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THE HIGHER ORDER q-DOLAN-GRADY RELATIONS AND QUANTUM INTEGRABLE SYSTEMS

Thi Thao Vu

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THÈSE présenté par :

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LES RELATIONS DE $q-$DOLAN-GRADY D'ORDRE SUPÉRIEUR ET CERTAINS SYSTÈMES INTÉGRABLES QUANTIQUES

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LES RELATIONS DE $q-$DOLAN-GRADY D'ORDRE SUPÉRIEUR ET CERTAINS SYSTÈMES INTÉGRABLES QUANTIQUES

Résumé

Dans cette thèse, la connexion entre certaines structures algébriques récentes (algèbres tridiagonales, algèbre $q-$Onsager, algèbres $q-$Onsager généralisées), la théorie des représentations (paire tridiagonale, paire de Leonard, polynômes orthogonaux), certaines des propriétés de ces algèbres et l'analyse de modèles intégrables quantiques sur le réseau (la chaîne de spin XXZ ouverte aux racines de l'unité) est considérée.

Le manuscript commence par une introduction brève sur le sujet. Dans le Chapitre 1 et 2, le matériel de base est rappelé: dans le Chapitre 1, les algèbres $q-$Onsager et tridiagonales, certaines relations avc l'algèbre quantique affine $U_q(\widehat{sl_2})$, avec l'algèbre de boucle $U_q(sl_2)$, avec l'algèbre de dimension infinie $\mathcal{A}_q$ et l'algèbre de réflexion sont présentées, ainsi que certains résultats connus de Terwilliger *et al.* concernant la théorie des représentation de l'algèbre $q-$Onsager (paire de Leonard, paire tridiagonale,polynômes orthogonaux); Dans le Chapitre 2, l'émergence de l'algèbre de Onsager et $q-$Onsager dans le contexte des systèmes intégrables quantiques est rappelée. En particulier, la découverte de l'algèbre $q-$Onsager dans ce contexte (Baseilhac *et al.*) et ses applications à la solution de la chaîne de spin XXZ ouverte avec conditions aux bords génériques sont revues. Le Chapitre 3 est dédié à la description des résultats principaux de cette thèse: (i) Pour la classe des algèbres $q-$Onsager associées à $\widehat{sl_2}$ et aux algèbres de Lie affines simplement lacées de type ADE, des relations d'ordre supérieur analogues à celles de Lusztig sont conjecturées; (ii) pour la chaîne de spin XXZ ouverte aux racines de l'unité, de nouveaux éléments (polynômes divisés des générateurs de l'algèbre $q-$Onsager) sont introduits et leurs propriétés étudiées. Ces deux éléments, en plus des deux générateurs de base de l'algèbre $q-$Onsager, forment une nouvelle algèbre. Celle-ci peut être comprise comme un analogue du groupe quantique de Lustzig pour l'algèbre $q-$Onsager. Dans le Chapitre 4, plusieurs problèmes ouverts sont proposés. Dans les Appendices, deux programmes en calcul symbolique MAPLE ayant servis sont donnés.

Mots clés : Algèbre tridiagonale; algèbre $q-$Onsager; algèbre $q-$Onsager généralisées; chaîne de spin XXZ ouverte; racine de l'unité.

THE HIGHER ORDER q-DOLAN-GRADY RELATIONS AND QUANTUM INTEGRABLE SYSTEMS

Abstract

In this thesis, the connection between recently introduced algebraic structures (tridiagonal algebra, q-Onsager algebra, generalized $q-$Onsager algebras), related representation theory (tridiagonal pair, Leonard pair, orthogonal polynomials), some properties of these algebras and the analysis of related quantum integrable models on the lattice (the XXZ open spin chain at roots of unity) is considered.

The manuscript starts with a brief introduction on the subject. In Chapter 1 and 2, some background material is recalled: In Chapter 1, the $q-$Onsager and tridiagonal algebras, some of the relations with the quantum affine algebra $U_q(\widehat{sl_2})$, the $U_q(sl_2)$-loop algebra, the infinite dimensional algebra $\mathcal{A}_q$ and the reflection equation algebra are presented, as well as some of the known results by Terwilliger *et al.* about the representation theory of the q-Onsager algebra (Leonard pair, tridiagonal pair, orthogonal polynomials); In Chapter 2, the emergence of the Onsager and $q-$Onsager algebras in the context of quantum integrable systems is recalled. In particular, the discovery of the $q-$Onsager algebra and its application to the solution of the open XXZ chain for generic values of q are reviewed. Chapter 3 is devoted to the description of the main results of the thesis: (i) for the class of $q-$Onsager algebras associated with $\widehat{sl_2}$ and ADE type simply-laced affine Lie algebras, higher order analogs of Lusztig's relations are conjectured; (ii) for the open XXZ spin chain at roots of unity, new elements (that are divided polynomials of $q-$Onsager generators) are introduced and some of their properties studied. These two elements together with the two basic elements of the $q-$Onsager algebra generate a new algebra, which can be understood as an analog of Lusztig's quantum group for the $q-$Onsager algebra. In Chapter 4, open problems are proposed. In the Appendices, computer programs (MAPLE) that were used are given.

Keywords : Tridiagonal algebra; Tridiagonal pair; q-Onsager algebra; Generalized q-Onsager algebra; XXZ open spin chain; root of unity.

Notations

Throughout this thesis, we use the following notations:

1. Let A, B denote generators, then

$$\begin{aligned} [A,B] &= AB - BA. \\ [A,B]_q &= qAB - q^{-1}BA. \end{aligned}$$

2. Let n, m be integers, then

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q!\,[n-m]_q!}\ , \qquad [n]_q! = \prod_{l=1}^{n} [l]_q\ , \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [0]_q = 1\ .$$

3. $\{x\}$ denotes the integer part of x. Let j, m, n be integers, write $j = \overline{m,n}$ for $j = m, m+1, ..., n-1, n$.

4. The Pauli matrices $\sigma_1, \sigma_2, \sigma_z, \sigma_\pm$:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

.

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Introduction

In the literature, the Onsager algebra and the Dolan-Grady relations first appeared in study of integrable systems (the XY, the Ising models,...) [O44, DG82, Dav91, Dav90]. Later on, they appeared in the context of mathematics in relation with certain subalgebras of $\widehat{sl_2}$ [DR99]. From 2003, a $q-$deformed analog of the Dolan-Grady relations appeared in the context of mathematics as a special case of tridiagonal algebras [Terwilliger et al.]. Almost simultaneously, the $q-$Dolan-Grady relations appeared in the context of quantum integrable systems on the lattice and continuum: the $q-$Onsager algebra was defined (which $q-$Dolan-Grady relations are the defining relations) in relation with the quantum reflection equation, as an algebra generating a large class of quantum integrable systems on the lattice or continuum [Baseilhac et al.].

There is now a rather vast literature on the subject of tridiagonal algebras [Ter93III], the representation theory of tridiagonal pairs [ITT99] and Leonard pairs [Ter03], the $q-$Onsager algebra and its generalizations [Ter01, Bas0404], the connections with coideal subalgebras of $U_q(\widehat{sl_2})$ [BB12] and with a new infinite dimensional algebra called $\mathcal{A}_q$ [BS09]. In the context of mathematical physics, these structures and the explicit analysis of some of their properties lead to several new exact non-perturbative results for the open XXZ spin chain [BK07], for the half-infinite XXZ spin chain [BB12, BK14], for the open affine Toda field theories [BB09, BF11]. From a general point of view, a new approach called '$q-$Onsager approach' has emerged as an alternative to existing ones in quantum integrable systems (Bethe ansatz [Bet31, FST80], separation of variable [Sk92], $q-$vertex operators [JM95, JKKKMW94]). Since 2007, this approach has been currently developed in different directions.

In this thesis, we explore one direction which overlaps between mathematics and physics. Namely, we investigate in detail some properties of the $q-$Onsager algebras (in particular the existence and explicit construction of higher order relations between monomials of the fundamental generators) which will find application in the analysis of the open XXZ spin chain at roots of unity (characterization of the symmetry of the Hamiltonian at roots of unity). At the moment, the results of this thesis have been published in two articles [BV13, BV1312]. There is another article in preparation [BGSV15].

The manuscript of the thesis is divided into three main Chapters.

Chapter 1. *We summarize without proofs the relevant material on tridiagonal algebras, the q-Onsager algebra and some aspects of its representation theory: Leonard pairs,*

tridiagonal pairs and orthogonal polynomials.

In the first part, tridiagonal algebras are defined by generators and relations. Several special cases corresponding to particular parameter sequences of the tridiagonal relations are mentioned such as the q-Serre relations or the Dolan-Grady relations.

The q-Onsager algebra is introduced in a second part. Its defining relations are the $q-$Dolan-Grady relations: these are ρ_i-deformed analogues of the q-Serre relations, and correspond to a special parameter sequence of the tridiagonal algebra. In connection with the quantum affine algebra $U_q(\widehat{sl_2})$ and the $U_q(sl_2)$-loop algebra, we thus recall homomorphisms from the q-Onsager algebra to these algebras. Finally, we recall the reflection equation algebra and indicate its relation with the q-Onsager algebra: the reflection equation algebra is defined by generators which are entries of the solution of the "RKRK" equations for the $U_q(\widehat{sl_2})$ R-matrix. The isomorphisms between the reflection equation algebra, the current algebra $O_q(\widehat{sl_2})$, and the infinite dimensional algebra $\mathcal{A}_q$ generated by $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1} | k \in \mathbb{Z}_+\}$ are recalled. We also recall the construction of a coaction map for the q-Onsager algebra and the defining relations of the K-matrix as an intertwiner of irreducible finite dimensional representations of the q-Onsager algebra. Thus, a quotient of the q-Onsager algebra is isomorphic to a quotient of the reflection equation algebra.

The last part recalls some aspects of the representation theory of tridiagonal algebras (including the case of the $q-$Onsager algebra), in particular the results of Terwilliger et al. about irreducible finite dimensional representations and the concept of tridiagonal pairs. For convenience, Leonard pairs (a subclass of the tridiagonal pairs) are introduced first. We recall the notion of Leonard pair, Leonard system as well as modification of a given Leonard pair in several ways. The relation between the Leonard pair and the tridiagonal algebra is also clarified. Namely, there exists a scalar sequence such that the Leonard pair satisfies the corresponding Askey-Wilson relations. Inversely, a pair of linear transformations satisfying the Askey-Wilson relations allows to define a Leonard pair under certain conditions. One more important result is the classification of Leonard pairs, it is asserted that a sequence of scalars satisfying conditions $(i) - (v)$ in Theorem (1.3.7) is necessary and sufficient to obtain a Leonard pair. In addition, we show that Leonard pairs arise naturally in relation with the Lie algebra sl_2 and $U_q(sl_2)$. A more general object than the concept of Leonard pair, namely the concept of tridiagonal pair, is also introduced and described in some details. We first recall the concept of a tridiagonal pair, of a tridiagonal system as well as properties of its (dual) eigenvalue sequence, the corresponding (dual) eigenspace sequence. It is asserted that the tridiagonal pair of q-Racah type satisfies the tridiagonal relations, inversely a tridiagonal pair can be obtained from a tridiagonal algebra under several conditions. We also describe some special classes of tridiagonal pairs, such as Leonard pairs, tridiagonal pairs of the q-Serre type, mild tridiagonal pairs, sharp tridiagonal pairs. Especially, the classification of the sharp tridiagonal pairs is clarified [INT10]. Last but not least, the relation between tridiagonal algebras and the theory of orthogonal polynomials is briefly described. We recall hypergeometric orthogonal polynomials, and describe the connection between the theory of Leonard pairs and the Askey-scheme of orthogonal polynomials [Ter0306]. Note that the extension to the theory of tridiagonal pairs leads to hypergeometric polynomials of several variables defined on a discrete support

(Gasper-Rahman), as recently discovered in [BM15].

Chapter 2. *We recall the known presentations of the Onsager algebra and how the so-called $q-$Onsager algebra appeared in the context of mathematical physics. We briefly recall the 'q−Onsager approach'.*

First, we provide a historical background about the two different known presentations of the Onsager algebra: either the original presentation with generators A_n, G_m [O44] or the presentation in terms of the Dolan-Grady relations [DG82]. We also recall the relation with the loop algebra of sl_2. Secondly, we recall how the $q-$Onsager algebra surprisingly appeared in 2004 in the context of quantum integrable systems, through an analysis of solutions of the reflection equation. In particular, we recall how the new infinite dimensional algebra $\mathcal{A}_q$ arises and how it is related with the $q-$Onsager algebra. It is explained how its connections with the quantum loop algebra of sl_2 and with $U_q(\widehat{sl_2})$ naturally appear from the Yang-Baxter and reflection equation algebra formulation. Then, the so-called '$q-$Onsager approach' is briefly recalled.

Chapter 3. *The three main results of the thesis are presented in some details. (i) and (ii): For the family of $q-$Onsager algebras ($\widehat{sl_2}$ and ADE type), analogs of the higher order relations of Lusztig are conjectured and supporting evidence is presented in detail; (iii) The open XXZ spin chain is considered at roots of unity in the framework of the $q-$Onsager approach. A new algebra, an analog of Lusztig's quantum group, naturally arises. For a class of finite dimensional representations, explicit generators and relations are described. With respect to the new algebra, symmetries of the Hamiltonian are explored.*

Recall the homomorphism from the $q-$Onsager algebra to the quantum affine Lie algebra $U_q(\widehat{sl_2})$ [Bas0408, BB12]. Recall the homomorphism from the generalized $q-$Onsager algebras (higher rank generalizations of the $q-$Onsager algebra) to the quantum affine Lie algebra $U_q(\widehat{g})$ [BB09, Kol12]. By analogy with Lusztig's higher order relations [Lusz93] which arise for any quantum affine Lie algebra, it is thus expected that higher order relations are satisfied. Successively, we obtained:

1. *The higher order relations for the q-Onsager algebra (the $\widehat{sl_2}$ case)* [BV13]

   Let A, A^* be the standard generators of the $q-$Onsager algebra. The $r-$th higher order relations for the q-Onsager algebra are conjectured. First, a generalization of the conjecture is proven for the case of tridiagonal pairs (i.e. certain irreducible finite dimensional representations on which A, A^* act). Two-variable polynomials which determine the relations are given. Then, the special case of the $q-$Onsager algebra is considered in details. The conjecture is proven for $r = 2, 3$. For r generic, the conjecture is studied recursively. A Maple software program is used to check the conjecture, which is confirmed for $r \leq 10$. Also, for a special case, the higher order relations of Lusztig are recovered.

2. *The higher order relations for the generalized q-Onsager algebra (the ADE serie)*

[BV1312]

For each affine Lie algebra, a generalized q-Onsager algebra has been defined in [BB09]. Let A_i, $i = 0, 1, ..., rank(g)$ be the standard generators of this algebra. By analogy with the $\widehat{sl_2}$ case, for any simply-laced affine Lie algebra analogues of Lusztig's higher order relations are conjectured. The conjecture is proven for $r \leq 5$. For r generic, the conjecture is studied recursively. A Maple software program is used to check the conjecture, which is confirmed for $r \leq 10$. According to the parity of r, two new families of two-variable polynomials are proposed, which determine the structure of the higher order relations. Several independent checks are done, which support the conjecture.

3. *The XXZ open spin chain at roots of unity* [BGSV15]

   Inspired by the fact that the XXZ periodic spin chain at roots of unity enjoys a sl_2 loop algebra symmetry in certain sectors of the spectrum [DFM99], the aim is to settle an algebraic framework for the analysis of the open XXZ spin chain at roots of unity within the $q-$Onsager approach. First, the two basic generators of the $q-$Onsager algebra are recalled, and their properties are studied for q a root of unity (spectrum, structure of the eigenspaces and action). They form a new object that we call a 'cyclic tridiagonal pair'. Secondly, two new operators that are divided polynomials of the fundamental generators of the q-Onsager algebra are introduced. We study some of their properties (spectrum, structure of the eigenspaces and action). The relations satisfied by the four operators are described in details. They generate an explicit realization and first example of an analog of Lusztig quantum group for the $q-$Onsager algebra. Finally, we briefly discuss the conditions on the boundary parameters such that the Hamiltonian of the open XXZ spin chain commutes with some of the generators.

In the end of this thesis, three families of open problems are presented in Chapter 4 and appendices are reported in Chapter 5.

Chapter 1

Mathematics: background

The aim of this Chapter is to present various aspects of the tridiagonal and q-Onsager algebras, their relations with other types of quantum algebras, their relations with the theory of Leonard and tridiagonal pairs, and finally the remarkable connection with the theory of orthogonal polynomials and Askey-scheme.

In the first part, the definition of the tridiagonal algebra is given in terms of generators and relations. Several special cases of tridiagonal relations corresponding to the particular parameter sequences are recalled such as the q-Serre relations, the Dolan-Grady relations.

In the second part, the q-Onsager algebra is defined as the ρ_i-deformed analogues of the q-Serre relations. Indeed, it is a special case of the tridiagonal algebra. Furthermore, the connection between the q-Onsager algebra and other algebras such that the quantum enveloping algebra $U_q(\widehat{sl_2})$, the quantum loop algebra $U_q(\mathcal{L}(sl_2)$, the reflection equation algebra and the current algebra denoted $O_q(\widehat{sl_2})$ recently introduced is briefly described.

In the last part, several results due to Terwilliger et al. about the representation theory of tridiagonal algebras - in particular the $q-$Onsager algebra - are recalled. Finite dimensional irreducible representations of the tridiagonal algebras and of the $q-$Onsager algebras have been considered in details by Terwilliger *et al.*. Provided two matrices A, A^* satisfy the defining relations of the tridiagonal algebra, are diagonalizable on the vector space and the representation is irreducible, it implies the following: in the basis in which the first matrix A is diagonal with degeneracies, the other matrix A^* takes a block tridiagonal structure. Furthermore, there exists another basis with respect to which the first matrix A transforms into a block tridiagonal matrix, whereas the second one transforms into a diagonal matrix with degeneracies. In the simplest case (no degeneracy in the spectra), tridiagonal pairs are called Leonard pairs, which definition and properties will be first recalled. Then, we will describe definition and properties of tridiagonal pairs based on Terwilliger and collaborators' investigations. In addition, the connection between the theory of special functions and orthogonal polynomial and the $q-$Onsager algebra is briefly described. Note that this subject was one of the motivations for studying tridiagonal algebras and the theory of tridiagonal pairs. For the simplest examples of tridiagonal pairs, namely the Leonard pairs, Askey-Wilson polynomials arise.

Note that some of the material presented in this Chapter will play a crucial role in the

analysis of the higher-order $q-$Dolan-Grady relations of the $q-$Onsager algebra, that will be considered in Chapter 3 entitled "MAIN RESULTS".

1.1 Tridiagonal algebras

Tridiagonal algebras come up in the theory of Q-polynomial distance-regular graphs [Ter93III, Lemma 5.4] and tridiagonal pairs [ITT99, Theorem 10.1], [Ter99, Theorem 3.10]. A tridiagonal algebra has a presentation by two generators and two relations as follows

Definition 1.1.1 *[Ter99] Let $\mathbb{K}$ denote a field, and let $\beta, \gamma, \gamma^*, \delta, \delta^*$ denote a sequence of scalars taken from $\mathbb{K}$. The corresponding tridiagonal algebra T is the associative $\mathbb{K}$-algebra with 1 generated by the generators A, A^* subject to the relations*

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] = 0, \tag{1.1.1}$$
$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A] = 0. \tag{1.1.2}$$

These relations are called the tridiagonal relations.

The algebra generated by A and A^* satisfy (1.1.1), (1.1.2) is known as the subconstituent algebra or the Terwilliger algebra [Ter92, Ter93II, Ter93III]. Furthermore, relations (1.1.1) and (1.1.2) are satisfied by the generators of both the classical and quantum 'Quadratic Askey-Wilson algebra' introduced by Granovskii et al. [GLZ92].

A tridiagonal algebra depends on the parameter sequence $\beta, \gamma, \gamma^*, \delta, \delta^*$. Now we consider some cases of the tridiagonal algebras that correspond to the particular parameter sequences.

A special case of (1.1.1) and (1.1.2) occurs in the context of quantum groups. For $\beta = q^2 + q^{-2}, \gamma = \gamma^* = 0, \delta = \delta^* = 0$, the tridiagonal relations are the q-Serre relations [CP91].

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0, \tag{1.1.3}$$
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0. \tag{1.1.4}$$

Note that equations (1.1.3) and (1.1.4) are among the defining relations for the quantum affine algebra $U_q(\widehat{sl_2})$.

Another special case of (1.1.1) and (1.1.2) has come up in the context of exactly solvable models in statistical mecanics. For $\beta = 2, \gamma = \gamma^* = 0, \delta = b^2, \delta^* = b^{*2}$, the tridiagonal relations are the so-called Dolan-Grady relations [DG82]

$$[A, [A, [A, A^*]]] = b^2[A, A^*], \tag{1.1.5}$$
$$[A^*, [A^*, [A^*, A]]] = b^{*2}[A^*, A]. \tag{1.1.6}$$

One more example of a tridiagonal algebra is the q-Onsager algebra. It will be considered in the following part.

1.2 The q-Onsager algebra from different points of view

In [Ter01, Bas0404, Bas0408], the q-Onsager algebra has been defined by standard generators and relations which are called the 'q-deformed' Dolan-Grady relations. Applications of the q-Onsager algebra to tridiagonal pairs can be found in [ITT99, IT08, IT09, Ter99, Ter01]. The q-Onsager algebra has applications to quantum integrable models [Bas0404, Bas0408, Bas06, Bas0604, BK13, BK0507, BK07, BS09, BB12, BK13] and quantum symmetric pairs [Kol12].

Definition 1.2.1 *[Ter01, Bas0404, Bas0408] Let $\mathbb{K}$ denote a field. The q-Onsager algebra is the associative algebra with unit and standard generators A, A^* subject to the following relations*

$$\sum_{k=0}^{3}(-1)^k \begin{bmatrix} 3 \\ k \end{bmatrix}_q A^{3-k}A^*A^k = \rho_0[A, A^*], \tag{1.2.1}$$

$$\sum_{k=0}^{3}(-1)^k \begin{bmatrix} 3 \\ k \end{bmatrix}_q A^{*3-k}AA^{*k} = \rho_1[A^*, A], \tag{1.2.2}$$

where q is a deformation parameter, and ρ_0, ρ_1 are fixed scalars in $\mathbb{K}$.

The relations (1.2.1), (1.2.2) can be seen as ρ_i-deformed analogues of the q-Serre relations because for $\rho_0 = \rho_1 = 0$ these relations are reduced to the q-Serre relations.
Clearly, the generators A, A^* of the q-Onsager algebra satisfy the defining relations of the tridiagonal algebra corresponding to the scalars $\beta = q^2 + q^{-2}, \gamma = \gamma^* = 0$, and $\delta = \rho_0, \delta^* = \rho_1$. This parameter sequence is said to be reduced.
For simplicity, the q-Onsager algebra can be defined by generators A, A^* subject to the relations

$$[A, A^2A^* - \beta AA^*A + A^*A^2] = \rho[A, A^*], \tag{1.2.3}$$

$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2}] = \rho[A^*, A], \tag{1.2.4}$$

where $\beta = q^2 + q^{-2}$ and $\rho_0 = \rho_1 = \rho$ [IT09]. Relations (1.2.3) and (1.2.4) can be regarded as a q-analogue of the Dolan-Grady relations.

The q-Onsager algebra (1.2.3), (1.2.4) has a basis as follows [IT09].

Let r denote a positive integer. Set

$$\begin{aligned} \Lambda_r &= \{\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^{r+1} | \lambda_0 \geq 0, \lambda_i \geq 1 (1 \leq i \leq r)\}, \\ \Lambda &= \bigcup_{r \in \mathbb{N} \cup \{0\}} \Lambda_r. \end{aligned}$$

If there exists an integer i $(0 \leq i \leq r)$ such that

$$\lambda_0 < \lambda_1 < \cdots < \lambda_i \geq \lambda_{i+1} \geq \cdots \geq \lambda_r,$$

then $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ is said to be irreducible.

Denote $\Lambda^{irr} = \{\lambda \in \Lambda | \lambda \text{ is irreducible}\}$.

Let X, Y denote noncommuting indeterminates. For $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r) \in \Lambda$, put

$$\omega_\lambda(X, Y) = \begin{cases} X^{\lambda_0} Y^{\lambda_1} \ldots X^{\lambda_r} & \text{if } r \text{ is even} \\ X^{\lambda_0} Y^{\lambda_1} \ldots Y^{\lambda_r} & \text{if } r \text{ is odd} \end{cases},$$

where $X^{\lambda_0} = 1$ if $\lambda_0 = 0$.

Theorem 1.2.1 *[IT09] The following set is a basis of the q-Onsager algebra as a $\mathbb{C}$-vector space:*

$$\{\omega_\lambda(A, A^*) | \lambda \in \Lambda^{irr}\}.$$

1.2.1 The q-Onsager algebra and $U_q(\widehat{sl_2})$

There is an explicit relationship between the q-Onsager algebra and the quantum affine algebra $U_q(\widehat{sl_2})$. There are homomorphisms from the q-Onsager algebra into $U_q(\widehat{sl_2})$ [Bas0408, Bas0604, BB12, Kol12]. Firstly, recall the definition of $U_q(\widehat{sl_2})$ in Drinfeld-Jimbo presentation.

Definition 1.2.2 *[CP91] Let $\mathbb{K}$ denote an algebraically closed field. The quantum affine algebra $U_q(\widehat{sl_2})$ is the associative $\mathbb{K}$-algebra with unit 1, defined by generators $e_i^\pm, K_i^{\pm 1}, i \in \{0, 1\}$ and the following relations:*

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_0 K_1 &= K_1 K_0, \\ K_i e_i^\pm K_i^{-1} &= q^{\pm 2} e_i^\pm, \\ K_i e_j^\pm K_i^{-1} &= q^{\mp 2} e_j^\pm, \ i \neq j, \\ [e_i^+, e_i^-] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ [e_0^\pm, e_1^\mp] &= 0, \end{aligned}$$

$$(e_i^\pm)^3 e_j^\pm - [3]_q (e_i^\pm)^2 e_j^\pm e_i^\pm + [3]_q e_i^\pm e_j^\pm (e_i^\pm)^2 - e_j^\pm (e_i^\pm)^3 = 0, \ i \neq j, \tag{1.2.5}$$

where the expression $[r, s]$ means $rs - sr$. We call $e_i^\pm, K_i^\pm, i \in \{0, 1\}$ the Chevalley generators for $U_q(\widehat{sl_2})$.

Theorem 1.2.2 *[BB12] There exist algebra homomorphisms φ, φ^* from the q-Onsager algebra to $U_q(\widehat{sl_2})$*

$$\varphi(A) = k_- e_0^+ + k_+ q^{-1} e_0^- K_0 + \epsilon_- K_0, \tag{1.2.6}$$

$$\varphi(A^*) = k_+ e_1^+ + k_- q^{-1} e_1^- K_1 + \epsilon_+ K_1, \tag{1.2.7}$$

where $k_+, k_-, \epsilon_+, \epsilon_- \in \mathbb{K}$, *and* $\rho = k_+k_-(q+q^{-1})^2$.
And

$$\varphi^*(A) = \bar{k}_- q^{-1} e_0^+ K_0^{-1} + \bar{k}_+ e_0^- + \bar{\epsilon}_+ K_0^{-1}, \tag{1.2.8}$$
$$\varphi^*(A^*) = \bar{k}_+ q^{-1} e_1^+ K_0^{-1} + \bar{k}_- e_1^- + \bar{\epsilon}_- K_1^{-1}, \tag{1.2.9}$$

where $\bar{k}_+, \bar{k}_-, \bar{\epsilon}_+, \bar{\epsilon}_- \in \mathbb{K}$, *and* $\rho = \bar{k}_+\bar{k}_-(q+q^{-1})^2$.

1.2.2 The q-Onsager algebra and the $U_q(sl_2)$-loop algebra

There are algebra homomorphisms from the q-Onsager algebra into the $U_q(sl_2)$-loop algebra [IT08, IT09].

Definition 1.2.3 *[CP91] The $U_q(sl_2)$-loop algebra $\mathcal{L}$ is the associative $\mathbb{C}$-algebra with 1 generated by $e_i^+, e_i^-, k_i, k_i^{-1}$ $(i = 0, 1)$ subject to the relations*

$$\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\
k_0 k_1 &= k_1 k_0 = 1, \\
k_i e_i^{\pm} k_i^{-1} &= q^{\pm 2} e_i^{\pm}, \\
k_i e_j^{\pm} k_i^{-1} &= q^{\mp 2} e_j^{\pm}, \ i \neq j, \\
[e_i^+, e_i^-] &= \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
[e_0^{\pm}, e_1^{\mp}] &= 0,
\end{aligned}$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \ i \neq j.$$

Note that if we replace $k_0k_1 = k_1k_0 = 1$ in the definition for $\mathcal{L}$ by $k_0k_1 = k_1k_0$ then we have the quantum affine algebra $U_q(\widehat{sl_2})$. Namely, $\mathcal{L}$ is isomorphic to the quotient algebra of $U_q(\widehat{sl_2})$ by the two-sided ideal generated by $k_0k_1 - 1$.

Theorem 1.2.3 *[IT0904] For arbitrary nonzero $s, t \in \mathbb{C}$, there exists an algebra homomorphism $\varphi_{s,t}$ from the q-Onsager algebra to $\mathcal{L}$ that sends A, A^* to*

$$\begin{aligned}
A_t(s) &= x(s) + tk(s) + t^{-1}k(s)^{-1}, \\
A_t^*(s) &= y(s) + t^{-1}k(s) + tk(s)^{-1},
\end{aligned}$$

respectively, where

$$\begin{aligned}
x(s) &= \alpha(se_0^+ + s^{-1}e_1^- k_1) \ \ \textit{with } \alpha = -q^{-1}(q-q^{-1})^2, \\
y(s) &= se_0^- k_0 + s^{-1}e_1^+, \\
k(s) &= sk_0, \\
\rho &= -(q^2 - q^{-2})^2.
\end{aligned}$$

Moreover $\varphi_{s,t}$ is injective.

Theorem 1.2.4 *[Bas0604] Let $k_+, k_-, \epsilon_+, \epsilon_-$ denote scalars in $\mathbb{K}$. There is an algebra homomorphism from the q-Onsager algebra to $\mathcal{L}$ such that*

$$A \mapsto k_- e_0^+ k_0^{1/2} + k_+ e_0^- k_0^{1/2} + \epsilon_- k_0, \tag{1.2.10}$$

$$A^* \mapsto k_+ e_1^+ k_1^{1/2} + k_- e_1^- k_1^{1/2} + \epsilon_+ k_1, \tag{1.2.11}$$

with $\rho_0 = \rho_1 = k_+ k_- (q + q^{-1})^2$.

1.2.3 The q-Onsager algebra, the reflection equation and the algebra $\mathcal{A}_q$

The aim of this part is to recall the relation between the reflection equation algebra [Cher84, Sk88] and the q-Onsager algebra (1.2.1), (1.2.2). In the literature, the connection between the two algebras appeared as follows: First, the structure of the solutions the K-operators satisfying "RKRK" defining relations for the $U_q(\widehat{sl_2})$ with R-matrix (1.2.12) had been studied in details in the case where the entries of the $K-$matrix act on an irreducible finite dimensional vector space. Recall that the entries of the $K-$matrix depends on a spectral parameter u. Expanding the entries in terms of a new spectral parameter $U = (qu^2 + q^{-1}u^{-2})/(q + q^{-1})$, it was observed that the first modes of the expansion of the diagonal entries of the $K-$matrix generate a $q-$Onsager algebra [BK0503]. This will be recalled in details in Chapter 2. This observation suggested that the reflection equation algebra and the $q-$Onsager algebra are closely related.

In [BS09], this connection was further studied. Let $K(u)$ be a solution of the reflection equation algebra with $U_q(\widehat{sl_2})$ with R-matrix (1.2.12). Assume q is not a root of unity. Let V be an irreducible finite dimensional vector space on which the entries of the $K-$matrix act. Then, it is known that the solution of the reflection equation is unique (up to an overall scalar factor). Independently, consider the $q-$Onsager algebra. Let δ denote a coaction map (see the definition below) that is introduced explicitly. Then, one introduces the $K-$matrix as the intertwiner between tensor product representations of the $q-$Onsager algera. By construction, the $K-$matrix automatically satisfies the reflection equation algebra. As a consequence, for V an irreducible finite dimensional vector space, the uniqueness of the solution K implies that a quotient of the $q-$Onsager algebra is isomorphic to a quotient of the reflection equation algebra.

Below, in a first part we recall the reflection equation algebra. Then, we recall the structure of the general $K-$matrix solutions of the reflection equation algebra for the $R-$matrix in terms of currents. The modes of these currents are denoted $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{W}}_{k+1} | k \in \mathbb{Z}_+\}$ and generate the algebra $\mathcal{A}_q$. The isomorphism between the current algebra and the reflection equation algebra is given (see Theorem 1.2.6). In a second part, we independently recall the construction of a coaction map of the $q-$Onsager algebra (see Proposition 1.2.1). The defining relations of the $K-$matrix as an intertwiner of tensor product representations of the $q-$Onsager algebra is mentioned. One of the main result of [BS09] is finally recalled: if the elements of the $q-$Onsager algebra act on an irreducible finite dimensional vector space (in which case a quotient of the $q-$Onsager algebra is considered), then it is isomorphic to a quotient of the algebra $\mathcal{A}_q$ or, equivalently, to a quotient of the reflection equation algebra.

Let $R : \mathbb{C}^* \to End(\mathcal{V} \otimes \mathcal{V})$ denote the intertwining operator between the tensor product of two fundamental representations $\mathcal{V} = \mathbb{C}^2$ associated with the algebra $U_q(\widehat{sl_2})$. The element $R(u)$ depends on the deformation parameter q and is defined by [Bax82]

$$R(u) = \begin{pmatrix} uq - u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u - u^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & u - u^{-1} & 0 \\ 0 & 0 & 0 & uq - u^{-1}q^{-1} \end{pmatrix}, \tag{1.2.12}$$

where u is called the special parameter.

By construction, $R(u)$ satisfies the Yang-Baxter equation in the space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$. Namely,

$$R_{12}(u/v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u/v), \quad \forall u, v. \tag{1.2.13}$$

Definition 1.2.4 *[Sk88] Reflection equation algebra $B_q(\widehat{sl_2})$ is an associative algebra with unit 1 and generators $K_{11}(u) \equiv A(u)$, $K_{12}(u) \equiv B(u)$, $K_{21}(u) \equiv C(u)$, $K_{22}(u) \equiv D(u)$ considered as the entries of the 2×2 square matrix $K(u)$ which obeys the defining relations*

$$R_{12}(u/v)(K(u) \otimes \mathbb{I})R_{12}(uv)(\mathbb{I} \otimes K(v)) = (\mathbb{I} \otimes K(v))R_{12}(uv)(K(u) \otimes \mathbb{I})R_{12}(u/v), \quad \forall u, v. \tag{1.2.14}$$

Definition 1.2.5 *[BS09] The current algebra $O_q(\widehat{sl_2})$ is an associative algebra with unit 1, current generators $\mathcal{W}_\pm(u)$, $\mathcal{G}_\pm(u)$ and parameter $\rho \in \mathbb{C}^*$. Define the formal variables $U = (qu^2 + q^{-1}u^{-2})/(q + q^{-1})$ and $V = (qv^2 + q^{-1}v^{-2})/(q + q^{-1})$, $\forall u, v$. The defining relations are:*

$$[\mathcal{W}_\pm(u), \mathcal{W}_\pm(v)] = 0, \tag{1.2.15}$$

$$[\mathcal{W}_+(u), \mathcal{W}_-(v)] + [\mathcal{W}_-(u), \mathcal{W}_+(v)] = 0, \tag{1.2.16}$$

$$\begin{aligned} (U - V)[\mathcal{W}_\pm(u), \mathcal{W}_\mp(v)] &= \frac{(q - q^{-1})}{\rho(q + q^{-1})}(\mathcal{G}_\pm(u)\mathcal{G}_\mp(v) - \mathcal{G}_\pm(v)\mathcal{G}_\mp(u)) \\ &+ \frac{1}{q + q^{-1}}(\mathcal{G}_\pm(u) - \mathcal{G}_\mp(u) + \mathcal{G}_\mp(v) - \mathcal{G}_\pm(v)), \end{aligned} \tag{1.2.17}$$

$$\begin{aligned} \mathcal{W}_\pm(u)\mathcal{W}_\pm(v) &- \mathcal{W}_\mp(u)\mathcal{W}_\mp(v) + \frac{1}{\rho(q^2 - q^{-2})}[\mathcal{G}_\pm(u), \mathcal{G}_\mp(v)] \\ &+ \frac{1 - UV}{U - V}(\mathcal{W}_\pm(u)\mathcal{W}_\mp(v) - \mathcal{W}_\pm(v)\mathcal{W}_\mp(u)) = 0, \end{aligned} \tag{1.2.18}$$

$$\begin{aligned} U[\mathcal{G}_\mp(v), \mathcal{W}_\pm(u)]_q &- V[\mathcal{G}_\mp(u), \mathcal{W}_\pm(v)]_q - (q - q^{-1})(\mathcal{W}_\mp(u)\mathcal{G}_\mp(v) - \mathcal{W}_\mp(v)\mathcal{G}_\mp(u)) \\ &+ \rho(U\mathcal{W}_\pm(u) - V\mathcal{W}_\pm(v) - \mathcal{W}_\mp(u) + \mathcal{W}_\mp(v)) = 0, \end{aligned} \tag{1.2.19}$$

$$\begin{aligned} U[\mathcal{W}_\mp(u), \mathcal{G}_\mp(v)]_q &- V[\mathcal{W}_\mp(v), \mathcal{G}_\mp(u)]_q - (q - q^{-1})(\mathcal{W}_\pm(u)\mathcal{G}_\mp(v) - \mathcal{W}_\pm(v)\mathcal{G}_\mp(u)) \\ &+ \rho(U\mathcal{W}_\mp(u) - V\mathcal{W}_\mp(v) - \mathcal{W}_\pm(u) + \mathcal{W}_\pm(v)) = 0, \end{aligned} \tag{1.2.20}$$

$$[\mathcal{G}_\epsilon(u), \mathcal{W}_\pm(v)] + [\mathcal{W}_\pm(u), \mathcal{G}_\epsilon(v)] = 0, \ \forall \epsilon = \pm, \tag{1.2.21}$$

$$[\mathcal{G}_\pm(u), \mathcal{G}_\pm(v)] = 0, \tag{1.2.22}$$

$$[\mathcal{G}_+(u), \mathcal{G}_-(v)] + [\mathcal{G}_-(u), \mathcal{G}_+(v)] = 0. \tag{1.2.23}$$

Theorem 1.2.5 *[BS09] The map $\Phi : B_q(\widehat{sl_2}) \longrightarrow O_q(\widehat{sl_2})$ defined by*

$$A(u) = uq\mathcal{W}_+(u) - u^{-1}q^{-1}\mathcal{W}_-(u), \tag{1.2.24}$$

$$D(u) = uq\mathcal{W}_-(u) - u^{-1}q^{-1}\mathcal{W}_+(u), \tag{1.2.25}$$

$$B(u) = \frac{1}{k_-(q+q^{-1})}\mathcal{G}_+(u) + \frac{k_+(q+q^{-1})}{q-q^{-1}}, \tag{1.2.26}$$

$$C(u) = \frac{1}{k_+(q+q^{-1})}\mathcal{G}_-(u) + \frac{k_-(q+q^{-1})}{q-q^{-1}} \tag{1.2.27}$$

is an algebra isomorphism, where $q \neq 1, u \neq q^{-1}$ and $k_\pm \in \mathbb{C}^*$.

Definition 1.2.6 *[BS09] $\mathcal{A}_q$ is an associative algebra with parameter $\rho \in \mathbb{C}^*$, unit 1 and generators $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1} | k \in \mathbb{Z}_+\}$ satisfying the following relations:*

$$[\mathcal{W}_0, \mathcal{W}_{k+1}] = [\mathcal{W}_{-k}, \mathcal{W}_1] = \frac{1}{q+q^{-1}}(\tilde{\mathcal{G}}_{k+1} - \mathcal{G}_{k+1}), \tag{1.2.28}$$

$$[\mathcal{W}_0, \mathcal{G}_{k+1}]_q = [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_0]_q = \rho\mathcal{W}_{-k-1} - \rho\mathcal{W}_{k+1}, \tag{1.2.29}$$

$$[\mathcal{G}_{k+1}, \mathcal{W}_1]_q = [\mathcal{W}_1, \tilde{\mathcal{G}}_{k+1}]_q = \rho\mathcal{W}_{k+2} - \rho\mathcal{W}_{-k}, \tag{1.2.30}$$

$$[\mathcal{W}_{-k}, \mathcal{W}_{-l}] = 0, \quad [\mathcal{W}_{k+1}, \mathcal{W}_{l+1}] = 0, \tag{1.2.31}$$

$$[\mathcal{W}_{-k}, \mathcal{W}_{l+1}] + [\mathcal{W}_{k+1}, \mathcal{W}_{-l}] = 0, \tag{1.2.32}$$

$$[\mathcal{W}_{-k}, \mathcal{G}_{l+1}] + [\mathcal{G}_{k+1}, \mathcal{W}_{-l}] = 0, \tag{1.2.33}$$

$$[\mathcal{W}_{-k}, \tilde{\mathcal{G}}_{l+1}] + [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{-l}] = 0, \tag{1.2.34}$$

$$[\mathcal{W}_{k+1}, \mathcal{G}_{l+1}] + [\mathcal{G}_{k+1}, \mathcal{W}_{l+1}] = 0, \tag{1.2.35}$$

$$[\mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{l+1}] + [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{l+1}] = 0, \tag{1.2.36}$$

$$[\mathcal{G}_{k+1}, \mathcal{G}_{l+1}] = 0, \quad [\tilde{\mathcal{G}}_{k+1}, \tilde{\mathcal{G}}_{l+1}] = 0, \tag{1.2.37}$$

$$[\tilde{\mathcal{G}}_{k+1}, \mathcal{G}_{l+1}] + [\mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{l+1}] = 0, \tag{1.2.38}$$

where $\mathbb{Z}_+$ is the set of all nonnegative integers.

By analogy with Drinfeld's construction, we are now looking for an infinite dimensional set of elements (the so-called 'modes') of an algebra in terms of which the currents $\mathcal{W}_\pm(u)$, $\mathcal{G}_\pm(u)$ are expanded.

Theorem 1.2.6 *[BS09] Define the formal variable $U = (qu^2 + q^{-1}u^{-2})/(q+q^{-1})$. Let $\Psi : O_q(\widehat{sl_2}) \longrightarrow \mathcal{A}_q$ be the map defined by*

$$\mathcal{W}_+(u) = \sum_{k\in\mathbb{Z}_+} \mathcal{W}_{-k}U^{-k-1}, \quad \mathcal{W}_-(u) = \sum_{k\in\mathbb{Z}_+} \mathcal{W}_{k+1}U^{-k-1}, \tag{1.2.39}$$

$$\mathcal{G}_+(u) = \sum_{k\in\mathbb{Z}_+} \mathcal{G}_{k+1}U^{-k-1}, \quad \mathcal{G}_-(u) = \sum_{k\in\mathbb{Z}_+} \tilde{\mathcal{G}}_{k+1}U^{-k-1}. \tag{1.2.40}$$

Then, Ψ is an algebra isomorphism between $O_q(\widehat{sl_2})$ and $\mathcal{A}_q$.

In [J85, J86], Jimbo pointed out that intertwiners R of quantum loop algebra lead to trigonometric solutions of the quantum Yang-Baxter equation (1.2.13). Any tensor product of two evaluation representations with generic evaluation parameters u and v being indecomposable, by Schur's lemma the solution R is unique up to an overall scalar factor. Consider the quantum affine algebra $U_q(\widehat{sl_2})$ the construction of the solution $R(u)$ given by (1.2.12) goes below. Similar arguments apply to solutions of the reflection equation algebra. If the vector space on which the entries of the $K-$matrix act is irreducible, then the solution is unique (up to an overall factor).

We now turn to the construction of intertwiners between representations of the $q-$Onsager algebra. First we recall some basic ingredients that are necessary for the discussion.

Recall the realization of the quantum affine algebra $U_q(\widehat{sl_2})$ in the Chevalley presentation $\{H_j, E_j, F_j\}, j \in \{0,1\}$ [CP91]

Definition 1.2.7 *Define the extended Cartan matrix* $\{a_{ij}\}$ $(a_{ii} = 2, a_{ij} = -2$ *for* $i \neq j)$. *The quantum affine algebra* $U_q(\widehat{sl_2})$ *is generated by the elements* $\{H_j, E_j, F_j\}, j \in \{0,1\}$ *which satisfy the defining relations*

$$[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \tag{1.2.41}$$

$$[H_i, F_j] = -a_{ij} F_j, \quad [E_i, F_j] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \tag{1.2.42}$$

together with the q-Serre relations

$$[E_i, [E_i, [E_i, E_j]_q]_{q^{-1}}] = 0, \tag{1.2.43}$$

$$[F_i, [F_i, [F_i, F_j]_q]_{q^{-1}}] = 0. \tag{1.2.44}$$

Clearly, this realization of $U_q(\widehat{sl_2})$ is equivalent to the one in Definition 1.2.2. We endow the quantum group with a Hopf algebra structure that is ensured by the existence of a coproduct $\Delta : U_q(\widehat{sl_2}) \longrightarrow U_q(\widehat{sl_2}) \otimes U_q(\widehat{sl_2})$, an antipode $\mathcal{S} : U_q(\widehat{sl_2}) \longrightarrow U_q(\widehat{sl_2})$ and a counit $\mathcal{E} : U_q(\widehat{sl_2}) \longrightarrow \mathbb{C}$ with

$$\Delta(E_i) = E_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i, \tag{1.2.45}$$

$$\Delta(F_i) = F_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes F_i, \tag{1.2.46}$$

$$\Delta(H_i) = H_i \otimes \mathbb{I} + \mathbb{I} \otimes H_i, \tag{1.2.47}$$

$$\mathcal{S}(E_i) = -E_i q^{H_i}, \quad \mathcal{S}(F_i) = -q^{H_i} F_i, \quad \mathcal{S}(H_i) = -H_i, \quad \mathcal{S}(\mathbb{I}) = 1, \tag{1.2.48}$$

and

$$\mathcal{E}(E_i) = \mathcal{E}(F_i) = \mathcal{E}(H_i) = 0, \quad \mathcal{E}(\mathbb{I}) = 1. \tag{1.2.49}$$

Note that the opposite coproduct Δ' can be similarly defined with $\Delta' \equiv \sigma \circ \Delta$ where the permutation map $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in U_q(\widehat{sl_2})$ is used.

Recall that, by definition the intertwiner $R(u/v) : \mathcal{V}_u \otimes \mathcal{V}_v \longrightarrow \mathcal{V}_v \otimes \mathcal{V}_u$ between two fundamental $U_q(\widehat{sl_2})$- evaluation representation obeys

$$R(u/v)(\pi_u \times \pi_v)[\Delta(x)] = (\pi_u \times \pi_v)[\Delta'(x)] R(u/v) \quad \forall x \in U_q(\widehat{sl_2}), \tag{1.2.50}$$

where the (evaluation) homomorphism $\pi_u : U_q(\widehat{sl_2}) \longrightarrow \mathrm{End}(\mathcal{V}_u)$ is chosen such that ($\mathcal{V} \equiv \mathbb{C}^2$)

$$\pi_u(E_1) = uq^{1/2}\sigma_+, \quad \pi_u(E_0) = uq^{1/2}\sigma_-, \tag{1.2.51}$$
$$\pi_u(F_1) = u^{-1}q^{-1/2}\sigma_-, \quad \pi_u(F_0) = u^{-1}q^{-1/2}\sigma_+, \tag{1.2.52}$$
$$\pi_u(q^{H_1}) = q^{\sigma_z}, \quad \pi_u(q^{H_0}) = q^{-\sigma_z}. \tag{1.2.53}$$

It is not difficult to check the matrix $R(u)$ given by (1.2.12) indeed satisfies the required conditions (1.2.50). The intertwiner $R(u)$ is unique (up to an overall scalar factor) and satisfies the Yang-Baxter algebra (1.2.13).

By analogy with the construction described above for the R-matrix and along the lines described in [DeG02, DeM01], an intertwiner for the q-Onsager algebra can be easily constructed. Before, we need to introduce the concept of the comodule algebra using the analogue of the Hopf's algebra coproduct action called the coaction map.

Definition 1.2.8 *[CP91] Given a Hopf algebra $\mathcal{H}$ with comultiplication Δ and counit $\mathcal{E}$, $\mathcal{I}$ is called a left $\mathcal{H}$-comodule if there exists a left coaction map $\delta : \mathcal{I} \longrightarrow \mathcal{H} \otimes \mathcal{I}$ such that*

$$(\Delta \times id) \circ \delta = (id \times \delta) \circ \delta, \quad (\mathcal{E} \times id) \circ \delta \cong id. \tag{1.2.54}$$

Right $\mathcal{H}$-comodules are defined similarly.

Proposition 1.2.1 *[Bas0404] Let $k_\pm \in \mathbb{C}^*$ and set $\rho_1 = \rho_2 = k_+k_-(q+q^{-1})^2$. The q-Onsager algebra $\mathbb{T}$ is a left $U_q(\widehat{sl_2})$-comodule algebra with coaction map $\delta : \mathbb{T} \longrightarrow U_q(\widehat{sl_2}) \otimes \mathbb{T}$ such that*

$$\delta(A) = (k_+E_1q^{H_1/2} + k_-F_1q^{H_1/2}) \otimes \mathbb{I} + q^{H_1} \otimes A, \tag{1.2.55}$$
$$\delta(A^*) = (k_-E_0q^{H_0/2} + k_+F_0q^{H_0/2}) \otimes \mathbb{I} + q^{H_0} \otimes A^*. \tag{1.2.56}$$

Proposition 1.2.2 *[BS09] Let $\pi_u : U_q(\widehat{sl_2}) \longrightarrow End(\mathcal{V}_u)$ be the evaluation homomorphism for $\mathcal{V} \equiv \mathbb{C}^2$. Let W denote a vector space over $\mathbb{C}$ on which the elements of the q-Onsager algebra $\mathbb{T}$ act. Assume the tensor product $\mathcal{V}_u \otimes W$ is irreducible. Then, there exists an intertwiner*

$$K(u) : \mathcal{V}_u \otimes W \longrightarrow \mathcal{V}_{u^{-1}} \otimes W, \tag{1.2.57}$$

such that

$$K(u)(\pi_u \times id)[\delta(a)] = (\pi_{u^{-1}} \times id)[\delta(a)]K(u), \quad \forall a \in \mathbb{T}. \tag{1.2.58}$$

It is unique (up to an overall scalar factor), and it satisfies the reflection equation (1.2.14).

According to Proposition (1.2.2), $K(u)$ is the unique intertwiner of the q-Onsager algebra $\mathbb{T}$ satisfying (1.2.58). By construction, it satisfies the reflection equation algebra (1.2.14). This implies that a quotient of the q-Onsager algebra is isomorphic to a quotient of the reflection equation algebra or, alternatively, a quotient of infinite dimensional algebra $\mathcal{A}_q$. Combining these results, it suggests that the q-Onsager algebra with $\rho_0 = \rho_1 = \rho$ has two different realizations: one in terms of the reflection equation algebra for the $U_q(\widehat{sl_2})$ R-matrix and another one in terms of the current algebra $O_q(\widehat{sl_2}) \cong \mathcal{A}_q$.

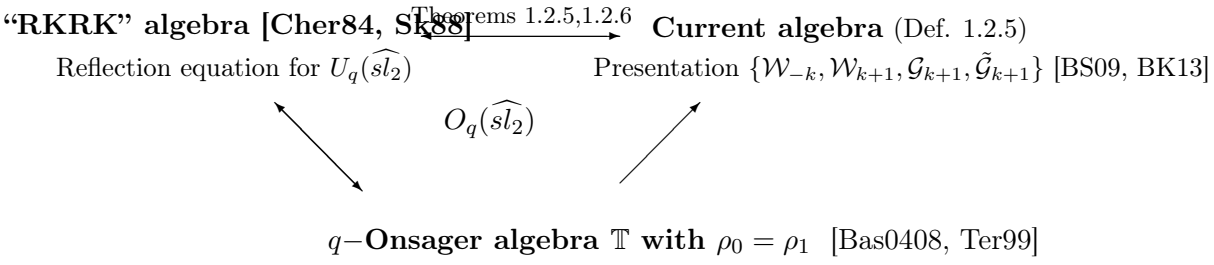

Figure 1.1: Presentations of $O_q(\widehat{sl_2})$

1.3 Representation theory

The concept of a tridiagonal pair originated in algebraic graph theory, or more precisely, the theory of Q-polynomial distance-regular graphs. The standard generators for the subconstituent algebra (Terwilliger algebra) of a P- and Q- polynomial association scheme give rise to the concept of tridiagonal pair when they are restricted to an irreducible submodule of the standard module [ITT99, Example 1.4], [Ter92, Lemmas 3.9, 3.12]. This fact motivates the ongoing investigation of the tridiagonal pairs [ITT99, INT10, IT07, IT03, NT08, Ter99].

It is now well understood that the representation theory of tridiagonal algebras - in particular the $q-$Onsager algebra - is intimately connected with the theory of tridiagonal pairs. For instant, for an algebraically closed field and no restrictions on q, note that a classification of tridiagonal pairs is given in [INT10] (see also [IT08]). In this Section, we first recall what is a Leonard pair, and then turn to tridiagonal pairs. The material is taken from the work of Terwilliger et al.

1.3.1 Leonard pairs

We recall the notion of a Leonard pair, a special case of the tridiagonal pair [ITT99, Ter03, Ter01, Ter99] and illustrate how Leonard pairs arise in representation theory, combinatorics, and the theory of orthogonal polynomials [Ter03].

Definition 1.3.1 *[Ter01] Let V denote a finite dimensional vector space over $\mathbb{K}$. By a Leonard pair we mean an ordered pair of linear transformations $A : V \to V$, and $A^* : V \to V$ that satisfy the following conditions.*

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

Note that a matrix is called tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be irre-

ducible if all entries on the subdiagonal and superdiagonal are nonzero.

Suppose that V has dimension $d+1$. Write $End(V)$ is the $\mathbb{K}$-algebra consisting of linear transformations from V to V. Now let $A \in End(V)$, suppose that A has $(n+1)$ distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_n$. If $n = d$ and the eigenvalues $\theta_0, \theta_1, \ldots, \theta_d \in \mathbb{K}$ then A is said to be multiplicity-free. Assume A is multiplicity-free, put

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \quad 0 \le i \le d,$$

where I denotes the identity of $End(V)$.

Apparently, by the elementary algebra,

$$AE_i = E_i A = \theta_i E_i, \quad 0 \le i \le d, \tag{1.3.1}$$

$$E_i E_j = \delta_{ij} E_i, \quad 0 \le i, j \le d, \tag{1.3.2}$$

$$\sum_{i=0}^{d} E_i = I, \qquad \sum_{i=0}^{d} \theta_i E_i = A, \tag{1.3.3}$$

$$V_i = E_i V \text{ is eigenspace of } A \text{ corresponding with the eigenvalue } \theta_i, \tag{1.3.4}$$

$$E_0, E_1, \ldots, E_d \text{ is a basic of subalgebra of } End(V) \text{ generated by } A. \tag{1.3.5}$$

We call E_i the primitive idempotent of A associated with θ_i. As a result,

$$V = E_0 V + E_1 V + \cdots + E_d V \qquad \text{(direct sum)}.$$

Lemma 1.3.1 *[Ter01] Let V denote a finite dimensional vector space over a field $\mathbb{K}$. Let (A, A^*) denote a Leonard pair on V. Then each of A, A^* is multiplicity-free.*

When working with a Leonard pair, it is often convenient to consider a closely related object called a Leonard system.

Definition 1.3.2 *[Ter01] Let d denote a nonnegative integer. Let V denote a finite dimensional vector space over a field $\mathbb{K}$ with dimension $d+1$. By a Leonard system on V, we mean a sequence*

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*),$$

that satisfy the following conditions
(i) A, A^* are both multiplicity-free elements.
(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of A.
(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of A^*.
(iv) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \le i, j \le d).$
(v) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \le i, j \le d).$

Ones refer to d as the diameter of Φ, and say Φ is over $\mathbb{K}$.

Remark 1 *Let $\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$ denote a Leonard system on V, and let φ denote an automorphism of $End(V)$. Then $\Psi = (\varphi(A); \varphi(E_0), \varphi(E_1), \ldots, \varphi(E_d); \varphi(A^*); \varphi(E_0^*), \varphi(E_1^*), \ldots, \varphi(E_d^*))$ is also a Leonard system on V.*

The Leonard pairs Φ and Ψ are said to be isomorphic whenever there exists an automorphism of $End(V)$ such that $\varphi(\Phi) = \Psi$.

A given Leonard system can be modified in several ways to get a new Leonard system. For instant, let $\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$ denote a Leonard system on V, let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in $\mathbb{K}$ such that $\alpha \neq 0, \alpha^* \neq 0$. Then,

$$(\alpha A + \beta I; E_0, E_1, \ldots, E_d; \alpha^* A^* + \beta^* I; E_0^*, E_1^*, \ldots, E_d^*)$$

is a Leonard system on V. Also,

$$\Phi^* := (A^*; E_0^*, E_1^*, \ldots, E_d^*; A; E_0, E_1, \ldots, E_d), \tag{1.3.6}$$

$$\Phi^{\downarrow} := (A; E_0, E_1, \ldots, E_d; A^*; E_d^*, E_{d-1}^*, \ldots, E_0^*), \tag{1.3.7}$$

$$\Phi^{\Downarrow} := (A; E_d, E_{d-1}, \ldots, E_0; A^*; E_0^*, E_1^*, \ldots, E_d^*) \tag{1.3.8}$$

are Leonard systems on V.

From the conditions (ii) and (iv) of Definition 1.3.2, with respect to an appropriate basis consisting of eigenvectors for A, the matrix representing A^* is irreducible tridiagonal. Similarly, from the conditions (iii) and (v) of Definition 1.3.2, with respect to an appropriate basis consisting of eigenvectors for A^*, the matrix representing A is irreducible tridiagonal. It means that in the Leonard system Φ, (A, A^*) is a Leonard pair. Inversely, if (A, A^*) is a Leonard pair, and $E_i, i = 0, \ldots, d$ (resp.E_i^*) denotes an ordering of the primitive idempotents of A (resp.A^*) corresponding with the basis of V in the condition (i) (resp.(ii)) of Definition 1.3.1, then $\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$ is a Leonard system.

Let (A, A^*) be a Leonard pair on V. Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote eigenvalues of A (resp. A^*) such that the corresponding eigenvectors $v_0, v_1, \ldots, v_d$ (resp. $v_0^*, v_1^*, \ldots, v_d^*$) satisfy the condition *(i)* (resp. *(ii)*) in Definition 1.3.1. We refer to θ_0,, θ_1, $\ldots$, θ_d as the eigenvalue sequence of Φ, and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ as the dual eigenvalue sequence of Φ. Let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*), $0 \leq i \leq d$. Let $a_0, a_1, \ldots, a_d$ be the diagonal of the matrix representing A with respect to the basis $v_0^*, v_1^*, \ldots, v_d^*$. Let $a_0^*, a_1^*, \ldots, a_d^*$ be the diagonal of the matrix representing A^* with respect to the basis $v_0, v_1, \ldots, v_d$. As we know, $a_i = tr(E_i^* A)$, $a_i^* = tr(E_i A^*)$ for $0 \leq i \leq d$.

Theorem 1.3.1 *[Ter0306] Let Φ denote a Leonard system from Definition 1.3.2. Then the elements $A^r E_0^* A^s$ $(0 \leq r, s \leq d)$ form a basis for the $\mathbb{K}$-vector space $End(V)$.*

Corollary 1.3.1 *Let Φ denote a Leonard system from Definition 1.3.2. Then the elements A, E_0^* together generate $End(V)$. Moreover the elements A, A^* together generate $End(V)$.*

Corollary 1.3.2 *Let Φ denote a Leonard system from Definition 1.3.2. Let D denote the subalgebra of $End(V)$ generated by A. Let $X_0, X_1, \ldots, X_d$ denote a basis for the $\mathbb{K}$-vector space D. Then the elements $X_r E_0^* X_s$ $(0 \leq r, s \leq d)$ form a basis for the $\mathbb{K}$-vector space $End(V)$.*

Lemma 1.3.2 *For $0 \leq i, j \leq d$, the following hold*

(i) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1, \\ a_i E_i^* & \text{if } i = j. \end{cases}$

(ii) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1, \\ a_i^* E_i & \text{if } i = j. \end{cases}$

where $a_i = tr(E_i^* A), a_i^* = tr(E_i A^*)$ $(0 \leq i \leq d)$

Lemma 1.3.3 *[TV03] For $0 \leq r, s, i, j \leq d$, the following hold*

(i) $E_i^* A^r E_j^* = \begin{cases} 0 & \text{if } 0 \leq r < |i-j|, \\ \neq 0 & \text{if } r = |i-j|. \end{cases}$

(ii) $E_i^* A^r A^* A^s E_j^* = \begin{cases} \theta_{j+s}^* E_i^* A^{r+s} E_j^*, & \text{if } i-j = r+s, \\ \theta_{j-s}^* E_i^* A^{r+s} E_j^*, & \text{if } j-i = r+s, \\ 0, & \text{if } |i-j| > r+s. \end{cases}$

Obviously, the role of A and A^* in the Leonard pair (A, A^*) are the same, hence the properties similar to Lemma 1.3.3 are also given for A^*.

Based on the relations between A, A^* and the primitive idempotents E_i, E_i^*, the eigenvalues θ_i, θ_i^*, P. Terwilliger found elements of the algebra generated by A, A^* commuting with elements in the subalgebra of $End(V)$ generated by A and elements in the subalgebra of $End(V)$ generated by A^*. As a result, A, A^* satisfy the defining relations of a tridiagonal algebra.

Theorem 1.3.2 *[Ter99] Let V denote a finite dimensional vector space over a field $\mathbb{K}$. Let (A, A^*) denote a Leonard pair on V. There exists a sequence of scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in $\mathbb{K}$ such that A, A^* satisfy the tridiagonal relations (1.1.1), (1.1.2). Furthermore, if the dimension of V is at least 3, then the sequence is uniquely determined.*

The eigenvalues θ_i (resp. θ_i^*) of A (resp. A^*) satisfy recurrence relations below

(i) The expressions $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}$ and $\frac{\theta_{i-2}^*-\theta_{i+1}^*}{\theta_{i-1}^*-\theta_i^*}$ are independent of i and equal to $\beta + 1$, for all $2 \leq i \leq d-1$.

(ii) $\theta_{i+1} - \beta\theta_i + \theta_{i-1} = \gamma, \quad \theta_{i+1}^* - \beta\theta_i^* + \theta_{i-1}^* = \gamma^*, \quad 1 \leq i \leq d.$

(iii) $\theta_i^2 - \beta\theta_i\theta_{i-1} + \theta_{i-1}^2 - \gamma(\theta_i + \theta_{i-1}) = \delta, \quad \theta^{*2}_i - \beta\theta_i^*\theta_{i-1}^* + \theta^{*2}_{i-1} - \gamma^*(\theta_i^* + \theta_{i-1}^*) = \delta^*, \quad 1 \leq i \leq d.$

Theorem 1.3.2 is extended as follows.

Theorem 1.3.3 *[TV03] Let V denote a finite dimensional vector space over a field $\mathbb{K}$. Let (A, A^*) denote a Leonard pair on V. There exists a sequence of scalars $\beta, \gamma, \gamma^*, \delta, \delta^*, \omega, \eta, \eta^*$ in $\mathbb{K}$ such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = \gamma^*A^2 + \omega A + \eta I, \quad (1.3.9)$$
$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^*A = \gamma A^{*2} + \omega A^* + \eta^* I. \quad (1.3.10)$$

If the dimension of V is at least 4, then the sequence is uniquely determined.

The relations (1.3.9), (1.3.10) are called the Askey-Wilson relations.

There is no doubt that if A, A^* is a Leonard pair then A, A^* satisfy the tridiagonal relations and the scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in Theorem 1.3.3 coincide with the scalars of the tridiagonal relations. If the dimension of V is at least 4, then they are uniquely defined by the recurrent relations. By [TV03] for the scalars ω, η, η^*, they are obtained by the following
(i) $\omega = a_i^*(\theta_i - \theta_{i+1}) + a_{i-1}^*(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_i + \theta_{i-1}), \quad 2 \le i \le d-1.$
(ii) $\omega = a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_{i-2}^*) - \gamma(\theta_i^* + \theta_{i-1}^*), \quad 2 \le i \le d-1.$
(iii) $\eta = a_i^*(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i, \quad 1 \le i \le d-1.$
(iv) $\eta^* = a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma\theta^{*2}_i - \omega\theta_i^*, \quad 1 \le i \le d-1.$

We now show conditions in order to get a Leonard pair. Indeed, it is that a kind of converse of Theorem 1.3.3.

Theorem 1.3.4 *[TV03] Let V denote a finite dimensional vector space on a field $\mathbb{K}$. Let A, A^* denote linear transformations from V to V. Suppose that:*
(i) There exists a sequence of scalars $\beta, \gamma, \gamma^*, \delta, \delta^*, \omega, \eta, \eta^*$ in $\mathbb{K}$ which satisfy (1.3.9), (1.3.10).
(ii) There exists a scalar q not a root of unity such that $q + q^{-1} = \beta$.
(iii) Each of A, A^* is multiplicity-free.
(iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W, A^*W \subseteq W$ other than 0 and V.
Then (A, A^*) is a Leonard pair on V.

Lemma 1.3.4 *[Nom05] Let A, A^* denote linear transformations from V to V. Suppose*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = P(A)$$

holds for some polynomial $P(x)$ in $\mathbb{K}(x)$, and for some scalars β, γ, δ in $\mathbb{K}$. The eigenspaces of A and the eigenspaces of A^* all have one common dimension.
Furthermore if $\mathbb{K}$ is algebraically closed or q is not a root of unity, where $\beta = q + q^{-1}$, then the eigenspaces of A and the eigenspaces of A^* all have dimension 1.

Theorem 1.3.5 *[Nom05] Suppose $\mathbb{K}$ is algebraically closed. Then the following are equivalent.*
i) (A, A^*) is a Leonard pair on V.
ii) A, A^* satisfy the Askey-Wilson relations (1.3.9)-(1.3.10) for some scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$, ω, η, η^*.

Now we move on to important properties about the classification of the Leonard pairs.

Theorem 1.3.6 *[Ter01] Let V denote a finite dimensional vector space over a field $\mathbb{K}$. Let $\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$ be a Leonard system. There exists an automorphism ψ of $End(V)$, and there exist scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathbb{K}$ such that*

$$\psi(A) = \begin{bmatrix} \theta_0 & & & & & 0 \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_d \end{bmatrix}, \quad \psi(A^*) = \begin{bmatrix} \theta_0^* & \varphi_1 & & & & 0 \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta_d^* \end{bmatrix}.$$

The sequence $\psi, \varphi_1, \ldots, \varphi_d$ is uniquely determined by Φ. Moreover, $\varphi_i \neq 0$ for $1 \leq i \leq d$.

The Leonard system $\psi(\Phi) = (\psi(A); \psi(E_0), \psi(E_1), \ldots, \psi(E_d); \psi(A^*); \psi(E_0^*), \psi(E_1^*), \ldots, \psi(E_d^*))$ is called the split canonical form of Φ, and the sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ is called the first split sequence of Φ.

Lemma 1.3.5 *Let Φ and Φ' denote Leonard systems over $\mathbb{K}$. Then the following are equivalent:*
(i) Φ and Φ' are isomorphic.
(ii) Φ and Φ' share the same eigenvalue sequence, dual eigenvalue sequence, and first split sequence.

Let $\phi_1, \phi_2, \ldots, \phi_d$ denote the first split sequence of $\Phi^{\Downarrow}$. We call $\phi_1, \phi_2, \ldots, \phi_d$ the second split sequence of Φ.

Lemma 1.3.6 *Let Φ and Φ' denote Leonard systems over $\mathbb{K}$. Then the following are equivalent:*
(i) Φ and Φ' are isomorphic.
(ii) Φ and Φ' share the same eigenvalue sequence, dual eigenvalue sequence, and second split sequence.

Theorem 1.3.7 *[Ter01] Let d denote a nonnegative integer, let $\mathbb{K}$ denote a field, and let*

$$\theta_0, \theta_1, \ldots, \theta_d; \quad \theta_0^*, \theta_1^*, \ldots, \theta_d^*;$$

$$\varphi_1, \varphi_2, \ldots, \varphi_d; \quad \phi_1, \phi_2, \ldots, \phi_d$$

denote scalars in $\mathbb{K}$. Then there exists a Leonard system Φ over $\mathbb{K}$ with eigenvalues sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$ if and only if (i)-(v) hold below:
(i) $\varphi_i \neq 0, \ \phi_i \neq 0 \ \ (1 \leq i \leq d)$.
(ii) $\theta_i \neq \theta_j, \ \theta_i^* \neq \theta_j^*, \ \ \text{if } i \neq j \ \ (0 \leq i, j \leq d)$.
(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \ \ (1 \leq i \leq d)$.

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \le i \le d)$.
(v) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \le i \le d-1$.
Moreover, if (i)-(v) hold above, then Φ is unique up to isomorphism of Leonard systems.

We display the 'most general' solution for the parameters in Theorem 1.3.7.

For $0 \le i \le d$,

$$\theta_i = \theta_0 + \frac{h(1-q^i)(1-sq^{i+1})}{q^i}, \tag{1.3.11}$$

$$\theta_i^* = \theta_0^* + \frac{h^*(1-q^i)(1-s^*q^{i+1})}{q^i}. \tag{1.3.12}$$

For $1 \le i \le d$,

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i), \tag{1.3.13}$$

$$\phi_i = \frac{hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1-s^*q^i)(r_2-s^*q^i)}{s^*}, \tag{1.3.14}$$

where $q, h, h^*, r_1, r_2, s, s^*$ are nonzero scalars in the algebraic closure of $\mathbb{K}$ such that $r_1r_2 = ss^*q^{d+1}$. For this solution the common value of (v) in Theorem 1.3.7 equals $q + q^{-1} + 1$.

Corollary 1.3.3 *Let Φ denote a Leonard system over $\mathbb{K}$ with diameter $d \ge 3$, eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$. Consider a sequence $\mathcal{L}$ of nine parameters consisting of $\theta_0, \theta_1, \theta_2, \theta_3, \theta_0^*, \theta_1^*, \theta_2^*, \theta_3^*$ and one of the parameters $\varphi_1, \varphi_d, \phi_1, \phi_d$. Then the isomorphism class of Φ as a Leonard system over $\mathbb{K}$ is determined by $\mathcal{L}$.*

Corollary 1.3.4 *Let d denote a nonnegative integer, and let A and A^* denote linear transformations in $End(V)$ of the form*

$$A = \begin{bmatrix} \theta_0 & & & & & 0 \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_d \end{bmatrix}, \quad A^* = \begin{bmatrix} \theta_0^* & \varphi_1 & & & & 0 \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta_d^* \end{bmatrix}.$$

Then the following are equivalent:
(i) (A, A^*) is a Leonard pair on V.
(ii) There exists a sequence of scalars $\phi_1, \phi_2, \ldots, \phi_d$ taken from $\mathbb{K}$ such that (i)-(v) hold in Theorem 1.3.7.

Suppose (i) and (ii) hold above. Then

$$(A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$$

is a Leonard system on V, where E_i (resp. E_i^*) denotes the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*), for $0 \leq i \leq d$. The Leonard system has eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and sequence split sequence $\phi_1, \phi_2, \ldots, \phi_d$.

Leonard pairs arise naturally in Lie algebra. Consider the remark below.

Definition 1.3.3 *Let $\mathbb{K}$ denote an algebraically closed field with characteristic zero. The algebra $sl_2(\mathbb{K})$ is a Lie algebra which has a basis e, f, h satisfying*

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h,$$

where $[,]$ denotes the Lie bracket.

Lemma 1.3.7 *[Kas95] There exists a family V_d, $d = 0, 1, 2 \ldots$ of irreducible finite dimensional $sl_2(\mathbb{K})$-modules such that the module V_d has a basis $v_0, v_1, \ldots, v_d$ satisfying*

$$hv_i = (d-2i)v_i \text{ for } 0 \leq i \leq d, \tag{1.3.15}$$

$$fv_i = (i+1)v_{i+1} \text{ for } 0 \leq i \leq d-1, \quad fv_d = 0, \tag{1.3.16}$$

$$ev_i = (d-i+1)v_{i-1} \text{ for } 1 \leq i \leq d, \quad ev_0 = 0. \tag{1.3.17}$$

Every irreducible finite dimensional $sl_2(\mathbb{K})$-module is isomorphic to exactly one of the modules $V_d, d = 0, 1, 2 \ldots$.

Remark 2 *[ITT99] Let $\mathbb{K}$ be an algebraically closed field with characteristic 0, let A, A^* be semi-simple (diagonalizable) elements in the Lie algebra $sl_2(\mathbb{K})$, and let V be an irreducible finite dimensional $sl_2(\mathbb{K})$-module. Assume $sl_2(\mathbb{K})$ is generated by the elements A, A^*, then the pair A, A^* acts on V as a Leonard pair.*

Definition 1.3.4 *Let $\mathbb{K}$ denote an algebraically closed field. The algebra $U_q(sl_2)$ is an associative $\mathbb{K}$-algebra with unit 1 and is generated by the elements e, f, k, k^{-1} that satisfy the following relations*

$$kk^{-1} = k^{-1}k = 1, \tag{1.3.18}$$

$$ke = q^2 ek, \quad kf = q^{-2} fk, \tag{1.3.19}$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}. \tag{1.3.20}$$

Lemma 1.3.8 *[Ter03, Lemma 6.2] Let d denote a nonnegative integer. There exists a family $V_{\epsilon,d}$, $\epsilon \in \{1, -1\}$ of irreducible finite dimensional $U_q(sl_2)$-modules such that the module $V_{\epsilon,d}$ has a basis $u_0, u_1, \ldots, u_d$ satisfying*

$$ku_i = \epsilon q^{d-2i} u_i, \ 0 \leq i \leq d, \tag{1.3.21}$$

$$fu_i = [i+1]_q u_{i+1}, \ 0 \leq i \leq d-1, \quad fu_d = 0, \tag{1.3.22}$$

$$eu_i = \epsilon[d-i+1]_q u_{i-1}, \ 1 \leq i \leq d, \quad eu_0 = 0. \tag{1.3.23}$$

Each irreducible finite dimensional $U_q(sl_2)$-module is isomorphic to exactly one of the modules $V_{\epsilon,d}$.

Theorem 1.3.8 *[Ter03, Example 6.3] Let d denote a nonnegative integer, and choose $\epsilon \in \{1, -1\}$. Let e, f, k, k^{-1} denote generators of the algebra $U_q(sl_2)$. Let α, β denote scalars in $\mathbb{K}$. Define $A = \alpha f + \frac{k}{q-q^{-1}}$, $A^* = \beta e + \frac{k^{-1}}{q-q^{-1}}$. Assume $\epsilon\alpha\beta$ is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$, then the pair A, A^* acts on the irreducible finite dimensional $U_q(sl_2)$-module $V_{\epsilon,d}$ as a Leonard pair.*

1.3.2 Tridiagonal pairs

A more general object called a tridiagonal pair is now considered. The concept of a tridiagonal pair is implicit in [BI84, page 263], [Leo82] and more explicit in [Ter92, Theorem 2.1]. A systematic study began in [ITT99]. As research progressed, connections were found to representation theory [AC04, Bas0404, IT07, IT03, IT08, IT09, Ter99, TV03], partially ordered sets [Ter03], the bispectral problem [GH96, GH9699, GLZ92, Zhe91]. Parallel with these progress, tridiagonal pairs appeared in statistical mechanical models [Bas0404, Bas06, BK0503, BK0507, BK07, Bas0604] and classical mechanics [ZK02].

Let V denote a vector space over a field $\mathbb{K}$ with finite positive dimension. Let $End(V)$ denote the $\mathbb{K}-$algebra of all linear transformations from V to V. Let A denote a diagonalizable element of $End(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A. For $0 \le i \le d$, define $E_i \in End(V)$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \ne i (0 \le i \le d)$. Here I denotes the identity of $End(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i).

Lemma 1.3.9 *The sequence $\{E_i\}_{i=0,\ldots,d}$ satisfies the following properties,*
(i) $I = \sum_{i=0}^d E_i$;
(ii) $E_iE_j = \delta_{i,j}E_i \quad (0 \le i, j \le d)$;
(iii) $V_i = E_iV \quad (0 \le i \le d)$;
(iv) $A = \sum_{i=0}^d \theta_i E_i$;
(v) $E_0, E_1, \ldots, E_d$ *is a basic for subalgebra of $End(V)$ generated by A.*

Proof. For all $v \in V$, there exist $a_j \in \mathbb{K}, v_j \in V_j, j = 0, \ldots, d$ such that $v = \sum_{j=0}^d a_jv_j$.
(i) $\sum_{i=1}^d E_iv = \sum_{i=1}^d E_i \sum_{j=0}^d a_jv_j = \sum_{i=1}^d E_i(a_iv_i) = \sum_{i=0}^d a_iv_i = v$.
(ii) $E_iE_jv = E_i(E_j(\sum_{k=0}^d a_kv_k)) = E_i(a_jv_j) = \delta_{ij}E_i(\sum_{k=0}^d a_kv_k) = \delta_{ij}E_iv$.
(iv) $Av = A\sum_{j=0}^d a_jv_j = \sum_{j=0}^d a_j\theta_jv_j$; $\sum_{i=0}^d \theta_iE_iv = \sum_{i=0}^d \theta_iE_i \sum_{j=0}^d a_jv_j = \sum_{j=0}^d a_j\theta_jv_j$.
(v) Let D denote the subalgebra of $End(V)$ generated by A. We have $D = \{\sum\limits_{i=0}^n a_iA^i | n \in \mathbb{N}, a_i \in K, 0 \le i \le n\}$. Since A is diagonalizable, and A has $d+1$ eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$, then the order of the minimal polynomial of A is $d+1$. Hence, $D = \{\sum\limits_{i=0}^d a_iA^i | a_i \in K, 0 \le i \le d\}$. Moreover, $E_0, E_1, \ldots, E_d \in D$, and they are independent. The proof is straightforward. ∎

Definition 1.3.5 *[ITT99, Definition 1.1] Let V denote a vector space over a field $\mathbb{K}$ with finite positive dimension. By a tridiagonal pair (or TD pair) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.*

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \le i \le d, \tag{1.3.24}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \le i \le \delta, \tag{1.3.25}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Let (A, A^*) denote a TD pair on V, the integers d and δ from (ii), (iii) are equal and called the diameter of the pair. We will prove it later. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1.3.24) (resp. (1.3.25)). Let $\{V_i\}_{i=0}^d$ (resp. $\{V_j^*\}_{j=0}^\delta$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \le i \le d, 0 \le j \le \delta$, let θ_i (resp. θ_j^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_j^*). By [ITT99], for $0 \le i \le d$ the spaces V_i, V_i^* have the same dimension; we denote this common dimension by ρ_i. By the construction $\rho_i \neq 0, i = 1, \ldots, d$, and the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \le i \le d$ and $\rho_{i-1} \le \rho_i$ for $1 \le i \le d/2$. We refer to the sequence $(\rho_0, \rho_1, \ldots, \rho_d)$ as the shape vector of A, A^*. In particular, the shape vector of A, A^* is independent of the choice of standard orderings of the eigenspaces of A, A^*.

Now let (A, A^*) denote a TD pair on V. An ordering of the primitive idempotents of A (resp. A^*) is said to be standard whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 1.3.6 *[ITT99, Definition 2.1] Let V denote a vector space over $\mathbb{K}$ with finite positive dimention. By a tridiagonal system (or TD system) on V we mean a sequence*

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta)$$

that satisfies (i)–(iii) below.

(i) (A, A^*) is a TD pair on V.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^{\delta}$ *is a standard ordering of the primitive idempotents of* A^*.

We say that Φ *is* over $\mathbb{K}$.

Actually, E_i can be written as follows,

$$E_i = \prod_{\substack{0\le j\le d\\ j\neq i}} \frac{A-\theta_j I}{\theta_i-\theta_j}.$$

We have that for all $v \in V_i$,

$$E_i v = (\prod_{\substack{0\le j\le d\\ j\neq i}} \frac{A-\theta_j I}{\theta_i-\theta_j})v = v \prod_{\substack{0\le j\le d\\ j\neq i}} \frac{\theta_i-\theta_j}{\theta_i-\theta_j} = v.$$

And for all $v \in V_j, j \neq i$, $E_i v = 0$. Therefore, E_i is the projection of V on V_i.
Our proof is complete with the observation that there exists one and only one linear transformation such that it is the projection of V onto V_i.

Lemma 1.3.10 *[INT10] Let* $(A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta})$ *denote a TD system. Then the following hold for* $0 \le i, j, r \le d, 0 \le h, k, r \le \delta$.

(i) $E_h^* A^r E_k^* = 0$ *if* $|h-k| > r$.

(ii) $E_i A^{*r} E_j = 0$ *if* $|i-j| > r$.

Proof. We give only the proof of (i), for (ii) it is also proved similarly.
For all $a \in V$, we have

$$\begin{aligned} E_h^* A^r E_k^*(a) &\in E_h^* A^r V_k^* \\ &\subset E_h^* A^{r-1}(V_{k-1}^* + V_k^* + V_{k+1}^*) \\ &\subset E_h^*(V_{k-r}^* + V_{k-r+1}^* + \cdots + V_{k+r-1}^* + V_{k+r}^*) \\ &= 0. \ \text{(Since } |h-k| > r). \end{aligned}$$

It follows directly that $E_h^* A^r E_k^* = 0$ if $|h-k| > r$. ■

Lemma 1.3.11 *[ITT99] Let* (A, A^*) *denote a TD pair on* V*, the integers* d *and* δ *from (ii), (iii) of Definition (1.3.5) are equal.*

Proof. Assume without loss of generality that $\delta \le d$. Set $V_{i,j} = (\sum_{h=0}^{i} E_h^* V) \cap (\sum_{k=j}^{d} E_k V)$ for all integers i, j, with convention that $\sum_{h=0}^{i} E_h^* V = \begin{cases} 0 & \text{if } i < 0 \\ V & \text{if } i > \delta \end{cases}$ and $\sum_{k=j}^{d} E_k V = \begin{cases} V & \text{if } j < 0 \\ 0 & \text{if } j > d \end{cases}$.

Since $E_1^* + \cdots + E_\delta^* = I$, then $AE_h^*V = (E_1^* + \cdots + E_\delta^*)AE_h^*V$. By (i) of Lemma 1.3.10, $AE_h^*V \subset E_{h-1}^*V + E_h^*V + E_{h+1}^*V$. Consequently, $(A - \theta_j I)\sum_{h=0}^{i} E_h^*V \subset \sum_{h=0}^{i+1} E_h^*V$. Moreover,

$$\begin{aligned}(A-\theta_j I)\sum_{k=j}^{d} E_kV &= \sum_{k=j}^{d}(AE_kV - \theta_j E_kV) = \sum_{k=j}^{d}(\theta_k E_kV - \theta_j E_kV)\\ &= \sum_{k=j+1}^{d}(\theta_k - \theta_j)E_kV = \sum_{k=j+1}^{d} E_kV.\end{aligned}$$

It follows that $(A - \theta_j I)V_{i,j} \subset \left(\sum_{h=0}^{i+1} E_h^*V\right) \cap \left(\sum_{k=j+1}^{d} E_kV\right) = V_{i+1,j+1}$. From that, we obtain $AV_{i,j} \subset V_{i,j} + V_{i+1,j+1}$.

Similarly, we get $A^*V_{i,j} \subset V_{i,j} + V_{i-1,j-1}$.

Put $W = V_{0,r} + V_{1,r+1} + \cdots + V_{d-r,d}$ for $0 < r \leq d$. By the above,

$$\begin{aligned}AW &= A(V_{0,r} + \cdots + V_{d-r,d}) \subset V_{0,r} + V_{1,r+1} + \cdots + V_{d-r,d} + V_{d-r+1,d+1} = W,\\ A^*W &= A^*(V_{0,r} + \cdots + V_{d-r,d}) \subset V_{-1,r-1} + V_{0,r} + \cdots + V_{d-r-1,d-1} + V_{d-r,d} = W.\end{aligned}$$

Use the condition (iv) of Definition 1.3.5, $W = 0$ or $W = V$.

Clearly, $V_{0,r}, \ldots, V_{d-r,d}$ are subsets of $\sum_{h=0}^{d-r} E_h^*V$, then $W \subset \sum_{h=0}^{d-r} E_h^*V$. Therefore $W \neq V$ ($\sum_{h=0}^{d-r} E_h^*V$ is the proper subset of V).
Suppose that $\delta < d$, put $0 < r = d - \delta \leq d$, we have $W = V_{0,r} + V_{1,r+1} + \cdots + V_{\delta,d} = 0$. However, $V_{\delta,d} = E_dV \leq 0$. This is a contradiction. Thus, we conclude that $\delta = d$. ■

Theorem 1.3.9 *[ITT99, Theorem 10.1] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension, and let A, A^* denote a TD pair on $\mathbb{K}$. There exist scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in $\mathbb{K}$ such that the tridiagonal relations are satisfied*

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] = 0\ , \tag{1.3.26}$$
$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A] = 0\ . \tag{1.3.27}$$

Moreover, these scalars are unique if the diameter of the pair is at least 3.

Proof. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system associated with (A, A^*). Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (resp. the dual eigenvalue sequence) of Φ.

Put $U_i = \sum_{h=0}^{i} E_h^*V \cap \sum_{k=i}^{d} E_kV$, for all $0 \leq i \leq d$, and $W = U_0 + U_1 + \cdots + U_d$. We first prove that V is the direct sum of $U_0, U_1, \ldots, U_d$. Similarly, in the proof of Lemma 1.3.11, $AW \subset A, A^*W \subset W$, thus $W = 0$, or $W = V$. Moreover, $0 \neq U_0 = E_0^*V \subset W$. Clearly, $V = W$. On the other hand, for all $0 \leq i \leq d-1$, $\sum_{l=0}^{i} U_l \subset \sum_{h=0}^{i} E_h^*V, U_{i+1} \subset \sum_{i+1}^{d} E_kV$. By

the proof of Lemma 1.3.11, $\sum_{l=0}^{i} U_l \cap U_{i+1} \subset \sum_{h=0}^{i} E_h^* V \cap \sum_{k=i+1}^{d} E_k V = \emptyset$. This gives that V is the direct sum of $U_0, U_1, \ldots, U_d$.

Let F_i denote the projection map of V onto U_i, for all $0 \leq i \leq d$. Put $R = A - \sum_{i=0}^{d} \theta_i F_i, L = A^* - \sum_{i=0}^{d} \theta_i^* F_i$.

Write $D = \{\sum_{k=0}^{n} a_k A^k | n \in \mathbb{N}, a_k \in \mathbb{K}\}$. Since A is diagonalizable, and A has $d+1$ eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$, then the minimal polynomial of A is $\prod_{i=0}^{d} (A - \theta_i I)$. Hence, we can rewrite $D = \{\sum_{k=0}^{d} a_k A^k | a_k \in \mathbb{K}\}$.

For all $0 \leq i \leq d$, by Lemma 1.3.10, $E_i A^* = E_i A^* (\sum_{j=0}^{d} E_j) = E_i A^* E_{i-1} + E_i A^* E_i + E_i A^* E_{i+1}$, with $E_{-1} = E_{d+1} = 0$. Similarly, $A^* E_i = E_{i-1} A^* E_i + E_i A^* E_i + E_{i+1} A^* E_i$. Put $L_l = \sum_{i=0}^{l} E_i$, for all $0 \leq l \leq d$,

$$L_l A^* - A^* L_l = \sum_{i=0}^{l} E_i A^* - A^* \sum_{i=0}^{l} E_i = E_l A^* E_{l+1} - E_{l+1} A^* E_l.$$

In fact, D is spanned by $E_0, E_1, \ldots, E_d$ (the property (v) of $\{E_i, i = 0, \ldots, d\}$), then $L_0, L_1, \ldots, L_d$ is also a basic of D.

$$\begin{aligned}
\mathrm{Span}\{XA^*Y - YA^*X | X, Y \in D\} &= \mathrm{Span}\{E_i A^* E_j - E_j A^* E_i | 0 \leq i, j \leq d\} \\
&= \mathrm{Span}\{E_i A^* E_{i+1} - E_{i+1} A^* E_i | 0 \leq i \leq d\} \\
&= \mathrm{Span}\{L_i A^* - A^* L_i | 0 \leq i \leq d\} \\
&= \{XA^* - A^*X | X \in D\}.
\end{aligned}$$

It follows immediately that there exist uniquely scalars $a_i \in \mathbb{K}$ $(i = 0, 1, \ldots, d)$ such that

$$A^2 A^* A - A A^* A^2 = \sum_{i=0}^{d} a_i (A^i A^* - A^* A^i) \tag{1.3.28}$$

Assume $d \geq 3$, put $t = \max\{i \in \{0, 1, \ldots, d\} | a_i \neq 0\}$. Suppose that $t \geq 4$, multiply two sides of the equation (1.3.28) on the left by F_t, and on the right by F_0. By Lemma 7.3 of [ITT99], we have

$$0 = a_t (F_t A^t A^* F_0 - F_t A^* A^t F_0) = a_t (\theta_0^* - \theta_t^*) R^t F_0$$

By Corollary 6.7 of [ITT99], $R^t F_0 \neq 0$, then $a_t(\theta_0^* - \theta_t^*) = 0$, it is a contradiction. Hence $t \leq 3$, the equation (1.3.28) can be rewritten

$$A^2 A^* A - A A^* A^2 = a_3 (A^3 A^* - A^* A^3) + a_2 (A^2 A^* - A^* A^2) + a_1 (A A^* - A^* A). \tag{1.3.29}$$

Suppose $a_3 = 0$, multiply the two sides of (1.3.29) on the left by F_3 and on the right by F_0, we get $(\theta_1^* - \theta_2^*)R^3F_0 = 0$. This is again a contradiction. So $a_3 \neq 0$. We can put $\beta = \frac{1}{a_3} - 1, \gamma = -\frac{a_2}{a_3}, \delta = -\frac{a_1}{a_3}$. The equation (1.3.29) becomes

$$(A^3A^* - A^*A^3) - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A) = 0$$

or $[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] = 0$. Multiply each term in (1.3.26) on the left by F_{i-2} and on the right by F_{i+1}, for all $2 \leq i \leq d-1$, we find $(\theta_{i-2}^* - \theta_{i+1}^* - (\beta+1)(\theta_{i-1}^* - \theta_i^*))R^3F_{i-2} = 0$. Consequently, $\theta_{i-2}^* - \theta_{i+1}^* - (\beta+1)(\theta_{i-1}^* - \theta_i^*) = 0$. It follows $\theta_{i-2}^* - \beta\theta_{i-1}^* + \theta_i^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}$, for all $2 \leq i \leq d$. So the expression $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$ does not depend on i. Let us put $\gamma* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$. Consider $(\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta\theta_i + \theta_{i=1} - \gamma^*) = 0$, then ${\theta_{i-1}^*}^2 - \beta\theta_{i-1}^*\theta_i^* + {\theta_i^*}^2 - \gamma^*(\theta_{i-1}^* + \theta_i^*) = {\theta_i^*}^2 - \beta\theta_i^*\theta_{i+1}^* + {\theta_{i+1}^*}^2 - \gamma^*(\theta_i^* + \theta_{i+1}^*)$. We can also put $\delta^* = {\theta_i^*}^2 - \beta\theta_i^*\theta_{i+1}^* + {\theta_{i+1}^*}^2 - \gamma^*(\theta_i^* + \theta_{i+1}^*)$ because ${\theta_i^*}^2 - \beta\theta_i^*\theta_{i+1}^* + {\theta_{i+1}^*}^2 - \gamma^*(\theta_i^* + \theta_{i+1}^*)$ is independent of i. According to Lemma 9.3 of [ITT99], we get (1.3.27). Since β, γ, θ are determined uniquely, and γ^*, δ^* satisfy (1.3.27) then γ^*, δ^* are also determined uniquely. We already finish proving for d at least equals 3, then β, γ, δ are determined uniquely such that (1.3.26), (1.3.27).

If $d = 2$, let β be any scalar in $\mathbb{K}$, put $\gamma = \theta_0 - \beta\theta_1 + \theta_2$, and $\delta = \theta_0^2 - \beta\theta_0\theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1)$. If $d = 1$, let β, γ be any scalars in $\mathbb{K}$, put $\delta = \theta_0^2 - \beta\theta_0\theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1)$. If $d = 0$, let β, γ, δ be any scalars in $\mathbb{K}$. By Lemma 9.3 of [ITT99], we get (1.3.26). Similarly, there exist scalars γ^*, δ^* for (1.3.27). ■

Corollary 1.3.5 *[Ter99] Let (A, A^*) be a TD pair on V, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Then the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}; \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{1.3.30}$$

are equal and independent of i for $2 \leq i \leq d-1$

Proof. This lemma is straightforward from the equations $\theta_{i-2} - \theta_{i+1} - (\beta+1)(\theta_{i-1} - \theta_i) = 0$, and $\theta_{i-2}^* - \theta_{i+1}^* - (\beta+1)(\theta_{i-1}^* - \theta_i^*) = 0$ in the proof of the above theorem. ■

Corollary 1.3.6 *Let (A, A^*) be a TD pair on V, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. For all $0 \leq i \leq d$, there exist scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^*$ in $\mathbb{K}$ and q in the algebraic closure of $\mathbb{K}$, $q \neq 0, q \neq \pm 1$ such that*

$$\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i}, \tag{1.3.31}$$

$$\theta_i^* = \alpha_1^* + \alpha_2^* q^i + \alpha_3^* q^{-i}. \tag{1.3.32}$$

Proof. By Corollary 1.3.5 and Theorem 1.3.9, there exists scalar β in $\mathbb{K}$, such that $\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \beta + 1$, for all $2 \leq i \leq d-1$. Let q in the algebraic closure of $\mathbb{K}$, such that $q + q^{-1} = \beta$. We have the equation

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = q + q^{-1} + 1.$$

It is equivalent to

$$\frac{\theta_{i-2}-\theta_{i-1}+\theta_{i-1}-\theta_i+\theta_i-\theta_{i+1}}{\theta_{i-1}-\theta_i}=q+q^{-1}+1.$$

Put $u_i=\theta_{i-2}-\theta_{i-1}$, it yields

$$u_{i+2}-(q+q^{-1})u_{i+1}+u_i=0.$$

Since $q\neq\pm1$, there exist scalars $a,b\in\mathbb{K}:u_i=aq^i+bq^{-i}$, or $\theta_{i-2}-\theta_{i-1}=aq^i+bq^{-i}$. Solving this equation, there exist scalars $\alpha_1,\alpha_2,\alpha_3\in\mathbb{K}$ such that $\theta_i=\alpha_1+\alpha_2q^i+\alpha_3q^{-i}$. By the same arguments, there also exist scalars $\alpha_1^*,\alpha_2^*,\alpha_3^*\in\mathbb{K}$ such that $\theta_i^*=\alpha_1^*+\alpha_2^*q^i+\alpha_3^*q^{-i}$. ■

Note that the following equations are inferred directly from Corollary 1.3.6. If $\beta=q^2+q^{-2}$, $q^4\neq1$, there exist scalars α,α^* in $\mathbb{K}$, and b,b^*,c,c^* in the algebraic closure of $\mathbb{K}$ such that

$$\theta_i=\alpha+bq^{2i-d}+cq^{d-2i}\qquad 0\leq i\leq d\ , \tag{1.3.33}$$
$$\theta_j^*=\alpha^*+b^*q^{2j-\delta}+c^*q^{\delta-2j}\qquad 0\leq j\leq\delta\ , \tag{1.3.34}$$

where b,b^*,c,c^* are nonzero scalars.

Lemma 1.3.12 *For each positive integer s, there exist scalars $\beta_s,\gamma_s,\gamma_s^*,\delta_s,\delta_s^*$ in the algebraic cloture of $\mathbb{K}$ such that*

$$\theta_i^2-\beta_s\theta_i\theta_j+\theta_j^2-\gamma_s(\theta_i+\theta_j)-\delta_s = 0 \quad \textit{if} \quad |i-j|=s\ .\ (0\leq i,j\leq d), \tag{1.3.35}$$
$$\theta_h^{*2}-\beta_s\theta_h^*\theta_k^*+\theta_k^{*2}-\gamma_s^*(\theta_h^*+\theta_k^*)-\delta_s^* = 0 \quad \textit{if} \quad |h-k|=s\ .\ (0\leq h,k\leq\delta). \tag{1.3.36}$$

Proof. There is no loss of generality of assuming $j=i+s$. Substituting (1.3.33) into (1.3.35) gives

$$\begin{aligned}0= \quad & b^2q^{2s-2d}(q^{2s}+q^{-2s}-\beta_s)q^{4i}+c^2q^{2d-2s}(q^{2s}+q^{-2s}-\beta_s)q^{-4i}\\ + \quad & b(q^{2s-d}+q^{-d})(2\alpha-\beta_s\alpha-\gamma_s)q^{2i}+c(q^d+q^{d-2s})(2\alpha-\beta_s\alpha-\gamma_s)q^{-2i}\\ + \quad & 2\alpha^2+4bc-\beta_s(\alpha^2+bc(q^{2s}+q^{-2s}))-2\alpha\gamma_s-\delta_s\end{aligned}$$

If $b,c,q\neq0$ and $q^{2s}+q^{-2s}\neq-2$, then (1.3.35) is satisfied for all $0\leq i\leq d$ if and only if

$$\begin{cases} q^{2s}+q^{-2s}-\beta_s=0\\ 2\alpha-\beta_s\alpha-\gamma_s=0\\ 2\alpha^2+4bc-\beta_s(\alpha^2+bc(q^{2s}+q^{-2s}))-2\alpha\gamma_s-\delta_s=0\end{cases}$$

Therefore,

$$\begin{cases} \beta_s=q^{2s}+q^{-2s}\\ \gamma_s=\alpha(2-q^{2s}-q^{-2s})\\ \delta_s=\alpha^2(q^s-q^{-s})^2-bc(q^{2s}-q^{-2s})^2\end{cases}$$

Similarly, there exist scalars γ_s^*,δ_s^* s.t (1.3.36) ■

Remark 3 *Let V denote a finite dimensional vector space over $\mathbb{K}$, and (A, A^*) be a tridiagonal pair on V. Let r, r^*, s, s^* denote scalars in $\mathbb{K}$ such that r, r^* are nonzero, then the ordered pair $(rA + sI, r^*A^* + s^*I)$ is also a tridiagonal pair on V. Moreover, if $\beta, \gamma, \gamma^*, \delta, \delta^*$ is a sequence of parameters in the tridiagonal relations of A, A^*, then $\beta, r\gamma + s(2-\beta), r^*\gamma^* + s^*(2-\beta), r^2\delta - 2rs\gamma + s^2(\beta-2), r^{*2}\delta^* - 2r^*s^*\gamma^* + s^{*2}(\beta-2)$ is the sequence of parameters in the tridiagonal relations of $rA + sI, r^*A^* + s^*I$.*

Theorem 1.3.10 *[Ter99] Let $\beta, \gamma, \gamma^*, \delta, \delta^*$ be scalars in $\mathbb{K}$. Let T denote a tridiagonal algebra over $\mathbb{K}$ with the parameters $\beta, \gamma, \gamma^*, \delta, \delta^*$, and generators A, A^*. Let V denote an irreducible finite dimensional T-module. Assume that q is not a root of unity such that $\beta = q + q^{-1}$, and A, A^* are diagonalizable on V. Then (A, A^*) acts on V as a tridiagonal pair.*

By Theorem 1.3.9 and Theorem 1.3.10, there exists a relationship between the tridiagonal pair and the generators of the q-Onsager algebra as follows.

Remark 4 *If A, A^* are generators of the q-Onsager algebra, V is an irreducible finite dimensional vector space on which A, A^* act, and A, A^* are diagonalizable on V then (A, A^*) is a tridiagonal pair on V. Inversely, if (A, A^*) is a tridiagonal pair on V, and the parameters $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$ then A, A^* satisfy the defining relations of the q-Onsager algebra.*

The conditions making a tridiagonal pair become a Leonard pair are established by the next theorem

Theorem 1.3.11 *[Ter99] Let V denote a finite dimensional vector space over $\mathbb{K}$. Let A, A^* be linear transformations from V to V. Then the following are equivalent.*
(i) (A, A^*) is a Leonard pair on V.
(ii) (A, A^*) is a tridiagonal pair, and for each of A, A^* all eigenspaces have dimension 1.

Lemma 1.3.13 *Let (A, A^*) denote a tridiagonal pair on V. If the shape vector of A, A^* satisfies $\rho_0 = \rho_1 = 1$, then A, A^* is Leonard pair.*

Corollary 1.3.7 *Let (A, A^*) denote a tridiagonal pair on V. If (A, A^*) is not a Leonard pair, the shape vector of (A, A^*) satisfies $\rho_i \geq 2$ $(1 \leq i \leq d-1)$.*

Theorem 1.3.12 *[CP91] The quantum affine algebra $U_q(\widehat{sl_2})$ is isomorphic to the unital*

associative $\mathbb{K}$*-algebra* $\mathbb{U}$ *with generators* $y_i^{\pm}, k_i^{\pm}, i \in \{0,1\}$ *and the following relations:*

$$\begin{aligned}
k_i k_i^{-1} = k_i^{-1} k_i &= 1, \\
k_0 k_1 \quad \textit{is central}, & \\
\frac{q y_i^+ k_i - q^{-1} k_i y_i^+}{q - q^{-1}} &= 1, \\
\frac{q k_i y_i^- - q^{-1} y_i^- k_i}{q - q^{-1}} &= 1, \\
\frac{q y_i^- y_i^+ - q^{-1} y_i^+ y_i^-}{q - q^{-1}} &= 1, \\
\frac{q y_i^+ y_i^- - q^{-1} y_i^- y_i^+}{q - q^{-1}} &= k_0^{-1} k_1^{-1}, \quad i \neq j,
\end{aligned}$$

$$(y_i^{\pm})^3 y_j^{\pm} - [3]_q (y_i^{\pm})^2 y_j^{\pm} y_i^{\pm} + [3]_q y_i^{\pm} y_j^{\pm} (y_i^{\pm})^2 - y_j^{\pm} (y_i^{\pm})^3 = 0, \ i \neq j.$$

An isomorphism from $\mathbb{U}$ *to* $U_q(\widehat{sl_2})$ *is given by:*

$$\begin{aligned}
k_i^{\pm} &\mapsto K_i^{\pm}, \\
y_i^- &\mapsto K_i^{-1} + e_i^-, \\
y_i^+ &\mapsto K_i^{-1} - q(q - q^{-1})^2 K_i^{-1} e_i^+.
\end{aligned}$$

The inverse of this isomorphsim is given by:

$$\begin{aligned}
K_i^{\pm} &\mapsto k_i^{\pm}, \\
e_i^- &\mapsto y_i^- - k_i^{-1}, \\
e_i^+ &\mapsto \frac{1 - k_i y_i^+}{q(q - q^{-1})^2}.
\end{aligned}$$

Theorem 1.3.13 *[IT07] Let* V *denote a vector space over* $\mathbb{K}$ *with finite positive dimension and let* (A, A^*) *denote a tridiagonal pair on* V*. Let* $\theta_0, \theta_1, \ldots, \theta_d$ *(resp.* $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$*) denote a standard ordering of the eigenvalues of* A *(resp.* A^**). We assume there exist nonzero scalars* a, a^* *in* $\mathbb{K}$ *such that* $\theta_i = aq^{2i-d}$ *and* $\theta_i^* = a^* q^{d-2i}$ *for* $0 \leq i \leq d$*. Then with reference to Theorem 1.3.12 there exists a unique* $U_q(\widehat{sl_2})$*-module structure on* V *such that* $a y_1^-$ *acts as* A *and* $a^* y_0^-$ *acts as* A^**. Moreover there exists a unique* $U_q(\widehat{sl_2})$*-module structure on* V *such that* $a y_0^+$ *acts as* A *and* $a^* y_1^+$ *acts as* A^**. Both* $U_q(\widehat{sl_2})$*-module structures are irreducible.*

Definition 1.3.7 *Let* V *denote a vector space over* $\mathbb{K}$ *with finite positive dimension. Let* d *denote a nonnegative integer. By a decomposition of* V *of length* d*, we mean a sequence* $U_0, U_1, \ldots, U_d$ *consisting of nonzero subspaces of* V *such that*

$$V = U_0 + U_1 + \cdots + U_d \quad (\textit{direct sum}).$$

Now we are concerned about six decompositions of a finite dimensional vector space V. Recall that (A, A^*) is the tridiagonal pair on V. Let $V_0, V_1, \ldots, V_d$ (resp. $V_0^*, V_1^*, \ldots, V_d^*$) denote a standard order of the eigenspaces of A (resp. A^*). For $0 \leq i \leq d$, let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*)

Lemma 1.3.14 *[IT07] The sequence of subspaces $U_0, U_1, \ldots, U_d$ of V is a decomposition of V if one of the six following is satisfied*
i) $U_i = V_i, \quad i = 0, 1, \ldots, d,$
ii) $U_i = V_i^*, \quad i = 0, 1, \ldots, d$
iii) $U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d), \quad i = 0, 1, \ldots, d,$
iv) $U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}), \quad i = 0, 1, \ldots, d,$
v) $U_i = (V_{d-i}^* + V_{d-i+1}^* + \cdots + V_d^*) \cap (V_0 + V_1 + \cdots + V_{d-i}), \quad i = 0, 1, \ldots, d,$
vi) $U_i = (V_{d-i}^* + V_{d-i+1}^* + \cdots + V_d^*) \cap (V_i + V_{i+1} + \cdots + V_d), \quad i = 0, 1, \ldots, d.$

Lemma 1.3.15 *[IT07] Let $U_0, U_1, \ldots, U_d$ denote one of the six decompositions of V associated with the tridiagonal pair (A, A^*) in Lemma 1.3.14. For $0 \leq i \leq d$, let ρ_i denote the dimension of U_i. Then the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is independent of the decomposition. Moreover the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is unimodal and symmetric.*

Lemma 1.3.16 *[IT07] Let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V associated with a tridiagonal pair (A, A^*) in Lemma 1.3.14. Then for $0 \leq i \leq d$ the action of A and A^* on U_i is described as follows.*
i) If $U_i = V_i$ then $(A - \theta_i I)U_i = 0$, $A^* U_i \subseteq U_{i-1} + U_i + U_{i+1}$,
ii) If $U_i = V_i^*$ then $AU_i \subseteq U_{i-1} + U_i + U_{i+1}$, $(A^* - \theta^* I)U_i = 0$,
iii) If $U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d)$ then $(A - \theta_i I)U_i \subseteq U_{i+1}$, $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$,
iv) If $U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i})$ then $(A - \theta_{d-i} I)U_i \subseteq U_{i+1}$, $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$,
v) If $U_i = (V_{d-i}^* + V_{d-i+1}^* + \cdots + V_d^*) \cap (V_0 + V_1 + \cdots + V_{d-i})$ then $(A - \theta_{d-i} I)U_i \subseteq U_{i+1}$, $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$,
vi) If $U_i = (V_{d-i}^* + V_{d-i+1}^* + \cdots + V_d^*) \cap (V_i + V_{i+1} + \cdots + V_d)$ then $(A - \theta_i I)U_i \subseteq U_{i+1}$, $(A^* - \theta_{d-i}^* I)U_i \subseteq U_{i-1}$.

Theorem 1.3.14 *[IT07] Let $B : V \to V$ denote the unique linear transformation such that for $0 \leq i \leq d$,*

$$(V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i})$$

is an eigenspace of B with eigenvalues bq^{2i-d}.
Let $B^* : V \to V$ denote the unique linear transformation such that for $0 \leq i \leq d$,

$$(V_{d-i}^* + V_{d-i+1}^* + \cdots + V_d^*) \cap (V_i + V_{i+1} + \cdots + V_d)$$

is an eigenspace of B^* with eigenvalue $b^* q^{d-2i}$.
Then (B, B^*) is a tridiagonal pair on V. The sequence $bq^{2i-d} (0 \leq i \leq d)$ is a standard ordering of the eigenvalues of B and the sequence $b^* q^{d-2i} (0 \leq i \leq d)$ is a standard ordering of the eigenvalues of B^*.

Now we give two relations involving the tridiagonal pair (A, A^*) which has a standard ordering of the eigevalues $\theta_i, i = 0, 1, \ldots, d$ of A satisfying $\theta_i = aq^{2i-d}$ and a standard ordering of the eigenvalues $\theta_i^*, i = 0, 1, \ldots, d$ of A^* satisfying $\theta_i^* = a^* q^{d-2i}$ where a, a^* are nonzero scalars in $\mathbb{K}$.

Theorem 1.3.15 *[IT07] Let (A, A^*) denote a tridiagonal pair on V, let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote a standard ordering of the eigenvalues of A (resp. A^*). Assume that $\theta_i = aq^{2i-d}, \theta_i^* = a^* q^{d-2i}, i = 0, 1, \ldots, d$ where a, a^* are nonzero scalars in $\mathbb{K}$ then A, A^* satisfy the q-Serre relations*

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0, \tag{1.3.37}$$
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0. \tag{1.3.38}$$

In fact, the result of the above theorem is expressed more efficiently as follows

Theorem 1.3.16 *[Ter99] Let (A, A^*) denote a tridiagonal pair on V of diameter d. Then the following are equivalent:*
i) A, A^* satisfy the q-Serre relations.
ii) There exist eigenvalue and dual eigenvalue sequences for A, A^* which satisfy

$$\theta_i = q^{2i}\theta, \quad \theta_i^* = q^{2d-2i}\theta^* \quad (0 \le i \le d),$$

for some nonzero scalars θ, θ^*.

Theorem 1.3.17 *[IT07] Let $\mathbb{K}$ denote an algebraically closed field with characteristic 0. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let (A, A^*) denote a tridiagonal pair on V of diameter d. Let $(\rho_0, \rho_1, \ldots, \rho_d)$ denote the corresponding shape vector. Assume A, A^* satisfy the q-Serre relations then the entries in this shape vector are upper bounded by binomial coefficients as follows*

$$\rho_i \le \begin{pmatrix} d \\ i \end{pmatrix} \qquad (0 \le i \le d).$$

In particular, $\rho_0 = \rho_d = 1$ for such tridiagonal pairs.

Lemma 1.3.17 *[AC04] Let (A, A^*) denote a tridiagonal pair on V, and let D and D^* denote the subalgebras of $End(V)$ generated by A and A^*, respectively. Fix standard orderings of the eigenspaces $V_0, V_1, \ldots, V_d$ and of the eigenspaces $V_0^*, V_1^*, \ldots, V_d^*$ of A^*. Then the following are equivalent:*
i) (A, A^*) is a Leonard pair.
ii) $V = Dv^*$ for some nonzero $v^* \in V_0^*$.
iii) $V = D^*v$ for some nonzero $v \in V_d$.

The preceding result suggests the following generalization of a Leonard pair, from which the case of Leonard pair is excluded to focus on what is new.

Definition 1.3.8 *Let (A, A^*) denote a tridiagonal pair on V. Let D and D^* denote the subalgebras of $End(V)$ generated by A and A^*, respectively. We say that (A, A^*) is mild whenever (A, A^*) is not a Leonard pair, but $\rho_0 = \rho_d = 1$ and $V = Dv^* + D^*v$ for some nonzero $v^* \in V_0^*$ and $v \in V_d$.*

Theorem 1.3.18 *[AC04] Suppose that (A, A^*) is mild, and pick nonzero vectors $v^* \in V_0^*$ and $v \in V_d$. Define*

$$v_i^* = (A - \theta_{i-1}I)\dots(A - \theta_1 I)(A - \theta_0 I)v^* \quad (0 \le i \le d-1),$$

$$v_i = (A^* - \theta_{i+1}^* I)\dots(A^* - \theta_{d-1}^* I)(A^* - \theta_d^* I)v \quad (1 \le i \le d).$$

Then $\{v_0^*, v_1, v_1^*, v_2, v_2^*, \dots, v_{d-1}, v_{d-1}^*, v_d\}$ is basis for V.

Definition 1.3.9 *Let (A, A^*) denote a tridiagonal pair on V. Then (A, A^*) is said to be of q-Serre type whenever the following hold:*

$$\begin{aligned} A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 &= 0, \\ A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} &= 0. \end{aligned}$$

The action of a mild tridiagonal pair of q-Serre type on the basis of Theorem 1.3.18 was studied more deeply by H. Alnajjar and B. Curtin.

Theorem 1.3.19 *[AC04] Let (A, A^*) denote a mild tridiagonal pair on V of q-Serre type with diameter $d \ge 3$. Fix standard orderings of the eigenspaces of A and A^* for which the corresponding eigenvalue and dual eigenvalue sequences satisfy ii) of Theorem 1.3.16 for some nonzero scalars θ, θ^*. Define v_i^* and v_i $(0 \le i \le d)$ as in Theorem 1.3.18. Then there exist nonzero scalars $\lambda, \mu, \mu^* \in \mathbb{K}$ such that*

$$\begin{aligned} Av_i^* &= q^{2i}\theta v_i^* + v_{i+1}^* \ (0 \le i \le d-2), \\ Av_{d-1}^* &= q^{2d-2}\theta v_{d-1}^* + \gamma_d \mu^* v_d, \\ Av_i &= q^{2i}\theta v_i + \lambda_i v_{i+1} + \gamma_{d-i}\mu v_{i+1}^* \ (1 \le i \le d-2), \\ Av_{d-1} &= q^{2d-2}\theta v_{d-1} + (\lambda_{d-1} + \gamma_{d-1}[2]_q \mu\mu^*)v_d, \\ Av_d &= q^{2d}\theta v_d, \\ A^*v_i &= q^{2d-2i}\theta^* v_i + v_{i-1} \ (2 \le i \le d), \\ A^*v_1 &= q^{2d-2}\theta^* v_1 + \gamma_d \mu v_0^*, \\ A^*v_i^* &= q^{2d-2i}\theta^* v_i^* + \lambda_{d-i} v_{i-1}^* + \gamma_i \mu^* v_{i-1} \ (2 \le i \le d-1), \\ A^*v_1^* &= q^{2d-2}\theta^* v_1^* + (\lambda_{d-1} + \gamma_{d-1}[2]_q \mu\mu^*)v_0^*, \\ A^*v_0^* &= q^{2d}\theta^* v_0^*, \end{aligned}$$

where

$$\begin{aligned} \lambda_i &= [i]_q[d-i]_q \lambda \ (1 \le i \le d-1), \\ \gamma_i &= \frac{[i]_q![d-1]_q!}{[2]_q[d-i+1]_q}\lambda^{i-2} \ (2 \le i \le d). \end{aligned}$$

Recall that V is irreducible as an (A, A^*)-module whenever there is no subalgebra W of V such that both $AW \subseteq W$ and $A^*W \subseteq W$, other than 0 and V.

Theorem 1.3.20 *[AC04]*

Let $\theta, \theta^*, q, \lambda, \mu^*$, and μ be nonzero scalars in $\mathbb{K}$. Pick any integer $d \geq 3$, and let V be a vector space of dimension $2d$. Let $A : V \to V$ and $A^* : V \to V$ denote linear transformations which act on some basis $v_0^*, v_1, v_1^*, \ldots, v_{d-1}, v_{d-1}^*, v_d$ as in Theorem 1.3.19. Further suppose that V is irreducible as an (A, A^*)-module. Then (A, A^*) is a mild tridiagonal pair on V of q-Serre type.

Let (A, A^*) denote a tridiagonal pair on V. Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote a standard ordering of the eigenvalues of A (resp. A^*) By [ITT99] there exists a unique sequence $U_0, U_1, \ldots, U_d$ consisting of subspaces of V suth that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)},$$

$$(A - \theta_i I)U_i \subseteq U_{i+1} \ \ (0 \leq i < d), \ (A - \theta_d I)U_d = 0,$$
$$(A^* - \theta_i^* I)U_i \subseteq U_{i-1} \ \ (0 < i \leq d), \ (A^* - \theta_0^* I)U_0 = 0.$$

We call the sequence $U_0, U_1, \ldots, U_d$ the split decomposition for (A, A^*) with respect to the orderings $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$.
For $0 \leq i \leq d$, let $F_i : V \to V$ denote the linear transformation which satisfies both

$$(F_i - I)U_i = 0,$$

$$F_i U_j = 0 \text{ if } j \neq i, \quad (0 \leq j \leq d).$$

In other words, F_i is the projection map from V onto U_i. We observe

$$F_i F_j = \delta_{ij} F_i \ \ (0 \leq i, j \leq d),$$

$$F_0 + F_1 + \cdots + F_d = I,$$
$$F_i V = U_i \ \ (0 \leq i \leq d).$$

Define

$$R = A - \sum_{h=0}^{d} \theta_h F_h,$$

$$L = A^* - \sum_{h=0}^{d} \theta_h^* F_h.$$

We obtain that

$$RU_i \subseteq U_{i+1} \ \ (0 \leq i < d), \ RU_d = 0,$$
$$LU_i \subseteq U_{i-1} \ \ (0 < i \leq d), \ LU_0 = 0,$$

and

$$R^{d+1} = 0, \quad L^{d+1} = 0.$$

We call R (resp. L) the raising map (resp. lowering map) for (A, A^*) with respect to $U_0, U_1, \ldots, U_d$.
Now we consider that (A, A^*) is a tridiagonal pair and satisfies the q-Serre relations. Hence

there exist nonzero scalars a, a^* such that $\theta_i = aq^{2i-d}$, $\theta_i^* = a^*q^{d-2i}$ $(0 \le i \le d)$. By [ITT99] the maps R, L satisfy the q-Serre relations

$$\begin{aligned} R^3L - [3]_q R^2LR + [3]_q RLR^2 - LR^3 &= 0, \\ L^3R - [3]_q L^2RL + [3]_q LRL^2 - RL^3 &= 0. \end{aligned}$$

Theorem 1.3.21 *[IT03] Let (A, A^*) denote a tridiagonal pair on V satisfying the q-Serre relations. Let $U_0, U_1, \ldots, U_d$ denote the split decomposition for (A, A^*) and let R, L denote the corresponding raising and lowering maps. Let v denote a nonzero vector in U_0. Then V is spanned by the vectors of the forms $L^{i_1}R^{i_2}L^{i_3}R^{i_4}\ldots R^{i_n}v$, where $i_1, i_2, \ldots, i_n$ ranges over all sequences such that n is a nonnegative even integer, and $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \le i_1 < i_2 < \cdots < i_n \le d$.*

Definition 1.3.10 *Let (A, A^*) denote a tridiagonal pair on V. Let $(\rho_0, \rho_1, \ldots, \rho_d)$ denote the shape vector of (A, A^*). The pair (A, A^*) is said to be sharp whenever $\rho_0 = 1$.*

Theorem 1.3.22 *[NT08] A tridiagonal pair over an algebraically closed field is sharp.*

Theorem 1.3.23 *[NT08] Let (A, A^*) denote a sharp tridiagonal pair on V. Then there exists a nonzero bilinear form $\langle , \rangle$ on V such that $\langle Au, v\rangle = \langle u, Av\rangle$ and $\langle A^*u, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in V$. This form is unique up to multiplication by a nonzero scalar in $\mathbb{K}$. This form is nondegenerate and symmetric.*

Let λ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{K}$. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ denote scalars in $\mathbb{K}$. Then for $0 \le i \le d$ we define the following polynomials in $\mathbb{K}[\lambda]$:

$$\begin{aligned} \tau_i &= (\lambda - \theta_0)(\lambda - \theta_1)\ldots(\lambda - \theta_{i-1}), \\ \tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\ldots(\lambda - \theta_{i-1}^*), \\ \eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1})\ldots(\lambda - \theta_{d-i+1}), \\ \eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*)\ldots(\lambda - \theta_{d-i+1}^*). \end{aligned}$$

Note that each of $\tau_i, \tau_i^*, \eta_i, \eta_i^*$ is monic with degree i.

Definition 1.3.11 *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a tridiagonal system on V. We say Φ is sharp whenever the tridiagonal pair A, A^* is sharp.*

Definition 1.3.12 *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a tridiagonal system over $\mathbb{K}$, with the standard ordering of the eigenvalues $\{\theta_i\}_{i+0}^d$ (resp. $\{\theta_i^*\}_{i+0}^d$) of A (resp. A^*). By [NT08], for $0 \le i \le d$ there exists a unique $\zeta \in \mathbb{K}$ such that*

$$E_0^*\tau_i(A)E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\ldots(\theta_0^* - \theta_i^*)} .$$

Note that $\zeta_0 = 1$. We call $\{\zeta_i\}_{i=0}^d$ the split sequence of the tridiagonal system.

Definition 1.3.13 *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d$ denote a sharp tridiagonal system. By the parameter array of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the standard ordering of the eigenvalues of A (resp. A^*) and $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ.*

Definition 1.3.14 *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d$ denote a tridiagonal system on V and let $\Phi' = (A'; \{E'_i\}_{i=0}^d;\ A^{*\prime}; \{E_i^{*\prime}\}_{i=0}^d)$ denote a tridiagonal system on V'. We say Φ and Φ' are isomorphic whenever there exists an isomorphism of $\mathbb{K}$-vector spaces $\gamma : V \to V'$ such that $\gamma A = A'\gamma, \gamma A^* = A^{*\prime}\gamma$ and $\gamma E_i = E_i{}'\gamma, \gamma E_i^* = E_i^{*\prime}\gamma$ for $0 \le i \le d$.*

The following result shows the significance of the parameter array.

Theorem 1.3.24 *[NT08] Two sharp tridiagonal systems over $\mathbb{K}$ are isomorphic if and only if they have the same parameter array.*

Definition 1.3.15 *Let d denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{K}$. We call this sequence q-Racah whenever the following i), ii) hold.*
i) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \le i, j \le d)$.
ii) There exist $q, a, b, c, a^*, b^*, c^*$ that satisfy

$$\theta_i = a + bq^{2i-d} + cq^{d-2i} \quad (0 \le i \le d),$$

$$\theta_i^* = a^* + b^*q^{2i-d} + c^*q^{d-2i} \quad (0 \le i \le d),$$

$$q, a, b, c, a^*, b^*, c^* \in \overline{\mathbb{K}},$$

$$q \neq 0,\ q^2 \neq 1,\ q^2 \neq -1,\ bb^*cc^* \neq 0.$$

where $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$.

Theorem 1.3.25 *[IT08] Assume the field $\mathbb{K}$ is algebraically closed and let d denote a nonnegative integer. Let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a q-Racah sequence of scalars of $\mathbb{K}$ and let $\{\zeta_i\}_{i=0}^d$ denote any sequence of scalars in $\mathbb{K}$. Then the following are equivalent:*
i) There exists a tridiagonal system Φ over $\mathbb{K}$ that has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$
ii) $\zeta_0 = 1, \zeta_d \neq 0$, and

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*)\zeta_i.$$

Suppose i), ii) hold. Then Φ is unique up to isomorphism of tridiagonal systems.

Theorem 1.3.26 *[INT10] Let d denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d) \tag{1.3.39}$$

denote a sequence of scalars taken from $\mathbb{K}$. Then there exists a sharp tridiagonal system Φ over $\mathbb{K}$ with parameter (1.3.39) if and only if i)-iii) hold below.

i) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$.
ii) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i *for* $2 \leq i \leq d-1$.
iii) $\zeta_0 = 1, \zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*)\zeta_i.$$

Suppose i)-iii) hold. Then Φ *is unique up to isomorphism of tridiagonal systems.*

There is no doubt that if $\mathbb{K}$ is an algebraically closed field, then there exists a tridiagonal system over $\mathbb{K}$ with parameter array (1.3.39) if and only if this array satisfies the conditions i), ii) and iii) of Theorem 1.3.26.

1.3.3 Connection with the theory of orthogonal polynomials

In the literature, the first connection between the theory of orthogonal polynomials and Leonard pairs arises in the analysis of finite dimensional representation of Zhedanov's algebra [Zhe91] (also called the Askey-Wilson algebra). It is shown that the Askey-Wilson polynomials are the overlap coefficients between the 'dual' basis in which two generators are diagonalized respectively.

Furthermore, the theory of Leonard pairs gives a nice algebraic framework for the orthogonal polynomials of the Askey-scheme. Historically, there is a theorem due to Leonard [Leo82], [BI84, page 260] that gives a characterization of the q-Racah polynomials and some related polynomials in the Askey scheme [ARS95, AW79, AW85, KS96, Koo86]. In this Section, the connection between the theory of Leonard pairs and the Askey-scheme of orthogonal polynomials is recalled [Ter99, Ter01, Ter0306].

First, we recall hypergeometric orthogonal polynomials which appeared in [KS96]. Define q-analogue of the Pochhammer-symbol $(a)_k$

$$(a)_0 = 1 \text{ and } (a)_k = a(a+1)(a+2)\dots(a+k-1), \ k = 1, 2, 3, \dots.$$

This q-extension is given by

$$(a;q)_0 = 1 \text{ and } (a;q)_k = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{k-1}), \ k = 1, 2, 3 \dots.$$

It is clear that

$$\lim_{q \to 1} \frac{(q^\alpha; q)_k}{(1-q)^k} = (\alpha)_k.$$

The symbols $(a;q)_k$ are called q-shifted factorials. They can also be defined for negative values of k as

$$(a;q)_k = \frac{1}{(1-aq^{-1})(1-aq^{-2})\dots(1-aq^k)}, \ a \neq q, q^2, q^3, \dots, q^{-k}, k = -1, -2, -3, \dots.$$

We can also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

This implies that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

and, for any complex number λ,

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty}$$

The hypergeometric series ${}_rF_s$ is defined by

$${}_rF_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array}\middle| z\right) = \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r)_k}{(b_1,\dots,b_s)_k}\frac{z^k}{k!},$$

where

$$(a_1,\dots,a_r)_k = (a_1)_k\dots(a_r)_k.$$

The hypergeometric series ${}_rF_s$ is called balanced if $r=s+1, z=1$ and $a_1+a_2+\dots+a_{s+1}+1 = b_1+b_2+\dots+b_s$.
The basic hypergeometric series (or q-hypergeometric series) ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array}\middle| q;z\right) = \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\begin{pmatrix} k \\ 2 \end{pmatrix}}\frac{z^k}{(q;q)_k},$$

where

$$(a_1,\dots,a_r;q)_k = (a_1;q)_k\dots(a_r;q)_k.$$

The special case $r=s+1$ reads

$${}_{s+1}\phi_s\left(\begin{array}{c} a_1,\dots,a_{s+1} \\ b_1,\dots,b_s \end{array}\middle| q;z\right) = \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}\frac{z^k}{(q;q)_k}.$$

A basic hypergeometric series is called balanced if $z=q$ and $a_1a_2\dots a_{s+1}q = b_1b_2\dots b_s$.

There is a natural correspondence between Leonard pairs and a family of orthogonal polynomials. The following material is taken from [Ter99, Ter01].

Let λ denote an indeterminant, and let $\mathbb{K}[\lambda]$ denote a $\mathbb{K}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{K}$. Let $\Phi = (A;E_0,E_1,\dots,E_d;A^*;E_0^*,E_1^*,\dots,E_d^*)$ denote a Leonard system over $\mathbb{K}$. Then there exists a unique sequence of monic polynomials $p_0,p_1,\dots,p_{d+1};\ p_0^*,p_1^*,\dots,p_{d+1}^*$ in $\mathbb{K}[\lambda]$ such that

$$deg(p_i) = i, \quad deg(p_i^*) = i \quad (0\le i\le d+1),$$

$$p_i(A)E_0^* = E_i^* A^i E_0^*, \quad p_i^*(A^*)E_0 = E_i A^{*i} E_0 \quad (0 \le i \le d),$$
$$p_{d+1}(A) = 0, \quad p_{d+1}^*(A^*) = 0.$$

These polynomials satisfy

$$p_0 = 1, \quad p_0^* = 1, \tag{1.3.40}$$
$$\lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \le i \le d), \tag{1.3.41}$$
$$\lambda p_i^* = p_{i+1}^* + a_i^* p_i^* + x_i^* p_{i-1}^* \quad (0 \le i \le d), \tag{1.3.42}$$

where $x_0, x_0^*, p_{-1}, p_{-1}^*$ are all 0, and where

$$a_i = tr(E_i^* A), \quad a_i^* = tr(E_i A^*) \quad (0 \le i \le d),$$

$$x_i = tr(E_i^* A E_{i-1}^* A), \quad x_i^* = tr(E_i A^* E_{i-1} A^*).$$

In fact

$$x_i \ne 0, \quad x_i^* \ne 0 \quad (1 \le i \le d). \tag{1.3.43}$$

We call $p_0, p_1, \ldots, p_{d+1}$ the monic polynomial sequence of Φ, and $p_0^*, p_1^*, \ldots, p_{d+1}^*$ the dual monic polynomial sequence of Φ.
Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ, so that

$$\theta_i \ne \theta_j, \quad \theta_i^* \ne \theta_j^* \quad \text{if } i \ne j \quad (0 \le i, j \le d), \tag{1.3.44}$$
$$p_{d+1}(\theta_i) = 0, \quad p_{d+1}^*(\theta_i^*) = 0 \quad (0 \le i \le d). \tag{1.3.45}$$

Then

$$p_i(\theta_0) \ne 0, \quad p_i^*(\theta_0^*) \ne 0 \quad (0 \le i \le d), \tag{1.3.46}$$

and

$$\frac{p_i(\theta_j)}{p_i(\theta_0)} = \frac{p_j^*(\theta_i^*)}{p_j^*(\theta_0^*)} \quad (0 \le i, j \le d). \tag{1.3.47}$$

Conversely, given polynomials

$$p_0, p_1, \ldots, p_{d+1}, \tag{1.3.48}$$
$$p_0^*, p_1^*, \ldots, p_{d+1}^* \tag{1.3.49}$$

in $\mathbb{K}[\lambda]$ satisfying (1.3.40)-(1.3.42), and given scalars

$$\theta_0, \theta_1, \ldots, \theta_d, \tag{1.3.50}$$
$$\theta_0^*, \theta_1^*, \ldots, \theta_d^* \tag{1.3.51}$$

in $\mathbb{K}$ satisfying (1.3.44)-(1.3.47), there exists a Leonard system Φ over $\mathbb{K}$ with monic polynomial sequence (1.3.48), dual monic polynomial sequence (1.3.49), eigenvalue sequence (1.3.50), and dual eigenvalue sequence (1.3.51). The system Φ is unique up to isomorphism of Leonard systems.

Hence, there is a bijection between the Leonard systems and systems (1.3.48)-(1.3.51) satisfying (1.3.40)-(1.3.47). For $\mathbb{K} = \mathbb{R}$, the systems (1.3.48)-(1.3.51) satisfying (1.3.40)-(1.3.47) were classified by Leonard, Bannai, and Ito. They found the polynomials involved are q-Racah polynomials or related polynomials from the Askey scheme.

We define the polynomials

$$u_i = \frac{p_i}{p_i(\theta_0)}, \quad u_i^* = \frac{p_i^*}{p_i^*(\theta_0^*)} \quad (0 \le i \le d).$$

By (1.3.47),

$$u_i(\theta_j) = u_j^*(\theta_i^*) \quad (0 \le i, j \le d),$$

then u_i and u_i^* are dual sequences of normalized polynomials [Leo82].

Furthermore, there exists a unique sequence of scalars $c_1, c_2, \ldots, c_d;\ b_0, b_1, \ldots, b_{d-1}$ in $\mathbb{K}$ such that:

$$\begin{aligned} x &= b_{i-1}c_i \quad (1 \le i \le d), \\ \theta_0 &= c_i + a_i + b_i \quad (0 \le i \le d), \end{aligned}$$

where $c_0 = 0, b_d = 0$. Then

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \quad (0 \le i \le d-1),$$

and $\lambda u_d - c_d u_{d-1} - a_d u_d$ vanishes on each of $\theta_0, \theta_1, \ldots, \theta_d$, where λ is an indeterminant. By the main theorem of [Leo82], we have

$$b_i = \varphi_{i+1}\frac{\tau_i^*(\theta_i^*)}{\tau_{i+1}^*(\theta_{i+1}^*)}, \quad b_i^* = \varphi_{i+1}\frac{\tau_i(\theta_i)}{\tau_{i+1}(\theta_{i+1})} \quad (0 \le i \le d-1),$$

$$c_i = \phi_i\frac{\eta_{d-i}^*(\theta_i^*)}{\eta_{d-i+1}^*(\theta_{i-1}^*)}, \quad c_i^* = \phi_i\frac{\eta_{d-i}(\theta_i)}{\eta_{d-i+1}(\theta_{i-1})} \quad (1 \le i \le d),$$

where

$$\begin{aligned} \tau_i(\lambda) &= (\lambda - \theta_0)(\lambda - \theta_1)\ldots(\lambda - \theta_{i-1}), \\ \tau_i^*(\lambda) &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\ldots(\lambda - \theta_{i-1}^*), \\ \eta_i(\lambda) &= (\lambda - \theta_d)(\lambda - \theta_{d-1})\ldots(\lambda - \theta_{d-i+1}), \\ \eta_i^*(\lambda) &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*)\ldots(\lambda - \theta_{d-i+1}^*). \end{aligned}$$

Hence, the polynomials u_i, u_i^* $(0 \le i \le d)$ are given by

$$u_i = \sum_{h=0}^{i} \frac{\tau_h^*(\theta_i^*)}{\varphi_1\varphi_2\ldots\varphi_h}\tau_h, \quad u_i^* = \sum_{h=0}^{i} \frac{\tau_h(\theta_i)}{\varphi_1\varphi_2\ldots\varphi_h}\tau_h^*.$$

We find that for $0 \le i, j \le d$, the common value of $u_i(\theta_j), u_j^*(\theta_i^*)$

$$\sum_{n=0}^{d} \frac{(q^{-i};q)_n (s^* q^{i+1};q)_n (q^{-j};q)_n (sq^{j+1};q)_n q^n}{(r_1 q;q)_n (r_2 q;q)_n (q^{-d};q)_n (q;q)_n} \tag{1.3.52}$$

In fact, (1.3.52) is the basic hypergeometric series

$${}_4\theta_3\left(\begin{matrix} q^{-i}, s^* q^{i+1}, q^{-j}, sq^{j+1} \\ r_1 q, r_2 q, q^{-d} \end{matrix} \middle| q, q \right),$$

where r_1, r_2, s, s^* are parameters in (1.3.11)-(1.3.14) and it is balanced because $r_1 r_2 = ss^* q^{d+1}$. Then u_i, u_i^* are q-Racah polynomials.

Now we consider orthogonality of p_i, p_i^* and u_i, u_i^*.

Put $m_i = tr(E_i E_0^*)$, $m_i^* = tr(E_i^* E_0), 0 \le i \le d$. Then each of m_i, m_i^* is nonzero $(0 \le i \le d)$, and the orthogonality for the p_i is

$$\sum_{r=0}^{d} p_i(\theta_r) p_j(\theta_r) m_r = \delta_{ij} x_1 x_2 \dots x_i \quad (0 \le i, j \le d),$$

$$\sum_{i=0}^{d} \frac{p_i(\theta_r) p_i(\theta_s)}{x_1 x_2 \dots x_i} = \delta_{rs} m_r^{-1} \quad (0 \le r, s \le d).$$

Observe that $m_0 = m_0^*$, let υ denote the multiplicative inverse of this common value, and set

$$k_i = m_i^* \upsilon \quad (0 \le i, j \le d).$$

The orthogonality for the u_i is

$$\sum_{r=0}^{d} u_i(\theta_r) u_j(\theta_r) m_r = \delta_{ij} k_i^{-1} \quad (0 \le i, j \le d),$$

$$\sum_{i=0}^{d} u_i(\theta_r) u_i(\theta_s) k_i = \delta_{rs} m_r^{-1} \quad (0 \le r, s \le d).$$

Remark that

$$k_i = \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i},$$

and

$$\upsilon = k_0 + k_1 + \dots + k_d.$$

As an example, we now consider an infinite family of Leonard pairs and find the relation with hypergeometric series. For any nonnegative integer d, the pair

$$A = \begin{pmatrix} 0 & d & & & & 0 \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ 0 & & & & d & 0 \end{pmatrix}, \quad A^* = \mathrm{diag}(d, d-2, d-4, \dots, -d)$$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d. One shows $P^2 = 2^d I$ and $AP = PA^*$, where P denotes the matrix with ij entry

$$P_{ij} = \begin{pmatrix} d \\ j \end{pmatrix} \sum_{n=0}^{d} \frac{(-i)_n(-j)_n 2^n}{(-d)_n n!} \quad (0 \le i, j \le d).$$

Further P_{ij} can be written in form of hypergeometric series

$$P_{ij} = {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| 2 \right).$$

There exist Leonard pairs similar to the one above in which the series of type ${}_2F_1$ is replaced by a series of one of the following types: ${}_3F_2$, ${}_4F_3$, ${}_2\phi_1$, ${}_3\phi_2$, ${}_4\phi_3$.

Let $\mathbb{K}$ denote a field with characteristic 0. Let d denote a nonnegative integer. Put $\Omega = \{d - 2i | i = 0, 1, \ldots, d\}$. Let V denote a vector space over $\mathbb{K}$ consisting of all functions from Ω to $\mathbb{K}$. Since the cardinality of Ω is $d + 1$ the dimension of V is $d + 1$.

Define two transformations A, A^* from V to V as follows.

For all $f \in V, \theta \in \Omega$,

$$(Af)(\theta) = \theta f(\theta), \tag{1.3.53}$$

$$(A^* f)(\theta) = \frac{d+\theta}{2} f(\theta - 2) + \frac{d-\theta}{2} f(\theta + 2). \tag{1.3.54}$$

It is easy seen that A, A^* are linear.

For $j = 0, \ldots, d$ let K_j denote the element in V satisfying

$$K_j(\theta_i) = \frac{d!}{j!(d-j)!} \, {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| 2 \right),$$

where $\theta_i = d - 2i, 0 \le i \le d$.
Observe that $K_j(\theta)$ is a polynomial of degree j in θ. The polynomials $K_0, K_1, \ldots, K_d$ are Krawtchouk polynomials and form a basis for V. With respect to this basis the matrices representing A and A^* are

$$A = \begin{pmatrix} 0 & d & & & & 0 \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ 0 & & & & d & 0 \end{pmatrix}, \quad A^* = \mathrm{diag}(d, d-2, d-4, \ldots, -d).$$

Since the pair (A, A^*) is a Leonard pair on V, there exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible

tridiagonal. Now we display this basis. For $0 \leq j \leq d$ let K_j^* denote the element in V which satisfies

$$K_j^*(\theta_i) = \delta_{ij} \quad (0 \leq i \leq d),$$

where δ_{ij} denotes the Kronecker delta. The sequence $K_0^*, K_1^*, \ldots, K_d^*$ forms a basis for V. With respect to this basis the matrices representing A and A^* are

$$A^* = \begin{pmatrix} 0 & d & & & & 0 \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ 0 & & & & d & 0 \end{pmatrix}, \quad A^* = \mathrm{diag}(d, d-2, d-4, \ldots, -d).$$

We have then shown how the above Krawtchouk polynomials correspond to the Leonard pairs.

The polynomials in the following table are related to Leonard pairs in a similar fashion.

Type	Polynomial
${}_4F_3$	Racah
${}_3F_2$	Hahn, dual Hahn
${}_2F_1$	Krawtchouk
${}_4\phi_3$	q-Racah
${}_3\phi_2$	q-Hahn, dual q-Hahn
${}_2\phi_1$	q-Krawtchouk (classical, affine, quantum, dual)

The above polynomials are defined in Koekoek and Swarttouw [KS96], and the connection to Leonard pairs is given in [Ter01, BI84]. Indeed, these polynomials exhaust all Leonard pairs for which $q \neq -1$. For $\mathbb{K} = \mathbb{R}$, the classification of Leonard pairs amounts to a "linear algebraic version" of Leonard's theorem [Leo82, BI84].

Chapter 2

Mathematical Physics: background

In the first part, we recall the two known presentations of the Onsager algebra. The second presentation (the original one) is given by an infinite set of elements $\{A_n\}, n = 0, \pm 1, \pm 2, \ldots,$ $\{G_m\}, m = 1, 2, \ldots$ that satisfy the relations (2.1.1)-(2.1.3) [O44]. The first presentation introduced by Dolan-Grady is given by A_0, A_1 satisfying the relations (2.1.12)-(2.1.13) [DG82]. It is also explained that the elements of an Abelian subalgebra of the Onsager algebra provide examples of mutually commuting quantities (see (2.1.6)) that generate integrable systems. Also, the connection with $\widehat{sl_2}$ (more precisely, with the loop algebra of sl_2) is described. For irreducible finite dimensional representation of the Onsager algebra, it is known that the generators of the second representation satisfy additional relations (2.1.17)-(2.1.18). These relations are usually called the Davies' relations [Dav91, Dav90].

In the second and third parts, by analogy with the undeformed case discussed in the first part, it is explained that the $q-$Onsager algebra admits two presentations. The second presentation is given by the generators $\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}$ that satisfy the infinite dimensional algebra $\mathcal{A}_q$ (see Definition 1.2.6) [BS09]. The first presentation is given by the standard generators $\mathcal{W}_0, \mathcal{W}_1$ that satisfy the $q-$Dolan-Grady relations [Ter01, Bas0404, Bas0408]. It is explained how mutually commuting quantities (2.3.5) that generate an Abelian subalgebra of the $q-$Onsager algebra are derived by using Sklyanin's formalism. Also, the connection between the two presentations and $U_q(\widehat{sl_2})$ (more precisely, the quantum loop algebra of sl_2) is described. For most of the examples considered in the literature [Bas0404, BK0503, BK07], the vector space on which the elements act is finite dimensional. As a consequence, quantum analogs of Davies relations naturally appear, see (2.2.39)-(2.2.42).

In the last part, we briefly recall how the open XXZ spin chain with generic boundary conditions and generic values of q can be formulated using the $q-$Onsager approach.

2.1 Historical background

The exact solution of the planar Ising model in zero magnetic field which was obtained by Onsager [O44] has provided a considerable source of developments in the theory of

exactly solvable systems of statistical mechanics, or quantum field theory in two dimensions. Onsager's successful approach was originally based on the so-called Onsager algebra and its representation theory. In progress of solving the two dimensional Ising model, he established the transfer matrix of the model in terms of an infinite set of elements $\{A_n\}, n = 0, \pm 1, \pm 2, \ldots,$ $\{G_m\}, m = 1, 2, \ldots$ that generate the so-called Onsager algebra.

Definition 2.1.1 *[O44] The Onsager algebra is a Lie algebra which has generators A_n, G_m, $n = 0, \pm 1, \pm 2, \ldots, m = 1, 2, \ldots$ such that they satisfy the following relations*

$$[A_n, A_l] = 4G_{n-l}, \tag{2.1.1}$$
$$[G_m, A_n] = 2A_{n+m} - 2A_{n-m}, \tag{2.1.2}$$
$$[G_m, G_l] = 0. \tag{2.1.3}$$

Use algebraic methods, Onsager derived the largest eigenvalue and the corresponding eigenvector of the transfer matrix of the Ising model. Despite the important role of the Onsager algebra, it received less attention in the following years than the star triangular relations which originated in [O44], [Wan45] and led to the Yang-Baxter equations, the theory of quantum groups, as well as the quantum inverse scattering method.

In the 1980s the Onsager algebra appeared [Dav91], [Dav90], [Perk87] to be closely related with the quantum integrable structure discovered by Dolan and Grady in [DG82]. In fact, Dolan and Grady considered a self-dual quantum Hamiltonian of the form

$$H = \kappa A + \kappa^* \tilde{A}, \tag{2.1.4}$$

where κ, κ^* are coupling constants and $\tilde{A}$ is the operator dual to A such that both operators satisfy the condition

$$[A, [A, [A, \tilde{A}]]] = 16[A, \tilde{A}]. \tag{2.1.5}$$

As a consequence of this relation, there exists an infinite set of commuting conserved self-dual charges

$$\mathcal{I}_{2n} = \kappa(W_{2n} - \tilde{W}_{2n-2}) + \kappa^*(\tilde{W}_{2n} - W_{2n-2}),\ n = 1, 2, \ldots, \tag{2.1.6}$$

where

$$W_{2n+2} \equiv -\frac{1}{8}[A, [\tilde{A}, W_{2n}]] - \tilde{W}_{2n},\ n = 1, 2, \ldots, \tag{2.1.7}$$

$W_0 \equiv A, \mathcal{I}_0 \equiv H$, and the sequence $\{W_{2n}\},\ n = 1, 2, \ldots$ can be extended to $n < 0$ by defining $W_{-2n} \equiv -\tilde{W}_{2n-2}$.
It was showed that $[\tilde{W}_{2l}, W_{2n-2l-2}] = [\tilde{W}_{2l-2}, W_{2n-2l}]$, for all $n \geq 0,\ l \geq 0$, then it followed

$$[H, \mathcal{I}_{2n}] = 0 \ \text{ and } \ [\mathcal{I}_{2n}, \mathcal{I}_{2m}] = 0 \tag{2.1.8}$$

Clearly, Dolan and Grady [DG82] showed the the Dolan-Grady relations are sufficient to guarantee that there is an infinite sequence of commuting operators of the Hamiltonian H. Based on this result the integrability does not depend on the dimension of the system or the nature of the space-time manifold, i.e, lattice, continuum or loop space. Integrable systems

are characterized by the existence of a sufficient number of constants of motion, i.e. equal to the number of degrees of freedom. The rigorous connection between simple self-duality and a set of commuting conserved charges was established. For a finite lattice, the set is finite. For an infinite system, the set is infinite. In the XY and Ising modes, the charges coincide with known results. Moreover, von Gehlen and Rittenberg [GR85] considered some Z_n symmetric quantum spin chain Hamiltonians (2.1.4), and presented strong numerical evidence that they exhibit Ising-like behavior in their spectra. Namely, the Hamiltonians have an infinite set of commuting conserved charges based on the Dolan-Grady relation

$$[A,[A,[A,\tilde{A}]]] = n^2[A,\tilde{A}]. \tag{2.1.9}$$

Obviously, if we put $B = 4n^{-1}A,\ \tilde{B} = 4n^{-1}\tilde{A}$ the equation (2.1.9) becomes the equation (2.1.5) for $B, \tilde{B}$. It means that

$$[B,[B,[B,\tilde{B}]]] = 16[B,\tilde{B}]. \tag{2.1.10}$$

In the early 1990s, Davies [Dav91], [Dav90] obtained the relation between the Onsager algebra and the Dolan Grady relations. Actually, Davies did not require the self-duality of operators A_0, A_1 in the Hamiltonian

$$H = \kappa A_0 + \kappa^* A_1, \tag{2.1.11}$$

where κ, κ^* are coupling constants, but he gave a pair of conditions

$$[A_0,[A_0,[A_0,A_1]]] = 16[A_0,A_1], \tag{2.1.12}$$
$$[A_1,[A_1,[A_1,A_0]]] = 16[A_1,A_0]. \tag{2.1.13}$$

We first recall the way to identify an Onsager algebra from the Dolan-Grady relations [Dav91]. Let A_0, A_1 denote generators satisfying the Dolan-Grady relations, define the sequences $A_n, G_m, n = 0, \pm1, \pm2, \ldots, m = 1, 2, \ldots$ by the recursion relations

$$G_1 = \frac{1}{4}[A_1, A_0], \tag{2.1.14}$$
$$A_{n+1} - A_{n-1} = \frac{1}{2}[G_1, A_n], \tag{2.1.15}$$
$$G_n = \frac{1}{4}[A_n, A_0]. \tag{2.1.16}$$

The generators A_n, G_m satisfy the defining relations of the Onsager algebra [Dav91].

Inversely, if $A_n, G_m, n = 0, \pm1, \pm2, \ldots, m = 1, 2, \ldots$ are generators of an Onsager algebra, then A_0, A_1 satisfy the Dolan-Grady relations. In fact, every adjacent pairs A_k, A_{k+1} of the sequence $\{A_n\}$ satisfy the Dolan-Grady relations. As a consequence, the Onsager algebra admits two presentations. One given by (2.1.1) - (2.1.3) and one given by (2.1.12) - (2.1.13). The proof of isomorphism between the two presentations is detailed in [Dav91]. More recently, see also [El10]

Around the same time, a relation between the Onsager algebra and the loop algebra of sl_2 was exhibited by Davies [Dav90]. It is argued that if the Onsager algebra acts on a finite dimensional vector space, then there exists some value of h such that the sequences $\{A_n\}, \{G_m\}$ satisfy the linear recurrence relations of length $(2h+1)$, namely

$$\sum_{k=-h}^{h} \alpha_k A_{k-l} = 0, \tag{2.1.17}$$

$$\sum_{k=-h}^{h} \alpha_k G_{k-l} = 0 \tag{2.1.18}$$

where l is arbitrary. Then, Davies showed that the finite-dimensional Onsager algebra is the direct sum of h copies of the algebra sl_2,

$$A_n = 2\sum_{j=1}^{h} (z_j^n E_j^+ + z_j^{-n} E_j^-), \tag{2.1.19}$$

$$G_m = \sum_{j=1}^{h} (z_j^m - z_j^{-m}) H_j \tag{2.1.20}$$

where $[E_j^+, E_k^-] = \delta_{jk} H_k, \quad [H_j, E_k^\pm] = \pm 2\delta_{jk} E_k^\pm$ are the generators of the sl_2 algebra and z_j are called the evaluation parameters of the representation.

Furthermore, after using the expression of A_n in terms of $E_j^\pm$ (2.1.19), the eigenvalues of the Hamiltonian (2.1.11) in the sector that is the direct product of n factors of dimension $d_j,\ 1 \leq j \leq n$ fit the general form

$$\lambda(\kappa, \kappa^*) = \kappa\alpha + \kappa^*\beta + \sum_{j=1}^{n} 4m_j \sqrt{\kappa^2 + \kappa^{*2} + 2\kappa\kappa^* \cos\theta_j}, \quad m_j = -s_j, -s_j+1, \ldots, s_j, \tag{2.1.21}$$

where $z_j = e^{-i\theta_j}$; α, β is a pair of eigenvalues of A_0, A_1; and $d_j = (2s_j+1)$ is the dimension of an irreducible representation of sl_2 associated with the pair z_j, z_j^{-1}.

2.2 Sklyanin's formalism and the $q-$Onsager algebra

Among the known examples of quadratic algebraic structures, one finds the Yang-Baxter algebra. For further analysis, let us first recall some known results. This algebra consists of a couple $R(u), L(u)$ where the R-matrix solves the Yang-Baxter equation

$$R_{\mathcal{V}_0}(u) R_{\mathcal{V}_0\mathcal{V}_0'}(uv) R_{\mathcal{V}_0'}(v) = R_{\mathcal{V}_0'}(v) R_{\mathcal{V}_0\mathcal{V}_0'}(uv) R_{\mathcal{V}_0}(u) \tag{2.2.1}$$

and the so-called L-operator satisfies the quadratic relation

$$R_{\mathcal{V}_0\mathcal{V}_0'}(u/v)(L_{\mathcal{V}_0}(u) \otimes L_{\mathcal{V}_0'}(v)) = (L_{\mathcal{V}_0'}(v) \otimes L_{\mathcal{V}_0}(u)) R_{\mathcal{V}_0\mathcal{V}_0'}(u/v) \tag{2.2.2}$$

where $\mathcal{V}_0, \mathcal{V}_0'$ denote finite dimensional auxiliary space representations. Here, the entries of the L-operators act on a quantum space denoted $\mathcal{V}$. If one considers a two-dimensional

(spin-$\frac{1}{2}$) representation for $\mathcal{V}_0$ and $\mathcal{V}_0'$, a solution $R(u)$ of the Yang-Baxter equation and the L-operator can be written in the form

$$R(u) = \sum_{i,j\in\{0,z,\pm\}} \omega_{ij}(u)\ \sigma_i \otimes \sigma_j, \tag{2.2.3}$$

$$L(u) = \sum_{i,j\in\{0,z,\pm\}} \omega_{ij}(u)\ \sigma_i \otimes S_j, \tag{2.2.4}$$

where $\sigma_z, \sigma_\pm$ are Pauli matrices, $\sigma_0 = \mathbb{I}$ and $\omega_{ij}(u)$ are some combinations of functions

$$\omega_{00}(u) = \frac{1}{2}(q+1)(u - q^{-1}u^{-1}), \tag{2.2.5}$$

$$\omega_{zz}(u) = \frac{1}{2}(q-1)(u + q^{-1}u^{-1}), \tag{2.2.6}$$

$$\omega_{+-}(u) = \omega_{-+}(u) = q - q^{-1}. \tag{2.2.7}$$

Note that the defining relations of the algebra generated by elements $\{S_j\}$ are determined by the equation (2.2.2). The elements $\{S_j\}$ act on the quantum space $\mathcal{V}$. The corresponding algebra is known as the Sklyanin algebra [Sk88]. It admits a trigonometric degeneration such that the elements $\{S_j\}$ are identified with the generators $\{S_\pm, s_3\}$ of the quantum enveloping algebra $U_q(sl_2)$

$$S_0 = \frac{q^{s_3} + q^{-s_3}}{q^{1/2} + q^{-1/2}}, \tag{2.2.8}$$

$$S_z = \frac{q^{s_3} - q^{-s_3}}{q^{1/2} - q^{-1/2}}, \tag{2.2.9}$$

where $[s_3, S_\pm] = \pm S_\pm$ and $[S_+, S_-] = \frac{q^{2s_3} - q^{-2s_3}}{q - q^{-1}}$, together with the Casimir operator

$$w = qq^{2s_3} + q^{-1}q^{-2s_3} + (q - q^{-1})^2 S_- S_+. \tag{2.2.10}$$

Following [Bas0404], [Bas0408], let us consider the reflection equation which was first introduced by Cherednik [Cher84] (see also [Sk88]):

$$R(u/v)(K(u) \otimes \mathbb{I})R(uv)(\mathbb{I} \otimes K(v)) = (\mathbb{I} \otimes K(v))R(uv)(K(u) \otimes \mathbb{I})R(u/v). \tag{2.2.11}$$

This equation arises, for instance, in the context of the quantum integrable systems with boundaries [Sk88]. Similarly to (2.2.4), in the spin-$\frac{1}{2}$ we introduce a K-matrix of the form:

$$K(u) = \sum_{j\in\{0,z,\pm\}} \sigma_j \otimes \Omega_j(u). \tag{2.2.12}$$

By [Bas0404], any solutions of the reflection equation (2.2.11) of degree $-2 \leq d \leq 2$ in the spectral parameter u - with non-commuting entries - can be written in the form (2.2.12)

where

$$\Omega_0(u) = \frac{(A+A^*)(qu-q^{-1}u^{-1})}{2}, \tag{2.2.13}$$

$$\Omega_z(u) = \frac{(A-A^*)(qu+q^{-1}u^{-1})}{2}, \tag{2.2.14}$$

$$\Omega_+(u) = -\frac{qu^2+q^{-1}u^{-2}}{c_0c_1(q^2-q^{-2})} - c_1[A^*,A]_q + c_2, \tag{2.2.15}$$

$$\Omega_-(u) = -\frac{qu^2+q^{-1}u^{-2}}{c_1(q^2-q^{-2})} - c_0c_1[A,A^*]_q + c_0c_2 \tag{2.2.16}$$

where the parameters $c_0, c_1 \neq 0$, c_2 are arbitrary, and A, A^* have to satisfy the Askey-Wilson relations (1.3.9)-(1.3.10) with particular values of the parameters

$$\rho = \rho^* = \frac{1}{c_0c_1^2}, \tag{2.2.17}$$

$$\omega = -\frac{c_2}{c_1}(q-q^{-1}), \tag{2.2.18}$$

$$\gamma = \gamma^* = \eta = \eta^* = 0. \tag{2.2.19}$$

This explicit relation between the Askey-Wilson algebra (1.3.9)- (1.3.10) and the reflection equation algebra through the analysis of $K-$operators suggested to investigate further this new connection.

The generalization of the above connection goes as follows. From the results of [Sk88], for any parameter v it is for instance known that

$$K^{(L)}(u) = L_n(uv)\dots L_1(uv)K^{(0)}(u)L_1(uv^{-1})\dots L_n(uv^{-1}), \tag{2.2.20}$$

the so-called Sklyanin's operator, gives a family of solutions to (2.2.11). Here $L_j(u)$ is the Lax operator given by (2.2.4), and we choose (for simplicity) the trivial solution of (2.2.11) to be $K^{(0)}(u) = (\sigma_+/c_0 + \sigma_-)/(q-q^{-1})$. For these choices, the Sklyanin operator acts on the quantum space $\otimes_{j=1}^L \mathcal{V}_j \otimes \mathcal{V}_0$.

Then, the Sklyanin's operator $K^{(L)}$ can be written as follows

$$K^{(L)}(u) = \sum_{j\in\{0,z,\pm\}} \sigma_j \otimes \Omega_j^{(L)}(u), \tag{2.2.21}$$

where the operators $\Omega_j^{(L)}$ are combinations of Laurent polynomials of degree $-2L \leq d \leq 2L$ in the spectral parameter u and operators acting solely on $\otimes_{j=1}^L U_q(sl_2)$. Baseilhac and Koizumi [BK0503] obtained the following result

Theorem 2.2.1 *[BK0503] For generic values of L, the operators $\Omega_j^{(L)}(u)$ are given by:*

$$\Omega_0^{(L)}(u)+\Omega_3^{(L)}(u) = uq\sum_{k=0}^{L-1}P_{-k}^{(L)}(u)\mathcal{W}_{-k}^{(L)}-u^{-1}q^{-1}\sum_{k=0}^{L-1}P_{-k}^{(L)}(u)\mathcal{W}_{k+1}^{(L)}, \tag{2.2.22}$$

$$\Omega_0^{(L)}(u)-\Omega_3^{(L)}(u) = uq\sum_{k=0}^{L-1}P_{-k}^{(L)}(u)\mathcal{W}_{k+1}^{(L)}-u^{-1}q^{-1}\sum_{k=0}^{L-1}P_{-k}^{(L)}(u)\mathcal{W}_{-k}^{L}, \tag{2.2.23}$$

$$\Omega_+^{(L)}(u) = \frac{qu^2+q^{-1}u^{-2}}{c_0(q-q^{-1})}P_0^{(L)}(u)+\frac{1}{q+q^{-1}}\sum_{k=0}^{L-1}P_{-k}^{(L)}(u)\mathcal{G}_{k+1}^{(L)}+\omega_0^{(L)}, \tag{2.2.24}$$

$$\Omega_-^{(L)}(u) = \frac{qu^2+q^{-1}u^{-2}}{q-q^{-1}}P_0^{(L)}(u)+\frac{c_0}{q+q^{-1}}\sum_{k=0}^{L-1}P_{-k}^{(L)}(u)\tilde{\mathcal{G}}_{k+1}^{(L)}+c_0\omega_0^{(L)}, \tag{2.2.25}$$

where $P_{-k}^{(L)}(u)$ are Laurent polynomials defined by

$$P_{-k}^{(L)}(u) = -\frac{1}{q+q^{-1}}\sum_{n=k}^{L-1}\left(\frac{qu^2+q^{-1}u^{-2}}{q+q^{-1}}\right)^{n-k}C_{-n}^{(L)}, \tag{2.2.26}$$

$$C_{-n}^{(L)} = (q+q^{-1})^{n+1}(-1)^{L-n}\left(\frac{(v^2+v^{-2})w_0^{(j)}}{q+q^{-1}}\right)^{L-n+1}\frac{L!}{(n+1)!(L-n-1)!}, \tag{2.2.27}$$

$$\omega_0^{(L+1)} = -\frac{v^2+v^{-2}}{q+q^{-1}}w_0^{(j)}\omega_0^{(L)}, \tag{2.2.28}$$

$$\omega_0^{(1)} = -\frac{v^2+v^{-2}}{c_0(q-q^{-1})}w_0^{(j)}, \tag{2.2.29}$$

$$w_0^{(j)} = q^{2j+1}+q^{-2j-1}, \tag{2.2.30}$$

provided the generators $\mathcal{W}_{-k}^{(L)},\mathcal{W}_{k+1}^{(L)},\mathcal{G}_{k+1}^{(L)},\tilde{\mathcal{G}}_{k+1}^{(L)}$ act on L-tensor product evaluation representation of $U_q(sl_2)$:

$$\mathcal{W}_0^{(L)} = \frac{1}{c_0}vq^{1/4}S_+q^{s_3/2}\otimes\mathbb{I}+v^{-1}q^{-1/4}S_-q^{s_3/2}\otimes\mathbb{I}+q^{s_3}\otimes\mathcal{W}_0^{(L-1)}, \tag{2.2.31}$$

$$\mathcal{W}_1^{(L)} = \frac{1}{c_0}v^{-1}q^{-1/4}S_+q^{-s_3/2}\otimes\mathbb{I}+vq^{1/4}S_-q^{-s_3/2}\otimes\mathbb{I}+q^{-s_3}\otimes\mathcal{W}_1^{(L-1)}\ , \tag{2.2.32}$$

$$\begin{aligned}\mathcal{G}_1^{(L)} &= (q-q^{-1})S_-^2\otimes\mathbb{I}-\frac{(q^{1/2}+q^{-1/2})}{c_0(q^{1/2}-q^{-1/2})}(v^2q^{s_3}+v^{-2}q^{-s_3})\otimes\mathbb{I}+\mathbb{I}\otimes\mathcal{G}_1^{(L-1)}\\ &\quad+(q-q^{-1})\left(vq^{-1/4}S_-q^{s_3/2}\otimes\mathcal{W}_0^{(L-1)}+v^{-1}q^{1/4}S_-q^{-s_3/2}\otimes\mathcal{W}_1^{(L-1)}\right)\\ &\quad+\frac{(v^2+v^{-2})w_0^{(j)}}{c_0(q^{1/2}-q^{-1/2})}\otimes\mathbb{I},\end{aligned} \tag{2.2.33}$$

$$\begin{aligned}\tilde{\mathcal{G}}_1^{(L)} &= \frac{(q-q^{-1})}{c_0^2}S_+^2\otimes\mathbb{I}-\frac{(q^{1/2}+q^{-1/2})}{c_0(q^{1/2}-q^{-1/2})}(v^2q^{-s_3}+v^{-2}q^{s_3})\otimes\mathbb{I}+\mathbb{I}\otimes\tilde{\mathcal{G}}_1^{(L-1)}\\ &+\frac{(q-q^{-1})}{c_0}\left(v^{-1}q^{1/4}S_+q^{s_3/2}\otimes\mathcal{W}_0^{(L-1)}+vq^{-1/4}S_+q^{-s_3/2}\otimes\mathcal{W}_1^{(L-1)}\right)\\ &+\frac{(v^2+v^{-2})w_0^{(j)}}{c_0(q^{1/2}-q^{-1/2})}\otimes\mathbb{I},\end{aligned}\tag{2.2.34}$$

$$\begin{aligned}\mathcal{W}_{-k-1}^{(L)} &= \frac{(w_0^{(j)}-(q^{1/2}+q^{-1/2})q^{s_3})}{(q^{1/2}+q^{-1/2})}\otimes\mathcal{W}_{k+1}^{(L-1)}-\frac{(v^2+v^{-2})}{(q^{1/2}+q^{-1/2})}\mathbb{I}\otimes\mathcal{W}_{-k}^{(L-1)}\\ &+\frac{(q^{1/2}-q^{-1/2})}{(q^{1/2}+q^{-1/2})^2}\left(vq^{1/4}S_+q^{s_3/2}\otimes\mathcal{G}_{k+1}^{(L-1)}+c_0v^{-1}q^{-1/4}S_-q^{s_3/2}\otimes\tilde{\mathcal{G}}_{k+1}^{(L-1)}\right)\\ &+\frac{(v^2+v^{-2})w_0^{(j)}}{(q^{1/2}+q^{-1/2})^2}\mathcal{W}_{-k}^{(L)}+q^{s_3}\otimes\mathcal{W}_{-k-1}^{(L-1)}\ ,\end{aligned}\tag{2.2.35}$$

$$\begin{aligned}\mathcal{W}_{k+2}^{(L)} &= \frac{(w_0^{(j)}-(q^{1/2}+q^{-1/2})q^{-s_3})}{(q^{1/2}+q^{-1/2})}\otimes\mathcal{W}_{-k}^{(L-1)}-\frac{(v^2+v^{-2})}{(q^{1/2}+q^{-1/2})}\mathbb{I}\otimes\mathcal{W}_{k+1}^{(L-1)}\\ &+\frac{(q^{1/2}-q^{-1/2})}{(q^{1/2}+q^{-1/2})^2}\left(v^{-1}q^{-1/4}S_+q^{-s_3/2}\otimes\mathcal{G}_{k+1}^{(L-1)}+c_0vq^{1/4}S_-q^{-s_3/2}\otimes\tilde{\mathcal{G}}_{k+1}^{(L-1)}\right)\\ &+\frac{(v^2+v^{-2})w_0^{(j)}}{(q^{1/2}+q^{-1/2})^2}\mathcal{W}_{k+1}^{(L)}+q^{-s_3}\otimes\mathcal{W}_{k+2}^{(L-1)},\end{aligned}\tag{2.2.36}$$

$$\begin{aligned}\mathcal{G}_{k+2}^{(L)} &= \frac{c_0(q^{1/2}-q^{-1/2})^2}{(q^{1/2}+q^{-1/2})}S_-^2\otimes\tilde{\mathcal{G}}_{k+1}^{(L-1)}-\frac{1}{(q^{1/2}+q^{-1/2})}(v^2q^{s_3}+v^{-2}q^{-s_3})\otimes\mathcal{G}_{k+1}^{(L-1)}\\ &+\mathbb{I}\otimes\mathcal{G}_{k+2}^{(L-1)}+\frac{(v^2+v^{-2})w_0^{(j)}}{(q^{1/2}+q^{-1/2})^2}\mathcal{G}_{k+1}^{(L)}\\ &+(q-q^{-1})\Big(vq^{-1/4}S_-q^{s_3/2}\otimes\big(\mathcal{W}_{-k-1}^{(L-1)}-\mathcal{W}_{k+1}^{(L-1)}\big)\\ &\qquad+v^{-1}q^{1/4}S_-q^{-s_3/2}\otimes\big(\mathcal{W}_{k+2}^{(L-1)}-\mathcal{W}_{-k}^{(L-1)}\big)\Big),\end{aligned}\tag{2.2.37}$$

$$\begin{aligned}\tilde{\mathcal{G}}_{k+2}^{(L)} &= \frac{(q^{1/2}-q^{-1/2})^2}{c_0(q^{1/2}+q^{-1/2})}S_+^2\otimes\mathcal{G}_{k+1}^{(L-1)}-\frac{1}{(q^{1/2}+q^{-1/2})}(v^2q^{-s_3}+v^{-2}q^{s_3})\otimes\tilde{\mathcal{G}}_{k+1}^{(L-1)}\\ &+\mathbb{I}\otimes\tilde{\mathcal{G}}_{k+2}^{(L-1)}+\frac{(v^2+v^{-2})w_0^{(j)}}{(q^{1/2}+q^{-1/2})^2}\tilde{\mathcal{G}}_{k+1}^{(L)}\\ &+\frac{(q-q^{-1})}{c_0}\Big(v^{-1}q^{1/4}S_+q^{s_3/2}\otimes\big(\mathcal{W}_{-k-1}^{(L-1)}-\mathcal{W}_{k+1}^{(L-1)}\big)\\ &\qquad+vq^{-1/4}S_+q^{-s_3/2}\otimes\big(\mathcal{W}_{k+2}^{(L-1)}-\mathcal{W}_{-k}^{(L-1)}\big)\Big),\end{aligned}\tag{2.2.38}$$

for $k\in\{0,1,...,L-2\}$,

and satisfy the analog of Davies relations

$$c_0(q-q^{-1})\omega_0^{(L)}\mathcal{W}_0^{(L)} - \sum_{k=1}^{L} C_{-k+1}^{(L)}\mathcal{W}_{-k}^{(L)} = 0, \tag{2.2.39}$$

$$c_0(q-q^{-1})\omega_0^{(L)}\mathcal{W}_1^{(L)} - \sum_{k=1}^{L} C_{-k+1}^{(L)}\mathcal{W}_{k+1}^{(L)} = 0, \tag{2.2.40}$$

$$c_0(q-q^{-1})\omega_0^{(L)}\mathcal{G}_1^{(L)} - \sum_{k=1}^{L} C_{-k+1}^{(L)}\mathcal{G}_{k+1}^{(L)} = 0, \tag{2.2.41}$$

$$c_0(q-q^{-1})\omega_0^{(L)}\tilde{\mathcal{G}}_1^{(L)} - \sum_{k=1}^{L} C_{-k+1}^{(L)}\tilde{\mathcal{G}}_{k+1}^{(L)} = 0 \tag{2.2.42}$$

with $C_{-k+1}^{(L)}$ given by (2.2.27).

As a consequence of the reflection equation algebra and the fact that the vector space on which the elements act is finite dimensional, the generators $\mathcal{W}_{-k}^{(L)}, \mathcal{W}_{k+1}^{(L)}, \mathcal{G}_{k+1}^{(L)}, \tilde{\mathcal{G}}_{k+1}^{(L)}$ generate a quotient of the infinite dimensional algebra $\mathcal{A}_q$ (1.2.28)-(1.2.38) by the relations (qDavies) (2.2.39)-(2.2.42).

Importantly, in [BK0507] it was shown that the first two elements $\mathcal{W}_0^{(L)}, \mathcal{W}_1^{(L)}$ of the family of generators $\mathcal{W}_{-k}^{(L)}, \mathcal{W}_{k+1}^{(L)}, \mathcal{G}_{k+1}^{(L)}, \tilde{\mathcal{G}}_{k+1}^{(L)}$ of the quotient of $\mathcal{A}_q$ satisfy the q-deformed Dolan-Grady relations

$$[\mathcal{W}_0^{(L)}, [\mathcal{W}_0^{(L)}, [\mathcal{W}_0^{(L)}, \mathcal{W}_1^{(L)}]_q]_{q^{-1}}] = \rho\, [\mathcal{W}_0^{(L)}, \mathcal{W}_1^{(L)}], \tag{2.2.43}$$

$$[\mathcal{W}_1^{(L)}, [\mathcal{W}_1^{(L)}, [\mathcal{W}_1^{(L)}, \mathcal{W}_0^{(L)}]_q]_{q^{-1}}] = \rho\, [\mathcal{W}_1^{(L)}, \mathcal{W}_0^{(L)}], \tag{2.2.44}$$

with

$$\rho = (q+q^{-1})^2 k_+ k_-.$$

For all these reasons, in the literature on the subject the algebra $\mathcal{A}_q$ is sometimes called the $q-$deformed analog of the Onsager algebra, and the relations (2.2.39)-(2.2.42) are called the $q-$deformed analog of Davies relations.

2.3 Commuting quantities and the $q-$Dolan-Grady hierarchy

For the Onsager algebra (2.1.1)-(2.1.3), the explicit construction of mutually commuting quantities that generate an Abelian subalgebra of the Onsager algebra (2.1.6) has been considered in details in [O44, DG82]. For the $q-$Onsager algebra, an analogous construction is, in general, a rather complicated problem. However, using the connection between the reflection equation algebra and the infinite dimensional algebra $\mathcal{A}_q$ or alternatively the $q-$Onsager algebra, this problem can be handled. Indeed, it is known that starting from solutions of the reflection equation algebra, a transfer matrix that generates mutually

commuting quantities can be constructed. Namely, following the analysis of [Sk88], for any values of the spectral parameters u, v

$$[t^{(L)}(u), t^{(L)}(v)] = 0 \quad \text{where} \quad t^{(L)}(u) = tr_0\{K_+(u)K^{(L)}(u)\} \tag{2.3.1}$$

where tr_0 denotes the trace over the two-dimensional auxiliary space. The idea of [BK0503] was then to consider the expansion of (2.3.1) in order to extract mutually commuting quantities expressed in terms of the generators of the infinite dimensional algebra $\mathcal{A}_q$. To this end, let us plug the $c-$number solution of the dual reflection equation[1] [IK94], [GZ94] given by

$$K_+(u) = \begin{pmatrix} uq\kappa + u^{-1}q^{-1}\kappa^* & \kappa_-(q+q^{-1})(q^2u^2 - q^{-2}u^{-2})/c_0 \\ \kappa_+(q+q^{-1})(q^2u^2 - q^{-2}u^{-2}) & uq\kappa^* + u^{-1}q^{-1}\kappa \end{pmatrix} \tag{2.3.2}$$

where $\kappa^* = \kappa^{-1}$ for any $\kappa \in \mathbb{C}$, and $\kappa_\pm \in \mathbb{C}$. Substitute (2.3.2), and (2.2.21) with the generators $\mathcal{W}_{-k}^{(L)}, \mathcal{W}_{k+1}^{(L)}, \mathcal{G}_{k+1}^{(L)}, \tilde{\mathcal{G}}_{k+1}^{(L)}$ of Theorem 2.2.1 into (2.3.1). By straightforward calculations, it follows [BK0503]

$$t^{(L)}(u) = \sum_{k=0}^{L-1} (q^2u^2 - q^{-2}u^{-2}) P_{-k}^{(L)}(u) \mathcal{I}_{2k+1}^{(L)} + \mathcal{F}(u)\mathbb{I} \tag{2.3.3}$$

with (2.2.26) and

$$\mathcal{F}(u) = \frac{(q+q^{-1})(q^2u^2 - q^{-2}u^{-2})}{c_0} \left(\frac{(qu^2 + q^{-1}u^{-2})P_0^{(L)}(u)}{q - q^{-1}} + c_0\omega_0^{(L)} \right) (\kappa_+ + \kappa_-). \tag{2.3.4}$$

Here, we have introduced the generators $\mathcal{I}_{2k+1}^{(L)}$ which can be written in terms of the generators of the quotient of $\mathcal{A}_q$. Explicitly, they read:

$$\mathcal{I}_{2k+1}^{(L)} = \kappa\mathcal{W}_{-k}^{(L)} + \kappa^*\mathcal{W}_{k+1}^{(L)} + \kappa_+\mathcal{G}_{k+1}^{(L)} + \kappa_-\tilde{\mathcal{G}}_{k+1}^{(L)}, \tag{2.3.5}$$

for $k \in \{0, 1, \ldots, L-1\}$. As a consequence of the property (2.3.1), it leads to

$$[\mathcal{I}_{2k+1}^{(L)}, \mathcal{I}_{2l+1}^{(L)}] = 0 \quad \text{for all} \quad k, l \in \{0, 1, \ldots, L-1\}. \tag{2.3.6}$$

The mutually commuting quantities (2.3.5) are the $q-$deformed analog of the mutually commuting quantities (2.1.6) of the Onsager algebra.

Clearly, an interesting problem is to write these quantities solely in terms of the fundamental generators of the $q-$Onsager algebra, in view of the connection between the quotient of $\mathcal{A}_q$ and the $q-$Onsager algebra. Such problem was first considered in details in [BK0503], and generalized in [BB10], where it is shown that the mutually conserved quantities are polynomials of the fundamental generators $\mathcal{W}_0^{(L)}, \mathcal{W}_1^{(L)}$. For instance,

$$\mathcal{G}_1^{(L)} = [\mathcal{W}_1^{(L)}, \mathcal{W}_0^{(L)}]_q, \quad \tilde{\mathcal{G}}_1^{(L)} = [\mathcal{W}_0^{(L)}, \mathcal{W}_1^{(L)}], \tag{2.3.7}$$

$$\mathcal{W}_{-1}^{(L)} = \frac{1}{\rho}\left((q^2+q^{-2})\mathcal{W}_0^{(L)}\mathcal{W}_1^{(L)}\mathcal{W}_0^{(L)} - {\mathcal{W}_0^{(L)}}^2\mathcal{W}_1^{(L)} - \mathcal{W}_1^{(L)}{\mathcal{W}_0^{(L)}}^2 \right) + \mathcal{W}_1^{(L)}, \tag{2.3.8}$$

$$\mathcal{W}_2^{(L)} = \frac{1}{\rho}\left((q^2+q^{-2})\mathcal{W}_1^{(L)}\mathcal{W}_0^{(L)}\mathcal{W}_1^{(L)} - {\mathcal{W}_1^{(L)}}^2\mathcal{W}_0^{(L)} - \mathcal{W}_0^{(L)}{\mathcal{W}_1^{(L)}}^2 \right) + \mathcal{W}_0^{(L)}. \tag{2.3.9}$$

[1] This equation is obtained from (2.2.11) by changing $u \to u^{-1}, v \to v^{-1}$ and $K(u)$ in its transpose

2.4 The open XXZ spin chain and the q-Onsager algebra

Since Sklyanin's work [Sk88] on spin chains with integrable boundary conditions, finding exact results such as the energy spectrum of a model, corresponding eigenstates and correlation functions for elementary excitations has remained an interesting problem in connection with condensed matter or high-energy physics. Among the known integrable open spin chains that have been considered in details, one finds the open XXZ spin chain.

For generic boundary conditions, the Hamiltonian of the XXZ spin chain reads

$$\begin{aligned} H_{XXZ}^{(L)} &= \sum_{k=1}^{L-1}\Big(\sigma_1^{k+1}\sigma_1^k+\sigma_2^{k+1}\sigma_2^k+\Delta\sigma_z^{k+1}\sigma_z^k\Big) \\ &+ \frac{(q-q^{-1})}{2}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_z^1+\frac{2}{(\epsilon_++\epsilon_-)}\big(k_+\sigma_+^1+k_-\sigma_-^1\big) \\ &+ \frac{(q-q^{-1})}{2}\frac{(\bar\epsilon_+-\bar\epsilon_-)}{(\bar\epsilon_++\bar\epsilon_-)}\sigma_z^L+\frac{2}{(\bar\epsilon_++\bar\epsilon_-)}\big(\bar k_+\sigma_+^L+\bar k_-\sigma_-^L\big) \end{aligned} \tag{2.4.1}$$

where L is the number of sites, $\Delta=\frac{q+q^{-1}}{2}$ denotes the anisotropy parameter, and $\sigma_\pm$, σ_1, σ_2, σ_z are the usual Pauli matrices.

Besides the anisotropy parameter, $\epsilon_\pm, k_\pm$ (resp. $\bar\epsilon_\pm, \bar k_\pm$) denote the right (resp. left) boundary parameters associated with the right (resp. left) boundary. Considering a gauge transformation, note that one parameter might be removed. For symmetry reasons, we however keep the boundary parametrization as defined above.

For generic values of q and generic non-diagonal boundary parameters, the Hamiltonian H can be formulated within the so-called Sklyanin's formalism (boundary quantum inverse scattering) [Sk88]. In this standard approach, a generating function for all mutually commuting quantities besides H is introduced. It is built from a $R-$matrix solution of the Yang-Baxter equation and two $K-$matrices solutions of the reflection equation [Sk88]. However, the algebraic setting based on $U_q(\widehat{sl_2})$ is obscured in this formulation, and the standard algebraic Bethe ansatz fails to apply for generic boundary parameters. Instead, an alternative formulation has been proposed [BK0507], which is based on an analog of Onsager's approach for the two-dimensional Ising model: the transfer matrix - denoted $t^{(L)}(\zeta)$ below - is written in terms of mutually commuting quantities $\mathcal{I}_{2k+1}$ that generate a $q-$deformed analog of the Onsager-Dolan-Grady's hierarchy[2]. Namely,

$$t^{(L)}(\zeta)=\sum_{k=0}^{L-1}\mathcal{F}_{2k+1}(\zeta)\,\mathcal{I}_{2k+1}+\mathcal{F}_0(\zeta)\mathbb{I}\quad\text{with}\qquad\big[\mathcal{I}_{2k+1},\mathcal{I}_{2l+1}\big]=0 \tag{2.4.2}$$

for all $k,l\in 0,...,L-1$ where ζ is the so-called spectral parameter,

$$\mathcal{I}_{2k+1}=\bar\epsilon_+\mathcal{W}_{-k}+\bar\epsilon_-\mathcal{W}_{k+1}+\frac{1}{q^2-q^{-2}}\Big(\frac{\bar k_-}{k_-}\mathcal{G}_{k+1}+\frac{\bar k_+}{k_+}\tilde{\mathcal{G}}_{k+1}\Big) \tag{2.4.3}$$

[2]The Onsager's (also called Dolan-Grady [DG82]) hierarchy is an Abelian algebra with elements of the form $\mathcal{I}_{2n+1}=\bar\epsilon_+(A_n+A_{-n})+\bar\epsilon_-(A_{n+1}+A_{-n+1})+\kappa G_{n+1}$ with $n\in\mathbb{Z}_+$, generated from the Onsager algebra with defining relations $\big[A_n,A_m\big]=4G_{n-m}$, $\big[G_m,A_n\big]=2A_{n+m}-2A_{n-m}$ and $\big[G_n,G_m\big]=0$ for any $n,m\in\mathbb{Z}$ [O44].

and $\mathcal{F}_{2k+1}(\zeta)$ are given in [BK0507]

$$\begin{aligned}
\mathcal{F}_{2k+1}(\zeta) &= -\frac{1}{q+q^{-1}}\frac{(q^2\zeta^4+q^{-2}\zeta^{-4}-q^2-q^{-2})}{(q^2+q^{-2}-\zeta^2-\zeta^{-2})^L}\sum_{n=k}^{L-1}(\frac{q\zeta^2+q^{-1}\zeta^{-2}}{q+q^{-1}})^{n-k}C_{-n}^{(L)},\\
\mathcal{F}_0(\zeta) &= (\frac{(q^2\zeta^4+q^{-2}\zeta^{-4}-q^2-q^{-2})(q+q^{-1})}{(q^2+q^{-2}-\zeta^2-\zeta^{-2})})(\frac{k_+k_-}{q-q^{-1}})(\frac{\bar{k}_+}{k_+}+\frac{\bar{k}_-}{k_-})\\
&\times\left(\frac{\omega_0^{(L)}(q-q^{-1})}{k_+k_-}-\sum_{n=0}^{L-1}(\frac{q\zeta^2+q^{-1}\zeta^{-2}}{q+q^{-1}})^{n+1}C_{-n}^{(L)}\right)\\
&+\frac{((q\zeta^2+q^{-1}\zeta^{-2})\bar{\epsilon}_++(q+q^{-1})\bar{\epsilon}_-)\epsilon_+^{(L)}}{(q^2+q^{-2}-\zeta^2-\zeta^{-2})^L}\\
&+\frac{((q\zeta^2+q^{-1}\zeta^{-2})\bar{\epsilon}_-+(q+q^{-1})\bar{\epsilon}_+)\epsilon_-^{(L)}}{(q^2+q^{-2}-\zeta^2-\zeta^{-2})^L}
\end{aligned}$$

The explicit expressions for the coefficients $C_{-n}^{(L)}$ and $\omega_0^{(L)},\epsilon_\pm^{(L)}$ essentially depend on the choice of quantum space representations at each site (two dimensional for (2.4.1)) of the spin chain

$$C_{-n}^{(L)}=(-1)^{L-n}(q+q^{-1})^{n+1}\sum_{k_1<\cdots<k_{L-n-1}=1}^{L}\alpha_{k_1}\ldots\alpha_{k_{L-n-1}} \tag{2.4.4}$$

where

$$\alpha_1 = \frac{2(q^2+q^{-2})}{(q+q^{-1})}+\frac{\epsilon_+\epsilon_-(q-q^{-1})^2}{k_+k_-(q+q^{-1})}, \tag{2.4.5}$$

$$\alpha_k = \frac{2(q^2+q^{-2})}{q+q^{-1}} \quad \text{for} \quad k=2,\ldots,L, \tag{2.4.6}$$

and for arbitrary values for L

$$\epsilon_\pm^{(L)} = (q^2+q^{-2})\epsilon_\mp^{(L-1)}-2\epsilon_\pm^{(L-1)},\quad \epsilon_\pm^{(0)}=\epsilon_\pm, \tag{2.4.7}$$

$$\omega_0^{(L)} = (-1)^L\frac{k_+k_-}{q-q^{-1}}\left(\frac{2(q^2+q^{-2})}{q+q^{-1}}\right)^{L-1}\left(\frac{2(q^2+q^{-2})}{q+q^{-1}}+\frac{\epsilon_+\epsilon_-(q-q^{-1})^2}{k_+k_-(q+q^{-1})}\right) \tag{2.4.8}$$

Note that the parameters $\epsilon_\pm$ of the right boundary - which do not appear explicitly in the formula (2.4.3)- are actually hidden in the definition of the elements $\mathcal{W}_{-k},\mathcal{W}_{k+1},\mathcal{G}_{k+1},\tilde{\mathcal{G}}_{k+1}$ [BK0507, BK07].

Since the generators $\mathcal{W}_{-k},\mathcal{W}_{k+1},\mathcal{G}_{k+1},\tilde{\mathcal{G}}_{k+1}$ of the q-deformed analogue of the Onsager algebra possess the block diagonal and block tridiagonal structure in the eigenbasis of $\mathcal{W}_0$ or the eigenbasis of $\mathcal{W}_1$ [BK07], the general spectral problem of all nonlocal commuting operators $\mathcal{I}_{2k+1}^{(L)}$ is solved

$$\mathcal{I}_{2k+1}^{(L)}\Psi_{\Lambda_1^{(L)}}^{(L)}=\Lambda_{2k+1}^{(L)}\Psi_{\Lambda_1^{(L)}}^{(L)} \quad \text{for } k=0,1,\ldots,L-1. \tag{2.4.9}$$

It follows that the eigenvalues $\Lambda_{XXZ}(\zeta)$ of the transfer matrix (2.4.2) are given by:

$$\Lambda_{XXZ}(\zeta) = \sum_{k=0}^{L-1} \mathcal{F}_{2k+1}(\zeta)\Lambda_{2k+1}^{(L)} + \mathcal{F}_0(\zeta). \tag{2.4.10}$$

The spectrum E of the Hamiltonian (2.4.1) immediately follows:

$$\begin{aligned} E &= \frac{(q-q^{-1})(q+q^{-1})^{-1}}{2(\bar{\epsilon}_+ + \bar{\epsilon}_-)(\epsilon_+ + \epsilon_-)} \left(\sum_{k=0}^{L-1} \frac{d\mathcal{F}_{2k+1}(\zeta)}{d\zeta}|_{\zeta=1}\ \Lambda_{2k+1}^{(L)} + \frac{d\mathcal{F}_0(\zeta)}{d\zeta}|_{\zeta=1} \right) \\ &\quad - \left(L\Delta + \frac{(q-q^{-1})^2}{2(q+q^{-1})} \right). \end{aligned} \tag{2.4.11}$$

Indeed, it is known that the Hamiltonian of the XXZ open spin chain with generic integrable boundary conditions (2.4.1) is obtained as follows:

$$\frac{d}{d\zeta} ln(t^{(L)}(\zeta))|_{\zeta=1} = \frac{2}{(q-q^{-1})}H + \Big(\frac{(q-q^{-1})}{(q+q^{-1})} + \frac{2L}{(q-q^{-1})}\Delta\Big)\mathbb{I}\ . \tag{2.4.12}$$

More generally, higher mutually commuting local conserved quantities, say H_n with $H_1 \equiv H$, can be derived similarly by taking higher derivatives of the transfer matrix (2.4.2).

To resume, in the $q-$Onsager approach of the open XXZ spin chain with generic integrable boundary conditions [Bas0404, BK07] the Hamiltonian is written in terms of the elements (2.4.3). These elements generate an Abelian subalgebra of the $q-$deformed analog of the Onsager algebra. For the open XXZ spin chain and q generic, the vector space on which the elements act is irreducible finite dimensional. So, stricly speaking the elements $\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}$ generate a quotient of the infinite dimensional algebra $\mathcal{A}_q$ (1.2.28)-(1.2.38) by the relations (2.2.39)-(2.2.42).

In [Bas0604], recall that two (dual) bases of the vector space were constructed, on which the operators $\mathcal{W}_0, \mathcal{W}_1$ act as a tridiagonal pair. Remarkably, in these bases the more general operators (called descendants) $\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}$ of the q-deformed analogue of the Onsager algebra act as block diagonal or block tridiagonal matrices. In [BK07], the spectrum and eigenstates of the Hamiltonian (2.4.1) were derived using this remarkable property. It was the first solution proposed in the literature for the open XXZ spin chain with generic integrable boundary conditions.

Chapter 3

Main Results

In this Chapter, we review the three main results of the thesis. The first two results have their own interest in the context of tridiagonal algebras, $q-$Onsager algebra, coideal subalgebras of $U_q(\widehat{sl_2})$ and their higher rank generalizations: analogs of Lusztig's higher order relations are conjectured, and supporting evidences for these are described. In particular, the theory of tridiagonal pairs recalled in Chapter 1 plays a central role in the analysis for the $q-$Onsager algebra associated with $\widehat{sl_2}$. The third main result of this thesis overlaps between mathematics and physics. In mathematics, quantum universal enveloping algebras have been studied at roots of unity in the literature [Lusz93]. The introduction of the divided powers of the Chevalley generators plays a central role in the construction. Here, for the $q-$Onsager algebra, analogs of these elements are introduced and studied in details at roots of unity at least for a certain class of irreducible finite dimensional representations that finds applications in physics. These elements are divided polynomials of the two fundamental generators of the $q-$Onsager algebra. For a special case, they satisfy a pair of relations that share some similarity with the higher order $q-$Dolan-Grady relations previously conjectured. All together, the $q-$Onsager generators and the divided polynomials generate a new quantum algebra. Based on this construction, the commutation relations between the Hamiltonian of the open XXZ chain at roots of unity and the four generators are studied.

3.1 Higher order relations for the q-Onsager algebra

3.1.1 Motivation

Consider the quantum universal enveloping algebras for arbitrary Kac-Moody algebras $\widehat{g}$ introduced by Drinfeld [Dr87], and Jimbo [J85]. Let $\{a_{ij}\}$ be the extended Cartan matrix of $\widehat{g}$. Fix coprime integers d_i such that $d_i a_{ij}$ is symmetric. Define $q_i = q^{d_i}$. The quantum universal enveloping algebra $U_q(\widehat{g})$ is generated by the elements $\{h_j, e_j, f_j\}$, $j =$

$0, 1, ..., rank(g)$, which satisfy the defining relations:

$$
\begin{aligned}
&[h_i, h_j] = 0 \ , \quad [h_i, e_j] = a_{ij} e_j \ , \quad [h_i, f_j] = -a_{ij} f_j \ , \\
&[e_i, f_j] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}, \quad i, j = 0, 1, ..., rank(g) \ , \\
&e_i \, e_j = e_j \, e_i \ , \qquad f_i \, f_j = f_j \, f_i \qquad \text{for} \qquad |i - j| > 1 \ ,
\end{aligned}
\tag{3.1.1}
$$

together with the so-called *quantum Serre relations* ($i \neq j$)

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} e_i^{1-a_{ij}-k} \, e_j \, e_i^k \quad = \quad 0 \ , \tag{3.1.2}
$$

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} f_i^{1-a_{ij}-k} \, f_j \, f_i^k \quad = \quad 0 \ , \tag{3.1.3}
$$

where δ_{ij} denotes Kronecker delta.

In the mathematical literature [Lusz93], generalizations of the relations (3.1.2)-(3.1.3) - the so-called *higher order quantum (q−)Serre relations* - have been proposed. For $\widehat{g} = \widehat{sl_2}$, they read[1] [Lusz93]:

$$
\sum_{k=0}^{2r+1} (-1)^k \left[\begin{array}{c} 2r + 1 \\ k \end{array} \right]_{q} e_i^{2r+1-k} e_j^r e_i^k \quad = \quad 0 \ , \tag{3.1.4}
$$

$$
\sum_{k=0}^{2r+1} (-1)^k \left[\begin{array}{c} 2r + 1 \\ k \end{array} \right]_{q} f_i^{2r+1-k} f_j^r f_i^k \quad = \quad 0 \qquad \text{for} \quad i \neq j, \quad i, j = 0, 1 \ . \tag{3.1.5}
$$

Consider the relations (3.1.2)-(3.1.3) for $g = sl_2$, we observe that the q-Onsager algebra is closely related with $U_q(\widehat{sl_2})$. In particular, there exists an homomorphism from the q-Onsager algebra to a coideal subalgebras of $U_q(\widehat{sl_2})$ (see Chapter 1).

Recall that the $q-$Onsager algebra is a special case of the tridiagonal algebra: it corresponds to the *reduced* parameter sequence $\gamma = 0, \gamma^* = 0$, $\beta = q^2 + q^{-2}$ and $\rho = \rho_0$, $\rho^* = \rho_1$ which exhibits all interesting properties that can be extended to more general parameter sequences. The defining relations of the $q-$Onsager algebra read

$$
[A, [A, [A, A^*]_q]_{q^{-1}}] = \rho_0 [A, A^*] \ , \qquad [A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = \rho_1 [A^*, A] \ , \tag{3.1.6}
$$

which can be seen as ρ_i-deformed analogues of the $q-$Serre relations (3.1.2)-(3.1.3) associated with $\widehat{g} \equiv \widehat{sl_2}$. For $q = 1$, $\rho_0 = \rho_1 = 16$, note that they coincide with the Dolan-Grady relations [DG82].

In the study of tridiagonal algebras and the representation theory associated with the special case $\rho_0 = \rho_1 = 0$, higher order $q-$Serre relations (3.1.4), (3.1.5) play an important

[1] For $\widehat{g} = \widehat{sl_2}$, recall that $a_{ii} = 2$, $a_{ij} = -2$ with $i, j = 0, 1$.

role [IT03] in the construction of a basis of the corresponding vector space. As suggested in [IT03, Problem 3.4], for $\rho_0 \neq 0$, $\rho_1 \neq 0$ finding analogues of the higher order $q-$Serre relations for the $q-$Onsager algebra is an interesting problem. Another interest for the construction of higher order relations associated with the $q-$Onsager algebra comes from the theory of quantum integrable systems with boundaries. Indeed, by analogy with the case of periodic boundary conditions [DFM99], such relations or similar relations should play a role in the identification of the symmetry of the Hamiltonian of the XXZ open spin chain at q a root of unity and for special boundary parameters. Motivated by these open problems, below we focus on the explicit construction of higher order tridiagonal relations and the special case associated with the q-Onsager algebra.

3.1.2 Conjecture about the higher order relations of the q-Onsager algebra

By analogy with Lusztig's higher order relations (3.1.4), (3.1.5) for quantum universal enveloping algebras, it is natural to expect higher order relations for the $q-$Onsager algebra.

Conjecture 1 *Let A, A^* be the fundamental generators of the $q-$Onsager algebra (3.1.6), then A, A^* satisfy the higher order q-Dolan-Grady relations as follows:*

$$\sum_{p=0}^{r}\sum_{j=0}^{2r+1-2p}(-1)^{j+p}\rho_0^p\, c_j^{[r,p]}\, A^{2r+1-2p-j}A^{*r}A^j \;=\; 0 \;, \tag{3.1.7}$$

$$\sum_{p=0}^{r}\sum_{j=0}^{2r+1-2p}(-1)^{j+p}\rho_1^p\, c_j^{[r,p]}\, A^{*2r+1-2p-j}A^{r}A^{*j} \;=\; 0 \tag{3.1.8}$$

where $c^{[r,p]}_{2(r-p)+1-j} = c_j^{[r,p]}$ *and*

$$c_j^{[r,p]} = \sum_{k=0}^{j}\frac{(r-p)!}{(\{\frac{j-k}{2}\})!(r-p-\{\frac{j-k}{2}\})!}\sum_{\mathcal{P}_k}[s_1]_{q^2}^2\ldots[s_p]_{q^2}^2\frac{[2s_{p+1}]_{q^2}\ldots[2s_{p+k}]_{q^2}}{[s_{p+1}]_{q^2}\ldots[s_{p+k}]_{q^2}}, \tag{3.1.9}$$

$$\textit{with}\quad \begin{cases} j=\overline{0,r-p}\;, \quad s_i\in\{1,2,...,r\}\;, \\ \mathcal{P}_k: \quad s_1<\cdots<s_p\;; \quad s_{p+1}<\cdots<s_{p+k}\;, \\ \qquad \{s_1,\ldots,s_p\}\cap\{s_{p+1},\ldots,s_{p+k}\}=\emptyset \end{cases} .$$

Below we give several supporting evidences for this conjecture. First, it is shown that a generalized version of the conjecture holds for every TD pair (A, A^*) of $q-$Racah type. As a special case, the higher order relations for the $q-$Onsager algebra are obtained and have the form of (3.1.9). Then, the conjecture is explicitely proven for $r = 2, 3$. Finally, the conjecture is studied recursively. Using a computer program, up to $r = 10$ it is checked that the conjecture holds.

3.1.3 Higher order relations and tridiagonal pairs

Let A, A^* act on an irreducible finite dimensional vector space V. In this Section, it is shown that A, A^* satisfy higher order relations which are generalizations (3.1.9). For

$r = 1$, these relations are the defining relations of the q-Onsager algebra. This is proven using the properties of tridiagonal pairs described in Chapter 1.

Let (A, A^*) denote a tridiagonal pair on V, the sequence $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote a standard ordering of the eigenvalues of A (resp. A^*). For each positive integer s, let $\beta_s, \gamma_s, \gamma_s^*, \delta_s, \delta_s^*$ in $\overline{\mathbb{K}}$ satisfying (1.3.35), (1.3.36).

Definition 3.1.1 *Let x, y denote commuting indeterminates. For each positive integer r, we define the polynomials $p_r(x, y), p_r^*(x, y)$ as follows:*

$$p_r(x, y) = (x - y) \prod_{s=1}^{r} (x^2 - \beta_s xy + y^2 - \gamma_s(x + y) - \delta_s) \ , \tag{3.1.10}$$

$$p_r^*(x, y) = (x - y) \prod_{s=1}^{r} (x^2 - \beta_s xy + y^2 - \gamma_s^*(x + y) - \delta_s^*) \ . \tag{3.1.11}$$

We observe $p_r(x, y)$ and $p_r^*(x, y)$ have a total degree $2r + 1$ in x, y.

Lemma 3.1.1 *For each positive integer r, $p_r(\theta_i, \theta_j) = 0$, and $p_r^*(\theta_i^*, \theta_j^*) = 0$ if $|i - j| \leq r, (0 \leq i, j \leq d)$.*

Proof. Let two integers i, j such that $|i - j| \leq r$, $(0 \leq i, j \leq d)$.

If $i = j$, then $\theta_i = \theta_j, \theta_i^* = \theta_j^*$. Hence, $p_r(\theta_i, \theta_j) = p_r^*(\theta_i^*, \theta_j^*) = 0$.

If $i \neq j$, since $0 < |i - j| \leq r$, there exists an integer $1 \leq s \leq r$ such that $|i - j| = s$.

By Lemma (1.3.12),

$$\theta_i^2 - \beta_s \theta_i \theta_j + \theta_j^2 - \gamma_s(\theta_i + \theta_j) - \delta_s = 0,$$

$$\theta_i^{*2} - \beta_s^* \theta_i^* \theta_j^* + \theta_j^{*2} - \gamma_s^*(\theta_i^* + \theta_j^*) - \delta_s^* = 0.$$

Hence, $p_r(\theta_i, \theta_j) = p_r^*(\theta_i^*, \theta_j^*) = 0$. ■

Theorem 3.1.1 *For each positive integer r,*

$$\sum_{i,j=0}^{i+j \leq 2r+1} a_{ij} A^i A^{*r} A^j = 0 \ , \qquad \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij}^* A^{*i} A^r A^{*j} = 0 \ , \tag{3.1.12}$$

where the scalars a_{ij}, a_{ij}^* are defined by:

$$p_r(x, y) = \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij} x^i y^j \quad \text{and} \quad p_r^*(x, y) = \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij}^* x^i y^j \ . \tag{3.1.13}$$

Proof. Let Δ_r denote the expression of the left-hand side of the first equation of (3.1.12). We show $\Delta_r = 0$.

Let $\{E_i\}_{i=0}^d$ denote the standard ordering of the primitive idempotents of A corresponding to $\{\theta_i\}_{i=0}^d$.

For $0 \leq i, j \leq d$, one finds $E_i \Delta_r E_j = p_r(\theta_i, \theta_j)\ E_i A^{*r} E_j$ with (3.1.13). According to Lemma 3.1.1 and Lemma 1.3.10, it follows $\Delta_r = 0$. Similar arguments are used to show the second equation of (3.1.12). ■

As a straightforward application, below we focus on the $r-th$ higher order tridiagonal relations associated with the $q-$Onsager algebra, a special case of the tridiagonal algebra.

Definition 3.1.2 *Consider a TD pair (A, A^*) of $q-$Racah type with eigenvalues such that $\alpha = \alpha^* = 0$ (see (1.3.33)-(1.3.34)). Assume $r = 1$. The corresponding tridiagonal relations (1.3.26), (1.3.27) are called the $q-$Dolan-Grady relations.*

Example 1 *For a TD pair (A, A^*) of $q-$Racah type with eigenvalues such that $\alpha = \alpha^* = 0$, the parameter sequence is given by $\beta_1 = q^2 + q^{-2}$, $\gamma_1 = \gamma_1^* = 0$, $\delta_1 = -bc(q^2 - q^{-2})^2$, $\delta_1^* = -b^* c^* (q^2 - q^{-2})^2$. Define $\delta_1 = \rho_0$, $\delta_1^* = \rho_1$. The $q-$Dolan-Grady relations are given by:*

$$\sum_{j=0}^{3} (-1)^j \begin{bmatrix} 3 \\ j \end{bmatrix}_q A^{3-j} A^* A^j - \rho_0 (AA^* - A^* A) \quad = \quad 0 \ , \tag{3.1.14}$$

$$\sum_{j=0}^{3} (-1)^j \begin{bmatrix} 3 \\ j \end{bmatrix}_q A^{*3-j} A A^{*j} - \rho_1 (A^* A - AA^*) \quad = \quad 0 \ . \tag{3.1.15}$$

Remark 5 *The relations (3.1.14), (3.1.15) are the defining relations of the $q-$Onsager algebra.*

Definition 3.1.3 *Consider a TD pair (A, A^*) of $q-$Racah type with eigenvalues such that $\alpha = \alpha^* = 0$. For any positive integer r, the corresponding higher order tridiagonal relations (3.1.12) are called the higher order $q-$Dolan-Grady relations.*

Theorem 3.1.2 *For a TD pair (A, A^*) of q-Racah type with eigenvalues such that $\alpha = \alpha^* = 0$, the higher order q-Dolan-Grady relations are given by (3.1.7)-(3.1.8) with the identification $\rho_0 = \delta_1$ and $\rho_1 = \delta_1^*$,*

Proof. For $\alpha = \alpha^* = 0$ in (1.3.33), (1.3.34), from Lemma 1.3.12 one finds $\beta_s = q^{2s} + q^{-2s}$, $\gamma_s = \gamma_s^* = 0$, $\delta_s = -bc(q^{2s} - q^{-2s})^2$, $\delta_s^* = -b^* c^* (q^{2s} - q^{-2s})^2$. Then, the first polynomial generating function (3.3.36) reads:

$$p_r(x, y) = (x - y) \prod_{s=1}^{r} \left(x^2 - \frac{[2s]_{q^2}}{[s]_{q^2}} xy + y^2 - [s]_{q^2}^2 \rho_0 \right) \ , \tag{3.1.16}$$

where the notation $\beta_s = [2s]_{q^2} / [s]_{q^2}$ and $\delta_s / \rho_0 = [s]_{q^2}^2$ has been introduced. Expanding the polynomial in the variables x, y as (3.1.13), one shows that the coefficients a_{ij} in (3.1.13) take the form:

$$a_{2r+1-2p-j\ j} = (-1)^{j+p} \rho_0^p c_j^{[r,p]} \tag{3.1.17}$$

where $c_j^{[r,p]}$ solely depend on q, and are vanishing otherwise. By induction, one finds that they are given by (3.1.9). Replacing $\rho_0 \to \rho_1$, the second relation (3.1.8) follows. ■

For $r = 2, 3$, the higher order $q-$Dolan-Grady relations (3.1.7), (3.1.8) can be constructed in a straightforward manner:

Example 2 *The first example of higher order $q-$Dolan-Grady relations is given by (3.1.7), (3.1.8) for $r = 2$ with:*

$$\begin{aligned}
c_0^{[2,0]} &= 1,\ c_1^{[2,0]} = 1 + [2]_{q^2} + \frac{[4]_{q^2}}{[2]_{q^2}} \equiv \left[\begin{array}{c} 5 \\ 1 \end{array}\right]_q,\ c_2^{[2,0]} = 2 + [2]_{q^2} + [4]_{q^2} + \frac{[4]_{q^2}}{[2]_{q^2}} \equiv \left[\begin{array}{c} 5 \\ 2 \end{array}\right]_q, \\
c_0^{[2,1]} &= 1 + [2]_{q^2}^2 \equiv q^4 + q^{-4} + 3\ , \quad c_1^{[2,1]} = 1 + [2]_{q^2}^2 + \frac{[4]_{q^2}}{[2]_{q^2}} + [2]_{q^2}^3 \equiv [5]_q[3]_q\ , \\
c_0^{[2,2]} &= [2]_{q^2}^2 \equiv (q^2 + q^{-2})^2\ .
\end{aligned}$$

Example 3 *The second example of higher order $q-$Dolan-Grady relations is given by (3.1.7), (3.1.8) for $r = 3$ with:*

$$\begin{aligned}
c_j^{[3,0]} &= \left[\begin{array}{c} 7 \\ j \end{array}\right]_q, \quad j = 0, ..., 7\ , \\
c_0^{[3,1]} &= 1 + [2]_{q^2}^2 + [3]_{q^2}^2\ , \\
c_1^{[3,1]} &= 1 + [2]_{q^2}^2 + [3]_{q^2}^2 + (1 + [3]_{q^2}^2)\frac{[4]_{q^2}}{[2]_{q^2}} + ([2]_{q^2}^2 + [3]_{q^2}^2)[2]_{q^2} + (1 + [2]_{q^2}^2)\frac{[6]_{q^2}}{[3]_{q^2}}\ , \\
c_2^{[3,1]} &= 2(1 + [2]_{q^2}^2 + [3]_{q^2}^2) + (1 + [3]_{q^2}^2)\frac{[4]_{q^2}}{[2]_{q^2}} + ([2]_{q^2}^2 + [3]_{q^2}^2)[2]_{q^2} + (1 + [2]_{q^2}^2)\frac{[6]_{q^2}}{[3]_{q^2}} \\
&\quad + \frac{[4]_{q^2}[6]_{q^2}}{[2]_{q^2}[3]_{q^2}} + \frac{[2]_{q^2}^3[6]_{q^2}}{[3]_{q^2}} + [3]_{q^2}^2[4]_{q^2}\ , \\
c_0^{[3,2]} &= [2]_{q^2}^2 + [3]_{q^2}^2 + [2]_{q^2}^2[3]_{q^2}^2\ , \\
c_1^{[3,2]} &= [2]_{q^2}^2 + [3]_{q^2}^2 + [2]_{q^2}^2[3]_{q^2}^2 + [2]_{q^2}^2\frac{[6]_{q^2}}{[3]_{q^2}} + [3]_{q^2}^2\frac{[4]_{q^2}}{[2]_{q^2}} + [2]_{q^2}^3[3]_{q^2}^2, \\
c_0^{[3,3]} &= [2]_{q^2}^2[3]_{q^2}^2\ .
\end{aligned}$$

To end up this Section, let us consider the family of relations satisfied by a TD pair of $q-$Racah type such that[2] $\rho_0 = \rho_1 = 0$.

We remind a powerful theorem called the q-Binomial Theorem in studying the binomial coefficients.

Theorem 3.1.3 *For all $n \geq 1$,* $\prod_{j=1}^{n}(1 + xq^j) = \sum_{k=0}^{n} q^{k(k+1)/2}\left[\begin{array}{c} n \\ k \end{array}\right]_q^* x^k$,

where $\left[\begin{array}{c} n \\ k \end{array}\right]_q^* = \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})...(1-q)}$.

[2]For instance, choose $b, b^* = 0$ and/or $c, c^* = 0$ in (1.3.33), (1.3.34).

Proof. By induction ■

Theorem 3.1.4 *For $\rho_0 = \rho_1 = 0$, the higher order $q-$Dolan-Grady relations satisfied by the corresponding TD pair simplify to the well-known Lusztig's higher order $q-$Serre relations [Lusz93]:*

$$\sum_{j=0}^{2r+1}(-1)^j \begin{bmatrix} 2r+1 \\ j \end{bmatrix}_q A^{2r+1-j}A^{*r}A^j = 0 \ , \tag{3.1.18}$$

$$\sum_{j=0}^{2r+1}(-1)^j \begin{bmatrix} 2r+1 \\ j \end{bmatrix}_q A^{*2r+1-j}A^{r}A^{*j} = 0 \ . \tag{3.1.19}$$

Proof. Since $\rho_0 = 0$, (3.1.16) can be written by:

$$\begin{aligned} p_r(x,y) &= x^{r+1}(1-\frac{y}{x})\prod_{s=1}^{r}(1-(q^{2s}+q^{-2s})\frac{y}{x}+(\frac{y}{x})^2) \\ &= x^{2r+1}(1-\frac{y}{x})\prod_{s=1}^{r}(1-q^{2s}\frac{y}{x})(1-q^{-2s}\frac{y}{x}) \\ &= x^{2r+1}\prod_{s=-r}^{r}(1-q^{2s}\frac{y}{x}). \end{aligned}$$

Put $t = s+r+1$, $p_r(x,y) = x^{2r+1}\prod_{t=1}^{2r+1}(1+q^{2t}(-q^{-2(r+1)}\frac{y}{x}))$.
Applying the q-Binomial Theorem yields

$$p_r(x,y) = x^{2r+1}\sum_{j=0}^{2r+1}(-1)^j q^{j(j+1)} \begin{bmatrix} 2r+1 \\ j \end{bmatrix}^*_{q^2} (q^{-2(r+1)}\frac{y}{x})^j.$$

Using the relation between $\begin{bmatrix} n \\ j \end{bmatrix}_q$ and $\begin{bmatrix} n \\ j \end{bmatrix}^*_{q^2}$ for all $n \geq j$, $\begin{bmatrix} n \\ j \end{bmatrix}_q = q^{j(j-n)} \begin{bmatrix} n \\ j \end{bmatrix}^*_{q^2}$.
We have

$$p_r(x,y) = \sum_{j=0}^{2r+1}(-1)^j \begin{bmatrix} 2r+1 \\ j \end{bmatrix}_q x^{2r+1-j}y^j.$$

■

3.1.4 Recursion for generating the coefficients of the higher order q-Dolan-Grady relations

In the previous Section, it was shown that every TD pair of $q-$Racah type such that $\alpha = \alpha^* = 0$ satisfies the $r-th$ higher order $q-$Dolan-Grady relations (3.1.7), (3.1.8) with (3.1.9). For the special case $r = 1$, these relations coincide with the defining relations of the $q-$Onsager algebra (1.2.1), (1.2.2). This strongly suggests that the conjecture holds in general. Below, the conjecture for $r = 2, 3$ will be proved. Then, using an inductive

argument we will study the general structure and derive recursion formulae - independently of the results of the previous part - for the coefficients $c_j^{[r,p]}$.

Let A, A^* be the fundamental generators of the $q-$Onsager algebra (1.2.1), (1.2.2). By analogy with Lusztig's higher order $q-$Serre relations, we are interested in more complicated linear combinations of monomials of the type $A^n A^{*r} A^m$, $n+m = 2r+1, 2r-1, ..., 1$, that are vanishing. The defining relations (1.2.1), (1.2.2) correspond to the case $r = 1$ of (3.1.7), (3.1.8). Below, successively we derive the relations (3.1.7), (3.1.8) for $r = 2, 3$ and study the generic case by induction.

3.1.4.1 Proof of the relations for $r = 2$

Consider the simplest example beyond (1.2.1): we are looking for a linear relation between monomials of the type $A^n A^{*2} A^m$, $n + m = 5, 3, 1$. According to the defining relations (1.2.1), note that the monomial $A^3 A^*$ can be written as:

$$A^3 A^* = \alpha A^2 A^* A - \alpha A A^* A^2 + A^* A^3 + \rho_0 (A A^* - A^* A) \quad \text{with} \quad \alpha = [3]_q \ . \tag{3.1.20}$$

Multiplying from the left by A or A^2, the corresponding expressions can be ordered as follows: each time a monomial of the form $A^n A^* A^m$ with $n \geq 3$ arises, it is reduced using (3.1.20). It follows:

$$\begin{aligned} A^4 A^* &= (\alpha^2 - \alpha) A^2 A^* A^2 + (1 - \alpha^2) A A^* A^3 + \alpha A^* A^4 \\ &\quad + \rho_0 \left(A^2 A^* - \alpha A^* A^2 + (\alpha - 1) A A^* A \right) \ , \\ A^5 A^* &= (\alpha^3 - 2\alpha^2 + 1) A^2 A^* A^3 + \alpha(-\alpha^2 + \alpha + 1) A A^* A^4 + \alpha(\alpha - 1) A^* A^5 \\ &\quad + \rho_0 \left((2\alpha - 1) A^2 A^* A + \alpha(\alpha - 3) A A^* A^2 - (\alpha^2 - \alpha - 1) A^* A^3 \right) \\ &\quad + \rho_0^2 (A A^* - A^* A) \ . \end{aligned}$$

For our purpose, four different types of monomials may be now considered: $A^5 A^{*2}$, $A^4 A^{*2} A$, $A^3 A^{*2} A^2$ and $A^3 A^{*2}$. Following the ordering prescription, each of these monomials can be reduced as a combination of monomials of the type $(n, m, p, s, t \geq 0)$:

$$\begin{aligned} &A^n A^{*2} A^m && \text{with} \quad n \leq 2 \ , \ n + m = 5, 3, 1 \ , \\ &A^p A^* A^s A^* A^t && \text{with} \quad p \leq 2 \ , \ 1 \leq s \leq 2 \ , \ p + s + t = 5, 3, 1 \ . \end{aligned} \tag{3.1.21}$$

For instance, the monomial A^5A^{*2} is reduced to:

$$\begin{aligned}
A^5A^{*2} \;=\;& (\alpha^3-2\alpha^2+1)\Big(\alpha A^2A^*A^2A^*A-\alpha A^2A^*AA^*A^2+A^2A^{*2}A^3\Big)\\
&-(\alpha^3-\alpha^2-\alpha)\Big((\alpha^2-\alpha)AA^*A^2A^*A^2+(1-\alpha^2)AA^*AA^*A^3+\alpha AA^{*2}A^4\Big)\\
&+(\alpha^2-\alpha)\Big((\alpha^3-2\alpha^2+1)A^*A^2A^*A^3-(\alpha^3-\alpha^2-1)A^*AA^*A^4+\alpha(\alpha-1)A^{*2}A^5\Big)\\
&+\rho_0(\alpha^3-2\alpha^2+2\alpha)\left(A^2A^*AA^*-AA^*A^2A^*+A^*A^2A^*A\right)\\
&+\rho_0(-\alpha^3+2\alpha^2-1)\Big(A^2A^{*2}A+\alpha AA^*AA^*A\Big)\\
&+\rho_0(\alpha^4-\alpha^3-\alpha^2)\Big(AA^{*2}A^2\Big)\\
&+\rho_0(\alpha^4-3\alpha^3+2\alpha^2-\alpha)\left(A^*AA^*A^2\right)\\
&+\rho_0(-\alpha^4+2\alpha^3-\alpha^2+1)\Big(A^{*2}A^3\Big)\\
&+\rho_0^2\big[A,A^{*2}\big]\ .
\end{aligned}$$

The two other monomials $A^4A^{*2}A$, $A^3A^{*2}A^2$ are also ordered using (3.1.20). One obtains:

$$\begin{aligned}
A^4A^{*2}A \;=\;& (\alpha^2-\alpha)\left(A^2A^*A^2A^*A+\alpha A^*A^2A^*A^3\right)+\ \alpha^2A^{*2}A^5\\
&+(\alpha^2-1)\Big(\alpha AA^*AA^*A^3-\alpha AA^*A^2A^*A^2-\alpha A^*AA^*A^4-AA^{*2}A^4\Big)\\
&+\rho_0\Big(A^2A^{*2}A-(1-\alpha^2)AA^{*2}A^2-\alpha^2A^{*2}A^3\Big)\\
&+\rho_0\left(\alpha^2-\alpha\right)\left(A^*AA^*A^2-AA^*AA^*A\right)\ ,\\
A^3A^{*2}A^2 \;=\;& \alpha\left(A^2A^*AA^*A^2-AA^*A^2A^*A^2+A^*A^2A^*A^3-A^*AA^*A^4\right)+A^{*2}A^5\\
&+\rho_0(AA^{*2}A^2-A^{*2}A^3)\ .
\end{aligned}$$

The ordered expression for the fourth monomial A^3A^{*2} directly follows from (3.1.20). Having the explicit ordered expressions of A^5A^{*2}, $A^4A^{*2}A$, $A^3A^{*2}A^2$ and A^3A^{*2} in terms of monomials of the type (3.1.21), let us consider the combination:

$$f_2(A,A^*)=c_0^{[2,0]}A^5A^{*2}-c_1^{[2,0]}A^4A^{*2}A+c_2^{[2,0]}A^3A^{*2}A^2-\rho_0c_0^{[2,1]}A^3A^{*2} \tag{3.1.22}$$

with unknown coefficients $c_j^{[2,0]}$, $j=1,2$, $c_0^{[2,1]}$, and normalization $c_0^{[2,0]}=1$. After simplifications, the combination takes the ordered form:

$$f_2(A,A^*)=c_3^{[2,0]}A^2A^{*2}A^3-c_4^{[2,0]}AA^{*2}A^4+c_5^{[2,0]}A^{*2}A^5\ +\ g_2(A,A^*) \tag{3.1.23}$$

where

$$\begin{aligned}
&c_3^{[2,0]}=\alpha^3-2\alpha^2+1\ ,\quad c_4^{[2,0]}=\alpha^2(\alpha^2-\alpha-1)+c_1^{[2,0]}(1-\alpha^2)\ ,\\
&c_5^{[2,0]}=(\alpha^2-\alpha)^2-\alpha^2c_1^{[2,0]}+c_2^{[2,0]}\ .
\end{aligned} \tag{3.1.24}$$

Inspired by the structure of Lusztig's higher order $q-$Serre relations, consider the conditions under which the combination $g_2(A,A^*)$ never contains monomials of the form $A^pA^*A^sA^*A^t$

($p \leq 2,\ 1 \leq s \leq 2$). At the lowest order in ρ_0, given a particular monomial the condition under which its coefficient in $g_2(A, A^*)$ is vanishing is given by:

$$\begin{aligned}
&A^2A^*A^2A^*A:\ \alpha^3 - 2\alpha^2 + 1 - c_1^{[2,0]}(\alpha - 1) = 0\ ,\\
&A^2A^*AA^*A^2:\ -\alpha^3 + 2\alpha^2 - 1 + c_2^{[2,0]} = 0\ ,\\
&AA^*A^2A^*A^2:\ (\alpha - 1)(-\alpha^3 + \alpha^2 + \alpha) - c_1^{[2,0]}(1 - \alpha^2) - c_2^{[2,0]} = 0\ ,\\
&AA^*AA^*A^3:\ (1 - \alpha^2)(-\alpha^3 + \alpha^2 + \alpha) + c_1^{[2,0]}\alpha(1 - \alpha^2) = 0\ ,\\
&A^*A^2A^*A^3:\ (\alpha - 1)(\alpha^3 - 2\alpha^2 + 1) - c_1^{[2,0]}(\alpha^2 - \alpha) + c_2^{[2,0]} = 0\ ,\\
&A^*AA^*A^4:\ (\alpha - 1)(-\alpha^3 + \alpha^2 + \alpha) - c_1^{[2,0]}(1 - \alpha^2) - c_2^{[2,0]} = 0\ .
\end{aligned}$$

Recall that $\alpha = [3]_q$. The solution $\{c_j^{[2,0]},\ j = 1, 2\}$ to this system of equations exists, and it is unique. In terms of $q-$binomials, it reads:

$$c_1^{[2,0]} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}_q\ , \qquad c_2^{[2,0]} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q\ . \tag{3.1.25}$$

At the next order ρ_0, the conditions such that monomials of the type $A^pA^*A^sA^*A^t$ with $p + s + t \leq 3$ and $1 \leq s \leq 2$ are vanishing yield to:

$$c_0^{[2,1]} = q^4 + q^{-4} + 3\ .$$

All the other coefficients of the monomials $A^nA^{*2}A^m$ for $n + m = 5, 3, 1$ are explicitly determined in terms of $c_j^{[2,0]}$ ($j = 0, 1, 2$), $c_0^{[2,1]}$, ρ_0 and ρ_0^2. Based on these results, we conclude that the $q-$Dolan-Grady relation (1.2.1) implies the existence of a unique linear relation between monomials of the type $A^nA^{*2}A^m$ with $n + m = 5, 3, 1$. This relation can be seen as a ρ_0-deformed analogue of the simplest higher order $q-$Serre relation. Explicitly, one finds:

$$\begin{aligned}
\sum_{j=0}^{5}(-1)^j \begin{bmatrix} 5 \\ j \end{bmatrix} A^{5-j}A^{*2}A^j \;=\;& \rho_0((q^4 + q^{-4} + 3)(A^3A^{*2} - A^{*2}A^3)\\
&- [5]_q\,[3]_q\,(A^2A^{*2}A - AA^{*2}A^2))\\
&- \rho_0^2\big(q^2 + q^{-2}\big)^2\,\big(AA^{*2} - A^{*2}A\big)\ .
\end{aligned} \tag{3.1.26}$$

Using the automorphism $A \leftrightarrow A^*$ and $\rho_0 \leftrightarrow \rho_1$ which exchanges (1.2.1) and (1.2.2), the second relation generalizing (1.2.2) is obtained. The coefficients coincide with the ones given in Example 2. For the special undeformed case $\rho_0 = \rho_1 = 0$, note that both relations reduce to the simplest examples of higher order $q-$Serre relations.

3.1.4.2 Proof of the relations for $r = 3$

Following a similar analysis, the next example of higher order $q-$Dolan-Grady relations can be also derived. To this end, one is looking for a linear relation between monomials of

the type $A^n A^{*3} A^m$, $n+m = 7,5,3,1$. Assume the $q-$Dolan-Grady relation (1.2.1) and its simplest consequence (3.1.26). Write the four monomials:

$$A^7 A^{*3} = (A^7 A^{*2}) A^* \ , \quad A^6 A^{*3} A = (A^6 A^{*2}) A^* A \ ,$$
$$A^5 A^{*3} A^2 = (A^5 A^{*2}) A^* A^2 \ , \quad A^5 A^{*3} = (A^5 A^{*2}) A^* \ .$$

Using (3.1.26) and then (3.1.20), they can be expressed solely in terms of monomials of the type:

$$\begin{aligned} &A^n A^{*3} A^m && \text{with} \quad n \leq 4 \ , \ n+m = 7,5,3,1 \ , \\ &A^p A^{*2} A^s A^* A^t && \text{with} \quad p \leq 4 \ , \ 1 \leq s \leq 2 \ , \ p+s+t = 7,5,3,1 \ . \end{aligned} \tag{3.1.27}$$

Then, introduce the combination

$$f_3(A, A^*) = c_0^{[3,0]} A^7 A^{*3} - c_1^{[3,0]} A^6 A^{*3} A + c_2^{[3,0]} A^5 A^{*3} A^2 - \rho_0 c_0^{[3,1]} A^5 A^{*3} \tag{3.1.28}$$

with unknown coefficients $c_j^{[3,0]}$, $(j = 1,2)$, $c_0^{[3,1]}$ and normalization $c_0^{[3,0]} = 1$. By straightforward calculations using the ordered expressions of $A^7 A^{*3}$, $A^6 A^{*3} A$, $A^5 A^{*3} A^2$ and $A^5 A^{*3}$, $f_3(A, A^*)$ is reduced to a combination of monomials of the type (3.1.27). Note that the coefficients of the monomials $A^n A^{*3} A^m$ for $n+m = 5,3,1$ are of order $\rho_0, \rho_0^2, \rho_0^3$, respectively. Identifying the conditions under which the coefficient of any monomial of the form

$$A^p A^{*2} A^s A^* A^t \quad \text{with} \quad p \leq 4 \ , \ 1 \leq s \leq 2 \ , \quad p+s+t = 7,5,3,1 \ ,$$

is vanishing, one obtains a system of equations for the coefficients, which solution is unique. Simplifying (3.1.28) according to the explicit solutions $c_j^{[3,0]}$, $j = 1,2$ and $c_0^{[3,1]}$, one ends up with the next example of higher order $q-$Dolan-Grady relations. Using the automorphism $A \leftrightarrow A^*$ and $\rho_0 \leftrightarrow \rho_1$, the second relation follows. One finds:

$$\begin{aligned} &\sum_{p=0}^{3} \rho_0^p \sum_{j=0}^{7-2p} (-1)^{j+p} c_j^{[3,p]} A^{7-2p-j} A^{*3} A^j = 0 \ , \\ &\sum_{p=0}^{3} \rho_1^p \sum_{j=0}^{7-2p} (-1)^{j+p} c_j^{[3,p]} A^{*7-2p-j} A^3 A^{*j} = 0 \end{aligned} \tag{3.1.29}$$

where $c_j^{[3,p]} = c_{7-2p-j}^{[3,p]}$, $c_j^{[3,0]} = \begin{bmatrix} 7 \\ j \end{bmatrix}_q$ and

$$\begin{aligned} c_0^{[3,1]} &= \left(q^8 + 3q^4 + 6 + 3q^{-4} + q^{-8}\right), c_1^{[3,1]} = [7]_q \left(q^6 + q^4 + q^2 + 4 + q^{-2} + q^{-4} + q^{-6}\right), \\ c_2^{[3,1]} &= [7]_q \left(q^2 - 1 + q^{-2}\right)\left(q^2 + q^{-2}\right)\left(q^4 + 2q^2 + 4 + 2q^{-2} + q^{-4}\right) \ , \\ c_0^{[3,2]} &= \left(q^6 + 2q^2 + 3q^{-2} + q^{-6}\right)\left(q^6 + 3q^2 + 2q^{-2} + q^{-6}\right) \ , \\ c_1^{[3,2]} &= [7]_q \left(q^8 + q^6 + 4q^4 + q^2 + 7 + q^{-2} + 4q^{-4} + q^{-6} + q^{-8}\right) \ , \\ c_0^{[3,3]} &= [2]_{q^2}^2 [3]_{q^2}^2 \ . \end{aligned}$$

It is straightforward to compare the coefficients above with the ones obtained from the expansion of the polynomial generating function $p_3(x, y)$. Although the coefficients above look different, they coincide exactly with the ones reported in Example 3.

3.1.4.3 Relations for r generic

To Look for a linear relation between monomials of the type $A^n A^{*r} A^m$, $n+m = 2r+1, 2r-1, ..., 1$, for $r \geq 1$, relations of the form

$$\sum_{p=0}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+1} (-1)^{j+p} c_j^{[r,p]} A^{2(r-p)+1-j} A^{*r} A^j = 0 \ , \tag{3.1.30}$$

$$\sum_{p=0}^{r} \rho_1^p \sum_{j=0}^{2(r-p)+1} (-1)^{j+p} c_j^{[r,p]} A^{*2(r-p)+1-j} A^{r} A^{*j} = 0 \tag{3.1.31}$$

are expected, where the elements A, A^* satisfy the $q-$Dolan-Grady relations (1.2.1) and (1.2.2). Our aim is now to study these relations in details and obtain recursive formulae for the coefficients $c_j^{[r,p]}$.

In order to study the higher order $q-$Dolan-Grady relations (3.1.30) for generic values of r, we proceed by induction. First, assume the basic relation (1.2.1) holds and implies all relations (3.1.30) up to r which explicit coefficients $c_j^{[r,p]}$ in terms of q are assumed to be known. It is the case for $r = 2, 3$ as shown above. Our aim is to construct the higher order relation associated with $r+1$ and express the coefficients $c_j^{[r+1,p]}$, $(j = 0, 1, ...2r+3-2p$, $p = 0, ..., r+1)$ in terms of $c_{j'}^{[r,p']}$ $(j' = 0, 1, ...2r+1-2p'$, $p' = 0, ..., r)$. Following the steps described for $r = 2, 3$, from the relation (3.1.30) we first deduce:

$$\begin{aligned} A^{2r+1} A^{*r} &= -\sum_{j=1}^{2r+1} (-1)^j c_j^{[r,0]} A^{2r+1-j} A^{*r} A^j \\ &\quad -\sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+1} (-1)^{j+p} c_j^{[r,p]} A^{2(r-p)+1-j} A^{*r} A^j, \end{aligned} \tag{3.1.32}$$

$$\begin{aligned} A^{2r+2} A^{*r} &= -\sum_{j=2}^{2r+2} (-1)^j M_j^{(r,0)} A^{2r+2-j} A^{*r} A^j \\ &\quad -\sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+2} (-1)^{j+p} M_j^{(r,p)} A^{2(r-p)+2-j} A^{*r} A^j, \end{aligned} \tag{3.1.33}$$

$$\begin{aligned} A^{2r+3} A^{*r} &= -\sum_{j=3}^{2r+3} (-1)^j N_j^{(r,0)} A^{2r+3-j} A^{*r} A^j \\ &\quad -\sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+3} (-1)^{j+p} N_j^{(r,p)} A^{2(r-p)+3-j} A^{*r} A^j \end{aligned} \tag{3.1.34}$$

where the coefficients $M_j^{(r,p)}$, $N_j^{(r,p)}$ are determined recursively in terms of $c_j^{[r,p]}$ (see Appendix A). Now, write the four monomials:

$$A^{2r+3} A^{*r+1} = (A^{2r+3} A^{*r}) A^* \ , \quad A^{2r+2} A^{*r+1} A = (A^{2r+2} A^{*r}) A^* A \ ,$$
$$A^{2r+1} A^{*r+1} A^2 = (A^{2r+1} A^{*r}) A^* A^2 \ , \quad A^{2r+1} A^{*r+1} = (A^{2r+1} A^{*r}) A^* \ .$$

Using (3.1.32)-(3.1.34), they can be expressed solely in terms of:

$$A^n A^{*r+1} A^m \text{ with } \quad n \leq 2r \ , \ n+m = 2r+3, 2r+1, ..., 1 \ , \tag{3.1.35}$$

$$A^p A^{*r} A^s A^* A^t \text{ with } \quad p \leq 2r \ , \ 1 \leq s \leq 2 \ , \ p+s+t = 2r+3, 2r+1, ..., 1. \tag{3.1.36}$$

It is however clear from (3.3.2.1)-(3.1.34) that each monomial $A^{2r+3}A^{*r+1}$, $A^{2r+2}A^{*r+1}A$, $A^{2r+1}A^{*r+1}A^2$ and $A^{2r+1}A^{*r+1}$ can be further reduced using (1.2.1). For instance,

$$\begin{aligned}(A^{2r+3}A^{*r})A^* &= -\sum_{j=3}^{2r+3}(-1)^j N_j^{(r,0)} A^{2r+3-j} A^{*r} \underbrace{A^j A^*}_{\text{reducible}} \\ &\quad -\sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+3} (-1)^{j+p} N_j^{(r,p)} A^{2(r-p)+3-j} A^{*r} \underbrace{A^j A^*}_{\text{reducible if } j \geq 3} \ .\end{aligned}$$

According to (1.2.1), observe that the monomials $A^j A^*$ (for j even or odd) can be written as:

$$A^{2n+2}A^* = \sum_{k=0}^{n}\sum_{i=0}^{2} \rho_0^{n-k} \eta_{k,i}^{(2n+2)} A^{2-i} A^* A^{2k+i} \ , \tag{3.1.37}$$

$$A^{2n+3}A^* = \sum_{k=1}^{n+1}\sum_{i=0}^{2} \rho_0^{n+1-k} \eta_{k,i}^{(2n+3)} A^{2-i} A^* A^{2k-1+i} + \rho_0^{n+1}(AA^* - A^*A) \ , \tag{3.1.38}$$

where the coefficients $\eta_{k,i}^{(2n+2)}, \eta_{k,i}^{(2n+3)}$ are determined recursively in terms of q (see Appendix A). It follows:

$$\begin{aligned}A^{2r+3}A^{*r+1} &= \sum_{i=1}^{r+1} N_{2i+1}^{(r,0)} A^{2(r-i)+2} A^{*r} \Big(\sum_{k=1}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k-1+j} + \rho_0^i (AA^* - A^*A)\Big) \\ &\quad - \sum_{i=1}^{r} N_{2i+2}^{(r,0)} A^{2(r-i)+1} A^{*r} \Big(\sum_{k=0}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+2)} A^{2-j} A^* A^{2k+j}\Big) \\ &\quad - \sum_{p=1}^{r} (-\rho_0)^p \big(N_0^{(r,p)} A^{2(r-p)+3} A^{*r+1} - N_1^{(r,p)} A^{2(r-p)+2} A^{*r} AA^* + N_2^{(r,p)} A^{2(r-p)+1} A^{*r} A^2 A^*\big) \\ &\quad + \sum_{p=1}^{r} (-\rho_0)^p \sum_{i=1}^{r-p+1} N_{2i+1}^{(r,p)} A^{2(r-p-i)+2} A^{*r} \Big(\sum_{k=1}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k-1+j} + \rho_0^i (AA^* - A^*A)\Big) \\ &\quad - \sum_{p=1}^{r-1} (-\rho_0)^p \sum_{i=1}^{r-p} N_{2i+2}^{(r,p)} A^{2(r-p-i)+1} A^{*r} \Big(\sum_{k=0}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+2)} A^{2-j} A^* A^{2k+j}\Big) \\ &\quad - (-\rho)^{r+1} \big(N_0^{(r,r+1)} AA^{*r+1} - N_1^{(r,r+1)} A^{*r} AA^*\big) \ .\end{aligned} \tag{3.1.39}$$

The three other monomials $A^{2r+2}A^{*r+1}A$, $A^{2r+1}A^{*r+1}A^2$ and $A^{2r+1}A^{*r+1}$ are also further reduced. For simplicity, corresponding expressions are reported in Appendix B. Now, introduce the combination

$$\begin{aligned}f_{r+1}(A, A^*) &= c_0^{[r+1,0]} A^{2r+3} A^{*r+1} - c_1^{[r+1,0]} A^{2r+2} A^{*r+1} A \\ &\quad + c_2^{[r+1,0]} A^{2r+1} A^{*r+1} A^2 - \rho_0 c_0^{[r+1,1]} A^{2r+1} A^{*r+1}\end{aligned} \tag{3.1.40}$$

with unknown coefficients $c_j^{[r+1,0]}$, $(j = 1,2)$, $c_0^{[r+1,1]}$ and normalization $c_0^{[r+1,0]} = 1$. Combining all reduced expressions for $A^{2r+3}A^{*r+1}$, $A^{2r+2}A^{*r+1}A$, $A^{2r+1}A^{*r+1}A^2$ and $A^{2r+1}A^{*r+1}$ reported in Appendix B, one observes that $f_{r+1}(A, A^*)$ generates monomials either of the type (3.1.35) or (3.1.36). First, consider monomials of the type (3.1.36) which occur at the lowest order in ρ_0, namely $A^{2r}A^{*r}A^2A^*A$ and $A^{2r}A^{*r}AA^*A^2$. The conditions under which their coefficients are vanishing read:

$$\begin{aligned} A^{2r}A^{*r}A^2A^*A: &\qquad N_3^{(r,0)}\eta_{1,0}^{(3)} + c_1^{[r+1,0]}M_2^{(r,0)} = 0\ , \\ A^{2r}A^{*r}AA^*A^2: &\qquad N_3^{(r,0)}\eta_{1,1}^{(3)} + c_2^{[r+1,0]}c_1^{[r,0]} = 0\ . \end{aligned}$$

Using the explicit expressions for $N_3^{(r,0)}$, $\eta_{1,0}^{(3)}$ and $\eta_{1,1}^{(3)}$ given in Appendices A,B, it is easy to solve these these equations. It yields to:

$$c_1^{[r+1,0]} = \left[\begin{array}{c} 2r+3 \\ 1 \end{array}\right]_q, \qquad c_2^{[r+1,0]} = \left[\begin{array}{c} 2r+3 \\ 2 \end{array}\right]_q. \tag{3.1.41}$$

The conditions under which the coefficients of other unwanted monomials of the type (3.1.36) are vanishing have now to be considered. In particular, similarly to the case $r = 2,3$ the coefficients $c_1^{[r+1,0]}$, $c_2^{[r+1,0]}$ arise in the following set of conditions:

$$\begin{aligned} A^{2r-1}A^{*r}AA^*A^3: &\qquad N_4^{(r,0)}\eta_{1,1}^{(4)} + c_1^{[r+1,0]}M_3^{(r,0)}\eta_{1,1}^{(3)} = 0, \\ A^{2r-1}A^{*r}A^2A^*A^2: &\qquad N_4^{(r,0)}\eta_{1,0}^{(4)} + c_1^{[r+1,0]}M_3^{(r,0)}\eta_{1,0}^{(3)} + c_2^{[r+1,0]}c_2^{[r,0]} = 0, \\ A^{2(r-i)}A^{*r}AA^*A^{2i+2}: &\qquad N_{2i+3}^{(r,0)}\eta_{i+1,1}^{(2i+3)} + c_1^{[r+1,0]}M_{2i+2}^{(r,0)}\eta_{i,1}^{(2i+2)} + c_2^{[r+1,0]}c_{2i+1}^{[r,0]}\eta_{i,1}^{(2i+1)} = 0,\ i = \overline{1,r}, \\ A^{2(r-i)}A^{*r}A^2A^*A^{2i+1}: &\qquad N_{2i+3}^{(r,0)}\eta_{i+1,0}^{(2i+3)} + c_1^{[r+1,0]}M_{2i+2}^{(r,0)}\eta_{i,0}^{(2i+2)} + c_2^{[r+1,0]}c_{2i+1}^{[r,0]}\eta_{i,0}^{(2i+1)} = 0,\ i = \overline{1,r}, \\ A^{2(r-i)+1}A^{*r}AA^*A^{2i+1}: &\qquad N_{2i+2}^{(r,0)}\eta_{i,1}^{(2i+2)} + c_1^{[r+1,0]}M_{2i+1}^{(r,0)}\eta_{i,1}^{(2i+1)} + c_2^{[r+1,0]}c_{2i}^{[r,0]}\eta_{i-1,1}^{(2i)} = 0,\ i = \overline{2,r}, \\ A^{2(r-i)+1}A^{*r}A^2A^*A^{2i}: &\qquad N_{2i+2}^{(r,0)}\eta_{i,0}^{(2i+2)} + c_1^{[r+1,0]}M_{2i+1}^{(r,0)}\eta_{i,0}^{(2i+1)} + c_2^{[r+1,0]}c_{2i}^{[r,0]}\eta_{i-1,0}^{(2i)} = 0\ \ i = \overline{2,r}. \end{aligned}$$

Using the recursion relations in Appendices A,B, we have checked that all above equations are satisfied, as expected.

More generally, one determines all other coefficients $c_j^{[r+1,0]}$ for $j \geq 3$. One finds:

$$\begin{aligned} c_3^{[r+1,0]} &= N_3^{(r,0)}\eta_{1,2}^{(3)} = \left[\begin{array}{c} 2r+3 \\ 3 \end{array}\right]_q, \\ c_4^{[r+1,0]} &= N_4^{(r,0)}\eta_{1,2}^{(4)} + c_1^{[r+1,0]}M_3^{(r,0)}\eta_{1,2}^{(3)} = \left[\begin{array}{c} 2r+3 \\ 4 \end{array}\right]_q, \\ c_{2k+1}^{[r+1,0]} &= N_{2k+1}^{(r,0)}\eta_{k,2}^{(2k+1)} + c_1^{[r+1,0]}M_{2k}^{r,0}\eta_{k-1,2}^{(2k)} + c_2^{[r+1,0]}c_{2k-1}^{[r,0]}\eta_{k-1,2}^{(2k-1)}, \quad k = \overline{2,r+1}, \\ c_{2k+2}^{[r+1,0]} &= N_{2k+2}^{(r,0)}\eta_{k,2}^{(2k+2)} + c_1^{[r+1,0]}M_{2k+1}^{(r,0)}\eta_{k,2}^{(2k+1)} + c_2^{[r+1,0]}c_{2k}^{[r,0]}\eta_{k-1,2}^{(2k)}, \quad k = \overline{2,r}. \end{aligned}$$

For any $j \geq 0$, one finds that the coefficient $c_j^{[r+1,0]}$ can be simply expressed as a $q-$binomial:

$$c_j^{[r+1,0]} = \left[\begin{array}{c} 2r+3 \\ j \end{array}\right]_q. \tag{3.1.42}$$

All coefficients $c_j^{[r+1,0]}$ being obtained, at the lowest order in ρ_0 one has to check that the coefficients of any unwanted term of the type (3.1.36) with $p+s+t = 2r+1, 2r-1, ..., 1$ are systematically vanishing. Using the recursion relations given in Appendices A,B, this has been checked in details. Then, following the analysis for $r = 3$ it remains to determine the coefficient $c_0^{[r+1,1]}$ which contributes at the order ρ_0. The condition such that the coefficient of the monomial $A^{2r}A^{*r}AA^*$ is vanishing yields to:

$$c_0^{[r+1,1]} = {c_1^{[r,0]}}^2 - 2c_2^{[r,0]} + \frac{c_3^{[r,0]}}{c_1^{[r,0]}} - \frac{c_1^{[r,1]}}{c_1^{[r,0]}} + 2c_0^{[r,1]} \ . \tag{3.1.43}$$

Using the explicit expression for $c_j^{[r+1,0]}, j = 0, 1, 2$ and $c_0^{[r+1,1]}$, we have checked in details that $f_{r+1}(A, A^*)$ reduces to a combination of monomials of the type (3.1.35) only. The reduced expression $f_{r+1}(A, A^*)$ determines uniquely all the remaining coefficients $c_j^{[r+1,p]}$ for $p \geq 1$. For r generic, in addition to (3.1.42) and (3.1.43) one finally obtains:

$$\begin{aligned} c_0^{[r+1,r+1]} &= c_0^{[r+1,1]}c_0^{[r,r]} + N_0^{(r,r+1)}, \\ c_0^{[r+1,p]} &= N_0^{(r,p)} + c_0^{[r+1,1]}c_0^{[r,p-1]}, \quad p = \overline{2,r}, \end{aligned} \tag{3.1.44}$$

$$\begin{aligned} c_1^{[r+1,1]} &= N_3^{(r,0)} + c_1^{[r+1,0]}M_0^{(r,1)}, \\ c_1^{[r+1,2]} &= -N_5^{(r,0)} + N_3^{(r,1)} + c_0^{[r+1,1]}c_3^{[r,0]} + c_1^{[r+1,0]}M_0^{(r,2)}, \\ c_1^{[r+1,r+1]} &= \sum_{p=0}^{r}(-1)^{r+p}N_{2(r-p)+3}^{(r,p)} + c_0^{[r+1,1]}\sum_{p=0}^{r-1}(-1)^{r+p+1}c_{2(r-p)+1}^{[r,p]}, \\ c_1^{[r+1,p]} &= \sum_{j=0}^{p-1}(-1)^{j+p+1}N_{2(p-j)+1}^{(r,j)} + c_1^{[r+1,0]}M_0^{(r,p)} \\ &\quad + c_0^{[r+1,1]}\sum_{j=0}^{p-2}(-1)^{j+p}c_{2(p-j)-1}^{[r,j]}, \quad p = \overline{3,r}, \end{aligned}$$

$$\begin{aligned} c_2^{[r+1,1]} &= -N_4^{(r,0)}\eta_{0,2}^{(4)} + c_1^{[r+1,0]}M_3^{(r,0)} + c_2^{[r+1,0]}c_0^{[r,1]}, \\ c_2^{[r+1,2]} &= N_6^{(r,0)}\eta_{0,2}^{(6)} - N_4^{(r,1)}\eta_{0,2}^{(4)} + c_1^{[r+1,0]}(M_5^{(r,0)} - M_3^{(r,1)}) - c_2^{[r+1,0]}c_0^{[r,2]} + c_0^{[r+1,1]}c_4^{[r,0]}\eta_{0,2}^{(4)}, \\ c_2^{[r+1,p]} &= \sum_{j=0}^{p-1}(-1)^{j+p}N_{2(p-j)+2}^{(r,j)}\eta_{0,2}^{(2(p-j)+2)} + c_1^{[r+1,0]}\sum_{j=0}^{p-1}(-1)^{j+p+1}M_{2(p-j)+1}^{(r,j)} \\ &\quad + c_2^{[r+1,0]}c_0^{[r,p]} + c_0^{[r+1,1]}\sum_{j=0}^{p-2}(-1)^{j+p+1}c_{2(p-j)}^{[r,j]}\eta_{0,2}^{(2(p-j))}, \quad p = \overline{3,r}, \end{aligned}$$

$$
\begin{aligned}
c_3^{[r+1,1]} &= -(N_5^{(r,0)}\eta_{1,2}^{(5)} - N_3^{(r,1)}\eta_{1,2}^{(3)}) - c_1^{[r+1,0]}M_4^{(r,0)}\eta_{0,2}^{(4)} + c_0^{[r+1,1]}c_3^{[r,0]}\eta_{1,2}^{(3)} + c_2^{[r+1,0]}c_3^{[r,0]}, \\
c_3^{[r+1,p]} &= \sum_{j=0}^{p}(-1)^{j+p}N_{2(p-j)+3}^{(r,j)}\eta_{1,2}^{(2(p-j)+3)} + c_1^{[r+1,0]}\sum_{j=0}^{p-1}(-1)^{j+p}M_{2(p-j)+2}^{(r,j)}\eta_{0,2}^{(2(p-j)+2)} \\
&\quad + c_2^{[r+1,0]}\sum_{j=0}^{p-1}(-1)^{j+p+1}c_{2(p-j)+1}^{[r,j]} + c_0^{[r+1,1]}\sum_{j=0}^{p-1}(-1)^{j+p+1}c_{2(p-j)+1}^{[r,j]}\eta_{1,2}^{(2(p-j)+1)}, \quad j=\overline{2,r},
\end{aligned}
$$

$$
\begin{aligned}
c_4^{[r+1,1]} &= -(N_6^{(r,0)}\eta_{1,2}^{(6)} - N_4^{(r,1)}\eta_{1,2}^{(4)}) - c_1^{[r+1,0]}(M_5^{(r,0)}\eta_{1,2}^{(5)} - M_3^{(r,1)}\eta_{1,2}^{(3)}) \\
&\quad - c_2^{[r+1,0]}c_4^{[r,0]}\eta_{0,2}^{(4)} + c_0^{[r+1,1]}c_4^{[r,0]}\eta_{1,2}^{(4)}, \\
c_4^{[r+1,p]} &= \sum_{j=0}^{p}(-1)^{j+p}N_{2(p-j)+4}^{(r,j)}\eta_{1,2}^{(2(p-j)+4)} + c_1^{[r+1,0]}\sum_{j=0}^{p}(-1)^{j+p}M_{2(p-j)+3}^{(r,j)}\eta_{1,2}^{(2(p-j)+3)} \\
&\quad + c_2^{[r+1,0]}\sum_{j=0}^{p-1}(-1)^{j+p}c_{2(p-j)+2}^{[r,j]}\eta_{0,2}^{(2(p-j)+2)} \\
&\quad + c_0^{[r+1,1]}\sum_{j=0}^{p-1}(-1)^{j+p+1}c_{2(p-j)+2}^{[r,j]}\eta_{1,2}^{(2(p-j)+2)}, \quad p=\overline{2,r-1},
\end{aligned}
$$

$$
\begin{aligned}
c_{2k+3}^{[r+1,j-k]} &= \sum_{p=0}^{j-k}(-1)^{p+j+k}N_{2(j-p)+4}^{(r,p)}\eta_{k+1,2}^{(2(j-p)+4)} + c_1^{[r+1,0]}\sum_{p=0}^{j-k}(-1)^{p+j+k}M_{2(j-p)+2}^{(r,p)}\eta_{k,2}^{(2(j-p)+2)} \\
&\quad + c_2^{[r+1,0]}\sum_{p=0}^{j-k}(-1)^{p+j+k}c_{2(j-p)+1}^{[r,p]}\eta_{k,2}^{(2(j-p)+1)} \\
&\quad + c_0^{[r+1,1]}\sum_{p=0}^{j-k-1}(-1)^{p+j+k+1}c_{(2(j-p)+1)}^{[r,p]}\eta_{k+1,2}^{(2(j-p)+1)}, \quad j=\overline{3,r}, \qquad k=\overline{1,j-2},
\end{aligned}
$$

$$
\begin{aligned}
c_{2j+1}^{[r+1,1]} &= -(N_{2j+3}^{(r,0)}\eta_{j,2}^{(2j+3)} - N_{2j+1}^{(r,1)}\eta_{j,2}^{(2j+1)}) - c_1^{[r+1,0]}(\eta_{j-1,2}^{(2j+2)}M_{2j+2}^{(r,0)} - M_{2j}^{(r,1)}\eta_{j-1,2}^{(2j)}) \\
&\quad - c_2^{[r+1,0]}(c_{2j+1}^{[r,0]}\eta_{j-1,2}^{(2j+1)} - c_{2j-1}^{[r,1]}\eta_{j-1,2}^{(2j-1)}) + c_0^{[r+1,1]}c_{2j+1}^{[r,0]}\eta_{j,2}^{(2j+1)}, \quad j=\overline{2,r},
\end{aligned}
$$

$$
\begin{aligned}
c_{2k+2}^{[r+1,j-k]} &= \sum_{p=0}^{j-k}(-1)^{p+j+k}N_{2(j-p)+2}^{(r,p)}\eta_{k,2}^{(2(j-p)+2)} + c_1^{[r+1,0]}\sum_{p=0}^{j-k}(-1)^{p+j+k}M_{2(j-p)+1}^{(r,p)}\eta_{k,2}^{(2(j-p)+1)} \\
&\quad + c_2^{[r+1,0]}\sum_{p=0}^{j-k}(-1)^{p+j+k}c_{2(j-p)}^{[r,p]}\eta_{k-1,2}^{(2(j-p))} \\
&\quad + c_0^{[r+1,1]}\sum_{p=0}^{j-k-1}(-1)^{p+j+k+1}c_{(2(j-p))}^{[r,p]}\eta_{k,2}^{(2(j-p))}, \quad j=\overline{4,r}, \qquad k=\overline{2,j-2},
\end{aligned}
$$

$$\begin{aligned} c_{2j}^{[r+1,1]} &= c_0^{[r+1,1]} c_{2j}^{[r,0]} \eta_{j-1,2}^{(2j)} - N_{2j+2}^{(r,0)} \eta_{j-1,2}^{(2j+2)} + N_{2j}^{(r,1)} \eta_{j-1,2}^{(2j)} \\ &\quad - c_1^{[r+1,0]} (M_{2j+1}^{(r,0)} \eta_{j-1,2}^{(2j+1)} - M_{2j-1}^{(r,1)} \eta_{j-1,2}^{(2j-1)}) \\ &\quad - c_2^{[r+1,0]} (c_{2j}^{[r,0]} \eta_{j-2,2}^{(2j)} - c_{2j-2}^{[r,1]} \eta_{j-2,2}^{(2j-2)}), \quad j = \overline{3,r}. \end{aligned}$$

According to above results and using the automorphism $A \leftrightarrow A^*$ and $\rho_0 \leftrightarrow \rho_1$, we conclude that if A, A^* satisfy the defining relations (1.2.1), (1.2.2), then the higher order $q-$Dolan-Grady relations (3.1.30), (3.1.31) are such that the coefficients $c_j^{[r,p]}$ are determined recursively by (3.1.42), (3.1.43) and (3.1.44). For $p \geq 1$, they can be computed for practical purpose, for $r = 2, 3$ the coefficients $c_j^{[r,p]}$ are proportional to $[2r+1]_q$ iff $j \neq 0$ or $2r+1$; for a large number of values $r \geq 4$, this property holds too. As a consequence, the relations (3.1.7) drastically simplify for $q^{2r+1} = \pm 1$. This case is however not considered here. In particular, one observes that $c_j^{[r,p]} = c_{2(r-p)+1-j}^{[r,p]}$. For $r = 4, 5, ... \leq 10$, using a computer program we have checked in details that $r-th$ higher order relations of the form (3.1.7) hold, and that the coefficients satisfy above recursive formula.

3.1.5 Algorithm

A Maple software program has been constructed to calculate the expressions for the coefficients $c_j^{[r+1,p]}$ by induction on r. For $r = 2, 3, ..., 10$, the expressions have been compared with the exact expressions for the coefficients given in (3.1.9). Both expressions agree, thus giving a strong support to the conjecture. The Maple program is reported in Appendix D. Here, we sketch the algorithm.

- Input: r, $\eta_{1,j}^{(3)}, j = \overline{0,2}$; $\eta_{k,j}^{(4)}, k = \overline{0,1}, j = \overline{0,2}$
  $c_j^{[1,p]}, p = \overline{0,1}, j = \overline{0,3-2p}$; $c_j^{[2,p]}, p = \overline{0,2}, j = \overline{0,5-2p}$

- Output: $c_j^{[r+1,p]}$, $p = 0, \ldots, r+1$; $j = 0, \ldots, 2r+3-2p$

- Algorithm:

  Step 1. Compute the coefficients $\eta_{k,j}^{(m)}, m = \overline{5, 2r+3}, j = \overline{0,2}$,
  $$k = \begin{cases} \overline{0, [\frac{m-1}{2}]} & \text{If } m \text{ is even} \\ \overline{1, [\frac{m-1}{2}]} & \text{If } m \text{ is odd} \end{cases}$$

  Step 2. Compute $M_j^{(r,0)}, j = \overline{2, 2r+2}$; $M_j^{(r,p)}, p = \overline{1,r}, j = \overline{0, 2(r-p)+2}$

  Step 3. Compute $N_j^{(r,0)}, j = \overline{3, 2r+3}$; $N_j^{(r,p)}, p = \overline{1,r}, j = \overline{0, 2(r-p)+3}$;

  Step 4. Compute $c_0^{[r+1,0]}, c_1^{[r+1,0]}, c_2^{[r+1,0]}, c_0^{[r+1,1]}$

  Step 5. Compute $f_{r+1}(A, A^*)$ in the equation (3.1.40).

3.2 Higher order relations for the generalized $q-$Onsager algebra

3.2.1 Introduction

Introduced in [BB09], the generalized $q-$Onsager algebra $\mathcal{O}_q(\widehat{g})$ associated with the affine Lie algebra $\widehat{g}$ is a higher rank generalization of the so-called $q-$Onsager algebra [Ter99, Bas0404]. The usual $q-$Onsager algebra corresponds to the choice $\widehat{g} = \widehat{sl_2}$. The defining relations are determined by the entries of the Cartan matrix of the algebra considered. For $\widehat{g} = a_n^{(1)}$, it can be understood as a $q-$deformation of the sl_{n+1}-Onsager algebra introduced by Uglov and Ivanov [UI95]. By analogy with the $\widehat{sl_2}$ case [Bas0404, IT07], an algebra homomorphism from $\mathcal{O}_q(\widehat{g})$ to a certain coideal subalgebra of the Drinfeld-Jimbo [Dr87, J85] quantum universal enveloping algebra $\mathcal{U}_q(\widehat{g})$ is known [BB09]. From a general point of view, generalized $q-$Onsager algebras appear in the theory of quantum affine symmetric pairs [Kol12]. Note that realizations in terms of finite dimensional quantum algebras may be also considered: for instance, coideal subalgebras of $\mathcal{U}_q(g)$ studied by Letzter [Letz99] or the non-standard $\mathcal{U}'_q(so_n)$ introduced by Klimyk, Gavrilik and Iorgov [GI97, Klim01].

Besides the definition of the generalized $q-$Onsager algebra in terms of generators and relations [BB09, Definition 2.1], most of its properties remain to be studied.

Definition 3.2.1 *Let $\{a_{ij}\}$ be the extended Cartan matrix of the affine Lie algebra $\hat{g}$. Fix coprime integers d_i such that $d_i a_{ij}$ is symmetric. The generalized q-Onsager algebra $O_q(\hat{g})$ is an associative algebra with unit 1, generators A_i and scalars $\rho_{ij}^k, \gamma_{ij}^{kl} \in \mathbb{C}$ with $i,j \in \{0,1,\dots,n\}, k \in \{0,1,\dots,[-\frac{a_{ij}}{2}]-1\}$ and $l \in \{0,1,\dots,-a_{ij}-1-2k\}$ (k and l are positive integers). The defining relations are:*

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} A_i^{1-a_{ij}-r} A_j A_i^r = \sum_{k=0}^{\left[-\frac{a_{ij}}{2}\right]-1} \rho_{ij}^k \sum_{l=0}^{-2k-a_{ij}-1} (-1)^l \gamma_{ij}^{kl} A_i^{-2k-a_{ij}-1-l} A_j A_i^l, \tag{3.2.1}$$

Generalized $q-$Onsager algebras are extensions of the $q-$Onsager algebra to *higher rank* affine Lie algebras [BB09]. Inspired by the analysis of [BV13], analogues of Lusztig's higher order relations for $\mathcal{O}_q(\widehat{g})$ can be conjectured. First, recall some basic definitions.

Definition 3.2.2 *Let the simply-laced affine Lie algebra $\hat{g}$, the generalized $q-$Onsager algebra $\mathcal{O}_q(\widehat{g})$ is an associative algebra with unit 1, generators A_i and scalars ρ_i. The defining relations are:*

$$\begin{aligned} \sum_{k=0}^{2} (-1)^k \begin{bmatrix} 2 \\ k \end{bmatrix}_q A_i^{2-k} A_j A_i^k - \rho_i A_j &= 0 \quad \text{if} \quad i,j \quad \text{are linked}\,, \\ \left[A_i, A_j\right] &= 0 \quad \text{otherwize}\,. \end{aligned} \tag{3.2.2}$$

Remark 6 *For $\rho_i = 0$ the relations (3.2.2) reduce to the $q-$Serre relations of $U_q(\widehat{g})$.*

Remark 7 *For $q = 1$, the relations (3.2.2) coincide with the defining relations of the so-called $sl_{n+1}-$Onsager's algebra for $n > 1$ introduced by Uglov and Ivanov [UI95].*

3.2.2 Conjecture about the higher order relations of the generalized q-Onsager algebra

By analogy with the $\widehat{sl_2}$ case discussed in details in the previous parts, we expect the following form for the higher order relations

Conjecture 2 *Let $\{A_i\}$ be the fundamental generators of the generalized $q-$Onsager algebra (3.2.2), then $\{A_i\}$ satisfy the higher order relations as follows:*

$$\sum_{p=0}^{\{\frac{r+1}{2}\}} \sum_{k=0}^{r-2p+1} (-1)^{k+p} \rho_i^p \, c_k^{[r,p]} \, A_i^{r-2p+1-k} A_j^r A_i^k = 0 \qquad \text{if} \quad i,j \quad \text{are linked} \tag{3.2.3}$$

where the coefficients are given by:

$$c_k^{[r,p]} = \sum_{l=0}^{\{k/\alpha\}} \frac{(\{\frac{r+1}{2}\} - k + \alpha l - p)!}{(\{\frac{\alpha l}{2}\})!(\{\frac{r+1}{2}\} - k + \alpha l - p - \{\frac{\alpha l}{2}\})!} \sum_{\mathcal{P}_l} [s_1]_q^2 \cdots [s_p]_q^2 \frac{[2s_{p+1}]_q \cdots [2s_{p+k-\alpha l}]_q}{[s_{p+1}]_q \cdots [s_{p+k-\alpha l}]_q} \tag{3.2.4}$$

$$\text{with} \quad \begin{cases} k = \overline{0, \{\frac{r+1}{2}\}} \ , \quad s_i \in \{r - 2\{\frac{r-1}{2}\}, \ldots, r-2, r\} \ , \\ \mathcal{P}_l : \quad \begin{array}{c} s_1 < \cdots < s_p \ ; \quad s_{p+1} < \cdots < s_{p+k-\alpha l} \ , \\ \{s_1, \ldots, s_p\} \cap \{s_{p+1}, \ldots, s_{p+k-\alpha l}\} = \emptyset \end{array} \ , \\ \alpha = \begin{cases} 1 & \text{if } r \text{ is even,} \\ 2 & \text{if } r \text{ is odd} \end{cases} \end{cases} .$$

Although it is highly expected that the concept of tridiagonal pair for the $q-$Onsager algebra could be extended to the higher rank generalizations of the $q-$Onsager algebras, in the mathematical literature such object has not been introduced yet. For this reason, the conjecture for the higher order relations (3.2.3) can not be checked for every irreducible finite dimensional vector space on which the generators A_i act. Still, below we provide several supporting evidences for the conjecture. First, the conjecture is proven for $r \leq 5$. Secondly, recursive relations for the coefficients are derived. Using a Maple software, it is found that the coefficients computed from the recursion relations coincide exactly with the ones conjectured. Other checks of the conjecture are considered, thus giving independent supporting evidences.

3.2.3 Proof of the higher order relations for $r \leq 5$

For $r = 1$, the relations (3.2.3) are the defining relations of the generalized $q-$Onsager algebra $\mathcal{O}_q(\widehat{g})$. Assume A_i are the fundamental generators of $\mathcal{O}_q(\widehat{g})$. To derive the first example of higher order relations, we are looking for a linear relations between monomials of the type $A_i^n A_j^2 A_i^m$ with $n + m = 3, 1$. Suppose it is of the form (3.2.3) for $r = 2$ with yet unknown coefficients $c_k^{[r,p]}$. We show $c_k^{[r,p]}$ are uniquely determined. First, according to the defining relations (3.2.2) the monomial $A_i^2 A_j$ can be ordered as:

$$A_i^2 A_j = [2]_q A_i A_j A_i - A_j A_i^2 + \rho_i A_j \ . \tag{3.2.5}$$

Multiplying from the left and/or right by A_i, A_j $(i \neq j)$, the new monomials can be ordered as follows: each time a monomial of the form $A_i^n A_j^2 A_i^m$ with $n \geq 2$ arises, it is reduced using (3.2.5). For instance, one has:

$$A_i^3 A_j = ([2]_q^2 - 1) A_i A_j A_i^2 - [2]_q A_j A_i^3 + \rho_i ([2]_q A_j A_i + A_i A_j) \ . \tag{3.2.6}$$

Now, observe that the first two monomials in (3.2.3) for $r = 2$ can be written as $A_i^3 A_j^2 \equiv (A_i^3 A_j) A_j$ and $A_i^2 A_j^2 A_i \equiv (A_i^2 A_j) A_j A_i$. Following the ordering prescription, each of these monomials can be reduced as a combination of monomials of the type:

$$A_i^n A_j^2 A_i^m \qquad \text{with} \quad n \leq 1 \ , \ n + m = 3, 1 \ , \tag{3.2.7}$$

$$A_i^p A_j A_i A_j A_i^t \quad \text{with} \quad p \leq 1 \ , \ p + t = 2, 0 \ . \tag{3.2.8}$$

Plugging the reduced expressions of $A_i^3 A_j^2$ and $A_i^2 A_j^2 A_i$ in (3.2.3) for $r = 2$, one finds that all monomials of the form (3.2.8) cancel provided a simple system of equations for the coefficients $c_k^{[r,p]}$ is satisfied. The solution of this system is unique, given by:

$$c_k^{[2,0]} = \left[\begin{array}{c} 3 \\ k \end{array} \right]_q \quad \text{for} \quad k = 0, 1, 2, 3 \ , \quad \text{and} \quad c_0^{[2,1]} = c_1^{[2,1]} = q^2 + q^{-2} + 2 \ .$$

For $r = 3, 4, 5$, we proceed similarly: the monomials entering in the relations (3.2.3) are ordered according to the prescription described above. Given r, the reduced expression of the corresponding relation (3.2.3) holds provided the coefficients $c_k^{[r,p]}$ satisfy a system of equation which solution is unique. In each case, one finds:

$$c_k^{[r,0]} = \left[\begin{array}{c} r+1 \\ k \end{array} \right]_q \quad \text{for} \quad k = 0, ..., r+1 \ , \ r = 3, 4, 5 \ , \tag{3.2.9}$$

whereas for $p \geq 1$, the other coefficients are such that $c_k^{[r,p]} = c_{r-2p+1-k}^{[r,p]}$, given by:

Case $r = 3$: $c_0^{[3,1]} = q^4 + 2q^2 + 4 + 2q^{-2} + q^{-4}$, $c_1^{[3,1]} = [4]_q(q^2 + q^{-2} + 3)$,
$c_0^{[3,2]} = (q^2 + q^{-2} + 1)^2$;

Case $r = 4$: $c_0^{[4,1]} = (q^4 + 3 + q^{-4})[2]_q^2$, $c_1^{[4,1]} = [5]_q[3]_q[2]_q^2$,
$c_0^{[4,2]} = (q^2 + q^{-2})^2[2]_q^4$;

Case $r = 5$:

$$\begin{aligned}
c_0^{[5,1]} &= q^8 + 2q^6 + 4q^4 + 6q^2 + 9 + 6q^{-2} + 4q^{-4} + 2q^{-6} + q^{-8}, \\
c_1^{[5,1]} &= [6]_q[3]_q^{-1}(q^8 + 4q^6 + 8q^4 + 14q^2 + 16 + 14q^{-2} + 8q^{-4} + 4q^{-6} + q^{-8}), \\
c_2^{[5,1]} &= [6]_q[2]_q^{-1}[5]_q(q^4 + 3q^2 + 6 + 3q^{-2} + q^{-4}), \\
c_0^{[5,2]} &= q^{12} + 4q^{10} + 11q^8 + 20q^6 + 31q^4 + 40q^2 + 45 \\
&\quad +40q^{-2} + 31q^{-4} + 20q^{-6} + 11q^{-8} + 4q^{-10} + q^{-12}, \\
c_1^{[5,2]} &= [6]_q[3]_q^{-1}(q^{10} + 6q^8 + 17q^6 + 32q^4 + 47q^2 + 53 \\
&\quad +47q^{-2} + 32q^{-4} + 17q^{-6} + 6q^{-8} + q^{-10}), \\
c_0^{[5,3]} &= [3]_q^2[5]_q^2 \ .
\end{aligned}$$

3.2.4 Recursion relations of the coefficients of the higher order relations in generic case r

Above examples suggest that higher order relations of the form (3.2.3) exist for generic values of r. To derive the coefficients recursively, one first assumes that given r, the relation (3.2.3) exists and that all coefficients $c_k^{[r,p]}$ are already known in terms of q. The relation (3.2.3) for $r \to r+1$ is then considered. In this case, the combination

$$f_r^{ADE}(A_i, A_j) = A_i^{r+2}A_j^{r+1} - c_1^{[r+1,0]}A_i^{r+1}A_j^{r+1}A_i \tag{3.2.10}$$

is introduced. Following the steps described in details in [BV13], the monomials $A_i^{r+2}A_j^{r+1}$ and $A_i^{r+1}A_j^{r+1}A_i$ are reduced using (3.2.2) and (3.2.3). The ordered expression of the first monomial follows:

$$\begin{aligned}
A_i^{r+2}A_j^{r+1} &= \sum_{k=2}^{r+2}(-1)^{k+1}M_k^{(r,0)}A_i^{r+2-k}A_j^r \underbrace{A_i^kA_j}_{\text{reducible}} \\
&\quad + \sum_{p=1}^{\{\frac{r+1}{2}\}}\sum_{k=0}^{r+2-2p}(-1)^{p+k+1}\rho_i^pM_k^{(r,p)}A_i^{r+2-2p-k}A_j^r \underbrace{A_i^kA_j}_{\text{reducible if } k\geq 2} ,
\end{aligned}$$

where the recursive relations for the coefficients $M_k^{(r,p)}$ are reported in Appendix C. Obviously, an ordered expression for the second monomial immediately follows from (3.2.3). Then, the whole combination can be further reduced $f_r^{ADE}(A_i, A_j)$. As an intermediate

step, one uses (3.2.2) to obtain:

$$\begin{aligned} A_i^{2n+1}A_j &= \sum_{p=0}^{n} \rho_i^p(\eta_0^{(2n+1,p)}A_iA_jA_i^{2n-2p} + \eta_1^{(2n+1,p)}A_jA_i^{(2n-2p+1)}) \ , \\ A_i^{2n+2}A_j &= \sum_{p=0}^{n} \rho_i^p(\eta_0^{(2n+2,p)}A_iA_jA_i^{2n+1-2p} + \eta_1^{(2n+2,p)}A_jA_i^{2n+2-2p}) + \rho_i^{n+1}A_j \ , \end{aligned}$$

where the coefficients $\eta_j^{(r,p)}$ are given in Appendix C. The ordered expression of $f_r^{ADE}(A_i, A_j)$ is then studied. A detailed analysis shows that all coefficients of monomials of the type $A_i^pA_j^rA_iA_jA_i^t$ (with $p+t = r-1, ..., 0$ if r is odd, and $p+t = r-1, ..., 1$ if r is even) vanish provided the coefficients $c_k^{[r,p]}$ satisfy a system of equations. According to the parity of r, one finds:

Case r odd: For $r = 2t+1$ and $p = 0$:

$$\begin{aligned} c_2^{[2t+1,0]} &= M_2^{(2t,0)}\eta_1^{(2,0)}, \\ c_{2h}^{[2t+1,0]} &= M_{2h}^{(2t,0)}\eta_1^{(2h,0)} + c_1^{[2t+1,0]}c_{2h-1}^{[2t,0]}\eta_1^{(2h-1,0)}, \qquad h = \overline{2, t+1}, \\ c_{2h+1}^{[2t+1,0]} &= M_{2h+1}^{(2t,0)}\eta_1^{(2h+1,0)} + c_1^{[2t+1,0]}c_{2h}^{[2t,0]}\eta_1^{(2h,0)}, \qquad h = \overline{1, t}. \end{aligned}$$

Using the recursion relations given in Appendix C, it is possible to show that these coefficients can be simply written in terms of $q-$binomials:

$$c_k^{[r,0]} = \left[\begin{array}{c} r+1 \\ k \end{array} \right]_q . \tag{3.2.11}$$

Other coefficients $c_0^{[2t+1,p]}$ for $p \geq 1$ are determined by the following recursion relations:

$$\begin{aligned} c_0^{[2t+1,t+1]} &= \sum_{p=0}^{t} (-1)^{p+t+1} M_{2(t+1-p)}^{(2t,p)}, \\ c_0^{[2t+1,1]} &= -M_2^{(2t,0)} + M_0^{(2t,1)}, \\ c_0^{[2t+1,h]} &= \sum_{p=0}^{h} (-1)^{p+h} M_{2(h-p)}^{(2t,p)}, \qquad h = \overline{2, t}, \end{aligned}$$

$$\begin{aligned} c_1^{[2t+1,1]} &= -(M_3^{(2t,0)}\eta_1^{(3,1)} - c_1^{[2t+1,0]}(-c_2^{[2t,0]} + c_0^{[2t,1]})), \\ c_1^{[2t+1,h]} &= \sum_{p=0}^{h-1} (-1)^{p+h} M_{2(h-p)+1}^{(2t,p)}\eta_1^{(2(h-p)+1,h-p)} \\ &\quad + c_1^{[2t+1,0]} \sum_{p=0}^{h} (-1)^{p+h} c_{2(h-p)}^{[2t,p]}, \qquad h = \overline{2, t}, \end{aligned}$$

$$
\begin{aligned}
c_2^{[2t+1,1]} &= -M_4^{(2t,0)}\eta_1^{(4,1)} + M_2^{(2t,1)}\eta_1^{(2,0)} - c_1^{[2t+1,0]}c_3^{[2t,0]}\eta_1^{(3,1)}, \\
c_{2h-2l+1}^{[2t+1,l]} &= \sum_{p=0}^{l}(-1)^{p+l}M_{2(h-p)+1}^{(2t,p)}\eta_1^{(2(h-p)+1,l-p)} \\
&\quad + c_1^{[2t+1,0]}\sum_{p=0}^{l}(-1)^{p+l}c_{2(h-p)}^{[2t,p]}\eta_1^{(2(h-p),l-p)}, \qquad h=\overline{2,t}, \quad l=\overline{1,h-1}, \\
c_{2h-2l}^{[2t+1,l]} &= \sum_{p=0}^{l}(-1)^{p+l}M_{2(h-p)}^{(2t,p)}\eta_1^{(2(h-p),l-p)} \\
&\quad + c_1^{[2t+1,0]}\sum_{p=0}^{min(l,h-2)}(-1)^{p+l}c_{2(h-p)-1}^{[2t,p]}\eta_1^{(2(h-p)-1,l-p)}, \qquad h=\overline{3,t+1}, \quad l=\overline{1,h-1}.
\end{aligned}
$$

<u>Case r even:</u> For $r=2t+2$, the coefficients $c_j^{[2t+2,0]}$ are given by:

$$
\begin{aligned}
c_2^{[2t+2,0]} &= M_2^{(2t+1,0)}\eta_1^{(2,0)}, \\
c_{2h+1}^{[2t+2,0]} &= M_{2h+1}^{(2t+1,0)}\eta_1^{(2h+1,0)} + c_1^{[2t+2,0]}c_{2h}^{[2t+1,0]}\eta_1^{(2h,0)}, \qquad h=\overline{1,t+1}, \\
c_{2h}^{[2t+2,0]} &= M_{2h}^{(2t+1,0)}\eta_1^{(2h,0)} + c_1^{[2t+2,0]}c_{2h-1}^{[2t+1,0]}\eta_1^{(2h-1,0)}, \qquad h=\overline{2,t+1}.
\end{aligned}
$$

According to the relations in Appendix C, one shows that $c_k^{[r,0]}$ simplify to $q-$binomials (3.2.11). For $p\geq 1$, the recursive formulae for all other coefficients are given by:

$$
\begin{aligned}
c_0^{[2t+2,1]} &= -M_2^{(2t+1,0)} + M_0^{(2t+1,1)}, \\
c_0^{[2t+2,h]} &= \sum_{p=0}^{h}(-1)^{p+h}M_{2(h-p)}^{(2t+1,p)}, \qquad h=\overline{2,t+1}, \\
c_1^{[2t+2,1]} &= -M_3^{(2t+1,0)}\eta_1^{(3,1)} + c_1^{[2t+2,0]}(-c_2^{[2t+1,0]} + c_0^{[2t+1,1]}),
\end{aligned}
$$

$$
\begin{aligned}
c_1^{[2t+2,h]} &= \sum_{p=0}^{h-1}(-1)^{p+h}M_{2(h-p)+1}^{(2t+1,p)}\eta_1^{(2(h-p)+1,h-p)} \\
&\quad + c_1^{[2t+2,0]}\sum_{p=0}^{h}(-1)^{p+h}c_{2(h-p)}^{[2t+1,p]}, \qquad h=\overline{2,t+1}, \\
c_2^{[2t+2,1]} &= -M_4^{(2t+1,0)}\eta_1^{(4,1)} + M_2^{(2t+1,1)}\eta_1^{(2,0)} - c_1^{[2t+2,0]}c_3^{[2t+1,0]}\eta_1^{(3,1)},
\end{aligned}
$$

$$
\begin{aligned}
c_{2h-2l}^{[2t+2,l]} &= \sum_{p=0}^{l} (-1)^{p+l} M_{2(h-p)}^{(2t+1,p)} \eta_1^{(2(h-p),l-p)} \\
&\quad + c_1^{[2t+2,0]} \sum_{p=0}^{min(l,h-2)} (-1)^{p+l} c_{2(h-p)-1}^{[2t+1,p]} \eta_1^{(2(h-p)-1,l-p)}, \qquad h=\overline{3,t+1}, l=\overline{1,h-1}, \\
c_{2h-2l+1}^{[2t+2,l]} &= \sum_{p=0}^{l} (-1)^{p+l} M_{2(h-p)+1}^{(2t+1,p)} \eta_1^{(2(h-p)+1,l-p)}) \\
&\quad + c_1^{[2t+2,0]} \sum_{p=0}^{l} (-1)^{p+l} c_{2(h-p)}^{[2t+1,p]} \eta_1^{(2(h-p),l-p)}, \qquad h=\overline{2,t+1}, l=\overline{1,h-1}.
\end{aligned}
$$

All coefficients $c_k^{[r,p]}$ entering in the higher order relations (3.2.3) can be computed recursively for any positive integer r. Note that setting $\rho_i = 0$, the relations (3.2.3) reproduce the higher order $q-$Serre relations (3.1.4) of $\mathcal{U}_q(\widehat{g})$ [Lusz93]. Using a computer program, up to $r = 10$ we have checked that the results for the coefficients derived from the recursion relations coincide exactly with the ones conjectured in (3.2.3).

The algorithm for the computation of the coefficients is the following (Appendix E)

- Input: $r+1,\ \eta_{0,0}^{(2)},\ \eta_{0,1}^{(2)},\ c_0^{[1,0]},\ c_1^{[1,0]},\ c_2^{[1,0]},\ c_0^{[1,1]}$
- Output: $c_k^{[r+1,p]},\ p=0,\dots,\{\frac{r+2}{2}\},\quad k=0,\dots,r+2-2p$
- Algorithm:

  Step 1. Compute the coefficients $\eta_{p,i}^{(h)},\ h=3,\dots,r+2,\ p=0,\dots,\{\frac{h-1}{2}\},\ i=0,1$

  Step 2. Compute $M_k^{(r,p)}$
  If $p=0$, then $k=2,\dots,r+2$.
  If $p=1,\dots,\{\frac{r+1}{2}\}$, then $k=1,\dots,r+2-2p$

  Step 3. Compute $c_1^{[r+1,0]}$

  Step 4. Compute $f_{r+1}(A,A^*)$ in the equation (3.2.10).

3.2.5 A two-variable polynomial generating function

For the $q-$Onsager algebra, it was shown that the coefficients entering in the $r-th$ higher order relations can be derived from a two-variable generating function. Here, for any simply-laced affine Lie algebras we propose a two-variable generating function for the coefficients $c_k^{[r,p]}$.

Definition 3.2.3 *Let $r \in \mathbb{Z}^+$. Let x, y be commuting indeterminates and ρ a scalar. To any simply-laced affine Lie algebra $\widehat{g}$, we associate the polynomial generating function*

$p_r^{ADE}(x,y)$ *such that:*

$$p_{2t+1}^{ADE}(x,y) = \prod_{l=1}^{t+1}\left(x^2 - \frac{[4l-2]_q}{[2l-1]_q}xy + y^2 - \rho[2l-1]_q^2\right) , \tag{3.2.12}$$

$$p_{2t+2}^{ADE}(x,y) = (x-y)\prod_{l=1}^{t+1}\left(x^2 - \frac{[4l]_q}{[2l]_q}xy + y^2 - \rho[2l]_q^2\right) . \tag{3.2.13}$$

Lemma 3.2.1 *The polynomial $p_r^{ADE}(x,y)$ can be expanded as:*

$$p_r^{ADE}(x,y) = \sum_{p=0}^{\{\frac{r+1}{2}\}} \sum_{k=0}^{r-2p+1} (-1)^{k+p}\rho^p\, c_k^{[r,p]}\, x^{r-2p+1-k}y^k \tag{3.2.14}$$

where the coefficients $c_k^{[r,p]}$ are given by (3.2.4).

Proof. By induction. ■

We claim that the two-variable polynomial (3.2.14) is the generating function for the coefficients $c_k^{[r,p]}$ entering in the higher order relations (3.2.3) in view of the following observations:

- For $r \leq 5$, it is an exercise to check that the coefficients $c_k^{[r,p]}$ from (3.2.4) coincide exactly with the ones derived in the previous Section (see cases $r = 2,3,4,5$);
- Using the $q-$binomial theorem, for r generic it is easy to check that the coefficients $c_k^{[r,0]}$ obtained from (3.2.4) are the $q-$binomials (3.2.11). Namely,

  For $\rho = 0$,

$$\begin{aligned} p_{2t+1}^{ADE}(x,y) &= \prod_{l=1}^{t+1}(x^2 - \frac{[4l-2]_q}{[2l-1]_q}xy + y^2) \\ &= x^{2(t+t)}\prod_{l=-(t+1)}^{t}(1 - q^{2l}(q\frac{y}{x})). \end{aligned}$$

  Put $k = l+t+2$, $p_{2t+1}^{ADE}(x,y) = x^{2(t+1)}\prod_{k=1}^{2(t+1)}(1 - q^{2k}(q^{-2t-3}\frac{y}{x}))$. Apply the q-binomial theorem

$$p_{2t+1}^{ADE}(x,y) = \sum_{k=1}^{2(t+1)}(-1)^k \begin{bmatrix} 2t+2 \\ k \end{bmatrix}_q x^{2t+2-k}y^k.$$

- For $r \geq 6$ and $p \geq 1$, the comparison is more involved. However, using a computer program we have checked that the coefficients derived from the recursive formulae coincide exactly with the ones given by (3.2.4) for a large number of values $r \geq 6$;

- Let $\{c_i, \overline{c}_i, w_i\} \in \mathbb{C}$. Let $\widehat{g} = a_n^{(1)}(n > 1), d_n^{(1)}, e_6^{(1)}, e_7^{(1)}, e_8^{(1)}$. There exists an algebra homomorphism: $\mathcal{O}_q(\widehat{g}) \to \mathcal{U}_q(\widehat{g})$ [BB09] given by

$$A_i \mapsto \mathcal{A}_i = c_i\, e_i q_i^{\frac{h_i}{2}} + \overline{c}_i\, f_i q_i^{\frac{h_i}{2}} + w_i q_i^{h_i} \tag{3.2.15}$$

iff the parameters w_i are subject to the constraints: $w_i \left(w_j^2 + \frac{c_j\, \overline{c}_j}{q+q^{-1}-2}\right) = 0$, $w_j \left(w_i^2 + \frac{c_i\, \overline{c}_i}{q+q^{-1}-2}\right) = 0$ where i, j are simply linked and $\rho_i \to c_i\, \overline{c}_i$. Let V be the so-called evaluation representation of $\mathcal{U}_q(\widehat{g})$ on which $\mathcal{A}_i$ act (see e.g. [J86, Proposition 1] for $\widehat{g} = a_n^{(1)}$). For generic parameters $c_i, \overline{c}_i, q$, V is irreducible and $\mathcal{A}_i$ is diagonalizable on V. Let $\theta_k^{(i)}$, $k = 0, 1, ...$ denote the (possibly degenerate) corresponding eigenvalues of $\mathcal{A}_i$. For instance, for the fundamental representation[3] of $\mathcal{U}_q(a_n^{(1)})$, the eigenvalues take the simple form:

$$\theta_k^{(i)} = C^{(i)}(vq^k + v^{-1}q^{-k}) \ , \tag{3.2.16}$$

where $v, C^{(i)}$ are scalar and $C^{(i)}$ depend on $c_i, \overline{c}_i, q$. Let $E_k^{(i)}$ be the projector on the eigenspace associated with the eigenvalue $\theta_k^{(i)}$. Denote $\Delta_1^{(i)}$ as the l.h.s of the first equation in (3.2.2). The relation (3.2.2) implies that it must exist integers k, l such that:

$$E_k^{(i)} \Delta_1^{(i)} E_l^{(i)} = 0 \quad \Rightarrow \quad p_1^{ADE}(\theta_k^{(i)}, \theta_l^{(i)}) E_k^{(i)} \mathcal{A}_j E_l^{(i)} = 0 \quad \text{with} \quad \rho \equiv \rho_i$$

For generic parameters $c_i, \overline{c}_i, q$, $E_k^{(i)} \mathcal{A}_j E_l^{(i)} \neq 0$. It implies $p_1^{ADE}(\theta_k^{(i)}, \theta_l^{(i)}) = 0$ which, using (3.2.16), is consistent with the structure (3.2.12) for $t = 0$ provided $l = k \pm 1$. The same observation about the structure of the two-variable polynomial can be generalized as follows. Denote $\Delta_r^{(i)}$ as the l.h.s of (3.2.3). If the relation (3.2.3) with (3.2.4) holds, then it must exist integers k, l such that:

$$E_k^{(i)} \Delta_r^{(i)} E_l^{(i)} = 0 \quad \Rightarrow \quad p_r^{ADE}(\theta_k^{(i)}, \theta_l^{(i)}) E_k^{(i)} \mathcal{A}_j^r E_l^{(i)} = 0 \quad \text{with} \quad \rho \equiv \rho_i \ .$$

For generic $c_i, \overline{c}_i, q$, $E_k^{(i)} \mathcal{A}_j^r E_l^{(i)} \neq 0$. It implies $p_r^{ADE}(\theta_k^{(i)}, \theta_l^{(i)}) = 0$ which leads to the following constraints of the integers k, l:

$$\begin{array}{lll} k = l \pm 1 \ , \ l \pm 3 \ , \ l \pm 5 \ , \cdots \ , \ l \pm r & \text{for} & r \ \text{odd} \ , \\ k = l \ , \ l \pm 2 \ , \ l \pm 4 \ , \cdots \ , \ l \pm r & \text{for} & r \ \text{even} \ . \end{array}$$

Again, this is in perfect agreement with the factorized form (3.2.12), (3.2.13). Thus, for $\widehat{g} = a_n^{(1)}$ the structure of the two-variable polynomial (3.2.3) is consistent with the spectral properties of $\mathcal{A}_i$.

[3] For $\widehat{g} = a_n^{(1)}$, $n > 1$, see [J86, Proposition 1]. For $\widehat{g} = d_n^{(1)}$, see for instance [DeG02].

3.3 The XXZ open spin chain at roots of unity

In the literature, the XXZ spin chain with periodic boundary conditions at roots of unity $q = e^{i\pi/N}$, $N \in \mathbb{N}\backslash\{0\}$, is known to enjoy a sl_2 loop algebra symmetry in certain sectors of the spectrum [DFM99]. A rather natural question is whether such phenomena occurs for the open XXZ spin chain and for which class of boundary conditions.

In this Section, we describe the third main result of the thesis. We consider the open XXZ spin chain within the framework of the $q-$Onsager algebra and its representation theory. Starting from the basic operators that generate the $q-$Onsager algebra, two new operators are introduced at roots of unity. These operators can be understood as analogues of the divided powers of the Chevalley generators (that occur in Lusztig's analysis of $U_q(\widehat{sl_2})$ at roots of unity) acting on a finite dimensional vector space. Some properties of the operators are studied. For a special class of parameters, it is shown that the new operators satisfy a pair of relations that can be understood as a higher order generalization of the two basic (Dolan-Grady [DG82]) defining relations of the classical Onsager algebra. The 'mixed relations' between the basic operators and the divided polynomials are also constructed. They can be seen as a different higher order generalization of the Dolan-Grady relations. All relations together provide, to our knowledge, the first example in the literature of an analog of Lusztig quantum group for the $q-$Onsager algebra. As an application in physics, we study some of the symmetries of the Hamiltonian with respect to the generators of the new algebra.

3.3.1 A background: the XXZ periodic spin chain at roots of unity

Recall that the Hamiltonian of the XXZ spin chain with periodic boundary conditions and L sites reads:

$$H_0 = \frac{1}{2}\sum_{j=1}^{L}(\sigma_1^j\sigma_1^{j+1} + \sigma_2^j\sigma_2^{j+1} + \Delta\sigma_z^j\sigma_z^{j+1}), \tag{3.3.1}$$

Here $\Delta = \frac{q+q^{-1}}{2}$ denotes the anisotropy parameter. By construction, the Hamiltonian acts on a 2^L finite dimensional vector space:

$$\mathcal{V}^{(L)} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots\cdots\cdots \otimes \mathbb{C}^2}_{L \quad times}. \tag{3.3.2}$$

For q a root of unity and L finite, it is known [Bax73] that additional degeneracies occur in the spectrum of the Hamiltonian. Such degeneracies are associated with the existence of an additional sl_2-loop algebra symmetry of the Hamiltonian, as shown in [DFM99]. Let us now recall the main steps and results of [DFM99].

In the works of Jimbo [J86], five basic operators satisfying the $U_q(\widehat{sl_2})$ defining relations

naturally occur in the study of the XXZ spin chain for q generic:

$$\begin{aligned} S^z &= \frac{1}{2}\sum_{j=1}^{L}\sigma_z^j, \\ S^{\pm} &= \sum_{j=1}^{L} q^{\sigma_z/2}\otimes\cdots\otimes q^{\sigma_z/2}\otimes\sigma_j^{\pm}\otimes q^{-\sigma_z/2}\otimes\cdots\otimes q^{-\sigma_z/2}, \\ T^{\pm} &= \sum_{j=1}^{L} q^{-\sigma_z/2}\otimes\cdots\otimes q^{-\sigma_z/2}\otimes\sigma_j^{\pm}\otimes q^{\sigma_z/2}\otimes\cdots\otimes q^{\sigma_z/2}. \end{aligned}$$

By straightforward calculations, $N-$th powers of the basic operators $S_{\pm}, T_{\pm}$ are shown to be proportional to $[N]_q!$. Explicitly, one derives:

$$\begin{aligned} (S^{\pm})^N &= [N]_q! \sum_{1\leq j_1<\cdots<j_N\leq L} q^{\frac{N}{2}\sigma_z}\otimes\cdots\otimes q^{\frac{N}{2}\sigma_z}\otimes\sigma_{j_1}^{\pm}\otimes q^{\frac{(N-2)}{2}\sigma_z}\otimes\cdots\otimes q^{\frac{(N-2)}{2}\sigma_z} \\ &\qquad \otimes\sigma_{j_2}^{\pm}\otimes q^{\frac{(N-4)}{2}\sigma_z}\otimes\cdots\otimes q^{\frac{(N-4)}{2}\sigma_z}\otimes\sigma_{j_N}^{\pm}\otimes q^{\frac{-N}{2}\sigma_z}\otimes\cdots\otimes q^{\frac{-N}{2}\sigma_z}, \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} (T^{\pm})^N &= [N]_q! \sum_{1\leq j_1<\cdots<j_N\leq L} q^{-\frac{N}{2}\sigma_z}\otimes\cdots\otimes q^{-\frac{N}{2}\sigma_z}\otimes\sigma_{j_1}^{\pm}\otimes q^{-\frac{(N-2)}{2}\sigma_z}\otimes\cdots\otimes q^{-\frac{(N-2)}{2}\sigma_z} \\ &\qquad \otimes\sigma_{j_2}^{\pm}\otimes q^{-\frac{(N-4)}{2}\sigma_z}\otimes\cdots\otimes q^{\frac{-(N-4)}{2}\sigma_z}\otimes\sigma_{j_N}^{\pm}\otimes q^{\frac{N}{2}\sigma_z}\otimes\cdots\otimes q^{\frac{N}{2}\sigma_z}. \end{aligned} \tag{3.3.4}$$

For $q^{2N}\to 1$, it implies $(S^{\pm})^N = (T^{\pm})^N = 0$.

Following Lusztig's works, the authors [DFM99] introduce the non-trivial divided powers:

$$S^{\pm(N)} = \lim_{q^{2N}\to 1}(S^{\pm})^N/[N]_q!, \quad T^{\pm(N)} = \lim_{q^{2N}\to 1}(T^{\pm})^N/[N]_q!$$

and study their commutation relations by using, for instance, the so-called Lusztig's higher order $q-$Serre relations [Lusz93] (3.1.4)-(3.1.5).

Let $\theta_1^{(m)} = (S^+)^m/[m]_q!$, $\theta_2^{(m)} = (T^-)^m/[m]_q!$. According to Lusztig's work [Lusz93], the following relations can be derived from the higher order $q-$Serre relations:

$$\theta_1^{(3N)}\theta_2^{(N)} = \sum_{s'=N}^{3N}\gamma_{s'}\,\theta_1^{(3N-s')}\theta_2^{(N)}\theta_1^{(s')} \tag{3.3.5}$$

where

$$\gamma_{s'} = (-1)^{s'+1}q^{s'(N-1)}\sum_{l=0}^{N-1}(-1)^l q^{l(1-s')}\begin{bmatrix} s' \\ l \end{bmatrix}_q. \tag{3.3.6}$$

Define

$$s' = Ns + p \qquad \text{with} \qquad p = 0, 1, ..., N-1.$$

Observe

$$\theta^{(Ns)} = \frac{[N]_q!^s}{[Ns]_q!}\theta^{(N)s}.$$

Then, one shows:

$$\lim_{q^{2N}\to 1} \gamma_{Ns+p} = \delta_{p,0}(-1)^{s+1} \qquad \text{and} \qquad \lim_{q^{2N}\to 1} \frac{[N]_q!^s}{[Ns]_q!} = \frac{q^{N^2}}{s!}.$$

It follows that $S^{+(N)}, T^{-(N)}$ satisfy the Serre relations of the sl_2-loop algebra namely:

$$\sum_{s=0}^{3} \frac{(-1)^{s+1}}{s!(3-s)!}(S^{+(N)})^{(3-s)}(T^{-(N)})(S^{+(N)})^s = 0 \tag{3.3.7}$$

Provided the change $S^+ \to S^-$, $T^- \to T^+$, the same relation holds for S^-, T^+.

Either based on straightforward calculations or using some formula[4] in [DeCK90], the authors [DFM99] derive successively the following relations for $q^{2N} = 1$:

$$[S^{+(N)}, T^{+(N)}] = [S^{-(N)}, T^{-(N)}] = 0, \tag{3.3.9}$$

$$[S^{\pm(N)}, S^z] = \pm N S^{\pm(N)}, \quad [T^{\pm(N)}, S^z] = \pm N T^{\pm(N)} \tag{3.3.10}$$

and in the sector $S^z \equiv 0 (\text{mod } N)$

$$[S^{+(N)}, S^{-(N)}] = [T^{+(N)}, T^{-(N)}] = -(-q)^N \frac{2}{N} S^z. \tag{3.3.11}$$

According to the above analysis, define

$$E_0 = S^{+(N)},\ F_0 = S^{-(N)},\ E_1 = T^{-(N)},\ F_1 = T^{+(N)},\ T_0 = -T_1 = -(-q)^N S^z/N.$$

Then, the operators $\{E_i, F_i, T_i\}$ satisfy the defining relations of the loop algebra of sl_2. When $S^z \equiv 0$ (mod N), the Hamiltonian (3.3.1) commutes with $S^{\pm(N)}, T^{\pm(N)}$ at $q^{2N} = 1$ [DFM99]. One has:

$$[E_0, H] = [E_1, H] = 0, \tag{3.3.12}$$

$$[F_0, H] = [F_1, H] = 0. \tag{3.3.13}$$

Also, by (3.3.11) and (3.3.12)-(3.3.13):

$$[T_0, H] = [T_1, H] = 0. \tag{3.3.14}$$

Hence, the Hamiltonian of the XXZ periodic spin chain at q a root of unity enjoys a sl_2-loop algebra invariance in the sector $S^z \equiv 0$.

[4]

$$[(S^+)^m, (S^-)^n] = \sum_{j=1}^{\min(m,n)} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} [j]!(S^-)^{n-j}(S^+)^{m-j} \prod_{k=0}^{j-1} \frac{q^{2S^z+m-n-k} - q^{-(2S^z+m-n-k)}}{q - q^{-1}}. \tag{3.3.8}$$

3.3.2 The case of the open XXZ spin chain

The purpose of this Section is to study the open XXZ spin chain with non-diagonal integrable boundary conditions for an anisotropy parameter $(q+q^{-1})/2$ evaluated at roots of unity $q = e^{i\pi/N}$. Inspired by the analysis done for the XXZ spin chain with periodic boundary conditions using the quantum algebra $U_q(\widehat{sl_2})$ [DFM99], it is thus natural to start from the framework of the $q-$Onsager algebra, a coideal subalgebra of $U_q(\widehat{sl_2})$ (see Chapter 1). For generic boundary conditions, recall that the Hamiltonian of the XXZ open spin chain reads

$$\begin{aligned} H_{XXZ}^{(L)} &= \sum_{k=1}^{L-1}\Big(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_z^{k+1}\sigma_z^k\Big) \\ &+ \frac{(q-q^{-1})}{2}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_z^1 + \frac{2}{(\epsilon_++\epsilon_-)}\big(k_+\sigma_+^1 + k_-\sigma_-^1\big) \\ &+ \frac{(q-q^{-1})}{2}\frac{(\bar\epsilon_+-\bar\epsilon_-)}{(\bar\epsilon_++\bar\epsilon_-)}\sigma_z^L + \frac{2}{(\bar\epsilon_++\bar\epsilon_-)}\big(\bar k_+\sigma_+^L + \bar k_-\sigma_-^L\big) \end{aligned} \tag{3.3.15}$$

where L is the number of sites, $\Delta = \frac{q+q^{-1}}{2}$ denotes the anisotropy parameter, and $\sigma_\pm, \sigma_1, \sigma_2, \sigma_z$ are the usual Pauli matrices.

3.3.2.1 The basic operators and the divided polynomials

Following [BK0507], the two operators $\mathcal{W}_0, \mathcal{W}_1$ of the q-Onsager algebra that naturally occur in the analysis of the XXZ open spin chain with non-diagonal boundary conditions are known explicitly (see Chapter 2). Denote:

$$w_0 = k_+\sigma_+ + k_-\sigma_-. \tag{3.3.16}$$

They are given by:

$$\mathcal{W}_0 = \sum_{j=1}^{L} q^{\sigma_z}\otimes\cdots\otimes q^{\sigma_z}\otimes w_{0_j}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I} + \epsilon_+ q^{\sigma_z}\otimes\cdots\otimes q^{\sigma_z}, \tag{3.3.17}$$

$$\mathcal{W}_1 = \sum_{j=1}^{L} q^{-\sigma_z}\otimes\cdots\otimes q^{-\sigma_z}\otimes w_{0_j}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I} + \epsilon_- q^{-\sigma_z}\otimes\cdots\otimes q^{-\sigma_z}. \tag{3.3.18}$$

with the parameters $\rho = \rho^* = (q+q^{-1})^2 k_+k_-$.

The finite dimensional module $\mathcal{V}^{(L)}$ on which they act is of dimension 2^L. Indeed, both operators are diagonalizable on the finite 2^L- dimensional vector space $\mathcal{V}^{(L)}$ [Bas0604]. For convenience, denote $\epsilon_0 = \epsilon_+$, $\epsilon_1 = \epsilon_-$.

Lemma 3.3.1 *[Bas0604] For generic values of q, the operator $\mathcal{W}_i$, $i = 0, 1$, has $L+1$ distinct eigenvalues. They read:*

$$\theta_n^{(i)} = a_2^{(i)}(q)q^{L-2n} + a_3^{(i)}(q)q^{-L+2n} \qquad \textit{for} \quad n = 0, 1, ..., L\ . \tag{3.3.19}$$

If $k_\pm \neq 0$ and use the convenient parametrization

$$\epsilon_i = \cosh(\alpha_i) \quad \textit{and} \quad k_+ = (k_-)^\dagger = -(q-q^{-1})e^{i\eta}/2 \quad \textit{with} \quad \alpha_i, \eta \in \mathbb{C} \tag{3.3.20}$$

then $a_2^{(i)}(q) = \frac{e^{\alpha_i}}{2}$, $a_3^{(i)}(q) = \frac{e^{-\alpha_i}}{2}$.

If $k_- = 0$, then $a_2^{(i)}(q) = \epsilon_i$, $a_3^{(i)}(q) = 0$.

Let $V_0, V_1, \ldots, V_L$ denote the eigenspaces of $\mathcal{W}_0$ with eigenvalues $\theta_0^{(0)}, \theta_1^{(0)}, \ldots, \theta_L^{(0)}$ given in Lemma 3.3.1. On $\{V_n\}$, the basic operator $\mathcal{W}_1$ acts as:

$$\mathcal{W}_1 V_n \subseteq V_{n-1} + V_n + V_{n+1} \qquad (0 \leq n \leq L), \tag{3.3.21}$$

where $V_{-1} = 0$, $V_{L+1} = 0$. On the other hand, let $V_0^*, V_1^*, \ldots, V_L^*$ denote the eigenspaces of $\mathcal{W}_1$ with eigenvalues $\theta_0^{(1)}, \theta_1^{(1)}, \ldots, \theta_L^{(1)}$. The basic operator $\mathcal{W}_0$ acts as:

$$\mathcal{W}_0 V_s^* \subseteq V_{s-1}^* + V_s^* + V_{s+1}^* \qquad (0 \leq s \leq L), \tag{3.3.22}$$

where $V_{-1}^* = 0$, $V_{L+1}^* = 0$. Note that the eigenbasis in which $\mathcal{W}_0$ (resp. $\mathcal{W}_1$) is diagonalized has been constructed and described in details in [Bas0604], as well as the entries of the block tridiagonal matrix $\mathcal{W}_1$ (resp. $\mathcal{W}_0$) in the same basis.

Note that $\mathrm{Trace}(\mathcal{W}_i) = \sum_{n=0}^{L} \dim(V_n) \left(a_2^{(i)}(q) q^{L-2n} + a_3^{(i)}(q) q^{-L+2n} \right)$, and $\mathrm{Trace}(\mathcal{W}_i) = \epsilon_i (q+q^{-1})^L$, then $\dim(V_n) = \begin{pmatrix} L \\ n \end{pmatrix}$, and $a_2^{(i)}(q) + a_3^{(i)}(q) = \epsilon_i$ if $q^2 \neq -1$. In addition, $a_2^{(i)}(q) a_3^{(i)}(q) = -\frac{k_+ k_-}{(q-q^{-1})^2}$. Therefore, if $q^2 \neq -1$, then

$$a_2^{(i)}(q), a_3^{(i)}(q) \in \left\{ \frac{1}{2}\left(\epsilon_i + \sqrt{\epsilon_i^2 + \frac{4k_+k_-}{(q-q^{-1})^2}} \right), \frac{1}{2}\left(\epsilon_i - \sqrt{\epsilon_i^2 + \frac{4k_+k_-}{(q-q^{-1})^2}} \right) \right\}.$$

Up to now, the parameter q is generic. Let us now consider the case $q = e^{i\pi/N}$. Starting from the known results for q generic, for $q = e^{i\pi/N}$ one observes that additional degeneracies in the spectra of $\mathcal{W}_0, \mathcal{W}_1$ occur.

Lemma 3.3.2 *For $q = e^{i\pi/N}, N > 2$, the operator $\mathcal{W}_i$, $i = 0, 1$, has N distinct eigenvalues. They read:*

$$\theta_t^{(i)} = a_2^{(i)}(q) q^{L-2t} + a_3^{(i)}(q) q^{-L+2t} \qquad \textit{for} \quad t = 0, 1, ..., N-1. \tag{3.3.23}$$

If $k_\pm \neq 0$ and use the parametrization (3.3.20), then $a_2^{(i)}(q) = \frac{e^{\alpha_i}}{2}$, $a_3^{(i)}(q) = \frac{e^{-\alpha_i}}{2}$.

If $k_- = 0$, then $a_2^{(i)}(q) = \epsilon_i$, $a_3^{(i)}(q) = 0$.

Furthermore, let $V_t^{(N)}$ (resp. $V_t^{*(N)}$) denote the eigenspace associated with $\theta_t^{(0)}$ (resp. $\theta_t^{(1)}$), one has the decomposition:

$$V_t^{(N)} = \bigoplus_{k=0}^{\{\frac{L}{N}\}} V_{t+kN} \qquad \textit{and} \qquad V_t^{*(N)} = \bigoplus_{k=0}^{\{\frac{L}{N}\}} V_{t+kN}^*. \tag{3.3.24}$$

Proof. For generic parameters α_i, η, the eigenvectors with explicit formulae given in [Bas0604] remain well-defined and linearly independent for $q = e^{i\pi/N}$. Given n, there exist uniquely integers t, k such that $n = t + kN$ with $0 \leq n \leq L$, and $0 \leq t \leq N-1$. From (3.3.19), one immediately finds (3.3.23). We now show (3.3.24). Let t be fixed, the eigenspace denoted $V_t^{(N)}$ (resp. $V_t^{*(N)}$) is spanned by the eigenvectors of $V_t, V_{t+N}, ...$ (resp. $V_t^*, V_{t+N}^*, ...$) evaluated at $q = e^{i\pi/N}$. ■

The dimension of the eigenspaces $V_t^{(N)}$, $V_t^{*(N)}$ is computed based on the above decomposition. It follows:

$$dim(V_t^{(N)}) = dim(V_t^{*(N)}) = \sum_{k=0}^{\{\frac{L}{N}\}} \left(\begin{array}{c} L \\ t+kN \end{array} \right). \tag{3.3.25}$$

From above results, using (3.3.21), (3.3.22) it follows:

Corollary 3.3.1 *Let $V_t^{(N)}, V_t^{*(N)}$ be defined as in Lemma 3.3.2.*

$$\begin{aligned} &\mathcal{W}_0 V_t^{(N)} \subseteq V_t^{(N)}, \\ &\mathcal{W}_1 V_t^{(N)} \subseteq V_{t+1}^{(N)} + V_t^{(N)} + V_{t-1}^{(N)} \qquad (0 \leq t \leq N-1), \end{aligned} \tag{3.3.26}$$

where $V_{-1}^{(N)} = V_{N-1}^{(N)}$ and $V_N^{(N)} = V_0^{(N)}$ and

$$\begin{aligned} &\mathcal{W}_1 V_t^{*(N)} \subseteq V_t^{*(N)}, \\ &\mathcal{W}_0 V_t^{*(N)} \subseteq V_{t+1}^{*(N)} + V_t^{*(N)} + V_{t-1}^{*(N)} \qquad (0 \leq t \leq N-1), \end{aligned} \tag{3.3.27}$$

where $V_{-1}^{*(N)} = V_{N-1}^{*(N)}$ and $V_N^{*(N)} = V_0^{*(N)}$.

For q a root of unity, besides the two fundamental operators of the $q-$Onsager algebra, two additional operators that are called the *divided polynomials* can be introduced[5] for the analysis of the spin chain. In particular, the spectrum of these operators is described in details below. These new operators are introduced as follows. Recall that the fundamental generators $S^{\pm}, T^{\pm}$ of the quantum loop algebra of sl_2 become nilpotent, for q a root of unity. Namely, $(S^{\pm})^N = (T^{\pm})^N = 0$ for $q^{2N} = 1$ [DFM99]. For non-vanishing values of the non-diagonal boundary parameters $k_{\pm}$ entering in (3.3.17)-(3.3.18), such property doesn't hold for the fundamental generators $\mathcal{W}_0, \mathcal{W}_1$ of the $q-$Onsager algebra. Instead, given N, certain polynomials of $\mathcal{W}_0, \mathcal{W}_1$ are vanishing for $q^{2N} = 1$. These polynomials can be constructed as follows.

First, for small values of N, by straightforward calculations using the explicit expressions of $\mathcal{W}_0$ and $\mathcal{W}_1$ (see Appendix F) it is not difficult to construct polynomials such that:

$$P_N^{(L,i)}(\mathcal{W}_i) = 0, \quad i = 0, 1. \tag{3.3.28}$$

For instance[6]:

[5] These operators can be seen as analogs of the divided powers of the Chevalley elements that generate the so-called *full* or *Lusztig* quantum group $U_q(\widehat{sl_2})$ with $q = e^{i\pi/N}$, acting on $\mathcal{V}^{(L)}$.

[6] For $N = 1$, $q^2 = 1$ there is no polynomial. In this special case, the operators $\mathcal{W}_0, \mathcal{W}_1$ however reduce to elements that generate the undeformed Onsager algebra.

- For $N = 2$, $q^4 = 1$,

$$P_2^{(L,i)}(x) = x^2 + (-1)^{L+1}\left(\epsilon_i^2 - \frac{(1+(-1)^{L+1})}{2}k_+k_-\right)\mathbb{I}.$$

- For $N = 3$, $q^6 = 1$,

$$P_3^{(L,i)}(x) = x^3 - k_+k_-x + (-1)^{L+1}\epsilon_i(\epsilon_i^2 - k_+k_-)\mathbb{I}.$$

- For $N = 4$, $q^8 = 1$,

$$P_4^{(L,i)}(x) = x^4 - 2k_+k_-x^2 + (-1)^{L+1}\left(\epsilon_i^4 - 2\epsilon_i^2k_+k_- + \frac{(1+(-1)^{L+1})}{2}k_+^2k_-^2\right)\mathbb{I}.$$

- For $N = 5$, $q^{10} = 1$, recall $\rho = k_+k_-(q+q^{-1})^2$,

$$\begin{aligned} P_5^{(L,i)}(x) &= x^5 - \rho(q^4+q^{-4}+3)x^3 + \rho^2(q^2+q^{-2})^2x \\ &\quad +(-1)^{L+1}\frac{(\sqrt{5}+3)}{4}\epsilon_i(\epsilon_i^2 - k_+k_-)(3\epsilon_i^2 - \sqrt{5}\epsilon_i^2 - 2k_+k_-)\mathbb{I}. \end{aligned}$$

- For $N = 6$, $q^{12} = 1$,

$$\begin{aligned} P_6^{(L,i)}(x) &= x^6 - 6k_+k_-x^4 + 9k_+^2k_-^2x^2 \\ &\quad +(-1)^{L+1}\left(\epsilon_i^6 - 6k_+k_-\epsilon_i^4 + 9k_+^2k_-^2\epsilon_i^2 - 2(1+(-1)^{L+1})k_+^3k_-^3\right)\mathbb{I}. \end{aligned}$$

Recall that $\mathcal{W}_i$, $i = 0, 1$ are both diagonalizable on $\mathcal{V}^{(L)}$. As above mentioned, for $q = e^{i\pi/N}$, there are exactly N distinct eigenvalues denoted $\theta_n^{(i)}$, $n = 0, 1, ..., N-1$. Then, observe that the above polynomials $P_N^{(L,i)}(x)$ for $N = 2, 3, ..., 6$ are nothing but the minimal polynomials associated with $\mathcal{W}_i$, $i = 0, 1$. More generally, it follows:

Proposition 3.3.1 *Let $e_k(\theta_0, \cdots, \theta_{N-1})$, $k = 0, 1, ..., N-1$ denote the so-called elementary symmetric polynomials in the variables θ_n, given by:*

$$e_k(\theta_0, \cdots, \theta_{N-1}) = \sum_{0\le j_1<j_2<\cdots<j_k\le N-1} \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} \quad \textit{with} \quad e_0(\theta_0, \cdots, \theta_{N-1}) = 1. \tag{3.3.29}$$

For $N \geq 2$ generic:

$$P_N^{(L,i)}(x) = \sum_{k=0}^{N} e_k(\theta_0^{(i)}, \cdots, \theta_{N-1}^{(i)})x^{N-k}\ . \tag{3.3.30}$$

Proof. Recall that the distinct eigenvalues are given by $\theta_0^{(i)}, ..., \theta_{N-1}^{(i)}$ in (3.3.23). Introduce the minimal polynomials of $\{\mathcal{W}_i\}_{i=0,1}$:

$$P_N^{(L,i)}(x) = \prod_{n=0}^{N-1}(x - \theta_n^{(i)}). \tag{3.3.31}$$

According to the Cayley-Hamilton theorem, one has $P_N^{(L,i)}(\mathcal{W}_i) = 0$. Expanding the factorized expression above in x, one obtains (3.3.30). ■

For q a root of unity, recall that the *small* or *restricted* quantum loop algebra of sl_2 can be defined: it is the quotient of the quantum loop algebra of sl_2 by the relations $(S^{\pm})^N = (T^{\pm})^N = 0$ and $S^z = 0$ mod N for $q^{2N} = 1$. Then, the *full* or *Lusztig* quantum group can be introduced. It is generated by the fundamental Chevalley elements and additionally by the divided powers $(S^{\pm})^N/[N]_q!$, $(T^{\pm})^N/[N]_q!$. The corresponding defining relations can be found in [DFM99]. By analogy, let us introduce the divided polynomials:

$$\overset{[N]}{\mathcal{W}}_i = \lim_{q \to e^{\frac{I\pi}{N}}} \frac{P_N^{(L,i)}(\mathcal{W}_i)}{[N]_q!} \qquad \text{with} \qquad i = 0, 1. \tag{3.3.32}$$

Note that using the formulae of Appendix F, it is possible to derive the generic expression of $\overset{[N]}{\mathcal{W}}_i$, $i = 0, 1$ for arbitrary N.

According to the spectral properties of the fundamental operators $\mathcal{W}_0, \mathcal{W}_1$, the eigenvalues of the divided polynomials (3.3.32) can be easily obtained.

Lemma 3.3.3 *Let $N > 2$ denote an integer. Let $\tilde{\theta}_n^{(i)}$, $n = 0, 1, ..., L$ denote the eigenvalues of the divided polynomials $\overset{[N]}{\mathcal{W}}_i$, $i = 0, 1$. For all $n = 0, 1, ..., L$ there exist uniquely k, t non-negative integers such that $n = t + kN$, $0 \leq t \leq N - 1$. For $n = 0, 1, ..., L$, the eigenvalues read:*

$$\tilde{\theta}_n^{(i)} = C_0^{(n,i)} \Big(\partial_q(a_2^{(i)})(q^{L-2n+1} - q^{-L+2n+1}) + (L - 2n)(a_2^{(i)}(q)q^{L-2n} - a_3^{(i)}(q)q^{-L+2n}) \Big) \tag{3.3.33}$$

where $q = e^{I\pi/N}$ and

$$C_0^{(n,i)} = -\frac{(q - q^{-1})}{2N[N-1]_q!} \prod_{j=0, j\neq t}^{N-1} (\theta_t^{(i)} - \theta_j^{(i)}).$$

The eigenspace of $\overset{[N]}{\mathcal{W}}_0$ (resp. $\overset{[N]}{\mathcal{W}}_1$) corresponding to $\tilde{\theta}_n^{(0)}$ (resp. $\tilde{\theta}_n^{(1)}$) is V_n (resp. V_n^*) at $q = e^{I\pi/N}$.

Proof. By Lemma 3.3.1, recall that $\mathcal{W}_i, i = 0, 1$ has $L + 1$ distinct eigenvalues $\theta_0^{(i)}(q)$, $\theta_1^{(i)}(q), \ldots, \theta_L^{(i)}(q)$, and $L+1$ eigenspaces $V_0^{(i)}(q), V_1^{(i)}(q), \ldots, V_L^{(i)}(q)$, respectively. Here the notations $\theta_n^{(i)}(q), V_n^{(i)}(q), n = 0, \ldots, L$ are used in order to emphasize that the eigenvalues and eigenspaces of $\mathcal{W}_i$ depend on q. Since $\mathcal{W}_0, \mathcal{W}_1$ is a tridiagonal pair, there exist scalars $a_2^{(i)} \neq 0, a_3^{(i)} \neq 0$ such that

$$\theta_n^{(i)}(q) = a_2^{(i)}(q)q^{L-2n} + a_3^{(i)}(q)q^{-L+2n}, \qquad n = 0, \ldots, L.$$

Clearly, if $q = e^{\frac{I\pi}{N}}$, then $\mathcal{W}_i$ has N distinct eigenvalues

$$\theta_0^{(i)} = \lim_{q\to e^{\frac{I\pi}{N}}} \theta_0^{(i)}(q), \quad \theta_1^{(i)} = \lim_{q\to e^{\frac{I\pi}{N}}} \theta_1^{(i)}(q), \ldots, \quad \theta_{N-1}^{(i)} = \lim_{q\to e^{\frac{I\pi}{N}}} \theta_{N-1}^{(i)}(q).$$

Now, define (3.3.32). Then, $\overset{[N]}{\mathcal{W}_i}$ has $L+1$ eigenvalues denoted $\widetilde{\theta}_n^{(i)}$ and $V_n = \lim_{q\to e^{\frac{I\pi}{N}}} V_n^{(i)}(q)$ denote the corresponding eigenspaces. Explicitely, $\widetilde{\theta}_n^{(i)}$ can be written as follows:

$$\widetilde{\theta}_n^{(i)} = \lim_{q\to e^{\frac{I\pi}{N}}} \frac{(\theta_n^{(i)}(q) - \theta_0^{(i)})(\theta_n^{(i)}(q) - \theta_1^{(i)}) \ldots (\theta_n^{(i)}(q) - \theta_{N-1}^{(i)})}{[N]_q!}, \qquad n = 0, \ldots, L.$$

For all $n = 0, \ldots, L$, there exist uniquely k, t non-negative integers, such that $n = kN + t, \quad 0 \le t \le N-1$. It yields to:

$$\widetilde{\theta}_n^{(i)} = \frac{(q - q^{-1}) \prod_{j=0, j\neq t}^{N-1} (\theta_t^{(i)} - \theta_j^{(i)})}{[N-1]_q!} \lim_{q\to e^{\frac{I\pi}{N}}} \frac{\theta_n^{(i)}(q) - \theta_t^{(i)}}{q^N - q^{-N}}.$$

Using the L'Hopital's rule, we have

$$\begin{aligned}\lim_{q\to e^{\frac{I\pi}{N}}} \frac{\theta_n^{(i)}(q) - \theta_t^{(i)}}{q^N - q^{-N}} &= \lim_{q\to e^{\frac{I\pi}{N}}} \frac{(\theta_n^{(i)}(q))'}{-2Nq^{-1}} \\ &= \left. \frac{\partial_q(a_2^{(i)}) \left(q^{L-2n+1} - q^{-L+2n+1}\right) + (L-2n)\left(a_2^{(i)}(q)q^{L-2n} - a_3^{(i)}(q)q^{-L+2n}\right)}{-2N} \right|_{q=e^{I\pi/N}}.\end{aligned}$$

It simplifies to

$$\tilde{\theta}_n^{(i)} = C_0^{(n,i)} \left(\partial_q(a_2^{(i)})(q^{L-2n+1} - q^{-L+2n+1}) + (L-2n)(a_2^{(i)}(q)q^{L-2n} - a_3^{(i)}(q)q^{-L+2n})\right)$$

where $q = e^{I\pi/N}$ and

$$C_0^{(n,i)} = -\frac{(q - q^{-1})}{2N[N-1]_q!} \prod_{j=0, j\neq t}^{N-1} (\theta_t^{(i)} - \theta_j^{(i)}).$$

■

Remark 8 *If $N = 2$, then $\tilde{\theta}_n^{(i)} = (-1)^L (L-2n)a$ where $a = \epsilon_i \sqrt{k_+k_- - \epsilon_i^2}$.*

Proof. For $q = e^{I\pi/2}$, $\mathcal{W}_i$ has the two eigenvalues $\theta_0^{(i)}, \theta_1^{(i)}$ such that

$$\theta_0^{(i)} = -\theta_1^{(i)} \in \left\{ \sqrt{(-1)^L \left(\epsilon_i^2 - \frac{1 + (-1)^{L+1}}{2} k_+k_-\right)}, -\sqrt{(-1)^L \left(\epsilon_i^2 - \frac{1 + (-1)^{L+1}}{2} k_+k_-\right)} \right\}.$$

We compute

$$\begin{aligned}
\tilde{\theta}_n^{(i)} &= \lim_{q\to e^{\frac{I\pi}{2}}} \frac{\left(a_2^{(i)}(q)q^{L-2n} + a_3^{(i)}(q)q^{-L+2n}\right)^2 - \theta_0^{(i)\,2}}{q+q^{-1}} \\
&= (L-2n)\left(a_2^{(i)}(q)q^{L-2n} + a_3^{(i)}(q)q^{-L+2n}\right)\left(a_2^{(i)}(q)q^{L-2n-1} - a_3^{(i)}(q)q^{-L+2n-1}\right)\Big|_{q=e^{I\pi/2}} \\
&= (-1)^L(L-2n)\epsilon_i\sqrt{k_+k_- - \epsilon_i^2}
\end{aligned}$$

■

Remark 9 *For $k_\pm \neq 0$ and the parametrization (3.3.20), one has the identification $a_2^{(i)}(q) = \frac{e^{\alpha_i}}{2}$, $a_3^{(i)}(q) = \frac{e^{-\alpha_i}}{2}$. For $k_- = 0$, one has $a_2^{(i)}(q) = \epsilon_i$, $a_3^{(i)}(q) = 0$*

Using the explicit expression (3.3.32) of the divided polynomials $\overset{[N]}{\mathcal{W}}_0, \overset{[N]}{\mathcal{W}}_1$ in terms of the basic operators $\mathcal{W}_0, \mathcal{W}_1$, the action of the basic operators in each eigenbasis implies:

Proposition 3.3.2 *On the eigenspaces V_n (resp. V_s^*), the divided polynomials $\overset{[N]}{\mathcal{W}}_0, \overset{[N]}{\mathcal{W}}_1$ act as*

$$\begin{aligned}
&\overset{[N]}{\mathcal{W}}_0 V_n \subseteq V_n\ , && (3.3.34)\\
&\overset{[N]}{\mathcal{W}}_1 V_n \subseteq V_{n-N} + \cdots + V_n + \cdots + V_{n+N} && (0 \le n \le L),\\
&\overset{[N]}{\mathcal{W}}_1 V_s^* \subseteq V_s^*\ , \\
&\overset{[N]}{\mathcal{W}}_0 V_s^* \subseteq V_{s-N}^* + \cdots + V_s^* + \cdots + V_{s+N}^* && (0 \le s \le L).
\end{aligned}$$

where $V_{-1} = \cdots = V_{-N} = 0, V_{L+1} = \cdots = V_{L+N} = 0$, and $V_{-1}^* = \cdots = V_{-N}^* = 0, V_{L+1}^* = \cdots = V_{L+N}^* = 0$.

Explicitly, in the basis which diagonalizes $\mathcal{W}_0$ (resp. $\mathcal{W}_1$) , the matrix representing $\overset{[N]}{\mathcal{W}}_1$ (resp. $\mathcal{W}_0$) is $(2N+1)$-block diagonal.

3.3.2.2 The algebra generated by $\mathcal{W}_0, \mathcal{W}_1, \overset{[N]}{\mathcal{W}}_0, \overset{[N]}{\mathcal{W}}_1$ for $k_- = 0$ or $k_+ = 0$.

Let us focus on the special class of parameters $k_+ = 0$ or $k_- = 0$. In this case, the spectrum of the tridiagonal pair $\mathcal{W}_0, \mathcal{W}_1$ simplifies: $a_2^{(i)}(q) = \epsilon_i$, $a_3^{(i)}(q) = 0$. This has the following consequence on the spectrum of the divided polynomials $\overset{[N]}{\mathcal{W}}_0, \overset{[N]}{\mathcal{W}}_1$:

Lemma 3.3.4 *Let $\tilde{\theta}_n^{(i)}$, $n = 0, 1, ..., L$ denote the distinct eigenvalues of the divided polynomials $\overset{[N]}{\mathcal{W}}_i$, $i = 0, 1$. Assume $a_2^{(i)}(q) = \epsilon_i$ and $a_3^{(i)}(q) = 0$. Then*

$$\tilde{\theta}_n^{(i)} = \epsilon_i^N \frac{(q-q^{-1})^N}{2N}(-1)^{\frac{N+1}{2}+L}(L-2n). \quad n = 0, 1, ..., L. \tag{3.3.35}$$

For this special choice of parameters, note that the spectrum of the divided polynomials has an arithmetic progression with respect to the integer $n = 0, 1, 2, ..., L$. In this special case, it is not difficult to show that the divided polynomials satisfy a pair of polynomial relations. Indeed, define

Definition 3.3.1 *Let x, y denote commuting indeterminates. For each positive integer N, we define the two two-variable polynomials $p_N^{(i)}(x, y)$, $i = 0, 1$, as follows:*

$$p_N^{(i)}(x, y) = (x - y) \prod_{s=1}^{N} (x^2 - 2xy + y^2 - \rho_i s^2) \tag{3.3.36}$$

with

$$\rho_i = (-1)^{N+1} \epsilon_i^{2N} \frac{(q - q^{-1})^{2N}}{N^2}, \quad q = e^{i\pi/N}.$$

Observe $p_N^{(i)}(x, y)$ have a total degree $2N + 1$ in x, y.

Lemma 3.3.5 *Assume $a_2^{(i)} = \epsilon_i$ and $a_3^{(i)} = 0$. For any $i = 0, 1$, one has:*

$$p_N^{(i)}(\tilde{\theta}_r^{(i)}, \tilde{\theta}_s^{(i)}) = 0 \qquad \textit{for} \qquad |r - s| \leq N \ .$$

Proof. For $r = s$, the relation obvioulsy holds. Then, suppose $|r - s| = k$ with $k \leq N$. Due to the arithmetic progression of $\tilde{\theta}_s$ for $a_2 = \epsilon_i$ and $a_3 = 0$, one has $s^2 - 2r(s \pm k) + (s \pm k)^2 = k^2$. The relation then follows. ■

Let us now introduce a new algebra. The defining relations of the new algebra can be understood as a generalization of the Dolan-Grady relations (3.1.6) which are recovered for $N = 1$. For this reason, we decide to call this new algebra the *$N - th$ higher order generalization of the Onsager algebra.*

Proposition 3.3.3 *Let $k_- = 0$ or $k_+ = 0$. Then, the divided polynomials $\overset{[N]}{\mathcal{W}}_i$, $i = 0, 1$ satisfy the defining relations of the $N - th$ higher order generalization of the Onsager algebra:*

$$\begin{aligned} \sum_{p=0}^{N} \sum_{k=0}^{2N+1-2p} (-1)^{k+p} \rho_0^p \, c_k^{[N,p]} \left(\overset{[N]}{\mathcal{W}}_0 \right)^{2N+1-2p-k} \overset{[N]}{\mathcal{W}}_1 \left(\overset{[N]}{\mathcal{W}}_0 \right)^k &= 0, \\ \sum_{p=0}^{N} \sum_{k=0}^{2N+1-2p} (-1)^{k+p} \rho_1^p \, c_k^{[N,p]} \left(\overset{[N]}{\mathcal{W}}_1 \right)^{2N+1-2p-k} \overset{[N]}{\mathcal{W}}_0 \left(\overset{[N]}{\mathcal{W}}_1 \right)^k &= 0, \end{aligned} \tag{3.3.37}$$

where $c_{2(N-p)+1-k}^{[N,p]} = c_k^{[N,p]}$ and

$$c_k^{[N,p]} = \begin{pmatrix} 2N + 1 - 2p \\ k \end{pmatrix} \sum_{1 \leq s_1 < \cdots < s_p \leq N} s_1^2 s_2^2 \ldots s_p^2 \tag{3.3.38}$$

with $k = \overline{0, N-p}$. *Also, one has:*

$$\rho_i = \epsilon_i^{2N}(-1)^{N+1}\frac{(q-q^{-1})^{2N}}{N^2}, \quad q = e^{i\pi/N}.$$

Proof. Let $\Delta^{[N]}$ denote the expression of the left-hand side of the first equation (3.3.37). We show $\Delta^{[N]} \equiv 0$. Let E_n denote the projector on the eigenspace V_n associated with the eigenvalue $\tilde{\theta}_n^{(0)}$. For $0 \leq n \leq m \leq N$, observe $E_n \Delta^{[N]} E_m = p_N^{(0)}(\tilde{\theta}_n^{(0)}, \tilde{\theta}_m^{(0)}) E_n \overset{[N]}{\mathcal{W}}_1 E_m$ where the two-variable polynomial (3.3.36) has been expanded in a straigthforward manner as power series in the variable x, y. According to Proposition 3.3.2, one has $E_n \overset{[N]}{\mathcal{W}}_1 E_m = 0$ for $|n-m| > N$. Using Lemma 3.3.5, it follows $\Delta^{[N]} = 0$. Similar arguments are used to show the second relation in (3.3.37) ■

Note that the defining relations above do *not* coincide with the higher order $q-$Dolan-Grady relations introduced in the previous Section, except for $N = r = 1$.

It remains to consider the mixed relations between the fundamental operators and the divided polynomials. The derivation of the relations below follows the same steps as the ones for the derivation of the higher $q-$Dolan-Grady relations. For this reason, we skip the details.

Proposition 3.3.4 *Let* $k_- = 0$ *or* $k_+ = 0$. *Then, the operators* $\mathcal{W}_i$ *and the divided polynomials* $\overset{[N]}{\mathcal{W}}_i$, $i = 0, 1$ *satisfy the mixed relations:*

$$\sum_{k=0}^{2N-1}(-1)^k \begin{bmatrix} 2N-1 \\ k \end{bmatrix}_{q=e^{i\pi/N}} \mathcal{W}_i^{2N-1-k} \overset{[N]}{\mathcal{W}}_{i+1} \mathcal{W}_i^k = 0, \tag{3.3.39}$$

$$\left(\overset{[N]}{\mathcal{W}}_i\right)^3 \mathcal{W}_{i+1} - 3\left(\overset{[N]}{\mathcal{W}}_i\right)^2 \mathcal{W}_{i+1} \overset{[N]}{\mathcal{W}}_i + 3 \overset{[N]}{\mathcal{W}}_i \mathcal{W}_{i+1} \left(\overset{[N]}{\mathcal{W}}_i\right)^2 - \mathcal{W}_i \left(\overset{[N]}{\mathcal{W}}_i\right)^3 = \rho_i \left[\overset{[N]}{\mathcal{W}}_i, \mathcal{W}_{i+1}\right],$$

where ρ_i *is defined in Proposition 3.3.3*

To resume, for $k_- = 0$ or $k_+ = 0$ the algebra generated by the basic operators $\mathcal{W}_i$ of the $q-$Onsager algebra and the divided polynomials $\overset{[N]}{\mathcal{W}}_i$ has defining relations (2.2.43), (3.3.37) and (3.3.39).

3.3.2.3 The algebra generated by $\mathcal{W}_0, \mathcal{W}_1, \overset{[2]}{\mathcal{W}}_0, \overset{[2]}{\mathcal{W}}_1$ for generic parameters.

For the special case $q = i$, the algebra generated by the basic operators together with the divided polynomials can be easily identified. The generators $\mathcal{W}_i$ satisfy obviously the defining relations of the $q-$Onsager algebra at $q = e^{i\pi/2}$ or, because $\rho = 0$, the q-Serre relations at $q^2 = -1$ together with the "nilpotency" relations

$$\mathcal{W}_i^2 = (-1)^L\big(\epsilon_i^2 - \frac{(1-(-1)^L)}{2}k_+k_-\big)\mathbb{I}. \tag{3.3.40}$$

For the mixed relations between $\mathcal{W}_i$ and $\overset{[2]}{\mathcal{W}}_{i+1}$, $i=0,1$ we obtain the defining relations of the Onsager algebra for $i=0,1$:

$$\mathcal{W}_i^3\,\overset{[2]}{\mathcal{W}}_{i+1} - 3\mathcal{W}_i^2\,\overset{[2]}{\mathcal{W}}_{i+1}\,\mathcal{W}_i + 3\mathcal{W}_i\,\overset{[2]}{\mathcal{W}}_{i+1}\,\mathcal{W}_i^2 - \overset{[2]}{\mathcal{W}}_{i+1}\,\mathcal{W}_i^3 = \rho_i^{(1)}\Big[\mathcal{W}_i, \overset{[2]}{\mathcal{W}}_{i+1}\Big], \tag{3.3.41}$$

$$\Big(\overset{[2]}{\mathcal{W}}_{i+1}\Big)^3\mathcal{W}_i - 3\Big(\overset{[2]}{\mathcal{W}}_{i+1}\Big)^2\mathcal{W}_i\,\overset{[2]}{\mathcal{W}}_{i+1} + 3\,\overset{[2]}{\mathcal{W}}_{i+1}\,\mathcal{W}_i\Big(\overset{[2]}{\mathcal{W}}_{i+1}\Big)^2 - \mathcal{W}_i\Big(\overset{[2]}{\mathcal{W}}_{i+1}\Big)^3 = \rho_{i+1}^{(2)}\Big[\overset{[2]}{\mathcal{W}}_{i+1}, \mathcal{W}_i\Big], \tag{3.3.42}$$

where we set $\epsilon_0=\epsilon_+$, $\epsilon_1=\epsilon_-$ and

$$\rho_i^{(1)} = 4(-1)^L\big(\epsilon_i^2 - \frac{1-(-1)^L}{2}k_+k_-\big), \qquad \rho_i^{(2)} = 4\epsilon_i^2(k_+k_- - \epsilon_i^2). \tag{3.3.43}$$

Finally, for the divided polynomials $\overset{[2]}{\mathcal{W}}_0$ and $\overset{[2]}{\mathcal{W}}_1$ we have

$$\Bigg[\overset{[2]}{\mathcal{W}}_i\,, \Big(\overset{[2]}{\mathcal{W}}_i\Big)^4\,\overset{[2]}{\mathcal{W}}_{i+1} - 4\Big(\overset{[2]}{\mathcal{W}}_i\Big)^3\,\overset{[2]}{\mathcal{W}}_{i+1}\overset{[2]}{\mathcal{W}}_i + 6\Big(\overset{[2]}{\mathcal{W}}_i\Big)^2\,\overset{[2]}{\mathcal{W}}_{i+1}\,\Big(\overset{[2]}{\mathcal{W}}_i\Big)^2 - 4\,\overset{[2]}{\mathcal{W}}_i\overset{[2]}{\mathcal{W}}_{i+1}\,\Big(\overset{[2]}{\mathcal{W}}_i\Big)^3 + \overset{[2]}{\mathcal{W}}_{i+1}\,\Big(\overset{[2]}{\mathcal{W}}_i\Big)^4$$
$$- 5\rho_i^{(2)}\Big\{\Big(\overset{[2]}{\mathcal{W}}_i\Big)^2\,\overset{[2]}{\mathcal{W}}_{i+1} - 2\,\overset{[2]}{\mathcal{W}}_i\overset{[2]}{\mathcal{W}}_{i+1}\overset{[2]}{\mathcal{W}}_i + \overset{[2]}{\mathcal{W}}_{i+1}\,\Big(\overset{[2]}{\mathcal{W}}_i\Big)^2\Big\} + 4\big(\rho_i^{(2)}\big)^2\,\overset{[2]}{\mathcal{W}}_{i+1}\Bigg] = 0. \tag{3.3.44}$$

3.3.3 Observations about the symmetries of the Hamiltonian

In this subsection, we study the commutation relations between the basic generators of the $q-$Onsager algebra or the divided polynomials in the basic generators and the Hamiltonian of the open XXZ spin chain for a certain choice of boundary conditions and q a root of unity. Basically, let us consider the Hamiltonian:

$$\begin{aligned} H_{XXZ}^{(L)} &= \sum_{k=1}^{L-1}\Big(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_z^{k+1}\sigma_z^k\Big) + \frac{(q-q^{-1})}{2}h_3\sigma_z^1 + h_+\sigma_+^1 + h_-\sigma_-^1 \\ &\qquad + \frac{(q-q^{-1})}{2}\bar{h}_3\sigma_z^L + \bar{h}_+\sigma_+^L + \bar{h}_-\sigma_-^L. \end{aligned}$$

• The special case $k_-=0$.
In this case, the operators satsify the relations considered in the previous Section. We find:

$$[H_{XXZ}^{(L)}, \overset{[N]}{\mathcal{W}}_i] = 0$$

if and only if

$$\begin{cases} \bar{h}_+ = \bar{h}_- = h_- = 0, \\ \bar{h}_3 = (-1)^i, \\ h_+ : \text{free} \\ h_3 = \frac{(-1)^i(-k_+ + \epsilon_i h_+)}{k_+}. \end{cases} \tag{3.3.45}$$

Note that the conditions $[H^{(L)}_{XXZ}, \mathcal{W}_i] = 0$ are satisfied provided the above relations are satisfied too. From the results above, we conclude that the Hamiltonian cannot commute simultaneously with both $\overset{[N]}{\mathcal{W}}_0$ and $\overset{[N]}{\mathcal{W}}_1$.

However, recall that for the XXZ spin chain with periodic boundary conditions, in some sectors of the spin chain the Hamiltonian commutes with the generators of the sl_2 loop algebra [DFM99] (see the first section of this part). By analogy, we are currently investigating the existence of sectors of the spin chain in which a non-Abelian symmetry associated with a subalgebra of the algebra generated $\overset{[N]}{\mathcal{W}}_0, \overset{[N]}{\mathcal{W}}_1$ may occur.

Note that the results reported here are part of an ongoing project.

Chapter 4

Perspectives

This thesis opens at least three new and promising directions for further research. To our knowledge, none of these have been discussed in the literature.

4.1 A family of new integrable hierarchies

Let A, A^* be the generators of the $q-$Onsager algebra. As shown by Dolan and Grady [DG82], it is possible to construct a hierarchy of mutually commuting quantities that are polynomials in A, A^*, of the form:

$$\begin{aligned} I_1 &= \kappa A + \kappa^* A^*, \\ I_3 &= \kappa \left(\frac{1}{\rho}((q^2+q^{-2})AA^*A - A^2A^* - A^*A^2) + A^* \right) \\ &\quad + \kappa^* \left(\frac{1}{\rho}((q^2+q^{-2})A^*AA^* - A^{*2}A - AA^{*2}) + A \right) \end{aligned} \tag{4.1.1}$$

In Chapter 2, we explained that examples of such hierarchy can be generated from the transfer matrix formalism, where the transfer matrix is built from solutions to the reflection equation algebra (the so-called Sklyanin's operator). In this thesis, it is conjectured (with several supporting evidences) that the $q-$Onsager algebra automatically implies the existence of higher order relations satisfied by A, A^*, the so-called $r-$th higher order $q-$Dolan-Grady relations.

Having this in mind and taking a more general point of view, let us introduce a new associative algebra with generators X, Y that satisfy *only* the $r-$th higher order Dolan-Grady relation (3.1.7)-(3.1.8) (i.e. we do *not* assume the $k-$th higher order $q-$Dolan-Grady relations for $1 \leq k \leq r-1$). We call this new algebra $O_q^{(r)}$. The following problems may be considered:

Problem 1: For $r = 1$, the $q-$Onsager algebra and the reflection equation algebra are closely related (see Chapters 1,2). Is there an $r-th$ analogue of the reflection equation algebra that would correspond to this new algebra? What is the interpretation of this new algebra in the context of scattering theory?

Problem 2: For $r = 1$, the quantities (4.1.1) generate the $q-$Dolan-Grady integrable hierarchy. For $r \neq 1$, what is the structure of the new integrable hierarchy ? Find an algorithm to construct explicitly its first few elements.

Problem 3: For $r = 1$, there is an homomorphism from the $q-$Onsager algebra to a coideal subalgebra of $U_q(\widehat{sl_2})$ (see Chapter 1). What happens for $r \neq 1$?

Problem 4: For $r = 1$, there are numerous examples of quantum integrable systems that are generated by the $q-$Onsager algebra for $q = 1$ or $q \neq 1$: Ising, superintegrable chiral Potts model, XY, open XXZ spin chain,... For $r \neq 1$, construct explicit examples of Hamiltonian integrable systems with applications to condensed matter physics.

4.2 Cyclic tridiagonal pairs

It is clear that one of the main ingredients in this thesis is the concept of tridiagonal pairs, provided q is not a root of unity. For details, we refer the reader to Chapter 2. In particular, the definition of tridiagonal pairs is given in [ITT99]. Among the known examples of tridiagonal pairs, for generic values of q one finds the basic operators that appear in the open XXZ spin chain, see (3.3.17)-(3.3.18) and found application to the solution of this model [BK07].

For q a root of unity, as shown in Chapter 3, the basic operators of the open XXZ spin chain, see (3.3.17)-(3.3.18) still satisfy the $q-$Onsager algebra. However, the decomposition of the vector space on which they act is different for q a root of unity compared to the case q generic. Indeed, for q a root of unity the spectrum of the operators admits additional degeneracies. In this case, to each eigenvalue one associates an eigenspace as defined by (3.3.24). Recall that for q generic, the ordered eigenspaces are such that $V_{-1} = V_{d+1} = 0$; $V^*_{-1} = V^*_{d+1} = 0$ (see Definition 1.3.5). However, for q a root of unity, one has: $V^{(N)}_{-1} = V^{(N)}_{N-1}$, $V^{(N)}_0 = V^{(N)}_N$; $V^{*(N)}_{-1} = V^{*(N)}_{N-1}$, $V^{*(N)}_0 = V^{*(N)}_N$ (see Corollary 3.3.1)

In other words, for q a root of unity the action of the two basic operators is 'cyclic'.

Based on this example, a new concept may be introduced that we call 'cyclic tridiagonal pairs'. For this object, we propose the following definition:

Definition 4.2.1 *Let V denote a vector space over a field $\mathbb{K}$ with finite positive dimension. By a cyclic tridiagonal pair on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.*

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \leq i \leq d, \tag{4.2.1}$$

where $V_{-1} = V_{d-1}$ and $V_0 = V_d$.

(iii) There exists an ordering $\{V^*_i\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \qquad 0 \leq i \leq \delta, \tag{4.2.2}$$

where $V^*_{-1} = V^*_{\delta-1}$ *and* $V^*_0 = V^*_\delta$.

(iv) There does not exist a subspace W *of* V *such that* $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

According to this definition, the following problems may be considered:

Problem 1: Classify irreducible finite dimensional representations (up to isomorphisms) of cyclic tridiagonal pairs.

Problem 2: Construct explicit examples of cyclic tridiagonal pairs using the connection between the solutions of the reflection equation algebra and the $q-$Onsager algebra at roots of unity.

Problem 3: Solve quantum integrable systems using the representation theory of the $q-$Onsager algebra at roots of unity and the properties of cyclic tridiagonal pairs.

4.3 An analog of Lusztig quantum group for the $q-$Onsager algebra at roots of unity

Let p be an positive integer. Consider the quantum group $U_q(sl_2)$ at root of unity $q = e^{i\pi/p}$, denote $\overline{U}_q(sl_2)$. It is sometimes referred as the 'small quantum group'. The generators are given by E, F and $K^{\pm 1}$ satisfying the standard defining relations of $U_q(sl_2)$.

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}, \tag{4.3.1}$$

with additional relations,

$$E^p = F^p = 0, \quad K^{2p} = 1. \tag{4.3.2}$$

It can be endowed with a Hopf algebra structure. The comultiplication is given by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K. \tag{4.3.3}$$

This associative algebra is finite-dimensional, namely $\dim \overline{U}_q(sl_2) = 2p^3$.

In the literature, the so-called full or Lusztig quantum group has been introduced. For $q = e^{i\pi/p}$ and for any integer $p \geq 2$, it is generated by E, F, $K^{\pm 1}$ together with h such that $K = e^{\alpha h}$ (α is a scalar) and the so-called Lusztig's divided powers $f \sim F^p/[p]!$ and $e \sim E^p/[p]!$. The defining relations are given by (4.3.1),

$$[h,e] = e, \qquad [h,f] = -f, \qquad [e,f] = 2h$$

and the 'mixed' relations

$$[h,K] = 0, \qquad [E,e] = 0, \qquad [K,e] = 0, \qquad [F,f] = 0, \qquad [K,f] = 0,$$
$$[F,e] = \frac{1}{[p-1]!}K^p\frac{qK-q^{-1}K^{-1}}{q-q^{-1}}E^{p-1}, \qquad [E,f] = \frac{(-1)^{p+1}}{[p-1]!}F^{p-1}\frac{qK-q^{-1}K^{-1}}{q-q^{-1}},$$
$$[h,E] = \frac{1}{2}EA, \quad [h,F] = -\frac{1}{2}AF,$$

where

$$A = \sum_{s=1}^{p-1} \frac{(u_s(q^{-s-1}) - u_s(q^{s-1}))K + q^{s-1}u_s(q^{s-1}) - q^{-s-1}u_s(q^{-s-1})}{(q^{s-1} - q^{-s-1})u_s(q^{-s-1})u_s(q^{s-1})} u_s(K)e_s.$$

Here, the polynomials $u_s(K) = \prod_{n=1,\, n\neq s}^{p-1}(K - q^{s-1-2n})$, and e_s are central primitive idempotents (see [FGST05])

The Lusztig quantum group can be endowed with a Hopf algebra structure. For instance, the comultiplication is given by:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$\Delta(e) = e \otimes 1 + K^p \otimes e + \frac{1}{[p-1]!} \sum_{r=1}^{p-1} \frac{q^{r(p-r)}}{[r]} K^p E^{p-r} \otimes E^r K^{-r},$$

$$\Delta(f) = f \otimes 1 + K^p \otimes f + \frac{(-1)^p}{[p-1]!} \sum_{s=1}^{p-1} \frac{q^{-s(p-s)}}{[s]} K^{p+s} F^s \otimes F^{p-s}.$$

In Chapter 3, we have seen that the operators $\mathcal{W}_0, \mathcal{W}_1, \overset{[N]}{\mathcal{W}}_0, \overset{[N]}{\mathcal{W}}_1$ satisfy certain relations at least for certain choices of the parameters $k_\pm$. According to these results, it is thus natural to consider the following problems:

Problem 1: Define an analog of the small quantum group for the $q-$Onsager algebra. Classify irreducible finite dimensional representations.

Problem 2: Define an analog of the Lusztig quantum group for the $q-$Onsager algebra. Classify irreducible finite dimensional representations.

Problem 3: What is the analog of the Hopf algebra structure?

Note that some results are already obtained in relation to Problem 1 and 2, although not published yet.

Chapter 5

Appendices

5.1 APPENDIX A: Coefficients $\eta_{k,j}^{(m)}$, $M_j^{(r,p)}$, $N_j^{(r,p)}$

The coefficients that appear in eqs. (3.1.37), (3.1.38) are such that:

$$\begin{aligned}
\eta_{1,0}^{(3)} &= [3]_q, \qquad \eta_{1,1}^{(3)} = -[3]_q, \qquad \eta_{1,2}^{(3)} = 1, \\
\eta_{0,0}^{(4)} &= 1, \qquad \eta_{0,1}^{(4)} = q^2 + q^{-2}, \qquad \eta_{0,2}^{(4)} = -[3]_q, \\
\eta_{1,0}^{(4)} &= (q^2 + q^{-2})[3]_q, \qquad \eta_{1,1}^{(4)} = -(q^2 + q^{-2})[2]_q^2, \qquad \eta_{1,2}^{(4)} = [3]_q.
\end{aligned}$$

The recursion relations for $\eta_{k,j}^{(m)}$ are such that:

$$\begin{aligned}
\eta_{0,0}^{(2n+2)} &= 1, \\
\eta_{k,0}^{(2n+2)} &= [3]_q \eta_{k,0}^{(2n+1)} + \eta_{k,1}^{(2n+1)}, \qquad 1 \le k \le n, \\
\eta_{0,1}^{(2n+2)} &= \eta_{1,0}^{(2n+1)} - 1, \\
\eta_{k,1}^{(2n+2)} &= -[3]_q \eta_{k,0}^{(2n+1)} + \eta_{k+1,0}^{(2n+1)} + \eta_{k,2}^{(2n+1)}, \qquad 1 \le k \le n-1, \\
\eta_{n,1}^{(2n+2)} &= -[3]_q \eta_{n,0}^{(2n+1)} + \eta_{n,2}^{(2n+1)}, \\
\eta_{0,2}^{(2n+2)} &= -\eta_{1,0}^{(2n+1)}, \\
\eta_{k,2}^{(2n+2)} &= \eta_{k,0}^{(2n+1)} - \eta_{k+1,0}^{(2n+1)}, \qquad 1 \le k \le n-1, \\
\eta_{n,2}^{(2n+2)} &= \eta_{n,0}^{(2n+1)},
\end{aligned}$$

and

$$\begin{aligned}
\eta_{k,0}^{(2n+3)} &= [3]_q \eta_{k-1,0}^{(2n+2)} + \eta_{k-1,1}^{(2n+2)}, \qquad 1 \le k \le n+1, \\
\eta_{k,1}^{(2n+3)} &= -[3]_q \eta_{k-1,0}^{(2n+2)} + \eta_{k,0}^{(2n+2)} + \eta_{k-1,2}^{(2n+2)}, \qquad 1 \le k \le n, \\
\eta_{n+1,1}^{(2n+3)} &= -[3]_q \eta_{n,0}^{(2n+2)} + \eta_{n,2}^{(2n+2)}, \\
\eta_{k,2}^{(2n+3)} &= \eta_{k-1,0}^{(2n+2)} - \eta_{k,0}^{(2n+2)}, \qquad 1 \le k \le n, \\
\eta_{n+1,2}^{(2n+3)} &= \eta_{n,0}^{(2n+2)}.
\end{aligned}$$

The coefficients that appear in eqs. (3.1.33), (3.1.34) are such that:

$$\begin{aligned}
M_j^{(r,0)} &= c_j^{[r,0]} - c_1^{[r,0]} c_{j-1}^{[r,0]}, \qquad j = \overline{2, 2r+1}, \\
M_{2r+2}^{(r,0)} &= -c_1^{[r,0]} c_{2r+1}^{[r,0]}, \\
M_0^{(r,p)} &= c_0^{[r,p]}, p = \overline{1, r}, \\
M_j^{(r,p)} &= c_j^{[r,p]} - c_1^{[r,0]} c_{j-1}^{[r,p]}, \qquad p = \overline{1, r}, \quad j = \overline{1, 2(r-p)+1}, \\
M_{2(r-p)+2}^{(r,p)} &= -c_1^{[r,0]} c_{2(r-p)+1}^{[r,p]}, \qquad p = \overline{1, r},
\end{aligned}$$

and

$$\begin{aligned}
N_j^{(r,0)} &= c_j^{[r,0]} - c_1^{[r,0]} c_{j-1}^{[r,0]} + ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{j-2}^{[r,0]}, \qquad j = \overline{3, 2r+1}, \\
N_{2r+2}^{(r,0)} &= -c_1^{[r,0]} c_{2r+1}^{[r,0]} + ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{2r}^{[r,0]}, \\
N_{2r+3}^{(r,0)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{2r+1}^{[r,0]}, \\
N_0^{(r,1)} &= 0, \\
N_j^{(r,1)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{j-2}^{[r,1]} - c_1^{[r,0]} c_{j-1}^{[r,1]} + c_j^{[r,1]} - c_0^{[r,1]} c_j^{[r,0]}, \qquad j = \overline{2, 2r-1}, \\
N_1^{(r,1)} &= -c_1^{[r,0]} c_0^{[r,1]} + c_1^{[r,1]} - c_0^{[r,1]} c_1^{[r,0]},
\end{aligned}$$

$$\begin{aligned}
N_{2r}^{(r,1)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{2r-2}^{[r,1]} - c_1^{[r,0]} c_{2r-1}^{[r,1]} - c_0^{[r,1]} c_{2r}^{[r,0]}, \\
N_{2r+1}^{(r,1)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{2r-1}^{[r,1]} - c_0^{[r,1]} c_{2r+1}^{[r,0]}, \\
N_j^{(r,r+1)} &= -c_0^{[r,1]} c_j^{[r,r]}, \qquad j = \overline{0, 1}.
\end{aligned}$$

For $2 \leq p \leq r$,

$$\begin{aligned}
N_j^{(r,p)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{j-2}^{[r,p]} - c_1^{[r,0]} c_{j-1}^{[r,p]} + c_j^{[r,p]} - c_0^{[r,1]} c_j^{[r,p-1]}, \quad j = \overline{2, 2(r-p)+1}, \\
N_0^{(r,p)} &= c_0^{[r,p]} - c_0^{[r,1]} c_0^{[r,p-1]}, \\
N_1^{(r,p)} &= -c_1^{[r,0]} c_0^{[r,p]} + c_1^{[r,p]} - c_0^{[r,1]} c_1^{[r,p-1]}, \\
N_{2(r-p)+2}^{(r,p)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{2(r-p)}^{[r,p]} - c_1^{[r,0]} c_{2(r-p)+1}^{[r,p]} - c_0^{[r,1]} c_{2(r-p)+2}^{[r,p-1]}, \\
N_{2(r-p)+3}^{(r,p)} &= ({c_1^{[r,0]}}^2 - c_2^{[r,0]}) c_{2(r-p)+1}^{[r,p]} - c_0^{[r,1]} c_{2(r-p)+3}^{[r,p-1]}.
\end{aligned}$$

5.2 APPENDIX B: $A^{2r+2}A^{*r+1}A$, $A^{2r+1}A^{*r+1}A^2$, $A^{2r+1}A^{*r+1}$

In addition to (3.1.39), the other monomials can be written as:

$$
\begin{aligned}
A^{2r+2}A^{*r+1}A = &-\sum_{i=1}^{r} M_{2i+2}^{(r,0)} A^{2(r-i)} A^{*r} (\sum_{k=0}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+2)} A^{2-j} A^* A^{2k+j+1}) \\
&+ \sum_{i=1}^{r} M_{2i+1}^{(r,0)} A^{2(r-i)+1} A^{*r} (\sum_{k=1}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k+j} + \rho_0^i (AA^*A - A^*A^2)) \\
&- \sum_{p=1}^{r} (-\rho_0)^p (M_0^{(r,p)} A^{2(r-p)+2} A^{*r+1} A - M_1^{(r,p)} A^{2(r-p)+1} A^{*r} AA^*A + M_2^{(r,p)} A^{2(r-p)} A^{*r} A^2 A^* A) \\
&- M_2^{(r,0)} A^{2r} A^{*r} A^2 A^* A - \sum_{p=1}^{r-1} (-\rho_0)^p \sum_{i=1}^{r-p} M_{2i+2}^{(r,p)} A^{2(r-p-i)} A^{*r} (\sum_{k=0}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+2)} A^{2-j} A^* A^{2k+j+1}) \\
&+ \sum_{p=1}^{r-1} (-\rho_0)^p \sum_{i=1}^{r-p} M_{2i+1}^{(r,p)} A^{2(r-p-i)+1} A^{*r} (\sum_{k=1}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k+j} + \rho_0^i (AA^*A - A^*A^2)) \ ,
\end{aligned}
$$

$$
\begin{aligned}
A^{2r+1}A^{*r+1}A^2 = &\ c_1^{[r,0]} A^{2r} A^{*r} AA^* A^2 - c_2^{[r,0]} A^{2r-1} A^{*r} A^2 A^* A^2 \\
&- \sum_{p=1}^{r-1} (-\rho_0)^p (c_0^{[r,p]} A^{2(r-p)+1} A^{*r+1} A^2 - c_1^{[r,p]} A^{2(r-p)} A^{*r} AA^* A^2 + c_2^{[r,p]} A^{2(r-p)-1} A^{*r} A^2 A^* A^2) \\
&- (-\rho_0)^r (c_0^{[r,r]} AA^{*r+1} A^2 - c_1^{[r,r]} A^{*r} AA^* A^2) \\
&+ \sum_{i=1}^{r} c_{2i+1}^{[r,0]} A^{2(r-i)} A^{*r} (\sum_{k=1}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k+1+j} + \rho_0^i (AA^*A^2 - A^*A^3)) \\
&- \sum_{i=1}^{r-1} c_{2i+2}^{[r,0]} A^{2(r-i)-1} A^{*r} (\sum_{k=0}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+2)} A^{2-j} A^* A^{2k+j+2}) \\
&+ \sum_{p=1}^{r-1} (-\rho_0)^p \sum_{i=1}^{r-p} c_{2i+1}^{[r,p]} A^{2(r-p-i)} A^{*r} (\sum_{k=1}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k+1+j} + \rho_0^i (AA^*A^2 - A^*A^3)) \\
&- \sum_{p=1}^{r-2} (-\rho_0)^p \sum_{i=1}^{r-p-1} c_{2i+2}^{[r,p]} A^{2(r-p-i)-1} A^{*r} (\sum_{k=0}^{i}\sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+2)} A^{2-j} A^* A^{2k+j+2}) \ ,
\end{aligned}
$$

$$
\begin{aligned}
A^{2r+1}A^{*r+1} &= c_1^{[r,0]}A^{2r}A^{*r}AA^* - c_2^{[r,0]}A^{2r-1}A^{*r}A^2A^* \\
&+ \sum_{k=1}^{r} c_{2k+1}^{[r,0]}A^{2(r-k)}A^{*r}\Big(\sum_{i=1}^{k}\sum_{j=0}^{2}\mu_{i,j}^{(2k+1)}A^{2-j}A^*A^{2i-1+j} + \rho^k(AA^* - A^*A)\Big) \\
&- \sum_{k=1}^{r-1} c_{2k+2}^{[r,0]}A^{2(r-k)-1}A^{*r}\Big(\sum_{i=0}^{k}\sum_{j=0}^{2}\mu_{i,j}^{(2k+2)}A^{2-j}A^*A^{2i+j}\Big) \\
&- \rho^r(c_0^{[r,r]}AA^{*r+1} - c_1^{[r,r]}A^{*r}AA^*) \\
&- \sum_{p=1}^{r-1}\rho^p(c_0^{[r,p]}A^{2(r-p)+1}A^{*r+1} - c_1^{[r,p]}A^{2(r-p)}A^{*r}AA^* + c_2^{[r,p]}A^{2(r-p)-1}A^{*r}A^2A^*) \\
&+ \sum_{p=1}^{r-1}\rho^p\sum_{k=1}^{r-p} c_{2k+1}^{[r,p]}A^{2(r-p-k)}A^{*r}\Big(\sum_{i=1}^{k}\sum_{j=0}^{2}\mu_{i,j}^{(2k+1)}A^{2-j}A^*A^{2i-1+j} + \rho^k(AA^* - A^*A)\Big) \\
&- \sum_{p=1}^{r-2}\rho^p\sum_{k=1}^{r-p-1} c_{2k+2}^{[r,p]}A^{2(r-p-k)-1}A^{*r}\Big(\sum_{i=0}^{k}\sum_{j=0}^{2}\mu_{i,j}^{(2k+2)}A^{2-j}A^*A^{2i+j}\Big).
\end{aligned}
$$

5.3 APPENDIX C: Coefficients $\eta_{k,j}^{(m)}$, $M_j^{(r,p)}$

The initial values of $\eta_{k,j}^{(m)}$ are given by:

$$\eta_{0,0}^{(2)} = [2]_q, \qquad \eta_{0,1}^{(2)} = -1.$$

The recursion relations for $\eta_{k,j}^{(m)}$ and $M_j^{(r,p)}$ read:

$$\begin{aligned}
\eta_{p,0}^{(2n+1)} &= [2]_q \eta_{p,0}^{(2n)} + \eta_{p,1}^{(2n)}, \qquad p = \overline{0, n-1}, \\
\eta_{n,0}^{(2n+1)} &= 1, \\
\eta_{p,1}^{(2n+1)} &= -\eta_{p,0}^{(2n)} + \eta_{p-1,0}^{(2n)}, \qquad p = \overline{1, n-1}, \\
\eta_{0,1}^{(2n+1)} &= -\eta_{0,0}^{(2n)}, \\
\eta_{n,1}^{(2n+1)} &= \eta_{n-1,0}^{(2n)}, \\
\eta_{p,0}^{(2n+2)} &= [2]_q \eta_{p,0}^{(2n+1)} + \eta_{p,1}^{(2n+1)}, \qquad p = \overline{0, n},
\end{aligned}$$

$$\begin{aligned}
\eta_{p,1}^{(2n+2)} &= \eta_{p-1,0}^{(2n+1)} - \eta_{p,0}^{(2n+1)}, \qquad p = \overline{1, n}, \\
\eta_{0,1}^{(2n+2)} &= -\eta_{0,0}^{(2n+1)}.
\end{aligned}$$

$$\begin{aligned}
M_0^{(2t,p)} &= c_0^{[2t,p]}, \\
M_{2t+2-2p}^{(2t,p)} &= -c_1^{[2t,0]} c_{2t+1-2p}^{[2t,p]}, \\
M_k^{(2t,p)} &= c_k^{[2t,p]} - c_1^{[2t,0]} c_{k-1}^{[2t,p]}, \qquad k = \overline{1, 2t+1-2p}, \\
M_0^{(2t+1,t+1)} &= c_0^{[2t+1,t+1]}, \\
M_1^{(2t+1,t+1)} &= -c_0^{[2t+1,t+1]} c_1^{[2t+1,0]}, \\
M_k^{(2t+1,p)} &= -c_1^{[2t+1,0]} c_{k-1}^{[2t+1,p]} + c_k^{[2t+1,p]}, \qquad k = \overline{1, 2t+2-2p}, \\
M_0^{(2t+1,p)} &= c_0^{[2t+1,p]}, \\
M_{2t+3-2p}^{(2t+1,p)} &= -c_1^{[2t+1,0]} c_{2t+2-2p}^{[2t+1,p]}.
\end{aligned}$$

5.4 APPENDIX D: Algorithms for the q-Onsager algebra

```
# Compute the coefficients eta[m,k,j] of A^(2n+2)A* and A^(2n+3)A*
Funct := proc(n)
  local i, k;
  global eta, alpha, q;
  # Input the initial values
  alpha := q^2+1+1/q^2;
  eta[3, 1, 0] := alpha;
  eta[3, 1, 1] := -alpha;
  eta[3, 1, 2] := 1;
  eta[4, 1, 0] := alpha^2-alpha;
  eta[4, 1, 1] := 1-alpha^2;
  eta[4, 1, 2] := alpha;
  eta[4, 0, 1] := alpha-1;
  eta[4, 0, 0] := 1;
  eta[4, 0, 2] := -alpha;
  eta[5, 2, 0] := alpha^3-2*alpha^2+1;
  eta[5, 2, 1] := -alpha^3+alpha^2+alpha;
  eta[5, 2, 2] := alpha*(alpha-1); eta[5, 1, 0] := 2*alpha-1;
  eta[5, 1, 1] := alpha^2-3*alpha;
  eta[5, 1, 2] := -alpha^2+alpha+1;
  for i from 2 to n do
     eta[2*i+2, 0, 0] := 1;
     for k from 1 to i do
        eta[2*i+2, k, 0] := simplify(alpha*eta[2*i+1, k, 0]+eta[2*i+1, k, 1]);
     end do;
     eta[2*i+2, 0, 1] := simplify(eta[2*i+1, 1, 0]-1);
     for k from 1 to i-1 do
        eta[2*i+2, k, 1] := simplify(-alpha*eta[2*i+1, k, 0]
                                  +eta[2*i+1, k+1, 0]+eta[2*i+1, k, 2]);
     end do;
     eta[2*i+2, i, 1] := simplify(-alpha*eta[2*i+1, i, 0]+eta[2*i+1, i, 2]);
     eta[2*i+2, 0, 2] := simplify(-eta[2*i+1, 1, 0]);
     for k from 1 to i-1 do
        eta[2*i+2, k, 2] := simplify(eta[2*i+1, k, 0]-eta[2*i+1, k+1, 0]);
     end do;
     eta[2*i+2, i, 2] := simplify(eta[2*i+1, i, 0]);
     for k from 1 to i+1 do
        eta[2*i+3, k, 0] := simplify(alpha*eta[2*i+2, k-1, 0]+eta[2*i+2, k-1, 1]);
     end do;
     for k from 1 to i do
        eta[2*i+3, k, 1] := simplify(-alpha*eta[2*i+2, k-1, 0]+eta[2*i+2, k, 0]
                                  +eta[2*i+2, k-1, 2]);
     end do;
```

```
      eta[2*i+3, i+1, 1] := simplify(-alpha*eta[2*i+2, i, 0]+eta[2*i+2, i, 2]);
      for k from 1 to i do
         eta[2*i+3, k, 2] := simplify(eta[2*i+2, k-1, 0]-eta[2*i+2, k, 0]);
      end do;
      eta[2*i+3, i+1, 2] := simplify(eta[2*i+2, i, 0])
   end do;
end proc;

Result1 := proc (n)
   local i, p, d, l, F, M, N, t, tam, ta1, ta2, ta3;
   global A, c, alpha, rho, q;
   # Input the initial values
   alpha := q^2+1+1/q^2;
   c[1, 0, 0] := 1;
   c[1, 0, 3] := 1;
   c[1, 0, 1] := alpha;
   c[1, 0, 2] := alpha;
   c[1, 1, 0] := -1;
   c[1, 1, 1] := -1;
   c[2, 0, 0] := 1;
   c[2, 0, 5] := 1;
   c[2, 0, 1] := expand(simplify(alpha^2-alpha-1));
   c[2, 0, 4] := expand(simplify(alpha^2-alpha-1));
   c[2, 0, 2] := expand(simplify(alpha^3-2*alpha^2+1));
   c[2, 0, 3] := expand(simplify(alpha^3-2*alpha^2+1));
   c[2, 1, 0] := expand(simplify(-alpha^2+2*alpha-2));
   c[2, 1, 3] := expand(simplify(-alpha^2+2*alpha-2));
   c[2, 1, 1] := expand(simplify(-alpha*(alpha^2-alpha-1)));
   c[2, 1, 2] := expand(simplify(-alpha*(alpha^2-alpha-1)));
   c[2, 2, 0] := expand(simplify((alpha-1)^2));
   c[2, 2, 1] := expand(simplify((alpha-1)^2));
   # Use the convention that the first factor A is denoted A[1],
   # the second factor A^* is denoted A[2], and the third factor A is denoted A[3]
   # Input the q-Dolan-Grady relations
   if n = 0 then
      F[1] := A[1]^3*A[2]-alpha*A[1]^2*A[2]*A[3]+alpha*A[1]*A[2]*A[3]^2-A[2]*A[3]^3
             +rho*(-A[1]*A[2]+A[2]*A[3]) = 0
   end if;
   # Input the higher order q-Dolan-Grady relations for r = 2
   if n = 1 then
      F[2] := A[1]^5*A[2]^2-(alpha^2-alpha-1)*A[1]^4*A[2]^2*A[3]
             +(alpha^3-2*alpha^2+1)*A[1]^3*A[2]^2*A[3]^2
             -(alpha^3-2*alpha^2+1)*A[1]^2*A[2]^2*A[3]^3
             +(alpha^2-alpha-1)*A[1]*A[2]^2*A[3]^4-A[2]^2*A[3]^5
             +rho*(-(alpha^2-2*alpha+2)*A[1]^3*A[2]^2
                   +alpha*(alpha^2-alpha-1)*A[1]^2*A[2]^2*A[3]
```

```
                -alpha*(alpha^2-alpha-1)*A[1]*A[2]^2*A[3]^2
                +(alpha^2-2*alpha+2)*A[2]^2*A[3]^3)
          +rho^2*((alpha-1)^2*A[1]*A[2]^2-(alpha-1)^2*A[2]^2*A[3]) = 0
end if;
if 1 < n then
   # Compute M[n, p, j], p = 0,..., n; j= 2,..., 2n+2 if p=0,
           #                     j = 0,..., 2(n-p)+2 if p = 1,...,n
   for i from 2 to n do
      M[i, 0, 2*i+2] := simplify(-c[i, 0, 1]*c[i, 0, 2*i+1]);
      for l from 2 to 2*i+1 do
         M[i, 0, l] := simplify(c[i, 0, l]-c[i, 0, 1]*c[i, 0, l-1]);
      end do;
      for d to i do
         M[i, d, 0] := c[i, d, 0];
         for l to 2*i-2*d+1 do
            M[i, d, l] := simplify(c[i, d, l]-c[i, 0, 1]*c[i, d, l-1]);
         end do;
         M[i, d, 2*i-2*d+2] := simplify(-c[i, 0, 1]*c[i, d, 2*i-2*d+1]);
      end do;
      Funct(i);
      # Expand A^{2n+2}{A^*}^{r+1}A
      t[2] := simplify(-(sum(M[i, 0, 2*h+2]*A[1]^(2*i-2*h)*A[2]^i
                         *(sum(sum(eta[2*h+2, k, j]*A[3]^(2-j)*A[4]*A[5]^(2*k+j+1),
                                   j = 0 .. 1)+eta[2*h+2, k, 2]*A[2]*A[3]^(2*k+3),
                                   k = 0 .. h)), h = 1 .. i))
                      +sum(M[i, 0, 2*h+1]*A[1]^(2*i-2*h+1)*A[2]^i*
                          (sum(eta[2*h+1, k, 2]*A[2]*A[3]^(2*k+2)
                              +sum(eta[2*h+1, k, j]*A[3]^(2-j)*A[4]*A[5]^(2*k+j),
                                 j = 0 .. 1),k = 1 .. h)
                                 +rho^h*(A[3]*A[4]*A[5]-A[2]*A[3]^2)), h = 1 .. i)
                     -M[i, 0, 2]*A[1]^(2*i)*A[2]^i*A[3]^2*A[4]*A[5]
                     -(sum(rho^p*(M[i, p, 0]*A[1]^(2*i-2*p+2)*A[2]^(i+1)*A[3]
                          +sum((-1)^j*M[i, p, j]*A[1]^(2*i-2*p+2-j)*A[2]^i*A[3]^j
                          *A[4]*A[5],j = 1 .. 2)), p = 1 .. i))
                     -(sum(rho^p*(sum(M[i, p, 2*h+2]*A[1]^(2*i-2*p-2*h)*A[2]^i
                                 *(sum(sum(eta[2*h+2, k, j]*A[3]^(2-j)*A[4]
                                 *A[5]^(2*k+j+1),j= 0 .. 1)
                                 +eta[2*h+2, k, 2]*A[2]*A[3]^(2*k+3), k = 0 .. h)),
                                 h = 1 .. i-p)), p = 1 .. i-1))
                     +sum(rho^p*(sum(M[i, p, 2*h+1]*A[1]^(2*i-2*p-2*h+1)*A[2]^i
                                  *(sum(eta[2*h+1, k, 2]*A[2]*A[3]^(2*k+2)
                                      +sum(eta[2*h+1, k, j]*A[3]^(2-j)*A[4]
                                      *A[5]^(2*k+j),j = 0 .. 1), k = 1 .. h)
                                      +rho^h*(A[3]*A[4]*A[5]-A[2]*A[3]^2)),
                                       h = 1 .. i-p)), p = 1 .. i-1));
```

```
# Compute N[n, p, j], p = 0,...,n;
#           j = 3,...,2n+2 if p =0
#           j = 0,...,2(n-p)+3 if p = 1,...,n
N[i, 0, 2*i+2] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 0, 2*i]
                           -c[i, 0, 1]*c[i, 0, 2*i+1]);
N[i, 0, 2*i+3] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 0, 2*i+1]);
for l from 3 to 2*i+1 do
   N[i, 0, l] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 0, l-2]
                       -c[i, 0, 1]*c[i, 0, l-1]+c[i, 0, l]);
end do;
N[i, 1, 1] := simplify(-2*c[i, 0, 1]*c[i, 1, 0]+c[i, 1, 1]);
N[i, 1, 2*i] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 1, 2*i-2]
                    -c[i, 0, 1]*c[i, 1, 2*i-1]-c[i, 1, 0]*c[i, 0, 2*i]);
N[i, 1, 2*i+1] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 1, 2*i-1]
                        -c[i, 1, 0]*c[i, 0, 2*i+1]);
N[i, 1, 0] := 0;
for l from 2 to 2*i-1 do
   N[i, 1, l] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 1, l-2]
                       -c[i, 0, 1]*c[i, 1, l-1]+c[i, 1, l]
                       -c[i, 1, 0]*c[i, 0, l]);
end do;
for d from 2 to i do
   for l from 2 to 2*i-2*d+1 do
      N[i, d, l] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, d, l-2]
                         -c[i, 0, 1]*c[i, d, l-1]+c[i, d, l]
                         -c[i, 1, 0]*c[i, d-1, l]);
   end do;
   N[i, d, 0] := simplify(c[i, d, 0]-c[i, 1, 0]*c[i, d-1, 0]);
   N[i, d, 1] := simplify(-c[i, 0, 1]*c[i, d, 0]+c[i, d, 1]
                          -c[i, 1, 0]*c[i, d-1, 1]);
   N[i, d, 2*i-2*d+2] := simplify((c[i, 0, 1]^2-c[i, 0, 2])
                            *c[i, d, 2*i-2*d]
                            -c[i, 0, 1]*c[i, d, 2*i-2*d+1]
                            -c[i, 1, 0]*c[i, d-1, 2*i-2*d+2]);
   N[i, d, 2*i-2*d+3] := simplify((c[i, 0, 1]^2-c[i, 0, 2])
                               *c[i, d, 2*i-2*d+1]
                               -c[i, 1, 0]*c[i, d-1, 2*i-2*d+3]);
end do;
for l from 0 to 1 do
   N[i, i+1, l] := simplify(-c[i, 1, 0]*c[i, i, l]);
end do;
# Expand A^{2n+3}{A^*}^{n+1}
t[1] := simplify(sum(N[i, 0, 2*h+1]*A[1]^(2*i+2-2*h)*A[2]^i
                  *(sum(eta[2*h+1, k, 2]*A[2]*A[3]^(2*k+1)
                       +sum(eta[2*h+1, k, j]*A[3]^(2-j)
```

```
                                *A[4]*A[5]^(2*k-1+j),
                                j = 0 .. 1), k = 1 .. h)
                     +rho^h*(A[3]*A[4]-A[2]*A[3])), h = 1 .. i+1)
                -(sum(N[i, 0, 2*h+2]*A[1]^(2*i-2*h+1)*A[2]^i
                     *(sum(eta[2*h+2, k, 2]*A[2]*A[3]^(2*k+2)
                          +sum(eta[2*h+2, k, j]*A[3]^(2-j)*A[4]*A[5]^(2*k+j),
                          j = 0 .. 1), k = 0 .. h)), h = 1 .. i))
                -(sum(rho^p*(N[i, p, 0]*A[1]^(2*i-2*p+3)*A[2]^(i+1)
                     -N[i, p, 1]*A[1]^(2*i-2*p+2)*A[2]^i*A[3]*A[4]
                     +N[i, p, 2]*A[1]^(2*i-2*p+1)*A[2]^i
                             *A[3]^2*A[4]), p = 1 .. i))
                +sum(rho^p*(sum(N[i, p, 2*h+1]*A[1]^(2*i-2*p-2*h+2)*A[2]^i
                               *(sum(eta[2*h+1, k, 2]*A[2]*A[3]^(2*k+1)
                                    +sum(eta[2*h+1, k, j]*A[3]^(2-j)
                                         *A[4]*A[5]^(2*k-1+j),
                                         j = 0 .. 1), k = 1 .. h)
                          +rho^h*(A[3]*A[4]-A[2]*A[3])), h = 1 .. i-p+1)),
                                                      p = 1 .. i)
                -(sum(rho^p*(sum(N[i, p, 2*h+2]*A[1]^(2*i-2*p-2*h+1)*A[2]^i
                                *(sum(eta[2*h+2, k, 2]*A[2]*A[3]^(2*k+2)
                                     +sum(eta[2*h+2, k, j]*A[3]^(2-j)
                                     *A[4]*A[5]^(2*k+j),j = 0 .. 1), k = 0 .. h)),
                                      h = 1 .. i-p)), p = 1 .. i-1))
                -rho^(i+1)*(N[i, i+1, 0]*A[1]*A[2]^(i+1)
                            -N[i, i+1, 1]*A[2]^i*A[3]*A[4]));
# Expand A^{2n+1}{A^*}^{n+1}A^2
t[3] := simplify(c[i, 0, 1]*A[1]^(2*i)*A[2]^i*A[3]*A[4]*A[5]^2
               -c[i, 0, 2]*A[1]^(2*i-1)*A[2]^i*A[3]^2*A[4]*A[5]^2
               -(sum(rho^p*(c[i, p, 0]*A[1]^(2*i-2*p+1)*A[2]^(i+1)*A[3]^2
                    -c[i, p, 1]*A[1]^(2*i-2*p)*A[2]^i*A[3]*A[4]*A[5]^2
                    +c[i, p, 2]*A[1]^(2*i-2*p-1)*A[2]^i*A[3]^2*A[4]*A[5]^2),
                  p = 1 .. i-1))
               -rho^i*(c[i, i, 0]*A[1]*A[2]^(i+1)*A[3]^2
                      -c[i, i, 1]*A[2]^i*A[3]*A[4]*A[5]^2)
               +sum(c[i, 0, 2*h+1]*A[1]^(2*i-2*h)*A[2]^i
                    *(sum(eta[2*h+1, k, 2]*A[2]*A[3]^(2*k+3)
                          +sum(eta[2*h+1, k, j]*A[3]^(2-j)*A[4]*A[5]^(2*k+j+1),
                      j = 0 .. 1), k = 1 .. h)
                      +rho^h*(A[3]*A[4]*A[5]^2-A[2]*A[3]^3)), h = 1 .. i)
               -(sum(c[i, 0, 2*h+2]*A[1]^(2*i-2*h-1)*A[2]^i
                    *(sum(eta[2*h+2, k, 2]*A[2]*A[3]^(2*k+4)
                          +sum(eta[2*h+2, k, j]*A[3]^(2-j)*A[4]*A[5]^(2*k+j+2),
                  j = 0 .. 1), k = 0 .. h)), h = 1 .. i-1))
               +sum(rho^p*(sum(c[i, p, 2*h+1]*A[1]^(2*i-2*p-2*h)*A[2]^i
                              *(sum(eta[2*h+1, k, 2]*A[2]*A[3]^(2*k+3)
```

```
                                        +sum(eta[2*h+1, k, j]*A[3]^(2-j)*A[4]
                                        *A[5]^(2*k+j+1),
                                    j = 0 .. 1), k = 1 .. h)
                    +rho^h*(A[3]*A[4]*A[5]^2-A[2]*A[3]^3)),

                      h = 1 .. i-p)),p = 1 .. i-1));
# Expand A^{2n+1}{A^*}^{n+1}
t[4] := c[i, 0, 1]*A[1]^(2*i)*A[2]^i*A[3]*A[4]
        -c[i, 0, 2]*A[1]^(2*i-1)*A[2]^i*A[3]^2*A[4]
    +sum(c[i, 0, 2*k+1]*A[1]^(2*i-2*k)*A[2]^i*(sum(eta[2*k+1, h, 2]
            *A[2]*A[3]^(2*h+1)
        +sum(eta[2*k+1, h, j]*A[3]^(2-j)*A[4]*A[5]^(2*h-1+j),
           j = 0 .. 1), h = 1 .. k)
        +rho^k*(A[3]*A[4]-A[2]*A[3])), k = 1 .. i)
    -(sum(c[i, 0, 2*k+2]*A[1]^(2*i-2*k-1)*A[2]^i
          *(sum(eta[2*k+2, h, 2]*A[2]*A[3]^(2*h+2)
                +sum(eta[2*k+2, h, j]*A[3]^(2-j)
                   *A[4]*A[5]^(2*h+j), j = 0 .. 1),
            h = 0 .. k)), k = 1 .. i-1))
    -rho^i*(c[i, i, 0]*A[1]*A[2]^(i+1)-c[i, i, 1]*A[2]^i*A[3]*A[4])
    -(sum(rho^p*(c[i, p, 0]*A[1]^(2*i-2*p+1)*A[2]^(i+1)
         -c[i, p, 1]*A[1]^(2*i-2*p)*A[2]^i*A[3]*A[4]
         +c[i, p, 2]*A[1]^(2*i-2*p-1)*A[2]^i*A[3]^2*A[4]),
               p = 1 .. i-1))
    +sum(rho^p*(sum(c[i, p, 2*k+1]*A[1]^(2*i-2*p-2*k)*A[2]^i
          *(sum(eta[2*k+1, h, 2]*A[2]*A[3]^(2*h+1)
          +sum(eta[2*k+1, h, j]*A[3]^(2-j)*A[4]*A[5]^(2*h-1+j),
               j = 0 .. 1),h = 1 .. k)
              +rho^k*(A[3]*A[4]-A[2]*A[3])),
               k = 1 .. i-p)), p = 1 .. i-1);
if 2 < i then
   t[4] := t[4]-(sum(rho^p*(sum(c[i, p, 2*k+2]
                     *A[1]^(2*i-2*p-2*k-1)*A[2]^i
                  *(sum(eta[2*k+2, h, 2]*A[2]*A[3]^(2*h+2)
                     +sum(eta[2*k+2, h, j]*A[3]^(2-j)*A[4]*A[5]^(2*h+j),
                     j = 0 .. 1),h = 0 .. k)),
                     k = 1 .. i-p-1)), p = 1 .. i-2));
   t[3] := t[3]-(sum(rho^p*(sum(c[i, p, 2*h+2]*A[1]^(2*i-2*p-2*h-1)
                   *A[2]^i*(sum(eta[2*h+2, k, 2]*A[2]*A[3]^(2*k+4)
                     +sum(eta[2*h+2, k, j]*A[3]^(2-j)*A[4]*A[5]^(2*k+j+2),
                     j = 0 .. 1),k = 0 .. h)),
                     h = 1 .. i-p-1)), p = 1 .. i-2));
end if;
t[3] := simplify(t[3]);
t[4] := simplify(t[4]);
```

```
        c[i+1, 0, 0] := 1;
        c[i+1, 0, 1] := expand(simplify((q^(2*i+3)-q^(-2*i-3))/(q-1/q)));
        c[i+1, 0, 2] := expand(simplify((q^(2*i+3)-q^(-2*i-3))*
                        (q^(2*i+2)-q^(-2*i-2))/((q-1/q)*(q^2-1/q^2))));
        c[i+1, 1, 0] := expand(simplify(-c[i, 0, 1]^2+2*c[i, 0, 2]
                            -c[i, 0, 3]/c[i, 0, 1]-c[i, 1, 1]/c[i, 0, 1]
                            +2*c[i, 1, 0]));
        # Compute the higher order q-Dolan-Grady relations F[i+1]
        F[i+1] := expand(simplify(c[i+1, 0, 0]*t[1]-c[i+1, 0, 1]*t[2]
                               +c[i+1, 0, 2]*t[3]+rho*c[i+1, 1, 0]*t[4]));
        # Rewrite F[i+1] in the distributed form of A[1], A[2], A[3] and rho
        F[i+1] := collect(F[i+1], [A[1], A[2], A[3], rho], distributed);
        # Extract the coefficient in A[1]and A[3] in the sum of
        # all elements containing A[2]^{i+1} of F[i+1]
        tam := coeff(F[i+1], A[2], i+1);
        for l from 3 to 2*i+3 do
           # Extract the coefficient in A[1] in the sum of
           # all elements containing A[3]^l of tam
           ta1[l] := coeff(tam, A[3], l);
           # Extract the coefficient of A[1]^{2*i+3-l} in ta1
           ta1[l] := coeff(ta1[l], A[1], 2*i+3-l);
           # Define the coefficient c[i+1, 0, l]
           c[i+1, 0, l] := ta1[l]*(-1)^(l+1);
        end do;
        for l to 2*i+1 do
           # Extract the coefficient in A[3] in the sum
           # of all elements containing A[1]^{2*i+1-l}
           ta2[l] := coeff(tam, A[3], l);
           ta2[l] := coeff(ta2[l], A[1], 2*i+1-l);
           # Define the coefficient c[i+1, 1, l]
           c[i+1, 1, l] := ta2[l]*(-1)^(l+1)/rho;
        end do;
        for d from 2 to i+1 do
           for l from 0 to 2*i-2*d+3 do
              ta3[l] := coeff(tam, A[3], l);
              ta3[l] := coeff(ta3[l], A[1], 2*i-2*d+3-l);
              c[i+1, d, l] := ta3[l]*(-1)^(l+1)*rho^(-d);
           end do
        end do
     end do
  end if;
  F[n+1]
end proc;

# Compute the coefficients c[n+1, p, j]
Result2 := proc (n)
```

```
local i, p, d, l, M, N;
global A, c, alpha, q;
# Input the innitial values
alpha := q^2+1+1/q^2;
c[1, 0, 0] := 1;
c[1, 0, 3] := 1;
c[1, 0, 1] := alpha;
c[1, 0, 2] := alpha;
c[1, 1, 0] := 1;
c[1, 1, 1] := 1;
c[2, 0, 0] := 1;
c[2, 0, 5] := 1;
c[2, 0, 1] := expand(simplify(alpha^2-alpha-1));
c[2, 0, 4] := expand(simplify(alpha^2-alpha-1));
c[2, 0, 2] := expand(simplify(alpha^3-2*alpha^2+1));
c[2, 0, 3] := expand(simplify(alpha^3-2*alpha^2+1));
c[2, 1, 0] := expand(simplify(alpha^2-2*alpha+2));
c[2, 1, 3] := expand(simplify(alpha^2-2*alpha+2));
c[2, 1, 1] := expand(simplify(alpha*(alpha^2-alpha-1)));
c[2, 1, 2] := expand(simplify(alpha*(alpha^2-alpha-1)));
c[2, 2, 0] := expand(simplify((alpha-1)^2));
c[2, 2, 1] := expand(simplify((alpha-1)^2));
if 1 < n then
   for i from 2 to n do
      # Compute M[n, p, j], p = 0,..., n; j= 2,..., 2n+2 if p=0,
      #                     j = 0,..., 2(n-p)+2 if p = 1,...,n
      M[i, 0, 2*i+2] := simplify(-c[i, 0, 1]*c[i, 0, 2*i+1]);
      for l from 2 to 2*i+1 do
         M[i, 0, l] := simplify(c[i, 0, l]-c[i, 0, 1]*c[i, 0, l-1]);
      end do;
      for d from 1 to i do
         M[i, d, 0] := c[i, d, 0];
            for l from 1 to 2*i-2*d+1 do
               M[i, d, l] := simplify(c[i, d, l]-c[i, 0, 1]*c[i, d, l-1]);
            end do;
         M[i, d, 2*i-2*d+2] := simplify(-c[i, 0, 1]*c[i, d, 2*i-2*d+1]);
      end do;
      Funct(i);
      # Compute N[n, p, j], p = 0,...,n;
      #           j = 3,...,2n+2 if p =0
      #           j = 0,...,2(n-p)+3 if p = 1,...,n
      N[i, 0, 2*i+2] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 0, 2*i]
                                 -c[i, 0, 1]*c[i, 0, 2*i+1]);
      N[i, 0, 2*i+3] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 0, 2*i+1]);
      for l from 3 to 2*i+1 do
```

```
   N[i, 0, l] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 0, l-2]
                          -c[i, 0, 1]*c[i, 0, l-1]+c[i, 0, l]);
end do;
N[i, 1, 1] := simplify(-2*c[i, 0, 1]*c[i, 1, 0]+c[i, 1, 1]);
N[i, 1, 2*i] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 1, 2*i-2]
                          -c[i, 0, 1]*c[i, 1, 2*i-1]
                          -c[i, 1, 0]*c[i, 0, 2*i]);
N[i, 1, 2*i+1] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 1, 2*i-1]
                           -c[i, 1, 0]*c[i, 0, 2*i+1]);
N[i, 1, 0] := 0;
for l from 2 to 2*i-1 do
   N[i, 1, l] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, 1, l-2]
                          -c[i, 0, 1]*c[i, 1, l-1]+c[i, 1, l]
                          -c[i, 1, 0]*c[i, 0, l]);
end do;
for d from 2 to i do
   for l from 2 to 2*i-2*d+1 do
      N[i, d, l] := simplify((c[i, 0, 1]^2-c[i, 0, 2])*c[i, d, l-2]
                             -c[i, 0, 1]*c[i, d, l-1]+c[i, d, l]
                             -c[i, 1, 0]*c[i, d-1, l]);
   end do;
   N[i, d, 0] := simplify(c[i, d, 0]-c[i, 1, 0]*c[i, d-1, 0]);
   N[i, d, 1] := simplify(-c[i, 0, 1]*c[i, d, 0]+c[i, d, 1]
                          -c[i, 1, 0]*c[i, d-1, 1]);
   N[i, d, 2*i-2*d+2] := simplify((c[i, 0, 1]^2-c[i, 0, 2])
                                  *c[i, d, 2*i-2*d]
                                  -c[i, 0, 1]*c[i, d, 2*i-2*d+1]
                                  -c[i, 1, 0]*c[i, d-1, 2*i-2*d+2]);
   N[i, d, 2*i-2*d+3] := simplify((c[i, 0, 1]^2-c[i, 0, 2])
                                  *c[i, d, 2*i-2*d+1]
                                  -c[i, 1, 0]*c[i, d-1, 2*i-2*d+3])
end do;
for l from 0 to 1 do
   N[i, i+1, l] := simplify(-c[i, 1, 0]*c[i, i, l]);
end do;
# Compute the coefficients c[n+1, p, j], p =0,...,n+1; j = 0,...,2n+3-2p.
c[i+1, 0, 0] := 1;
c[i+1, 0, 1] := expand(simplify((q^(2*i+3)-q^(-2*i-3))/(q-1/q)));
c[i+1, 0, 2] := expand(simplify((q^(2*i+3)-q^(-2*i-3))
                *(q^(2*i+2)-q^(-2*i-2))/((q^2-1/q^2)*(q-1/q))));
c[i+1, 0, 3] := expand(simplify(N[i, 0, 3]*eta[3, 1, 2]));
c[i+1, 0, 4] := expand(simplify(N[i, 0, 4]*eta[4, 1, 2]
                +c[i+1, 0, 1]*M[i, 0, 3]*eta[3, 1, 2]));
for l from 2 to i+1 do
   c[i+1, 0, 2*l+1] := expand(simplify(N[i, 0, 2*l+1]*eta[2*l+1, l, 2]
```

```
                      +c[i+1, 0, 1]*M[i, 0, 2*l]*eta[2*l, l-1, 2]
                      +c[i+1, 0, 2]*c[i, 0, 2*l-1]*eta[2*l-1, l-1, 2]));
end do;
for l from 2 to i do
   c[i+1, 0, 2*l+2] := expand(simplify(N[i, 0, 2*l+2]*eta[2*l+2, l, 2]
                      +c[i+1, 0, 1]*M[i, 0, 2*l+1]*eta[2*l+1, l, 2]
                      +c[i+1, 0, 2]*c[i, 0, 2*l]*eta[2*l, l-1, 2]));
end do;
c[i+1, 1, 0] := expand(simplify(c[i, 0, 1]^2-2*c[i, 0, 2]
                +c[i, 0, 3]/c[i, 0, 1]
                -c[i, 1, 1]/c[i, 0, 1]+2*c[i, 1, 0]));
c[i+1, i+1, 0] := expand(simplify(N[i, i+1, 0]+c[i+1, 1, 0]
                                  *c[i, i, 0]));
for d from 2 to i do
   c[i+1, d, 0] := expand(simplify(N[i, d, 0]+c[i+1, 1, 0]
                                  *c[i, d-1, 0]));
end do;
c[i+1, 1, 1] := expand(simplify(N[i, 0, 3]+c[i+1, 0, 1]*M[i, 1, 0]));
c[i+1, 2, 1] := expand(simplify(-N[i, 0, 5]+N[i, 1, 3]
                +c[i+1, 1, 0]*c[i, 0, 3]+c[i+1, 0, 1]*M[i, 2, 0]));
c[i+1, i+1, 1] := expand(simplify((-1)^i*(N[i, 0, 2*i+3]
                +sum((-1)^p*N[i, p, 2*i-2*p+3], p = 1 .. i))
                +(-1)^(i+1)*c[i+1, 1, 0]*(c[i, 0, 2*i+1]
                +sum((-1)^p*c[i, p, 2*i-2*p+1], p = 1 .. i-1))));
if 2 < i then
   for d from 3 to i do
      c[i+1, d, 1] := expand(simplify((-1)^(d+1)*(N[i, 0, 2*d+1]
                +sum((-1)^j*N[i, j, 2*d-2*j+1], j = 1 .. d-1))
                +c[i+1, 0, 1]*M[i, d, 0]+(-1)^d
                *c[i+1, 1, 0]*(c[i, 0, 2*d-1]
                +sum((-1)^j*c[i, j, 2*d-2*j-1], j = 1 .. d-2))));
      c[i+1, d, 2] := expand(simplify((-1)^d*(eta[2*d+2, 0, 2]
                       *N[i, 0, 2*d+2]
                +sum((-1)^j*N[i, j, 2*d-2*j+2]*eta[2*d-2*j+2, 0, 2],
                    j = 1 .. d-1))
                +(-1)^(d+1)*c[i+1, 0, 1]*(M[i, 0, 2*d+1]
                +sum((-1)^j*M[i, j, 2*d-2*j+1], j = 1 .. d-1))
                      +c[i+1, 0, 2]*c[i, d, 0]
                +(-1)^(d+1)*c[i+1, 1, 0]*(c[i, 0, 2*d]*eta[2*d, 0, 2]
                +sum((-1)^j*c[i, j, 2*d-2*j]*eta[2*d-2*j, 0, 2],
                      j = 1 .. d-2))));
   end do;
   for d from 2 to i-1 do
      c[i+1, d, 4] := expand(simplify((-1)^d*(N[i, 0, 2*d+4]
                       *eta[2*d+4, 1, 2]
```

```
          +sum((-1)^j*N[i, j, 2*d-2*j+4]*eta[2*d-2*j+4, 1, 2], j = 1 .. d))
          +(-1)^d*c[i+1, 0, 2]*(c[i, 0, 2*d+2]*eta[2*d+2, 0, 2]
          +sum((-1)^j*c[i, j, 2*d-2*j+2]*eta[2*d-2*j+2, 0, 2], j = 1 .. d-1))
          +(-1)^d*c[i+1, 0, 1]*(M[i, 0, 2*d+3]*eta[2*d+3, 1, 2]
          +sum((-1)^j*M[i, j, 2*d-2*j+3]*eta[2*d-2*j+3, 1, 2], j = 1 .. d))
          +(-1)^(d+1)*c[i+1, 1, 0]*(c[i, 0, 2*d+2]*eta[2*d+2, 1, 2]
          +sum((-1)^j*c[i, j, 2*d-2*j+2]*eta[2*d-2*j+2, 1, 2], j = 1 .. d-1))));
      end do;
      for d from 3 to i do
         for l to d-2 do
            c[i+1, d-l, 2*l+3] := expand(simplify((-1)^(d+l)
                                          *(N[i, 0, 2*d+3]*eta[2*d+3, l+1, 2]
                     +sum((-1)^j*N[i, j, 2*d-2*j+3]*eta[2*d-2*j+3, l+1, 2],
                            j = 1 .. d-l))
                     +(-1)^(d+l)*c[i+1, 0, 2]*(c[i, 0, 2*d+1]*eta[2*d+1, l, 2]
                     +sum((-1)^j*c[i, j, 2*d-2*j+1]*eta[2*d-2*j+1, l, 2],
                            j = 1 .. d-l))
                     +(-1)^(d+l)*c[i+1, 0, 1]*(eta[2*d+2, l, 2]*M[i, 0, 2*d+2]
                     +sum((-1)^j*M[i, j, 2*d-2*j+2]*eta[2*d-2*j+2, l, 2],
                            j = 1 .. d-l))
                     +(-1)^(d+l+1)*c[i+1, 1, 0]*(c[i, 0, 2*d+1]*eta[2*d+1, l+1, 2]
                     +sum((-1)^j*c[i, j, 2*d-2*j+1]*eta[2*d-2*j+1, l+1, 2],
                             j = 1 .. d-l-1))));
         end do
      end do;
      for l from 3 to i do
         c[i+1, 1, 2*l] := expand(simplify(c[i+1, 1, 0]
                              *c[i, 0, 2*l]*eta[2*l, l-1, 2]
                    -N[i, 0, 2*l+2]*eta[2*l+2, l-1, 2]
                    +N[i, 1, 2*l]*eta[2*l, l-1, 2]
                    -c[i+1, 0, 1]*(M[i, 0, 2*l+1]*eta[2*l+1, l-1, 2]
                    -M[i, 1, 2*l-1]*eta[2*l-1, l-1, 2])
                    -c[i+1, 0, 2]*(c[i, 0, 2*l]*eta[2*l, l-2, 2]
                    -c[i, 1, 2*l-2]*eta[2*l-2, l-2, 2])));
      end do
   end if;
   for d from 2 to i do
      c[i+1, d, 3] := expand(simplify((-1)^d*c[i+1, 0, 1]
                         *(eta[2*d+2, 0, 2]*M[i, 0, 2*d+2]
                       +sum((-1)^j*M[i, j, 2*d-2*j+2]*eta[2*d-2*j+2, 0, 2],
                            j = 1 .. d-1))
                       +(-1)^(d+1)*c[i+1, 0, 2]*(c[i, 0, 2*d+1]
                       +sum((-1)^j*c[i, j, 2*d-2*j+1], j = 1 .. d-1))
                       +(-1)^d*(N[i, 0, 2*d+3]*eta[2*d+3, 1, 2]
                       +sum((-1)^j*N[i, j, 2*d-2*j+3]*eta[2*d-2*j+3, 1, 2],
```

```
                        j = 1 .. d))
                    +(-1)^(d+1)*c[i+1, 1, 0]*(c[i, 0, 2*d+1]*eta[2*d+1, 1, 2]
                    +sum((-1)^j*c[i, j, 2*d-2*j+1]*eta[2*d-2*j+1, 1, 2],
                        j = 1 .. d-1))));
end do;
for l from 2 to i do
   c[i+1, 1, 2*l+1] := expand(simplify(-N[i, 0, 2*l+3]*eta[2*l+3, l, 2]
                    +N[i, 1, 2*l+1]*eta[2*l+1, l, 2]
                    -c[i+1, 0, 2]*(c[i, 0, 2*l+1]*eta[2*l+1, l-1, 2]
                    -c[i, 1, 2*l-1]*eta[2*l-1, l-1, 2])
                    -c[i+1, 0, 1]*(eta[2*l+2, l-1, 2]*M[i, 0, 2*l+2]
                    -M[i, 1, 2*l]*eta[2*l, l-1, 2])
                    +c[i+1, 1, 0]*c[i, 0, 2*l+1]*eta[2*l+1, l, 2]));
end do;
if 3 < i then
   for d from 4 to i do
      for l from 2 to d-2 do
         c[i+1, d-l, 2*l+2] := expand(simplify((-1)^(d+l)*(N[i, 0, 2*d+2]
                                 *eta[2*d+2, l, 2]
               +sum((-1)^j*N[i, j, 2*d-2*j+2]*eta[2*d-2*j+2, l, 2],
                    j = 1 .. d-l))
               +(-1)^(d+l)*c[i+1, 0, 1]*(M[i, 0, 2*d+1]*eta[2*d+1, l, 2]
               +sum((-1)^j*M[i, j, 2*d-2*j+1]*eta[2*d-2*j+1, l, 2],
                    j = 1 .. d-l))
               +(-1)^(d+l)*c[i+1, 0, 2]*(c[i, 0, 2*d]*eta[2*d, l-1, 2]
               +sum((-1)^j*c[i, j, 2*d-2*j]*eta[2*d-2*j, l-1, 2],
                    j = 1 .. d-l))
               +(-1)^(d+l+1)*c[i+1, 1, 0]*(c[i, 0, 2*d]*eta[2*d, l, 2]
               +sum((-1)^j*c[i, j, 2*d-2*j]*eta[2*d-2*j, l, 2],
                    j = 1 .. d-l-1))));
      end do
   end do
end if;
c[i+1, 1, 4] := expand(simplify(-N[i, 0, 6]*eta[6, 1, 2]
               +N[i, 1, 4]*eta[4, 1, 2]
               -eta[4, 0, 2]*c[i, 0, 4]*c[i+1, 0, 2]-c[i+1, 0, 1]
                  *(M[i, 0, 5]*eta[5, 1, 2]
               -M[i, 1, 3]*eta[3, 1, 2])+c[i+1, 1, 0]*c[i, 0, 4]
                  *eta[4, 1, 2]));
c[i+1, 1, 3] := expand(simplify(-c[i+1, 0, 1]*eta[4, 0, 2]*M[i, 0, 4]
               +c[i+1, 1, 0]*c[i, 0, 3]*eta[3, 1, 2]
               +c[i+1, 0, 2]*c[i, 0, 3]
               -N[i, 0, 5]*eta[5, 1, 2]+N[i, 1, 3]*eta[3, 1, 2]));
c[i+1, 1, 2] := expand(simplify(-eta[4, 0, 2]*N[i, 0, 4]
               +c[i+1, 0, 1]*M[i, 0, 3]
```

```
                        +c[i+1, 0, 2]*c[i, 1, 0]));
        c[i+1, 2, 2] := -expand(simplify(-eta[6, 0, 2]*N[i, 0, 6]
                        +N[i, 1, 4]*eta[4, 0, 2]
                        -c[i+1, 0, 1]*(-M[i, 0, 5]+M[i, 1, 3])
                        -c[i+1, 0, 2]*c[i, 2, 0]
                        +c[i+1, 1, 0]*c[i, 0, 4]*eta[4, 0, 2]));
     end do
  end if;
  # Display the coefficients c[n+1, p, j] = coef[n+1, p, j]
  for d from 0 to n+1 do
     for l from 0 to 2*n+3-2*d do
        print(coef[n+1, d, l], "=", c[n+1, d, l]);
     end do
  end do
end proc;
```

The coefficients c[n+1, p, j] which are obtained from the two programming Result1 and Result2 completely coincide together.

5.5 APPENDIX E: Algorithms for the generalized q-Onsager algebra

```
# Compute the coefficients eta[m,k,j] of A^{2n+1}A^* and A^{2n+2}A^*
Funct := proc (n)
  local p, i, j;
  global eta;
  # Input the initial values
  eta[2, 0, 0] := q+1/q;
  eta[2, 0, 1] := -1;
  for i to n do
     for p from 0 to i-1 do
        eta[2*i+1, p, 0] := expand(simplify(eta[2*i, p, 0]*(q+1/q)
                                 +eta[2*i, p, 1]));
     end do;
     eta[2*i+1, i, 0] := 1;
     for p to i-1 do
        eta[2*i+1, p, 1] := expand(simplify(-eta[2*i, p, 0]
                                 +eta[2*i, p-1, 0]));
     end do;
     eta[2*i+1, 0, 1] := expand(simplify(-eta[2*i, 0, 0]));
     eta[2*i+1, i, 1] := expand(simplify(eta[2*i, i-1, 0]));
     for p from 0 to i do
        eta[2*i+2, p, 0] := expand(simplify(eta[2*i+1, p, 0]*(q+1/q)
                                              +eta[2*i+1, p, 1]));
     end do;
     for p to i do
       eta[2*i+2, p, 1] := expand(simplify(eta[2*i+1, p-1, 0]
                                            -eta[2*i+1, p, 0]));
     end do;
     eta[2*i+2, 0, 1] := expand(simplify(-eta[2*i+1, 0, 0]))
  end do
end proc;

# Compute A^{r+2}{A^*}^{r+1}-c[r+1,0,1]A^{r+1}{A^*}^{r+1}A
# and output the higher order relations of the generalized q-Onsager algebra
Result1 := proc (r)
  local k, i, t, p, F, f, tam, ta2, ta1, B;
  global M, c, A, eta;
  # Input the initial values
  c[1, 0, 0] := 1;
  c[1, 0, 1] := q+1/q;
  c[1, 0, 2] := 1;
  c[1, 1, 0] := 1;
  c[2, 0, 0] := 1;
```

```
c[2, 0, 1] := q^2+1+1/q^2;
c[2, 0, 2] := q^2+1+1/q^2;
c[2, 0, 3] := 1;
c[2, 1, 0] := q^2+2+1/q^2;
c[2, 1, 1] := q^2+2+1/q^2;
c[3, 0, 0] := 1;
c[3, 0, 1] := expand((q^4-1/q^4)/(q-1/q));
c[3, 0, 2] := expand((q^2+1/q^2)*(q^2+1+1/q^2));
c[3, 0, 3] := expand((q^4-1/q^4)/(q-1/q));
c[3, 0, 4] := 1;
c[3, 1, 0] := q^4+2*q^2+4+2/q^2+1/q^4;
c[3, 1, 1] := expand((q+1/q)*(q^2+1/q^2+3)*(q^2+1/q^2));
c[3, 1, 2] := q^4+2*q^2+4+2/q^2+1/q^4;
c[3, 2, 0] := expand(q^2+1+1/q^2);
if r = 0 then
   F[1] := A[1]^2*A[2]-(q+1/q)*A[1]*A[2]*A[3]+A[2]*A[3]^2-rho*A[2] = 0
end if;
if r = 1 then
   F[2] := A[1]^3*A[2]^2-(q^2+1+1/q^2)*A[1]^2*A[2]^2*A[3]
          +(q^2+1+1/q^2)*A[1]*A[2]^2*A[3]^2
          -A[2]^2*A[3]^3-rho*(q^2+2+1/q^2)*A[1]*A[2]^2
          +rho*(q^2+2+1/q^2)*A[2]^2*A[3] = 0
end if;
if 1 < r then

   for i from 2 to r do
      t := trunc((1/2)*i);
      # Compute the coefficients M[r,p,j]
      for k from 2 to i+1 do
         M[i, 0, k] := -c[i, 0, 1]*c[i, 0, k-1]+c[i, 0, k];
      end do;
      M[i, 0, i+2] := -c[i, 0, 1]*c[i, 0, i+1];
      if i = 2*t then
         for p to t do
            for k to i-2*p+1 do
               M[i, p, k] := -c[i, 0, 1]*c[i, p, k-1]+c[i, p, k];
            end do;
            M[i, p, 0] := c[i, p, 0];
            M[i, p, i+2-2*p] := -c[i, 0, 1]*c[i, p, i-2*p+1];
         end do
      else
         for p to t do
            M[i, p, 0] := c[i, p, 0];
            M[i, p, i+2-2*p] := -c[i, 0, 1]*c[i, p, i-2*p+1];
            for k to i-2*p+1 do
```

```
            M[i, p, k] := -c[i, 0, 1]*c[i, p, k-1]+c[i, p, k];
         end do
      end do;
      M[i, t+1, 0] := c[i, t+1, 0];
      M[i, t+1, 1] := -c[i, t+1, 0]*c[i, 0, 1];
   end if;
   Funct(i);
   if i = 2*t then
      # Expand A^{2r+2}{A^*}^{r+1}A
      f[1] := simplify(-(sum(M[2*t, 0, 2*h]*A[1]^(2*t+2-2*h)*A[2]^(2*t)
                 *(rho^h*A[2]+sum(rho^l*(eta[2*h, l, 0]
                   *A[3]*A[4]*A[5]^(2*h-1-2*l)
                    +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l)),
                    l = 0 .. h-1)), h = 1 .. t+1))
            +sum(M[2*t, 0, 2*h+1]*A[1]^(2*t+1-2*h)*A[2]^(2*t)
                 *(sum(rho^l*(eta[2*h+1, l, 0]*A[3]*A[4]*A[5]^(2*h-2*l)
                    +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
                    l = 0 .. h)), h = 1 .. t)
            -(sum(rho^d*((-1)^d*M[2*t, d, 0]*A[1]^(2*t+2-2*d)
               *A[2]^(2*t+1)+(-1)^(d+1)
            *M[2*t, d, 1]*A[1]^(2*t+1-2*d)*A[2]^(2*t)*A[3]*A[4]),
               d = 1 .. t))
            -(sum(rho^d*(sum((-1)^d*M[2*t, d, 2*h]
                  *A[1]^(2*t+2-2*d-2*h)*A[2]^(2*t)
                      *(rho^h*A[2]+sum(rho^l*(eta[2*h, l, 0]
                                *A[3]*A[4]*A[5]^(2*h-1-2*l)
                      +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l)),
                 l = 0 .. h-1)), h = 1 .. t+1-d)), d = 1 .. t))
            -(sum(rho^d*(sum((-1)^(d+1)*M[2*t, d, 2*h+1]
                     *A[1]^(2*t+1-2*d-2*h)
                 *A[2]^(2*t)*(sum(rho^l*(eta[2*h+1, l, 0]
                     *A[3]*A[4]*A[5]^(2*h-2*l)
                     +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
             l = 0 .. h)), h = 1 .. t-d)), d = 1 .. t)));
      # Expand A^{2r+1}{A^*}^{r+1}A^2
      f[2] := simplify(c[2*t, 0, 1]*A[1]^(2*t)*A[2]^(2*t)*A[3]*A[4]*A[5]
           -(sum(c[2*t, 0, 2*h]*A[1]^(2*t+1-2*h)*A[2]^(2*t)
                *(rho^h*A[2]*A[3]
               +sum(rho^l*(eta[2*h, l, 0]*A[3]*A[4]*A[5]^(2*h-2*l)
                  +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
                    l = 0 .. h-1)), h = 1 .. t))
           +sum(c[2*t, 0, 2*h+1]*A[1]^(2*t-2*h)*A[2]^(2*t)
               *(sum(rho^l*(eta[2*h+1, l, 0]*A[3]*A[4]*A[5]^(2*h-2*l+1)
                    +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+2)),
                       l = 0 .. h)), h = 1 .. t)
```

```
            -(sum(rho^d*((-1)^d*c[2*t, d, 0]*A[1]^(2*t+1-2*d)
               *A[2]^(2*t+1)*A[3]+(-1)^(d+1)
                 *c[2*t, d, 1]*A[1]^(2*t-2*d)*A[2]^(2*t)
                 *A[3]*A[4]*A[5]), d = 1 .. t)));
      if 2 <= t then
         f[2] := simplify(f[2]-(sum(rho^d*(sum((-1)^d*c[2*t, d, 2*h]
                        *A[1]^(2*t+1-2*d-2*h)*A[2]^(2*t)*(rho^h*A[2]*A[3]
                              +sum(rho^l*(eta[2*h, l, 0]*A[3]
                                  *A[4]*A[5]^(2*h-2*l)
                                +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
                                l = 0 .. h-1)),
                         h = 1 .. t-d)), d = 1 .. t-1))
                     -(sum(rho^d*(sum((-1)^(d+1)*c[2*t, d, 2*h+1]
                           *A[1]^(2*t-2*d-2*h)
                        *A[2]^(2*t)*(sum(rho^l*(eta[2*h+1, l, 0]
                              *A[3]*A[4]*A[5]^(2*h-2*l+1)
                         +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+2)),
                     l = 0 .. h)),h = 1 .. t-d)), d = 1 .. t-1)));
      end if
   else
      f[1] := simplify(-(sum(M[2*t+1, 0, 2*h]*A[1]^(2*t+3-2*h)
                  *A[2]^(2*t+1)*(rho^h*A[2]
                     +sum(rho^l*(eta[2*h, l, 0]*A[3]*A[4]*A[5]^(2*h-1-2*l)
                        +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l)),
                        l = 0 .. h-1)),h = 1 .. t+1))
                  +sum(M[2*t+1, 0, 2*h+1]*A[1]^(2*t+2-2*h)*A[2]^(2*t+1)
                    *(sum(rho^l*(eta[2*h+1, l, 0]*A[3]*A[4]*A[5]^(2*h-2*l)
                     +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
                     l = 0 .. h)), h = 1 .. t+1)
                  -(sum(rho^d*((-1)^d*M[2*t+1, d, 0]*A[1]^(2*t+3-2*d)
                       *A[2]^(2*t+2)
                       +(-1)^(d+1)*M[2*t+1, d, 1]*A[1]^(2*t+2-2*d)
                       *A[2]^(2*t+1)*A[3]*A[4]), d = 1 .. t+1))
                  -(sum(rho^d*(sum((-1)^d*M[2*t+1, d, 2*h]
                         *A[1]^(2*t+3-2*d-2*h)
                     *A[2]^(2*t+1)*(rho^h*A[2]
                     +sum(rho^l*(eta[2*h, l, 0]*A[3]*A[4]
                        *A[5]^(2*h-1-2*l)+eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l)),
                     l = 0 .. h-1)), h = 1 .. t+1-d)), d = 1 .. t))
                     -(sum(rho^d*(sum((-1)^(d+1)*M[2*t+1, d, 2*h+1]
                          *A[1]^(2*t+2-2*d-2*h)
                       *A[2]^(2*t+1)*(sum(rho^l*(eta[2*h+1, l, 0]
                       *A[3]*A[4]*A[5]^(2*h-2*l)
                       +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
                        l = 0 .. h)), h = 1 .. t+1-d)), d = 1 .. t)));
```

```
   f[2] := simplify(c[2*t+1, 0, 1]*A[1]^(2*t+1)*A[2]^(2*t+1)*A[3]*A[4]*A[5]
         -(sum(c[2*t+1, 0, 2*h]*A[1]^(2*t+2-2*h)
              *A[2]^(2*t+1)*(rho^h*A[2]*A[3]
            +sum(rho^l*(eta[2*h, l, 0]*A[3]*A[4]*A[5]^(2*h-2*l)
            +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
             l = 0 .. h-1)), h = 1 .. t+1))
         +sum(c[2*t+1, 0, 2*h+1]*A[1]^(2*t+1-2*h)*A[2]^(2*t+1)
            *(sum(rho^l*(eta[2*h+1, l, 0]*A[3]*A[4]
            *A[5]^(2*h-2*l+1)+eta[2*h+1, l, 1]
              *A[2]*A[3]^(2*h-2*l+2)), l = 0 .. h)), h = 1 .. t)
         -rho^(t+1)*(-1)^(t+1)*c[2*t+1, t+1, 0]*A[2]^(2*t+2)*A[3]
         -(sum(rho^d*((-1)^(d+1)*c[2*t+1, d, 1]*A[1]^(2*t+1-2*d)
            *A[2]^(2*t+1)*A[3]*A[4]*A[5]+(-1)^d*c[2*t+1, d, 0]
            *A[1]^(2*t+2-2*d)*A[2]^(2*t+2)*A[3]), d = 1 .. t))
         -(sum(rho^d*(sum((-1)^d*c[2*t+1, d, 2*h]
               *A[1]^(2*t+2-2*d-2*h)*A[2]^(2*t+1)
             *(rho^h*A[2]*A[3]+sum(rho^l*(eta[2*h, l, 0]
             *A[3]*A[4]*A[5]^(2*h-2*l)
              +eta[2*h, l, 1]*A[2]*A[3]^(2*h-2*l+1)),
               l = 0 .. h-1)), h = 1 .. t+1-d)), d = 1 .. t))
         -(sum(rho^d*(sum((-1)^(d+1)*c[2*t+1, d, 2*h+1]
         *A[1]^(2*t+1-2*d-2*h)*A[2]^(2*t+1)
             *(sum(rho^l*(eta[2*h+1, l, 0]*A[3]*A[4]*A[5]^(2*h-2*l+1)
               +eta[2*h+1, l, 1]*A[2]*A[3]^(2*h-2*l+2)),
             l = 0 .. h)), h = 1 .. t-d)), d = 1 .. t)));
end if;
c[i+1, 0, 0] := 1;
c[i+1, 0, 1] := expand(simplify((q^(i+2)-q^(-i-2))/(q-1/q)));
# Compute A^{i+2}{A^*}^{i+1}-c[i+1,0,1]*A^{i+1}{A^*}^{i+1}A
B[i+1] := expand(simplify(f[1]-c[i+1, 0, 1]*f[2]));
# Rewrite B[i+1] in the distributed form of A[1], A[2], A[3] and rho
B[i+1] := collect(B[i+1], [A[1], A[2], A[3], rho], distributed);
# Extract the coefficient in A[1] and A[3] in the sum
# of all elements containing A[2]^{i+1} of B[i+1]
tam := coeff(B[i+1], A[2], i+1);
for k from 2 to i+2 do
   # Extract the coefficient in A[1] in the sum
   # of all elements containing A[3]^k of tam
   ta1[k] := coeff(tam, A[3], k);
   # Extract the coefficient of A[1]^{i+2-k} in ta1[k]
   ta1[k] := coeff(ta1[k], A[1], i+2-k);
   # Extract the coefficient c[i+1, 0, k] of
    (-1)^{k+l}A[1]^{i+2-k}A[2]^{i+1}A[3]^k
   c[i+1, 0, k] := ta1[k]*(-1)^(k+1);
end do;
```

```
        for p to trunc((1/2)*i+1) do
           for k from 0 to i+2-2*p do
              # Extract the coefficient in A[1] in the sum
              # of all elements containing A[3]^k of tam
              ta2[k] := coeff(tam, A[3], k);
              # Extract the coefficient of A[1]^{i+2-2*p-k} in ta2k
              ta2[k] := coeff(ta2[k], A[1], i+2-2*p-k);
              # Extract the coefficient c[i+1,p,k]
              # of (-1)^{k+p+1}*rho^{p}A[1]^{i+2-2*p-k}A[2]^{i+1}A[3]^k
              c[i+1, p, k] := ta2[k]*rho^(-p)*(-1)^(k+p+1);
           end do
        end do;
        # Compute the higher order relations of
        # the generalized q-Onsager algebra for r = i+1
        F[i+1] := A[1]^(i+2)*A[2]^(i+1)
                  -c[i+1, 0, 1]*A[1]^(i+1)*A[2]^(i+1)*A[3]-B[i+1] =0
     end do
  end if;
  # Output the higher order relations of the
  # generalized q-Onsager algebra for r+1
  F[r+1];
end proc;

# Print the coefficients c[r+1, p, k] of the higher
# order relations of the generalized q-Onsager algebra
pri := proc (r)
  local p, k, coef;
  global c;
  for p from 0 to trunc((1/2)*r+1) do
     for k from 0 to r+2-2*p do
        print(coef[r+1, p, k], "=", c[r+1, p, k])
     end do
  end do
end proc;

# Compute the coefficients c[n+1,p,j] by recursion relations
Result2 := proc (n)
  local p, i, j, k, t, coef;
  global eta, c, M;
  # Input the initial values
  c[1, 0, 0] := 1;
  c[1, 0, 1] := q+1/q;
  c[1, 0, 2] := 1;
  c[1, 1, 0] := 1;
  c[2, 0, 0] := 1;
  c[2, 0, 1] := q^2+1+1/q^2;
```

```
c[2, 0, 2] := q^2+1+1/q^2;
c[2, 0, 3] := 1;
c[2, 1, 0] := q^2+2+1/q^2;
c[2, 1, 1] := q^2+2+1/q^2;
c[3, 0, 0] := 1;
c[3, 0, 1] := expand((q^4-1/q^4)/(q-1/q));
c[3, 0, 2] := expand((q^2+1/q^2)*(q^2+1+1/q^2));
c[3, 0, 3] := expand((q^4-1/q^4)/(q-1/q));
c[3, 0, 4] := 1;
c[3, 1, 0] := q^4+2*q^2+4+2/q^2+1/q^4;
c[3, 1, 1] := expand((q+1/q)*(q^2+1/q^2+3)*(q^2+1/q^2));
c[3, 1, 2] := q^4+2*q^2+4+2/q^2+1/q^4;
c[3, 2, 0] := expand(q^2+1+1/q^2);
Function(n);
if 1 < n then
   for i from 2 to n do
      t := trunc((1/2)*i);
      # Compute the coefficients M[r, p, j] of the expansion
      #  of A^{r+2}{A^*}^{r+1}
      for k from 2 to i+1 do
         M[i, 0, k] := -c[i, 0, 1]*c[i, 0, k-1]+c[i, 0, k];
      end do;
      M[i, 0, i+2] := -c[i, 0, 1]*c[i, 0, i+1];
      if i = 2*t then
         for p to t do
            for k to i-2*p+1 do
               M[i, p, k] := -c[i, 0, 1]*c[i, p, k-1]+c[i, p, k];
            end do;
            M[i, p, 0] := c[i, p, 0];
            M[i, p, i+2-2*p] := -c[i, 0, 1]*c[i, p, i-2*p+1];
         end do
      else
         for p to t do
            M[i, p, 0] := c[i, p, 0];
            M[i, p, i+2-2*p] := -c[i, 0, 1]*c[i, p, i-2*p+1];
            for k to i-2*p+1 do
               M[i, p, k] := -c[i, 0, 1]*c[i, p, k-1]+c[i, p, k];

            end do
         end do;
         M[i, t+1, 0] := c[i, t+1, 0];
         M[i, t+1, 1] := -c[i, t+1, 0]*c[i, 0, 1]
      end if;
      # Compute the coefficients c[i+1, p, j]
      c[i+1, 0, 0] := 1;
```

```
c[i+1, 0, 1] := expand(simplify((q^(i+2)-q^(-i-2))/(q-1/q)));
if i = 2*t then
   c[i+1, 0, 2] := expand(simplify(M[i, 0, 2]*eta[2, 0, 1]));
   for k from 2 to t+1 do
      c[i+1, 0, 2*k] := expand(simplify(M[i, 0, 2*k]*eta[2*k, 0, 1]
                          +c[i+1, 0, 1]*c[i, 0, 2*k-1]*eta[2*k-1, 0, 1]));
   end do;
   for k to t do
      c[i+1, 0, 2*k+1] := expand(simplify(M[i, 0, 2*k+1]*eta[2*k+1, 0, 1]
                          +c[i+1, 0, 1]*c[i, 0, 2*k]*eta[2*k, 0, 1]));
   end do;
   c[i+1, t+1, 0] := expand(simplify((-1)^t*(-M[i, 0, i+2]
                          -(sum((-1)^l*M[i, l, i+2-2*l], l = 1 .. t)))));
   c[i+1, 1, 0] := expand(simplify(-M[i, 0, 2]+M[i, 1, 0]));
   if 2 <= t then
      for k from 2 to t do
         c[i+1, k, 0] := expand(simplify((-1)^k*(sum((-1)^l
                            *M[i, l, 2*k-2*l], l = 0 .. k))));
      end do
   end if;
   c[i+1, 1, 1] := expand(simplify(-M[i, 0, 3]*eta[3, 1, 1]
                           +c[i+1, 0, 1]*(-c[i, 0, 2]+c[i, 1, 0])));
   if 2 <= t then
      for k from 2 to t do
         c[i+1, k, 1] := expand(simplify((-1)^k*(-(sum((-1)^(l+1)
         *M[i, l, 2*k-2*l+1]
                              *eta[2*k-2*l+1, k-l, 1], l = 0 .. k-1))
                         +c[i+1, 0, 1]*(sum((-1)^l*c[i, l, 2*k-2*l],
                         l = 0 .. k)))));
      end do
   end if;
   c[i+1, 1, 2] := expand(simplify(-M[i, 0, 4]*eta[4, 1, 1]
                   +M[i, 1, 2]*eta[2, 0, 1]
                   -c[i+1, 0, 1]*c[i, 0, 3]*eta[3, 1, 1]));
   if 2 <= t then
      for k from 2 to t do
         for p to k-1 do
            c[i+1, p, 2*k-2*p+1] := expand(simplify((-1)^p*(sum((-1)^l
                              *M[i, l, 2*k-2*l+1]*eta[2*k-2*l+1, p-l, 1],
                                                        l = 0 .. p)
                              +c[i+1, 0, 1]*(sum((-1)^l*c[i, l, 2*k-2*l]
                                 *eta[2*k-2*l, p-l, 1], l = 0 .. p)))));
         end do
      end do;
      for k from 3 to t+1 do
```

```
            for p to k-1 do
               c[i+1, p, 2*k-2*p] := expand(simplify((-1)^p*(sum((-1)^l
                    *M[i, l, 2*k-2*l]
                                    *eta[2*k-2*l, p-l, 1], l = 0 .. p)
                              +c[i+1, 0, 1]*(sum((-1)^l*c[i, l, 2*k-2*l-1]
                         *eta[2*k-2*l-1, p-l, 1], l = 0 .. min(p, k-2))))));
            end do
         end do
      end if
   else
      c[i+1, 0, 2] := expand(simplify(M[i, 0, 2]*eta[2, 0, 1]));
      for k to t+1 do
         c[i+1, 0, 2*k+1] := expand(simplify(M[i, 0, 2*k+1]
                           *eta[2*k+1, 0, 1]
                          +c[i+1, 0, 1]*c[i, 0, 2*k]*eta[2*k, 0, 1]));
      end do;
      for k from 2 to t+1 do
         c[i+1, 0, 2*k] := expand(simplify(M[i, 0, 2*k]*eta[2*k, 0, 1]
                          +c[i+1, 0, 1]*c[i, 0, 2*k-1]*eta[2*k-1, 0, 1]));
      end do;
      c[i+1, 1, 0] := expand(simplify(-M[i, 0, 2]+M[i, 1, 0]));
      for k from 2 to t+1 do
         c[i+1, k, 0] := expand(simplify((-1)^k*(sum((-1)^l
                      *M[i, l, 2*k-2*l],l = 0 .. k))));
      end do;
      c[i+1, 1, 1] := expand(simplify(-M[i, 0, 3]*eta[3, 1, 1]
                                    +c[i+1, 0, 1]*(-c[i, 0, 2]+c[i, 1, 0])));
      for k from 2 to t+1 do
         c[i+1, k, 1] := expand(simplify((-1)^k*(sum((-1)^l*M[i, l, 2*k-2*l+1]
                                  *eta[2*k-2*l+1, k-l, 1], l = 0 .. k-1)
                            +c[i+1, 0, 1]*(sum((-1)^l*c[i, l, 2*k-2*l],
                            l = 0 .. k)))));
      end do;
      c[i+1, 1, 2] := expand(simplify(-M[i, 0, 4]*eta[4, 1, 1]
                          +M[i, 1, 2]*eta[2, 0, 1]
                          -c[i+1, 0, 1]*c[i, 0, 3]*eta[3, 1, 1]));
      if 2 <= t then
         for k from 3 to t+1 do
            for p to k-1 do
               c[i+1, p, 2*k-2*p] := expand(simplify((-1)^p
                  *(sum((-1)^l*M[i, l, 2*k-2*l]
                                  *eta[2*k-2*l, p-l, 1], l = 0 .. p)
                                  +c[i+1, 0, 1]*(sum((-1)^l
                                     *c[i, l, 2*k-2*l-1]
                                     *eta[2*k-2*l-1, p-l, 1],
```

```
                                        l = 0 .. min(p, k-2))))));
                end do
             end do
          end if;
          for k from 2 to t+1 do
             for p to k-1 do
                c[i+1, p, 2*k-2*p+1] := expand(simplify((-1)^p*(sum((-1)^l
                                            *M[i, l, 2*k-2*l+1]
                                         *eta[2*k-2*l+1, p-l, 1], l = 0 .. p)
                                          +c[i+1, 0, 1]*(sum((-1)^l*c[i, l, 2*k-2*l]
                                            *eta[2*k-2*l, p-l, 1],l = 0 .. p)))));
            end do
          end do
       end if
    end do
  end if;
  for p from 0 to trunc((1/2)*n+1) do
     for k from 0 to n+2-2*p do
        print(coef[n+1, p, k], "=", c[n+1, p, k])
     end do
  end do
end proc;
```

5.6 APPENDIX F: Powers of $q-$Onsager generators

The explicit expressions of the divided polynomials in terms of tensor products of Pauli matrices can be derived using (3.3.30) and computing powers $\mathcal{W}_i^n$, $i = 0, 1$ (n is a positive integer) of the fundamental generators of the $q-$Onsager algebra (3.3.17)-(3.3.18). According to the parity of n, the expressions slightly differ. For simplicity, let us first study the case $\epsilon_\pm = 0$. To get the results below, we used:

$$w_0^n = \begin{cases} (k_+k_-)^{\frac{n}{2}}\mathbb{I} & \text{if } n \text{ is even,} \\ (k_+k_-)^{\frac{n-1}{2}} w_0 & \text{if } n \text{ is odd.} \end{cases}$$

For $N = 2$ and $\epsilon_\pm = 0$:

$$\begin{aligned} \mathcal{W}_0^2 &= [2]! \sum_{1\leq j_1<j_2\leq L} q^{2\sigma_z} \otimes \cdots \otimes q^{2\sigma_z} \otimes w_{0_{j_1}} \otimes q^{\sigma_z} \otimes \cdots \otimes q^{\sigma_z} \otimes w_{0_{j_2}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \\ &+ \sum_{1\leq j\leq L} q^{2\sigma_z} \otimes \cdots \otimes q^{2\sigma_z} \otimes w_{0_j}^2 \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}. \end{aligned}$$

For $N = 3$ and $\epsilon_\pm = 0$:

$$\begin{aligned} \mathcal{W}_0^3 &= [3]! \sum_{1\leq j_1<j_2<j_3\leq L} \Big(q^{3\sigma_z} \otimes \cdots \otimes q^{3\sigma_z} \otimes w_{0_{j_1}} \otimes q^{2\sigma_z} \otimes \cdots \otimes q^{2\sigma_z} \otimes w_{0_{j_2}} \otimes \\ &\qquad \otimes q^{\sigma_z} \otimes \cdots \otimes q^{\sigma_z} \otimes w_{0_{j_3}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \Big) \\ &+ \sum_{1\leq j\leq L} q^{3\sigma_z} \otimes \cdots \otimes q^{3\sigma_z} \otimes w_{0_j}^3 \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \\ &+ [3] \sum_{1\leq j_1<j_2\leq L} q^{3\sigma_z} \otimes \cdots \otimes q^{3\sigma_z} \otimes w_{0_{j_1}} \otimes q^{2\sigma_z} \otimes \cdots \otimes q^{2\sigma_z} \otimes w_{0_{j_2}}^2 \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \\ &+ [2] \sum_{1\leq j_1<j_2\leq L} q^{3\sigma_z} \otimes \cdots \otimes q^{3\sigma_z} \otimes w_{0_{j_1}}^2 \otimes q^{\sigma_z} \otimes \cdots \otimes q^{\sigma_z} \otimes w_{0_{j_2}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \\ &+ \sum_{1\leq j_1<j_2\leq L} q^{3\sigma_z} \otimes \cdots \otimes q^{3\sigma_z} \otimes q_{j_1}^{\sigma_z} w_{0_{j_1}}^2 \otimes q^{\sigma_z} \otimes \cdots \otimes q^{\sigma_z} \otimes w_{0_{j_2}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}. \end{aligned}$$

More generally, by induction one finds the general structure:

- Case n is odd and $\epsilon_{\pm} = 0$:

$$
\begin{aligned}
\mathcal{W}_0^n \;=\; & \sum_{h=1}^{n} \sum_{\substack{0<i_1,i_2,\ldots,i_h \\ i_1+i_2+\cdots+i_h=n}} \sum_{1\le j_1<j_2<\cdots<j_h\le L} \Bigg(C^{(h)}_{\substack{i_1,\ldots,i_h \\ j_1,\ldots,j_h}} \, q^{n\sigma_z} \otimes \cdots \otimes q^{n\sigma_z} \otimes w^{i_1}_{0_{j_1}} \otimes \\
& \otimes\, q^{(n-i_1)\sigma_z} \otimes \cdots \otimes q^{(n-i_1)\sigma_z} \otimes w^{i_2}_{0_{j_2}} \otimes q^{(n-i_1-i_2)\sigma_z} \otimes \cdots \otimes q^{(n-i_1-i_2)\sigma_z} \otimes \ldots \\
& \otimes\, q^{(n-i_1-\cdots-i_{h-1})\sigma_z} \otimes \cdots \otimes q^{(n-i_1-\cdots-i_{h-1})\sigma_z} \otimes w^{i_h}_{0_{j_h}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \Bigg) \\
+ \; & \sum_{h=2}^{n-1} \sum_{u=1}^{[\frac{n-1}{2}]-|h-[\frac{n+1}{2}]|} \sum_{\substack{0<x_1,x_2,\ldots,x_u \\ x_1+x_2+\cdots+x_u=u}} \sum_{\substack{0<i_{u+1},i_{u+2},\ldots,i_h \\ i_{u+1}+i_{u+2}+\cdots+i_h=n-2(x_1+\cdots+x_u)}} \sum_{P} \\
& \Bigg(c^{(h)}_{\substack{t_1,x_1,j_1;\ldots;t_u,x_u,j_u \\ i_{u+1},i_{u+2},\ldots,i_h;j'_{u+1},\ldots,j'_h}} \, q^{n\sigma_z} \otimes \cdots \otimes q^{n\sigma_z} \otimes w^{i_{u+1}}_{0_{j'_{u+1}}} \otimes q^{(n-i_{u+1})\sigma_z} \otimes \cdots \otimes \\
& \otimes\, q^{(n-i_{u+1})\sigma_z} \otimes q^{t_1}_{j_1} w^{2x_1}_{0_{j_1}} \otimes q^{(n-i_{u+1}-2x_1)\sigma_z} \otimes \cdots \otimes q^{(n-i_{u+1}-\cdots-i_{h-1}-2(x_1+\cdots+x_u))\sigma_z} \otimes \\
& \otimes\, w^{i_h}_{0_{j'_h}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \Bigg)
\end{aligned}
$$

- Case n is even and $\epsilon_{\pm}=0$:

$$
\begin{aligned}
\mathcal{W}_0^n &= \sum_{h=1}^{n} \sum_{\substack{0<i_1,i_2,\ldots,i_h \\ i_1+i_2+\cdots+i_h=n}} \sum_{1\le j_1<j_2<\ldots<j_h\le L} \Bigg(C^{(h)}_{\substack{i_1,\ldots,i_h \\ j_1,\ldots,j_h}} q^{n\sigma_z}\otimes\cdots\otimes q^{n\sigma_z}\otimes w^{i_1}_{0_{j_1}}\otimes \\
&\otimes\ q^{(n-i_1)\sigma_z}\otimes\cdots\otimes q^{(n-i_1)\sigma_z}\otimes w^{i_2}_{0_{j_2}}\otimes q^{(n-i_1-i_2)\sigma_z}\otimes\cdots\otimes q^{(n-i_1-i_2)\sigma_z}\otimes\ldots \\
&\otimes\ q^{(n-i_1-\cdots-i_{h-1})\sigma_z}\otimes\cdots\otimes q^{(n-i_1-\cdots-i_{h-1})\sigma_z}\otimes w^{i_h}_{0_{j_h}}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}\Bigg) \\
&+ \sum_{h=2}^{n/2}\sum_{u=1}^{h-1} \sum_{\substack{0<x_1,x_2,\ldots,x_u \\ x_1+x_2+\cdots+x_u=u}}^{x_1+x_2+\cdots+x_u=Min\{\left[\frac{n-1}{2}\right],h\}} \sum_{\substack{0<i_{u+1},i_{u+2},\ldots,i_h \\ i_{u+1}+i_{u+2}+\cdots+i_h=n-2(x_1+\cdots+x_u)}} \sum_{P} \\
&\Bigg(c^{(h)}_{\substack{t_1,x_1,j_1;\ldots;t_u,x_u,j_u \\ i_{u+1},i_{u+2},\ldots,i_h;j'_{u+1},\ldots,j'_h}} q^{n\sigma_z}\otimes\cdots\otimes q^{n\sigma_z}\otimes w^{i_{u+1}}_{0_{j'_{u+1}}}\otimes q^{(n-i_{u+1})\sigma_z}\otimes\cdots\otimes \\
&\otimes\ q^{(n-i_{u+1})\sigma_z}\otimes q^{t_1}_{j_1} w^{x_1}_{0_{j_1}}\otimes q^{(n-i_{u+1}-2x_1)\sigma_z}\otimes\cdots\otimes q^{(n-i_{u+1}-\cdots-i_{h-1}-2(x_1+\cdots+x_u))}\otimes \\
&\otimes\ w^{i_h}_{0_{j'_h}}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}\Bigg) \\
&+ \sum_{h=n/2+1}^{n-1}\sum_{u=1}^{n-h} \sum_{\substack{0<x_1,x_2,\ldots,x_u \\ x_1+x_2+\cdots+x_u=u}}^{x_1+x_2+\cdots+x_u=Min\{\left[\frac{n-1}{2}\right],h\}} \sum_{\substack{0<i_{u+1},i_{u+2},\ldots,i_h \\ i_{u+1}+i_{u+2}+\cdots+i_h=n-2(x_1+\cdots+x_u)}} \sum_{P} \\
&\Bigg(c^{(h)}_{\substack{t_1,x_1,j_1;\ldots;t_u,x_u,j_u \\ i_{u+1},\ldots,i_h;j'_{u+1},\ldots,j'_h}} q^{n\sigma_z}\otimes\cdots\otimes q^{n\sigma_z}\otimes w^{i_{u+1}}_{0_{j'_{u+1}}}\otimes q^{(n-i_{u+1})\sigma_z}\otimes\cdots\otimes \\
&\otimes\ q^{(n-i_{u+1})\sigma_z}\otimes q^{t_1}_{j_1} w^{2x_1}_{0_{j_1}}\otimes q^{(n-i_{u+1}-2x_1)\sigma_z}\otimes\cdots\otimes q^{(n-i_{u+1}-\cdots-i_{h-1}-2(x_1+\cdots+x_u))}\otimes \\
&\otimes\ w^{i_h}_{0_{j'_h}}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}\Bigg)
\end{aligned}
$$

where

$$
P=\begin{cases} 1\le j_1<\ldots<j_u\le L, \\ 1\le j'_{u+1}<\ldots<j'_h\le L, \\ \{j_1,\ldots,j_u\}\cap\{j'_{u+1},\ldots,j'_h\}=\emptyset, \\ t_1\ge t_2\ge\cdots\ge t_u>0, \\ t_k\le \sum\limits_{v>k} 2x_v+\sum\limits_{j'_v>j_k} i_v, \quad k=1,\ldots,u. \end{cases}
$$

For $n=1,2,3,4$, the explicit expressions for the coefficients are:

- For $n=1$: $\quad C^{(1)}_{\substack{1\\ j_1}}=1$;

- For $n=2$: $\quad C^{(2)}_{\substack{1,1\\ j_1,j_2}}=q+q^{-1}=[2]!, \quad C^{(1)}_{\substack{2\\ j_1}}=1$, where $j_1<j_2$;

- For $n=3$:
  $C^{(3)}_{\substack{1,1,1\\ j_1,j_2,j_3}} = [3]!, \quad C^{(2)}_{\substack{1,2\\ j_1,j_2}} = [3], \quad C^{(2)}_{\substack{2,1\\ j_1,j_2}} = [2], \quad C^{(1)}_{\substack{3\\ j_1}} = 1,$

  $c^{(2)}_{\substack{1,2,j_1\\ 1,j_2}} = 1$, where $j_1 < j_2 < j_3$;

- For $n=4$:
  $C^{(4)}_{\substack{1,1,1,1\\ j_1,j_2,j_3,j_4}} = [4]!, \quad C^{(3)}_{\substack{2,1,1\\ j_1,j_2,j_3}} = [3]!, \quad C^{(3)}_{\substack{1,2,1\\ j_1,j_2,j_3}} = [2][4], \quad C^{(3)}_{\substack{1,1,2\\ j_1,j_2,j_3}} = [3][4],$

  $C^{(2)}_{\substack{3,1\\ j_1,j_2}} = 2[2], \quad C^{(2)}_{\substack{1,3\\ j_1,j_2}} = [4], \quad C^{(2)}_{\substack{2,2\\ j_1,j_2}} = [3], \quad C^{(1)}_{\substack{4\\ j_1}} = 1,$

  $c^{(3)}_{\substack{1,2,j_2\\ 1,1;j_1,j_3}} = [4], \quad c^{(3)}_{\substack{1,2,j_1\\ 1,1;j_2,j_3}} = [2]^2, \quad c^{(3)}_{\substack{2,2,j_1\\ 1,1;j_2,j_3}} = [2], \quad c^{(2)}_{\substack{1,2,j_1\\ 2,j_2}} = [2], \quad c^{(2)}_{\substack{2,2,j_1\\ 2,j_2}} = 1\ ,$

  where $j_1 < j_2 < j_3 < j_4$.

Remark 10 *Powers of $\mathcal{W}_1$ for $\epsilon_- = 0$ are readily obtained through the substitutions $q \to q^{-1}$ in above expressions.*

The results above correspond to the special choice $\epsilon_\pm = 0$. For $\epsilon_\pm \neq 0$, a similar analysis yields to:

$$
\begin{aligned}
\mathcal{W}_0^n &= \mathcal{W}_0^n|_{\epsilon_+=0} + \epsilon_+^n q^{n\sigma_z} \otimes \cdots \otimes q^{n\sigma_z} \\
&+ \sum_{m=1}^{n-1}\sum_{h=1}^{m}\sum_{P_1}\Bigg(\epsilon_+^{n-m} C^{(m,h)}_{\substack{i_1,\ldots,i_h\\ j_1,\ldots,j_h}} q^{n\sigma_z} \otimes \cdots \otimes q^{n\sigma_z} \otimes w^{i_1}_{0_{j_1}} \otimes q^{(n-i_1)\sigma_z} \otimes \cdots \otimes w^{i_h}_{0_{j_h}} \otimes \\
&\qquad \otimes q^{(n-m)\sigma_z} \otimes \cdots \otimes q^{(n-m)\sigma_z}\Bigg) \\
&+ \sum_{m=2}^{n-1}\sum_{h=1}^{m-1}\sum_{u=1}^{Min([\frac{m}{2}],h)} \sum_{\substack{0<x_1,x_2,\ldots,x_u\\ x_1+x_2+\cdots+x_u=u}}^{x_1+x_2+\cdots+x_u=[\frac{m}{2}]} \sum_{\substack{0<i_{u+1},i_{u+2},\ldots,i_h\\ i_{u+1}+i_{u+2}+\cdots+i_h=m-2(x_1+\cdots+x_u)}} \sum_{P_2} \\
&\Bigg(\epsilon_+^{n-m} c^{(m,h)}_{\substack{t_1,x_1,j_1;\ldots;t_u,x_u,j_u\\ i_{u+1},i_{u+2},\ldots,i_h;j'_{u+1},\ldots,j'_h}} q^{n\sigma_z} \otimes \cdots \otimes q^{n\sigma_z} \otimes w^{i_{u+1}}_{0_{j'_{u+1}}} \otimes q^{(n-i_{u+1})\sigma_z} \otimes \cdots \otimes q^{(n-i_{u+1})\sigma_z} \otimes \\
&\quad \otimes q^{t_1}_{j_1} w^{2x_1}_{0_{j_1}} \otimes q^{(n-i_{u+1}-2x_1)\sigma_z} \otimes \cdots \otimes q^{(n-i_{u+1}-2x_1)\sigma_z} \otimes \cdots \otimes q^{(n-m)\sigma_z} \otimes \cdots \otimes q^{(n-m)\sigma_z}\Bigg) \\
&+ \sum_{t=1}^{[\frac{n-1}{2}]}\sum_{h=1}^{t} \sum_{\substack{0<x_1,x_2,\ldots,x_h\\ x_1+x_2+\cdots+x_h=t}} \sum_{P_3}\Bigg(\epsilon_+^{n-2t} c^{(2t,h)}_{t_1,x_1,j_1;\ldots;t_h,x_h,j_h} q^{n\sigma_z} \otimes \cdots \otimes q^{n\sigma_z} \otimes q^{t_1\sigma_z}_{j_1} w^{2x_1}_{0_{j_1}} \otimes \\
&\quad \otimes q^{(n-2x_1)\sigma_z} \otimes \cdots \otimes q^{(n-2x_1)\sigma_z} \otimes q^{t_2\sigma_z}_{j_2} w^{2x_2}_{0_{j_2}} \otimes q^{(n-2x_1-2x_2)\sigma_z} \otimes \\
&\quad \otimes \cdots \otimes q^{(n-2x_1-2x_2)\sigma_z} \otimes \cdots \otimes q^{t_h\sigma_z}_{j_h} w^{2x_h}_{0_{j_h}} \otimes q^{(n-2t)\sigma_z} \otimes \cdots \otimes q^{(n-2t)\sigma_z}\Bigg)
\end{aligned}
$$

where

$$P_1 = \left\{ \begin{array}{l} i_1 + \cdots + i_h = m, \\ 0 < j_1 < j_2 < \cdots < j_h \leq L \end{array} \right. ;$$

$$P_2 = \left\{ \begin{array}{l} 1 \leq j_1 < \cdots < j_u \leq L, \\ 1 \leq j'_{u+1} < \cdots < j'_h \leq L, \\ \{j_1, \ldots, j_u\} \cap \{j'_{u+1}, \ldots, j'_h\} = \emptyset, \\ t_1 \geq t_2 \geq \cdots \geq t_u > 0, \\ t_k \leq n - 2 \sum\limits_{i \leq k} x_i - \sum\limits_{j'_v < j_k} i_v, \quad k = 1, \ldots, u. \end{array} \right.$$

$$P_3 = \left\{ \begin{array}{l} 1 \leq j_1 < \cdots < j_h \leq L, \\ t_1 \geq t_2 \geq \cdots \geq t_h > 0, \\ t_k \leq n - 2 \sum\limits_{i \leq k} x_i, \quad k = 1, \ldots, h. \end{array} \right.$$

For $n = 2, 3, 4$, the explicit expressions for the coefficients are:

- For $n = 2$: $\quad C^{(1,1)}_{\substack{1 \\ j_1}} = [2]$;
- For $n = 3$:
  $C^{(2,2)}_{\substack{1,1 \\ j_1,j_2}} = [3]!, \quad C^{(2,1)}_{\substack{2 \\ j_1}} = [2], \quad C^{(1,1)}_{\substack{1 \\ j_1}} = [3],$

  $c^{(2,1)}_{1,1,j_1} = 1$, where $j_1 < j_2$;
- For $n = 4$:
  $C^{(3,3)}_{\substack{1,1,1 \\ j_1,j_2,j_3}} = [4]!, \quad C^{(3,2)}_{\substack{2,1 \\ j_1,j_2}} = [3]!, \quad C^{(3,2)}_{\substack{1,2 \\ j_1,j_2,j_3}} = [2][4], \quad C^{(3,1)}_{\substack{3 \\ j_1}} = 2[2],$

  $C^{(2,2)}_{\substack{1,1 \\ j_1,j_2}} = [3][4], \quad C^{(2,1)}_{\substack{2 \\ j_1}} = [3], \quad C^{(1,1)}_{\substack{1 \\ j_1}} = [4],$

  $c^{(3,2)}_{\substack{1,1,j_2 \\ 1;j_1}} = [4], \quad c^{(3,2)}_{\substack{2,1,j_1 \\ 1;j_2}} = [2], \quad c^{(3,2)}_{\substack{1,1,j_1 \\ 1;j_2}} = [2]^2,$

  $c^{(2,1)}_{1,1,j_1} = [2], \quad c^{(2,1)}_{2,1,j_1} = 1$,

  where $j_1 < j_2 < j_3$.

Remark 11 *Powers of $\mathcal{W}_1$ for $\epsilon_- \neq 0$ are readily obtained through the substitutions $q \to q^{-1}, \epsilon_+ \to \epsilon_-$ in above expressions.*

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