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# Théorèmes limites de la théorie des probabilités dans les systèmes dynamiques

Davide Giraud

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# THÈSE DE DOCTORAT

pour l'obtention du grade de

Docteur de l'université de Rouen

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Présentée et soutenue publiquement par

Davide Giraud

le 4 décembre 2015

## **Théorèmes limites de la théorie des probabilités dans les systèmes dynamiques**

**Après les rapports de**

Richard C. Bradley et Jérôme Dedecker

**Devant le jury composé par**

Pierre Calka	Président du jury
Jérôme Dedecker	Rapporteur
Herold Dehling	Examineur
Sébastien Gouëzel	Examineur
Nadine Guillotin-Plantard	Examinatrice
Alfredas Račkauskas	Examineur
Charles Suquet	Examineur
Dalibor Volný	Directeur de thèse



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**Théorèmes limites de la théorie des probabilités dans les systèmes dynamiques**  
— **Limit theorems of probability theory in dynamical systems**

**Résumé**

Cette thèse se consacre aux théorèmes limites pour les suites et les champs aléatoires strictement stationnaires, principalement sur le théorème limite central et sa version fonctionnelle, appelée principe d'invariance.

Dans un premier temps, nous montrons à l'aide d'un contre-exemple que pour les processus strictement stationnaires  $\beta$ -mélangeants, le théorème limite central peut avoir lieu sans que ce ne soit le cas pour la version fonctionnelle. Nous montrons également que le théorème limite central n'a pas nécessairement lieu pour les sommes partielles d'une suite strictement stationnaire  $\beta$ -mélangeante à valeurs dans un espace de Hilbert de dimension infinie, même en supposant l'uniforme intégrabilité de la suite des sommes partielles normalisées.

Puis nous étudions le principe d'invariance dans l'espace des fonctions hölderiennes. Nous traitons le cas des suites strictement stationnaires  $\tau$ -dépendantes (au sens de Dedecker, Priour, 2005) ou  $\rho$ -mélangeantes. Nous donnons également une condition suffisante sur la loi d'une suite strictement stationnaire d'accroissements d'une martingale et la variance quadratique garantissant le principe d'invariance dans l'espace des fonctions hölderiennes, et nous démontrons son optimalité à l'aide d'un contre-exemple. Ensuite, nous déduisons grâce à une approximation par martingales des conditions dans l'esprit de celles de Hannan (1979), et Maxwell et Woodrooffe (2000).

Nous discutons ensuite de la décomposition martingale/cobord. Dans le cas des suites, nous fournissons des conditions d'intégrabilité sur la fonction de transfert et le cobord pour que ce dernier ne perturbe pas le principe d'invariance, la loi des logarithmes itérés ou bien la loi forte des grands nombres si ceux-ci ont lieu pour la martingale issue de la décomposition. Dans le cas des champs, nous formulons une condition suffisante pour une décomposition ortho-martingale/cobord.

Enfin, nous établissons des inégalités sur les queues des maxima des sommes partielles d'un champ aléatoire de type ortho-martingale ou bien d'un champ qui s'exprime comme une fonctionnelle d'un champ i.i.d. Ces inégalités permettent d'obtenir un principe d'invariance dans les espaces hölderiens pour ces champs aléatoires.

**Mots-clés :**

Processus stationnaires; Théorème limite central; Principe d'invariance faible; Approximation par martingales; Critères projectifs; Systèmes dynamiques; Suites mélangeantes; Espaces de Hölder; Champs aléatoires; Ortho-martingales; Champs bernoulliens ; Inégalités de queues.

**Classification AMS :**

28D05; 37A05; 60B12; 60F05; 60F15; 60F17; 60G10; 60G42; 60G48; 60G60.



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**Abstract**

This thesis is devoted to limit theorems for strictly stationary sequences and random fields. We concentrate essentially on the central limit theorem and its invariance principle.

First, we show with the help of a counter-example that for a strictly stationary absolutely regular sequence, the central limit theorem may hold but not the invariance principle. We also show that the central limit theorem does not take place for partial sums of a Hilbert space valued, strictly stationary and absolutely regular sequence, even if we assume that the normalized partial sums form a uniformly integrable family.

Second, we investigate the Holderian invariance principle. We treat the case of  $\tau$ -dependent (Dedecker, Prieur, 2005) and  $\rho$ -mixing strictly stationary sequences. We provide a sufficient condition on the law of a strictly stationary martingale difference sequence and the quadratic variance which guarantee the invariance principle in a Hölder space. We construct a counter-example which shows its sharpness. We derive conditions in the spirit of Hannan (1979), and Maxwell and Woodroffe (2000) by a martingale approximation.

We then discuss the martingale/coboundary decomposition. In dimension one, we provide sharp integrability conditions on the transfer function and the coboundary for which the latter does not spoil the invariance principle, the law of the iterated logarithm or the strong law of large numbers if these theorems take place for the martingale involved in the decomposition. We also provide a sufficient condition for an orthomartingale/coboundary decomposition for strictly stationary random fields.

Lastly, we establish tails inequalities for orthomartingale and Bernoulli random fields. We prove an invariance principle in Hölder spaces for these random fields using such inequalities.

**Keywords:**

Stationary processes; Central limit theorem; Weak invariance principle; Martingale approximation; Projective criteria; Dynamical systems; Mixing sequences; Hölder spaces; Random fields; Orthomartingales; Bernoulli random fields ; Tail inequalities.

**AMS classification:**

28D05; 37A05; 60B12; 60F05; 60F15; 60F17; 60G10; 60G42; 60G48; 60G60.





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# Introduction

## Avant-propos

La compréhension du comportement asymptotique des sommes partielles d'un processus aléatoire est un enjeu important en théorie des probabilités. Les résultats décrivant ce comportement sont appelés "théorèmes limite" car on s'intéresse à une limite, que ce soit en loi ou presque sûre.

## Théorèmes limites

Dans toute cette thèse, les suites impliquées seront strictement stationnaires, ce qui signifie que la loi de la suite est invariante par translation du temps. Soit  $(\Omega, \mathcal{F}, \mu)$  un espace probabilisé. Étant donnée une suite strictement stationnaire  $(f \circ T^j)_{j \geq 0}$  où  $T: \Omega \rightarrow \Omega$  préserve la mesure, on s'intéresse au comportement de la suite  $(S_n(f))_{n \geq 1} = \left( \sum_{j=0}^{n-1} f \circ T^j \right)_{n \geq 1}$  lorsque  $n$  est grand. Même lorsque  $(f \circ T^j)_{j \geq 0}$  est indépendante et identiquement distribuée (i.i.d.), la suite  $((S_n(f))_{n \geq 1})$  ne converge pas (sauf si  $f$  est dégénérée). Il faut donc normaliser la suite des sommes partielles pour espérer une convergence. Si la fonction  $f$  est intégrable, alors on sait que pour presque tout  $\omega \in \Omega$ , la convergence  $S_n(f)(\omega)/n \rightarrow \mathbb{E}[f]$  a lieu. En supposant toujours la suite  $(f \circ T^j)_{j \geq 0}$  i.i.d., la normalisation de la  $n$ -ème somme partielle par  $\sqrt{n}$  donne lieu à une convergence vers une loi non dégénérée si  $f$  est centrée et de variance non nulle : la loi normale. C'est ce que l'on appelle *théorème limite central*. On peut chercher également une version plus fine de ce résultat, appelée *théorème limite central fonctionnel* ou encore *principe d'invariance*. Ceci consiste à étudier la convergence en loi du processus donné par (1.3.4) dans l'espace des fonctions sur  $[0, 1]$  vers un mouvement brownien. Les fonctions aléatoires considérées  $S_n^{\text{pl}}(f, \cdot)$  tiennent compte, à  $\omega$  fixé, de toutes les sommes partielles  $S_i(f)(\omega)$  pour les  $i \leq n$ . Ceci contraste avec le théorème limite central où l'on ne tient compte que de la  $n$ -ème somme partielle, sans se soucier de l'historique, c'est-à-dire des valeurs prises par les sommes partielles précédentes. Le principe d'invariance reste valide pour les suites i.i.d. centrée à variance finie (Donsker, [Don51]). L'hypothèse d'indépendance de la suite  $(f \circ T^j)_{j \geq 0}$  fournit un cadre idéal pour les théorèmes limites. Cependant, cette hypothèse n'est pas nécessairement vérifiée. Il est donc nécessaire de sortir du cadre i.i.d. Nous traiterons le cas stationnaire non nécessairement indépendant.

## Approximations par martingales

Les martingales de carré intégrable à accroissements strictement stationnaires (voir Définition 1.1.9) jouent un rôle clé pour l'obtention de théorèmes limites. En effet, celles-ci jouissent de bonnes inégalités sur les maxima des sommes partielles (voir la Proposition 1.1.1 ou encore le Théorème 11.1.1). Celles-ci sont très pratiques pour vérifier les critères de tension du processus défini par (1.3.4) dans différents espaces fonctionnels. Le Théorème 1.2.2 donne le théorème

limite central pour de telles suites, et le même argument fournit la convergence des lois finidimensionnelles. De nombreux résultats concernant les théorèmes limites pour les martingales figurent dans [HH80].

Si on dispose d'une suite strictement stationnaire  $(f \circ T^j)_{j \in \mathbb{Z}}$ , on peut donc chercher à savoir si les sommes partielles sont proches dans un certain sens d'une martingale à accroissements strictement stationnaires. Gordin [Gor69] fut le premier à exploiter cette idée. Il existe des conditions suffisantes mettant en jeu l'espérance conditionnelle de  $f \circ T^k$  par rapport à ce qui est mesurable avant le temps initial (voir la Sous-section 1.2.2 pour les conditions exactes).

## Contexte

Ce travail de recherche se décline en deux volets indépendants : le premier est dédié aux théorèmes limites pour les suites strictement stationnaires, le second pour les champs aléatoires strictement stationnaires. On étudie une action de  $\mathbb{Z}^d$  qui préserve la mesure sur un espace probabilisé, avec  $d = 1$  pour les suites. Souvent, les suites ou champs considérés seront ergodiques.

## Théorèmes limites pour les suites de variables aléatoires

Comme nous l'avons vu dans la section précédente, le premier théorème limite central fonctionnel est dû à Donsker [Don51]. Il concerne les suites indépendantes et identiquement distribuées. De puis, l'extension de ce résultat aux suites strictement stationnaires non nécessairement indépendantes est devenue un enjeu important. En général, lorsque l'on suppose que  $\|S_n(f)\|_2 \rightarrow \infty$ , on fera des hypothèses sur la dépendance et/ou sur les moments.

Dans [Ros56], Rosenblatt propose une manière de quantifier la dépendance d'une suite de variables aléatoires à l'aide des coefficients dits d' $\alpha$ -mélange. Il existe d'autres types de coefficients de mélange que nous exposons dans le Chapitre 2. Comme pour le  $\alpha$ -mélange (cf. [DMR94]), il existe des résultats optimaux pour les suites  $\rho$ -mélangeantes [Pel87] ou  $\rho^*$ -mélangeantes [Bra92]. De même, les conditions de type Hannan, Maxwell-Woodrooffe et Dedecker-Rio sont optimales parmi les conditions mettant en jeu les mêmes quantités. De nombreux travaux ont donné des inégalités sur les queues ou les moments des sommes partielles pour les suites mélangeantes : [Rio00] pour le  $\alpha$ -mélange, [Vie97] pour le  $\beta$ -mélange, [Sha95] pour le  $\rho$ -mélange, [PG99] pour le  $\rho^*$ -mélange et [DP04] pour la  $\tau$ -dépendance. Le cas des conditions projectives a été bien étudié également : [Wu07] pour la condition de Hannan, [PUW07] pour la condition de Maxwell-Woodrooffe (1.2.10), [MP13] pour un autre critère projectif mettant en jeu l'espérance conditionnelle de  $S_n(f)^2$  sachant le passé et [Rio00] sous une condition mettant en jeu les quantités dans (1.2.9). D'autres structures de dépendance ont été considérées dans la littérature, comme les suites associées [LS02]

Cependant, d'autres théorèmes limites ont bénéficié d'une attention moindre, comme le principe d'invariance dans les espaces hölderiens. Des résultats optimaux ont été trouvés [RS04b] pour les suites i.i.d. ainsi que pour les champs i.i.d. [RSZ07]. Cependant, peu de résultats concernant les suites ou champs strictement stationnaires dépendants sont connus. Le résultat d'Hamadouche [Ham00] concernant les suites  $\alpha$ -mélangeantes ne redonne pas le résultat optimal lorsqu'on l'applique aux suites i.i.d. (ce-dernier est postérieur aux travaux d'Hamadouche).

Les applications concernent la détection de point de rupture, comme expliqué dans [RS07a, RS07b, RS06]. L'idée est de tester si un des paramètres d'un échantillon supposé indépendant a changé au cours du temps. L'utilisation de statistiques liées à la norme hölderienne du processus sommes partielles permet de détecter un changement de paramètre.

## Théorèmes limites pour les champs aléatoires

Les théorèmes limites pour les champs aléatoires ont été activement étudiés. Pour ces processus, l'ensemble d'indexation est l'ensemble des  $d$ -uplets d'entiers naturels ou relatifs. Celui-ci peut être muni d'une relation d'ordre total, mais en dimension plus grande que deux, celui-ci n'est pas aussi utile dans l'étude des sommes de champs aléatoires. En particulier, contrairement à la dimension un, il est délicat de définir la notion de passé et futur d'un champ aléatoire. La théorie des temps d'arrêts, qui donne des résultats puissants pour les martingales (voir par exemple [Bur73]), ne fonctionne pas en l'état. De plus, de nombreux résultats classiques pour les suites comme le lemme de Kronecker ou le théorème de convergence des trois séries de Kolmogorov ne s'étendent pas directement aux champs aléatoires. En particulier, la démonstration de la loi forte des grands nombres pour les champs i.i.d. n'est pas une conséquence immédiate ou une simple adaptation du résultat en dimension un. Il en va de même des autres théorèmes limites comme la loi des logarithmes itérés ou bien le théorème limite central fonctionnel.

Dans [Kue68, Wic69], la question de la convergence du processus (3.3.3) pour un champ aléatoire i.i.d. est traitée. L'une des richesses des champs aléatoires est que la formule (3.3.2) peut se généraliser, en considérant à la place de  $[\mathbf{0}, \mathbf{t}]$  un borélien de  $[0, 1]^d$  appartenant à une certaine classe  $\mathcal{A}$ . Le théorème limite central a été étudié dans ce contexte (voir par exemple [Bas85, AP86, Ded98, EVW13]). Dans le mémoire, nous nous concentrerons sur le cas particulier où  $\mathcal{A}$  est la classe des quadrants  $[\mathbf{0}, \mathbf{t}]$ ,  $\mathbf{t} \in [0, 1]^d$ . Dans [Ded01], Dedecker donne un principe d'invariance sous une condition projective. Cependant, l'obtention du principe d'invariance exige des moment d'ordre strictement plus grand que deux, ce qui n'est pas le cas du résultat obtenu par Basu et Dorea [BD79]. La question du théorème limite central pour les ortho-martingales au sens de Cairoli [Cai69] est longtemps restée sans réponse. Les généralisations des conditions de Hannan [VW14] et Maxwell-Woodrooffe [WW13] ne concernent que les champs aléatoires qui sont fonctions de champs i.i.d. En effet, contrairement au cas des suites, le théorème limite central n'est pas acquis pour les martingales comme le montre l'exemple de la Remarque 3.3.4. Le problème vient de l'absence d'ergodicité de toutes les transformations  $T_i$ ,  $1 \leq i \leq d$ . Cependant, si on suppose que l'une des transformations est ergodique, alors le théorème limite central a lieu (voir le Théorème 3.3.5 extrait de [Vol15] pour un énoncé précis). Les résultats de [WW13, VW14] ont donc lieu dans un contexte plus général que celui des champs bernoulliens. D'autres types de champs aléatoires jouant le rôle de martingales multidimensionnelles ont été étudiés dans [NP92].

La liste des champs aléatoires strictement stationnaires pour lesquels le principe d'invariance (pour les quadrants) a été étudié n'est pas exhaustive, les champs associés [Sha08, Bal05] ou mélangeants [KS95a, KS95b, Ton11] ont été considérés. L'attention a également été portée sur d'autres théorèmes limites comme la loi des grands nombres [Gut78] ou la loi des logarithmes itérés [Bas85, Leo76].

## Résultats et plan du mémoire

Notre mémoire est composé de cinq parties. La première est dédiée à une présentation des objets mathématiques qui interviennent dans les résultats : les suites strictement stationnaires, les martingales, les suites mélangeantes, et les champs aléatoires de type accroissements d'ortho-martingale ou bien bernoulliens. Nous y définissons également les théorèmes limites considérés dans le mémoire, et nous rappelons les résultats existants les concernant. Dans les parties suivantes, nous donnons les résultats obtenus.



## Contre-exemples au théorème limite central et théorème limite central fonctionnel

En toute généralité, le principe d'invariance dans l'espace des fonctions continues sur l'intervalle unité implique le théorème limite central classique. Si on dispose d'une suite strictement stationnaire qui vérifie le théorème limite central, il est raisonnable de se demander si elle vérifie de plus le principe d'invariance. Il est possible que cette suite ne vérifie pas les conditions suffisantes classiques d'approximation par martingale ou bien de mélange. Si la suite  $(Y_n)_{n \geq 1} := (\max_{1 \leq j \leq n} |f \circ T^j| / \sigma_n(f))_{n \geq 1}$  ne tend pas vers zéro en probabilité, on sait que le principe d'invariance ne peut pas avoir lieu. Herrndorf montre dans [Her83b] que la condition  $Y_n \rightarrow 0$  en probabilité n'est pas nécessairement vérifiée par une suite  $\phi$ -mélangeante de carré intégrable. Herrndorf a également montré que si  $Y_n \rightarrow 0$  en probabilité, la suite  $(f \circ T^j)_{j \geq 0}$  est  $\phi$ -mélangeante et vérifie le théorème limite central avec la normalisation  $\sigma_n(f)$ , alors le principe d'invariance a lieu. On remarque que ce résultat n'exige pas des moments finis d'ordre plus grand que deux pour  $f$ . Le  $\phi$ -mélange étant une condition restrictive, il est naturel de se demander si le résultat s'étend à des classes plus larges de processus mélangeants. Dans [GV14b], en collaboration avec Dalibor Volný, nous donnons un exemple de suite  $\beta$ -mélangeante, à variance linéaire, admettant des moments finis de tout ordre, qui vérifie le théorème limite central mais pas le principe d'invariance dans l'espace des fonctions continues sur l'intervalle unité. En particulier, avec les notations ci-dessus,  $Y_n \rightarrow 0$  en probabilité. Nous nous sommes également intéressés à la vitesse de convergence des coefficients de  $\beta$ -mélange. La version du Théorème A dans [GV14b] permet de construire, pour un  $\delta > 0$  fixé, le contre-exemple avec  $\beta(N)$  de l'ordre de  $N^{\delta-1/2}$ . Ceci a été amélioré dans [Gir15a], où les taux sont de l'ordre  $N^{\delta-1}$  à  $\delta > 0$  fixé. Au vu du résultat de [DMR94], on ne peut pas espérer des taux de mélange de l'ordre de  $N^{-q}$  pour  $q > 1$ . Dans le Théorème A', nous construisons le contre-exemple de manière à avoir une convergence arbitrairement rapide le long d'une sous-suite, *i.e.*, pour une suite  $(c_N)_{N \geq 1}$  qui décroît vers 0, on peut trouver un sous-ensemble infini  $I$  de l'ensemble des entiers naturels tel que  $\beta(N) \leq c_N$  pour tout  $N \in I$ . Puisque le processus construit possède des moments de tout ordre finis, on ne peut obtenir l'inégalité précédente pour tous les entiers naturels. En effet, le résultat d'Ibragimov [Ibr62] impliquerait le principe d'invariance si par exemple  $c_N = N^{-q}$  pour  $q$  strictement plus grand que 1.

Dans le chapitre 5, nous avons traité la question du théorème limite central en dimension infinie. Les articles [Den86, MY86, Vol88] montrent que pour une suite strictement stationnaire à valeurs dans  $\mathbb{R}^d$ ,  $d \geq 1$ , centrée et  $\alpha$ -mélangeante  $(f \circ T^i)_{i \geq 0}$  telle que  $\|S_n(f)\|_2 \rightarrow +\infty$ , la convergence en loi de la suite  $(S_n(f) / \|S_n(f)\|_2)_{n \geq 1}$  vers une loi normale centrée réduite est équivalente à l'uniforme intégrabilité de la famille  $\{S_n(f)^2 / \|S_n(f)\|_2^2, n \geq 1\}$ . On peut se demander si ce résultat peut s'étendre en dimension infinie. La validité du théorème limite central pour une suite i.i.d. de variables aléatoires à valeurs dans un espace de Banach dépend de sa géométrie et de son type. C'est pourquoi nous restreignons notre recherche aux espaces de Hilbert. Nous les supposons séparables afin d'éviter les problèmes de mesurabilité. Le chapitre 5 reprend l'article [GV14a], dont le résultat principal est le Théorème B. Dans celui-ci, nous construisons un processus strictement stationnaire  $\beta$ -mélangeant, à valeurs dans l'espace de Hilbert des suites réelles de carré sommable, qui admet des moments de tout ordre, dont l'espérance du carré de la norme de la  $n$ -ème somme partielle  $S_n(\mathbf{f})$ , notée  $\sigma_n(\mathbf{f})^2$ , tend vers l'infini lorsque  $n$  tend vers l'infini, et la suite  $(\|S_n(\mathbf{f})\|^2 / \sigma_n(\mathbf{f})^2)_{n \geq 1}$  est uniformément intégrable mais pour laquelle la suite  $(S_n(\mathbf{f}) / \sigma_n(\mathbf{f}))_{n \geq 1}$  n'admet aucune sous-suite équi-tendue. De plus, on montre qu'aucune normalisation ne convient : si  $(a_n)_{n \geq 1}$  est une suite qui tend vers l'infini, ou bien  $S_n(\mathbf{f}) / a_n$  tend vers 0 en loi, ou bien la suite  $(S_n(\mathbf{f}) / a_n)_{n \geq 1}$  n'est pas équi-tendue. Pour un  $\delta > 0$  donné, on peut construire un processus vérifiant les propriétés précédentes avec  $\beta(N) = O(N^{\delta-1})$ . Comme dans le Chapitre 4, nous pouvons construire le processus avec des taux de mélange arbitrairement rapides le long d'une sous-suite (voir le Théorème B'). De

plus, on peut modifier la construction de manière à ce que la quantité  $\sigma_N(\mathbf{f})$  tende vers l'infini arbitrairement lentement lorsque  $N$  tend vers l'infini.

## Principe d'invariance dans les espaces hölderiens pour les suites

Dans la partie [III](#), nous étudions le théorème limite central fonctionnel dans les espaces hölderiens. Définissons  $\mathcal{H}_\alpha[0, 1]$  l'espace des fonctions  $x: [0, 1] \rightarrow \mathbb{R}$  telles que  $\|x\|_{\mathcal{H}_\alpha} := \sup_{0 \leq s < t \leq 1} |x(t) - x(s)| / |t - s|^\alpha + |x(0)|$  soit fini. Pour tout  $\alpha$  strictement compris entre 0 et 1/2, le mouvement brownien standard a (presque sûrement) des trajectoires hölderiennes d'exposant  $\alpha$ . Les fonctions aléatoires définies par [\(1.3.4\)](#) sont  $\alpha$ -hölderiennes pour les mêmes  $\alpha$ . La question du principe d'invariance dans l'espace  $\mathcal{H}_\alpha$  est légitime. Lamperti [[Lam62](#)] fut le premier à traiter cette question en vue d'applications statistiques.

Étant donnée une suite  $(f \circ T^i)_{i \geq 0}$  qui vérifie le théorème limite central fonctionnel dans l'espace des fonctions continues sur l'intervalle unité, on peut se demander si ceci a lieu dans l'espace  $\mathcal{H}_\alpha[0, 1]$ . Puisque l'espace  $\mathcal{H}_\alpha[0, 1]$  n'est pas séparable, nous allons plutôt travailler avec le sous-espace  $\mathcal{H}_\alpha^o[0, 1]$  des fonctions  $x: [0, 1] \rightarrow \mathbb{R}$  telle que  $\sup_{|t-s| < \delta} |x(t) - x(s)| / |t - s|^\alpha \rightarrow 0$  lorsque  $\delta$  converge vers 0, qui muni de la norme  $\|\cdot\|_\alpha$ , est séparable. Les lois fini-dimensionnelles caractérisant une mesure de probabilité sur  $\mathcal{H}_\alpha^o[0, 1]$ , la démonstration du théorème limite central fonctionnel dans cet espace repose sur l'équi-tension de la suite  $(a_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$ . Lorsque  $a_n = \sqrt{n}$ , une condition nécessaire pour l'équi-tension est [\(1.3.23\)](#). Dans le cas i.i.d., ceci est équivalent à  $t^{1/(1/2-\alpha)} \mu\{|f| > t\} \rightarrow 0$  quand  $t \rightarrow +\infty$ . Comme  $1/(1/2 - \alpha)$  est strictement supérieur à 2, la fonction  $f$  a des moments finis d'ordre strictement plus grand que 2, donc il y a un certain "prix à payer" pour obtenir l'équi-tension de  $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$  dans  $\mathcal{H}_\alpha^o[0, 1]$ .

Nous disposons d'un critère de tension (voir [Théorème 1.3.7](#)) qui donne une condition suffisante sur les moments des sommes partielles. Il permet de retrouver l'approche utilisée par Lamperti [[Lam62](#)]. Cependant, il s'avère qu'il ne fournit pas le résultat optimal dans le cas i.i.d. trouvé par Račkauskas et Suquet [[RS03](#)], qui montre que la condition  $t^{1/(1/2-\alpha)} \mu\{|f| > t\} \rightarrow 0$  quand  $t \rightarrow +\infty$  est nécessaire et suffisante pour la convergence en loi de  $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$  vers un mouvement brownien standard. Leur approche utilise un critère de tension qui repose sur une décomposition de Schauder de l'espace  $\mathcal{H}_\alpha^o[0, 1]$ .

Dans le [Chapitre 6](#), nous reprenons les résultats de [[Gir15b](#)]. Nous commençons par donner un critère de tension général dans  $\mathcal{H}_\alpha^o[0, 1]$ ,  $0 < \alpha < 1/2$ , pour le processus sommes partielles associé à une suite strictement stationnaire (voir [\(6.1.3\)](#)). Lorsque l'on normalise ce processus par  $\sqrt{n}$ , ce critère a une forme plus simple (cf. [\(6.1.4\)](#)). Les inégalités de Nagaev pour les suites i.i.d. [[Nag79](#)] couplées au critère [\(6.1.4\)](#) permettent de retrouver le résultat de Račkauskas et Suquet. Nous étendons celui-ci au cas des suites  $\tau$ -dépendantes ainsi que celui des suites  $\rho$ -mélangeantes. Pour les suites  $\tau$ -dépendantes, nous obtenons une condition mettant en jeu la fonction de quantile de la fonction  $f$  ainsi que l'inverse généralisée des coefficients de  $\tau$ -dépendance (cf. [Théorème 6.2.1](#)). En utilisant une comparaison entre ces derniers et les coefficients d' $\alpha$ -mélange, nous obtenons une condition dans l'esprit de celle de Doukhan, Massart et Rio (cf. [[DMR94](#)]), donnée dans le [Corollaire 6.2.2](#). Nous montrons également qu'en gardant la même hypothèse sur la loi de  $f$  que dans le cas i.i.d., mais en remplaçant l'hypothèse d'indépendance par la convergence de la série  $\sum \rho(2^n)$ , le principe d'invariance a lieu dans  $\mathcal{H}_\alpha[0, 1]$ . Enfin, nous montrons qu'il est en général préférable de travailler avec des inégalités sur les queues des sommes partielles plutôt que sur les moments. En effet, si  $(f \circ T^i)_{i \geq 0}$  est une suite strictement stationnaire telle que la suite  $(n^{-1/2} S_n(f))_{n \geq 1}$  est bornée dans  $\mathbb{L}^p$  pour un  $p > 2$ , on peut montrer grâce au [Théorème 1.3.7](#) que  $f$  vérifie le principe d'invariance dans  $\mathcal{H}_\beta[0, 1]$  pour tout  $\beta < 1/2 - 1/p$ . En revanche, l'exemple donné dans la démonstration du [Théorème 6.2.6](#) montre qu'il n'est pas possible d'en déduire le principe d'invariance dans  $\mathcal{H}_{1/2-1/p}[0, 1]$ . La stratégie qui consiste à montrer le caractère borné dans  $\mathbb{L}^p$  de la famille des sommes partielles normalisées ne donne donc pas de résultat optimaux. En résumé, à moins que les inégalités de moments pour la suite considérée ne soient "très bonnes" (c'est-à-dire de type Rosenthal), il vaut mieux travailler avec des inégalités sur les queues des sommes partielles. Comme cette approche fonctionne aussi

dans le cas des suites vérifiant des inégalité de moments de type Rosenthal, nous privilégierons l'approche basée sur des inégalités sur les queues des sommes partielles.

Outre les suites mélangeantes ou faiblement dépendantes, les suites strictement stationnaires d'accroissements d'une martingale ont un rôle prépondérant dans la compréhension des théorèmes limites pour les suites strictement stationnaires. Afin de déterminer si une fonction  $f$  vérifie le principe d'invariance dans l'espace  $\mathcal{H}_\alpha[0, 1]$ , on peut, comme pour l'espace des fonctions continues, essayer de voir si on peut approximer dans un certain sens le processus sommes partielles par celui associé à une martingale. Cependant, il est nécessaire de savoir quelle type de condition imposer sur les moments de la suite d'accroissements d'une martingale  $(m \circ T^i)_{i \geq 0}$ . Comme les suites i.i.d. de loi commune intégrable et centrée sont des accroissements d'une martingale, il est nécessaire que  $t^{1/(1/2-\alpha)} \mu\{|m| > t\}$  tende vers 0 lorsque  $t$  tend vers l'infini. Contrairement au cas i.i.d., il s'avère que cette condition n'est plus suffisante. Nous construisons un contre-exemple dans la démonstration du Théorème 7.2.1. Cependant on obtient la conclusion voulue en imposant de plus une condition d'intégrabilité sur la variance quadratique (voir le Théorème 7.2.2). Comme la variance quadratique est constante dans le cas i.i.d., on retrouve le résultat de Račkauskas et Suquet. La démonstration repose sur le contrôle de la norme hölderienne du processus sommes partielles à l'aide de  $m$  et de la variance quadratique. Enfin, nous donnons une condition de type Hannan qui garantit le principe d'invariance dans  $\mathcal{H}_\alpha^2[0, 1]$ . Ces résultats ont fait l'objet d'un article [Gir15c].

Enfin, dans le Chapitre 8, nous donnons une condition dans l'esprit de celle de Maxwell et Woodroffe (cf. (1.2.10)). Dans le cas des suites i.i.d., le principe d'invariance dans l'espace  $\mathcal{H}_\alpha[0, 1]$  ( $0 < \alpha < 1/2$ ) impose une condition sur la queue de la loi commune qui est plus restrictive que la finitude du moment d'ordre deux. On ne peut donc pas espérer que (1.2.10) en l'état donne le principe d'invariance dans ces espaces. Au vu de ce que la condition de Hannan (1.2.8) pour le principe d'invariance dans l'espace des fonctions continues est devenue pour l'obtention du principe d'invariance dans les espaces hölderiens (voir (7.2.8)), il est raisonnable de conjecturer que remplacer la norme  $\mathbb{L}^2$  dans (1.2.10) par la norme  $\mathbb{L}^{1/(1/2-\alpha)}$  donne le principe d'invariance dans  $\mathcal{H}_\alpha[0, 1]$ . Dans le Théorème 8.1.1, nous démontrons cette conjecture *via* une approximation par martingales. Nous traitons également le cas non-adapté. Enfin, comme l'ont fait Peligrad et Utev [PU05], nous justifions l'optimalité de la condition trouvée parmi celles faisant intervenir les quantités  $\|\mathbb{E}[S_n(f) | \mathcal{M}]\|_{1/(1/2-\alpha)}$  en reprenant l'exemple de [PUW07]. Ces résultats ont fait l'objet d'un article soumis [Gir15d].

## Décomposition (ortho-)martingale cobord

L'obtention de théorèmes limites pour une suite strictement stationnaire  $(f \circ T^i)_{i \geq 0}$  peut se faire *via* une approximation par une martingale à accroissements strictement stationnaires. Un cas particulier est le suivant : la fonction  $f$  ne diffère de  $m$ , où  $(m \circ T^i)_{i \geq 0}$  est une suite d'accroissements d'une martingale, que d'un cobord, c'est-à-dire d'une fonction de la forme  $g - g \circ T$ , où  $g$  est une fonction mesurable. Dans ce cas, pour tout entier  $n \geq 1$ , l'égalité  $S_n(f) = S_n(m) + g - g \circ T^n$  a lieu. Il est donc raisonnable d'espérer que les bonnes propriétés de la suite  $(n^{-1/2} S_n(m))_{n \geq 1}$  se transmettent à la suite  $(n^{-1/2} S_n(f))_{n \geq 1}$ . Cependant, les choses ne sont pas aussi simples. Certes, le fait que la suite  $(n^{-1/2}(g - g \circ T^n))_{n \geq 1}$  converge vers 0 en probabilité permet de déduire le théorème limite central si  $m$  est de carré intégrable. Cependant, l'obtention du principe d'invariance dans l'espace des fonctions continues sur  $[0, 1]$  ne peut se faire que si  $(n^{-1/2} \max_{1 \leq j \leq n} |g \circ T^j|)$  tend vers 0 en probabilité. Ceci n'a pas nécessairement lieu, même si la fonction  $g - g \circ T$  est de carré intégrable (voir [VS00]). Similairement, la loi des logarithmes itérés et la loi forte des grands nombres ne sont pas garanties. Il est néanmoins possible de formuler des conditions d'intégrabilité sur les fonction  $g$  et  $g - g \circ T$  garantissant ces théorèmes limites. Plus précisément, si  $g \in \mathbb{L}^p$  et  $g - g \circ T \in \mathbb{L}^r$ , où  $1 < p < 2 < r$ , quelle condition imposer sur  $p$  et  $r$  pour que  $n^{-1/2} \max_{1 \leq j \leq n} |g \circ T^j| \rightarrow 0$  en probabilité ? Une réponse a été donnée par Volný et Samek dans [VS00]. Les résultats de l'article ne couvrent

pas les cas de tous les couples  $(p, r)$  : certains sont traités par la condition suffisante, d'autres par un contre-exemple, et si  $(r-1)/(r-3/2) \leq p < (r+2)/r$  on ne peut pas conclure. Dans le Chapitre 9, nous montrons que si  $p \geq r/(r-1)$ ,  $g \in \mathbb{L}^p$  et  $g - g \circ T \in \mathbb{L}^r$ , alors les suites  $(n^{-1/2} \max_{1 \leq j \leq n} |g \circ T^j|)_{n \geq 1}$  et  $((n \log \log n)^{-1/2} g \circ T^n)_{n \geq 1}$  tendent vers 0 en probabilité. Si  $p < r/(r-1)$ , nous fournissons un contre-exemple. Nous obtenons des résultats similaires pour la loi forte des grands nombres.

Le Chapitre 10 est dédié à la décomposition ortho-martingale cobord pour des champs aléatoires strictement stationnaires. Les résultats ont été obtenus en collaboration avec Mohamed El Machkouri et font l'objet d'un article [EMG]. Dans le cas des suites, la décomposition martingale/cobord permet d'approximer les sommes partielles par celles d'une martingale, pourvu que l'on arrive à montrer que celles du cobord sont négligeables. La question de l'extension de cette approche à  $\mathbb{Z}^d$  où  $d$  est un entier supérieur ou égal à 2 est naturelle. Cependant, nous nous heurtons à plusieurs difficultés. D'abord, il faut définir ce qui joue le rôle de martingale. Ensuite, il faut déterminer le type de filtration considéré. Dans notre travail, nous nous intéresserons aux ortho-martingales au sens de Cairoli [Cai69] (voir aussi Khoshnevisan [Kho02]) par rapport à des filtrations complètement commutantes (voir Définition 3.1.7). Dans le Théorème 10.2.4, nous obtenons une condition suffisante (cf. (10.2.2)) pour obtenir une décomposition de type ortho-martingale/cobord, donnée par (10.2.3). Elle est constituée de  $2^d$  termes. Le premier est une "vraie" différence d'ortho-martingales, le dernier est un "vraie" cobord. Les autres termes sont "hybrides", au sens suivant : si  $J$  est un sous-ensemble de  $\{1, \dots, d\}$  non vide et différent de  $\{1, \dots, d\}$ , le terme associé à ce sous-ensemble est une ortho-martingale par rapport aux applications  $T_j$ ,  $j \in \{1, \dots, d\} \setminus J$ , et un cobord par rapport aux opérateurs  $T_j$ ,  $j \in J$ . Un résultat similaire a été obtenu par Gordin [Gor09]. Son approche repose sur la résolution d'équations de Poisson, et fait intervenir les adjoints des opérateurs de Koopman. La décomposition ortho-martingale/cobord permet de traiter efficacement les sommations sur les rectangles, pour lesquelles nous établissons une inégalité de moments (voir la Proposition 10.2.8). L'idée est que pour chaque terme de la décomposition, les coordonnées associées au cobord jouent souvent un rôle négligeables si le terme  $m_J$  (cf. (10.2.3)) a de bonnes propriétés d'intégrabilité, tandis que la sommation par rapport aux autres coordonnées peut être traitée par la propriété d'ortho-martingale. La condition suffisante peut être facilement vérifiée par un processus linéaire (voir la Proposition 10.2.14). Nous retrouvons également le principe d'invariance de Volný et Wang [VW14]. Cependant, notre condition est en général plus restrictive. Nous proposons une application au principe d'invariance dans les espaces hölderiens (voir le Théorème 10.2.16). Ce résultat sera amélioré dans le Chapitre 12.

## Inégalités de queue pour les champs aléatoires

Les inégalités sur les queues des sommes partielles de suites strictement stationnaires sont très utiles pour établir des théorèmes limites. Elles interviennent dans les critères de tension. Après avoir traité le principe d'invariance dans les espaces hölderiens pour des suites, la question de l'obtention de résultats similaires pour les champs aléatoires est naturelle. Nous étudierons deux types de champs aléatoires : les ortho-martingales (au sens de la Définition 3.1.1) à accroissements strictement stationnaires et les champs qui sont des fonctionnelles de champs i.i.d., dits bernoulliens (voir la Définition 3.2.1).

Dans un premier temps, nous redémontrons le Théorème 1 de [Nag03]. Il s'applique aussi aux sous-martingales, mais la démonstration peut être simplifiée dans le cas des martingales. La Proposition 11.1.2 contient une majoration uniforme en  $n$  de la queue de la variable aléatoire  $n^{-1/2} \max_{1 \leq i \leq n} |S_i(M)|$  pour une martingale  $(S_i(M))_{i \geq 1}$  à accroissements strictement stationnaires. On peut donner une inégalité à l'aide des queues de la variance quadratique  $\mathbb{E}[M^2 | T\mathcal{M}]$  ou bien simplement en fonctions de celles de  $M$ . En utilisant le fait que pour tout  $q \in \{1, \dots, d\}$ , et pour tous  $\{n_l\}_{l \neq q} \subset \mathbb{N}$  fixés, la suite  $(M_n)_{n_q \in \mathbb{N}}$  est une martingale (à un paramètre) par rapport à la filtration  $\mathcal{F}^{(q)}$  (donnée par (3.1.4)), on obtient une majoration de la queue des maxima des

sommes partielles d'un champ aléatoire de type ortho-martingale à accroissements strictement stationnaires. Là encore, on dispose d'un résultat mettant en jeu la queue d'une des variances quadratiques (l'inégalité (11.1.36)) et l'autre seulement la queue de  $m$  (voir (11.1.37)). Il est possible d'obtenir un contrôle de la norme  $L^p$  du maximum des sommes partielles d'une ortho-martingale à accroissements strictement stationnaires pour  $p > 2$ . Pour les champs bernoulliens, nous exprimons l'inégalité de queue à l'aide de projecteurs définis par (3.2.10). La démonstration est inspirée de [LXW13]. Nous utilisons une décomposition des sommes partielles à l'aide de ces projecteurs, des propriétés de martingales pour traiter les maxima et les inégalités sur les queues de sommes de suites  $m$ -dépendantes.

Dans le Chapitre 12, nous fournissons un critère de tension dans l'espace des fonctions hölderiennes de  $[0, 1]^d$  dans  $\mathbb{R}$  pour le processus sommes partielles défini par (3.3.2) (voir la Proposition 12.1.2). Celui-ci met en jeu les maxima des sommes partielles sur les rectangles, et les inégalités (11.1.36) et (11.2.10) permettent de contrôler ces quantités. Ceci mène au Théorème 12.1.5. Nous montrons que dans le cas des champs de type ortho-martingale à accroissements strictement stationnaires, la condition sur la loi commune du cas i.i.d. n'est pas suffisante (Théorème 12.1.4). Enfin, nous donnons des conditions de type Hannan : une à l'aide des opérateurs de projection donnés par (3.1.10) (voir (12.1.20)), une autre avec ceux donnés par (3.2.10) (voir (12.1.23)). La première ne permet pas de retrouver le résultat du cas i.i.d. car elle ne prend pas en compte les variances quadratiques. Cependant, elle s'applique à un champ aléatoire linéaire (cf. (3.2.2)) où les innovations  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  ne sont pas nécessairement i.i.d., mais des accroissements strictement stationnaires d'ortho-martingale. La condition (12.1.23) redonne le résultat de [RSZ07] concernant les champs i.i.d.

**Part I**  
**Préliminaires**



# Chapitre 1

## Théorèmes limites pour les suites strictement stationnaires

Dans ce premier chapitre, nous définissons les principaux objets qui interviennent dans ce mémoire. Nous commençons par présenter les suites strictement stationnaires, qui interviendront dans les Parties II et III ainsi que le Chapitre 9. Une attention particulière est donnée aux suites strictement stationnaires d'accroissements d'une martingale. Nous définissons ensuite les théorèmes limites étudiés dans cette thèse. Nous rappelons les résultats existants qui les concernent et qui sont en lien avec ceux de la Partie III et le Chapitre 9.

### 1.1 Suites strictement stationnaires

#### 1.1.1 Une représentation générale

Dans tout le manuscrit, nous noterons par  $(\Omega, \mathcal{F}, \mu)$  un espace probabilisé, ce qui signifie que  $\Omega$  est un ensemble non vide muni d'une tribu  $\mathcal{F}$  et d'une mesure de probabilités  $\mu$ .

**Définition 1.1.1.** *La suite de variables aléatoires  $(X_i)_{i \geq 0}$  est dite strictement stationnaire si pour tout entier  $d$ , les vecteurs  $(X_0, \dots, X_d)$  et  $(X_1, \dots, X_{d+1})$  ont la même loi.*

Si  $T: \Omega \rightarrow \Omega$  est une application préservant la mesure, on appelle  $(\Omega, \mathcal{F}, \mu, T)$  un système dynamique.

Pour toute fonction  $f: \Omega \rightarrow \mathbb{R}$ , la suite  $(f \circ T^i)_{i \geq 1}$  est strictement stationnaire. Réciproquement, toute suite strictement stationnaire  $(X_i)_{i \geq 0}$ , il existe un système dynamique  $(\Omega, \mathcal{F}, \mu, T)$  tel que les suites  $(X_i)_{i \geq 0}$  et  $(f \circ T^i)_{i \geq 0}$  ont la même loi (voir p. 178 de [CFS82] ou bien [Doo53], p. 456, deuxième paragraphe).

Signalons qu'il existe aussi une représentation à l'aide des fonctionnelles de chaînes de Markov (par exemple, voir [Vol10]). Celle-ci est particulièrement pratique lors de l'étude des théorèmes limites dits *quenched*, mais nous n'étudions pas ce type de résultats dans cette thèse. La représentation *via* les systèmes dynamiques sera utilisée dans la grande majorité des cas. Ceci permettra d'utiliser des outils de la théorie ergodique.

Nous continuons cette sous-section par des résultats connus sur les systèmes dynamiques et qui seront utilisés tout au long de la thèse. Nous nous fixons donc une fois pour toute un système dynamique  $(\Omega, \mathcal{F}, \mu, T)$ .

**Définition 1.1.2.** *Soit  $f: \Omega \rightarrow \mathbb{R}$  une fonction mesurable. On note  $U_T(f)$  la fonction définie sur  $\Omega$ , à valeurs réelles, telle que  $U_T(f)(\omega) = f(T(\omega))$ . On définit ainsi un opérateur  $U_T$  sur l'espace vectoriel des fonctions mesurables. Cet opérateur est appelé opérateur de Koopman.*



Lorsqu'il n'y aura pas d'ambiguïté, on notera simplement  $U$ . Celui-ci est lié à l'espérance conditionnelle de la manière suivante.

**Lemme 1.1.3.** *Soit  $\mathcal{M}$  une sous-tribu de  $\mathcal{F}$  et  $f$  une fonction intégrable. Alors*

$$U\mathbb{E}[f \mid \mathcal{M}] = \mathbb{E}[Uf \mid T^{-1}\mathcal{M}]. \quad (1.1.1)$$

Si  $f: \Omega \rightarrow \mathbb{R}$  est une fonction mesurable et  $\theta: \Omega \rightarrow \Omega$  une application qui preserve la mesure  $\mu$ , on définit les sommes partielles par

$$S_n(\theta, f) := \sum_{j=0}^{n-1} f \circ \theta^j = \sum_{j=0}^{n-1} U_\theta^j f. \quad (1.1.2)$$

Lorsque le  $\theta$  considéré est  $T$ , on notera simplement cette quantité  $S_n(f)$ . Nous allons nous intéresser au comportement asymptotique de ces sommes partielles.

L'un des premiers résultats dans cette direction est le théorème ergodique de Birkhoff.

Avant de l'énoncer, rappelons la définition la tribu des invariants, notée  $\mathcal{I}$  : il s'agit de la collections des éléments  $A \in \mathcal{F}$  tels que  $T^{-1}A = A$ .

**Définition 1.1.4** (Fonction maximale). *Pour une fonction  $f$  mesurable et  $\theta: \Omega \rightarrow \Omega$  une application qui preserve la mesure  $\mu$ , on note*

$$M^*(\theta, f) := \sup_{n \geq 1} \frac{|S_n(\theta, f)|}{n}. \quad (1.1.3)$$

Là encore, on notera simplement  $M^*(f)$  dans le cas  $\theta = T$ .

**Théorème 1.1.5.** *Soit  $f: \Omega \rightarrow \mathbb{R}$  une fonction intégrable. Alors la suite  $(S_n(f)/n)_{n \geq 1}$  converge dans  $\mathbb{L}^1$  et presque sûrement vers  $\mathbb{E}[f \mid \mathcal{I}]$ .*

*De plus, pour tout  $\lambda$  strictement positif, l'inégalité suivante a lieu :*

$$\lambda \mu \{M^*(f) > \lambda\} \leq \int_{\{M^*(f) > \lambda\}} f d\mu. \quad (1.1.4)$$

La première partie du Théorème 1.1.5 est due à Birkhoff (voir [Bir31]), la seconde à Wiener [Wie39], Yosida et Kakutani [YK39].

**Définition 1.1.6.** *Le système dynamique  $(\Omega, \mathcal{F}, \mu, T)$  est dit ergodique si les éléments de la tribu  $\mathcal{I}$  sont tous de mesure 0 ou 1.*

Si le système dynamique est ergodique, alors  $S_n(f)/n \rightarrow \mathbb{E}[f]$  presque sûrement.

**Définition 1.1.7.** *Un système dynamique de Lebesgue est dit apériodique si pour tout entier  $k \neq 0$ ,*

$$\mu \{\omega \in \Omega \mid T^k \omega = \omega\} = 0. \quad (1.1.5)$$

Il existe une définition plus générale pour des systèmes dynamiques généraux.

**Lemme 1.1.8** ([Hal56]). *Soit  $(\Omega, \mathcal{F}, \mu, T)$  un système dynamique ergodique et apériodique. Pour tout entier  $n$  et tout réel positif  $\varepsilon$ , il existe un ensemble mesurable  $A = A(n, \varepsilon)$  tel que les ensembles  $(T^i A)_{i=0}^{n-1}$  soient deux à deux disjoints et  $\mu \left( \bigcup_{i=0}^{n-1} T^i A \right) > 1 - \varepsilon$ .*

Ce résultat a été démontré dans le cas où l'espace probabilisé est non-atomique par Kakutani et Rokhlin respectivement dans [Kak43] et [Roh48]. Dans la suite, nous désignerons le Lemme 1.1.8 par « lemme de Rokhlin ».

### 1.1.2 Accroissements de martingales

Commençons par rappeler la définition de martingale en toute généralité.

**Définition 1.1.9.** Soit  $(\mathcal{F}_i)_{i \geq 0}$  une suite croissante de tribus et  $(S_i)_{i \geq 0}$  une suite de variables aléatoires intégrables. On dit que la suite  $(S_i)_{i \geq 0}$  est une martingale par rapport à la filtration  $(\mathcal{F}_i)_{i \geq 0}$  si pour tout  $i \geq 1$ , la variable aléatoire  $S_i$  est  $\mathcal{F}_i$ -mesurable et  $\mathbb{E}[S_i | \mathcal{F}_{i-1}] = S_{i-1}$ .

Pour le moment, nous ne faisons aucune hypothèse de stationnarité. Même dans ce contexte, les martingales jouissent de la propriété remarquable suivante concernant la queue de leur maxima. Celle-ci est liée à la queue des maxima des accroissements (notés  $X_i := S_i - S_{i-1}$  pour  $i \geq 1$  et  $X_0 = S_0$ ), ainsi qu'à celle des variances quadratiques  $\mathbb{E}[X_i^2 | \mathcal{F}_{i-1}]$ .

**Proposition 1.1.1.** Soit  $(S_i)_{i \geq 0}$  une martingale par rapport à la filtration  $(\mathcal{F}_i)_{i \geq 0}$ . Alors pour tous réels  $\lambda > 0$ ,  $\beta > 1$  et  $0 < \delta < \beta - 1$  et pour tout entier  $n \geq 1$ ,

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq N} |S_i| > \beta \lambda \right\} &\leq \frac{\delta^2}{(\beta - \delta - 1)^2} \mu \left\{ \max_{1 \leq i \leq N} |S_i| > \lambda \right\} + \\ &+ \mu \left\{ \sum_{i=1}^N \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] > \delta^2 \lambda^2 \right\} + \mu \left\{ \max_{1 \leq i \leq N} |X_i| > \delta \lambda \right\} \end{aligned} \quad (1.1.6)$$

Ce résultat figure dans l'article [Bur73], équation (11.4) ainsi que dans [HH80] page 28. Il permet de contrôler les moments des maxima de la martingale à l'aide de ceux des maxima des accroissements et des sommes des variances quadratiques. En effet, en multipliant des deux côtés de (1.1.6) par  $p\lambda^{p-1}$  pour  $p > 1$ , il vient

$$\begin{aligned} \frac{1}{\beta^p} \mathbb{E} \left[ \max_{1 \leq i \leq N} |S_i|^p \right] &\leq \frac{\delta^2}{(\beta - \delta - 1)^2} \mathbb{E} \left[ \max_{1 \leq i \leq N} |S_i|^p \right] \\ &+ \frac{1}{\delta^{2p}} \mathbb{E} \left[ \left( \sum_{i=1}^N \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] \right)^{p/2} \right] + \frac{1}{\delta^p} \mathbb{E} \left[ \max_{1 \leq i \leq N} |X_i|^p \right]. \end{aligned} \quad (1.1.7)$$

En choisissant  $\delta$  tel que  $\delta^2 < (\beta - \delta - 1)^2 \beta^p$ , on obtient que pour une constante  $C_p$  ne dépendant que de  $p$ ,

$$\mathbb{E} \left[ \max_{1 \leq i \leq N} |S_i|^p \right] \leq C_p \mathbb{E} \left[ \left( \sum_{i=1}^N \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] \right)^{p/2} \right] + C_p \mathbb{E} \left[ \max_{1 \leq i \leq N} |X_i|^p \right]. \quad (1.1.8)$$

Dans le cas particulier où la suite  $(X_i)_{i \geq 1}$  est indépendante et à moyenne nulle, le terme  $\mathbb{E}[X_i^2 | \mathcal{F}_{i-1}]$  est  $\mathbb{E}[X_i^2]$  donc (1.1.8) devient

$$\mathbb{E} \left[ \max_{1 \leq i \leq N} |S_i|^p \right] \leq C_p \mathbb{E} \left[ \left( \sum_{i=1}^N \mathbb{E}[X_i^2] \right)^{p/2} \right] + C_p \mathbb{E} \left[ \max_{1 \leq i \leq N} |X_i|^p \right]. \quad (1.1.9)$$

*Démonstration de la Proposition 1.1.1.* Fixons un réel strictement positif  $\lambda$  et posons

$$E_k := \left\{ \lambda < \max_{1 \leq i \leq k-1} |S_i| \leq \beta \lambda; \max_{1 \leq i \leq k-1} |X_i| \leq \delta \lambda; \sum_{i=1}^k \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] \leq \delta^2 \lambda^2 \right\}.$$

Posons  $T_i := \sum_{k=1}^i \mathbf{1}_{E_k} X_k$ . L'inclusion suivante a lieu :

$$\left\{ \max_{1 \leq i \leq N} |S_i| > \beta \lambda; \sum_{i=1}^N \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] \leq \delta \lambda \right\} \subset \left\{ \max_{1 \leq i \leq n} |T_i| > (\beta - \delta - 1) \lambda \right\}.$$

Soit en effet  $\omega$  dans le membre de gauche et  $i_0$  l'indice  $i$  minimal pour lequel  $|S_i| > \beta\lambda$ . Puisque

$$T_i = S_{i_0} - X_{i_0} + \sum_{k=1}^{i_0-1} \mathbf{1}_{E_k^c \cap \{X > \beta\lambda; Y \leq \delta\lambda\}} X_k,$$

alors

$$|T_i| \geq \beta\lambda - \delta\lambda - \left| \sum_{k=1}^{i_0-1} X_k \right|.$$

puisque  $\omega \notin E_k$  pour  $k \leq i_0 - 1$ .

Remarquons que  $E_k \in \mathcal{F}_{k-1}$ , ainsi  $(T_n, \mathcal{F}_n)_{n \geq 1}$  est une martingale. On en déduit, par l'ingal-  
tité maximale de Doob et l'inclusion précédente

$$\begin{aligned} \mu \left( \left\{ \max_{1 \leq i \leq N} |S_i| > \beta\lambda \right\} \cap \left\{ \max \left\{ \sum_{i=1}^N \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}], \max_{1 \leq i \leq n} |X_i| \right\} \leq \delta\lambda \right\} \right) &\leq \\ &\leq (\beta - \delta - 1)^{-2} \lambda^{-2} \mathbb{E}(T_n^2). \end{aligned} \quad (1.1.10)$$

En utilisant encore le fait que  $E_k \in \mathcal{F}_{k-1}$ , il vient

$$\mathbb{E}(T_n^2) = \mathbb{E} \left[ \sum_{k=1}^N \mathbf{1}_{E_k} \mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}] \right].$$

Posons  $f_k := \sum_{i=1}^k \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}]$  et  $F_k := \{f_k \leq \delta^2 \lambda^2\}$ , avec  $f_0 := 0$ .

Puisque

$$E_k \subset \left\{ \max_{1 \leq i \leq N} |S_i| \geq \lambda \right\} \cap F_k,$$

alors

$$\sum_{k=1}^N \mathbf{1}_{E_k} \mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}] \leq \mathbf{1} \left\{ \max_{1 \leq i \leq N} |S_i| \geq \lambda \right\} \sum_{k=1}^N \mathbf{1}_{F_k} (f_k - f_{k-1}) \quad (1.1.11)$$

$$= \mathbf{1} \left\{ \max_{1 \leq i \leq N} |S_i| \geq \lambda \right\} \sum_{j=1}^N \mathbf{1}_{F_j} f_j - \sum_{j=1}^{N-1} \mathbf{1}_{F_{j+1}} f_j \quad (1.1.12)$$

$$= \mathbf{1} \left\{ \max_{1 \leq i \leq N} |S_i| \geq \lambda \right\} \left( \mathbf{1}_{F_N} f_N + \sum_{j=1}^{N-1} (\mathbf{1}_{F_j} - \mathbf{1}_{F_{j+1}}) f_j \right). \quad (1.1.13)$$

Comme  $F_{j+1} \subset F_j$  pour tout  $j \in \{1, \dots, N-1\}$  et  $f_j \leq \delta^2 \lambda^2$ , il vient

$$\sum_{j=1}^{N-1} (\mathbf{1}_{F_j} - \mathbf{1}_{F_{j+1}}) f_j = \sum_{j=1}^{N-1} \mathbf{1}_{F_j \setminus F_{j+1}} f_j \leq \delta^2 \lambda^2 \sum_{j=1}^{N-1} \mathbf{1}_{F_j \setminus F_{j+1}} = \delta^2 \lambda^2 (\mathbf{1}_{F_1} - \mathbf{1}_{F_N}). \quad (1.1.14)$$

En reportant ceci dans (1.1.13), nous déduisons

$$\sum_{k=1}^N \mathbf{1}_{E_k} \mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}] \leq \delta^2 \lambda^2 \mathbf{1} \left\{ \max_{1 \leq i \leq N} |S_i| \geq \lambda \right\}, \quad (1.1.15)$$

donc

$$\mathbb{E} [T_n^2] \leq \delta^2 \lambda^2 \mu \left\{ \max_{1 \leq i \leq N} |S_i| \geq \lambda \right\}. \quad (1.1.16)$$

Au vu de (1.1.13), nous obtenons (1.1.6).  $\square$

On considère un système dynamique  $(\Omega, \mathcal{F}, \mu, T)$ .

**Définition 1.1.10.** *On dit que la filtration  $(\mathcal{F}_i)_{i \geq 0}$  est stationnaire s'il existe une sous-tribu  $\mathcal{M}$  vérifiant  $\mathcal{F}_i = T^{-i}\mathcal{M}$  pour tout  $i \geq 0$ .*

Considérons une filtration stationnaire  $(T^{-i}\mathcal{M})_{i \geq 0}$  et une fonction  $m: \Omega \rightarrow \mathbb{R}$  que l'on suppose  $\mathcal{M}$ -mesurable. Si  $\mathbb{E}[m | T\mathcal{M}] = 0$ , alors le Lemme 1.1.3 permet de déduire que la suite  $(S_n(f))_{n \geq 0}$  est une martingale par rapport à la filtration  $(T^{-n}\mathcal{M})_{n \geq 0}$ .

## 1.2 Théorème limite central

### 1.2.1 Définition et cas des martingales

Lorsque  $(X_i)_{i \geq 1}$  est une suite indépendante identiquement distribuée telle que  $\mathbb{E}[X_1] = 0$  et  $\mathbb{E}[X_1^2]$  est finie, alors la suite  $(n^{-1/2} \sum_{i=1}^n X_i)_{n \geq 1}$  converge en loi vers une variable aléatoire normale centrée réduite.

Plaçons nous à présent dans le contexte de la Section 1.1.

**Définition 1.2.1.** *On dit que la suite strictement stationnaire  $(f \circ T^i)_{i \geq 0}$  vérifie le théorème limite central avec normalisation  $(a_n)_{n \geq 1}$  s'il existe une suite  $(b_n)_{n \geq 1}$  telle que*

$$\frac{1}{a_n} (S_n(f) - b_n) \rightarrow N \text{ en loi quand } n \rightarrow \infty. \quad (1.2.1)$$

Une suite i.i.d. vérifie le théorème limite central avec la normalisation  $(\sqrt{n})_{n \geq 1}$ . C'est également le cas des martingales à accroissements strictement stationnaires ergodique.

**Théorème 1.2.2.** *Soit  $(m \circ T^k)_{k \geq 0}$  une suite ergodique d'accroissements d'une martingale telle que  $\sigma^2 := \mathbb{E}[m^2]$  soit fini et strictement positif. Alors la suite  $(S_n(m)/(\sqrt{n}\sigma))_{n \geq 1}$  converge en loi vers une loi normale centrée réduite lorsque  $n \rightarrow +\infty$ .*

Ce résultat fut démontré indépendamment par Billingsley [Bil61] et Ibragimov [Ibr63].

### 1.2.2 Approximation par martingales

Le résultat de Billingsley et Ibragimov permet de déduire le théorème limite central pour des processus dont les sommes partielles sont proches de celles d'une suite strictement stationnaire d'accroissements d'une martingale. Le premier résultat dans cette direction est dû à Gordin (voir [Gor69]). Il considère un système dynamique ergodique muni d'une filtration stationnaire  $(\mathcal{F}_k)$  et note  $H_k$  la collection des fonctions  $\mathcal{F}_k$ -mesurables et de carré intégrable, et  $Q$  l'ensemble des fonctions  $Y$  telles qu'il existe des entiers  $i < k$  pour lesquels  $Y \in H_i \ominus H_k$ .

**Théorème 1.2.3.** *Si  $f \in \mathbb{L}^2$  est centrée et si*

$$\inf_{h \in Q} \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[ (S_n(f - h))^2 \right] = 0, \quad (1.2.2)$$

*alors  $(S_n(f)/\sqrt{n})_{n \geq 1}$  converge en loi vers une variable aléatoire normale centrée.*

**Définition 1.2.4.** *On dit que la fonction  $f$  admet une décomposition martingale-cobord s'il existe une filtration stationnaire  $(T^{-i}\mathcal{M})_{i \in \mathbb{Z}}$  et des fonctions  $m, g$  telles que*

$$f = m + g - g \circ T, \quad (1.2.3)$$

*où  $m$  est une fonction  $\mathcal{M}$ -mesurable vérifiant  $\mathbb{E}[m | T\mathcal{M}] = 0$ .*

*Le terme  $g - g \circ T$  est appelé cobord, et la fonction  $g$  est dite « fonction de transfert ».*

Supposons que  $f$  admette la décomposition (1.2.3) avec  $m \in \mathbb{L}^2$ . Alors  $S_n(f) = S_n(m) + g - g \circ T^n$ . Comme  $T^n$  préserve la mesure  $\mu$ ,

$$\mu \{ |g - g \circ T^n| / \sqrt{n} > 2\varepsilon \} \leq 2\mu \{ |g| > \varepsilon\sqrt{n} \}, \quad (1.2.4)$$

et on peut déduire du Théorème 1.2.2 que  $f$  vérifie le théorème limite central avec la normalisation  $(\sqrt{n})_{n \geq 1}$ .

**Définition 1.2.5.** Soit  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  une filtration stationnaire. On dit que la fonction  $f$  est régulière si elle est mesurable pour la tribu engendrée par les  $\mathcal{F}_i$ ,  $i \in \mathbb{Z}$  et si  $\mathbb{E}[f | \bigcap_{i \in \mathbb{Z}} \mathcal{F}_i] = 0$ .

**Théorème 1.2.6** (Volný, [Vol93, Vol06a]). Soit  $f$  une fonction régulière par rapport à une filtration stationnaire. Pour tout  $p \in [1, \infty[$ , les conditions suivantes sont équivalentes :

1.  $f$  admet la décomposition (1.2.3) où  $m$  et  $g$  appartiennent à  $\mathbb{L}^p$  ;
2. les séries

$$\sum_{n=0}^{+\infty} \mathbb{E}[U^n f | \mathcal{F}_0] \quad \text{et} \quad \sum_{n=0}^{+\infty} (U^{-n} f - \mathbb{E}[U^{-n} f | \mathcal{F}_0]) \quad (1.2.5)$$

convergent dans  $\mathbb{L}^p$ .

On dispose d'un critère permettant de déterminer si une suite strictement stationnaire ergodique d'accroissements d'une martingale est de carré intégrable.

**Théorème 1.2.7** (Esseen et Janson, [EJ85]). Soit  $(m \circ T^i)_{i \geq 0}$  une suite ergodique d'accroissements d'une martingale. Les conditions suivantes sont équivalentes :

1. la fonction  $m$  est de carré intégrable ;
2.  $\liminf_{n \rightarrow \infty} n^{-1/2} \mathbb{E}|S_n(m)|$  est finie.

En rassemblant les résultats précédents, on obtient la condition suffisante suivante pour le théorème limite central avec normalisation  $(\sqrt{n})_{n \geq 1}$  :

$$f \text{ admet une décomposition martingale-cobord dans } \mathbb{L}^1 \text{ et } \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathbb{E}|S_n(f)| < \infty. \quad (1.2.6)$$

Dans la suite de cette sous-section, nous considérons une filtration stationnaire  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  et nous supposons que le système dynamique est ergodique. Définissons les opérateurs de projection

$$P_j(f) := \mathbb{E}[f | \mathcal{F}_j] - \mathbb{E}[f | \mathcal{F}_{j-1}]. \quad (1.2.7)$$

Par les articles [Han73, Hey74], on sait que si  $f$  est régulière et centrée, la condition

$$\sum_{i \in \mathbb{Z}} \|P_i(f)\|_2 < \infty \quad (1.2.8)$$

est suffisante pour obtenir le théorème limite central avec la normalisation  $(\sqrt{n})_{n \geq 1}$ .

Si  $f$  est une variable aléatoire centrée et  $\mathcal{F}_0$  mesurable, Dedecker et Rio ont montré dans [DR00] que si

$$f \mathbb{E}[S_n(f) | \mathcal{F}_0] \text{ converge dans } \mathbb{L}^1, \quad (1.2.9)$$

alors le théorème limite central a lieu avec la normalisation  $(\sqrt{n})_{n \geq 1}$ .

Enfin Maxwell et Woodroffe (cf. [MW00]) ont démontré que pour une fonction  $f$  centrée et  $\mathcal{F}_0$ -mesurable, le théorème limite central a lieu avec la normalisation  $(\sqrt{n})_{n \geq 1}$  sous la condition

$$\sum_{n=1}^{+\infty} \frac{\|\mathbb{E}[S_n(f) | \mathcal{F}_0]\|_2}{n^{3/2}} < \infty. \quad (1.2.10)$$

Nous n'avons mentionné que les conditions suffisantes qui interviendront dans le cadre de cette thèse. Il existe d'autres moyens d'obtenir le théorème limite central dans le contexte des systèmes dynamiques (voir par exemple [Der06]).

### 1.3 Théorème limite central fonctionnel

Dans [EK46], Erdős et Kac ont remarqué que la loi limite du maxima normalisé des sommes partielles d'une suite i.i.d.  $(X_i)$  ne dépend pas de la loi commune (pourvu que  $X_1$  soit centrée et de carré intégrable).

Ceci a été généralisé par Donsker [Don51] :

**Théorème 1.3.1.** *Soit  $(X_i)_{i \geq 1}$  une suite i.i.d. où  $X_1$  est de moyenne nulle et de variance finie  $\sigma^2$ . Alors le processus*

$$\left( \frac{1}{\sigma\sqrt{n}} \left( \sum_{i=1}^{[nt]} X_i + (nt - [nt])X_{[nt]+1} \right) \right)_{t \in [0,1]} \quad (1.3.1)$$

converge en loi vers la mesure de Wiener dans l'espace des fonctions continues sur l'intervalle  $[0, 1]$  muni de la norme uniforme.

On rappelle qu'un processus  $\{W(t), t \in [0, 1]\}$  est distribué suivant la mesure de Wiener si ses accroissements sont indépendants et pour tout  $t \in [0, 1]$ ,  $W(t)$  suit une loi normale centrée de variance  $t$ . On dira aussi que  $(W(t))_{t \in [0,1]}$  est un mouvement brownien standard.

Là encore, le processus limite est le même quelle que soit la loi de  $X_1$ , pourvu que  $X_1$  soit centrée et de carré intégrable. La convergence du processus (1.3.1) en loi vers la mesure de Wiener sera par conséquent appelée principe d'invariance. On emploiera aussi le terme de théorème limite central fonctionnel.

Cette section est consacrée à l'extension du Théorème 1.3.1 au contexte des systèmes dynamiques, et à l'étude de tels processus dans des espaces fonctionnels différents.

#### 1.3.1 Espace de Skorohod

On se donne un système dynamique  $(\Omega, \mathcal{F}, \mu, T)$ . Pour une fonction  $f: \Omega \rightarrow \mathbb{R}$ , on pose

$$S_n^{\text{ps}}(f, t) := S_{[nt]}(f), t \in [0, 1], \quad (1.3.2)$$

où  $[x]$  désigne la partie entière inférieure du réel  $x$ . Les lettres « ps » signifient « partial sum ».

Pour tout  $\omega \in \Omega$  fixé, la fonction  $t \mapsto S_n^{\text{ps}}(f, t)$  est continue à droite et admet une limite à gauche en tout point de  $]0, 1]$  (càdlàg). La collection des fonctions càdlàg peut être munie d'une métrique qui en fait un espace métrique séparable complet, noté  $D[0, 1]$  (voir [Bil68]). Le Théorème 1.3.1 reste valide dans ce contexte, en remplaçant le processus (1.3.1) par

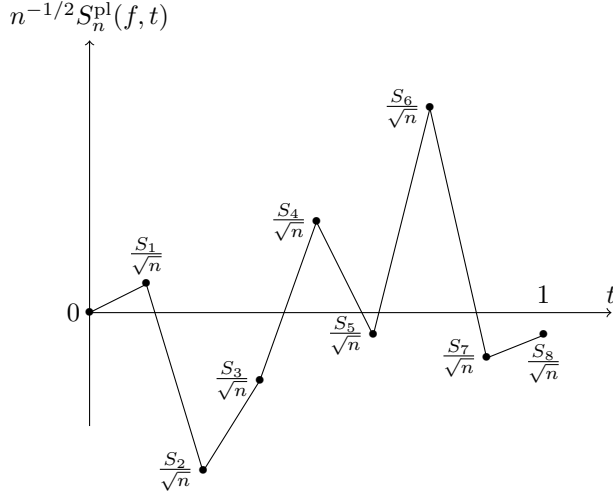
$$\left( \frac{1}{\sigma\sqrt{n}} \left( \sum_{i=1}^{[nt]} X_i \right) \right)_{t \in [0,1]} . \quad (1.3.3)$$

#### 1.3.2 Espace des fonctions continues

Considérons un système dynamique ergodique  $(\Omega, \mathcal{F}, \mu, T)$ . Comme le mouvement brownien possède des trajectoires (presque sûrement) continues, on peut faire l'étude du théorème limite central fonctionnel dans l'espace des fonctions continues sur l'intervalle  $[0, 1]$ . On munit cet espace vectoriel de la norme uniforme. L'extension de la définition du processus (1.3.1) aux suites strictement stationnaires est la suivante :

$$S_n^{\text{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}, t \in [0, 1]. \quad (1.3.4)$$

Les lettres « pl » signifient « polygonal line » car pour tout  $\omega \in \Omega$  fixé, le graphe de la fonction  $t \mapsto S_n^{\text{pl}}(f, t)(\omega)$  est affine par morceaux.

FIGURE 1.1 – La fonction  $t \mapsto n^{-1/2}S_n^{\text{pl}}(f, t)$  pour  $n = 8$ 

**Définition 1.3.2.** Soit  $(E, \|\cdot\|_E)$  un espace de fonctions à valeurs réelles définies sur l'intervalle  $[0, 1]$ , muni d'une norme  $\|\cdot\|_E$ . On dit que la fonction  $f$  vérifie le principe d'invariance dans  $E$  avec la normalisation  $(a_n)_{n \geq 1}$  si la convergence

$$\frac{1}{a_n} S_n^{\text{pl}}(f, \cdot) \rightarrow W \text{ en loi dans } E \text{ quand } n \rightarrow +\infty \quad (1.3.5)$$

a lieu, où  $W$  est un mouvement brownien standard. Autrement dit, pour toute fonction  $F: E \rightarrow \mathbb{R}$  continue et bornée, la convergence en loi

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ F \left( \frac{1}{a_n} S_n^{\text{pl}}(f, \cdot) \right) \right] = \mathbb{E} [F(W)] \quad (1.3.6)$$

a lieu dans  $\mathbb{R}^d$ .

Dans la section précédente, nous avons vu des conditions suffisantes pour le théorème limite central. Elles fournissent également la convergence des lois fini-dimensionnelles du processus  $(n^{-1/2}S_n^{\text{pl}}(f, t), t \in [0, 1])_{n \geq 1}$ , c'est-à-dire que pour tout entier  $d \geq 1$  et tous  $t_1, \dots, t_d \in [0, 1]$ , la convergence

$$\left( n^{-1/2} S_n^{\text{pl}}(f, t_i) \right)_{i=1}^d \rightarrow (W(t_i))_{i=1}^d \quad (1.3.7)$$

a lieu. Comme les lois fini-dimensionnelles caractérisent une mesure de probabilité sur  $C[0, 1]$ , il faut montrer l'équi-tension de la suite  $(n^{-1/2}S_n^{\text{pl}}(f, t))_{n \geq 1}$  dans l'espace  $C[0, 1]$ , ce qui se fait à l'aide d'inégalités de moments sur les maxima des carrés des sommes partielles.

Supposons que  $f$  admette la décomposition (1.2.3) avec  $m$  et  $g$  de carré intégrable. Heyde a démontré (cf. [Hey75]) que dans ce cas, la suite  $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$  converge en loi vers un mouvement brownien. Il s'avère qu'il n'est pas nécessaire que  $g$  soit de carré intégrable : si

$$f = m + g - g \circ T, m \in \mathbb{L}^2 \text{ et } \frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} |g \circ T^j| \rightarrow 0 \text{ en probabilité,} \quad (1.3.8)$$

alors la même conclusion a lieu.

Hannan a montré que la condition (1.2.8) est suffisante pour obtenir le principe d'invariance dans  $C[0, 1]$  (cf. [Han73, Han79]) avec les restrictions suivantes : une hypothèse de mélange sur le système dynamique et le fait que le processus soit adapté. La première a été levée dans [DM03],

et le cas non-adapté a été traité dans [DMV07]. Si la fonction  $f$  est régulière, la condition (1.2.8) est donc suffisante pour le principe d'invariance dans  $C[0, 1]$ . La démonstration dans le cas adapté repose sur l'inégalité

$$\mathbb{E} \left[ \max_{1 \leq i \leq n} S_i(f)^2 \right] \leq 4n \left( \sum_{i=0}^{+\infty} \|P_i(f)\|_2 \right)^2. \quad (1.3.9)$$

Le caractère suffisant pour le principe d'invariance de la condition (1.2.9) a été établi dans [DR00]. Ils ont établi et utilisé l'inégalité maximale suivante :

$$\mathbb{E} \left[ \max_{1 \leq i \leq n} S_i(f)^2 \right] \leq 8n \mathbb{E} [f^2] + 16 \sum_{k=2}^n \mathbb{E} |f \mathbb{E} [S_k(f) | \mathcal{F}_0]|. \quad (1.3.10)$$

Dans [PU05], Peligrad et Utev ont montré que le principe d'invariance a lieu dans  $D[0, 1]$  sous la condition (1.2.10) pour une fonction  $f$  adaptée et centrée. Ils ont utilisé une approximation par martingale, qui fonctionne grâce à l'inégalité maximale suivante. Pour tous entiers  $n$  et  $r$  tels que  $2^{r-1} \leq n < 2^r$ ,

$$\mathbb{E} \left[ \max_{1 \leq i \leq n} S_i(f)^2 \right] \leq n \left( 2 \|f\|_2 + (1 + \sqrt{2}) \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E} [S_{2^j}(f) | \mathcal{F}_0]\|_2 \right). \quad (1.3.11)$$

Ces arguments permettent aussi d'établir le principe d'invariance dans l'espace  $C[0, 1]$ . Le cas non-adapté a été traité dans [Vol07] en supposant que la fonction régulière  $f$  vérifie en plus de (1.2.10) la condition

$$\sum_{n=1}^{+\infty} \frac{\|S_n(f) - \mathbb{E} [S_n(f) | \mathcal{F}_n]\|_2}{n^{3/2}} < \infty. \quad (1.3.12)$$

Gordin et Peligrad (cf. [GP11]) ont donné une approche unificatrice de ces résultats à l'aide des inégalités (1.3.9), (1.3.10) et (1.3.11) et d'un critère permettant de déterminer si pour une fonction  $f$ , il existe une martingale  $(M_n)_{n \geq 1}$  à accroissements strictement stationnaires telle que

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[ \max_{1 \leq j \leq n} (S_j(f) - M_j)^2 \right] = 0. \quad (1.3.13)$$

Nous énonçons leur résultat. Pour une fonction  $f: \Omega \rightarrow \mathbb{R}$ , on pose

$$A_m(f) := \frac{1}{m} \mathbb{E} [S_m(f) | \mathcal{F}_0], \quad (1.3.14)$$

et pour une fonction  $h: \Omega \rightarrow \mathbb{R}$ ,

$$\|h\|_+ := \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \max_{1 \leq k \leq n} |S_k(h)| \right\|_2. \quad (1.3.15)$$

**Théorème 1.3.3** (Théorème 1, [GP11]). *La fonction  $f$  admet l'approximation par martingales (1.3.13) avec  $(M_n)_{n \geq 1}$  à accroissements strictement stationnaires si et seulement si la suite  $(\|A_m(f)\|_+)$  converge vers 0 quand  $m \rightarrow +\infty$ .*

Dans la partie 3 de [GP11], les auteurs retrouvent les démonstrations du principe d'invariance dans  $C[0, 1]$  sous les conditions de Hannan, Dedecker-Rio et Maxwell et Woodroffe en majorant  $\|A_m(f)\|_+$  grâce aux inégalités (1.3.9), (1.3.10) et (1.3.11).

On notera que dans (1.3.13), on normalise par  $\sqrt{n}$ . Il est possible d'approximer par martingales avec d'autres choix de normalisation, voir par exemple [Vol06a].



### 1.3.3 Espaces de Hölder

Il est connu que le mouvement brownien possède des trajectoire presque sûrement hölderiennes d'exposant  $\alpha$  pour tout  $\alpha \in (0, 1/2)$ . Comme les fonctions aléatoires définies par (1.3.4) sont elles aussi hölderiennes, on peut s'intéresser à la convergence dans les espaces de Hölder de  $(n^{-1/2}S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$ .

**Définition 1.3.4** (Espaces de Hölder). *Soit  $0 < \alpha < 1$ . On note  $\mathcal{H}_\alpha[0, 1]$  l'espace vectoriel des fonctions  $x: [0, 1] \rightarrow \mathbb{R}$  telles que la quantité*

$$\|x\|_\alpha := |x(0)| + \sup_{0 \leq s < t \leq 1} \frac{|x(t) - x(s)|}{|t - s|^\alpha} \quad (1.3.16)$$

soit finie.

L'espace  $\mathcal{H}_\alpha[0, 1]$  muni de la norme  $\|\cdot\|_\alpha$  n'est pas séparable. En effet, posons pour  $a \in [0, 1]$ ,  $x_a(t) := |x - a|^\alpha$ . Alors pour  $a \neq b$ ,

$$\|x_b - x_a\|_\alpha \geq \frac{|x_b(b) - x_a(b) - x_b(a) + x_a(a)|}{|b - a|^\alpha} = 2. \quad (1.3.17)$$

Il est préférable de travailler avec des espaces vectoriels normés séparables.

**Définition 1.3.5.** *Soient  $0 < \alpha < 1$ ,  $x \in \mathcal{H}_\alpha[0, 1]$  et  $\delta \in ]0, 1[$ . On définit le module de régularité hölderienne, noté  $\omega_\alpha(x, \delta)$ , par*

$$\omega_\alpha(x, \delta) := \sup_{\substack{0 \leq s < t \leq 1 \\ t - s < \delta}} \frac{|x(t) - x(s)|}{|t - s|^\alpha}. \quad (1.3.18)$$

On remarque que pour tout  $\delta \in ]0, 1[$ , l'application  $x \in \mathcal{H}_\alpha[0, 1] \mapsto \omega_\alpha(x, \delta)$  est continue. À l'aide de ce module de régularité hölderienne, on peut définir un sous-espace séparable de  $\mathcal{H}_\alpha$ .

**Définition 1.3.6.** *Soit  $0 < \alpha < 1$ . On note  $\mathcal{H}_\alpha^o[0, 1]$  l'espace vectoriel des fonctions  $x: [0, 1] \rightarrow \mathbb{R}$  telles que*

$$\lim_{\delta \rightarrow 0} \omega_\alpha(x, \delta) = 0. \quad (1.3.19)$$

On munit également  $\mathcal{H}_\alpha^o[0, 1]$  de la topologie induite par la norme  $\|\cdot\|_\alpha$ .

À l'instar des démonstrations du théorème limite central fonctionnel dans l'espace  $C[0, 1]$ , on aimerait utiliser la stratégie suivante pour obtenir le principe d'invariance dans  $\mathcal{H}_\alpha[0, 1]$  :

- montrer la convergence des loi fini-dimensionnelles vers celles d'un mouvement brownien (voir (1.3.7) ;
- montrer l'équi-tension dans  $\mathcal{H}_\alpha[0, 1]$  de la suite  $(n^{-1/2}S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$ .

On verra qu'il faudra faire des hypothèses plus restrictives que la finitude du moment d'ordre deux pour obtenir un principe d'invariance dans  $\mathcal{H}_\alpha[0, 1]$ . En particulier, ces hypothèses entraînent le principe d'invariance dans l'espace des fonctions continues, donc la convergence des loi fini-dimensionnelles. L'équi-tension sera établie dans l'espace  $\mathcal{H}_\alpha^o[0, 1]$ . Ce n'est pas restrictif car l'injection canonique de  $\mathcal{H}_\alpha^o[0, 1]$  dans  $\mathcal{H}_\alpha[0, 1]$  est continue. La raison principale de ce changement d'espace est que  $\mathcal{H}_\alpha^o[0, 1]$  est séparable. En particulier, la tribu borélienne de  $\mathcal{H}_\alpha^o[0, 1]$  est la tribu engendrée par les formes linéaires continues sur  $\mathcal{H}_\alpha^o[0, 1]$ . Par conséquent, on peut démontrer que deux mesures de probabilité sur  $\mathcal{H}_\alpha^o[0, 1]$  ayant les mêmes lois fini-dimensionnelles sont égales. Pour plus de détails, on pourra consulter la démonstration de la Proposition 3 dans [Ham00].

En résumé, la difficulté principale pour obtenir le théorème limite central fonctionnel dans les espaces de Hölder réside dans la démonstration de l'équi-tension de la suite  $(n^{-1/2}S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  dans  $\mathcal{H}_\alpha^o[0, 1]$ .

Avant de voir les critères de tension, commençons par identifier le type de restriction que le principe d'invariance dans  $\mathcal{H}_\alpha^o[0, 1]$  impose. Supposons donc que  $f$  soit une fonction telle que la suite  $(n^{-1/2}S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  converge vers un mouvement brownien dans  $\mathcal{H}_\alpha^o[0, 1]$ . Pour un  $\delta \in ]0, 1[$  fixé, la fonction  $x \mapsto \omega_\alpha(x, \delta)$  est continue, ce qui implique que

$$\omega_\alpha \left( \frac{1}{\sqrt{n}} S_n^{\text{pl}}(f, \cdot), \delta \right) \rightarrow \omega_\alpha(W(\cdot), \delta). \quad (1.3.20)$$

Remarquons que pour  $n > 1/\delta + 1$ , l'inégalité

$$\omega_\alpha \left( \frac{1}{\sqrt{n}} S_n^{\text{pl}}(f, \cdot), \delta \right) \geq \frac{1}{n^{1/p}} \max_{1 \leq k \leq n} |f \circ T^k| \quad (1.3.21)$$

a lieu, et on déduit par (1.3.20) que pour tout  $\delta > 0$ ,

$$\limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{n^{1/p}} \max_{1 \leq k \leq n} |f \circ T^k| > 1 \right\} \leq \mu \{ \omega_\alpha(W(\cdot), \delta) > 1 \}. \quad (1.3.22)$$

Comme  $W$  est un élément de  $\mathcal{H}_\beta$  pour un  $\beta \in ]\alpha, 1/2[$ , le membre de droite de (1.3.22) converge vers 0 lorsque  $\delta$  tend vers 0. En conclusion, on obtient que

$$\lim_{n \rightarrow \infty} \mu \left\{ \frac{1}{n^{1/p}} \max_{1 \leq k \leq n} |f \circ T^k| > 1 \right\} = 0. \quad (1.3.23)$$

Si la suite  $(f \circ T^k)_{k \geq 0}$  est indépendante, alors (1.3.23) est équivalente à

$$\lim_{t \rightarrow +\infty} t^p \mu \{ |f| > t \} = 0. \quad (1.3.24)$$

Nous disposons d'un premier critère de tension.

**Théorème 1.3.7** (Kerkyacharian et Roynette, [KR91]). *Soit  $(\xi_n(t))_{n \geq 1, t \in [0, 1]}$  une suite de processus s'annulant en 0. On suppose qu'il existe  $\gamma > 0$ ,  $\delta > 0$  et  $c > 0$  tels que pour tout  $\lambda > 0$  et tous  $s, t \in [0, 1]$ ,*

$$\mu \{ |\xi_n(t) - \xi_n(s)| > \lambda \} \leq \frac{c}{\lambda^\gamma} |t - s|^{1+\delta}. \quad (1.3.25)$$

Alors pour tout  $\alpha \in ]0, \delta/\gamma[$ , la suite  $(\xi_n(t))_{n \geq 1, t \in [0, 1]}$  est équi-tendue dans  $\mathcal{H}_\alpha^o[0, 1]$ .

Comme le processus sommes partielles donné par (1.3.4) s'annule en 0, il est possible d'utiliser ce résultat avec  $\xi_n(t) := n^{-1/2}S_n^{\text{pl}}(f, t)$ . Prenons dans un premier temps  $t := k/n$  et  $s := j/n$ . Le membre de gauche de (1.3.25) vaut  $\mu \{ |S_k - S_j| > \lambda\sqrt{n} \}$ . En utilisant la stationnarité, il faut d'abord voir s'il existe des  $\gamma > 0$ ,  $\delta > 0$  et  $c > 0$  tels que pour tout  $\lambda$ , tout entier  $n$  et tout  $i \in \{1, \dots, n\}$ ,

$$\mu \{ |S_i| > \lambda\sqrt{n} \} \leq \frac{c}{\lambda^\gamma} \left( \frac{i}{n} \right)^{1+\delta}. \quad (1.3.26)$$

En prenant  $i = n$ , il faudrait donc que pour tout entier  $n$ ,

$$\mu \{ |S_n| > \lambda\sqrt{n} \} \leq \frac{c}{\lambda^\gamma}. \quad (1.3.27)$$

Si tel est le cas, alors

$$\mu \{ |S_i| > \lambda\sqrt{n} \} \leq \frac{c}{(\lambda\sqrt{n}/\sqrt{i})^\gamma} = \frac{1}{\lambda^\gamma} \left( \frac{i}{n} \right)^{\gamma/2}, \quad (1.3.28)$$

et si  $\gamma/2 \geq 1 + \delta$ , alors (1.3.26) a lieu. La vérification de (1.3.25) pour les autres  $s$  et  $t$  peut se faire de la manière suivante. D'abord, si  $j/n \leq s < (j+1)/n$ , on majore la quantité  $\mu \{|\xi_n((j+1)/n) - \xi_n(s)| > \lambda\}$  en utilisant l'expression (1.3.4). Puis dans le cas où  $j/n \leq s < (j+1)/n \leq k/n \leq t < (k+1)/n$ , on écrit

$$\begin{aligned} \mu \{|\xi_n(t) - \xi_n(s)| > 3\lambda\} &\leq \mu \{|\xi_n(t) - \xi_n(k/n)| > \lambda\} + \\ &+ \mu \{|\xi_n(k/n) - \xi_n((j+1)/n)| > \lambda\} + \mu \{|\xi_n((j+1)/n) - \xi_n(s)| > \lambda\} \end{aligned} \quad (1.3.29)$$

Soit  $\gamma$  vérifiant (1.3.27) pour tout  $n$ . La seule restriction trouvée sur  $\delta$  pour que (1.3.25) soit vérifiée est que  $\gamma/2 \geq 1 + \delta$ . Au vu de la conclusion du Théorème 1.3.7, il est préférable de choisir  $\delta$  aussi grand que possible, donc  $\delta := \gamma/2 - 1$ .

En résumé, on vient d'obtenir le

**Corollaire 1.3.8.** *Soit  $(f \circ T^j)_{j \geq 0}$  une suite strictement stationnaire telle que pour un  $\gamma > 2$ ,*

$$\sup_n \sup_{\lambda > 0} \lambda^\gamma \mu \{|S_n(f)| > \lambda\} < \infty. \quad (1.3.30)$$

Alors pour tout  $\alpha \in ]0, 1/2 - 1/\gamma[$ , la suite  $(n^{-1/2} S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  est équi-tendue dans  $\mathcal{H}_\alpha^o[0, 1]$ .

Remarquons que la condition (1.3.30) implique que la suite  $(\|S_n(f)/\sqrt{n}\|_p)_{n \geq 1}$  est bornée pour tout  $p \in [2, \gamma[$ . Si la suite  $(\|S_n(f)/\sqrt{n}\|_p)_{n \geq 1}$  est bornée, alors la conclusion du Corollaire 1.3.8 a lieu, puisqu'on peut l'appliquer pour tous les  $\gamma'$  strictement compris entre 2 et  $\gamma$ . Comme cette condition est vérifiée pour les suites i.i.d. ayant un moment d'ordre  $\gamma$  fini, Kerkycharian et Roynette ont ainsi retrouvé le résultat de Lamperti [Lam62] :

**Théorème 1.3.9.** *Soit  $\alpha \in ]0, 1/2[$  et  $(X_i)_{i \geq 1}$  une suite i.i.d. centrée, vérifiant  $\mathbb{E}|X_1|^p < \infty$  pour un certain  $p > 1/(1/2 - \alpha)$ . Alors le processus défini par (1.3.1) converge en loi vers  $\mathbb{E}[X_1^2]W$  dans l'espace  $\mathcal{H}_\alpha^o[0, 1]$ .*

Pour récapituler : soient  $\alpha \in ]0, 1/2[$  et  $p(\alpha) := 1/(1/2 - \alpha)$  et  $(f \circ T^i)_{i \geq 0}$  une suite i.i.d. centrée.

- Si  $\mathbb{E}|f|^q < \infty$  pour  $q > p(\alpha)$ , alors  $f$  vérifie le principe d'invariance dans  $\mathcal{H}_\alpha^o[0, 1]$ .
- Si  $f$  vérifie le principe d'invariance dans  $\mathcal{H}_\alpha^o[0, 1]$ , alors  $\lim_{t \rightarrow \infty} t^{p(\alpha)} \mu \{|f| > t\} = 0$ .

Avec ces résultats, nous ne pouvons pas déterminer si une suite i.i.d.  $(f \circ T^i)_{i \geq 0}$  telle que  $\lim_{t \rightarrow \infty} t^{p(\alpha)} \mu \{|f| > t\} = 0$  vérifie le principe d'invariance.

La condition nécessaire et suffisante fut donnée par Račkauskas et Suquet [RS03] :

**Théorème 1.3.10.** *Soient  $\alpha \in ]0, 1/2[$  et  $p(\alpha) := 1/(1/2 - \alpha)$  et  $(f \circ T^i)_{i \geq 0}$  une suite i.i.d. centrée. La fonction  $f$  vérifie le principe d'invariance dans  $\mathcal{H}_\alpha^o[0, 1]$  si et seulement si*

$$\lim_{t \rightarrow +\infty} t^{p(\alpha)} \mu \{|f| > t\} = 0. \quad (1.3.31)$$

La démonstration de ce résultat repose sur un critère de tension qui ne fait intervenir que les maxima des sommes partielles. Nous suivons l'approche de [Suq99].

## 1.4 Théorèmes limites presque sûrs

Dans les deux sections précédentes, nous avons évoqué le théorème limite central et sa version fonctionnelle dans différents espaces. Ceux-ci concernent la loi de certaines fonctionnelles des sommes partielles et non leur comportement presque sûr. Il existe des versions conditionnelles de ces théorèmes, mais nous ne les évoquerons pas dans la suite. Nous ferons référence à deux théorèmes limites classiques : la loi forte des grands nombres et la loi des logarithmes itérés.

### 1.4.1 Loi forte des grands nombres

En 1963, Baum et Katz ont démontré que si  $p > 1$  et si  $(f \circ T^i)_{i \geq 0}$  est une suite i.i.d. vérifiant  $\mathbb{E}[f] = 0$  et  $\mathbb{E}|f|^p < \infty$ , alors

$$\sum_{n=1}^{+\infty} n^{p-2} \mu \left\{ \sup_{k \geq n} |S_k|/k > \varepsilon \right\} < \infty \text{ pour tout } \varepsilon > 0 \quad (1.4.1)$$

(voir le Théorème 1 de [BK63]). Ceci donne une vitesse de convergence pour la loi forte des grands nombres. En effet, si  $1 < p < 2$ , alors  $n^{p-1} \mu \left\{ \sup_{k \geq n} |S_k|/k > \varepsilon \right\} \rightarrow 0$  lorsque  $n$  tend vers l'infini.

Dans [BK65], les mêmes auteurs ont montré que si  $1 \leq p < 2$ ,  $f$  est centrée et  $\mathbb{E}|f|^p \log^+ |f| < \infty$ , alors

$$\sum_{n=1}^{+\infty} \frac{1}{n} \mu \left\{ \sup_{k \geq n} \frac{|S_k|}{k^{1/p}} > \varepsilon \right\} < \infty \text{ pour tout } \varepsilon > 0. \quad (1.4.2)$$

Ceci entraîne que la suite  $(S_n/n^{1/p})_{n \geq 1}$  converge vers 0 presque sûrement.

**Définition 1.4.1.** Soit  $1 < p < 2$ . On dit que la fonction  $f$  vérifie la  $p$ -loi forte des grands nombres si pour tout  $\alpha \in [1/p, 1]$ , la convergence

$$\sum_{n=1}^{+\infty} n^{\alpha p-2} \mu \left\{ \max_{1 \leq k \leq n} |S_k(f)| \geq \varepsilon n^\alpha \right\} < +\infty \quad (1.4.3)$$

a lieu pour tout  $\varepsilon$  strictement positif.

Ce type de résultats fut étendu aux martingales à valeurs dans certains espaces de Banach par Woyczyński dans [Woy76].

### 1.4.2 Loi des logarithmes itérés

Soit  $(\Omega, \mathcal{F}, \mu, T)$  un système dynamique.

**Définition 1.4.2.** On dit que la fonction  $f: \Omega \rightarrow \mathbb{R}$  vérifie la loi des logarithmes itérés s'il existe une constante  $C(f)$  telle que pour presque tout  $\omega \in \Omega$ ,

$$\liminf_{n \rightarrow \infty} (\sqrt{n \log \log n})^{-1} S_n(f) = -C(f) \text{ et } \limsup_{n \rightarrow \infty} (\sqrt{n \log \log n})^{-1} S_n(f) = C(f). \quad (1.4.4)$$

On dit que la fonction  $f: \Omega \rightarrow \mathbb{R}$  vérifie la loi des logarithmes itérés fonctionnelle si la suite  $((\sqrt{n \log \log n})^{-1} S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  est relativement compacte dans  $C[0, 1]$  et l'ensemble des valeurs d'adhérence coïncide avec l'ensemble des fonctions absolument continues  $x \in C[0, 1]$  telles que  $x(0) = 0$  et  $\int_0^1 (x'(t))^2 dt \leq 1$ , où  $x'$  désigne la dérivée par rapport à la mesure de Lebesgue.

La loi des logarithmes itérés fonctionnelle pour une suite i.i.d. centrée, à moyenne nulle à variance finie fut établie par Strassen [Str64]. Son extension aux accroissements ergodiques d'une martingale de carré intégrable est due à Basu [Bas73]. Comme pour le principe d'invariance, une approximation par martingales est possible. Par exemple, si la décomposition (1.2.3) a lieu avec  $m$  et  $g$  dans  $\mathbb{L}^2$ , alors la fonction  $f$  vérifie la loi des logarithmes itérés (voir [HH80]). La condition de Hannan (1.2.8) donne la même conclusion. Zhao et Woodrooffe ont donné une condition dans l'esprit de (1.2.10) (cf. [ZW08]).



# Chapitre 2

## Suites faiblement dépendantes

Dans ce chapitre, nous nous intéressons au théorème limite central et à sa version fonctionnelle pour des suites faiblement dépendantes. Nous allons considérer les suites mélangeantes classiques  $(\alpha, \beta, \rho, \rho^*)$  ainsi que les suites  $\tau$ -dépendantes et  $\theta$ -dépendantes. L'idée générale est de quantifier la dépendance d'une suite en mesurant la dépendance entre la tribu engendrée par les variables de la suite avant un instant  $i$  et celle engendrée par les variables après un instant  $i + n$ . Cela forme une suite indexée par  $n$  (nulle lorsque la suite est indépendante) et décroissante. La suite stationnaire considérée aura des propriétés d'autant plus proches de celles d'une suite indépendante que les coefficients de dépendance tendent rapidement vers 0. En combinant les hypothèses sur les coefficients avec celles sur la queue de la loi commune, il est possible d'obtenir de bonnes inégalités de covariance et de déviation du maximum des valeurs absolues des sommes partielles.

Ce chapitre est organisé de la manière suivante : dans une première section, nous donnons les définitions des coefficients de dépendance et leur comparaison ; dans la seconde section, nous rappelons les résultats concernant le théorème limite central pour les suites mélangeantes aussi bien à valeurs réelles que dans un espace de Hilbert séparable. Finalement, nous évoquons les résultats concernant le principe d'invariance dans  $C[0, 1]$  pour les suites faiblement dépendantes à valeurs réelles.

### 2.1 Définitions et comparaison

#### 2.1.1 Coefficients de mélange

La définition des coefficients de mélange repose sur la mesure de dépendance de deux tribus. Si  $(\Omega, \mathcal{F}, \mu)$  est un espace probabilisé et  $\mathcal{A}$  et  $\mathcal{B}$  sont deux sous-tribus de  $\mathcal{F}$ , une première façon de mesurer la dépendance est de définir

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \}. \quad (2.1.1)$$

Ce coefficient, dit d' $\alpha$ -mélange, a été introduit par Rosenblatt dans [Ros56]. Volkonskii et Rozanov [VR59] ont défini les coefficients de  $\beta$ -mélange par l'égalité

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

où le supremum est pris sur les partitions finies  $\{A_i, 1 \leq i \leq I\}$  et  $\{B_j, 1 \leq j \leq J\}$  de  $\Omega$  ne contenant respectivement que des éléments de  $\mathcal{A}$  et  $\mathcal{B}$ .

Il est également possible de comparer les corrélations entre les fonctions  $\mathcal{A}$ -mesurables et  $\mathcal{B}$ -mesurables. Il s'agit de l'approche de Hirschfeld [Hir35] ; plus précisément,

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{ |\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B}) \}, \quad (2.1.2)$$

où  $\text{Corr}(f, g) := [\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)] [ \|f - \mathbb{E}(f)\|_{\mathbb{L}^2} \|g - \mathbb{E}(g)\|_{\mathbb{L}^2} ]^{-1}$ .

Ibragimov [Ibr59] a introduit les coefficients de  $\phi$ -mélange, définis par la formule :

$$\phi(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(B | A) - \mu(B)|, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0 \}.$$

On notera que contrairement aux coefficients d' $\alpha$ ,  $\beta$  ou  $\rho$ -mélange, le rôle des tribus  $\mathcal{A}$  et  $\mathcal{B}$  n'est pas symétrique dans le cas du  $\phi$ -mélange. En effet, reprenons l'exemple p. 101 de [Bra07]. On considère  $\varepsilon \in ]0, 1/2]$ . Soient  $X$  et  $Y$  des variables aléatoires de loi jointe  $\mu \{(X, Y) = (0, 0)\} = 1 - \varepsilon$ ,  $\mu \{(X, Y) = (0, 1)\} = 0$ ,  $\mu \{(X, Y) = (1, 0)\} = \varepsilon - \varepsilon^2$  et  $\mu \{(X, Y) = (1, 1)\} = \varepsilon^2$ . En posant  $\mathcal{A} := \sigma(X)$  et  $\mathcal{B} := \sigma(Y)$ , on peut vérifier que  $\phi(\mathcal{A}, \mathcal{B}) = \varepsilon - \varepsilon^2$  et  $\phi(\mathcal{B}, \mathcal{A}) = 1 - \varepsilon$ .

Les inégalités suivantes ont lieu pour toutes les sous-tribus  $\mathcal{A}$  et  $\mathcal{B}$  de  $\mathcal{F}$  :

$$2\alpha(\mathcal{A}, \mathcal{B}) \leq \beta(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}), \quad \alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}) \leq 2\sqrt{\phi(\mathcal{A}, \mathcal{B})}. \quad (2.1.3)$$

Nous renvoyons le lecteur à la Proposition 3.11 de [Bra07] pour une démonstration.

L'exemple suivant (p. 100 de [Bra07]) montre que les mesures de dépendance  $\alpha$  et  $\beta$  peuvent être petites sans que  $\rho$  et  $\phi$  ne le soient. Pour  $\varepsilon \in ]0, 1/2]$ , on considère des variables aléatoires  $X$  et  $Y$  dont la loi jointe est donnée par  $\mu \{X = Y = 0\} = 1 - \varepsilon$  et  $\mu \{X = Y = 1\} = \varepsilon$ . En posant  $\mathcal{A} := \sigma(X)$  et  $\mathcal{B} := \sigma(Y)$ , les égalités

$$\alpha(\mathcal{A}, \mathcal{B}) = \varepsilon(1 - \varepsilon); \beta(\mathcal{A}, \mathcal{B}) = 2\varepsilon(1 - \varepsilon); \rho(\mathcal{A}, \mathcal{B}) = 1 \text{ et } \phi(\mathcal{A}, \mathcal{B}) = 1 - \varepsilon \quad (2.1.4)$$

ont lieu.

Tout ceci concerne la dépendance entre deux tribus. Pour mesurer la dépendance d'une suite  $(X_k)_{k \in \mathbb{Z}}$ , on définit pour  $-\infty \leq J \leq L \leq +\infty$  la tribu

$$\mathcal{F}_J^L := \sigma(X_k, J \leq k \leq L). \quad (2.1.5)$$

Puis, pour un entier positif  $n$ , on pose

$$\alpha(n) := \sup_{J \in \mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^{+\infty}), \quad (2.1.6)$$

et de la même manière

$$\beta(n) := \sup_{J \in \mathbb{Z}} \beta(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^{+\infty}); \quad (2.1.7)$$

$$\rho(n) := \sup_{J \in \mathbb{Z}} \rho(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^{+\infty}); \quad (2.1.8)$$

$$\phi(n) := \sup_{J \in \mathbb{Z}} \phi(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^{+\infty}). \quad (2.1.9)$$

La suite  $(X_k)_{k \in \mathbb{Z}}$  n'a pas été supposée strictement stationnaire dans les définitions précédentes. Dans le cas stationnaire, celles-ci se réduisent à

$$\alpha(n) := \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{+\infty}); \quad (2.1.10)$$

$$\beta(n) := \beta(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{+\infty}); \quad (2.1.11)$$

$$\rho(n) := \rho(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{+\infty}); \quad (2.1.12)$$

$$\phi(n) := \phi(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{+\infty}). \quad (2.1.13)$$

Dans ces définitions, la suite de variables aléatoires considérée est indexée par les entiers relatifs. On peut définir les coefficients de mélange de la suite  $(X_k)_{k \geq 1}$  en posant  $X'_k = X_k$  pour  $k \geq 1$  et  $X'_k = 0$  pour  $k \leq 0$ , et en définissant  $\alpha(n)$  comme le  $n$ -ème coefficient d' $\alpha$ -mélange de la suite  $(X'_k)_{k \in \mathbb{Z}}$ . Les coefficients de  $\beta$ ,  $\rho$  et  $\phi$ -mélange sont définis de la même manière.

Une autre condition de mélange est le  $\rho$ -mélange entrelacé, qui est définie différemment des autres conditions. Si  $(X_k)_{k \in \mathbb{Z}}$  est une suite de variables (non nécessairement stationnaire), on définit  $\mathcal{F}_I := \sigma(X_k, k \in I)$  et

$$\rho^*(n) := \sup \left\{ \rho(\mathcal{F}_I, \mathcal{F}_J) \mid \inf_{i \in I, j \in J} |i - j| \geq n, I, J \subset \mathbb{Z} \right\}. \quad (2.1.14)$$

Pour les coefficients de  $\rho$ -mélange classiques, on restreint le supremum aux ensembles  $I$  et  $J$  de la forme  $] - \infty, J]$ ,  $[J + n, +\infty[$  donc pour tout entier  $n$ ,  $\rho(n) \leq \rho^*(n)$ .

**Définition 2.1.1.** *On dit que la suite  $(X_k)_{k \in \mathbb{Z}}$  est  $\alpha$ -mélangeante (respectivement  $\beta$ ,  $\rho$ ,  $\phi$  et  $\rho^*$ -mélangeante) si  $\lim_{n \rightarrow +\infty} \alpha(n) = 0$  (respectivement  $\lim_{n \rightarrow +\infty} \beta_X(n) = 0$ ,  $\lim_{n \rightarrow +\infty} \rho_X(n) = 0$  and  $\lim_{n \rightarrow +\infty} \phi_X(n) = 0$ ).*

Notons qu'une suite indépendante est nécessairement mélangeante (que ce soit au sens  $\phi$  ou  $\rho^*$ , et donc  $\alpha$ ,  $\beta$  ou  $\rho$ ). La littérature contient des éléments de comparaison des différentes types de suites mélangeantes.

- Dans [Bra99], Bradley construit une suite strictement stationnaire, gaussienne, centrée, qui est à la fois  $\rho$  et  $\beta$ -mélangeante mais pas  $\rho^*$ -mélangeante.
- Dans [Ros71] (p. 214, ligne 3 à p. 215, ligne 15), Rosenblatt donne un exemple de suite strictement stationnaire  $\rho$ -mélangeante mais non  $\beta$ -mélangeante. L'exemple 7.16 de [Bra07] donne un exemple de suite strictement stationnaire  $\rho^*$ -mélangeante (donc  $\rho$ -mélangeante) mais non  $\beta$ -mélangeante.
- L'exemple 7.11 de [Bra07] donne un exemple de chaîne de Markov strictement stationnaire sur un espace d'états dénombrable,  $\beta$ -mélangeante mais non  $\rho$ -mélangeante.
- Dans [Bra97], il est conjecturé que toute suite  $\phi$ -mélangeante est  $\rho^*$ -mélangeante. Cette conjecture n'est à notre connaissance pas résolue, même pour les chaînes de Markov strictement stationnaires.

Parmi ces conditions de mélange, la moins restrictive est l' $\alpha$ -mélange. Cependant, il existe des exemples de chaînes de Markov qui ne sont pas  $\alpha$ -mélangeantes. Par exemple, Andrews [And84] a montré que si  $(\varepsilon_i)_{i \geq 1}$  est une suite i.i.d. où  $\varepsilon_1$  suit une loi de Bernoulli, alors la solution stationnaire  $(X_i)_{i \geq 0}$  de l'équation

$$X_n = \frac{1}{2}(X_{n-1} + \varepsilon_n), X_0 \text{ est indépendante de } (\varepsilon_i)_{i \geq 1} \quad (2.1.15)$$

vérifie  $\alpha(n) \geq \alpha(\sigma(X_0), \sigma(X_n)) = 1/4$ .

### 2.1.2 Suites $\theta$ et $\tau$ -dépendantes

Il existe des mesures de dépendance définies à l'aide des lois conditionnelles : les coefficients de  $\theta$  et  $\tau$ -dépendance. Les premiers furent introduits dans [DL99], les seconds dans [DP04]. Celles-ci permettent de conserver la plupart des propriétés des suites mélangeantes et présentent l'avantage de s'appliquer à une classe de processus plus grande.

Pour définir les coefficients de  $\theta$  et  $\tau$ -dépendance d'une suite strictement stationnaire, nous avons d'abord besoin d'un lemme sur les lois conditionnelles (voir le Théorème 33.3 de [Bil95]).



**Lemme 2.1.2.** Soient  $(\Omega, \mathcal{F}, \mu)$  un espace probabilisé,  $\mathcal{M}$  une sous-tribu de  $\mathcal{F}$  et  $X$  une variable aléatoire à valeurs réelle de loi  $\mu_X$ . Il existe une fonction  $\mu_{X|\mathcal{M}}$  de  $\mathcal{B}(\mathbb{R}) \times \Omega$  dans  $[0, 1]$  telle que

1. pour tout  $\omega \in \Omega$ ,  $\mu_{X|\mathcal{M}}(\cdot, \omega)$  est une mesure de probabilité sur  $\mathcal{B}(\mathbb{R})$ .
2. Pour tout  $A \in \mathcal{B}(\mathbb{R})$ ,  $\mu_{X|\mathcal{M}}(A, \cdot)$  est une version de  $\mathbb{E}[\mathbf{1}_{\{X \in A\}} | \mathcal{M}]$ .

À présent, introduisons les coefficients de  $\tau$ -dépendance comme dans [DP05]. Notons  $\Lambda_1(\mathbb{R})$  la collection des fonctions 1-Lipschitz de  $\mathbb{R}$  dans  $\mathbb{R}$  et définissons la quantité

$$W(\mu_{X|\mathcal{M}}) := \sup \left\{ \left| \int f(x) \mu_{X|\mathcal{M}}(dx) - \int f(x) \mu_X(dx) \right|, f \in \Lambda_1(\mathbb{R}) \right\}.$$

Pour une variable aléatoire intégrable  $X$  et une sous-tribu  $\mathcal{M}$ , le coefficient  $\tau$  est défini par

$$\tau(\mathcal{M}, X) = \|W(\mu_{X|\mathcal{M}})\|_1. \quad (2.1.16)$$

Cette définition peut s'étendre aux variables aléatoires à valeurs dans un espace vectoriel de dimension finie. Si  $d$  est un entier naturel strictement positif, on munit  $\mathbb{R}^d$  de la norme  $\|x - y\| := \sum_{j=1}^d |x_j - y_j|$  et on note  $\Lambda_1(\mathbb{R}^d)$  la collection des fonctions 1-Lipschitz de  $\mathbb{R}^d$  dans  $\mathbb{R}$ .

**Définition 2.1.3.** Soit  $(\Omega, \mathcal{F}, \mu)$  un espace probabilisé,  $\mathcal{M}$  une sous-tribu de  $\mathcal{F}$  et  $X$  une variable aléatoire à valeurs dans  $\mathbb{R}^d$ . On définit

$$\tau(\mathcal{M}, X) := \sup \{ \|\mathbb{E}[f(X) | \mathcal{M}] - \mathbb{E}[f(X)]\|_1, f \in \Lambda_1(\mathbb{R}^d) \}; \quad (2.1.17)$$

$$\tau(\mathcal{M}, X) := \sup \{ \tau(\mathcal{M}, f(X)), f \in \Lambda_1(E) \}; \quad (2.1.18)$$

$$\theta(\mathcal{M}, X) := \sup \{ \tau(\mathcal{M}, f(X)), f \in \Lambda_1(E), \|f\|_\infty \leq 1 \}. \quad (2.1.19)$$

On peut à présent introduire les coefficients de  $\theta$  et  $\tau$ -dépendance pour une suite de variables aléatoires à valeurs réelles.

**Définition 2.1.4.** Soient  $(X_i)_{i \geq 1}$  suite de variables aléatoires à valeurs réelles et  $(\mathcal{M}_i)_{i \geq 1}$  une suite de sous-tribus de  $\mathcal{F}$ . Pour un entier  $k$  strictement positif, on définit

$$\tau(i) := \max_{p, l \geq 1} \frac{1}{l} \sup \{ \tau(\mathcal{M}_p, (X_{j_1}, \dots, X_{j_l})), p + i \leq j_1 < \dots < j_l \}. \quad (2.1.20)$$

On définit  $\theta(i)$  de manière similaire.

Dans la suite, on se concentrera sur le cas  $\mathcal{M}_i := \sigma(X_k, k \leq i)$ .

*Notation 2.1.1.* Soit  $X: \Omega \rightarrow \mathbb{R}$  une variable aléatoire. Notons  $Q_X(\cdot)$  la fonction de quantile, c'est-à-dire  $Q_X(u) := \inf \{t, \mu\{|X| > t\} \leq u\}$ . Si  $(f \circ T^j)_{j \geq 0}$  est une suite strictement stationnaire et  $(\alpha(n))_{n \geq 1}$  la suite de ces coefficients d' $\alpha$ -mélange, on désigne par  $\alpha^{-1}(u)$  le nombre d'indices  $n$  pour lesquels  $\alpha(n) \geq u$ . Plus généralement, si  $(\delta_i)_{i \geq 0}$  est une suite décroissante de nombres positifs, on définit  $\delta^{-1}(u) := \inf \{k \in \mathbb{N}, \delta_k \leq u\}$ .

Il est possible de comparer les coefficients de  $\theta$  et  $\tau$ -dépendance avec ceux d' $\alpha$ -mélange. Nous énonçons le Lemme 7 de [DP04] dans le cas strictement stationnaire.

**Lemme 2.1.5.** Soit  $(f \circ T^j)_{j \geq 0}$  une suite strictement stationnaire. Alors pour tout entier  $i$ , les inégalités suivantes ont lieu :

$$\theta(i) \leq \tau(i) \leq 2 \int_0^{2\alpha(i)} Q_f(u) du. \quad (2.1.21)$$

Dans [DP04], "Application 1 : causal linear processes" (p. 871), Dedecker et Prieur ont fourni un exemple de processus dont les coefficients de  $\tau$ -dépendance convergent vers 0 comme du  $2^{-i}$  mais  $\alpha(i) = 1/4$  pour tout entier  $i$ .

## 2.2 Théorème limite central pour les suites mélangeantes

Dans la Section 1.2 du Chapitre 1, nous avons donné des conditions suffisantes pour qu'une suite strictement stationnaire vérifie le théorème limite central. La même question peut se poser pour les suites mélangeantes.

### 2.2.1 Cas de la dimension finie

Commençons par le cas des suites  $\alpha$ -mélangeantes. Il convient de donner une caractérisation des limites en loi potentielles des sommes partielles normalisées d'une suite strictement stationnaire  $\alpha$ -mélangeante. On rappelle qu'une mesure de probabilité  $\mu$  sur  $\mathbb{R}$  est dite infiniment divisible si pour tout entier positif  $m$ , il existe une mesure de probabilité  $\nu_m$  sur  $\mathbb{R}$  telle que  $\nu_m * \dots * \nu_m = \mu$ , où  $\nu_m$  apparaît  $m$  fois.

**Théorème 2.2.1.** *Soit  $(X_k)_{k \in \mathbb{Z}}$  une suite strictement stationnaire telle que  $\alpha(n) \rightarrow 0$  quand  $n \rightarrow \infty$ . Supposons que*

- $\mu$  est une mesure de probabilité sur  $\mathbb{R}$  ;
- $Q$  est un sous-ensemble infini de l'ensemble des entiers naturels ;
- pour tout  $n \in Q$ ,  $a_n$  est un nombre réel et  $b_n$  est strictement positif ;
- $b_n \rightarrow +\infty$  lorsque  $n \rightarrow \infty, n \in Q$  ; et
- $(S_n - a_n)/b_n \rightarrow \mu$  en loi lorsque  $n \rightarrow \infty, n \in Q$ .

Alors  $\mu$  est infiniment divisible.

Ce résultat est contenu dans [Cog60] (p. 122, Théorème 12).

Lorsque la convergence en loi  $(S_n - a_n)/b_n \rightarrow \mu$  a lieu pour toute la suite, la loi limite  $\mu$  est nécessairement de loi stable. On rappelle qu'une mesure  $\mu$  est dite de loi stable si pour tout entier  $n$ , il existe  $s > 0$  et  $t \in \mathbb{R}$  tels que pour toute variable aléatoire  $Y$  de loi  $\mu^{(n)}$ , la loi de  $sY + t$  est  $\mu$ . Dans ce cas, en notant  $\phi$  la fonction caractéristique de  $\mu$ , il existe des uniques constantes  $c > 0$  et  $p \in (0, 2]$  telles que pour tout  $t \in \mathbb{R}$ ,  $|\phi(t)| = \exp(-c|t|^p)$ . Le nombre  $p$  est appelé exposant de la loi stable (non dégénérée)  $\mu$ .

**Définition 2.2.2.** *Une suite  $(h(n))_{n \geq 1}$  de nombres réels strictement positifs est dite à variation lente si*

$$\lim_{m \rightarrow \infty} \frac{\min_{m \leq n \leq 2m} h(n)}{\max_{m \leq n \leq 2m} h(n)} = 1. \quad (2.2.1)$$

**Théorème 2.2.3.** *Soit  $(X_k)_{k \in \mathbb{Z}}$  une suite strictement stationnaire telle que  $\alpha(n) \rightarrow 0$  quand  $n \rightarrow \infty$ . Supposons que*

- $\mu$  est une mesure de probabilité non-dégénérée sur  $\mathbb{R}$  ;
- pour tout  $n \in \mathbb{N}$ ,  $a_n$  est un nombre réel et  $b_n$  est strictement positif ;
- $b_n \rightarrow +\infty$  lorsque  $n \rightarrow \infty$  ; et
- $(S_n - a_n)/b_n \rightarrow \mu$  en loi lorsque  $n \rightarrow \infty$ .

Alors  $\mu$  est de loi stable. De plus, en notant  $p$  l'exposant de  $\mu$ ,  $b_n = n^{1/p}h(n)$  où  $(h(n))_{n \geq 1}$  est à variation lente.

La première partie est due à Ibragimov (voir [Ibr62]), la seconde à Philipp (cf. [Phi80]), qui a étendu le résultat aux variables aléatoires à valeurs dans un espace de Banach séparable.

On note  $\sigma_n^2(f) := \mathbb{E}[S_n(f)^2]$ ,  $N$  une variable aléatoire de loi normale et  $N(0, 1)$  une variable aléatoire de loi normale centrée et réduite.

**Théorème 2.2.4.** Soit  $(f \circ T^k)_{k \in \mathbb{Z}}$  une suite strictement stationnaire de variables aléatoires centrées, à variance finie, telles que  $\mathbb{E}[S_n(f)^2] \rightarrow \infty$  lorsque  $n \rightarrow +\infty$  et  $\alpha(n) \rightarrow 0$ . Les assertions suivantes sont équivalentes :

1. la famille  $\{S_n(f)^2/\mathbb{E}[S_n(f)^2], n \in \mathbb{N}\}$  est uniformément intégrable ;
2.  $S_n(f)/\sigma_n(f) \rightarrow N(0, 1)$  quand  $n \rightarrow \infty$ .

De plus, si l'une des assertions 1 ou 2 ont lieu, alors  $\sigma_n^2 = n \cdot h(n)$  où  $(h(n))_{n \geq 1}$  est à variation lente.

L'équivalence est contenue dans [Cog60], Théorème 13 et [DDP86]. Le Théorème 2.2.4 fut démontré dans [Den86]. et [MY86].

**Théorème 2.2.5** (Herrndorf, [Her83a]). Soit  $(g(n))_{n \geq 1}$  une suite décroissante de nombres réels strictement positifs telle que  $g(n) \rightarrow 0$  lorsque  $n \rightarrow \infty$ . Il existe une suite strictement stationnaire  $(X_k)_{k \in \mathbb{Z}}$  telle que

1.  $\mathbb{E}[X_0] = 0$  et  $\mathbb{E}[X_0^2] = 1$  ;
2. pour tout  $n \geq 1$ ,  $\mathbb{E}[X_0 X_n] = 0$  ;
3. pour tout  $n \geq 1$ ,  $\alpha(n) \leq \beta(n) \leq g(n)$  ;
4.  $\inf_{n \geq 1} \mu\{S_n = 0\} > 0$  ;
5. la suite  $(S_n)_{n \geq 1}$  est équi-tendue.

**Théorème 2.2.6.** Soit  $(X_k)_{k \in \mathbb{Z}}$  une suite strictement stationnaire de variables aléatoires telle que  $\mathbb{E}[X_0] = 0$ ,  $\mathbb{E}[X_0^2] < \infty$ ,  $\alpha(n) \rightarrow 0$  lorsque  $n \rightarrow \infty$  et

$$\sum_{n=1}^{+\infty} \int_0^{\alpha(n)} Q_{|X_0|}^2(u) du < \infty. \quad (2.2.2)$$

Alors :

1.  $\sigma^2 := \mathbb{E}[X_0^2] + 2 \sum_{n=1}^{\infty} \mathbb{E}[X_0 X_n]$  existe dans  $[0, +\infty[$  et la série est absolument convergente.
2. De plus, si  $\sigma^2 > 0$ , alors  $(S_n/(\sqrt{n}\sigma))$  converge en loi vers une loi normale centrée réduite.

Le premier point se trouve dans [Rio93], le second dans [DMR94].

**Théorème 2.2.7** (Peligrad, [Pel87]). Soit  $(X_k)_{k \in \mathbb{Z}}$  une suite strictement stationnaire telle que  $\mathbb{E}[X_0] = 0$ ,  $\mathbb{E}[X_0^2] < \infty$ ,  $\sigma_n^2 = \mathbb{E}[S_n^2] \rightarrow \infty$  quand  $n \rightarrow \infty$  et  $\rho(n) \rightarrow 0$ . Supposons qu'il existe une fonction croissante  $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  telle que

1.  $\mathbb{E}[X_0^2 g(|X_0|)] < \infty$  et
2. pour un certain  $A > 2$ ,

$$\limsup_{n \rightarrow +\infty} \frac{1}{g(n^{1/2})} \exp \left( A \sum_{j=1}^{\lfloor \log_2 n \rfloor} \rho(2^j) \right) < +\infty \text{ lorsque } n \rightarrow \infty. \quad (2.2.3)$$

Alors  $S_n/\sigma_n \rightarrow N(0, 1)$  en loi quand  $n \rightarrow \infty$ .

*Remarque 2.2.8.* • Supposons que  $\sum_{j=1}^{+\infty} \rho(2^j)$  soit finie. Dans ce cas, les hypothèses du Théorème 2.2.7 sont vérifiées car il suffit de prendre une fonction  $g \equiv 1$ .

- Supposons que  $\mathbb{E} [|X_0|^{2+\delta}]$  soit finie pour un certain  $\delta$  strictement positif. Sous les hypothèses du Théorème 2.2.7, la condition (2.2.3) est vérifiée.

Concernant le  $\rho^*$ -mélange, les Théorèmes 1, 3 et 4 de [Bra92] donnent le résultat suivant.

**Théorème 2.2.9.** *Soit  $(X_k)_{k \in \mathbb{Z}}$  une suite de variables aléatoires strictement stationnaire telle que  $\mathbb{E}[X_0] = 0$ ,  $\mathbb{E}[X_0^2] < \infty$  et  $\rho^*(n) \rightarrow \infty$  quand  $n \rightarrow \infty$ .*

1. Alors  $\sigma^2 := \lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[S_n^2]$  existe dans  $[0, \infty[$ .
2. Si de plus  $\mathbb{E}[S_n^2] \rightarrow \infty$  quand  $n \rightarrow \infty$ , alors  $\sigma^2 > 0$  et la suite  $S_n/(\sqrt{n}\sigma) \rightarrow N(0, 1)$  en loi quand  $n \rightarrow \infty$ .

Peligrad [Pel96] a montré qu'en remplaçant  $\rho^*(n) \rightarrow 0$  par  $\rho^*(n) < 1$  pour un certain  $n$  et  $\alpha(k) \rightarrow 0$ ,  $k \rightarrow \infty$ , la suite  $(S_n/\sigma_n)$  converge en loi vers une loi normale centrée réduite.

Dans les années 60, Ibragimov a conjecturé l'énoncé suivant, connu sous le nom de « conjecture d'Ibragimov » (voir [IL65, IL71], p. 393, Problem 3) :

Si  $(X_k)_{k \in \mathbb{Z}}$  est une suite strictement stationnaire,  $\phi$ -mélangeante de variables aléatoires telle que  $\mathbb{E}[X_0^2] < \infty$  et  $\text{Var}(S_n) \rightarrow \infty$  quand  $n \rightarrow +\infty$ , alors  $S_n$  est asymptotiquement normale quand  $n \rightarrow +\infty$ .

Peligrad a démontré dans [Pel85] que si on suppose la suite centrée et que de plus,  $\liminf_{n \rightarrow +\infty} \sigma_n^2/n$  est strictement positive, alors  $S_n/\sigma_n \rightarrow N(0, 1)$  en loi lorsque  $n \rightarrow \infty$ , où  $\sigma_n^2 = \mathbb{E}[S_n^2]$ .

Pour une suite décroissante  $\delta = (\delta_i)_{i \geq 0}$ , la fonction  $u \mapsto \delta^{-1}(u)$  a été définie dans la Notation 2.1.1. Posons pour une variable aléatoire intégrable  $Y$  la fonction  $G_Y$  comme étant l'inverse généralisée de  $x \mapsto \int_0^x Q_Y(u) du$ . Dans [DD03], la finitude de la quantité

$$D(\delta, X) := \int_0^{\|X\|_1} \delta^{-1}(u) Q_X \circ G_X(u) du \quad (2.2.4)$$

est considérée. Le Corollaire 1 de [DD03] combiné avec (2.1.21) fournit les implications

$$D(\tau, X_0) < \infty \Rightarrow D(\theta, X_0) < \infty \Rightarrow (1.2.9), \quad (2.2.5)$$

où  $(X_k)_{k \in \mathbb{Z}}$  est une suite strictement stationnaire.

Le théorème limite central a donc lieu sous la condition  $D(\tau, X_0) < \infty$ .

## 2.2.2 Espaces de Hilbert

Dans [Ton11], Tone a établi théorème limite central pour des champs aléatoires à valeurs dans  $\mathcal{H}$  sous une condition de  $\rho^*$ -mélange.

Politis et Romano [PR94] ont montré que si pour un  $\delta > 0$ ,  $\mathbb{E} \|\mathbf{X}_1\|_{\mathcal{H}}^{2+\delta}$  est fini et  $\sum_j \alpha_{\mathbf{X}}(j)^{\frac{\delta}{2+\delta}}$  converge, alors la suite  $(n^{-1/2} \sum_{j=1}^n \mathbf{X}_j)$  converge vers une variable aléatoire gaussienne  $\mathcal{N}$ , dont l'opérateur de covariance  $S$  vérifie

$$\mathbb{E} [\langle \mathcal{N}, h \rangle^2] = \langle Sh, h \rangle_{\mathcal{H}} = \text{Var}(\langle \mathbf{X}_1, h \rangle) + 2 \sum_{i=1}^{+\infty} \text{Cov}(\langle \mathbf{X}_1, h \rangle, \langle \mathbf{X}_{1+i}, h \rangle).$$

Un résultat similaire fut obtenu par Dehling [Deh83].

Rappelons l'inégalité de Rio [Rio93] : étant données deux variables aléatoires réelles  $X$  et  $Y$  ayant un moment d'ordre deux fini,

$$|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]| \leq 2 \int_0^{\alpha(\sigma(X), \sigma(Y))} Q_X(u) Q_Y(u) du.$$

Cette inégalité fut étendue par Merlevède et al. [MPU97] : si  $\mathbf{X}$  et  $\mathbf{Y}$  sont des variables aléatoires à valeurs dans  $\mathcal{H}$ , de fonctions de quantile respectives  $Q_{\|\mathbf{X}\|_{\mathcal{H}}}$  et  $Q_{\|\mathbf{Y}\|_{\mathcal{H}}}$ , alors

$$|\mathbb{E}[\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathcal{H}}] - \langle \mathbb{E}[\mathbf{X}], \mathbb{E}[\mathbf{Y}] \rangle_{\mathcal{H}}| \leq 18 \int_0^\alpha Q_{\|\mathbf{X}\|_{\mathcal{H}}} Q_{\|\mathbf{Y}\|_{\mathcal{H}}} du,$$

où  $\alpha := \alpha(\sigma(\mathbf{X}), \sigma(\mathbf{Y}))$ .

De cette inégalité, ils déduisent un théorème limite central pour une suite  $(\mathbf{X}_j)_{j \in \mathbb{Z}}$  de variables aléatoires centrées à valeurs dans  $\mathcal{H}$  sous la condition

$$\int_0^1 \alpha^{-1}(u) Q_{\|\mathbf{X}_0\|_{\mathcal{H}}}^2(u) du < \infty, \quad (2.2.6)$$

où  $\alpha^{-1}$  est la fonction inverse de  $x \mapsto \alpha_{\mathbf{X}}(\lfloor x \rfloor)$ .

Le Corollaire 1.2 de [Rio00] montre que ce résultat implique celui de Politis et Romano.

## 2.3 Principe d'invariance

### 2.3.1 Conditions suffisantes

Dans la section précédente, nous avons vu des conditions suffisantes pour le théorème limite central en terme de coefficients de mélange. Il est naturel de se demander si celles-ci garantissent aussi le principe d'invariance dans  $D[0, 1]$  ou  $C[0, 1]$ .

Le principe d'invariance sous les hypothèses du Théorème 2.2.7 a été démontré par Shao dans [Sha89].

Peligrad a montré que le principe d'invariance dans  $D[0, 1]$  reste valide sous les conditions du Théorème 2.2.9 ainsi que sous les hypothèses mentionnées à la suite de ce résultat (voir [Pel98]).

### 2.3.2 Implication du principe d'invariance par le théorème limite central

Soit  $(f \circ T^i)_{i \geq 0}$  une suite strictement stationnaire. On suppose que  $(a_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$  converge en loi vers un mouvement brownien  $W$ . Comme la fonctionnelle  $\pi_1 : x \mapsto x(1)$  est continue sur  $C[0, 1]$ , la suite  $(a_n^{-1} S_n(f))_{n \geq 1}$  converge en loi vers une loi normale. En utilisant la fonctionnelle  $\pi_{t_1, \dots, t_d}(x) := (x_{t_i})_{i=1}^d$ , on observe que la convergence des lois fini-dimensionnelles a lieu. En utilisant la continuité sur  $C[0, 1]$  des fonctionnelles  $x \mapsto w(x, \delta)$ , on peut montrer que

$$\frac{1}{a_n} \max_{1 \leq j \leq n} |f \circ T^j| \rightarrow 0 \text{ en probabilité quand } n \rightarrow \infty. \quad (2.3.1)$$

En effet, pour  $n > 1/\delta$ , nous avons

$$\frac{1}{a_n} \max_{1 \leq j \leq n} |f \circ T^j| \leq w(a_n^{-1} S_n^{\text{pl}}(f), 1/j). \quad (2.3.2)$$

Comme l'union des points de discontinuité des fonction de répartition de  $w(W, 1/l)$   $l \geq 1$  est au plus dénombrable, il suffit de montrer que  $\mu \{ \max_{1 \leq j \leq n} |f \circ T^j| > \varepsilon a_n \} \rightarrow 0$  pour chaque  $\varepsilon > 0$  qui est un point de continuité des fonctions de répartition de tous  $w(W, 1/l)$   $l \geq 1$ .

Pour de tels  $\varepsilon$ , nous déduisons par continuité de  $x \mapsto w(x, 1/j)$  l'inégalité

$$\mu \left\{ \max_{1 \leq j \leq n} |f \circ T^j| > \varepsilon a_n \right\} \leq \mu \{ w(W, 1/j) > \varepsilon \}. \quad (2.3.3)$$

En faisant tendre  $j$  vers 0, nous obtenons (2.3.1).

En résumé, le théorème limite central fonctionnel dans  $C[0, 1]$  implique la convergence des lois fini-dimensionnelles et une condition sur les maxima des accroissements des sommes partielles (2.3.1). La question de la réciproque survient alors :

**Question.** Soit  $(f \circ T^k)_{k \geq 0}$  une suite strictement stationnaire centrée ayant une variance finie. Supposons qu'il existe une suite  $(a_n)_{n \geq 1}$  de réels strictement positifs telle que les lois fini-dimensionnelles de  $(a_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$  convergent en loi vers celles d'un mouvement brownien  $W$  et vérifiant (2.3.1). Quelle condition sur le mélange faut-il imposer pour garantir le principe d'invariance dans  $C[0, 1]$  ?

Cette condition ne doit pas garantir à elle seule le principe d'invariance.

Le  $\phi$ -mélange est une réponse à la question. Herrndorf a montré dans [Her83b] que si  $S_n(f)/\sigma_n$  converge vers une loi normale centrée réduite,  $\mathbb{E}[f] = 0$  et  $\mathbb{E}[f^2] < \infty$  et (2.3.1) a lieu, alors  $f$  vérifie le principe d'invariance dans  $D[0, 1]$  avec la normalisation  $\sigma_n$ . L'idée clé est d'utiliser le fait que  $\phi(m) < 1$  pour un certain  $m$ , et de contrôler la quantité  $\mu \{ \max_{1 \leq j \leq n} |S_j| a_n > 3\lambda \}$  par  $\mu \{ |S_n| a_n > \lambda \}$ ,  $\mu \{ \max_{1 \leq j \leq n} |f \circ T^j| a_n > \lambda \}$  et  $\phi(m)$ . Plus précisément, le Lemme 3.1 écrit dans le cas stationnaire de la manière suivante : pour tout entier  $q \geq 1$ , tout  $a > 0$  et  $r \geq q + 1$ ,

$$\begin{aligned} \left( 1 - \phi(q) - \max_{q \leq j \leq r} \mu \{ |S_r - S_j| > a \} \right) \mu \left\{ \max_{1 \leq j \leq r} |S_j| > 3a \right\} &\leq \\ &\leq \mu \{ |S_r| > a \} + \mu \left\{ (q-1) \max_{1 \leq j \leq r} |f \circ T^j| > a \right\}. \end{aligned} \quad (2.3.4)$$

L'inégalité (2.3.4) est ensuite utilisée pour établir l'équi-tension de la suite  $(\sigma_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$  dans l'espace  $D[0, 1]$ .

Nous traiterons le cas des suites  $\beta$ -mélangeantes dans le Chapitre 4.



# Chapitre 3

## Principe d'invariance pour les champs aléatoires strictement stationnaires

Dans ce chapitre, nous rappelons les définitions et résultats existants concernant le théorème limite central et le principe d'invariance pour des champs aléatoires strictement stationnaires.

### 3.1 Champs aléatoires de type différence d'orthomartingale

La notion de martingales a été introduite dans le Chapitre 1. Lorsque les accroissements sont strictement stationnaires, on a vu que le théorème limite central fonctionnel a lieu sous une condition de variance finie. Une stratégie pour démontrer un théorème limite consiste donc à approximer les sommes partielles par des martingales à accroissements strictement stationnaires. On aimerait suivre cette approche dans le cadre des champs aléatoires, mais on se heurte à une première difficulté : comment définir une martingale ? La définition de martingale en dimension un repose sur l'ordre naturel sur les entiers relatifs. Dans cette thèse, nous nous concentrerons sur les ortho-martingales.

#### 3.1.1 Définitions générales

Soit  $(\Omega, \mathcal{F}, \mu)$  un espace probabisé et  $T_q, q \in \{1, \dots, d\}$  des applications de  $\Omega$  dans lui-même qui préservent la mesure. On suppose que ces applications commutent, *i.e.*,  $T_i \circ T_j = T_j \circ T_i$  pour tous  $i, j \in \{1, \dots, d\}$ . On supposera de plus que ces applications sont bijective et bi-mesurables. On note pour  $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$ ,

$$T^{\mathbf{i}} := T_1^{i_1} \circ \dots \circ T_d^{i_d}. \quad (3.1.1)$$

On dira que  $T$  est une action de  $\mathbb{Z}^d$  sur  $\Omega$  qui préserve la mesure.

**Définition 3.1.1.** On note  $\mathcal{I}$  la tribu des invariants, *c'est-à-dire* la collection des ensembles  $A \in \mathcal{F}$  tels que pour tout  $\mathbf{k} \in \mathbb{Z}^d$ ,  $T^{\mathbf{k}}A = A$ .

**Définition 3.1.2.** On dit que  $(\Omega, \mathcal{F}, \mu, T)$  est ergodique si tous les éléments de  $\mathcal{I}$  sont de mesure nulle ou égale à 1.

On note également pour une fonction mesurable  $f: \Omega \rightarrow \mathbb{R}$  et  $\omega \in \Omega$

$$(U^{\mathbf{i}}f)(\omega) := f(T^{\mathbf{i}}\omega). \quad (3.1.2)$$



**Définition 3.1.3.** *Le champ aléatoire  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  est dit strictement stationnaire si pour tout  $(\mathbf{k}, n) \in \mathbb{Z}^d \times \mathbb{N}^*$  et tous  $(\mathbf{i}_1, \dots, \mathbf{i}_n) \in \mathbb{Z}^{dn}$ , les vecteurs  $(X_{\mathbf{i}_1}, \dots, X_{\mathbf{i}_n})$  et  $(X_{\mathbf{i}_1+\mathbf{k}}, \dots, X_{\mathbf{i}_n+\mathbf{k}})$  ont la même loi.*

Avec ces notations, le champ aléatoire  $(U^{\mathbf{j}}f)_{\mathbf{j} \in \mathbb{Z}^d}$  est strictement stationnaire.

### 3.1.2 Filtrations commutantes

Étant donnés deux éléments  $\mathbf{i}$  et  $\mathbf{j}$  de  $\mathbb{Z}^d$ , on dit que  $\mathbf{i} \preceq \mathbf{j}$  si  $i_q \leq j_q$  pour tout  $q \in \{1, \dots, d\}$ .

**Définition 3.1.4.** *La collection de tribus  $(\mathcal{F}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  est une filtration si  $\mathcal{F}_{\mathbf{i}} \subset \mathcal{F}_{\mathbf{j}}$  pour tous  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$  vérifiant  $\mathbf{i} \preceq \mathbf{j}$ . Cette filtration sera dite commutante si de plus, pour tous  $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$  et toute variable aléatoire intégrable et  $\mathcal{F}_1$ -mesurable  $Y$ ,*

$$\mathbb{E}[Y \mid \mathcal{F}_{\mathbf{k}}] = \mathbb{E}[Y \mid \mathcal{F}_{\mathbf{k} \wedge \mathbf{l}}] \text{ presque sûrement.} \quad (3.1.3)$$

Le terme « commutante » peut s'expliquer de la manière suivante : pour  $q \in \{1, \dots, d\}$ , considérons la tribu

$$\mathcal{F}_l^{(q)} = \bigvee_{\substack{\mathbf{i} \in \mathbb{Z}^d \\ i_q \leq l}} \mathcal{F}_{\mathbf{i}}, \quad l \in \mathbb{Z}. \quad (3.1.4)$$

où  $(\mathcal{F}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  est une filtration commutante. Alors pour toute permutation  $\pi$  de  $\{1, \dots, d\}$  et toute variable aléatoire intégrable  $Y$ , l'égalité

$$\mathbb{E} \left[ \dots \mathbb{E} \left[ \mathbb{E} \left[ Y \mid \mathcal{F}_{i_{\pi(1)}}^{(\pi(1))} \right] \mid \mathcal{F}_{i_{\pi(2)}}^{(\pi(2))} \right] \dots \mid \mathcal{F}_{i_{\pi(d)}}^{(\pi(d))} \right] = \mathbb{E}[Y \mid \mathcal{F}_{\mathbf{i}}] \quad (3.1.5)$$

a lieu pour tout  $\mathbf{i} \in \mathbb{Z}^d$  (voir [Kho02] page 36, Corollaire 3.4.1).

Un exemple important de filtrations commutantes est celui de filtrations engendrées par un champs i.i.d.  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$ . En effet, si  $\mathcal{F}_{\mathbf{i}} := \sigma(\varepsilon_{\mathbf{j}}, \mathbf{j} \preceq \mathbf{i})$ , alors la filtration  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est commutante. Pour le vérifier, on peut s'appuyer sur le

**Lemme 3.1.5.** *Soient  $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$  des sous-tribus indépendantes. Alors pour toute variable aléatoire intégrable  $X$ , l'égalité*

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}_1 \vee \mathcal{G}_2] \mid \mathcal{G}_2 \vee \mathcal{G}_3] = \mathbb{E}[X \mid \mathcal{G}_2] \quad (3.1.6)$$

a lieu.

Nous avons aussi dans le même esprit le lemme suivant, qui ne sera cependant pas utilisé dans la suite de cette sous-section.

**Lemme 3.1.6.** *Soient  $\mathcal{G}_i$ ,  $i \in \{1, 2, 3\}$  des sous-tribus de  $\mathcal{F}$  telles que  $\mathcal{G}_3$  est indépendante de  $\mathcal{G}_1 \vee \mathcal{G}_2$ . Alors pour toute variable aléatoire intégrable  $X$ , l'égalité*

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}_1] \mid \mathcal{G}_2 \vee \mathcal{G}_3] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}_1] \mid \mathcal{G}_2]. \quad (3.1.7)$$

a lieu.

Étant données des applications  $T_q$ ,  $1 \leq q \leq d$  qui commutent et préservent la mesure, on aimerait pouvoir former une filtration commutante à l'aide d'une tribu  $\mathcal{M}$  telle que pour tout  $q \in \{1, \dots, d\}$ ,  $T_q \mathcal{M} \subset \mathcal{M}$ .

**Définition 3.1.7.** *Soient  $T_q$ ,  $1 \leq q \leq d$  des applications bijectives, bi-mesurables de  $\Omega$  dans lui-même, qui préservent la mesure et commutent deux à deux. Soit  $\mathcal{M}$  une sous-tribu de  $\mathcal{F}$  telle que pour tout  $q \in \{1, \dots, d\}$ ,  $T_q \mathcal{M} \subset \mathcal{M}$ . On dit que la filtration  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d} := (T^{-\mathbf{i}} \mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  est complètement commutante si elle vérifie (3.1.3) pour tout  $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$  toute variable aléatoire intégrable et  $\mathcal{F}_1$ -mesurable  $Y$ .*

### 3.1.3 Définitions, opérateurs de projection

Afin d'énoncer la définition des orthomartingales, nous aurons besoin des notations suivantes :

$$\mathbb{E}_{\mathbf{j}}[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_{\mathbf{j}}], \mathbf{j} \in \mathbb{Z}^d \text{ et } \mathbb{E}_l^{(q)}[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_l^{(q)}], q \in \{1, \dots, d\}, l \in \mathbb{Z}. \quad (3.1.8)$$

**Définition 3.1.1.** La collection de variables aléatoires  $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$  est appelée ortho-martingale par rapport à la filtration commutante  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}_+^d}$  si :

- pour tout  $\mathbf{n} \in \mathbb{N}^d$ , la variable aléatoire  $M_{\mathbf{n}}$  est  $\mathcal{F}_{\mathbf{n}}$ -mesurable, intégrable et
- pour tous  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$  tels que  $\mathbf{i} \preceq \mathbf{j}$ ,

$$\mathbb{E}[M_{\mathbf{j}} \mid \mathcal{F}_{\mathbf{i}}] = M_{\mathbf{i}}. \quad (3.1.9)$$

De manière équivalente, étant donnée une filtration commutante  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}_+^d}$ , une collection de variables aléatoires  $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$  forme une ortho-martingale si pour tout  $q \in \{1, \dots, d\}$ , et pour tous  $\{n_l\}_{l \neq q} \subset \mathbb{N}$  fixés, la suite  $(M_{\mathbf{n}})_{n_q \in \mathbb{N}}$  est une martingale (à un paramètre) par rapport à la filtration  $(\mathcal{F}_{n_q}^{(q)})_{n_q \in \mathbb{N}}$  (voir [Kho02], p. 37, Théorème 3.5.1).

La définition est donnée pour une filtration commutante. Cependant, la restriction aux filtrations complètement commutantes est plus adaptée au contexte des champs strictement stationnaires.

Pour conclure cette sous-section, nous définissons les opérateurs de projection par

$$P_{\mathbf{j}} := \prod_{q=1}^d P_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d \quad (3.1.10)$$

où pour  $l \in \mathbb{Z}$ ,  $P_l^{(q)} : \mathbb{L}^1(\mathcal{F}) \rightarrow \mathbb{L}^1(\mathcal{F})$  est donné par

$$P_l^{(q)}(f) = \mathbb{E}_l^{(q)}[f] - \mathbb{E}_{l-1}^{(q)}[f]. \quad (3.1.11)$$

## 3.2 Champs aléatoires bernoulliens

### 3.2.1 Définition et exemples

Nous appellerons « champs aléatoires bernoulliens » les champs aléatoires strictement stationnaires qui sont des fonctionnelles de champs i.i.d.

**Définition 3.2.1.** On dit que le champ aléatoire  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est bernoullien s'il peut s'écrire sous la forme

$$X_{\mathbf{i}} = g(\varepsilon_{\mathbf{i}-\mathbf{s}}, \mathbf{s} \in \mathbb{Z}^d), \quad \mathbf{i} \in \mathbb{Z}^d, \quad (3.2.1)$$

où  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est un champs aléatoire i.i.d. et  $g : \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$  est une fonction mesurable.

Donnons des exemples de tels champs aléatoires.

- **Champs aléatoires linéaires.** Étant donné un champ aléatoire i.i.d.  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  où  $\varepsilon_{\mathbf{0}}$  est centrée et de carré intégrable, le champ aléatoire linéaire  $X = (X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  est défini par

$$X_{\mathbf{k}} := \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \varepsilon_{\mathbf{k}-\mathbf{i}}, \quad (3.2.2)$$

où  $(a_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est une famille de carré sommable. Ici, la fonction considérée est  $g : (x_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d} \mapsto \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} x_{\mathbf{i}}$ . Notons que par le Théorème 5.2 de [Kle14], le terme de droite dans (3.2.2) converge presque sûrement au sens où la limite  $\lim_{\min \mathbf{n} \rightarrow \infty} \sum_{-\mathbf{n} \preceq \mathbf{i} \preceq \mathbf{n}} a_{\mathbf{i}} \varepsilon_{\mathbf{k}-\mathbf{i}}$  existe presque sûrement.

- **Champs de Volterra.** Les processus de Volterra du second ordre sont définis de la manière suivante : soit  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  un champ aléatoire i.i.d. centré et de carré intégrable. On pose

$$X_{\mathbf{k}} := \sum_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^d} a_{\mathbf{s}_1, \mathbf{s}_2} \varepsilon_{\mathbf{k} - \mathbf{s}_1} \varepsilon_{\mathbf{k} - \mathbf{s}_2}, \quad (3.2.3)$$

où la famille  $(a_{\mathbf{s}_1, \mathbf{s}_2})_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^d}$  est de carré sommable telle que  $a_{\mathbf{s}_1, \mathbf{s}_2} = 0$  si  $\mathbf{s}_1 = \mathbf{s}_2$ . Nous utilisons aussi le Théorème 5.2 de [Kle14] pour garantir l'existence du membre de droite de (3.2.3).

### 3.2.2 Mesures de dépendance

Dans l'optique d'étudier les sommes partielles de champs aléatoires bernoulliens, il convient de quantifier la dépendance. Ceci peut se faire à l'aide de la mesure de dépendance physique introduite par Wu [Wu05]. Nous considérons un champs aléatoire i.i.d.  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  et une copie indépendante  $(\varepsilon'_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ . Pour  $\mathbf{i} \in \mathbb{Z}^d$ , on définit la version couplée  $X_{\mathbf{i}}^*$  par

$$X_{\mathbf{i}}^* := g(\varepsilon_{\mathbf{i}-\mathbf{s}}^*; \mathbf{s} \in \mathbb{Z}^d), \quad (3.2.4)$$

où, pour tout  $\mathbf{j} \in \mathbb{Z}^d$ ,

$$\varepsilon_{\mathbf{j}}^* = \begin{cases} \varepsilon_{\mathbf{j}} & \text{si } \mathbf{j} \neq \mathbf{0}, \\ \varepsilon'_{\mathbf{0}} & \text{si } \mathbf{j} = \mathbf{0}. \end{cases} \quad (3.2.5)$$

**Définition 3.2.2** (Mesure de dépendance physique). *Soit  $p \geq 1$  et  $\mathbf{i} \in \mathbb{Z}^d$ . Si  $X_{\mathbf{i}} \in \mathbb{L}^p$ , on pose*

$$\delta_{\mathbf{i}, p} := \|X_{\mathbf{i}} - X_{\mathbf{i}}^*\|_p. \quad (3.2.6)$$

**Définition 3.2.3.** *On dit que le champ bernoullien  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est  $p$ -stable si*

$$\Delta_p := \sum_{\mathbf{i} \in \mathbb{Z}^d} \delta_{\mathbf{i}, p} < +\infty. \quad (3.2.7)$$

Soit  $p \geq 2$ . Supposons que  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est un champ aléatoire i.i.d. centré ayant un moment d'ordre  $p$  fini. Pour les exemples de champs bernoulliens donnés dans la sous-section précédente, on peut donner une majoration des coefficients de dépendance physique.

- Considérons le champ linéaire (3.2.2). Comme  $\delta_{\mathbf{i}, p} = |a_{\mathbf{i}}| \|\varepsilon_{\mathbf{0}} - \varepsilon'_{\mathbf{0}}\|_p$ , le champ aléatoire défini par (3.2.2) est  $p$ -stable si la famille  $(a_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est sommable.
- Considérons le champ de Volterra (3.2.3). Posons pour  $\mathbf{k} \in \mathbb{Z}^d$ ,

$$A_{\mathbf{k}} := \sum_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^d} (a_{\mathbf{s}_1, \mathbf{k}}^2 + a_{\mathbf{k}, \mathbf{s}_2}^2) \text{ et } B_{\mathbf{k}} := \sum_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^d} (|a_{\mathbf{s}_1, \mathbf{k}}|^p + |a_{\mathbf{k}, \mathbf{s}_2}|^p). \quad (3.2.8)$$

Une application de l'inégalité de Rosenthal fournit la majoration

$$\delta_{\mathbf{k}, p} = \|X_{\mathbf{k}} - X_{\mathbf{k}}^*\|_p \leq C_p A_{\mathbf{k}}^{1/2} \|\varepsilon_{\mathbf{0}}\|_2 \|\varepsilon_{\mathbf{0}}\|_p + C_p B_{\mathbf{k}}^{1/p} \|\varepsilon_{\mathbf{0}}\|_p^2, \quad (3.2.9)$$

donc le champ de Volterra est  $p$ -stable si les familles  $(A_{\mathbf{k}}^{1/2})_{\mathbf{k} \in \mathbb{Z}^d}$  et  $(B_{\mathbf{k}}^{1/p})_{\mathbf{k} \in \mathbb{Z}^d}$  sont sommables.

- Plus généralement, si  $K: \mathbb{R} \rightarrow \mathbb{R}$  est une fonction lipschitzienne et  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  un processus  $p$ -stable, alors le processus  $(Y_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  défini par  $Y_{\mathbf{k}} := K(X_{\mathbf{k}})$  est  $p$ -stable.

En dimension 1, il est connu (cf. [Wu05], Théorème 1) que pour tout  $i \in \mathbb{Z}$ ,  $\|P_i(f)\|_2 \leq \delta_{i,2}$ , où  $P_i$  est donné par (1.2.7). En général, la condition de Hannan (1.2.8) est donc moins restrictive que la 2-stabilité. De plus, la Proposition 6.1 de [VW14] fournit un exemple de suite strictement stationnaire d'accroissements d'une martingale qui ne vérifie pas la condition de 2-stabilité. Ceci explique pourquoi il est préférable de travailler avec une condition de type Hannan plutôt qu'avec la 2-stabilité.

Pour mesurer la dépendance, nous allons utiliser des projecteurs, dont la définition diffère de (3.1.10). Nous considérons  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  un champ i.i.d., une fonction  $g: \mathbb{R}^{\mathbb{N}^d} \rightarrow \mathbb{R}$  et un champ aléatoire  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  défini par  $X_{\mathbf{k}} := g(\varepsilon_{\mathbf{k}-\mathbf{s}}, \mathbf{s} \in \mathbb{N}^d)$ . On pose

$$Q_{\mathbf{j}} := \prod_{q=1}^d Q_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d \quad (3.2.10)$$

où l'opérateur  $Q_{j_q}^{(q)}: \mathbb{L}^1 \rightarrow \mathbb{L}^1$  est défini par

$$Q_{j_q}^{(q)}(h) := \mathbb{E}[h \mid \sigma(\varepsilon_{\mathbf{i}}, i_q \geq -j_q)] - \mathbb{E}[h \mid \sigma(\varepsilon_{\mathbf{i}}, i_q \geq -j_q + 1)]. \quad (3.2.11)$$

Il s'agit en fait des opérateurs  $P_{\mathbf{i}}$  (donnés par (3.1.10)) relatifs à la filtration  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  définie par  $\mathcal{F}_{\mathbf{i}} := \sigma(\varepsilon_{\mathbf{j}}, \mathbf{j} \succcurlyeq -\mathbf{i})$ .

### 3.3 Théorème limite central fonctionnel

Pour une suite strictement stationnaire  $(f \circ T^i)_{i \geq 0}$ , nous pouvons définir un processus sommes partielles par (1.3.4). Dans cette section, nous rappelons la définition de l'extension aux champs aléatoires ainsi que certains résultats concernant les champs aléatoires strictement stationnaires dépendants.

#### 3.3.1 Principe d'invariance dans $C([0, 1]^d)$

Notons pour  $\mathbf{i} \succcurlyeq \mathbf{1}$ ,

$$R_{\mathbf{i}} := \prod_{q=1}^d [i_q - 1, i_q]. \quad (3.3.1)$$

Pour  $\mathbf{n} \succcurlyeq \mathbf{1}$ ,  $f: \Omega \rightarrow \mathbb{R}$  et  $\mathbf{t} \in [0, 1]^d$ , on pose

$$S_{\mathbf{n}}(f, \mathbf{t}) := \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} \lambda([0, \mathbf{n} \cdot \mathbf{t}] \cap R_{\mathbf{i}}) U^{\mathbf{i}} f, \quad (3.3.2)$$

où  $\lambda$  désigne la mesure de Lebesgue sur  $\mathbb{R}^d$  et  $[0, \mathbf{n} \cdot \mathbf{t}] = \prod_{q=1}^d [0, n_q t_q]$ . Il s'agit d'un processus dont les trajectoires se trouvent dans l'espace des fonctions continues sur  $[0, 1]^d$ .

Pour  $\mathbf{n} \in \mathbb{N}^d$ , notons  $|\mathbf{n}| := \prod_{q=1}^d n_q$  et  $\min \mathbf{n} := \min_{1 \leq q \leq d} n_q$ .

**Question.** À quelle(s) condition(s) le processus

$$\left( \frac{1}{\sqrt{|\mathbf{n}|}} S_{\mathbf{n}}(f, \mathbf{t}) \right)_{\mathbf{t} \in [0, 1]^d} \quad (3.3.3)$$

converge en loi dans  $C([0, 1]^d)$  lorsque  $\min \mathbf{n} \rightarrow +\infty$  ?

Ici, l'espace  $C([0, 1]^d)$  est muni de la norme uniforme.

Comme en dimension 1, on va supposer que le champ est centré, *i.e.*,  $\mathbb{E}[f] = 0$ . Kuelbs [Kue68] a étudié le problème lorsque le champ  $(f \circ T^{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^d}$  est indépendant et identiquement distribué. Il a obtenu une condition qui met en jeu les moments d'ordre 6 des variables aléatoires

tronquées  $f\mathbf{1}_{\{|f| \leq R\}}$  (qui est impliquée par la finitude de moments d'ordre  $3 + \delta$  pour un  $\delta > 0$ ). Puis Wichura [Wic69] a montré que la finitude du moment d'ordre 2 de  $f$  est une condition suffisante pour que le processus défini par (3.3.3) converge en loi dans  $C([0, 1]^d)$  le long de la suite  $(n\mathbf{1})_{n \geq 1}$  vers un drap brownien : un processus  $(B_{\mathbf{t}})_{\mathbf{t} \in [0, 1]^d}$  tel que  $B_{\mathbf{t}}$  est de loi normale centrée et  $\text{Cov}(B_{\mathbf{s}}, B_{\mathbf{t}}) = \prod_{q=1}^d \min\{s_q, t_q\}$  pour tous  $\mathbf{s}, \mathbf{t} \in [0, 1]^d$ . Le résultat a lieu si l'on considère la convergence lorsque  $\min \mathbf{n} \rightarrow +\infty$ . Nous avons vu que la finitude du moment d'ordre 2 de  $f$  est suffisante pour que le processus défini par (3.3.3) converge dans  $C([0, 1]^d)$  vers un drap brownien lorsque  $\min \mathbf{n} \rightarrow +\infty$ . Comme dans le cas des suites, la question de l'extension aux champs aléatoires stationnaires mais non indépendants est naturelle.

Basu et Dorea ont donné une définition de champs de type d'accroissements d'une martingale. Soit  $(\xi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$  un champ aléatoire. Si  $\mathbf{n} = (n_1, \dots, n_d)$  et  $n_i \geq 1$ , on pose

$$\mathcal{G}_{\mathbf{n}} := \sigma(\xi_{\mathbf{p}}, p_j \geq 1, 1 \leq j \leq d, \text{ et il existe } i \in \{1, \dots, d\} \text{ tel que } p_i \leq n_i).$$

**Théorème 3.3.1** (Basu et Dorea, 1979). *Soit  $(\xi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$  un champ aléatoire strictement stationnaire ergodique, de carré intégrable, tel que*

$$\mathbb{E}[\xi_{\mathbf{n}} \mid \mathcal{G}_{\mathbf{m}}] = 0 \text{ p.s. si } m < n. \quad (3.3.4)$$

*Alors le principe d'invariance a lieu.*

On pose

$$\Delta_{d,p}(f) := \sum_{\mathbf{k} \in \mathbb{N}^d} \frac{\|f \circ T^{\mathbf{k}} \mid \mathcal{F}_{\mathbf{1}}\|_p}{\prod_{i=1}^d k_i^{1/2}},$$

où  $\mathcal{F}_{\mathbf{k}} = \sigma(\varepsilon_{\mathbf{i}}, \mathbf{i} \preceq \mathbf{k})$  et  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est un champ i.i.d.

**Théorème 3.3.2** (Wang, Woodroffe, [WW13]). *Si  $\Delta_{d,2}(f)$  est fini alors  $(f \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d}$  vérifie le théorème limite central.*

*Si  $\Delta_{d,p}(f)$  est fini pour un  $p > 2$ , alors  $(f \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d}$  vérifie le principe d'invariance.*

Puis ce résultat a été amélioré.

**Théorème 3.3.3** (Volný, Wang, [VW14]). *Supposons que  $\mathcal{F}_j$  soit engendrée par un champ i.i.d.,  $f$  est  $\mathcal{M}$ -mesurable et que*

$$\sum_{\mathbf{j} \in \mathbb{N}^d} \|P_{\mathbf{j}}(f)\|_2 < +\infty. \quad (3.3.5)$$

*Alors le principe d'invariance a lieu.*

*Remarque 3.3.4.* On notera que ces résultats ne concernent que le cas des champs ergodiques. En général, on ne peut pas espérer un théorème limite central pour les champs de type accroissements d'une martingale. Si  $(f_1 \circ T_1^k)_{k \geq 0}$  et  $(f_2 \circ T_2^k)_{k \geq 0}$  sont deux suites i.i.d., indépendantes entre elles et de loi normale centrée réduite, le champ  $(U^{\mathbf{k}}(f_1 f_2))_{\mathbf{k} \in \mathbb{Z}^2}$  est un champ de type accroissements d'ortho-martingale. La convergence de  $(m^{-1/2} n^{-1/2} S_{n,m}(f_1 f_2))$  a lieu vers un produit de gaussiennes indépendantes donc la limite en loi n'est pas gaussienne. L'action de  $T_{i,j}(x, y) := (T_1^i x, T_2^j x)$  sur  $\mathbb{Z}^2$  est ergodique alors que les transformations  $T_{1,0}$  et  $T_{0,1}$  ne le sont pas.

**Théorème 3.3.5** ([Vol15]). *Soit  $f \in \mathbb{L}^2$  une fonction telle que  $(f \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  est un champ aléatoire de type accroissements d'orthomartingale par rapport à une filtration complètement commutante  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ . Si l'une des transformations  $T_q$ ,  $1 \leq q \leq d$  est ergodique, alors le théorème limite central a lieu, i.e., quand  $n_1, \dots, n_d \rightarrow \infty$ ,*

$$\frac{1}{\sqrt{|\mathbf{n}|}} S_{\mathbf{n}}(f) \quad (3.3.6)$$

*converge vers une loi normale centrée de variance  $\|f\|_2^2$ .*

Dans la Remarque 1 de [Vol15], il est précisé que le résultat de [VW14] reste valide pourvu que l'action de  $T$  vérifie les conditions du Théorème 3.3.5.

Les travaux de Dedecker [Ded98, Ded01] fournissent une condition projective qui reste valable sans aucune hypothèse d'ergodicité. Le processus limite n'est pas nécessairement un drap brownien. Nous reprenons les notations de ces articles. L'ordre lexicographique est considéré : si  $\mathbf{i}$  et  $\mathbf{j}$  sont des éléments distincts de  $\mathbb{Z}^d$ , on dira que  $\mathbf{i} <_{\text{lex}} \mathbf{j}$  si, soit  $i_1 < j_1$ , soit il existe  $p \in \{2, \dots, d\}$  tel que  $i_p < j_p$  et  $i_q = j_q$  pour  $1 \leq q < p$ .

Pour  $\mathbf{i} \in \mathbb{Z}^d$  et  $k \in \mathbb{N}^*$ , on définit

$$V_{\mathbf{i}}^1 := \{j \in \mathbb{Z}^d, \mathbf{j} <_{\text{lex}} \mathbf{i}\} \quad (3.3.7)$$

et pour  $k \geq 2$ ,

$$V_{\mathbf{i}}^k := V_{\mathbf{i}}^1 \cap \left\{ \mathbf{j} \in \mathbb{Z}^d, \max_{1 \leq q \leq d} |i_q - j_q| \geq k \right\}. \quad (3.3.8)$$

Pour  $\Gamma \subset \mathbb{Z}^d$ , on pose  $\mathcal{F}_{\Gamma} := \sigma(X_{\mathbf{k}}, \mathbf{k} \in \Gamma)$  et si  $h(X_{\mathbf{i}})$  est intégrable, on écrit

$$\mathbb{E}_k[h(X_{\mathbf{i}})] = \mathbb{E}[h(X_{\mathbf{i}}) \mid \mathcal{F}_{V_{\mathbf{i}}^k}]. \quad (3.3.9)$$

**Théorème 3.3.6** (Dedecker, [Ded01]). *Soit  $(f \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  un champs aléatoire strictement stationnaire centré. On suppose qu'il existe  $p > 1$  tel que que  $\|X_{\mathbf{0}}^2\|_p$  soit finie et*

$$\sum_{\mathbf{k} \in V_{\mathbf{0}}^1} \|X_{\mathbf{k}} \mathbb{E}_{\max_{1 \leq q \leq d} |k_q|} [X_{\mathbf{0}}]\|_p < +\infty. \quad (3.3.10)$$

Alors :

1. la série  $\sum_{\mathbf{k} \in \mathbb{Z}^d} \|\mathbb{E}[X_{\mathbf{0}} X_{\mathbf{k}} \mid \mathcal{I}]\|_p$  converge ;
2. notons  $\eta$  la variable aléatoire définie par l'égalité  $\eta := \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{E}[X_{\mathbf{0}} X_{\mathbf{k}} \mid \mathcal{I}]$ . Alors la suite  $(n^{-d/2} S_{n\mathbf{1}}(f, \mathbf{t}), \mathbf{t} \in [0, 1]^d)_{n \geq 1}$  converge en loi dans  $C([0, 1]^d)$  vers  $\sqrt{\eta} W$ , où  $W$  est un drap brownien indépendant de  $\eta$ .

On peut se demander ce qu'il est possible de déduire lorsque  $p = 1$  dans (3.3.10). Dans la Remarque 1 de [Ded01], les observations suivantes sont faites.

- Si  $d = 1$ , alors la conclusion du Théorème 3.3.6 a lieu.
- Les martingales au sens de Basu et Dorea (voir la condition (3.3.4)) de carré intégrable vérifient (3.3.10) avec  $p = 1$ .
- La question de la validité de la conclusion du Théorème 3.3.6 avec  $p = 1$  reste ouverte.

### 3.3.2 Principe d'invariance dans les espace hölderiens

Nous définissons pour  $d \geq 2$  et  $\alpha \in ]0, 1]$  les espaces  $\mathcal{H}_{\alpha}([0, 1]^d)$  et  $\mathcal{H}_{\alpha}^o([0, 1]^d)$  de manière analogue au cas uni-dimensionnel.

Erickson [Eri81] a montré que pour  $\alpha \in ]0, 1/2[$ , les trajectoires d'un drap brownien appartiennent à l'espace  $\mathcal{H}_{\alpha}([0, 1]^d)$ . L'étude du principe d'invariance dans  $\mathcal{H}_{\alpha}^o([0, 1]^d)$  a donc du sens.

**Question.** À quelle condition sur  $f$  le processus défini par (3.3.3) converge en loi vers un drap brownien dans  $\mathcal{H}_{\alpha}^o([0, 1]^d)$  quand  $\min \mathbf{n} \rightarrow +\infty$  ?

La convergence en loi dans ce cadre est équivalente à la convergence

$$\lim_{\min \mathbf{n} \rightarrow +\infty} F \left( \frac{1}{\sqrt{\mathbf{n}}} S_{\mathbf{n}}(f, \cdot) \right) = F(W) \quad (3.3.11)$$

pour toute fonctionnelle  $F: \mathcal{H}_\alpha^o([0, 1]^d) \rightarrow \mathbb{R}$  continue et bornée. En particulier, (3.3.11) a lieu pour toute fonctionnelle  $F: C([0, 1]^d) \rightarrow \mathbb{R}$  continue et bornée. Les hypothèses faites sur  $f$  devront donc au minimum impliquer le principe d'invariance dans l'espace  $C([0, 1]^d)$ .

Comme en dimension 1, notons pour  $\alpha \in ]0, 1[$ ,  $p(\alpha) := (1/2 - \alpha)^{-1}$  (remarquons que  $p(\alpha) > 2$ ). Dans [Eri81], Erickson a montré que si  $(f \circ T^i)_{i \in \mathbb{Z}^d}$  est un champ i.i.d. centré, la finitude du moment d'ordre  $q$  pour un  $q > d \cdot p(\alpha)$  garantit la convergence (3.3.11) pour toute fonctionnelle  $F: \mathcal{H}_\alpha^o([0, 1]^d) \rightarrow \mathbb{R}$  continue et bornée.

La même approche que dans le cas uni-dimensionnel montre que si  $f$  vérifie le principe d'invariance dans  $\mathcal{H}_\alpha^o([0, 1]^d)$ , alors

$$n_1^{-1/p(\alpha)} \prod_{j=2}^d n_j^{-1/2} \max_{\mathbf{1} \preccurlyeq \mathbf{j} \preccurlyeq \mathbf{n}} |f \circ T^{\mathbf{j}}| \rightarrow 0 \text{ en probabilité quand } \min \mathbf{n} \rightarrow +\infty. \quad (3.3.12)$$

Dans le cas où  $(f \circ T^i)_{i \in \mathbb{Z}^d}$  est indépendant, (3.3.12) est équivalente à

$$\sup_{t>0} t^{p(\alpha)} \mu \{|f| > t\} < +\infty. \quad (3.3.13)$$

Comme  $p(\alpha)$  est strictement supérieur à 2, la condition (3.3.13) implique l'existence de moments d'ordre  $q$  pour  $2 < q < p(\alpha)$ .

Si  $d = 2$ , et  $(f \circ T^i)_{i \in \mathbb{Z}^2}$  est indépendant, Račkauskas et Zemlys ont montré que (3.3.13) est également suffisante (cf. [RZ05]). Le caractère nécessaire et suffisant de (3.3.13) fut étendu aux dimensions supérieures ou égales à deux par Račkauskas, Suquet et Zemlys dans [RSZ07].

Deuxième partie

**Absolutely regular  
counter-examples to the  
(functional) central limit theorem**





# Chapter 4

## A strictly stationary $\beta$ -mixing process satisfying the central limit theorem but not the weak invariance principle

In 1983, N. Herrndorf proved that for a  $\phi$ -mixing sequence satisfying the central limit theorem and  $\liminf_{n \rightarrow \infty} \sigma_n^2/n > 0$ , the weak invariance principle takes place. The question whether for strictly stationary sequences with finite second moments and a weaker type  $(\alpha, \beta, \rho)$  of mixing the central limit theorem implies the weak invariance principle remained open.

We construct a strictly stationary  $\beta$ -mixing sequence with finite moments of any order and linear variance for which the central limit theorem takes place but not the weak invariance principle.

### 4.1 Introduction and notations

Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. If  $T: \Omega \rightarrow \Omega$  is one-to-one, bi-measurable and measure preserving (in sense that  $\mu(T^{-1}(A)) = \mu(A)$  for all  $A \in \mathcal{F}$ ), then the sequence  $(f \circ T^k)_{k \in \mathbb{Z}}$  is strictly stationary for any measurable  $f: \Omega \rightarrow \mathbb{R}$ . Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable  $f: \Omega \rightarrow \mathbb{R}$ , we define  $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$ ,  $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$  and  $S_n^{\text{pl}}(f, t) := S_{\lfloor nt \rfloor}(f) + (nt - \lfloor nt \rfloor)f \circ T^{\lfloor nt \rfloor}$ , where  $\lfloor x \rfloor$  is the greatest integer which is less than or equal to  $x$  (where "pl" stands for polygonal line).

We say that  $(f \circ T^j)_{j \geq 1}$  satisfies the *central limit theorem with normalization*  $a_n$  if the sequence  $(a_n^{-1} S_n(f))_{n \geq 1}$  converges weakly to a standard normal distribution. Let  $C[0, 1]$  denote the space of continuous functions on the unit interval endowed with the norm  $\|g\|_\infty := \sup_{t \in [0, 1]} |g(t)|$ .

Let  $D[0, 1]$  be the space of real valued functions which have left limits and are continuous-from-the-right at each point of  $[0, 1)$ . We endow it with Skorohod metric (cf. [Bil68]). We define  $S_n^{\text{ps}}(f, t) := S_{\lfloor nt \rfloor}(f)$  (where "ps" stands for partial sums), which gives a random element of  $D[0, 1]$ .

We shall say that the strictly stationary sequence  $(f \circ T^j)_{j \geq 0}$  satisfies the *weak invariance principle in*  $C[0, 1]$  *with normalization*  $a_n$  (respectively *in*  $D[0, 1]$ ) if the sequence of  $C[0, 1]$  (of  $D[0, 1]$ ) valued random variables  $(a_n^{-1} S_n^{\text{ps}}(f, \cdot))_{n \geq 1}$  (resp.  $(a_n^{-1} S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$ ) weakly converges

to a Brownian motion process in the corresponding space.

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two sub- $\sigma$ -algebras of  $\mathcal{F}$ , where  $(\Omega, \mathcal{F}, \mu)$  is a probability space. We define the  $\alpha$ -mixing coefficients as introduced by Rosenblatt in [Ros56]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \}. \quad (4.1.1)$$

Define the  $\beta$ -mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

where the supremum is taken over the finite partitions  $\{A_i, 1 \leq i \leq I\}$  and  $\{B_j, 1 \leq j \leq J\}$  of  $\Omega$  of elements of  $\mathcal{A}$  (respectively of  $\mathcal{B}$ ). They were introduced by Volkonskii and Rozanov [VR59].

The  $\rho$ -mixing coefficients were introduced by Hirschfeld [Hir35] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{ |\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B}) \}, \quad (4.1.2)$$

where  $\text{Corr}(f, g) := [\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)] [ \|f - \mathbb{E}(f)\|_{\mathbb{L}^2} \|g - \mathbb{E}(g)\|_{\mathbb{L}^2} ]^{-1}$ .

Ibragimov [Ibr59] introduced for the first time  $\phi$ -mixing coefficients, which are given by the formula

$$\phi(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(B | A) - \mu(B)|, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0 \}.$$

Taking  $f$  and  $g$  as characteristic functions of elements of  $\mathcal{F}_{-\infty}^0$  and  $\mathcal{F}_n^{+\infty}$  respectively, one can see that  $\alpha(n) \leq \rho(n)$ , hence  $\rho$ -mixing condition is more restrictive than  $\alpha$ -mixing condition.

To sum up, we have the inequalities

$$2\alpha(\mathcal{A}, \mathcal{B}) \leq \beta(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}), \quad \alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}) \leq 2\sqrt{\phi(\mathcal{A}, \mathcal{B})}. \quad (4.1.3)$$

For a strictly stationary sequence  $(X_k)_{k \in \mathbb{Z}}$  and  $n \geq 0$  we define

$$\alpha_X(n) := \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{\infty}) \quad (4.1.4)$$

where  $\mathcal{F}_u^v$  is the  $\sigma$ -algebra generated by  $X_k$  with  $u \leq k \leq v$  (if  $u = -\infty$  or  $v = \infty$ , the corresponding inequality is strict). In the same way we define coefficients  $\beta_X(n)$ ,  $\rho_X(n)$ ,  $\phi_X(n)$ .

**Definition 4.1.1.** We say that the sequence  $(X_k)_{k \in \mathbb{Z}}$  is  $\alpha$ -mixing (respectively  $\beta$ ,  $\rho$ , and  $\phi$ -mixing) if  $\lim_{n \rightarrow +\infty} \alpha_X(n) = 0$  (respectively  $\lim_{n \rightarrow +\infty} \beta_X(n) = 0$ ,  $\lim_{n \rightarrow +\infty} \rho_X(n) = 0$  and  $\lim_{n \rightarrow +\infty} \phi_X(n) = 0$ ). The term "absolutely regular" will refer to  $\beta$ -mixing sequences.

$\alpha$ ,  $\beta$  and  $\phi$ -mixing sequences were considered in the mentioned references, while  $\rho$ -mixing sequences first appeared in [KR60].

## 4.2 Main results

If  $(a_N)_{N \geq 1}$  and  $(b_N)_{N \geq 1}$  are two sequences of positive real numbers, we write  $a_N \asymp b_N$  if there exists a positive constant  $C$  such that for each  $N$ ,  $C^{-1}a_N \leq b_N \leq Ca_N$ .

The main results are

**Theorem A.** Let  $\delta$  be a positive real number. There exists a strictly stationary real valued process  $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$  satisfying the following conditions:

- a) the central limit theorem with normalization  $\sqrt{n}$  takes place;
- b) the weak invariance principle with normalization  $\sqrt{n}$  does not hold;

- c) we have  $\sigma_N(f)^2 \asymp N$ ;
- d) we have for some positive  $C$ ,  $\beta_Y(N) \leq C/N^{1-\delta}$ ;
- e)  $Y_0 \in \mathbb{L}^p$  for any  $p > 0$ .

Alternatively, we can construct the process in order to have a control of the mixing coefficients on a subsequence.

**Theorem A'.** *Let  $(c_j)_{j \geq 0}$  be a decreasing sequence of positive numbers. Then there exists a strictly stationary real valued process  $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$  satisfying conditions a), b), c), e) in Theorem A, and:*

- d') *there is an increasing sequence  $(m_k)_{k \geq 1}$  of integers such that for each  $k$ ,  $\beta_Y(m_k) \leq c_{m_k}$ .*

*Remark 4.2.1.* Herrndorf proved ([Her83b], Theorem 2.13) that if  $(\xi_n)_{n \geq 1}$  is a strictly stationary  $\phi$ -mixing sequence for which  $\sigma_n \rightarrow \infty$ ,  $S_n/\sigma_n$  converges in distribution to a standard normal distribution and  $\sigma_n^{-1} \max_{1 \leq i \leq n} |\xi_i| \rightarrow 0$  in probability, then the weak invariance principle takes place. So Herrndorf's result does not extend to  $\beta$ -mixing sequences.

*Remark 4.2.2.* Rio *et al.* proved in [Rio00] that the condition  $\int_0^1 \alpha^{-1}(u) Q^2(u) du < \infty$  implies the weak invariance principle, where  $\alpha^{-1}(u) := \inf \{k, \alpha(k) \leq u\}$  and  $Q$  is the right-continuous inverse of the quantile function  $t \mapsto \mu \{|X_0| > t\}$ . If the process is strictly stationary, with finite moments of order  $2+r$ ,  $r > 0$ , the latter condition is satisfied whenever  $\sum_{n=1}^{\infty} (n+1)^{2/r} \alpha(n) < \infty$  (Ibragimov [Ibr62] found the condition  $\sum_{n=1}^{\infty} \alpha(n)^{1-2/r} < \infty$ ). Since  $Y_0 \in L^p$  for all  $p < \infty$ , we have that  $\sum_N \alpha(N)^r = +\infty$  for any  $r < 1$ , hence in Theorem A' we can thus hardly get such a bound as in d') for the whole sequence.

*Remark 4.2.3.* Ibragimov proved that for a strictly stationary  $\rho$ -mixing sequence with finite moments of order  $2 + \delta$  for some positive  $\delta$ , the weak invariance principle holds, cf. [Ibr75]. In particular, this proves that our construction does not give a  $\rho$ -mixing process. Shao also showed in [Sha89] that the condition  $\sum_n \rho(2^n) < \infty$  is sufficient in order to guarantee the weak invariance principle in  $D[0, 1]$  for stationary sequences having order two moments. So a potential  $\rho$ -mixing counter-example has to adhere to restrictions on the moments as well as on the mixing rates.

### About the method of proof

In proving the result we will use properties of coboundaries  $h = g - g \circ T$  ( $g$  is called a transfer function). For a positive integer  $N$  and a measurable function  $v$ , we denote  $S_N(v) := \sum_{j=0}^{N-1} U^j v$

(Here and below,  $U^j v := v \circ T^j$ ). Because  $S_n(g - g \circ T) = g - g \circ T^n$ , for any sequence  $a_n \rightarrow \infty$  we have  $(a_n)^{-1} S_n(g - g \circ T) \rightarrow 0$  in probability hence adding a coboundary does not change validity of the central limit theorem. If, moreover,  $g \in \mathbb{L}^2$  then  $n^{-1/2} \|S_n^{\text{ps}}(g - g \circ T)\|_{\infty} \rightarrow 0$  a.s. hence adding of such coboundary does not change validity of the invariance principle (if norming by  $\sqrt{n}$  or by  $\sigma_n$  with  $\liminf_n \sigma_n^2/n > 0$ ), cf. [HH80], pages 140-141. On the other hand, if  $g \notin \mathbb{L}^2$ , adding a coboundary can spoil tightness even if  $g - g \circ T$  is square integrable, cf. [VS00]. A similar idea was used in [DMV07]. In the proof of Theorem A and A' we will find a coboundary  $g - g \circ T$  which is  $\beta$ -mixing and spoils tightness. The coboundary has all finite moments but the transfer function is not integrable. We then add an  $m$  such that  $(m \circ T^i)_{i \in \mathbb{Z}}$  and  $(h \circ T^i)_{i \in \mathbb{Z}}$  are independent (enlarging the probability space), and  $m \circ T^i$  is i.i.d. with moments of any order (in particular, it satisfies the weak invariance principle).

The proof uses the fact that  $|\mu(A \cap B) - \mu(A)\mu(B)| \leq \mu(A)$ . The method does not seem to apply to processes which are  $\rho$ -mixing and for this kind of processes the problem remains open.

## 4.3 Proofs

### 4.3.1 Construction of $h$

Let us consider an increasing sequence of positive integers  $(n_k)_{k \geq 1}$  such that

$$n_1 \geq 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty, \quad (4.3.1)$$

and for each integer  $k \geq 1$ , let  $A_k^-, A_k^+$  be disjoint measurable sets such that  $\mu(A_k^-) = 1/(2n_k^2) = \mu(A_k^+)$ .

Let the random variables  $e_k$  be defined by

$$e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise.} \end{cases} \quad (4.3.2)$$

We can choose the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  and the sets  $A_k^+, A_k^-$  in such a way that the family  $(e_k \circ T^i)_{k \geq 1, i \in \mathbb{Z}}$  is independent. We define  $A_k := A_k^+ \cup A_k^-$  and

$$h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k. \quad (4.3.3)$$

Since  $\mu\{h_k \neq 0\} \leq 2/n_k$ , the function  $h$  is almost everywhere well-defined (by the Borel-Cantelli lemma).

It will be useful to express, for  $N \geq n_k$ , the sum  $S_N(h_k)$  as a linear combination of  $U^p e_k$ . Denote  $s_k := \sum_{j=0}^{n_k-1} U^{-j} e_k$ . As  $N \geq n_k$  and  $h_k = s_k - U^{-n_k} s_k$ , we have

$$\begin{aligned} S_N(h_k) &= \sum_{j=0}^{N-1} (U^j s_k - U^{j-n_k} s_k) \\ &= \sum_{j=0}^{N-1} U^j s_k - \sum_{j=-n_k}^{N-n_k-1} U^j s_k \\ &= - \sum_{j=-n_k}^{-1} U^j s_k + U^N \sum_{j=-n_k}^{-1} U^j s_k. \end{aligned}$$

We also have

$$\begin{aligned} \sum_{j=-n_k}^{-1} U^j s_k &= \sum_{j=1}^{n_k} U^{-j} s_k \\ &= \sum_{i=0}^{n_k-1} U^{-i-1} \sum_{j=0}^{n_k-1} U^{-j} e_k \end{aligned} \quad (4.3.4)$$

$$\begin{aligned} &= U^{-1} \sum_{i=0}^{n_k-1} \sum_{j=0}^{n_k-1} U^{-(i+j)} e_k \\ &= U^{1-2n_k} \left( \sum_{j=1}^{n_k} j U^{j-1} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{n_k+j-1} e_k \right) \end{aligned} \quad (4.3.5)$$

$$\sum_{j=-n_k}^{-1} U^j s_k = U^{-2n_k} \left( \sum_{j=1}^{n_k} j U^j e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{n_k+j} e_k \right). \quad (4.3.6)$$

The previous equations yield

$$\begin{aligned} S_N(h_k) &= \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k \\ &\quad - \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k. \end{aligned} \quad (4.3.7)$$

Each  $h_k$  is a coboundary, as if we define  $v_k := \sum_{j=0}^{n_k-1} U^{-j} s_k$ , then  $v_k - U^{-1} v_k = s_k - U^{-n_k} s_k = h_k$  (so in this case the transfer function is  $-U^{-1} v_k$ ).

Since  $\mu \{v_k \neq 0\} \leq 2/n_k$ , Borel-Cantelli's lemma shows that the function  $g := -\sum_{k=1}^{+\infty} U^{-1} v_k$  is almost everywhere well defined under our assumption that  $\sum_k 1/n_k$  is convergent. Because  $h = g - Ug$ ,  $h$  is a coboundary.

### 4.3.2 Mixing rates

We show that the process  $(U^i f)_{i \in \mathbb{Z}}$  is  $\beta$ -mixing. In doing so we use the following proposition (cf. [Bra07], Theorem 6.2).

**Proposition 4.3.1.** *Let  $(X_{k,i})_i$ ,  $k = 1, 2, \dots$  be mutually independent strictly stationary processes with respective mixing coefficients  $\beta_k(n)$ , let  $X_i = \sum_{k=1}^{\infty} X_{k,i}$  converge. The process  $(X_i)_i$  is strictly stationary with mixing coefficients  $\beta(n) \leq \sum_{k=1}^{\infty} \beta_k(n)$ .*

This reduces the proof of  $\beta$ -mixing of  $(U^i f)_{i \in \mathbb{Z}}$  (in Theorems A and A') to that of  $(U^i h)_{i \in \mathbb{Z}}$  and thereby to that of  $(U^i h_k)_{i \in \mathbb{Z}}$  for  $k \geq 1$ .

In the following text we denote by  $\beta_k(n)$  the mixing coefficients of the process  $(h_k \circ T^i)_{i \in \mathbb{Z}}$ .

**Lemma 4.3.2.** *For  $k \geq 1$ , we have the estimate  $\beta_k(0) \leq 4/n_k$ .*

*Proof.* Suppose  $k$  is a positive integer. For  $-\infty \leq j \leq l \leq \infty$ , let  $\mathcal{H}_j^l$  denote the  $\sigma$ -field generated by  $U^i h_k$ ,  $j \leq i \leq l$ , ( $i \in \mathbb{Z}$ ), and let  $\mathcal{G}_j^l$  denote the  $\sigma$ -field generated by  $U^i e_k$ ,  $j \leq i \leq l$ , ( $i \in \mathbb{Z}$ ). Define the  $\sigma$ -fields  $\mathcal{B}_1 := \mathcal{G}_{-\infty}^{-2n_k}$ ,  $\mathcal{B}_2 := \mathcal{G}_{-2n_k+1}^0$  and  $\mathcal{B}_3 := \mathcal{G}_1^{\infty}$ . Now  $\mathcal{H}_{-\infty}^0 \subset \mathcal{B}_1 \vee \mathcal{B}_2$  and

$\mathcal{H}_0^\infty \subset \mathcal{B}_2 \vee \mathcal{B}_3$ . Therefore  $\beta_k(0) \leq \beta(\mathcal{B}_1 \vee \mathcal{B}_2, \mathcal{B}_2 \vee \mathcal{B}_3)$ . The  $\sigma$ -fields  $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ , are independent; hence the  $\sigma$ -fields  $\mathcal{B}_1 \vee \mathcal{B}_3$  and  $\mathcal{B}_2 \vee \mathcal{B}_2$  (with index 2 in both places) are independent; this implies by a result given e.g. in [Bra07, Theorem 6.2],

$$\beta(\mathcal{B}_1 \vee \mathcal{B}_2, \mathcal{B}_2 \vee \mathcal{B}_3) \leq \beta(\mathcal{B}_1, \mathcal{B}_3) + \beta(\mathcal{B}_2, \mathcal{B}_2) = 0 + \beta(\mathcal{B}_2, \mathcal{B}_2).$$

Thus  $\beta_k(0) \leq \beta(\mathcal{B}_2, \mathcal{B}_2)$ . Also, the  $\sigma$ -field  $\mathcal{B}_2$  has an atom  $P_0 := \bigcap_{i=-2n_k+1}^0 \{U^i e_k = 0\}$  that satisfies  $\mu(P_0) \geq 1 - 2/n_k$  (since  $\mu(U^i e_k \neq 0) = 1/n_k^2$  for each  $i$ ). By Lemma 2.2 of [Bra85], if  $\mathcal{B}$  is a  $\sigma$ -field which has an atom  $D$ , then  $\beta(\mathcal{B}, \mathcal{B}) \leq 2[1 - \mu(D)]$ . Hence  $\beta(\mathcal{B}_2, \mathcal{B}_2) \leq 2[1 - \mu(P_0)] \leq 4/n_k$ . This finishes the proof of Lemma 5.  $\square$

Denoting by  $\beta(N)$  the mixing coefficients of the sequence  $(h \circ T^i)_{i \in \mathbb{Z}}$ , Proposition 4.3.1, Lemma 4.3.2 and the fact that  $\beta_k(N) = 0$  when  $N \geq 2n_k$  yield

**Corollary 4.3.3.** *For each integer  $k$ , we have*

$$\beta(N) \leq \sum_{j \geq 1} \beta_j(N) \leq \sum_{j: 2n_j \geq N} \frac{4}{n_j}. \quad (4.3.8)$$

Now we can prove d) and d'). Let  $i(N)$  denote the unique integer such that  $n_{i(N)} \leq N < n_{i(N)+1}$ . For sequences  $(u_N)_{N \geq 1}, (v_N)_{N \geq 1}$  of positive numbers,  $u_N \lesssim v_N$  means that there is  $C > 0$  such that for each  $N$ ,  $u_N \leq C \cdot v_N$ .

**Proposition 4.3.4.** *Let  $\delta \in (0, 1)$  be fixed. Let  $\eta$  be such that  $1/(1 + \eta) = 1 - \delta$ . With the choice  $n_k := \lfloor 2^{(1+\eta)^k + 1} \rfloor$ , we have d).*

*Proof.* We deduce from (4.3.8)

$$\beta(2n_k) \leq \sum_{j \geq k} \frac{4}{n_j} \lesssim \sum_{j \geq 0} \lfloor 2^{(1+\eta)^{k+j} + 1} \rfloor \lesssim 2^{(1+\eta)^{k+1}}.$$

Consequently,

$$\beta(2N) \leq \beta(2n_{i(N)}) \lesssim \frac{4}{n_{i(N)}} \lesssim \frac{1}{n_{i(N)+1}^{1/(1+\eta)}} \lesssim \frac{1}{N^{1/(1+\eta)}}.$$

$\square$

Hence d) is fulfilled.

**Proposition 4.3.5.** *Given  $(c_k)_{k \geq 1}$  as in Theorem A', one can recursively choose a sequence  $(n_k)_{k \geq 1}$  growing to infinity arbitrarily fast, such that for the construction given above, one has that for each  $k \geq 1$ ,  $\beta(2n_k) \leq c_{2n_k}$ .*

*Proof.* Suppose that the sequence  $(n_k)_{k \geq 1}$  satisfies

$$n_{k+1} \geq \frac{8}{c_{2n_k}} \quad \text{and} \quad n_{k+1} \geq 2n_k, \quad k \geq 1.$$

Then

$$\beta(2n_k) \leq \sum_{j=k+1}^{\infty} \beta_j(2n_k) \leq \sum_{j=k+1}^{\infty} \beta_j(0),$$

and, by Lemma 4.3.2 and the condition  $n_{j+l} \geq 2^l n_j$  for  $j, l \geq 1$ , we derive

$$\beta(2n_k) \leq \sum_{j=k+1}^{\infty} \frac{4}{n_j} \leq \frac{8}{n_{k+1}}.$$

The assumption  $n_{k+1} \geq 8/c_{2n_k}$  yields  $\beta(2n_k) \leq c_{2n_k}$  for each  $k \geq 1$ .

$\square$

This proves d') with  $m_k := 2n_k$ .

*Remark 4.3.6.* The sequence of integers  $(n_k)_{k \geq 1}$  can be chosen to meet all other conditions imposed in this chapter.

### 4.3.3 Proof of non-tightness

**Lemma 4.3.7.** *There exists  $N_0$  such that*

$$\mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > 1/4 \quad (4.3.9)$$

whenever  $n_k \geq N_0$ .

*Proof.* For  $2n_k \leq N \leq n_k^2$ , thanks to (4.3.7), we have

$$\{|S_N(h_k)| = n_k\} \supset \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in I} \{U^j e_k = 0\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\},$$

where  $I = [1 - 2n_k, -1] \cap \mathbb{Z}$  and  $J_N = ([N - 2n_k + 1, N - 1 - n_k] \cup [N + 1 - n_k, N - 1]) \cap \mathbb{Z}$ . We define

$$\begin{aligned} B_{N,k} &:= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j=1-2n_k}^{-1-n_k} \{U^{N+j} e_k = 0\} \cap \bigcap_{j=1-n_k}^{-1} \{U^{j+N} e_k = 0\} \\ &= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\}. \end{aligned}$$

We have  $|S_N(h_k)| = n_k$  on  $\bigcap_{j \in I} \{U^j e_k = 0\} \cap B_{N,k}$  and the sets  $\bigcap_{j \in I} \{U^j e_k = 0\}$ ,  $\bigcup_{N=2n_k}^{n_k^2} B_{N,k}$  belong to independent  $\sigma$ -algebras. Therefore

$$\mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} \geq \left(1 - \frac{1}{n_k^2}\right)^{2n_k} \mu \left( \bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right). \quad (4.3.10)$$

Recall Bonferroni's inequality, which states that for any integer  $n$  and any events  $A_j, j \in \{1, \dots, n\}$ , we have

$$\mu \left( \bigcup_{j=1}^n A_j \right) \geq \sum_{j=1}^n \mu(A_j) - \sum_{1 \leq i < j \leq n} \mu(A_i \cap A_j). \quad (4.3.11)$$

It can be proved by induction. Notice that

$$\mu(B_{N,k}) = \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k-2} \geq \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right)$$

and for  $i \neq j$

$$\mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \leq \mu\{|U^{i+n_k-1} e_k| = 1\} \mu\{|U^{j+n_k-1} e_k| = 1\} = \frac{1}{n_k^4}$$

hence

$$\begin{aligned} \mu \left( \bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right) &= \mu \left( \bigcup_{N=1}^{(n_k-1)^2} B_{N+2n_k-1,k} \right) \\ &\geq \sum_{N=1}^{(n_k-1)^2} \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right) - \sum_{1 \leq i < j \leq (n_k-1)^2} \mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \\ &\geq \left(1 - \frac{2}{n_k}\right)^3 - \frac{1}{2} \end{aligned}$$



which together with (4.3.10) and the inequality  $(1 - 1/n_k^2)^{2n_k} \geq 1 - 2/n_k$  implies that

$$\mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > \frac{1}{4}, \quad (4.3.12)$$

whenever  $n_k \geq N_0$ , where  $N_0 \geq 3$  is such that  $(1 - 2/n) [(1 - 2/n)^3 - 1/2] > 1/4$  for  $n \geq N_0$ .  $\square$

**Proposition 4.3.8.** *Assume that the sequence  $(n_k)_{k \geq 1}$  is such that*

$$\text{there exists } \eta > 0 \text{ such that for each } k, \quad n_{k+1} \geq n_k^{1+\eta}. \quad (4.3.13)$$

*Then we have for  $k$  large enough*

$$\mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq 1/2 \right\} \geq 1/8. \quad (4.3.14)$$

*Proof.* Let us fix an integer  $k$ . Let us define the events

$$A := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\}, \quad (4.3.15)$$

$$B := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \text{ and} \quad (4.3.16)$$

$$C := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \leq k-1} h_j \right) \right| \leq \frac{1}{2} \right\}. \quad (4.3.17)$$

Since the family  $\{e_k \circ T^i, k \geq 1, i \in \mathbb{Z}\}$  is independent, the events  $B$  and  $C$  are independent. Notice that  $B \cap C \subset A$  hence

$$\mu(A) = \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \geq \mu(B)\mu(C).$$

In order to give a lower bound for  $\mu(B)$ , we define  $E_k := \bigcup_{N=2n_k}^{n_k^2} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$ ; then

$$\mu(B) \geq \mu(B \cap E_k^c) \quad (4.3.18)$$

$$= \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c \right) \quad (4.3.19)$$

$$\geq \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \right) - \mu(E_k). \quad (4.3.20)$$

Let us give an estimate of the probability of  $E_k$ . Fix  $j > k$ . Writing  $S_N(h_j) = \sum_{i=0}^{N-1} U^i s_j - \sum_{i=-n_j}^{N-n_j-1} U^i s_j$ , where  $s_j := \sum_{i=0}^{n_j-1} U^{-i} e_j$ , we can see that

$$\bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \subset \bigcup_{i=-2n_j+1}^{n_k^2} T^{-i} A_j \quad (4.3.21)$$

takes place for  $j > k$ , hence

$$\mu \left( \bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \right) \leq \frac{n_k^2 + 2n_j}{n_j^2}, \quad (4.3.22)$$

and it follows that

$$\mu(E_k) \leq \sum_{j=k+1}^{+\infty} \frac{2n_k}{n_j}. \quad (4.3.23)$$

By (4.3.13), we have  $n_k \leq n_j^{1/(1+\eta)}$  for  $j > k$ , hence by (4.3.23),

$$\mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} n_j^{-\frac{\eta}{1+\eta}}. \quad (4.3.24)$$

As condition (4.3.13) implies that  $n_k \geq 2^k$  for  $k$  large enough, we conclude that the following inequality holds for  $k$  large enough:

$$\mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} 2^{-j \frac{\eta}{1+\eta}}. \quad (4.3.25)$$

Thus, by Lemma 4.3.7 and (4.3.25), we have for  $k$  large enough

$$\begin{aligned} & \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \\ & \geq \left( \frac{1}{4} - 2 \sum_{j=k+1}^{+\infty} 2^{-j \frac{\eta}{1+\eta}} \right) \left( 1 - \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \leq k-1} h_j \right) \right| > \frac{1}{2} \right\} \right). \end{aligned} \quad (4.3.26)$$

Defining  $c_k := \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \leq k-1} h_j \right) \right| > \frac{1}{2} \right\}$ , it is enough to prove that

$$\lim_{k \rightarrow \infty} c_k = 0. \quad (4.3.27)$$

Using (4.3.7) (accounting  $N \geq 2n_k \geq n_j$  for  $j < k$ ), we get the inequalities

$$c_k \leq \sum_{j=1}^{k-1} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_j)| > \frac{1}{2(k-1)} \right\} \quad (4.3.28)$$

$$\leq \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \quad (4.3.29)$$

$$+ \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} U^N \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} +$$

$$+ \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} U^N \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\}$$

$$\leq n_k^2 \left( \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \right). \quad (4.3.30)$$

Notice that for each  $j \leq k - 1$ ,

$$\mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \leq \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\} + \mu \left\{ |n_j U^{n_j} e_j| > \frac{n_k}{16k} \right\}. \quad (4.3.31)$$

Condition (4.3.13) implies the inequality  $16k \cdot n_{k-1} < n_k$  for  $k$  large enough, hence keeping in mind that  $U^{n_j} e_j$  is bounded by 1, inequality (4.3.31) becomes for such  $k$ 's,

$$\mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \leq \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\}. \quad (4.3.32)$$

Combining (4.3.30) with (4.3.32), we obtain for each  $x > 0$ ,

$$c_k \leq 2n_k^2 \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\} \quad (4.3.33)$$

$$\leq 2n_k^2 \frac{(16k)^p}{n_k^p} \sum_{j=1}^{k-1} \mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p, \quad (4.3.34)$$

where  $p > 2 + 1/\eta$ . By Rosenthal's inequality (see [Ros70], Theorem 1), we have

$$\mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p \leq C_p \left( \sum_{i=1}^{n_j} i^p \mathbb{E} |e_j| + \left( \sum_{i=1}^{n_j} \mathbb{E} [i^2 e_j^2] \right)^{p/2} \right) \quad (4.3.35)$$

$$\leq C_p (n_j^{p+1-2} + n_j^{3p/2}/n_j^p) \quad (4.3.36)$$

$$\leq 2C_p n_j^{p-1} \quad (4.3.37)$$

as  $p > 2$ . Therefore, for some constant  $K$  depending only on  $p$ ,

$$c_k \leq K \cdot n_k^{2-p} k^p \sum_{j=1}^{k-1} n_j^{p-1} \leq K \cdot k^{p+1} \frac{n_{k-1}^{p-1}}{n_k^{p-2}}, \quad (4.3.38)$$

and by (4.3.13),

$$c_k \leq K \cdot k^{p+1} n_{k-1}^{p-1-(p-2)(1+\eta)}. \quad (4.3.39)$$

Since  $p - 1 - (p - 2)(1 + \eta) = 1 - (p - 2)\eta < 0$  and  $n_{k-1} \geq 2^{k-1}$  for each  $k \geq 2$ , we get

$$c_k \leq K \cdot k^{p+1} 2^{(1-(p-2)\eta)(k-1)}. \quad (4.3.40)$$

This concludes the proof of Proposition 4.3.8 hence that of b).  $\square$

The previous lemma yields together with Theorems 8.1 and 15.1 of [Bil68] and the convergence to 0 of the finite dimensional distributions of  $(N^{-1/2} S_N^{\text{ps}}(h))_{N \geq 1}$  and  $(N^{-1/2} S_N^{\text{pl}}(h))_{N \geq 1}$  the following corollary.

**Corollary 4.3.9.** *If  $(n_k)_{k \geq 1}$  satisfies (4.3.1) and , then the sequences  $(N^{-1/2} S_N^{\text{ps}}(h, \cdot))_{N \geq 1}$  and  $(N^{-1/2} S_N^{\text{pl}}(h, \cdot))_{N \geq 1}$  are not tight in their respective spaces.*

Let  $\delta > 0$ . Then the choice  $n_k := \lfloor 2^{(1+\eta)k+1} \rfloor$  satisfies the conditions (4.3.1) and .

Under assumptions of Proposition 4.3.4 (the choice of  $n_k$ ) we get d) in A and because (4.3.1), (4.3.13) are satisfied, we get b) in A.

By Remark 4.3.6, we can construct in Proposition 4.3.5 the sequence  $(n_k)_{k \geq 1}$  in such a way that it also satisfies (4.3.1) and (4.3.13); this yields b) in Theorem A', and of course from Proposition 4.3.5 itself, property d') in Theorem A' also holds.

### 4.3.4 Proof of the central limit theorem and linear variance

Let us denote by  $\sigma_N^2$  the variance of  $S_N(h)$ , that is,  $\mathbb{E}[S_N(h)^2]$ .

**Proposition 4.3.10.** *Under the conditions (4.3.1) and (4.3.13), we have  $\sigma_N^2 \lesssim N$ .*

*Proof.* From (4.3.7) and (4.3.13), we deduce that

$$\sum_{j=1}^{i(N)} \mathbb{E}[S_N(h_j)^2] \lesssim \sum_{j=1}^{i(N)} n_j \leq 2n_{i(N)} \lesssim N. \quad (4.3.41)$$

Recall that  $h_k = (I - U^{-n_k})s_k$  with  $s_k := \sum_{i=0}^{n_k-1} U^{-i}e_k$ . Therefore, when  $n_k \geq N$ , we have by a similar computation as for (4.3.6),

$$S_N(h_k) = (I - U^{-n_k}) \sum_{j=1}^{N-1} j(U^{j-n_k} + U^{N-j})e_k + N(I - U^{-n_k}) \sum_{j=N}^{n_k} U^{j-n_k}e_k. \quad (4.3.42)$$

The first term has a variance of order  $N^3/n_k^2$ , and the variance of the second term is (at most) of order  $N^2n_k/n_k^2$ . We thus have that for  $n_k \geq N$ ,  $\mathbb{E}[S_N(h_k)^2] \lesssim N^2/n_k$ , hence by (4.3.13),

$$\begin{aligned} \sum_{k \geq i(N)+1} \mathbb{E}[S_N(h_k)^2] &\lesssim \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \\ &= \frac{N^2}{n_{i(N)+1}} + \sum_{k \geq i(N)+2} \frac{N^2}{n_k} \leq \frac{N^2}{n_{i(N)+1}} \left( 1 + \sum_{j \geq i(N)+2} \frac{1}{j^2} \right), \end{aligned}$$

therefore,

$$\sum_{k \geq i(N)+1} \mathbb{E}[S_N(h_k)^2] \lesssim N. \quad (4.3.43)$$

Combining (4.3.41) and (4.3.43), and using (for a fixed  $N$ ) the independence of the sequence  $(S_N(h_j))_{j \geq 1}$ , we conclude that  $\sigma_N^2(h) = \sigma_N^2(g - g \circ T) \lesssim N$ .

When we add a mean-zero nondegenerate independent sequence  $(m \circ T^i)_{i \in \mathbb{Z}}$  with moments of any order greater than 2, the variance of the  $N$ th partial sum of  $((m + h) \circ T^i)_{i \geq 1}$  is bounded above and below by a quantity proportional to  $N$ , hence **c)** is satisfied in Theorems **A** and **A'**. By the observation in the paragraph "About the method of proof", **a)** holds.  $\square$

### 4.3.5 Moments of the coboundary and the transfer function

One can wonder to which  $\mathbb{L}^p$  space can  $g$  and  $g - g \circ T$  belong.

**Proposition 4.3.11.** *Under the conditions (4.3.1) and (4.3.13), we have  $g \in \mathbb{L}^p$  for  $0 < p < 1$  and  $g - g \circ T \in \mathbb{L}^p$  for each  $p > 0$ .*

*Proof.* Let  $g_k := U^{-1}v_k$ , where  $v_k = \sum_{j=0}^{n_k-1} U^{-j}s_k$  and  $s_k = \sum_{j=0}^{n_k-1} U^{-j}e_k$ . Recall that  $g = -\sum_{k=1}^{\infty} g_k$ . For  $0 < p < 1$  and any two non-negative real numbers  $a$  and  $b$ , we have  $(a + b)^p \leq$

$a^p + b^p$ . This gives, using (4.3.6),

$$\begin{aligned} \mathbb{E} |g_k|^p &\leq \left( \sum_{j=1}^{n_k} j^p \mathbb{E} |U^{-j} e_k| + \sum_{j=1}^{n_k-1} (n_k - j)^p \mathbb{E} |U^{-j+n_k} e_k| \right) \\ &= \frac{1}{n_k^2} \left( n_k^p + 2 \sum_{j=1}^{n_k-1} j^p \right) \\ &\leq \frac{1}{n_k^2} (n_k^p + 2n_k^{p+1}) \\ &\leq 3n_k^{p-1}. \end{aligned}$$

By (4.3.13), we have  $n_k \geq k! \cdot n_1$  hence the series  $\sum_{k \geq 1} \mathbb{E} |g_k|^p$  is convergent. This proves that  $g \in \mathbb{L}^p$  for  $0 < p < 1$ .

Corollary 2.4. in [BT10] states the following: given positive integers  $t$  and  $p$ ,  $X_1, \dots, X_t$  independent random variables such that  $\mu\{0 \leq X_j \leq 1\} = 1$  for each  $j \in \{1, \dots, t\}$ , then

$$\mathbb{E}(X)^p \leq B_p \cdot \max\{\mathbb{E}(X), (\mathbb{E}(X))^p\}, \quad (4.3.44)$$

where  $B_p$  is the  $p$ -th Bell's number (defined by the recursion relation  $B_{p+1} = \sum_{k=0}^p \binom{p}{k} B_k$  and  $B_0 = B_1 = 1$ ) and  $X := \sum_{j=1}^t X_j$ .

We shall show that the series  $\sum_{k \geq 1} \|h_k\|_p$  is convergent for any integer  $p$ . Fix  $k \geq 1$ , and let  $t := 2n_k$ ,  $X_j := |U^{j-2n_k} e_k|$ . Applying (4.3.44), we get

$$\|h_k\|_p^p \leq B_p \cdot \max\{2n_k^{-1}, (2n_k^{-1})^p\} = 2B_p \cdot n_k^{-1},$$

hence  $\|h_k\|_p \leq (2B_p)^{1/p} \cdot n_k^{-1/p}$  and condition (4.3.13) guarantees the convergence of the series  $\sum_k n_k^{-1/p}$ . One could also use Rosenthal's inequality.  $\square$

Since the added process has moments of any order, Proposition 4.3.11 proves e) in Theorems A and A'.

**Proposition 4.3.12.** *The transfer function  $g$  does not belong to  $\mathbb{L}^1$ .*

*Proof.* Fix an integer  $k$ , and define for  $1 \leq j \leq n_k$ :

$$E_j := \{|U^{-j} e_k| = 1\} \cap \bigcap_{i \in \{1, \dots, 2n_k-1\} \setminus \{j\}} \{U^{-i} e_k = 0\} \cap \bigcap_{l \neq k} \{g_l = 0\}.$$

Since these sets are pairwise disjoint,  $g = \sum_{k \geq 1} g_k$ , with  $g_k := U^{-1} \sum_{i=0}^{n_k-1} U^{-i} \left[ \sum_{h=0}^{n_k-1} U^{-h} e_k \right]$  and  $g_l(\omega) = 0$  if  $l \neq k$  and  $\omega \in \bigcup_{j=1}^{n_k} E_j$ , we have the equality of functions

$$\begin{aligned} |g_k| \cdot \mathbf{1} \left( \bigcup_{j=1}^{n_k} E_j \right) &= \sum_{j=1}^{n_k} |g_k| \cdot \mathbf{1}(E_j) \\ &= \sum_{j=1}^{n_k} \mathbf{1}(E_j) \cdot \left| U^{-1} \sum_{i=0}^{n_k-1} U^{-i} \left[ \sum_{h=0}^{n_k-1} U^{-h} e_k \right] \right| \\ &= \sum_{j=1}^{n_k} \mathbf{1}(E_j) \cdot \left| \sum_{i=0}^{n_k-1} \sum_{h=0}^{n_k-1} U^{-1-(i+h)} e_k \right| = \sum_{j=1}^{n_k} \mathbf{1}(E_j) |jU^{-j} e_k| \\ &= \sum_{j=1}^{n_k} j \cdot \mathbf{1}(E_j) \end{aligned}$$

and hence

$$\left\| |g_k| \cdot \mathbf{1} \left( \bigcup_{j=1}^{n_k} E_j \right) \right\|_1 = \sum_{j=1}^{n_k} j \cdot \mu(E_j). \quad (4.3.45)$$

As  $(1 - 1/n_k^2)^{-1} \rightarrow 1$  for  $k \rightarrow +\infty$  and  $\prod_{j \geq 1} (1 - 1/n_j^2)^{2n_j}$  is positive, we get

$$\mu(E_j) \geq \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k-1} \prod_{l \neq k} \left(1 - \frac{1}{n_l^2}\right)^{2n_l} \geq \frac{c}{n_k^2} \quad (4.3.46)$$

for some positive constant  $c$  independent of  $k$  and  $j$ .

Let us define  $F_k := \bigcup_{j=1}^{n_k} E_j$  for  $k \geq 1$ . Notice that the event  $F_k$ ,  $k \geq 1$  are pairwise disjoint because  $F_k \subset \{g_k \neq 0\} \cap \bigcap_{l \neq k} \{g_l = 0\}$ . Therefore, combining (4.3.45) and (4.3.46), we obtain for each  $k \geq 1$ ,

$$\mathbb{E}[|g| \mathbf{1}(F_k)] = \mathbb{E}[|g_k| \mathbf{1}(F_k)] \geq c/2.$$

It then follows that

$$\mathbb{E}|g| \geq \sum_k \mathbb{E}[|g_k| \mathbf{1}(F_k)] = \infty,$$

proving Proposition 4.3.12. □

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# Chapter 5

## A counter-example to the central limit theorem in Hilbert spaces under a strong mixing condition

We show that in a separable infinite dimensional Hilbert space, uniform integrability of the square of the norm of normalized partial sums of a strictly stationary sequence, together with a strong mixing condition, does not guarantee the central limit theorem.

### 5.1 Introduction and notations

Let  $(V, \|\cdot\|)$  be a separable normed space. We can represent a strictly stationary sequence  $(X_j)_j$  by  $X_j = f \circ T^j$ , where  $T: \Omega \rightarrow \Omega$  is measurable and measure preserving, that is,  $\mu(T^{-1}(S)) = \mu(S)$  for all  $S \in \mathcal{F}$  (see [Doo53], p. 456, second paragraph).

Given an integer  $N$ , we define  $S_N(f) := \sum_{j=0}^{N-1} f \circ T^j$  and  $(\sigma_N(f))^2 := \mathbb{E} [\|S_N(f)\|^2]$ . The mixing coefficients were defined in Section 4.1.

When  $V = \mathbb{R}$  it is well-known that if  $(f \circ T^j)_{j \geq 0}$  satisfies the following assumptions:

1.  $\lim_{N \rightarrow +\infty} \sigma_N(f) = +\infty$ ;
2.  $\int f d\mu = 0$ ;
3.  $\lim_{n \rightarrow +\infty} \alpha(n) = 0$ ;
4. the family  $\left\{ \frac{\|S_N(f)\|^2}{(\sigma_N(f))^2}, N \geq 1 \right\}$  is uniformly integrable,

then  $\left( \frac{1}{\sigma_N(f)} S_N(f) \right)_{N \geq 1}$  converges in distribution to a Gaussian law. It was established for  $d = 1$  by Denker [Den86], Mori and Yoshihara [MY86] using a blocking argument and Volný [Vol88].

When  $\mathbb{R}$  is replaced by  $\mathbb{R}^d$ ,  $d \in \mathbb{N}^*$ , the sequence  $\left( \frac{1}{\sigma_N(f)} S_N(f) \right)_{N \geq 1}$  does not necessarily converge to a Gaussian, but the limiting behavior involves "wobbling limit law" (cf. [Bra12]).

A natural question would be: what if we replace  $\mathbb{R}^d$  by another normed space?

First, we restrict ourselves to separable normed spaces in order to avoid measurability issues of sums of random variables. Corollary 10.9. in [LT91] asserts that a separable Banach space  $B$



with norm  $\|\cdot\|_B$  is isomorphic to a Hilbert space if and only if for all random variables  $X$  with values in  $B$ , the conditions  $\mathbb{E}[\mathbf{X}] = 0$  and  $\mathbb{E}[\|\mathbf{X}\|_B^2] < \infty$  are necessary and sufficient for  $X$  to satisfy the central limit theorem. By " $\mathbf{X}$  satisfies the CLT", we mean that if  $(\mathbf{X}_j)_{j \geq 1}$  is a sequence of independent random variables, with the same law as  $X$ , the sequence  $\left(n^{-1/2} \sum_{j=1}^n \mathbf{X}_j\right)_{n \geq 1}$  weakly converges in  $B$ . Hence we cannot expect a generalization in a class larger than separable Hilbert spaces. Such a space is necessarily isomorphic to  $\mathcal{H} := \ell^2(\mathbb{R})$ , the space of square summable sequences  $(x_n)_{n \geq 1}$  endowed with the inner product  $\langle x, y \rangle_{\mathcal{H}} := \sum_{n=1}^{+\infty} x_n y_n$ . We shall denote by  $\mathbf{e}_n$  the sequence whose all terms are 0, except the  $n$ -th which is 1. Bold letters denote both random variables taking their values in  $\mathcal{H}$  and elements of this space.

General considerations about probability measures and central limit theorem in Banach spaces are contained in Araujo and Giné's book [AG80].

## 5.2 Main results

*Notation 5.2.1.* If  $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$  are sequences of non-negative real numbers,  $a_n \lesssim b_n$  means that  $a_n \leq C b_n$ , where  $C$  does not depend on  $n$ . In an analogous way, we define  $a_n \gtrsim b_n$ . When  $a_n \lesssim b_n \lesssim a_n$ , we simply write  $a_n \asymp b_n$ .

Our main results are

**Theorem B.** *There exists a probability space  $(\Omega, \mathcal{F}, \mu)$  such that given  $0 < q < 1$ , we can construct a strictly stationary sequence  $\mathbf{X} = (\mathbf{f} \circ T^k) = (\mathbf{X}_k)_{k \in \mathbb{N}}$  defined on  $\Omega$ , taking its values in  $\mathcal{H}$ , such that:*

- a)  $\mathbb{E}[\mathbf{f}] = 0$ ,  $\mathbb{E}[\|\mathbf{f}\|_{\mathcal{H}}^p]$  is finite for each  $p$ ;
- b) the limit  $\lim_{N \rightarrow \infty} \sigma_N(\mathbf{f})$  is infinite;
- c) the process  $(\mathbf{X}_k)_{k \in \mathbb{N}}$  is  $\beta$ -mixing, more precisely,  $\beta_{\mathbf{X}}(j) = O\left(\frac{1}{j^q}\right)$ ;
- d) the family  $\left\{ \frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})}, N \geq 1 \right\}$  is uniformly integrable;
- e) if  $I \subset \mathbb{N}$  is infinite, the family  $\left\{ \frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \in I \right\}$  is not tight in  $\mathcal{H}$ ; furthermore, given a sequence  $(c_N)_{N \geq 1}$  of real numbers going to infinity, we have either

- $\lim_{N \rightarrow +\infty} \frac{\sigma_N(\mathbf{f})}{c_N} = 0$ , hence  $\left(\frac{S_N(\mathbf{f})}{c_N}\right)_{N \geq 1}$  converges to  $\mathbf{0}_{\mathcal{H}}$  in distribution, or
- $\limsup_{N \rightarrow +\infty} \frac{\sigma_N(\mathbf{f})}{c_N} > 0$ , and in this case the collection  $\left\{ \frac{S_N(\mathbf{f})}{c_N}, N \geq 1 \right\}$  is not tight.

**Theorem B'.** *Let  $(b_N)_{N \geq 1}$  and  $(h_N)_{N \geq 1}$  be sequences of positive real numbers, with  $\lim_{N \rightarrow \infty} b_N = 0$  and  $\lim_{N \rightarrow \infty} h_N = \infty$ . Then there exists a strictly stationary sequence  $\mathbf{X} := (\mathbf{f} \circ T^k)_{k \in \mathbb{N}} = (\mathbf{X}_k)_{k \in \mathbb{N}}$  of random variables with values in  $\mathcal{H}$  such that a), d), e) of Theorem B and the following two properties hold:*

- b') we have  $\sigma_N^2(\mathbf{f}) \lesssim N \cdot h_N$  and  $\sigma_N^2(\mathbf{f})/N \rightarrow \infty$ ;
- c') the process  $(\mathbf{X}_k)_{k \in \mathbb{N}}$  is  $\beta$ -mixing, and there is an increasing sequence  $(n_k)_{k \geq 1}$  of integers such that for each  $k$ ,  $\beta_{\mathbf{X}}(n_k) \leq b_{n_k}$ .

*Remark 5.2.2.* Theorem B shows that Denker's result does not remain valid in its full generality in the context of Hilbert space valued random variables.

Furthermore, a careful analysis of the proof of Proposition 5.3.3 shows that for the construction given in Theorem B, we have  $\sigma_N^2(f) = N \cdot h(N)$  with  $h$  slowly varying in the strong

sense. Theorem 1 of [MP06] does not remain valid in the Hilbert space setting. Indeed, the arguments given in pages 654-655 show that the conditions of Denker's theorem together with the assumption that  $\sigma_N^2 = N \cdot h(N)$  with  $h$  slowly varying in the strong sense imply those of Theorem 1. These arguments are still true in the Hilbert space setting.

*Remark 5.2.3.* Theorem B' gives a control of the mixing coefficients on a subsequence. When  $b_N := N^{-2}$  for example, the construction gives a better estimation for the considered subsequence than what we get by Theorem B.

Tone has established in [Ton11] a central limit theorem for strictly stationary random fields with values in  $\mathcal{H}$  under  $\rho'$ -mixing conditions. For sequences, these coefficients are defined by

$$\rho'(n) := \sup \left\{ \frac{|\mathbb{E}[\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}}] - \langle \mathbb{E}[\mathbf{f}], \mathbb{E}[\mathbf{g}] \rangle_{\mathcal{H}}|}{\|\mathbf{f}\|_{\mathbb{L}^2(\mathcal{H})} \|\mathbf{g}\|_{\mathbb{L}^2(\mathcal{H})}} \right\},$$

where the supremum is taken over all the non-zero functions  $\mathbf{f}$  and  $\mathbf{g}$  such that  $\mathbf{f}$  and  $\mathbf{g}$  are respectively  $\sigma(X_j, j \in S_1)$  and  $\sigma(X_j, j \in S_2)$ -measurable, where  $S_1$  and  $S_2$  are such that  $\min_{s \in S_1, t \in S_2} |s - t| \geq n$ , while  $\mathbb{L}^2(\mathcal{H})$  denote the collection of equivalence classes of random variables  $\mathbf{X}: \Omega \rightarrow \mathcal{H}$  such that  $\|\mathbf{X}\|_{\mathcal{H}}^2$  is integrable.

So "interlaced index sets" can be considered, which is not the case for  $\alpha$  and  $\beta$ -mixing coefficient. Taking  $f$  and  $g$  as characteristic functions of elements of  $\mathcal{F}_{-\infty}^0$  and  $\mathcal{F}_n^{+\infty}$  respectively, one can see that  $\alpha(n) \leq \rho'(n)$ , hence  $\rho'$ -mixing condition is more restrictive than  $\alpha$ -mixing condition.

A partial generalization of the finite dimensional result was proved by Politis and Romano [PR94], namely, the conditions  $\mathbb{E} \|\mathbf{X}_1\|_{\mathcal{H}}^{2+\delta}$  finite for some positive  $\delta$  and  $\sum_j \alpha_{\mathbf{X}}(j)^{\frac{\delta}{2+\delta}}$  guarantees the convergence of  $n^{-1/2} \sum_{j=1}^n \mathbf{X}_j$  to a Gaussian random variable  $\mathcal{N}$ , whose covariance operator  $S$  satisfies

$$\mathbb{E}[\langle \mathcal{N}, h \rangle^2] = \langle Sh, h \rangle_{\mathcal{H}} = \text{Var}(\langle \mathbf{X}_1, h \rangle) + 2 \sum_{i=1}^{+\infty} \text{Cov}(\langle \mathbf{X}_1, h \rangle, \langle \mathbf{X}_{1+i}, h \rangle).$$

Similar results were obtained by Dehling [Deh83].

Rio's inequality [Rio93] asserts that given two real valued random variables  $X$  and  $Y$  with finite two order moments,

$$|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]| \leq 2 \int_0^{\alpha(\sigma(X), \sigma(Y))} Q_X(u)Q_Y(u)du.$$

It was extended by Merlevède et al. [MPU97], namely, if  $\mathbf{X}$  and  $\mathbf{Y}$  are two random variables with values in  $\mathcal{H}$ , with respective quantile function  $Q_{\|\mathbf{X}\|_{\mathcal{H}}}$  (that is,  $Q_{\mathbf{X}}(u) := \inf \{t \in \mathbb{R}, \mu \{\|\mathbf{X}\|_{\mathcal{H}} > t\} \leq u\}$ ) and  $Q_{\|\mathbf{Y}\|_{\mathcal{H}}}$ , then

$$|\mathbb{E}[\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathcal{H}}] - \langle \mathbb{E}[\mathbf{X}], \mathbb{E}[\mathbf{Y}] \rangle_{\mathcal{H}}| \leq 18 \int_0^{\alpha} Q_{\|\mathbf{X}\|_{\mathcal{H}}} Q_{\|\mathbf{Y}\|_{\mathcal{H}}} du,$$

where  $\alpha := \alpha(\sigma(\mathbf{X}), \sigma(\mathbf{Y}))$ .

From this inequality, they deduce a central limit theorem for a stationary sequence  $(\mathbf{X}_j)_{j \in \mathbb{Z}}$  of  $\mathcal{H}$ -valued zero-mean random variables satisfying

$$\int_0^1 \alpha^{-1}(u) Q_{\|\mathbf{X}_0\|_{\mathcal{H}}}^2(u) du < \infty, \quad (5.2.1)$$

where  $\alpha^{-1}$  is the inverse function of  $x \mapsto \alpha_{\mathbf{X}}(\lfloor x \rfloor)$ .

Discussion after Corollary 1.2 in [Rio00] proves that the later result implies Politis' one.

Relative optimality of condition (5.2.1) (cf. [DMR94]) can give a finite-dimensional counter-example to the central limit theorem when this condition is not satisfied. Here, the condition of uniform integrability prevents such counter-examples.

Defining  $\alpha_{2,\mathbf{X}}(n) := \sup_{i \geq j \geq n} \alpha(\mathcal{F}_{-\infty}^0, \sigma(\mathbf{X}_i, \mathbf{X}_j))$  and  $Q_{\mathbf{X}_0}$  the right-continuous inverse of the function  $t \mapsto \mu\{\|\mathbf{X}_0\|_{\mathcal{H}} > t\}$ , Dedecker and Merlevède have shown in [DM10] that under the assumption

$$\sum_{k=1}^{+\infty} \int_0^{\alpha_{2,\mathbf{X}}(k)} Q_{\mathbf{X}_0}^2(u) du < \infty,$$

we can find a sequence  $(\mathbf{Z}_i)_{i \in \mathbb{N}}$  of Gaussian random variables with values in  $\mathcal{H}$  such that almost surely,

$$\left\| \mathbf{S}_n - \sum_{i=1}^n \mathbf{Z}_i \right\|_{\mathcal{H}} = o\left(\sqrt{n \log \log n}\right).$$

## 5.3 The proof

### 5.3.1 Construction of $f$

In order to construct a counter-example, we shall need the following lemma, which will be proved later.

We will denote  $U$  the Koopman operator associated to  $T$ , which acts on measurable functions by  $U(f)(x) := f(T(x))$ .

**Lemma 5.3.1.** *Let  $(u_k)_{k \geq 1} \subset (0, 1)$  be a sequence of numbers. Then there exists a dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  and a sequence of random variables  $(\xi_k)_{k \geq 1}$  such that*

1. for each  $k \geq 1$ ,  $\mu(\xi_k = 1) = \mu(\xi_k = -1) = \frac{u_k}{2}$  and  $\mu(\xi_k = 0) = 1 - u_k$ ;
2. the random variables  $(U^i \xi_k, k \geq 1, i \in \mathbb{Z})$  are mutually independent.

Recall that  $\mathbf{e}_k$  is the  $k$ -th element of the canonical orthonormal system of  $\mathcal{H} = \ell^2(\mathbb{R})$ . We define

$$f_k := \sum_{i=0}^{n_k-1} U^{-i} \xi_k \text{ and } \mathbf{f} := \sum_{k=1}^{+\infty} f_k \mathbf{e}_k, \tag{5.3.1}$$

where the  $\xi_i$ 's are constructed using to Lemma 5.3.1 taking  $u_k := n_k^{-2}$ . Conditions on the increasing sequence of integers  $(n_k)_{k \geq 1}$  will be specified latter.

Then  $\mathbf{X}_k := \mathbf{f} \circ T^k$  is a strictly stationary sequence. Note that  $\|\mathbf{f}\|_{\mathcal{H}}^2$  is an integrable random variable whenever  $\sum_k \frac{1}{n_k}$  is convergent. In the sequel, the choice of  $n_k$  will guarantee this condition.

### 5.3.2 Preliminary results

We express  $S_N(f_k)$  as a linear combination of independent random variables. By direct computations, we get

$$f_k = n_k \xi_k + (I - U) \sum_{i=1-n_k}^{-1} (n_k + i) U^i \xi_k, \tag{5.3.2}$$

hence

$$S_N(f_k) = n_k \sum_{j=0}^{N-1} U^j \xi_k + \sum_{i=1-n_k}^{-1} (n_k + i) U^i \xi_k - \sum_{i=N-n_k+1}^{N-1} (n_k + i - N) U^i \xi_k.$$

This formula can be simplified if we distinguish the cases  $N \geq n_k$  and  $n_k < N$  (we break the third sum at the index  $i = 0$  if necessary). This gives

$$S_N(f_k) = \sum_{j=0}^{N-1} (N-j)U^j \xi_k + \sum_{j=1-n_k}^{N-n_k} (n_k+j)U^j \xi_k + N \sum_{j=1+N-n_k}^{-1} U^j \xi_k, \quad \text{if } N < n_k, \quad (5.3.3)$$

$$S_N(f_k) = n_k \sum_{j=0}^{N-n_k} U^j \xi_k + \sum_{j=N-n_k+1}^{N-1} (N-j)U^j \xi_k + \sum_{j=1-n_k}^{-1} (n_k+j)U^j \xi_k, \quad \text{if } N \geq n_k. \quad (5.3.4)$$

The computation of the expectation of the square of partial sums gives

$$\sigma_N^2(f_k) = \begin{cases} \frac{1}{n_k^2} \left( 2 \sum_{j=1}^N j^2 + (n_k - N - 1)N^2 \right) & \text{if } N < n_k, \\ \frac{1}{n_k^2} \left( n_k^2(N - n_k + 1) + 2 \sum_{j=1}^{n_k-1} j^2 \right) & \text{if } N \geq n_k. \end{cases} \quad (5.3.5)$$

*Notation 5.3.2.* If  $N$  is a positive integer and  $(n_k)_{k \geq 1}$  is an increasing sequence of integers, denote by  $i(N)$  the unique integer for which  $n_{i(N)} \leq N < n_{i(N)+1}$ .

**Proposition 5.3.3.** *Assume that  $(n_k)_{k \geq 1}$  satisfies the condition*

$$\text{there is } p > 1 \text{ such that for each } k, \quad n_{k+1} \geq n_k^p. \quad (\text{C})$$

*Then  $\sigma_N^2(\mathbf{f}) \asymp N \cdot i(N)$ .*

*Proof.* Using (5.3.5), the fact that  $M^3 \asymp \sum_{j=1}^M j^2$  and  $\sigma_N^2(\mathbf{f}) = \sum_{k \geq 1} \sigma_N^2(f_k)$ , we have

$$\sigma_N^2(\mathbf{f}) \geq \sum_{k=1}^{i(N)} \sigma_N^2(f_k) \asymp N \sum_{j=1}^{i(N)} 1 = N \cdot i(N). \quad (5.3.6)$$

From (5.3.5) in the case  $n_k \geq N$ , we deduce

$$\sum_{k \geq i(N)+1} \sigma_N^2(f_k) \lesssim \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \leq \frac{N^2}{n_{i(N)+1}} + \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \frac{1}{n_k^{p-1}}. \quad (5.3.7)$$

Since  $n_{i(N)+1} \geq N$  and the series  $\sum_{k \geq 1} n_k^{1-p}$  is convergent (by the ratio test), we obtain

$$\sum_{k \geq i(N)+1} \sigma_N^2(f_k) \lesssim N + N \sum_{k \geq i(N)+1} \frac{1}{n_k^{p-1}} \lesssim N. \quad (5.3.8)$$

Combining (5.3.6) and (5.3.8), we get

$$N \cdot i(N) \lesssim \sigma_N^2(\mathbf{f}) \lesssim \sum_{k=1}^{i(N)} \sigma_N^2(f_k) + \sum_{k \geq i(N)+1} \sigma_N^2(f_k) \lesssim N \cdot i(N) + N \lesssim N \cdot i(N). \quad (5.3.9)$$

□

**Proposition 5.3.4.** *Assume that  $\sum_k n_k^{-a}$  is convergent for any positive real number  $a$ . Then for each integer  $p$ ,  $\|\mathbf{f}\|_{\mathcal{H}}$  has a finite moment of order  $p$ .*

*Proof.* We shall use Rosenthal's inequality (Theorem 3, [Ros70]): there exists a constant  $C$  depending only on  $q$  such that if  $M$  is an integer,  $Y_1, \dots, Y_M$  are independent real valued zero mean random variables for which  $\mathbb{E}|Y_i|^q < \infty$  for each  $i$ , then

$$\mathbb{E} \left| \sum_{j=1}^M Y_j \right|^q \leq C \left( \sum_{j=1}^M \mathbb{E}|Y_j|^q + \left( \sum_{j=1}^M \mathbb{E}[Y_j^2] \right)^{q/2} \right). \quad (5.3.10)$$

If  $q = 2p$  is given then we have

$$\mathbb{E}|f_k|^{2p} \lesssim n_k^{-1} + n_k^{-p} \lesssim n_k^{-1}. \quad (5.3.11)$$

□

We provide a sufficient condition for the uniform integrability of the family

$$\mathcal{S} := \left\{ \frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})}, N \geq 1 \right\}. \quad (5.3.12)$$

**Proposition 5.3.5.** *If  $(n_k)_{k \geq 1}$  satisfies (C), then  $\mathcal{S}$  is uniformly integrable.*

*Proof.* Let  $i_0$  be such that for each integer  $k$ ,  $n_{k+i_0} \geq n_k^2$ . For  $N \geq 1$ , we have:

$$\frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})} = \sum_{j=1}^{i(N)-1} \frac{|S_N(f_j)|^2}{\sigma_N^2(\mathbf{f})} + \frac{|S_N(f_{i(N)})|^2}{\sigma_N^2(\mathbf{f})} + \frac{|S_N(f_{i(N)+1})|^2}{\sigma_N^2(\mathbf{f})} + \sum_{j \geq i(N)+2} \frac{|S_N(f_j)|^2}{\sigma_N^2(\mathbf{f})},$$

Since  $\mathbb{E} \left[ \sum_{j=i-i_0}^{i(N)-1} \frac{|S_N(f_j)|^2}{\sigma_N^2(\mathbf{f})} \right] \leq \frac{i_0}{i(N)}$  it is enough to prove that the families

$$\begin{aligned} \mathcal{S}_1 &:= \left\{ \sum_{k=1}^{i(N)-i_0} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})}, N \geq 1 \right\}, \\ \mathcal{S}_2 &:= \left\{ \frac{|S_N(f_{i(N)})|^2}{\sigma_N^2(\mathbf{f})}, N \geq 1 \right\} =: \{u_N, N \geq 1\}, \\ \mathcal{S}_3 &:= \left\{ \frac{|S_N(f_{i(N)+1})|^2}{\sigma_N^2(\mathbf{f})}, N \geq 1 \right\} =: \{v_N, N \geq 1\}, \text{ and} \\ \mathcal{S}_4 &:= \left\{ \sum_{k \geq i(N)+2} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})}, N \geq 1 \right\} \end{aligned}$$

are uniformly integrable. For  $\mathcal{S}_1$  and  $\mathcal{S}_4$ , we shall show that these families are bounded in  $\mathbb{L}^p$  for  $p \in (1, 2]$  as in (C).

- for  $\mathcal{S}_1$ : using the expression in (5.3.4) and (5.3.10) with  $q := 2p > 2$ , we have

$$\begin{aligned} \mathbb{E} \left[ |S_N(f_k)|^{2p} \right] &\leq C \left( 2 \sum_{j=1}^{n_k} \frac{j^{2p}}{n_k^2} + \frac{n_k^{2p}(N-n_k)}{n_k^2} \right) + C \left( 2 \sum_{j=1}^{n_k} \frac{j^2}{n_k^2} + \frac{(N-n_k)n_k^2}{n_k^2} \right)^p \\ &\lesssim \frac{1}{n_k^2} \left( n_k^{2p+1} + (N-n_k)n_k^{2p} \right) + \frac{1}{n_k^{2p}} \left( n_k^3 + (N-n_k)n_k^2 \right)^p \\ &= \frac{Nn_k^{2p}}{n_k^2} + \frac{N^p n_k^{2p}}{n_k^{2p}} \\ &= Nn_k^{2(p-1)} + N^p \end{aligned}$$

hence

$$\|S_N(f_k)^2\|_p \lesssim N^{1/p} n_k^{2\frac{p-1}{p}} + N,$$

which gives

$$\begin{aligned} \left\| \sum_{k=1}^{i(N)-i_0} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})} \right\|_p &\lesssim \frac{\sum_{k=1}^{i(N)-i_0} (N^{1/p} n_k^{2\frac{p-1}{p}} + N)}{\sigma_N^2(\mathbf{f})} \\ &\lesssim \frac{N^{1/p} i(N) n_{i(N)-i_0}^{2\frac{p-1}{p}} + N i(N)}{\sigma_N^2(\mathbf{f})} \\ &\lesssim \left( \frac{n_{i(N)-i_0}^2}{N} \right)^{1-1/p} + 1 \end{aligned}$$

From (5.3.6), we get

$$\left\| \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})} \right\|_p \lesssim \frac{n_{i(N)}^{2\frac{p-1}{p}}}{n_{i(N)}} + 1 = n_{i(N)}^{\frac{p-2}{p}} + 1.$$

Since  $p-2 \leq 0$ , we obtain that  $\mathcal{S}_1$  is bounded in  $\mathbb{L}^p$  hence uniformly integrable.

- for  $\mathcal{S}_2$ : using (5.3.4) in the case  $n_k \leq N$  and Proposition 5.3.3, we get

$$\|u_N\|_1 \lesssim \frac{N}{\sigma_N^2(\mathbf{f})} \lesssim \frac{1}{i(N)}. \quad (5.3.13)$$

Since  $\|u_N\|_1 \rightarrow 0$  and  $u_N \in \mathbb{L}^1$  for each  $N$ , the family  $\mathcal{S}_2$  is uniformly integrable.

- for  $\mathcal{S}_3$ : using (5.3.3) in the case  $n_k > N$  and Proposition 5.3.3, we get

$$\|v_N\|_1 \lesssim \frac{N^2}{n_{i(N)+1} \sigma_N^2(\mathbf{f})} \lesssim \frac{N}{N \cdot i(N)}. \quad (5.3.14)$$

Since  $\|v_N\|_1 \rightarrow 0$  and  $v_N \in \mathbb{L}^1$  for each  $N$ , the family  $\mathcal{S}_3$  is uniformly integrable.

- for  $\mathcal{S}_4$ : as for  $\mathcal{S}_1$ , we shall show that this family is bounded in  $\mathbb{L}^p$  with  $p \in (1, 2]$ . We have, using (5.3.3) and (5.3.10)

$$\begin{aligned} \mathbb{E} \left[ |S_N(f_k)|^{2p} \right] &\lesssim \frac{1}{n_k^2} (N^{2p+1} + N^{2p}(n_k - N)) + \frac{1}{n_k^{2p}} (N^3 + (n_k - N)N^2)^p \\ &= \frac{N^{2p}}{n_k} + \frac{N^{2p}}{n_k^p} \\ &\lesssim \frac{N^{2p}}{n_k} \end{aligned}$$

as  $N \leq n_k$ . We thus get that

$$\left\| \sum_{k \geq i(N)+2} |S_N(f_k)|^2 \right\|_p \lesssim N^2 \sum_{k \geq i(N)+2} \frac{1}{n_k^{1/p}}.$$

Also, using (5.3.5), we have

$$\sigma_N^2(\mathbf{f}) \gtrsim N^2 \sum_{k \geq i(N)+1} \frac{1}{n_k}.$$

The condition  $n_{k+1} \geq n_k^p$  gives boundedness in  $\mathbb{L}^p$  of  $\mathcal{S}_4$ .

This concludes the proof of **d**. □

**Proposition 5.3.6.** *Assume that  $(n_k)_{k \geq 1}$  is such that  $\mathcal{S}$  is uniformly integrable and  $\sum_k n_k^{-1}$  is convergent. Then for each  $I \subset \mathbb{N}$  infinite, the collection  $\left\{ \frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \in I \right\}$  is not tight in  $\mathcal{H}$ . Its finite-dimensional distributions converge to 0 in probability.*

*Furthermore, if  $(c_N)_{N \geq 0}$  is a sequence of positive numbers going to infinity, we have either*

- $\lim_{N \rightarrow +\infty} \frac{\sigma_N(\mathbf{f})}{c_N} = 0$ , hence  $\left( \frac{S_N(\mathbf{f})}{c_N} \right)_{N \geq 1}$  converges to  $\mathbf{0}_{\mathcal{H}}$  in distribution, or
- $\limsup_{N \rightarrow +\infty} \frac{\sigma_N(\mathbf{f})}{c_N} > 0$ , and in this case the sequence  $\left\{ \frac{S_N(\mathbf{f})}{c_N}, N \geq 1 \right\}$  is not tight.

*Proof.* We first prove that the finite dimensional distributions of  $\frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}$  converge weakly to 0.

For each  $d \in \mathbb{N}$ , we have  $\frac{\langle S_N(\mathbf{f}), \mathbf{e}_d \rangle_{\mathcal{H}}}{\sigma_N(\mathbf{f})} \rightarrow 0$  in distribution. Indeed, we have by (5.3.2) that  $\langle S_N(\mathbf{f}), \mathbf{e}_d \rangle_{\mathcal{H}} = n_d \sum_{i=0}^{N-1} U^i \xi_d + (I - U^N) \sum_{i=1}^{-1} (n_d + i) U^i \xi_d$ . We conclude noticing that  $\sigma_N(\mathbf{f})^{-1} (I - U^N) \sum_{i=1}^{-1} (n_d + i) U^i \xi_d$  goes to 0 in probability as  $N$  goes to infinity, using Proposition 5.3.3 and the estimate

$$\mathbb{E} \left( n_d \sum_{i=0}^{N-1} U^i \xi_d \right)^2 = N \lesssim \frac{\sigma_N^2(\mathbf{f})}{i(N)}$$

This can be extended replacing  $\mathbf{e}_d$  by any  $\mathbf{v} \in \mathcal{H}$  by an application of Theorem 4.2. in [Bil68]. By Proposition 4.15 in [AG80], the only possible limit is the Dirac measure at  $\mathbf{0}_{\mathcal{H}}$ .

Assume that the sequence  $\left\{ \frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \geq 1 \right\}$  is tight. The sequence  $\left( \frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})} \right)_{N \geq 1}$  is a uniformly integrable sequence of random variables of mean 1. A weakly convergent subsequence would go to  $\mathbf{0}_{\mathcal{H}}$ . According to Theorem 5.4 in [Bil68], we should have that the limit random variable has expectation 1. This contradiction gives the result when  $I = \mathbb{N} \setminus \{0\}$ . Applying this reasoning to subsequences, one can see that for any infinite subset  $I$  of  $\mathbb{N} \setminus \{0\}$ , the family  $\left\{ \frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \in I \right\}$  is not tight.

Let  $(c_N)_{N \geq 1}$  be a sequence of positive real numbers such that  $\lim_{N \rightarrow +\infty} c_N = +\infty$ .

- first case:  $\frac{\sigma_N(\mathbf{f})}{c_N}$  converges to 0. In this case, the sequence  $\left( \frac{\|S_N(\mathbf{f})\|^2}{c_N^2} \right)_{N \geq 1}$  converges to 0 in  $\mathbb{L}^1$ , hence the sequence  $\left( \frac{S_N(\mathbf{f})}{c_N} \right)_{N \geq 1}$  converges in distribution to  $\mathbf{0}_{\mathcal{H}}$ .
- second case:  $\limsup_{N \rightarrow \infty} \frac{\sigma_N(\mathbf{f})}{c_N} > 0$ . Hence there is some  $r > 0$  and a sequence of integers  $l_i \uparrow \infty$  such that for each  $i$ ,  $\frac{\sigma_{l_i}(\mathbf{f})}{c_{l_i}} \geq \frac{1}{r}$ , that is,  $c_{l_i} \leq r \sigma_{l_i}(\mathbf{f})$ .

Assume that the family  $\left\{ \frac{S_{l_i}(\mathbf{f})}{c_{l_i}}, i \geq 1 \right\}$  is tight. This means that given a positive  $\varepsilon$ , one can find a compact set  $K = K(\varepsilon)$  such that for each  $i$ ,  $\mu \left\{ \frac{S_{l_i}(\mathbf{f})}{c_{l_i}} \in K \right\} > 1 - \varepsilon$ . We can assume that this compact set is convex and contains 0 (we consider the closed convex hull of  $K \cup \{0\}$ , which is compact by Theorem 5.35 in [AB06]). Then we have

$$\begin{aligned} \left\{ \frac{S_{l_i}(\mathbf{f})}{c_{l_i}} \in K \right\} &= \left\{ \frac{S_{l_i}(\mathbf{f})}{\sigma_{l_i}(\mathbf{f})} \in \frac{c_{l_i}}{\sigma_{l_i}(\mathbf{f})} K \right\} \\ &\subset \left\{ \frac{S_{l_i}(\mathbf{f})}{\sigma_{l_i}(\mathbf{f})} \in rK \right\}, \end{aligned}$$

and we would deduce tightness of  $\left\{ \frac{S_{l_i}(\mathbf{f})}{\sigma_{l_i}(\mathbf{f})}, i \geq 1 \right\}$ , which cannot happen.

*Remark 5.3.7.* In the second case, it may happen that the finite dimensional distributions does not converge to degenerate ones, for example with  $c_N := N$ .

□

### 5.3.3 Proof of Theorem B

Notice that if  $n_{k+1} \geq n_k^p$  for some  $p > 1$  and  $n_1 = 2$ , then  $n_k \geq 2^{p^k}$ , hence the condition of Proposition 5.3.4 is fulfilled. We get a) since each  $f_k$  has expectation 0.

We denote  $[x] := \sup\{k \in \mathbb{Z}, k \leq x\}$  the integer part of the real number  $x$ .

**Proposition 5.3.8.** *Let  $p > 1$ . With  $n_k := [2^{p^k}]$  (which satisfies (C)), we have for each positive integer  $l$ ,*

$$\beta_{\mathbf{X}}(l) \lesssim \frac{1}{l^{1/p}}.$$

*Proof.* We define  $\beta_k(n)$  as the  $n$ -th  $\beta$ -mixing coefficient of the sequence  $(f_k \circ T^i)_{i \geq 0}$ .

By Lemma 5 of [GV14c], we have the estimate  $\beta_k(0) \leq 4n_k^{-1}$  for each  $k$ . Using then Proposition 4 of this paper (cf. [Bra07] for a proof), we get that  $\beta_{\mathbf{X}}(n_k) \lesssim \sum_{j \geq k} \frac{1}{n_j}$  for each integer  $k$ . Since  $p^i \geq i$  for  $i$  large enough,

$$\sum_{j \geq k} \frac{1}{n_j} = \sum_{i=0}^{+\infty} \frac{1}{2^{p^{i+k}}} = \sum_{i=0}^{+\infty} \frac{1}{2^{p^i p^k}} \lesssim \sum_{i=0}^{+\infty} \frac{1}{2^i} \frac{1}{2^{p^k}} = \frac{2}{2^{p^k}},$$

we get

$$\beta_{\mathbf{X}}(N) \leq \beta_{\mathbf{X}}(n_{i(N)}) \lesssim \frac{1}{n_{i(N)}} = \frac{1}{n_{i(N)+1}^{1/p}} \leq \frac{1}{N^{1/p}}.$$

□

This proves c). For any  $p$ , the choice  $n_k := [2^{p^k}]$  satisfies the condition of Proposition 5.3.5, which proves d). We conclude the proof by Proposition 5.3.6.

*Remark 5.3.9.* For each of these choices,  $\sigma_N^2(\mathbf{f})$  behaves asymptotically like  $N \log \log N$ . Theorem B' shows that we can construct a process which satisfies the same asymptotic behavior of partial sums and has a variance close to a linear one.

A question would be: can we construct a strictly stationary sequence with all the properties of Theorem B, except b) which is replaced by an assumption of linear variance?

### 5.3.4 Proof of Theorem B'

Let  $(h_N)_{N \geq 1}$  be the sequence involved in Theorem B'. We define for an integer  $u$  the quantity  $h^{-1}(u) := \inf\{j \in \mathbb{N}, h_j \geq u\}$ .

If  $(b_k)_{k \geq 1}$  is the given sequence (that can be assumed decreasing), we define inductively

$$n_{k+1} := \max \left\{ n_k^2, \left\lfloor \frac{2^k}{b_{n_k}} \right\rfloor, h^{-1}(k) \right\}. \quad (5.3.15)$$

Let  $N$  be an integer. We assume without loss of generality that the growth of the sequence  $(h_N)_{N \geq 1}$  is slow enough in order to guarantee that there exists  $k$  such that  $N = h^{-1}(k)$ . We then have  $i(N) \leq k + 1 \leq h_N + 1$ , hence using Proposition 5.3.3, we get b').

We have  $n_k \geq 2^{2^k}$  hence by a similar argument as in the proof of Theorem B, a) is satisfied.

By a similar argument as in [GV14c], we get  $\beta_{\mathbf{X}}(n_k) \leq b_{n_k}$ , hence c') holds.

*Remark 5.3.10.* By (5.2.1), we cannot expect the relationship  $\beta_{\mathbf{X}}(\cdot) \leq b$  for the whole sequence.



Since for each  $k$ ,  $n_{k+1} \geq n_k^2$ , Proposition 5.3.5 and 5.3.6 apply. This concludes the proof of Theorem B'.

*Proof of Lemma 5.3.1.* Let  $\Omega := [0, 1]^{\mathbb{N}^* \times \mathbb{Z}}$ , where  $[0, 1]$  is endowed with Borel  $\sigma$ - algebra and Lebesgue measure, and  $\Omega$  with the product structure.

For  $(k, j) \in \mathbb{N}^* \times \mathbb{Z}$  and  $S \subset [0, 1]$ , let  $P_{k,j}(S) := \prod_{(i_1, i_2) \in \mathbb{N}^* \times \mathbb{Z}} S_{i_1, i_2}$ , where  $S_{i_1, i_2} = S$  if  $(i_1, i_2) = (k, j)$  and  $[0, 1]$  otherwise. Then we define

$$A_{k,j}^+ := P_{k,j}([0, 2^{-1}(u_k)^{-1}]),$$

$$A_{k,j}^- := P_{k,j}([2^{-1}(u_k)^{-1}, (u_k)^{-1}]),$$

$$A_{k,j}^{(0)} := P_{k,j}([(u_k)^{-1}, 1]),$$

the map  $T$  by  $T \left( (x_{k,j})_{(k,j) \in \mathbb{N}^* \times \mathbb{Z}} \right) := (x_{k,j+1})_{(k,j) \in \mathbb{N}^* \times \mathbb{Z}}$ , and

$$\xi_k := \mathbf{1}_{A_{k,0}^+} - \mathbf{1}_{A_{k,0}^-}.$$

□

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## Part III

# Holderian weak invariance principle for strictly stationary sequences



# Chapter 6

## Holderian weak invariance principle for strictly stationary mixing sequences

### 6.1 Introduction

#### 6.1.1 Context and notations

Let  $(X_j)_{j \geq 0}$  be a strictly stationary sequence of real valued random variables with zero mean and finite variance, and for an integer  $n \geq 1$ ,  $S_n := \sum_{j=1}^n X_j$  denotes the  $n$ -th partial sum. Its variance is denoted by  $\sigma_n^2$ . Let us consider the partial sum process

$$S_n^{\text{pl}}(t) := \sum_{j=1}^{[nt]} X_j + (nt - [nt])X_{[nt]+1}, \quad n \geq 1, t \in [0, 1]. \quad (6.1.1)$$

We are interested in the asymptotic behavior of  $\sigma_n^{-1} S_n^{\text{pl}}(\cdot)$  viewed as a random function in some function spaces.

*Notation 6.1.1.* If  $T: \Omega \rightarrow \Omega$  is a bi-measurable measure preserving map, we define for  $f: \Omega \rightarrow \mathbb{R}$  and a positive integer  $n$  the  $n$ th partial sum  $S_n(f) := \sum_{j=1}^n f \circ T^j$  and  $\sigma_n^2(f) := \mathbb{E}[S_n^2(f)] - (\mathbb{E}[S_n(f)])^2$  denotes its variance. We shall also consider  $S_n^{\text{pl}}(f)$  defined in a similar way as in (6.1.1), that is,

$$S_n^{\text{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]+1}, \quad (6.1.2)$$

and  $W_n(f, t) := n^{-1/2} S_n^{\text{pl}}(f, t)$ .

In all the paper, the involved sequences of random variable are assumed to be strictly stationary.

When  $(X_j)_{j \geq 0}$  is an independent identically distributed sequence with unit variance, Donsker showed (cf. [Don51]) that  $(n^{-1/2} S_n^{\text{pl}})_{n \geq 1}$  converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion  $W$ . An intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [MPU06] for the main theorems in this area.

In this chapter, we rather focus on the convergence in distribution of the partial sum in other function spaces.

#### 6.1.2 Hölder spaces

It is well known that standard Brownian motion's paths are almost surely Hölder regular of exponent  $\alpha$  for each  $\alpha \in (0, 1/2)$ , hence it is natural to consider the random function defined

in (6.1.2) as an element of  $\mathcal{H}_\alpha[0, 1]$  and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for  $\alpha \in (0, 1)$  the Hölder space  $\mathcal{H}_\alpha[0, 1]$  of functions  $x: [0, 1] \rightarrow \mathbb{R}$  such that  $\sup_{s \neq t} |x(s) - x(t)| / |s - t|^\alpha$  is finite. The analogue of the continuity modulus in  $C[0, 1]$  is  $w_\alpha$ , defined by

$$w_\alpha(x, \delta) = \sup_{0 < |t-s| < \delta} \frac{|x(t) - x(s)|}{|t - s|^\alpha}.$$

We then define  $\mathcal{H}_\alpha^o[0, 1]$  by  $\mathcal{H}_\alpha^o[0, 1] := \{x \in \mathcal{H}_\alpha[0, 1], \lim_{\delta \rightarrow 0} w_\alpha(x, \delta) = 0\}$ . We shall essentially work with the space  $\mathcal{H}_\alpha^o[0, 1]$  which, endowed with  $\|\cdot\|_\alpha: x \mapsto w_\alpha(x, 1) + |x(0)|$ , is a separable Banach space (while  $\mathcal{H}_\alpha[0, 1]$  is not). Since the canonical embedding  $\iota: \mathcal{H}_\alpha^o[0, 1] \rightarrow \mathcal{H}_\alpha[0, 1]$  is continuous, each convergence in distribution in  $\mathcal{H}_\alpha^o[0, 1]$  also takes place in  $\mathcal{H}_\alpha[0, 1]$ .

In order to prove such a convergence, we need a tightness criterion. Combining Theorem 14 in [Suq99] in the particular case of the partial sum process (6.1.2) with Lemma 3.3 in [MR10], the condition

$$\forall \varepsilon > 0, \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\delta} \sum_{j=1}^{\log[n\delta]} 2^j \mu \left\{ \max_{1 \leq k \leq [n\delta]2^{-j}} |S_k(f)| > \frac{\sigma_n}{n^\alpha} \varepsilon ([n\delta]2^{-j})^\alpha \right\} = 0 \quad (6.1.3)$$

is sufficient for tightness of the sequence  $(\sigma_n^{-1}(f)S_n^{\text{pl}}(f))_{n \geq 1}$  in  $\mathcal{H}_\alpha^o[0, 1]$ . Here and in the sequel,  $\log$  denotes the binary logarithm.

In the particular case of linear variance (that is,  $\sigma_n^2 \sim cn$  as  $n \rightarrow \infty$  for some constant  $c$ ), we will normalize by  $\sqrt{n}$ . Using the change of indexes  $k = \log[n\delta] - j$  (so that  $2^{-j} = 2^k/[n\delta]$ ), this leads to the following tightness criterion for  $(W_n(f))_{n \geq 1}$  in  $\mathcal{H}_\alpha^o[0, 1]$ :

$$\forall \varepsilon > 0, \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu \left\{ \max_{1 \leq i \leq 2^k} |S_i(f)| > \varepsilon 2^{k\alpha} n^{1/p} \right\} = 0, \quad (6.1.4)$$

where  $\alpha = 1/2 - 1/p$ .

As mentioned before, the random function defined in (6.1.1) can be viewed as an element of  $\mathcal{H}_\alpha[0, 1]$ ,  $\alpha \in (0, 1/2)$  and we can try to establish the weak convergence of the sequence  $(\sigma_n^{-1}S_n^{\text{pl}}(f))_{n \geq 1}$  to a standard Brownian motion in this function space. To the best of our knowledge, it seems that the study of this kind of convergence was not as intensive as in the space of continuous functions or the Skorohod space. The first result in this direction was established by Lamperti in [Lam62]: if  $(X_j)_{j \geq 0}$  is an i.i.d. sequence with  $\mathbb{E}[X_0] = 0$ ,  $\mathbb{E}[X_0^2] = 1$  and  $\mathbb{E}|X_0|^p$  is finite, then the sequence  $(n^{-1/2}S_n^{\text{pl}})_{n \geq 1}$  converges to a standard Brownian motion in  $\mathcal{H}_\gamma^o[0, 1]$  for each  $\gamma < 1/2 - 1/p$ . Later, Račkauskas and Suquet improved this result (cf. [RS03]), showing that for an i.i.d. zero mean sequence, a necessary and sufficient condition to obtain the invariance principle in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$  is  $\lim_{t \rightarrow \infty} t^p \mu\{|X_0| > t\} = 0$  (in [RS04b] they considered the case of more general Hölder spaces, where the role of  $t \mapsto t^\alpha$  is played by  $t \mapsto t^\alpha L(t)$  with some conditions on  $L$ ).

Thus, establishing the weak convergence of the partial sum process in Hölder spaces requires, even in the independent case, finite moment of order greater than 2 and the moment condition depends on the exponent of the considered Hölder space. It is a natural question to ask about generalizations of the result by Račkauskas and Suquet for dependent sequences. In this paper, we focus on strictly stationary sequences satisfying some mixing conditions (see next section).

### 6.1.3 Mixing conditions

We present the mixing conditions involved in the paper.

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two sub- $\sigma$ -algebras of  $\mathcal{F}$ , where  $(\Omega, \mathcal{F}, \mu)$  is a probability space. We define the  $\alpha$ -mixing coefficients as introduced by Rosenblatt in [Ros56]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \}.$$

The  $\rho$ -mixing coefficients were introduced by Hirschfeld [Hir35] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{ |\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B}), f \neq 0, g \neq 0 \},$$

where  $\text{Corr}(f, g) := [\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)] [ \|f - \mathbb{E}(f)\|_{\mathbb{L}^2} \|g - \mathbb{E}(g)\|_{\mathbb{L}^2} ]^{-1}$ .

The coefficients are related by the inequalities

$$4\alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}). \quad (6.1.5)$$

For a strictly stationary sequence  $(X_k)_{k \in \mathbb{Z}}$  and  $n \geq 0$  we define  $\alpha_X(n) = \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$  where  $\mathcal{F}_u^v$  is the  $\sigma$ -algebra generated by  $X_k$  with  $u \leq k \leq v$  (if  $u = -\infty$  or  $v = \infty$ , the corresponding inequality is strict). In the same way we define coefficients  $\rho_X(n)$ .

When there will be no ambiguity, we shall simply write  $\alpha(n)$  and  $\rho(n)$ . We say that the sequence  $(X_k)_{k \in \mathbb{Z}}$  is  $\alpha$ -mixing if  $\lim_{n \rightarrow +\infty} \alpha(n) = 0$ , and similarly we define  $\rho$ -mixing sequences.

$\alpha$ -mixing sequences were considered in the mentioned references, while  $\rho$ -mixing sequences first appeared in [KR60]. Inequality (6.1.5) translated in terms of mixing coefficients of a sequence states that for each positive integer  $n$ ,

$$4\alpha(n) \leq \rho(n).$$

In particular, a  $\rho$ -mixing sequence is  $\alpha$ -mixing.

#### 6.1.4 $\tau$ -dependence coefficient

In order to define the  $\tau$ -dependence coefficients of a stationary sequence, we first need a result about conditional probability (see Theorem 33.3 of [Bil95]).

**Lemma 6.1.2.** *Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space,  $\mathcal{M}$  a sub- $\sigma$ -algebra of  $\mathcal{F}$  and  $X$  a real-valued random variable with distribution  $\mu_X$ . There exists a function  $\mu_{X|\mathcal{M}}$  from  $\mathcal{B}(\mathbb{R}) \times \Omega$  to  $[0, 1]$  such that*

1. *For any  $\omega \in \Omega$ ,  $\mu_{X|\mathcal{M}}(\cdot, \omega)$  is a probability measure on  $\mathcal{B}(\mathbb{R})$ .*
2. *For any  $A \in \mathcal{B}(\mathbb{R})$ ,  $\mu_{X|\mathcal{M}}(A, \cdot)$  is a version of  $\mathbb{E}[\mathbf{1}_{\{X \in A\}} | \mathcal{M}]$ .*

We now introduce the  $\tau$ -dependence coefficients as in [DP05]. We denote by  $\Lambda_1(\mathbb{R})$  the collection of 1-Lipschitz functions from  $\mathbb{R}$  to  $\mathbb{R}$  and define the quantity

$$W(\mu_{X|\mathcal{M}}) := \sup \left\{ \left| \int f(x) \mu_{X|\mathcal{M}}(dx) - \int f(x) \mu_X(dx) \right|, f \in \Lambda_1(\mathbb{R}) \right\}.$$

For an integrable random variable  $X$  and a sub- $\sigma$ -algebra  $\mathcal{M}$ , the coefficient  $\tau$  is defined by

$$\tau(\mathcal{M}, X) = \|W(\mu_{X|\mathcal{M}})\|_1. \quad (6.1.6)$$

This definition can be extended to random variables with values in finite dimensional vector spaces. If  $d$  is a positive integer, we endow  $\mathbb{R}^d$  with the norm  $\|x - y\| := \sum_{j=1}^d |x_j - y_j|$  and define  $\Lambda_1(\mathbb{R}^d)$  as the set of 1-Lipschitz functions from  $\mathbb{R}^d$  to  $\mathbb{R}$ .

**Definition 6.1.3.** *Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space,  $\mathcal{M}$  a sub- $\sigma$ -algebra of  $\mathcal{F}$  and  $X$  an  $\mathbb{R}^d$ -valued random variable. We define*

$$\tau(\mathcal{M}, X) := \sup \{ \tau(\mathcal{M}, f(X)), f \in \Lambda_1(\mathbb{R}^d) \}. \quad (6.1.7)$$

We can now introduce the  $\tau$ -mixing coefficient for a sequence of real-valued random variables.

**Definition 6.1.4.** Let  $(X_i)_{i \geq 1}$  be a sequence of random variables and  $(\mathcal{M}_i)_{i \geq 1}$  a sequence of sub- $\sigma$ -algebras of  $\mathcal{F}$ . For any positive integer  $k$ , define

$$\tau(i) := \max_{p, l \geq 1} \frac{1}{l} \sup \{ \tau(\mathcal{M}_p, (X_{j_1}, \dots, X_{j_l})), p + i \leq j_1 < \dots < j_l \}. \quad (6.1.8)$$

In the sequel, we shall focus on the case  $\mathcal{M}_i := \sigma(X_k, k \leq i)$ .

*Notation 6.1.5.* Let  $X: \Omega \rightarrow \mathbb{R}$  be a random variable. We denote  $Q_X(\cdot)$  the inverse function defined by  $Q_X(u) := \inf \{ t, \mu \{ |X| > t \} \leq u \}$ . If  $(f \circ T^j)_{j \geq 0}$  is a strictly stationary sequence and  $(\alpha(n))_{n \geq 1}$  is its sequence of  $\alpha$ -mixing coefficients, we denote by  $\alpha^{-1}(u)$  the number of indices  $n$  for which  $\alpha(n) \geq u$ . More generally, if  $(\delta_i)_{i \geq 0}$  is a non-increasing sequence of non-negative numbers, we define  $\delta^{-1}(u) := \inf \{ k \in \mathbb{N}, \delta_k \leq u \}$ .

We can compare the  $\tau$ -dependence coefficient with the  $\alpha$ -mixing coefficients. The following is a simplified version of Lemma 7 of [DP04].

**Lemma 6.1.6.** Let  $(f \circ T^j)_{j \geq 0}$  be a strictly stationary sequence. Then for each integer  $i$ , the following inequality holds:

$$\tau(i) \leq 2 \int_0^{2\alpha(i)} Q_f(u) du. \quad (6.1.9)$$

In [DP04], "Application 1: causal linear processes" (p. 871), Dedecker and Priour provide an example of a process whose  $\tau$ -dependence coefficients converge to 0 as fast as  $2^{-i}$  but  $\alpha_i = 1/4$  for each positive integer  $i$ .

## 6.2 Main results

### 6.2.1 Mixing sequences

In this subsection, we give sufficient mixing conditions which guarantee the convergence of the sequence  $(W_n(f))_{n \geq 1}$  to a Brownian motion in the space  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ ,  $p > 2$ .

We refer the reader to Notations 6.1.5 and 6.4.2.

**Theorem 6.2.1.** Let  $p > 2$  and let  $(f \circ T^j)_{j \geq 0}$  be a strictly stationary centered sequence such that

$$\lim_{t \rightarrow \infty} t^{p-1} \int_0^1 Q_f(u) \mathbf{1} \left\{ \left( (\tau/2)^{-1} \circ G_f^{-1} \right) (u) Q_f(u) > t \right\} du = 0. \quad (6.2.1)$$

Then

$$W_n(f) \rightarrow \sigma(f)W \text{ in distribution in } \mathcal{H}_{1/2-1/p}^o[0, 1], \quad (6.2.2)$$

where  $\sigma^2(f) = \text{Var}(f) + 2 \sum_{k=1}^{\infty} \text{Cov}(f, f \circ T^k)$ .

Using the comparison between  $\alpha$  and  $\tau$ , we can deduce a condition in the spirit of that of Doukhan, Massart and Rio (see [DMR95]). One can also derive it from the tightness criterion (6.1.4) and Theorem 6.2 of [Rio00].

**Corollary 6.2.2.** Let  $p > 2$  and let  $(f \circ T^j)_{j \geq 0}$  be a strictly stationary centered sequence such that

$$\lim_{t \rightarrow \infty} t^{p-1} \int_0^1 Q_f(u) \mathbf{1} \{ \alpha^{-1}(u) Q_f(u) > t \} du = 0. \quad (6.2.3)$$

Then

$$W_n(f) \rightarrow \sigma(f)W \text{ in distribution in } \mathcal{H}_{1/2-1/p}^o[0, 1], \quad (6.2.4)$$

where  $\sigma^2(f) = \lim_{n \rightarrow \infty} \sigma_n^2(f)/n$ .

*Remark 6.2.3.* Assume that the sequence  $(f \circ T^j)_{j \geq 0}$  is independent and that  $t^p \mu \{|f| > t\} \rightarrow 0$ . Then the condition of Theorem 6.2.1 is satisfied. Indeed, since  $Q_f(U)$  is distributed as  $|f|$  if  $U$  is uniformly distributed on  $[0, 1]$ , both conditions (6.2.2) and (6.2.4) read

$$\lim_{t \rightarrow \infty} t^{p-1} \mathbb{E}[|f| \mathbf{1}\{|f| > t\}] = 0. \quad (6.2.5)$$

As

$$\begin{aligned} t^{p-1} \mathbb{E}[|f| \mathbf{1}\{|f| > t\}] &= t^{p-1} \int_0^\infty \mu\{|f| > \max\{u, t\}\} du \\ &= t^p \mu\{|f| > t\} + t^{p-1} \int_t^\infty \mu\{|f| > u\} du \\ &\leq t^p \mu\{|f| > t\} + \sup_{s \geq t} s^p \mu\{|f| > s\} / (p-1), \end{aligned}$$

condition (6.2.5) is satisfied hence we can derive the result by Račkauskas and Suquet in the i.i.d. case from Theorem 6.2.1. This contrasts with Theorem 17 of [Ham00], from which we can only deduce the result by Lamperti (cf. [Lam62]) in the i.i.d. case.

*Remark 6.2.4.* Assume that  $Q_f(u) \leq C u^{-1/a}$  for some  $a > p$  (this is the case if  $f$  admits a finite weak moment of order  $a$ ). If  $\alpha(k) = o(k^{-a(p-1)/(a-p)})$  or  $\tau(k) = o(k^{-(a-1)(p-1)/(a-p)})$ , then condition (6.2.1) holds. If  $f$  is bounded, these sufficient conditions can be weakened respectively to  $\alpha(k) = o(k^{-(p-1)})$  and  $\tau(k) = o(k^{-(p-1)})$ .

We conclude this subsection by a result on  $\rho$ -mixing sequences.

**Theorem 6.2.5.** *Let  $p > 2$  and let  $(f \circ T^j)_{j \geq 0}$  be a strictly stationary centered sequence such that  $t^p \mu\{|f| > t\} \rightarrow 0$  as  $t \rightarrow +\infty$  and  $\sum_{i=0}^\infty \rho(2^i) < \infty$ . Then*

$$W_n(f) \rightarrow \sigma(f)W \text{ in distribution in } \mathcal{H}_{1/2-1/p}^o[0, 1], \quad (6.2.6)$$

where  $\sigma^2(f) = \lim_{n \rightarrow \infty} \sigma_n^2(f)/n$ .

## 6.2.2 A counter-example

In this subsection, we show that boundedness of the sequence of  $p$ th moments of the normalized partial sums is not enough to guarantee tightness in  $\mathcal{H}_{1/2-1/p}[0, 1]$ .

**Theorem 6.2.6.** *Let  $p > 2$ . There exists a strictly stationary sequence  $(f \circ T^j)_{j \geq 0}$  such that*

- *the finite dimensional distributions of  $(W_n(f))_{n \geq 1}$  converge to those of a standard Brownian motion,*
- *the sequence  $(\mathbb{E}|S_n(f)|^p / n^{p/2})_{n \geq 1}$  is bounded and*
- *the process  $(W_n(f))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}[0, 1]$ .*

*Remark 6.2.7.* The constructed process has no reason to be  $\alpha$ -mixing. However, this proves that in general, establishing tightness in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$  of  $(W_n(f))_{n \geq 1}$  cannot be done by proving boundedness in  $\mathbb{L}^p$  of the sequence  $(W_n(f))_{n \geq 1}$ . Thus other methods need to be used.

Let us recall that a sequence  $(c_n)_{n \geq 1}$  is *slowly varying* if there exists a continuous function  $h: \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  such that  $c_n = h(n)$  for each positive integer  $n$  and for each positive  $x$ ,  $\lim_{t \rightarrow \infty} h(tx)/h(t) = 1$ .

*Remark 6.2.8.* If  $p > 2$  and  $(f \circ T^j)_{j \geq 0}$  is a strictly stationary centered sequence such that the finite dimensional distributions of  $(\sigma_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$  converge to those of a standard Brownian motion, the sequence  $(\sigma_n^2(f)/n)_{n \geq 1}$  is slowly varying, and the sequence  $(\mathbb{E}|S_n(f)|^p / \sigma_n^p)_{n \geq 1}$  is bounded, then for each  $\gamma < 1/2 - 1/p$  the sequence  $(\sigma_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$  converges in distribution in  $\mathcal{H}_\alpha^o[0, 1]$  to a standard Brownian motion. This can be seen using tightness criterion (6.1.3), Markov's inequality and boundedness in  $\mathbb{L}^p$  of  $(\sigma_n^{-1} \max_{1 \leq j \leq n} |S_j(f)|)_{n \geq 1}$  (by Serfling arguments, see [Ser70]).



### 6.3 Proofs

*Proof of Theorem 6.2.1.* Notice that (6.2.1) implies finiteness of  $\int_0^1 Q_f^2(u)(\tau/2)^{-1} \circ G_f^{-1}(u)du$ , hence condition (5.5) in [DP04]. This implies the convergence of  $(\sigma_n^2(f)/n)_{n \geq 1}$  to  $\sigma(f)$ . Since  $\theta(k)$  is smaller than  $\tau(k)$ , Corollary 1 in [DD03] shows that the function  $f$  satisfies the projective criterion by Dedecker and Rio (see [DR00]), from which the convergence of the finite dimensional distributions follows. It remains to check tightness of  $(W_n(f))_{n \geq 1}$  in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ . We shall check that (6.1.4) is satisfied. To this aim, we apply Theorem 6.4.3 for each  $k \in \{1, \dots, \log[n\delta]\}$  with some  $r > p$ ,  $N := 2^k$  and  $\lambda := \varepsilon 2^{k\alpha} n^{1/p}$ . This gives

$$\begin{aligned} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu \left\{ \max_{1 \leq i \leq 2^k} |S_i(f)| > 5\varepsilon 2^{k\alpha} n^{1/p} \right\} &\leq n \sum_{k=1}^{\log[n\delta]} 2^{-k} 4r^{r/2} s_{2^k}(f)^r \left( \varepsilon 2^{k\alpha} n^{1/p} \right)^{-r} + \\ &+ n \sum_{k=1}^{\log[n\delta]} (\varepsilon 2^{k\alpha} n^{1/p})^{-1} \int_0^1 Q(u) \mathbf{1} \left\{ R(u) \geq \varepsilon 2^{k\alpha} n^{1/p}/r \right\} du. \end{aligned}$$

By (6.2.1), the quantity  $C := \int_0^{\|f\|_1} (\tau/2)^{-1}(u) Q_f \circ G_f(u) du$  is finite. In view of (6.4.3), we thus have

$$\begin{aligned} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu \left\{ \max_{1 \leq i \leq 2^k} |S_i(f)| > 5\varepsilon 2^{k\alpha} n^{1/p} \right\} &\leq 4 \cdot (4r)^{r/2} C^{r/2} n \sum_{k=1}^{\log[n\delta]} 2^{-k} 2^{kr/2} \left( \varepsilon 2^{k\alpha} n^{1/p} \right)^{-r} + \\ &+ n^{1-1/p} \sum_{k=1}^{\log[n\delta]} (\varepsilon 2^{k\alpha})^{-1} \int_0^1 Q(u) \mathbf{1} \left\{ R(u) \geq \varepsilon 2^{k\alpha} n^{1/p}/r \right\} du =: (I) + (II). \end{aligned} \quad (6.3.1)$$

A simple computation shows that

$$(I) \leq K(r, p, \varepsilon) \delta^{r/p-1}, \quad (6.3.2)$$

and for the second term, we have the upper bound

$$(II) \leq K(\alpha, \varepsilon) n^{(p-1)/p} \int_0^1 Q(u) \mathbf{1} \left\{ (\tau/2)^{-1} \circ G_f^{-1}(u) Q(u) \geq \varepsilon n^{1/p}/r \right\} du. \quad (6.3.3)$$

Since  $r > p$ , the condition (6.1.4) is satisfied in view of (6.3.1), (6.3.2), (6.3.3) and (6.2.1).  $\square$

*Proof of Corollary 6.2.2.* It suffices to check that condition (6.2.3) implies (6.2.1). Notice that by (6.1.9), we have for a fixed  $v$ ,

$$\inf \{i \mid \tau(i)/2 \leq v\} \leq \inf \{i \mid G^{-1}(2(\alpha(i))) \leq v\} = \inf \{i \mid \alpha(i) \leq G(v)/2\} \quad (6.3.4)$$

hence  $(\tau/2)^{-1}(v) \leq \alpha^{-1}(G(v)/2)$ . Taking  $v = G^{-1}(u)$  for a fixed  $u$ , we get

$$(\tau/2)^{-1} \circ G^{-1}(u) \leq \alpha^{-1}(u/2). \quad (6.3.5)$$

Since the function  $u \mapsto \alpha^{-1}(u)$  is non-increasing, the inclusion

$$\left\{ (\tau/2)^{-1} \circ G^{-1}(u) Q_f(u) > t \right\} \subset \left\{ \alpha^{-1}(u/2) Q_f(u/2) > t \right\}.$$

takes place. As a consequence, we obtain

$$\begin{aligned} t^{p-1} \int_0^1 Q_f(u) \mathbf{1} \left\{ \left( (\tau/2)^{-1} \circ G_f^{-1} \right) (u) Q_f(u) > t \right\} du &\leq \\ &t^{p-1} \int_0^1 Q_f(u/2) \left\{ \alpha^{-1}(u/2) Q_f(u/2) > t \right\} du, \end{aligned} \quad (6.3.6)$$

which concludes the proof of Corollary 6.2.2.  $\square$

*Proof of Theorem 6.2.5.* Theorem 4.1. of [Pel82] guarantees the existence of the limit of the sequence  $(\sigma_n^2/n)_{n \geq 1}$  and [Sha88] gives the convergence of the finite dimensional distributions. Therefore, the proof will be finished if we check the convergence (6.1.4). We apply Theorem 6.4.4 for each  $1 \leq k \leq \log[n\delta]$  with a  $q > p$ ,  $N := 2^k$ ,  $x := \varepsilon 2^{k\alpha} n^{1/p}$  and  $A := 2^{k\alpha} n^{1/p} \eta$ , where  $\eta$  is fixed (notice that since

$$\mathbb{E}[|f| \mathbf{1}\{|f| \geq A\}] = A\mu\{|f| \geq A\} + \int_A^{+\infty} \mu\{|f| \geq t\} dt \leq C(p, f)A^{1-p}, \quad (6.3.7)$$

we have for  $n \geq n(\eta, p, \varepsilon, f)$  and  $1 \leq k \leq \log[n\delta]$ ,

$$2 \cdot 2^k \cdot \mathbb{E}[|f| \mathbf{1}\{|f| \geq A\}] \leq 2C(p, f)(\eta 2^{k\alpha} n^{1/p})^{1-p} \leq \varepsilon 2^{k\alpha} n^{1/p} = x, \quad (6.3.8)$$

hence (6.4.5) is satisfied). This yields

$$\begin{aligned} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu \left\{ \max_{1 \leq i \leq 2^k} |S_i(f)| \geq \varepsilon 2^{k\alpha} n^{1/p} \right\} &\leq n \sum_{k=1}^{\log[n\delta]} \mu \left\{ |f| \geq 2^{k\alpha} n^{1/p} \right\} + \\ &+ K \exp \left( K \sum_{i=0}^{\infty} \rho(2^i) \right) n \sum_{k=1}^{\log[n\delta]} 2^{-k} 2^{kq/2} (\varepsilon 2^{k\alpha} n^{1/p})^{-q} \|f\|_2^q + \\ &+ K n \sum_{k=1}^{\log[n\delta]} \exp \left( K \sum_{i=0}^k \rho^{2/q}(2^i) \right) (\varepsilon 2^{k\alpha} n^{1/p})^{-q} \left\| f \mathbf{1}\{|f| \leq \eta 2^{k\alpha} n^{1/p}\} \right\|_q^q. \end{aligned} \quad (6.3.9)$$

Since for some constant  $C$  depending only on  $f$  and  $p$ , the bound

$$\left\| f \mathbf{1}\{|f| \leq \eta 2^{k\alpha} n^{1/p}\} \right\|_q^q \leq C(\eta 2^{k\alpha} n^{1/p})^{q-p} \quad (6.3.10)$$

is valid, we derive from (6.3.9) the inequality

$$\begin{aligned} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu \left\{ \max_{1 \leq i \leq 2^k} |S_i(f)| \geq \varepsilon 2^{k\alpha} n^{1/p} \right\} &\leq \varepsilon^{-p} \eta^{-p} \sup_{t \geq n^{1/p}} t^p \mu\{|f| \geq t\} \sum_{k=1}^{\infty} 2^{-kp\alpha} + \\ &+ K \exp \left( K \sum_{i=0}^{\infty} \rho(2^i) \right) \delta^{q/p-1} \cdot \frac{1}{2^{q/p-1}} \varepsilon^{-q} \|f\|_2^q + \\ &+ KC \varepsilon^{-q} \eta^{q-p} \sum_{k=1}^{\log[n\delta]} \exp \left( K \sum_{i=0}^k \rho^{2/q}(2^i) \right) 2^{-kp\alpha}. \end{aligned}$$

Since  $\lim_{t \rightarrow \infty} t^p \mu\{|f| > t\} = 0$ , we obtain for each  $\delta$  and  $\eta$

$$\begin{aligned} \limsup_{n \rightarrow \infty} n \sum_{k=1}^{\log[n\delta]} 2^{-k} \mu \left\{ \max_{1 \leq i \leq 2^k} |S_i(f)| > \varepsilon 2^{k\alpha} n^{1/p} \right\} &\leq \\ &\leq K \exp \left( K \sum_{i=0}^{\infty} \rho(2^i) \right) (2\delta)^{q/p-1} \varepsilon^{-q} \|f\|_2^q + KC \varepsilon^{-q} \eta^{q-p} \sum_{k=1}^{\infty} \exp \left( K \sum_{i=0}^k \rho^{2/q}(2^i) \right) 2^{-kp\alpha}, \end{aligned}$$

from which (6.1.4) follows (the convergence of the last series is ensured by the ratio test and the convergence to 0 of  $\rho^{2/q}(2^k)$ ).  $\square$

*Proof of Theorem 6.2.6.* We assume that  $(\Omega, \mathcal{F}, \mu, T)$  is a non-atomic invertible measure preserving system. We shall first construct a function  $g$  such that:

1. the sequence  $(\mathbb{E}|S_n(g - g \circ T)|^p / n^{p/2})_{n \geq 1}$  is bounded and
2. the process  $(W_n(g - g \circ T))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ .

We then consider  $f := m + g - g \circ T$ , where  $m$  is such that  $(m \circ T^j)_{j \geq 0}$  is a martingale difference sequence with  $m \in \mathbb{L}^p$  and  $m \neq 0$ . This will guarantee the convergence of the finite dimensional distributions of  $(W_n(f))_{n \geq 1}$  to those of a scalar multiple of a standard Brownian motion, and Burkholder's inequality ensures boundedness of the sequence  $(\mathbb{E}|S_n(f)|^p / n^{p/2})_{n \geq 1}$ . We use a construction similar to that given in [VS00]. Let us consider two increasing sequences of integer  $(K_l)_{l \geq 1}$  and  $(N_l)_{l \geq 1}$  satisfying for each  $l \geq 2$ :

$$\lim_{l \rightarrow +\infty} N_l \sum_{l' > l} K_{l'} / N_{l'} = 0; \quad (6.3.11)$$

$$4N_l^{-1/p} \cdot l \cdot N_{l-1} < 1; \quad (6.3.12)$$

$$\sum_{i=1}^l K_i^{1/2} \leq K_{l+1}^{1/2}; \quad (6.3.13)$$

$$\sum_{l=1}^{+\infty} \frac{K_l}{K_{l+1}^{1/2}} < \infty. \quad (6.3.14)$$

We also assume that  $4K_l \leq N_l$  for each  $l$ .

Let us fix an integer  $l$ . Using Rokhlin's lemma, we can find a set  $A_l \in \mathcal{F}$  such that the set  $T^i A_l$ ,  $0 \leq i \leq N_l - 1$  are pairwise disjoint and  $\mu\left(\bigcup_{i=0}^{N_l-1} T^i A_l\right) \geq 1/2$ . We define

$$h_l := \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \cdot \mathbf{1}\left(\bigcup_{j=1}^{K_l} T^{N_l-j} A_l\right);$$

$$g_l := \sum_{j=0}^{K_l-1} h_l \circ T^j = \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left( \sum_{j=1}^{K_l} j \mathbf{1}(T^{N_l-j} A_l) + \sum_{j=K_l+1}^{2K_l-1} (2K_l - j) \mathbf{1}(T^{N_l-j} A_l) \right);$$

$$g := \sum_{l=1}^{+\infty} g_l.$$

Assume that  $\omega \in A_l$  and  $N_l - K_l \leq i \leq N_l - 1$ . Then  $g_l \circ T^i(\omega) = N_l - i$ . Consequently, for  $i, i' \in \{N_k - K_l, \dots, N_l - 1\}$ ,

$$\left| g_l \circ T^i - g_l \circ T^{i'} \right| \geq \frac{N_l^{1/p}}{K_l^{1/2+1/p}} |i' - i| \mathbf{1}(A_l).$$

Applying  $U^{-k}$  on both sides of the previous inequality for  $0 \leq k \leq N_l - K_l$  and taking the maximum over these  $k$ , we obtain

$$\max_{0 \leq k \leq N_l - K_l} \left| g_l \circ T^{i-k} - g_l \circ T^{i'-k} \right| \geq |i' - i| \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \mathbf{1}\left(\bigcup_{k=0}^{N_k - K_l} T^k(A_l)\right).$$

This implies

$$\frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|g_l \circ T^{i'} - g_l \circ T^i|}{(i' - i)^{1/2-1/p}} \geq \mathbf{1}\left(\bigcup_{k=0}^{N_k - K_l} T^k(A_l)\right). \quad (6.3.15)$$

For  $l' < l$ , noticing that  $|g_{l'}| \leq N_{l'}^{1/p} K_{l'}^{1/2-1/p}$ , the following inequality takes place:

$$\max_{1 \leq i < j \leq N_l} |g_{l'} \circ T^j - g_{l'} \circ T^i| \leq 2N_{l'}^{1/p} K_{l'}^{1-1/p},$$

therefore,

$$\frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' < l} g_{l'} \circ T^{i'} - \sum_{l' < l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \leq \frac{1}{N_l^{1/p}} \sum_{l' < l} 2N_{l'}^{1/p} K_{l'}^{1-1/p} \leq 2N_l^{-1/p} l N_{l-1}.$$

By condition (6.3.12), we conclude that

$$\frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' < l} g_{l'} \circ T^{i'} - \sum_{l' < l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \leq 1/2. \quad (6.3.16)$$

Moreover, notice that

$$\begin{aligned} & \mu \left\{ N_l^{-1/p} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' > l} g_{l'} \circ T^{i'} - \sum_{l' > l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \neq 0 \right\} \\ & \leq \sum_{l' > l} \mu \left\{ \max_{1 \leq i < i' \leq N_l} |g_{l'} \circ T^{i'} - g_{l'} \circ T^i| \neq 0 \right\} \leq N_l \sum_{l' > l} \mu \{g_{l'} \neq 0\}, \end{aligned}$$

hence

$$\mu \left\{ \frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' > l} g_{l'} \circ T^{i'} - \sum_{l' > l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \neq 0 \right\} \leq 2N_l \sum_{l' > l} K_{l'}/N_{l'}. \quad (6.3.17)$$

By (6.3.16) and (6.3.17), we get

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|g \circ T^{i'} - g \circ T^i|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p}/2 \right\} \geq \\ & \geq \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' \geq l} (g_{l'} \circ T^{i'} - g_{l'} \circ T^i)|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p} \right\} \\ & \geq \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|(g_l \circ T^{i'} - g_l \circ T^i)|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p} \right\} - 2N_l \sum_{l' > l} K_{l'}/N_{l'}. \end{aligned}$$

Combining the previous inequality with (6.3.15), we obtain for each integer  $l$ ,

$$\mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|g \circ T^{i'} - g \circ T^i|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p}/2 \right\} \geq \frac{1}{2} - \frac{K_l}{2N_l} - 2N_l \sum_{l' > l} K_{l'}/N_{l'}, \quad (6.3.18)$$

and by (6.3.11), the inequality

$$\mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|g \circ T^{i'} - g \circ T^i|}{(i' - i)^{1/2-1/p}} \geq \frac{N_l^{1/p}}{2} \right\} \geq \frac{1}{8} \quad (6.3.19)$$

holds for  $l$  large enough. We deduce that for such integers  $l$  and each  $\delta \in (0, 1)$ ,

$$\mu \left\{ w_{1/2-1/p} \left( \frac{1}{\sqrt{N_l}} S_{N_l}^{\text{pl}}(g - g \circ T), \delta \right) \geq 1/2 \right\} \geq \frac{1}{8}, \quad (6.3.20)$$

hence the process  $(W_n(g - g \circ T))_{n \geq 1}$  cannot be tight in  $\mathcal{H}_{1/2-1/p}[0, 1]$ .

It remains to show that the sequence  $(n^{-1/2}(g - g \circ T^n))_{n \geq 1}$  is bounded in  $\mathbb{L}^p$ . Notice that for a fixed integer  $l \geq 1$ , the equalities

$$|g_l - g_l \circ T| = |h_l - h_l \circ T^{K_l}| = \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \cdot \mathbf{1} \left( \bigcup_{j=1}^{2K_l} T^{N_l-j} A_l \right)$$

take place. This implies that

$$\begin{aligned} \|g_l - g_l \circ T\|_p &= \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \cdot \left[ \mu \left( \bigcup_{j=1}^{2K_l} T^{N_l-j} A_l \right) \right]^{1/p} \\ &\leq \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left( \frac{2K_l}{N_l} \right)^{1/p} = 2^{1/p} K_l^{-1/2}, \end{aligned}$$

hence for each integer  $n \geq 1$ ,  $\|g_l - g_l \circ T^n\|_p \leq 2^{1/p} n K_l^{-1/2}$ . Let us define

$$\tilde{g}_l := \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \sum_{j=1}^{K_l} j \mathbf{1}(T^{N_l-j} A_l).$$

If  $K_l \leq n$ , then

$$\begin{aligned} \tilde{g}_l - \tilde{g}_l \circ T^n &= \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left( \sum_{j=1}^{K_l} j \mathbf{1}(T^{N_l-j} A_l) - \sum_{j=1}^{K_l} j \mathbf{1}(T^{N_l-j-n} A_l) \right) \\ &= \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left( \sum_{j=1}^{K_l} j \mathbf{1}(T^{N_l-j} A_l) - \sum_{j=n+1}^{n+K_l} (j-n) \mathbf{1}(T^{N_l-j} A_l) \right), \end{aligned}$$

hence

$$|\tilde{g}_l - \tilde{g}_l \circ T^n| \leq \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left( \sum_{j=1}^{K_l} j \mathbf{1}(T^{N_l-j} A_l) + \sum_{j=n+1}^{n+K_l} (j-n) \mathbf{1}(T^{N_l-j} A_l) \right),$$

and the following upper bound follows:

$$\mathbb{E} |\tilde{g}_l - \tilde{g}_l \circ T^n|^p \leq 2^{p-1} \sum_{j=1}^{K_l} j^p K_l^{-1-p/2} \leq 2^{p-1} K_l^{p/2}.$$

Treating in a similar manner the function  $g_l - \tilde{g}_l$ , we observe that the following inequality holds:

$$\|g_l - g_l \circ T^n\|_p \leq C_p \begin{cases} n K_l^{-1/2} & \text{if } K_l > n \\ K_l^{1/2} & \text{otherwise,} \end{cases} \quad (6.3.21)$$

where  $C_p$  depends only on  $p$  (neither on  $n$ , nor on  $l$ ). For a fixed integer  $n$ , we denote by  $i(n)$  the unique integer satisfying the inequalities  $K_{i(n)} \leq n < K_{i(n)+1}$ .

By (6.3.21), we have

$$\begin{aligned}
\|g - g \circ T^n\|_p &\leq \sum_{l=1}^{+\infty} \|g_l - g_l \circ T^n\|_p \\
&\leq C_p \left( \sum_{l=1}^{i(n)-1} K_l^{1/2} + K_{i(n)}^{1/2} + nK_{i(n)+1}^{-1/2} + \sum_{l=i(n)+2}^{+\infty} nK_l^{-1/2} \right) \\
&\leq 3C_p\sqrt{n} + C_p\sqrt{n} \sum_{l=i(n)+1}^{+\infty} \frac{K_l}{K_{l+1}^{1/2}} \\
&\leq C_p \left( 3 + \sum_{l=1}^{+\infty} \frac{K_l}{K_{l+1}^{1/2}} \right) \sqrt{n},
\end{aligned}$$

where we used (6.3.13) in the second inequality and condition (6.3.14) ensures finiteness of the right hand side in this inequality.

This concludes the proof of Theorem 6.2.6.  $\square$

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## 6.4 Appendix

For the reader's convenience, we state deviation inequalities for  $\tau$ -dependent and  $\rho$ -mixing sequences.

*Notation 6.4.1.* If  $(f \circ T^j)_{j \geq 0}$  is a (strictly stationary) sequence of random variables, we define

$$s_N^2(f) := \sum_{i=1}^N \sum_{j=1}^N |\text{Cov}(f \circ T^i, f \circ T^j)|. \quad (6.4.1)$$

*Notation 6.4.2.* Let  $Y$  be an integrable random variable. We denote by  $G_Y$  the generalized inverse of  $x \mapsto \int_0^x Q_Y(u) du$ .

The following Fuk-Nagaev inequality was established in Theorem 2 of [DP04].

**Theorem 6.4.3.** *Let  $(f \circ T^j)_{j \geq 0}$  be a strictly stationary sequence of centered and square integrable random variables. Let  $R := ((\tau/2)^{-1} \circ G_f^{-1})Q_f$  and  $S = R^{-1}$ . For any  $\lambda > 0$ , any integer  $N \geq 1$  and any  $r \geq 1$ ,*

$$\mu \left\{ \max_{1 \leq i \leq N} |S_i(f)| \geq 5\lambda \right\} \leq 4 \left( 1 + \frac{\lambda^2}{rs_N^2(f)} \right)^{-r/2} + \frac{4N}{\lambda} \int_0^{S(\lambda/r)} Q_f(u) du, \quad (6.4.2)$$

and

$$s_N^2(f) \leq 4N \int_0^{\|f\|_1} (\tau/2)^{-1}(u) Q_f \circ G_f(u) du. \quad (6.4.3)$$

For  $\rho$ -mixing sequences, Shao (Theorem 1.2, [Sha95]) showed the following inequality.

**Theorem 6.4.4.** *Let  $(f \circ T^j)_{j \geq 0}$  be a strictly stationary sequence of centered random variables and  $q \geq 2$ . Then there exists a constant  $K$  depending only on  $q$  and the sequence  $(\rho(n))_{n \geq 1}$  such that for each integer  $N$  and  $x > 0$ ,*

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq N} |S_i(f)| \geq x \right\} &\leq N \mu \{|f| \geq A\} + K x^{-q} \left( N^{q/2} \exp \left( K \sum_{i=0}^{\lfloor \log N \rfloor} \rho(2^i) \right) \|f\|_2^q + \right. \\ &\quad \left. + N \exp \left( K \sum_{i=0}^{\lfloor \log N \rfloor} \rho^{2/q}(2^i) \right) \|f \mathbf{1}_{\{|f| \leq A\}}\|_q^q \right). \end{aligned} \quad (6.4.4)$$

where  $A$  satisfies

$$2N \cdot \mathbb{E} [|f| \mathbf{1}_{\{|f| \geq A\}}] \leq x. \quad (6.4.5)$$

# Holderian weak invariance principle under a Hannan type condition

## 7.1 Introduction

One of the main problems in probability theory is the understanding of the asymptotic behavior of Birkhoff sums  $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$ , where  $(\Omega, \mathcal{F}, \mu, T)$  is a dynamical system and  $f$  a map from  $\Omega$  to the real line.

One can consider random functions constructed from the Birkhoff sums

$$S_n^{\text{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]+1}, \quad t \in [0, 1]. \quad (7.1.1)$$

and investigate the asymptotic behaviour of the sequence  $(S_n^{\text{pl}}(f, t))_{n \geq 1}$  seen as an element of a function space. Donsker showed (cf. [Don51]) that the sequence  $(n^{-1/2}(\mathbb{E}(f^2))^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$  converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion  $W$  when the sequence  $(f \circ T^i)_{i \geq 0}$  is i.i.d. and zero mean. Then an intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [MPU06] for the main theorems in this direction.

Our purpose is to investigate the weak convergence of the sequence  $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$  in Hölder spaces when  $(f \circ T^i)_{i \geq 0}$  is a strictly stationary sequence. A classical method for showing a limit theorem is to use a martingale approximation, which allows to deduce the corresponding result if it holds for martingale differences sequences provided that the approximation is good enough. To the best of our knowledge, no result about the invariance principle in Hölder space for stationary martingale difference sequences is known.

### 7.1.1 The Hölder spaces

It is well known that standard Brownian motion's paths are almost surely Hölder regular of exponent  $\alpha$  for each  $\alpha \in (0, 1/2)$ , hence it is natural to consider the random function defined in (7.1.1) as an element of  $\mathcal{H}_\alpha[0, 1]$  and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for  $\alpha \in (0, 1)$  the Hölder space  $\mathcal{H}_\alpha[0, 1]$  of functions  $x: [0, 1] \rightarrow \mathbb{R}$  such that  $\sup_{s \neq t} |x(s) - x(t)| / |s - t|^\alpha$  is finite. The analogue of the continuity modulus in  $C[0, 1]$  is  $w_\alpha$ , defined by

$$w_\alpha(x, \delta) = \sup_{0 < |t-s| < \delta} \frac{|x(t) - x(s)|}{|t - s|^\alpha}. \quad (7.1.2)$$



We then define  $\mathcal{H}_\alpha^0[0, 1]$  by  $\mathcal{H}_\alpha^o[0, 1] := \{x \in \mathcal{H}_\alpha[0, 1], \lim_{\delta \rightarrow 0} w_\alpha(x, \delta) = 0\}$ . We shall essentially work with the space  $\mathcal{H}_\alpha^o[0, 1]$  which, endowed with  $\|x\|_\alpha := w_\alpha(x, 1) + |x(0)|$ , is a separable Banach space (while  $\mathcal{H}_\alpha[0, 1]$  is not separable). Since the canonical embedding  $\iota: \mathcal{H}_\alpha^o[0, 1] \rightarrow \mathcal{H}_\alpha[0, 1]$  is continuous, each convergence in distribution in  $\mathcal{H}_\alpha^o[0, 1]$  also takes place in  $\mathcal{H}_\alpha[0, 1]$ .

Let us denote by  $D_j$  the set of dyadic numbers in  $[0, 1]$  of level  $j$ , that is,

$$D_0 := \{0, 1\}, \quad D_j := \{(2l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}, j \geq 1. \quad (7.1.3)$$

If  $r \in D_j$  for some  $j \geq 0$ , we define  $r^+ := r + 2^{-j}$  and  $r^- := r - 2^{-j}$ . For  $r \in D_j$ ,  $j \geq 1$ , let  $\Lambda_r$  be the function whose graph is the polygonal path joining the points  $(0, 0)$ ,  $(r^-, 0)$ ,  $(r, 1)$ ,  $(r^+, 0)$  and  $(1, 0)$ . We can decompose each  $x \in C[0, 1]$  as

$$x = \sum_{r \in D} \lambda_r(x) \Lambda_r = \sum_{j=0}^{+\infty} \sum_{r \in D_j} \lambda_r(x) \Lambda_r, \quad (7.1.4)$$

and the convergence is uniform on  $[0, 1]$ . The coefficients  $\lambda_r(x)$  are given by

$$\lambda_r(x) = x(r) - \frac{x(r^+) + x(r^-)}{2}, \quad r \in D_j, j \geq 1, \quad (7.1.5)$$

and  $\lambda_0(x) = x(0)$ ,  $\lambda_1(x) = x(1)$ .

Ciesielski proved (cf. [Cie60]) that  $\{\Lambda_r; r \in D\}$  is a Schauder basis of  $\mathcal{H}_\alpha^o[0, 1]$  and the norms  $\|\cdot\|_\alpha$  and the sequential norm defined by

$$\|x\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(x)|, \quad (7.1.6)$$

are equivalent.

Considering the sequential norm, we can show (see Theorem 3 in [Suq99]) that a sequence  $(\xi_n)_{n \geq 1}$  of random elements of  $\mathcal{H}_\alpha^o$  vanishing at 0 is tight if and only if for each positive  $\varepsilon$ ,

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(\xi_n)| > \varepsilon \right\} = 0. \quad (7.1.7)$$

*Notation 7.1.1.* In the sequel, we will denote  $r_{k,j} := k2^{-j}$  and  $u_{k,j} := [nr_{k,j}]$  (or  $r_k$  and  $u_k$  for short). Notice that  $u_{k+1,j} - u_{k,j} = [nr_{k+1,j} + n2^{-j}] - u_{k,j} \leq 2n2^{-j}$  if  $j \leq \log n$ , where  $\log n$  denotes the binary logarithm of  $n$  and for a real number  $x$ ,  $[x]$  is the unique integer for which  $[x] \leq x < [x] + 1$ .

*Remark 7.1.2.* Since for each  $x \in \mathcal{H}_{1/2-1/p}[0, 1]$ , each  $j \geq 1$  and each  $r \in D_j$ ,

$$|\lambda_r(x)| \leq \frac{|x(r^+) - x(r)|}{2} + \frac{|x(r) - x(r^-)|}{2} \leq \max\{|x(r^+) - x(r)|, |x(r) - x(r^-)|\}, \quad (7.1.8)$$

for a function  $f$ , the sequential norm of  $n^{-1/2}S_n^{\text{pl}}(f)$  does not exceed

$$\sup_{j \geq 1} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} |S_n^{\text{pl}}(f, r_{k+1,j}) - S_n^{\text{pl}}(f, r_{k,j})|. \quad (7.1.9)$$

Now, we state the result obtained by Račkauskas and Suquet in [RS03].

**Theorem 7.1.3.** *Let  $p > 2$  and let  $(f \circ T^j)_{j \geq 0}$  be an i.i.d. centered sequence with unit variance. Then the condition*

$$\lim_{t \rightarrow \infty} t^p \mu \{|f| > t\} = 0 \quad (7.1.10)$$

*is equivalent to the weak convergence of the sequence  $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$  to a standard Brownian motion in the space  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ .*

### 7.1.2 Some facts about the $\mathbb{L}^{p,\infty}$ spaces

In the rest of the paper,  $\mathbf{1}$  denotes the indicator function. Let  $p > 2$ . We define the  $\mathbb{L}^{p,\infty}$  space as the collection of functions  $f: \Omega \rightarrow \mathbb{R}$  such that the quantity

$$\|f\|_{p,\infty}^p := \sup_{t>0} t^p \mu\{|f| > t\} < \infty. \quad (7.1.11)$$

This quantity is denoted like a norm, while it is not a norm (the triangle inequality may fail, for example if  $X = [0, 1]$  endowed with the Lebesgue measure,  $f(x) := x^{-1/p}$  and  $g(x) := f(1-x)$ ; in this case  $\|f+g\|_{p,\infty} \geq 2^{1+1/p}$  but  $\|f\|_{p,\infty} + \|g\|_{p,\infty} = 2$ ). However, there exists a constant  $\kappa_p$  such that for each  $f$ ,

$$\|f\|_{p,\infty} \leq \sup_{A:\mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \mathbf{1}_A] \leq \kappa_p \|f\|_{p,\infty} \quad (7.1.12)$$

and  $N_p(f) := \sup_{A:\mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \mathbf{1}_A]$  defines a norm. The first inequality in (7.1.12) can be seen from the estimate  $t\mu\{|f| > t\} \leq \mathbb{E}[|f| \mathbf{1}_{\{|f| > t\}}]$ ; for the second one, we write

$$\mathbb{E}[|f| \mathbf{1}_A] = \int_0^{+\infty} \mu(\{|f| > t\} \cap A) dt \leq \int_0^{+\infty} \min\{\mu\{|f| > t\}, \mu(A)\} dt, \quad (7.1.13)$$

and we bound the integrand by  $\min\{t^{-p} \|f\|_{p,\infty}^p, \mu(A)\}$ .

A function  $f$  satisfies (7.1.10) if and only if it belongs to the closure of bounded functions with respect to  $N_p$ . Indeed, if  $f$  satisfies (7.1.10), then the sequence  $(f \mathbf{1}_{|f| < n})_{n \geq 1}$  converges to  $f$  in  $\mathbb{L}^{p,\infty}$ . If  $N_p(f-g) < \varepsilon$  with  $g$  bounded, then

$$\limsup_{t \rightarrow \infty} t^p \mu\{|f| > t\} \leq \limsup_{t \rightarrow \infty} t^p \mu\{|f-g| > t/2\} \leq 2^p \varepsilon. \quad (7.1.14)$$

We now provide two technical lemmas about  $\mathbb{L}^{p,\infty}$  spaces. The first one will be used in the proof of the weak invariance principle for martingales, since we will have to control the tail function of the random variables involved in the construction of the truncated martingale (cf.(7.3.111)). The second one will provides an estimation of the  $\mathbb{L}^{p,\infty}$  norm of a simple function, which will be used in the proof of Theorem 9.2.6, since the function  $m$  is constructed as a series of simple functions.

**Lemma 7.1.4.** *If  $\lim_{t \rightarrow \infty} t^p \mu\{|f| > t\} = 0$ , then for each sub- $\sigma$ -algebra  $\mathcal{A}$ , we have*

$$\lim_{t \rightarrow \infty} t^p \mu\{\mathbb{E}[|f| \mid \mathcal{A}] > t\} = 0. \quad (7.1.15)$$

*Proof.* For simplicity, we assume that  $f$  is non-negative. For a fixed  $t$ , the set  $\{\mathbb{E}[f \mid \mathcal{A}] > t\}$  belongs to the  $\sigma$ -algebra  $\mathcal{A}$ , hence

$$t\mu\{\mathbb{E}[f \mid \mathcal{A}] > t\} \leq \mathbb{E}[\mathbb{E}[f \mid \mathcal{A}] \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}}] = \mathbb{E}[f \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}}]. \quad (7.1.16)$$

By definition of  $N_p$ ,

$$\mathbb{E}[f \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}}] \leq N_p(f \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}}) \mu\{\mathbb{E}[f \mid \mathcal{A}] > t\}^{1-1/p}, \quad (7.1.17)$$

hence

$$t^p \mu\{\mathbb{E}[f \mid \mathcal{A}] > t\} \leq N_p(f \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}})^p. \quad (7.1.18)$$

Notice that

$$\forall s > 0, \quad N_p(f \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}}) \leq s\mu\{\mathbb{E}[f \mid \mathcal{A}] > t\}^{1/p} + N_p(f \mathbf{1}_{\{f > s\}}), \quad (7.1.19)$$

hence

$$\limsup_{t \rightarrow \infty} \mathbb{E}[f \mathbf{1}_{\{\mathbb{E}[f \mid \mathcal{A}] > t\}}] \leq N_p(f \mathbf{1}_{\{f > s\}}) \leq \kappa_p \sup_{x \geq s} x^p \mu\{f > x\}. \quad (7.1.20)$$

By the assumption on the function  $f$ , the right hand side goes to 0 as  $s$  goes to infinity, which concludes the proof of the lemma.  $\square$

**Lemma 7.1.5.** *Let  $f := \sum_{i=0}^N a_i \mathbf{1}(A_i)$ , where the family  $(A_i)_{i=0}^N$  is pairwise disjoint and  $0 \leq a_N < \dots < a_0$ . Then*

$$\|f\|_{p,\infty}^p \leq \max_{0 \leq j \leq N} a_j^p \sum_{i=0}^j \mu(A_i). \quad (7.1.21)$$

*Proof.* We have the equality

$$\mu\{f > t\} = \sum_{j=0}^N \mathbf{1}_{(a_{j+1}, a_j]}(t) \sum_{i=0}^j \mu(A_i), \quad (7.1.22)$$

where  $a_{N+1} := 0$ , therefore

$$t^p \mu\{f > t\} \leq \max_{0 \leq j \leq N} a_j^p \sum_{i=0}^j \mu(A_i). \quad (7.1.23)$$

□

## 7.2 Main results

The goal of the paper is to give a sharp sufficient condition on the moments of a strictly stationary martingale difference sequence which guarantees the weak invariance principle in  $\mathcal{H}_\alpha^o[0, 1]$  for a fixed  $\alpha$ .

We first show that Theorem 7.1.3 does not extend to strictly stationary ergodic martingale difference sequences, that is, sequences of the form  $(m \circ T^i)_{i \geq 0}$  such that  $m$  is  $\mathcal{M}$  measurable and  $\mathbb{E}[m | T\mathcal{M}] = 0$  for some sub- $\sigma$ -algebra  $\mathcal{M}$  satisfying  $T\mathcal{M} \subset \mathcal{M}$ .

An application of Kolmogorov's continuity criterion shows that if  $(m \circ T^i)_{i \geq 0}$  is a martingale difference sequence such that  $m \in \mathbb{L}^{p+\delta}$  for some positive  $\delta$  and  $p > 2$ , then the partial sum process  $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$  is tight in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$  (see [KR91]).

We provide a condition on the quadratic variance which improves the previous approach (since the previous condition can be replaced by  $m \in \mathbb{L}^p$ ). Then using martingale approximation we can provide a Hannan type condition which guarantees the weak invariance principle in  $\mathcal{H}_\alpha^o[0, 1]$ .

**Theorem 7.2.1.** *Let  $p > 2$  and  $(\Omega, \mathcal{F}, \mu, T)$  be a dynamical system with positive entropy. There exists a function  $m: \Omega \rightarrow \mathbb{R}$  and a  $\sigma$ -algebra  $\mathcal{M}$  for which  $T\mathcal{M} \subset \mathcal{M}$  such that:*

- *the sequence  $(m \circ T^i)_{i \geq 0}$  is a martingale difference sequence with respect to the filtration  $(T^{-i}\mathcal{M})_{i \geq 0}$ ;*
- *the convergence  $\lim_{t \rightarrow +\infty} t^p \mu\{|m| > t\} = 0$  takes place;*
- *the sequence  $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ .*

**Theorem 7.2.2.** *Let  $(\Omega, \mathcal{F}, \mu, T)$  be a dynamical system,  $\mathcal{M}$  a sub- $\sigma$ -algebra of  $\mathcal{F}$  such that  $T\mathcal{M} \subset \mathcal{M}$  and  $\mathcal{I}$  the collection of sets  $A \in \mathcal{F}$  such that  $T^{-1}A = A$ .*

*Let  $p > 2$  and let  $(m \circ T^j, T^{-j}\mathcal{M})_{j \geq 0}$  be a strictly stationary martingale difference sequence. Assume that  $t^p \mu\{|m| > t\} \rightarrow 0$  and  $\mathbb{E}[m^2 | T\mathcal{M}] \in \mathbb{L}^{p/2}$ . Then*

$$n^{-1/2} S_n^{\text{pl}}(m) \rightarrow \eta \cdot W \text{ in distribution in } \mathcal{H}_{1/2-1/p}^o[0, 1], \quad (7.2.1)$$

where the random variable  $\eta$  is given by

$$\eta = \lim_{n \rightarrow \infty} \mathbb{E}[S_n^2 | \mathcal{I}] / n \text{ in } \mathbb{L}^1 \quad (7.2.2)$$

and  $\eta$  is independent of the process  $(W_t)_{t \in [0, 1]}$ .

In particular, (7.2.1) takes place if  $m$  belongs to  $\mathbb{L}^p$ .

The key point of the proof of Theorem 7.2.2 is an inequality in the spirit of Doob's one, which gives  $n^{-1}\mathbb{E}[\max_{1 \leq j \leq n} S_j(m)^2] \leq 2\mathbb{E}[m^2]$ . It is used in order to establish tightness of the sequence  $(n^{-1/2}S_n^{\text{pl}}(m))_{n \geq 1}$  in the space  $C[0, 1]$ .

**Proposition 7.2.3.** *Let  $p > 2$ . There exists a constant  $C_p$  depending only on  $p$  such that if  $(m \circ T^i)_{i \geq 1}$  is a martingale difference sequence, then the following inequality holds:*

$$\sup_{n \geq 1} \left\| \left\| n^{-1/2} S_n^{\text{pl}}(m) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty}^p \leq C_p \left( \|m\|_{p, \infty}^p + \mathbb{E}(\mathbb{E}[m^2 | T\mathcal{M}]^{p/2}) \right). \quad (7.2.3)$$

*Remark 7.2.4.* As Theorem 7.2.1 shows, the condition  $\lim_{t \rightarrow \infty} t^p \mu\{|m| > t\} = 0$  alone for martingale difference sequences is not sufficient to obtain the weak convergence of  $n^{-1/2}S_n^{\text{pl}}(m)$  in  $\mathcal{H}_\alpha^o[0, 1]$  for  $\alpha = 1/2 - 1/p$ . For the constructed  $m$  in Theorem 7.2.1, the quadratic variance is  $\kappa m^2$  for some constant  $\kappa$  and  $m$  does not belong to the  $\mathbb{L}^p$  space.

By Lemma A.2 in [MSR12], the Hölder norm of a polygonal line is reached at two vertices, hence, for a function  $g$ ,

$$\left\| n^{-1/2} S_n^{\text{pl}}(g - g \circ T) \right\|_{\mathcal{H}_{1/2-1/p}^o} = n^{-1/p} \max_{1 \leq i < j \leq n} \frac{|g \circ T^j - g \circ T^i|}{(j-i)^{1/2-1/p}} \quad (7.2.4)$$

$$\leq 2n^{-1/p} \max_{1 \leq j \leq n} |g \circ T^j|. \quad (7.2.5)$$

As a consequence, if  $g$  belongs to  $\mathbb{L}^p$ , then the sequence  $\left( \left\| n^{-1/2} S_n^{\text{pl}}(g - g \circ T) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right)_{n \geq 1}$  converges to 0 in probability. Therefore, we can exploit a martingale-coboundary decomposition in  $\mathbb{L}^p$ .

**Corollary 7.2.5.** *Let  $p > 2$  and let  $f$  be an  $\mathcal{M}$ -measurable function which can be written as*

$$f = m + g - g \circ T, \quad (7.2.6)$$

where  $m, g \in \mathbb{L}^p$  and  $(m \circ T^i)_{i \geq 0}$  is a martingale difference sequence for the filtration  $(T^{-i}\mathcal{M})_{i \geq 0}$ . Then  $n^{-1/2}S_n^{\text{pl}}(f) \rightarrow \eta W$  in distribution in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ , where  $\eta$  is given by (7.2.2) and independent of  $W$ .

We define for a function  $h$  the operators  $\mathbb{E}_k(h) := \mathbb{E}[h | T^k\mathcal{M}]$  and  $P_i(h) := \mathbb{E}_i(h) - \mathbb{E}_{i+1}(h)$ . The condition  $\sum_{i=0}^{\infty} \|P_i(f)\|_2$  was introduced by Hannan in [Han73] in order to deduce a central limit theorem. It actually implies the weak invariance principle (see Corollary 2 in [DMV07]).

**Theorem 7.2.6.** *Let  $p > 2$  and let  $f$  be an  $\mathcal{M}$ -measurable function such that*

$$\mathbb{E} \left[ f \mid \bigcap_{i \in \mathbb{Z}} T^i \mathcal{M} \right] = 0 \text{ and} \quad (7.2.7)$$

$$\sum_{i \geq 0} \|P_i(f)\|_p < \infty. \quad (7.2.8)$$

Then  $n^{-1/2}S_n^{\text{pl}}(m) \rightarrow \eta W$  in distribution in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ , where  $\eta$  is given by (7.2.2) and independent of  $W$ .

## 7.3 Proofs

### 7.3.1 Proof of Theorem 7.2.1

We need a result about dynamical systems of positive entropy for the construction of a counterexample.

**Lemma 7.3.1.** *Let  $(\Omega, \mathcal{A}, \mu, T)$  be an ergodic probability measure preserving system of positive entropy. There exists two  $T$ -invariant sub- $\sigma$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  of  $\mathcal{A}$  and a function  $g: \Omega \rightarrow \mathbb{R}$  such that:*

- *the  $\sigma$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  are independent;*
- *the function  $g$  is  $\mathcal{B}$ -measurable, takes the values  $-1, 0$  and  $1$ , has zero mean and the process  $(g \circ T^n)_{n \in \mathbb{Z}}$  is independent;*
- *the dynamical system  $(\Omega, \mathcal{C}, \mu, T)$  is aperiodic.*

This is Lemma 3.8 from [LV01].

We consider the following four increasing sequences of positive integers  $(I_l)_{l \geq 1}$ ,  $(J_l)_{l \geq 1}$ ,  $(n_l)_{l \geq 1}$  and  $(L_l)_{l \geq 1}$ . We define  $k_l := 2^{I_l + J_l}$  and impose the conditions:

$$\sum_{l=1}^{\infty} \frac{1}{L_l} < \infty; \quad (7.3.1)$$

$$\lim_{l \rightarrow \infty} J_l \cdot \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} = 1; \quad (7.3.2)$$

$$\lim_{l \rightarrow \infty} J_l 2^{-I_l/2} = 0; \quad (7.3.3)$$

$$\lim_{l \rightarrow \infty} n_l \sum_{i>l} \frac{k_i}{n_i} = 0; \quad (7.3.4)$$

$$\text{for each } l, \quad \sum_{i=1}^{l-1} \frac{k_l}{L_i} \left( \frac{n_i}{2^{I_i}} \right)^{1/p} < \frac{n_l^{1/p}}{2}. \quad (7.3.5)$$

Here  $\mathcal{N}$  denotes a random variable whose distribution is standard normal. Such sequences can be constructed as follows: first pick a sequence  $(L_l)_{l \geq 1}$  satisfying (7.3.1), for example  $L_l = l^2$ . Then construct  $(J_l)_{l \geq 1}$  such that (7.3.2) holds. Once the sequence  $(J_l)_{l \geq 1}$  is constructed, define  $(I_l)_{l \geq 1}$  satisfying (7.3.3). Now the sequence  $(k_l)_{l \geq 1}$  is completely determined. Noticing that (7.3.4) is satisfied if the series  $\sum_i k_i n_{i-1}/n_i$  converges, we construct the sequence  $(n_l)_{l \geq 1}$  by induction; once  $n_i, i \leq l-1$  are defined, we choose  $n_l$  such that  $n_l \geq l^2 k_l n_{l-1}$  and (7.3.5) holds.

Using Rokhlin's lemma, we can find for any integer  $l \geq 1$  a measurable set  $C_l \in \mathcal{C}$  such that the sets  $T^{-i}C_l, i = 0, \dots, n_l - 1$  are pairwise disjoint and  $\mu \left( \bigcup_{i=0}^{n_l-1} T^{-i}C_l \right) > 1/2$ .

For a fixed  $l$ , we define

$$k_{l,j} := 2^{I_l + J_l - j}, \quad 0 \leq j \leq J_l, \quad (7.3.6)$$

$k_{l,j} := 2^{I_l + J_l - j}, \quad 0 \leq j \leq J_l$  and

$$\begin{aligned} f_l := \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j}-1} T^{-i}C_l \right) + \\ + \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=0}^{k_{l,J_l}-1} T^{-i}C_l \right), \end{aligned} \quad (7.3.7)$$

$$f := \sum_{l=1}^{+\infty} f_l, \quad m := g \cdot f, \quad (7.3.8)$$

where  $g$  is the function obtained by Lemma 7.3.1.

**Proposition 7.3.2.** *We have the estimate  $\|f_l\|_{p,\infty} \leq \kappa'_p L_l^{-1}$  for some constant  $\kappa'_p$  depending only on  $p$ . As a consequence,  $\lim_{t \rightarrow \infty} t^p \mu \{|m| > t\} = 0$ .*

*Proof.* Notice that

$$\left\| \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right) \right\|_{p,\infty}^p = \frac{1}{L_l^p} \frac{n_l}{k_{l,J_l}} k_{l,J_l} \cdot \mu(C_l) \leq \frac{1}{L_l^p}. \quad (7.3.9)$$

Next, using Lemma 7.1.5 with

$$N := J_l - 1, a_j := \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \text{ and } A_j := \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l, \quad (7.3.10)$$

we obtain

$$\begin{aligned} \left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l \right) \right\|_{p,\infty}^p &\leq \max_{0 \leq j \leq J_l-1} \left( \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \right)^p \sum_{i=0}^j \mu(A_j) \\ &\leq \frac{1}{L_l^p} \max_{0 \leq j \leq J_l-1} \frac{n_l}{k_{l,J_l-j}} \sum_{i=0}^j \frac{k_{l,J_l-i}}{n_l} \end{aligned} \quad (7.3.11)$$

$$= \frac{1}{L_l^p} \max_{0 \leq j \leq J_l-1} \sum_{i=0}^j \frac{2^{J_l+i}}{2^{J_l+j}} \quad (7.3.12)$$

$$\leq \frac{2}{L_l^p}, \quad (7.3.13)$$

hence by (7.1.12), (7.3.9) and (7.3.13),

$$\|f_l\|_{p,\infty} \leq N_p \left( \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right) \right) + \quad (7.3.14)$$

$$+ N_p \left( \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l \right) \right) \quad (7.3.15)$$

$$\leq \kappa_p \left\| \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right) \right\|_{p,\infty} + \quad (7.3.16)$$

$$+ \kappa_p \left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \mathbf{1} \left( \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l \right) \right\|_{p,\infty} \quad (7.3.17)$$

$$\leq \frac{1}{L_l} \kappa_p (1 + 2^{1/p}) \quad (7.3.18)$$

We thus define  $\kappa'_p := \kappa_p (1 + 2^{1/p})$ .

We fix  $\varepsilon > 0$ ; using (7.3.1), we can find an integer  $l_0$  such that  $\sum_{l>l_0} 1/L_l < \varepsilon$ . Since the

function  $\sum_{l=1}^{l_0} g f_l$  is bounded, we have,

$$\limsup_{t \rightarrow \infty} t^p \mu \{ |m| > t \} \leq \limsup_{t \rightarrow \infty} t^p \mu \left\{ \left| \sum_{l=1}^{l_0} g f_l \right| > \frac{t}{2} \right\} + 2^p \left\| \sum_{l > l_0} g f_l \right\|_{p, \infty}^p \quad (7.3.19)$$

$$= 2^p \left\| \sum_{l > l_0} g f_l \right\|_{p, \infty}^p \quad (7.3.20)$$

$$\leq \left( 2 \sum_{l > l_0} N_p(f_l) \right)^p \quad (7.3.21)$$

$$\leq \kappa'_p \left( \sum_{l > l_0} \frac{1}{L_l} \right)^p \quad (7.3.22)$$

$$\leq \kappa'_p \varepsilon^p, \quad (7.3.23)$$

where the second inequality comes from inequalities (7.1.12). Since  $\varepsilon$  is arbitrary, the proof of Lemma 7.3.2 is complete.  $\square$

We denote by  $\mathcal{M}$  the  $\sigma$ -algebra generated by  $\mathcal{C}$  and the random variables  $g \circ T^k$ ,  $k \leq 0$ . It satisfies  $\mathcal{M} \subset T^{-1}\mathcal{M}$ .

**Proposition 7.3.3.** *The sequence  $(m \circ T^i)_{i \geq 0}$  is a (stationary) martingale difference sequence with respect to the filtration  $(T^{-i}\mathcal{M})_{i \geq 0}$ .*

*Proof.* We have to show that  $\mathbb{E}[m \mid T\mathcal{M}] = 0$ . Since the  $\sigma$ -algebra  $\mathcal{C}$  is  $T$ -invariant, we have  $T\mathcal{M} = \sigma(\mathcal{C} \cup \sigma(g \circ T^k, k \leq -1))$ . This implies

$$\mathbb{E}[m \mid T\mathcal{M}] = \mathbb{E}[g f \mid T\mathcal{M}] = f \cdot \mathbb{E}[g \mid T\mathcal{M}]. \quad (7.3.24)$$

Since  $g$  is centered and independent of  $T\mathcal{M}$ , Proposition 7.3.3 is proved.  $\square$

It remains to prove that the process  $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}^0[0, 1]$ .

**Proposition 7.3.4.** *Under conditions (6.3.13), (6.3.12) and (6.3.11), there exists an integer  $l_0$  such that for  $l \geq l_0$*

$$P_l := \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(g f_l) - S_u(g f_l)|}{v^{1/2-1/p}} \geq 1 \right\} \geq \frac{1}{16}. \quad (7.3.25)$$

*Proof.* Let us fix an integer  $l \geq 1$ . Assume that  $\omega \in T^{-s}C_l$ , where  $k_l \leq s \leq n_l - 1$ . Since  $T^u\omega$  belongs to  $T^{-(s-u)}C_l$  we have for  $s - n_l \leq u \leq s$

$$(f_l \circ T^u)(\omega) = \begin{cases} \frac{1}{L_l} \left( \frac{n_l}{k_{l, J_l}} \right)^{1/p}, & \text{if } s - k_{l, J_l} < u \leq s; \\ \frac{1}{L_l} \left( \frac{n_l}{k_{l, j}} \right)^{1/p}, & \text{if } s - k_{l, j-1} < u \leq s - k_{l, j}, \text{ and } 1 \leq j \leq J_l; \\ 0, & \text{if } s - n_l \leq u < s - k_l. \end{cases} \quad (7.3.26)$$

As a consequence,

$$\begin{aligned} T^{-s}C_l \cap \left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq j \leq J_l} \frac{|S_{s-k_{l, j-1}+1}(g f_l) - S_{s-k_{l, j}}(g f_l)|}{(k_{l, j-1} - 1 - k_{l, j})^{1/2-1/p}} \geq 1 \right\} \\ = T^{-s}C_l \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{s-k_{l, j-1}+1}(g) - S_{s-k_{l, j}}(g)|}{(k_{l, j} - 1)^{1/2-1/p} k_{l, j-1}^{1/p}} \geq L_l \right\}. \end{aligned} \quad (7.3.27)$$

Since for  $k_l + 1 \leq s \leq n_l - k_l$  and  $1 \leq j \leq J_l$ , we have  $1 \leq s - k_{l,j} \leq n_l - k_l$  and  $1 \leq k_{l,j-1} - 1 - k_{l,j} \leq k_l$ , the inequality

$$\begin{aligned} \mathbf{1}(T^{-s}C_l) \cdot \max_{1 \leq j \leq J_l} \frac{|S_{s-k_{l,j-1}+1}(gfl) - S_{s-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \\ \leq \mathbf{1}(T^{-s}(C_l)) \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gfl) - S_u(gfl)|}{v^{1/2-1/p}} \end{aligned} \quad (7.3.28)$$

takes place and since the sets  $(T^{-s}C_l)_{s=0}^{n_l-1}$  are pairwise disjoint, we obtain the lower bound

$$P_l \geq \sum_{s=1}^{n_l-2k_l} \mu \left( T^{-(s+k_l)}(C_l) \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{s+k_l-k_{l,j-1}+1}(gfl) - S_{s+k_l-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \geq 1 \right\} \right). \quad (7.3.29)$$

Using the fact that  $T$  is measure-preserving, this becomes

$$P_l \geq (n_l - 2k_l) \cdot \mu \left( T^{-k_l}(C_l) \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_l-k_{l,j-1}-1}(gfl) - S_{k_l-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \geq 1 \right\} \right), \quad (7.3.30)$$

and plugging (7.3.27) in the previous estimate, we get

$$P_l \geq (n_l - 2k_l) \mu \left( T^{-k_l}(C_l) \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_l-k_{l,j-1}-1}(g) - S_{k_l-k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\} \right). \quad (7.3.31)$$

The sets  $\left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_l-k_{l,j-1}+1}(g) - S_{k_l-k_{l,j}}(g)|}{(k_{l,j-1})^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}$  and  $T^{-k_l}C_l$  belong to the independent sub- $\sigma$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  respectively, hence using the fact that the sequences  $(g \circ T^i)_{i \geq 0}$  and  $(g \circ T^{-i})_{i \geq 0}$  are identically distributed, we obtain

$$P_l \geq (n_l - 2k_l) \mu(C_l) \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}. \quad (7.3.32)$$

By construction, we have  $n_l \cdot \mu(C_l) = \mu \left( \bigcup_{i=0}^{n_l-1} T^{-i}C_l \right) > 1/2$ , hence

$$P_l \geq \frac{1}{2} \left( 1 - 2 \frac{k_l}{n_l} \right) \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}. \quad (7.3.33)$$

It remains to find a lower bound for

$$P'_l := \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}. \quad (7.3.34)$$

Let us define the set

$$E_j := \left\{ \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\} \quad (7.3.35)$$

Since the sequence  $(g \circ T^i)_{i \in \mathbb{Z}}$  is independent, the family  $(E_j)_{1 \leq j \leq J_l}$  is independent, hence

$$P'_l \geq 1 - \prod_{j=1}^{J_l} (1 - \mu(E_j)). \quad (7.3.36)$$



We define the quantity

$$c_j := \mu \left\{ |\mathcal{N}| \geq \frac{L_l}{\|g\|_2} \left( \frac{k_{l,j-1}}{k_{l,j} - 1} \right)^{1/p} \right\} \quad (7.3.37)$$

(we recall that  $\mathcal{N}$  denotes a standard normally distributed random variable). By the Berry-Esseen theorem, we have for each  $j \in \{1, \dots, J_l\}$ ,

$$|\mu(E_j) - c_j| \leq \frac{1}{\|g\|_2^3} \frac{1}{(k_{l,j-1} - 1)^{1/2}} \leq \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}. \quad (7.3.38)$$

Plugging the estimate (7.3.38) into (7.3.36) and noticing that for an integer  $N$  and  $(a_n)_{n=1}^N$ ,  $(b_n)_{n=1}^N$  two families of numbers in the unit interval,

$$\left| \prod_{n=1}^N a_n - \prod_{n=1}^N b_n \right| \leq \sum_{n=1}^N |a_n - b_n|, \quad (7.3.39)$$

we obtain

$$P'_l \geq 1 - \prod_{j=1}^{J_l} (1 - \mu(E_j)) + \prod_{j=1}^{J_l} (1 - c_j) - \prod_{j=1}^{J_l} (1 - c_j) \quad (7.3.40)$$

$$\geq 1 - \prod_{j=1}^{J_l} (1 - c_j) - \sum_{j=1}^{J_l} |\mu(E_j) - c_j| \quad (7.3.41)$$

$$\geq 1 - \prod_{j=1}^{J_l} (1 - c_j) - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}. \quad (7.3.42)$$

Notice that

$$1 - \prod_{j=1}^{J_l} (1 - c_j) \geq 1 - \max_{1 \leq j \leq J_l} (1 - c_j)^{J_l} \quad (7.3.43)$$

and since  $(I_l)_{\geq}$  is increasing and  $I_1 \geq 1$ , we have

$$\frac{k_{l,j-1}}{k_{l,j} - 1} = \frac{2}{1 - k_{l,j}^{-1}} \leq \frac{2}{1 - 2^{-I_l}} \leq 4 \quad (7.3.44)$$

it follows by (7.3.37) that  $c_j \geq \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\}$  for  $1 \leq j \leq J_l$ . We thus have

$$P'_l \geq 1 - \left( 1 - \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} \right)^{J_l} - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}. \quad (7.3.45)$$

Using the elementary inequality

$$1 - (1 - t)^n \geq nt - \frac{n(n-1)}{2} t^2 \quad (7.3.46)$$

valid for a positive integer  $n$  and  $t \in [0, 1]$ , we obtain

$$P'_l \geq J_l \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} - \frac{J_l^2}{2} \left( \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} \right)^2 - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}. \quad (7.3.47)$$

By conditions (6.3.12) and (6.3.13), there exists an integer  $l'_0$  such that if  $l \geq l'_0$ , then

$$\mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\} \geq \frac{1}{4}. \quad (7.3.48)$$

Combining (7.3.33) with (7.3.48), we obtain for  $l \geq l'_0$

$$P_l \geq \frac{1}{8} \left( 1 - 2 \frac{k_l}{n_l} \right). \quad (7.3.49)$$

By condition (6.3.11), we thus get that  $P_l \geq 1/16$  for  $l \geq l_0$ , where  $l_0 \geq l'_0$  and  $k_l/n_l \leq 1/4$  if  $l \geq l_0$ .

This concludes the proof of Proposition 7.3.4.  $\square$

**Proposition 7.3.5.** *Under conditions (7.3.1), (7.3.2), (7.3.3), (7.3.4) and (7.3.5), we have for  $l$  large enough*

$$\mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m) - S_u(m)|}{v^{1/2-1/p}} \geq \frac{1}{2} \right\} \geq \frac{1}{32}. \quad (7.3.50)$$

Since the Hölder modulus of continuity of a piecewise linear function is reached at vertices, we derive the following corollary.

**Corollary 7.3.6.** *If  $l \geq l_0$ , then*

$$\mu \left\{ \omega_{1/2-1/p} \left( \frac{1}{\sqrt{n_l}} S_{n_l}^{\text{pl}}(m), \frac{k_l}{n_l} \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32}. \quad (7.3.51)$$

Therefore, for each positive  $\delta$ , we have

$$\limsup_{n \rightarrow \infty} \mu \left\{ \omega_{1/2-1/p} \left( \frac{1}{\sqrt{n}} S_n^{\text{pl}}(m), \delta \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32}, \quad (7.3.52)$$

and the process  $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ .

*Proof of Proposition 7.3.5.* Let  $l_0$  be the integer given by Proposition 7.3.4 and let  $l \geq l_0$ . We define  $m'_l := \sum_{i=1}^{l-1} g f_i$  and  $m''_l := \sum_{i=l+1}^{+\infty} g f_i$ .

We define for  $i \geq 1$ ,

$$M_{l,i} := \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(g f_i) - S_u(g f_i)|}{v^{1/2-1/p}}. \quad (7.3.53)$$

Let  $i$  be an integer such that  $i < l$ . Notice that for  $1 \leq u \leq n_l - k_l$  and  $v \leq k_l$ , we have

$$|S_{u+v}(g f_i) - S_u(g f_i)| = U^u(|S_v(g f_i)|), \quad (7.3.54)$$

where  $U(h)(\omega) = h(T(\omega))$  and since

$$|S_v(g f_i)| \leq v \|g f_i\|_\infty \leq \frac{k_l}{L_i} \left( \frac{n_i}{2^{I_i}} \right)^{1/p}, \quad (7.3.55)$$

the estimate

$$M_{l,i} \leq \frac{k_l}{L_i n_l^{1/p}} \left( \frac{n_i}{2^{I_i}} \right)^{1/p} \quad (7.3.56)$$

holds. Since

$$\frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leq \sum_{i=1}^{l-1} M_{l,i}, \quad (7.3.57)$$

we have by (7.3.56),

$$\frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leq \sum_{i=1}^{l-1} \frac{k_l}{L_i n_l^{1/p}} \left( \frac{n_i}{2^{I_i}} \right)^{1/p}. \quad (7.3.58)$$

By (6.3.14), the following bound takes place:

$$\frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leq \frac{1}{2}. \quad (7.3.59)$$

The following set inclusions hold

$$\left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m''_l) - S_u(m''_l)|}{v^{1/2-1/p}} \neq 0 \right\} \subset \bigcup_{i>l} \{M_{l,i} \neq 0\} \quad (7.3.60)$$

$$\subset \bigcup_{i>l} \bigcup_{u=1}^{n_i} \{U^u(gf_i) \neq 0\}. \quad (7.3.61)$$

We thus have

$$\mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m''_l) - S_u(m''_l)|}{v^{1/2-1/p}} \neq 0 \right\} \leq \sum_{i>l} n_l \cdot \mu \{gf_i \neq 0\} \quad (7.3.62)$$

$$\leq n_l \sum_{i>l} \mu \{f_i \neq 0\} \quad (7.3.63)$$

$$= n_l \sum_{i>l} (k_i + 1) \mu(C_i) \quad (7.3.64)$$

$$\leq 2n_l \sum_{i>l} \frac{k_i}{n_i}. \quad (7.3.65)$$

and by (6.3.11), it follows that

$$\mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m''_l) - S_u(m''_l)|}{v^{1/2-1/p}} \neq 0 \right\} \leq \frac{1}{32} \quad (7.3.66)$$

Accounting (7.3.59), we thus have

$$\begin{aligned} & \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m) - S_u(m)|}{v^{1/2-1/p}} \geq \frac{1}{2} \right\} \\ & \geq \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gf_l + m''_l) - S_u(gf_l + m''_l)|}{v^{1/2-1/p}} \geq 1 \right\} \\ & \geq \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gf_l) - S_u(gf_l)|}{v^{1/2-1/p}} \geq 1 \right\} \\ & \quad - \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m''_l) - S_u(m''_l)|}{v^{1/2-1/p}} \neq 0 \right\}, \quad (7.3.67) \end{aligned}$$

hence combining Proposition 7.3.4 with (7.3.66), we obtain the conclusion of Proposition 7.3.5.  $\square$

Theorem 9.2.6 follows from Corollary 7.3.6 and Propositions 7.3.2 and 7.3.3.

### 7.3.2 Proof of Theorem 7.2.2 and Proposition 7.2.3

*Proof of Proposition 7.2.3.* Let us fix a positive  $t$ . Recall the equivalence between  $\|x\|_\alpha$  and  $\|x\|_\alpha^{\text{seq}}$  and Notation 7.1.1. By Remark 7.1.2, we have to show that for some constant  $C$  depending only on  $p$  and each integer  $n \geq 1$ ,

$$\begin{aligned} P(n, t) &:= t^p \mu \left\{ \sup_{j \geq 1} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} |S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j})| > t \right\} \leq \\ &\leq C \left( \|m\|_{p, \infty}^p + \mathbb{E}(\mathbb{E}[m^2 | \mathcal{TM}])^{p/2} \right) \end{aligned} \quad (7.3.68)$$

In the proof, we shall denote by  $C_p$  a constant depending only on  $p$  which may change from line to line.

We define

$$P_1(n, t) := \mu \left\{ \sup_{1 \leq j \leq \log n} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} |S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j})| > t \right\}, \text{ and} \quad (7.3.69)$$

$$P_2(n, t) := \mu \left\{ \sup_{j > \log n} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} |S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j})| > t \right\}, \quad (7.3.70)$$

hence

$$P(n, t) \leq t^p P_1(n, t/2) + t^p P_2(n, t/2). \quad (7.3.71)$$

We estimate  $P_2(n, t)$ . For  $j > \log n$ , we have the inequality

$$r_{k+1, j} - r_{k, j} = (k+1)2^{-j} - k2^{-j} = 2^{-j} < 1/n, \quad (7.3.72)$$

hence if  $r_{k, j}$  belongs to the interval  $[l/n, (l+1)/n)$  for some  $l \in \{0, \dots, n-1\}$ , then

- either  $r_{k+1, j} \in [l/n, (l+1)/n)$ , and in this case,

$$|S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j})| = |m \circ T^{l+1}| 2^{-j} n \leq 2^{-j} n \max_{1 \leq l \leq n} |U^l(m)|; \quad (7.3.73)$$

- or  $r_{k+1, j}$  belongs to the interval  $[(l+1)/n, (l+2)/n)$ . The estimates

$$\begin{aligned} |S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j})| &\leq |S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, (l+1)/n)| + \\ &+ |S_n^{\text{pl}}(m, (l+1)/n) - S_n^{\text{pl}}(m, r_{k, j})| \leq 2^{1-j} n \max_{1 \leq l \leq n} |U^l(m)| \end{aligned} \quad (7.3.74)$$

hold.

Considering these two cases, we obtain

$$P_2(n, t) \leq \mu \left\{ \sup_{j > \log n} 2^{\alpha j} n 2^{1-j} n^{-1/2} \max_{1 \leq l \leq n} |U^l(m)| > t \right\} \quad (7.3.75)$$

$$\leq \mu \left\{ 2n^{\alpha-1/2} \max_{1 \leq l \leq n} |U^l(m)| > t \right\} \quad (7.3.76)$$

$$\leq n\mu \left\{ 2n^{-1/p} |m| > t \right\} \quad (7.3.77)$$

$$\leq \frac{2^p}{t^p} \sup_{x>0} x^p \mu \{ |m| > x \}. \quad (7.3.78)$$

Therefore, establishing inequality (7.3.68) reduces to find a constant  $C$  depending only on  $p$  such that

$$\sup_n \sup_t t^p P_1(n, t) \leq C \left( \|m\|_{p, \infty}^p + \mathbb{E}(\mathbb{E}[m^2 | \mathcal{TM}])^{p/2} \right) \quad (7.3.79)$$

We define  $u_{k,j} := \lceil nr_{k,j} \rceil$  for  $k < 2^j$  and  $j \geq 1$  (see Notation 7.1.1).

Notice that the inequalities

$$|S_{u_{k,j}}(m) - S_n^{\text{pl}}(m, r_{k,j})| \leq |U^{u_{k,j}+1}(m)| \quad \text{and} \quad (7.3.80)$$

$$|S_n^{\text{pl}}(m, r_{k+1,j}) - S_{u_{k+1,j}}(m)| \leq |U^{u_{k+1,j}+1}(m)| \quad (7.3.81)$$

take place because if  $j \leq \log n$ , then

$$u_{k,j} \leq nr_{k,j} \leq u_{k,j} + 1 \leq u_{k+1,j} \leq nr_{k+1,j} \leq u_{k+1,j} + 1. \quad (7.3.82)$$

Therefore,  $P_1(n, t) \leq P_{1,1}(n, t) + P_{1,2}(n, t)$ , where

$$P_{1,1}(n, t) := \mu \left\{ \max_{1 \leq j \leq \log n} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} |S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)| > t/2 \right\}, \quad (7.3.83)$$

$$P_{1,2}(n, t) := \mu \left\{ \max_{1 \leq j \leq \log n} 2^{\alpha j} n^{-1/2} \max_{1 \leq l \leq n} |U^l(m)| > t/4 \right\}. \quad (7.3.84)$$

Notice that

$$P_{1,2}(n, t) \leq \mu \left\{ n^{\alpha-1/2} \max_{1 \leq l \leq n} |U^l(m)| > t/4 \right\} \quad (7.3.85)$$

$$\leq n\mu \left\{ |m| > n^{1/p} t/4 \right\} \quad (7.3.86)$$

$$\leq 4^p t^{-p} \sup_{x \geq 0} x^p \mu \{ |m| > x \}, \quad (7.3.87)$$

hence (7.3.79) will follow from the existence of a constant  $C$  depending only on  $p$  such that

$$\sup_n \sup_t t^p P_{1,1}(n, t) \leq C \left( \|m\|_{p,\infty}^p + \mathbb{E} (\mathbb{E}[m^2 | T\mathcal{M}])^{p/2} \right). \quad (7.3.88)$$

We estimate  $P_{1,1}(n, t)$  in the following way:

$$P_{1,1}(n, t) \leq \sum_{j=1}^{\log n} 2^j \max_{0 \leq k < 2^j} \mu \left\{ |S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)| > tn^{1/2} 2^{-1-\alpha j} \right\} \quad (7.3.89)$$

We define for  $1 \leq j \leq \log n$  and  $0 \leq k < 2^j$  the quantity

$$P(n, j, k, t) := \mu \left\{ |S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)| > tn^{1/2} 2^{-1-\alpha j} \right\}. \quad (7.3.90)$$

If  $(f \circ T^j)_{j \geq 0}$  is a strictly stationary sequence, we define

$$Q_{f,n}(u) := \mu \left\{ \max_{1 \leq j \leq n} |f \circ T^j| > u \right\} + \mu \left\{ \left( \sum_{i=1}^n U^i \mathbb{E}[f^2 | T\mathcal{M}] \right)^{1/2} > u \right\}. \quad (7.3.91)$$

The following inequality is Theorem 1 of [Nag03]. It allows us to express the tail function of a martingale by that of the increments and the quadratic variance.

**Theorem 7.3.7.** *Let  $m$  be an  $\mathcal{M}$ -measurable function such that  $\mathbb{E}[m | T\mathcal{M}] = 0$ . Then for each positive  $y$  and each integer  $n$ ,*

$$\mu \{ |S_n(m)| > y \} \leq c(q, \eta) \int_0^1 Q_{m,n}(\varepsilon_q u \cdot y) u^{q-1} du, \quad (7.3.92)$$

where  $q > 0$ ,  $\eta > 0$ ,  $\varepsilon_q := \eta/q$  and  $c(q, \eta) := q \exp(3\eta e^{\eta+1} - \eta - 1)/\eta$ .

We shall use (7.3.92) with  $q := p + 1$ ,  $\eta = 1$  and  $y := n^{1/2}2^{-1-\alpha j}t$  in order to estimate  $P(n, j, k, t)$ :

$$\begin{aligned} P(n, j, k, t) &\leq C_p \int_0^1 \mu \left\{ \max_{1 \leq i \leq u_{k+1,j} - u_{k,j}} |U^i(m)| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\} u^p du \\ &\quad + C_p \int_0^1 \mu \left\{ \left( \sum_{i=u_{k,j}+1}^{u_{k+1,j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\} u^p du. \end{aligned} \quad (7.3.93)$$

Exploiting the inequality  $u_{k+1,j} - u_{k,j} \leq 2n2^{-j}$ , we get from the previous bound

$$\begin{aligned} P(n, j, k, t) &\leq C_p \int_0^1 \mu \left\{ \max_{1 \leq i \leq 2n2^{-j}} |U^i(m)| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\} u^p du \\ &\quad + C_p \int_0^1 \mu \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\} u^p du. \end{aligned} \quad (7.3.94)$$

We define for  $j \leq \log n$ ,  $t \geq 0$  and  $u \in (0, 1)$ ,

$$P'(n, j, t, u) := \mu \left\{ \max_{1 \leq i \leq 2n2^{-j}} |U^i(m)| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\}, \quad \text{and} \quad (7.3.95)$$

$$P''(n, j, t, u) := \mu \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\}. \quad (7.3.96)$$

Using the fact that the random variables  $U^i(m)$ ,  $1 \leq i \leq 2n2^{-j}$  are identically distributed, we derive the bound

$$P'(n, j, t, u) \leq 2n2^{-j} \mu \left\{ |m| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1} \right\}, \quad (7.3.97)$$

hence

$$\begin{aligned} P'(n, j, t, u) &\leq 2n2^{-j} (n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1})^{-p} \|m\|_{p,\infty}^p \\ &= 2^{p+1} \varepsilon_{p+1}^{-p} n^{1-p/2} 2^{j(-1+p\alpha)} t^{-p} u^{-p} \|m\|_{p,\infty}^p. \end{aligned} \quad (7.3.98)$$

Since  $\alpha$  and  $p$  are linked by the relationship  $1/2 - 1/p = \alpha$ , we have  $p\alpha = p/2 - 1$  hence

$$\int_0^1 P'(n, j, t, u) u^p du \leq C_p t^{-p} n^{1-p/2} 2^{j(p/2-2)} \|m\|_{p,\infty}^p. \quad (7.3.99)$$

Notice the following set equalities:

$$\begin{aligned} &\left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t \right\} \\ &= \left\{ \frac{1}{2n2^{-j}} \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\} \end{aligned} \quad (7.3.100)$$

and that  $n2^{-j} \geq 1$  (because  $j \leq \log n$ ), hence

$$\left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t \right\} \subseteq \bigcup_{N \geq 2} \left\{ \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\}, \quad (7.3.101)$$

from which it follows

$$\mu \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t \right\} \leq \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\}. \quad (7.3.102)$$

Combining (7.3.99) and (7.3.102), we obtain

$$\begin{aligned} \max_{0 \leq k < 2^j} P(n, j, k, t) &\leq C_p t^{-p} n^{1-p/2} 2^{j(p/2-2)} \|m\|_{p, \infty}^p \\ &\quad + C_p \int_0^1 \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\} u^p du, \end{aligned} \quad (7.3.103)$$

hence by (7.3.89) and (7.3.90),

$$\begin{aligned} P_{1,1}(n, t) &\leq C_p t^{-p} \|m\|_{p, \infty}^p \sum_{j=1}^{\log n} 2^j 2^{j(p/2-2)} n^{1-p/2} \\ &\quad + C_p \int_0^1 \sum_{j=1}^{\log n} 2^j \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\} u^p du. \end{aligned} \quad (7.3.104)$$

From the elementary bounds

$$\sum_{j=1}^{\log n} 2^j 2^{j(p/2-1)} n^{1-p/2} \leq (1 - 2^{1-p/2})^{-1} \quad (7.3.105)$$

$$\sum_{j \geq 1} 2^j \mu \left\{ |g| > 2^{2j/p} \right\} \leq 2 \mathbb{E} |g|^{p/2}, \quad \text{for any non-negative function } g, \quad (7.3.106)$$

with

$$g := 2^3 \varepsilon_{p+1}^{-2} u^{-2} \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]), u \in (0, 1) \quad (7.3.107)$$

we obtain

$$P_{1,1}(n, t) \leq C_p t^{-p} \|m\|_{p, \infty}^p + C_p t^{-p} \left\| \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right\|_{p/2}^{p/2}. \quad (7.3.108)$$

As the Koopman operator  $U$  is an  $\mathbb{L}^1$ - $\mathbb{L}^\infty$  contraction, Theorem 1 of [Ste61] gives the existence of a constant  $A_p$  such that for each  $h \in \mathbb{L}^{p/2}$ ,

$$\left\| \sup_{N \geq 1} \frac{1}{N} \sum_{j=1}^N U^j(h) \right\|_{p/2} \leq A_p \|h\|_{p/2}. \quad (7.3.109)$$

Applying (7.3.109) with  $h := \mathbb{E}[m^2 \mid T\mathcal{M}]$ , we get by (7.3.108)

$$P_{1,1}(n, t) \leq C_p t^{-p} \|m\|_{p, \infty}^p + C_p t^{-p} \mathbb{E}(\mathbb{E}[m^2 \mid T\mathcal{M}])^{p/2}, \quad (7.3.110)$$

which establishes (7.3.79). This concludes the proof of Proposition 7.2.3.  $\square$

*Proof of Theorem 7.2.2.* The convergence of finite dimensional distributions can be proved using the result of [Bil61]. Its proof works for filtrations of the form  $(T^{-i}\mathcal{M})_{i \geq 0}$  where  $T\mathcal{M} \subset \mathcal{M}$  and also in the non-ergodic setting by considering the ergodic components.

We deduce tightness in Theorem 7.2.2 from Proposition 7.2.3 by a truncation argument. For a fixed  $R$ , we define

$$m_R := m \mathbf{1}\{|m| \leq R\} - \mathbb{E}[m \mathbf{1}\{|m| \leq R\} \mid T\mathcal{M}] \quad \text{and} \quad (7.3.111)$$

$$m'_R := m \mathbf{1}\{|m| > R\} - \mathbb{E}[m \mathbf{1}\{|m| > R\} \mid T\mathcal{M}]. \quad (7.3.112)$$

In this way, the sequences  $(m_R \circ T^i)_{i \geq 0}$  and  $(m'_R \circ T^i)_{i \geq 0}$  are martingale differences sequences and  $m = m_R + m'_R$ .

Since  $|m_R| \leq 2R$  and  $(m_R \circ T^i)_{i \geq 0}$  is a martingale difference sequence, the sequence  $(n^{-1/2} S_n^{\text{pl}}(m_R))_{n \geq 1}$  is tight in  $\mathcal{H}_{1/2-1/p}^0[0, 1]$ . Consequently, for each positive  $\varepsilon$ , the following convergence takes place:

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(m_R))| > \varepsilon n^{1/2} \right\} = 0. \quad (7.3.113)$$

Using Proposition 7.2.3, we derive the following bound, valid for each  $\varepsilon$  and each  $R$ ,

$$\begin{aligned} & \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(m))| > \varepsilon n^{1/2} \right\} \leq \\ & \leq C_p \varepsilon^{-p} \left( \sup_{t \geq 0} t^p \mu \{ |m| \mathbf{1}\{|m| > R\} > t \} + \sup_{t \geq 0} t^p \mu \{ \mathbb{E}[|m| \mathbf{1}\{|m| > R\} \mid T\mathcal{M}] > t \} \right) + \\ & \quad + \varepsilon^{-p} C_p \mathbb{E} \left( (\mathbb{E}[m^2 \mathbf{1}\{|m| > R\} \mid T\mathcal{M}])^{p/2} \right). \end{aligned} \quad (7.3.114)$$

The first term is  $\sup_{t \geq R} t^p \mu \{|m| > t\}$ , which goes to 0 as  $R$  goes to infinity.

The second term can be bounded by  $\sup_{t \geq R} t^p \mu \{\mathbb{E}[|m| \mid T\mathcal{M}] > t\}$ . Indeed, if  $t \geq R$ , we use the inclusion

$$\{\mathbb{E}[|m| \mathbf{1}\{|m| > R\} \mid T\mathcal{M}] > t\} \subset \{\mathbb{E}[|m| \mid T\mathcal{M}] > t\}, \quad (7.3.115)$$

and if  $t < R$ , then accounting the fact that the random variable  $\mathbb{E}[|m| \mathbf{1}\{|m| > R\} \mid T\mathcal{M}]$  is greater than  $R$ , we get

$$\begin{aligned} \mathbb{E}[|m| \mathbf{1}\{|m| > R\} \mid T\mathcal{M}] &= \mathbb{E}[|m| \mathbf{1}\{|m| > R\} \mid T\mathcal{M}] \mathbf{1}\{\mathbb{E}[|m| \mid T\mathcal{M}] > R\} \\ &\leq \mathbb{E}[|m| \mid T\mathcal{M}] \mathbf{1}\{\mathbb{E}[|m| \mid T\mathcal{M}] > R\}, \end{aligned} \quad (7.3.116)$$

from which it follows that

$$t^p \mu \{\mathbb{E}[|m| \mathbf{1}\{|m| > R\} \mid T\mathcal{M}] > t\} \leq R^p \mu \{\mathbb{E}[|m| \mid T\mathcal{M}] > R\}. \quad (7.3.117)$$

By Lemma 7.1.4, the convergence

$$\lim_{R \rightarrow \infty} \sup_{t \geq R} t^p \mu \{\mathbb{E}[|m| \mid T\mathcal{M}] > t\} = 0 \quad (7.3.118)$$

takes place.

The third term of (7.3.114) converges to 0 as  $R$  goes to infinity by monotone convergence.

This concludes the proof of tightness in Theorem 7.2.2.  $\square$



### 7.3.3 Proof of Theorem 7.2.6

By (7.2.7), the equality  $f = \sum_{i \geq 0} P_i(f)$  holds almost surely. For a fixed integer  $K$ , we define  $f_K := \sum_{i=0}^K P_i(f)$ . Then  $f_K$  satisfies the conditions of Corollary 7.2.5.

Indeed, we have the equalities

$$P_i(f) - P_0(U^i f) = \mathbb{E}[f | T^i \mathcal{M}] - \mathbb{E}[U^i f | \mathcal{M}] - \mathbb{E}[f | T^{i+1} \mathcal{M}] + \mathbb{E}[U^i f | T \mathcal{M}] \quad (7.3.119)$$

$$= (I - U^i) \mathbb{E}[f | T^i \mathcal{M}] - (I - U^i) \mathbb{E}[f | T^{i+1} \mathcal{M}] \quad (7.3.120)$$

and the later term can be expressed as a coboundary noticing that  $(I - U^i) = (I - U) \sum_{k=0}^{i-1} U^k$ . Since  $P_i(f)$  belongs to the  $\mathbb{L}^p$  space, we may write  $f_K - \sum_{i=0}^K P_0(U^i f)$  as  $(I - U)g_K$  where  $g_K$  belongs to the  $\mathbb{L}^p$  space. Defining  $m_K := \sum_{i=0}^K P_0(U^i(f))$ , the sequence  $(m_K \circ T^i)_{i \geq 0}$  is a martingale difference sequence hence for each positive  $\varepsilon$ ,

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f_K))| > \varepsilon n^{1/2} \right\} = 0. \quad (7.3.121)$$

Now, we have to show that the convergence in (7.3.121) holds if  $f_K$  is replaced by  $f - f_K$ . To this aim, we use the inclusion

$$\begin{aligned} \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f - f_K))| > \varepsilon n^{1/2} \right\} &\subseteq \\ &\subseteq \left\{ \sup_{j \geq 1} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f - f_K))| > \varepsilon n^{1/2} \right\}, \end{aligned} \quad (7.3.122)$$

hence

$$\begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f - f_K))| > \varepsilon n^{1/2} \right\} &\leq \varepsilon^{-p} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}}(f - f_K) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty}^p \\ &= \varepsilon^{-p} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}} \left( \sum_{i \geq K+1} P_i(f) \right) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty}^p, \end{aligned} \quad (7.3.123)$$

from which it follows that

$$\begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f - f_K))| > \varepsilon n^{1/2} \right\} \\ \leq \varepsilon^{-p} \left( \sum_{i \geq K+1} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}}(P_i(f)) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty} \right)^p. \end{aligned} \quad (7.3.124)$$

Notice that for a fixed  $i$ , the sequence  $(U^l(P_i(f)))_{l \geq 1}$  is a martingale difference sequence (with respect to the filtration  $(T^{-i-l} \mathcal{M})_{l \geq 0}$ ). Therefore, by Proposition 7.2.3, we obtain

$$\left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}}(P_i(f)) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty} \leq C_p \|P_i(f)\|_p. \quad (7.3.125)$$

Plugging this estimate into (7.3.124), we obtain that for some constant  $C$  depending only on  $p$ ,

$$\mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f - f_K))| > \varepsilon n^{1/2} \right\} \leq C \varepsilon^{-p} \left( \sum_{i \geq K+1} \|P_i(f)\|_p \right)^p. \quad (7.3.126)$$

Combining (7.3.121) and (7.3.126), we obtain for each  $K$ :

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha_j} \max_{r \in D_j} |\lambda_r(S_n^{\text{pl}}(f))| > n^{1/2} \varepsilon \right\} \leq C \varepsilon^{-p} \left( \sum_{i \geq K+1} \|P_i(f)\|_p \right)^p. \quad (7.3.127)$$

Since  $K$  is arbitrary, we conclude the proof of Theorem 7.2.6 thanks to assumption (7.2.8).

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# Holderian weak invariance principle under Maxwell and Woodroffe condition

## 8.1 Introduction and main results

Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space and let  $T: \Omega \rightarrow \Omega$  be a measure-preserving bijective and bi-measurable function. Let  $\mathcal{M}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$  such that  $T\mathcal{M} \subset \mathcal{M}$ . If  $\theta$  is a measure preserving operator and  $f: \Omega \rightarrow \mathbb{R}$  a measurable function, we denote  $S_n(\theta, f) := \sum_{j=0}^{n-1} f \circ \theta^j$  and

$$W(n, f, \theta, t) := S_{[nt]}(f) + (nt - [nt])f \circ \theta^{[nt]}. \quad (8.1.1)$$

When  $\theta = T$  we shall often write  $S_n(f)$  and  $W(n, f, t)$ . We denote  $\mathcal{M}_\infty$  the  $\sigma$ -algebra generated by  $\bigcup_{i \in \mathbb{Z}} T^i \mathcal{M}$  and  $\mathcal{M}_{-\infty} := \bigcap_{i \in \mathbb{Z}} T^i \mathcal{M}$ . We say that the function  $f \in \mathbb{L}^1$  is *regular* if  $f$  is  $\mathcal{M}_\infty$ -measurable and  $\mathbb{E}[f | \mathcal{M}_{-\infty}] = 0$ .

An important problem in probability theory is the understanding of the asymptotic behavior of the process  $(n^{-1/2}W(n, f, t), t \in [0, 1])_{n \geq 1}$ . Conditions on the quantities  $\mathbb{E}[S_n(f) | T\mathcal{M}]$  and  $S_n(f) - \mathbb{E}[S_n(f) | T^{-n}\mathcal{M}]$  have been investigated. The first result in this direction was obtained by Maxwell and Woodroffe [MW00]: if  $f$  is regular,  $\mathcal{M}$  measurable and

$$\sum_{n=1}^{+\infty} \frac{\|\mathbb{E}[S_n(f) | T\mathcal{M}]\|_2}{n^{3/2}} < \infty, \quad (8.1.2)$$

then  $(n^{-1/2}S_n(f))_{n \geq 1}$  converges in distribution to  $\eta^2 N$ , where  $N$  is normally distributed and independent of  $\eta$ . Then Volný [Vol06a] proposed a method to treat the nonadapted case. Peligrad and Utev [PU05] proved the weak invariance principle under condition (8.1.2). The nonadapted case was addressed in [Vol07]. Peligrad and Utev also showed that condition (8.1.2) is optimal among conditions on the growth of the sequence  $(\|S_n(f) | T\mathcal{M}\|_2)_{n \geq 1}$ : if

$$\sum_{n=1}^{+\infty} a_n \frac{\|\mathbb{E}[S_n(f) | T\mathcal{M}]\|_2}{n^{3/2}} < \infty \quad (8.1.3)$$

for some sequence  $(a_n)_{n \geq 1}$  converging to 0, the sequence  $(n^{-1/2}S_n(f))_{n \geq 1}$  is not necessarily stochastically bounded (Theorem 1.2. of [PU05]). Volný constructed [Vol10] an example satisfying (8.1.3) and such that the sequence  $(\|S_n(f)\|_2^{-1} S_n(f))_{n \geq 1}$  admits two subsequences which converge weakly to two different distributions.

We denote by  $\mathcal{H}_\alpha$  the space of Hölder continuous functions, that is, the functions  $x: [0, 1] \rightarrow \mathbb{R}$  such that

$$\|x\|_{\mathcal{H}_\alpha} := \sup_{0 \leq s < t \leq 1} |x(t) - x(s)| / (t - s)^\alpha + |x(0)| \quad (8.1.4)$$

is finite. Since the paths of Brownian motion belong almost surely to  $\mathcal{H}_\alpha$  for each  $\alpha \in (0, 1/2)$  as well as  $W(n, f, \cdot)$ , we can investigate the weak convergence of the sequence  $(n^{-1/2}W(n, f, \cdot))_{n \geq 1}$  in the space  $\mathcal{H}_\alpha$ , for  $0 < \alpha < 1/2$ . The case of i.i.d. sequences and stationary martingale difference sequences have been addressed respectively by Račkauskas and Suquet (Theorem 1 of [RS03]) and Giraudo (Theorem 2.2 of [Gir15c]). In this chapter, we focus on conditions on the sequences  $(\mathbb{E}[S_n(f) | T\mathcal{M}])_{n \geq 1}$  and  $(S_n(f) - \mathbb{E}[S_n(f) | T^{-n}\mathcal{M}])_{n \geq 1}$ .

**Theorem 8.1.1.** *Let  $p > 2$  and  $f \in \mathbb{L}^p$  be a regular function. If*

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}[S_k(f) | T\mathcal{M}]\|_p}{k^{3/2}} < \infty, \quad \sum_{k=1}^{\infty} \frac{\|S_k(f) - \mathbb{E}[S_k(f) | T^{-k}\mathcal{M}]\|_p}{k^{3/2}} < \infty, \quad (8.1.5)$$

then the sequence  $(n^{-1/2}W(n, f))_{n \geq 1}$  converges weakly to the process  $\eta^2 W$  in  $\mathcal{H}_{1/2-1/p}$ , where  $W$  is the Brownian motion and the random variable  $\eta$  is independent of  $W$ .

The expression of  $\eta$  is given in Theorem 1 of [MTK08]. Of course, if  $f$  is  $\mathcal{M}$ -measurable, all the terms of the second series vanish and we only have to check the convergence of the first series.

*Remark 8.1.2.* If the sequence  $(f \circ T^j)_{j \geq 0}$  is a martingale difference sequence with respect to the filtration  $(T^{-i}\mathcal{M})$ , then condition (8.1.5) is satisfied if and only if the function  $f$  belongs to  $\mathbb{L}^p$ , hence we recover the result of [Gir15c]. However, if the sequence  $(f \circ T^j)_{j \geq 0}$  is independent, (8.1.5) is stronger than the sufficient condition  $t^p \mu\{|f| > t\} \rightarrow 0$ . This can be explained by the fact that the key maximal inequality (8.2.7) does not include the quadratic variance term which appears in the martingale inequality. In Remark 1 after Theorem 1 in [PUW07], a version of (8.2.7) with this term is obtained. In our context it seems that it does not follow from an adaptation of the proof.

*Remark 8.1.3.* In [Gir15c], the conclusion of Theorem 8.1.1 was obtained under the condition

$$\sum_{i=1}^{\infty} \|\mathbb{E}[f | T^i\mathcal{M}] - \mathbb{E}[f | T^{i+1}\mathcal{M}]\|_p < \infty. \quad (8.1.6)$$

Using the construction given in [DV08, Dur09], in any ergodic dynamic system of positive entropy one can construct a function satisfying condition (8.1.5) but not (8.1.6) and vice versa.

*Remark 8.1.4.* For the  $\rho$ -mixing coefficient defined by

$$\rho(n) = \sup \{ \text{Cov}(X, Y) / (\|X\|_2 \|Y\|_2), X \in \mathbb{L}^2(\sigma(f \circ T^i, i \leq 0)), Y \in \mathbb{L}^2(\sigma(f \circ T^i, i \geq n)) \},$$

Lemma 1 of [PUW07] shows that for an adapted process, condition (8.1.5) is satisfied if the series  $\sum_{n=1}^{\infty} \rho^{2/p}(2^n)$  converges. However, the conclusion of Theorem 8.1.1 holds if  $t^p \mu\{|f| > t\} \rightarrow 0$  and  $\sum_{n=1}^{\infty} \rho(2^n)$  converges (see Theorem 2.3, [Gir15b]), which is less restrictive.

It turns out that even in the adapted case, condition (8.1.5) is sharp among conditions on  $\|\mathbb{E}[S_k(f) | T\mathcal{M}]\|_p$  in the following sense.

**Theorem 8.1.5.** *For each sequence  $(a_n)_{n \geq 1}$  converging to 0 and each real number  $p > 2$ , there exists a strictly stationary sequence  $(f \circ T^j)_{j \geq 0}$  and a sub- $\sigma$ -algebra  $\mathcal{M}$  such that  $T\mathcal{M} \subset \mathcal{M}$ ,*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}} \|\mathbb{E}[S_n(f) | T\mathcal{M}]\|_p < \infty, \quad (8.1.7)$$

but the sequence  $(n^{-1/2}W(n, f, t))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}$ .

*Remark 8.1.6.* Using the inequalities in [PUW07] in order to bound  $\|\mathbb{E}[S_n(f) \mid T\mathcal{M}]\|_2$ , we can see that the constructed  $f$  in the proof of Theorem 8.1.5 satisfies the classical Maxwell and Woodroffe condition (8.1.2) (the fact that  $p$  is strictly greater than 2 is crucial), hence the weak invariance principle in the space of continuous functions takes place.

However, it remains an open question whether condition (8.1.7) implies the central limit theorem or the weak invariance principle (in the space of continuous functions).

## 8.2 Proofs

The proof of Theorem 8.1.1 will follow the same strategy as in [PU05]. We start by the adapted case. We want to approximate the partial sum process  $(n^{-1/2}W(n, f))_{n \geq 1}$  by a similar process associated to a stationary martingale difference. The approximating martingale is the same as in Section 2.4 of [PU05], and we have to check that it approximates  $(n^{-1/2}W(n, f))_{n \geq 1}$  in the sense of the topology of  $\mathcal{H}_{1/2-1/p}$ . To this aim, we establish a maximal inequality which allows to control the  $\mathbb{L}^{p, \infty}$ -norm of the Hölderian norm of the function  $t \mapsto W(n, f, T)$ . We then exploit ideas of [KV07] to address the non-adapted case.

Notice that condition (8.1.5) implies by Theorem 1 of [PUW07] that the sequence  $(n^{-1/2}S_n(f))_{n \geq 1}$  is bounded in  $\mathbb{L}^p$ ; nevertheless the counter-example given in Theorem 2.5 of [Gir15b] shows that we cannot deduce the weak invariance principle from this.

### 8.2.1 A maximal inequality

For  $p > 2$ , we define

$$\|h\|_{p, \infty} := \sup_{\substack{A \in \mathcal{F} \\ \mu(A) > 0}} \frac{1}{\mu(A)^{1-1/p}} \mathbb{E}[|h| \mathbf{1}_A]. \quad (8.2.1)$$

This norm is linked to the tail function of  $h$  by the following inequalities:

$$\left( \sup_{t > 0} t^p \mu\{|h| > t\} \right)^{1/p} \leq \|h\|_{p, \infty} \leq \frac{p}{p-1} \left( \sup_{t > 0} t^p \mu\{|h| > t\} \right)^{1/p}. \quad (8.2.2)$$

As a consequence, if  $N$  is an integer and  $h_1, \dots, h_n$  are functions, then

$$\left\| \max_{1 \leq j \leq N} |h_j| \right\|_{p, \infty} \leq \frac{p}{p-1} N^{1/p} \max_{1 \leq j \leq N} \|h_j\|_{p, \infty}. \quad (8.2.3)$$

For a positive  $n \geq 1$ , a function  $f: \Omega \rightarrow \mathbb{R}$  and a measure-preserving map  $\theta$ , we define

$$M(n, f, \theta) := \max_{0 \leq i < j \leq n} \frac{|S_j(\theta, f) - S_i(\theta, f)|}{(j-i)^{1/2-1/p}}. \quad (8.2.4)$$

By Lemma A.2 of [MSR12], the Hölderian norm of polygonal line is reached at two vertices, hence

$$M(n, f, \theta) = n^{1/2-1/p} \|W(n, f, \theta, \cdot)\|_{\mathcal{H}_{1/2-1/p}} \quad (8.2.5)$$

Applying Proposition 2.3 of [Gir15c], we can find for each  $p > 2$  a constant  $C_p$  depending only on  $p$  such that if  $(m \circ T^i)_{i \geq 1}$  is a martingale difference sequence, then for each  $n$ ,

$$\|M(n, m, T)\|_{p, \infty} \leq C_p n^{1/p} \|m\|_p. \quad (8.2.6)$$

In the sequel, fix such a constant  $C_p$  and define  $K_p := 4 + 2^{1/p}$ .

The goal of this subsection is to establish the following maximal inequality.

**Proposition 8.2.1.** *Let  $r$  be a positive integer. For each measure-preserving map  $T: \Omega \rightarrow \Omega$  bijective and bi-measurable, each sub- $\sigma$ -algebra  $\mathcal{M}$  satisfying  $T\mathcal{M} \subset \mathcal{M}$ , each  $\mathcal{M}$ -measurable function  $f: \Omega \rightarrow \mathbb{R}$  and each integer  $n$  satisfying  $2^{r-1} \leq n < 2^r$ ,*

$$\|M(n, f, T)\|_{p, \infty} \leq C_p n^{1/p} \left( \|f - \mathbb{E}[f | T\mathcal{M}]\|_p + K_p \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E}[S_{2^j}(f) | T\mathcal{M}]\|_p \right). \quad (8.2.7)$$

The proof is in the same spirit as the proof of Theorem 1 of [PUW07], which is done by dyadic induction. To do so, we start from the following lemma:

**Lemma 8.2.2.** *For each positive integer  $n$ , each function  $h: \Omega \rightarrow \mathbb{R}$  and each measure-preserving map  $T: \Omega \rightarrow \Omega$ , the following inequality holds:*

$$M(n, h, T) \leq 6 \max_{0 \leq k \leq n} |h \circ T^k| + \frac{1}{2^{1/2-1/p}} M\left(\left\lfloor \frac{n}{2} \right\rfloor, h + h \circ T, T^2\right). \quad (8.2.8)$$

*Proof.* First, notice that if  $1 \leq j \leq n$ , then  $j = 2 \lfloor \frac{j}{2} \rfloor$  or  $j = 2 \lfloor \frac{j}{2} \rfloor + 1$ , hence

$$\left| S_j(h) - S_{2 \lfloor \frac{j}{2} \rfloor}(h) \right| \leq \max_{0 \leq k \leq n} |h \circ T^k|. \quad (8.2.9)$$

Similarly, we have

$$\left| S_i(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right| \leq 2 \max_{0 \leq k \leq n} |h \circ T^k|. \quad (8.2.10)$$

It thus follows that

$$M(n, h, T) \leq 4 \max_{0 \leq k \leq n} |h \circ T^k| + \max_{0 \leq i < j \leq n} \frac{\left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|}{(j-i)^{1/2-1/p}}. \quad (8.2.11)$$

Notice that if  $j \geq i + 4$ , then

$$1 \leq \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{i+2}{2} \right\rfloor \leq \frac{j-i}{2}, \quad (8.2.12)$$

and we derive the bound

$$\begin{aligned} \max_{0 \leq i < j \leq n} \frac{\left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|}{(j-i)^{1/2-1/p}} &\leq \frac{1}{2^{1/2-1/p}} \max_{0 \leq u < v \leq \lfloor \frac{n}{2} \rfloor} \frac{\left| S_v(T^2, h + h \circ T) - S_u(T^2, h + h \circ T) \right|}{(v-u)^{1/2-1/p}} + \\ &+ \max_{\substack{0 \leq i < j \leq n \\ j \leq i+4}} \left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|. \end{aligned}$$

Since for  $j \leq i + 4$ , the number of terms of the form  $h \circ T^q$  involved in  $S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h)$  is at most 2, we conclude that

$$\begin{aligned} \max_{0 \leq i < j \leq n} \frac{\left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|}{(j-i)^{1/2-1/p}} &\leq \frac{1}{2^{1/2-1/p}} M\left(\left\lfloor \frac{n}{2} \right\rfloor, h + h \circ T, T^2\right) + \\ &+ 2 \max_{0 \leq k \leq n} |h \circ T^k|. \end{aligned}$$

Combining this inequality with (8.2.11), we obtain (8.2.8), hence the proof of Lemma 8.2.2 is complete.  $\square$

Now, we establish inequality (8.2.7) by induction on  $r$ .

*Proof of Proposition 8.2.1.* We check the case  $r = 1$ . Then necessarily  $n = 1$  and the expression  $M(n, f, t)$  reduces to  $f$ . Since  $C_p$  and  $K_p$  are greater than 1, the result is a simple consequence of the triangle inequality applied to  $f - \mathbb{E}[f | T\mathcal{M}]$  and  $\mathbb{E}[f | T\mathcal{M}]$ .

Now, assume that Proposition 8.2.1 holds for some  $r$  and let us show that it takes place for  $r + 1$ . We thus consider an integer  $n$  such that  $2^r \leq n < 2^{r+1}$ , a function  $f: \Omega \rightarrow \mathbb{R}$ , for each measure-preserving map  $T: \Omega \rightarrow \Omega$  bijective and bi-measurable, and a sub- $\sigma$ -algebra  $\mathcal{M}$  satisfying  $T\mathcal{M} \subset \mathcal{M}$  and we have to show that (8.2.7) holds with  $r + 1$  instead of  $r$ . First, using inequality  $M(n, f, t) \leq M(n, f - \mathbb{E}[f | T\mathcal{M}]) + M(n, \mathbb{E}[f | T\mathcal{M}])$  and Lemma 8.2.2 with  $h := \mathbb{E}[f | T\mathcal{M}]$ , we derive

$$M(n, f, T) \leq M(n, f - \mathbb{E}[f | T\mathcal{M}], T) + 6 \max_{0 \leq k \leq n} |\mathbb{E}[f | T\mathcal{M}] \circ T^k| + \frac{1}{2^{1/2-1/p}} M\left(\left[\frac{n}{2}\right], \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T, T^2\right), \quad (8.2.13)$$

hence taking the norm  $\|\cdot\|_{p,\infty}$ , we obtain by (8.2.2) that

$$\|M(n, f, T)\|_{p,\infty} \leq \|M(n, f - \mathbb{E}[f | T\mathcal{M}], T)\|_{p,\infty} + 6(n+1)^{1/p} \frac{p}{p-1} \|\mathbb{E}[f | T\mathcal{M}]\|_p + \frac{1}{2^{1/2-1/p}} \left\| M\left(\left[\frac{n}{2}\right], \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T, T^2\right) \right\|_{p,\infty}. \quad (8.2.14)$$

By inequality (8.2.6) and accounting the fact that  $6 \cdot (n+1)^{1/p} p / (p-1) \leq C_p n^{1/p}$ , we obtain

$$\|M(n, f, T)\|_{p,\infty} \leq C_p n^{1/p} \|f - \mathbb{E}[f | T\mathcal{M}]\|_p + C_p n^{1/p} \|\mathbb{E}[f | T\mathcal{M}]\|_p + \frac{1}{2^{1/2-1/p}} \left\| M\left(\left[\frac{n}{2}\right], \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T, T^2\right) \right\|_{p,\infty}. \quad (8.2.15)$$

Since  $2^{r-1} \leq [n/2] < 2^r$ , we may apply the induction hypothesis to the integer  $[n/2]$ , the function  $h := \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T$ , the operator  $T^2$  and the  $\sigma$ -algebra  $T\mathcal{M}$ . This gives

$$\begin{aligned} \left[\frac{n}{2}\right]^{-1/p} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty} &\leq C_p \|h - \mathbb{E}[h | T^3\mathcal{M}]\|_p + \\ &+ C_p K_p \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E}[S_{2^j}(T^2, h) | T^3\mathcal{M}]\|_p. \end{aligned} \quad (8.2.16)$$

Notice that  $\|h - \mathbb{E}[h | T^3\mathcal{M}]\|_p \leq 2\|h\|_p \leq 4\|\mathbb{E}[f | T\mathcal{M}]\|_p$ , and that  $S_{2^j}(T^2, h) = S_{2^{j+1}}(\mathbb{E}[f | T\mathcal{M}])$ , hence using the fact that  $T^3\mathcal{M} \subset T\mathcal{M}$ , we derive

$$\begin{aligned} \left[\frac{n}{2}\right]^{-1/p} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty} &\leq 4C_p \|\mathbb{E}[f | T\mathcal{M}]\|_p + \\ &+ C_p K_p \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E}[S_{2^{j+1}}(T, \mathbb{E}[f | T\mathcal{M}]) | T\mathcal{M}]\|_p \\ &= 4C_p \|\mathbb{E}[f | T\mathcal{M}]\|_p \\ &+ 2^{1/2} C_p K_p \sum_{j=1}^r 2^{-j/2} \|\mathbb{E}[S_{2^j}(T, \mathbb{E}[f | T\mathcal{M}]) | T\mathcal{M}]\|_p \end{aligned}$$

and we infer

$$\begin{aligned} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty} &\leq \left(\frac{n}{2}\right)^{1/p} (4 - K_p) C_p \|\mathbb{E}[f | T\mathcal{M}]\|_p \\ &+ n^{1/p} 2^{1/2-1/p} C_p K \sum_{j=0}^r 2^{-j/2} \|\mathbb{E}[S_{2^j}(T, f) | T\mathcal{M}]\|_p. \end{aligned} \quad (8.2.17)$$



Plugging this into (8.2.15), we derive

$$\begin{aligned} \|M(n, f, T)\|_{p, \infty} &\leq C_p n^{1/p} \|f - \mathbb{E}[f | T\mathcal{M}]\|_p + C_p n^{1/p} (1 + (4 - K_p)2^{-1/p}) \|\mathbb{E}[f | T\mathcal{M}]\|_p + \\ &\quad + n^{1/p} C_p K_p \sum_{j=0}^r 2^{-j/2} \|\mathbb{E}[S_{2^j}(T, f) | T\mathcal{M}]\|_p \end{aligned} \quad (8.2.18)$$

The definition of  $K_p$  implies that  $1 + (4 - K_p)2^{-1/p} = 0$ , hence (8.2.7) is established. This concludes the proof of Proposition 8.2.1.  $\square$

## 8.2.2 Martingale approximation

In this section, we recall the construction of the approximating martingale given in [PU05] and we shall derive tightness of  $(n^{-1/2}W(n, f, T))_{n \geq 1}$  in  $\mathcal{H}_{1/2-1/p}$ .

For a fixed positive integer  $r$  we define the functions

$$f_r := \sum_{j=0}^{r-1} f \circ T^j, \quad m_r := \frac{1}{\sqrt{r}}(f_r - \mathbb{E}[f_r | T^r \mathcal{M}]). \quad (8.2.19)$$

Notice that  $\mathbb{E}[m_r | T^r \mathcal{M}] = 0$ , hence the sequence  $(m_r \circ T^{ir})_{i \geq 0}$  is a strictly stationary martingale difference sequence for the filtration  $(T^{-ir} \mathcal{M})_{i \geq 0}$ . Therefore, by Theorem 2.2 of [Gir15c], the process  $n^{-1/2}W(n, m_r, T^r)$  converges in distribution in  $\mathcal{H}_{1/2-1/p}$  to  $\eta_r W$ , where  $\eta_r$  is independent of the Wiener process  $W$ . By the arguments after equation (12) in [PU05], the convergence  $\lim_{r \rightarrow \infty} \|\sqrt{\eta_r} - \sqrt{\eta}\|_2 = 0$  takes place. Therefore, we have to check in view of Theorem 4.2 of [Bil68] that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} c(r, n) = 0, \quad (8.2.20)$$

where

$$c(r, n) := \left\| \left\| \frac{1}{\sqrt{n}} W(n, f, T) - \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor}} W\left(\left\lfloor \frac{n}{r} \right\rfloor, m_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} \quad (8.2.21)$$

For each  $r, n \geq 1$ , we have

$$\begin{aligned} c(r, n) &\leq \left(1 - \sqrt{\frac{n}{\lfloor \frac{n}{r} \rfloor r}}\right) \frac{1}{\sqrt{n}} \left\| \|W(n, f, T)\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} + \\ &\quad + \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \|W(n, f, T) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, \sqrt{r} \cdot m_r, T^r\right)\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty}. \end{aligned} \quad (8.2.22)$$

By Proposition 8.2.1, we have

$$\begin{aligned} &\left(1 - \sqrt{\frac{n}{\lfloor \frac{n}{r} \rfloor r}}\right) \frac{1}{\sqrt{n}} \left\| \|W(n, f, T)\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} \leq \\ &\leq \left(1 - \sqrt{\frac{n}{\lfloor \frac{n}{r} \rfloor r}}\right) C_p \left( \|f - \mathbb{E}[f | T\mathcal{M}]\|_p + K_p \sum_{j=0}^{+\infty} 2^{-j/2} \|\mathbb{E}[S_{2^j}(f) | T\mathcal{M}]\|_p \right), \end{aligned} \quad (8.2.23)$$

hence it suffices to show that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} c'(r, n) = 0, \quad (8.2.24)$$

where

$$c'(r, n) := \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \left\| W(n, f, T) - W\left(\left[\frac{n}{r}\right], \sqrt{r} \cdot m_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty}. \quad (8.2.25)$$

By (8.2.19), we have  $\sqrt{r} \cdot m_r = f_r - \mathbb{E}[f_r \mid T^r \mathcal{M}]$  hence

$$\begin{aligned} c'(r, n) &\leq \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \left\| W(n, f, T) - W\left(\left[\frac{n}{r}\right], f_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} + \\ &+ \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \left\| W\left(\left[\frac{n}{r}\right], \mathbb{E}[f_r \mid T^r \mathcal{M}], T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} =: a(r, n) + b(r, n). \end{aligned} \quad (8.2.26)$$

We shall use the following elementary lemma several times.

**Lemma 8.2.3.** *Let  $p > 2$  and let  $f$  be a function in  $\mathbb{L}^p$ . Then*

$$\lim_{n \rightarrow \infty} n^{-1/p} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty} = 0. \quad (8.2.27)$$

*Proof.* Let  $R > 0$  be fixed; then

$$\max_{0 \leq k \leq n} |f \circ T^k| \leq R + \max_{0 \leq k \leq n} |(f \mathbf{1}_{\{|f| > R\}}) \circ T^k|,$$

hence using inequality (8.2.3), we get

$$n^{-1/p} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty} \leq \frac{R}{n^{1/p}} + \frac{p}{p-1} \left(\frac{n+1}{n}\right)^{1/p} \|f \mathbf{1}_{\{|f| > R\}}\|_{p, \infty}.$$

Taking the lim sup as  $n$  goes to infinity and using the fact that  $\|g\|_{p, \infty} \leq \|g\|_p$  for any function  $g$ , we infer that

$$\limsup_{n \rightarrow \infty} n^{-1/p} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty} \leq \frac{p}{p-1} \|f \mathbf{1}_{\{|f| > R\}}\|_p.$$

We conclude by monotone convergence as  $R$  is arbitrary.  $\square$

- *Control of  $a(r, n)$ :* let us define for  $n \geq r$  the sets

$$I := \left\{ \frac{i}{n} \mid i \in \{0, \dots, n\} \right\} \quad \text{and} \quad J := \left\{ \frac{j}{\lfloor \frac{n}{r} \rfloor} \mid j \in \{0, \dots, \lfloor \frac{n}{r} \rfloor\} \right\}. \quad (8.2.28)$$

Notice that the random function  $W(n, f, T) - W(\lfloor n/r \rfloor, f_r, T^r)$  is piecewise linear, and the vertices of its graph are at points of abscissa in  $I \cup J$ , hence

$$\begin{aligned} a(r, n) &= \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ s \neq t}} \left\{ \frac{|W(n, f, T, t) - W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, t)|}{|s - t|^{1/2-1/p}} - \right. \right. \\ &\quad \left. \left. - \frac{|W(n, f, T, s) - W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, s)|}{|s - t|^{1/2-1/p}} \right\} \right\|_{p, \infty} =: \max \{a'(r, n), a''(r, n)\}, \end{aligned} \quad (8.2.29)$$

where in  $a'(r, n)$  (respectively  $a''(r, n)$ ), the supremum is restricted to the  $s, t \in I \cup J$  such that  $|t - s| \geq 1/n$  (respectively  $< 1/n$ ), which entails

$$a'(r, n) \leq 2 \frac{n^{1/2-1/p}}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \sup_{t \in I \cup J} \left\| W(n, f, T, t) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r, t\right) \right\|_{p, \infty}. \quad (8.2.30)$$

For each  $i \in \{0, \dots, n\}$ , we have

$$\left| W\left(n, f, T, \frac{i}{n}\right) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r, \frac{i}{n}\right) \right| = \left| \sum_{l=0}^{i-1} f \circ T^l - \sum_{l=0}^{\lfloor \frac{n}{r} \rfloor \frac{i}{n} - 1} f_r \circ T^{lr} - \right. \quad (8.2.31)$$

$$\left. - \left( \left\lfloor \frac{n}{r} \right\rfloor \frac{i}{n} - \left\lfloor \frac{n}{r} \right\rfloor \frac{i}{n} \right) f_r \circ T^{r \lfloor \frac{n}{r} \rfloor \frac{i}{n}} \right| \\ \leq \left| \sum_{l=r \lfloor \frac{n}{r} \rfloor \frac{i}{n}}^{i-1} f \circ T^l \right| + r \max_{0 \leq k \leq n} |f \circ T^k|, \quad (8.2.32)$$

and since the number of indices in the sum is at most  $r(1 + 1/n) \leq 2r$ , we derive that

$$\sup_{t \in I} \left| W(n, f, T, t) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r, t\right) \right| \leq 3r \max_{0 \leq k \leq n} |f \circ T^k|. \quad (8.2.33)$$

Treating the supremum over  $J$  in a similar way, we obtain, in view of (8.2.30),

$$a'(r, n) \leq 6\sqrt{r} \cdot \frac{n^{1/2-1/p}}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty}, \quad (8.2.34)$$

hence

$$\limsup_{n \rightarrow \infty} a'(r, n) \leq 6r \limsup_{n \rightarrow \infty} \frac{1}{n^{1/p}} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty}. \quad (8.2.35)$$

By Lemma 8.2.3, it follows that

$$\limsup_{n \rightarrow \infty} a'(r, n) = 0. \quad (8.2.36)$$

Next, we bound  $a''(r, n)$  by

$$\frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ |t-s| \leq 1/n}} \frac{|W(n, f, T, t) - W(n, f, T, s)|}{|s-t|^{1/2-1/p}} \right\|_{p, \infty} \\ + \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ |t-s| \leq 1/n}} \frac{|W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, t) - W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, s)|}{|s-t|^{1/2-1/p}} \right\|_{p, \infty}. \quad (8.2.37)$$

Let  $s, t \in I \cup J$  be such that  $0 < t - s < 1/n$ . Then we either have  $k \leq ns < nt \leq k + 1$  or  $k - 1 \leq ns \leq k < nt < k + 1$  for some  $k \in \{0, \dots, n - 1\}$ . In the first case,

$$\frac{|W(n, f, T, t) - W(n, f, T, s)|}{|s-t|^{1/2-1/p}} = (nt - ns) \frac{|f \circ T^k|}{|s-t|^{1/2-1/p}} \\ \leq n(t-s)^{1-(1/2-1/p)} |f \circ T^k| \\ \leq n^{1/2-1/p} \max_{0 \leq j \leq n} |f \circ T^j|,$$

and in the second one, we have

$$\begin{aligned} \frac{|W(n, f, t) - W(n, f, s)|}{|s - t|^{1/2-1/p}} &\leq \frac{|W(n, f, t) - W(n, f, k/n)|}{|t - k/n|^{1/2-1/p}} + \frac{|W(n, f, k/n) - W(n, f, s)|}{|k/n - s|^{1/2-1/p}} \\ &= \frac{(nt - k) |f \circ T^k|}{|t - k/n|^{1/2-1/p}} + \frac{(k - ns) |f \circ T^{k-1}|}{(k/n - s)^{1/2-1/p}} \\ &\leq 2n^{1/2-1/p} \max_{0 \leq j \leq n} |f \circ T^j|. \end{aligned}$$

As a consequence, the following inequality holds:

$$\frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ |t-s| \leq 1/n}} \frac{|W(n, f, t) - W(n, f, s)|}{|s - t|^{1/2-1/p}} \right\|_{p, \infty} \leq \frac{2\sqrt{n}}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \max_{0 \leq j \leq n} |f \circ T^j| \right\|_{p, \infty} n^{-1/p}. \quad (8.2.38)$$

Using a similar bound for the second term in (8.2.37), we obtain by Lemma 8.2.3, that for each  $r \geq 1$ ,

$$\lim_{n \rightarrow \infty} a''(r, n) = 0. \quad (8.2.39)$$

By (8.2.29), (8.2.36) and (8.2.39), we finally obtain

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} a(r, n) = 0. \quad (8.2.40)$$

- *Control of  $b(r, n)$* : by Proposition 8.2.1, we have the following upper bound:

$$b(n, r) \leq C_p \left( \frac{2}{\sqrt{r}} \|\mathbb{E}[S_r(f) | \mathcal{TM}]\|_p + \frac{K_p}{\sqrt{r}} \sum_{j=0}^{+\infty} 2^{-j/2} \|\mathbb{E}[S_{r2^j}(f) | \mathcal{TM}]\|_p \right). \quad (8.2.41)$$

To conclude, we recall Lemma 2.8 of [PU05]:

**Lemma 8.2.4.** *Let  $(V_n)_{n \geq 1}$  be a subadditive sequence such that  $\sum_{n=1}^{\infty} V_n n^{-3/2} < \infty$ . Then*

$$\lim_{r \rightarrow \infty} \frac{1}{\sqrt{r}} \sum_{k=0}^{+\infty} \frac{V_r 2^k}{2^{k/2}} = 0. \quad (8.2.42)$$

In particular,  $V_r/\sqrt{r} \rightarrow 0$  as  $r \rightarrow \infty$ .

Since the sequence  $(\|\mathbb{E}[S_n(f) | \mathcal{TM}]\|_p)_{n \geq 1}$  is subadditive, from inequality (8.2.41) and Lemma 8.2.4 we derive

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} b(n, r) = 0. \quad (8.2.43)$$

Combining (8.2.40) with (8.2.43), we obtain (8.2.24).

This concludes the proof of Theorem 8.1.1 in the adapted case.

### 8.2.3 The non-adapted case

In [Vol06a], a method to prove the central limit theorem under the condition

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}[S_k(f) | \mathcal{TM}]\|_2}{k^{3/2}} < \infty, \quad \sum_{k=1}^{\infty} \frac{\|S_k(f) - \mathbb{E}[S_k(f) | T^{-k}\mathcal{M}]\|_2}{k^{3/2}} < \infty \quad (8.2.44)$$

is proposed. The idea is the following: one writes  $f = f' + f''$ , where  $f' = \mathbb{E}[f \mid T\mathcal{M}]$  and applies a transformation  $V$  to the process  $(U^i f'')$  in such a way that  $(U^i V f'')$  is a adapted sequence. The mapping  $V$  is defined as

$$Vh := \sum_{i \in \mathbb{Z}} U^{-i} P_0 U^{-i} h, \quad (8.2.45)$$

where  $P_0(h) := \mathbb{E}[h \mid \mathcal{M}] - \mathbb{E}[h \mid T\mathcal{M}]$ .

Notice that the operator  $V$  is not necessarily a point mapping (see section 3 of [KV07]). Therefore, deducing the non-adapted case from the adapted one is not immediate.

Volný proved the functional central limit theorem under (8.2.44) in [Vol07]. The idea is to write the maximal inequality (5) in [PU05] with the notion of contraction. We follow this approach and begin by recalling the definition of contraction operators. Let  $H$  be a subspace of  $\mathbb{L}^p$  for which  $UH \subset H$ . We associate to the operator  $U$  a semigroup of contraction operators  $(P_{T^k})_{k \geq 1}$  on  $H$  which satisfy:

1.  $P_{T^k} = P_T^k$  for each  $k \geq 1$ ;
2.  $P_T U = I$  (where  $I$  is the identity operator);
3. if  $P_T f = 0$ , then  $(U^i f)_{i \geq 0}$  is a martingale difference sequence.

Writing  $P_T =: P$ , we are able to write Proposition 8.2.1 in a more general form.

**Proposition 8.2.5.** *There exist constants  $C_p$  and  $K_p$  depending only on  $p$  such that for each  $f \in H$  and each  $n \geq 1$ ,*

$$\|M(n, f, T)\|_{p, \infty} \leq C_p n^{1/p} \left( \|f - U^{-1} P(f)\|_p + K_p \sum_{j=1}^{\lfloor \log_2 n \rfloor} 2^{-j/2} \left\| \sum_{i=0}^{2^j-1} P^i f \right\|_p \right). \quad (8.2.46)$$

The proof can be done in a similar way as that of Proposition 8.2.1. The later corresponds to the particular case  $H = \mathbb{L}^p(\mathcal{M})$ ,  $P_T(f) = \mathbb{E}[Uf \mid \mathcal{M}]$  and the operator  $U$  is then replaced by  $U^{-1}$ . From this, we may deduce the following:

**Corollary 8.2.6.** *Let  $f \in H$  be such that*

$$\sum_{n=1}^{\infty} \frac{\|\sum_{i=1}^n P_T^i f\|_p}{n^{3/2}} < \infty. \quad (8.2.47)$$

*Then the sequence  $(n^{-1/2} W(n, f, T))_{n \geq 1}$  is tight in  $\mathcal{H}_{1/2-1/p}$ . In particular, if  $f \in \mathbb{L}^p$  is  $\mathcal{M}_\infty$ -measurable,  $\mathbb{E}[f \mid \mathcal{M}] = 0$  and*

$$\sum_{k=1}^{\infty} \frac{\|S_k(f) - \mathbb{E}[S_k(f) \mid T^{-k}\mathcal{M}]\|_p}{k^{3/2}} < \infty, \quad (8.2.48)$$

*then the sequence  $(n^{-1/2} W(n, f, T))_{n \geq 1}$  is tight in  $\mathcal{H}_{1/2-1/p}$ .*

*Proof.* We define  $H := \{h \in \mathbb{L}^p(\mathcal{M}_\infty), \mathbb{E}[h \mid \mathcal{M}] = 0\}$  and  $P_{T^k} h := U^{-k} h - \mathbb{E}[U^{-k} h \mid \mathcal{M}]$ . It is checked in the proof of Proposition 2 of [Vol07] that such a  $P_T$  satisfies the conditions (1)-(2) and (3) of the definition of a semigroup of contractions. We then conclude in a similar way as in the adapted case.  $\square$

*End of the proof of Theorem 8.1.1.* The proof of the convergence of the finite-dimensional distributions under condition (8.2.44) is addressed in Theorem 1 of [Vol07]. It remains to check tightness. We define  $f' := \mathbb{E}[f \mid \mathcal{M}]$  and  $f'' := f - \mathbb{E}[f \mid \mathcal{M}]$  and we have to check that the sequences  $(n^{-1/2} W(n, f', T))_{n \geq 1}$  and  $(n^{-1/2} W(n, f'', T))_{n \geq 1}$  are tight in  $\mathcal{H}_{1/2-1/p}$ . Tightness of the first sequence follows from the results of Subsection 8.2.2. That of the second sequence is a consequence of Corollary 8.2.6. This concludes the proof of Theorem 8.1.1.  $\square$

### 8.2.4 Counter-example

We take a similar construction as in the proof of Proposition 1 of [PUW07]. We consider a non-negative sequence  $(a_n)_{n \geq 1}$ , and a sequence  $(u_k)_{k \geq 1}$  of real numbers such that

$$u_1 = 1, u_2 = 2, u_k^{p/2+1} + 1 < u_{k+1} \text{ for } k \geq 3 \text{ and } a_t \leq k^{-2} \text{ for } t \geq u_k. \quad (8.2.49)$$

Notice that since  $p > 2$ , the conditions (8.2.49) are more restrictive than that of the proof of Proposition 1 of [PUW07]. If  $i = u_j$  for some  $j \geq 1$ , then we define  $p_i := cj/u_j^{1+p/2}$  and  $p_i = 0$  otherwise. Let  $(Y_k)_{k \geq 0}$  be a discrete time Markov chain with the state space  $\mathbb{Z}^+$  and transition matrix given by  $p_{k,k-1} = 1$  for  $k \geq 1$  and  $p_{0,j-1} := p_j$ ,  $j \geq 1$ . We shall also consider a random variable  $\tau$  which takes its values among non-negative integers, and whose distribution is given by  $\mu(\tau = j) = p_j$ . Then the stationary distribution exists and is given by

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i, j \geq 1, \text{ where } \pi_0 = 1/\mathbb{E}[\tau]. \quad (8.2.50)$$

We start from the stationary distribution  $(\pi_j)_{j \geq 0}$  and we take  $g(x) := \mathbf{1}_{x=0} - \pi_0$ , where  $\pi_0 = \mu\{Y_0 = 0\}$ . We then define  $f \circ T^j = X_j := g(Y_j)$ .

It is already checked in [PUW07] that the sequence  $(X_j)_{j \geq 0}$  satisfies (8.1.7), where  $\mathcal{M} = \sigma(X_k, k \leq j)$  and  $S_n = \sum_{j=1}^n X_j$ . To conclude the proof, it remains to check that the sequence  $(n^{-1/2}W(n, f, T))_{n \geq 1}$  is not tight in  $\mathcal{H}_{1/2-1/p}$ . To this aim, we define

$$T_0 = 0, T_k = \min\{t > T_{k-1} \mid Y_t = 0\}, \quad \tau_k = T_k - T_{k-1}, k \geq 1. \quad (8.2.51)$$

Then  $(\tau_k)_{k \geq 1}$  is an independent sequence and each  $\tau_k$  is distributed as  $\tau$  and

$$S_{T_k} = \sum_{j=1}^k (1 - \pi_0 \tau_j) = k - \pi_0 T_k. \quad (8.2.52)$$

Let us fix some integer  $K$  greater than  $\mathbb{E}[\tau]$ . Let  $\delta > 0$  be fixed and  $n$  an integer such that  $1/n < \delta$ . Then the inequality

$$\begin{aligned} \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} &\geq \frac{1}{(nK)^{1/p}} \mathbf{1}\{T_n \leq Kn\} \times \\ &\times \max_{1 \leq k \leq n} \frac{|S_{T_k} - S_{T_{k-1}}|}{(T_k - T_{k-1})^{1/2-1/p}} \mathbf{1}\{|T_k - T_{k-1}| \leq n\delta\} \end{aligned} \quad (8.2.53)$$

takes place. By (8.2.51) and (8.2.52), this can be rewritten as

$$\begin{aligned} \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} &\geq \frac{1}{(nK)^{1/p}} \mathbf{1}\{T_n \leq Kn\} \times \\ &\times \max_{1 \leq k \leq n} \frac{|1 - \pi_0 \tau_k|}{\tau_k^{1/2-1/p}} \mathbf{1}\{\tau_k \leq n\delta\}. \end{aligned} \quad (8.2.54)$$

Defining for a fixed  $C$  the event

$$A_n(C) := \left\{ \frac{|1 - \pi_0 \tau|}{\tau^{1/2-1/p}} \geq C(Kn)^{1/p} \right\} \cap \{\tau \leq n\delta\}, \quad (8.2.55)$$

we obtain by the remark before equation (8.2.52)

$$\mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq C \right\} \geq 1 - (1 - \mu(A_n(C)))^n - \mu\{T_n > Kn\}. \quad (8.2.56)$$

By the law of large numbers, we obtain, accounting  $K > \mathbb{E}[\tau]$ , that

$$\limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq C \right\} \geq \limsup_{n \rightarrow \infty} 1 - (1 - \mu(A_n(C)))^n. \quad (8.2.57)$$

We choose  $C := \pi_0/(2K^{1/p})$ . Considering the integers  $n$  of the form  $[u_j^{(p+2)/2}]$ , we obtain in view of (8.2.57) :

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq \frac{\pi_0}{2K^{1/p}} \right\} &\geq \\ &\geq \limsup_{j \rightarrow \infty} 1 - \left( 1 - \mu(A_{[u_j^{(p+2)/2}]}(\pi_0/(2K^{1/p}))) \right)^{u_j^{(p+2)/2}}. \end{aligned} \quad (8.2.58)$$

Since  $\tau \geq 1$  almost surely, the following inclusions take place for  $n > (2/\pi_0)^p$ :

$$\begin{aligned} A_n(\pi_0/(2K^{1/p})) &\supset \left\{ \pi_0 \tau^{1/2+1/p} - \tau^{-1/2+1/p} \geq \pi_0/(2K^{1/p})(Kn)^{1/p} \right\} \cap \{\tau \leq n\delta\} \\ &\supset \left\{ \tau^{1/2+1/p} \geq \frac{1 + \pi_0 n^{1/p}/2}{\pi_0} \right\} \cap \{\tau \leq n\delta\} \\ &\supset \left\{ \tau^{1/2+1/p} \geq n^{1/p} \right\} \cap \{\tau \leq n\delta\} \\ &= \left\{ n^{2/(p+2)} \leq \tau \leq n\delta \right\}. \end{aligned}$$

Consequently, for  $j$  large enough,

$$\mu(A_{[u_j^{(p+2)/2}]}(\pi_0/(2K^{1/p}))) \geq \mu \left\{ [u_j^{(p+2)/2}]^{2/(p+2)} \leq \tau \leq [u_j^{(p+2)/2}] \delta \right\} \quad (8.2.59)$$

Since  $\tau$  take only integer values among  $u_l$ 's and  $[u_j^{(p+2)/2}] \delta < u_{j+1}$  (by (8.2.49)), we obtain in view of (8.2.58), that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq \frac{\pi_0}{2K^{1/p}} \right\} &\geq \\ &\geq \limsup_{j \rightarrow \infty} 1 - (1 - \mu\{\tau = u_j\})^{[u_j^{(p+2)/2}]}. \end{aligned} \quad (8.2.60)$$

If a sequence  $(x_k)_k$  is such that  $x_k/k \rightarrow 0$  and  $x_k \rightarrow \infty$  as  $k$  goes to infinity, then  $(1 - x_k/k)^k$  converges to 0. Consequently, the right hand side of (8.2.60) is equal to 1, which finishes the proof of Theorem 8.1.5.

## Part IV

# Orthomartingale-coboundary decomposition





# Integrability conditions on coboundary and transfer function for limit theorems

## 9.1 Introduction and notations

Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space and  $T: \Omega \rightarrow \Omega$  be a bijective bi-measurable and measure preserving map. We assume that the dynamical system is ergodic (that is, if  $T^{-1}A = A$  for some  $A \in \mathcal{F}$ , then  $\mu(A) \in \{0, 1\}$ ). If  $n \geq 1$  is an integer and  $f: \Omega \rightarrow \mathbb{R}$ , we denote  $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$  and for a fixed  $t$ , define

$$S_n^{\text{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}, t \in [0, 1], \quad (9.1.1)$$

where  $[x]$  denote the integer part of the real number  $x$ . Then for each  $\omega \in \Omega$  and each integer  $n \geq 1$ , the map  $t \mapsto S_n^{\text{pl}}(f, t)$  is an element of the space of continuous functions in  $[0, 1]$ , denoted by  $C[0, 1]$ .

Let us state the limit theorems we are interested in.

**Definition 9.1.1.** *Let  $f: \Omega \rightarrow \mathbb{R}$  be a measurable function.*

- We say that the function  $f$  satisfies the invariance principle if the sequence  $(S_n^{\text{pl}}(f, \cdot)/\sqrt{n})_{n \geq 1}$  weakly converges in  $C[0, 1]$  (endowed with the topology given by the supremum norm  $\|x\|_\infty := \sup_{t \in [0, 1]} |x(t)|$ ) to a scalar multiple of a standard Brownian motion.
- We say that the function  $f$  satisfies the law of the iterated logarithm if there exists a constant  $C(f)$  such that for almost every  $\omega \in \Omega$ ,

$$\limsup_{n \rightarrow +\infty} \frac{S_n(f)(\omega)}{\sqrt{n \log \log n}} = C(f) \text{ and } \liminf_{n \rightarrow +\infty} \frac{S_n(f)(\omega)}{\sqrt{n \log \log n}} = -C(f). \quad (9.1.2)$$

- We say that the function  $f$  satisfies the functional law of the iterated logarithm if the sequence  $((\sqrt{n \log \log n})^{-1} S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  is relatively compact and there exists a constant  $C(f)$  such that the set of its limit points coincides with the set of all absolutely continuous functions  $x \in C[0, 1]$  such that  $x(0) = 0$  and  $\int_0^1 (x'(t))^2 dt \leq C(f)$ , where  $x'$  denotes the derivative with respect to the Lebesgue measure.
- Let  $1 < p < 2$ . We say that the function  $f$  satisfies the  $p$ -strong law of large numbers if for any  $\alpha \in [1/p, 1]$  if

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{+\infty} n^{\alpha p - 2} \mu \left\{ \max_{1 \leq k \leq n} |S_k(f)| \geq \varepsilon n^\alpha \right\} < +\infty. \quad (9.1.3)$$

If it is possible to find a decomposition of the function  $f$

$$f = m + g - g \circ T, \quad (9.1.4)$$

where  $g: \Omega \rightarrow \mathbb{R}$  is a measurable function and  $m$  satisfies one of the previous limit theorems, then one can wonder if we can deduce the same result for  $f$ .

A known situation is when the sequence  $(m \circ T^i)_{i \geq 0}$  is a square-integrable martingale difference sequence. A necessary and sufficient condition to have (9.1.4) with such an  $m$  and a square integrable  $g$  is known (see Theorem 2 in [Vol93]). If  $(m \circ T^i)_{i \geq 0}$  is a square-integrable martingale difference sequence, then the functional law of the iterated logarithm and the invariance principle take place.

If  $1 < p < 2$  and  $m \in \mathbb{L}^p$ , then Theorem 5 by Dedecker and Merlevède [DM07] implies that  $m$  satisfies the  $p$ -strong law of large numbers. Actually, their results holds in a more general setting than strictly stationary sequences, as they only require a stochastic domination on the martingale difference sequence  $(X_j)_{j \geq 0}$ . A similar result as (9.1.3) takes place for  $\alpha = 1$  if we require a conditional stochastic domination (see [BQ15], Theorem 2.2). A necessary and sufficient condition for (9.1.4) to hold with  $m, g \in \mathbb{L}^p$ ,  $1 < p < 2$ , is given by Volný in Theorem 1 of [Vol06b], and in this case, (9.1.3) is satisfied (see Theorem 6 of [DM07]).

We call a *coboundary* a function of the form  $g - g \circ T$ , where  $g: \Omega \rightarrow \mathbb{R}$  is a measurable function. The function  $g$  is called a *transfer function*. The following result is Theorem 1 of [VS00]. It gives a necessary and sufficient condition on the transfer function to preserve the limit theorems mentioned in the previous definition. Sufficiency for the invariance principle and the law of the iterated logarithm was established in [HH80], pages 140-142).

**Theorem 9.1.2** (The equivalence theorem, [VS00]). *Let us suppose that for the process  $(m \circ T^i)_{i \in \mathbb{Z}}$  the invariance principle, the law of the iterated logarithm (functional law of the iterated logarithm) respectively, holds true. Let  $g$  be a measurable function and*

$$f = m + g - g \circ T. \quad (9.1.5)$$

Then for the process  $(f \circ T^i)_{i \in \mathbb{Z}}$

- the invariance principle holds if and only if

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} |g \circ T^k| \xrightarrow[n \rightarrow \infty]{} 0 \text{ in probability}; \quad (9.1.6)$$

- the law of the iterated logarithm as well as the functional law of the iterated logarithm holds if and only if

$$\frac{1}{\sqrt{n \log \log n}} g \circ T^n \xrightarrow[n \rightarrow \infty]{} 0 \text{ a.s.} \quad (9.1.7)$$

Both conditions (9.1.6) and (9.1.7) take place when the function  $g$  is square-integrable. If  $1 < p < 2$ , Theorem 6 in [DM07] shows that (9.1.3) holds if  $g$  belongs to  $\mathbb{L}^p$ .

However, it may happen that we obtain a decomposition (9.1.4) where  $m \in \mathbb{L}^2$  but the function  $g$  is only integrable (see [VS00] for explicit counter-examples, and [EJ85] for a condition which guarantees the square integrability of  $m$ ) and in this case, the weak invariance principle does not need to hold. We investigate conditions on the functions  $t \mapsto \mu \{|g| > t\}$  and  $t \mapsto \mu \{|g - g \circ T| > t\}$  which guarantee (9.1.6) or (9.1.7). In order to state these conditions in a more concise way, we introduce the so-called weak  $\mathbb{L}^q$ -spaces.

**Definition 9.1.3.** *Let  $q$  be a real number strictly greater than 1. We denote by  $\mathbb{L}^{q,\infty}$  the space of functions  $h: \Omega \rightarrow \mathbb{R}$  such that*

$$\|h\|_{q,\infty}^q := \sup_{t>0} t^q \mu \{|h| > t\} \text{ is finite.} \quad (9.1.8)$$

The subspace of  $\mathbb{L}^{q,\infty}$  which consists of functions  $h$  such that  $\lim_{t \rightarrow +\infty} t^q \mu \{|h| > t\} = 0$  is denoted by  $\mathbb{L}_0^{q,\infty}$ .

## 9.2 Main results

In this section, we state the main results of this note. In the first subsection, we give a sufficient condition on the functions  $t \mapsto \mu\{|g| > t\}$  and  $t \mapsto \mu\{|g - g \circ T| > t\}$  in order to preserve the weak invariance principle, the law of the iterated logarithm and the  $p$ -strong law of large numbers respectively. In the second subsection, we construct counter-examples which show that the found conditions are sharp when the considered dynamical system is aperiodic.

Volný and Samek showed in [VS00] that the conclusion of Theorems 9.2.1 and 9.2.2 (see the next subsections) holds when  $p \geq (r+2)/r$  and that of Theorem 9.2.6 when  $p < (r-1)/(r-3/2)$ . In the case  $r > 2$ , we cannot conclude if  $(r-1)/(r-3/2) \leq p < (r+2)/r$ .

In this section, we assume that  $(m \circ T^i)_{i \geq 0}$  is a square-integrable martingale difference sequence.

### 9.2.1 Sufficient conditions

**Theorem 9.2.1.** *Let  $1 < p < 2 < r$  be such that  $p \geq r/(r-1)$  and let  $g: \Omega \rightarrow \mathbb{R}$  be a function such that  $g \in \mathbb{L}_0^{p,\infty}$  and  $g - g \circ T \in \mathbb{L}_0^{r,\infty}$ . Then the function  $f = m + g - g \circ T$  satisfies the weak invariance principle in  $C[0, 1]$ .*

A similar result has been obtained for the quenched functional central limit theorem (see [BPP15], Corollary 7).

**Theorem 9.2.2.** *Let  $1 < p < 2 < r$  and let  $g: \Omega \rightarrow \mathbb{R}$  be a function.*

- (i) *If  $p > r/(r-1)$ ,  $g \in \mathbb{L}^{p,\infty}$  and  $g - g \circ T \in \mathbb{L}^{r,\infty}$ , then the function  $f = m + g - g \circ T$  satisfies the law of the iterated logarithm;*
- (ii) *if  $p = r/(r-1)$ ,  $g \in \mathbb{L}^p$  and  $g - g \circ T \in \mathbb{L}^r$ , then the function  $f = m + g - g \circ T$  satisfies the law of the iterated logarithm.*

**Theorem 9.2.3.** *Let  $1 < q < p < r < 2$  be real numbers and let  $g: \Omega \rightarrow \mathbb{R}$  be a function such that  $g \in \mathbb{L}^q$  and  $g - g \circ T \in \mathbb{L}^r$ . If  $q \geq (p-1)r/(r-1)$ , then the function  $f = m + g - g \circ T$  satisfies (9.1.3).*

*Remark 9.2.4.* In [VS00],  $\mathbb{L}^q$  spaces are involved. It turns out that in the setting of Theorems 9.2.1 and 9.2.2, (i), we may work with weak  $\mathbb{L}^q$ -spaces. For the case (ii) in Theorem 9.2.2, it is an open question to determine whether strong moments are actually needed.

*Remark 9.2.5.* Ergodicity of the dynamical system is required for the "only if" direction in the equivalence involving the law of the iterated logarithm of Theorem 9.1.2. Therefore, the results of this subsection remain valid in the non-ergodic setting.

### 9.2.2 Counter-examples

**Theorem 9.2.6.** *Assume that the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  is aperiodic. Let  $1 \leq p < 2 \leq r$  be real numbers such that  $p < r/(r-1)$ . Then there exists a function  $g \in \mathbb{L}^p$  such that  $g - g \circ T \in \mathbb{L}^r$  and the function  $f = m + g - g \circ T$  satisfies neither the invariance principle nor the law of the iterated logarithm.*

**Theorem 9.2.7.** *Assume that the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  is aperiodic. Let  $1 < p < 2$  and let  $1 \leq q < p < r$  be real numbers such that  $q < (p-1)r/(r-1)$ . Then there exists a function  $g \in \mathbb{L}^q$  such that  $g - g \circ T \in \mathbb{L}^r$  but the sequence  $(n^{-1/p} S_n(g - g \circ T))_{n \geq 1}$  does not converge almost surely to 0.*

### 9.3 Proofs

If  $h: \Omega \rightarrow \mathbb{R}$  is a measurable function, we define  $M^*(h) := \sup_{N \geq 1} N^{-1} |S_N(h)|$ .

The following lemma about Birkhoff averages will be used in the proof.

**Lemma 9.3.1.** *Let  $q > 1$  and let  $h: \Omega \rightarrow \mathbb{R}$  be a measurable function.*

- (i) *If  $h$  belongs to  $\mathbb{L}_0^{q,\infty}$  then the function  $M^*(h)$  belongs to  $\mathbb{L}_0^{q,\infty}$ ;*
- (ii) *if  $h$  belongs to  $\mathbb{L}^q$ , then so does  $M^*(h)$ .*

*Proof.* By the maximal ergodic theorem, we have for each positive  $t$ ,

$$t \cdot \mu \{M^*(h) \geq t\} \leq \mathbb{E} [|h| \cdot \mathbf{1} \{M^*(h) \geq t\}]. \quad (9.3.1)$$

- (i) The expectation can be bounded by

$$\int_0^{+\infty} \min \left\{ \mu(A_t), s^{-q} \|h \cdot \mathbf{1}(A_t)\|_{q,\infty} \right\} ds, \quad (9.3.2)$$

where

$$A_t := \{M^*(h) \geq t\}. \quad (9.3.3)$$

Therefore, we infer the bound

$$\mathbb{E} [|h| \cdot \mathbf{1}(A_t)] \leq \mu(A_t)^{1-1/q} \|h \mathbf{1}(A_t)\|_{q,\infty} + \quad (9.3.4)$$

$$\begin{aligned} &+ \|h \cdot \mathbf{1}(A_t)\|_{q,\infty}^q \int_{\|h \cdot \mathbf{1}(A_t)\|_{q,\infty} \mu(A_t)^{-1/q}}^{+\infty} s^{-q} ds \\ &= \mu(A_t)^{1-1/q} \|h \cdot \mathbf{1}(A_t)\|_{q,\infty} + \end{aligned} \quad (9.3.5)$$

$$\begin{aligned} &+ \|h \cdot \mathbf{1}(A_t)\|_{q,\infty}^q \frac{(\|h \cdot \mathbf{1}(A_t)\|_{q,\infty} \mu(A_t)^{-1/q})^{1-q}}{q-1} \\ &= \mu(A_t)^{1-1/q} \|h \cdot \mathbf{1}(A_t)\|_{q,\infty} + \frac{\|h \cdot \mathbf{1}(A_t)\|_{q,\infty} \mu(A_t)^{1-1/q}}{q-1} \end{aligned} \quad (9.3.6)$$

$$= \frac{q}{q-1} \mu(A_t)^{1-1/q} \|h \cdot \mathbf{1}(A_t)\|_{q,\infty}. \quad (9.3.7)$$

Plugging this into (9.3.1), we obtain

$$t \cdot \mu \{M^*(h) \geq t\}^{1/q} \leq \frac{q}{q-1} \|h \cdot \mathbf{1}(A_t)\|_{q,\infty}, \quad (9.3.8)$$

hence it is enough to prove that

$$\lim_{t \rightarrow +\infty} \sup_{s \geq 0} s^q \mu(\{|h| \geq s\} \cap A_t) = 0. \quad (9.3.9)$$

To this aim, fix a positive  $\varepsilon$ ; by assumption, there exists a positive real number  $s_0$  such that for  $s \geq s_0$ , we have  $s^q \mu(\{|h| \geq s\}) \leq \varepsilon$ , hence

$$\sup_{s \geq 0} s^q \mu(\{|h| \geq s\} \cap A_t) \leq \max \{\varepsilon, s_0^q \mu(A_t)\}, \quad (9.3.10)$$

which is smaller than  $\varepsilon$  for  $t$  large enough.

- (ii) This follows by multiplying (9.3.1) by  $t^{q-2}$ , integrating over  $[0, +\infty)$  with respect to the Lebesgue measure and switching the integrals.

This concludes the proof of Lemma 9.3.1.  $\square$

We now give the proofs of the main results, which combine Lemma 9.3.1 with the ideas of [VS00].

### 9.3.1 Proof of sufficient conditions

*Proof of Theorem 9.2.1.* In view of Theorem 9.1.2, we have to show that the sequence

$$\left( n^{-1/2} \max_{1 \leq j \leq n} |g \circ T^j| \right)_{n \geq 1} \quad (9.3.11)$$

converges to 0 in probability.

Let  $\varepsilon$  be a positive fixed number. Let  $k, n$  be positive integers such that  $k < n$ . Denoting  $p_n := \mu \{ \max_{1 \leq j \leq n} |g \circ T^j| > 2\varepsilon n^{1/2} \}$ , the following estimates take place:

$$\begin{aligned} p_n &\leq \mu \left\{ \max_{1 \leq i \leq \left[ \frac{n}{k} \right] + 1} \max_{ik \leq j < (i+1)k} |g \circ T^{ik}| + |g \circ T^j - g \circ T^{ik}| > 2\varepsilon n^{1/2} \right\} \\ &\leq \mu \left\{ \max_{1 \leq i \leq \left[ \frac{n}{k} \right] + 1} |g \circ T^{ik}| > \varepsilon \sqrt{n} \right\} + \sum_{i=1}^{\left[ \frac{n}{k} \right] + 1} \mu \left\{ \max_{ik \leq j < (i+1)k} |g \circ T^j - g \circ T^{ik}| > \varepsilon n^{1/2} \right\} \\ &\leq \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ \max_{0 \leq j < k} |g \circ T^j - g| > \varepsilon n^{1/2} \right\} \\ &\leq \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ \max_{0 \leq j < k} \frac{1}{j} |S_j(g - g \circ T)| > \varepsilon \frac{n^{1/2}}{k} \right\}. \end{aligned}$$

This yields

$$p_n \leq \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ M^*(g - g \circ T) > \varepsilon \frac{n^{1/2}}{k} \right\}. \quad (9.3.12)$$

By the assumption on  $p$  and  $r$ , the inequality

$$\frac{r/2 - 1}{r - 1} = \frac{r - 1 - r/2}{r - 1} = 1 - \frac{r}{2(r - 1)} \geq 1 - \frac{p}{2} \quad (9.3.13)$$

takes place, hence we may choose a number  $\alpha$  such that

$$1 - \frac{p}{2} \leq \alpha \leq \frac{r/2 - 1}{r - 1}. \quad (9.3.14)$$

We now use (9.3.12) with  $k := [n^\alpha]$ . This yields, for some constant  $c$  depending only on  $p$  and  $r$ :

$$p_n \leq c \cdot n^{1-\alpha} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + c \cdot n^{1-\alpha} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\}, \quad (9.3.15)$$

and, by (9.3.14),

$$p_n \leq cn^{p/2} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \quad (9.3.16)$$

$$\begin{aligned} &+ cn^{1-\alpha-r(1/2-\alpha)} n^{(1/2-\alpha)r} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\} \\ &= cn^{p/2} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \quad (9.3.17) \end{aligned}$$

$$\begin{aligned} &+ cn^{1-r/2+\alpha(r-1)} n^{(1/2-\alpha)r} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\} \\ &\leq cn^{p/2} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + cn^{(1/2-\alpha)r} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\}. \quad (9.3.18) \end{aligned}$$

Since  $g \in \mathbb{L}_0^{p,\infty}$  and  $g - g \circ T \in \mathbb{L}_0^{r,\infty}$ , we conclude by item (i) of Lemma 9.3.1 that the sequence  $(p_n)_{n \geq 1}$  converges to 0.

This concludes the proof of Theorem 9.2.1.  $\square$

*Proof of Theorem 9.2.2.* In view of Theorem 9.1.2, we have to prove that the convergence in (9.1.7) takes place. Let  $\alpha \in (0, 1)$  be a number which will be specified later. We define

$$m_j := \sum_{i=1}^{j-1} [i^\alpha], j \geq 1. \quad (9.3.19)$$

Notice that for some constant  $\kappa$  depending only on  $\alpha$ , we have

$$\frac{j^{\alpha+1}}{\kappa} \leq m_j \leq \kappa j^{\alpha+1}, j \geq 1. \quad (9.3.20)$$

By the Borel-Cantelli lemma, we have to prove the convergence of the series

$$\sum_{j=1}^{+\infty} p_j, \text{ with } p_j := \mu \left\{ \max_{0 \leq i \leq [j^\alpha]} \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^{m_j+i}| > \varepsilon \right\} \quad (9.3.21)$$

for each positive  $\varepsilon$ . To this aim, we start from the inequalities

$$\begin{aligned} p_j &\leq \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^{m_j}| > \varepsilon/2 \right\} + \\ &+ \mu \left\{ \max_{0 \leq i \leq [j^\alpha]} \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^{m_j+i} - T^{m_j}| > \varepsilon/2 \right\} \\ &= \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g| > \varepsilon/2 \right\} + \\ &+ \mu \left\{ \max_{0 \leq i \leq [j^\alpha]} \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^i - g| > \varepsilon/2 \right\}, \end{aligned}$$

from which we infer

$$\begin{aligned} p_j &\leq \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g| > \varepsilon/2 \right\} + \\ &+ \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} M^*(g - g \circ T) > \frac{\varepsilon}{2[j^\alpha]} \right\}. \quad (9.3.22) \end{aligned}$$

(i) Assume that  $p > r/(r-1)$ . Using the definition of  $\|\cdot\|_{p,\infty}$  and inequality (9.3.20), we obtain

$$\mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g| > \varepsilon/2 \right\} \leq c(p, \varepsilon, \alpha) \kappa^{p/2} \|g\|_{p,\infty}^p \frac{1}{j^{(\alpha+1)p/2}}, \quad (9.3.23)$$

where  $c(p, \varepsilon, \alpha)$  is independent of  $j$ . Using (9.3.20) and (9.3.8), we derive

$$\begin{aligned} \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} M^*(g - g \circ T) > \frac{\varepsilon}{2[j^\alpha]} \right\} &\leq \\ &\leq c(r, \varepsilon, \alpha) \kappa^{p/2} \|g - g \circ T\|_{r,\infty}^r j^{\alpha r - (\alpha+1)r/2}, \quad (9.3.24) \end{aligned}$$

where  $c(r, \varepsilon, \alpha)$  is independent of  $j$ .

Combining (9.3.22), (9.3.23) and (9.3.24), we deduce the upper bound

$$p_j \leq c(p, r, \alpha, \varepsilon, g) \left( \frac{1}{j^{(\alpha+1)p/2}} + j^{(1-\alpha)r/2} \right). \quad (9.3.25)$$

We have to take  $\alpha$  such that

$$(\alpha + 1)p/2 > 1 \text{ and } (1 - \alpha)r/2 > 1. \quad (9.3.26)$$

This is equivalent to

$$\alpha > 2/p - 1 \text{ and } \alpha < 1 - 2/r. \quad (9.3.27)$$

Since  $2/p - 1 < 1 - 2/r$ , inequalities (9.3.26) are satisfied and in view of (9.3.25), the series defined by (9.3.21) is convergent for any positive  $\varepsilon$ . This concludes part (i) of Theorem 9.2.2.

- (ii) Assume that  $p = r/(r - 1)$ . We pick  $\alpha := 2/p - 1 = 1 - 2/r$ . In this case, for some constant  $c$  depending only on  $p$  and  $r$ , the inequality

$$p_j \leq \mu \left\{ |g| > c\varepsilon j^{1/p} \right\} + \mu \left\{ M^*(g - g \circ T) > \varepsilon c j^{-\alpha + (\alpha + 1)/2} \right\} \quad (9.3.28)$$

$$= \mu \left\{ |g| > c\varepsilon j^{1/p} \right\} + \mu \left\{ M^*(g - g \circ T) > \varepsilon c j^{(1 - \alpha)/2} \right\} \quad (9.3.29)$$

$$= \mu \left\{ |g| > c\varepsilon j^{1/p} \right\} + \mu \left\{ M^*(g - g \circ T) > \varepsilon c j^{1/r} \right\} \quad (9.3.30)$$

takes place. By item (ii) of Lemma 9.3.1, we conclude that the series defined by (9.3.21) is convergent for any positive  $\varepsilon$  and this concludes part (ii) of Theorem 9.2.2, hence the proof of Theorem 9.2.2.  $\square$

*Proof of Theorem 9.2.3.* Let us fix a positive  $\varepsilon$  and  $\alpha \in [1/p, 1]$ . Let  $1 \leq k < n$  be integers. By similar inequalities which led to (9.3.12) (we replace the exponent  $1/2$  by  $\alpha$ ), we derive

$$\begin{aligned} \mu \left\{ \max_{1 \leq j \leq n} |g - g \circ T^j| > \varepsilon n^\alpha \right\} &\leq 2 \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ |g| > \varepsilon n^\alpha / 4 \right\} + \\ &+ 2 \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ M^*(g - g \circ T) > \frac{\varepsilon n^\alpha}{2k} \right\}. \end{aligned} \quad (9.3.31)$$

Let us choose  $k := \lceil n^\beta \rceil$ , where

$$(p - q)\alpha \leq \beta \leq \alpha(r - p)/(r - 1) \quad (9.3.32)$$

(the existence of such a  $\beta$  is guaranteed by the assumptions on  $p$ ,  $q$  and  $r$ ). Then we have to check that for each positive constant  $c$ , the series

$$\Sigma_1 := \sum_{n=1}^{+\infty} n^{p\alpha - 1 - \beta} \mu \left\{ |g| \geq cn^\alpha \right\} \text{ and} \quad (9.3.33)$$

$$\Sigma_2 := \sum_{n=1}^{+\infty} n^{p\alpha - 1 - \beta} \mu \left\{ M^*(g - g \circ T) \geq cn^{\alpha - \beta} \right\} \quad (9.3.34)$$

are convergent. The convergence of  $\Sigma_1$  is equivalent to the integrability of the function  $|g|^{p - \beta/\alpha}$ ; this holds since (9.3.32) implies  $q \geq p - \beta/\alpha$ .

Note that the second series converges if

$$\mathbb{E} \left[ (M^*(g - g \circ T))^{\frac{p\alpha - \beta}{\alpha - \beta}} \right] < +\infty. \quad (9.3.35)$$

Notice that inequality (9.3.32) implies that  $(p\alpha - \beta)/(\alpha - \beta) \leq r$ , hence we derive the convergence of  $\Sigma_2$  by item (ii) of Lemma 9.3.1 (with the exponent  $(p\alpha - \beta)/(\alpha - \beta) > 1$  since  $p > 1$ ).

This concludes the proof of Theorem 9.2.3.  $\square$



### 9.3.2 Counter-examples

*Proof of Theorem 9.2.6.* We recall the construction given in the proof of Theorem 3 of [VS00]. We choose a real number  $\alpha$  such that

$$\frac{r-2}{2(r-1)} < \alpha < 1 - \frac{p}{2}. \quad (9.3.36)$$

This is possible because

$$\begin{aligned} 1 - \frac{p}{2} - \frac{r-2}{2(r-1)} &= \frac{1}{2} \left( 2 - p - \frac{r-1-1}{r-1} \right) = \\ &= \frac{1}{2} \left( 1 - p + \frac{1}{r-1} \right) = \frac{1}{2} \left( \frac{r}{r-1} - p \right) > 0. \end{aligned} \quad (9.3.37)$$

For each  $i \geq 1$ , we define  $n_i := 2^i$  and  $k_i := \lceil 2^{i\alpha} \rceil$ . By the Rokhlin lemma, one can find a set  $A_i \in \mathcal{F}$  such that

$$\text{sets } A_i, TA_i, \dots, T^{n_i-1}A_i \text{ are pairwise disjoint and} \quad (9.3.38)$$

$$\mu \left( \bigcup_{j=0}^{n_i-1} T^j A_i \right) > 1/2. \quad (9.3.39)$$

In particular, the quantity  $\mu(A_i)$  can be bounded as follows:

$$\frac{1}{2n_i} \leq \mu(A_i) \leq \frac{1}{n_i}. \quad (9.3.40)$$

We then define for  $i \geq 1$ ,

$$g_i := \frac{\sqrt{n_i \log \log n_i}}{k_i} \left( \sum_{j=1}^{k_i} j \mathbf{1}(T^{n_i-j} A_i) + \sum_{j=k_i+1}^{2k_i-1} (2k_i - j) \mathbf{1}(T^{n_i-j} A_i) \right), \quad (9.3.41)$$

and  $g := \sum_{i=1}^{+\infty} g_i$ .

Since it has been shown in [VS00] that the function  $f$  satisfies neither the invariance principle nor the law of the iterated logarithm, it remains to prove that the constructed function  $g$  belongs to  $\mathbb{L}^p$  and that the coboundary  $g - g \circ T$  belongs to  $\mathbb{L}^r$ .

By (9.3.38) and (9.3.41), the equality

$$|g_i|^p = \left( \frac{\sqrt{n_i \log \log n_i}}{k_i} \right)^p \left( \sum_{j=1}^{k_i} j^p \mathbf{1}(T^{n_i-j} A_i) + \sum_{j=k_i+1}^{2k_i-1} (2k_i - j)^p \mathbf{1}(T^{n_i-j} A_i) \right) \quad (9.3.42)$$

takes place, hence integrating and accounting (9.3.40), we derive the estimates

$$\mathbb{E} |g_i|^p \leq \left( \frac{\sqrt{n_i \log \log n_i}}{k_i} \right)^p \left( \sum_{j=1}^{k_i} j^p + \sum_{j=k_i+1}^{2k_i-1} (2k_i - j)^p \right) \frac{1}{n_i} \quad (9.3.43)$$

$$\leq 2 \frac{n_i^{p/2-1} (\log \log n_i)^{p/2}}{k_i^p} k_i^{p+1} \quad (9.3.44)$$

$$= 2n_i^{p/2-1} (\log \log n_i)^{p/2} k_i, \quad (9.3.45)$$

hence

$$\|g_i\|_p \leq 2^{1/p} n_i^{1/2-1/p} (\log \log n_i)^{1/2} k_i^{1/p}. \quad (9.3.46)$$

By definition of  $n_i$  and  $k_i$ , one can find a constant  $c$  depending only on  $\alpha$  (hence on  $p$  and  $r$ ) such that for  $i$  large enough,

$$\|g_i\|_p \leq c \cdot 2^{i(1/2-1/p)} (\log i)^{1/2} \cdot 2^{i\alpha/p} = c \cdot (\log i)^{1/2} \cdot 2^{i(\alpha-1+p/2)/p}. \quad (9.3.47)$$

By (9.3.36), the series  $\sum_{i=1}^{+\infty} (\log i)^{1/2} \cdot 2^{i(\alpha-1+p/2)/p}$  is convergent, and we conclude by (9.3.47) that  $g$  belongs to  $\mathbb{L}^p$ .

It is proved in [VS00] that by construction, the equality

$$|g_i - g_i \circ T| = \frac{\sqrt{n_i \log \log n_i}}{k_i} \cdot \mathbf{1} \left( \bigcup_{j=1}^{2k_i} T^{n_i-j} A_i \right) \quad (9.3.48)$$

holds. By (9.3.38) and (9.3.41), we have

$$\|g_i - g_i \circ T\|_r \leq \frac{\sqrt{n_i \log \log n_i}}{k_i} \left( \frac{2k_i}{n_i} \right)^{1/r}, \quad (9.3.49)$$

hence by the definition of  $n_i$  and  $k_i$ , we have for  $i$  large enough and a constant  $c$  depending only on  $\alpha$ ,

$$\|g_i - g_i \circ T\|_r \leq c \cdot 2^{i(1/2-1/r)} 2^{i\alpha(1/r-1)} (\log i)^{1/2}, \quad (9.3.50)$$

from which we infer (by (9.3.36)) the convergence of the series  $\sum_{i=1}^{+\infty} \|g_i - g_i \circ T\|_r$  hence the fact that the function  $g - g \circ T$  belongs to  $\mathbb{L}^r$ .

This concludes the proof of Theorem 9.2.6.  $\square$

*Proof of Theorem 9.2.7.* The construction is similar to that of the proof of Theorem 9.2.6.

For each  $i \geq 1$ , we define  $n_i := 2^i$  and  $k_i := \lceil 2^{i\beta} \rceil$ , where  $\beta$  satisfies

$$\frac{r-p}{p(r-1)} < \beta < 1 - \frac{q}{p}. \quad (9.3.51)$$

Such a choice is possible since

$$p - q - \frac{r-p}{r-1} > p - (p-1) \frac{r}{r-1} - \frac{r-p}{r-1} = \frac{p(r-1) - (p-1)r - r + p}{r-1} = 0. \quad (9.3.52)$$

We take a set  $A_i \in \mathcal{F}$  such that (9.3.38) and (9.3.39) hold. We then define for  $i \geq 1$ ,

$$g_i := \frac{n_i^{1/p}}{k_i} \left( \sum_{j=1}^{k_i} j \mathbf{1}(T^{n_i-j} A_i) + \sum_{j=k_i+1}^{2k_i-1} (2k_i-j) \mathbf{1}(T^{n_i-j} A_i) \right), \quad (9.3.53)$$

and  $g := \sum_{i=1}^{+\infty} g_i$ .

The proof will be complete if we show the following three assertions:

1. the function  $g$  belongs to  $\mathbb{L}^q$ ;
2. the function  $g - g \circ T$  belongs to  $\mathbb{L}^r$ ;
3. the sequence  $(n^{-1/p} g \circ T^n)_{n \geq 1}$  does not converge almost surely to 0.

The first two items follow by completely similar computations as in the proof of Theorem 9.2.6. To show the last item, we notice that the sequence  $(2^{-i/p} \max_{2^i \leq l \leq 2^{i+1}} g_i \circ T^l)_{i \geq 1}$  does not converge to 0 in probability. To see this, one can note that

$$\mu \left\{ 2^{-i/p} \max_{2^i \leq l \leq 2^{i+1}} g_i \circ T^l \geq 1 \right\} \geq \mu \left( \bigcup_{j=1}^{n_i-k_i} T^j(A_i) \right) \geq \frac{1}{2}. \quad (9.3.54)$$

This finishes the proof of Theorem 9.2.7.  $\square$



# Chapter 10

## Orthomartingale-coboundary decomposition for strictly stationary random fields

The chapter consists of the paper [EMG]. We provide a new projective condition for a stationary real random field indexed by the lattice  $\mathbb{Z}^d$  to be well approximated by an orthomartingale in the sense of Cairoli (1969). Our main result can be viewed as a multidimensional version of the martingale-coboundary decomposition method whose idea goes back to Gordin (1969). It is a powerful tool for proving limit theorems or large deviations inequalities for stationary random fields when the corresponding result is valid for orthomartingales.

### 10.1 Introduction and notations

In probability theory, a powerful approach for proving limit theorems for stationary sequences of random variables is to find a way to approximate such sequences by martingales. This idea goes back to Gordin [Gor69]. It is a powerful method for proving the central limit theorem (CLT) and the weak invariance principle (WIP) for stationary sequences of dependent random variables satisfying a projective condition (see (10.1.1) in Theorem C below). More precisely, let  $(X_k)_{k \in \mathbb{Z}}$  be a sequence of real random variables defined on the probability space  $(\Omega, \mathcal{F}, \mu)$ . We assume that  $(X_k)_{k \in \mathbb{Z}}$  is stationary in the sense that its finite-dimensional laws are invariant by translations and we denote by  $\nu$  the law of  $(X_k)_{k \in \mathbb{Z}}$ . Let  $f : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$  be defined by  $f(\omega) = \omega_0$  and  $T : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}^{\mathbb{Z}}$  by  $(T\omega)_k = \omega_{k+1}$  for any  $\omega$  in  $\mathbb{R}^{\mathbb{Z}}$  and any  $k$  in  $\mathbb{Z}$ . Then the sequence  $(f \circ T^k)_{k \in \mathbb{Z}}$  defined on the probability space  $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \nu)$  is stationary with law  $\nu$ . So, without loss of generality, we can assume that  $X_k = f \circ T^k$  for any  $k$  in  $\mathbb{Z}$ . For any  $p \geq 1$  and any  $\sigma$ -algebra  $\mathcal{M} \subset \mathcal{F}$ , we denote by  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$  the space of  $p$ -integrable real random variables defined on  $(\Omega, \mathcal{M}, \mu)$  and we consider the norm  $\|\cdot\|_p$  defined by  $\|Z\|_p^p = \int_{\Omega} |Z(\omega)|^p d\mu(\omega)$  for any  $Z$  in  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ . We denote also by  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu) \ominus \mathbb{L}^p(\Omega, \mathcal{M}, \mu)$  the space of all  $Z$  in  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$  such that  $\mathbb{E}[Z | \mathcal{M}] = 0$  a.s.

**Theorem C** (Gordin, 1969). Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space and let  $T : \Omega \rightarrow \Omega$  be a measurable function such that  $\mu = T\mu$ . Let also  $p \geq 1$  and  $\mathcal{M} \subset \mathcal{F}$  be a  $\sigma$ -algebra such that  $\mathcal{M} \subset T^{-1}\mathcal{M}$ . If  $f$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{i \in \mathbb{Z}} T^{-i}\mathcal{M}, \mu)$  such that

$$\sum_{k \geq 0} \|\mathbb{E}[f | T^k \mathcal{M}]\|_p < \infty \quad (10.1.1)$$

then there exist  $m$  in  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T\mathcal{M}, \mu)$  and  $g$  in  $\mathbb{L}^p(\Omega, T\mathcal{M}, \mu)$  such that

$$f = m + g - g \circ T. \quad (10.1.2)$$

The term  $g - g \circ T$  in (10.1.2) is called a coboundary and equation (10.1.2) is called the martingale-coboundary decomposition of  $f$ . Moreover, the stationary sequence  $(m \circ T^i)_{i \in \mathbb{Z}}$  is a martingale-difference sequence with respect to the filtration  $(T^{-i}\mathcal{M})_{i \in \mathbb{Z}}$  (see Definition 10.2.1 below) and for any positive integer  $n$ ,

$$S_n(f) = S_n(m) + g - g \circ T^n \quad (10.1.3)$$

where  $S_n(h) = \sum_{i=0}^{n-1} h \circ T^i$  for any function  $h : \Omega \rightarrow \mathbb{R}$ . Combining (10.1.3) with the Billingsley-Ibragimov CLT for martingales (see [Bil61] or [Ibr63]), one obtain the CLT for the stationary sequence  $(f \circ T^k)_{k \in \mathbb{Z}}$  when the projective condition (10.1.1) holds. Similarly, combining (10.1.3) with the WIP for martingales (see [Bil68]), we derive the WIP for the stationary sequence  $(f \circ T^k)_{k \in \mathbb{Z}}$ . Thus, Gordin's method provides a sufficient condition for proving limit theorems for stationary sequences when such a limit theorem holds for martingale-difference sequences. Our aim in this work is to provide an extension of Theorem C for random fields indexed by the lattice  $\mathbb{Z}^d$  where  $d$  is a positive integer (see Theorem 10.2.4).

## 10.2 Main results

**Definition 10.2.1.** We say that a sequence  $(X_k)_{k \in \mathbb{Z}}$  of real random variables defined on a probability space  $(\Omega, \mathcal{F}, \mu)$  is a martingale-difference (MD) sequence if there exists a filtration  $(\mathcal{G}_k)_{k \in \mathbb{Z}}$  such that  $\mathcal{G}_k \subset \mathcal{G}_{k+1} \subset \mathcal{F}$  and  $X_k$  belongs to  $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_{k-1}, \mu)$  for any  $k$  in  $\mathbb{Z}$ .

The concept of MD sequences can be extended to the random field setting. One can refer for example to Basu and Dorea [BD79] or Nahapetian [NP92] where MD random fields are defined in two different ways and limit theorems are obtained. In this paper, we are interested by orthomartingale-difference random fields in the sense of Cairoli [Cai69]. A good introduction to this concept is done in the book by Khoshnevisan [Kho02]. Let  $d$  be a positive integer. We denote by  $\langle d \rangle$  the set  $\{1, \dots, d\}$ . For any  $\mathbf{s} = (s_1, \dots, s_d)$  and any  $\mathbf{t} = (t_1, \dots, t_d)$  in  $\mathbb{Z}^d$ , we write  $\mathbf{s} \preceq \mathbf{t}$  (resp.  $\mathbf{s} \prec \mathbf{t}$ ,  $\mathbf{s} \succeq \mathbf{t}$  and  $\mathbf{s} \succ \mathbf{t}$ ) if and only if  $s_k \leq t_k$  (resp.  $s_k < t_k$ ,  $s_k \geq t_k$  and  $s_k > t_k$ ) for any  $k$  in  $\langle d \rangle$  and we denote also  $\mathbf{s} \wedge \mathbf{t} = (s_1 \wedge t_1, \dots, s_d \wedge t_d)$ .

**Definition 10.2.2.** Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. A family  $(\mathcal{G}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  of  $\sigma$ -algebras is a commuting filtration if  $\mathcal{G}_{\mathbf{i}} \subset \mathcal{G}_{\mathbf{j}} \subset \mathcal{F}$  for any  $\mathbf{i}$  and  $\mathbf{j}$  in  $\mathbb{Z}^d$  such that  $\mathbf{i} \preceq \mathbf{j}$  and

$$\mathbb{E}[\mathbb{E}[Z \mid \mathcal{G}_{\mathbf{s}}] \mid \mathcal{G}_{\mathbf{t}}] = \mathbb{E}[Z \mid \mathcal{G}_{\mathbf{s} \wedge \mathbf{t}}] \quad a.s. \quad (10.2.1)$$

for any  $\mathbf{s}$  and  $\mathbf{t}$  in  $\mathbb{Z}^d$  and any bounded random variable  $Z$ .

Definition 10.2.2 is known as the "F4 condition".

**Definition 10.2.3.** Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. A random field  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  is an orthomartingale-difference (OMD) random field if there exists a commuting filtration  $(\mathcal{G}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  such that  $X_{\mathbf{k}}$  belongs to  $\mathbb{L}^1(\Omega, \mathcal{G}_{\mathbf{k}}, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_{\mathbf{l}}, \mu)$  for any  $\mathbf{l} \preceq \mathbf{k}$  and  $\mathbf{k}$  in  $\mathbb{Z}^d$ .

Let  $\mathbf{k}$  be fixed in  $\mathbb{Z}^d$  and  $S_{\mathbf{k}} = \sum_{\mathbf{0} \preceq \mathbf{i} \preceq \mathbf{k}} X_{\mathbf{i}}$  where  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  is an OMD random field with respect to a commuting filtration  $(\mathcal{G}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ . Then  $S_{\mathbf{k}}$  belongs to  $\mathbb{L}^1(\Omega, \mathcal{G}_{\mathbf{k}}, \mu)$  and  $\mathbb{E}[S_{\mathbf{k}} \mid \mathcal{G}_{\mathbf{l}}] = S_{\mathbf{l}}$  for any  $\mathbf{l} \preceq \mathbf{k}$ . We say that  $(S_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  is an orthomartingale (OM) random field.

Arguing as above, without loss of generality, every stationary real random field  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  can be written as  $(f \circ T^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  where  $f : \Omega \rightarrow \mathbb{R}$  is a measurable function and for any  $\mathbf{k}$  in  $\mathbb{Z}^d$ ,  $T^{\mathbf{k}} : \Omega \rightarrow \Omega$  is a measure-preserving operator satisfying  $T^{\mathbf{i}} \circ T^{\mathbf{j}} = T^{\mathbf{i} + \mathbf{j}}$  for any  $\mathbf{i}$  and  $\mathbf{j}$  in  $\mathbb{Z}^d$ . For

any  $s$  in  $\langle d \rangle$ , we denote  $T_s = T^{\mathbf{e}_s}$  where  $\mathbf{e}_s = (e_s^{(1)}, \dots, e_s^{(d)})$  is the unique element of  $\mathbb{Z}^d$  such that  $\mathbf{e}_s^{(s)} = 1$  and  $\mathbf{e}_s^{(i)} = 0$  for any  $i$  in  $\langle d \rangle \setminus \{s\}$  and  $U_s$  is the operator defined by  $U_s h = h \circ T_s$  for any function  $h: \Omega \rightarrow \mathbb{R}$ . We define also  $U_J$  as the product operator  $\prod_{s \in J} U_s$  for any  $\emptyset \subsetneq J \subset \langle d \rangle$  and we write simply  $U$  for  $U_{\langle d \rangle} = U_1 \circ U_2 \circ \dots \circ U_d$ . For any  $\emptyset \subsetneq J \subset \langle d \rangle$ , we denote also by  $|J|$  the number of elements in  $J$  and by  $J^c$  the set  $\langle d \rangle \setminus J$ . Finally, the set of nonnegative integers will be denoted by  $\mathbb{N}$ . The main result of this chapter is the following.

**Theorem 10.2.4.** *Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space and let  $T^1: \Omega \rightarrow \Omega$  be a measure-preserving operator for any  $\mathbf{l}$  in  $\mathbb{Z}^d$  such that  $T^{\mathbf{i}} \circ T^{\mathbf{j}} = T^{\mathbf{i}+\mathbf{j}}$  for any  $\mathbf{i}$  and  $\mathbf{j}$  in  $\mathbb{Z}^d$ . Let  $p \geq 1$  and let  $\mathcal{M} \subset \mathcal{F}$  be a  $\sigma$ -algebra such that  $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  is a commuting filtration. If  $f$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \bigcap_{\mathbf{k} \in \mathbb{N}^d} T^{\mathbf{k}}\mathcal{M}, \mu)$  and*

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[f | T^{\mathbf{k}}\mathcal{M}]\|_p < +\infty \tag{10.2.2}$$

then  $f$  admits the decomposition

$$f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) m_J + \prod_{s=1}^d (I - U_s) g, \tag{10.2.3}$$

where  $m$ ,  $g$  and  $m_J$  belong to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ ,  $\mathbb{L}^p(\Omega, \prod_{s=1}^d T_s \mathcal{M}, \mu)$  and  $\mathbb{L}^p(\Omega, \prod_{s \in J} T_s \mathcal{M}, \mu)$  respectively and  $(U^{\mathbf{i}}m)_{\mathbf{i} \in \mathbb{Z}^d}$  and  $(U_{J^c}^{\mathbf{i}}m_J)_{\mathbf{i} \in \mathbb{Z}^{d-|J|}}$  are OMD random fields for  $\emptyset \subsetneq J \subsetneq \langle d \rangle$ .

*Remark 10.2.5.* One can notice that condition (10.2.2) is exactly Gordin's condition (10.1.1) when  $d = 1$ . It is well known that condition (10.1.1) is not necessary for  $f$  to admit a martingale-coboundary decomposition. In fact, in dimension  $d = 1$ , a necessary and sufficient condition for  $f$  to admit the martingale-coboundary decomposition (10.1.2) is the convergence in  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$  for  $p \geq 1$  of the series  $\sum_{k \geq 0} \mathbb{E}[U^k f | \mathcal{M}]$  (see [Vol93], Theorem 2, condition (7)). So, let  $(\delta_j)_{j \geq 0}$  be a decreasing sequence of real numbers such that  $\sum_{j \geq 0} \delta_j^2 < \infty$  and define  $a_{2j} = \delta_j$  and  $a_{2j+1} = -\delta_j$  for any  $j \geq 0$ . If  $(\varepsilon_i)_{i \in \mathbb{Z}}$  is a sequence of iid real random variables with zero-mean and unit variance and  $f \circ T^k = \sum_{i \geq 0} a_j \varepsilon_{k-j}$  for any  $k$  in  $\mathbb{Z}$  (we say that  $(f \circ T^k)_{k \geq 1}$  is a linear process) then  $\sum_{k \geq 1} \mathbb{E}(U^k f | \mathcal{M})$  converges in  $\mathbb{L}^2(\Omega, \mathcal{F}, \mu)$  while the decay of the sequence  $\left(\sum_{j \geq k} a_j^2\right)_{k \geq 1}$  can be arbitrarily slow such that the series  $\sum_{k \geq 1} \|\mathbb{E}(U^k f | \mathcal{M})\|_2$  does not converge. That is,  $f = \sum_{i \geq 0} a_i \varepsilon_{-i}$  is a function which admits the martingale-coboundary decomposition (10.1.2) even if Gordin's condition (10.1.1) does not hold. Finally, it will be interesting to investigate a necessary and sufficient condition for the orthomartingale-coboundary decomposition (10.2.3) when  $d \geq 2$ . This question is still an open problem and will be considered elsewhere.

*Remark 10.2.6.* If  $d = 2$  then (10.2.3) reduces to

$$f = m + (I - U_1)m_1 + (I - U_2)m_2 + (I - U_1)(I - U_2)g, \tag{10.2.4}$$

where  $m$ ,  $m_1$ ,  $m_2$  and  $g$  belong to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$  such that  $(U^i m)_{i \in \mathbb{Z}}$  is an OMD random field and  $(U_2^k m_1)_{k \in \mathbb{Z}}$  and  $(U_1^k m_2)_{k \in \mathbb{Z}}$  are MD sequences. If  $d = 3$  then (10.2.3) becomes

$$\begin{aligned} f = & m + (I - U_1)m_1 + (I - U_2)m_2 + (I - U_3)m_3 \\ & + (I - U_1)(I - U_2)m_{\{1,2\}} + (I - U_1)(I - U_3)m_{\{1,3\}} + (I - U_2)(I - U_3)m_{\{2,3\}} \\ & + (I - U_1)(I - U_2)(I - U_3)g \end{aligned}$$

where  $m$ ,  $m_1$ ,  $m_2$ ,  $m_3$ ,  $m_{\{1,2\}}$ ,  $m_{\{1,3\}}$ ,  $m_{\{2,3\}}$  and  $g$  belong to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$  such that  $(U^{\mathbf{i}}m)_{\mathbf{i} \in \mathbb{Z}^3}$ ,  $(U_{\{2,3\}}^{\mathbf{i}}m_1)_{\mathbf{i} \in \mathbb{Z}^2}$ ,  $(U_{\{1,3\}}^{\mathbf{i}}m_2)_{\mathbf{i} \in \mathbb{Z}^2}$  and  $(U_{\{1,2\}}^{\mathbf{i}}m_3)_{\mathbf{i} \in \mathbb{Z}^2}$  are OMD random fields and  $(U_1^k m_{\{2,3\}})_{k \in \mathbb{Z}}$ ,  $(U_2^k m_{\{1,3\}})_{k \in \mathbb{Z}}$  and  $(U_3^k m_{\{1,2\}})_{k \in \mathbb{Z}}$  are MD sequences.

*Remark 10.2.7.* A decomposition similar to (10.2.3) was obtained by Gordin [Gor69] but with reversed orthomartingales and under an assumption on the so-called Poisson equation.

**Proposition 10.2.8.** *Let  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be an OMD random field. For any  $p \geq 2$  and any  $\mathbf{n}$  in  $\mathbb{N}^d$ ,*

$$\left\| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right\|_p \leq p^{d/2} \left( \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} \|X_{\mathbf{k}}\|_p^2 \right)^{1/2} \quad (10.2.5)$$

and the constant  $p^{d/2}$  in (10.2.5) is optimal in the following sense: there exists a stationary OMD random field  $(Z_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  with  $\|Z_{\mathbf{0}}\|_{\infty} = 1$  and a positive constant  $\kappa$  such that for any  $p \geq 2$

$$\inf \left\{ C > 0 ; \left\| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} Z_{\mathbf{k}} \right\|_p \leq C \left( \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} \|Z_{\mathbf{k}}\|_p^2 \right)^{1/2} \quad \forall \mathbf{n} \in \mathbb{N}^d \right\} \geq \kappa p^{d/2}. \quad (10.2.6)$$

Combining Proposition 10.2.8 and Theorem 10.2.4, we obtain the following result.

**Proposition 10.2.9.** *Let  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be a stationary real random field defined on a probability space  $(\Omega, \mathcal{F}, \mu)$  and  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be a commuting filtration such that  $X_{\mathbf{i}}$  is  $\mathcal{F}_{\mathbf{i}}$ -measurable for each  $\mathbf{i}$  in  $\mathbb{Z}^d$ . If there exists  $p \geq 2$  such that  $X_{\mathbf{0}}$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{F}_{\mathbf{0}}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{\mathbf{k} \in \mathbb{N}^d} \mathcal{F}_{-\mathbf{k}}, \mu)$  and*

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\mathbf{k}}]\|_p < \infty \quad (10.2.7)$$

then for any  $\mathbf{n} = (n_1, \dots, n_d)$  in  $\mathbb{N}^d$ ,

$$\left\| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right\|_p \leq C_d p^{d/2} |\mathbf{n}|^{d/2} \sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\mathbf{k}}]\|_p \quad (10.2.8)$$

where  $|\mathbf{n}| = \prod_{i=1}^d n_i$  and  $C_d$  is a positive constant depending only on  $d$ .

*Remark 10.2.10.* A Young function  $\psi$  is a real convex nondecreasing function defined on  $\mathbb{R}^+$  which satisfies  $\lim_{t \rightarrow \infty} \psi(t) = \infty$  and  $\psi(0) = 0$ . We denote by  $\mathbb{L}_{\psi}(\Omega, \mathcal{F}, \mu)$  the Orlicz space associated to the Young function  $\psi$ , that is the space of real random variables  $Z$  defined on  $(\Omega, \mathcal{F}, \mu)$  such that  $\mathbb{E}[\psi(|Z|/c)] < \infty$  for some  $c > 0$ . The Orlicz space  $\mathbb{L}_{\psi}(\Omega, \mathcal{F}, \mu)$  equipped with the so-called Luxemburg norm  $\|\cdot\|_{\psi}$  defined for any real random variable  $Z$  by  $\|Z\|_{\psi} = \inf\{c > 0; \mathbb{E}[\psi(|Z|/c)] \leq 1\}$  is a Banach space. For any  $p \geq 1$ , if  $\varphi_p$  is the function defined by  $\varphi_p(x) = x^p$  for any nonnegative real  $x$  then  $\varphi_p$  is a Young function and the Orlicz space  $\mathbb{L}_{\varphi_p}(\Omega, \mathcal{F}, \mu)$  reduces to  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$  equipped with the norm  $\|\cdot\|_p$ . For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [KR61]. Combining (10.2.8) and Lemma 4 in [EVW13], we obtain Kahane-Khintchine inequalities: for any  $0 < q < 2/d$ , there exists a positive constant  $C$  depending only on  $d$  and  $q$  such that for any  $\mathbf{n}$  in  $\mathbb{N}^d$ ,

$$\left\| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right\|_{\psi_q} \leq C |\mathbf{n}|^{d/2} \sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\mathbf{k}}]\|_{\psi_{\beta(q)}} \quad (10.2.9)$$

where  $\beta(q) = 2q/(2-dq)$  and  $\psi_{\alpha}$  is the Young function defined for any  $x \in \mathbb{R}^+$  by

$$\psi_{\alpha}(x) = \exp((x + h_{\alpha})^{\alpha}) - \exp(h_{\alpha}^{\alpha}) \quad \text{with} \quad h_{\alpha} = ((1-\alpha)/\alpha)^{1/\alpha} \mathbf{1}\{0 < \alpha < 1\} \quad (10.2.10)$$

for any real  $\alpha > 0$ . Using Markov inequality and the definition of the Luxembourg norm, we derive the following large deviations inequalities: for any  $0 < q < 2/d$ , there exists a positive constant  $C$  depending only on  $d$  and  $q$  such that for any  $\mathbf{n}$  in  $\mathbb{N}^d$  and any positive real  $x$ ,

$$\begin{aligned} \mu \left( \left| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right| > x \right) &\leq \\ &\leq \left( 1 + e^{h_q} \right) \exp \left( - \left( \frac{x}{C |\mathbf{n}|^{d/2} \sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\mathbf{k}}]\|_{\psi_{\beta(q)}}} + h_q \right)^q \right). \end{aligned} \quad (10.2.11)$$

Finally, one can check that (10.2.9) and (10.2.11) still hold for  $q = 2/d$  if  $X_{\mathbf{0}}$  is bounded.

**Proposition 10.2.11.** *Let  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be a stationary real random field defined on a probability space  $(\Omega, \mathcal{F}, \mu)$  and  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be a commuting filtration such that  $X_{\mathbf{i}}$  is  $\mathcal{F}_{\mathbf{i}}$ -measurable for each  $\mathbf{i}$  in  $\mathbb{Z}^d$ . If  $X_{\mathbf{0}}$  belongs to  $\mathbb{L}^2(\Omega, \mathcal{F}_{\mathbf{0}}, \mu) \ominus \mathbb{L}^2(\Omega, \cap_{\mathbf{k} \in \mathbb{N}^d} \mathcal{F}_{-\mathbf{k}}, \mu)$  and  $\sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\mathbf{k}}]\|_2 < \infty$  then  $\sum_{\mathbf{k} \in \mathbb{Z}^d} |\mathbb{E}[X_{\mathbf{0}} X_{\mathbf{k}}]| < \infty$  and*

$$\lim_{\min \mathbf{n} \rightarrow +\infty} |\mathbf{n}|^{-1} \mathbb{E} \left( \left( \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right)^2 \right) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{E}[X_{\mathbf{0}} X_{\mathbf{k}}] \quad (10.2.12)$$

where  $\min \mathbf{n} \rightarrow +\infty$  means that  $\min_{1 \leq i \leq d} n_i \rightarrow +\infty$ .

Now, we are able to investigate the WIP for random fields. Let  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be a stationary real random field defined on a probability space  $(\Omega, \mathcal{F}, \mu)$ . Let also  $\mathcal{A}$  be a collection of Borel subsets of  $[0, 1]^d$  and consider the process  $\{S_n(A); A \in \mathcal{A}\}$  defined by

$$S_n(A) = \sum_{\mathbf{i} \in \langle n \rangle^d} \lambda(nA \cap R_{\mathbf{i}}) X_{\mathbf{i}} \quad (10.2.13)$$

where  $R_{\mathbf{i}} = ]i_1 - 1, i_1] \times \cdots \times ]i_d - 1, i_d]$  is the unit cube with upper corner at  $\mathbf{i} = (i_1, \dots, i_d)$  in  $\langle n \rangle^d$  and  $\lambda$  is the Lebesgue measure on  $\mathbb{R}^d$ . The collection  $\mathcal{A}$  is equipped with the pseudo-metric  $\rho$  defined by  $\rho(A, B) = \sqrt{\lambda(A \Delta B)}$  for any  $A$  and  $B$  in  $\mathcal{A}$ . Let  $\varepsilon > 0$  and let  $H(\mathcal{A}, \rho, \varepsilon)$  be the logarithm of the smallest number  $N(\mathcal{A}, \rho, \varepsilon)$  of open balls of radius  $\varepsilon$  with respect to  $\rho$  which form a covering of  $\mathcal{A}$ . The function  $H(\mathcal{A}, \rho, \cdot)$  is called the metric entropy of the class  $\mathcal{A}$  and allows us to control the size of the collection  $\mathcal{A}$ . Let  $(\mathcal{C}(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$  be the Banach space of continuous real functions on  $\mathcal{A}$  equipped with the uniform norm  $\|\cdot\|_{\mathcal{A}}$  defined by  $\|f\|_{\mathcal{A}} = \sup_{A \in \mathcal{A}} |f(A)|$ . A standard Brownian motion indexed by  $\mathcal{A}$  is a mean zero Gaussian process  $W$  with sample paths in  $\mathcal{C}(\mathcal{A})$  such that  $\text{Cov}(W(A), W(B)) = \lambda(A \cap B)$  and we know from Dudley [Dud73] that such a process is well defined if  $\int_0^1 \sqrt{H(\mathcal{A}, \rho, \varepsilon)} d\varepsilon < \infty$ . We say that the WIP holds if the sequence of processes  $\{n^{-d/2} S_n(A); A \in \mathcal{A}\}$  converges in distribution in  $\mathcal{C}(\mathcal{A})$  to a mixture of  $\mathcal{A}$ -indexed Brownian motion. The first weak convergence results for  $\mathcal{Q}_d$ -indexed partial sum processes were established for i.i.d. real random fields where  $\mathcal{Q}_d$  is the collection  $\{[\mathbf{0}, \mathbf{t}]; \mathbf{t} \in [0, 1]^d\}$  of lower-left quadrants in  $[0, 1]^d$ . They were proved by Wichura [Wic69] under a finite variance condition and earlier by Kuelbs [Kue68] under additional moment restrictions. If  $d = 1$ , these results coincide with the original invariance principle of Donsker [Don51]. Many others WIP have been established for dependent random fields indexed by large classes of sets. One can refer for example to [Ded01], [EM02], [EMO06] or [EVW13]. In the sequel, we are going to apply Theorem 10.2.4 in order to establish a WIP (Theorem 10.2.12) for  $\mathcal{Q}_d$ -indexed partial sum dependent random fields. Let  $n$  be a positive integer. For simplicity, we denote  $S_n(\mathbf{t}) = S_n([\mathbf{0}, \mathbf{t}])$  for any  $[\mathbf{0}, \mathbf{t}]$  in  $\mathcal{Q}_d$ . More precisely, for any  $\mathbf{t}$  in  $[0, 1]^d$ ,

$$S_n(\mathbf{t}) = \sum_{\mathbf{i} \in \langle n \rangle^d} \lambda([0, n\mathbf{t}] \cap R_{\mathbf{i}}) X_{\mathbf{i}} \quad (10.2.14)$$



Recall that the standard  $d$ -parameter Brownian sheet on  $[0, 1]^d$  denoted by  $\mathbb{B} = (\mathbb{B}_t)_{t \in [0, 1]^d}$  is a mean-zero Gaussian random field such that  $\text{Cov}(\mathbb{B}_s, \mathbb{B}_t) = \prod_{i=1}^d s_i \wedge t_i$  for any  $\mathbf{s} = (s_1, \dots, s_d)$  and  $\mathbf{t} = (t_1, \dots, t_d)$  in  $[0, 1]^d$ . Since the CLT does not hold for general OMD random fields (see [WW13], example 1, page 12), we restrict ourselves to the case of a filtration generated by iid random variables which is necessarily a commuting filtration (see Proposition 8.1 in [WW13]).

**Theorem 10.2.12.** *Let  $(\varepsilon_j)_{j \in \mathbb{Z}^d}$  be an i.i.d. real random field defined on a probability space  $(\Omega, \mathcal{F}, \mu)$ . Denote by  $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$  the commuting filtration where  $\mathcal{F}_i$  is the  $\sigma$ -algebra generated by  $\varepsilon_j$  for  $\mathbf{j} \preceq \mathbf{i}$  and  $\mathbf{i}$  in  $\mathbb{Z}^d$ . Let  $(X_i)_{i \in \mathbb{Z}^d}$  be a stationary real random field such that  $X_0$  belongs to  $\mathbb{L}^2(\Omega, \mathcal{F}_0, \mu) \ominus \mathbb{L}^2(\Omega, \cap_{\mathbf{k} \in \mathbb{N}^d} \mathcal{F}_{-\mathbf{k}}, \mu)$  and (10.2.7) holds for  $p = 2$ . Then the sequence of processes  $\{n^{-d/2} S_n(\mathbf{t}); \mathbf{t} \in [0, 1]^d\}$  converges in distribution in  $\mathcal{C}(\mathcal{Q}_d)$  to  $\sqrt{\eta} \mathbb{B}$  where  $\mathbb{B}$  is a standard  $d$ -Brownian sheet and  $\eta = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{E}[X_0 X_{\mathbf{k}}]$ .*

*Remark 10.2.13.* El Machkouri et al. [EVW13] and Wang and Woodroffe [WW13] obtained also a WIP for random fields  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  which can be expressed as a functional of i.i.d. real random variables but under the more restrictive condition that  $X_0$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$  with  $p > 2$ . In a recent work, Wang and Volný [VW14] obtained the WIP for  $p = 2$  under a multi-dimensional version of the so-called Hannan's condition for time series. Their condition is less restrictive than (10.2.7) but condition (10.2.7) gives also an orthomartingale approximation for the considered random field which is of independent interest (see Theorem 10.2.4). In particular, (10.2.7) provides not only a WIP but also large deviations inequalities (see Proposition 10.2.9 and Remark 10.2.10).

**Proposition 10.2.14.** *Let  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  be an i.i.d. real random field defined on a probability space  $(\Omega, \mathcal{F}, \mu)$  such that  $\varepsilon_0$  has zero mean and belongs to  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$  for some  $p \geq 2$ . Consider the linear random field  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  defined for any  $\mathbf{k}$  in  $\mathbb{Z}^d$  by  $X_{\mathbf{k}} = \sum_{\mathbf{j} \in \mathbb{N}^d} a_{\mathbf{j}} \varepsilon_{\mathbf{k} - \mathbf{j}}$  where  $(a_{\mathbf{j}})_{\mathbf{j} \in \mathbb{N}^d}$  is a family of real numbers satisfying  $\sum_{\mathbf{j} \in \mathbb{N}^d} a_{\mathbf{j}}^2 < \infty$ . Then the condition (10.2.7) holds if and only if*

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \sqrt{\sum_{\mathbf{j} \succ \mathbf{k}} a_{\mathbf{j}}^2} < \infty. \quad (10.2.15)$$

*Remark 10.2.15.* Proposition 10.2.14 ensures that the conclusion of Theorem 10.2.12 still hold for linear random fields with iid innovations under assumption (10.2.15). Let  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  be a linear random field defined as in Proposition 10.2.14. In [WW13], Wang and Woodroffe obtained a WIP for  $(X_k)_{k \in \mathbb{Z}^d}$  under a weaker condition than (10.2.15) but again with the additional assumption that  $\varepsilon_0$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$  with  $p > 2$ . In [BD14], Biermé and Durieu obtained also a WIP under the so-called stability condition  $\sum_{\mathbf{k} \in \mathbb{N}^d} |a_{\mathbf{k}}| < \infty$  which is less restrictive than (10.2.15). In fact, let  $a_{\mathbf{k}} := k_1^{-\alpha} \dots k_d^{-\alpha}$  for  $1 < \alpha < 3/2$  and  $\mathbf{k} = (k_1, \dots, k_d)$  in  $\mathbb{N}^d$  then the linear process  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  satisfies the stability condition but (10.2.15) does not hold. Indeed, (10.2.15) is equivalent to the convergence of the series  $\sum_{j \geq 0} \sqrt{\sum_{l \geq j} l^{-2\alpha}}$  and this last one is not convergent since there exists a positive constant  $C_\alpha$  such that  $\sum_{l \geq j} l^{-2\alpha} \geq C_\alpha j^{1-2\alpha}$  for each  $j \geq 0$ . Nevertheless, (10.2.15) provides the orthomartingale-coboundary decomposition (10.2.3) while it is not the case for the stability condition even when  $d = 1$ . In fact, let  $(b_k)_{k \geq 0}$  be a decreasing sequence of real numbers such that  $b_k$  goes to zero as  $k$  goes to infinity and  $\sum_{k \geq 0} b_k^2 = +\infty$  and let  $a_k = b_k - b_{k+1}$  for any  $k \geq 0$ . So, we have  $\sum_{k \geq 0} |a_k| < \infty$  but if  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\varepsilon_j$  for  $j \preceq 0$  then  $\left\| \sum_{i=1}^N \mathbb{E}(X_i | \mathcal{F}_0) \right\|_2^2 = \sum_{l \geq 0} (b_{l+1} - b_{l+N})^2$  for any positive integer  $N$ . Since, for any positive integer  $L$ , we have

$$\sup_N \sum_{l=0}^L (b_{l+1} - b_{l+N})^2 \geq \sum_{l=0}^L b_{l+1}^2,$$

we obtain  $\sup_N \left\| \sum_{i=1}^N \mathbb{E}(X_i | \mathcal{M}) \right\|_2^2 = +\infty$ . Consequently, the martingale-coboundary decomposition (10.1.2) does not hold.

We now provide an application of Theorem 10.2.4 to the WIP in Hölder spaces. We consider for  $0 < \gamma \leq 1$  the space  $\mathbb{H}_\gamma([0, 1]^d)$  as the space of all continuous functions  $g$  for which there exists a constant  $K$  such that  $|g(\mathbf{s}) - g(\mathbf{t})| \leq K\|\mathbf{s} - \mathbf{t}\|^\gamma$  for each  $\mathbf{s}$  and  $\mathbf{t}$  in  $[0, 1]^d$  where  $\|\cdot\|$  denotes the Euclidian norm on  $\mathbb{R}^d$ . We endow this function space with the norm  $\|g\| := |g(\mathbf{0})| + \sup_{\mathbf{s}, \mathbf{t} \in [0, 1]^d, \mathbf{s} \neq \mathbf{t}} |g(\mathbf{t}) - g(\mathbf{s})|/\|\mathbf{t} - \mathbf{s}\|^\gamma$  and we consider the partial sum process  $(S_n(\mathbf{t}))_{\mathbf{t} \in [0, 1]^d}$  defined by (10.2.14) as an element of  $\mathbb{H}_\gamma([0, 1]^d)$ .

**Theorem 10.2.16.** *If the assumptions of Theorem 10.2.12 hold with  $p > 4 \times (\log_2(4d/(4d - 3)))^{-1}$  then the sequence of processes  $\{n^{-d/2}S_n(t); t \in [0, 1]^d\}_{n \geq 1}$  converges in distribution in  $\mathbb{H}_\gamma([0, 1]^d)$  to  $\sqrt{\eta}\mathbb{B}$  for each  $\gamma < 1/2 - d/p$  where  $\mathbb{B}$  is a standard  $d$ -Brownian sheet and  $\eta = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{E}[X_{\mathbf{0}}X_{\mathbf{k}}]$ .*

*Remark 10.2.17.* In [RSZ07], a necessary and sufficient condition was obtained for iid random fields to satisfy the WIP in Hölder spaces. Our result provides a sufficient condition for stationary real random fields which can be expressed as a functional of iid real random variables.

## 10.3 Proofs

In this section, the letter  $\kappa$  will denote a universal positive constant which the value may change from line to line. The proof of Theorem 10.2.4 will be done by induction on  $d$ . We shall need the following lemma.

**Lemma 10.3.1.** *Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. Let  $d$  be a positive integer and  $T^1: \Omega \rightarrow \Omega$  be a measure-preserving operator for any  $\mathbf{l}$  in  $\mathbb{Z}^{d+1}$  such that  $T^{\mathbf{i}} \circ T^{\mathbf{j}} = T^{\mathbf{i}+\mathbf{j}}$  for any  $\mathbf{i}$  and  $\mathbf{j}$  in  $\mathbb{Z}^{d+1}$ . Let  $p \geq 1$  and  $\mathcal{M} \subset \mathcal{F}$  be a  $\sigma$ -algebra such that  $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^{d+1}}$  is a commuting filtration. Assume that  $f$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{\mathbf{k} \in \mathbb{N}^{d+1}} T^{\mathbf{k}}\mathcal{M}, \mu)$  and*

$$\sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \|\mathbb{E}[f | T^{\mathbf{k}}\mathcal{M}]\|_p < \infty. \quad (10.3.1)$$

*Then there exist  $M \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_{d+1}\mathcal{M}, \mu)$  and  $G \in \mathbb{L}^p(\Omega, T_{d+1}\mathcal{M}, \mu)$  such that*

$$f = M + G - G \circ T_{d+1} \quad (10.3.2)$$

*and*

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \left\| \mathbb{E}[M | T^{(\mathbf{k}, 0)}\mathcal{M}] \right\|_p + \left\| \mathbb{E}[G | T^{(\mathbf{k}, 0)}\mathcal{M}] \right\|_p < \infty. \quad (10.3.3)$$

*Proof of Lemma 10.3.1.* First, the decomposition (10.3.2) is a direct consequence of Theorem A (see section 1). Moreover, a careful reading of the proof of Theorem A (see Volný [Vol93]) gives the following expressions of  $M$  and  $G$ :

$$M = \sum_{j \geq 0} \mathbb{E}[U_{d+1}^j f | \mathcal{M}] - \mathbb{E}[U_{d+1}^j f | T_{d+1}\mathcal{M}] \quad \text{and} \quad G = \sum_{j \geq 0} \mathbb{E}[U_{d+1}^j f | T_{d+1}\mathcal{M}]. \quad (10.3.4)$$

Let  $\mathbf{k} = (k_1, \dots, k_d)$  be fixed in  $\mathbb{N}^d$ . Since

$$\mathbb{E}[M | T^{(\mathbf{k}, 0)}\mathcal{M}] = \sum_{j \geq 0} \mathbb{E}[U_{d+1}^j f | T^{(\mathbf{k}, 0)}\mathcal{M}] - \sum_{j \geq 0} \mathbb{E}[U_{d+1}^j f | T^{(\mathbf{k}, 1)}\mathcal{M}],$$

we derive

$$\left\| \mathbb{E}[M | T^{(\mathbf{k}, 0)}\mathcal{M}] \right\|_p \leq 2 \sum_{j \geq 0} \left\| \mathbb{E}[U_{d+1}^j f | T^{(\mathbf{k}, 0)}\mathcal{M}] \right\|_p = 2 \sum_{j \geq 0} \left\| \mathbb{E}[f | T^{(\mathbf{k}, j)}\mathcal{M}] \right\|_p.$$

Finally, using (10.3.1), we obtain

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \left\| \mathbb{E} \left[ M \mid T^{(\mathbf{k},0)} \mathcal{M} \right] \right\|_p \leq 2 \sum_{\mathbf{k} \in \mathbb{N}^d} \sum_{j \geq 0} \left\| \mathbb{E} [f \mid T^{(\mathbf{k},j)} \mathcal{M}] \right\|_p < \infty. \quad (10.3.5)$$

Similarly, we have also  $\sum_{\mathbf{k} \in \mathbb{N}^d} \left\| \mathbb{E} [G \mid T^{(\mathbf{k},0)} \mathcal{M}] \right\|_p < \infty$ . The proof of Lemma 10.3.1 is complete.  $\square$

*Proof of Theorem 10.2.4.* We are going to prove Theorem 10.2.4 by induction on the dimension  $d$ . First, for  $d = 1$ , the result reduces to Gordin's martingale-difference coboundary decomposition (see Theorem C above). Let  $d$  be a positive integer. We assume that our result is true for  $d$  and we have to show that it is true for  $d + 1$ . We thus consider a measure-preserving operator  $T^{\mathbf{l}}: \Omega \rightarrow \Omega$  for any  $\mathbf{l}$  in  $\mathbb{Z}^{d+1}$  such that  $T^{\mathbf{i}} \circ T^{\mathbf{j}} = T^{\mathbf{i}+\mathbf{j}}$  for any  $\mathbf{i}$  and  $\mathbf{j}$  in  $\mathbb{Z}^{d+1}$ . Let  $p \geq 1$  and let  $\mathcal{M} \subset \mathcal{F}$  be a  $\sigma$ -algebra such that  $(T^{-\mathbf{i}} \mathcal{M})_{\mathbf{i} \in \mathbb{Z}^{d+1}}$  is a commuting filtration. Assume that  $f$  belongs to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$  and satisfies (10.3.1). By Lemma 10.3.1, there exist  $M \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_{d+1} \mathcal{M}, \mu)$  and  $G \in \mathbb{L}^p(\Omega, T_{d+1} \mathcal{M}, \mu)$  such that (10.3.2) and (10.3.3) hold. So, by the induction hypothesis, we have

$$M = m' + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) m'_J + \prod_{s=1}^d (I - U_s) g', \quad (10.3.6)$$

$$G = m'' + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) m''_J + \prod_{s=1}^d (I - U_s) g'' \quad (10.3.7)$$

where

- $m'$  and  $m''$  belong to  $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_i \mathcal{M}, \mu)$  for each  $i$  in  $\langle d \rangle$ .
- $m'_J$  and  $m''_J$  belong to  $\mathbb{L}^p(\Omega, \prod_{s \in J} T_s \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_i \prod_{s \in J} T_s \mathcal{M}, \mu)$  for each  $i$  in  $\langle d \rangle \setminus J$ .
- $g'$  and  $g''$  belong to  $\mathbb{L}^p(\Omega, \prod_{s=1}^d T_s \mathcal{M}, \mu)$ ;

Since  $\mathbb{E}[M \mid T_{d+1} \mathcal{M}] = 0$  and using (10.3.6), we derive

$$\begin{aligned} \mathbb{E}[m' \mid T_{d+1} \mathcal{M}] &= - \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \mathbb{E} \left[ \prod_{s \in J} (I - U_s) m'_J \mid T_{d+1} \mathcal{M} \right] - \\ &\quad - \mathbb{E} \left[ \prod_{s=1}^d (I - U_s) g' \mid T_{d+1} \mathcal{M} \right]. \end{aligned} \quad (10.3.8)$$

Let  $\emptyset \subsetneq J \subsetneq \langle d \rangle$  be fixed and recall that  $|A|$  is the number of elements of  $A$  for any  $A \subset J$ . So,

$$\begin{aligned} \mathbb{E} \left[ \prod_{s \in J} (I - U_s) m'_J \mid T_{d+1} \mathcal{M} \right] &= \mathbb{E} \left[ \sum_{A \subset J} (-1)^{|A|} \prod_{s \in A} U_s m'_J \mid T_{d+1} \mathcal{M} \right] \\ &= \sum_{A \subset J} (-1)^{|A|} \mathbb{E} \left[ \prod_{s \in A} U_s m'_J \mid T_{d+1} \mathcal{M} \right] \\ &= \sum_{A \subset J} (-1)^{|A|} \prod_{s \in A} U_s \mathbb{E} \left[ m'_J \mid \prod_{s \in A} T_s T_{d+1} \mathcal{M} \right] \end{aligned}$$

where we used the convention  $\prod_{s \in \emptyset} U_s = I$  and the property  $\mathbb{E}[U_s h \mid \mathcal{G}] = U_s \mathbb{E}[h \mid T_s \mathcal{G}]$  for any  $s$  in  $\langle d \rangle$ , any  $\sigma$ -algebra  $\mathcal{G} \subset \mathcal{F}$  and any integrable function  $h$ . Let  $A \subset J$  be fixed. Since  $(T^{-\mathbf{k}} \mathcal{M})_{\mathbf{k} \in \mathbb{Z}^{d+1}}$  is a commuting filtration, we have

$$\mathbb{E} \left[ m'_J \mid \prod_{s \in A} T_s T_{d+1} \mathcal{M} \right] = \mathbb{E} \left[ \mathbb{E} \left[ m'_J \mid \prod_{s \in A} T_s \mathcal{M} \right] \mid T_{d+1} \mathcal{M} \right].$$

Using the measurability of  $m'_J$  with respect to  $\prod_{s \in A} T_s \mathcal{M}$ , we obtain

$$\mathbb{E} \left[ m'_J \mid \prod_{s \in A} T_s T_{d+1} \mathcal{M} \right] = \mathbb{E}[m'_J \mid T_{d+1} \mathcal{M}].$$

Consequently,

$$\mathbb{E} \left[ \prod_{s \in J} (I - U_s) m'_J \mid T_{d+1} \mathcal{M} \right] = \prod_{s \in J} (I - U_s) \mathbb{E}[m'_J \mid T_{d+1} \mathcal{M}]. \quad (10.3.9)$$

Similarly, since  $g'$  is  $\prod_{s=1}^d T_s \mathcal{M}$ -measurable, we have also

$$\mathbb{E} \left[ \prod_{s=1}^d (I - U_s) g' \mid T_{d+1} \mathcal{M} \right] = \prod_{s=1}^d (I - U_s) \mathbb{E}[g' \mid T_{d+1} \mathcal{M}]. \quad (10.3.10)$$

Using (10.3.8), we derive

$$\mathbb{E}[m' \mid T_{d+1} \mathcal{M}] = - \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) \mathbb{E}[m'_J \mid T_{d+1} \mathcal{M}] - \prod_{s=1}^d (I - U_s) \mathbb{E}[g' \mid T_{d+1} \mathcal{M}]. \quad (10.3.11)$$

So, denoting  $m := m' - \mathbb{E}[m' \mid T_{d+1} \mathcal{M}]$  and combining (10.3.6) and (10.3.11) we obtain

$$\begin{aligned} M = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) (m'_J - \mathbb{E}[m'_J \mid T_{d+1} \mathcal{M}]) + \\ + \prod_{s=1}^d (I - U_s) (g' - \mathbb{E}[g' \mid T_{d+1} \mathcal{M}]). \end{aligned} \quad (10.3.12)$$

Moreover,  $m$  is  $\mathcal{M}$ -measurable and  $\mathbb{E}[m \mid T_s \mathcal{M}] = 0$  for each  $s$  in  $\langle d+1 \rangle$ . Combining (10.3.7) and (10.3.12), one can write

$$\begin{aligned} f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) (m'_J - \mathbb{E}[m'_J \mid T_{d+1} \mathcal{M}]) + \prod_{s=1}^d (I - U_s) (g' - \mathbb{E}[g' \mid T_{d+1} \mathcal{M}]) \\ + (I - U_{d+1}) \left( m'' + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) m''_J + \prod_{s=1}^d (I - U_s) g'' \right), \end{aligned}$$

which is of the form (10.2.3) for  $d+1$  instead of  $d$ . Indeed, let  $\emptyset \subsetneq J \subsetneq \langle d+1 \rangle$  be fixed. If  $d+1 \in J$ , we denote

$$m_J = \begin{cases} m'' & \text{if } J = \{d+1\} \\ m''_{J \setminus \{d+1\}} & \text{if } J \setminus \{d+1\} \neq \emptyset \end{cases} \quad (10.3.13)$$

and if  $d+1 \notin J$ , we denote

$$m_J = \begin{cases} m'_J - \mathbb{E}[m'_J \mid T_{d+1} \mathcal{M}] & \text{if } J \neq \langle d \rangle \\ g' - \mathbb{E}[g' \mid T_{d+1} \mathcal{M}] & \text{if } J = \langle d \rangle. \end{cases} \quad (10.3.14)$$

Finally, denoting  $g = g''$ , we obtain

$$f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d+1 \rangle} \prod_{s \in J} (I - U_s) m_J + \prod_{s=1}^{d+1} (I - U_s) g,$$

The proof of Theorem 10.2.4 is complete.  $\square$

*Proof.* Proof of Proposition 10.2.8 For simplicity, we consider only the case  $d = 2$ . Let  $(X_{i,j})_{(i,j) \in \mathbb{Z}^2}$  be an OMD random field with respect to a commuting filtration  $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$ . Let  $(n_1, n_2)$  be fixed in  $\mathbb{N}^2$  and consider  $(Y_i)_{i \in \mathbb{Z}}$  defined for any  $i$  in  $\mathbb{Z}$  by  $Y_i = \sum_{j=0}^{n_2} X_{i,j}$ . One can notice that  $(Y_i)_{i \in \mathbb{Z}}$  is a MD sequence with respect to the filtration  $(\vee_{j \in \mathbb{Z}} \mathcal{F}_{i,j})_{i \in \mathbb{Z}}$ . Consequently, by Rio's inequality (cf. [Rio09]), we have

$$\left\| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} X_{i,j} \right\|_p \leq \kappa \sqrt{p} \left( \sum_{i=0}^{n_1} \|Y_i\|_p^2 \right)^{1/2}.$$

Moreover, since for any  $i$  in  $\mathbb{Z}$ ,  $(X_{i,j})_{j \in \mathbb{Z}}$  is a MD sequence with respect to the filtration  $(\vee_{i \in \mathbb{Z}} \mathcal{F}_{i,j})_{j \in \mathbb{Z}}$ , we have also

$$\|Y_i\|_p = \left\| \sum_{j=0}^{n_2} X_{i,j} \right\|_p \leq \kappa \sqrt{p} \left( \sum_{j=0}^{n_2} \|X_{i,j}\|_p^2 \right)^{1/2}.$$

Consequently, we obtain

$$\left\| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} X_{i,j} \right\|_p \leq \kappa p \left( \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \|X_{i,j}\|_p^2 \right)^{1/2}. \quad (10.3.15)$$

In order to prove the optimality of the constant  $p$  in (10.3.15), arguing as in Wang and Woodroffe [WW13] (Example 1, page 12), we consider a sequence  $(\eta_i)_{i \in \mathbb{Z}}$  of i.i.d. real random variables satisfying  $\mu(\eta_0 = 1) = \mu(\eta_0 = -1) = 1/2$ . Let also  $(\eta'_i)_{i \in \mathbb{Z}}$  be an independent copy of  $(\eta_i)_{i \in \mathbb{Z}}$  and consider the filtrations  $(\mathcal{G}_k)_{k \in \mathbb{Z}}$  and  $(\mathcal{H}_k)_{k \in \mathbb{Z}}$  defined for any  $k$  in  $\mathbb{Z}$  by  $\mathcal{G}_k = \sigma(\eta_s; s \leq k)$  and  $\mathcal{H}_k = \sigma(\eta'_s; s \leq k)$ . For any  $(i, j)$  in  $\mathbb{Z}^2$ , we denote  $Z_{i,j} = \eta_i \eta'_j$ . Then  $(Z_{i,j})_{(i,j) \in \mathbb{Z}^2}$  is an OMD random field with respect to the commuting filtration  $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$  defined by  $\mathcal{F}_{i,j} = \mathcal{G}_i \vee \mathcal{H}_j$  for any  $(i, j)$  in  $\mathbb{Z}^2$ . Let  $C$  be a positive constant such that for any  $(n_1, n_2)$  in  $\mathbb{N}^2$ ,

$$\left\| \sum_{i=0}^{n_1} \eta_i \right\|_p \times \left\| \sum_{j=0}^{n_2} \eta'_j \right\|_p = \left\| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} Z_{i,j} \right\|_p \leq C \left( \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \|Z_{i,j}\|_p^2 \right)^{1/2} \leq C \sqrt{n_1 n_2}.$$

Applying the CLT for i.i.d. real random variables, we derive  $C \geq \|N\|_p^2$  where  $N$  is a standard normal random variable. Since there exists  $\kappa > 0$  such that  $\|N\|_p^2 \geq \kappa p$ , we derive (10.2.6). The proof of Proposition 10.2.8 is complete.  $\square$

*Proof of Proposition 10.2.9.* We start with the following lemma.

**Lemma 10.3.2.** *If  $f$  is a function satisfying the assumptions of Theorem 10.2.4 for some  $p \geq 2$  then there exists a constant  $C_d$  depending only on  $d$  such that*

$$\max \left\{ \|m\|_p, \|m_J\|_p, \|g\|_p \right\} \leq C_d \Delta_d(f, p), \quad (10.3.16)$$

where  $m$ ,  $g$  and  $m_J$  are defined by (10.2.3) and  $\Delta_d(f, p) := \sum_{k \in \mathbb{N}^d} \|\mathbb{E}[f \mid T^k \mathcal{M}]\|_p$ .

*Proof of Lemma 10.3.2.* We prove this lemma by induction on  $d$ . The case  $d = 1$  is a direct consequence of (10.3.4). Let  $d$  be a positive integer and let  $p \geq 2$ . We assume that Lemma 10.3.2 is true for  $d$  and we are going to prove that it is true for  $d + 1$ . Assume that  $\Delta_{d+1}(f, p)$  is finite. Using Lemma 10.3.1 and arguing as in the proof of Theorem 10.2.4, we have

$$f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d+1 \rangle} \prod_{s \in J} (I - U_s) m_J + \prod_{s=1}^{d+1} (I - U_s) g$$

where  $m_J$  is given by (10.3.13) and (10.3.14),  $g = g''$  and  $m := m' - \mathbb{E}[m' \mid T_{d+1}\mathcal{M}]$  (see the last part of the proof of Theorem 10.2.4). Keeping in mind (10.3.2) and arguing as in the proof of Lemma 10.3.1 (see (10.3.5)), we derive

$$\max \{ \Delta_d(M, p), \Delta_d(G, p) \} \leq 2\Delta_{d+1}(f, p). \quad (10.3.17)$$

The induction hypothesis yields  $\|m'\|_p \leq C_d \Delta_d(M, p)$ . Since  $\|m\|_p \leq 2\|m'\|_p$ , using (10.3.17), we obtain

$$\|m\|_p \leq 4C_d \Delta_{d+1}(f, p).$$

Similarly, we have

$$\|g\| = \|g''\|_p \leq C_d \Delta_d(G, p) \leq 2C_d \Delta_{d+1}(f, p).$$

Let  $J$  be a nonempty subset of  $\langle d + 1 \rangle$ .

- If  $d + 1 \in J$  then using (10.3.13) and the induction hypothesis, we have  $\|m_J\|_p \leq C_d \Delta_d(G, p)$ . Hence by (10.3.17),

$$\|m_J\|_p \leq 2C_d \Delta_{d+1}(f, p).$$

- Similarly, using (10.3.14), if  $d + 1 \notin J$  and  $J \neq \langle d \rangle$  then

$$\|m_J\|_p \leq 2\|m'_J\|_p \leq 2C_d \Delta_d(M, p) \leq 4C_d \Delta_{d+1}(f, p)$$

$$\text{and } \|m_{\langle d \rangle}\|_p \leq 2\|g'\|_p \leq 4C_d \Delta_{d+1}(f, p).$$

Finally, it suffices to define  $C_{d+1} := 4C_d$ . The proof of Lemma 10.3.2 is complete.

Without loss of generality, one can write  $X_{\mathbf{i}} = f \circ T^{\mathbf{i}}$  for any  $\mathbf{i}$  in  $\mathbb{Z}^d$  where  $f = X_{\mathbf{0}}$  and  $(T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  is a family of measure-preserving operators on  $\Omega$  such that  $T^{\mathbf{k}} \circ T^{\mathbf{l}} = T^{\mathbf{k}+\mathbf{l}}$  for any  $\mathbf{k}$  and  $\mathbf{l}$  in  $\mathbb{Z}^d$ . Let  $\mathbf{n}$  be fixed in  $\mathbb{N}^d$ . In the sequel, we denote  $\lambda_{\mathbf{n}} = \{\mathbf{i} \in \mathbb{N}^d; \mathbf{0} \preceq \mathbf{i} \preceq \mathbf{n}\}$  and  $S_{\mathbf{n}}(h) = \sum_{\mathbf{i} \in \lambda_{\mathbf{n}}} h \circ T^{\mathbf{i}}$  for any function  $h$  defined on  $\Omega$ . Applying Theorem 10.2.4, we have

$$S_{\mathbf{n}}(f) = S_{\mathbf{n}}(m) + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} S_{\mathbf{n}} \left( \prod_{s \in J} (I - U_s) m_J \right) + S_{\mathbf{n}} \left( \prod_{s=1}^d (I - U_s) g \right). \quad (10.3.18)$$

Let  $\emptyset \subsetneq J \subsetneq \langle d \rangle$  and  $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$  be fixed and define  $\mathbf{k}^{(J)} = (k_i)_{i \in \langle d \rangle \setminus J}$ . We have

$$S_{\mathbf{n}} \left( \prod_{s \in J} (I - U_s) m_J \right) = \prod_{s \in J} (I - U_s^{n_s+1}) \sum_{\mathbf{0} \preceq \mathbf{k}^{(J)} \preceq \mathbf{n}^{(J)}} \prod_{i \in \langle d \rangle \setminus J} U_i^{k_i} m_J.$$

Since the operator  $\prod_{s \in J} (I - U_s^{n_s+1})$  may be written as a sum of  $2^{|J|}$  isometries, the inequality

$$\left\| S_{\mathbf{n}} \left( \prod_{s \in J} (I - U_s) m_J \right) \right\|_p \leq 2^{|J|} \left\| \sum_{\mathbf{0} \preceq \mathbf{k}^{(J)} \preceq \mathbf{n}^{(J)}} \prod_{i \in \langle d \rangle \setminus J} U_i^{k_i} m_J \right\|_p \quad (10.3.19)$$

takes place and an application of Proposition 10.2.8 yields

$$\left\| \sum_{\mathbf{0} \preccurlyeq \mathbf{k}^{(J)} \preccurlyeq \mathbf{n}^{(J)}} \prod_{i \in \langle d \rangle \setminus J} U_i^{k_i} m_J \right\|_p \leq \kappa p^{d/2} \left( \prod_{s \in J} n_s \right)^{1/2} \|m_J\|_p. \quad (10.3.20)$$

Combining (10.3.19) and (10.3.20), we get

$$\left\| S_{\mathbf{n}} \left( \prod_{s \in J} (I - U_s) m_J \right) \right\|_p \leq 2^{|J|} \kappa p^{d/2} |\mathbf{n}|^{1/2} \|m_J\|_p. \quad (10.3.21)$$

Moreover, since

$$S_{\mathbf{n}} \left( \prod_{s=1}^d (I - U_s) g \right) = \prod_{s=1}^d (I - U_s^{n_s+1}) g$$

and  $\prod_{s=1}^d (I - U_s^{n_s+1})$  is a sum of  $2^d$  isometries, it follows that

$$\left\| S_{\mathbf{n}} \left( \prod_{s=1}^d (I - U_s) g \right) \right\|_p \leq 2^d \kappa p^{d/2} |\mathbf{n}|^{1/2} \|g\|_p. \quad (10.3.22)$$

By Proposition 10.2.8, we have also

$$\|S_{\mathbf{n}}(m)\|_p \leq \kappa p^{d/2} |\mathbf{n}|^{1/2} \|m\|_p. \quad (10.3.23)$$

Combining (10.3.18), (10.3.21), (10.3.22) and (10.3.23), we obtain

$$\|S_{\mathbf{n}}(f)\|_p \leq \kappa p^{d/2} |\mathbf{n}|^{1/2} \left( \|m\|_p + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} 2^{|J|} \|m_J\|_p + 2^d \|g\|_p \right). \quad (10.3.24)$$

Finally, applying Lemma 10.3.2 yields

$$\|S_{\mathbf{n}}(f)\|_p \leq \kappa p^{d/2} |\mathbf{n}|^{1/2} C_d \Delta_d(f, p) \left( 1 + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} 2^{|J|} + 2^d \right).$$

The proof of Proposition 10.2.9 is complete.

*Proof.* Proof of Proposition 10.2.11 First, since  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  is stationary, we have

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} |\mathbb{E}[X_{\mathbf{0}} X_{\mathbf{k}}]| \leq 2^d \sum_{\mathbf{k} \in \mathbb{N}^d} |\mathbb{E}[X_{\mathbf{0}} X_{\mathbf{k}}]| \leq \|X_{\mathbf{0}}\|_2 \sum_{\mathbf{k} \in \mathbb{N}^d} \|\mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{-\mathbf{k}}]\|_2 < \infty.$$

Let  $\mathbf{n} = (n_1, \dots, n_d)$  be fixed in  $\mathbb{N}^d$ . Then,

$$|\mathbf{n}|^{-1} \mathbb{E} \left( \left( \sum_{\mathbf{k} \in \Lambda_{\mathbf{n}}} X_{\mathbf{k}} \right)^2 \right) = \sum_{\mathbf{k} \in \mathbb{Z}^d} |\mathbf{n}|^{-1} |\Lambda_{\mathbf{n}} \cap (\Lambda_{\mathbf{n}} - \mathbf{k})| \mathbb{E}(X_{\mathbf{0}} X_{\mathbf{k}})$$

where  $\Lambda_{\mathbf{n}} = \{\mathbf{i} \in \mathbb{N}^d; \mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}\}$  and  $\Lambda_{\mathbf{n}} - \mathbf{k} = \{\mathbf{i} - \mathbf{k}; \mathbf{i} \in \Lambda_{\mathbf{n}}\}$  for any  $\mathbf{k}$  in  $\mathbb{Z}^d$ . Moreover,

$$|\mathbf{n}|^{-1} |\Lambda_{\mathbf{n}} \cap (\Lambda_{\mathbf{n}} - \mathbf{k})| \mathbb{E}(X_{\mathbf{0}} X_{\mathbf{k}}) \leq |\mathbb{E}(X_{\mathbf{0}} X_{\mathbf{k}})|$$

and  $\lim_{|\mathbf{n}| \rightarrow +\infty} |\mathbf{n}|^{-1} |\Lambda_{\mathbf{n}} \cap (\Lambda_{\mathbf{n}} - \mathbf{k})| = 1$  for any  $\mathbf{k}$  in  $\mathbb{Z}^d$ . Finally, it suffices to apply the Lebesgue convergence theorem. The proof of Proposition 10.2.11 is complete.  $\square$

*Proof.* Proof of Theorem 10.2.12

Let  $(\Omega, \mathcal{F}, \mu, \{T^k\}_{k \in \mathbb{Z}^d})$  be a dynamical system (that is,  $(\Omega, \mathcal{F}, \mu)$  is a probability space and  $T^k: \Omega \rightarrow \Omega$  is a measure-preserving transformation for any  $k$  in  $\mathbb{Z}^d$  satisfying  $T^i \circ T^j = T^{i+j}$  for any  $i$  and any  $j$  in  $\mathbb{Z}^d$ ) and let  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  be a field of i.i.d. real random variables defined on  $(\Omega, \mathcal{F}, \mu)$ . Let  $\mathcal{M} \subset \mathcal{F}$  be the  $\sigma$ -algebra generated by the random variables  $\varepsilon_i$  for  $i \preceq 0$  and let  $f: \Omega \rightarrow \mathbb{R}$  be  $\mathcal{M}$ -measurable. We consider the stationary real random field  $(f \circ T^i)_{i \in \mathbb{Z}^d}$  and the partial sum process  $\{S_n(f, t); t \in [0, 1]^d\}_{n \geq 1}$  defined for any integer  $n \geq 1$  and any  $t$  in  $[0, 1]^d$  by

$$S_n(f, t) = \sum_{i \in \langle n \rangle^d} \lambda([0, nt] \cap R_i) f \circ T^i \quad (10.3.25)$$

where  $\lambda$  is the Lebesgue measure on  $\mathbb{R}^d$  and  $R_i = ]i_1 - 1, i_1] \times \cdots \times ]i_d - 1, i_d]$  is the unit cube with upper corner  $i = (i_1, \dots, i_d)$  in  $\langle n \rangle^d$ . As usual, we have to prove the convergence of the finite-dimensional laws and the tightness of the sequence of processes  $\{n^{-d/2} S_n(f, t); t \in [0, 1]^d\}_{n \geq 1}$ . We start with the tightness property: it suffices to establish for any  $\varepsilon > 0$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^d \\ |s - t| < \delta}} n^{-d/2} |S_n(f, s) - S_n(f, t)| > \varepsilon \right) = 0$$

where  $|x| = \max_{k \in \langle d \rangle} |x_k|$  for any  $x = (x_1, \dots, x_d)$  in  $[0, 1]^d$ . For simplicity, we are going to consider only the case  $d = 2$ . By Theorem 10.2.4, we have

$$f = m + (I - U_1)m_1 + (I - U_2)m_2 + (I - U_1)(I - U_2)g, \quad (10.3.26)$$

where  $m, m_1, m_2$  and  $g$  are square-integrable functions defined on  $\Omega$  such that  $(U^i m)_{i \in \mathbb{Z}^2}$  is an OMD random field and  $(U_2^k m_1)_{k \in \mathbb{Z}}$  and  $(U_1^k m_2)_{k \in \mathbb{Z}}$  are MD sequences. In the sequel, for any real  $x$ , we denote by  $[x]$  the integer part of  $x$ . Let  $n \geq 1$  and  $t = (t_1, t_2)$  in  $[0, 1]^2$ . For any  $1 \leq i \leq [nt_1] + 1$  and any  $1 \leq j \leq [nt_2] + 1$ , we denote  $\lambda_{i,j}(t) = \lambda([0, nt] \cap R_{(i,j)})$ . We have

$$S_n((I - U_1)m_1, t) = \sum_{i=1}^{[nt_1]+1} \sum_{j=1}^{[nt_2]+1} \lambda_{i,j}(t) U^{(i,j)} (I - U_1)m_1 \quad (10.3.27)$$

$$= \sum_{j=1}^{[nt_2]+1} U_2^j \sum_{i=1}^{[nt_1]+1} \lambda_{i,j}(t) (U_1^i m_1 - U_1^{i+1} m_1). \quad (10.3.28)$$

Using Abel's transformation and noting that  $\lambda_{i+1,j}(t) = \lambda_{i,j}(t)$  for any  $1 \leq i \leq [nt_1] - 1$  and any  $1 \leq j \leq [nt_2] + 1$ , we obtain that  $S_n((I - U_1)m_1, t)$  equals

$$\begin{aligned} & \sum_{j=1}^{[nt_2]+1} U_2^j \left\{ \lambda_{[nt_1]+1,j}(t) (U_1 m_1 - U_1^{[nt_1]+2} m_1) - \sum_{i=1}^{[nt_1]} (U_1 m_1 - U_1^{i+1} m_1) (\lambda_{i+1,j}(t) - \lambda_{i,j}(t)) \right\} \\ &= \sum_{j=1}^{[nt_2]+1} U_2^j \left\{ \lambda_{[nt_1]+1,j}(t) (U_1 m_1 - U_1^{[nt_1]+2} m_1) - \right. \\ & \quad \left. - (U_1 m_1 - U_1^{[nt_1]+1} m_1) (\lambda_{[nt_1]+1,j}(t) - \lambda_{[nt_1],j}(t)) \right\} \\ &= U_1 (I - U_1^{[nt_1]+1}) \sum_{j=1}^{[nt_2]+1} \lambda_{[nt_1]+1,j}(t) U_2^j m_1 - \\ & \quad - U_1 (I - U_1^{[nt_1]}) \sum_{j=1}^{[nt_2]+1} (\lambda_{[nt_1]+1,j}(t) - \lambda_{[nt_1],j}(t)) U_2^j m_1. \end{aligned}$$



Moreover, since  $\lambda_{i,j}(t) = \lambda_{i,1}(t)$  for any  $1 \leq i \leq [nt_1] + 1$  and any  $1 \leq j \leq [nt_2]$ , we derive

$$\begin{aligned} S_n((I - U_1)m_1, t) &= U_1(I - U_1^{[nt_1]+1})\lambda_{[nt_1]+1,1}(t) \sum_{j=1}^{[nt_2]} U_2^j m_1 \\ &\quad + U_1(I - U_1^{[nt_1]+1})\lambda_{[nt_1]+1,[nt_2]+1}(t) U_2^{[nt_2]+1} m_1 \\ &\quad - U_1(I - U_1^{[nt_1]}) (\lambda_{[nt_1]+1,1}(t) - \lambda_{[nt_1],1}(t)) \sum_{j=1}^{[nt_2]} U_2^j m_1 \\ &\quad - U_1(I - U_1^{[nt_1]}) (\lambda_{[nt_1]+1,[nt_2]+1}(t) - \lambda_{[nt_1],[nt_2]+1}(t)) U_2^{[nt_2]+1} m_1. \end{aligned}$$

So, we obtain

$$\sup_{t \in [0,1]^2} |S_n((I - U_1)m_1, t)| \leq 4 \max_{1 \leq l, k \leq n+2} U_1^l U_2^k |m_1| + 4 \max_{1 \leq l, k \leq n+2} U_1^l \left| \sum_{j=1}^k U_2^j m_1 \right|. \quad (10.3.29)$$

Let  $x > 0$  be fixed. Since  $m_1 \in \mathbb{L}^2(\Omega, \mathcal{F}, \mu)$ , we have

$$\mu \left( \max_{1 \leq l, k \leq n+2} U_1^l U_2^k |m_1| > nx \right) \leq \kappa n^2 \mu(m_1^2 > n^2 x^2) \rightarrow 0. \quad (10.3.30)$$

In the other part,

$$\begin{aligned} \mu \left( \max_{1 \leq l, k \leq n+2} U_1^l \left| \sum_{j=1}^k U_2^j m_1 \right| > xn \right) &= \\ &= \mu \left( \max_{1 \leq l \leq n+2} U_1^l \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2 > nx^2 \right). \end{aligned} \quad (10.3.31)$$

**Lemma 10.3.3.** *Let  $(Z_n)_{n \geq 1}$  be a uniformly integrable sequence of real random variables. For any  $s$  in  $\langle d \rangle$ ,*

$$\limsup_{n \rightarrow \infty} \mu \left( \max_{1 \leq i_1, \dots, i_s \leq n} U_1^{i_1} \dots U_s^{i_s} |Z_n| > n^s \right) = 0.$$

*Proof of Lemma 10.3.3.* Let  $n$  be a positive integer. For any  $s$  in  $\langle d \rangle$ , we denote

$$p_n(s) := \mu \left( \max_{1 \leq i_1, \dots, i_s \leq n} U_1^{i_1} \dots U_s^{i_s} |Z_n| > n^s \right).$$

Let  $R$  be a positive real number. We have

$$p_n(s) \leq \frac{2R}{n^s} + n^s \mu \left( |Z_n| \mathbf{1}\{|Z_n| > R\} > \frac{n^s}{2} \right) \leq \frac{2R}{n^s} + 2 \sup_{k \geq 1} \mathbb{E} [|Z_k| \mathbf{1}\{|Z_k| > R\}].$$

Consequently,  $\limsup_{n \rightarrow \infty} p_n(s) \leq 2 \sup_{k \geq 1} \mathbb{E} [|Z_k| \mathbf{1}\{|Z_k| > R\}] \xrightarrow{R \rightarrow \infty} 0$ . The proof of Lemma 10.3.3 is complete.

**Lemma 10.3.4.** *The sequence  $\left\{ \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2; n \geq 1 \right\}$  is uniformly integrable.*

*Proof of Lemma 10.3.4.* Since  $(U_2^k m_1)_{k \in \mathbb{Z}}$  is a MD sequence, using Doob's inequality, we derive

$$\left\| \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right\|_2 \leq 2 \left\| \sum_{j=1}^{n+2} U_2^j m_1 \right\|_2 \leq \kappa \sqrt{n} \|m_1\|_2.$$

So,  $\left\{ \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2 ; n \geq 1 \right\}$  is bounded in  $\mathbb{L}^1(\Omega, \mathcal{F}, \mu)$ . Let  $M$  be a fixed positive constant. We have  $m_1 = m'_1 + m''_1$  where

$$\begin{aligned} m'_1 &= m_1 \mathbf{1} \{|m_1| \leq M\} - \mathbb{E} [m_1 \mathbf{1} \{|m_1| \leq M\} | T_2 \mathcal{M}] \\ m''_1 &= m_1 \mathbf{1} \{|m_1| > M\} - \mathbb{E} [m_1 \mathbf{1} \{|m_1| > M\} | T_2 \mathcal{M}]. \end{aligned}$$

Moreover, if  $A$  belongs to  $\mathcal{F}$  then

$$\begin{aligned} \int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2 d\mu &\leq 2 \int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m'_1 \right| \right)^2 d\mu \\ &\quad + 2 \int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m''_1 \right| \right)^2 d\mu. \end{aligned}$$

Since  $(U_2^k m'_1)_{k \in \mathbb{Z}}$  and  $(U_2^k m''_1)_{k \in \mathbb{Z}}$  are MD sequences, using Schwarz's inequality, we obtain

$$\begin{aligned} \int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2 d\mu &\leq 2 \left\| \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m'_1 \right| \right\|_4^2 \sqrt{\mu(A)} \\ &\quad + 2 \left\| \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m''_1 \right| \right\|_2^2. \end{aligned}$$

Keeping in mind that  $m'_1$  is bounded by  $M$  and using again Doob's inequality, there exists a positive constant  $\kappa_0$  such that

$$\int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2 d\mu \leq \kappa_0 \left( M^2 \sqrt{\mu(A)} + \mathbb{E} [m_1^2 \mathbf{1} \{|m_1| > M\}] \right).$$

Let  $\varepsilon > 0$  be fixed and let  $M > 0$  such that  $\kappa_0 \mathbb{E} [m_1^2 \mathbf{1} \{|m_1| > M\}] \leq \frac{\varepsilon}{2}$ . One can choose the measurable set  $A$  in  $\mathcal{F}$  such that  $\kappa_0 M^2 \sqrt{\mu(A)} \leq \frac{\varepsilon}{2}$  and consequently

$$\sup_{n \geq 1} \int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n+2} \left| \sum_{j=1}^k U_2^j m_1 \right| \right)^2 d\mu \leq \varepsilon.$$

The proof of Lemma 10.3.4 is complete.

Combining (10.3.29), (10.3.30), (10.3.31), Lemma 10.3.3 and Lemma 10.3.4, we obtain

$$\limsup_{n \rightarrow \infty} \mu \left( \sup_{t \in [0,1]^2} |S_n((I - U_1)m_1, t)| > xn \right) = 0. \quad (10.3.32)$$

In a similar way, we derive also

$$\limsup_{n \rightarrow \infty} \mu \left( \sup_{t \in [0,1]^2} |S_n((I - U_2)m_2, t)| > xn \right) = 0. \quad (10.3.33)$$

Now, noting that  $\lambda_{i,j}(t) = \lambda_{i,1}(t)$  for any  $1 \leq i \leq [nt_1] + 1$  and any  $1 \leq j \leq [nt_2]$ , we have  $S_n((I - U_1)(I - U_2)g, t)$  equals

$$\begin{aligned} & \sum_{i=1}^{[nt_1]+1} \sum_{j=1}^{[nt_2]+1} \lambda_{i,j}(t) U^{(i,j)}(I - U_1)(I - U_2)g \\ &= \sum_{i=1}^{[nt_1]+1} U_1^i(I - U_1) \left( \lambda_{i,1}(t) \sum_{j=1}^{[nt_2]} (U_2^j - U_2^{j+1})g + \lambda_{i,[nt_2]+1}(t) U_2^{[nt_2]+1}(I - U_2)g \right) \\ &= U_2(I - U_2^{[nt_2]}) \sum_{i=1}^{[nt_1]+1} \lambda_{i,1}(t) (U_1^i - U_1^{i+1})g + U_2^{[nt_2]+1}(I - U_2) \sum_{i=1}^{[nt_1]+1} \lambda_{i,[nt_2]+1}(t) (U_1^i - U_1^{i+1})g. \end{aligned}$$

Since  $\lambda_{i,j}(t) = \lambda_{1,j}(t)$  for any  $1 \leq i \leq [nt_1]$  and any  $1 \leq j \leq [nt_2] + 1$ , we derive

$$\begin{aligned} S_n((I - U_1)(I - U_2)g, t) &= \lambda_{1,1}(t) U_2(I - U_2^{[nt_2]}) U_1(I - U_1^{[nt_1]})g \\ &\quad + \lambda_{[nt_1]+1,1}(t) U_2(I - U_2^{[nt_2]}) U_1^{[nt_1]+1}(I - U_1)g \\ &\quad + \lambda_{1,[nt_2]+1}(t) U_2^{[nt_2]+1}(I - U_2) U_1(I - U_1^{[nt_1]+1})g \\ &\quad + \lambda_{[nt_1]+1,[nt_2]+1}(t) U_2^{[nt_2]+1}(I - U_2) U_1^{[nt_1]+1}(I - U_1)g. \end{aligned}$$

Thus

$$\sup_{t \in [0,1]^2} |S_n((I - U_1)(I - U_2)g, t)| \leq \kappa \max_{1 \leq k, l \leq n+2} U_1^k U_2^l |g|$$

and for any positive  $x$ ,

$$\lim_{n \rightarrow \infty} \mu \left( \sup_{t \in [0,1]^2} |S_n((I - U_1)(I - U_2)g, t)| > xn \right) \leq \kappa n^2 \mu(g^2 > n^2 x^2) = 0. \quad (10.3.34)$$

Now, it suffices to prove the tightness of the process  $\{\frac{1}{n} S_n(m, t); t \in [0, 1]^2\}_{n \geq 1}$ . That is, for any positive  $x$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0,1]^2 \\ |s-t| < \delta}} |S_n(m, s) - S_n(m, t)| > xn \right) = 0. \quad (10.3.35)$$

Let  $n$  be a positive integer and let  $s = (s_1, s_2)$  and  $t = (t_1, t_2)$  be fixed in  $[0, 1]^2$ . We denote  $\Delta_n(s, t) = S_n(m, s) - S_n(m, t)$  and for any  $i$  and  $j$  in  $\langle n \rangle$ ,

$$\beta_{i,j} = \lambda_{i,j}(s) - \lambda_{i,j}(t) = \lambda([0, ns] \cap R_{(i,j)}) - \lambda([0, nt] \cap R_{(i,j)}).$$

Noting that  $\beta_{i,j} = 0$  for any  $1 \leq i \leq [n(s_1 \wedge t_1)]$  and any  $1 \leq j \leq [n(s_2 \wedge t_2)]$ , we have  $\Delta_n(s, t) = \Delta'_n(s, t) + \Delta''_n(s, t)$  where

$$\Delta'_n(s, t) = \sum_{i=[n(s_1 \wedge t_1)]+1}^{[n(s_1 \vee t_1)]+1} \sum_{j=1}^{[n(s_2 \wedge t_2)]+1} \beta_{i,j} U^{(i,j)} m \quad \text{and} \quad (10.3.36)$$

$$\Delta''_n(s, t) = \sum_{i=1}^{[n(s_1 \wedge t_1)]+1} \sum_{j=[n(s_2 \wedge t_2)]+1}^{[n(s_2 \vee t_2)]+1} \beta_{i,j} U^{(i,j)} m. \quad (10.3.37)$$

Moreover,  $\Delta'_n(s, t) = \Delta'_{1,n}(s, t) + \Delta'_{2,n}(s, t) + \Delta'_{3,n}(s, t) + \Delta'_{4,n}(s, t)$  where

$$\begin{aligned}\Delta'_{1,n}(s, t) &= \sum_{i=[n(s_1 \wedge t_1)]+2}^{[n(s_1 \vee t_1)]} \sum_{j=1}^{[n(s_2 \wedge t_2)]} \beta_{i,j} U^{(i,j)} m \\ \Delta'_{2,n}(s, t) &= \sum_{j=1}^{[n(s_2 \wedge t_2)]} \beta_{[n(s_1 \vee t_1)]+1,j} U^{([n(s_1 \vee t_1)]+1,j)} m \\ \Delta'_{3,n}(s, t) &= \sum_{j=1}^{[n(s_2 \wedge t_2)]} \beta_{[n(s_1 \wedge t_1)]+1,j} U^{([n(s_1 \wedge t_1)]+1,j)} m \\ \Delta'_{4,n}(s, t) &= \sum_{i=[n(s_1 \wedge t_1)]+1}^{[n(s_1 \vee t_1)]+1} \beta_{i,[n(s_2 \wedge t_2)]+1} U^{(i,[n(s_2 \wedge t_2)]+1)} m.\end{aligned}$$

Let  $\alpha$  in  $\{-1, +1\}$  such that  $\beta_{i,j} = \alpha$  if  $[n(s_1 \wedge t_1)] + 2 \leq i \leq [n(s_1 \vee t_1)]$  and  $1 \leq j \leq [n(s_2 \wedge t_2)]$ . So,

$$\Delta'_{1,n}(s, t) = \alpha \sum_{i=[n(s_1 \wedge t_1)]+2}^{[n(s_1 \vee t_1)]} \sum_{j=1}^{[n(s_2 \wedge t_2)]} U^{(i,j)} m$$

and for any positive  $x$ ,

$$\begin{aligned}\mu \left( \sup_{\substack{s,t \in [0,1]^2 \\ |s-t| < \delta}} |\Delta'_{1,n}(s, t)| > nx \right) &\leq \sum_{k=0}^{\lfloor \frac{1}{\delta} \rfloor} \mu \left( \max_{\substack{1 \leq p \leq n \\ r \in [0, \delta]}} \left| \sum_{i=[nk\delta]+2}^{[n(k\delta+r)]} \sum_{j=1}^p U^{(i,j)} m \right| > nx \right) \\ &= \sum_{k=0}^{\lfloor \frac{1}{\delta} \rfloor} \mu \left( \max_{\substack{1 \leq p \leq n \\ r \in [0, \delta]}} \left| \sum_{i=1}^{[n(k\delta+r)]-[nk\delta]-1} \sum_{j=1}^p U^{(i,j)} m \right| > nx \right).\end{aligned}$$

Since  $[n(k\delta+r)] - [nk\delta] - 1$  is an integer smaller than  $[nr]$ , we obtain

$$\begin{aligned}\mu \left( \sup_{\substack{s,t \in [0,1]^2 \\ |s-t| < \delta}} |\Delta'_{1,n}(s, t)| > nx \right) &\leq \left(1 + \frac{1}{\delta}\right) \mu \left( \max_{\substack{1 \leq p \leq n \\ 1 \leq q \leq [n\delta]}} \left| \sum_{i=1}^q \sum_{j=1}^p U^{(i,j)} m \right| > nx \right) \\ &= \left(1 + \frac{1}{\delta}\right) \mu \left( \max_{\substack{1 \leq p \leq n \\ 1 \leq q \leq [n\delta]}} \left( \frac{1}{n\sqrt{\delta}} \sum_{i=1}^q \sum_{j=1}^p U^{(i,j)} m \right)^2 > \frac{x^2}{\delta} \right) \\ &\leq \left( \frac{1+\delta}{x^2} \right) \mathbb{E}_{\frac{x^2}{\delta}} \left( \max_{\substack{1 \leq p \leq n \\ 1 \leq q \leq [n\delta]}} \left( \frac{1}{n\sqrt{\delta}} \sum_{i=1}^q \sum_{j=1}^p U^{(i,j)} m \right)^2 \right)\end{aligned}$$

where we used the notation  $\mathbb{E}_A(Z) = \mathbb{E}[Z \mathbf{1}\{|Z| > A\}]$  for any  $A > 0$  and any  $Z$  in  $\mathbb{L}^1(\Omega, \mathcal{F}, \mu)$ .

**Lemma 10.3.5.** *The family  $\left\{ \max_{\substack{1 \leq p \leq n \\ 1 \leq q \leq [n\delta]}} \left( \frac{1}{n\sqrt{\delta}} \sum_{i=1}^q \sum_{j=1}^p U^{(i,j)} m \right)^2 ; n \geq 1, \delta > 0 \right\}$  is uniformly integrable.*

*Proof of Lemma 10.3.5.* The proof follows the same lines as the proof of Lemma 10.3.4 using Cairoli's maximal inequality for orthomartingales (see [Kho02], Theorem 2.3.1) instead of Doob's inequality for martingales. It is left to the reader.

So, we obtain

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{1,n}(s, t)| > nx \right) = 0. \quad (10.3.38)$$

In the other part, since  $\beta_{[n(s_1 \vee t_1)]+1, j} = \beta_{[n(s_1 \vee t_1)]+1, 1}$  for any  $1 \leq j \leq [n(s_2 \wedge t_2)]$ , we have

$$\Delta'_{2,n}(s, t) = \beta_{[n(s_1 \vee t_1)]+1, 1} U_1^{[n(s_1 \vee t_1)]+1} \sum_{j=1}^{[n(s_2 \wedge t_2)]} U_2^j m$$

and consequently

$$\sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{2,n}(s, t)| \leq \max_{\substack{1 \leq k \leq n+1 \\ 1 \leq l \leq n}} U_1^k \left| \sum_{j=1}^l U_2^j m \right|.$$

So,

$$\mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{2,n}(s, t)| > nx \right) \leq \mu \left( \max_{1 \leq k \leq n+1} U_1^k \left( \max_{1 \leq l \leq n} \frac{1}{\sqrt{n}} \left| \sum_{j=1}^l U_2^j m \right| \right)^2 > nx^2 \right). \quad (10.3.39)$$

Since  $(U_2^k m)_{k \in \mathbb{Z}}$  is a MD sequence, arguing as in Lemma 10.3.4, the sequence

$$\left\{ \left( \max_{1 \leq l \leq n} \frac{1}{\sqrt{n}} \left| \sum_{j=1}^l U_2^j m \right| \right)^2 \right\}_{n \geq 1} \quad (10.3.40)$$

is uniformly integrable. Combining (10.3.39) and Lemma 10.3.3, we derive that for any  $\delta > 0$ ,

$$\limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{2,n}(s, t)| > nx \right) = 0. \quad (10.3.41)$$

Similarly, we have also

$$\limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{3,n}(s, t)| > nx \right) = 0 \quad (10.3.42)$$

for any  $\delta > 0$ . Moreover, for any  $[n(s_1 \wedge t_1)] + 1 \leq i \leq [n(s_1 \vee t_1)]$ , we have  $\beta_{i, [n(s_2 \wedge t_2)]+1} = \beta_{[n(s_1 \wedge t_1)]+1, [n(s_2 \wedge t_2)]+1}$  and consequently

$$\begin{aligned} \Delta'_{4,n}(s, t) &= \beta_{[n(s_1 \wedge t_1)]+1, [n(s_2 \wedge t_2)]+1} U_2^{[n(s_2 \wedge t_2)]+1} \sum_{i=[n(s_1 \wedge t_1)]+1}^{[n(s_1 \vee t_1)]} U_1^i m \\ &\quad + \beta_{[n(s_1 \vee t_1)]+1, [n(s_2 \wedge t_2)]+1} U^{([n(s_1 \vee t_1)]+1, [n(s_2 \wedge t_2)]+1)} m \end{aligned}$$

and

$$\begin{aligned} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{4,n}(s, t)| > nx \right) &\leq \mu \left( \max_{1 \leq k \leq n+1} U_2^k \left( \max_{1 \leq l \leq [n\delta]} \frac{1}{\sqrt{n\delta}} \left| \sum_{j=1}^l U_1^j m \right| \right)^2 > \frac{nx^2}{2\delta} \right) \\ &\quad + 2n^2 \mu \left( m^2 > \frac{n^2 x^2}{4} \right) \end{aligned}$$

Arguing as in Lemma 10.3.3, the family  $\left\{ \left( \max_{1 \leq l \leq [n\delta]} \frac{1}{\sqrt{n\delta}} \left| \sum_{j=1}^l U_1^j m \right| \right)^2 ; n \geq 1, \delta > 0 \right\}$  is uniformly integrable since  $(U_1^k m)_{k \in \mathbb{Z}}$  is a MD sequence. By Lemma 10.3.4, we obtain for any  $\delta > 0$ ,

$$\limsup_{n \rightarrow \infty} \mu \left( \max_{1 \leq k \leq n+1} U_2^k \left( \max_{1 \leq l \leq [n\delta]} \frac{1}{\sqrt{n\delta}} \left| \sum_{j=1}^l U_1^j m \right| \right)^2 > \frac{nx^2}{2\delta} \right) = 0.$$

Moreover,  $n^2 \mu \left( m^2 > \frac{n^2 x^2}{4} \right)$  goes to zero as  $n$  goes to infinity since  $m$  belongs to  $\mathbb{L}^2(\Omega, \mathcal{F}, \mu)$ . Consequently, for any  $\delta > 0$ ,

$$\limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_{4,n}(s, t)| > nx \right) = 0. \quad (10.3.43)$$

Combining (10.3.38), (10.3.41), (10.3.42) and (10.3.43), we obtain

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta'_n(s, t)| > nx \right) = 0. \quad (10.3.44)$$

Similarly, one can check that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mu \left( \sup_{\substack{s, t \in [0, 1]^2 \\ |s-t| < \delta}} |\Delta''_n(s, t)| > nx \right) = 0. \quad (10.3.45)$$

Finally, keeping in mind  $\Delta_n(s, t) = \Delta'_n(s, t) + \Delta''_n(s, t)$  and combining (10.3.44) and (10.3.45), we obtain (10.3.35). Now, we are going to prove the convergence of the finite-dimensional laws. In fact, combining (10.2.3), (10.3.32), (10.3.33) and (10.3.34), we have

$$\limsup_{n \rightarrow \infty} \mu \left( \sup_{t \in [0, 1]^2} |S_n(f - m, t)| > xn \right) = 0. \quad (10.3.46)$$

Let  $(t, n)$  be fixed in  $[0, 1]^2 \times \mathbb{N}^2$  and denote  $\Lambda_n(t) = [0, nt] \cap \mathbb{N}^2$ . We have

$$S_n(m, t) - \sum_{i \in \Lambda_n(t)} m \circ T^i = \sum_{i \in W_n(t)} a_i m \circ T^i \quad (10.3.47)$$

where  $a_i = \lambda([0, nt] \cap R_i) - \mathbf{1}_{\{i \in \Lambda_n(t)\}}$  and  $W_n(t)$  is the set of all  $i$  in  $\langle n \rangle^d$  such that  $R_i \cap [0, nt] \neq \emptyset$  and  $R_i \cap (\mathbb{R}^2 \setminus [0, nt]) \neq \emptyset$ . Noting that  $|a_i| \leq 1$  and combining (10.3.47) and Proposition 10.2.8, we obtain

$$\left\| S_n(m, t) - \sum_{i \in \Lambda_n(t)} m \circ T^i \right\|_2 \leq C \|m\|_2 \left( \sum_{i \in W_n(t)} a_i^2 \right)^{1/2} \leq C \|m\|_2 \sqrt{|W_n(t)|} \quad (10.3.48)$$

where  $C$  is a positive constant and  $|W_n(t)|$  denotes the number of elements in  $W_n(t)$ . Since  $|W_n(t)| = O(n)$ , we derive

$$n^{-1} \left\| S_n(m, t) - \sum_{i \in \Lambda_n(t)} m \circ T^i \right\|_2 = O\left(\frac{1}{\sqrt{n}}\right). \quad (10.3.49)$$

Finally, combining (10.3.46), (10.3.49) and Proposition 10.2.11 and arguing as in the proof of Theorem 4.1 by Wang and Woodroffe [WW13], we derive the convergence of the finite dimensional laws of  $\{n^{-1}S_n(f, \mathbf{t}); \mathbf{t} \in [0, 1]^2\}$ . The proof of Theorem 10.2.12 is complete.  $\square$

*Proof.* Proof of Proposition 10.2.14 Since  $X_{\mathbf{0}} = \sum_{\mathbf{j} \in \mathbb{N}^d} a_{\mathbf{j}} \varepsilon_{-\mathbf{j}}$  where  $(a_{\mathbf{j}})_{\mathbf{j} \in \mathbb{N}^d}$  is a family of real numbers satisfying  $\sum_{\mathbf{j} \in \mathbb{N}^d} a_{\mathbf{j}}^2 < \infty$  and  $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  is an i.i.d. zero-mean real random field, we have

$$\mathbb{E}(X_{\mathbf{0}} \mid \mathcal{F}_{-\mathbf{k}}) = \sum_{\mathbf{j} \succ \mathbf{k}} a_{\mathbf{j}} \varepsilon_{-\mathbf{j}} \quad (10.3.50)$$

where  $\mathbf{k} \in \mathbb{N}^d$  and  $\mathcal{F}_{-\mathbf{k}}$  is the  $\sigma$ -algebra generated by  $\varepsilon_{\mathbf{i}}$  for any  $\mathbf{i} \preccurlyeq -\mathbf{k}$ . We recall the Rosenthal's inequality ([HH80], Theorem 2.12): for any  $p \geq 2$ , there exists a positive constant  $C$  depending only on  $p$  such that if  $(Y_j)_{j \geq 1}$  is a sequence of independent zero-mean random variables and  $n$  is a positive integer then

$$\frac{1}{C} \left( \sum_{j=1}^n \mathbb{E}[Y_j^2] \right)^{p/2} + \frac{1}{C} \sum_{j=1}^n \mathbb{E}|Y_j|^p \leq \mathbb{E} \left| \sum_{j=1}^n Y_j \right|^p \leq C \left( \sum_{j=1}^n \mathbb{E}[Y_j^2] \right)^{p/2} + C \sum_{j=1}^n \mathbb{E}|Y_j|^p. \quad (10.3.51)$$

Combining (10.3.50) and (10.3.51), we obtain that (10.2.7) holds if and only if (10.2.15) holds. The proof of Proposition 10.2.14 is complete.  $\square$

*Proof.* *Proof of Theorem 10.2.16* We shall use Theorem 1 in [Kli07] which states that if a sequence of random processes  $\{Y_n(t); t \in [0, 1]^d\}_{n \geq 1}$  whose finite dimensional distributions are weakly convergent and for some constants  $\alpha, \beta$  and  $K$  such that

$$\beta \in (0, 1] \quad \text{and} \quad \alpha\beta > \frac{2}{\log_2 \left( \frac{4d}{4d-3} \right)}$$

and

$$\mu \{ |Y_n(t) - Y_n(s)| \geq \varepsilon \} \leq \frac{K}{\varepsilon^\alpha} \|s - t\|^{\alpha\beta} \quad (10.3.52)$$

for any  $s$  and  $t$  in  $[0, 1]^d$ , any  $\varepsilon > 0$  and any positive integer  $n$  then  $(Y_n(\cdot))_{n \geq 1}$  converges weakly to some process in  $\mathbb{H}_\gamma([0, 1]^d)$  where  $0 < \gamma < \beta - d/\alpha$ . Since the finite-dimensional laws of the process  $\{n^{-d/2}S_n(t); t \in [0, 1]^d\}_{n \geq 1}$  are weakly convergent (cf. Theorem 10.2.12), it suffices to convert the moment inequality given by Proposition 10.2.9 into an inequality involving  $\mu \{ |S_n(t) - S_n(s)| \geq n^{d/2}\varepsilon \}$  in order to check that condition (10.3.52) is satisfied with  $\alpha = p$ ,  $\beta = 1/2$  and  $Y_n(t) = n^{-d/2}S_n(t)$ . We shall do the proof for  $d = 2$ . Let  $s = (s_1, s_2)$  and  $t = (t_1, t_2)$  be fixed in  $[0, 1]^2$  and  $n$  be a positive integer. Without loss of generality, we assume that  $s_1 > t_1$  and  $s_2 < t_2$  (similar arguments can be used to treat the general case). Let  $s'_1 = k_1/n$  and  $t'_1 = (l_1 + 1)/n$  where  $(k_1, l_1)$  is the unique element of  $\langle n \rangle^2$  such that  $k_1/n \leq s_1 < (k_1 + 1)/n$  and  $l_1/n \leq t_1 < (l_1 + 1)/n$ . In other words, keeping in mind that  $[\cdot]$  denotes the integer part function, we have  $s'_1 = [ns_1]/n$  and  $t'_1 = ([nt_1] + 1)/n$  and similarly, we define  $s'_2 = ([ns_2] + 1)/n$  and  $t'_2 = [nt_2]/n$ . With these notations, we have

$$\begin{aligned} |S_n(\mathbf{t}) - S_n(\mathbf{s})| &= |S_n(t_1, t_2) - S_n(s_1, s_2)| \\ &\leq |S_n(t_1, t_2) - S_n(t_1, t'_2)| + |S_n(t'_1, t'_2) - S_n(t_1, t'_2)| \\ &\quad + |S_n(t'_1, t'_2) - S_n(s'_1, s'_2)| + |S_n(s'_1, s'_2) - S_n(s'_1, s_2)| \\ &\quad + |S_n(s'_1, s_2) - S_n(s_1, s_2)|. \end{aligned}$$

Since

$$|S_n(t_1, t_2) - S_n(t_1, t'_2)| = (t_2 - t'_2) \left| \sum_{i=1}^{[nt_1]} X_{i, [nt_2]} + (t'_1 - t_1) X_{[nt_1]+1, [nt_2]} \right|$$

and  $t_2 - t'_2 \leq 1/n$ , by Proposition 10.2.9, there exists a positive constant  $C$  such that

$$\mathbb{E} |S_n(t_1, t_2) - S_n(t_1, t'_2)|^p \leq C p^p (t_2 - t'_2)^p n^{p/2} \leq C p^p (t_2 - t'_2)^{p/2}. \quad (10.3.53)$$

Similarly,

$$\mathbb{E} |S_n(t'_1, t'_2) - S_n(t_1, t'_2)|^p \leq C p^p (t'_1 - t_1)^{p/2} \quad (10.3.54)$$

$$\mathbb{E} |S_n(s'_1, s'_2) - S_n(s'_1, s_2)|^p \leq C p^p (s'_2 - s_2)^{p/2} \quad (10.3.55)$$

$$\mathbb{E} |S_n(s'_1, s_2) - S_n(s_1, s_2)|^p \leq C p^p (s_1 - s'_1)^{p/2}. \quad (10.3.56)$$

Moreover, from Proposition 10.2.9, for any positive integer  $n$  and any  $i$  and  $j$  in  $\langle n \rangle^2$ , we have

$$\mathbb{E} \left| \frac{1}{n} S_n \left( \frac{i}{n} \right) - \frac{1}{n} S_n \left( \frac{j}{n} \right) \right|^p \leq C p^p \left\| \frac{i}{n} - \frac{j}{n} \right\|^{p/2}. \quad (10.3.57)$$

Combining (10.3.53), (10.3.54), (10.3.55), (10.3.56) and (10.3.57) and using the elementary convexity inequality  $(a_1 + a_2 + a_3 + a_4 + a_5)^p \leq 5^{p-1}(a_1^p + a_2^p + a_3^p + a_4^p + a_5^p)$  for any nonnegative  $a_1, a_2, a_3, a_4$  and  $a_5$ , we derive

$$\mathbb{E} |S_n(t) - S_n(s)|^p \leq \kappa \|s - t\|^{p/2}.$$

Finally, using Markov's inequality, we obtain (10.3.52). The proof of Theorem 10.2.16 is complete.  $\square$

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## Part V

# Probability inequalities for strictly stationary random fields and applications



# Chapter 11

## Probability inequalities for stationary random fields

In this chapter, we establish some probability inequalities for the maxima of partial sums of some classes of strictly stationary random fields: orthomartingales with respect to a completely commuting filtration and Bernoulli random fields. For the first class, the probability is bounded by the tail function of the increment and that of quadratic variances. In the second case, the bound is expressed in terms of projection operators.

### 11.1 A maximal inequality for stationary orthomartingale difference random fields

#### 11.1.1 The one dimensional case

A lot of inequalities for martingales involve the maxima of increments and the quadratic variance. Let us mention the following result by Nagaev, which links the tail function of the maxima of a martingale with that of the increments and the quadratic variance.

**Theorem 11.1.1** (Theorem 1, [Nag03]). *Let  $q > 0$  and  $C(q) := qe^{3qe^{q+1}}/qe^{q+1}$  and let  $(S_n, \mathcal{F}_n)$  be a martingale. Then*

$$\mu \left\{ \max_{1 \leq k \leq n} S_k \geq t \right\} \leq C(q)t^{-q} \int_0^t Q(u) u^{q-1} du, \quad (11.1.1)$$

where

$$Q(u) := \mu \left\{ \max_{1 \leq k \leq n} |X_k| > u \right\} + \mu \left\{ \left( \sum_{k=1}^n \mathbb{E}[X_k^2 | \mathcal{F}_{k-1}] \right)^{1/2} > u \right\}. \quad (11.1.2)$$

Let us focus on the stationary case. If  $q$  is a real number greater than 2, we take  $\delta = \delta(q) > 0$  such that  $\delta(1 + \delta + \sqrt{\delta})^q = 1$ .

We define on the positive real line the functions

$$g(q, v) := \frac{3}{(q-2)\delta^3} v^{q-1} \mathbf{1}_{(0,1)}(v) + v \mathbf{1}_{[1,\infty)}(v); \quad (11.1.3)$$

$$h(q, v) := \left( \frac{3}{\delta^{3+q}} + \frac{12 \cdot 2^q}{\delta^{3+q}(q-2)^2} \right) v^{q-1} \mathbf{1}_{(0, \delta/2)}(v) + v \left( \frac{48}{\delta^{q+5}(q-2)} (\log v + \log(2/\delta) + 1) + \frac{3}{\delta} \right) \mathbf{1}_{[\delta/2, \infty)}. \quad (11.1.4)$$

**Proposition 11.1.2.** *Let  $T: \Omega \rightarrow \Omega$  be a measure-preserving map. Assume that  $\mathcal{F}$  is a sub- $\sigma$ -algebra such that  $T\mathcal{F} \subset \mathcal{F}$  and that  $(M \circ T^i)_{i \geq 1}$  is a martingale difference sequence with respect to the filtration  $(T^{-i}\mathcal{F})_{i \geq 0}$ . Then for each real number  $t$ , each  $q > 2$  and each positive integer  $n$ , the following inequalities hold:*

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq 2\sqrt{nt}/\delta \right\} \leq \frac{3}{\delta^3} n \int_0^1 \mu \{ |M| \geq 2\sqrt{nu} \} u^{q-1} du + \int_0^\infty g(q, v) \mu \left\{ (\mathbb{E}[m^2 | T\mathcal{F}])^{1/2} > \sqrt{2vt} \right\} dv; \quad (11.1.5)$$

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq \sqrt{nt} \right\} \leq \int_0^{+\infty} \mu \{ |M| \geq tv \} h(q, v) dv. \quad (11.1.6)$$

While this can be deduced from a combination of Theorem 11.1.1 and the maximal ergodic theorem, we shall give a complete proof for the following two reasons. First, Nagaev's result also applies for supermartingales and in the martingale case, his proof can be simplified. Second, the constant  $C(q)$  obtained in the general case can be improved when restricted to martingales.

We shall use the following lemma.

**Lemma 11.1.3.** *Assume that  $X$  and  $Y$  are two random non-negative variables such that for each positive  $\lambda$ , we have  $\lambda \mu \{ X > \lambda \} \leq \mathbb{E}[Y \mathbf{1} \{ X \geq \lambda \}]$ .*

*Then for each  $t$ , we have*

$$\mu \{ X > 2t \} \leq \int_1^{+\infty} \mu \{ Y > st \} ds. \quad (11.1.7)$$

*Proof of Lemma 11.1.3.* Rewriting the expectation as

$$\mathbb{E}[Y \mathbf{1} \{ X \geq 2t \}] = \int_0^{+\infty} \mu \{ Y \mathbf{1} \{ X \geq 2t \} > u \} du \leq t \mu \{ X \geq 2t \} + \int_t^{+\infty} \mu \{ Y > u \} du, \quad (11.1.8)$$

we derive by the assumption the bound

$$2t \mu \{ X > 2t \} \leq t \mu \{ X \geq 2t \} + \int_t^{+\infty} \mu \{ Y > u \} du. \quad (11.1.9)$$

We conclude using the substitution  $ts := u$ .  $\square$

*Proof of Proposition 11.1.2.* If  $(S_i = \sum_{j \leq i} X_j, \mathcal{F}_i)$  is a martingale, then for each  $\beta > 1$ ,  $\delta \in (0, \beta - 1)$  and  $\lambda > 0$ , and each integer  $N \geq 1$ , the inequality

$$\mu \left\{ \max_{1 \leq i \leq N} |S_i| > \beta \lambda \right\} \leq \frac{\delta^2}{(\beta - \delta - 1)^2} \mu \left\{ \max_{1 \leq i \leq N} |S_i| > \lambda \right\} + \mu \left\{ \sum_{i=1}^N \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] > \delta^2 \lambda^2 \right\} + \mu \left\{ \max_{1 \leq i \leq N} |X_i| > \delta \lambda \right\} \quad (11.1.10)$$

takes place (see [HH80], p. 28). Let us consider a positive  $\delta$  such that  $\delta(1 + \delta + \sqrt{\delta})^q = 1$ .

For fixed  $x$ , and a non-negative integer  $m$ , we define

$$y_m := (1 + \delta + \sqrt{\delta})^m x. \quad (11.1.11)$$

By (11.1.10) applied with  $\beta = 1 + \delta + \sqrt{\delta}$ , it follows that for each  $m$ ,

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \delta \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_{m-1} \right\} + Q(\delta y_{m-1}), \quad (11.1.12)$$

where  $Q$  is defined by (11.1.2) (with  $X_k = M \circ T^k$  and  $\mathcal{F}_i = T^{-i}\mathcal{F}$ ). Denoting by  $a_m$  the quantity  $a_m := \delta^{-m} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\}$ , we have  $a_m \leq a_{m-1} + Q(\delta y_{m-1})$ , hence

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \sum_{i=0}^{m-1} Q(\delta y_i) \delta^{m-i-1} + \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq x \right\} \delta^m. \quad (11.1.13)$$

Since  $Q(\delta y_i) \leq 2$  and  $\delta \leq 1$ , we derive the bound

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \sum_{i=1}^{m-1} Q(\delta y_i) \delta^{m-i-1} + 3\delta^m. \quad (11.1.14)$$

Since the function  $u \mapsto Q(u)$  is non-increasing, we have

$$Q(\delta y_i) \int_{y_{i-1}}^{y_i} u^{q-1} du \leq \int_{y_{i-1}}^{y_i} Q(\delta u) u^{q-1} du,$$

hence

$$Q(\delta y_i) \leq \int_{y_{i-1}}^{y_i} Q(\delta u) u^{q-1} du \cdot \left( (y_i - y_{i-1}) y_i^{q-1} \right)^{-1}. \quad (11.1.15)$$

Plugging this estimate and (11.1.11) into (11.1.14), we obtain

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \frac{1}{y_m^q} \sum_{i=1}^{m-1} \int_{y_{i-1}}^{y_i} Q(\delta u) u^{q-1} du \cdot y_m^q \left( (y_i - y_{i-1}) y_i^{q-1} \right)^{-1} \delta^{m-i-1} + 3\delta^m. \quad (11.1.16)$$

Now, since  $\delta(1 + \delta + \sqrt{\delta})^q = 1$ , it follows that

$$\begin{aligned} y_m^q \left( (y_i - y_{i-1}) y_i^{q-1} \right)^{-1} \delta^{m-i-1} &= \frac{y_m^q}{y_i^q} \frac{y_i}{y_i - y_{i-1}} \\ &= (1 + \delta + \sqrt{\delta})^{(m-i)q} \frac{(1 + \delta + \sqrt{\delta}) y_{i-1}}{(\delta + \sqrt{\delta}) y_{i-1}} \delta^{m-i-1} \\ &= \frac{(1 + \delta + \sqrt{\delta})}{\delta(\delta + \sqrt{\delta})} \\ &\leq \frac{3}{\delta^2}, \end{aligned}$$

hence by (11.1.16), we derive

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \frac{3}{y_m^q \delta^2} \int_0^{y_{m-1}} Q(\delta u) u^{q-1} du + 3\delta^m. \quad (11.1.17)$$

Let  $y$  be such that  $y_m \leq y < y_{m+1}$ ; then

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y \right\} &\leq \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \\ &\leq \frac{3}{y_m^q \delta^2} \int_0^{y_{m-1}} Q(\delta u) u^{q-1} du + 3\delta^m \\ &\leq \frac{3}{y^q \delta^2} \left( \frac{y_{m+1}}{y_m} \right)^q \int_0^y Q(\delta u) u^{q-1} du + 3\delta^m \\ &= \frac{3}{y^q \delta^2} (1 + \delta + \sqrt{\delta})^q \int_0^y Q(\delta u) u^{q-1} du + 3\delta^m, \end{aligned}$$

and using again the equality  $\delta(1 + \delta + \sqrt{\delta})^q = 1$ , we derive

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y \right\} \mathbf{1}_{[x, \infty)}(y) &\leq \frac{3}{y^q \delta^3} \int_0^y Q(\delta u) u^{q-1} du \cdot \mathbf{1}_{[x, \infty)}(y) + \\ &\quad + 3 \sum_{m=0}^{\infty} \delta^m \cdot \mathbf{1}_{[x(1+\delta+\sqrt{\delta})^m, x(1+\delta+\sqrt{\delta})^{m+1})}(y). \end{aligned} \quad (11.1.18)$$

Since  $x$  is arbitrary and  $\delta \in (0, 1)$ , the second term converges to 0 as  $x$  goes to 0, hence

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y \right\} \leq \frac{3}{y^q \delta^3} \int_0^y Q(\delta u) u^{q-1} du. \quad (11.1.19)$$

By definition of  $Q$  and the fact that  $T$  is measure preserving, we obtain

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq 2\sqrt{nt}/\delta \right\} &\leq \frac{3}{\delta^3} n \int_0^1 \mu \left\{ |M| \geq 2\sqrt{nu}y \right\} u^{q-1} du + \\ &\quad + \frac{3}{\delta^3} \int_0^1 \mu \left\{ n^{-1} \sum_{j=0}^{n-1} U^j (\mathbb{E}[M^2 | T\mathcal{F}]) > 4u^2 t^2 \right\} u^{q-1} du. \end{aligned} \quad (11.1.20)$$

By the maximal ergodic theorem, inequality (11.1.8) holds with  $X = \sup_{n \geq 1} n^{-1} S_n (\mathbb{E}[M^2 | T\mathcal{F}])$  and  $Y = \mathbb{E}[M^2 | T\mathcal{F}]$ , hence by Lemma 11.1.3 the estimate

$$\mu \left\{ n^{-1} \sum_{j=1}^n U^j (\mathbb{E}[M^2 | T\mathcal{F}]) > 4u^2 t^2 \right\} \leq \int_1^{+\infty} \mu \left\{ \mathbb{E}[M^2 | T\mathcal{F}] > 2u^2 t^2 s \right\} ds \quad (11.1.21)$$

is valid for any  $n$ . We can deduce from inequalities (11.1.20) and (11.1.21) that (11.1.5) is satisfied (after switching the integrals).

By Markov's inequality, one can see that (11.1.8) is satisfied with  $X = \mathbb{E}[M^2 | T\mathcal{F}]$  and  $Y = M^2$ , and using again Lemma 11.1.3, it follows that

$$\mu \left\{ n^{-1} \sum_{j=1}^n U^j (\mathbb{E}[M^2 | T\mathcal{F}]) > 4u^2 t^2 \right\} \leq \int_1^{+\infty} \int_1^{+\infty} \mu \left\{ M^2 > u^2 t^2 s_1 s_2 \right\} ds_1 ds_2, \quad (11.1.22)$$

which, combined with (11.1.20), yields

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq 2\sqrt{nt}/\delta \right\} &\leq \frac{3}{\delta^3} n \int_0^1 \mu \left\{ |M| \geq 2\sqrt{nut} \right\} u^{q-1} du + \\ &\quad + \frac{3}{\delta^3} \int_0^1 \int_1^{+\infty} \int_1^{+\infty} \mu \left\{ |M| > ut\sqrt{s_1 s_2} \right\} ds_1 ds_2 u^{q-1} du. \end{aligned} \quad (11.1.23)$$

We now simplify the second term. Using the substitution  $v = u\sqrt{s_1}\sqrt{s_2}$  and switching the integrals, we obtain

$$\begin{aligned} & \int_0^1 \int_1^{+\infty} \int_1^{+\infty} \mu \{ |M| > ut\sqrt{s_1}\sqrt{s_2} \} ds_1 ds_2 u^{q-1} du = \\ & = \iint_{[1,+\infty)^2} \int_0^{+\infty} \mathbf{1} \{ \sqrt{s_1 s_2} \geq v \} (s_1 s_2)^{-q/2} \mu \{ |M| \geq tv \} v^{q-1} dv ds_1 ds_2. \end{aligned} \quad (11.1.24)$$

Define for a fixed  $v$  the quantity  $J(v) := \iint_{[1,+\infty)^2} \int_0^{\sqrt{s_1 s_2}} \mathbf{1} \{ \sqrt{s_1 s_2} \geq v \} (s_1 s_2)^{-q/2} ds_1 ds_2$ . If  $v \in (0, 1)$ , then  $J(v) = \left( \int_1^{+\infty} s^{-q/2} ds \right)^2 = (q/2 - 1)^{-2}$  and if  $v \in [1, \infty)$ , then

$$J(v)/4 = \int_1^{+\infty} \int_{\max\{1, v/x_1\}}^{+\infty} \frac{1}{x_1^{q-1} x_2^{q-1}} dx_1 dx_2 \quad (11.1.25)$$

$$= \int_1^v \int_{v/x_1}^{+\infty} \frac{1}{x_1^{q-1} x_2^{q-1}} dx_1 dx_2 + \int_v^{+\infty} \int_1^{+\infty} \frac{1}{x_1^{q-1} x_2^{q-1}} dx_1 dx_2 \quad (11.1.26)$$

$$= \int_1^v (v/x_1)^{2-q} / (q-2) x_1^{1-q} dx_1 + \int_1^{+\infty} x_1^{1-q} dx_1 \cdot v^{2-q} \quad (11.1.27)$$

$$= v^{2-q} \log v / (q-2) + v^{2-q} / (q-2), \quad (11.1.28)$$

hence

$$\begin{aligned} & \iint_{[1,+\infty)^2} \int_0^{+\infty} \mathbf{1} \{ \sqrt{s_1 s_2} \geq v \} (s_1 s_2)^{-q/2} \mu \{ |M| \geq tv \} v^{q-1} dv ds_1 ds_2 = \\ & = \frac{4}{q-2} \left( \frac{1}{q-2} \int_0^1 \mu \{ |M| \geq tv \} dv + \int_1^{+\infty} v(\log v + 1) \mu \{ |M| \geq tv \} dv \right). \end{aligned} \quad (11.1.29)$$

In view of (11.1.23), (11.1.24) and (11.1.29), we have

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq 2\sqrt{nt}/\delta \right\} \leq \frac{3}{\delta^3} n \int_0^1 \mu \{ |M| \geq 2\sqrt{nut} \} u^{q-1} du + \\ & + \frac{12}{\delta^3(q-2)} \left( \frac{1}{q-2} \int_0^1 \mu \{ |M| \geq tv \} v^{q-1} dv + \int_1^{+\infty} v(\log v + 1) \mu \{ |M| \geq tv \} dv \right), \end{aligned} \quad (11.1.30)$$

which can be rewritten as

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq \sqrt{nt} \right\} \leq \frac{3}{\delta^3} n \int_0^1 \mu \{ |M| \geq 2\sqrt{nut}\delta/2 \} u^{q-1} du + \\ & + \frac{12}{\delta^3(q-2)} \left( \frac{1}{q-2} \int_0^1 \mu \{ |M| \geq t\delta v/2 \} v^{q-1} dv \right. \\ & \left. + \int_1^{+\infty} v(\log v + 1) \mu \{ |M| \geq t\delta v/2 \} dv \right). \end{aligned} \quad (11.1.31)$$

In the first term of the right hand side of (11.1.31), we use the substitution  $v := \sqrt{nu}\delta$ , and in



the second and third terms, we use  $w := \delta v/2$ . This gives

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq \sqrt{nt} \right\} &\leq \frac{3}{\delta^3} n \left( \frac{1}{\delta \sqrt{n}} \right)^q \int_0^{\sqrt{n}\delta} \mu \{|M| \geq tv\} v^{q-1} dv + \\ &+ \frac{24}{\delta^4(q-2)} \left( \frac{2^{q-1}}{\delta^{q-1}(q-2)} \int_0^{\delta/2} \mu \{|M| \geq tw\} w^{q-1} dw + \right. \\ &\quad \left. \frac{2}{\delta} \int_{\delta/2}^{+\infty} w (\log w + \log(2/\delta) + 1) dw \right). \end{aligned} \quad (11.1.32)$$

We now bound the first term independently of  $n$ . If  $v \leq \delta/2$ , then  $n^{1-q/2} v^{q-1} \mathbf{1}\{v < \sqrt{n}\delta\} \leq v^{q-1}$ , and if  $v > \delta/2$ , then  $n^{1-q/2} v^{q-1} \mathbf{1}\{v < \sqrt{n}\delta\} \leq v \delta^{q-2}$ . Plugging these bounds into (11.1.32), we obtain

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq \sqrt{nt} \right\} &\leq \left( \frac{3}{\delta^{3+q}} + \frac{12 \cdot 2^q}{\delta^{3+q}(q-2)^2} \right) \int_0^{\delta/2} \mu \{|M| \geq tv\} v^{q-1} dv + \\ &+ \int_{\delta/2}^{+\infty} \mu \{|M| \geq tv\} v \left( \frac{48}{\delta^{q+5}(q-2)} (\log v + \log(2/\delta) + 1) + \frac{3}{\delta} \right) dv \end{aligned} \quad (11.1.33)$$

which concludes the proof of Proposition 11.1.2.  $\square$

### 11.1.2 The general case

We briefly recall the definition of orthomartingale random fields. Let  $(T_q)_{q=1}^d$  be bijective, bi-measurable and measure preserving transformations on  $(\Omega, \mathcal{F}, \mu)$ . Assume that  $T_q \circ T_{q'} = T_{q'} \circ T_q$  for each  $q, q' \in \{1, \dots, d\}$ . Let  $\mathcal{M}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$  such that for each  $q \in \{1, \dots, d\}$ ,  $\mathcal{M} \subset T_q^{-1} \mathcal{M}$ . In this way,  $\mathcal{F}_{\mathbf{i}} := T^{-\mathbf{i}} \mathcal{M}$ ,  $\mathbf{i} \in \mathbb{Z}^d$ , yields a filtration. If for each  $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$  and each integrable and  $\mathcal{F}_{\mathbf{l}}$ -measurable random variable  $Y$ ,

$$\mathbb{E}[Y | \mathcal{F}_{\mathbf{k}}] = \mathbb{E}[Y | \mathcal{F}_{\mathbf{k} \wedge \mathbf{l}}] \text{ almost surely,} \quad (11.1.34)$$

the transformations  $(T_q)_{q=1}^d$  are said to be completely commuting.

The collection of random variables  $\{M_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is said to be an orthomartingale with respect to the completely commuting filtration  $(T^{-\mathbf{i}} \mathcal{M})$  if for each  $\mathbf{n} \in \mathbb{N}^d$ ,  $M_{\mathbf{n}}$  is  $\mathcal{F}_{\mathbf{n}}$ -measurable, integrable and for each  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$  such that  $\mathbf{i} \preceq \mathbf{j}$ ,

$$\mathbb{E}[M_{\mathbf{j}} | \mathcal{F}_{\mathbf{i}}] = M_{\mathbf{i}}. \quad (11.1.35)$$

Here  $\mathbf{i} \preceq \mathbf{j}$  means that  $i_q \leq j_q$  for each  $q \in \{1, \dots, d\}$ .

Recall the definitions of the function  $g$  and  $h$  in (11.1.3) and (11.1.4) respectively.

**Proposition 11.1.4.** *Let  $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  be a strictly stationary orthomartingale difference random field with respect to the completely commuting filtration  $(T^{-\mathbf{i}} \mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  and  $q > 2$ . Then for each*

$t$ , each  $\mathbf{n} \in \mathbb{N}^d$  and each  $j \in \{1, \dots, d\}$ , the following inequalities take place:

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} |S_{\mathbf{i}}(m)| \geq 4^d \prod_{l=1}^d \sqrt{n_l} t \right\} &\leq \\ &\leq \frac{3}{\delta^3} n_j \int_{\mathbf{R}_+^d} \mu \left\{ |m| > tv 2\sqrt{n_j} \prod_{l=1, l \neq j}^d w_l \right\} (\log v)^{d-2} \prod_{l=1}^d h(q, w_l) d\mathbf{w} dv \\ &\quad + \int_{\mathbf{R}_+^d} \int_1^{+\infty} \mu \left\{ (\mathbb{E}[m^2 | T_j \mathcal{M}])^{1/2} > tv \sqrt{2} \prod_{l=1}^d w_l \right\} \\ &\quad (\log v)^{d-2} g(q, w_j) \prod_{\substack{l=1, \\ l \neq j}}^d h(q, w_l) g(q, w_j) d\mathbf{w} dv, \end{aligned} \quad (11.1.36)$$

where  $\mathbf{w} = (w_1, \dots, w_d)$  and

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} |S_{\mathbf{i}}(m)| \geq 4^d \prod_{l=1}^d \sqrt{n_l} t \right\} &\leq \\ &\leq \int_{\mathbf{R}_+^d} \int_1^{+\infty} \mu \left\{ |m| > tv \prod_{l=1}^d w_l \right\} (\log v)^{d-2} \prod_{l=1}^d h(q, w_l) d\mathbf{w} dv. \end{aligned} \quad (11.1.37)$$

**Corollary 11.1.5.** *Let  $p > 2$  and  $m$  a function as in Proposition 11.1.4 and such that  $\mathbb{E}|m|^p$  is finite. Then the family*

$$\mathcal{F} := \left\{ |\mathbf{n}|^{-p/2} \max_{\mathbf{1} \preceq \mathbf{j} \preceq \mathbf{n}} |S_{\mathbf{j}}(m)|^p, \mathbf{n} \succcurlyeq \mathbf{1} \right\} \quad (11.1.38)$$

is uniformly integrable.

*Remark 11.1.6.* For a fixed  $p > 2$ , one can deduce multidimensional Burkholder's inequality (see [Faz05]) by applying Proposition 11.1.4 with  $q > p$ . Indeed, we multiply on both sides of inequality (11.1.36) by  $pt^{p-1}$  and integrate on  $[0, +\infty)$  using the convergence of  $\int_0^1 v^{q-p-1} dv$  and  $\int_1^\infty v \log v \cdot v^{-p} dv$  (because  $2 < p < q$ ).

However, in this way, the obtained constant is certainly not optimal.

*Remark 11.1.7.* It seems that Proposition 11.1.4 is not efficient for second moments. In particular, the uniform integrability result in [VW14] does not appear as a consequence of (11.1.37). This is due to the fact that the integral  $\int_0^\infty v^{-2} h(q, v) dv$  is divergent.

*Proof of Proposition 11.1.4.* We shall do the proof when  $j = 1$ ; the general case can be deduced by switching the roles of the operators  $T_i$ . Observe that for each fixed  $i \in \{2, \dots, d\}$ , the sequence

$\left( \max_{\substack{\mathbf{1} \preceq \mathbf{j}_i \preceq \mathbf{n}_i \\ l < i}} |S_{j_1, \dots, j_{i-1}, n_i, \dots, n_d}| \right)_{n_i \geq 1}$  is a sub-martingale. Therefore, using repeatedly Lemma 11.1.3,

we derive

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} |S_{\mathbf{i}}(m)| \geq 4^d \prod_{l=1}^d \sqrt{n_l} t \right\} &\leq \\ &\leq \int_{[1, \infty)^{d-1}} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{i}_d \preceq \mathbf{n}_d} |S_{n_1, \dots, n_{d-1}, i_d}| > t 2^d s_1 \dots s_{d-1} \right\} ds_1 \dots ds_{d-1}. \end{aligned} \quad (11.1.39)$$

Using the substitution  $v := s_1 \dots s_{d-1}$  for fixed  $s_1, \dots, s_{d-2}$ , we obtain

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq 4^d \prod_{l=1}^d \sqrt{n_l t} \right\} \leq \\ & \leq \int_0^\infty \mu \left\{ \max_{1 \leq i_d \leq n_d} |S_{n_1, \dots, n_{d-1}, i_d}| > tv \right\} \int_{[1, \infty)^{d-2}} \mathbf{1} \left\{ \prod_{j=1}^{d-2} s_j \leq v \right\} \frac{1}{\prod_{j=1}^{d-2} s_j} ds dv. \end{aligned} \quad (11.1.40)$$

Since  $s_j \geq 1$  for each  $j \in \{1, \dots, d-1\}$ , we have the upper bound

$$\begin{aligned} & \int_{[1, \infty)^{d-2}} \mathbf{1} \{s_1 \dots s_{d-2} \leq v\} \frac{1}{s_1 \dots s_{d-2}} ds_1 \dots ds_{d-2} \leq \\ & \leq \int_{[1, \infty)^{d-2}} \prod_{j=1}^{d-2} \left( \mathbf{1} \{s_j \leq v\} \frac{1}{s_j} ds_j \right) = (\log v)^{d-2}, \end{aligned} \quad (11.1.41)$$

hence by (11.1.39), we have

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq 4^d \prod_{l=1}^d \sqrt{n_l t} \right\} \leq \\ & \leq \int_0^\infty \mu \left\{ \max_{1 \leq i_d \leq n_d} |S_{n_1, \dots, n_{d-1}, i_d}| > t2^d v \right\} (\log v)^{d-2} dv. \end{aligned} \quad (11.1.42)$$

Now, by Proposition 11.1.2 applied with  $M := S_{n_1, \dots, n_{d-1}, 0}(m)$ , we get

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq 4^d \prod_{l=1}^d \sqrt{n_l t} \delta \right\} \leq \\ & \leq \int_0^\infty \int_0^\infty \mu \left\{ |S_{n_1, \dots, n_{d-1}, 0}| > t2^{d-1} \prod_{l=1}^{d-1} \sqrt{n_l} v w \right\} (\log v)^{d-2} h(q, w) dv dw. \end{aligned} \quad (11.1.43)$$

Using again Proposition 11.1.2, we derive that

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq (4/\delta)^d \prod_{l=1}^d \sqrt{n_l t} \right\} \leq \\ & \leq \int_{\mathbf{R}_+^d} \mu \left\{ |S_{n_1, 0, \dots, 0}| > tv2\sqrt{n_1} \prod_{l=2}^d w_l / \delta \right\} (\log v)^{d-2} \prod_{l=2}^d h(q, w_l) dw_2 \dots dw_d dv. \end{aligned} \quad (11.1.44)$$

It remains to apply inequalities (11.1.5) (respectively (11.1.6)) in order to obtain (11.1.36) (respectively (11.1.37)).  $\square$

*Proof of Corollary 11.1.5.* The uniform integrability of  $\mathcal{F}$  is equivalent to

$$\lim_{k \rightarrow \infty} \sup_{\mathbf{n} \geq \mathbf{1}} \mathbb{E} \left[ |\mathbf{n}|^{-p/2} \max_{1 \leq j \leq \mathbf{n}} |S_j(m)|^p \mathbf{1} \left\{ \max_{1 \leq j \leq \mathbf{n}} |S_j(m)| \geq 2^{k/p} |\mathbf{n}|^{1/2} \right\} \right] = 0. \quad (11.1.45)$$

Writing  $\left\{ \max_{1 \leq j \leq \mathbf{n}} |S_j(m)| \geq 2^{k/p} |\mathbf{n}|^{1/2} \right\}$  as the disjoint union of the sets

$$\left\{ |\mathbf{n}|^{1/2} \max_{1 \leq j \leq \mathbf{n}} |S_j(m)| \in [2^{l/p}, 2^{(l+1)/p}) \right\}, \quad l \geq k \quad (11.1.46)$$

we notice that it is sufficient to prove that

$$\lim_{k \rightarrow \infty} \sup_{\mathbf{n} \succ \mathbf{1}} \sum_{l \geq k} 2^l \mu \left\{ \max_{1 \leq j \leq \mathbf{n}} |S_j(m)| \geq 2^{l/p} |\mathbf{n}|^{1/2} \right\} = 0. \quad (11.1.47)$$

To this aim, we apply inequality (11.1.37) with  $t = 4^{-d} 2^{l/p}$  and a  $q > p$ . This gives

$$\begin{aligned} \mu \left\{ \max_{1 \leq j \leq \mathbf{n}} |S_j(m)| \geq 2^{l/p} |\mathbf{n}|^{1/2} \right\} &\leq \\ &\leq \int_{\mathbf{R}_+^d} \int_1^\infty \mu \left\{ |m| \geq 4^{-d} 2^{l/p} v \prod_{i=1}^d w_i \right\} \prod_{i=1}^d h(q, w_i) \log(v)^{d-2} dv d\mathbf{w}, \end{aligned} \quad (11.1.48)$$

and since the right hand side is independent of  $\mathbf{n}$ , it suffices to establish the convergence of the series

$$\sum_{l=1}^{\infty} 2^l \int_{\mathbf{R}_+^d} \int_1^\infty \mu \left\{ |m| \geq 4^{-d} 2^{l/p} v \prod_{i=1}^d w_i \right\} \prod_{i=1}^d h(q, w_i) \log(v)^{d-2} dv d\mathbf{w}. \quad (11.1.49)$$

Since for a non-negative random variable  $Y$ , we have

$$\sum_{l=1}^{\infty} 2^l \mu \left\{ |Y| \geq 2^{l/p} \right\} \leq 2\mathbb{E}(Y^p), \quad (11.1.50)$$

the term (11.1.49) does not exceed

$$2 \cdot 4^{pd} \int_{\mathbf{R}_+^d} \int_1^\infty \left( v \prod_{i=1}^d w_i \right)^{-p} \prod_{i=1}^d h(q, w_i) \log(v)^{d-2} dv d\mathbf{w}, \quad (11.1.51)$$

which is finite since  $\int_{\mathbf{R}} v^{-p} h(q, v) dv$  and  $\int_1^{+\infty} v^{-p} (\log v)^d dv$  are finite. This concludes the proof of Lemma 11.1.5.  $\square$

## 11.2 A tail inequality for Bernoulli random fields

### 11.2.1 Notations

Assume that  $\{\varepsilon_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d\}$  is an i.i.d. collection of centered random variable with finite  $p$ th moment and  $\mathbf{N} \in \mathbb{N}^d$ . Using the fact that the sequence  $\left( \max_{\mathbf{0} \leq \mathbf{n}' \leq \mathbf{N}'} \left| \sum_{\mathbf{0} \leq \mathbf{i}' \leq \mathbf{n}'} \sum_{j=0}^{i_d} \varepsilon_{\mathbf{i}', j} \right| \right)_{i_d \geq 0}$  is a sub-martingale, we obtain by Doob's inequality that

$$\left\| \max_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{N}} \left\| \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}} \varepsilon_{\mathbf{i}} \right\|_p \right\| \leq \frac{p}{p-1} \left\| \max_{\mathbf{0} \leq \mathbf{n}' \leq \mathbf{N}'} \left\| \sum_{\mathbf{0} \leq \mathbf{i}' \leq \mathbf{n}'} \sum_{j=0}^{N_d} \varepsilon_{\mathbf{i}', j} \right\|_p \right\|. \quad (11.2.1)$$

Repeating this argument, we get

$$\left\| \max_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{N}} \left\| \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}} \varepsilon_{\mathbf{i}} \right\|_p \right\| \leq \left( \frac{p}{p-1} \right)^d \left\| \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{N}} \varepsilon_{\mathbf{i}} \right\|_p. \quad (11.2.2)$$

By Rosenthal's inequality, it follows that

$$\left\| \max_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{N}} \left\| \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}} \varepsilon_{\mathbf{i}} \right\|_p \right\| \leq C(p, d) (N_1 \dots N_d)^{1/p} \|\varepsilon_{\mathbf{0}}\|_p + C(p, d) (N_1 \dots N_d)^{1/2} \|\varepsilon_{\mathbf{0}}\|_2. \quad (11.2.3)$$

This inequality is used in [RSZ07] for Hilbert space valued i.i.d. random field in order to establish tightness of the partial sum process.

Now, let us consider a random field  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  of the form  $X_{\mathbf{k}} = g(\varepsilon_{\mathbf{k}-\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d)$  where  $g: \mathbb{R}^{\mathbb{N}^d} \rightarrow \mathbb{R}$  is measurable. One would like to extend inequality (11.2.3) to partial sums of  $X_{\mathbf{k}}$ 's. Proposition 1 of [EVW13] gives a moment inequality for weighted sums of a strictly stationary random field which can be expressed as a functional of i.i.d. However, when applied to rectangles and an i.i.d. random field, we do not recover (11.2.3). In the one dimensional case, Liu et al. extended Rosenthal inequality to stationary sequences which are functional of i.i.d. ([LXW13], Theorem 1). Their result is expressed in terms of physical dependence measure defined in [Wu05]. The idea of the proof is to approximate the partial sums by that of an  $m$ -dependent process which is also a martingale difference sequence. A tail inequality is also obtained. Since the tightness criterion for stationary sequences is expressed in term of tails of maxima of partial sums, it would be preferable to obtain a probability inequality instead of a moment inequality. Using successive applications of Lemma 11.1.3, we have

$$\mu \left\{ \max_{\mathbf{0} \preceq \mathbf{n} \preceq \mathbf{N}} \left| \sum_{\mathbf{0} \preceq \mathbf{i} \preceq \mathbf{n}} \varepsilon_{\mathbf{i}} \right| \geq 2^d t \right\} \leq \int_{[1, +\infty)^d} \mu \left\{ \left| \sum_{\mathbf{0} \preceq \mathbf{i} \preceq \mathbf{N}} \varepsilon_{\mathbf{i}} \right| \geq s_1 \dots s_d t \right\} ds_1 \dots ds_d. \quad (11.2.4)$$

Then using inequality (11.1.19), it follows that

$$\begin{aligned} \mu \left\{ \max_{\mathbf{0} \preceq \mathbf{n} \preceq \mathbf{N}} \left| \sum_{\mathbf{0} \preceq \mathbf{i} \preceq \mathbf{n}} \varepsilon_{\mathbf{i}} \right| \geq 2^d t \right\} &\leq \frac{3}{\delta_q^3} N_1 \dots N_d \int_{[1, +\infty)^d} \int_0^1 \mu \{ |\varepsilon_{\mathbf{0}}| \geq s_1 \dots s_d u t \} u^{q-1} du ds \\ &+ \frac{3}{t^q \delta_q^3} (N_1 \dots N_d)^{q/2} \left( \int_{[1, \infty)} x^{-q} dx \right)^d \mathbb{E} [\varepsilon_{\mathbf{0}}^2], \end{aligned} \quad (11.2.5)$$

where  $ds := ds_1 \dots ds_d$ .

## 11.2.2 The result and its proof

The main result of this subsection is an extension of (11.2.5) to random fields  $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  of the form  $X_{\mathbf{k}} = g(\varepsilon_{\mathbf{k}-\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d)$  where  $g: \mathbb{R}^{\mathbb{N}^d} \rightarrow \mathbb{R}$  is measurable. We define

$$Q_{\mathbf{j}} := \prod_{q=1}^d Q_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d \quad (11.2.6)$$

where the operator  $Q_{j_q}^{(q)}: \mathbb{L}^1 \rightarrow \mathbb{L}^1$  is defined by

$$Q_{j_q}^{(q)}(h) := \mathbb{E}[h \mid \sigma(\varepsilon_{\mathbf{i}}, i_q \geq -j_q)] - \mathbb{E}[h \mid \sigma(\varepsilon_{\mathbf{i}}, i_q \geq -j_q + 1)]. \quad (11.2.7)$$

Actually, these operators are linked to the  $P_i$ 's for the filtration  $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  given by  $\mathcal{F}_{\mathbf{i}} := \sigma(\varepsilon_{\mathbf{j}}, \mathbf{j} \succeq -\mathbf{i})$ , where

$$P_{\mathbf{j}} := \prod_{q=1}^d P_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d, \quad (11.2.8)$$

where for  $l \in \mathbb{Z}$ ,  $P_l^{(q)}: \mathbb{L}^1(\mathcal{F}) \rightarrow \mathbb{L}^1(\mathcal{F})$  is defined by

$$P_l^{(q)}(f) = \mathbb{E}_l^{(q)}[f] - \mathbb{E}_{l-1}^{(q)}[f]. \quad (11.2.9)$$

**Theorem 11.2.1.** Let  $\{\varepsilon_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d\}$  be a collection of independent identically distributed random variables and  $U^{\mathbf{k}}f = X_{\mathbf{k}} = g(\varepsilon_{\mathbf{k}-\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d)$  where  $g: \mathbb{R}^{\mathbb{N}^d} \rightarrow \mathbb{R}$  is measurable and  $\mathbf{k} \in \mathbb{N}^d$ . Assume that  $r > 2$ ,  $X_{\mathbf{0}}$  is centered and has a finite weak  $r$ th moment. Then for each positive  $d$  and each  $\mathbf{N} \in \mathbb{N}^d$ , the inequality

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}} \left| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right| \geq t \right\} &\leq C(r, d) t^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{\mathbf{i} \in E_{\mathbf{N}}} \|Q_{\mathbf{i}}(f)\|_{r, \infty} \right)^r + \\ &+ C(r, d) t^{-r} \prod_{q=1}^d N_q \left( \sum_{\mathbf{0} \preceq \mathbf{l} \preceq \mathbf{N}} \left( \|Q_{\mathbf{l}}(f)\|_{r, \infty}^r \prod_{q=1}^d \max\{l_q, 1\}^{r/2-1} \right)^{1/(r+1)} \right)^{r+1} + \\ &+ C(r, d) t^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{\mathbf{0} \preceq \mathbf{l} \preceq \mathbf{N}} \|Q_{\mathbf{l}}(f)\|_2^{r/(r+1)} \prod_{q=1}^d (\max\{l_q, 1\})^{\frac{2-r}{2(r+1)}} \right) \\ &\quad \left( \sum_{\mathbf{0} \preceq \mathbf{l} \preceq \mathbf{N}} \left( \|Q_{\mathbf{l}}(f)\|_{r, \infty}^r \prod_{q=1}^d (\max\{l_q, 1\})^{r/2-1} \right)^{1/(r+1)} \right)^r \end{aligned} \quad (11.2.10)$$

takes place, where  $C(r, d)$  depends only on  $r$  and  $d$  and

$$E_{\mathbf{N}} := \{\mathbf{i} \in \mathbb{Z}^d \mid i_q \geq N_q + 1 \text{ for some } q \in \langle d \rangle\}. \quad (11.2.11)$$

In particular, the following inequality holds

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}} \left| \sum_{\mathbf{0} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \right| \geq t \right\} &\leq C(r, d) t^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{\mathbf{l} \in E_{\mathbf{N}}} \|Q_{\mathbf{l}}(f)\|_{r, \infty} \right)^r + \\ &+ C(r, d) t^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{\mathbf{0} \preceq \mathbf{l} \preceq \mathbf{N}} \left( \|Q_{\mathbf{l}}(f)\|_{r, \infty}^r \prod_{q=1}^d (\max\{l_q, 1\})^{r/2-1} \right)^{1/(r+1)} \right)^{r+1}. \end{aligned} \quad (11.2.12)$$

*Proof.* We start from the decomposition

$$f = \sum_{\mathbf{i} \succeq \mathbf{0}} Q_{\mathbf{i}}(f) \quad (11.2.13)$$

$$= \sum_{\mathbf{i} \in E_{\mathbf{N}}} Q_{\mathbf{i}}(f) + \sum_{\mathbf{0} \preceq \mathbf{i} \preceq \mathbf{N}} Q_{\mathbf{i}}(f) \quad (11.2.14)$$

$$= \sum_{\mathbf{i} \in E_{\mathbf{N}}} Q_{\mathbf{i}}(f) + \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{N}} Q_{\mathbf{i}}(f) + \sum_{J \subset \langle d \rangle} \sum_{\substack{\mathbf{1} \preceq k_j \preceq n_j \\ j \in J}} Q_{k^J}(f), \quad (11.2.15)$$

where for  $J = \{j_1, \dots, j_{|J|}\} \subset \langle d \rangle$ , and  $k \in \mathbb{Z}^{|J|}$ ,  $k^J$  is the element of  $\mathbb{Z}^d$  whose  $q$ th coordinate is  $k_s$  if  $q = j_s$  for some  $s \in \{1, \dots, |J|\}$  and 0 otherwise. We would like to use martingale properties in order to remove the maxima. But by definition of the operators  $Q_{\mathbf{i}}$ , the random field  $(U^{\mathbf{j}}Q_{\mathbf{i}}(f))_{\mathbf{j} \in \mathbb{Z}^d}$  is a *reversed* orthomartingale-difference. To this aim, we proceed in a similar way as in [LXW13]. We use the inequality

$$\max_{\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}} |S_{\mathbf{n}}(Q_{\mathbf{i}}(f))| \leq \sum_{J \subset \langle d \rangle} \max_{\substack{\mathbf{1} \preceq n_j \preceq N_{j-1} \\ j \in J}} \sum_{\substack{N_j - n_j \preceq k_j \preceq N_j \\ j \in J}} |S_{(k_q \mathbf{1}_{\{q \in J\}} + N_q \mathbf{1}_{\{q \in J^c\}})}(Q_{\mathbf{i}}(f))|. \quad (11.2.16)$$

To see this, we start from the inequality

$$\begin{aligned} \max_{\mathbf{1} \preceq (\mathbf{n}', n_d) \preceq (\mathbf{N}', N_d)} |S_{\mathbf{n}', n_d}(Q_i(f))| &\leq \max_{\mathbf{1} \preceq \mathbf{n}' \preceq \mathbf{N}'} |S_{\mathbf{n}', N_d}(Q_i(f))| + \\ &+ \max_{\mathbf{1} \preceq \mathbf{n}' \preceq \mathbf{N}'} \max_{0 \leq j_d \leq N_d - 1} \left| \sum_{k_d = N_d - j_d}^{N_d} U_d^{k_d} S_{\mathbf{n}', 0}(Q_i(f)) \right|, \end{aligned} \quad (11.2.17)$$

and treat in a similar way the two terms. We also obtain

$$\begin{aligned} \max_{\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}} \left| S_{\mathbf{n}} \left( \sum_{J \subsetneq \langle d \rangle} \sum_{\substack{1 \leq k_j \leq N_j \\ j \in J}} Q_{k^J}(f) \right) \right| &\leq \\ &\leq \sum_{J \subsetneq \langle d \rangle} \sum_{I \subset J} \max_{\substack{1 \leq n_i \leq N_i - 1 \\ i \in I}} \sum_{\substack{N_i - n_i \leq l_i \leq N_i \\ i \in I}} \left| S_{(l_q \mathbf{1}\{q \in I\} + N_q \mathbf{1}\{q \in I^c\})} \left( \sum_{\substack{1 \leq k_j \leq n_j \\ j \in J}} Q_{k^J}(f) \right) \right| \end{aligned} \quad (11.2.18)$$

Combining (11.2.15), (11.2.16) and (11.2.18), we derive

$$\begin{aligned} \max_{\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}} |S_{\mathbf{n}}(f)| &\leq \sum_{\mathbf{i} \in E_{\mathbf{N}}} \sum_{J \subset \langle d \rangle} \max_{\substack{1 \leq n_j \leq N_j - 1 \\ j \in J}} \sum_{\substack{N_j - n_j \leq k_j \leq N_j \\ j \in J}} |S_{(k_q \mathbf{1}\{q \in J\} + N_q \mathbf{1}\{q \in J^c\})}(Q_i(f))| + \\ &+ \sum_{J \subset \langle d \rangle} \max_{\substack{1 \leq n_j \leq N_j - 1 \\ j \in J}} \sum_{\substack{N_j - n_j \leq k_j \leq N_j \\ j \in J}} \left| S_{(k_q \mathbf{1}\{q \in J\} + N_q \mathbf{1}\{q \in J^c\})} \left( \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{N}} Q_i(f) \right) \right| + \\ &+ \sum_{J \subsetneq \langle d \rangle} \sum_{I \subset J} \max_{\substack{1 \leq n_i \leq N_i - 1 \\ i \in I}} \sum_{\substack{N_i - n_i \leq l_i \leq N_i \\ i \in I}} \left| S_{(l_q \mathbf{1}\{q \in I\} + N_q \mathbf{1}\{q \in I^c\})} \left( \sum_{\substack{1 \leq k_j \leq n_j \\ j \in J}} Q_{k^J}(f) \right) \right|, \end{aligned} \quad (11.2.19)$$

hence for each positive  $t$ ,

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}} |S_{\mathbf{n}}(f)| \geq (1 + 2 \cdot 6^d)t \right\} &\leq t^{-r} \left( 1 + \frac{r}{r-1} \right)^d \left( \sum_{\mathbf{i} \in E_{\mathbf{N}}} \|S_{\mathbf{N}}(Q_i(f))\|_{r, \infty} \right)^r + \\ &+ \sum_{J \subset \langle d \rangle} \mu \left\{ \max_{\substack{1 \leq n_j \leq N_j - 1 \\ j \in J}} \sum_{\substack{N_j - n_j \leq k_j \leq N_j \\ j \in J}} \left| S_{(k_q \mathbf{1}\{q \in J\} + N_q \mathbf{1}\{q \in J^c\})} \left( \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{N}} Q_i(f) \right) \right| \geq 4^d t \right\} + \\ &+ \sum_{J \subsetneq \langle d \rangle} \sum_{I \subset J} \mu \left\{ \max_{\substack{1 \leq n_i \leq N_i - 1 \\ i \in I}} \sum_{\substack{N_i - n_i \leq l_i \leq N_i \\ i \in I}} \left| S_{(l_q \mathbf{1}\{q \in I\} + N_q \mathbf{1}\{q \in I^c\})} \left( \sum_{\substack{1 \leq k_j \leq n_j \\ j \in J}} Q_{k^J}(f) \right) \right| \geq 2^d t \right\}. \end{aligned}$$

Using repeatedly Lemma 11.1.3, it follows that

$$\begin{aligned} \mu \left\{ \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} |S_{\mathbf{n}}(f)| \geq (1 + 2 \cdot 6^d)t \right\} &\leq t^{-r} \left( 1 + \frac{r}{r-1} \right)^d \left( \sum_{\mathbf{i} \in E_{\mathbf{N}}} \|S_{\mathbf{N}}(Q_{\mathbf{i}}(f))\|_{r,\infty} \right)^r + \\ &+ \sum_{J \subset \langle d \rangle} \int_{[1,\infty)} \mu \left\{ \left| S_{\mathbf{N}} \left( \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} Q_{\mathbf{i}}(f) \right) \right| \geq ts \right\} (\log s)^{\max\{|J|,1\}-1} ds + \\ &+ \sum_{J \subset \langle d \rangle} \int_{[1,\infty)} \mu \left\{ \left| S_{\mathbf{N}} \left( \sum_{\substack{1 \leq k_j \leq n_j \\ j \in J}} Q_{k^J}(f) \right) \right| \geq ts \right\} (\log s)^{\max\{|J|,1\}-1} ds. \end{aligned} \quad (11.2.20)$$

We now have to bound the three terms in (11.2.20).

*First term.* We use the fact that  $(U^{\mathbf{j}}Q_{\mathbf{i}}(f))_{\mathbf{j} \in \mathbb{N}^d}$  is a reversed orthomartingale difference random field in order to derive that

$$\|S_{\mathbf{N}}(Q_{\mathbf{i}}(f))\|_{r,\infty} \leq C(r, d) \prod_{q=1}^d \sqrt{N_q} \|Q_{\mathbf{i}}(f)\|_{r,\infty} \quad (11.2.21)$$

for some constant  $C(r, d)$  depending only on  $r$  and  $d$ . As a consequence, we obtain

$$\begin{aligned} t^{-r} \left( 1 + \frac{r}{r-1} \right)^d \left( \sum_{\mathbf{i} \in E_{\mathbf{N}}} \|S_{\mathbf{N}}(Q_{\mathbf{i}}(f))\|_{r,\infty} \right)^r &\leq \\ &\leq C(r, d) t^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{\mathbf{i} \in E_{\mathbf{N}}} \|Q_{\mathbf{i}}(f)\|_{r,\infty} \right)^r. \end{aligned} \quad (11.2.22)$$

*Second term.* Let  $(\lambda_{\mathbf{i}})_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}}$  be a family of positive real numbers such that  $\sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} \lambda_{\mathbf{i}} \leq 1$  (which will be specified later). Then

$$\mu \left\{ \left| S_{\mathbf{N}} \left( \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} Q_{\mathbf{i}}(f) \right) \right| \geq ts \right\} \leq \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} \mu \{ |S_{\mathbf{N}}(Q_{\mathbf{i}}(f))| \geq ts \lambda_{\mathbf{i}} \}. \quad (11.2.23)$$

Let us fix  $\mathbf{i}$  such that  $\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}$ . We define  $l_q := [N_q/i_q] + 1$  and

$$Y_{\mathbf{j}} := \sum_{\mathbf{1} + (\mathbf{j}-\mathbf{1}) \mathbf{i} \preccurlyeq \mathbf{k} \preccurlyeq \min\{\mathbf{i}, \mathbf{N}\}} U^{\mathbf{k}}(Q_{\mathbf{i}}(f)), \quad \mathbf{1} \preccurlyeq \mathbf{j} \preccurlyeq \mathbf{l}, \quad (11.2.24)$$

where the product is taken coordinatewise. We now write the set  $\{\mathbf{j} \mid \mathbf{1} \preccurlyeq \mathbf{j} \preccurlyeq \mathbf{l}\}$  as the disjoint union of the following set

$$F_J := \{\mathbf{j} \mid \mathbf{1} \preccurlyeq \mathbf{j} \preccurlyeq \mathbf{l} \text{ and } j_q \text{ is even if and only if } q \in J\}, \quad J \subset \{1, \dots, d\}. \quad (11.2.25)$$

With this notation, we have

$$S_{\mathbf{N}}(Q_{\mathbf{i}}(f)) = \sum_{J \subset \langle d \rangle} \sum_{\mathbf{j} \in F_J} Y_{\mathbf{j}}, \quad (11.2.26)$$

hence

$$\mu \{ |S_{\mathbf{N}}(Q_{\mathbf{i}}(f))| \geq 2^d ts \lambda_{\mathbf{i}} \} \leq \sum_{J \subset \langle d \rangle} \mu \left\{ \left| \sum_{\mathbf{j} \in F_J} Y_{\mathbf{j}} \right| \geq ts \lambda_{\mathbf{i}} \right\}. \quad (11.2.27)$$



Observing that for a fixed  $J \subset \langle d \rangle$ , the collection of random variables  $(Y_j)_{j \in F_J}$  is independent, we obtain by that

$$\mu \left\{ \left| \sum_{j \in F_J} Y_j \right| \geq ts \lambda_i \right\} \leq C(r, d) (ts \lambda_i)^{-r} \left( \sum_{j \in F_J} \|Y_j\|_{r, \infty}^r + \left( \sum_{j \in F_J} \|Y_j\|_2^2 \right)^{r/2} \right). \quad (11.2.28)$$

By Burkholder's inequality, we have

$$\begin{aligned} \|Y_j\|_{r, \infty} &\leq C(r, d) \prod_{q=1}^d (\min \{i_q j_q, N\} - i_q(j_q - 1))^{1/2} \|Q_i(f)\|_{r, \infty} \leq \\ &\leq C(r, d) \prod_{q=1}^d i_q^{1/2} \|Q_i(f)\|_{r, \infty} \quad \text{and} \quad (11.2.29) \end{aligned}$$

$$\begin{aligned} \|Y_j\|_2 &\leq C(r, d) \prod_{q=1}^d (\min \{i_q j_q, N\} - i_q(j_q - 1))^{1/2} \|Q_i(f)\|_2 \leq \\ &\leq C(r, d) \prod_{q=1}^d i_q^{1/2} \|Q_i(f)\|_2, \quad (11.2.30) \end{aligned}$$

hence by (11.2.27) and (11.2.28) (and the fact that the cardinality of  $F_J$  is smaller than  $2^d \prod_{q=1}^d N_d / i_d$ ),

$$\begin{aligned} \mu \{ |S_{\mathbf{N}}(Q_i(f))| \geq 2^d ts \lambda_i \} &\leq C(r, d) (ts \lambda_i)^{-r} \left( \prod_{q=1}^d N_q i_q^{r/2-1} \|Q_i(f)\|_{r, \infty}^r + \right. \\ &\quad \left. + \prod_{q=1}^d N_q^{r/2} \|Q_i(f)\|_2^r \right). \quad (11.2.31) \end{aligned}$$

Plugging this estimate into (11.2.23), we obtain

$$\begin{aligned} \mu \left\{ \left| S_{\mathbf{N}} \left( \sum_{\mathbf{1} \leq i \leq \mathbf{N}} Q_i(f) \right) \right| \geq ts \right\} &\leq \sum_{\mathbf{1} \leq i \leq \mathbf{N}} C(r, d) (ts \lambda_i)^{-r} \left( \prod_{q=1}^d N_q i_q^{r/2-1} \|Q_i(f)\|_{r, \infty}^r + \right. \\ &\quad \left. + \prod_{q=1}^d N_q^{r/2} \|Q_i(f)\|_2^r \right). \quad (11.2.32) \end{aligned}$$

We now choose

$$\lambda_i := \left( \|Q_i(f)\|_{r, \infty}^r \prod_{q=1}^d i_q^{r/2-1} \right)^{1/(r+1)} \cdot \left( \sum_{\mathbf{1} \leq \mathbf{n}} \left( \|Q_{\mathbf{n}}(f)\|_{r, \infty}^r \prod_{q=1}^d i_q^{r/2-1} \right)^{1/(r+1)} \right)^{-1}; \quad (11.2.33)$$

in such a way,  $\sum_{\mathbf{1} \leq i \leq \mathbf{n}} \lambda_i \leq 1$  and we can assume that  $Q_i(f) \neq 0$  (otherwise we remove it in

$\sum_{1 \leq i \leq N} Q_i(f)$ , hence each  $\lambda_i$  is positive. In view of (11.2.32), this choice entails

$$\begin{aligned} \mu \left\{ \left| S_{\mathbf{N}} \left( \sum_{1 \leq i \leq N} Q_i(f) \right) \right| \geq ts \right\} &\leq \\ &\leq C(r, d)(st)^{-r} \prod_{q=1}^d N_q \left( \sum_{l \geq 1} \left( \|Q_1(f)\|_{r, \infty}^r \prod_{q=1}^d l_q^{r/2-1} \right)^{1/(r+1)} \right)^{r+1} + \\ &+ C(r, d)(st)^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{l \geq 1} \|Q_1(f)\|_2^{r^2/(r+1)} \prod_{q=1}^d l_q^{\frac{2-r}{2(r+1)}} \right) \left( \sum_{l \geq 1} \|Q_1(f)\|_{r, \infty}^r \prod_{q=1}^d l_q^{r/2-1} \right)^r. \end{aligned} \quad (11.2.34)$$

We finally obtain

$$\begin{aligned} \sum_{J \subset [d]} \int_{[1, \infty)} \mu \left\{ \left| S_{\mathbf{N}} \left( \sum_{1 \leq i \leq N} Q_i(f) \right) \right| \geq ts \right\} (\log s)^{\max\{|J|, 1\}-1} ds &\leq \\ &\leq C(r, d)t^{-r} \prod_{q=1}^d N_q \left( \sum_{l \geq 1} \left( \|Q_1(f)\|_{r, \infty}^r \prod_{q=1}^d l_q^{r/2-1} \right)^{1/(r+1)} \right)^{r+1} + \\ &+ C(r, d)t^{-r} \prod_{q=1}^d N_q^{r/2} \left( \sum_{l \geq 1} \|Q_1(f)\|_2^{r^2/(r+1)} \prod_{q=1}^d l_q^{\frac{2-r}{2(r+1)}} \right) \left( \|Q_1(f)\|_{r, \infty}^r \prod_{q=1}^d l_q^{r/2-1} \right)^r. \end{aligned} \quad (11.2.35)$$

*Third term.* We follow the same strategy as in the bound of the second term of (11.2.20). This explains the presence of the terms  $\max\{l_q, 1\}$  instead of  $l_q$ .

The estimates of the three terms of (11.2.20) yield (11.2.10), which concludes the proof of Theorem 11.2.1.  $\square$



# Chapter 12

## Application to the invariance principle in Hölder spaces

In this chapter, we investigate the weak invariance principle in Hölder spaces for strictly stationary random fields. We shall study first orthomartingale random fields and then Bernoulli random fields. We establish a tightness criterion for the partial sum processes defined by

$$S_{\mathbf{n}}(f, \mathbf{t}) := \sum_{1 \leq i \leq \mathbf{n}} \lambda([\mathbf{0}, \mathbf{t}] \cap R_i) U^i f, \quad (12.0.1)$$

where  $\lambda$  denotes Lebesgue measure on  $\mathbb{R}^d$ ,  $[\mathbf{0}, \mathbf{t}] = \prod_{q=1}^d [0, t_q]$  and  $R_i := \prod_{q=1}^d [i_q - 1, i_q]$ . We establish a sufficient condition for the asymptotic tightness of the net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \geq 1}$  in the space of Hölder continuous functions  $\mathcal{H}_\alpha([0, 1]^d)$ .

### 12.1 Notations and results

#### 12.1.1 The Hölder space

In this subsection, we recall the properties of the Hölder spaces. For  $0 < \alpha < 1$ , we denote by  $\mathcal{H}_\alpha^o([0, 1]^d)$  the vector space of function  $x: [0, 1]^d \rightarrow \mathbb{R}$  such that

$$\|x\|_\alpha := |x(\mathbf{0})| + \omega_\alpha(x, 1) < \infty, \quad (12.1.1)$$

with

$$\omega_\alpha(x, \delta) := \sup_{0 < |\mathbf{t} - \mathbf{s}| \leq \delta} \frac{|x(\mathbf{t}) - x(\mathbf{s})|}{|\mathbf{t} - \mathbf{s}|^\alpha} \rightarrow 0 \text{ as } \delta \rightarrow 0. \quad (12.1.2)$$

Set for  $j \geq 0$ ,

$$W_j := \{k2^{-j}, 0 \leq k \leq 2^j\} \quad (12.1.3)$$

and

$$V_0 := W_0, \quad V_j := W_j \setminus W_{j-1}, j \geq 1. \quad (12.1.4)$$

We define for  $\mathbf{v} \in V_j$  the pyramidal function  $\Lambda_{j, \mathbf{v}}$  by

$$\Lambda_{j, \mathbf{v}}(\mathbf{t}) := \Lambda(2^j(\mathbf{t} - \mathbf{v})), \quad \mathbf{t} \in [0, 1]^d, \quad (12.1.5)$$

where

$$\Lambda(\mathbf{t}) := \max \left\{ 0, 1 - \max_{t_i < 0} |t_i| - \max_{t_i > 0} |t_i| \right\}, \quad \mathbf{t} = (t_i)_{i=1}^d \in [-1, 1]^d. \quad (12.1.6)$$

For  $x \in \mathcal{H}_\alpha^o([0, 1]^d)$ , we define the coefficients  $\lambda_{j, \mathbf{v}}(x)$  by  $\lambda_{0, \mathbf{v}}(x) = x(\mathbf{v})$ ,  $v \in V_0$  and for  $j \geq 1$  and  $v \in V_j$ ,

$$\lambda_{j, \mathbf{v}}(x) := x(\mathbf{v}) - \frac{1}{2} (x(\mathbf{v}^-) + x(\mathbf{v}^+)), \quad (12.1.7)$$

where  $\mathbf{v}^+$  and  $\mathbf{v}^-$  are defined in the following way. Each  $\mathbf{v} \in V_j$  is represented in a unique way by  $\mathbf{v} = (k_i 2^{-j})_{i=1}^d$ . Then  $\mathbf{v}^+ := (v_i^+)_{i=1}^d$  and  $\mathbf{v}^- := (v_i^-)_{i=1}^d$  are defined by

$$\mathbf{v}_i^- := \begin{cases} v_i - 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even} \end{cases} \quad \mathbf{v}_i^+ := \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even.} \end{cases} \quad (12.1.8)$$

The sequential norm is defined by

$$\|x\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(x)|, \quad x \in \mathcal{H}_\alpha^o([0, 1]^d). \quad (12.1.9)$$

By [RS04a], the norms  $\|\cdot\|_\alpha^{\text{seq}}$  is equivalent to  $\|\cdot\|_\alpha$  on  $\mathcal{H}_\alpha^o([0, 1]^d)$ .

### 12.1.2 Tightness of the partial sum proces

In this subsection, we give a sufficient condition for tightness of the partial sum process associated to a strictly stationary random field. No other assumption is done but of course, some dependence will be required for this condition to be satisfied. A general tightness criterion is available.

**Theorem 12.1.1** (Theorem 6, [RSZ07]). *Let  $\{\zeta_n, n \in \mathbb{N}^d\}$  and  $\zeta$  be random elements with values in the space  $\mathcal{H}_\alpha([0, 1]^d)$ . Assume that the following conditions are satisfied.*

1. *For each dyadic  $t \in [0, 1]^d$ , the net  $\{\zeta_n(t), n \in \mathbb{N}^d\}$  is asymptotically tight on  $\mathbb{R}$ .*
2. *For each positive  $\varepsilon$ ,*

$$\lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(\zeta_{\mathbf{n}})| > \varepsilon \right\} = 0. \quad (12.1.10)$$

*Then the net  $\{\zeta_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is asymptotically tight in the space  $\mathcal{H}_\alpha([0, 1]^d)$ .*

**Proposition 12.1.2.** *Let  $f: \Omega \rightarrow \mathbb{R}$  be a measurable function. Then for each  $n \in \mathbb{N}^d$ ,  $J \geq 1$  and  $x > 0$ , the following inequality holds*

$$\begin{aligned} & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > x \right\} \leq \\ & \leq \sum_{i=1}^d n_i \mu \left\{ \max_{\substack{0 \leq j_l \leq n_l \\ l \neq i}} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} f \right| > C x n_i^{-\alpha} \prod_{l \in \langle d \rangle \setminus \{i\}} \sqrt{n_l} \right\} + \\ & + 2 \sum_{i=1}^d \sum_{j=J}^{n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{1 \leq s \leq 1} U^s f \right| > C x \prod_{q=1}^d \sqrt{n_q} \right\}, \end{aligned} \quad (12.1.11)$$

where  $C$  depends only on  $d$ .

Therefore, if  $f$  is such that for each  $i \in \{1, \dots, d\}$ ,

$$\limsup_{\min \mathbf{n} \rightarrow \infty} n_i \mu \left\{ \max_{\substack{0 \leq j_l \leq n_l, l \neq i}} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} f \right| > \prod_{l=1}^d \sqrt{n_l} n_i^{-\alpha} \right\} = 0, \quad (12.1.12)$$

and that for each positive  $\varepsilon$

$$\lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \sum_{j=J}^{\log n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \leq \mathbf{s} \leq \mathbf{1}} U^{\mathbf{s}} f \right| > \varepsilon \prod_{q=1}^d \sqrt{n_q} \right\} = 0, \quad (12.1.13)$$

then the net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))$  is asymptotically tight in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$ .

*Remark 12.1.3.* 1. Assume that  $f$  is such that for some  $q > p$ ,  $\|\max_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}} |S_{\mathbf{i}}(f)|\|_q \leq C |\mathbf{n}|^{1/2}$  for each  $\mathbf{n} \in \mathbb{N}^d$  (where  $C$  is independent of  $\mathbf{n}$ ). Then the partial sum process  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, t))_{\mathbf{n} \in \mathbb{N}^d}$  is asymptotically tight in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$ . This can be seen by a direct application of Proposition 12.1.2, where conditions (12.2.41) and (12.2.42) can be checked by Markov's inequality.

2. If we merely assume uniform integrability of the family

$$\mathcal{S} := \left\{ |\mathbf{n}|^{-p/2} \max_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{n}} |S_{\mathbf{j}}(f)|^p, \mathbf{n} \succcurlyeq \mathbf{1} \right\}, \quad (12.1.14)$$

then (12.1.12) is automatically satisfied in view of the elementary estimate

$$\begin{aligned} n_i \mu \left\{ \max_{0 \leq j_i \leq n_i, l \neq i} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} f \right| > n_i^{-\alpha} |\mathbf{n}|^{1/2} \right\} &\leq \\ &\leq \mathbb{E} \left[ |\mathbf{n}|^{-p/2} \max_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{n}} |S_{\mathbf{j}}(f)|^p \mathbf{1} \left\{ |\mathbf{n}|^{-1/2} \max_{\mathbf{1} \leq \mathbf{j} \leq \mathbf{n}} |S_{\mathbf{j}}(f)| \geq n_i^{1/2-\alpha} \right\} \right]. \end{aligned} \quad (12.1.15)$$

On the other hand, the link between uniform integrability of the normalized maxima of partial sums and condition (12.1.13) does not look obvious.

### 12.1.3 Hölderian invariance principle by martingale approximation

When we try to extend a limit theorem for i.i.d. random to strictly stationary orthomartingale difference random fields, it is natural to determine whether the sufficient condition on the common law in the i.i.d. still is sufficient. In the case of the Hölderian weak invariance principle, like in the one dimensional case, the answer is negative.

**Theorem 12.1.4.** *Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space,  $T$  a  $\mathbb{Z}^d$ -measure preserving action. We assume that the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  is ergodic and of positive entropy. For each  $p > 2$ , there exists a sub- $\sigma$ -algebra  $\mathcal{M}$  such that  $(T^{-i}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  is a commuting filtration, a function  $m$  and an increasing sequence of positive integers  $(N_l)_{l \geq 1}$  such that*

1.  $\lim_{t \rightarrow \infty} t^p \mu \{|m| > t\} = 0$ ;
2.  $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  is an orthomartingale difference random field with respect to the filtration  $(T^{-i}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ ;
3. the sequence  $((N_l l^{d-1})^{-1/2} S_{N_l, l, \dots, l}(m, \cdot))$  is not tight in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$ .

**Theorem 12.1.5.** *Assume that  $T: \Omega \rightarrow \Omega$  is a  $\mathbb{Z}^d$ -measure preserving action with  $T_1$  ergodic and  $\mathcal{M}$  is a sub- $\sigma$ -algebra such that  $(T^{-i}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  is a commuting filtration. If  $m$  is an orthomartingale martingale difference random field with respect to  $(T^{-i}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  such that for some  $p > 2$ ,*

1.  $\lim_{t \rightarrow +\infty} t^p \mu \{|m| > t\} = 0$ ;
2. for each  $q \in \{1, \dots, d\}$ ,  $\mathbb{E}[m^2 | T_q \mathcal{M}] \in \mathbb{L}^{p/2}$ ,

then the convergence

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{\mathbf{n}}(m, \cdot) \xrightarrow{\min \mathbf{n} \rightarrow \infty} \|m\|_2 W \text{ in distribution in } \mathcal{H}_{1/2-1/p}([0, 1]^d) \quad (12.1.16)$$

takes place. In particular, if  $m \in \mathbb{L}^p$ , then the convergence (12.1.16) holds.

*Remark 12.1.6.* Ergodicity of the map  $T_1$  is not needed to guarantee asymptotic tightness of  $\left(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m, \cdot)\right)_{\mathbf{n} \in (\mathbb{N} \setminus \{0\})^d}$  in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$ . This is used for the convergence of the finite dimensional distribution.

*Remark 12.1.7.* If  $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d} =: (\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$  is an i.i.d. random field, then we do not exactly recover Theorem 2 of [RSZ07]. Indeed, in this case, the conditions of Theorem 12.1.5 are equivalent to  $\mu \{|m| > t\} = o(t^{-p})$ , while that of Theorem 2 of [RSZ07] is  $\mu \{|m| > t\} = O(t^{-p})$ .

Using a martingale approximation argument, we may formulate a sufficient condition for the multiindexed Hölderian invariance principle.

$$P_{\mathbf{j}} := \prod_{q=1}^d P_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d, \quad (12.1.17)$$

where for  $l \in \mathbb{Z}$ ,  $P_l^{(q)} : \mathbf{L}^1(\mathcal{F}) \rightarrow \mathbf{L}^1(\mathcal{F})$  is defined by

$$P_l^{(q)}(f) = \mathbb{E}_l^{(q)}[f] - \mathbb{E}_{l-1}^{(q)}[f] \quad (12.1.18)$$

and

$$\mathbb{E}_l^{(q)}[\cdot] = \mathbb{E} \left[ \cdot \mid \bigvee_{\substack{\mathbf{i} \in \mathbb{Z}^d \\ i_q \leq l}} \mathcal{F}_{\mathbf{i}} \right], \quad q \in \{1, \dots, d\}, l \in \mathbb{Z}. \quad (12.1.19)$$

**Theorem 12.1.8.** Let  $p > 2$  and let  $f$  be a  $\mathcal{F}_{\infty, \dots, \infty}$ -measurable function such  $\mathbb{E}[f | \mathcal{F}_{\mathbf{i}}] \rightarrow 0$  in  $\mathbb{L}^p$  as  $\min_{q \in \langle d \rangle} i_q \rightarrow -\infty$ . Assume that one of the transformations  $T_q$ ,  $q \in \{1, \dots, d\}$  is ergodic and that

$$\sum_{\mathbf{i} \in \mathbb{Z}^d} \|P_{\mathbf{i}}(f)\|_p < \infty. \quad (12.1.20)$$

Then

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{\mathbf{n}}(f, \cdot) \xrightarrow{\min \mathbf{n} \rightarrow \infty} \sigma^2 W \text{ in distribution in } \mathcal{H}_{1/2-1/p}([0, 1]^d), \quad (12.1.21)$$

where

$$\sigma^2 = \mathbb{E} \left[ \left( \sum_{\mathbf{i} \in \mathbb{Z}^d} P_{\mathbf{0}}(U^{\mathbf{i}} f) \right)^2 \right]. \quad (12.1.22)$$

### 12.1.4 Bernoulli random fields

**Theorem 12.1.9.** *Let  $(\varepsilon_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d)$  be i.i.d. random variables,  $p > 2$  and let  $U^{\mathbf{k}}f := X_{\mathbf{k}} = g(\varepsilon_{\mathbf{k}-\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d)$ ,  $\mathbf{k} \in \mathbb{Z}^d$ , where  $g: \mathbb{R}^{\mathbb{N}^d} \rightarrow \mathbb{R}$  is a measurable function, be centered random variables. Assume that*

$$\sum_{\mathbf{l} \neq \mathbf{0}} \left( \|Q_{\mathbf{l}}(f)\|_{p, \infty}^p \prod_{q=1}^d (\max\{l_q, 1\})^{p/2-1} \right)^{1/(p+1)} < \infty. \quad (12.1.23)$$

Then the net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \geq \mathbf{1}}$  converges to  $\sigma W$  in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$  as  $\min \mathbf{n}$  goes to infinity, where

$$\sigma^2 = \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov}(X_{\mathbf{0}}, X_{\mathbf{k}}). \quad (12.1.24)$$

When we apply this result to  $g: ((x_j)_{j \in \mathbb{N}^d}) \mapsto x_0 - \mathbb{E}[\varepsilon_{\mathbf{0}}]$ , we recover the result by Račkauskas, Suquet and Zemlys [RSZ07].

## 12.2 Proofs

### 12.2.1 Tightness criterion

Consider for  $\mathbf{s} := (s_2, \dots, s_d) \in [0, 1]^{d-1}$  and  $t, t' \in [0, 1]$  the quantity

$$\Delta_n(t, t', \mathbf{s}) := |S_{\mathbf{n}}(f, (t', \mathbf{s})) - S_{\mathbf{n}}(f, (t, \mathbf{s}))| \quad (12.2.1)$$

We recall the following lemma:

**Lemma 12.2.1** (Lemma 11, [RSZ07]). *For any  $t', t \in [0, 1]$ ,  $t' > t$ , we have*

$$\begin{aligned} \sup_{\mathbf{s} \in [0, 1]^{d-1}} \Delta_n(t, t', \mathbf{s}) &\leq 3^d \mathbf{1} \left\{ t' - t \geq \frac{1}{n_1} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ 2 \leq l \leq d}} \left| \sum_{i_1=[n_1 t]+1}^{[n_1 t']} \sum_{\substack{1 \leq i_l \leq k_l \\ 2 \leq l \leq d}} U^{\mathbf{i}} f \right| + \\ &+ 3^d \min\{1, n_1(t' - t)\} \max_{1 \leq i_1 \leq n_1} \max_{\substack{1 \leq k_l \leq n_l \\ 2 \leq l \leq d}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ 2 \leq l \leq d}} U^{\mathbf{i}} f \right|. \end{aligned} \quad (12.2.2)$$

Now, if we define for  $q \in \langle d \rangle$  and  $\mathbf{s} = (s_l)_{l \in \langle d \rangle \setminus \{q\}} \in [0, 1]^{d-1}$ ,

$$\Delta_n^{(q)}(t, t', \mathbf{s}) := |S_{\mathbf{n}}(f, (s_1, \dots, s_{q-1}, t', s_{q+1}, \dots, s_d)) - S_{\mathbf{n}}(f, (s_1, \dots, s_{q-1}, t, s_{q+1}, \dots, s_d))|, \quad (12.2.3)$$

we have in view of Lemma 12.2.1 and

By definition of  $\lambda_{j, \mathbf{v}}$  and  $V_j$ , the inequality

$$\begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha_j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > 2^{d+1} 3^d x \right\} &\leq \\ &\sum_{q=1}^d \mu \left\{ \sup_{j \geq J} 2^{\alpha_j} \max_{\substack{0 \leq k < 2^j \\ 0 \leq l \leq 2^j}} \Delta_n^{(q)}(t_{k+1}, t_k; s_l) > 2 \cdot 3^d x \right\} \end{aligned} \quad (12.2.4)$$



take place, where  $t_k = k2^{-j}$  and  $(s_l)_{l \in \langle d \rangle \setminus \{q\}} = (l_i 2^{-j})_{l \in \langle d \rangle \setminus \{q\}}$ .

We have in view of Lemma 12.2.1 that for each  $q \in \langle d \rangle$ ,

$$\begin{aligned}
& \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\substack{0 \leq k < 2^j \\ 0 \leq l < 2^j}} \Delta_n^{(q)}(t_{k+1}, t_k; s_l) > 2x |\mathbf{n}|^{1/2} \right\} \\
& \leq \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| + \right. \\
& \quad \left. + \min \{1, n_q 2^{-j}\} \max_{1 \leq i_q \leq n_q} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > 2x |\mathbf{n}|^{1/2} \right\} \\
& \leq \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} + \\
& \quad + \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \min \{1, n_q 2^{-j}\} \max_{1 \leq i_q \leq n_q} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\}. \quad (12.2.5)
\end{aligned}$$

Since the indicator in the first term of the right hand side of (12.2.5) vanishes if  $j > \log n_q$ , we have

$$\begin{aligned}
& \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\
& \leq \mu \left\{ \sup_{J \leq j \leq \log n_q} \max_{0 \leq a < 2^j} 2^{\alpha j} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\
& \leq \sum_{j=J}^{\log n_q} 2^j \max_{0 \leq a < 2^j} \mu \left\{ 2^{\alpha j} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\}, \quad (12.2.6)
\end{aligned}$$

and by stationarity, it follows that

$$\begin{aligned}
& \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\
& \leq \sum_{j=J}^{\log n_q} 2^j \mu \left\{ 2^{\alpha j} \max_{0 \leq i_q \leq 2n_q 2^{-j}} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\}. \quad (12.2.7)
\end{aligned}$$

For the second term of the right hand side of (12.2.5), notice that

$$\sup_{j \geq J} 2^{\alpha j} \min \{1, n_q 2^{-j}\} \leq n_q^\alpha. \quad (12.2.8)$$

Indeed, if  $j \leq \log n_q$ , then  $2^j \leq n_q$  hence  $2^{\alpha j} \min \{1, n_q 2^{-j}\} \leq n_q^\alpha$ , and if  $j > \log n_q$ , then  $2^j > n_q$ , hence  $\min \{1, n_q 2^{-j}\} = n_q 2^{-j}$  and for such  $j$ s, we have  $2^{\alpha j} n_q 2^{-j} = n_q 2^{-(1-\alpha)j} \leq n_q^\alpha$ , since  $\alpha < 1$ . As a consequence, after having bounded the probability of the max over  $i_q$  by the sum of probabilities and used stationarity, we obtain

$$\begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \min \{1, n_q 2^{-j}\} \max_{1 \leq i_q \leq n_q} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^{i_l} f \right| > x |\mathbf{n}|^{1/2} \right\} \\ \leq n_q \mu \left\{ \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^{i_l} f \right| > x |\mathbf{n}|^{1/2} n_q^{-\alpha} \right\}. \end{aligned} \quad (12.2.9)$$

Combining inequalities (12.2.5) with (12.2.7) and (12.2.9), we obtain (12.1.11).

The second part of Proposition 12.1.2 follows by Theorem 12.1.1.

### 12.2.2 Counter-example

In this subsection, we prove Theorem 12.1.4. The construction will require a result on dynamical systems of positive entropy.

**Lemma 12.2.2** (Lemma 1, [EV03]). *There exist two  $T$ -invariant sub- $\sigma$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  of  $\mathcal{F}$  and a function  $g$  defined on  $\Omega$  such that*

- *the  $\sigma$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  are independent;*
- *the function  $g$  is  $\mathcal{B}$ -measurable, zero-mean, with values in  $\{-1, 0, 1\}$  and the random field  $(g \circ T^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  is independent (identically distributed);*
- *the dynamical system  $(\Omega, \mathcal{C}, \mu, T)$  is aperiodic: for each  $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$ ,*

$$\mu \{ \omega \in \Omega \mid T^{\mathbf{k}} \omega = \omega \} = 0. \quad (12.2.10)$$

Moreover, there exists  $0 < a \leq 1$  such that  $\mu \{g = 1\} = \mu \{g = -1\} = a/2$  and  $\mu \{g = 0\} = 1 - a$ .

We consider four increasing sequences of positive integers  $(I_l)_{l \geq 1}$ ,  $(J_l)_{l \geq 1}$ ,  $(L_l)_{l \geq 1}$  and  $(n_l)_{l \geq 1}$

such that

$$\sum_{l \geq 1} \frac{1}{L_l} \text{ is convergent ;} \quad (12.2.11)$$

$$\lim_{l \rightarrow \infty} J_l \mu \left\{ |N| > \frac{L_l}{\|g\|_2} \right\} = 0; \quad (12.2.12)$$

$$\lim_{l \rightarrow \infty} J_l 2^{-I_l/2} = 0; \quad (12.2.13)$$

$$\lim_{l \rightarrow \infty} n_l \sum_{i > l} \frac{2^{I_i+J_i} i^{d-1}}{n_i} = 0; \quad (12.2.14)$$

$$\text{for each } l \geq 1, \sum_{i=1}^{l-1} \frac{2^{I_i+J_i}}{L_i} \quad (12.2.15)$$

$$\left( \frac{n_i}{2^{I_i}} \right)^{1/p} i^{d-1} < \frac{n_l^{1/p}}{2} \text{ and} \quad (12.2.16)$$

$$4k_l \leq n_l. \quad (12.2.17)$$

By the multidimensional version of Rokhlin's lemma (see [Con73]), we can find for each  $l$  a measurable set  $C_l$  such that the family

$$(T^i C_l)_{\substack{0 \leq i_1 \leq n_l - 1 \\ 0 \leq i_q \leq l, q \geq 2}} \text{ is pairwise disjoint and} \quad (12.2.18)$$

$$\mu \left( \bigcup_{\substack{0 \leq i_1 \leq n_l - 1 \\ 0 \leq i_q \leq l, q \geq 2}} T^i C_l \right) > \frac{1}{2}. \quad (12.2.19)$$

We define

$$f_l := \frac{1}{L_l} \left( \frac{n_l}{2^{I_l}} \right)^{1/p} l^{d-1} \sum_{j=1}^{J_l} 2^{-j/p} \sum_{i_1=2^{I_l+j}}^{2^{I_l+j+1}} \sum_{0 \leq i_q \leq l, q \geq 2} \mathbf{1} \left( T_1^{n_l - i_1} \prod_{q=2}^d T_q^{l - i_q} C_l \right) \quad (12.2.20)$$

and the function  $m$  by the equalities

$$f := \sum_{l \geq 1} f_l, \quad m = gf, \quad (12.2.21)$$

where  $g$  is the function obtained in Lemma 12.2.2.

We define the  $\sigma$ -algebra  $\mathcal{M}$  by the equality

$$\mathcal{M} := \sigma(g \circ T^l, l \leq 0) \vee \mathcal{C}. \quad (12.2.22)$$

By Lemma 12.2.2, the  $\sigma$ -algebra  $\mathcal{C}$  is  $T$ -invariant, hence for each  $i, j \in \mathbb{Z}^d$  such that  $i \preceq j$ , the inclusion  $T^{-i} \mathcal{M} \subset T^{-j} \mathcal{M}$  takes place, hence  $(T^{-i} \mathcal{M})_{i \in \mathbb{Z}^d}$  is a stationary filtration.

**Proposition 12.2.3.** *The filtration  $(T^{-i} \mathcal{M})_{i \in \mathbb{Z}^d}$  is commuting and the random field  $(U^i m)_{i \in \mathbb{Z}^d}$  is an orthomartingale difference random field with respect to this filtration.*

*Proof.* Denoting for  $\mathbf{j} \in \mathbb{Z}^d$ ,  $\mathcal{F}_{\mathbf{j}} := T^{-\mathbf{j}} \mathcal{M}$ , we have to check that for each  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$  and each bounded random variable  $X$ ,

$$\mathbb{E} [\mathbb{E} [X | \mathcal{F}_{\mathbf{i}}] | \mathcal{F}_{\mathbf{j}}] = \mathbb{E} [X | \mathcal{F}_{\min\{\mathbf{i}, \mathbf{j}\}}]. \quad (12.2.23)$$

To this aim, we write  $\mathcal{F}_j = \mathcal{F}_{\min\{i,j\}} \vee \mathcal{F}'_{i,j}$ , where

$$\mathcal{F}'_{i,j} = \sigma(g \circ T^1, \mathbf{1} \preceq \mathbf{j}, l_q \geq \min\{i_q, j_q\} \text{ for some } q \in \langle d \rangle). \quad (12.2.24)$$

An application of Lemma 3.1.6 with  $\mathcal{G}_1 := \mathcal{F}_i$ ,  $\mathcal{G}_2 := \mathcal{F}_{\min\{i,j\}}$  and  $\mathcal{G}_3 := \mathcal{F}'_{i,j}$  yields

$$\mathbb{E}[\mathbb{E}[X | \mathcal{F}_i] | \mathcal{F}_j] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_i] | \mathcal{F}_{\min\{i,j\}} \vee \mathcal{F}'_{i,j}] \quad (12.2.25)$$

$$= \mathbb{E}[\mathbb{E}[X | \mathcal{F}_i] | \mathcal{F}_{\min\{i,j\}}] \quad (12.2.26)$$

which proves (12.2.23).

In order to prove the second part of Proposition 12.2.3, notice that for each  $q \in \langle d \rangle$ , we have by (12.2.21) and (12.2.22) that

$$\mathbb{E}[m | T_q \mathcal{M}] = \mathbb{E}[fg | \sigma(g \circ T^1, l \preceq -\mathbf{e}_q) \vee \mathcal{C}]. \quad (12.2.27)$$

Since the function  $f$  is  $\mathcal{C}$ -measurable and the function  $g$  is independent of  $\sigma(g \circ T^1, \mathbf{1} \preceq -\mathbf{e}_q) \vee \mathcal{C}$ , the terms in (12.2.27) are zero. This concludes the proof of Proposition 12.2.3.  $\square$

**Proposition 12.2.4.** *The function  $m$  satisfies  $\lim_{t \rightarrow +\infty} t^p \mu\{|m| > t\} = 0$ .*

This follows by adapting the proof of Proposition 7.3.2.

Now, we have to prove that the sequence  $\left((N_l l^{d-1})^{-1/2} S_{N_l, l, \dots, l}(m, \cdot)\right)$  is not tight in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$ . The first step in this direction is

**Proposition 12.2.5.** *For  $l$  large enough, we have*

$$\mu \left\{ \frac{1}{n_l^{1/p} l^{d-1}} \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ 2 \leq q \leq d}} \frac{|S_{u+v_1 \mathbf{e}_1}(gf_l) - S_u(gf_l)|}{v_1^{1/2-1/p}} \geq 1 \right\} \geq \frac{1}{16}. \quad (12.2.28)$$

*Proof.* We fix  $l \geq 1$ . Assume that  $\omega$  belongs to  $C_l$ . Let  $j \in \{1, \dots, J_l\}$ . If  $k$  is such that  $n_l - k_l + 1 - 2^{l+j+1} \leq k \leq n_l - k_l - 2^{l+j}$ , then  $T^{k+k_l}(\omega)$  belongs to the set

$$\bigcup_{i_1=2^{l+j}}^{2^{l+j+1}} \bigcup_{0 \leq i_q \leq l, q \geq 2} T_1^{n_l - i_1} \prod_{q=2}^d T_q^{2^l - i_q} C_l,$$

hence

$$\begin{aligned} S_{n_l - k_l - 2^{l+j+1}}(T_1, gf_l)(\omega) - S_{n_l - k_l + 1 - 2^{l+j+1}}(T_1, gf_l)(\omega) &= \sum_{k=n_l - k_l + 1 - 2^{l+j+1}}^{n_l - k_l - 2^{l+j}} (gf_l(T_1^k(\omega))) \\ &= \frac{1}{L_l} \left(\frac{n_l}{k_l}\right)^{1/p} 2^{j/p} l^{d-1} \sum_{k=n_l + 1 - 2^{l-k_l+j+1}}^{n_l - k_l - 2^{l-l+j}} g \circ T_1^k(\omega) \\ &= \frac{1}{L_l} \left(\frac{n_l}{k_l}\right)^{1/p} 2^{j/p} l^{d-1} (S_{n_l - k_l - 2^{l+j+1}}(T_1, g)(\omega) - S_{n_l - k_l + 1 - 2^{l+j+1}}(T_1, g)(\omega)). \end{aligned} \quad (12.2.29)$$

Consequently, for each  $j \in \{1, \dots, J_l\}$ , we have the equality

$$\begin{aligned} \mathbf{1}_{C_l} \cdot (S_{n_l - k_l - 2^{l+j+1}}(T_1, gf_l) - S_{n_l - k_l + 1 - 2^{l+j+1}}(T_1, gf_l)) &= \\ &= \mathbf{1}_{C_l} \cdot \frac{1}{L_l} \left(\frac{n_l}{k_l}\right)^{1/p} 2^{j/p} l^{d-1} (S_{n_l - k_l - 2^{l+j+1}}(T_1, g) - S_{n_l - k_l + 1 - 2^{l+j+1}}(T_1, g)), \end{aligned} \quad (12.2.30)$$

and applying the operator  $U_1^{-s_1} \prod_{q=2}^d U_q^{-s_q}$  for  $0 \leq s_1 \leq n_l - 2k_l$  and  $0 \leq s_q \leq l$  for  $q \geq 2$ , we obtain

$$\begin{aligned} & \mathbf{1}_{T^s C_l} \cdot U^{\mathbf{s}'} (S_{n_l - s_1 - k_l - 2^{I_l + j + 1}}(T_1, gfi) - S_{n_l - s_1 - k_l + 1 - 2^{I_l + j + 1}}(T_1, gfi)) = \\ & = \mathbf{1}_{T^s C_l} \cdot \frac{1}{L_l} \left( \frac{n_l}{k_l} \right)^{1/p} 2^{j/p} l^{d-1} U^{\mathbf{s}'} (S_{n_l - s_1 - k_l - 2^{I_l + j + 1}}(T_1, g) - S_{n_l - s_1 - k_l + 1 - 2^{I_l + j + 1}}(T_1, g)), \end{aligned} \quad (12.2.31)$$

where  $\mathbf{s}' := (0, s_2, \dots, s_d)$ . Since  $0 \leq n_l - s_1 - k_l + 1 - 2^{I_l + j + 1} \leq n_l - k_l + 1 - 2^{I_l + j + 1} \leq n_l - k_l$  and  $2^{I_l - l + j} \leq k_l$ , we have the inequality

$$\begin{aligned} & \max_{1 \leq j \leq J_l} \frac{|U^{\mathbf{s}'} (S_{n_l - s_1 - k_l - 2^{I_l + j + 1}}(T_1, gfi) - S_{n_l - s_1 - k_l + 1 - 2^{I_l + j + 1}}(T_1, gfi))|}{2^{(I_l + j)(1/2 - 1/p)}} \leq \\ & \leq \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ q \geq 2}} \frac{|S_{u+v_1 e_1}(gfi) - S_u(gfi)|}{v_1^{1/2 - 1/p}}. \end{aligned} \quad (12.2.32)$$

Combining (12.2.31) with (12.2.32), we derive that for each  $\mathbf{s} := (s_1, \dots, s_d)$  such that  $0 \leq s_1 \leq n_l - 2k_l$  and  $0 \leq s_q \leq l$  for  $q \geq 2$ ,

$$\begin{aligned} & \mathbf{1}_{T^s C_l} \cdot \frac{1}{L_l} \left( \frac{n_l}{k_l} \right)^{1/p} l^{d-1} \max_{1 \leq j \leq J_l} \frac{|U^{\mathbf{s}'} (S_{n_l - s_1 - k_l - 2^{I_l + j + 1}}(T_1, g) - S_{n_l - s_1 - k_l + 1 - 2^{I_l + j + 1}}(T_1, g))|}{2^{(I_l + j)/2}} \leq \\ & \leq \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ q \geq 2}} \frac{|S_{u+v_1 e_1}(gfi) - S_u(gfi)|}{v_1^{1/2 - 1/p}}. \end{aligned} \quad (12.2.33)$$

By (12.2.18), we obtain

$$\begin{aligned} & \mu \left\{ \frac{1}{n_l^{1/p} l^{d-1}} \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ 2 \leq q \leq d}} \frac{|S_{u+v_1 e_1}(gfi) - S_u(gfi)|}{v_1^{1/2 - 1/p}} \geq 1 \right\} \geq \\ & \geq (n_l - 2k_l) l^{d-1} \mu \left( C_l \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{n_l - k_l - 2^{I_l + j + 1}}(T_1, g) - S_{n_l - k_l + 1 - 2^{I_l + j + 1}}(T_1, g)|}{\sqrt{2^{I_l + j}}} \geq L_l \right\} \right), \end{aligned} \quad (12.2.34)$$

and since  $C_l$  and  $\left\{ \max_{1 \leq j \leq J_l} |S_{n_l - k_l - 2^{I_l + j + 1}}(T_1, g) - S_{n_l - k_l + 1 - 2^{I_l + j + 1}}(T_1, g)| \sqrt{2^{-I_l - j}} \geq L_l \right\}$  belong to independent  $\sigma$ -algebras, we get by (12.2.19):

$$\begin{aligned} & \mu \left\{ \frac{1}{n_l^{1/p} l^{d-1}} \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ 2 \leq q \leq d}} \frac{|S_{u+v_1 e_1}(gfi) - S_u(gfi)|}{v_1^{1/2 - 1/p}} \geq 1 \right\} \geq \\ & \geq \left( 1 - 2 \frac{k_l}{n_l} \right) \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{n_l - k_l - 2^{I_l + j + 1}}(T_1, g) - S_{n_l - k_l + 1 - 2^{I_l + j + 1}}(T_1, g)|}{\sqrt{2^{I_l + j}}} \geq L_l \right\}, \end{aligned} \quad (12.2.35)$$

and using (12.2.17), we derive

$$\mu \left\{ \frac{1}{n_l^{1/p} l^{d-1}} \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ 2 \leq q \leq d}} \frac{|S_{u+v_1 e_1}(gf_l) - S_u(gf_l)|}{v_1^{1/2-1/p}} \geq 1 \right\} \geq \\ \geq \frac{1}{4} \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{n_l - k_l - 2^{l+j+1}}(T_1, g) - S_{n_l - k_l + 1 - 2^{l+j+1}}(T_1, g)|}{\sqrt{2^{l+j}}} \geq L_l \right\}, \quad (12.2.36)$$

By similar arguments as in the proof of Theorem 7.2.1, we obtain that for  $l$  large enough,

$$\mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{n_l - k_l - 2^{l+j+1}}(T_1, g) - S_{n_l - k_l + 1 - 2^{l+j+1}}(T_1, g)|}{\sqrt{2^{l+j}}} \geq L_l \right\} \geq \frac{1}{4}. \quad (12.2.37)$$

We obtain (12.2.28) in view of (12.2.36) and (12.2.37), which concludes the proof of Proposition 12.2.5.  $\square$

**Corollary 12.2.6.** *For  $l$  large enough,*

$$\mu \left\{ \frac{1}{n_l^{1/p} l^{d-1}} \max_{\substack{0 \leq u_1 \leq n_l - k_l \\ 1 \leq v_1 \leq k_l}} \max_{\substack{0 \leq i_q \leq l \\ 2 \leq q \leq d}} \frac{|S_{u+v_1 e_1}(m) - S_u(m)|}{v_1^{1/2-1/p}} \geq 1 \right\} \geq \frac{1}{32}. \quad (12.2.38)$$

*Remark 12.2.7.* Since for each  $\mathbf{k} \in \mathbb{Z}^d$ ,  $\sigma(f \circ T^{\mathbf{i}}, \mathbf{i} \neq \mathbf{k}) \subset \sigma(g \circ T^{\mathbf{i}}, \mathbf{i} \neq \mathbf{k}) \vee \mathcal{C}$ , the random field  $(U^{\mathbf{i}}f)_{\mathbf{i} \in \mathbb{Z}^d}$  is strong martingale random field in the sense of Nahapetian and Petrosian.

### 12.2.3 Proof in the orthomartingale case

We already know by [Vol15] that the finite dimensional distributions of  $|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m, \cdot)$  converge to that of  $\|m\|_2 W$ . The rest of the proof is thus devoted to asymptotic tightness in  $\mathcal{H}_\alpha^o([0, 1]^d)$  of the net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m, \cdot))_{\mathbf{n} \geq \mathbf{1}}$ .

To this aim, we have to check conditions (12.1.12) and (12.1.13). Actually, we shall prove the following proposition, which links the Hölderian modulus of continuity of the partial sum process associated to an orthomartingale difference random field with its tails and those of the quadratic variance.

**Proposition 12.2.8.** *Assume that the conditions of Theorem 12.1.5 hold. Then for each  $\mathbf{n} \in \mathbb{N}^d$  and each positive real number  $x$ , the following inequality takes place:*

$$\mu \left\{ \sup_{j \geq 1} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(m, \cdot))| > x \prod_{l=1}^d n_l^{1/2} \right\} \leq \\ \leq C(p, d) x^{-p} \left( \|m\|_{p, \infty}^p + \max_{1 \leq i \leq d} \|\mathbb{E}[m^2 | T_i \mathcal{M}]\|_{p/2}^{p/2} \right) \quad (12.2.39)$$

where the constant  $C(p, d)$  depends only on  $p$  and  $d$ . In particular,

$$\left\| \left\| \prod_{l=1}^d n_l^{-1/2} S_{\mathbf{n}}(m, \cdot) \right\|_{\mathcal{H}_{1/2-1/p}([0, 1]^d)} \right\|_{p, \infty} \leq C(p, d) \|m\|_p. \quad (12.2.40)$$

**Corollary 12.2.9.** *The net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \geq \mathbf{1}}$  is asymptotically tight in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$ .*

*Proof of Proposition 12.2.8.* In view of the equivalence of the sequential and classical norm in  $\mathcal{H}_{1/2-1/p}([0, 1]^d)$  and inequality (12.1.11) with  $J = 1$ , we have to prove the existence of a constant  $C(p, d)$  such that for any  $\mathbf{n} \in \mathbb{N}^d$  and  $i \in \langle d \rangle$ , the following two inequalities take place:

$$n_i \mu \left\{ \max_{\substack{0 \leq j_l \leq n_l \\ l \neq i}} \left| \sum_{0 \leq i_l \leq j_l} U^i m \right| > x n_i^{1/p} \prod_{l \in \langle d \rangle \setminus \{i\}} \sqrt{n_l} \right\} \leq C(p, d) x^{-p} \|m\|_{p, \infty}^p; \quad (12.2.41)$$

$$\begin{aligned} \sum_{j=1}^{n_i} 2^j \mu \left\{ 2^{\alpha_j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{1 \leq s \leq l} U^s m \right| > x |\mathbf{n}|^{1/2} \right\} \leq \\ \leq C(p, d) x^{-p} \left( \|m\|_{p, \infty}^p + \|\mathbb{E}[m^2 | T_i \mathcal{M}]\|_{p/2}^{p/2} \right). \end{aligned} \quad (12.2.42)$$

- *Proof of (12.2.41).* For simplicity, we shall do the proof in the case  $i = d$ ; we switch the roles of the operators  $(T_q)_{q \in \langle d \rangle}$  to treat the other indexes. Since  $(U^{(i,0)} m)_{\mathbf{i} \in \mathbb{N}^{d-1}}$  is an orthomartingale difference random field with respect to the commuting filtration  $(T^{-(i,0)} \mathcal{M})_{\mathbf{i} \in \mathbb{N}^{d-1}}$ , we can apply inequality (11.1.36) in this setting with a  $q > p$ ; this gives

$$\begin{aligned} n_d \mu \left\{ \max_{\substack{0 \leq j_l \leq n_l \\ l < d}} \left| \sum_{0 \leq i_l \leq j_l} U^i m \right| > C x n_d^{1/p} \prod_{l=1}^{d-1} \sqrt{n_l} \right\} \leq \\ \leq n_d \int_{\mathbf{R}_+^{d-1}} \mu \left\{ |m| > x n_d^{1/p} n_{d-1}^{1/2} \prod_{l=1}^{d-2} w_l \right\} \log^+(w_{d-1})^{d-3} \prod_{l=1}^{d-2} h(q, w_l) dw_1 \dots dw_{d-1}, \end{aligned}$$

and using the bound

$$\mu \left\{ |m| > x n_d^{1/p} n_{d-1}^{1/2} \prod_{l=1}^{d-2} w_l \right\} \leq \left( x n_d^{1/p} n_{d-1}^{1/2} \prod_{l=1}^{d-2} w_l \right)^{-p} \|m\|_{p, \infty}^p, \quad (12.2.43)$$

we finally get

$$\begin{aligned} n_d \mu \left\{ \max_{\substack{0 \leq j_l \leq n_l \\ l < d}} \left| \sum_{0 \leq i_l \leq j_l} U^i m \right| > C x n_d^{1/p} \prod_{l=1}^{d-1} \sqrt{n_l} \right\} \leq \\ \leq C(q, d) x^{-p} n_{d-1}^{-p/2} \left( \int_{\mathbf{R}_+} h(q, w) w^{-p} dw \right)^{d-2} \cdot \int_{\mathbf{R}_+} (\log^+ v)^{d-2} \cdot v^{-p} dv = \\ \leq C(q, d) (\min \mathbf{n})^{-p/2}. \end{aligned} \quad (12.2.44)$$

(we used the convention that  $\prod_{l=1}^0 = 1$  in the case  $d = 2$ ). In view of Remark 11.1.6, the term  $\prod_{l=1}^{d-2} \int_{\mathbf{R}_+} h(q, w) w^{-p} dw \cdot \int_{\mathbf{R}_+} (\log^+ v)^{d-2} \cdot v^{-p} dv$  is bounded by a constant independent of  $\mathbf{n}$ . This concludes the proof of (12.2.41).

- *Proof of (12.2.42).* Using (11.1.36) with a  $q > p$ , we obtain

$$\begin{aligned}
& \sum_{j=1}^{\log n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{1 \leq s \leq l} U^s m \right| > 4^d x \prod_{q=1}^d \sqrt{n_q} \right\} \leq \\
& \leq n_i \sum_{j=1}^{\log n_i} \int_{\mathbf{R}_+^d} \mu \left\{ |m| > x 2^{-\alpha j} \sqrt{n_i} \prod_{l=1}^d w_l v \right\} (\log^+ v)^{d-2} \prod_{l=1}^d h(q, w_l) d\mathbf{w} dv + \\
& \quad + C(q, d) \sum_{j=1}^{\log n_i} 2^j \int_{\mathbf{R}_+^{d+1}} \mu \left\{ (\mathbb{E}[m^2 | T_i \mathcal{M}])^{1/2} \geq x v 2^{j/p} \prod_{l=1}^d w_l \right\} \cdot \\
& \quad \cdot (\log^+ v)^{d-2} g(q, w_i) \prod_{l=1, l \neq i}^d h(q, w_l) d\mathbf{w} dv. \quad (12.2.45)
\end{aligned}$$

Bounding  $\mu \left\{ |m| > x 2^{-\alpha j} \sqrt{n_i} \prod_{l=1, l \neq j}^d w_l v \right\}$  by  $\left( x 2^{-\alpha j} \sqrt{n_i} \prod_{l=1, l \neq i}^d w_l v \right)^{-p} \|m\|_{p, \infty}^p$ , we derive that

$$\begin{aligned}
& n_i \sum_{j=1}^{\log n_i} \int_{\mathbf{R}_+^d} \mu \left\{ |m| > x 2^{-\alpha j} \sqrt{n_i} \prod_{l=1}^d w_l v \right\} (\log^+ v)^{d-2} \prod_{l=1}^d h(q, w_l) d\mathbf{w} dv \leq \\
& \leq n_i^{1-p/2} x^{-p} \|m\|_{p, \infty}^p \sum_{j=1}^{\log n_i} 2^{j(p/2-1)} \int_{\mathbf{R}_+^d} \left( \prod_{l=1}^d w_l v \right)^{-p} (\log^+ v)^{d-2} \prod_{l=1}^d h(q, w_l) d\mathbf{w} dv \\
& \quad = C(p, d) x^{-p} \|m\|_{p, \infty}^p. \quad (12.2.46)
\end{aligned}$$

Using for a non-negative random variable  $Y$  the bound  $\sum_{j \geq 1} 2^j \mu \{ Y \geq 2^{2j/p} \} \leq 2\mathbb{E}[Y^{p/2}]$ , we derive

$$\begin{aligned}
& \sum_{j=J}^{\log n_i} 2^j \int_{\mathbf{R}_+^{d+1}} \mu \left\{ (\mathbb{E}[m^2 | T_i \mathcal{M}])^{1/2} \geq x v 2^{j/p} \prod_{l=1}^d w_l \right\} \cdot \\
& \quad \cdot (\log^+ v)^{d-2} g(q, w_i) \prod_{l=1, l \neq i}^d h(q, w_l) d\mathbf{w} dv \leq \\
& \quad \leq 2x^{-p} \|\mathbb{E}[m^2 | T_i \mathcal{M}]\|_{p/2}^{p/2} \cdot \\
& \quad \cdot \int_{\mathbf{R}_+^d} \int_1^\infty v^{-p} (\log v)^{d-2} g(q, w_i) \prod_{l=1}^d h(q, w_l) w_l^{-p} d\mathbf{w} dv. \quad (12.2.47)
\end{aligned}$$

Combining (12.2.46) and (12.2.47), we obtain (12.2.42) in view of (12.2.45).

This concludes the proof of Proposition 12.2.8.  $\square$

*Proof of Corollary 12.2.9.* We use a truncation argument. For a positive fixed constant  $K$ , we define

$$m_K := P_0(m \mathbf{1} \{|m| \leq K\}) \text{ and} \quad (12.2.48)$$

$$m'_K := P_0(m \mathbf{1} \{|m| > K\}). \quad (12.2.49)$$



In this way, the random fields  $(U^{\mathbf{i}}m_K)_{\mathbf{i} \in \mathbb{Z}^d}$  and  $(U^{\mathbf{i}}m'_K)_{\mathbf{i} \in \mathbb{Z}^d}$  are orthomartingale difference random fields with respect to the commuting filtration  $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  and  $m = m_K + m'_K$ , hence

$$\begin{aligned} & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j,\mathbf{v}}(S_{\mathbf{n}}(m, \cdot))| > 2\varepsilon \right\} \leq \\ & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j,\mathbf{v}}(S_{\mathbf{n}}(m_K, \cdot))| > \varepsilon \right\} + \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j,\mathbf{v}}(S_{\mathbf{n}}(m'_K, \cdot))| > \varepsilon \right\} \end{aligned} \quad (12.2.50)$$

Since  $(U^{\mathbf{i}}m_K)_{\mathbf{i} \in \mathbb{Z}^d}$  is an orthomartingale random field which admits moments of any order, we deduce by Remark 12.1.3 and Proposition (12.2.8) that

$$\begin{aligned} & \lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow +\infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j,\mathbf{v}}(S_{\mathbf{n}}(m, \cdot))| > 2\varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ & \leq C(p, d) \varepsilon^{-p} \left( \|m'_K\|_{p, \infty}^p + \max_{1 \leq i \leq d} \|\mathbb{E}[(m'_K)^2 | T_i \mathcal{M}]\|_{p/2}^{p/2} \right). \end{aligned} \quad (12.2.51)$$

□

## 12.2.4 Proof of the Hannan type condition

The convergence of the finite dimensional distributions of  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))$  to that of  $\sigma^2 W$  where  $\sigma$  is given by (12.1.22) is implied by the result of [VW14] and the Remark 1 of [Vol15].

Notice that the assumptions of Theorem 12.1.8 imply that

$$f = \sum_{\mathbf{i} \in \mathbb{Z}^d} P_{\mathbf{i}}(f), \quad (12.2.52)$$

where the summation converges in  $\mathbb{L}^p$ . For a fixed positive integer  $k$ , let us introduce the following functions:

$$m := \sum_{\mathbf{i} \in \mathbb{Z}^d} P_{\mathbf{0}}(U^{\mathbf{i}}f), \quad m^{(k)} := \sum_{-k \cdot \mathbf{1} \leq \mathbf{i} \leq k \cdot \mathbf{1}} P_{\mathbf{0}}(U^{\mathbf{i}}f) \text{ and} \quad (12.2.53)$$

$$f^{(k)} := \sum_{-k \cdot \mathbf{1} \leq \mathbf{i} \leq k \cdot \mathbf{1}} P_{\mathbf{i}}(f). \quad (12.2.54)$$

We shall prove the asymptotic tightness of the net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \geq \mathbf{1}}$  by using the following relationship:

$$f = f - f^{(k)} + f^{(k)} - m^{(k)} + m^{(k)} - m. \quad (12.2.55)$$

We divide the proof into three steps.

1. In this first step, we prove the

**Proposition 12.2.10.** *For each  $k \geq 1$ , the function  $f^{(k)} - m^{(k)}$  satisfies (12.1.12) and (12.1.13).*

*Proof.* By Theorem 4.1. of [VW14], the function  $f^{(k)} - m^{(k)}$  admits an orthomartingale-coboundary decomposition of the form (10.2.3) (here we work for a fixed  $k$  with the filtration  $\mathcal{G}_{\mathbf{i}}$  given by  $\mathcal{G}_{\mathbf{i}} := T^{-k \cdot \mathbf{1} - \mathbf{i}}$ ,  $\mathbf{i} \in \mathbb{Z}^d$  instead of the initial one). As a consequence, by Corollary 11.1.5, the family

$$\left\{ |\mathbf{n}|^{-p/2} \max_{1 \leq j \leq \mathbf{n}} \left| S_j \left( f^{(k)} - m^{(k)} \right) \right|^p, \mathbf{n} \geq \mathbf{1} \right\} \quad (12.2.56)$$

is uniformly integrable. It follows from item 2 of Remark 12.1.3 that condition (12.1.12) is satisfied. We have already seen that (12.1.13) holds for an orthomartingale. It thus remains to check it for the other terms of the orthomartingale-coboundary decomposition. For the last term of (10.2.3), we note that

$$\begin{aligned} \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} \prod_{q=1}^d (I - U_q) g \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} &\leq \\ &\leq 2^d \mu \left\{ \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} |g \circ T^{\mathbf{1}}| > \varepsilon 2^{-\alpha j} |\mathbf{n}|^{1/2} \right\} \\ &\leq 2^d |\mathbf{n}| 2^{-j} \mu \left\{ |g| > \varepsilon 2^{-\alpha j} |\mathbf{n}|^{1/2} \right\} \leq 2^d |\mathbf{n}|^{1-p/2} 2^{j(p\alpha-1)} \|g\|_p^p, \end{aligned} \quad (12.2.57)$$

hence

$$\begin{aligned} \sum_{j=J}^{n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} f \right| > \varepsilon \prod_{q=1}^d \sqrt{n_q} \right\} &\leq \\ &\leq 2^d \|g\|_p^p |\mathbf{n}|^{1-p/2} \sum_{j=J}^{n_i} 2^{j p \alpha} \\ &\leq 2^d \|g\|_p^p |\mathbf{n}|^{1-p/2} n_i^{p/2-1} = 2^d \|g\|_p^p \prod_{l=1, l \neq i}^d n_l^{1-p/2} \leq (\min \mathbf{n})^{(1-p/2)(d-1)}. \end{aligned} \quad (12.2.58)$$

Since  $d \geq 2$  and  $p > 2$ , we get (12.1.13).

Assume now that  $J \subsetneq \langle d \rangle$  is a non-empty set and consider  $i \in \langle d \rangle$  and  $m_J$  like in (10.2.3).

- If  $i$  belong to  $J$ , then

$$\begin{aligned} \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} \prod_{k \in J} (I - U_k) m_J \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} &\leq \\ &\leq 2^{|J|} n_i 2^{-j} \prod_{k \in J \setminus \{i\}} n_k \mu \left\{ \max_{\substack{1 \leq l_k \leq n_k \\ k \in \langle d \rangle \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right| > \varepsilon |\mathbf{n}|^{1/2} 2^{-\alpha j} \right\} \\ &\leq 2^{|J|} n_i 2^{-j} \prod_{k \in J \setminus \{i\}} n_k \left( \varepsilon |\mathbf{n}|^{1/2} 2^{-\alpha j} \right)^{-p} \\ &\mathbb{E} \left[ \max_{\substack{1 \leq l_k \leq n_k \\ k \in \langle d \rangle \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right|^p \mathbf{1} \left\{ \max_{\substack{1 \leq l_k \leq n_k \\ k \in \langle d \rangle \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right| > \varepsilon |\mathbf{n}|^{1/2} 2^{-\alpha j} \right\} \right], \end{aligned} \quad (12.2.59)$$

and accounting  $i \in J$  and  $j \leq \log n_i$ , we have the bound

$$\begin{aligned} |\mathbf{n}|^{-p/2} \mathbb{E} \left[ \max_{\substack{1 \leq l_k \leq n_k \\ k \in \langle d \rangle \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right|^p \mathbf{1} \left\{ \max_{\substack{1 \leq l_k \leq n_k \\ k \in \langle d \rangle \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right| > \varepsilon |\mathbf{n}|^{1/2} 2^{-\alpha j} \right\} \right] \\ \leq \prod_{k \in J} n_k^{-p/2} \delta \left( \varepsilon n_i^{1/p} \right) \end{aligned} \quad (12.2.60)$$

where

$$\delta(R) := \sup_{\mathbf{N} \in (\mathbb{N}^*)^{d-|J|}} \left\{ |\mathbf{N}|^{-p/2} \mathbb{E} \left[ \max_{\substack{1 \leq l_k \leq N_k \\ k \in (d) \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right|^p \right. \right. \\ \left. \left. \mathbf{1} \left\{ \max_{\substack{1 \leq l_k \leq N_k \\ k \in (d) \setminus J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right| > |\mathbf{N}|^{1/2} R \right\} \right] \right\}. \quad (12.2.61)$$

The combination of (12.2.59) and (12.2.60) yields

$$\sum_{j=J}^{\log n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} \prod_{k \in J} (I - U_k) m_J \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ \leq 2^{|J|} \prod_{k \in J} n_k \sum_{j=1}^{\log n_i} 2^{j p \alpha} \prod_{k \in J} n_k^{-p/2} \delta(\varepsilon n_i^{1/p}) \leq \\ \leq 2^{|J|} C(p) \prod_{k \in J} n_k^{1-p/2} n_i^{1-p/2} \delta(\varepsilon n_i^{1/p}) \leq 2^{|J|} C(p) \delta(\varepsilon (\min \mathbf{n})^{1/p}). \quad (12.2.62)$$

By Corollary 11.1.5,  $\varepsilon(R)$  goes to 0 as  $R$  goes to infinity hence (12.1.13) follows directly from (12.2.62).

- Assume that  $i \notin J$ . In this case, we write

$$\mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} \prod_{k \in J} (I - U_k) m_J \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ \leq \prod_{j \in J} n_j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \notin \{i\} \cup J}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} m_J \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} \quad (12.2.63)$$

and by (11.1.37), we have

$$\mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} \prod_{k \in J} (I - U_k) m_J \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ \leq \prod_{j \in J} n_j \int_{\mathbf{R}_+^d} \int_1^{+\infty} \mu \left\{ |m_J| > 4^{-d} \prod_{k \in J} n_k^{1/2} \varepsilon v \prod_{l=1}^d w_l \right\} (\log v)^{d-2} \prod_{l=1}^d h(q, w_l) d\mathbf{w} dv.$$

We now bound the quantity  $\mu \left\{ |m_J| > 4^{-d} \prod_{k \in J} n_k^{1/2} \varepsilon v \prod_{l=1}^d w_l \right\}$  by Markov's inequality in order to check that  $\prod_{k \in J} (I - U_k) m_J$  satisfies (12.1.13).

We thus obtained (12.1.13) for  $m^{(k)} - f^{(k)}$ . This concludes the proof of Proposition 12.2.10.  $\square$

2. By definition of  $m$  and  $m^{(k)}$  (see (12.2.53)),  $(U^{\mathbf{i}}(m - m^{(k)}))_{\mathbf{i} \in \mathbb{Z}^d}$  is an orthomartingale random field with respect to the commuting filtration  $(T^{-\mathbf{i}} \mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$  and  $m - m^{(k)}$  belongs to  $\mathbb{L}^p$ . By Theorem 12.1.5, the net  $\left( |\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m - m^{(k)}, \cdot) \right)_{\mathbf{n} \geq \mathbf{1}}$  is asymptotically tight in

$\mathcal{H}_{1/2-1/p}([0, 1]^d)$  hence in view of Propositions 12.1.2 and 12.2.10 and equality (12.2.55), we have for each positive integer  $k$ ,

$$\begin{aligned} & \lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > 3\varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ & \leq \lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f - f^{(k)}, \cdot))| > \varepsilon |\mathbf{n}|^{1/2} \right\}. \end{aligned} \quad (12.2.64)$$

3. At last, we have to estimate the right hand side of (12.2.64). To this aim, we start from the elementary inequality

$$\begin{aligned} & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f - f^{(k)}, \cdot))| > \varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ & \leq \varepsilon^{-p} \left\| \frac{1}{|\mathbf{n}|^{1/2}} \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f - f^{(k)}, \cdot))| \right\|_{p, \infty}^p, \end{aligned}$$

and using (12.2.52) and (12.2.54), we obtain

$$\begin{aligned} & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f - f^{(k)}, \cdot))| > \varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ & \leq \varepsilon^{-p} \left( \sum_{\mathbf{i} \in I_k} \left\| \frac{1}{|\mathbf{n}|^{1/2}} \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(P_{\mathbf{i}}(f), \cdot))| \right\|_{p, \infty} \right)^p, \end{aligned} \quad (12.2.65)$$

where  $I_k := \{\mathbf{i} \in \mathbb{Z}^d, |i_q| > k \text{ for some } q \in \langle d \rangle\}$ . Since  $(U^{\mathbf{j}}(P_{\mathbf{i}}(f)))_{\mathbf{j} \in \mathbb{Z}^d}$  is an orthomartingale difference random field with respect to a commuting filtration (with finite  $p$ th moments), we deduce by a combination of (12.2.40) and (12.2.65) that

$$\mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f - f^{(k)}, \cdot))| > \varepsilon |\mathbf{n}|^{1/2} \right\} \leq \varepsilon^{-p} C(p, d)^p \left( \sum_{\mathbf{i} \in I_k} \|P_{\mathbf{i}}(f)\|_p \right)^p, \quad (12.2.66)$$

hence, by (12.2.64),

$$\begin{aligned} & \lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > 3\varepsilon |\mathbf{n}|^{1/2} \right\} \leq \\ & \leq \varepsilon^{-p} C(p, d)^p \left( \sum_{\mathbf{i} \in I_k} \|P_{\mathbf{i}}(f)\|_p \right)^p. \end{aligned} \quad (12.2.67)$$

Since the estimate in (12.2.67) is valid for any  $k$ , the asymptotic tightness of the net  $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \geq \mathbf{1}}$  follows from (12.1.20).

This concludes the proof of Theorem 12.1.8.

### 12.2.5 Bernoulli random fields

The proof will be done by an  $m$ -dependent approximation. We define for a positive integer  $m$  and  $\mathbf{j} \in \mathbb{Z}^d$  the  $\sigma$ -algebra

$$\mathcal{G}_{\mathbf{j}}^{(m)} := \sigma(\varepsilon_{\mathbf{i}} : |j_q - i_q| \leq m \text{ for each } q \in \langle d \rangle), \quad (12.2.68)$$

and we introduce  $X_{\mathbf{j}}^{(m)} := \mathbb{E} [X_{\mathbf{j}} | \mathcal{G}_{\mathbf{j}}^{(m)}] = U^{\mathbf{j}} \mathbb{E} [X_{\mathbf{0}} | \mathcal{G}_{\mathbf{0}}^{(m)}]$ . In such a way, for a fixed  $m$ , the random field  $(X_{\mathbf{j}}^{(m)})_{\mathbf{j} \in \mathbb{Z}^d} =: (U^{\mathbf{j}} f^{(m)})_{\mathbf{j} \in \mathbb{Z}^d}$  is  $(2m+1)$ -dependent. We shall use the following notations:

$$Y_{\mathbf{n}}(\mathbf{t}) := \frac{1}{|\mathbf{n}|^{1/2}} S_{\mathbf{n}}(f, \mathbf{t}), \quad \mathbf{t} \in [0, 1]^d, \mathbf{n} \succcurlyeq \mathbf{1}, \quad (12.2.69)$$

and

$$Y_{\mathbf{n}}^{(m)}(t) := \frac{1}{|\mathbf{n}|^{1/2}} S_{\mathbf{n}} \left( \mathbb{E} [X_{\mathbf{0}} | \mathcal{G}_{\mathbf{0}}^{(m)}], t \right), \quad \mathbf{t} \in [0, 1]^d, \mathbf{n} \succcurlyeq \mathbf{1}. \quad (12.2.70)$$

We divide the proof into two parts: finite dimensional distributions and tightness.

- *Finite dimensional distributions*

We have to prove that for any integer  $k$  and any  $\mathbf{t}_1, \dots, \mathbf{t}_k \in [0, 1]^d$ , the convergence

$$(Y_{\mathbf{n}}(\mathbf{t}_i))_{i=1}^k \xrightarrow{\min \mathbf{n} \rightarrow \infty} \sigma(W(\mathbf{t}_i))_{i=1}^k \quad (12.2.71)$$

takes place. Notice that for each  $m$ , since the process  $(X_{\mathbf{j}}^{(m)})_{\mathbf{j} \in \mathbb{Z}^d}$  is  $(2m+1)$ -dependent, centered with finite variance, the convergence

$$(Y_{\mathbf{n}}^{(m)}(t_i))_{i=1}^k \xrightarrow{\min \mathbf{n} \rightarrow \infty} \sigma^{(m)}(W(t_i))_{i=1}^k \quad (12.2.72)$$

takes place for any  $m$ , where

$$\sigma^{(m)} = \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov} \left( \mathbb{E} [X_{\mathbf{0}} | \mathcal{G}_{\mathbf{0}}^{(m)}], \mathbb{E} [X_{\mathbf{k}} | \mathcal{G}_{\mathbf{k}}^{(m)}] \right). \quad (12.2.73)$$

We derive from the convergence of  $\sum_{\mathbf{i} \succcurlyeq \mathbf{0}} \|Q_{\mathbf{i}}\|_2$  that  $\lim_{m \rightarrow \infty} \sigma^{(m)} = \sigma$  and that for each  $\mathbf{t} \in [0, 1]^d$ ,  $\sup_{\mathbf{n} \succcurlyeq \mathbf{1}} \|Y_{\mathbf{n}}(\mathbf{t}) - Y_{\mathbf{n}}^{(m)}(\mathbf{t})\|_2 \rightarrow 0$  as  $m$  goes to infinity, from which the convergence of the finite dimensional distributions follows.

- *Asymptotic tightness*

We shall prove that  $f = X_{\mathbf{0}}$  satisfies conditions (12.1.12) and (12.1.13).

Since the random field  $(X_{\mathbf{j}}^{(m)})_{\mathbf{j} \in \mathbb{Z}^d}$  is strictly stationary,  $(2m+1)$ -dependent and centered, we can check in a similar way as in the independent case that (12.1.12) and (12.1.13) are satisfied when  $f$  is replaced by  $f^{(m)}$ .

The following lemma will help us to control the terms which appear in the right hand side of (11.2.12).

**Lemma 12.2.11.** *For each integer  $m$  and each  $\mathbf{i} \in \mathbb{Z}^d$ , we have*

$$Q_{\mathbf{i}}(f^{(m)}) = \mathbb{E} [Q_{\mathbf{i}}(f) | \mathcal{G}_{\mathbf{0}}^{(m)}]. \quad (12.2.74)$$

*In particular, the following inequalities hold:*

$$\|Q_{\mathbf{i}}(f - f^{(m)})\|_2 \leq 2 \|Q_{\mathbf{i}}(f)\|_2 \quad \text{and} \quad \|Q_{\mathbf{i}}(f - f^{(m)})\|_{p, \infty} \leq 2 \|Q_{\mathbf{i}}(f)\|_{p, \infty}. \quad (12.2.75)$$

– *Proof of (12.2.41)*: by the previous observation, we have for each positive integer  $m$ , each  $i \in \langle d \rangle$  and each positive  $R$  that

$$\begin{aligned} & \limsup_{R \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} R^p \mu \left\{ \max_{0 \leq j_l \leq n_l, l \neq i} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} f \right| > R \prod_{l=1, l \neq i}^d \sqrt{n_l} \right\} \leq \\ & \leq 2^p \limsup_{R \rightarrow \infty} \limsup_{m(n) \rightarrow \infty} R^p \mu \left\{ \max_{0 \leq j_l \leq n_l, l \neq i} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} (f - f^{(m)}) \right| > R \prod_{l=1, l \neq i}^d \sqrt{n_l} \right\}. \end{aligned} \quad (12.2.76)$$

Let us fix  $i \in \langle d \rangle$ . We now use inequality (11.2.12) with  $r := p$ ,  $N_j = n_j$  if  $j \neq i$  and  $N_i = 1$ , and  $t := R \prod_{l=1, l \neq i}^d \sqrt{n_l}$ :

$$\begin{aligned} R^p \mu \left\{ \max_{0 \leq j_l \leq n_l, l \neq i} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} (f - f^{(m)}) \right| > R \prod_{l=1, l \neq i}^d \sqrt{n_l} \right\} \leq \\ \leq C(p, d) \left( A(\mathbf{n}) + B^{(m)}(\mathbf{n}) \right), \end{aligned} \quad (12.2.77)$$

where

$$A(\mathbf{n}) := \left( \sum_{l \in E_{\mathbf{n} - (n_i - 1)\mathbf{e}_i}} \left\| Q_l(f - f^{(m)}) \right\|_{p, \infty} \right)^p, \quad (12.2.78)$$

$$B^{(m)}(\mathbf{n}) := \left( \sum_{\mathbf{0} \leq \mathbf{l} \leq \mathbf{N}} \left( \left\| Q_{\mathbf{l}}(f - f^{(m)}) \right\|_{r, \infty}^r |\max\{\mathbf{l}, \mathbf{1}\}|^{r/2-1} \right)^{\frac{1}{r+1}} \right)^{r+1}. \quad (12.2.79)$$

In view of (12.2.75),  $A(\mathbf{n})$  does not exceed  $2^p \left( \sum_{l \in E_{\mathbf{n} - (n_i - 1)\mathbf{e}_i}} \left\| Q_l(f) \right\|_{p, \infty} \right)^p$  and since  $d \geq 2$ , the later quantity converges to 0 as  $\min \mathbf{n}$  goes to infinity by (12.1.23). Therefore, equations (12.2.76) and (12.2.77) imply that for each  $m$ ,

$$\begin{aligned} \limsup_{R \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} R^p \mu \left\{ \max_{0 \leq j_l \leq n_l, l \neq i} \left| \sum_{0 \leq j'_l \leq j_l} U^{j'} f \right| > R \prod_{l=1, l \neq i}^d \sqrt{n_l} \right\} \leq \\ \leq \sup_{\mathbf{n} \geq \mathbf{1}} B^{(m)}(\mathbf{n}). \end{aligned} \quad (12.2.80)$$

Notice that  $\sum_{\mathbf{l} \geq \mathbf{0}} \left( \left\| Q_l(f) \right\|_{r, \infty}^r \prod_{q=1}^d (\max\{l_q, 1\})^{r/2-1} \right)^{1/(r+1)}$  is finite and for each  $\mathbf{l} \geq \mathbf{0}$ ,  $\left\| Q_{\mathbf{l}}(f - f^{(m)}) \right\|_{r, \infty} \rightarrow 0$  as  $m$  goes to infinity. By (12.2.75), we deduce by dominated convergence that  $\lim_{m \rightarrow \infty} \sup_{\mathbf{n} \geq \mathbf{1}} B^{(m)}(\mathbf{n}) = 0$ , hence, by (12.2.80), the function  $f$  satisfies (12.1.12).

– *Proof of (12.1.13)*: let us fix  $i \in \langle d \rangle$ . We recall that since the random field  $\left( X_{\mathbf{j}}^{(m)} \right)_{\mathbf{j} \in \mathbb{Z}^d}$  is strictly stationary,  $(2m+1)$ -dependent and centered, we can check in a similar way as in the independent case that (12.1.13) is satisfied when  $f$  is replaced by  $f^{(m)}$ . The

proof thus reduces to

$$\limsup_{m \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \sum_{j=1}^{\log n_i} 2^j \cdot \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preceq \mathbf{s} \preceq \mathbf{1}} U^{\mathbf{s}} (f - f^{(m)}) \right| > \varepsilon |\mathbf{n}|^{1/2} \right\} = 0,$$

and using the change of index  $u := [\log n_i] - j$ , this can be rewritten as

$$\limsup_{m \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} n_i \sum_{u=1}^{\log n_i} 2^{-u} \cdot \mu \left\{ \max_{1 \leq l_i \leq 2^u} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preceq \mathbf{s} \preceq \mathbf{1}} U^{\mathbf{s}} (f - f^{(m)}) \right| > \varepsilon |\mathbf{n}|^{1/2} n_i^{1/p-1/2} 2^{u\alpha} \right\} = 0. \quad (12.2.81)$$

We apply inequality (11.2.12) with  $r := p$ ,  $N_j := n_j$  if  $j \neq i$ ,  $N_i := 2^u$  and  $t := \varepsilon |\mathbf{n}|^{1/2} n_i^{1/p-1/2} 2^u$ , which gives

$$\begin{aligned} 2^{-u} \mu \left\{ \max_{1 \leq l_i \leq 2^u} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preceq \mathbf{s} \preceq \mathbf{1}} U^{\mathbf{s}} (f - f^{(m)}) \right| > \varepsilon |\mathbf{n}|^{1/2} n_i^{1/p-1/2} 2^{u\alpha} \right\} &\leq \\ &\leq C(p, d) 2^{-u} \left( \varepsilon |\mathbf{n}|^{1/2} n_i^{1/p-1/2} 2^{u\alpha} \right)^{-p} |n|^{p/2} n_i^{-p/2} 2^{up/2} \cdot \\ &\quad \cdot \left( \left( \sum_{\mathbf{l} \in E_{\mathbf{n}+(2^u-n_i)\mathbf{e}_i}} \|Q_{\mathbf{l}}(f - f^{(m)})\|_{p, \infty} \right)^p + \right. \\ &\quad \left. \left( \sum_{\mathbf{0} \preceq \mathbf{l} \preceq \mathbf{n}+(2^u-n_i)\mathbf{e}_i} \left( \|Q_{\mathbf{l}}(f - f^{(m)})\|_{p, \infty}^p |\max\{\mathbf{1}, \mathbf{1}\}|^{p/2-1} \right)^{\frac{1}{p+1}} \right)^{p+1} \right), \end{aligned} \quad (12.2.82)$$

By (12.2.75), and using the fact that  $p\alpha = p/2 - 1$ , this becomes

$$\begin{aligned} n_i \sum_{u=1}^{\log n_i} 2^{-u} \mu \left\{ \max_{1 \leq l_i \leq 2^u} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preceq \mathbf{s} \preceq \mathbf{1}} U^{\mathbf{s}} (f - f^{(m)}) \right| > \varepsilon |\mathbf{n}|^{1/2} n_i^{1/p-1/2} 2^{u\alpha} \right\} &= \\ &= 2^p \varepsilon^{-p} C(p, d) A(\mathbf{n}) + \varepsilon^{-p} C(p, d) B^{(m)}(\mathbf{n}), \end{aligned} \quad (12.2.83)$$

where

$$\begin{aligned} A(\mathbf{n}) &:= \sum_{u=1}^{\log n_i} \left( \sum_{\mathbf{l} \in E_{\mathbf{n}+(2^u-n_i)\mathbf{e}_i}} \|Q_{\mathbf{l}}(f)\|_{p, \infty} \right)^p, \\ B^{(m)}(\mathbf{n}) &:= \sum_{u=1}^{\log n_i} \left( \sum_{\mathbf{0} \preceq \mathbf{l} \preceq \mathbf{n}+(2^u-n_i)\mathbf{e}_i} \left( \|Q_{\mathbf{l}}(f - f^{(m)})\|_{p, \infty}^p |\max\{\mathbf{1}, \mathbf{1}\}|^{p/2-1} \right)^{\frac{1}{p+1}} \right)^{p+1}. \end{aligned}$$

Noticing the inclusion

$$E_{\mathbf{n}+(2^u-n_i)\mathbf{e}_i} \subset \{\mathbf{l} \in \mathbb{Z}^d, l_q \geq n_j \text{ for some } q \neq i\} =: E'_{\mathbf{n}}, \quad (12.2.84)$$

we may bound  $A(\mathbf{n})$  by  $\left(\sum_{\mathbf{l} \in E'_n} \|Q_{\mathbf{l}}(f)\|_{p,\infty}\right)^p$ , which converges to 0 as  $\min \mathbf{n}$  goes to infinity.

Since for each  $\mathbf{n}$ ,

$$B^{(m)}(\mathbf{n}) \leq \left( \sum_{\mathbf{l} \neq \mathbf{0}} \left( \|Q_{\mathbf{l}}(f - f^{(m)})\|_{p,\infty}^p |\max\{\mathbf{l}, \mathbf{1}\}|^{p/2-1} \right)^{1/(p+1)} \right)^{p+1} \quad (12.2.85)$$

we obtain by dominated convergence that  $\lim_{m \rightarrow \infty} \sup_{\mathbf{n} \geq \mathbf{1}} B^{(m)}(\mathbf{n}) = 0$ , hence the convergence (12.2.81) holds.

This finishes the proof of asymptotic tightness.





# Perspectives

## Contre-exemples mélangeants

### Processus $\rho$ -mélangeants

Les résultats obtenus dans le chapitre 4 montrent qu'une suite strictement stationnaire,  $\beta$ -mélangeante, vérifiant le théorème limite central et  $\max_{1 \leq i \leq n} |f \circ T^i| / \sqrt{n} \rightarrow 0$  en probabilité ne vérifie pas nécessairement le principe d'invariance. Bien sûr, puisqu'une suite  $\beta$ -mélangeante est  $\alpha$ -mélangeante, ceci reste valable pour les suites  $\alpha$ -mélangeantes. Concernant les suites  $\rho$ -mélangeantes, la question de l'implication du principe d'invariance par le théorème limite central et la convergence vers 0 en probabilité de  $\max_{1 \leq i \leq n} |f \circ T^i| / \sqrt{n}$  semble ouverte. Aucun des contre-exemples construits dans le chapitre 4 n'est  $\rho$ -mélangeant (sinon, la présence de moments d'ordre strictement plus grand que deux donnerait le principe d'invariance).

Les conditions pour le théorème limite central mentionnées dans le Théorème 2.2.7 donnent aussi le principe d'invariance. Il faudrait donc voir s'il existe une condition suffisante pour les suites  $\rho$ -mélangeantes qui garantit seulement le théorème limite central, et pas nécessairement le principe d'invariance.

### Variance linéaire dans le contre-exemple au TLC dans les espaces de Hilbert

Dans chaque contre-exemple construit dans le chapitre 5, nous avons  $\mathbb{E} [\|S_N(\mathbf{f})\|_{\mathcal{H}}^2] / N \rightarrow +\infty$  quand  $N$  tend vers l'infini.

On peut se demander si le Théorème B (respectivement B') reste vrai si on remplace  $\mathbf{b}$  (respectivement B') par

$$\lim_{N \rightarrow \infty} \frac{\sigma_N^2(\mathbf{f})}{N} \text{ existe et est strictement positive.} \quad (12.2.86)$$

## Principe d'invariance dans les espaces hölderiens

### Extension à des modules de régularité plus généraux

Soit  $\rho: [0, 1] \rightarrow \mathbb{R}$  une fonction croissante, nulle et continue à droite en zéro, et strictement positive sur  $]0, 1]$  On peut définir la quantité

$$\omega_\rho(x, \delta) := \sup_{\substack{s, t \in [0, 1] \\ 0 < t - s < \delta}} \frac{|x(t) - x(s)|}{\rho(t - s)} \quad (12.2.87)$$

pour toute fonction  $x: [0, 1] \rightarrow \mathbb{R}$ , ainsi que l'espace Hölderien

$$\mathcal{H}_\rho^\alpha := \left\{ x: [0, 1] \rightarrow \mathbb{R} \mid \lim_{\delta \rightarrow 0} \omega_\rho(x, \delta) = 0 \right\}. \quad (12.2.88)$$

Dans toute la partie III, nous avons étudié la convergence du processus  $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$  dans l'espace  $\mathcal{H}_\rho^\alpha$ , où  $\rho(h) = h^\alpha$  et  $0 < \alpha < 1/2$ . Dans [RS04b], Račkauskas et Suquet ont étudié ce problème dans le cas i.i.d. pour des modules de régularité plus généraux. Ils ont considéré la classe  $\mathcal{R}$  des fonctions  $\rho$  vérifiant

1. pour un certain  $0 < \alpha \leq 1/2$  et une fonction strictement positive  $L$  à variation lente

$$\rho(h) = h^\alpha L(1/h), \quad 0 < h \leq 1; \quad (12.2.89)$$

2.  $\theta: t \mapsto t^{1/2} \rho(1/t)$  est de classe  $C^1$  sur  $[1, +\infty[$ ;
3. il existe  $\beta > 1/2$  et un  $a > 1$  tels que la fonction  $t \mapsto \theta(t)(\ln(t))^{-\beta}$  est croissante sur  $[a, +\infty[$ .

Soit  $(f \circ T^i)_{i \geq 0}$  une suite i.i.d. centrée et  $\rho$  un élément de  $\mathcal{R}$ . Le Théorème 8 de [RS04b] fournit une condition nécessaire et suffisante pour que  $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$  converge en loi vers un mouvement brownien dans l'espace  $\mathcal{H}_\rho^\alpha$  :

$$\text{pour tout } A > 0, \quad \lim_{t \rightarrow +\infty} t \mu \left\{ |f| \geq At^{1/2} \rho \left( \frac{1}{t} \right) \right\} = 0. \quad (12.2.90)$$

En particulier, si  $\rho(h) = h^{1/2} (\log(c/h))^\beta$  avec  $\beta > 1/2$ , la condition (12.2.90) devient

$$\text{pour tout } d > 0, \quad \mathbb{E} \left[ \exp \left( d |f|^{1/\beta} \right) \right] < +\infty. \quad (12.2.91)$$

On peut envisager d'étendre ce résultat à des suites strictement stationnaires non indépendantes.

- *Accroissements de martingales.* On peut essayer de généraliser le Théorème 7.2.2 en remplaçant  $\rho: h \mapsto h^\alpha$  avec  $0 < \alpha < 1/2$  par un élément de la classe  $\mathcal{R}$ . Bien sûr, une condition suffisante doit être plus restrictive (12.2.90). Étant donné que cette condition n'est pas suffisante lorsque  $\rho(h) = h^\alpha$ ,  $0 < \alpha < 1/2$ , il doit être possible de donner un contre-exemple analogue à celui du Théorème 7.2.1 lorsque  $\rho(h) = h^\alpha L(1/h)$  pour  $0 < \alpha < 1/2$ . Il faudra donc faire une hypothèse sur la variance quadratique.

Pour les modules définis par  $\rho(h) = h^{1/2} (\log(c/h))^\beta$  avec  $\beta > 1/2$ , l'adaptation du contre-exemple du Théorème 7.2.1 ne semble pas fonctionner. Il faudrait donc voir dans un premier temps si la condition (12.2.91) est suffisante pour les accroissements d'une martingale.

- *Suites  $\tau$ -dépendantes,  $\alpha$ -mélangeantes.* Dans le Théorème 6.2.1 (respectivement le Corollaire 6.2.2), nous avons obtenu une condition mettant en jeu la fonction de quantile et l'inverse généralisée de la suite des coefficients de  $\tau$ -dépendance (respectivement d' $\alpha$ -mélange). Comme indiqué dans la Remarque 6.2.3, celles-ci redonnent la condition nécessaire et suffisante dans le cas i.i.d. On pourrait chercher une extension de ces conditions aux modules de la classe  $\mathcal{R}$ .
- *Suites  $\rho$ -mélangeantes.* Au vu du Théorème 6.2.5, il est raisonnable d'essayer de remplacer l'hypothèse d'indépendance par la convergence de la série  $\sum_i \rho(2^i)$  dans le Théorème 8 de [RS04b].

### Condition projective de Dedecker-Rio

D'après les résultats de la partie III, le principe d'invariance dans  $\mathcal{H}_{1/2-1/p}^o[0, 1]$  sous des conditions projectives de type Hannan ou Maxwell et Woodrooffe peut s'obtenir en remplaçant la norme  $\mathbb{L}^2$  par la norme  $\mathbb{L}^p$ . Il semble donc raisonnable de conjecturer que  $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$  converge vers  $\eta W$ , où  $\eta$  est une variable aléatoire indépendante de  $W$ , si la condition

$$(f \cdot \mathbb{E}[S_n(f) \mid \mathcal{M}])_{n \geq 1} \text{ converge dans } \mathbb{L}^{p/2} \tag{12.2.92}$$

est vérifiée. Comme on considère  $p > 2$ , (12.2.92) implique (1.2.9) et par conséquent la convergence des lois fini-dimensionnelles de  $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ . Le problème est donc d'établir la tension de la suite  $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$  dans l'espace  $\mathcal{H}_{1/2-1/p}^o[0, 1]$ . Pour cela, une possibilité serait d'établir une inégalité sur les queues des sommes partielles dans l'esprit du Théorème 11.1.1, où le terme  $\max_{1 \leq i \leq n} |X_i|$  serait remplacé par

$$\left( \max_{1 \leq i \leq n} \max_{i \leq l \leq n} \left| X_i \sum_{k=i}^l \mathbb{E}[X_k \mid \mathcal{F}_i] \right| \right)^{1/2}. \tag{12.2.93}$$

Cependant, il est possible qu'il ne soit pas nécessaire de passer par une inégalité sur les queues des sommes partielles, comme ce fut le cas pour la condition de Maxwell et Woodrooffe.

### Variables aléatoires à valeurs dans un espace de dimension infinie

Les résultats de la partie III ne concernent que les variables aléatoires à valeurs réelles.

Le Théorème 1 de l'article [DM06] contient une inégalité de type Fuk-Nagaev pour les suites  $\tau$ -dépendantes à valeurs dans un espace de Banach dit "(2, D)-smooth", voir [Pis75] pour la définition. Donc on peut obtenir un résultat semblable à celui obtenu dans le Théorème 6.2.1 pour des suites à valeurs dans de tels espaces de Banach (dont les espaces de Hilbert séparables font partie).

La question des martingales semble plus délicate. L'inégalité obtenue au Lemme 4.2 de [Pin94] (pour les espaces de Banach "(2, D)-smooth") permet de répondre à la question lorsque les accroissements  $m \circ T^j$  ont une loi conditionnelle symétrique par rapport à  $T^{-(j-1)}\mathcal{M}$ . Cependant, une généralisation du Théorème 7.2.2 à de tels espaces de Banach constitue un problème ouvert.

## Champs aléatoires

### Cas non-adapté de la décomposition orthomartingale-cobord

Notre résultat concernant la décomposition orthomartingale-cobord ne s'applique que si la fonction  $f$  est  $\mathcal{M}$ -mesurable. En dimension un, il est possible de formuler une condition nécessaire et suffisante pour la décomposition martingale-cobord (voir le Théorème 1.2.6) pour les suites régulières non-nécessairement adaptées. On peut essayer d'étendre la condition (10.2.2) au cas où  $f$  n'est pas nécessairement  $\mathcal{M}$ -mesurable. Même en cherchant une condition "avec la norme à l'intérieur", savoir ce qui joue le rôle des termes  $\|f - \mathbb{E}[f \mid T^{-k}\mathcal{M}]\|_p$  en dimension supérieure à deux n'est pas clair. On peut définir les opérateurs  $\mathbb{E}_{\mathbf{k}}(f) := \mathbb{E}[f \mid T^{\mathbf{k}}\mathcal{M}]$  pour  $k \in \mathbb{Z}^d$  et conjecturer que la condition

$$\text{pour tout } J \subset \langle d \rangle, \sum_{k \in \mathbb{N}^d} \left\| \prod_{j \in J} E_{k_j \mathbf{e}_j} \prod_{l \in \langle d \rangle \setminus J} (I - E_{k_l \mathbf{e}_l}) f \right\|_p < +\infty \tag{12.2.94}$$

est suffisante pour la décomposition (10.2.3) (sans la condition de  $\prod_{j \in J} T_j \mathcal{M}$ -mesurabilité de  $m_J$ ).

## Théorème limite central pour les rectangles

Lorsque la décomposition (10.2.3) a lieu dans  $\mathbb{L}^2$ , si l'une des transformation  $T_i$  est ergodique, alors le théorème limite central a lieu. Ceci peut aussi être déduit de la condition de Hannan (3.3.5) qui est moins restrictive que (10.2.2). En dimension 1, il est possible de déduire le théorème limite central à partir d'une décomposition martingale cobord dans  $\mathbb{L}^1$  (sans avoir la condition de Hannan). Pour cela, il suffit que la quantité  $\liminf_{n \rightarrow \infty} n^{-1/2} \mathbb{E} |S_n(f)|$  soit finie (voir le Théorème 1.2.7) afin de garantir l'appartenance à  $\mathbb{L}^2$  de la martingale. On aimerait obtenir un résultat similaire pour la dimension  $d$ . Plaçons-nous en dimension 2, et supposons que la condition (10.2.2) soit vérifiée pour  $p = 1$ . Si les transformation  $T_1$  et  $T_2$  sont ergodiques et la quantité  $\liminf_{n \rightarrow \infty} n^{-1/2} \mathbb{E} |S_n(T_i, f)|$  est finie pour  $i \in \{1, 2\}$  alors on peut déduire l'appartenance à  $\mathbb{L}^2$  des fonctions  $m + (I - U_1)m_1$  et  $m + (I - U_2)m_2$  dans (10.2.4). Il faudrait trouver une condition supplémentaire sur  $f$  pour garantir l'appartenance à  $\mathbb{L}^2$  de la fonction  $m$ .

## Approximation par orthomartingales pour le principe d'invariance

En dimension un, l'introduction de  $\|\cdot\|_+$  (voir (1.3.15)) permet de donner une condition nécessaire et suffisante pour pouvoir approximer les sommes partielles par une martingale à accroissement strictement stationnaires. Celle-ci peut se vérifier sous les conditions de type Hannan, Dedecker-Rio ou Maxwell-Woodroffe. Dans l'optique d'obtenir un principe d'invariance dans l'espace des fonctions continues sur  $[0, 1]^d$  pour les champs aléatoires strictement stationnaires, une approche pourrait consister à chercher un résultat analogue à celui de [GP11] en dimension  $d$ . Si on suppose que l'une des transformations parmi  $T_i$ ,  $i \in \{1, \dots, d\}$ , est ergodique, ceci permettrait d'obtenir le principe d'invariance.

## Loi des logarithmes itérés

La question de la loi des logarithmes itérés pour des champs i.i.d. a été étudiée par Wichura [Wic73]. Pour  $d \geq 2$ , la condition

$$\mathbb{E} \left[ f^2 \frac{(\log^+ |f|)^{d-1}}{\log^+ \log^+ |f|} \right] < +\infty \quad (12.2.95)$$

implique que

$$\limsup_{\mathbf{n} \rightarrow +\infty} \frac{|S_{\mathbf{n}}(f)|}{\sqrt{|\mathbf{n}| \log \log |\mathbf{n}|}} = \sqrt{d} (\mathbb{E} [f^2])^{1/2}. \quad (12.2.96)$$

Dans [LL07], d'autres normalisations que  $\sqrt{|\mathbf{n}| \log \log |\mathbf{n}|}$  ont été étudiées. Dans [Jia99], Jiang a traité le cas des champs aléatoire de type ortho-martingale en dimension deux. Ses résultats exigent la finitude de moments d'ordre strictement plus grand que deux. Il est donc naturel d'essayer de voir si c'est nécessaire en dimension deux, et d'essayer de formuler une extension aux dimensions arbitraires.

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**Théorèmes limites de la théorie des probabilités dans les systèmes dynamiques**  
— **Limit theorems of probability theory in dynamical systems**

**Résumé**

Cette thèse se consacre aux théorèmes limites pour les suites et les champs aléatoires strictement stationnaires, principalement sur le théorème limite central et sa version fonctionnelle, appelée principe d'invariance.

Dans un premier temps, nous montrons à l'aide d'un contre-exemple que pour les processus strictement stationnaires  $\beta$ -mélangeants, le théorème limite central peut avoir lieu sans que ce ne soit le cas pour la version fonctionnelle. Nous montrons également que le théorème limite central n'a pas nécessairement lieu pour les sommes partielles d'une suite strictement stationnaire  $\beta$ -mélangeante à valeurs dans un espace de Hilbert de dimension infinie, même en supposant l'uniforme intégrabilité de la suite des sommes partielles normalisées.

Puis nous étudions le principe d'invariance dans l'espace des fonctions hölderiennes. Nous traitons le cas des suites strictement stationnaires  $\tau$ -dépendantes (au sens de Dedecker, Prieur, 2005) ou  $\rho$ -mélangeantes. Nous donnons également une condition suffisante sur la loi d'une suite strictement stationnaire d'accroissements d'une martingale et la variance quadratique garantissant le principe d'invariance dans l'espace des fonctions hölderiennes, et nous démontrons son optimalité à l'aide d'un contre-exemple. Ensuite, nous déduisons grâce à une approximation par martingales des conditions dans l'esprit de celles de Hannan (1979), et Maxwell et Woodrooffe (2000).

Nous discutons ensuite de la décomposition martingale/cobord. Dans le cas des suites, nous fournissons des conditions d'intégrabilité sur la fonction de transfert et le cobord pour que ce dernier ne perturbe pas le principe d'invariance, la loi des logarithmes itérés ou bien la loi forte des grands nombres si ceux-ci ont lieu pour la martingale issue de la décomposition. Dans le cas des champs, nous formulons une condition suffisante pour une décomposition ortho-martingale/cobord.

Enfin, nous établissons des inégalités sur les queues des maxima des sommes partielles d'un champ aléatoire de type ortho-martingale ou bien d'un champ qui s'exprime comme une fonctionnelle d'un champ i.i.d. Ces inégalités permettent d'obtenir un principe d'invariance dans les espaces hölderiens pour ces champs aléatoires.

**Abstract**

This thesis is devoted to limit theorems for strictly stationary sequences and random fields. We concentrate essentially on the central limit theorem and its invariance principle.

First, we show with the help of a counter-example that for a strictly stationary absolutely regular sequence, the central limit theorem may hold but not the invariance principle. We also show that the central limit theorem does not take place for partial sums of a Hilbert space valued, strictly stationary and absolutely regular sequence, even if we assume that the normalized partial sums form a uniformly integrable family.

Second, we investigate the Holderian invariance principle. We treat the case of  $\tau$ -dependent (Dedecker, Prieur, 2005) and  $\rho$ -mixing strictly stationary sequences. We provide a sufficient condition on the law of a strictly stationary martingale difference sequence and the quadratic variance which guarantee the invariance principle in a Hölder space. We construct a counter-example which shows its sharpness. We derive conditions in the spirit of Hannan (1979), and Maxwell and Woodrooffe (2000) by a martingale approximation.

We then discuss the martingale/coboundary decomposition. In dimension one, we provide sharp integrability conditions on the transfer function and the coboundary for which the latter does not spoil the invariance principle, the law of the iterated logarithm or the strong law of large numbers if these theorems take place for the martingale involved in the decomposition. We

also provide a sufficient condition for an orthomartingale/coboundary decomposition for strictly stationary random fields.

Lastly, we establish tails inequalities for orthomartingale and Bernoulli random fields. We prove an invariance principle in Hölder spaces for these random fields using such inequalities.