical expression for the limit of synchronization frequency in the case when coupling strength is sufficiently large.

In Section 4, we introduce a novel linear model that possesses properties similar to those of the Kuramoto model and asymptotically it can be reduced to the Kuramoto model. Next we present necessary and sufficient conditions for the existence of phase locking solutions of the Kuramoto model with a weighted interconnection digraph and an arbitrary distribution of the natural frequencies.

3.2 Existence conditions for frequency synchronization

In this section, we present the necessary and sufficient conditions for frequency synchronization of the coupled oscillators model (3.1).

$$\dot{\theta}_i = \omega_i - \gamma \sum_{j=1}^N a_{ij} \sin(\theta_i - \theta_j) \quad i = 1 \dots N, \quad (3.2)$$

where the topology of the network $G = (V, E)$ is defined by the elements of adjacency matrix $\mathcal{A} = [a_{ij}]$ and γ is the coupling strength between pairs of connected oscillators.

As a first result in this direction, we present a theorem where we formulate both necessary and sufficient conditions for the existence of frequency locked solutions. We recall that system (3.2) are frequency locked if there exists a constant $\omega_s \in \mathbb{R}$ and initial conditions $\theta_o \in \mathbb{R}^N$ such that $\dot{\theta}_i(t, \theta_o) = \omega_s, \forall t \geq 0$ and for all $i \in \{1, \dots, N\}$ (see Definition 6, Section 2.3.2).

These conditions are formulated in terms of natural frequencies of the oscillators and properties of the adjacency matrix associated with underlying graph.

Theorem 6. *Consider a system of N Kuramoto oscillators (3.2) coupled via a digraph G with a given coupled strength $\gamma > 0$.*

Let $\tilde{\omega}_i = \omega_s - \omega_i$ and $a_{i,j}$ be the elements of adjacency matrix associated with the graph G . The oscillators (3.2) have a frequency locked solution if and only if the following conditions are satisfied :

1. *The following system of algebraic equations*

$$\begin{aligned} \sum_{j, j \neq i}^{N-1} \pm a_{i,j} \left(\xi_i \sqrt{1 - \xi_j^2} - \sqrt{1 - \xi_i^2} \xi_j \right) &= \frac{\tilde{\omega}_i}{\gamma}, i = 1 \dots N-1 \\ \sum_{j=1}^{N-1} a_{N,j} \xi_j &= \frac{\tilde{\omega}_N}{\gamma}, \end{aligned} \quad (3.3)$$

has at least one solution $[\xi_1^, \dots, \xi_{N-1}^*, \omega_s^*]$.*

2. *For all $i = 1 \dots N-1$, we have that $|\xi_i^*| \leq 1$.*

Proof.

Necessity : Let $\theta^*(t, \theta_o^*)$ be a phase locked solution of the system (3.2). Then from Definition 6 in Section 2.3.2, we have that all the oscillators have some constant common frequency ω_s . Similar to [4], let us define the change of variables :

$$\varphi_i = \theta_i - \omega_s t.$$

Using this change of coordinates and (3.2), we obtain that

$$\dot{\varphi}_i = \omega_i - \omega_s - \gamma \sum_{j=1}^N a_{ij} \sin(\varphi_i - \varphi_j), \quad i \in \{1, \dots, N\}. \quad (3.4)$$

Moreover, it is easy to see that every frequency locked solution of the system (3.2) corresponds to an equilibrium point of the system (3.4), that is, in new coordinates we have that for all $i \in \{1, \dots, N\}$

$$\dot{\varphi}_i(t, \varphi_o) \equiv 0,$$

where $\varphi_o = \theta_o^*$. Thus, in new coordinates, the frequency locked solution is defined by the following set of equations :

$$\begin{aligned} \omega_1 - \omega_s + \gamma \sum_{j=1}^N a_{1j} \sin(\varphi_1 - \varphi_j) &= 0 \\ &\vdots \\ \omega_N - \omega_s + \gamma \sum_{j=1}^N a_{Nj} \sin(\varphi_N - \varphi_j) &= 0 \end{aligned} \quad (3.5)$$

It is worth noting that the last system is a system of N equations depending on $N + 1$ unknown variables (ω_s and φ_i for $i = 1, \dots, N$).

Next, we consider a new change of variables

$$\begin{aligned} \xi_1 &= \sin(\varphi_1 - \varphi_N) \\ &\vdots \\ \xi_j &= \sin(\varphi_j - \varphi_N) \end{aligned} \quad (3.6)$$

Notice that for $j = 1 \dots N - 1$, we have $-1 \leq \xi_j \leq 1$.

Using these new variables, the differences $\sin(\varphi_i - \varphi_j)$ can be defined as

$$\sin(\varphi_i - \varphi_j) = \pm \xi_i \sqrt{1 - \xi_j^2} \mp \sqrt{1 - \xi_i^2} \xi_j \quad (3.7)$$

and we obtain the following set of systems which describes all the possible equilibrium points of (3.4)

$$\begin{aligned}
\omega_1 - \omega_s + \gamma \sum_{j=2}^{N-1} a_{1,j} (\pm \xi_1 \sqrt{1 - \xi_j^2} \mp \sqrt{1 - \xi_1^2} \xi_j) + \gamma a_{1,N} \xi_1 &= 0 \\
\vdots & \\
\omega_i - \omega_s + \gamma \sum_{j=1, j \neq i}^{N-1} a_{i,j} (\pm \xi_i \sqrt{1 - \xi_j^2} \mp \sqrt{1 - \xi_i^2} \xi_j) + \gamma a_{i,N} \xi_i &= 0 \\
\vdots & \\
\omega_{N-1} - \omega_s + \gamma \sum_{j=1}^{N-2} a_{N-1,j} (\pm \xi_{N-1} \sqrt{1 - \xi_j^2} \mp \sqrt{1 - \xi_{N-1}^2} \xi_j) + \gamma a_{N-1,N} \xi_{N-1} &= 0 \\
\omega_N - \omega_s - \gamma \sum_{j=1}^{N-1} a_{N,j} \xi_j &= 0
\end{aligned} \tag{3.8}$$

Sufficiency : To prove the sufficiency of these conditions, we assume that the system (3.3) has a finite solution $(\xi_1^*, \dots, \xi_{N-1}^*, \omega_s^*)$ with the property $|\xi_i^*| \leq 1$ for all $i = 1 \dots N-1$. Next, we define

$$\sin(\varphi_i^* - \varphi_j^*) = \xi_i^* \sqrt{1 - \xi_j^{*2}} - \sqrt{1 - \xi_i^{*2}} \xi_j^*,$$

that is we use the sign + in (3.7). Now, we consider the dynamics of our system (3.4) evaluated at $\varphi^{*\top} = [\varphi_1^*, \dots, \varphi_N^*]$ and we obtain

$$\omega_i - \omega_s^* - \gamma \sum_{j=1}^N a_{ij} \sin(\varphi_i^* - \varphi_j^*) = 0, \quad i = 1 \dots N, \tag{3.9}$$

that is we obtain that φ^* is an equilibrium point of system (3.4) and this implies that corresponding solutions of the system (3.2) are frequency locked.

Thus, we proved that the problem of existence of frequency locked solutions is equivalent to the problem of existence of solutions for a non-linear system of algebraic equations (3.3). ■

3.3 Estimation of synchronization frequency in terms of the coupling gain

In the case of undirected graph (i.e. symmetric topology), synchronization frequency is always equal to the average of natural frequencies [24],[54]. However, in the case of complex networks, numerical simulations show that synchronization frequency does not necessarily coincide with the average frequency. In particular in [31], the authors showed that the synchronization frequency is different from the average frequency for a particular case of weighted interconnection matrix. We show next that, in general, synchronization frequency ω_s depends on the digraph topology. Moreover, we additionally demonstrate that synchronization frequency can depend on the coupling strength.

3.3.1 Network topology effect on the synchronization frequency

The following result gives an analytical expression for the limit of synchronization frequency in the function of the coupling strength and underlying network topology.

Theorem 7. *Consider a network of oscillators (3.2) with a topology of interconnections defined by a strongly connected graph G and interconnection gain γ . If the system is synchronized, then as the coupling strength γ is increasing, the frequency of synchronization ω_s converges to $\frac{w_l^\top \Omega}{w_l^\top \mathbf{1}}$, where w_l is the left eigenvector of the Laplacian L associated with zero eigenvalue and Ω is the vector of natural frequencies, i.e.*

$$\lim_{\gamma \rightarrow \infty} \omega_s(\gamma) = \frac{w_l^\top \Omega}{w_l^\top \mathbf{1}}. \quad (3.10)$$

That is, in the limit as $\gamma \rightarrow \infty$, synchronization frequency ω_s is defined uniquely by the distribution of natural frequencies $\Omega = [\omega_1 \dots \omega_N]^\top$ and by the topology of the interconnection graph.

Proof. We start by introducing rescaling of time $\tau = \gamma t$ which leads to the following equation for θ_i

$$\frac{d\theta_i}{d\tau} = - \sum_{j=1}^N a_{ij} \sin(\theta_i - \theta_j) + \varepsilon_i, \quad i = 1 \dots N. \quad (3.11)$$

where $\varepsilon_i = \frac{\omega_i}{\gamma}$. Notice that $\varepsilon_i \rightarrow 0$ as $\gamma \rightarrow \infty$.

Assume that the system (3.11) is frequency synchronized. In this case, we have $\frac{d\theta_i}{d\tau} = \omega_s \forall i = 1 \dots N$, where ω_s is the frequency of synchronization to be defined later.

Using the following change of coordinates $\varphi_i = \theta_i - \omega_s \tau$; system (3.11) has this form :

$$\frac{d\varphi_i}{d\tau} = \sum_{j=1}^N a_{ij} \sin(\varphi_j - \varphi_i) + \rho_i \quad i = 1 \dots N, \quad (3.12)$$

where $\rho_i = \varepsilon_i - \frac{\omega_s}{\gamma}$.

Since the system is frequency synchronized we have that $\frac{d\varphi_i}{d\tau} = 0$; therefore the system (3.12) can be rewritten in the form

$$\sum_{j=1}^N a_{ij} \sin(\varphi_j - \varphi_i) = -\rho_i \quad i = 1 \dots N. \quad (3.13)$$

In the limit, as $\gamma \rightarrow \infty$ we can approximate (3.13) by the system

$$\sum_{j=1}^N a_{ij} \sin(\varphi_j - \varphi_i) = 0 \quad i = 1 \dots N \quad (3.14)$$

and $\boldsymbol{\varphi}^{*\top} = [0 \dots 0]$ is a solution of (3.14). Hence, for γ sufficiently large it is possible to analyze the system locally in the neighbourhood of the fixed point $\boldsymbol{\varphi}^* = 0$.

Let $\boldsymbol{\delta} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^*$ be a small perturbation away from $\boldsymbol{\varphi}^*$, so we can write (3.13) in the form

$$\sum_{j=1}^N a_{ij} \sin(\delta_j - \delta_i) + \rho_i = 0 \quad i = 1 \dots N. \quad (3.15)$$

Finally, linearizing the system we obtain

$$\sum_{j=1}^N a_{ij} (\delta_j - \delta_i) + \rho_i + o(\boldsymbol{\delta}) = 0 \quad i = 1 \dots N. \quad (3.16)$$

Using the definition of the Laplacian matrix, we can rewrite the above equation as

$$-L \boldsymbol{\Lambda} + \boldsymbol{\Psi} + o(\boldsymbol{\delta}) = 0, \quad (3.17)$$

where $\boldsymbol{\Lambda}^\top = [\delta_1, \dots, \delta_N]$ and $\boldsymbol{\Psi}^\top = (\boldsymbol{\omega}_s \mathbf{1} - \boldsymbol{\Omega})/\gamma$.

Multiplying (3.17) from the left by the left eigenvector w_l^\top of the Laplacian matrix L associated with the zero eigenvalue, we obtain that

$$w_l^\top \mathbf{1} \boldsymbol{\omega}_s - w_l^\top \boldsymbol{\Omega} = 0, \quad (3.18)$$

therefore,

$$\boldsymbol{\omega}_s = \frac{w_l^\top \boldsymbol{\Omega}}{w_l^\top \mathbf{1}} = \frac{\sum_{j=1}^N w_{l_j} \boldsymbol{\omega}_j}{\sum_{j=1}^N w_{l_j}}. \quad (3.19)$$

This limit frequency of synchronization corresponds to the solution of the system equation (3.4) with all positive signs. ■

Corollary 1. *If underlying graph $G = (V, E)$ is balanced then the limit of synchronization frequency $\boldsymbol{\omega}_s$ is equal to the average of natural frequencies of oscillators.*

$$\boldsymbol{\omega}_s = \frac{1}{N} \sum_{j=1}^N \boldsymbol{\omega}_j \quad (3.20)$$

Proof. Let $A = [a_{ij}]$ and $D = \text{diag}[d_{ii}]$ the adjacency and degree matrix of balanced graph $G = (V, E)$. By definition of balanced graph, all node $v_i \in V$ satisfies

$$\text{deg}_{out}(v_i) = \text{deg}_{in}(v_i) \quad i = 1 \dots N \quad (3.21)$$

It's also known that

$$\text{deg}_{out}(v_i) = d_{ii} \quad ; \quad \text{deg}_{in}(v_i) = \sum_{j:j \neq i}^N a_{ij} \quad i = 1 \dots N \quad (3.22)$$

Thus, every column sum of the Laplacian matrix $L = [l_{ij}]$ is equal to zero :

$$l_{ij} = l_{ii} + \sum_{j:j=i} l_{ij} = deg_{out}(v_i) - deg_{in}(v_i) = 0 \tag{3.23}$$

Consequently, we can conclude that $\mathbf{1}^T L = 0$ for all balanced graphs. According to the previous definition, the left eigenvector w_l of the Laplacian matrix L associated with the zero eigenvalue satisfies $w_l L = 0$. Then, we have $w_l = \mathbf{1}$ for all balanced graphs G . Using the equation (3.19), we thus obtain that the limit of synchronization frequency ω_s is equal to the average of the natural frequency of oscillators.

3.3.2 Simulation and validation of results : Examples of particular graphs

In this section, we present two examples of the Kuramoto model with strongly connected graphs, presented in Figure 3.1. In both cases the Laplacian matrix L is non symmetric, however, in the second example the interconnection graph G is balanced.

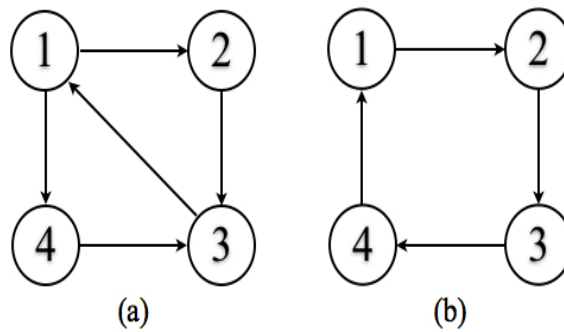


FIGURE 3.1 Two examples of a strongly connected graphs with four nodes, graph (b) is also balanced.

Example 1. Consider the Kuramoto system with strongly connected interconnection graphs G (see Figure 3.1(a)).

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^4 a_{ij} \sin(\theta_i - \theta_j) \quad i = 1 \dots 4 \tag{3.24}$$

where $A = [a_{ij}]$ is the adjacency matrix of G , and L is the corresponding Laplacian matrix :

$$A = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Bmatrix} ; L = \begin{Bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{Bmatrix}$$

We choose the vector of natural frequencies as $T = [12 \ 5 \ 8 \ 3]$. Starting from arbitrary initial conditions, each system quickly reaches a synchronization state as the coupling strength is sufficiently large. The evolution of system (3.24) is given in Figure 3.2, with the coupling strength $\gamma = 30$.

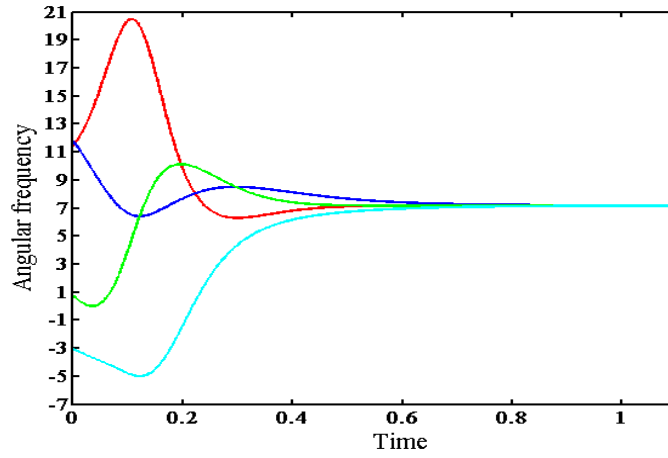


FIGURE 3.2 Synchronization of Kuramoto oscillators considered in Fig.3.1(a).

The resolution of the system's equations (3.4) defined in Theorem 6 show that the value of synchronization frequency ω_s is depending on coupling strength γ . Using Theorem 7, we can compute the limit of ω_s where γ is sufficiently large. In this example we have $w_s = \frac{w_l}{w_l \mathbb{1}} = 7.2$, such that the left eigenvector of the Laplacian matrix L associated with zero eigenvalue is $w_l = [\frac{1}{5} \ \frac{1}{5} \ \frac{2}{5} \ \frac{1}{5}]$. As the value of coupling strength γ increases, the frequency ω_s converges to ω_s given by theorem 2 (see Figure 3.3).

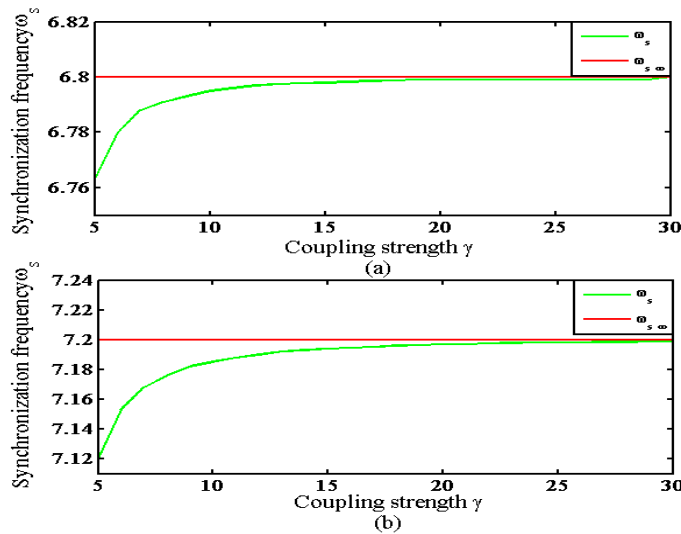


FIGURE 3.3 Variation of synchronization frequency ω_s in function of the coupling strength γ for two vectors of natural frequency $T_1 = [10 \ 8 \ 7 \ 2]$ and $T_2 = [12 \ 5 \ 8 \ 3]$. The red lines correspond to the limit of synchronization frequency given in theorem 2.

Example 2. Consider the Kuramoto system with a balanced interconnection graph G (Figure 3.1(b)). Numerical simulation of this system's dynamic is given in Figure 3.4 for the coupling strength $\kappa = 20$ and the vector of natural frequencies $\omega^T = [5 \ 13 \ 9 \ 3]$.

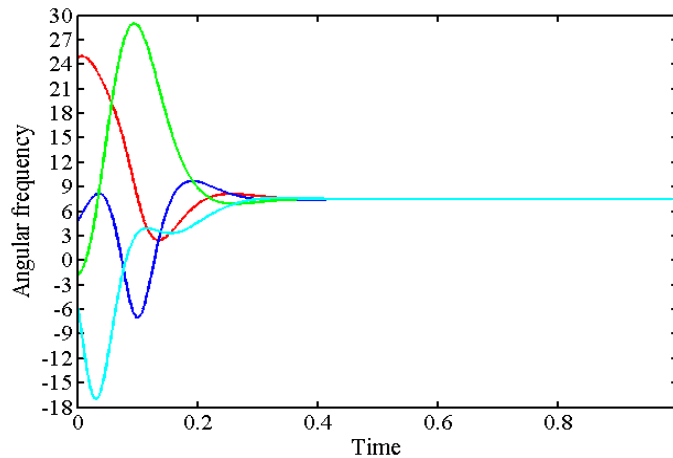


FIGURE 3.4 Synchronization of Kuramoto oscillators considered in Fig.3.1(b).

Evolution of synchronization frequency as a function of coupling strength κ , for the system with natural frequencies $\omega_1^T = [5 \ 13 \ 9 \ 3]$ and $\omega_2^T = [5 \ 10 \ 18 \ 7]$ is depicted in Figure 3.4. Numerical simulation shows that in the balanced case the limit frequency of synchronization coincides with the average of the oscillator's natural frequency, which agrees with the result of Corollary 1, while it is not the case in the first example.

3.4 Existence conditions of phase locked solutions

The above results give the existence's conditions for frequency locking solutions. These conditions are related to the adjacency matrix associated with the underlying graph. However in general, the resolution of the algebraic equations of the system (3.3) is very complex for a big dimension system. Several attempts were done to simplify the analysis. For example, in [106] a linear model was introduced that asymptotically can be reduced to the classical Kuramoto model (2.11). However, for a given structure of interconnections (all-to-all), the resulting reduced model obtained in [106] has a different structure of interconnections that can't be imposed beforehand. Starting with the same idea, in [25], the authors construct a family of auxiliary linear systems that preserve information on the natural frequencies and interconnection gains of the original Kuramoto model. We start our analysis by generalizing these results for the Kuramoto model with a complex network, and we show that the problem of existence of the phased locked solutions for

the Kuramoto model (3.2) with the arbitrary interconnection matrix $\mathcal{K} = [k_{ij}]$, can be recast as a problem of existence of a certain linear system of complex variables satisfying certain proprieties.

3.4.1 Linear presentation of the Kuramoto model

The reformulation of the Kuramoto model in terms of linear dynamics can be done, permitting its solution through an eigenvalue/eigenvector approach. The analysis in [106] is restricted to solving for the critical point of the fully synchronization transition and the order parameter beyond this transition for the classical Kuramoto model. Within this regime, the authors in [25] show that the classical Kuramoto model can be considered as an asymptotic projection of the linear system. In this section, we present the linear model corresponding to the Kuramoto model with complex network :

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^N k_{ij} \sin(\theta_j - \theta_i), \quad i = 1 \dots N, \quad (3.25)$$

where the topology of the network and the coupling strength between pairs of connected oscillators are defined by the adjacency matrix $\mathcal{K} = [k_{ij}]$.

The underlying idea of the approach can be summarized as follows. Consider a parametrized linear system (in terms of μ) in this form

$$\dot{x}_i = (i\omega_i - \mu_i)x_i + \sum_{j=1}^N k_{ij} x_j, \quad i = 1 \dots N, \quad (3.26)$$

where $x_i \in \mathbb{C}$, $\mu_i \in \mathbb{R}$ and k_{ij} are the elements of the interconnection matrix. Equivalently, this system can be written in the matrix form

$$\dot{\mathbf{x}} = (\mathcal{K} + i\Omega + \mathcal{M})\mathbf{x} \quad (3.27)$$

where $\mathcal{K} = [k_{ij}]$ is the matrix of the interconnections of the Kuramoto model (3.25), $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ and $\mathcal{M} = \text{diag}(\mu)$.

Using polar coordinates transformation $x_j = R_j e^{i\theta_j}$, the system (3.26) can be rewritten in the form

$$\dot{R}_i e^{i\theta_i} + iR_i e^{i\theta_i} \dot{\theta}_i = (i\omega_i - \mu_i)R_i e^{i\theta_i} + \sum_{j=1}^N k_{ij} R_j e^{i\theta_j} \quad (3.28)$$

By multiplying both sides in the last equation by $e^{-i\theta_j}$ and separating the real and imaginary parts, one obtains the following equations corresponding to the real and imaginary

parts

$$\begin{aligned}\frac{\dot{R}_i}{R_i} &= -\mu_i + \sum_{j=1}^N k_{ij} \frac{R_j}{R_i} \cos(\theta_j - \theta_i) \\ \dot{\theta}_i &= \omega_i + \sum_{j=1}^N k_{ij} \frac{R_j}{R_i} \sin(\theta_j - \theta_i)\end{aligned}$$

From the last equation (imaginary parts) it is easy to see that the existence of a constant $R > 0$ such that $\lim_{t \rightarrow \infty} R_i = R$ for all $i \in \{1, \dots, N\}$ would imply that asymptotically dynamics of θ can be described by the Kuramoto model (3.25).

The reformulated linear model provides an alternative coherent framework through which one can analytically solve synchronization problems that are not amenable with the original Kuramoto analysis. In addition, in the original version of the Kuramoto model the synchronization order parameter is only solvable in the continuum limit and then only implicitly. Using the linear formulation, we are able to find the explicit form of order parameter and for any number of oscillators. Furthermore, the linearity of the reformulation makes it possible to investigate the time evolution of a system's self-synchronization and allows itself to adapt to systems that exhibit asymmetric coupling between oscillators. This point is explored in greater detail in the next chapter. In the next section, we give the conditions on the matrix A that guarantee the existence of phase locked solutions for the Kuramoto model (3.25).

3.4.2 Reformulation of the conditions of phase locked existence

In this section, we show that the problem of existence of phase locked solutions for the model (3.25) can be recast as a problem of existence of a complex matrix A with off-diagonal elements given by coefficients k_{ij} and properties described below.

Namely, let parameter $\mu \in \mathbb{R}^N$ and consider the following family of parametrized linear systems of a complex variable $\mathbf{x} \in \mathbb{C}^N$

$$\dot{\mathbf{x}} = (\mathcal{K} + i\Omega + \mathcal{M})\mathbf{x} = A\mathbf{x}, \quad (3.29)$$

where $\mathcal{K} = [k_{ij}]$ is the matrix of the interconnections of the Kuramoto model (3.25), $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ and $\mathcal{M} = \text{diag}(\mu)$.

The structure of this system corresponds (up to the term $\mathcal{M}\mathbf{x}$) to the model (3.25). The following theorem shows that the existence of the phase locked solutions of (3.25) is equivalent to the existence of the matrix A , such that it has a pure imaginary eigenvalue and all elements of the corresponding eigenvector have the same absolute value.

Theorem 8. Consider the Kuramoto model (3.25), let $\mathcal{K} = [k_{ij}]$ be the corresponding adjacency matrix and $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ be the matrix of the natural frequencies.

Proof. $1 \Rightarrow 2$. Any complex matrix A can be presented in the Jordan form as $A = SJS^{-1}$, where S is a $N \times N$ matrix whose columns are eigenvectors and generalized eigenvectors of A . Then for any initial conditions \mathbf{x}_o , corresponding solution of this system \mathbf{x}_o can be written as

$$\mathbf{x}(t, \mathbf{x}_o) = e^{At} \mathbf{x}_o = Se^{Jt} S^{-1} \mathbf{x}_o. \quad (3.30)$$

Since by assumption multiplicity of the eigenvalue λ_o is equal to one, matrix A can be represented in the Jordan form as follows :

$$A = S \left[\begin{array}{c|c} \lambda_o & 0 \\ \hline 0 & J_1 \end{array} \right] S^{-1} = S \left[\begin{array}{c|c} i\omega_s & 0 \\ \hline 0 & J_1 \end{array} \right] S^{-1}$$

where $J_1 \in \mathbb{R}^{(N-1) \times (N-1)}$ is the other part of the Jordan form.

Now, for the particular choice of initial conditions $\mathbf{x}_o = v_o(A)$ we have

$$\mathbf{x}(t, \mathbf{x}_o) = Se^{Jt} S^{-1} \mathbf{x}_o = Se^{Jt} S v_o = Se^{Jt} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e^{i\omega_s t} S \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and since the first column of the matrix S is the vector $v_o(A)$, we obtain that $\mathbf{x}(t, \mathbf{x}_o) = e^{i\omega_s t} v_o$. By assumption we have that $v_o(A) = \text{col}(e^{\xi_1}, \dots, e^{\xi_N})$; therefore,

$$\mathbf{x}_j(t, \mathbf{x}_o) = e^{i(\xi_j + \omega_s t)} = e^{i\theta_j(t, \xi_j)}. \quad (3.31)$$

Let $\theta(t, \xi) = \text{col}(\theta_1(t), \dots, \theta_N(t))$ and define $e^{i\theta(t, \xi)} = \text{col}(e^{i\theta_1(t)}, \dots, e^{i\theta_N(t)})$, for thus defined vector $e^{i\theta(t, \xi)}$ we have that

$$i\dot{\theta} e^{i\theta(t, \xi)} = \frac{d}{dt} [e^{i\theta(t)}] = A e^{i\theta(t, \xi)}.$$

By multiplying the last equation on both sides by the matrix $\Phi = \text{diag}(e^{-i\theta(t)})$ and considering the imaginary part of thus obtained equation, we get

$$\begin{aligned} \dot{\theta} &= \text{Im} \left(\Phi A e^{i\theta(t)} \right) = \text{Im} \left(\Phi (\mathcal{K} + i\Omega + \mathcal{M}) e^{i\theta} \right) \\ &= \Phi \Omega e^{i\theta} + \text{Im} \left(\Phi \mathcal{K} e^{i\theta} \right) = \Omega \mathbf{1}_N + \text{Im} \left(\Phi \mathcal{K} e^{i\theta} \right), \end{aligned}$$

where in the last line we used the fact that Ω is a diagonal matrix and the equality $\Phi(t)e^{i\theta} = \mathbb{1}_N$. Writing explicitly expressions for the elements of $Im(\Phi\mathcal{K}e^{i\theta})$ we obtain that $\theta_i(t, \xi)$ satisfies the following equation

$$\dot{\theta}_i(t, \xi) = \omega_i + \sum_{j=1}^N k_{ij} \sin(\theta_j(t) - \theta_i(t)),$$

which implies that $\theta(t, \xi)$ is a solution of the Kuramoto model (3.25). From (3.31) it follows that $\theta_j(t) = \xi_j + \omega_s t$, thus the equalities $\theta_i(t, \xi) - \theta_j(t, \xi) = \xi_i - \xi_j$ are satisfied that for all $i, j = \{1, \dots, N\}$; therefore, $\theta(t, \xi)$ is a phase locked solution of the Kuramoto model (3.25).

2 \Rightarrow 1. Let $\theta_o \in \mathbb{R}$ be an initial condition corresponding to the phase locked solution $\theta(t, \theta_o)$ of the Kuramoto model (3.25). Then, it follows from the Definition 8 that there exists a continuous function $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}$ and such that $\theta_i(t, \theta_o) = \theta_{i_o} + \psi(t)$ for all $i = 1, \dots, N$. Moreover, using contradiction arguments, it is easy to show that there exists a constant ω_s such that

$$\begin{aligned} \dot{\theta}_i(t, \theta_o) &= \omega_i + \sum_{j=1}^N k_{ij} \sin(\theta_j(t) - \theta_i(t)) \\ &= \omega_i + \sum_{j=1}^N k_{ij} \sin(\theta_{j_o} - \theta_{i_o}) = \omega_s \quad \forall i = 1, \dots, N, \end{aligned}$$

hence the phased locked solutions $\theta_i(t, \theta_o)$ are synchronized with the constant frequency, i.e. $\psi(t) = c + \omega_s t$, where c is an arbitrary constant. Therefore, we can conclude that the phased locked solutions necessarily have the form $\theta(t, \theta_o) = \theta_o + \omega_s \mathbb{1}_N + c \mathbb{1}_N$.

Next, we define the vectors $\mathbf{x}_o = e^{i\theta_o}$, $\mathbf{x}(t, \mathbf{x}_o) = e^{i\theta(t, \theta_o)}$, constants $\mu_i = \sum_{j=1}^N \cos(\theta_{i_o} - \theta_{j_o})$, and the matrices $\mathcal{M} = \text{diag}(\mu_1, \dots, \mu_N)$, $\mathcal{K} + i\Omega + \mathcal{M}$. Reversing arguments of the first part of the proof it is easy to show that the vector $\mathbf{x}(t, \mathbf{x}_o)$ is a solution of a linear equation $\dot{\mathbf{x}} = A\mathbf{x}$, and moreover, the matrix $A = \mathcal{K} + i\Omega + \mathcal{M}$ has an eigenvalue $\lambda_o = \omega_s$ and equality (0.10) is satisfied. ■

Chapter 4

Analysis of Kuramoto oscillators with asymmetric weighted digraph

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4.1 Introduction

Dynamical systems with oscillatory motion are a basic component in the mathematical modelling of a large class of physical and biological phenomena. As we mentioned before, the Kuramoto model reproduces the main features of the emergence of coherent behavior found in more elaborate models of interacting oscillators. The basic process assumed to be the cause of the self-organization phenomenon is that a few elements in a network of interacting oscillators may synchronize if they have similar frequencies and their coupling is strong enough. Under suitable conditions, other oscillators may be attracted by this *averaged dynamic* and contribute to form a macroscopic oscillating cluster. Then,

the collective oscillations would appear as the result of coherent microscopic dynamics through the process of phase transition [139]. The transition to the synchronized dynamic results from the conflicting effect of two factors : the heterogeneity in the distribution of natural frequencies favours incoherent behaviour, while coupling induces the emergence of averaged dynamic [77]. Kuramoto oscillators provide a phenomenological description of complex systems whose collective evolution is driven by these synchronization processes. All real systems of coupled dynamical agents are characterized by a certain level of diversity both in their interaction and in the properties of their components. Heterogeneities are always present in macroscopic natural systems and, in particular, the coupling intensity between any pair of elements vary from pair to pair. A well known class of models with more complex interaction architectures is that of neural networks. In connection with a more realistic modelization of both artificial and natural systems, it is desirable to generalize standard models in order to envelop heterogeneity.

This situation differs from the globally coupled networks, considered in the case of the classical Kuramoto model, where connections are uniform and all interacting pairs have identical coupling strength. In the same spirit, coupled elements with heterogeneous interactions can be thought of as occupying the nodes of a network where links are present between those elements that may potentially interact with a particular intensity. In turn, each link can be weighted by a different coupling strength. Numerous presentations of the coupling parameters of the Kuramoto model have been proposed and studied, mostly via numerical experiments in the literature. It is not possible to review the totality of the existing articles, we refer to [3] for a global review on the subject.

In this chapter we analyze synchronization phenomena in networks of phase oscillators with two sources of heterogeneity added to the Kuramoto model. That is, we consider as in the previous chapter that natural frequencies of the oscillators are different, and additionally, we assume that the individual contributions of the oscillators to the mean field are weighted by two factors. These factors represent the input and output weights of the oscillator. In contrast to natural frequencies which determine the individual dynamics in the absence of coupling, the two new coupling weights affect the way in which each oscillator interacts with the ensemble. Consequently, they define a heterogeneous interaction in the system.

Motivated by the previous work on the local stability of phase-locked solutions for the "all-to-all" Kuramoto model with the uniform coupling [4] (see also [24] and [87]), we examine the effects of the heterogeneous interaction on the synchronization properties of the model. It turns out that the main ideas behind synchronization results of [4] can be

generalized in the case of weighted directed graphs, although tools used to compute explicit expressions for phase locked solutions are quite different, as some of the techniques used in the stability proofs.

4.2 Networks of oscillators with input-output weighted digraph

Consider the Kuramoto model (3.1). In this chapter, we consider the situation where the coupling gain k_{ij} is defined as a product of the *input* weight $w_{in}^j > 0$ of the j -th oscillator and its *output* weight $w_{out}^i > 0$, i.e.

$$k_{ij} = w_{out}^i w_{in}^j.$$

Let us define the vectors of input and output weights $W_{in}, W_{out} \in \mathbb{R}_+^N$ as

$$W_{in}^\top = [w_{in}^1, \dots, w_{in}^N] \quad (4.1)$$

$$W_{out}^\top = [w_{out}^1, \dots, w_{out}^N]. \quad (4.2)$$

Using these notations, the interconnection matrix $\mathcal{K} = [k_{ij}]$ can be represented as

$$\mathcal{K} = W_{out} W_{in}^\top. \quad (4.3)$$

Evidently, in this case the interconnection matrix is not necessarily symmetric.

Notice that in this section heterogeneity of the Kuramoto model comes from two different sources : variations in the natural frequencies of the oscillators and variations in the vectors of input and output weights. Such representation of the coupling matrix can appear in various applications, for example, in neuronal networks, the input and output weights can be associated with the conductivity of the dendrites and the axons respectively.

With this particular choice of gains, the Kuramoto model (2.11) can be written as

$$\dot{\theta}_i = \omega_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \mathbb{I}_N, \quad (4.4)$$

where the coupling strength between pairs of connected oscillators is defined by the coupling gain $K > 0$, the vectors of input weights W_{in} , and of output weights W_{out} .

In this chapter, we assume that all the interconnection gains are positive. In the next chapter, we will analyse the case of a signed weighted network where this assumption about positivity of the coupling gains will be relaxed.

4.3 Phase locked solutions

As we mentioned before, for some particular interconnection graphs conditions for the existence of phase locked solutions were formulated in the literature. In particular, for the Kuramoto model with the uniform all-to-all coupling explicit expressions for the asymptotic phase offset and conditions for local asymptotic stability were given in [4], where it was shown that the following assumption on the natural frequencies ω and the interconnection gain K is both necessary and sufficient for the existence of the phase locked solutions of the Kuramoto model (2.12) with uniform all-to-all coupling.

Assumption 1. *There exists a $r_\infty \in (0; 1]$ such that*

$$r_\infty = \frac{1}{N} \sum_{j=1}^N \pm \sqrt{1 - \left(\frac{\tilde{\omega}_j}{Kr_\infty}\right)^2}, \quad (4.5)$$

and for all $i = 1, \dots, N$ the following bounds are satisfied $|\tilde{\omega}_j| \leq Kr_\infty$, where $\tilde{\omega}_j = \omega_j - \omega_m$ and ω_m is the average of the natural frequencies.

The expression (4.5) is known in the literature as the consistency condition on r_∞ of the phase locked solutions. Similar results for the model with the ring interconnection structure were presented in [107], while for the case of the complete bipartite graph analysis of the Kuramoto model was presented in [134].

Also, we remark that the problem of frequency synchronization for the weighted Kuramoto model was considered in [24], and [54]; see also [31] and [122] for a detailed overview. However, neither the explicit structure of the synchronized solutions for weighted network were addressed in these papers.

4.3.1 Necessary and sufficient conditions

In this chapter, in the first step we consider the question of existence of phase locked solutions for the Kuramoto model (4.4).

In the previous chapter (Theorem 8), we proved that for the general case of the Kuramoto model (3.1) existence of the phase locked solutions is equivalent to the existence of a complex matrix with certain properties. Therefore, we start by finding such a matrix for

the particular choice of the matrix of interconnection gains \mathcal{K} given by (4.3).

With a slight abuse of notation, we can write the linear model (3.29) corresponding to the system (4.4) as

$$\dot{\mathbf{x}} = \left(\frac{K}{N} W_{out} W_{in}^\top - \mathcal{M} + i\Omega \right) \mathbf{x} = A \mathbf{x}. \quad (4.6)$$

where we recall that $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ is the diagonal matrix of natural frequencies, vectors W_{in} , W_{out} were defined in (4.1), and (4.2) and as in the previous chapter, $\mathcal{M} \in \mathbb{R}^{N \times N}$ is a diagonal matrix of unknown parameters, $\mathcal{M} = \text{diag}(\mu_1, \dots, \mu_N)$.

In the first step, we formulate conditions on the matrix \mathcal{M} ensuring that the matrix A defined in (4.6) has an imaginary eigenvalue $\lambda_1 = i\omega_s$ and the corresponding eigenvector $v_1(A)$ has the form $v_1(A) = [e^{i\phi_1}, \dots, e^{i\phi_N}]^\top$; that is we look for the diagonal matrix \mathcal{M} , constant $\omega_s \in \mathbb{R}$ and vector $v_1 \in \mathbb{C}^N$ such that

$$A v_1 = \frac{K}{N} W_{out} W_{in}^\top v_1 - (\mathcal{M} - i\Omega)v_1 = i\omega_s v_1. \quad (4.7)$$

From the last equation it is easy to see (4.7) that the vector v_1 has to satisfy the following equality

$$\frac{K}{N} W_{out} W_{in}^\top v_1 = \text{diag}(\mu_1 - i\tilde{\omega}_1, \dots, \mu_N - i\tilde{\omega}_N)v_1 = \text{diag}(\xi)v_1, \quad (4.8)$$

where $\tilde{\omega}_j = \omega_j - \omega_s$ and the vector $\xi \in \mathbb{C}^N$ with

$$\xi_j = \mu_j - i\tilde{\omega}_j. \quad (4.9)$$

Notice that until now exact value of ω_s is not specified, it will be defined later.

Next, we introduce a scalar quantity γ as follows

$$\gamma = W_{in}^\top v_1 = \sum_{j=1}^N w_{in}^j e^{i\phi_j}. \quad (4.10)$$

By using this notation we can rewrite the expression on the right hand side of (4.10) as

$$\frac{K}{N} W_{out} W_{in}^\top v_1 = \frac{K}{N} W_{out} \gamma, \quad (4.11)$$

and then equation (4.8) can be rewritten as follows :

$$\gamma \frac{K}{N} W_{out} = \text{diag}(\xi)v_1. \quad (4.12)$$

If v_1 is an eigenvector of A then, evidently, any vector $v = \eta v_1$ (where $\eta \in \mathbb{C}$) is also an eigenvector of A ; therefore, without loss of generality we can assume that $\gamma \in \mathbb{R}$, i.e.

$$\gamma = \sum_{j=1}^N w_{in}^j e^{i\phi_j} = \sum_{j=1}^N w_{in}^j \cos(\phi_j). \quad (4.13)$$

Since $\gamma \in \mathbb{R}$ then from (4.12) it follows that necessarily ξ_j has the form

$$\xi_j = \gamma \frac{K}{N} w_{out}^j e^{-i\phi_j}. \quad (4.14)$$

By denoting $K_j = K w_{out}^j$ and using (4.9) we can rewrite the last equation as

$$\mu_j - i\tilde{\omega}_j = \frac{\gamma K_j}{N} (\cos(\phi_j) - i \sin(\phi_j)). \quad (4.15)$$

Since $\gamma \in \mathbb{R}$, it trivially follows that

$$\sin(\phi_j) = \frac{N}{\gamma K_j} \tilde{\omega}_j, \quad (4.16)$$

while $\cos(\phi_j)$ can be defined as

$$\cos(\phi_j) = \frac{N}{\gamma K_j} \mu_j = \pm \sqrt{1 - \frac{\tilde{\omega}_j^2}{(K_j)^2 (\frac{\gamma}{N})^2}}, \quad (4.17)$$

therefore, the eigenvector $v_1 = \text{col}(e^{i\phi_1}, \dots, e^{i\phi_N})$ is completely defined.

Notice that the equations (4.16) yield a solution if and only if

$$-1 \leq \frac{N}{\gamma K_j} \tilde{\omega}_j \leq 1. \quad (4.18)$$

Combining equations (4.13) and (4.17), we obtain that

$$\gamma = \sum_{j=1}^N w_{in}^j \cos(\phi_j) = \sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \frac{\tilde{\omega}_j^2}{(K_j)^2 (\frac{\gamma}{N})^2}}. \quad (4.19)$$

By defining $r_\infty = \gamma/N$ we obtain that r_∞ has to satisfy the following equation

$$r_\infty = \frac{1}{N} \sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{K_j r_\infty}\right)^2}. \quad (4.20)$$

Thus, as in the case of the Kuramoto model with uniform all-to-all coupling, we obtained a transcendental equation in r_∞ which plays a key role in defining our matrix A . The important difference, however, is that up to now the synchronization frequency, or the parameter ω_s , is still not defined. Next, we show that ω_s is defined by the following expression

$$\omega_s = \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}} \quad (4.21)$$

Indeed, since γ is real, then from (4.19) it follows that

$$\sum_{j=1}^N w_{in}^j \sin(\phi_j) = 0, \quad (4.22)$$

and by substituting in this equation expressions for $\sin(\phi_j)$ given in (4.16) we obtain that

$$\sum_{j=1}^N w_{in}^j \sin(\phi_j) = \frac{1}{Kr_\infty} \sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \tilde{\omega}_j = 0 \quad (4.23)$$

Recalling that $\tilde{\omega}_j = \omega_j - \omega_s$, we can rewrite the last equation as

$$\frac{1}{Kr_\infty} \left(\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \omega_j - \omega_s \sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \right) = 0$$

from which we obtain that the synchronization frequency is defined by the following expression

$$\omega_s = \frac{\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \omega_j}{\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j}} = \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}}.$$

where the matrix $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$ and for simplicity of notations we denote by W_{out}^{-1} the following vector $W_{out}^{-1} = [\frac{1}{w_{out}^1}, \dots, \frac{1}{w_{out}^N}]$.

Next, similar to Assumption 1 used for the uniformly coupled Kuramoto model, let us introduce the following assumption :

Assumption 2. *There exists at least one solution $r_\infty > 0$ of the equations (4.20) such that for all $j = \{1, \dots, N\}$ the following bounds are satisfied $-1 \leq \frac{\tilde{\omega}_j}{K_j r_\infty} \leq 1$, where $K_j =$*

$$K \frac{w_{out}^j}{w_{in}^j}, \tilde{\omega}_j = \omega_j - \omega_s \text{ and } \omega_s = \frac{\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \omega_j}{\sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j}}.$$

If this assumption is satisfied, then using equation (4.17) and the relation $\gamma = Nr_\infty$, we can define the diagonal elements of matrix \mathcal{M} as

$$\mu_j = \pm K_j r_\infty \sqrt{1 - \frac{\tilde{\omega}_j^2}{(K_j r_\infty)^2}}. \quad (4.24)$$

Notice that in the case of uniform coupling, i.e. when $W_{in} = W_{out} = \mathbf{1}_N$, Assumption 2 reduces to Assumption 1; therefore, it can be seen as a weighted analog of Assumption 1 used for the Kuramoto model with uniform coupling.

Let Assumption 2 be satisfied then with elements of the matrix \mathcal{M} given by (4.24), we obtain that matrix A defined in (4.6) has the requested properties; namely, it has an imaginary eigenvalue

$$\lambda_1 = i\omega_s = i \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}}$$

and the corresponding eigenvector $v_1(A) = col(e^{i\phi_1}, \dots, e^{i\phi_N})$ is defined by (4.16), (4.17).

Then, Theorem 8 ensures the existence of phase locked solutions for the Kuramoto model (4.4) and we proved the sufficiency part of the following theorem

Theorem 9. *The Kuramoto model (4.4) has a phase locked solution if and only if Assumption 2 is satisfied.*

Furthermore, frequency of synchronization is defined by the following expression

$$\omega_s = \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}} \quad (4.25)$$

and the phase locked solutions have the form $\theta(t, \theta^*) = \theta^* + \omega_s t \mathbf{1}_n + c \mathbf{1}_n$, where $c \in \mathbb{R}$ is an arbitrary constant while the vector θ^* is defined by the equations

$$\cos(\theta_j^*) = \pm \sqrt{1 - \left(\frac{\tilde{\omega}_j}{K_j r_\infty}\right)^2} \quad ; \quad \sin(\theta_j^*) = \frac{\tilde{\omega}_j}{K_j r_\infty}. \quad (4.26)$$

The choice of the signs in the expressions for the cos corresponds to the choice of the signs in equation (4.20).

To prove necessity, it is enough to show that v_1 defined by (4.16)-(4.17) is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_1 = i \omega_s$. Direct computations show that this is indeed the case.

$$\begin{aligned} Av_1 &= \frac{K}{N} W_{out} W_{in}^\top v_1 - \text{diag}(\mu_1 - i\tilde{\omega}_1, \dots, \mu_N - i\tilde{\omega}_N) v_1 + i\omega_s v_1 \\ &= K r_\infty W_{out} - K r_\infty \text{diag}(w_{out}^1 e^{-i\phi_1}, \dots, w_{out}^N e^{-i\phi_N}) v_1 + i \omega_s v_1 \\ &= i\omega_s v_1. \end{aligned}$$

Remark 2. *Notice that given a choice of \pm signs in (4.20), every solution $r_\infty > 0$ corresponds to a different phase locked solution of (4.4). As it was remarked in [4] and [84], not all of 2^N equations, represented by (4.20), would have a solution for each fixed coupling gain K . On the other hand, some of the equations in (4.20) can have multiple solutions.*

In the following section, we will show that similarly to the classical Kuramoto model with uniform coupling, the only stable phase locked solution corresponds to all positive signs in the equation (4.20).

4.4 Stability analysis of the phase locking solutions

In this section, we analyse local stability properties of the phase locked solutions defined in the previous section. Let us consider dynamics of the Kuramoto model in the frame of coordinates rotating with the synchronization frequency ω_s . It is easy to see that in these new coordinates, the system (4.4) can be rewritten as follows

$$\dot{\theta}_i = f_i(\theta) = \tilde{\omega}_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \{1 \dots N\}, \quad (4.27)$$

where $\tilde{\omega}_i = \omega_i - \omega_s$.

Since the new synchronization frequency is equal to zero, we have that in the new frame of coordinates the phase locked solutions of the Kuramoto model correspond to the equilibrium points of the system (4.27).

Assume that the set of equation (4.20) has M solutions r_{∞_k} , $k = \{1, \dots, M\}$, satisfying Assumption 2. Notice that there is a set of equilibrium points $\theta_k^* \in \mathbb{R}^N$, $k = \{1, \dots, M\}$, which define phase locked solutions of the system (4.27).

Since the system (4.27) is invariant under the phase transformations $\theta \rightarrow \theta + c\mathbf{1}$, then we have that each equilibrium point $\theta_k^* \in \mathbb{R}^N$ of the system (4.27) belongs to a curve \mathcal{D}_k^* of equilibrium points defined by $\mathcal{D}_k^* = \{\theta \in \mathbb{R}^N : \theta = \theta_k^* + c\mathbf{1}, c \in \mathbb{R}\}$. We remark here that every of these curves corresponds to a phase locked solution of (4.4).

As it was mentioned in [4], asymptotic phase synchronization of solutions of (4.4), as define in definition 9, follows from the following two properties of the solutions of the system (4.27) :

- local stability of every equilibrium point on the curve \mathcal{D}_k^* .
- local asymptotic stability of the set \mathcal{D}_k^* .

4.4.1 Linearisation around an equilibrium point

The equilibrium points of (4.27) (or equivalently phase locked solutions of the Kuramoto model (4.4)) are defined by (4.26). Linearizing the system (4.27) around these equilibrium points and using the equations (4.20) and (4.23), we obtain that the off-diagonal elements

of the linearization are defined as

$$\begin{aligned} \frac{df_i}{d\theta_i} \Big|_{\theta=\theta^*} &= \frac{K}{N} w_{out}^i w_{in}^l \cos(\theta_i^* - \theta_i^{*l}) \\ &= \frac{K}{N} w_{out}^i w_{in}^l [\cos(\theta_i^*) \cos(\theta_i^{*l}) + \sin(\theta_i^*) \sin(\theta_i^{*l})] \\ &= \frac{K w_{out}^i}{N} \left[w_{in}^l \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_l}{w_{out}^l K r_\infty} \right)^2} \right) \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) + \frac{w_{in}^l \tilde{\omega}_l \tilde{\omega}_i}{w_{out}^i w_{out}^l (K r_\infty)^2} \right], \end{aligned} \quad (4.28)$$

while the diagonal elements are computed as follows

$$\begin{aligned} \frac{df_i}{d\theta_i} \Big|_{\theta=\theta^*} &= -\frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \cos(\theta_j^* - \theta_i^{*j}) \\ &= \frac{w_{out}^i K}{N} \left[-N r_\infty \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) + w_{in}^i \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} + \frac{w_{in}^i \tilde{\omega}_i^2}{(w_{out}^i K r_\infty)^2} \right]. \end{aligned} \quad (4.29)$$

Therefore, the Jacobian matrix J can be written in this form

$$J = \frac{K}{N} \Psi \Upsilon^{-1} (\Upsilon A + b b^\top + c c^\top) = \frac{K}{N} \mathcal{S} J_1 \quad (4.30)$$

where

$$J_1 = \Upsilon A + b b^\top + c c^\top ; \mathcal{S} = \Psi \Upsilon^{-1} \quad (4.31)$$

$$\Psi = \text{diag}(w_{out}^1, \dots, w_{out}^N) ; \Upsilon = \text{diag}(w_{in}^1, \dots, w_{in}^N)$$

$$A = -\text{diag}(a_1, \dots, a_N) , a_i = \pm N r_\infty \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \quad (4.32)$$

$$b = \begin{pmatrix} \pm w_{in}^1 \sqrt{1 - \left(\frac{\tilde{\omega}_1}{w_{out}^1 K r_\infty} \right)^2} \\ \vdots \\ \pm w_{in}^N \sqrt{1 - \left(\frac{\tilde{\omega}_N}{w_{out}^N K r_\infty} \right)^2} \end{pmatrix} ; c = \begin{pmatrix} w_{in}^1 \frac{\tilde{\omega}_1}{w_{out}^1 K r_\infty} \\ \vdots \\ w_{in}^N \frac{\tilde{\omega}_N}{w_{out}^N K r_\infty} \end{pmatrix}$$

Notice that matrix J_1 defined in (4.31) is symmetric.

Using these notations, the linearized system can be written as follows

$$\dot{\theta} = J \theta = \frac{K}{N} \mathcal{S} J_1 \theta, \quad (4.33)$$

where the constant matrix $J \in \mathbb{R}^{N \times N}$ is given by expressions (4.30 - 4.32).

For the linear system (4.33), let us consider the following change of coordinates

$$y = \mathcal{S}^{-1/2}\theta.$$

With this change of coordinates, we obtain

$$\dot{y} = \mathcal{S}^{-1/2}\dot{\theta} = \frac{K}{N}\mathcal{S}^{1/2}J_1\theta = \frac{K}{N}\mathcal{S}^{1/2}J_1\mathcal{S}^{1/2}y = \frac{K}{N}\tilde{J}y, \quad (4.34)$$

where $\tilde{J} = \mathcal{S}^{1/2}J_1\mathcal{S}^{1/2}$.

Using the fact that matrix \mathcal{S} is diagonal non-singular (with positive elements) and symmetry of the matrix J_1 we obtain that matrix \tilde{J} define above is symmetric and therefore stability properties of system (4.34) are defined by the properties of the matrix \tilde{J} .

Using Sylvester's law of inertia (see Appendix B), one can state that the matrices \tilde{J} and J_1 are congruent and therefore, they have the same inertia $In(\tilde{J}) = In(J_1)$. We recall that the inertia of a symmetric matrix \tilde{J} is the triplet of nonnegative integers $In(\tilde{J}) := (n_p, n_n, n_z)$, where n_p , n_n , and n_z are the number of positive, negative, and zero eigenvalues of \tilde{J} .

For that reason, it is enough to analyse the spectral properties of the matrix J_1 to study the stability properties of the linear system (4.34) or also the linearized system (4.33). The next section is devoted to the spectral analysis of the matrix J_1 .

4.4.2 Stability results

Analysis of the matrix J_1 that we propose in this section is similar to analysis of linearized model of uniform "all-to-all" Kuramoto model done in [4], [84], and follows the same steps as in those articles.

In order to simplify the stability analysis, we restrict our attention the case where

$$\frac{|\tilde{\omega}_i|}{w_{out}^i k r_\infty} < 1, \quad \forall i \in \{1, \dots, N\}. \quad (4.35)$$

This implies that all eigenvalues of A are different from zero and that $b_i \neq 0$, $\forall i \in \{1, \dots, N\}$.

First we show that matrix J_1 has at least one zero eigenvalue.

Lemma 1. *The matrix J_1 has an eigenvalue zero with the corresponding eigenvector $v = \mathbf{1}_N$.*

Proof. Let $v = \mathbf{1}_N$, then the i -th component of $J_1 \mathbf{1}$ can be calculated as follows

$$(J_1 v)_i = \frac{K}{N} \left[-w_{in}^i N r_\infty \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) + w_{in}^i \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) \right. \\ \left. \times \sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2} + \frac{w_{in}^i \tilde{\omega}_i}{w_{out}^i (K r_\infty)^2} \sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \tilde{\omega}_j \right]$$

Using the following relations in the last sum,

$$\sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2} = N r_\infty \quad ; \quad \sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \tilde{\omega}_j = 0,$$

we obtain that $Jv = 0$. ■

We show next that for the system (4.33) all, except one, equilibrium point are unstable. Therefore, if the linearization at an equilibrium point has at least one positive eigenvalue, then this equilibrium point is locally unstable for the system (4.27) and, therefore the corresponding phase locked solution is unstable as well. On the other hand, if J_1 contains one zero eigenvalue and all the other eigenvalues are negative, then the corresponding set \mathcal{D}^* of equilibrium points is locally stable [4], [84]. As we show in the following results, these two cases are linked with the choice of signs in the equation (4.20). Namely, solutions of (4.20) with at least one negative sign in the expression on the right-hand side correspond to locally unstable solutions while case of all positive signs in (4.20) corresponds to stable phase locking.

Theorem 10. *Fixed the coupling gain $K > 0$ and assume that Assumption 2 is satisfied. Let $\mathbb{A} = \{\theta_k^* \in \mathbb{R}^N, k = \{1, \dots, M\}\}$ be the set of all possible phase locked solutions defined by equations (4.20)-(4.26). The following statements are true :*

- 1) *There is at most one phase locked solution $\theta^* \in \mathbb{A}$ that is locally stable.*
- 2) *Let θ^+ be the phase locked solution defined by equation (4.26) and equation (4.20) taken with all positive signs, i.e*

$$r_\infty = \frac{1}{N} \sum_{j=1}^N w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2},$$

then, the corresponding set $\mathcal{D}^+ = \{\boldsymbol{\theta} \in \mathbb{R}^N : \boldsymbol{\theta} = \boldsymbol{\theta}^+ + c\mathbf{1}, c \in \mathbb{R}\}$ is locally asymptotically stable if and only if the following inequality holds

$$\sum_{j=1}^N \frac{w_{in}^j (1 - 2(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty})^2)}{\sqrt{1 - (\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty})^2}} > 0. \quad (4.36)$$

Proof. Let us define the matrix $A_1 = \Upsilon A$. The matrices bb^\top , cc^\top are positive semi-definite and A_1 is symmetric matrix. Applying Lemma 3 (See Appendix B) with $\tilde{A} = A_1 + bb^\top$ and $\tilde{B} = cc^\top$, we obtain that

$$\lambda_i(J_1) \geq \lambda_i(\frac{K}{N}(A_1 + bb^\top)), \forall i \in \{1, \dots, N\}. \quad (4.37)$$

In the same way, we have that

$$\lambda_i(A_1 + bb^\top) \geq \lambda_i(A_1), \forall i \in \{1, \dots, N\}. \quad (4.38)$$

Therefore,

$$\lambda_i(J_1) \geq \lambda_i(\frac{K}{N}A_1), \forall i \in \{1, \dots, N\}. \quad (4.39)$$

If the equilibrium point corresponds to a phase locked solution that satisfies the equation (4.5) with at least one minus sign present in its expression, at least one eigenvalue of A_1 is positive, and by (4.39), J_1 possesses also at least one positive eigenvalue.

From now on, we restrict our attention to studying the stability analysis of the phase locked solutions satisfying (4.5) with positive signs. The Jacobian corresponding to these phase locking solutions is given by equations (4.30 - 4.32) and matrix $A = \text{diag}(a_1, \dots, a_N)$ is negative definite, in this case :

$$a_i = -Nr_\infty \sqrt{1 - (\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty})^2}, \forall i \in \{1, \dots, N\}. \quad (4.40)$$

Using Lemma 4 (See Appendix B), we can prove that the matrices bb^\top and cc^\top each have one eigenvalue different from zero and positive. This implies that $(bb^\top + cc^\top)$ has, at most, two non-zero eigenvalue. Applying Lemma 3 with $\tilde{A} = -\frac{K}{N}(bb^\top + cc^\top)$ and $\tilde{B} = -\frac{K}{N}A_1$, we can prove that the matrix J_1 has at least $N - 2$ negative eigenvalues.

We know that the matrix J_1 has at least $N - 2$ eigenvalue smaller than zero and one eigenvalue equal to zero. Denote the zero-eigenvalue of J_1 by λ_N and the $N - 2$ eigenvalue smaller than zero by $\lambda_2 \dots \lambda_{N-1}$. Moreover, the eigenvalues of J_1 are the roots of $\det(A_1 - \lambda I + bb^\top + cc^\top)$. Using the same approach as used in the proof of Theorem 3 in [4], we

have

$$\det(A' + bb^\top + cc^\top) = \det(A')((1 + b^\top A'^{-1}b)(1 + c^\top A'^{-1}c) - (c^\top A'^{-1}b)^2), \quad (4.41)$$

with $A' = A_1 - \lambda I$.

The determinant $\det(A' + bb^\top + cc^\top)$ is a polynomial in λ and since J_1 has a zero-eigenvalue, the coefficients belonging to the zero-th order term is zero and the one belonging to the first order term are the product of all non-zero eigenvalues. Thus, we have

$$\det(A' + bb^\top + cc^\top) = -(\lambda_1 \lambda_2 \dots \lambda_{N-1})\lambda + O(\lambda^2). \quad (4.42)$$

By computing the Taylor series expansion in λ of (4.41) and replacing the expression of A , b and c , we obtain

$$\det(A' + bb^\top + cc^\top) = (\det(A_1) + O(\lambda))\left(\frac{1}{Nr_\infty^2}\lambda + O(\lambda^2)\right) (1 + c^\top A_1^{-1}c + O(\lambda)) - O(\lambda^2). \quad (4.43)$$

and therefore,

$$\det(A' + bb^\top + cc^\top) = (1 + c^\top A_1^{-1}c) \frac{\det(A_1)}{Nr_\infty^2} \lambda + O(\lambda^2). \quad (4.44)$$

Since (4.42) and (4.44) are equal, we have

$$-(\lambda_1 \lambda_2 \dots \lambda_{N-1}) = (1 + c^\top A_1^{-1}c) \frac{\det(A_1)}{Nr_\infty^2}. \quad (4.45)$$

We know that J_1 has $N-2$ negative eigenvalues. Denote these eigenvalues by $\lambda_2, \dots, \lambda_{N-1}$. The sign of the product of these eigenvalues is equal to the sign of $(-1)^{N-2}$. We also know that the sign of $\det(A_1)$ is equal to $(-1)^N$. Therefore, we have

$$\text{sgn}((-1)^{N-1}\lambda_1) = \text{sgn}((-1)^N(1 + c^\top A_1^{-1}c)), \quad (4.46)$$

hence,

$$\text{sgn}(\lambda_1) = -\text{sgn}((1 + c^\top A_1^{-1}c)). \quad (4.47)$$

Using the expression of A and c , we obtain

$$\begin{aligned}
 (1 + c^\top A_1^{-1} c) &= 1 - \sum_{j=1}^N \frac{w_{in}^j \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty}\right)^2}{N r_\infty \sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty}\right)^2}} \\
 &= \frac{1}{N r_\infty} \sum_{j=1}^N \frac{w_{in}^j \left(1 - 2\left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty}\right)^2\right)}{\sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty}\right)^2}}
 \end{aligned} \tag{4.48}$$

and, therefore, from the last equation we obtain that $\lambda_1(J_1) < 0$ if and only if the following condition is satisfied

$$\sum_{j=1}^N \frac{w_{in}^j \left(1 - 2\left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty}\right)^2\right)}{\sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty}\right)^2}} > 0. \tag{4.49}$$

If condition (4.49) is satisfied, then the phase locked solution corresponding to this equilibrium point is locally asymptotically stable. ■

4.5 Illustrative examples : numeric simulations

Consider Kuramoto system (4.4) of 10 oscillators with complete interconnection digraph and the following properties :

- vector of natural frequencies $\Omega^T = (20, 31, 27, 24, 33, 39, 21, 26, 34, 37)$;
- vector of input weights $W_{in}^\top = (1.5, 3, 2, 0.5, 1.7, 2, 2.5, 1, 0.5, 3.5)$;
- vector of output weights $W_{out}^\top = (0.5, 0.1, 0.7, 1, 0.7, 0.5, 0.4, 0.9, 1.2, 1.5)$.

The evolution of the system is given in Fig.4.1.

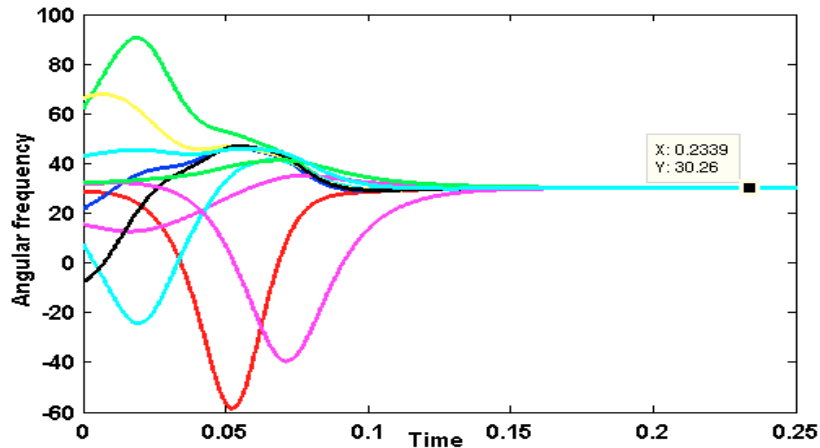


FIGURE 4.1 – Synchronization of angular frequencies of Kuramoto oscillators with the coupling strength $K = 100$.

From (4.25), we obtain that in this example synchronization frequency is equal to $\omega_s = 30.26$, which corresponds well with the simulation results presented in the Figure 4.1. In chapter 3, we show that the limit value (as $K \rightarrow \infty$) of synchronization frequency is given by the following expression

$$\lim_{K \rightarrow \infty} \omega_{s_\infty}(K) = \frac{w_l^\top \Omega}{w_l^\top \mathbf{1}}, \quad (4.50)$$

where w_l is the left eigenvector of the Laplacian matrix of the interconnection graph associated with a zero eigenvalue. Direct calculations show that this limit value coincides with the expression for ω_s given in (4.25).

For $K = 100$, the phase locked solution corresponding to system (4.4) with $N = 10$ is depicted in Fig.4.2(a). The phase differences of this solution are presented in Fig.4.2(b).

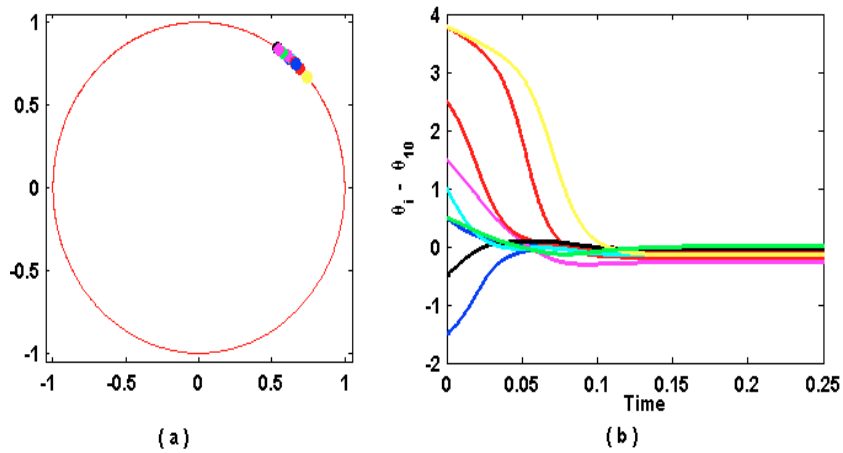


FIGURE 4.2 – (a) Phase locked solution of the system (4.4) with the coupling strength $K = 100$. (b) The evolution in time of the nine phase differences $(\theta_i - \theta_{10}), i = 1, \dots, 9$.

Chapter 5

Synchronization of the Kuramoto model with attractive and repulsive interactions

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5.1 Introduction

In the previous chapter, we considered the problem of phase synchronization for Kuramoto model with positive interconnection gains, that is only attractive interactions between oscillators were considered. However, the interplay of attractive and repulsive interactions can play an important role in the context of dynamical networks, as is the case, for example, in synthetic genetic circuits [41] and neuronal networks [45]. The crucial role of repulsive interactions in synchronization phenomena was studied analytically in [92] and confirmed experimentally in [141]. Recently, the combined action of attractive and repulsive couplings of Kuramoto oscillators was considered, mostly numerically, in [48], [70], [71]. In [47] and [143], the authors studied the influence of repulsive couplings

on the stability of full synchronization. Their results, however, are restricted only to case of identical oscillators.

In comparison with the classical Kuramoto model, little attention has been paid to stability analysis of the Kuramoto model with signed coupling strength, and the corresponding problem of the existence of phase locked solutions is still open. Still the inherent difference between activator and repressor interactions is clearly visible in the phase oscillator models. This particularly refers to anti-phase synchronization, where the oscillators gather in two groups having the opposite phases, i.e. towards being synchronized with a fixed phase difference π . Similar dynamics were observed in the case of a consensus problem with signed graph [5] (mostly called bipartite consensus), where all agents converge to a value which is the same for all in modulus but opposite in the sign.

In this chapter, we analyze phase and frequency synchronization of the Kuramoto model with a directed signed weighted graph of interconnection. The motivation for this work is to examine the effect of the heterogeneous coupling gains on the system's dynamics and on synchronization frequency. As in the previous chapter, we consider a scenario when the coupling coefficients of each pair can be separated into two different factors, or input-output weights. That is, we consider the case where interconnection matrix $\mathcal{K} = [k_{ij}]$ can be represented as $\mathcal{K} = W_{out} W_{in}^T$, where the vectors $W_{out}, W_{in} \in \mathbb{R}^N$. However unlike Chapter 4, here we do not assume that interconnection gains are necessarily positive; that is, some of the weights w_{in}^i and w_{out}^i can be negative. Such a representation of the coupling matrix can appear in various applications. For example, in neuroscience this could relate to the degree of excitation/inhibition that is going through the dendrites from neighbouring neurons to stimulate a neuron to fire.

It turns out that the ideas behind stability analysis of phase-locking for the *classical* Kuramoto model in [4] and [133] carry over to the case of weighted signed digraphs, although the actual evidence is different, as are some of the results. We show that in the case of weighted sign-symmetric graph of interconnection, existence and stability of phase locked solutions for the Kuramoto model with positive weights guarantees the existence of the whole family of Kuramoto models defined by the corresponding model with positive weights and by the family of gauge transformations. To the best of our knowledge, there are no similar results for the Kuramoto model with signed interconnections. We also give an exact expression for synchronization frequency which appears to be the same as in the case of the Kuramoto model with positive weights.

5.2 Phase locked solutions in the case of signed networks

Despite active research in the area of Kuramoto model synchronization (see e.g. recent review [31]), there are only few theoretical results on the stability of the Kuramoto model with mixed interaction. Even though synchronization for other models with mixed attractive and repulsive interactions has been explored by several authors [47], [48], [92]. In particular, for linear systems, a notion of consensus in presence of negative weights (repulsive links) was introduced in [5] and it was shown that under suitable conditions, a linear system $\dot{\mathbf{x}} = L \mathbf{x}$, where L is the Laplacian, can achieve a bipartite consensus if and only if the signed graph of the network is structurally balanced.

In the case of general signed digraphs, analysis of the Kuramoto model and is not a trivial task; that is why we concentrate on the case of structurally balanced (complete and sign-symmetric) network for which it appears possible to define phase locked solutions and analyze their stability properties.

In this section, we start by generalizing the conditions of existence of phase locked solutions for the Kuramoto model formulated in the previous chapter to the case of mixed (positive and negative) input-output weights. Therefore, in what follows, we consider the system

$$\dot{\theta}_i = \omega_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \mathbb{I}_N, \quad (5.1)$$

and we assume that the interconnection gains are not necessarily positive; i.e. some of the coefficients w_{in}^i and w_{out}^i can be negative.

However, we assume that the interconnection matrix $\mathcal{K} = W_{out} W_{in}^\top$ is sign-symmetric, i.e. $k_{ij}k_{ji} > 0$. The last inequality can be rewritten as $k_{ij}k_{ji} = w_{in}^i w_{out}^i w_{in}^j w_{out}^j > 0$ for all $i, j \in \mathbb{I}_N$, which can be satisfied only if w_{in}^i and w_{out}^i have the same sign for all $i \in \mathbb{I}_N$.

5.2.1 Existence of phase locked solutions for the Kuramoto model with signed weighted digraph

Similar to the study of the Kuramoto model with input-output weights, the following section we exploit the link between properties of the Kuramoto model and those of a linear system defined (up to a parameter) by the matrix of interconnections \mathcal{K} . Moreover, we use the properties of the structurally balanced signed digraph and the effect of a gauge transformation. We briefly recall here that a gauge transformation is a change of orthonormal order in \mathbb{R}^N introduced by a matrix $G = \text{diag}(\delta)$. The set of all gauge transformations in

\mathbb{R}^N is denoted as

$$\mathcal{G} = \{G \in \mathbb{R}^{N \times N}, G = \text{diag}(\delta), \delta = [\delta_1 \dots \delta_N], \delta_i \in \{\pm 1\}\}. \quad (5.2)$$

Detailed information on gauge transformations and their properties was presented in Chapter 2, Section 2.2.

As in the previous chapter, we start with the analysis of phase locked solutions and formulate the conditions that ensure existence. Namely, we have proved the following theorem :

Theorem 11. (*Phase locked solutions*)

The Kuramoto model (5.1) has a phase locked solution if and only if there exists a positive solution $r_\infty > 0$ of the following set of equations

$$r_\infty = \frac{1}{N} \sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{K_j r_\infty}\right)^2}, \quad (5.3)$$

and for all $j = \{1, \dots, N\}$ the following bounds are satisfied $-1 \leq \frac{\tilde{\omega}_j}{K_j r_\infty} \leq 1$, where $K_j = K w_{in}^j$, $\tilde{\omega}_j = \omega_j - \omega_s$ and frequency of synchronization ω_s is defined by

$$\omega_s = \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}}. \quad (5.4)$$

Furthermore, the phase locked solutions have the $\theta(t, \theta^*) = \theta^* + \omega_s t \mathbf{1}_n + c \mathbf{1}_n$ form, where $c \in \mathbb{R}$ is an arbitrary constant, while the vector θ^* is defined by the equations

$$\cos(\theta_j^*) = \pm \sqrt{1 - \left(\frac{\tilde{\omega}_j}{K_j r_\infty}\right)^2} \quad ; \quad \sin(\theta_j^*) = \frac{\tilde{\omega}_j}{K_j r_\infty} \quad (5.5)$$

and the choice of signs in the expressions for the cos corresponds to the choice of signs in the equation (5.3).

Proof : We recall the results of Chapter 3 (Theorem 8), it was shown that for the Kuramoto model, the problem of the existence of phase locked solutions is equivalent to the problem of the existence of a complex matrix \mathcal{A} with off-diagonal elements given by coefficients k_{ij} and properties described below.

Namely, let parameter $\mu \in \mathbb{R}^N$ and consider the following family of parametrized linear systems of a complex variable $\mathbf{x} \in \mathbb{C}^N$

$$\dot{\mathbf{x}} = (\mathcal{K} + i\Omega + \mathcal{M})\mathbf{x} = \mathcal{A}\mathbf{x}, \quad (5.6)$$

where $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$, $\mathcal{M} = \text{diag}(\mu)$ and $\mathcal{K} = [k_{ij}]$ is the coupling matrix of the Kuramoto model (2.15), i.e. the structure of this system corresponds (up to the term $\mathcal{M}\mathbf{x}$) to the model (2.15). It was shown that existence of the phase locked solutions of (2.15) is equivalent to existence of a matrix \mathcal{A} with a pure imaginary eigenvalue $\lambda(\mathcal{A}) = i\omega_s$ and all elements of the corresponding eigenvector have the same absolute value, i.e. $v(\mathcal{A}) = \text{col}(e^{i\phi_1}, \dots, e^{i\phi_N})$.

In Chapter 4 (Theorem 9), we proved that for the Kuramoto model with positive input/output weights (4.4), the elements of \mathcal{M} are defined as

$$\mu_j = \pm K_j r_\infty \sqrt{1 - \frac{\tilde{\omega}_j^2}{(K_j r_\infty)^2}} \quad (5.7)$$

and the eigenvector $v(\mathcal{A})$ corresponding to $\lambda(\mathcal{A}) = i\omega_s$ is defined by the equations (5.4)-(5.5).

Let us denote by $\overline{\mathcal{K}}$ the coupling matrix corresponding to the Kuramoto model with only positive weights, i.e. $\overline{k}_{ij} = |k_{ij}|$ for all $i, j = 1, \dots, N$. Since the digraph corresponding to the model (5.1) is sign balanced, then according to Definition 4 (Chapter 2) of structurally balanced digraph, it is always possible to represent the coupling matrix \mathcal{K} as $\overline{\mathcal{K}} = G\mathcal{K}G = G W_{out} W_{in}^\top G$ where all elements of $\overline{\mathcal{K}}$ non-negative and $G = \text{diag}(\delta_1, \dots, \delta_N)$ is the matrix of gauge transformation with $\delta_i = 1$ if $w_{in}^i > 0$ and $\delta_i = -1$ if $w_{in}^i < 0$.

Now, for the system (5.1), let us consider the matrix $\overline{\mathcal{A}} = \overline{\mathcal{K}} - \mathcal{M} + i\Omega$, where $\overline{\mathcal{K}}$ is defined as $\overline{\mathcal{K}} = G\mathcal{K}G$. Thus defined matrix $\overline{\mathcal{A}}$ can be written as follows :

$$\overline{\mathcal{A}} = G(\mathcal{K} - \mathcal{M} + i\Omega)G = G\mathcal{A}G.$$

It is easy to see that if matrix \mathcal{A} has an imaginary eigenvalue $\lambda_1 = i\omega_s$ and the corresponding eigenvector $v_1(\mathcal{A})$ has the form $v_1(\mathcal{A}) = \text{col}(e^{i\phi_1}, \dots, e^{i\phi_N})$, then the matrix $\overline{\mathcal{A}}$ has an eigenvalue $\lambda(\overline{\mathcal{A}}) = i\omega_s$ and the corresponding eigenvector is defined as $v = Gv_1(\mathcal{A})$.

The existence of phase locked solutions follows then from Theorem 9, Chapter 4. ■

5.2.2 Family of structurally balanced signed digraphs : intrinsic relationships between corresponding phase locked solutions

We show next that in the case of a weighted sign-symmetric graph of interconnections, existence of phase locked solutions for the Kuramoto model with positive weights guarantees existence of the whole family of Kuramoto models defined by the initial model and the family of gauge transformations. Consider the Kuramoto model with positive

weight only

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \bar{w}_{out}^i \sum_{j=1}^N \bar{w}_{in}^j \sin(\theta_j - \theta_i), \quad i \in \mathbb{I}_N, \quad (5.8)$$

where \bar{w}_{in}^i and \bar{w}_{out}^i present the positive input and output weights respectively.

Let, as in the previous section, $\bar{\mathcal{K}} = \bar{W}_{in} \bar{W}_{out}^\top$ be the adjacency matrix with positive elements. Notice that for each structurally balanced directed graph with positive weights there exists a family of structurally balanced *signed* directed graphs characterized by the same weights but with different signs. All these realizations of the signed graphs are related by gauge transformations. Let us define the set of all gauge equivalent adjacency matrices which contain at most 2^{N-1} distinct matrices, in the sequel we will denote this set as

$$\mathcal{K}_{\mathcal{G}} = \{ \mathcal{K}_g = G \bar{\mathcal{K}} G, G \in \mathcal{G} \}, \quad (5.9)$$

where \mathcal{G} is the set of all gauge transformations defined in (5.2).

In order to distinguish phase locked solutions of the system (5.8) and the phase locked solutions of the family of Kuramoto models defined by the interconnection matrices $\mathcal{K}_g \in \mathcal{K}_{\mathcal{G}}$, we will denote by θ^* and θ_g^* respectively the phase locked solutions of the Kuramoto model (5.8) and the Kuramoto model (5.1) with vectors of weights $W_{in}, W_{out} \in \mathbb{R}^N$ defined by relationships

$$W_{in} = G \bar{W}_{in} \quad W_{out} = G \bar{W}_{out} \quad (5.10)$$

Notice in this case the adjacency matrix \mathcal{K}_g is defined by the following relationship

$$\mathcal{K}_g = W_{out} W_{in}^\top = G \bar{W}_{out} \bar{W}_{in}^\top G = G \bar{\mathcal{K}} G.$$

Theorem 12. *Let $\mathcal{G} = \{G = \text{diag}(\delta_1, \dots, \delta_N), \delta_i \in \{\pm 1\}\}$ be the set of all gauge transformations in \mathbb{R}^N . Then, any Kuramoto model (5.1) with a sign-symmetric interconnections defined by the adjacency matrix $\mathcal{K}_g = G \bar{\mathcal{K}} G$ has a phase locked solution if and only if Assumption 2 is satisfied. Moreover, the synchronization frequency is defined by (5.4) and the phase locked solutions of (5.1) have the form*

$$\theta_g(t, \theta_g^*) = \theta_g^* + \omega_s t \mathbf{1}_n + c \mathbf{1}_n, \quad (5.11)$$

where $c \in \mathbb{R}$ is an arbitrary constant while the vector θ_g^* is defined by the relations :

$$\cos(\theta_g^*) = G \cos(\theta^*) \quad ; \quad \sin(\theta_g^*) = G \sin(\theta^*), \quad (5.12)$$

and the vector θ^* is the phase locked solution of the Kuramoto model (5.8), defined in (4.26).

Remark 3. *The theorem above states that any Kuramoto model defined by the adjacency matrix $\mathcal{K}_g = G\mathcal{K}G$ with $G \in \mathcal{G}$, has a phase locked solution if and only if the Kuramoto model (5.8), i.e. model with only positive weights, has a phase locked solution. Moreover, the angles θ_{g^*} , which correspond to negative entries of the gauge matrix G , have a π -shift with respect to corresponding angles of the model (5.8), i.e. $\theta_{g^*}^* = \theta_j^* + \pi$.*

Proof : We recall the results of Chapter 3 (Theorem 8) where it was shown that for the Kuramoto model, the problem of the existence of phase locked solutions can be recast as a problem of the existence of a complex matrix \mathcal{A} with off-diagonal elements given by coefficients k_{ij} and properties described below.

Let $\mu \in \mathbb{R}^N$ be a parameter and consider the following family of parametrized linear systems of a complex variable $\mathbf{x} \in \mathbb{C}^N$

$$\dot{\mathbf{x}} = (\mathcal{K} + i\Omega + \mathcal{M})\mathbf{x} = \mathcal{A}\mathbf{x}, \quad (5.13)$$

where $\Omega = \text{diag}(\omega_1, \dots, \omega_N)$, $\mathcal{M} = \text{diag}(\mu)$ and $\mathcal{K} = [k_{ij}]$ is the coupling matrix of the Kuramoto model (2.15), i.e. the structure of this system corresponds (up to the term $\mathcal{M}\mathbf{x}$) to the model (2.15). It was shown that the existence of the phase locked solutions of (2.15) is equivalent to existence of a matrix \mathcal{A} , such that it has a pure imaginary eigenvalue $\lambda(\mathcal{A}) = i\omega_s$ and all elements of the corresponding eigenvector have the same absolute value, i.e. $v(\mathcal{A}) = \text{col}(e^{i\phi_1}, \dots, e^{i\phi_N})$.

In particular, we proved that for the model (4.4) that the elements of \mathcal{M} are defined by

$$\mu_j = \pm K_j r_\infty \sqrt{1 - \frac{\tilde{\omega}_j^2}{(K_j r_\infty)^2}} \quad (5.14)$$

and the eigenvector $v(\mathcal{A})$ corresponding to $\lambda(\mathcal{A}) = i\omega_s$ is defined by the equations (5.4)-(5.5).

Now, for the system (5.1), let us consider the matrix $\mathcal{A}_g = \mathcal{K}_g - \mathcal{M} + i\Omega$, where \mathcal{K}_g is defined as $\mathcal{K}_g = G\mathcal{K}G$. Thus the defined matrix \mathcal{A}_g can be written as follows :

$$\mathcal{A}_g = G(\mathcal{K} - \mathcal{M} + i\Omega)G = G\mathcal{A}G.$$

It is easy to see that matrix \mathcal{A} has an imaginary eigenvalue $\lambda_1 = i\omega_s$ and the corresponding eigenvector $v_1(\mathcal{A})$ has the form $v_1(\mathcal{A}) = \text{col}(e^{i\phi_1}, \dots, e^{i\phi_N})$. Since the transformation $\mathcal{A} \rightarrow G\mathcal{A}G$ is a similarity transformation and as such it preserves the spectrum. Thus, matrix \mathcal{A}_g has an eigenvalue $\lambda(\mathcal{A}_g) = i\omega_s$ and the corresponding eigenvector is defined as $v = Gv_1(\mathcal{A})$ and the existence of the phase locked solutions follows from Theorem 9, Chapter 4. ■

5.3 Stability analysis of the phase locked solutions

In Section 5.2, we demonstrated that the Kuramoto model (5.1) can have one or multiple phase locked solutions if Assumption 2 is satisfied. In this section, we will analyse local stability properties of these phase locked solutions. Similar to the procedure used in Chapter 4, we start by changing the coordinate frame and consider the Kuramoto model (5.1) in the coordinate frame rotating with the frequency ω_s , where ω_s is defined by (5.4). In the new coordinates, the system (5.1) can be written as follows

$$\dot{\theta}_i = f_i(\theta) = \tilde{\omega}_i + \frac{K}{N} w_{out}^i \sum_{j=1}^N w_{in}^j \sin(\theta_j - \theta_i), \quad i \in \mathbb{I}_N, \quad (5.15)$$

where $\tilde{\omega}_i = \omega_i - \omega_s$.

Notice that in the new coordinates, the fixed points (i.e. points where $\dot{\theta}_{eq} = 0$) turn out to be just the values of θ_g^* , defined in Theorem 12.

Let Assumption 2 be satisfied, then linearizing the system (5.15) around the equilibrium point θ_g^* , we obtain the linearized system as follows

$$\dot{\theta} = J \theta, \quad (5.16)$$

where $\theta^\top = [\theta_1, \dots, \theta_N]$ and J is the Jacobian matrix, whose off-diagonal elements are defined as

$$\frac{df_i}{d\theta_l} \Big|_{\theta=\theta_g^*} = \frac{K}{N} w_{out}^i w_{in}^l \cos(\theta_l - \theta_i) = \frac{K}{N} w_{out}^i w_{in}^l \cos(\theta_{gl}^* - \theta_{gi}^*). \quad (5.17)$$

From Theorem 12 it follows that $\cos(\theta_{gj}^*) = \delta_j \cos(\theta_j^*)$ and $\sin(\theta_{gj}^*) = \delta_j \sin(\theta_j^*)$, at the same time we have that $w_{in}^j = \delta_j \bar{w}_{in}^j$ and $w_{out}^i = \delta_i \bar{w}_{out}^i$.

Substituting this equalities in (5.19) we obtain that

$$\frac{df_i}{d\theta_l} \Big|_{\theta=\theta_g^*} = \frac{K}{N} \bar{w}_{out}^i \bar{w}_{in}^l \cos(\theta_l^* - \theta_i^*). \quad (5.18)$$

In the same way, the diagonal elements of the matrix J can be calculated as follows

$$\frac{df_i}{d\theta_i} \Big|_{\theta=\theta_g^*} = -\frac{K}{N} w_{out}^i \sum_{j=1, j \neq i}^N w_{in}^j \cos(\theta_{gj}^* - \theta_{gi}^*). \quad (5.19)$$

Since the interconnection digraph is sign-symmetric, we have $w_{out}^i w_{in}^j = \delta_i \bar{w}_{out}^i \delta_j \bar{w}_{in}^j = \bar{w}_{out}^i \bar{w}_{in}^j$ for all $i, j = \{1, \dots, N\}$ and then we can equivalently write

$$\frac{df_i}{d\theta_i} \Big|_{\theta=\theta_g^*} = -\frac{K}{N} \bar{w}_{out}^i \sum_{j=1, j \neq i}^N \bar{w}_{in}^j (\cos(\theta_j^*) \cos(\theta_i^*) + \sin(\theta_j^*) \sin(\theta_i^*)). \quad (5.20)$$

Let us introduce the following notations :

$$\Psi = \begin{pmatrix} \bar{w}_{out}^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{w}_{out}^N \end{pmatrix}; \Upsilon = \begin{pmatrix} \bar{w}_{in}^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{w}_{in}^N \end{pmatrix} \quad (5.21)$$

that is Ψ and Υ are diagonal matrices with output and input weights on the diagonal.

Using these notations and explicit expressions of θ^* given by (5.5), we can write Jacobian matrix J in the form :

$$J = \frac{K}{N} \Psi \Upsilon^{-1} (\Upsilon A + bb^\top + cc^\top) = \frac{K}{N} \mathcal{S} J_1, \quad (5.22)$$

where

$$J_1 = \Upsilon A + bb^\top + cc^\top; \mathcal{S} = \Psi \Upsilon^{-1} \quad (5.23)$$

$$A = -diag(a_1, \dots, a_N), \quad a_i = \pm N r_\infty \sqrt{1 - \left(\frac{\tilde{\omega}_i}{K_i r_\infty}\right)^2},$$

$$b = \begin{pmatrix} \pm \bar{w}_{in}^1 \sqrt{1 - \left(\frac{\tilde{\omega}_1}{K_1 r_\infty}\right)^2} \\ \vdots \\ \pm \bar{w}_{in}^N \sqrt{1 - \left(\frac{\tilde{\omega}_N}{K_N r_\infty}\right)^2} \end{pmatrix}; c = \begin{pmatrix} \bar{w}_{in}^1 \frac{\tilde{\omega}_1}{K_1 r_\infty} \\ \vdots \\ \bar{w}_{in}^N \frac{\tilde{\omega}_N}{K_N r_\infty} \end{pmatrix}. \quad (5.24)$$

As in the case of Kuramoto model with attractive interactions (see Chapter 4) and in order to simplify the stability analysis, we restrict our attention to the case where $|\tilde{\omega}_j| < K_j r_\infty$ for all $j \in \mathbb{I}_N$. This last inequality implies that all eigenvalues of A are different from zero and $b_i \neq 0$ for all $i \in \mathbb{I}_N$.

Next, to analyse local stability of the linearized system (5.16), we use the following change of coordinates

$$y = \mathcal{S}^{-1/2} \theta.$$

Therefore, in new coordinates we obtain

$$\dot{y} = \mathcal{S}^{-1/2} \dot{\theta} = \frac{K}{N} \mathcal{S}^{1/2} J_1 \theta = \frac{K}{N} \mathcal{S}^{1/2} J_1 \mathcal{S}^{1/2} y, \quad (5.25)$$

As it was shown in the Chapter 4 (4.4.1), stability properties of the systems (5.25) and (5.16) are defined by the spectral properties of the symmetric matrix J_1 .

First, we show that the matrix J_1 has a zero eigenvalue. Namely, the following result is valid.

Lemma 2. *The matrix J_1 has an eigenvalue zero with corresponding eigenvector $v = \mathbf{1}_N$.*

Proof. Let $v = \mathbf{1}_N$, then the i -th component of $J_1 v$ can be calculated as follows

$$\begin{aligned} & \frac{K}{N} \left[-w_{in}^i N r_\infty \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) + w_{in}^i \left(\pm \sqrt{1 - \left(\frac{\tilde{\omega}_i}{w_{out}^i K r_\infty} \right)^2} \right) \right. \\ & \left. \sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2} + \frac{w_{in}^i \tilde{\omega}_i}{w_{out}^i (K r_\infty)^2} \sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \tilde{\omega}_j \right] = 0, \end{aligned}$$

where in the last line we used the following relations

$$\sum_{j=1}^N \pm w_{in}^j \sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2} = N r_\infty \quad ; \quad \sum_{j=1}^N \frac{w_{in}^j}{w_{out}^j} \tilde{\omega}_j = 0.$$

This implies that $J_1 v = 0$. ■

Similar to the results presented in Chapter 4, section 4.4 on stability of phase locked solution, here we formulate the following theorem which shows that among all the possible phased locked solutions of the system (5.1) there is only one which is asymptotically stable.

Theorem 13. Consider the Kuramoto models (4.4), (5.1). Let $\mathcal{K} = \overline{W}_{out} \overline{W}_{out}^\top$ and Assumption 2 be satisfied. Define the family of gauge equivalent adjacency matrices as $\mathcal{K}_G = \{ \mathcal{K}_g = G \mathcal{K} G, G \in \mathcal{G} \}$. Given G , let $\mathbb{A}_g = \{ \theta_{gk}^* \in \mathbb{R}^N, k = \{1, \dots, M\} \}$ be the set of all possible phase locked solutions for the system (5.1) defined in Theorem 12. The following statements are true :

- 1) There is, at most one phase locked solution $\theta_g^* \in \mathbb{A}_g$ which is stable.
- 2) Let θ^+ be a phase locked solution of the Kuramoto model with positive weights (4.4) defined by equations (4.20)-(4.26) taken with all positive signs and let θ_g^+ be the corresponding phase locked solution of (5.1) defined by (5.12). Then the set $\mathcal{D}_g^+ = \{ \theta \in \mathbb{R}^N : \theta = \theta_g^+ + c \mathbf{1}, c \in \mathbb{R} \}$ is locally asymptotically stable if and only if the following inequality holds

$$\sum_{j=1}^N \frac{|w_{in}^j| \left(1 - 2 \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2 \right)}{\sqrt{1 - \left(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty} \right)^2}} > 0. \quad (5.26)$$

Proof. Let us define the matrix $A_1 = \Upsilon A = -diag(\overline{w}_{in}^1 a_1, \dots, \overline{w}_{in}^N a_N)$, where $\Upsilon = diag(\overline{w}_{in}^1, \dots, \overline{w}_{in}^N)$. By construction, A_1 is a symmetric matrix and matrices bb^\top, cc^\top are symmetric and positive semi-definite. Applying Theorem 3 in [17] (Chapter 7, page 117) with $\tilde{A} = A_1 + bb^\top$ and $\tilde{B} = cc^\top$ we obtain that

$$\lambda_i(J_1) \geq \lambda_i \left(\frac{K}{N} (A_1 + bb^\top) \right), \quad \forall i \in \{1, \dots, N\} \quad (5.27)$$

and similarly,

$$\lambda_i(A_1 + bb^\top) \geq \lambda_i(A_1), \forall i \in \{1, \dots, N\}. \quad (5.28)$$

Therefore,

$$\lambda_i(J_1) \geq \lambda_i\left(\frac{K}{N}A_1\right), \forall i \in \{1, \dots, N\}. \quad (5.29)$$

Then it is easy to see that if the equilibrium point θ_g^* corresponds to a phase locked solution θ_g^* of (4.4) that satisfies equation (4.20) with at least one minus sign, then matrices A_1 and J_1 have at least one positive eigenvalue. Hence any equilibrium point, which corresponds to a phase locked solution with at least one negative sign in the equation for r_∞ , is locally unstable. Finally, using arguments similar to those of [35], we can show that the matrix J_1 has $N - 1$ negative eigenvalues if and only if the following condition is satisfied :

$$\sum_{j=1}^N \frac{|w_{in}^j| (1 - 2(\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty})^2)}{\sqrt{1 - (\frac{\tilde{\omega}_j}{w_{out}^j K r_\infty})^2}} > 0, \quad (5.30)$$

therefore, if (5.30) is satisfied, the corresponding phase locked solution is locally asymptotically stable. ■

Remark 4. Notice that the synchronization frequency ω_s defined by (5.4) depends on \bar{W}_{in} , \bar{W}_{out} and ω , and is independent of the signs of interconnection.

5.4 Simulation results

We illustrate the results presented in this chapter by means of a numerical example. Consider 8 complete coupled Kuramoto oscillators with the following parameters :

- vector of natural frequencies $\omega^\top = (25, 31, 20, 24, 33, 35, 21, 28)$;
- vector of input weights $\bar{W}_{in}^\top = (0.5, 0.7, 0.6, 1, 2, 1.2, 0.4, 0.9)$;
- vector of output weights $\bar{W}_{out}^\top = (2, 1.5, 1.2, 0.5, 0.7, 2, 2.5, 0.4)$.

To emphasize the effect of attractive and repulsive connections on the synchronization frequency and the phase differences, we consider three different scenarios of interconnection. The corresponding graphs for the three networks are presented in Figure 5.1. In the first case, all of the interaction weights are positive and there are three and four oscillators with different negative weights in the other two scenarios. The corresponding

gauge matrices G_1 , G_2 and G_3 are given as follows

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; G_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and the interconnection gain is chosen as $K = 50$.

In order to make clear the apparent links between the phase locked solutions of three different Kuramoto models, we do numerical simulations for three models in the same time interval, just switching from the first model to the second and then to the third one. In particular, we switch from one interconnection structure to another at the time instants $T_1 = 0.5$ and $T_2 = 1$. The corresponding graphs of the three networks are presented in Figure 5.1. Figures 5.2 and 5.3 presented below, represent frequencies of the

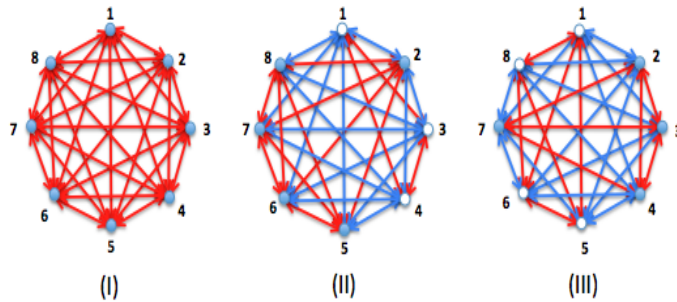


FIGURE 5.1 – Three networks of a set of eight oscillators corresponding respectively to the interconnection matrices inferred by the gauge matrices G_1 , G_2 and G_3 . Attractive and repulsive interactions are respectively represented by red and blue connections between nodes. The open and filled nodes represent oscillators with respectively negative and positive input weights.

oscillators and phase differences.

It is easy to see from Figure 5.2 that the frequencies of all 8 oscillators converge after a small transient to the common value $\omega_s = 28,66$. After both switches of network structure the oscillators synchronize once again to the same frequency $\omega_s = 28,66$, as it is ensured by Theorem 11. At the same time, a direct calculation of the synchronization frequency using expression (5.4), given as :

$$\omega_s = \frac{W_{in}^\top \Omega W_{out}^{-1}}{W_{in}^\top W_{out}^{-1}} = 28,66.$$

Indeed, for all presentations of the interconnection matrix inferred by a gauge transformation, the Kuramoto model (5.1) has the same synchronization frequency.

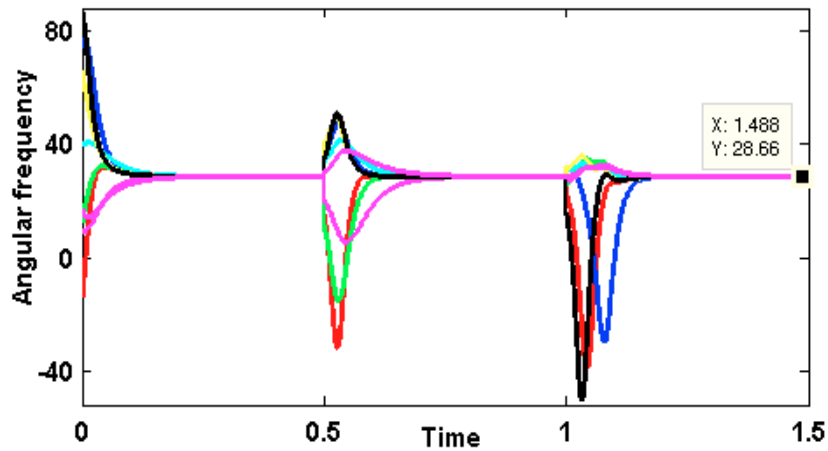


FIGURE 5.2 – Evolution of the angular frequencies $\dot{\theta}_i$ corresponding to the three interconnection graphs inferred by the gauge matrices G_1 , G_2 , and G_3 .

It is interesting to remark that this value of synchronization frequency coincides with the expression of synchronization frequency of the Kuramoto model with non-symmetric interconnection structure given in Chapter 3, Section 3.3.1 (see also [34]).

$$\lim_{K \rightarrow \infty} \omega_{s_\infty}(K) = \frac{w_l^\top \Omega}{w_l^\top \mathbf{1}},$$

where w_l is the left eigenvector of the Laplacian matrix of interconnection graph associated with zero eigenvalue. Direct calculations show that this limit value coincides with the expression for ω_s given in (5.4).

The phase differences of phase locked solution are presented in Figure 5.3 for the three interconnection matrices inferred by the gauge matrices G_1 , G_2 and G_3 .

A situation with the behaviour of phase differences presented in the Figure 5.3 is in accordance with the results of Theorem 13. Also, it should be remarked the similarity of the behaviour of the Kuramoto model with the case of consensus on networks with antagonistic interactions [5] where the agents converge to bipartite consensus. In particular,

when both attractive and repulsive interconnections are present, our system presents two sets of phase differences. Indeed, the positive coupling tends to pull the phases of the oscillators together, thus favouring the phase difference zero. However, the negative coupling pushes the phases apart and thus favours a phase difference of π .

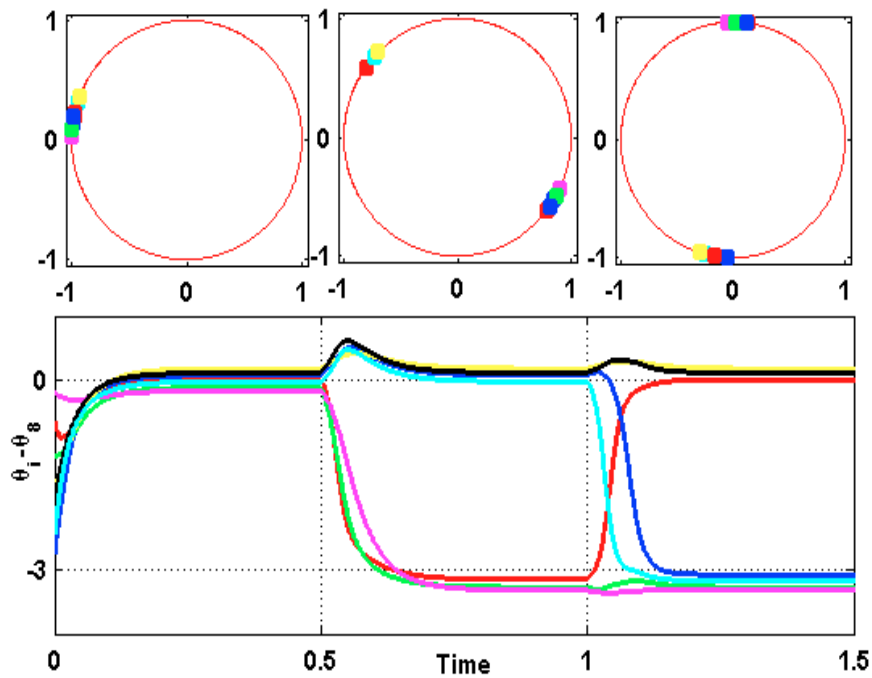


FIGURE 5.3 – (b) The time evolution of the seven phase differences $(\theta_i - \theta_8), i = 1, \dots, 7$ correspond respectively to the three networks (I, II, III) presented in Fig.5.1. (a) The instant positions of phase oscillators on unit circle illustrate the anti-phase synchronization.

Chapter 6

Emergence of limit cycle oscillators network

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6.1 Introduction

One of the key issues in synchronization analysis in networks of heterogeneous systems pertains to the emergence of new collective phenomena and the manner how individual dynamics, as well as the coupling architecture, affect the arising synchronized dynamics. For instance, the collective behavior of a network of coupled nonlinear oscillators is important to understand the complex dynamics of engineering and physical systems. Beyond engineering, problems of collective synchronization are of profound interest within a wide range of research disciplines such as physics, ecology, economics, biology *etc.* Particular examples include the synchronization of circadian rhythms, of heart cells as well as of neuronal activity [46], [131]. The latter are motivated by the recognized cause-effect of synchronization in neural disorders –see [90], [33], particularly Parkinson’s disease [76], [20]. The societal impact of the latter is one of the driving forces for the recent large-scale interest in the analysis and control of synchronization of complex interconnected systems.

In the preceding chapters, we have used coupled phase oscillators as a phenomenological approach to ensembles of interacting periodic elements. In fact, phase oscillators represent a good approximation to systems with limit-cycle orbits when the strength of coupling is small. This is the case of the classical Kuramoto model. On the other hand, as the coupling is strong enough, the phase approximation breaks down and it becomes necessary to take into account the full dynamics of each oscillatory element, including both its phase and its amplitude. The interplay of these two variables in ensembles of coupled limit-cycle oscillators gives rise to new phenomena in the collective evolution. Hence a more comprehensive model is required, where the dynamics of the amplitudes besides that of the phases is included.

Following the results of Andronov on limit-cycle oscillators, more generalized versions of the coupled oscillators model, including both phase and amplitude variations, have been published. Among these, the complex Stuart-Landau equation displays the amplitude equation derived from a general ordinary differential equation near an Andronov-Hopf bifurcation point –[29]. The Stuart-Landau oscillator is used in a wide range of applications; for instance, to describe chemical reaction diffusion systems [52], semiconductor lasers [22] as well as in neurophysiology [10]. In the case of finite gain and non-identical individual dynamics, the coupled Stuart-Landau oscillators are only frequency synchronized that is, the amplitudes of their oscillations do not coincide. Different tools, such as Dula’s theorem and Lyapunov exponents, have been used in the literature to study

stability properties of the limit cycle for a single Stuart-Landau oscillator [6], [65], [99] and for networks of such oscillators, see *e.g.*, [57], [130], Lyapunov type techniques were used to study stability for a network of identical oscillators [101]. However, in the general case of heterogeneous networks of Stuart-Landau oscillations, finding the synchronization frequency is a challenging and, to the best of our knowledge, is an open problem. In this chapter we give an approximate expression for this frequency which depends on the natural frequencies of the individual oscillators and on the matrix of the interconnections.

The remainder of this chapter is organized as follows. First, we introduce the general framework of our synchronization analysis and we present the model of interconnected Landau-Stuart oscillators and review its general properties. Next, transformation of the system in terms of interconnection matrix is given as well as approximate synchronization manifold. Afterwards, we analyze the stability of a network of diffusive coupled Stuart-Landau oscillators, based on the notion of practical stability which allows to study synchronization properties in case when only approximations of the limit cycle and synchronization frequency are available. The technical tools that support our findings are inspired by the work of [66], see also [23], [78], [129].

6.2 Model description and problem statement

In this chapter we consider a network composed of N heterogeneous diffusively coupled oscillators. In order to simplify the notations, in the sequel we will use the set $I = \{1, \dots, N\}$ to index the oscillators in the network.

Dynamics of individual oscillator

We assume that taken separately dynamics of each oscillator is described by Stuart-Landau equation

$$\dot{z}_i = -|z_i|^2 z_i + \mu_i z_i + u_i = f(z_i, \mu_i) + u_i, \quad i \in I \quad (6.1)$$

where $z_i, u_i \in \mathbb{C}$ are respectively the state and input of i -th oscillator, $\mu_i = \beta_i + i\omega_i \in \mathbb{C}$ is a complex parameter which defines asymptotic behavior of the i -th oscillator, as it was discussed in chapter 2, section 2.1. Later we introduce heterogeneity in the network by allowing parameters $\mu_i \in \mathbb{C}$ to be different for different oscillators.

Network structure and interconnections between the nodes

We assume that the graph of the network is connected and undirected, in this case interconnections between the nodes are defined by the adjacency matrix $D = [d_{ij}]_{i,j \in I_N}$ and moreover, $d_{ij} = d_{ji}$ for all $i, j \in I_N$. For the simplicity we assume that the interconnections weights are positive. i.e. $d_{ij} \in \mathbb{R}^+$ for all $i \in I_N$. In this case the corresponding Laplacian matrix is defined as

$$L = \begin{bmatrix} \sum_{i=2}^N d_{1i} & -d_{12} & \dots & -d_{1N} \\ -d_{21} & \sum_{i=1, i \neq 2}^N d_{2i} & \dots & -d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{N1} & -d_{N2} & \dots & \sum_{i=1}^{N-1} d_{Ni} \end{bmatrix} \quad (6.2)$$

where all row sums are equal to zero. Since the network is connected and undirected we have that matrix $-L$ has exactly one eigenvalue (λ_1) equal to zero, while others are negative, $0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_N$.

We assume that oscillators of the network are connected using diffusive coupling which represents a static relation between inputs and states of the oscillators, namely for the i -th oscillator the coupling is given by the following relation

$$u_i = -\gamma \left(d_{i1}(z_i - z_1) + d_{i2}(z_i - z_2) \dots + d_{iN}(z_i - z_N) \right), \quad (6.3)$$

where the scalar parameter $\gamma > 0$ corresponds to the coupling strength between the oscillators.

Dynamics of the overall network

Let us denote by $\mathbf{z} \in \mathbb{C}^N$ the overall network's state, that is $\mathbf{z} = [z_1, \dots, z_N]^\top \in \mathbb{C}^N$.

From (6.1) and expression for diffusive coupling (6.3), the overall network dynamics can be described by the following differential equation

$$\dot{\mathbf{z}} = F(\mathbf{z}) - \gamma L\mathbf{z}, \quad (6.4)$$

where we recall that $\gamma \in \mathbb{R}_+$ is the coupling gain, the matrix $L \in \mathbb{R}^{N \times N}$ is a Laplacian matrix defined in (6.2) and the function $F : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is given by

$$F(\mathbf{z}) = [f(z_i, \mu_i)]_{i \in I}. \quad (6.5)$$

As a first step in the analysis of the behavior of the diffusively couple network we formulate conditions that ensure that trajectories of the network of coupled Andronov-Hopf oscillators described by (6.4) are ultimately bounded, which, roughly speaking, means that all solutions eventually end up within some bounded domain. More precisely, following [59], we define ultimate boundedness of solutions in the following way.

Definition 14. *The solutions of the system $\dot{x} = f(x)$ are said to be ultimately bounded if there exist positive constants Δ_o and c such that for every $\Delta \in (0, \Delta_o)$, there exists a positive constant $T(\Delta)$ such that, for all $x_o \in \mathcal{B}_\Delta = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq \Delta\}$ they satisfy*

$$|x(t, x_o)| \leq c \quad \forall t \geq T.$$

If this bound holds for arbitrary large Δ , then the solutions are globally ultimately bounded.

Using simple Lyapunov arguments, similar to those of [101] and [80], one can establish that all trajectories of the network (6.4) are ultimately bounded for any interconnection gain $\gamma > 0$. To ensure ultimate boundedness of network trajectories we will use the following theorem which is a special case of Theorem 4.18 in [59].

Theorem 14. *Consider a system $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and $f(\cdot)$ is a continuous, locally Lipschitz function. Assume that the system is forward complete, there exists a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, continuous and positive definite function $W(x)$ and a constant $c > 0$ such that the following inequalities are satisfied*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{6.6}$$

$$\frac{\partial V}{\partial t} f(x) \leq -W(x) \quad \text{for all } |x| \geq c > 0 \tag{6.7}$$

Then, for every initial state $x(0) = x_o$ there exists a $T \geq 0$ (dependent on x_o and c) such that for all $t \geq T$ the following inequality holds

$$|x(t, x_o)| \leq \alpha_1^{-1}(\alpha_2(c)).$$

Using this theorem we can prove the following result.

Proposition 2. *Consider the system (6.4), (6.5) and let the graph of the network interconnections be undirected and connected, then solutions of the system (6.4), (6.5) are ultimately bounded and satisfy the following bound*

$$|\mathbf{z}(t, \mathbf{z}_o)| \leq \sqrt{2\mu N}. \tag{6.8}$$

Proof.

Consider the following Lyapunov function for the system (6.4), (6.5)

$$V(\mathbf{z}) = \frac{1}{2} \mathbf{z}^* \mathbf{z},$$

It is easy to see that thus defined Lyapunov function is positive definite and radially unbounded. Moreover, this Lyapunov function satisfies inequalities (6.6) with $\alpha_1(s) = \alpha_2(s) = \frac{1}{2} s^2$.

Taking the derivative of $V(\mathbf{z})$ along trajectories of the system (6.4), (6.5) we obtain that

$$\dot{V}(\mathbf{z}) = \frac{1}{2} \mathbf{z}^* (F(\mathbf{z}) - \gamma L \mathbf{z}) + \frac{1}{2} (F^*(\mathbf{z}) - \gamma \mathbf{z}^* L) \mathbf{z}.$$

Since the matrix L is symmetric, we have

$$\dot{V}(\mathbf{z}) = \frac{1}{2} (\mathbf{z}^* F(\mathbf{z}) + F^*(\mathbf{z}) \mathbf{z}) - \gamma \mathbf{z}^* L \mathbf{z}.$$

Next we use the fact that matrix L is a Laplacian matrix and therefore all eigenvalues of $-L$ are non-positive. Therefore we have

$$\dot{V}(\mathbf{z}) \leq \frac{1}{2} (\mathbf{z}^* F(\mathbf{z}) + F^*(\mathbf{z}) \mathbf{z})$$

Next, using (6.1), (6.5) and notations $C(\mathbf{z}) = \text{diag}(|z_1|^2, \dots, |z_N|^2)$ and $M = \text{diag}(\mu_1, \dots, \mu_N)$ we can write the terms $F(\mathbf{z})$ and $F^*(\mathbf{z})$ as

$$F(\mathbf{z}) = -C(\mathbf{z}) \mathbf{z} + M \mathbf{z}, \quad F^*(\mathbf{z}) = -\mathbf{z}^* C(\mathbf{z}) + \mathbf{z}^* M.$$

Using these notations we rewrite \dot{V} as

$$\dot{V}(\mathbf{z}) \leq -\mathbf{z}^* C(\mathbf{z}) \mathbf{z} + \frac{1}{2} (\mathbf{z}^* M \mathbf{z} + \mathbf{z}^* M^* \mathbf{z}) = -\sum_{i=1}^N |z_i|^4 + \mu \sum_{i=1}^N |z_i|^2 = -\sum_{i=1}^N |z_i|^4 + \mu |\mathbf{z}|^2, \quad (6.9)$$

where in the last line we used the notation $\mu = \max_{i \in I_N} \{\text{Re}(\mu_i), 0\}$ for the largest real part of the elements of the matrix $M = \text{diag}(\mu_1, \dots, \mu_N)$.

Next we use the triangular inequality to upperbound the term $\sum_{i=1}^N |z_i|^4$, we have

$$\begin{aligned} |\mathbf{z}|^4 &= \left(\sum_{i=1}^N |z_i|^2 \right)^2 = \sum_{j=1}^N \sum_{i=1}^N |z_i|^2 |z_j|^2 \leq \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^N (|z_i|^4 + |z_j|^4) \\ &= \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^N |z_i|^4 + \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^N |z_j|^4 = N \sum_{j=1}^N |z_j|^4 \end{aligned}$$

and therefore

$$\sum_{j=1}^N |z_j|^4 \geq \frac{1}{N} |\mathbf{z}|^4.$$

Substituting this last bound in (6.9) we obtain

$$\dot{V}(\mathbf{z}) \leq -\frac{1}{N} |\mathbf{z}|^4 + \mu |\mathbf{z}|^2 = -\left(\frac{1}{N} |\mathbf{z}|^2 - \mu\right) |\mathbf{z}|^2 = -\frac{1}{2N} |\mathbf{z}|^4 - \frac{1}{2N} (|\mathbf{z}|^2 - 2\mu N) |\mathbf{z}|^2$$

From the last inequality we obtain that $\dot{V}(\mathbf{z}) < -\frac{1}{2N} |\mathbf{z}|^4$ for all \mathbf{z} such that $|\mathbf{z}| > \sqrt{2\mu N}$. Then from theorem 14 we obtain that solutions are ultimately bounded and for any $R > 0$ there exists a $T(R)$ such that for all $t \geq T$ and all initial conditions $|\mathbf{z}_0| \leq R$ system trajectories satisfy the following bound

$$|\mathbf{z}(t, \mathbf{z}_0)| \leq \sqrt{2\mu N}. \quad \blacksquare$$

Problem statement - First iteration

In case of a network of identical oscillators (i.e. in the case when $\mu_i = \mu_j$ for all $i, j \in I$) synchronization is often described in terms of (asymptotic) identical evolution of the units and hence formulated as (asymptotic) stability of the synchronization manifold

$$\mathcal{S} = \{\mathbf{z} \in \mathbb{C}^N : \mathbf{z}_1 = \mathbf{z}_2 = \dots = \mathbf{z}_N\}, \quad (6.10)$$

which can be deduced using results and tools developed for semi-passive, incrementally passive or incrementally input-output stable systems [1], [101], [102], [103], [111], [112].

Behavior of the networks with non-identical units is more complex due to the fact that synchronization manifold \mathcal{S} does not necessarily exist due to the differences in the dynamics of the oscillators. Still it is well known from the literature on dynamical systems that such heterogeneous networks can exhibit some type of synchronization and collective behavior.

One of the possible approaches to this problem is to consider some kind of "practical" synchronization, when differences between the dynamical evolution of the different units are bounded and become smaller for larger values of interconnection gain γ . That is the approach that we will persist in this chapter.

In particular we will show that in the case of general heterogeneous network of oscillators, we can characterize evolution of the networks dynamics in terms of two different separate properties : synchronization and emergent dynamics of the network. To do so we first show that for large values of the gain γ the network behavior can be decomposed in

two parts, one of them corresponds to dynamics of some "averaged" ("mean-field") oscillator, while the other describes dynamics of the network units relative to the "averaged" dynamics.

Before formulating the synchronization problem in more precise terms, in the next section we define coordinate transformation that will allow us the network decomposition described above.

6.3 Coordinate transformation and model reformulation in new coordinates

To clarify our approach to the synchronization of the nonlinear network, we next rewrite this system in a new coordinate frame which is convenient for our purposes. To that end, first let us split the network dynamics into linear and nonlinear parts.

We introduce the following diagonal matrices

$$C(\mathbf{z}) = \begin{bmatrix} |\mathbf{z}_1|^2 & 0 & \dots & 0 \\ 0 & |\mathbf{z}_2|^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & |\mathbf{z}_N|^2 \end{bmatrix} \quad \text{and} \quad \mathcal{M} = \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mu_N \end{bmatrix}$$

Using these notations we can rewrite the system (6.4) as

$$\dot{\mathbf{z}} = (\mathcal{M} - \gamma L)\mathbf{z} - C(\mathbf{z}) \mathbf{z} = A_\gamma \mathbf{z} - C(\mathbf{z}) \mathbf{z}, \quad (6.11)$$

where

$$A_\gamma = \mathcal{M} - \gamma L. \quad (6.12)$$

6.3.1 Properties of the matrix A_γ

Notice that $A_\gamma \in \mathbb{C}^{N \times N}$ is a complex symmetric matrix, $A_\gamma = A_\gamma^\top$. It is well known, see e.g. [27], [50], that for any symmetric matrix C there exists a complex orthogonal matrix Q ($Q^{-1} = Q^\top$) such that matrix $Q^\top C Q$ is of bloc diagonal form

$$\begin{bmatrix} B_1 & 0 & 0 & \dots \\ 0 & B_2 & 0 & \dots \\ 0 & 0 & B_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

where each block is either

(i) a 1×1 submatrix

or

(ii) $B = \lambda I + M$, where M is a $q \times q$ matrix with all zero eigenvalues (for more details on the structure of the matrix M we refer to the theorem 4 [27]).

Next we remark that matrix A_γ can be viewed as

$$A_\gamma = \gamma(-L + \varepsilon \mathcal{M})$$

where $\varepsilon = 1/\gamma$.

Thus for sufficiently small values of ε (or equivalently for sufficiently large values of γ) we can see matrix A_γ as a scaled by ε -perturbation of the Laplacian matrix $-L$ and therefore we can use results from the perturbation theory for matrices (see e.g. [50], [91]) to characterize the eigenvalues and eigenvectors of the matrix A_γ in terms of ε and eigenvalues and eigenvectors of the matrix L .

Using [91, Theorem 2.1], [50], [67] we can estimate the eigenvalues of the matrix A_γ in terms of those of L , \mathcal{M} and ε .

In the general case let us define a small perturbation of a generic matrix A as

$$A_\varepsilon = A_0 + \varepsilon A_1, \quad \varepsilon \rightarrow 0.$$

If we denote with λ_1 a simple eigenvalue of A_0 and with λ_ε its induced perturbation, then for sufficiently small ε there exists the convergent power series representation

$$\lambda_\varepsilon = \lambda_1 + c_1 \varepsilon + o(\varepsilon), \quad (6.13)$$

where the first order term c_1 can be characterized as

$$c_1 = \frac{w^\top A_1 v}{w^\top v} \quad (6.14)$$

where w , v are the left and right eigenvectors of the unperturbed matrix A_0 associated to λ_1 , normalized with unitary norm, i.e. $\|w\| = \|v\| = 1$. This result is also applicable if the multiplicity of λ_1 is larger than one, provided that there exists a complete set of eigenvectors for the associated eigenspace [91].

Since in our case Laplacian matrix L is symmetric and corresponds to a connected graph, then it follows from the standard results [50] that it is diagonalizable and there exists a

real orthogonal matrix U such that

$$L = U \begin{bmatrix} \lambda_1(L) = 0 & & \\ & \ddots & \\ & & \lambda_N(L) \end{bmatrix} U^\top.$$

Moreover, left and right eigenvectors of the matrix L which correspond to $\lambda_1(L) = 0$ coincide and have the following form : $w = v = \frac{1}{\sqrt{N}}[1, \dots, 1]^\top = \frac{1}{\sqrt{N}} \mathbf{1}$. Then from (6.14) we have that in our case

$$c_1 = \frac{w^\top A_1 v}{w^\top v} = \frac{1}{N} \mathbf{1}^\top \mathcal{M} \mathbf{1} = \frac{1}{N} \sum_{i=1}^N \mu_i.$$

Then using (6.13) and the fact that $\varepsilon = \frac{1}{\gamma}$ we can approximate $\lambda_1(A_\gamma)$ as

$$\lambda_1(A_\gamma) = \gamma(-\lambda_1(L) + c_1 \varepsilon + o(\varepsilon)) = \gamma\left(c_1 \frac{1}{\gamma} + o\left(\frac{1}{\gamma}\right)\right) = c_1 + O(\varepsilon) = \frac{1}{N} \sum_{i=1}^N \mu_i + O(\varepsilon),$$

thus we have that $\lambda_1(A_\gamma)$ is bounded and converges to $\frac{1}{N} \sum_{i=1}^N \mu_i$ as the coupling strength $\gamma \rightarrow \infty$. Moreover, we can show that all $\lambda_j(A)$ ($j = 2, \dots, N$) are proportional to γ and $\text{Real}(\lambda_j(A)) \rightarrow -\infty$ as $\gamma \rightarrow \infty$.

In the following we will just assume that γ is sufficiently large so that matrix A_γ is diagonalizable, *i.e.* we impose the following assumption.

Assumption 3. *There exists a $\gamma^* > 0$ such that for all $\gamma \geq \gamma^*$ matrix A_γ defined in (6.12) can be factorized as*

$$A_\gamma = V_\gamma \Lambda_\gamma V_\gamma^{-1}, \quad (6.15)$$

where $\Lambda_\gamma \in \mathbb{C}^{N \times N}$ is a diagonal matrix whose diagonal elements are the eigenvalues of A_γ and $V_\gamma \in \mathbb{C}^{N \times N}$ is a complex orthogonal matrix, that is $V_\gamma^{-1} = V_\gamma^\top$ and there exists $k \in I_N$ such that $\text{Re}(\lambda_k) > \max_{j \in \{1, \dots, N\}, j \neq k} \text{Re}(\lambda_j)$.

Notice that matrix V_γ that we defined in (6.17) is a square matrix whose i^{th} column is the right eigenvector $\vartheta_{\mathbf{r}_i}$ of A_γ .

Without loss of generality, in what follows we assume that the eigenvalues of A_γ are ordered in decreasing order, that is, $\lambda_1(A_\gamma)$ has the largest real part and $\text{Re}(\lambda_1) > \text{Re}(\lambda_2) \geq \dots \geq \text{Re}(\lambda_N)$.

6.3.2 Decomposition of the matrix A_γ

We next decompose matrix A_γ in the following way. First we split matrix Λ_γ in two parts as follows

$$\Lambda = \lambda_1(A_\gamma)\mathbb{I} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2(A_\gamma) - \lambda_1(A_\gamma) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_N(A_\gamma) - \lambda_1(A_\gamma) \end{pmatrix} = \Lambda_1 + \Lambda_2. \quad (6.16)$$

Notice that if $\gamma > \gamma^*$, where γ^* satisfies assumption 3 then $(N-1)$ non-zero eigenvalues of the matrix Λ_2 have negative real parts and moreover we have that for $i = 2, \dots, N$ eigenvalues $\lambda_i(\Lambda_2) \rightarrow -\infty$ as $\gamma \rightarrow +\infty$.

Using these notations we can write matrix A_γ as

$$A = V\Lambda_1V^\top + V\Lambda_2V^\top = \lambda_1\mathbb{I} + D, \quad (6.17)$$

where $D = V\Lambda_2V^\top$. Notice that due to the properties of the matrix Λ_2 we have that $D \leq 0$ and moreover $N-1$ eigenvalues of this matrix have negative real parts and as for the matrix Λ_2 we have that for $i = 2, \dots, N$

$$\lambda_i(\Lambda_2) \rightarrow -\infty \quad \text{as} \quad \gamma \rightarrow +\infty.$$

Using these notations we can write the system (6.11) in the form

$$\dot{\mathbf{z}} = (\lambda_1\mathbb{I} - C(\mathbf{z})) \mathbf{z} + D \mathbf{z}. \quad (6.18)$$

Since by assumption, the matrix A_γ is diagonalizable then from [50, Theorem 4.4.13] (see also [27]), it follows that matrix V_γ is complex orthogonal, *i.e.*

$$V_\gamma^\top V_\gamma = \mathbb{I}_N. \quad (6.19)$$

Let $\vartheta_{\mathbf{r}}$ and $\vartheta_{\mathbf{l}}$ be the first columns of the matrices V_γ and V_γ^\top , *i.e.* the right and left eigenvectors of the matrices A_γ and D corresponding to the largest eigenvalues $\lambda_1(A_\gamma)$ and $\lambda_1(D) = 0$. Then we have

$$\begin{aligned} A_\gamma \vartheta_{\mathbf{r}} &= \lambda_1(A_\gamma) \vartheta_{\mathbf{r}}, \\ \vartheta_{\mathbf{l}}^* A_\gamma &= \lambda_1(A_\gamma) \vartheta_{\mathbf{l}}^* \end{aligned}$$

and

$$D\vartheta_r = 0, \quad \vartheta_1^* D = 0.$$

Moreover, due to the orthogonality of V_γ in the sense of equation (6.19) we have

$$[\vartheta_r]^2 = \vartheta_r^\top \vartheta_r = [\vartheta_1]^2 = \vartheta_1^\top \vartheta_1 = \mathbf{1}. \quad (6.20)$$

Since ϑ_r and ϑ_l are eigenvectors corresponding to the same eigenvalue we have that $\vartheta_l^* \vartheta_r = 1$. Combining (6.20) and the last equality we obtain that

$$\vartheta_l = (\vartheta_r^*)^\top.$$

Notice that equation (6.18) is just another form of writing the equations of the oscillators network (6.4). Thus the transformations that we presented in this section allow us to interpret the original network of heterogeneous oscillators as a network where all the nodes have identical dynamics and the interconnection matrix D has properties similar to a Laplacian matrix since its largest eigenvalue is equal to zero.

6.3.3 Coordinate transformation depending on the matrix A_γ

To clarify our approach to the synchronization of the oscillators network (6.4), or equivalently (6.18), we next rewrite the system (6.18) in a new coordinate frame which is convenient for our purposes.

We recall that the matrix D can be presented in the form

$$D = V_\gamma \Lambda_2 V_\gamma^\top = V_\gamma \begin{bmatrix} 0 & & & \\ & \lambda_2(D) & & 0 \\ & & \ddots & \\ 0 & & & \lambda_N(D) \end{bmatrix} V_\gamma^\top, \quad (6.21)$$

where λ_i ($i = 2, \dots, N$) are the eigenvalues of the matrix D and i -th column of the matrix V_γ is an eigenvector of D corresponding to the i -th eigenvalue $\lambda_i(D)$.

Denoting the first right eigenvector corresponding to $\lambda_1(D) = 0$ by $\vartheta_1(D)$ ¹, we can decompose the matrix V_γ in the following way

$$V_\gamma = [\vartheta_1(D), V_{\gamma_1}], \quad (6.22)$$

1. When it is clear from the context, here and afterwards we will use notation ϑ for the right eigenvector of a matrix instead of ϑ_r .

where $V_{\gamma_1} \in \mathbb{C}^{(N-1) \times N}$ is a matrix composed of $N - 1$ eigenvectors of D corresponding to $\lambda_2, \dots, \lambda_N$. From (6.19) we have that

$$\vartheta_1^\top(D)V_{\gamma_1} = 0, \quad V_{\gamma_1}^\top V_{\gamma_1} = I_{N-1}.$$

Based on the orthogonal matrix V_γ we introduce the following change of coordinates. Let

$$\bar{\mathbf{z}} = V_\gamma^\top \mathbf{z}. \quad (6.23)$$

Using (6.22) we can decompose new coordinates in two parts as follows :

$$\bar{\mathbf{z}} = \begin{bmatrix} \vartheta_1^\top \\ V_{\gamma_1}^\top \end{bmatrix} \mathbf{z} = \begin{bmatrix} \bar{\mathbf{z}}_1 \\ \bar{\mathbf{z}}_2 \end{bmatrix}, \quad (6.24)$$

where $\bar{\mathbf{z}}_1 \in \mathbb{C}$ and $\bar{\mathbf{z}}_2 \in \mathbb{C}^{N-1}$.

From the previous section it is clear that in the limit, as $\gamma \rightarrow \infty$, we have that

$$\vartheta_1^\top(D) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

and therefore, in the limit (as $\gamma \rightarrow \infty$) coordinate $\bar{\mathbf{z}}_1$ converges to

$$\mathbf{z}_a = \frac{1}{N} \sum_{i=1}^N \mathbf{z}_i$$

which, in the literature on nonlinear oscillators, is known as *averaged* or *mean-field* oscillator.

Thus we can see $\bar{\mathbf{z}}_1$ as a type of weighted average of \mathbf{z}_i and that is why we use the term "*average oscillator*" for $\bar{\mathbf{z}}_1$.

We now consider the vector $\bar{\mathbf{z}}_2$ and equality $\bar{\mathbf{z}}_2 = 0$ in particular. From (6.19) we have that

$$V_{\gamma_1} V_{\gamma_1}^\top = I_N - \vartheta_1 \vartheta_1^\top. \quad (6.25)$$

Then, multiplying $\bar{\mathbf{z}}_2$ by V_{γ_1} from the left and using (6.25) we obtain that $\bar{\mathbf{z}}_2$ is equal to zero if and only if

$$\mathbf{z} = \vartheta_1 \vartheta_1^\top \mathbf{z} = \vartheta_1 \bar{\mathbf{z}}_1,$$

or equivalently, if

$$\mathbf{e} = \mathbf{z} - \vartheta_1 \bar{\mathbf{z}}_1 = 0,$$

where $\mathbf{e} \in \mathbb{C}^N$.

Therefore we can see both $\bar{\mathbf{z}}_2$ and \mathbf{e} as a measure of synchrony between the oscillators in the network and the "averaged" oscillator.

Combining representations of $\bar{\mathbf{z}}_1$ and $\bar{\mathbf{z}}_2$ we see that this coordinate transformation allows to explicitly decompose the network behaviour in two parts with very different roles : coordinate $\bar{\mathbf{z}}_1$ corresponds to the "averaged" oscillator, while $\bar{\mathbf{z}}_2$ reflects the synchronization error between the oscillators in the network.

In order to simplify the notations, in the rest of this chapter for the stability analysis of the system (6.18) we will use the following notations

$$\mathbf{z}_m = \bar{\mathbf{z}}_1 = \vartheta_1^\top \mathbf{z} \quad (6.26)$$

and

$$\mathbf{e} = \mathbf{z} - \vartheta_1 \bar{\mathbf{z}}_1 = \mathbf{z} - \vartheta_1 \mathbf{z}_m \quad (6.27)$$

or equivalently

$$\mathbf{e} = \mathbf{z} - \vartheta_1 \bar{\mathbf{z}}_1 = \mathbf{z} - \vartheta_1 \vartheta_1^\top \mathbf{z} = \left(\mathbb{I} - \vartheta_1 \vartheta_1^\top \right) \mathbf{z} = \mathbf{P} \mathbf{z}, \quad (6.28)$$

where $\mathbf{P} = \mathbb{I} - \vartheta_1 \vartheta_1^\top$ is a projection matrix.

Notice that the vector \mathbf{e} can be considered as a vector of errors between the oscillator z_j and the scaled and rotated averaged oscillator z_m . In general, vector $\vartheta_{\mathbf{r}_1}$ does not necessarily have only rotational components since some of $\vartheta_{\mathbf{r}_1 i}$ can have an amplitude not equal to one. However, in the limit as $\gamma \rightarrow \infty$, we have that $\vartheta_{\mathbf{r}_1} \rightarrow \mathbb{1}$ and it is possible to show that for sufficiently large value of γ , components of vector $\vartheta_{\mathbf{r}_1}$ converge to $e^{i\varphi_j}$ for some $\varphi_j \in \mathbb{R}$. Then for sufficiently large values of γ , the right eigenvector $\vartheta_{\mathbf{r}_1} \in \mathbb{C}$ can be considered as a vector of rotations which correspond to the phase difference between the oscillators and the average oscillator.

6.4 Problem formulation

In the particular case when oscillators are completely decoupled (i.e. $\gamma = 0$), all oscillators in the network rotate at their individual (natural) frequencies. Actually, it was shown in [39] that this individual behavior persists in the case of weak coupling (i.e. for small values of γ). The effect of network synchronization, which appears in the case of strong coupling, is well documented in the literature.

- *Frequency synchronization*; for sufficiently large values of γ all the units tend asymptotically to oscillate at the same frequency, see e.g. [80].

- *Phase locking* : in addition to frequency synchronization we have that the phase differences between the oscillators tend to be constant and independent of initial conditions.

In the case of a network of identical oscillators ($\mu_i = \mu$ for all $i \in I$), all of the latter tend to oscillate at the same frequency and with zero phase differences. This effect, which is called *complete* network synchronization, can be analyzed using stability theory on the synchronization manifold

$$\mathcal{S}_0 = \{ \mathbf{z} \in \mathbb{C}^N : z_1 = z_2 = \dots = z_N \}.$$

Indeed, stability of this manifold can be deduced in a straightforward manner from the synchronization results for semi-passive, incrementally passive or incremental input-output systems, as it is done *e.g.*, in [101].

In the previous section we saw that in the case of general heterogeneous network with arbitrary N and large values of the parameter γ the network behavior can be decomposed in two parts :

- one of them describes dynamic behavior of an "averaged" oscillator \mathbf{z}_m
- another part describes dynamics of the network oscillators relative to the dynamics of the averaged one.

That is why we characterize synchronization properties of the network in terms of stability of two compact sets :

- the synchronization manifold $\mathcal{S}(\gamma)$ which we define in the following way

$$\mathcal{S}(\gamma) = \{ \mathbf{e} \in \mathbb{C}^N : \mathbf{e}_1 = \mathbf{e}_2 = \dots = \mathbf{e}_N = 0 \}. \quad (6.29)$$

- a compact invariant set \mathcal{W} corresponding to the averaged oscillator. It is composed of the origin and the limit cycle of z_m .

To analyse stability properties of the set $\mathcal{S}(\gamma)$ we introduce the following definition of practical stability of a set, which is similar to that of practical stability of an equilibrium point –see [129].

Consider a parametrized system of differential equations

$$\dot{x} = f(x, \varepsilon), \quad (6.30)$$

where $x \in \mathbb{R}^n$ and the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz and $\varepsilon > 0$ is a scalar parameter. We assume that $\varepsilon \in (0, \varepsilon_0]$. For such a family we introduce the notion of global practical uniform asymptotic stability with respect to closed, not necessarily compact, sets. Given a closed set $\mathcal{A} \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$.

Definition 15. For the system (6.30), the closed set \mathcal{A} is practically globally uniformly asymptotically stable (practical UGAS) if the system (6.30) is forward complete and

(1) for any given $r > 0$ there exists $R > 0$ and $\epsilon^* \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_*]$

$$|x_0| \leq r \quad \implies \quad |x(t, x_0, \epsilon)| \leq R, \quad \forall t \geq 0$$

(2) for any given (r, δ) with $0, \delta < r$, there exists $\epsilon^* \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_*]$

$$|x_0|_{\mathcal{A}} \leq \delta \quad \implies \quad |x(t, x_0, \epsilon)|_{\mathcal{A}} \leq r, \quad \forall t \geq 0$$

(3) for any given (r, δ) with $0, \delta < r$, there exists $T = T(r, \delta)$ and $\epsilon^* \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_*]$ and all $x_0 \in \mathbb{R}^n$ such that $|x_0|_{\mathcal{A}} \leq r$, we have

$$|x(t, x_0, \epsilon)|_{\mathcal{A}} \leq \delta \quad \forall t \geq T.$$

If the last inequality holds for all $x_0 \in \mathbb{R}^n$ then the set \mathcal{A} is practically globally uniformly asymptotically stable .

Remark 5. Similarly to the definition of UGAS of a set, the definition of practical UGAS include three properties : uniform boundedness of the solutions with respect to the set (part 1), uniform stability of the set (part 2) and and uniform practical convergence to the set for all initial conditions $x_0 \in \mathbb{R}^n$ (part 3).

Notice that in new coordinates synchronization problem that we formulated in the section 6.2 can be recast as a (practical) stability problem for the sets $\mathcal{S}(\gamma)$ and \mathcal{W} .

6.5 Network dynamics in new coordinates

In this section we reformulate dynamics of system (6.18) in terms of coordinates \mathbf{z}_m and \mathbf{e} , later in this chapter it will be used for stability analysis of the network.

6.5.1 Dynamics of the averaged oscillator

From the definition of the averaged unit $\mathbf{z}_m = \mathfrak{D}_1^\top \mathbf{z}$ and network dynamics (6.18) we obtain that

The derivative of z_m is given as follows

$$\begin{aligned}\dot{z}_m &= \vartheta_{\mathbf{1}}^* \dot{\mathbf{z}} = \vartheta_{\mathbf{1}}^* (\mathbf{A}\mathbf{z} - \mathbf{C}(\mathbf{z})\mathbf{z}) \\ &= \lambda_1 \vartheta_{\mathbf{r}\mathbf{1}}^\top \mathbf{z} - \vartheta_{\mathbf{r}\mathbf{1}}^\top \mathbf{C}(\mathbf{z})\mathbf{z} \\ &= \lambda_1 z_m - \vartheta_{\mathbf{r}\mathbf{1}}^\top \mathbf{C}(\mathbf{z})\mathbf{z}.\end{aligned}\tag{6.31}$$

Some simple but tedious calculations presented in Appendix A, show that dynamics of \mathbf{z}_m can be rewritten in terms of coordinates \mathbf{e} and \mathbf{z}_m as follows

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + f_2(z_m, \mathbf{e}),\tag{6.32}$$

where $c \in \mathbb{C}$ is a constant defined as

$$c = \sum_{i=1}^N \vartheta_{li}^* \vartheta_{ri}^* \vartheta_{ri}^2\tag{6.33}$$

and the function $f_2(z_m, \mathbf{e})$ is given by the following expression

$$\begin{aligned}f_2(z_m, \mathbf{e}) &= \\ & - \vartheta_{\mathbf{1}\mathbf{1}}^* \begin{pmatrix} |z_1|^2 + z_1^* v_{r1} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |z_N|^2 + z_N^* v_{rN} z_m \end{pmatrix} \mathbf{e} \\ & - \vartheta_{l_1}^* \begin{pmatrix} (v_{r1} z_m)^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{rN} z_m)^2 \end{pmatrix} \bar{\mathbf{e}},\end{aligned}\tag{6.34}$$

where $v_{r_{ij}}$ corresponds to the ij -th element of the matrix V_γ .

It is important to note here that dynamics of thus defined average unit is linearly independent of the gain of interconnections γ in the sense that parameter γ does not appear explicitly in the equation (6.31) even though eigenvalue $\lambda_1(D)$ and corresponding eigenvector $\vartheta_{r\mathbf{1}}$ do depend on the gain γ implicitly. However, it is easy to see from (6.13) that dependence of λ_1 on γ is inverse proportional, roughly speaking, $\lambda_1(D) = c + O(\frac{1}{\gamma})$, where constant c depends only on the matrix \mathcal{M} . The same type of relationship is also valid for the eigenvector $\vartheta_{r\mathbf{1}}$.

6.5.2 Dynamics of synchronization errors

From the definition of the synchronization error (see section 6.4)

$$\mathbf{e} = \mathbf{z} - \vartheta_1 \bar{\mathbf{z}}_m = (\mathbb{I} - \vartheta_1 \vartheta_1^\top) \mathbf{z} = P \mathbf{z}$$

and network dynamics (6.18) we obtain that the error dynamics of \mathbf{e} as follows

$$\begin{aligned} \dot{\mathbf{e}} &= P \dot{\mathbf{z}} = (\mathbb{I} - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{r}_1}^*) \dot{\mathbf{z}}, \\ &= P D \mathbf{z} + (\lambda_1 \mathbb{I} - P C(\mathbf{z})) \mathbf{z}, \end{aligned} \tag{6.35}$$

where we recall that matrix D is a complex analog of a Laplacian matrix it was defined in (6.17) as $D = V \Lambda_2 V^\top$, $\Lambda_2 = \text{diag}(0, \lambda_2 - \lambda_1, \dots, \lambda_N - \lambda_1)$.

Next we need the following properties of the left and right eigenvectors of D associated with the eigenvalue $\lambda_1(D) = 0$

$$D P = D \quad P D = D \tag{6.36}$$

which can be easily deduced recalling that matrix P is a projection matrix.

Indeed,

$$D P = D (\mathbb{I} - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{r}_1}^*) = D - \lambda_{D_1} D \vartheta_{\mathbf{r}_1}^* = D$$

$$P D = (\mathbb{I} - \vartheta_{\mathbf{r}_1} \vartheta_{\mathbf{r}_1}^*) D = D - \lambda_{D_1} \vartheta_{\mathbf{r}_1} D = D.$$

Then using the properties (6.36) equation (6.35) can be rewritten as

$$\begin{aligned} \dot{\mathbf{e}} &= P D \mathbf{e} + (\lambda_1 \mathbf{e} - P C(\mathbf{z}) \mathbf{z}) \\ &= D \mathbf{e} + (\lambda_1 \mathbf{e} - P C(\mathbf{e}, z_m)(\mathbf{e} + z_m \mathbf{1})). \end{aligned} \tag{6.37}$$

Thus, equations (6.32) and (6.37) represent dynamics of the network in new coordinates, namely in terms of the averaged oscillator \mathbf{z}_m and synchronization errors \mathbf{e} between each of the individual oscillators and \mathbf{z}_m .

The next section will be devoted to the stability analysis of the network and we will actively use it's representation in new coordinates \mathbf{e} and \mathbf{z}_m . For further convenience, we

regroup here equations describing dynamics of the network in new coordinates

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + f_2(z_m, \mathbf{e}), \quad (6.38)$$

$$\dot{\mathbf{e}} = D \mathbf{e} + (\lambda_1 \mathbf{e} - P C(\mathbf{e}, z_m)(\mathbf{e} + z_m \mathbf{1})). \quad (6.39)$$

In the following section two types of properties of the networked system (6.38) and (6.39) will be investigated. The first aims to show how the coupling gain affects synchronization of the network, while the second how network synchronization can contribute to appearance of common, emergent behavior. In the first case we'll be interested in the analysis of the subsystem (6.39), while in the second case, we'll analyse the subsystem (6.38).

6.6 Practical synchronization of a diffusively coupled network of Stuart-Landau oscillators

In this section we analyze the system (6.39) for the large values of the coupling gain γ , that is we suppose that assumption 3 is satisfied. Here we present stability analysis of the set $\mathcal{S}(\gamma)$ and formulate conditions that ensure practical global asymptotic stability of this set, which imply practical synchronization of the network.

The following result, which can be deduced from [26], will be one of the key tools in our subsequent analysis.

Theorem 15. *Consider a system $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and $f(\cdot)$ is a continuous, locally Lipschitz function. Assume that the system is forward complete, there exists a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ and a constant $c > 0$ such that the following inequalities are satisfied*

$$\alpha_1(|\mathbf{x}|_{\mathcal{A}}) \leq V(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|_{\mathcal{A}})$$

$$\dot{V} \leq -\alpha_3(|\mathbf{x}|_{\mathcal{A}}) + c.$$

Then for any $R, \varepsilon > 0$ there exists a $T = T(R, \varepsilon)$ such that for all $t \geq T$ and all \mathbf{x}_0 such that $|\mathbf{x}_0|_{\mathcal{A}} \leq R$

$$|\mathbf{x}(t, \mathbf{x}_0)|_{\mathcal{A}} \leq r + \varepsilon,$$

where $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(c)$.

Notice that even though the set $\mathcal{S}(\gamma)$ is not invariant for the oscillators network, we show next that it appears to be practically stable for the system (6.38) and (6.39). That is,

for the large values of γ the norm of error $\mathbf{e}(\mathbf{t})$ will be small and inversely proportional to γ .

Let \mathbf{z}_0 be initial conditions of the system with the property $|\mathbf{z}_0| \leq R$, where constant $R > 0$ is arbitrary.

We introduce next the following change of coordinates

$$\mathbf{e}_v = V_\gamma^\top \mathbf{e}, \quad (6.40)$$

where we recall that matrix V_γ is complex orthogonal and satisfies (6.19).

From the definition of the synchronization error \mathbf{e} we obtain that the vector \mathbf{e}_v has the following form

$$\mathbf{e}_v = V_\gamma^\top \mathbf{e} = V_\gamma^\top (\mathbf{z} - \vartheta_{\mathbf{r}_1} z_m) = V_\gamma^\top \mathbf{z} - V_\gamma^\top \vartheta_{\mathbf{r}_1} z_m = \begin{pmatrix} \vartheta_{\mathbf{r}_1}^\top \mathbf{z} \\ \vdots \\ \vdots \\ \vartheta_{\mathbf{r}_N}^\top \mathbf{z} \end{pmatrix} - \begin{pmatrix} z_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\mathbf{e}}_v \end{pmatrix}$$

where $\tilde{\mathbf{e}}_v = (\vartheta_{\mathbf{l}_2}^* \mathbf{z}, \dots, \vartheta_{\mathbf{l}_N}^* \mathbf{z})^\top \in \mathbb{C}^{N-1}$ and we used the property

$$V_\gamma^\top \vartheta_{\mathbf{r}_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Actually, vector $\tilde{\mathbf{e}}_v$ coincides with the vector $\bar{\mathbf{z}}_2$ introduced in section 6.3.3.

Using equation (6.37) for the dynamics of \mathbf{e} we obtain that derivative of \mathbf{e}_v is given as follows

$$\dot{\mathbf{e}}_v = \Lambda_2 \mathbf{e}_v + \lambda_1 \mathbf{e}_v - V_\gamma^\top \mathbf{P} C(\mathbf{z}) \mathbf{z}, \quad (6.41)$$

where matrix Λ_2 was defined in (6.16) as

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2(A_\gamma) - \lambda_1(A_\gamma) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_N(A_\gamma) - \lambda_1(A_\gamma) \end{pmatrix}$$

To prove practical stability of the set $\mathcal{S}(\gamma)$, we define the following Lyapunov function in terms of synchronization errors

$$V(\mathbf{e}_v) = \frac{1}{2} \mathbf{e}_v^* \mathbf{e}_v = \frac{1}{2} \tilde{\mathbf{e}}_v^* \tilde{\mathbf{e}}_v \quad (6.42)$$

Notice that thus defined Lyapunov function is positive definite in terms of $\tilde{\mathbf{e}}_v$ since $V(\mathbf{e}_v) = \frac{1}{2} |\tilde{\mathbf{e}}_v|^2$. Calculating derivative of the function $V(\mathbf{e}_v)$ along trajectories of the system (6.41) we obtain

$$\begin{aligned} \dot{V} &= \frac{1}{2} \mathbf{e}_v^* \left(\Lambda_2 \mathbf{e}_v + \lambda_1 \mathbf{e}_v - V_\gamma^\top P C(\mathbf{z}) \mathbf{z} \right) + \frac{1}{2} \left(\mathbf{e}_v^* \Lambda_2^* + \mathbf{e}_v^* \lambda_1^* - \mathbf{z}^* C^*(\mathbf{z}) P^* (V_\gamma^\top)^* \right) \mathbf{e}_v \\ &= \frac{1}{2} \mathbf{e}_v^* (\Lambda_2 + \Lambda_2^*) \mathbf{e}_v + \frac{1}{2} \mathbf{e}_v^* (\lambda_1 + \lambda_1^*) \mathbf{e}_v - \frac{1}{2} \left(\mathbf{e}_v^* V_\gamma^\top P C(\mathbf{z}) \mathbf{z} + (V_\gamma^\top P C(\mathbf{z}) \mathbf{z})^* \mathbf{e}_v \right) \\ &= \mathbf{e}_v^* \text{Real}(\Lambda_2) \mathbf{e}_v + \text{Real}(\lambda_1) |\mathbf{e}_v|^2 - g(\mathbf{e}_v, \mathbf{z}), \end{aligned} \quad (6.43)$$

where $g(\mathbf{e}_v, \mathbf{z}) = \mathbf{e}_v^* V_\gamma^\top P C(\mathbf{z}) \mathbf{z} + (V_\gamma^\top P C(\mathbf{z}) \mathbf{z})^* \mathbf{e}_v$.

In the section 6.2 we proved that solutions of the system (6.38) are ultimately bounded and for any initial conditions $|\mathbf{z}_o| \leq R$ there exists a time instant $T(R) > 0$ such that the following bound holds for all $t \geq T$.

$$|\mathbf{z}(t, \mathbf{z}_o)| \leq \sqrt{2\mu N}.$$

Ultimate boundedness of trajectories $z(t)$ implies ultimate boundedness of $\mathbf{z}_m(t)$ and synchronization errors $\mathbf{e}(t)$, $\mathbf{e}_v(t)$.

Using ultimate boundedness property we can also bound the term $g(\mathbf{e}_v, \mathbf{z})$ as follows

$$|g(\mathbf{e}_v, \mathbf{z})| \leq c_1 |\mathbf{e}_v|^2 + c |\mathbf{z}|^4 \leq c_1 |\mathbf{e}_v|^2 + c_2,$$

where the constants $c_1, c_2 \in \mathbb{R}^+$ depend on μ and are independent of γ .

Using thus obtained bounds we have that for all $t \geq T$

$$\dot{V} \leq \text{Real}(\lambda_2 - \lambda_1) |\mathbf{e}_v|^2 + \text{Real}(\lambda_1) |\mathbf{e}_v|^2 + |g(\mathbf{e}_v, \mathbf{z})|$$

Since \mathbf{e}_v is ultimately bounded we have that there exists constant $c = c(\mu)$ such that for all $t \geq T$

$$\text{Real}(\lambda_1) |\mathbf{e}_v|^2 + c_1 |\mathbf{e}_v|^2 + |g(\mathbf{e}_v, \mathbf{z})| \leq c$$

and therefore

$$\begin{aligned}\dot{V} &\leq \text{Real}(\lambda_2)|\mathbf{e}_v|^2 + \lambda_1|\mathbf{e}_v|^2 + c_1|\mathbf{e}_v|^2 + c_2 = (\text{Real}(\lambda_2 - \lambda_1)|\mathbf{e}_v|^2 + \mathbf{c}) \\ &\leq \frac{1}{2}(\text{Real}(\lambda_2 - \lambda_1)|\mathbf{e}_v|^2 + \frac{1}{2}(\text{Real}(\lambda_2 - \lambda_1) \left(|\mathbf{e}_v|^2 + \frac{2}{\text{Real}(\lambda_2 - \lambda_1)} \mathbf{c} \right))\end{aligned}$$

Since by construction we have that for all $\gamma \geq \gamma^*$ matrix Λ_2 is non-positive, that is $\text{Re}(\lambda_2 - \lambda_1) < 0$, then from theorem 15 it follows that there exists T^* such that

$$|\mathbf{e}_v(\mathbf{t})|^2 \leq \frac{2\mathbf{c}}{|\text{Real}(\lambda_2 - \lambda_1)|} \quad \text{for all } \mathbf{t} \geq \mathbf{T}^*.$$

From (6.40) and complex orthogonality of the matrix V_γ we have that $\mathbf{e} = V_\gamma \mathbf{e}_v$ and therefore for all $t \geq T^*$ we have that

$$|\mathbf{e}(t)| = |V_\gamma| |\mathbf{e}_v(\mathbf{t})| \leq \frac{2\mathbf{c}}{|\text{Real}(\lambda_2 - \lambda_1)|}.$$

Finally, we note that $\lim_{\gamma \rightarrow \infty} \text{Re}(\lambda_2) = -\infty$ and therefore we obtain practical stability of the set $\mathcal{S}(\gamma)$.

Theorem 16. *Consider the system (6.4), (6.5) and let assumption 3 be satisfied. Then the set $\mathcal{S}(\gamma)$ is globally practically stable for all $\gamma \geq \gamma^*$. Moreover, there exists $T^* > 0$ and constant $c > 0$ independent of γ such that synchronization errors $\mathbf{e}(\mathbf{t})$ satisfy the following bound for all $t \geq T^*$*

$$|\mathbf{e}(\mathbf{t})|^2 \leq \frac{2\mathbf{c}}{|\text{Real}(\lambda_2 - \lambda_1)|}. \quad (6.44)$$

From the proof of the theorem we see that the synchronization result is mostly based on only two properties of the networked system, namely, negative definiteness of the second smallest eigenvalue of the Laplacian matrix L and uniform boundedness of the trajectories of the network.

Theorem 16 implies that for given arbitrary large ball of initial conditions $B_r = \{\mathbf{z} \in \mathbb{C}^N : |\mathbf{z}| \leq R\}$ and arbitrary small constant $\delta > 0$, we can always find constants $\gamma(R, \delta)$ and $t^*(R, \delta)$ such that for the network of the Stuart-Landau oscillators with the coupling gain γ synchronization errors $\mathbf{e}(\mathbf{t}, \mathbf{z}(\mathbf{0}))$ satisfy the following bound

$$|\mathbf{e}(t, \mathbf{z}_o)| \leq \delta \quad \text{for all } t \geq t^*.$$

6.7 On practical stability of the invariant set for the averaged oscillator

Next, we consider behavior of z_m . To that end we first note that, from (6.32), (6.34), the dynamics of z_m may be seen as a perturbation of an Stuart-Landau oscillator with an input that linearly depends on $|\mathbf{e}|$ that is,

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + u, \quad (6.45)$$

where $u = f_2(z_m, \mathbf{e})$.

We remark here that this term is locally Lipschitz in terms of synchronization errors \mathbf{e} , while $\mathbf{z}_m(t)$ is ultimately bounded. Therefore we can consider equation (6.26) as a perturbation of a Stuart-Landau oscillator by bounded (and inverse proportional to γ) disturbances. That is why in the following next sections we will first analyse stability and robustness properties of a generalized Stuart-Landau oscillator and in the end will apply these results to the analysis of the system (6.45).

6.7.1 Dynamics of a generalized Stuart-Landau oscillator

Later in the chapter we would need to analyze behavior and robustness of a generalized Stuart-Landau oscillator, that is an oscillator whose dynamics is given by the following equation

$$\dot{z} = -\nu|z|^2 z + \mu z + u \quad (6.46)$$

where $z, u \in \mathbb{C}$ are respectively the state and input of the oscillator, $\nu = \nu_1 + i\nu_2 \in \mathbb{C}$ and $\mu = \mu_1 + i\mu_2 \in \mathbb{C}$ are complex parameters and we assume that $\nu_1 > 0$, since otherwise either solutions of the system will expose in finite time for $\nu_1 > 0$ or the oscillator will be just a linear system for $\nu_1 > 0$.

As it was mentioned in the chapter 2, analysis of such oscillators (with $u \equiv 0$) is well documented in the literature using both Lyapunov second method and Lyapunov exponents for the oscillators for the case $\nu_1 = 1$, however we were unable to find neither analysis of the system (6.46) using second Lyapunov method with $\nu_1 \neq 1$ nor analysis of (6.46) in the presence of disturbances, i.e. case $u \neq 0$. That is why we start this chapter with the analysis of a generalized Stuart-Landau oscillator (6.46).

Invariant set for the system (6.46)

Let us consider first the case when no perturbations affect the oscillator, i.e. when $u \equiv 0$. As it was mentioned in the chapter 2, invariant set $\mathcal{W} \in \mathbb{C}$ of the unperturbed oscillator (6.46) depends on the sign of μ_1 . Namely, if $\mu_1 \leq 0$, then the invariant set \mathcal{W} is just the origin, i.e. $\mathcal{W} = \{z = 0\}$, in case $\mu_1 > 0$, this invariant set is composed of two elements : origin and limit cycle of amplitude $\sqrt{\mu_1/\nu_1}$, that is

$$\mathcal{W} = \{z = 0\} \cup \left\{ z \in \mathbb{C} : z = 0 \text{ or } |z| = \sqrt{\frac{\mu_1}{\nu_1}} \right\}. \quad (6.47)$$

Stability analysis of the invariant set \mathcal{W}

Similar to analysis done in [101] for the case of $\nu_1=1$ and $\nu_2 = 0$, we can prove the following result for the general case of Stuart-Landau oscillator (6.46).

Theorem 17. *Consider a Stuart-Landau oscillator (6.46) with input $u \equiv 0$. The following statements hold for this system*

(1) *if $\text{Re}(\nu) \leq 0$ then the origin $z \equiv 0$ is globally asymptotically stable for the system (6.46).*

(2) *If $\text{Re}(\nu) > 0$ then the limit cycle $\mathcal{W}_1 = \{z : |z| = \sqrt{\nu_1/\mu_1}\}$ is almost globally asymptotically stable and the origin $z = 0$ is antistable for the system (6.46).*

Moreover, in this case frequency ω of oscillations on the limit cycle is defined by the following equation

$$\omega = \mu_2 - \frac{\nu_2}{\nu_1} \mu_1$$

Proof.

Part (i).

Proof of GAS of the origin $z = 0$ follows easily using Lyapunov function $V(z) = \frac{1}{2} z^* z = |z|^2$. Indeed, taking the derivative of $V(z)$ along trajectories of (6.46) we obtain

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2} \frac{d}{dt} z^* z = \frac{1}{2} (z^* \dot{z} + \dot{z}^* z) \\ &= ((-\bar{\nu}|z|^2 z^* + \bar{\mu} z^*)z + z^*(-\nu|z|^2 z + \mu z)) \\ &= -(\nu + \bar{\nu})|z|^4 + (\mu + \bar{\mu})|z|^2 \\ &= -\nu_1|z|^4 + \mu_1|z|^2. \end{aligned}$$

Since in this case we have that $\mu_1 \leq 0$, then trivially $\dot{V}(z) < 0$ for all $z \neq 0$ and GAS of the origin follows.

Part (ii).

First we note that antistability of the origin follows trivially using the same Lyapunov function $V(z) = \frac{1}{2}z^*z = |z|^2$ and linearization of the system (6.46) around the origin.

Next, to analyze stability properties of the limit cycle for the system (6.46) with $\mu_1 > 0$ we introduce the following Lyapunov function

$$V(z) = \frac{1}{4\nu_1} (z^*z - \alpha)^2, \quad (6.48)$$

where $\alpha = \mu_1/\nu_1 = \text{Re}(\mu)/\text{Re}(\nu)$.

Notice that $V(z) = 0$ for all z belonging to the limit cycle.

Taking derivative of this Lyapunov function along solutions of the system (6.46) we obtain

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2\nu_1} (z^*z - \alpha) \frac{d}{dt} z^*z \\ &= \frac{1}{2\nu_1} (z^*z - \alpha) (\dot{z}^*z + z^*\dot{z}) \\ &= \frac{1}{2\nu_1} (z^*z - \alpha) ((-\bar{\nu}|z|^2 z^* + \bar{\mu}z^*)z + z^*(-\nu|z|^2 z + \mu z)). \end{aligned}$$

Regrouping the terms in the last bracket we obtain that

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2\nu_1} (z^*z - \alpha) ((-\nu + \bar{\nu})|z|^4 + (\mu + \bar{\mu})|z|^2) \\ &= \frac{1}{\nu_1} (|z|^2 - \alpha) (-\nu_1|z|^4 + \mu_1|z|^2) = \frac{1}{\nu_1} (|z|^2 - \alpha) (-\nu_1|z|^2 + \mu_1)|z|^2 \\ &= -(|z|^2 - \alpha)^2 |z|^2, \end{aligned}$$

where in the last lines we used the definition of the constant α .

It is easy to see that derivative of this Lyapunov function is negative definite for all z except those belonging to the set \mathcal{W} , since the origin is antistable, *almost GAS* of the limit cycle follows. ■

Analysis of the invariant set \mathcal{W} in the presence of disturbances

In case of $\mu_1 > 0$ we just proved that Landau-Stuart oscillator with no external disturbances has a limit cycle which is *almost* globally asymptotically stable. Next we analyse

robustness of this system with respect to disturbances. Usually Input-to-state stability is an adequate tool for robustness analysis of nonlinear systems.

However, in our case in this classical input to state stability theory is inapplicable because for a single Stuart-Landau oscillator (see equation (6.1)), stability is ensured only for *almost all* initial conditions. Thus standard ISS tools can't be applied to analysis of the system (6.46) as they only apply in the case when attractor of the system is globally asymptotically stable.

Several results in the literature deal with this situation, in particular notion of *almost Input-to-State Stability* was introduced in [7], see also [9].

In order to analyze system (6.46) in the presence of disturbance we use the recently developed tools for Input-to-State Stability analysis of decomposable invariant sets [8].

Main advantage of this approach is that it allows to analyze robustness properties of the complex invariant sets without use of tools involving manifolds and dimensionality arguments and it is applicable when invariant set is compact and allows finite decomposition without cycles (see [8], section II) as it is the case for a single Stuart-Landau oscillator.

We briefly summarize below the definitions and tools from [8] required for the robustness analysis of the system (6.46).

Input-to-State Stability with respect to decomposable invariant sets

Consider a nonlinear system

$$\dot{x}(t) = f(x(t), d(t)), \quad (6.49)$$

where the map

$$f(x, d) : M \times D \rightarrow T_x M$$

is assumed to be of class C^1 , M is an n dimensional C^2 connected and orientable Riemannian manifold without boundary and D is a closed subset of \mathbb{R}^m containing the origin.

Let \mathcal{W} be a compact invariant set containing all α and ω limit sets of the system

$$\dot{x}(t) = f(x(t), 0).$$

The following assumption is imposed on \mathcal{W} :

Assumption 4. [8]. The set \mathcal{W} admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^k \mathcal{W}_i$, for some non-empty disjoint compact sets \mathcal{W}_i , which form a filtration ordering of \mathcal{W} , as detailed in definitions 4 and 5 of [8]².

Definition 16. [8]. We say that system (6.49) has the asymptotic gain (AG) property if there exist $\eta \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\Delta)$ solutions are defined for all $t \geq 0$ and the following holds :

$$\limsup_{t \rightarrow +\infty} |X(t, x; d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty). \quad (6.50)$$

Definition 17. [8]. We say that a C^1 function $V : M \rightarrow \mathbb{R}$ is an ISS-Lyapunov function for (6.49) if there exists \mathcal{K}_∞ functions α_1 , $[\alpha_2]$, α and γ , and a non-negative real $[c]$ such that :

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq [\alpha_2(|x|_{\mathcal{W}}) + c$$

the function V is constant on each \mathcal{W}_i and the following dissipation holds :

$$DV(x)f(x, d) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|).$$

As the authors of [8] remark, α_1 and c are in brackets since their existence follows (without any additional assumptions) by standard continuity arguments.

Another important issue underlined in [8] is that existence of an ISS Lyapunov function from the last definition is possible for a set \mathcal{W} admitting a decomposition without cycles only. Thus existence of such ISS Lyapunov function implies that assumption 4 is satisfied.

The following theorem (Theorem 1 from [8]) will be the main tool for robustness analysis of (6.46).

Theorem 18. [8] Consider a nonlinear system as in (6.49) and let \mathcal{W} be a compact invariant set containing all α and ω and limit sets of as in Assumption 1. Let \mathcal{W} be a compact invariant set containing all and limit sets of $\dot{x}(t) = f(x(t), 0)$ as in Assumption 4. Then the following facts are equivalent :

- The system (6.49) enjoys the AG property;
- The system(6.49) admits an ISS Lyapunov function.

2. We don't present here these definitions because, as we will see below, in case of Stuart-Landau oscillator this assumption is satisfied due to existence of an ISS-Lyapunov function

6.7.1.1 Robustness analysis of Stuart-Landau oscillator

Next we apply results presented above to analysis of the system (6.46). In this case the invariant set \mathcal{W} given by (6.47) is composed of two elements : the (antistable) origin and (almost GAS) limit cycle.

In this case norm $|\cdot|_{\mathcal{W}}$ is defined as follows

$$|z|_{\mathcal{W}} = \begin{cases} |z| & \text{if } |z| \leq \frac{1}{\sqrt{2}}\sqrt{\alpha} \\ \sqrt{|z|^2 - \frac{\mu_1}{\nu_1}} & \text{if } |z| \geq \frac{1}{\sqrt{2}}\sqrt{\alpha} \end{cases} \quad (6.51)$$

where we recall that $\frac{\mu_1}{\nu_1} = \alpha$.

The following result ensures that the system (6.46) enjoys the AG property, that is asymptotically the distance between the oscillator's trajectory and set \mathcal{W} is proportional to the size of perturbations $\|d\|_{\infty}$.

Theorem 19. *Consider the system (6.46) with initial conditions z_0 and let the set \mathcal{W} be defined by (6.47). Then the system (6.46) has the asymptotic gain property, i.e. the following upperbound holds*

$$\limsup_{t \rightarrow +\infty} |z(t, z_0; u)|_{\mathcal{W}} \leq \eta(\|u\|_{\infty}). \quad (6.52)$$

Proof.

For the system (6.46) consider the following candidate for the ISS Lyapunov function $V(z) = \frac{1}{4\nu_1} (z^*z - \alpha)^2$, which we used previously to prove almost GAS property for the system (6.46) – see (6.48).

Taking derivative of $V(z)$ along trajectories of the system (6.46) we obtain

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2\nu_1} (z^*z - \alpha) \frac{d}{dt} z^*z = \frac{1}{2\nu_1} (z^*z - \alpha) (\dot{z}^*z + z^*\dot{z}) \\ &= \frac{1}{2\nu_1} (z^*z - \alpha) ((-\bar{\nu}|z|^2 z^* + \bar{\mu}z^* + u^*)z + z^*(-\nu|z|^2 z + \mu z + u)) \\ &= \frac{1}{2\nu_1} (z^*z - \alpha) ((-\bar{\nu}|z|^2 z^* + \bar{\mu}z^*)z + z^*(-\nu|z|^2 z + \mu z)) + \frac{1}{2\nu_1} (z^*z - \alpha) (u^*z + z^*u). \end{aligned}$$

Using the derivations from the theorem 17 we obtain that

$$\begin{aligned}\dot{V}(z) &= -(|z|^2 - \alpha)^2 |z|^2 + \frac{1}{2\nu_1} (z^* z - \alpha) (u^* z + z^* u) \\ &\leq -(|z|^2 - \alpha)^2 |z|^2 + \frac{1}{\nu_1} (z^* z - \alpha) |z| |u| \\ &\leq -\frac{1}{2} (|z|^2 - \alpha)^2 |z|^2 + \frac{1}{2\nu_1} \|u\|^2,\end{aligned}$$

where we used triangular inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ in the last line.

Next we bound derivative of $V(z)$ in terms of $|z|_{\mathcal{W}}$. To bound the term $(|z|^2 - \alpha)^2 |z|^2$ let us use (6.51) and consider two cases separately.

Case 1. Let $|z| \leq \frac{1}{\sqrt{2}}\sqrt{\alpha}$ or equivalently $|z|^2 \leq \frac{1}{2}\alpha$, then we have that $|z|_{\mathcal{W}} = |z|$ and

$$-(|z|^2 - \alpha)^2 |z|^2 \leq -\left(\frac{1}{2}\alpha - \alpha\right)^2 |z|^2 \leq -\frac{1}{4}\alpha^2 |z|^2 \leq -\frac{1}{4}\alpha^2 |z|_{\mathcal{W}}^2$$

Case 2. Let $|z| \geq \frac{1}{\sqrt{2}}\sqrt{\alpha}$ or equivalently $|z|^2 \geq \frac{1}{2}\alpha$, then we have that $|z|_{\mathcal{W}} = \sqrt{|z|^2 - \alpha}$ and

$$-(|z|^2 - \alpha)^2 |z|^2 \leq -\frac{1}{2}\alpha (|z|^2 - \alpha)^2 = -\frac{1}{2}\alpha |z|_{\mathcal{W}}^2$$

Combining the two cases together and denoting $c_3 = \min\{\frac{1}{4}\alpha^2, \frac{1}{2}\alpha\}$ we obtain that

$$\dot{V}(z) \leq -c_3 |z|_{\mathcal{W}}^2$$

This proves that $V(z)$ is an ISS-Lyapunov function and by Theorem 18 the system enjoys AG property. ■

6.7.2 Practical stability of the invariant set

Finally let us again consider equation (6.45) which describes dynamics of the averaged oscillator

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + u,$$

where $u = f_2(z_m, \mathbf{e})$. Combining previous results on the robustness of the invariant set \mathcal{W} (theorem 19) and practical stability of synchronization errors (theorem 16) we obtain the following result

Theorem 20. *Consider the network of Stuart-Landau oscillators described by equations (6.4), (6.5) and averaged oscillator of the network defined by (6.26), whose dynamics is given by equation (6.45). Let assumption 3 be satisfied. Then the system (6.45) has an*

asymptotic gain property and moreover for any $\varepsilon > 0$ there exists a gain $\gamma \geq \gamma^*$ such that the following bound is satisfied

$$\limsup_{t \rightarrow +\infty} |z_m(t, z_o)|_{\mathcal{W}} \leq \varepsilon.$$

Proof.

Let $\gamma \geq \gamma^*$ and $R > 0$ be arbitrary and consider system (6.4), (6.5) with initial conditions \mathbf{z}_o such that $|\mathbf{z}_o| \leq R$.

From theorem 2 we have that solutions of the system (6.4), (6.5) are ultimately bounded and hence there exists a $T = T(R)$ such that bound (6.8) is satisfied for all $t \geq T$.

Now let us consider dynamics of the averaged oscillator (6.45) given by

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m + f_2(z_m, \mathbf{e}),$$

where function $f_2(z_m, \mathbf{e})$ was defined in (6.34).

From the expression (6.34) we have that function $f_2(z_m, \mathbf{e})$ is locally Lipschitz in \mathbf{e} and moreover due to the ultimate boundedness of solutions we have that there exists a constant $c_3 > 0$ such that for all $t \geq T$ we have that

$$|f_2(z_m(t), \mathbf{e}(t))| \leq c_3 |\mathbf{e}(t)|.$$

Thus applying theorem 19 with $u(t) = f_2(z_m(t), \mathbf{e}(t))$ and $t \geq T$ we have that solutions of equation (6.45) satisfy the asymptotical bound

$$\limsup_{t \rightarrow +\infty} |z_m(t, z_o)|_{\mathcal{W}} \leq \eta(\|\mathbf{e}\|_{\infty}).$$

At the same time, from theorem 16 we have that there exist a $T^* > T$ and a constant $c > 0$ independent of γ such that for all $t \geq T^*$ synchronization errors $\mathbf{e}(t)$ satisfy the following bound

$$|\mathbf{e}(t)|^2 \leq \frac{2c}{|\text{Real}(\lambda_2 - \lambda_1)|}.$$

Combining the last two bounds we obtain that for all $t \geq T^*$

$$\limsup_{t \rightarrow +\infty} |z_m(t, z_o)|_{\mathcal{W}} \leq \eta\left(\frac{2c}{|\text{Real}(\lambda_2 - \lambda_1)|}\right).$$

Repeating the same argument as before that

$$\lim_{\gamma \rightarrow \infty} \text{Real}(\lambda_2 - \lambda_1) = -\infty$$

and using the fact that $\eta \in \mathcal{K}_\infty$ we obtain that for any $\varepsilon > 0$ there exists a $\gamma > \gamma^*$ such that

$$\eta\left(\frac{2c}{|\text{Real}(\lambda_2 - \lambda_1)|}\right) \leq \varepsilon$$

and therefore

$$\limsup_{t \rightarrow +\infty} |z_m(t, z_o)|_{\mathcal{W}} \leq \varepsilon.$$

Thus we obtained that for the system (6.45) the invariant set \mathcal{W} is practically stable in the sense that increasing the interconnection gain γ we can make solutions $z_m(t, z_o)$ converge arbitrary close to the set \mathcal{W} . ■

Combining this result with theorem 16 we conclude that all solutions of the system (6.4), (6.5) practically converge to the set \mathcal{W} .

6.8 Example : network of four Stuart-Landau oscillators

For illustration, we consider a Stuart-Landau model with symmetric connected graph, as illustrated in Figure 6.1. The dynamics of the interconnected oscillators, for $\alpha = 4$, is given by

$$\frac{dz_j}{dt} = (\alpha - |z_j|^2 + i \omega_j) z_j + \gamma \sum_{i=1}^N a_{ji} (z_i - z_j) \quad j = 1 \dots 4, \quad (6.53)$$

where the Laplacian matrix $L = [l_{ij}]$ is

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Firstly, we re-write the system's equations (5.15) in polar coordinates with $z_j = R_j e^{i\theta_j}$, thus we have

$$\dot{R}_j = (\alpha - R_j^2)R_j + \gamma \sum_{i=1}^N a_{ji} (R_i \cos(\theta_i - \theta_j) - R_j), \quad j = 1 \dots 4, \quad (6.54)$$

$$\dot{\theta}_j = \omega_j R_j + \gamma \sum_{i=1}^N a_{ji} R_i \sin(\theta_i - \theta_j), \quad j = 1 \dots 4. \quad (6.55)$$

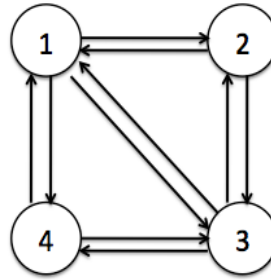


FIGURE 6.1 – Symmetrically connected graph with four nodes

Numerical simulations show that each oscillator rapidly reaches a common limit cycle when the coupling strength is sufficiently large. The phase portrait of the system is depicted in Fig.6.2 for four coupling strengths $\gamma \in \{2, 8, 20, 70\}$.

In Fig. 6.3, we can see clearly the full synchronization of the system (6.53) for a strong coupling strength ($\gamma = 70$) i.e. the limit-cycle oscillators (6.53) synchronize their amplitudes and also their frequencies and phases (phase locked). Fig. 6.3(b) represent the evolution of the radius of oscillators defined in (6.54). Fig. 6.3(c) display the temporal evolution of all individual phase values θ_j . Fig. 6.3(d) present the evolution of angular frequency of the oscillators defined via (6.55).

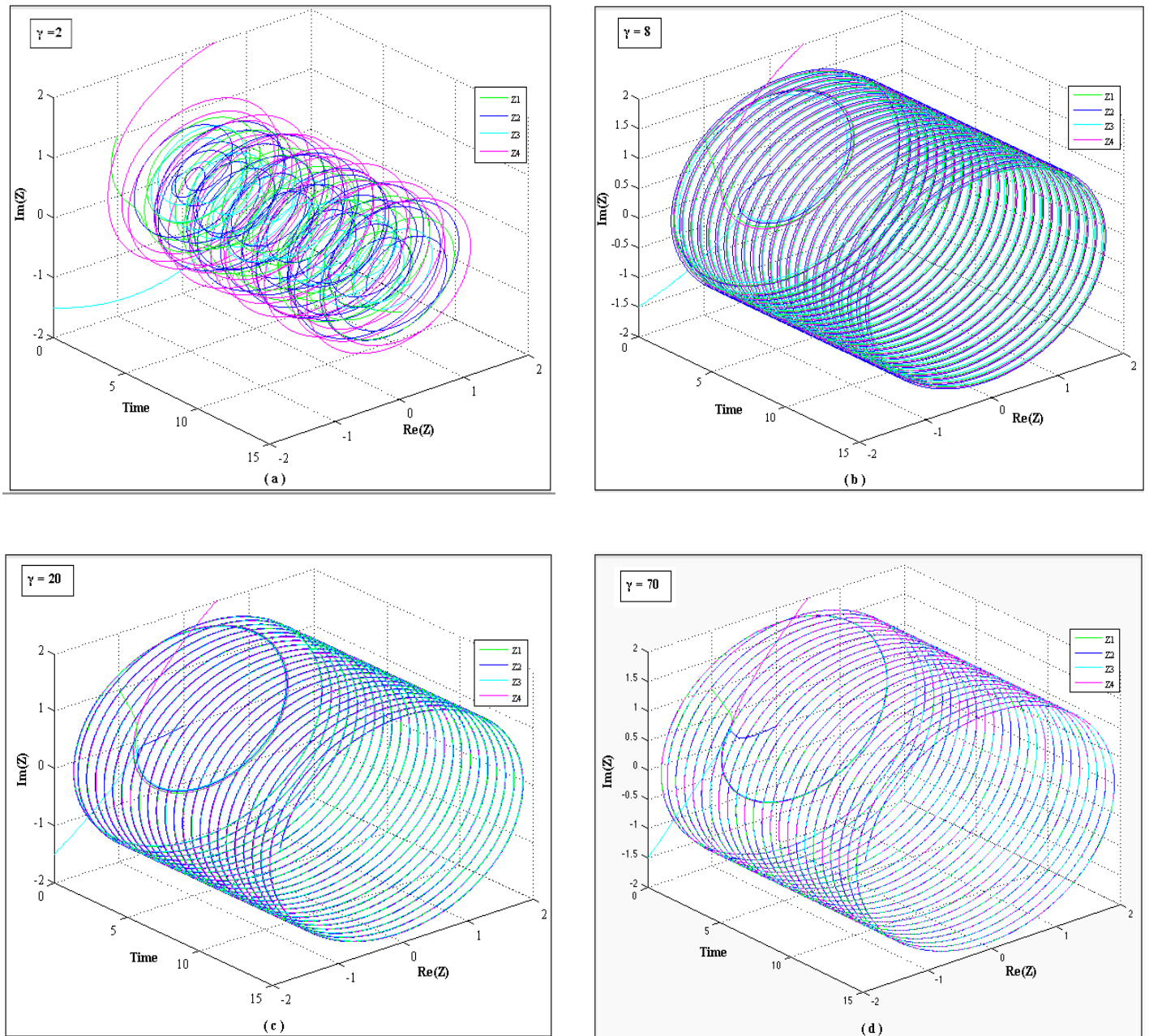


FIGURE 6.2 The phase portrait of the system (6.53) for four coupling strengths. For weak, Fig.(a) illustrates the incoherence of oscillators, when γ grows sufficiently we remark, from Fig.(d) the apparition of perfect synchronous limit cycle.

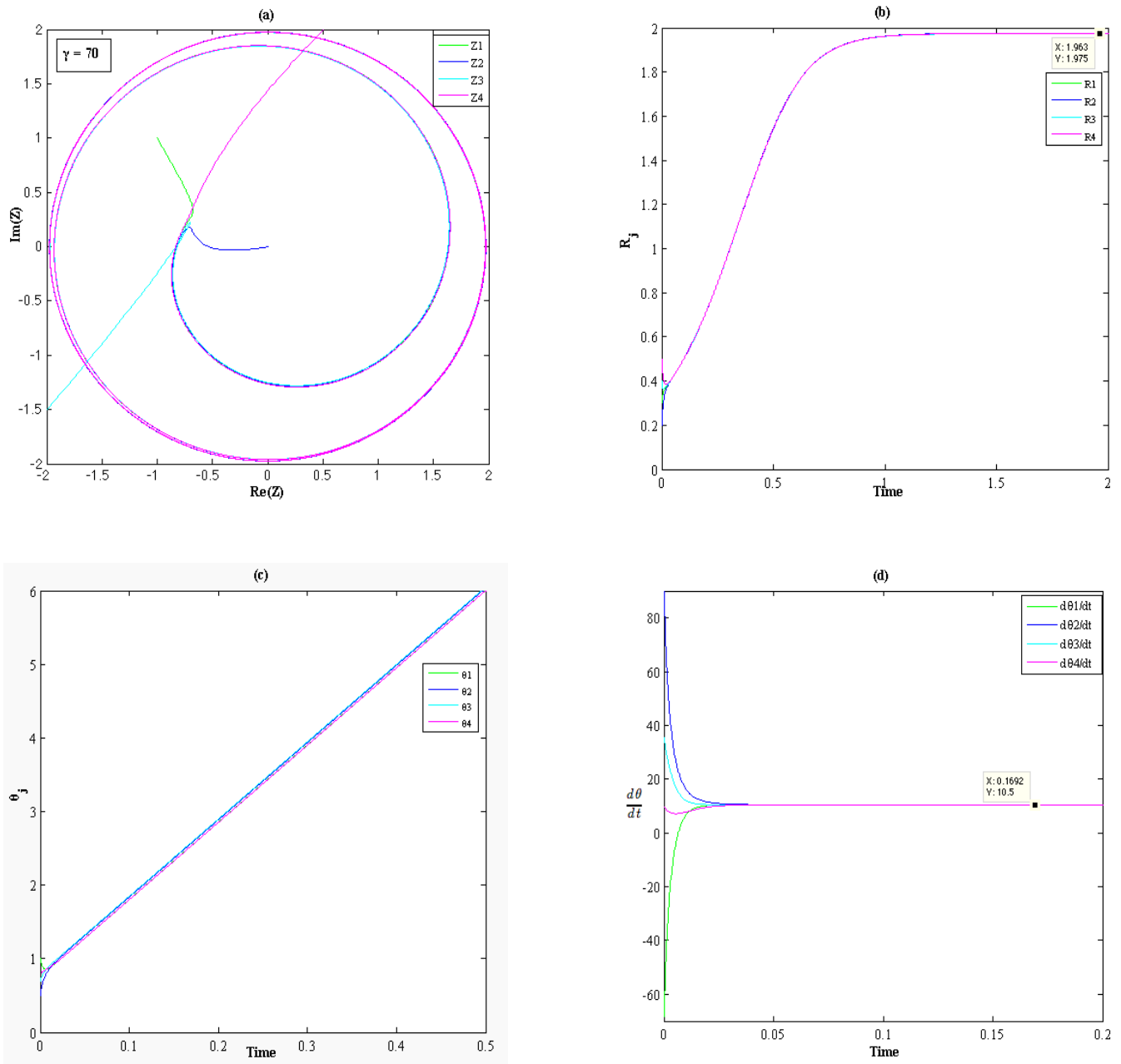


FIGURE 6.3 – Full synchronization of system (6.53) with the vector of natural frequencies $w^T = [5 \ 12 \ 18 \ 7]$ for strong coupling strength ($\gamma = 70$).

As it is proved in the previous sections, the synchronization frequency and the amplitude of synchronous limit cycle is completely defined by the first eigenvalue λ_1 of $A = \alpha \mathbb{I} - \gamma L + i\Omega$ and its corresponding eigenvectors. The synchronous limit cycle is defined by the Landau-Stuart equation

$$\dot{z}_m = (\lambda_1 - c|z_m|^2) z_m \quad (6.56)$$

with $c = \sum_{i=1}^N v_{l_i}^* v_{r_i}^* v_{r_i}^2$.

Therefore, the radius of the limit cycle is $R = \sqrt{\frac{\text{Real}(\lambda_1)\text{Real}(c)}{|c|}}$ and the frequency of oscillation $\omega_m = \text{Im}g(\lambda_1)$. In Fig. 6.4(a) we present the evolution of angular frequency of the oscillators defined via (6.53) with the vector of natural frequencies $w^\top = [5 \ 12 \ 18 \ 7]$ and the coupling strength $\gamma = 70$.

In Fig. 6.4(b), we present the evolution of $\text{Im}g(\lambda_1(A))$ in function of the coupling strength.

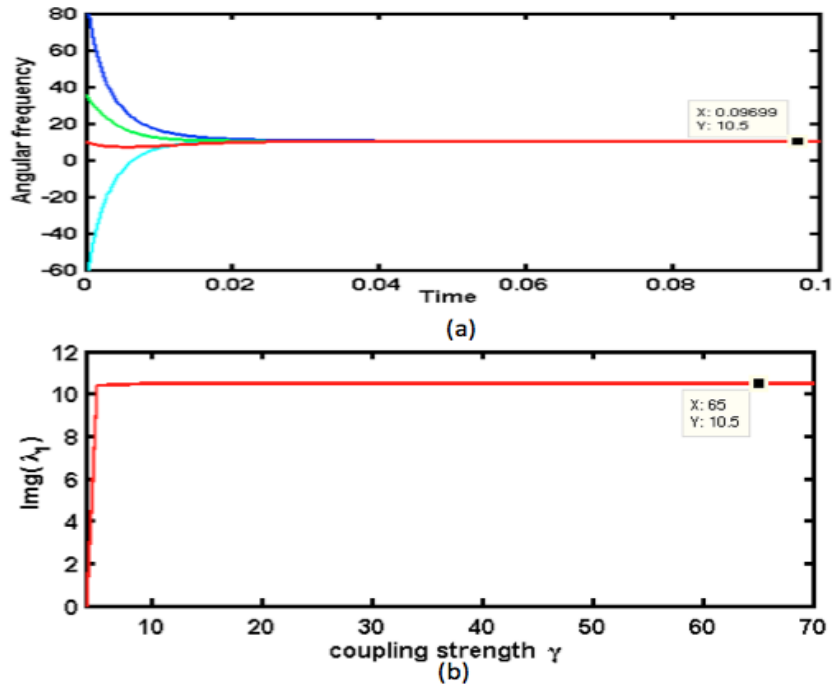


FIGURE 6.4 – Synchronization of angular frequencies for the system (6.53) (a) and the comparison of the synchronization frequency ω_m with the value of $\text{Im}g(\lambda_1(A))$ (b).

In Fig. 6.5 we compare the evolution of the radial component of the system (5.15) –Fig. 6.5 (a) against the evolution of the radius of the synchronous limit cycle R –Fig. 6.5(b).

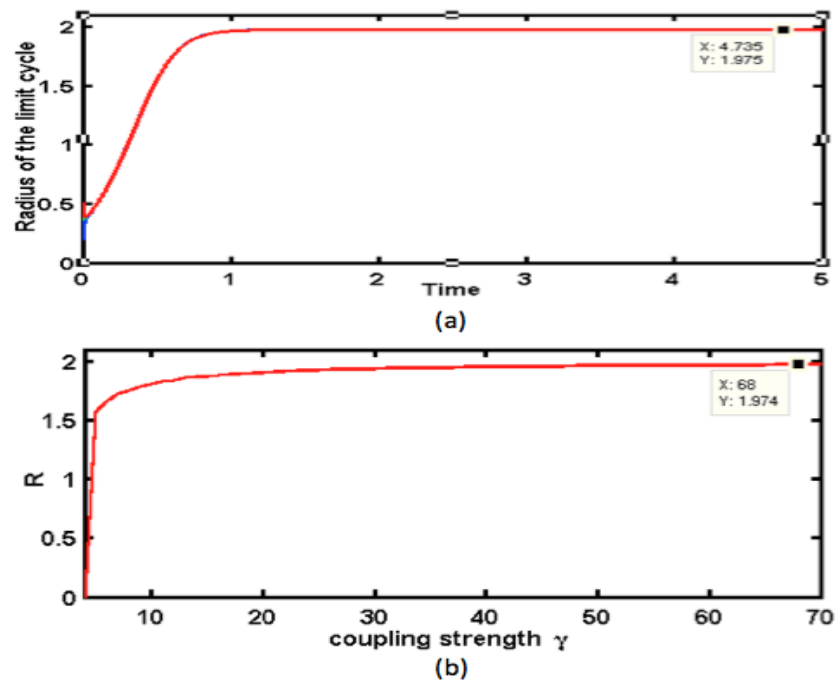


FIGURE 6.5 – The evolution of the radial component of the system (6.53) (a) and the radius of the averaged limit cycle R (b).

Chapter 7

Conclusions and perspectives

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Synchronization phenomena in large populations of coupled oscillatory systems are the subject of research in different domains and appear in physical, biological, chemical, and social systems. One approaches to analysing this phenomena consists of modelling each element of the population by a canonical model with few parameters. The most important benefit of using canonical models to study the synchronization phenomena in large oscillator networks is their simplicity. In this thesis, synchronization is analyzed for two of the most representative models of coupled oscillators, the Kuramoto model as phase oscillator model and Landau-Stuart model as limit-cycle oscillator model. The choice of models is motivated by two reasons. First, these models are not too complicated and are still tractable; and second, these models are rich enough to present a large variety of synchronization patterns which allows us to represent large classes of systems with very few assumptions and parameters.

By avoiding the complications generated by particular details of a given system, we also keep away from conjectural or specious results that may be caused by the specific features of the chosen model. Throughout this thesis we have studied a few systems of coupled heterogeneous oscillators and with various types of interactions. In particular, we studied heterogeneous phase oscillator networks with complete or complex interaction topologies. We have done this with the hope that the choice of the coupling parameters may help to advance the analysis of more general forms of system interactions.

In the following two sections, we summarize the contributions of this thesis and suggest few directions for future research.

7.1 Summary

In Chapter 1, we gave a general presentation of self-synchronization phenomena and a motivation for the study of complex oscillator networks. We emphasized the importance of the synchronization in various applications, and we outlined the contents and contributions of this thesis.

In Chapter 2, we reviewed the main features of the Kuramoto model and we roughly presented the phase reduction of coupled limit cycle oscillators. Additionally, we discussed different synchronization notions and introduced some definitions and results from algebraic graph theory and consensus protocols, which proved to be valuable tools throughout this thesis.

In Chapter 3, we analyzed heterogeneous oscillators interacting through a complex network; namely, a network of Kuramoto oscillators with a strongly connected interconnection graph was considered. Conditions for frequency synchronization were presented and it was shown that the synchronization frequency depends on the coupling strengths and the analytical expression for the limit of synchronization frequency was given for the case when coupling strength is sufficiently large. Additionally, we showed that the problem of existence of the phased locked solutions for the Kuramoto model with an arbitrary interconnection topology can be recast as a problem of existence of a linear system with certain properties, and defined the corresponding linear system of complex variables. Moreover, necessary and sufficient conditions for the existence of phased locked solutions were formulated.

In Chapter 4, a network of Kuramoto oscillators with non-uniform weights of interconnection was considered, and necessary and sufficient conditions for the existence of phase locked solutions were presented. In fact, we analyzed synchronization phenomena in networks of phase oscillators with two sources of heterogeneity. We have considered, as in the previous chapter, that natural frequencies of the oscillators are different and additionally, we have assumed that the individual contribution of oscillators to the mean field is weighted by two factors. These factors represent the input and output weight of the oscillators. For this particular input-output weight model, it was shown that the synchronization frequency depends on the interconnection weights and an analytical expression for the synchronization frequency was given. Local asymptotic stability of one phase locked solution was proven as well as instability of other phase locked solutions.

In the previous chapters, we considered the problem of phase synchronization of phase and frequency synchronization for the Kuramoto model with positive gains; that is, only

attractive interactions between oscillators were considered. However, interplay of attractive and repulsive interactions can play an important role in the context of dynamical networks. The crucial role of repulsive interactions in synchronization phenomena was studied analytically in [92] and confirmed experimentally in [141].

In Chapter 5, also as in Chapter 4, we considered a scenario when the coupling coefficient of each pair can be separated into two different factors, input-output weights. Thus, we have analysed phase and frequency synchronization of the Kuramoto model with a signed weighted digraph of interconnection. Motivation for this work has been to examine the effect of the repulsive coupling on the system's dynamics and on synchronization frequency. We have showed that in the case of a weighted sign-symmetric graph of interconnection, existence and stability of phase locked solutions for the Kuramoto model with positive weights guarantees the existence of the whole family of Kuramoto models defined by the initial model and a family of gauge transformations of interconnection matrix. We also gave an exact expression for synchronization frequency and presented local stability analysis.

Chapter 6 was devoted to the analysis of collective behaviour in populations of periodic oscillators. We considered a network of diffusively coupled Stuart-Landau oscillators with a directed interconnection graph. To analyse the stability of such a network we proposed using results on practical stability which allows the study of stability or synchronization properties in cases when only approximations of the limit cycle and synchronization frequency are available. Therefore, we have shown that if the coupling strength is sufficiently large, the networked system practically synchronizes to a common limit cycle. We also provided an analytical expression for the approximate synchronized dynamics and analysed its stability properties.

7.2 Future research

Despite the vast literature on applications, numerical and theoretical analysis of synchronization of oscillators networks, many interesting problems still remain. In the following section, we present few directions for future research which stem from the results of this thesis.

Kuromato model

- All stability results that were presented in Chapters 4-5 are local of nature. The extension of these results for, at least, some bounded domain of initial conditions similar to the results of [24], seems to be possible.

- Extension of the results of chapters 4 and 5 on input-output weighted complete interaction digraph to other cases of network interconnection topology, such as, for example, ring network [108] and small world network [56], [123].
- Robustness analysis of frequency and phase synchronization vis-a-vis bounded perturbations including particular cases of white and colored noise also represents an interesting direction of research.
- Control of the Kuramoto model, using pinning strategies or mean-field control can be done using the same framework as proposed in chapter 3-5. In both cases, the control objective can be either synchronization of initially non-synchronized model, as it is done in [119], or desynchronization of a synchronized Kuramoto model, similar to [39].

Landau-Stuart model

In Chapter 6 we analysed the synchronization properties of a network of Landau-Stuart oscillators under the following assumption that interconnections among the oscillators are linear and could be presented in the form of a symmetric Laplacian matrix with non-negative coefficients.

In the future, we plan to relax these assumptions in several directions. In particular,

- Extension of the stability results for the network of Landau-Stuart oscillators to the case where digraph of interconnections is signed can be done combining tools proposed in Chapter 6 with the approach proposed in [5] and used in Chapter 5.
- In a similar way, the extension of the results of Chapter 6 to the case of a non-symmetric Laplacian matrix seems possible, but will require more defined tools from linear algebra and theory of perturbation for eigenvalues in particular.
- Only the case of linear interconnections was considered in Chapter 6 ; an extension of these results to the case of more realistic non-linear interconnections seems to be a challenging but necessary step in the analysis of oscillatory networks.
- Finally, an analysis of this chapter was done for large values of interconnection gain with the objective to ensure a practical synchronization oscillators network. It is well known that for smaller gains synchronization of Landau-Stuart oscillators is not possible and network exhibits other types of behaviour, see [40]. It would be interesting to analyse such a situation using the tools developed in Chapter 6.

Annexe A

Derivation of the average oscillator model

Consider the function $F(\mathbf{z}) = C(\mathbf{z}) \mathbf{z}$, where we recall that $C(\mathbf{z}) = \text{diag}(|z_1|^2, \dots, |z_N|^2)$, $Z = \text{diag}(z_1, \dots, z_N)$ and $Z^* = \text{diag}(z_1^*, \dots, z_N^*)$. We can write the derivative of z_m as follows

$$\begin{aligned}
 \dot{z}_m &= \lambda_1 z_m - \vartheta_{l_1}^* F(\mathbf{z}), \\
 &= \lambda_1 z_m - \vartheta_{l_1}^* C(\mathbf{z}) \mathbf{e} - \vartheta_{l_1}^* \begin{pmatrix} z_1^* v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_N^* v_{r_{1N}} z_m \end{pmatrix} \mathbf{e} \\
 &\quad - \vartheta_{l_1}^* \begin{pmatrix} (v_{r_{11}} z_m)^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{r_{1N}} z_m)^2 \end{pmatrix} e^{*\top} - \alpha |z_m|^2 z_m, \\
 &= \lambda_1 z_m - \alpha |z_m|^2 z_m - \vartheta_{l_1}^* \begin{pmatrix} |z_1|^2 + z_1^* v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |z_N|^2 + z_N^* v_{r_{1N}} z_m \end{pmatrix} \mathbf{e} \\
 &\quad - \vartheta_{l_1}^* \begin{pmatrix} (v_{r_{11}} z_m)^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{r_{1N}} z_m)^2 \end{pmatrix} \bar{\mathbf{e}}, \\
 &= f_m(z_m) - \vartheta_{l_1}^* \begin{pmatrix} |z_1|^2 + z_1^* v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |z_N|^2 + z_N^* v_{r_{1N}} z_m \end{pmatrix} \mathbf{e} \\
 &\quad - \vartheta_{l_1}^* \begin{pmatrix} (v_{r_{11}} z_m)^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{r_{1N}} z_m)^2 \end{pmatrix} \bar{\mathbf{e}},
 \end{aligned}$$

Using the definition of the synchronization error $\mathbf{e} = \mathbf{z} - \vartheta_{r_1} z_m$ we can express $F(\mathbf{z})$ as a function of z_m and \mathbf{e} as follows

$$\begin{aligned}
F(\mathbf{z}) &= C(\mathbf{z}) \mathbf{z}, \\
&= C(\mathbf{z}) (\mathbf{z} - \vartheta_{r_1} \vartheta_{r_1}^* \mathbf{z}) + C(\mathbf{z}) \vartheta_{r_1} \vartheta_{r_1}^* \mathbf{z}, \\
&= C(\mathbf{z}) \mathbf{e} + C(\mathbf{z}) \vartheta_{r_1} z_m, \\
&= C(\mathbf{z}) \mathbf{e} + \mathbf{Z}^* \mathbf{Z} \vartheta_{r_1} z_m, \\
&= C(\mathbf{z}) \mathbf{e} + \mathbf{Z}^* \begin{pmatrix} z_1 - v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_N - v_{r_{1N}} z_m \end{pmatrix} \vartheta_{r_1} z_m + \\
&\quad \mathbf{Z}^* \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \vartheta_{r_1} z_m, \\
&= C(\mathbf{z}) \mathbf{e} + \mathbf{Z}^* \begin{pmatrix} e_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_N \end{pmatrix} \vartheta_{r_1} z_m + \\
&\quad \mathbf{Z}^* \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \vartheta_{r_1} z_m, \\
&= C(\mathbf{z}) \mathbf{e} + \mathbf{Z}^* \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \mathbf{e} + \\
&\quad \begin{pmatrix} z_1^* - v_{r_{11}}^* z_m^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_N^* - v_{r_{1N}}^* z_m^* \end{pmatrix} \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \vartheta_{r_1} z_m \\
&\quad + \begin{pmatrix} v_{r_{11}}^* z_m^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}}^* z_m^* \end{pmatrix} \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \vartheta_{r_1} z_m,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&= C(\mathbf{z}) \mathbf{e} + Z^* \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \mathbf{e} + \\
&\begin{pmatrix} e_1^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_N^* \end{pmatrix} \begin{pmatrix} v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} z_m \end{pmatrix} \vartheta_{r_1} z_m + \\
&\begin{pmatrix} v_{r_{11}}^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}}^* \end{pmatrix} \begin{pmatrix} v_{r_{11}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} \end{pmatrix} \vartheta_{r_1} |z_m|^2 z_m.
\end{aligned}$$

Let us define the constant $\alpha \in \mathbb{C}$ in this form

$$\begin{aligned}
\alpha &= \vartheta_{\mathbf{1}}^* \begin{pmatrix} v_{r_{11}}^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}}^* \end{pmatrix} \begin{pmatrix} v_{r_{11}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_{r_{1N}} \end{pmatrix} \vartheta_{\mathbf{1}}, \\
&= \sum_{i=1}^N v_{l_i}^* v_{r_i}^* v_{r_{1i}} v_{r_{1i}} = \sum_{i=1}^N v_{l_i}^* v_{r_i}^* v_{r_{1i}}^2.
\end{aligned}$$

Therefore, we have

$$\dot{z}_m = (\lambda_1 - \alpha |z_m|^2) z_m + f_2(z_m, \mathbf{e}),$$

with

$$f_2(z_m, \mathbf{e}) = -\vartheta_{\mathbf{1}}^* \begin{pmatrix} |z_1|^2 + z_1^* v_{r_{11}} z_m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |z_N|^2 + z_N^* v_{r_{1N}} z_m \end{pmatrix} \mathbf{e} - \vartheta_{\mathbf{1}}^* \begin{pmatrix} (v_{r_{11}} z_m)^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{r_{1N}} z_m)^2 \end{pmatrix} \bar{\mathbf{e}}.$$

Annexe B

Matrix properties

Lemma 3. *Let \tilde{A} and \tilde{B} be symmetric matrices in $\mathbb{R}^{N \times N}$, with \tilde{B} positive semi-definite. Then*

$$\lambda_i(\tilde{A} + \tilde{B}) \geq \lambda_i(\tilde{A}), \quad \forall i \in \{1, \dots, N\}.$$

If \tilde{B} is positive definite then

$$\lambda_i(\tilde{A} + \tilde{B}) > \lambda_i(\tilde{A}), \quad \forall i \in \{1, \dots, N\}.$$

Lemma 4. *Suppose that $A \in M_{m,n}$ and $B \in M_{n,m}$ with $m \leq n$. then BA has the same eigenvalues as AB , counting multiplicity, together with an additional $n - m$ eigenvalues equal to zero.*

Theorem 21. (Gershgorin Circle Theorem [50]) *Let A be a complex $(n \times n)$ matrix, with entries a_{ij} . For $i \in \{1, \dots, n\}$, let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the i -th row. Let $\mathcal{D}(a_{ii}, R_i)$ be the closed disc centred at a_{ii} with radius R_i . Such a disk is called a Gershgorin disk. Then, each eigenvalue of A is in at least one of the disks $\{z : |z - a_{ij}| \leq R_i\}$.*

Theorem 22. (Sylvester's Law of Inertia [55]) *Let $A, B \in M_n$ be Hermitian. Then A and B are congruent if and only if they have the same inertia.*

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